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DESIGN RESEARCH IN MATHEMATICS EDUCATION: ON MATHEMATICAL MODELLING AND PROBLEM-POSING

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Preface

In 2017, after my Master graduation in Mathematics, I attended a *Post Graduate Course in Methodology and Didactics of Mathematics*, organized by the Department of Mathematics “Tullio Levi-Civita”, University of Padova. In the same year I started teaching in a secondary high school. These experiences modified my view on the learning of mathematics and on mathematics as a discipline, from an abstract language with hardly any connections to the world around us, to a language which emerges from organizing phenomena and restructuring through reflection and generalization. After one year, I had the opportunity to start my PhD in mathematics in the area of Mathematics Education.

This thesis has one author, although the work is the result of many discussions with colleagues and relatives within and outside the Department. Unfortunately, it is not possible to thank all these persons, so let the following words of gratitude be dedicated to all of them.

First of all, I am grateful to have had the opportunity to carry out the research at the Department of Mathematics “Tullio Levi-Civita” of the University of Padova. It is the place where I moved my first steps as a mathematician, and where my interests in mathematics and its teaching and learning born. An inspiring Department with a multidisciplinary team of mathematicians. I thank all my colleagues for their support during the research.

Secondly, I want to express my gratitude to the teachers and students who were involved in the experiments. This study could not have been accomplished without their willingness to invest time in an alternative approach of teaching and learning. Special thanks go to the teachers Marzia Baroni, Alessandra Borgia, Antonella Bosatta, Tiziana Corso, Anna Gobitti, Maria Giovanna Idone, for their thinking along with the instructional design and their role in the teaching experiments.

Finally, I thank my supervisor prof. Cinzia Bonotto, who gave me the opportunity to start this journey in the field of Mathematics Education.

Giulia, you helped me in various ways and I am grateful for your very existence in my life. That is why I dedicate this book to you.

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1. Introduction

1.1 Reasons for the study

There is a strong discontinuity between in- and out-of- school mathematical competences, discontinuity that can be found in all school levels. Several researches identified one of the causes of this fracture in the stereotyped nature of the problems proposed by textbooks which, rather than serving as an interface between mathematics and reality, promote in students an exclusion of realistic considerations and a *suspension of sense-making* (Schoenfeld 1991); see Verschaffel, Greer and De Corte (2000) for an overview of these studies. In this situation, teachers' expectations (Bonotto 2007; Gravemeijer 1997) and their beliefs about the aims of mathematics education (Verschaffel, De Corte and Borghart 1997) often have a decisive role. Indeed, teachers usually identify mathematical problems with exercises in the four basic operations solved in a mechanical way, and not as catalyst instruments for a process of mathematization of reality, as instead desired by national and international curricula (DM 254/2012; NCTM 2000). Mathematical problems turned in stereotyped exercises in the four basic operations solved through the application of mechanical procedures, and students themselves seem to have established a set of rules of which include: i) any problem is solvable and makes sense; ii) there is a single, correct and precise (numerical) answer which must be obtained by performing one or more arithmetical operations with numbers given in the text; iii) violations of personal knowledge about the everyday would may be ignored (Greer, Verschaffel and Mukhopadhyay 2007). Those implicit teachers' expectations and students' rules are part of the construct of the *didactic contract* (Brousseau 1980). Indeed, the didactic contract regulates classroom activities, influencing both the teacher's behavior and students' learning processes. More than twenty years after the mentioned studies, current research confirms their results and they extend their validity also to the case of teachers in training (Bonotto and Passarella 2019b).

One of the main consequences of this situation is an increasing gap between mathematics and real-world (Gravemeijer 1997). Instead, realistic and less stereotyped problems that take into consideration the experiential world of students must be inserted in the school practice, in order to create a bridge between mathematics classroom activities and everyday-life experiences. In fact, encouraging students to relate mathematical problems with real-world scenarios may help them more closely associate mathematics with their everyday activities (De Corte, Verschaffel and Greer 2000). According to the Realistic Mathematics Education (RME) perspective, a connection between mathematics and reality in order to improve students' critical thinking and reasoning should be fostered with activities based on *realistic* and *rich* contexts (Gravemeijer and Doorman 1999). The teaching of mathematics might be seen as a human activity of *guided reinvention* (Freudenthal 1991), in which students are active participants in the learning process, in a balance between students' freedom of invention and the power of teacher's guidance.

In this direction, mathematical modelling and problem-posing could represent powerful educational strategies to improve the teaching of mathematics in a guided reinvention approach, offering students opportunities to attach meaning to the mathematical constructs they develop while solving problems. A modelling perspective, in fact, provides basic arguments for including authentic situations in the mathematics classroom (Maass, Doorman, Jonker and Wijers 2019) and represents a critical tool to understand the reality or society in general. Teaching students to interpret critically the communities they live in and to understand its codes and messages should be an important goal for education (Bonotto 2007), in order to give students not only mathematical competencies but also to prepare them to situations they will have to face in an increasingly complex world. Mathematical modelling naturally cooperates with problem-posing, which can be seen as the process by which students generate their own problems in addition to solving pre-formulated problems (English 1997; NCTM 2000; Silver and Cai 1996). Allowing students to write their own mathematical problems may help them to make connections between mathematics in the classroom and their real life (Kopparla et al. 2018). In conclusion, modelling and problem-posing should represent valuable strategies to support students in give sense to their mathematical activity filling the gap between in- and out-of-school mathematical competencies and experiences.

1.2 Context for the study

In the previous section we outlined the necessity of a paradigmatic change in the didactics of mathematics that aims to build a bridge between reality, in which intuition plays a fundamental role, and school life, in which exercise and memorization continue to play an important role. Despite the specificities of both in- and out-of-school contexts, it is believed that those conditions that make real-life learning more effective should be recreated in mathematics classrooms (Bonotto 2005).

> The idea is not only to motivate students with everyday-life contexts but also to look for contexts that are experientially real for the students as starting points for progressive mathematization. (Gravemeijer 1999, p. 162)

This process of mathematization of reality is desired also by national and international curricula. In the Italian context, the National Indications for the First Cycle of Education (DM 254/2012), emphasize how mathematical knowledge should offer skills for perceiving, interpreting and linking artifacts and daily-life events. Furthermore, students are required to analyze situations and to translate them into mathematical terms, recognize recurrent patterns, establish analogies with known models, choose the actions to be performed and chain them effectively to produce a solution to the problem. These indications are reflected in the European context. On May 22, 2018, the European Council reiterated the key competences for lifelong learning (2018/C189/01), already presented in the 2006 document (2006/962/EC). Among these competences, *mathematical competence* appears, seen as the ability to develop and apply mathematical thinking and the ability to use different representations (formulas, constructs, models, …) to solve problems starting from everyday situations, with particular attention to the process and activity, as well as knowledge.

In Italian schools, despite some experiences of innovation and reflection on the curriculum, teaching strategies and learning environments, still persists a resistance to abandoning traditional teaching models of transmission type (INNS 2017).

Instead, following both the European recommendations and the Italian National Indications, a valid methodology to reduce school and extra-school mathematical competences is that of modeling. Modeling seen not only as a process of solving real problems, but as a possibility to achieve a process of mathematization and reflection on mathematics that leads to the construction of new mathematical concepts and tools. As a consequence, mathematics teachers should become able of recognizing the mathematics incorporated in daily life. This requires knowing how to integrate pedagogical-didactic and disciplinary knowledge together, paying attention also to the particular school context and the cultural environment in which operating. This complex request is high demanding for teachers who must be supported in a continuous training.

This study is part of a research project of the University of Padova, concerning teachers' professional development. In the specific area of mathematics, the overall purpose is to provide mathematics teachers with methodological models and format of school practices based on mathematical modelling. This purpose is outlined in the following points:

- implementing some teaching experiments wherein connecting mathematics and daily-life experiences;

- start developing prototypes of significant didactics practices based on mathematical modelling ready to be transferred and implemented in different concrete school contexts;

- developing specific models for professional development courses based on mathematical modelling for mathematics teachers of every school level.

1.3 The Italian education system

In this section we remark some facts concerning the Italian education system and initial teachers' training programs.

1.3.1 The Italian education system

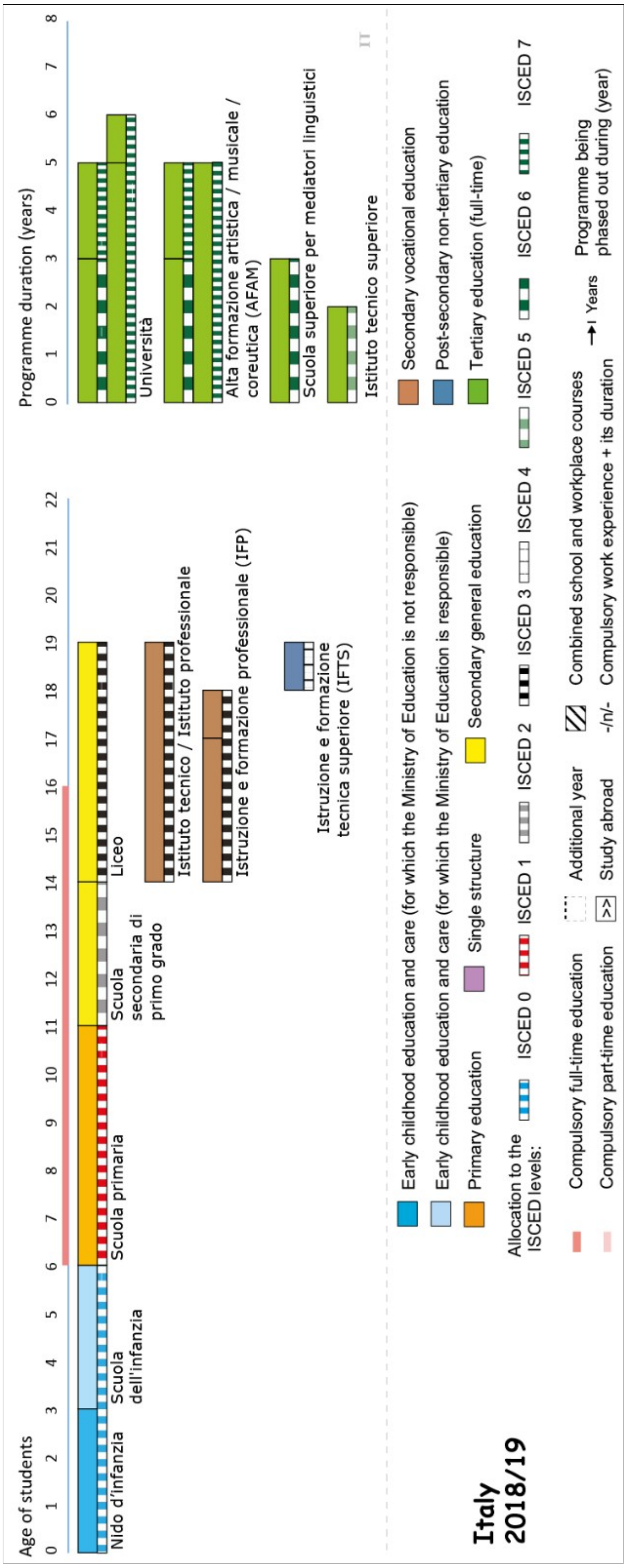
The Italian education system (Eurydice 2020a) is mainly a public State system. However, private subjects and public bodies can establish education institutions. The State competences concern exclusively legislative competences on the general organization of the education system, such as minimum standards of education, school staff, quality assurance, State financial resources.

Schools have a high degree of autonomy: they define curricula, widen the educational offer, organize teaching (school time and groups of pupils). Every three years, schools draw up their own *three-years educational offer plan* (*PTOF*). At higher education level, universities and institutions of Higher education for the fine arts, music and dance (*AFAM*) have statutory, regulatory, teaching and organizational autonomy.

Education at all levels must be open to everyone: Italian citizens as well as foreigner minors from both EU and non-EU countries. Compulsory education is free. The principle of inclusion also applies to pupils with disabilities, to pupils with social and economic disadvantages and to immigrant pupils. In such circumstances, measures focus on personalization and didactic flexibility and, in the case of immigrants with low levels of Italian, on linguistic support. The State also guarantees the right to education to students who are unable to attend school because hospitalized, detained or at home for a long illness.

The Italian education system includes the following five stages (Fig. 1):

- early childhood education and care (ECEC);
- primary education;
- secondary education;
- post-secondary education;
- higher education.



Italy
2018/19
Age of students
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22
Nido d'infanzia
Scuola dell'infanzia
Scuola primaria
Scuola secondaria di primo grado
Liceo
Istituto tecnico / Istituto professionale
Istruzione e formazione professionale (IFP)
Istruzione e formazione tecnica superiore (IFTS)
Programme duration (years)
0 1 2 3 4 5 6 7 8
Università
Alta formazione artistica / musicale / coreutica (AFAM)
Scuola superiore per mediatori linguistici
Istituto tecnico superiore
IT
Early childhood education and care (for which the Ministry of Education is not responsible)
Early childhood education and care (for which the Ministry of Education is responsible)
Primary education
Single structure
Secondary general education
Secondary vocational education
Post-secondary non-tertiary education
Tertiary education (full-time)
Allocation to the ISCED levels:
ISCED 0
ISCED 1
ISCED 2
ISCED 3
ISCED 4
ISCED 5
ISCED 6
ISCED 7
Compulsory full-time education
Compulsory part-time education
Additional year
Study abroad
Combined school and workplace courses
-/n/- Compulsory work experience + its duration
Years
Programme being phased out during (year)

Figure 1. The Italian education system. Source: Eurydice 2018/19

Education is compulsory for ten years, between 6 and 16 years of age, covering three levels of the education system: five years of primary education, three years of lower secondary education and the first two years of upper secondary education (law 296/2006). The final two years of compulsory education (from 14 to 16 years of age) can be undertaken through two different paths: the State upper secondary education (*liceo*, *technical institute* or *vocational institute*) or the regional vocational education and training system (law 133/2008). In addition to compulsory education, everyone has a right and a duty to receive education and training for at least 12 years within the education system or until they have obtained a three-years vocational qualification by the age of 18 (law 53/2003). Finally, 15-year-olds can also spend the last year of compulsory education on an apprenticeship, upon a specific arrangement between the Regions, the Ministry of Laboure, the Ministry of Education and trade unions (law 183/2010). Once they have completed compulsory education, those who do not continue with their studies receive a certificate of completion of compulsory education that also describes the skills they have acquired.

Early childhood education and care (ECEC)

ECEC is divided in two stages based on child age groups: 0-3 years and 3-6 years. ECEC for children aged less than 3 years is offered by *educational services* while ECEC for children aged from 3 to 6 years is available at *preprimary schools*. The two offers make up a single ECEC system, called *integrated system*, which is part of the education system and is not compulsory. Although being part of the same system, the ECEC 0-3 is organized by the Regions according to the single regional legislations, while the 3-6 offer is under the responsibility of the Ministry of Education.

Educational guidelines for this ECEC phase are published at central level and are included in the guidelines that apply to the first cycle of education (DM 254/2012).

Primary education

The compulsory first cycle of education is made up of primary and lower secondary education and its total length is 8 years. Comprehensive institutes group primary schools, lower secondary schools and pre-primary schools managed by a single school manager. The purpose of comprehensive institutes is to assure didactic continuity within the same cycle of education. However, although being part of the same cycle, primary and lower secondary education are considered separate levels of education with their own specificities.

Primary education is organized at *primary schools*. Primary education is compulsory, has an overall length of 5 years and is attended by pupils aged 6 to 11. The aim of primary education is to provide pupils with basic learning and the basic tools of active citizenship. It helps pupils to understand the meaning of their own experiences. Educational guidelines for primary education are published at central level and are included in the guidelines that apply to the first cycle of education (DM 254/2012).

Lower secondary education

The lower secondary level of education is organized at *first-level secondary schools*. Lower secondary education is compulsory, lasts for 3 years and is attended by pupils aged 11 to 14 years. Lower secondary school aims at fostering the ability to study autonomously and at strengthening the pupils' attitudes towards social interaction, at organizing and increasing knowledge and skills and at providing students with adequate instruments to continue their education and training activities. Educational guidelines for lower secondary education are published at central level and are included in the guidelines that apply to the first cycle of education (DM 254/2012).

Within the first cycle, students pass from one level to the next one without exams. At the end of the first cycle of education, students who pass the final state examination progress directly to the second cycle of education, the first two years of which are compulsory.

Upper secondary education

The second cycle of education is made up of two parallel paths:

i. State upper secondary education called *second-level upper secondary school;*
ii. *vocational education and training system* (IFP) organized at regional level.

Second level upper secondary schools offer general, technical and vocational education. The total length of studies at upper secondary level is 5 years (from 14 to 19 years of age).

The general path is organized at *licei*. It aims at preparing students to higher-level studies and to the labour world. It provides students with adequate competences and knowledge, as well as cultural and methodological instruments for developing their own critical and planning attitude.

Technical education is organized at *technical institutes* (*istituti tecnici*). It provides students with a strong scientific and technological background in the economic and technological professional sectors.

Vocational education is organized at *vocational institute* (*istituti professionali*). It provides students with a strong technical and vocational general background in the sectors of services, industry and handicraft, to facilitate access to the labour world.

At the end of upper secondary education students receive a certification that gives access to university, to the Higher education for the fine arts, music and dance (*Alta formazione artistica e musicale,* AFAM) and to the Higher technical education and training (ITS).

Educational guidelines for upper secondary education are published at central level and are differentiated for each specific path.

Regional vocational education and training

Regional vocational education and training (IFP) is organized into three and four-year courses. Courses can be organized by both accredited local training agencies and by vocational upper secondary schools in partnership with training agencies. The main characteristic of courses is a wider use of laboratories and of periods of work experiences. The aim is to faster access to the job market. At the end of courses, learners receive a vocational qualification that gives access to the second-level regional courses or, in case of the four-year programs and at certain conditions, to tertiary education.

Post-secondary non-tertiary education

The post-secondary non-tertiary level offers courses within the Higher technical education and training system (IFTS) and within the vocational training system managed by the Regions.

The Higher technical education and training system (IFTS) aims at developing professional specializations at post-secondary level that meet the requirements of the labour market, both in the public and private sectors. The Regions organize short vocational training courses (400-800 hours) addressed to those who hold a qualification obtained either in the regional or in the State vocational training system. They are also called *second-level vocational training courses*.

Higher education

The following types of institution offer higher education:

i. Universities and equivalent institutions;
ii. Institutes of Higher Education for the fine arts, music and dance (AFAM);
iii. Higher Technical Institutes (ITSs).

Universities and AFAM institutes offer programs of the first, second and third cycle according to the Bologna structure and issue the relevant qualifications. First and second-cycle courses at universities lead to qualifications called *laurea* (bachelor) and *laurea magistrale* (master). AFAM institutes release qualifications called *diploma accademico di primo livello* and *diploma accademico di secondo livello*. In addition, Universities and AFAM institutes organize courses leading to qualifications outside the Bologna structure. ITSs are highly specialized technical schools that offer short-cycle programs in the technical and technological sectors. In general, courses last 4 semesters and lead to the qualification of *higher technician* (*diploma tecnico superiore*).

Universities issue the following qualifications, corresponding to the Bologna Process structure (cycles):

i. bachelor's degree, corresponding to a first-cycle qualification (180 credits-CFU);
ii. master's degree, corresponding to a second-cycle qualification, issued at the end of a two-years course of study (120 credits - CFU) or to a 5-6-years single course (300-360 credits - CFU);
iii. PhD, corresponding to a third-cycle qualification.

In addition, universities may organize courses leading to the following qualifications: *first-level University masters* (addressed to holders of a Bachelor's degree and lead to a second-cycle qualification outside the Bachelor and Master structure); *specialization diploma* and *second-level university master* (addressed to holders of a Master's degree and lead to a third-cycle qualification outside the Bachelor and Master structure).

1.3.2 Initial teacher training

Initial teacher training for teachers of the pre-primary and primary levels and for teachers of the secondary level is organized differently (Eurydice 2020b). Both programs also aim at the acquisition of competences on ICT, languages (English language at least corresponding to the B2 level within the European common framework of references for languages) and didactic competences to help the integration at school of pupils with special educational needs.

Pre-primary and primary levels

Teachers of the pre-primary and primary levels obtain the second-cycle qualification after completion of a specific five-years single-cycle program, including traineeship activities. Admission to courses requires also the possession of an upper secondary qualification or any other equivalent qualification obtained abroad. Courses provide teachers with the necessary subject related competences and with the ability of adapting their teaching to different age groups and cultures and planning their teaching activities.

Programs are organized in general and specific training activities. The former aim at the acquisition of knowledge in the fields of pedagogy, didactic, psychology, sociology and anthropology. These studies correspond to 78 CFU credits. Specific activities aim at both the acquisition of subject-related knowledge and competences and the integration of pupils with special educational needs. These latter include studies in the fields of infantile neuropsychiatry, psychology, law and health. Studies correspond to 31 CFU credits. Future teachers acquire competences in the following subject areas: mathematics, physics, chemistry, biology, Italian language and literature, English language, history, geography, sports, arts, music, children's literature. Studies correspond to 135 CFU credits. The remaining 56 CFU credits come from traineeship activities, laboratories, the language qualification and the final exam. Traineeship activities are carried out starting from the second year of studies for a total duration of 600 hours (24 CFU credits). Courses end up with the discussion of a final work and of the final traineeship report. The discussion of the two reports makes up the final exam that also qualifies to teach at pre-primary and primary levels. Courses under this procedure started in academic year 2011/2012. At the end of the single-cycle University programs for teaching at pre-primary and primary level, successful students are awarded the master's degree (*laurea magistrale*) in primary education sciences.

Secondary level

Starting from 2018, education for teaching in secondary schools is organized into one single system that includes both initial education and access to the teaching post. All secondary teachers start their initial education by getting through an open competitive examination. To access the examination, candidates must hold:

- a master's degree from a University or AFAM qualification, or any other equivalent qualification;
- 24 CFU/CFA credits (equivalent to 24 ECTS), acquired either within or in addition to the main course of study in the field of anthropology, psychology, pedagogy and teaching methodology. At least 6 of the 24 credits must be

acquired for each of three among the following four sectors: pedagogy, special pedagogy and inclusion; psychology; anthropology; teaching methods.

Those who successfully pass the examination, start a three-years traineeship that includes both theoretical education, practical training and access to a post as teacher. Those who have received positive results in their periodic and final assessments during the three-years period of training become permanent-contract teachers.

All courses for training future teachers include:

- the acquisition of linguistic competences in English equivalent to the level B2 of the Common European Framework of Reference for Languages adopted in 1996 by the Council of Europe;

- the acquisition of digital competences as foreseen by the Recommendation of the European Parliament and Council of 18 December 2006. In particular, such competences refer to the capacity of using multimedia languages for representing and communicating knowledge, for using digital contents and, more in general, for using simulated environments and virtual labs;

- the acquisition of teaching competences suitable to favor the school integration of pupils with disabilities.

The final assessment of the traineeship takes into account the level of development of professional competences in relation with the methodological, didactical, relational and project-related aspects both in the class and in the school.

Under the previous legislation, teachers of the lower and upper secondary levels were required to hold a master's degree or AFAM qualification followed by a one-year traineeship period called *active formative traineeship* (TFA). Completion of this traineeship gave access to the open competitive examinations held to recruit and appoint new teachers.

1.4 Overview of the thesis

This thesis starts with discussing the theoretical background of the project. In chapter 2 the theory of Realistic Mathematics Education is recalled, that represents the instruction theory underlying the entire research project. Then, the main literature concerning mathematical modelling and problem-posing is reported. Attention is given also to real contexts for mathematics lessons. This discussion permits to formulate and specify the research questions, that are formulated at the end of the chapter.

In chapter 3 the research methodology to answer the research questions is described. The focus is on *design research*. This methodology is characterized by research cycles made by a design phase, a teaching experiment and a retrospective analysis. In our case, we developed two research cycles concerning the first research question about mathematical modelling, and two research cycles concerning the second research question about problem-posing.

Chapter 4 describes an exploratory study made before the design and implementation of the design research cycles. This exploratory study consisted in a questionnaire for mathematics teachers of primary and secondary school, and its aim was to have a first overview about the use and knowledge of modelling and problem-posing in the Italian context.

Chapters 5 to 8 describe the design research cycles. Each chapter is made by three main sections: design phase, teaching experiment and retrospective analysis.

In chapter 9 conclusions with respect the research questions are described. In addition, some recommendations about modelling and problem-posing in mathematics education are drawn.

2. Background and Research Questions

In this chapter we focus on the literature underlying the thesis. We start focusing on the theory of Realistic Mathematics Education (RME). The main characteristic of this theory is that mathematics is seen as an active process of mathematization. The core principles of RME will be described, together with the design heuristics of *guided reinvention*, *didactical phenomenology* and *emergent models*. In the second section we focus on mathematical modelling. Starting from its historical development, we report the main trends on the teaching and learning of mathematical modelling. Particular attention is given to two approaches: *emergent modelling* and *model eliciting*. In section three we focus on problem-posing, that represents an educational strategy directly linked to modelling. In particular, we explain what we mean with problem-posing and describe its relations with problem solving and creativity, and some schemes to assess students' and teachers' problem-posing performances. For the implementation of both modelling and problem-posing activities, the role of contexts for mathematical problems is fundamental. For this reason, in section four we describe two perspectives that define what is meant by a real context for a mathematical task: the perspective of RME, in which *realisti*c and *rich* contexts play a prominent role, and Palm's framework (2006) for real-life mathematical situations. The discussion about the literature permitted to formulate the research questions of this research, that are formulated at the end of this chapter.

2.1 Realistic Mathematics Education

Realistic Mathematics Education (RME) is a domain specific instruction theory for mathematics that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing instructional materials for mathematics education. RME was firstly developed by the Freudenthal Institute for Mathematics and Science Education of Utrecht as reaction to the limitations of a mechanistic and structuralist approach to mathematics education. Rich and realistic situations are given a prominent position in the learning process and represent a starting point for the development of mathematical concepts and applications. Realistic refers to problem situations that students can image and that are, at a certain stage, meaningful for them. Therefore, problems can come from the real world, but also from a fantasy world or from the formal world of mathematics, as long as the problems are experientially real in students' mind (Van den Heuvel-Panhuizen and Drijvers 2014). Students can experience an abstract mathematical problem as real when the mathematics of that problem is meaningful to them. Freudenthal's (1991) ideal was that mathematical learning should be an enhancement of common sense. Students should be allowed and encouraged to invent their own strategies and ideas, and they should learn mathematics on their own authority. At the same time, this process should lead to particular end goals. This raises the question that underlies much of the RME-based research, namely that of how to support this process of engaging students in meaningful mathematical problem solving and using students' contributions to reach certain end goals.

2.1.1 Core principles of RME

The core principles of RME were articulated originally by Treffers (1978) but were reformulated over the years (for instance Freudenthal (1983, 1991); Treffers 1987; De Lange 1987; Gravemeijer 1994; Van den Heuvel-Panhuizen 1996; Drijvers 2003). These tenets can be synthetized in six educational principles (see also Van den Heuvel-Panhuizen and Drijvers 2014):

activity principle students are active participants in the learning process, developing mathematical tools and insights by themselves, rather than being receivers of ready-made mathematics. Mathematics has not to be learned as a closed system, but as a human activity (Freudenthal 1991) of mathematizing reality and if possible, even that of mathematizing mathematics (Treffers 1987);

reality principle students should become able to apply mathematics in solving real-life problems. Mathematics education should start from realistic and rich contexts, i.e. problem situations that are meaningful to students and that offer them opportunities to attach meaning to the mathematical constructs they develop while solving problems. Teaching begins offering students the opportunity to face with contexts that can be mathematized. As a consequence, informal contexts represent a first step in the learning process, wherein students can develop their first strategies to solve a problem;

level principle students pass various levels of understanding in their learning process: from informal context-related solutions to acquiring insights into how concepts and strategies are related. Fundamental tools for bridging the gap between the informal, context-related mathematics and the more formal mathematics are models;

intertwinement pr. mathematical content domains must be heavily integrated and not considered as isolated chapters;

interactivity pr. learning mathematics is also a social activity. Whole class discussion and group work should be favoured, since they offer students the opportunity to share, reflect on and improve their strategies, reaching a higher level of understanding;

guidance principle teachers should have a pro-active role in students' learning, and educational programs should contain scenarios which have the potential to work as a lever to reach shifts in students' understanding.

2.1.2 Heuristics design of RME

Based on the core principles presented in the previous section, RME also offers heuristics design in mathematics education: *guided reinvention*, *didactical phenomenology*, and *emergent models* (Gravemeijer 1994a).

Guided reinvention

As advocated in the activity principle, the teaching of mathematics should be a human activity as opposed to a ready-made system (Freudenthal 1973; 1991). When students progressively mathematize their own mathematical activity (Treffers 1987) they can reinvent mathematics under the guidance of the teacher and the instructional design. This is the meaning of the first heuristic, *guided reinvention*: students should experience the learning of mathematics as a process similar to the process by which mathematics was invented (Gravemeijer 1994a). Consequently, the role of the designer is fundamental in RME, since she/he has to foster this process of guided reinvention. In so doing, the designer can use different methods: (i) *thought experiments,* in which the designer thinks of how she/he could have reinvented the mathematics at issue themselves; (ii) study the *history* of the topic at issue; (iii) use students' informal solution strategies as a source: could teachers and designers support students' solutions in getting closer to the end goal? (Bakker 2004).

Didactical phenomenology

Freudenthal (1983) distinguished thought objects (*nooumena*) and phenomena (*phainomena*). Mathematical concepts and tools serve to organize phenomena, both from daily life and from mathematics itself (Bakker 2004). A phenomenology of a mathematical concept is an analysis of that concept in relation to the phenomena it organizes. One of the possible ways to do this is offered by *didactical phenomenology*.

Didactical phenomenology is the study of concepts in relation to phenomena with a didactical interest. In this perspective the challenge is to find phenomena that beg to be organized by the concepts that are to be taught (Freudenthal 1983). In this research, the design of instructional materials followed a didactical phenomenology approach, in which the goal was to find problem situations that could provide the basis for the development of the mathematical concepts or tools we wanted students to develop. Such problem situations could lead to solutions that are first specific for that situation but can be generalized to other problem situations.

Emergent models

In the level principle it is suggested that fundamental tools for bridging the gap between the informal, context-related mathematics and the more formal mathematics are models. Indeed, in RME a *model of* a certain situation can become a *model for* more formal reasoning (Gravemeijer 1994a; 1999). The movement from situational to formal reasoning is described by the four levels in Fig. 2 (Gravemeijer, Cobb, Bowers, and Whitenack 2000):

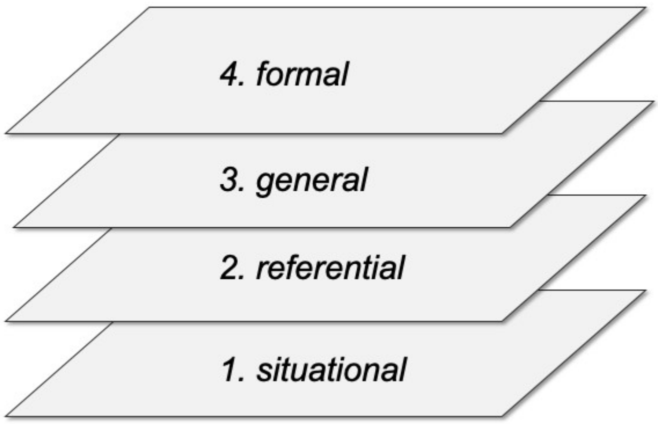


Figure 2. Movement from situational to formal reasoning

1. *Situational level:* activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting (often in out-of-school settings);

2. *Referential level*: referential activity, in which *models of* refer to activity in the setting described in instructional activities (mostly posed in school);

3. *General level*: general activity, in which *models for* enable a focus on interpretations and solutions independently of situation-specific imagery;

4. *Formal level*: reasoning with conventional symbolizations, which is no longer dependent on the support of *models for* mathematical activity.

This shift is at the basis of the notion of emergent modelling, that will be treated in the next section.

2.2 Mathematical modelling

The promotion of mathematical modelling is accepted as a central goal of mathematics education worldwide, especially if mathematics education aims to promote responsible citizenship (Kaiser 2017). Modelling is a creative process of making sense of the real world to describe, control, optimize aspects of a situation, interpret results, and make modifications to the model if it is not adequate for the situation.

> Mathematical proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has nor served its purpose. (NGA and CCSSO 2010, p. 7)

However, there still exists a substantial gap between the ideals expressed in educational debate and innovative curricula on the one hand, and everyday teaching practice on the other. In particular, genuine modelling activities are till rather rare in mathematics classrooms (Blum, Galbraith, Henn and Niss 2007).

In the last decades empirical research on mathematical modelling has improved considerably, in terms of both quality and quantity. Several studies introduced different teaching approaches that have been analyzed by quantitative and/or qualitative methods. The most two important principles that seem to be important for designing learning environments for modelling were student-centered teaching and prompting. However, comparison between different approaches is not common. Schukajlow, Kaiser and Stillman (2018) identified some future directions and open questions for empirical research on mathematical modelling. In particular, they suggest the need of: (i) more intervention studies to examine ways for teaching modelling and monitoring the development of modelling competencies; (ii) monitoring the development of pedagogical content knowledge of pre-service and in-service mathematics teachers on modelling; (iii) increasing the number of studies that use mixed methods for the analysis of research questions.

In the following sections, we will start presenting the historical development of different approaches to mathematical modelling. Then we report more modern trends concerning the teaching and learning of modelling, focusing particularly on the perspectives of emergent modelling and model eliciting. In conclusion we describe what is meant by modelling competencies and their teaching.

2.2.1 Historical development

How and why to include mathematical modelling in mathematics education has been the focus of many research studies until the half of the twentieth century. Mathematics education during the ninetieth was dominated by learning to execute algorithms without relation to the real-world (Kaiser-Messner 1986). This situation has changed beginning with the symposium *How to teach mathematics so as to be useful* (Freudenthal 1968;

Pollak 1968) in August 1967. Since then, differences started emerging, in particular concerning the ways to integrate mathematical modelling into classroom teaching. The development of mathematical modelling, and its teaching, was influenced until the middle of 1980s by two main perspectives (Kaiser-Messner 1986): a *pragmatic perspective* and a *scientific-humanistic perspective*. Despite these two perspectives shared the conviction of changing mathematics education including the real-world in the teaching of mathematics, significant differences can be identified.

Pragmatic perspective

In the pragmatic perspective, students should learn to apply mathematics to solve practical problems from the real world, focusing on utilitarian and pragmatic goals (Pollak 1968; 1969; 1979).

> Students should all develop the habit to see and enjoy the possibilities for interesting problems around them. (Pollak 1968, p. 26)

Concerning the way that teaching content should be determined:

> If we can agree that one of the main imperatives is to teach mathematics so as to be useful, then a first step is to find out how mathematics is used by people good at doing so and then to identify the mathematical contexts teachable in schools that will be helpful to most people in making them better at using mathematics. (Bell 1979, p.314)

This pragmatic view of modelling is reflected also in the way in which mathematical modelling, or the inclusion of real-world mathematics in mathematics education, is understood (Kaiser 2017). Mathematical modelling should be seen as a cyclic process (Pollak 1968), emphasizing the interplay between the real-world and different kinds of applicable mathematics. A cyclic way moving from the real-world to mathematics and back, as described by Pollak (1979, p.233) in his view of the modelling cycle (Fig. 3).

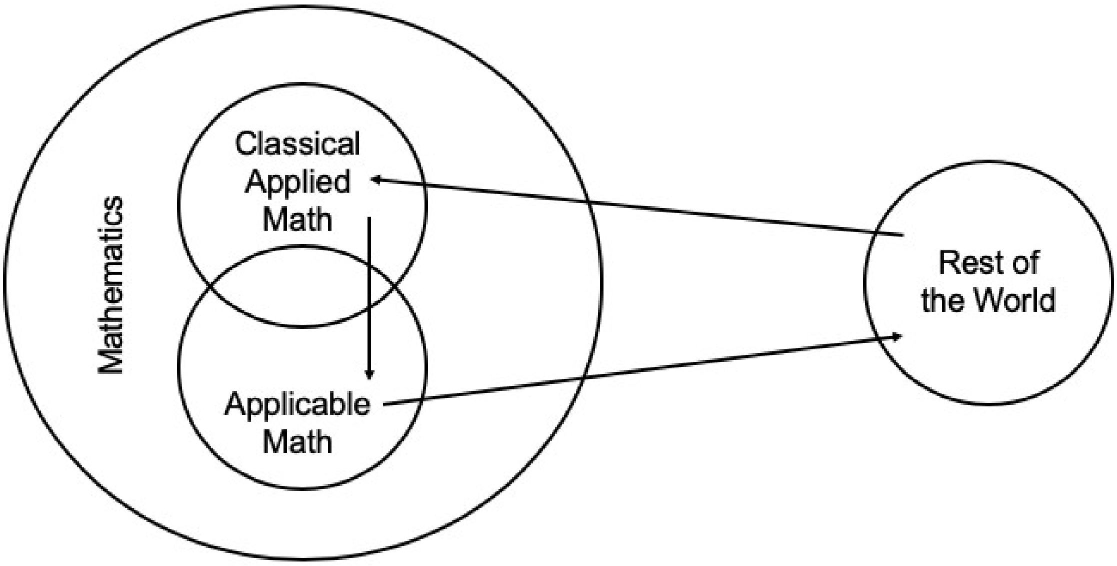


Figure 3. Pollak's (1979) modelling cycle

Scientific-humanistic perspective

The scientific-humanistic perspective is characterized by a twofold orientation (Kaiser 2017): (i) a focus on mathematics as a science, in the sense that it is a discipline characterized by formal and nonformal structures, and (ii) a focus on the humanistic ideals of education that emphasize the ability of learners to create relations between mathematics and the real-world.

> There are two extreme attitudes: to teach mathematics with no other relation to its use than the hope that students will be able to apply it whenever they need to. If anything, this hope has proved idle. … The opposite attitude would be to teach useful mathematics. It has not been tried too often, and you understand that this is not what I mean when speaking about mathematics being taught to be useful. The disadvantage of useful mathematics is that it may prove useful as long as the context does not change, and not a bit longer, and this is just the contrary of what true mathematics should be. Indeed, it is the marvelous power of mathematics to eliminate the context, and to put the remainder into a mathematical form in which it can be used time and again. (Freudenthal 1968, p. 5).

Mathematics is not a close system, but an activity, the activity of mathematizing reality and if possible even that of mathematizing mathematics, in which rich contexts have a central position in fostering this process (Freudenthal 1973). It is evident that this perspective was taken up within the subsequent approach of RME.

Concerning the way in which mathematical modelling, or the inclusion of real-world mathematics in mathematics education is understood, in the scientific-humanistic perspective modelling is interpreted as a complex interplay between mathematics and real world, based on various kinds of mathematization processes (Freudenthal 1968; De Lange 1987). Treffers (1987) differentiated *horizontal mathematization* from *vertical mathematization*. In horizontal mathematization, students use mathematical tools to organize and solve problems situated in real life. It involves going from the world of life into that of symbols and vice-versa. Vertical mathematization, instead, refers to the process of recognizing within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. Models exist only at the lowest level of mathematization, when a mathematical model is constructed for an extra-mathematical situation (Freudenthal 1973). Such an extra-mathematical situation, in the perspective of RME, refers not exclusively to real-world contexts, but also to realistic contexts, where

> …the term *realistic* refers more to the intention that students should be offered problem situations which they can image… than that it refers to the realness or authenticity of problems. However, the latter does not mean that the connection to real life is not important. It only implies that the contexts are not necessarily restricted to real-world situations. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for problems, as long as they are real in students' minds. (Van den Heuvel-Panhuizen 2003, p. 9-10)

In the scientific-humanistic perspective, mathematical modelling should be seen in a spiral way, as the relation between mathematics and the real-world (De Lange 1987, p.39), see Fig. 4.

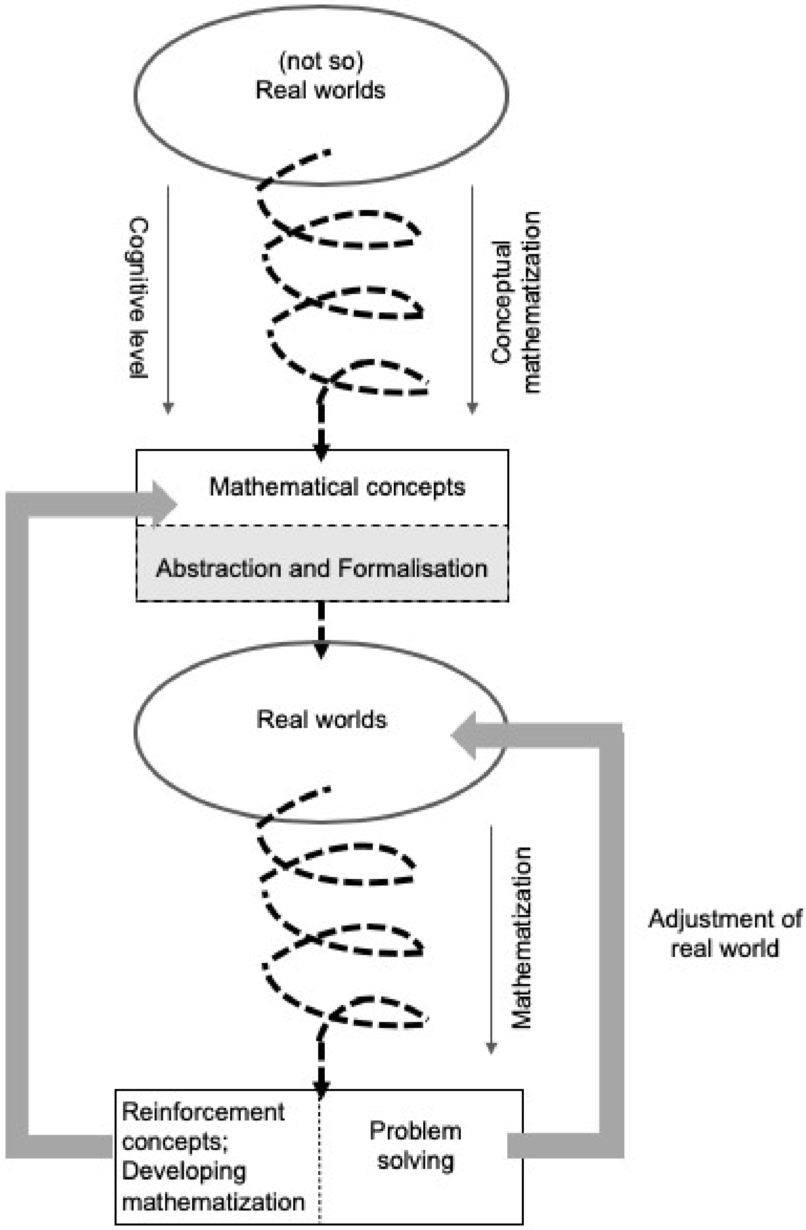


Figure 4. De Lange's (1987) modelling spiral

2.2.2 Recent trends in mathematical modelling

The historical perspectives on modelling developed in the 1970s have been refined to more joint perspectives on mathematical modelling. However, the perspectives have also become more differentiated, and new perspectives have evolved. Kaiser and Sriraman (2006) developed a framework for the description of the various approaches to mathematical modelling, which classifies the approaches according to their aims, types of mathematical modelling examples, epistemological background, and relation to the initial perspectives. In the following this classification is being described.

Realistic or applied modelling

The realistic or applied perspective follows the pragmatic approach for mathematical modelling, fostering the pragmatic and utilitarian goal that the application of mathematics should contribute to the understanding of the real-world and the solution of real-world problems (Haines and Couch 2007; Kaiser and Schwarz 2010). Two important points in this approach are that: (i) modelling processes are carried out as a whole and not as partial processes, similar to the work of applied mathematicians; (ii) the new development of mathematical concepts and algorithms is not the focus.

Epistemological or theoretical modelling

The epistemological or theoretical modelling pursues the tradition of the scientific-humanistic perspective for mathematical modelling, enhancing a theory-oriented goal. In this perspective applications of mathematics in the real-world should promote the development of mathematical concepts and algorithms. This approach relies on an epistemological framework based on the mathematical praxeologies of Chevallard (Chevallard 1985; Barquero, Bosch and Gascon 2007). The epistemological or theoretical perspective assigns low importance to the practical part (types of tasks, solution techniques) of a mathematical praxeology, fostering instead the complementary theoretical part (supporting theory, necessary technology).

> If the approach of praxeology becomes the main orientation, this leads to the fact that every mathematical activity is identified as modelling activity for which modelling is not limited to mathematising of non-mathematics issues. (Kaiser and Sriramn 2006, p.305)

Besides these two first perspectives (realistic or applied and epistemological or theoretical), that reflect the historical polarization between pragmatic and scientific-humanistic modelling, new perspectives developed assimilating aspects from both the perspectives.

Educational modelling

Educational modelling has strong links with the last development of the scientific-humanistic perspective (Freudenthal 1991; Treffers 1987; De Lange 1987), in which real-world examples and their relations to mathematics as central elements of the structure of teaching and learning processes are stressed. The perspective of educational modelling can be split in two facets: *didactical modelling* and *conceptual modelling*.

In didactical modelling the emphasis is on pedagogical goals. The concept of competencies and their promotion are heavily discussed. Moreover, developing communication and argumentation competencies and fostering social learning through modelling are emphasized (Blum 2011).

In conceptual modelling real-world examples are used to introduce new mathematical concepts and to enhance the understanding of mathematical concepts, posing the attention to the development of a deeper mathematical understanding via modelling examples or to the understanding of the modelling process itself (Maass 2006; Stillman 2011).

Contextual modelling or model eliciting perspective

Contextual modelling or model eliciting finds its roots in problem-solving and cognitive-psychological research (Kaiser 2013). This approach was introduced by Lesh and Doerr (2003). Model eliciting activities are problem-solving activities constricted by specific instructional design principles, in which students make sense of meaningful situations and invent, extend, and refine their own mathematical constructs. Here the challenge is to develop activities that motivate students to develop the mathematics needed to make sense of such situations.

Sociocritical and sociocultural modelling

Sociocritical and sociocultural modelling emphasize critical thinking about the role of mathematics in society, the role and nature of mathematical models, and the function of mathematical modelling in society.

> [the] promotion of critical understanding of modelling processes and models developed as overall goal connected with recognition of cultural dependency of modelling examples and modelling approaches developed. (Kaiser, Sriraman, Blomhoj and Garcia 2007, p. 2039)

In the teaching and learning process, the focus is on the promotion of students' critical thinking, based on reflective discussions amongst the students within the modelling process.

Cognitive modelling as metaperspective

The last approach in Kaiser and Sriramn (2006) classification of modelling approaches, can be considered as a metaperspective, due to its descriptive nature. The focus is on the analysis of students' modelling processes and the promotion of mathematical thinking processes. Modelling processes are analyzed on different types of modelling situations that vary in their degree of authenticity or mathematical complexity. The main goals are to reconstruct individual modelling routes (Borromeo Ferri 2011) or individual cognitive barriers and difficulties of students during their modelling activities (Stillman 2011).

2.2.3 The modelling cycle

The perspectives described in the previous section result in different characterization of the modelling process, emphasizing either the solution of the original problem or the development of mathematical concepts or ideas. Corresponding to the different perspectives on mathematical modelling, various modelling cycles developed with different emphases (see for an overview Borromeo Ferri 2006). In all the approaches, the idealized process of mathematical modelling is described as a cycle process to solve real problems using mathematics comprising different steps or phases (Kaiser 2017).

The modelling cycle is not only a theoretical model which characterizes the modelling processes, but actually it is a multi-purpose learning instrument for students and a diagnostic instrument for teachers (Borromeo Ferri 2018).

The development of most of the modelling cycles was influenced by Pollak's work (Fig. 3), who first separated reality and mathematics as two worlds. A first example of modelling cycle is the one developed by Blum (1985, p.200) and Kaiser-Messner (1986, p.3). Here, the starting point is a real situation, which is given through a real problem, that has to be idealized to build a real model, making assumptions and identifying influencing factors. Then, the real model is translated into a mathematical model through a process of mathematization. Investigation of the model simply means inner-mathematical working, and so getting mathematical results. The final step is the interpretation and validation of the mathematical results (Fig. 5). This cycle was named *didactical* or *pedagogical cycle* (Borromeo Ferri 2006). This cycle, indeed, was developed to focus on if, and how, the modeling cycle can be a tool to promote modeling competencies, and the understanding of modelling in general, of students in middle school, high-school and university (see Blum 2015; Maass 2007). The implementation of the cycle should offer students the opportunity to reflect what they had done while solving real problems while learning the notions of *real model* or *mathematical model.* Furthermore, this meta-level and the visualization of the modeling process through the cycle is helpful to get an idea of how modeling problems are different from routine problems, because of the transitions between reality and mathematics (Borromeo Ferri 2018).

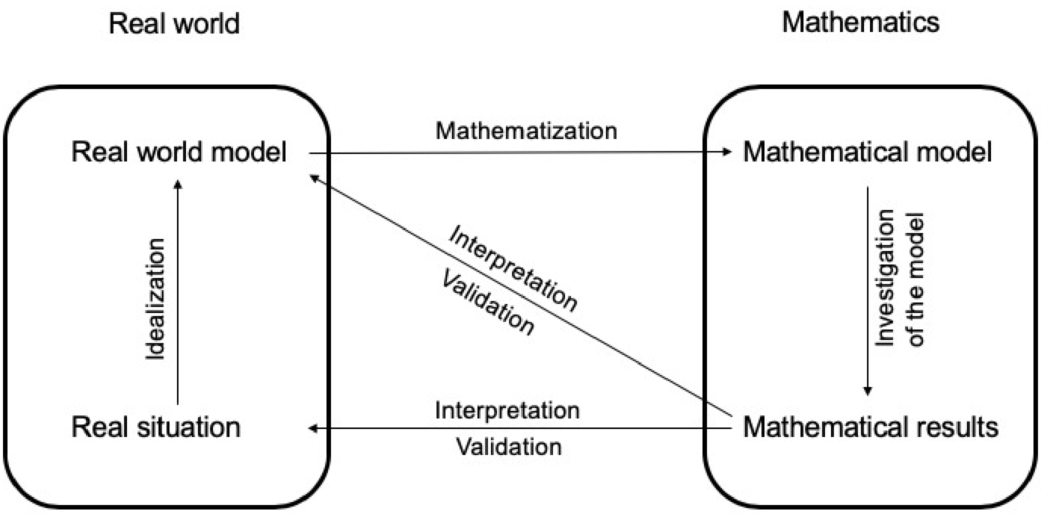


Figure 5. Modelling cycle from Blum (1985) and Kaiser-Messner (1986)

Another kind of modelling cycle founds its origin in psychology: the so-called *psychological modelling cycle* (Verschaffel, Greer and De Corte 2000). In this approach, the step between the real situation and the mathematical model is given by the *situation model* (Fig. 6). The term situation model is mainly used in connection with word problems (see Kintsch and Greeno 1985; Nesher, Hershkowitz and Novotna 2003) and has its origin in text linguistics. A situation model can be described as a mental representation of the situation that is given in the problem.

> The situation model includes inferences that are made using knowledge about the domain of the text information. It is a representation of the content of a text, independent of how the text was formulated and integrated with other relevant experiences. Its structure is adapted to the demands of whatever tasks the reader expects to perform. (Kintsch and Greeno 1985, p. 110)

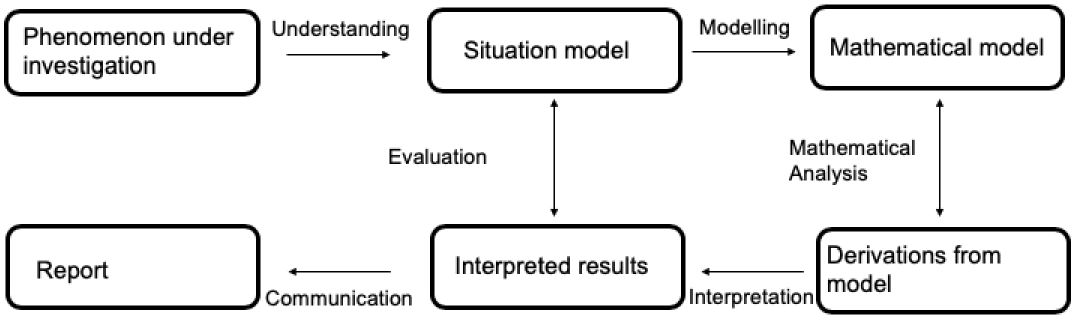


Figure 6. Modelling cycle from Verschaffel et al. (2000)

In this cycle it is clear there is no distinction between mathematics and reality. The aim of this cycle is not to be used in schools, but the relevance for including the situation model in the modeling cycle offered new ways for research and for practice.

From the previous approach, in fact, brought out the *diagnostic modelling cycle* (Fig. 7). In this cycle the focus is on the cognitive processes of individuals during modelling processes. The situation model represents the most important phase during the modeling process (Blum and Leiß 2007), being run by all individuals during modelling. That is because the transition between real situation and situation model is described as a phase of understanding the task. In this direction a contribution is given by Borromeo Ferri (2007), who used the phase of the situation model in an adaptation of the modelling cycle. Moreover, in order to better describe kind of internal processes an individual goes through to obtain a corresponding mental picture while/after reading the (complex) modelling task, instead of situation model the name *mental representation of the situation* was introduced.

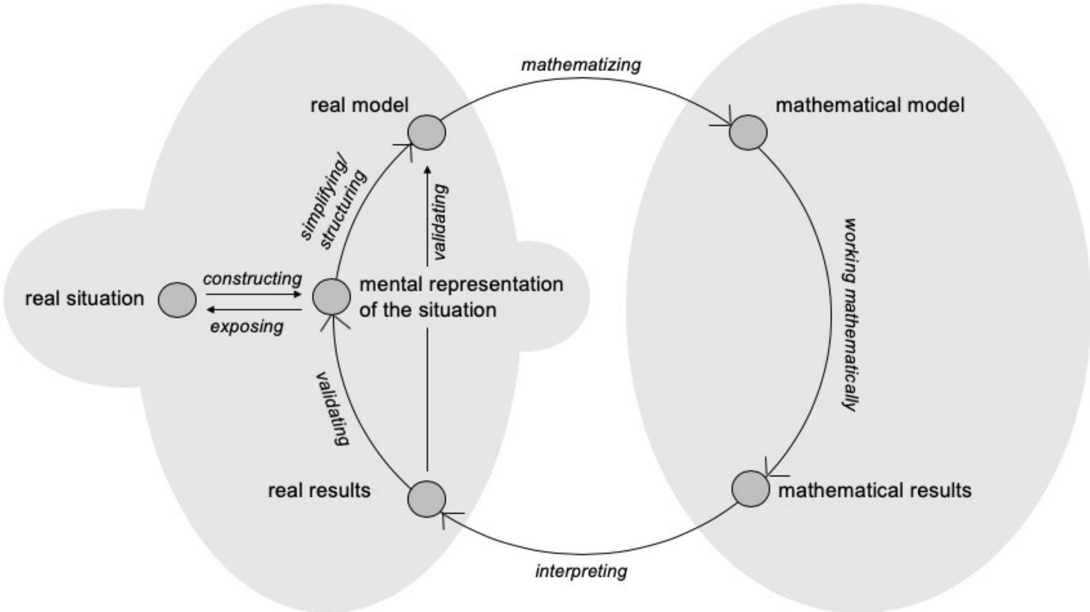


Figure 7. Modelling cycle from Borromeo Ferri (2007)

To conclude, the following synthesis given by Kaiser and Stender (2013, p.279) is reported (Fig.8).

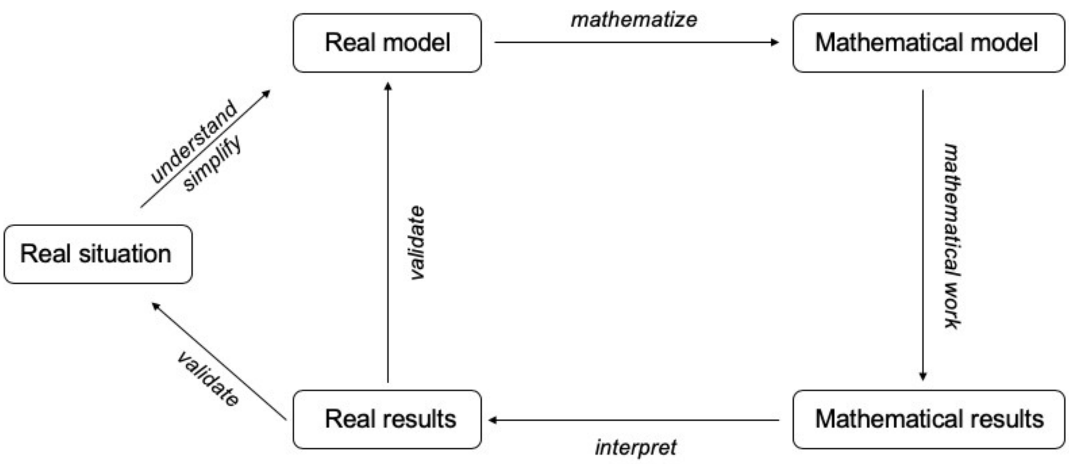


Figure 8. Modelling cycle from Kaiser and Stender (2013)

In the specific, the following characteristics are taken into account:

- the real situation is simplified in order to build a real model of the situation, making assumptions and identifying the central influencing factors;

- the real model has to be translated into mathematics in order to create a mathematical model. The process of developing a real model and a mathematical model are interwoven since the developed real problem is related to the mathematical knowledge of the modeler;

- mathematical results are worked out by using mathematics;

- after interpreting the mathematical results, the real results have to be validated. Then, parts or the whole process go through again.

We remark that this is an idealized cycle. Indeed, what happens is that mini cycles occur that are either worked out in linear sequential steps like the entire cycle or in a less ordered way. Most modelling processes include frequent switching between the different steps of the modelling cycles (Borromeo Ferri 2011). To conclude, in this cycle the distinction of the real-world situation from the real model and the mathematical model is present, and the emphasis is on the interpretation of the mathematical results to obtain real-world results, that also need to be validated.

In addition to the classification above, further descriptions of modelling cycles, which are used in school or higher education, can be found in Cirillo, Pelesko, Felton-Koestler and Rubel (2016).

In the next two sections two modelling perspective that will be considered in the following research cycles will be described in more details: emergent modelling and model eliciting.

2.2.4 Emergent modelling

Emergent modelling refers to *educational modelling* in Kaiser and Sriraman (2006) classification of modelling perspectives. Educational modelling is linked to the scientific-humanistic perspective in the version formulated by Freudenthal in his later years and its extensions developed by Treffers (1987) and DeLange (1987), who emphasized real-world examples and their relations to mathematics as central elements of the structure of teaching and learning process.

Emergent modelling was initially developed by Gravemeijer (1999) with the meaning of supporting the emergence of formal mathematical ways of knowing. The underlying educational theory is the one of RME, in which models have always employed to foster a process in which formal mathematics is re-invented by students themselves. Indeed, in this perspective modelling activities are used as a vehicle for the development, rather than applications, of mathematical concepts (Greer, Verschaffel and Mukhopadhyay 2007). Students, starting from a real context, begin to model their informal mathematical strategies and arrive to re-invent mathematical concepts and applications they need. These concepts and applications can be subsequently formalized in mathematical terms and generalized to other situations. As a consequence, the role of the model shifts during the learning process, from being situation-related to becoming more general. Emergent modelling can be seen as a long-term dynamic process from a *model of* students' situated informal mathematical strategies to a *model for* more formal mathematical reasoning (Gravemeijer and Doorman 1999), that favours understanding, reasoning and sense-making. This transition from *model of* to *model for* involves the constitution of a new mathematical reality (Streefland 1985) that can be denoted as *formal* in relation to the original starting points of the students. The movement from situational to formal reasoning is well described by the four levels in Fig. 2 (Gravemeijer, Cobb, Bowers, and Whitenack 2000).

Emergent modelling reinforces the vertical component of the *mathematization* process. Mathematization should be divided in two components: *horizontal mathematization* and *vertical mathematization* (Treffers 1987; Freudenthal 1991). In horizontal mathematization, students use mathematical tools to organize and solve problems situated in real life. It involves going from the world of life into that of symbols and vice-versa. Vertical mathematization, instead, refers to the process of recognizing within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. Vertical mathematization is stimulated by emergent modelling. In this situation, in fact, models help students in passing through several stages of understanding and reflection on formal mathematical concepts.

To conclude, in emergent modelling what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality (Gravemeijer 1999).

2.2.5 Model eliciting

The *model eliciting* perspective was firstly formulated by Lesh and Doerr (2003) and its theoretical basis are founded in psychological theories and pragmatism.

Model eliciting activities can be defined as simulations of real-life problem solving situations in which students develop a model going through iterative phases of invention, refinement and revision. Modelling, in fact, is a process of developing representational descriptions for specific purposes in specific situations, involving iterative testing and revision cycles (Lesh and Lehrer 2003). In the model eliciting approach, a modelling cycle is intended as a four-step process: i) formulation of a possible mathematical approach to solve a real-context problem; ii) test of the designed solving strategy; iii) interpretation and discussion of the testing results; iv) revision of the starting approach. As a consequence, a mathematical model is no longer the translation of a real-world problem in a mathematical symbolic formulation but becomes the result of many repeated modelling cycles in order to create the mathematical model that best describes the given situation.

The goal of a model eliciting activity is the process of model construction. This process is the key difference between a model eliciting activity and word problems (Leavitt and Ahn 2010). In the traditional practice of word problems, a set of mathematical constructs and procedures are introduced by the teacher to students, who apply these procedures to solve a problem. In model eliciting activities, instead, students struggle to create interpretations that fit their interpretations of the starting dilemma, discuss, make sense of meaningful situations and invent, extend and refine their own mathematical constructs (Kaiser 2017).

In order to create simulations of real-life problems, Lesh, Cramer, Doerr, Post and Zawojewski (2003) introduced six principles of instructional design. In Table 1, these principles and their relative purposes are reported.

Table 1. Model eliciting principles by Lesh, Cramer, Doerr, Post and Zawojewski. (2003).

Principle	Purpose
Personal meaningfulness	Ensure that the task could really happen in real life and that responses can be based on extensions of students' everyday knowledge and experiences.
Model construction	Students' involvement in repeated modelling cycles when engaged with the task.
Self-evaluation	Students' ability to clearly assess their work in relation to the purposes of the task.
Model externalization	Students' expression of their thinking about the situation and description of the developed steps to solve the task.
Simple prototype	The situation must be as simple as possible, while still creating the need for a significant model that could represent a prototype for interpreting similar situations.
Model generalization	Extension of the constructed conceptual tool to a broader range of situations.

To get the most instructional value out of model eliciting activities, Lesh, Cramer, Doerr, Post and Zawojewski (2003) developed a model development sequence whose components can easily be re-sequenced to suit the needs of researchers or teachers (Fig. 9). Such model development sequences are made by the following activities: *warm-up activities*; *model eliciting activities*; *model exploration activities*; *model adaptation activities*; *discussions about structural similarities*; *presentations and discussions*; *reflection and debriefing activities*; *follow-up activities*; *the on-line how toolkit*; *other high-quality resources and references*. A complete description of such activities can be found in Lesh, Cramer, Doerr, Post and Zawojewski (2003). Here the focus is only on the activities that will be considered in the rest of the thesis. In the specific, *warm-up* activities are usually given the day before students are expected to begin working on the model eliciting activity. Warm-up activities aimed at helping students to be confident with the context of the modelling activity and at introducing or testing eventually minimum prerequisites. In *model eliciting activities* students are engaged in performing modelling cycles to produce a model that describes the starting situation. *Presentations and discussions* are whole-class activities in which students make formal presentations about the results of their work. In *reflection and debriefing activities* students work individually, thinking back about their experiences during the whole modelling process.

Further research in model eliciting focused on the role of the teacher. Leavitt and Ahn (2010) proposed a teacher's guide to model eliciting activities, in order to help researchers and teachers in conducting investigations of students' or teachers' actions. These guidelines, that can be adapted to each specific environment, move around four categories: *group composition*; *relevant model eliciting activity sections*; *teachers' role during group work*; *group presentations and individual work*. Moreover, some additional recommendations are suggested. In particular, the starting problem context of a model eliciting activity should be based on students' familiar situations in which they can understand the need for the desired mathematical construct. Moreover, the teacher has to pay attention to anticipate the mathematics needed for the paths that the students might explore; resist guiding students toward one specific method; remind students to write down their reasoning.

Despite the goal of a model eliciting activity is to develop a conceptual tool that is also sharable and reusable, models are inherently provisional and are developed for specific purposes in specific situations (Lesh and Lehrer 2003).

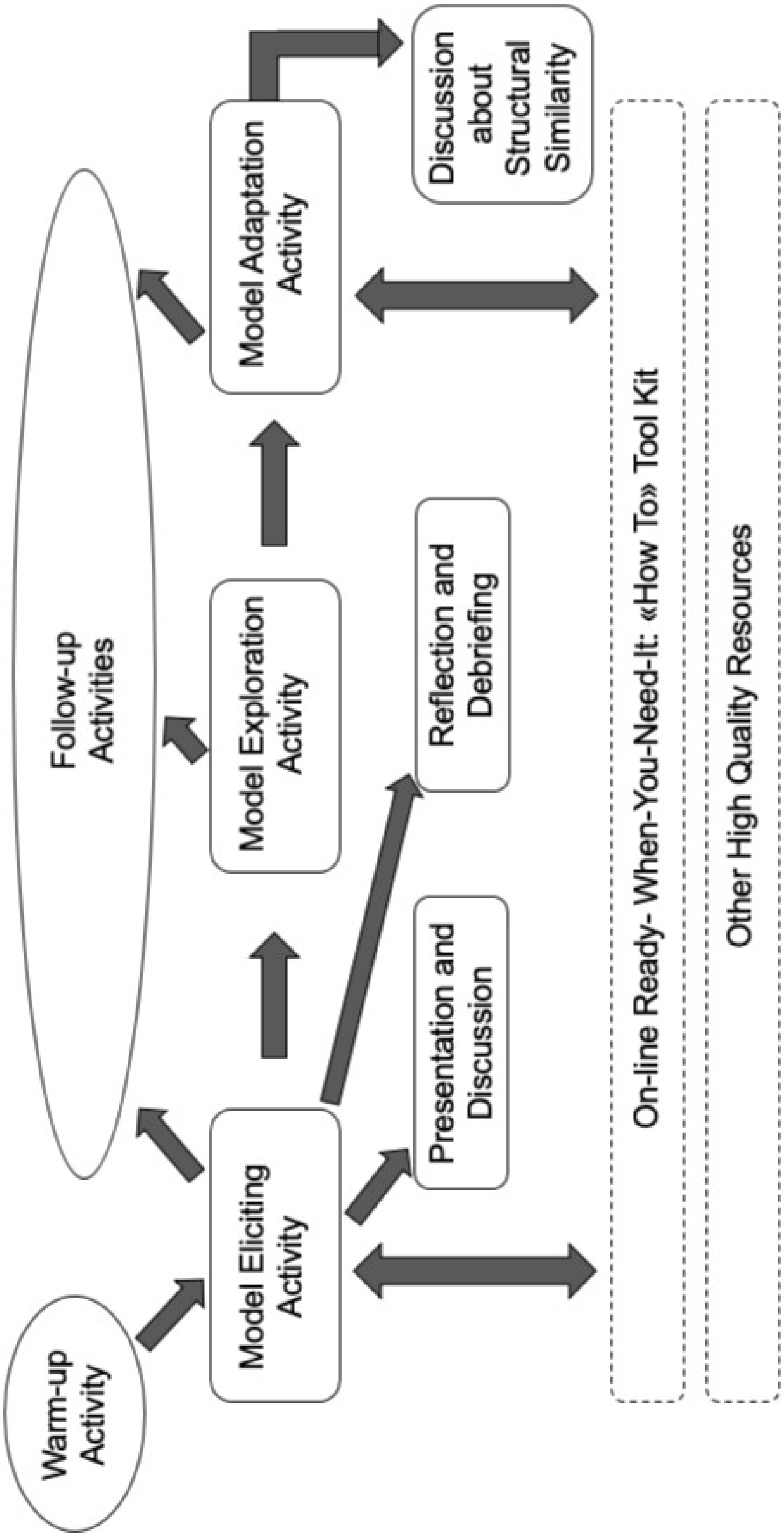

Warm-up Activity
Model Eliciting Activity
Follow-up Activities
Model Exploration Activity
Model Adaptation Activity
Presentation and Discussion
Reflection and Debriefing
Discussion about Structural Similarity
On-line Ready- When-You-Need-It: «How To» Tool Kit
Other High Quality Resources

Figure 9. Model development sequence

2.2.6 Mathematical modelling competencies and their teaching

The concept of modelling competencies has been included in many curricula all over the world. The relevance of modelling competencies was first emphasized at the start of the international conference series on the teaching of mathematical modelling and applications in 1983, later called the International Conference on the Teaching and Learning of Mathematical Modelling and Applications:

> The basic philosophy behind the approach… of the modelling workshop for higher education is that to become proficient in modelling, you must fully experience it- it is no good just watching somebody else do it, or repeat what somebody else has done- you must experience it yourself. I would liken it to the activity of swimming. You can watch others swim, you can practice exercises, but to swim, you must be in the water doing it yourself. (Burghes 1984, p. 13).

Some important strands of discussion with different emphases and foci shaped the debate on modelling competencies (Kaiser and Brand 2015). Here we report two of them: the introduction of modelling competencies in an overall comprehensive concept of competencies by the Danish KOM project; the development of a comprehensive concept of modelling competencies based on sub-competencies and its evaluation by the German discussion on modeling.

In 2002, the team members of the Danish KOM project developed a comprehensive approach to the definition of mathematical competencies. The mathematical competency was defined as

> … a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge. (Niss and Hojgaard 2011)

In the specific, eight mathematical competencies were distinguished (Fig. 10), including the modelling competency. These competences were not seen as independent sub-competences, but they describe the mathematical competency as a whole.

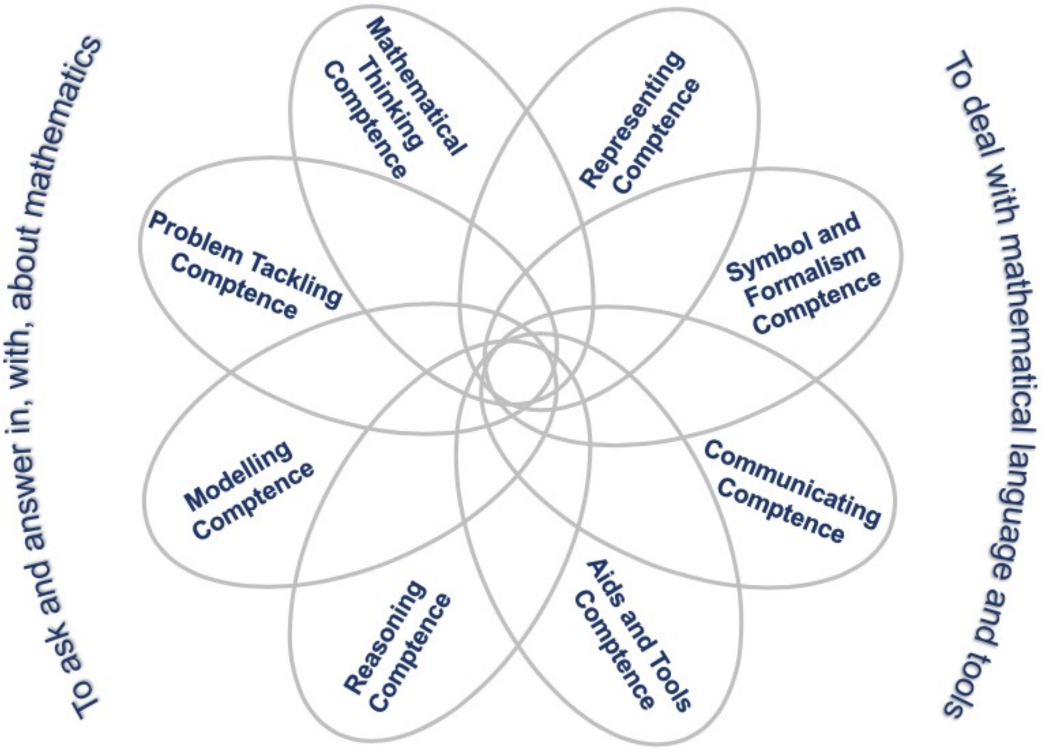


Figure 10. The KOM flower of the eight mathematical competencies (Niss and Jensen 2002).

We report the definition of modelling competency:

> This competency involves, on the one hand, being able to *analyze* the foundations and properties of existing models and being able to assess their range and validity. Belonging to this is the ability to *de-mathematize* existing mathematical models, i.e. being able to decode and interpret model elements and results in terms of the real area or situation which they are supposed to model. On the other hand, the competency involves being able to *perform active modelling* in given contexts, i.e. mathematising and applying it to situations beyond mathematics itself. (Niss and Hojgaard 2011, p. 58)

Active modelling includes different phases of the modelling process, such as structuring real-world situations that has to be modeled; mathematising, translating the situation into mathematical terms; working with the resulting model, finding solution to the mathematical problems; interpreting and validating these results in relation to the real starting situation; monitoring the entire process of modelling.

Another contribution on modelling competencies was given by the work of the German modelling group.

> Modelling competencies include, in contrast to modelling abilities, not only the ability, but also the willingness to work out problems, with mathematical aspects taken from reality, through mathematical modelling. (Kaiser 2007, p.110)

Within this strand a distinction is made between global modelling competencies and sub-competencies of mathematical modelling. Global competencies are abilities necessary to perform and reflect on the whole modeling process. The sub-competencies instead refer to the specific different competencies essential to performing single steps of the modeling cycle (Kaiser 2007), distinguished as:

- competency to solve at least partly a real-world problem through mathematical description developed by oneself;
- competency to reflect about the modeling process by activating meta-knowledge about modeling processes;
- insight into the connections between mathematics and reality;
- insight into the perception of mathematics as process and not merely as product;
- insight into the subjectivity of mathematical modeling, that is, the dependence of modeling processes on the aims and the available mathematical tools and students' competencies;
- social competencies such as the ability to work in a group and to communicate about and via mathematics.

Teaching mathematical modelling competencies

Besides the definition of mathematical modelling competence/ies, an important aspect is how to teach modelling competence/ies. As a consequence, teachers need to be supported in learning mathematical modelling, how to teach it and how to promote in students the development of such modelling competencies.

Mathematical modeling is a crucial part of teacher education. Four teaching competencies had been individuated being particularly important when teaching mathematical modeling (Borromeo Ferri and Blum 2009; Borromeo Ferri 2018): *theoretical competence*; *task competence*; *instruction competence*; *diagnostic competence* (Table 2).

Table 2. Competencies for teaching mathematical modelling (Borromeo Ferri and Blum 2009; Borromeo Ferri 2018)

Theoretical competence	Modelling cycles
	Aims and perspectives on modelling
	Types of modelling tasks
Task competence	Multiple solution of modelling tasks
	Cognitive analysis of modelling tasks
	Development of modelling tasks
Instruction competence	Planning lessons with modelling tasks
	Carrying out lessons with modelling tasks
	Interventions, support and feedback
Diagnostic competence	Recognize phases in modelling processes
	Recognizing difficulties and mistakes
	Marking modelling tasks

The theoretical competence refers to the necessary theoretical background to develop and implement modelling activities in school. In the specific, teachers must know the historical development of mathematical modelling, different perspectives and goals.

However, not only theory is sufficient to be able to develop modelling lessons in classrooms. In general, tasks are at the core of mathematics lessons. As a consequence, the selection and the quality of tasks for lessons are essential for mathematical understanding, for promoting students' mathematical practices and competencies, and can be the basis for structuring lessons using several teaching methods. In this direction, the task competency aims to help teachers in learning to solve, analyze and create modelling tasks.

How to plan and execute modelling lessons? The instruction competence concerns the ability to plan and execute modelling lessons and knowledge of appropriate interventions during the pupils' modelling processes. Borromeo Ferri (2018) identifies some principles for preparing and planning, and for executing and reflecting (Table 3) on modelling lessons.

Table3. Principles concerning modelling lessons

Preparing and Planning	Executing and Reflecting
Chose an adequate modeling problem for students, which has an interesting context, is problem-oriented, authentic, realistic and can be solved through the steps of the modeling cycle.	Students need time to understand what the problem is about, and they should know how they will work on this problem (with partners or in groups). Before group work starts, students need to know how the solution should be presented afterwards.
Solve the problem before to give it to students, going through all steps of the modeling cycle. Write down multiple solutions and possible	During group work go around and take notes about the students' modeling process. Look at groups that used different models for their

models/formulations of the problem. Think about potential cognitive barriers students could have when they work on this problem and have adequate interventions for stronger and weaker students prepared.	solution.
Make clear the central goal of the lesson: understanding of certain mathematical content; improving students' modeling sub-competencies; social goals, for instance observing students' teamwork and give them feedback and help so they can be more effective in coming to a result.	Is the central goal of the lesson being achieved?
Which tools are needed for the modeling problem? Should students use technology and if yes, how can they combine it with the modeling process?	Develop a lesson plan. At the end, reflect about what was successful in the lesson and what was not, and which aspects can be optimized in further lessons.
Think about the duration of the different phases for the lesson plan. Which method is suitable for the class?	Is there enough time for students to work on the problem and also for discussing the results? Do students need further materials?

The diagnostic competence concerns the ability to identify phases in pupils' modelling processes and to diagnose pupils' difficulties during such processes.

Learning how to assess and to grade modelling problems in school is one of the hardest dimensions to take into account. In literature there are some examples of schemas for assessing solutions to modelling problems (see for example Schukajlow et al. 2009).

2.3 Mathematical problem-posing

An educational strategy that is directly linked to mathematical modelling is problem-posing. Indeed, problem-posing forms and integral part of modelling: the problem and its formulation are an essential part of modelling, and a modelling process is a continual adjustment and refinement of the main problem (Hansen and Hana 2015).

> It is terribly important for students to have practice in seeing situations in which mathematics might be helpful, and in trying their hand at formulating useful problems. (Pollak 1969, p. 399)

The era of information and communication technology creates new social environments and needs, wherein young generations have to face unpredictable changes they should learn coping with (Singer, Ellerton and Cai 2015). As specified in the previous paragraph, the Council of European Union recommended as one of the key competencies for lifelong learning the *mathematical competence*, seen as the ability to develop and apply mathematical thinking and insight in order to solve a range of problems in everyday situations[1]. In order to help students to prepare to cope with such situations they have to face out of school, the type of problem-solving experiences they are engaged at school need to be rethought (Bonotto 2013). In particular, realistic and less stereotyped problems that take into consideration the experiential world of students must be inserted in the school practice, in order to create a bridge between mathematics classroom activities and everyday-life experiences. In fact, encouraging students to relate mathematical problems with real-world scenarios may help them more closely associate mathematics with their everyday activities (De Corte, Verschaffel and Greer 2000). To this end, allowing students to write their own mathematical problems may help them to make connections between mathematics in the classroom and their real life (Kopparla et al. 2018). In this direction, problem-posing, which can be seen as the

[1] Recommendation of the European Parliament and of the Council of 22 May 2018 on key competencies for lifelong learning (2018/C189/01).

process by which students generate their own problems in addition to solving pre-formulated problems (English 1997; NCTM 2000; Silver and Cai 1996), should represent a valuable strategy to support students in give sense to their mathematical activity filling the gap between in- and out-of-school mathematical competencies and experiences. Indeed, if the goal of education is to prepare students for the kinds of thinking they will need, problem-posing should be an important part of the curriculum (Singer, Ellerton and Cai 2015), as requested in many curricular and pedagogical innovation in mathematics education:

> Teaching mathematics from a problem-solving perspective entails more than solving nonroutine but often isolated problems or typical textbook type of problems. It involves the notion that the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining, and testing all aspects of problem solving. Tasks should be created and presented that are accessible to students and extend their knowledge of mathematics and problem solving. Students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem. (NCTM 1991, p.95)

The term problem-posing was introduced in education by Paulo Freire in 1970 in his book *Pedagogy of the Oppressed* as a metaphor for emphasizing critical thinking. Problem-posing extended to various domains of knowledge. In mathematics education, problem-posing has been identified as an important aspect (Christou, Mousoulides, Pittalis, Pitta-Pantazi, and Sriraman 2005; Freudenthal 1973; Polya 1954), and more in general as a critically important intellectual activity in scientific investigation. Einstein (Einstein and Infeld 1938) argued that the formulation of an interesting problem is often more important than its solution.

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, a new possibility, to regard old problems from a new angle, requires creative imagination and marks real advances in sciences. (Einstein and Infeld 1938, p. 92)

Since in real life problems must often be created by the solver, the formulation of a problem should be viewed not only as a goal of instruction but also as means of instruction (Killpatrick 1987). The advancement of mathematics, in fact, requires creative imagination, which is the result of raising new questions and viewing old questions from a new perspective (Ellerton and Clarkson 1996). Problem-posing, being the act of generating mathematical problems, is a process through which the importance of creativity and critical thinking are emphasized (NCTM 2000). In this perspective, students can actively construct meaning in both the natural and simulated worlds in classrooms. Moreover, teachers and students might create knowledge together in a variety of contexts and generate and address critical questions about the knowledge they produce. In Freire's version, all these could help to develop more democratic, diverse, critically thinking members of society (Singer, Ellerton and Cai 2015).

Problem-posing has been defined by researchers from different perspectives (Silver and Cai 1996), referring both to the generation of new problems and to the reformulation of given problems (Silver 1994). In this project problem-posing is considered as the process by which students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems (Stoyanova and Ellerton 1996). These concrete situations considered as starting point for the practice of problem-posing could be divided in three categories (Stoyanova & Ellerton 1996):

1. free situations, where students are asked to pose problems without restrictions;

2. semi-structured situations, where students are provided with an open situation and are invited to explore its structure and to complete it using their personal previous mathematical experience;

3. structured situations, where students pose problems reformulating or varying given problems.

In order to provide students with such meaningful contexts, a precious contribution is given by *cultural artefacts* (Bonotto 2013). Thanks to its complexity and richness in mathematical meaning, an artefact lives in both the world of symbols and the real one, creating a sort of hybrid space that connects mathematics and everyday contexts. A re-mathematization process is thereby favoured, wherein students are invited to unpack from artefacts the mathematics that has been hidden in them, in contrast with the de-mathematization process in which the need to understand mathematics that becomes embodied in artefacts disappears (Gellert & Jablonka 2007). As a consequence, movement from common use situations to mathematical structures and vice-versa is allowed, in agreement with the process of modelling. Moreover, by removing some data from an artefact, we can stimulate students to face out with new mathematical goals, such as create new concepts or applications (Bonotto 2005). Artefacts represent in this direction a valuable tool to offer students opportunities also in emergent-modelling, connecting modelling and problem-posing.

The theoretical arguments supporting the importance of problem-posing in school mathematics are supported by a growing body of empirical research. Various aspects of problem-posing had been studied in literature, such as examining thinking processes related to problem-posing (Brown and Walter 1990; Christou, Mousoulides, Pittalis, Pitta-Pantazi and Sriraman 2005), or including problem-posing in mathematics activities. In particular, several studies focused on the relations between problem-posing and problem solving (section 2.3.1) and/or between problem-posing and creativity (section 2.3.2).

2.3.1 Problem-posing and problem solving

One of the most important direction of research on problem-posing is studying its relations with problem solving.

As described by Silver (1994), problem-posing can occur at three stages in relation to problem solving. The first stage consists in problem-posing prior to solving a problem, when a problem is generated from a given situation, and the goal is not the solution itself but the creation of a new problem. The second stage concerns problem-posing while solving a problem, wherein a solver recreates a given problem in some ways to make it more accessible for solution. The third stage refers to problem-posing after having solved a problem, indeed one might explore the conditions of that problem to generate related problems.

In addition to the described relations between problem-posing and problem solving, it has been proved that students engaged in problem-posing activities improved their problem solving abilities (Van Harpen and Presmeg 2013; Cai and Hwang 2002; Silver 1996; Ellerton 1986). Indeed, problem-posing affords students the unique opportunity to improve their problem solving skills while developing their academic skills to encounter and solve problems in mathematics and beyond (Kopparla et al. 2018).

2.3.2 Problem-posing and creativity

Another aspect of problem-posing that was investigated in literature is its relationship with students' creativity. Problem-posing, in fact, is a form of creative activity that can operate within tasks involving rich situations (Freudenthal 1991), using real-life artefacts and human interactions (English 2009). Creativity is directly linked to the mathematical activity of problem-posing, being the act of creating mathematical problems in specific contexts (Bonotto and Dal Santo 2015).

Creativity started receiving attention in 1950, when Guilford (1959) proposed a distinction between two types of thought: *divergent thinking* and *convergent thinking*. In particular, divergent thinking was characterized by *fluency, flexibility* and *originality,* that represented the three aspects of creativity (Guilford 1959). Fluency in thinking refers to the quantity of output; flexibility in thinking refers to a change of some kind (meaning, interpretation, use of something, strategy); originality in thinking means the production of unusual, remote or clever responses.

The creative process in school mathematics may be encouraged by the presence of semi-structured situations (Stoyanova and Ellerton 1996). In particular the use of cultural artefacts can help creating such situations.

> Through the use of artefacts, children can be encouraged to recognize a great variety of situations as mathematical situations, or more precisely *mathematizable* situations, by

> asking them: (a) to select other artefacts from their everyday life; (b) to identify the mathematical facts associated with them; (c) to look for analogies and differences; (d) to generate problems. (Bonotto and Dal Santo 2015, pp. 109-110).

Several studies used problem-posing and problem solving to promote and assess creativity (Xie and Masingila 2017; Bonotto and Dal Santo 2015; Bonotto 2013; Yuan and Sriraman 2010; Leung 1997; Silver 1997, Leung and Silver 1997; Sriraman 2009), proving that an inquiry-oriented mathematics instruction, including problem-posing activities, could assist students to develop more creative approaches to mathematics.

However, given the value of problem-posing activities as opportunities for measuring students' creativity, or other mathematical learning outcomes, it is mandatory to develop and validate suitable problem-posing instruments, understanding which kind of problem-posing tasks best reveal students' creativity and their mathematical understandings (Cai, Hwang, Jiang and Silber 2015).

2.3.3 Problem-posing analytic schemes

Several analytic schemes have been created to evaluate students' or teachers' problem-posing performances.

A first analytic scheme to examine problem-posing of middle school students was developed by Silver and Cai (1996). Students' problem-posing responses were firstly categorized as *mathematical questions*, *non-mathematical questions* or *statements*. Then, mathematical questions were divided in *solvable* and *not-solvable*. In the specific, problems were considered to be not solvable if they lacked sufficient information or if they posed a goal that was incompatible with the given information. The last step involved examining the complexity of the posed problems. Complexity was considered under two perspectives: *syntactic complexity* and *semantic complexity*. Syntactic complexity consisted in the presence of *assignment*, *relational* or *conditional* propositions. In agreement with Mayer, Lewis and Hegarty (1992), the presence of conditional or relational propositions could be taken as an indication of problem complexity. Semantic complexity involved the number of semantic relations used. Such semantic relations had been taken from Marshall's (1995) five categories: *change*, *group*, *compare*, *restate*, *vary*. Silver and Cai (1996) scheme is shown in Fig. 11.

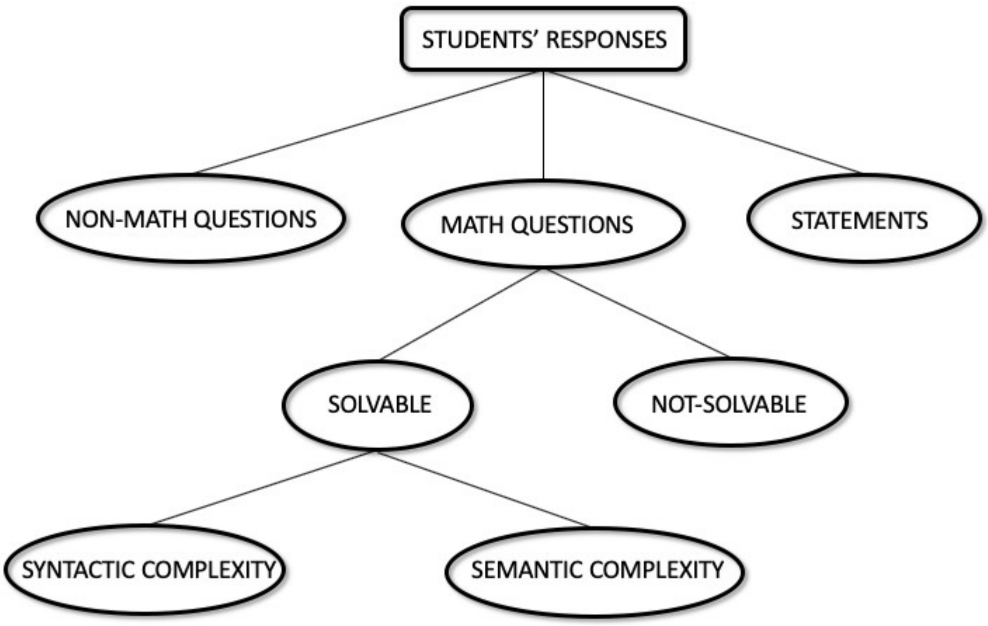


Figure 11. Silver and Cai (1996) scheme

Another significant analytic instrument for problem-posing was developed in Leung and Silver (1997): the *Test for Arithmetic Problem Posing* (TAPP). TAPP evaluated students' responses in semi-structured situations in terms of quality and complexity. Regarding the quality, a three-step process was implemented. First, each statement was classified as *mathematical* or *non-mathematical problem*. Next, each mathematical problem was classified as *plausible* or *implausible*. A problem was considered plausible if the initial state of the posed problem appeared to be feasible and no discrepant information could be found. Finally, plausible mathematical problems were analysed respect to the sufficiency of the information provided for solution of the posed problem. With respect to complexity, responses were classified according to the arithmetic complexity of the solution of the posed problem. In the specific, posed problems were judged on the basis of zero, single or multi-step for its solution. While in Silver and Cai (1996) scheme complexity was analysed in terms of the number of semantic relations and of the syntactic differences of problems, in this case complexity was studied only in terms of the number of steps required to solve a posed problem. In Fig. 12 Leung and Silver (1997) scheme is reported.

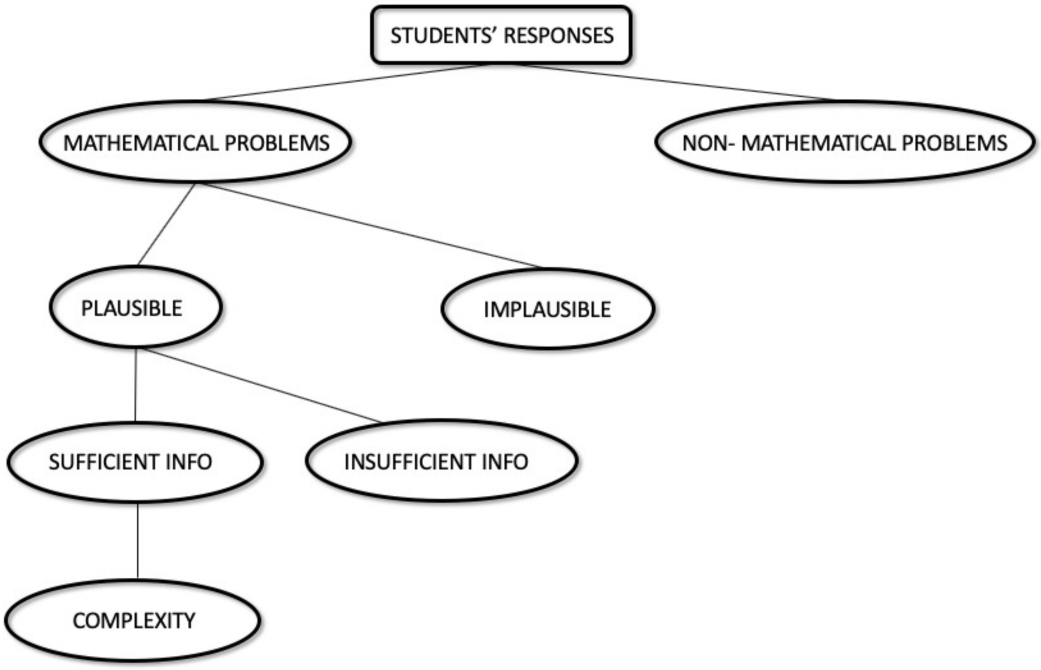


Figure 12. Leung and Silver (1997) scheme

Christou et al. (2005) proposed a model which enables young students' problem-posing thinking to be described by four processes: *editing*, *selecting*, *comprehending*, *translating*. Editing quantitative information involved tasks that requires students to pose a problem without any restriction from provided information, stories or prompts (Mamona-Downs 1993). Selecting quantitative information refered to tasks that require students to pose problems or questions that are appropriate to specific given answers. Comprehending quantitative information associated to tasks in which students pose problems from given mathematical equations or calculations, requiring understanding the meaning of the operations and an algorithm approach (English 1998). Translating quantitative information required students to pose problems or questions from graphs, diagrams or tables. Such a model incorporated semi-structured and structured situations (Stoyanova 1998).

More recently, Kopparla et al. (2018) developed a rubric to grade problem-posing questions. Problems posed by students, are analysed following three directions: *structure and context*; *mathematical expression*; *appropriateness of the design*. To each of these three categories can be associated an index of satisfaction from 1 to 4: 1=unsatisfactory, 2=minimal, 3=moderate, 4=satisfactory (Table 4).

Table 4. Kopparla et al. (2018) rubric to grade problem-posing questions

	1	**2**	**3**	**4**
structure-context	no complete problem posed	exact the same word choice as the given	moderate changes in numbers or scenario	all numbers and scenario original
mathematical expression	no setups to solve the problem	setup does not align to the question	problem statement aligns somewhat with student's setup to solve	problem statement aligns completely with student's setup to solve
appropriateness of design	scenario not realistic or solvable	scenario somewhat realistic or solvable with incorrect use of units	scenario somewhat realistic or solvable with partially correct use of units	scenario is realistic or solvable with accurate use of units

Researchers created and applied different schemes also to evaluate students' creativity in problem-posing. One of the most important schemes to measure participants' creativity was the *Torrence Test of Creative Thinking*, based on the four factors of fluency, flexibility, originality and elaboration (Leung and Silver 1997; Yuan and Sriraman 2011). The test consisted of four items as modifications of the tasks used by Getzels and Jackson (1962). The four items were obtained by crossing two problem contexts with two information content formats, respectively with or without specific numerical information.

More recently, Xie and Masingila (2017) proposed a scoring rubric to assess prospective teachers' problem-posing performances. In particular, in relation to a given problem, they analysed teachers' posed problems in terms of quality, complexity and creativity (Table 5).

Table 5. Xie and Masingila (2017) rubric to asses prospective teachers' problem-posing performance.

category of the posed problem	3 points	Solvable mathematical problem
	2 points	Unsolvable mathematical problem
	1 point	Not a mathematical problem
understanding of problem-posing and problem-posing strategy	3 points	Complete understanding
	2 points	Partially understanding
	1 point	Poor understanding
creativity of the posed problem	3 points	Completely different problem
	2 points	Somewhat different problem
	1 point	Comparable problem (with similar structure)
complexity of the posed problem	3 points	Need three or more operational steps to solve
	2 points	Need two operational steps or one operational step to solve
	1 point	No operational step needed

2.3.4 Emergent problem-posing

In this section we take another perspective concerning problem-posing, that is no long seen as an end in itself, but as a means to extend students' mathematical knowledge and skills (Klaassen and Doorman 2015). In this direction the focus on a particular aspect of problem-posing, defined as *emergent problem-posing.* To define and justify this new term we start making a connection with the notion of emergent modelling.

Recall that emergent modelling (Gravemijer 1999) is a long-time learning process in which students begin to model their informal mathematical strategies and arrive to re-invent mathematical concepts and applications they need (see section 2.2.4). These concepts and applications can be subsequently formalized in mathematical terms and generalized to other situations. As a consequence, emergent-modelling can be seen as a process in which a model develops from an informal, situated model (*model of*) into a generalizable mathematical structure (*model for*), that increases formal mathematical reasoning (Gravemeijer 1999) and sense-making. It is evident that emergent modelling was introduced with the meaning of supporting the emergence of formal mathematical ways of knowing. Indeed, in this perspective modelling activities are used as a vehicle for the development, rather than applications, of mathematical concepts (Greer, Verschaffel and Mukhopadhyay 2007). Students, starting from a real context, begin to model their informal mathematical strategies and arrive to re-invent (Freudenthal 1991) mathematical concepts and applications they need.

When generating a problem, students do not always take into account possible solving strategies related to that problem, instead often they are not able to solve the problems they have posed. In this situation, problems posed by students that require new mathematical knowledge for their solution can be used as a vehicle to introduce new mathematical concepts. Moreover, these new concepts assume meaning for students, because rooted in their personal experience and for the specific purpose of solving the problems posed by themselves. As a consequence, new mathematical knowledge should be not only introduced, but also re-invented (Freudenthal 1991) by students. Similarly to emergent modelling, we call this aspect of problem posing as *emergent problem-posing*, highlighting its aim to support the emergence of formal mathematical ways of knowing.

Emergent problem-posing is also connected to the process of *prospective learning* (Freudenthal 1991), in which informal contexts play a prominent role in offering students opportunities to improve their knowledge, before to deal with more systematicity and formalism. Emergent problem-posing should reinforce prospective learning, motivating and supporting students in creating new mathematical knowledge from informal contexts.

2.3.5 Directions for the future

To conclude this section, the main results concerning research on problem-posing in mathematics education could be sum up in four main themes (Ellerton, Singer and Cai 2015):

- problem-posing can transform attitudes towards mathematics so that the object of mathematics is the problem not just the solution of a problem;
- problem-posing can be an agent of change in the mathematics classroom;
- through purposeful planning, problem-posing can be integrated into school mathematics curricula;
- problem-posing can be seen as a natural link between formal mathematics instruction, problem solving and modelling;

However, when implementing problem-posing activities several difficulties can be encountered (Hansen and Hana 2015):

- *posing mathematically relevant problems*: distinguish between problems that are mathematical relevant and problems that are not is a competence that both students and teachers should become proficient;

- *posing mathematical suitable problems*: which problems are not too difficult neither nontrivial for the students? A fundamental skill is to be able to reformulate problems and choose such contexts that attain a reasonable degree of mathematical sophistication;

- *posing problems such that pupils feel ownership of the problems:* problem-posing is an ongoing process, where reformulations and adjustments are required also by students;

- *making problem-posing a relevant part of the learning trajectory*: if problem-posing should be seen as an integral part of mathematics classes, it must be connected to other mathematical activities in the classroom;

- *incorporating the teaching of mathematical content with problem-posing*: two main difficulties can be individuated. The first, not communicating the intent to students and second, posing-problems to a little-known mathematical topic, especially when the topic is a specific real-world situation.

Consequently, further research is needed for the future (Ellerton, Singer and Cai 2015), particularly on:

- the development of problem-posing skills for in-service and pre-service teachers' education;

- the possible connections between problem-posing and mathematical creativity;

- the links between problem-posing and problem solving;

- knowing more about the potential of problem-posing to support students' learning.

2.4 Real contexts

Contexts for mathematics lessons play a crucial role in bridging modelling and problem-posing to promote students' reasoning, critical thinking and give meaning to their mathematical activity. Which kind of contexts are suitable to foster students' reasoning? Which contexts could help in bridging the gap between in- and out-of-school mathematics? In our perspective, such contexts should be connoted as *real*. As a consequence, a real context needs to defined, as well as a real mathematical task. To pursue this goal, in literature several approaches emerged. We present two of them. The first one refers to *realistic* and *rich* contexts in the perspective of RME. The second one is a framework developed by Palm (2006) concerning concordance between mathematical school tasks and the corresponding out-of-school situations.

2.4.1 RME context problems

As we have seen in section 2.1 the core principle of RME is that mathematics education should take its point of departure primarily in mathematics as an activity, and not in mathematics as a ready-made-system (Freudenthal 1971; 1973; 1991). Students, in fact, are active participants in the learning process. Mathematics is a human activity (Freudenthal 1991): you do mathematics through mathematization (Treffers 1987). Therefore, learning becomes a constructed understanding through a continuous interaction between teacher and students, that can be synthetized, using Freudenthal's words, in teaching and learning as *guided reinvention*. Anchoring points for the reinvention of mathematics are offered by *context problems*. Context problems are problems of which the problem situation is experientially real to the student (Gravemeijer and Doorman 1999). As a consequence, problems should come from the real world, but also from a fantasy world or from the mathematics itself, until they are experientially real for the student (Van den Heuvel-Panhuizen and Drijvers 2014).

Therefore, in contexts problems the contexts on which the mathematical task is based can be defined as *realistic*, i.e. experientially significant for students. However, this realistic connotation is not sufficient to have a valuable mathematical problem. The context, indeed, must also be *rich* (Freudenthal 1991). A rich context is a context that promote a structuring process as a means of organizing phenomena, physical and mathematical, and even mathematics as a whole, i.e. contexts that give more opportunities in the mathematization process. In conclusion, in this perspective a real context is a realistic and rich context, and a real mathematical task is a mathematical task based on a realistic and rich context.

2.4.2 Palm's framework

Another perspective for real mathematical tasks was introduced by Palm (2006), who developed a framework to depict that aspects of real-life situations that should be considered important in their simulations (Fig. 13).

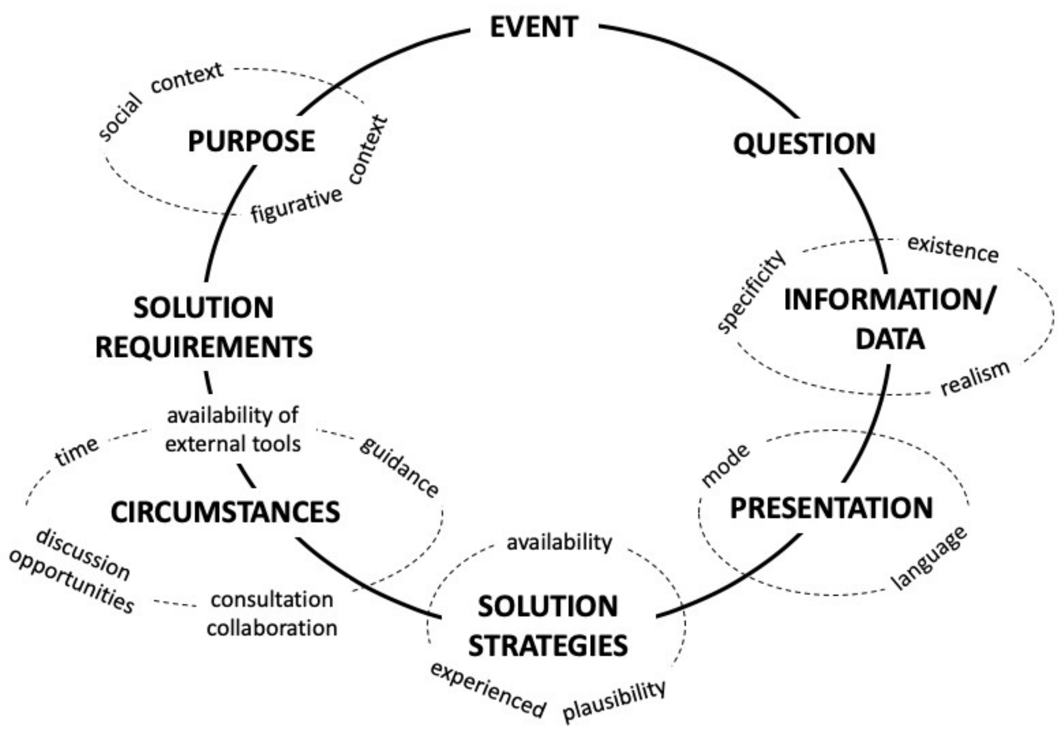


Figure 13. Palm's (2006) framework for simulations of real-life situations

In Table 6 for each aspect a key question that helps in reasoning about the concordance between mathematical school tasks and corresponding situations in real-life beyond the mathematics classroom are reported.

Table 6. Key questions associated to real-life aspects in Palm’s (2006) framework

Event	Could the event described in the school task take place in real-life?
Question	Does the questions in the school task might be posed in a corresponding real-life event?
Information/Data	
Existence	Do the information available in the school task exist in a corresponding real-life event?
Realism	Are the values given in the school task close to values in a corresponding real-life event?
Specificity	Can specifications of the school task context be compared to a reasonably extent to the corresponding out-of-school situation?
Presentation	
Mode	Is the problem communicated orally or in a written form? Are the information presented in diagrams, tables, graphs, …?
Language	Does the language used in the school task not negatively affect the possibilities for students to use the same mathematics as they would have used in a corresponding real-life event?
Solution strategies	
Availability	Do the solution strategies available to the students solving the school task match with those available to the persons described in the tasks as solving the corresponding tasks in real-life?
Experienced plausibility	Do the solution strategies experienced as plausible for solving the task in the school situation match

	with those experienced as plausible in a corresponding real-life event?
Circumstances	
Availability of external tools	Which concrete tools outside the mind can be used to solve the task?
Guidance	Which explicit or implicit hints are given?
Consultation and collaboration	Which kind of inputs from other people are took into account?
Discussion opportunities	Are there occasions for students to ask about and discuss the meaning and understanding of the task?
Time	Do time restrictions not cause significant differences in the possibilities of solving the school task compared with the corresponding real-life event?
Consequences of task solving success	Are there any efforts to promote motivation to solve the task?
Solution requirements	What is an appropriate solution in a corresponding real situation?
Purpose	
Figurative context	Is the purpose of the task in the figurative context as clear to the students as it is for the solver in the corresponding real event?
Social context	Is the purpose in the social context of the school situation permit similarities in actions between the in- and out-of-school situations?

If we look at the aspects described in Table 6, we can divide them in two categories. The first category deals with aspects that describe what to take into account to have a real mathematical task, in the sense of making it close as much as possible to a corresponding real-life situation. These aspects are: *event, questions, information and data, solution strategies, time, solution requirements, purpose*. The second category concerns some aspects that could increase the efficacy of an activity based on a real mathematical task. These aspects are: *specificity, presentation, circumstances, time, consequences, purpose.* The aspects *consequences* and *purpose* are shared between the two categories. As a consequence, to define what a real task is in this framework we focus on the first category. In order to define what could be a real context, the aspects of these category can be synthetized in three main connotations: *feasibility, availability* and *appropriateness*. With feasibility we refer to the fact that the context, and eventually the problem formulated starting from it, should actually occur in real-life. Availability means that students should have at their disposal information and strategies to solve the problem sufficiently close to real-life data. Appropriateness concerns with the fact that the purpose must be clear to students, and consequently the students are able to discern which solutions could be considered as appropriate in relation with the context of the problem. In conclusion, in this perspective a real context is a feasible, available and appropriate context, and a real mathematical task is a mathematical task based on such contexts. These real contexts are closed to the notion of artifacts (Bonotto 2013), that, thanks to their complexity and richness in mathematical meaning, live in both the world of symbols and the real one, creating a new dialectic between school mathematics and extra school experiences, by bringing students' everyday-life experiences and informal reasoning into play.

Comparing the two proposed frameworks, two main dichotomies can be observed. The first is between *realistic-rich* and *feasibility*. In RME, a realistic and rich context is a context which is meaningful both for the student and for mathematics. As a consequence, a context can come not only from the real-world, until it is significant for students and rich of mathematical stimulus. In Palm's framework, instead, the accent is given to the feasibility of the context, in the sense that both the event and the related problems described in the context of the mathematical task have to occur in real-life.

The second main difference is between *re-invention* and *availability*. Contexts described in RME as to be real, take their features in the view of the teaching of mathematics as a process of guided re-invention. As a consequence, real contexts should offer the opportunity to students to re-invent mathematical concepts, strategies, algorithms, reinforcing in this way prospective learning. In Palm's perspective, instead, students should have at their disposal solution strategies to solve the task, that have to match with those available to the persons solving the corresponding tasks in real-life. In conclusion, despite these differences, both the perspectives concord in the fact that students' motivation should be promoted when engaged in solving mathematical problems, and so in mathematical problems based on real contexts.

2.5 Research questions

In the previous section we outlined the context and theoretical background of the research. In the specific, we remarked how the teaching of mathematics still have a stereotyped nature in which students are requested to apply mechanical rules to solve standard problems. In the Italian contexts, some studies outlined how is common a persistence of situations of disorientation and uncertainty and a certain resistance to abandoning traditional teaching methods of mainly transmission type (INNS 2017). In this situation, this Univeristy project, whose our research is part of, aimed to providing mathematics teachers with methodological models and format of school practices based on mathematical modelling, in order to reduce the gap between in- and out-of- school mathematical competencies and to foster students' reasoning and sense-making.

The goal of this research is to design a re-invention process (Freudenthal 1991) to integrate mathematical modelling in the regular school practice in the Italian context. In the specific, we want to investigate how this process can be implemented and used to help students give sense to their mathematical activity. Thus, the main question of this project is:

How, and to what extent, can mathematical modelling be integrated in the teaching and learning of mathematics in a guided re-invention paradigm?

In section 2.1 the RME underlying theory of this project was presented. A fundamental characteristic of this theory is that learning occurs through experience, the experience of mathematizing experientially real situations, extending day-to-day reasoning to acquire new (mathematical) knowledge. In this project, we investigated how the design heuristics of didactical phenomenology and emergent modelling should support such a re-invention process. As a consequence, a central role will be covered by the choice of contexts for mathematical problems that must be experientially significant for students and able to evoke new mathematical concepts or strategies for their solution. Therefore, a learning trajectory that brings students to invent their mathematical principles in a modelling environment needs to be designed. To design such a re-invention process, our choices are the design heuristics of didactical phenomenology and emergent modelling, and the use of problem-posing in relation to such heuristics. How can these choices be used? Why and to what extent they work? Are students able in the designed learning process to re-create (their) new mathematics rooted in an informal day-to-day reasoning? As a consequence, our main research question was split in two more specific questions. The first research question deals with the design of modelling activities with the focus in the promotion of students' creation of new mathematical concepts or strategies to solve a real problem. In the specific, we investigated how activities designed following the MEA principles could foster the emergent nature of modelling. The second research question consists in start investigating the impact in the use of different contexts during problem-posing activities in terms of students' creativity and emergent problems. The resulting research questions are the following:

RQ1. *How can Model Eliciting Activities promote the process of emergent modelling?*

RQ2. *How do different contexts influence the process of problem-posing?*

Concerning the second research question, we focused on two aspects of the problem-posing process, namely its relations with creativity and emergent problem-posing. As a consequence, from the second research question, two more specific sub-questions emerged:

RQ2.1. How do different contexts influence students' creativity in problem-posing?

RQ2.2. How do different contexts influence emergent problem-posing?

In the following chapter the methodology of design research will be described.

3. Methodology

In this chapter the research methodology to answer to the research questions is described. The starting point consists in justifying the choice of the research method of *design research*.

As stated in the previous section, the aim of this research project consisted in understanding how to integrate mathematical modelling in the Italian school context, that is still characterized by traditional transmissive teaching methods. From the motivations for this study and the theoretical background, two specific research questions had been formulated in order to pursue our goal. The first research question consisted in investigating how can MEAs enhance the emergent nature of modelling, while the second research question consisted in investigating the use of different contexts in problem-posing activities, and their consequences in terms of creativity and emergent problem-posing.

To be able to answer to the research questions we had to create an instructional environment with which it could be possible to study how and to what extent the suggested processes could be fostered. An instructional sequence was therefore necessary to answer to the research questions. Moreover, the research had an explorative character, since the research questions aim at quite new aspects in mathematics education specifically for the Italian context, and consequently a research design that allowed for revising theories, hypothesis and instructional activities was needed. Furthermore, new teaching materials that support new types of learning must be developed, making the design process an integrated part of the research. A research approach that consists in planning and creating innovative educational settings and analyzing teaching and learning processes is represented by the methodology of design research.

In section 3.1 we present the main characteristics of the design research methodology. Section 3.2 describes the phases of design research applied to this research project.

3.1 Design research

Design research in education is research in which the design of new educational materials (learning activities, professional development programs, etc.) is a crucial part of the research (Bakker 2018). Design research does not describe or evaluate education as it currently is, but it is about education as it could be or even as it should be. Design research can be defined

> As the systematic study of designing, developing and evaluating educational interventions, - such as programs, teaching-learning strategies and materials, products and systems – as solutions to such problems, which also aims at advancing our knowledge about the characteristics of these interventions and the processes to design and develop them. (Plomp 2010, p.9)

Moreover, the design of instructional activities is more than a necessity for carrying out teaching experiments. The design process forces the researcher to make explicit choices, hypothesis and expectations that otherwise might have remained implicit.

> (…) design research explicitly exploits the design process as an opportunity to advance the researchers' understanding of teaching, learning, and educational systems. Design research may still incorporate the same types of outcome-based evaluation that characterize traditional theory testing, however, it recognizes design as an important approach to research in its own right. (Edelson 2002, p. 107)

In various countries different names to similar approaches had been given: *developmental research* (Freudenthal 1988; Gravemeijer 1994b; Lijnse 1995; Romberg 1973; Van den Akker 1999); *design experiments* (Brown 1992; Cobb, Confrey, diSessa, Lehrer and Schauble 2003; Collins 1990, 1992); *design based research* (Hoadley 2002); *educational design research* (McKenney and Reeves 2012; Plomp and Nieveen 2013;

Van den Akker, Gravemeijer, McKenney and Nieveen 2006); *formative experiments* (Reinking and Bradley 2008).

A key characteristic of design research is that educational ideas for students' or teachers' learning are formulated in the design but can be adjusted during the empirical testing of these ideas. As a consequence, due to its component of adaptation of the learning trajectory through the research, design research is particularly suitable in situations where a full theoretical framework is not yet available and where hypothesis is still to be developed (Drijvers 2003).

Design research includes a design or development function, but also an advisory function, consisting in giving theoretical insights into how particular ways of teaching and learning can be promoted.

Design research is very closed to action research. Action research focuses on solving a practical problem and aims to produce practical guidelines (Brandbury 2015; Denscombe 2014). Like action research, design research is interventionist and open, involves a reflective and cycle process and aims to bridge theory and practice (Opie and Sikes 2004). However, in design research, design is a crucial part of the research, whereas in action research the focus is on action and change.

Cobb et al. (2003) identified five key characteristics of design research:

1. the purpose of design research is to develop theories about learning and the means that are designed to support that learning;

2. design research has an interventionist nature. Changes and understanding are intertwined, because *if you want to change something you have to understand it, and if you want to understand something, you have to change it* (Bakker 2004);

3. design research has both prospective and reflective components. During the implementation of some learning hypothesis (*prospective part*) the researcher confronts conjectures with actual learning that is observed (*reflective part*). Such reflective analysis can lead to changes to the original plan for the next lesson;

4. design research has a cyclic nature, forming an iterative process of invention and revision. Each cycle typically consists of three phases: *preparation and design*; *teaching experiment*; *retrospective analysis*;

5. the theory under development has to do real work. Theory generated from design research is usually developed for a specific domain, but it must also be general enough to be applicable in different contexts such as classrooms in other schools or other countries.

To sum up, design research aims at generating empirically grounded theories. Its main result is not a design that works, but the reasons how, why and to what extent it works. The first point consists in developing an instructional design for investigating and generating theoretical conjectures. Then, in relations to the considered questions, the analysis of the teaching experiments focuses on various aspects of the design, of students' reasoning, classroom discussion and development of classroom norms (Cobb et al. 2003).

In this research, for each research question two cycles had been implemented (Fig. 14): cycles M-I and M-II concerning the first research question about mathematical modelling and cycles PP-I and PP-II concerning the second research question about problem-posing. Each cycle is considered in concatenation with the next one as a spiral. Those cycles had been preceded by a preparation phase, common to both the modelling and problem-posing cycles, that consisted in an analysis of the theoretical background concerning the research topic and an empirical study. The first phase of each cycle consisted in the design phase and included the development of a *Hypothetical Learning Trajectory* (HLT). This phase followed by the teaching experiment whose aim consists in providing empirically based arguments to justify or refute the hypothesis conjectured in the previous phase. The last phase is represented by the retrospective analysis. The reflection at the end of the first cycle led to adapting the conjectures and the teaching sequence, which became the starting point for the second cycle. This cyclic process aimed at empirically grounded answers to hypothesis concerning the research questions. Ideally, such instructional sequence should converge into a sequence that works best within the constraints of the educational setting in order to develop a local instructional theory. As a consequence, the sequence should be tried out and analyzed in various situations, as well as be discussed with other parties who play a role in educational innovation, such as teacher training institutes and educational publishers. However, we were not able to go through all these phases, and we let them for future work.

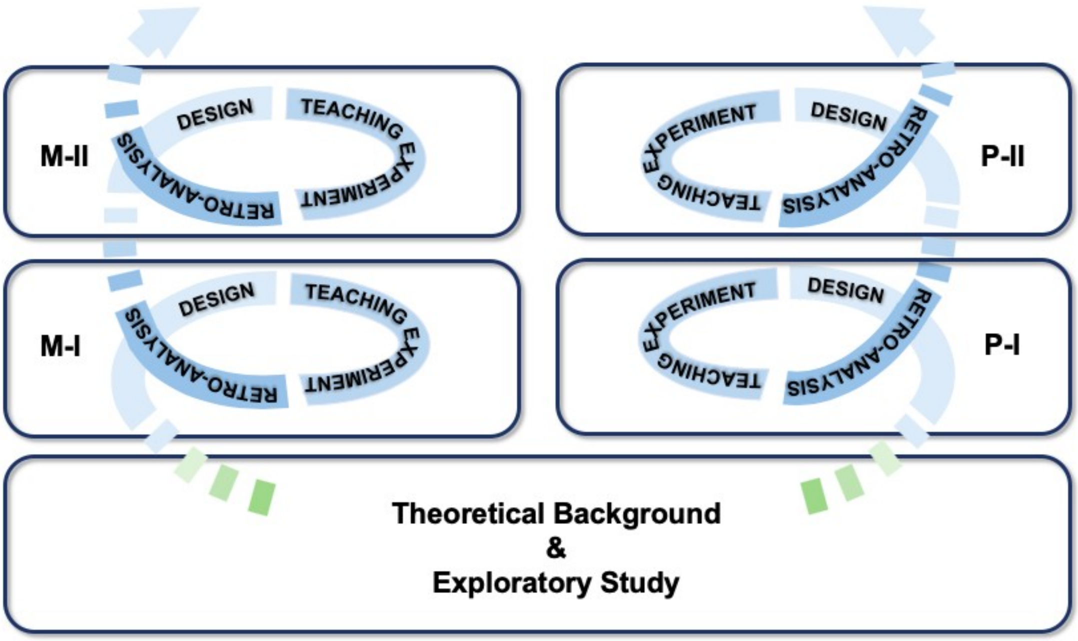


Figure 14. Design research cycles for this research project

3.2 Design Research Phases

As described in the previous section, each cycle of design research was made by three main phases: a design phase, a teaching experiment and a retrospective analysis. In this section the common features of each of those phases are described. In conclusion, section 3.3 will focus on validity and reliability in design research, and particularly for this thesis.

3.2.1 Design

In this research the design phase was explicated by the development of a hypothetical learning trajectory (HLT). This phase was preceded by an analysis of the theoretical background concerning the didactical problems we are addressing, and by an explorative study, that together constitute the preparation phase.

The first phase of each research cycle consisted in the development of an HLT. This term was introduced by Simon (1995), as a key part in developing an extension of the teaching experiment methodology, called *Mathematics Teaching Cycle*. It consists in a teachers' plan for classroom activities. HLT refers to teachers' prediction as to the path by which learning might proceed (Simon 1995). The HLT is made by three components (Fig. 15): the *learning goal*, that defines the direction; the *learning activities*; the *hypothetical learning process,* that is a prediction of how students' thinking and understanding will evolve in the context of the learning activities.

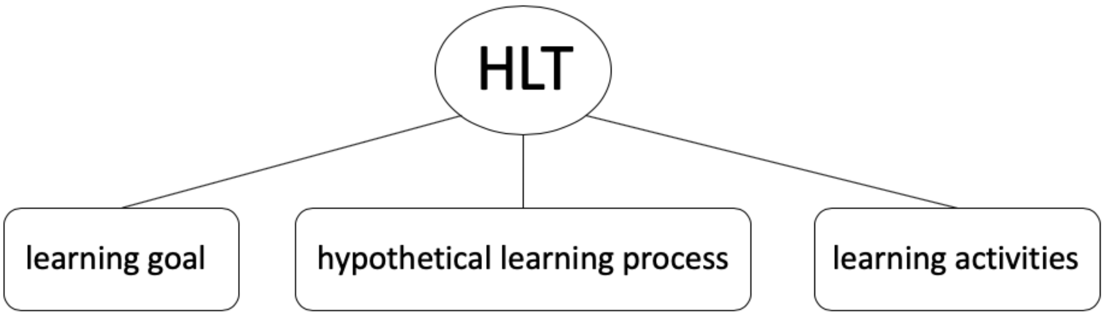


Figure 15. Components of a hypothetical learning trajectory

The development of the hypothetical learning process and of classroom activities are closely related: ideas for the learning activities depend on teachers' hypothesis about the development of students' thinking and learning, and vice-versa, the generation of hypotheses of students' conceptual development depends on the nature of the anticipated activities. The design of instructional activities in this study included the development of teacher guides, tasks, student booklets, artefacts, possible solutions to the assignments and tests. While designing these materials, choices and intentions were captured and motivated, to inform the teacher and to keep track of the development of the designer's insights.

While designing instructional activities, the key question is what meaningful problems may foster the students' cognitive development according to the goals of the HLT. The important criterion for selecting an activity was its potential role in the HLT towards the end goal of distribution. Would it possibly lead to types of reasoning that students could build upon towards that end goal? Would it be challenging? Would it be a meaningful context for students?

The design process was guided by the RME heuristic of didactical phenomenology. Didactical phenomenology aims at confronting students with phenomena that beg to be organized by means of mathematical structures. In this way, students are invited to develop mathematical concepts from meaningful contexts (de Lange 1987; Treffers 1987). As a consequence, the point is to find meaningful problem contexts that may foster the development of the targeted mathematical concepts. The heuristic of didactical phenomenology directly cooperates with another heuristic: guided reinvention. Guided reinvention involves reconstructing the natural way of developing a mathematical concept from a given problem situation. One way to do this is to try to *think how you might have figured it out yourself* (Gravemeijer 1994a, p. 179).

Together with the end goal and classroom activities, the development of an HLT involves also the assessment of the starting level of students' understanding. The classroom activities are designed to foster productive mental activities by the students, and are accompanied by the designer's description of why the instructional activity is supposed to work and what kind of mental development is expected to be elicited. This sequence of activities, motivations and expectations makes explicit the hypothetical learning process in terms of student activities and cognitive development (Drijvers 2003).

In designing a chain of activities, the designer makes use of his domain specific knowledge, his repertoire of activities and representations, his teaching experience, and his view of the teaching and learning of the topic. Such hypothetical trajectory is empirically tested by means of a teaching experiment, then the HLT could be adapted and changed. These changes, based on the experiences in the classroom, start a new round through the next research cycle.

HLT is not a rigid structure that must be followed by all, instead it represents a learning route that is broader than one single track and has a particular bandwidth, wherein students can go through it at different speeds.

The HLT can be seen as the link between an instruction theory and a concrete teaching experiment. Moreover, an HLT has different functions depending on the phase of the design research and continually develops through the different phases. Indeed:

- during the design phase, the HLT guides the design of instructional materials that have to be developed or adapted. The confrontation of a general rationale with concrete activities often leads to a more specific HLT, which means that the HLT usually develops during the design phase (Drijvers 2003);

- during the teaching experiment, the HLT functions as a guideline for the teacher and researcher what to focus on in teaching, interviewing, and observing. HLT should be adjusted for the next lesson, because of incidents in the classroom such as anticipations that have not come true, strategies that have not been foreseen, activities that were too difficult, and so on. In design research, changes in the HLT are made to create optimal conditions and are regarded as elements of the data corpus. This means that these changes have to be reported well and the information is stronger if changes are supported by theoretical considerations;

- during the retrospective analysis, the HLT functions as a guideline determining what the researcher should focus on in the analysis. Because predictions are made about students' learning, the researcher can contrast those anticipations with the observations made during the teaching experiment. Such an analysis of the interplay between the evolving HLT and empirical observations forms the basis for developing an instruction theory. After the retrospective analysis, the HLT can be reformulated, in an often more drastic way than during the teaching experiment, and the new HLT can guide a next design phase.

In this research HLT was used in the following way. The first phase of each cycle started with considering the specific classroom context, with choosing a mathematical topic on which focus on, with formulating a problem, with analyzing the concepts and with reflecting on possible solutions. The proposed solutions were made concrete in the form of student activities that defined the hypothetical learning trajectory. The HLT finally was condensed into a table that contains its components of learning goal, hypothetical learning process and learning activities.

3.2.2 Teaching experiments

The second phase of the design research cycle is the phase of the teaching experiment, in which the prior hypothesis embedded in the HLT and the instructional activities are confronted with classroom reality. The term *teaching experiment* is borrowed from Steffe (1983); see also Steffe & Thompson (2000). The word *experiment* does not refer to a comparison between an experimental group and a control group, but to an experimental classroom setting that is created as a result of the innovative teaching materials provided. In the teaching experiments the instructional sequence is carried out by the teacher and the students, and the overall goal is to understand and improve the initial design on the basis of students' reasoning with respect to the created educational setting (Doorman 2005).

Before the start of the teaching experiments we spoke with the teachers about the aims of the experiments, the teaching materials and the schedule of the teaching sequence. Since the teachers who participated with their classrooms to the research did not have any preparation on the strategies of mathematical modelling and problem-posing, we decided with them that the teaching experiments would be carried out by the researcher, with the help of the regular mathematics teacher in the classroom, joining in classroom discussions and explanations. Also, they approached students with questions concerning key concepts, or answered questions from students. The researcher made notes during all the lessons, evaluated each lesson and participated in students' group work, letting the students clarify what they were doing. We were aware that this participation influenced the students' learning processes, but we wanted to hear students express their thinking, and to create a classroom culture in which clarifying questions was part of the mathematical activities.

We focused on data that reflected that process and provided insight into the thinking of the students. The main sources of data, therefore, were observations of students behavior, the classroom discussions, written data from students, such as handed-in tasks, notebooks and tests.

The teaching experiments took place in Italian schools. In Table 7 an overview of the teaching experiments is reported, specifying: the classroom grade; the number of students; the characterization of the experiment; the types of data collection.

Table 7. Teaching experiments overview

Teaching experiment	School grade	Number of students	Mathematical topic	Data collection
M-I teaching experiment	2	19	Multiplication in N	pre-test; notes; students' materials; field observations; post-test
M-II teaching experiment	12	25	3D Euclidean geometry	pre-test; notes; students' materials; field observations; students' feedback; individual student report; teacher's report
PP-I teaching experiment	6	22	Fractions	pre-test; student performances in problem-posing and problem solving; field observations
PP-II teaching experiment	4	25	Decimal numbers	pre-test; student performances in

	problem-posing and problem solving; field observations

M-I cycle teaching experiment

The first teaching experiment concerning mathematical modelling, M-I cycle, was conducted in a second-grade class (age 7) composed by nineteen students during two weeks of regular mathematics lessons. The mathematical topic that was chosen for the modelling activity was multiplication in N. In relation to the research question RQ1, in this teaching experiment we wanted to study how emergent modelling can be fostered to help students understanding of some aspects of the multiplicative structure (distributivity property of multiplication over addition). Our hypothesis was that a modelling activity designed following a model eliciting sequence (Lesh et al. 2003) with the use of suitable artifacts could actually foster the emergent nature of modelling. The teaching experiment was divided in three sessions, designed by the author adapting the model development sequence of Lesh et al. (2003). In session one, students had to work before individually and after in couples to some comprehension questions concerning the next modelling activity. In session two, students in groups had to solve a modelling task, producing a final project. The third session consisted of group presentations of their projects. After those sessions, a post-test was administrated to students.

The research method for the data analysis was mixed quantitative and qualitative. The aim of the data analysis was to reconstruct the classroom progress, which resulted in an empirical grounded understanding of students' reasoning during the classroom activity. In order to be able to reconstruct the learning process and verify our hypothesis, different kinds of data were collected: pre-test; transcriptions of classroom dialogs; observations of group working; students' final projects; post-test.

The analysis of these data provided information on verifying our hypothesis, give first answers to the first research question and adjust the designed instructional sequence for the next research cycle.

This research cycle is described in chapter 5.

M-II cycle teaching experiment

The second teaching experiment concerning mathematical modelling, M-II cycle, was conducted in a twelfth-grade class (age 17) composed by 25 students. The mathematical topic for the modelling activity was 3D-Euclidean geometry. In relation to the first research question RQ1, the aims of the study were: to study how model eliciting activities could impact on the emergent nature of modelling; to provide teachers with design principles and materials based on mathematical modelling to be usable in their classrooms; to foster the understanding of some aspects of 3D-Euclidean geometry in a meaningful way. The teaching experiment was divided in four sessions, designed by the author adapting the model development sequence of Lesh et al. (2003). In session one, students were divided in groups. Each group had to work on some comprehension questions about the modelling task. In session two, the groups worked on the model eliciting activity, producing a final project in a multimedia presentation. The third session consisted of group presentations followed by a *Question & Answer* (Q&A) session. The final session consisted in an individual assignment on the whole activity. After those four sessions, a questionnaire was administrated to students, in order to collect some information concerning their perceptions on the entire modelling activity.

Data collection consisted of pre-test; students' projects; classroom observations; final students' individual reports; final questionnaire; teacher's report. Data analysis was mixed qualitative and quantitative, providing information on verifying our hypothesis, give first answers to the first research question and suggestions for future work.

This research cycle is described in chapter 6.

PP-I cycle teaching experiment

The first teaching experiment concerning problem-posing, PP-I cycle, was conducted in a sixth-grade class (age 12) composed by twenty-two students. At the moment of the intervention, students were working on fractions. The mathematical topic that was chosen for the problem-posing activity was fractions. In relation to the research question RQ2, the aim of the study was to investigate how different contexts should influence students' creativity in problem-posing. The teaching experiment was divided in two parts: the first represented by two problem-posing sessions, and the second that consisted in a problem solving activity based on some problems posed in the previous part.

Data consisted in students' pre-test; students' performances in the problem-posing sessions and results from the problem-solving activity. Problems posed by students were analysed referring to the starting context and level of creativity. An analytic scheme concerning students' creativity was developed.

The analysis of these data provided information on how to optimize the activities with respect to formulating student texts, contexts used, and information provided. Second, conjectures that paralleled the instructional sequence could be verified as far as the students were taught as intended, letting to adjustments to the sequence and our conjectured instruction theory. Third, this analysis led to new hypotheses concerning the choices made with respect to the research questions. The adjustments and the new hypotheses were objects of study in the research cycle P-II.

This research cycle is described in chapter 7.

PP-II cycle teaching experiment

The second teaching experiment concerning problem-posing, PP-II cycle, was conducted in a fourth-grade class (age 9) composed by twenty-five students. In this teaching experiment we started investigating how emergent problem-posing can actually be enhanced in the school practice. In particular we studied the impact of different contexts on emergent problem-posing. To pursue our goal, we conducted a teaching experiment in a primary school in which different contexts had been used as starting situations for problem-posing activities. The classroom involved in the study had never been engaged in problem-posing activities before the study. The mathematical topic was represented by decimal numbers, and in the specific addition between decimal numbers. The teaching experiment was split in two lessons: the first represented by two problem-posing sessions, and the second that consisted in a problem solving activity based on some problems posed in the previous part.

Data consisted in students' pre-test; students' performances in the problem-posing sessions and results from the problem solving activity. Problems posed by students were analysed in relation to the starting context and *emergent problems*.

Data analysis was mixed qualitative and quantitative, providing information on verifying our hypothesis, give first answers to the first research question and suggestions for future work.

This research cycle is described in chapter 8.

3.2.3 Retrospective analysis

The final phase of a design research cycle comprises the retrospective analysis. In the retrospective analysis the HLT is compared with students' actual learning, and on the basis of such analysis answers to the research questions can be formulated.

The first step consisted in selecting data from the teaching experiment. Criteria for selection were the relevance of the fragment to the research questions and to the HLT of this teaching experiment in particular. The relation between theory development and teaching experiments emphasizes that hypotheses are created and modified while interpreting the data available. The interpretation of data depends on the ability to reconstruct the learning and teaching process, that consists in understanding students' reasoning, on which ideas their reasoning builds and by which perspectives it is guided. The point is to reconstruct the classroom progress based on the data available, which should result in an empirically grounded understanding of what happened during the teaching experiments.

The data were organized into case studies, represented by the teaching experiments, of class discussions and of students' work during the mathematics lessons. These case studies were interpreted in terms of what preceded the lessons, the student activities, the teaching, and the tools provided. Interpretations were compared with other available data, such as students' written materials and data from another experiment in our research.

The results of the retrospective analysis formed the basis for adjusting the HLT and for answering the research questions.

3.3 Validity and reliability

This last section of this chapter addresses the concepts of *reliability* and *validity* of the research method of design research. In the specific, the question is how these criteria are met in this study based on design research, where observations and student materials are the main sources of data and interpretation and coding are the main techniques of analysis.

Internal validity

Internal validity refers to the quality of the data collections and the soundness of the reasoning that has led to the conclusions. To improve internal validity of this study we used several methods:

- in the retrospective analysis, we tested conjectures that were generated and tested at specific episodes at other episodes and other data material, such as field notes, tests, students' work (*source triangulation*);

- the succession of different cycles permitted to test the conjectures developed in earlier teaching experiments in later teaching experiments;

- different theoretical instruments were used to analyze single episodes (*theoretical triangulation*);

- theoretical claims are substantiated with transcripts to provide rich and meaningful contexts;

- in preparing the experiments, we had discussed our ideas and the instructional sequence with the teachers. Our ability to explain our intended goals for the experiment, their willingness to participate, their contributions to performing the teaching experiment, and their engagement during the actual experiments validate, to a certain extent, the teaching experiments in our project. A similar argument holds for the participating students.

External validity

External validity is mostly realized as the generalizability of the conclusions. The question is how we can generalize the results from these specific contexts as to be useful for other contexts. An important way to do so is by framing issues as instances of something more general (Cobb et al. 2003; Gravemeijer and Cobb 2001). The challenge is to present the results (instruction theory, HLT, instructional activities) in such a way that others can adjust them to their local contingencies (Barab and Kirshner 2002).

Additionally, we found patterns that occurred in several classes of our own teaching experiments. In addition to generalizability as a criterion for external validity we mention *transferability* (Maso and Smaling 1998). If lessons learned in one experiment are successfully applied in other experiments, this is a sign of successful generalization.

Also, the quality of the reasoning and the conclusions was controlled by means of submitting papers and conference contributions that were reviewed during the research period.

Internal reliability

Internal reliability concerns the reliability of the methods that were used within the research project. Our measures for obtaining internal reliability included systematically gathering data by means of prior identified key items in student activities, and processing the data using consistent coding systems. Crucial observations were shared with colleagues (*peer examination*).

External reliability

For the external reliability, the criterion is virtual replicability by means of *trackability* (Gravemeijer 1993, 1994b; Gravemeijer and Cobb 2001; Smaling 1987, 1992). This means that the research is reported in such a way that is can be reconstructed by others. The teaching experiments and data analysis resulted either in verification of the conjectures, or in adjustments or new conjectures for subsequent experiments. We described this process systematically to offer other researchers the possibility of virtually replicating it and retracing our conclusions through the cycles of data analyses and teaching experiments. This requires transparency and explicitness about the learning process of the researcher and justification of the choices that are made within the research project. Raw data should be made available. The following quotation addresses the need for trackability:

> [Developmental research means] experiencing the cyclic process of development and research so consciously and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience. (Freudenthal 1991, p. 161)

In this study we ensured the external reliability by reporting extensively on the research methodology, the process of data reduction and the learning process of the researcher, by means of justifying the decisions and by making available the raw data.

4. Exploratory Study

As part of the preparation phase of the design research cycles, this chapter presents the results of an exploratory study. This exploratory study, together with the theoretical background that was discussed in chapter 2, constitutes the preparation phase of the research, and represents a starting point for the development of an HLT for the firsts design research cycles. Indeed, to identify possible starting points for the research cycles, it was needed to know more about prior teachers' knowledge and practice concerning mathematical modelling and problem-posing. To pursue this goal, an exploratory questionnaire for mathematics teachers of primary and secondary schools was developed, in order to know if teachers include in their teaching some aspects of mathematical modelling and problem-posing.

4.1 Design of the exploratory study

As described in section 1.2, this research project is part of a University project whose overall aim consists in providing mathematics teachers with methodological models and format of school practices based on mathematical modelling, in order to reduce the gap between in- and out-of-school mathematical competencies and to foster students' reasoning and sense-making. Indeed, teachers education is crucial in the implementation of both modelling (Blum 2015) and problem-posing in mathematics classrooms (Osana and Pelczer 2015). In this exploratory study we investigated teachers' knowledge and practice in mathematical modelling and problem-posing. In the Italian context some studies outlined how in schools is common a resistance to abandoning traditional teaching methods of mainly transmission type (INNS 2017). Accordingly, the questions we wanted to address with this exploratory study were the following:

Q1. Do teachers include in their mathematics lessons some aspects of mathematical modelling?

Q2. Do teachers include in their mathematics lessons some aspects of problem posing?

Concerning the first question, only two aspects of the modelling cycle had been considered: (i) the use of real contexts as starting situations for mathematical activities and (ii) the presentation of (and the work with) mathematical applications.

To answer to those questions a questionnaire for in-service mathematics teachers of primary and secondary school was developed (Appendix A). The questionnaire was anonymous and made by closed and open questions and Likert-scales. In the first part of the questionnaire teachers were asked to give some personal details, concerning their higher degree of instruction, years of teaching and teaching level. The second part was dedicated to the investigation of teachers' educational practices. Regarding the first question Q1 about mathematical modelling, only two aspects of the modelling process had been considered, respectively the use of real contexts as starting situations for mathematics lesson and the work with mathematics applications. Since we wanted to explore these aspects, we decided to insert in the questionnaire two items in a five-Likert scale: the first dealing with the use of starting real situations for mathematics lessons and the second with mathematics applications. Regarding the second question Q2 about problem-posing, we split it in two questions: a closed question in which teachers were asked if they include or not problem-posing in their school practice and an open question wherein teachers that actually implemented problem-posing activities could report an example. In the questionnaire other questions were inserted, in order to analyze some relations between the use of specific educational strategies and/or tools and the implementation of problem-posing activities, and to explore what teachers believed indispensable to improve their teaching. In the specific: one question in seven items of a five-Likert scale about the performance frequency of some educational strategies (e.g.: individual work, group work, laboratories,…); one question in ten items of a five-Likert scale about the performance frequency of some educational tools (e.g.: textbooks, notes, software, artefacts,…). The questionnaire ended with an open question in which teachers could express some suggestions they believed indispensable to improve their teaching of mathematics.

The sample comprised one-hundred and seven primary school and seventy-two secondary school teachers from the North of Italy. The method of sampling was randomly stratified (Cohen, Manion and Morrison 2011): the teachers' population was divided into the two groups of primary and secondary teachers, and in each of them teachers who participated to the questionnaire had been randomly chosen. The 66% of the sample had a master's degree and the 34% a high school diploma. In Fig. 16, the distribution of the sample respect to years of teaching is reported. No one of the teachers that participated to the questionnaire had ever took part to a professional course on mathematical modelling or problem-posing before.

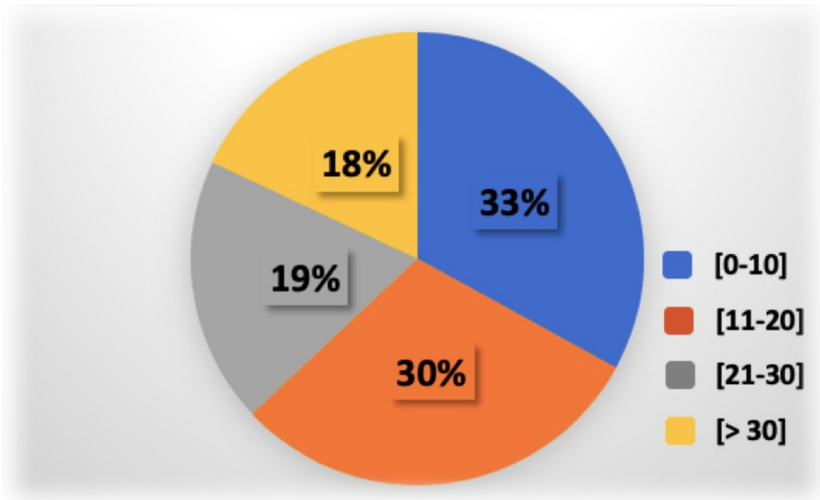


Figure 16. Distribution of the sample respect to years of teaching

The questionnaire was administrated directly by the researcher. The approach for the data analysis was mixed quantitative and qualitative. The coding of the answers to the open questions consisted in closing and grouping them in categories and families. Univariate and bivariate analysis had been performed.

4.2 Some results

In this section we report and analyse data from the questionnaires. In the specific, we focus on the results concerning the inclusion of some aspects of mathematical modelling and problem-posing in teachers' school practice. Recall that the question Q1 deals with teachers' inclusion in their school practice of some aspects of mathematical modelling and the second Q2 with teachers' knowledge of problem-posing and its implementation. In addition, answers to the last question of the questionnaire, concerning teachers' suggestions on how to improve their teaching will be presented and analysed in relation of the purpose of the study.

The results show that modelling is actually inserted by teachers in their school practice, in terms of real contexts as starting situations for mathematical activities and mathematical applications. However, teachers ask for more materials based on realistic situations in order to implement more meaningful modelling activities. Problem-posing, instead, is quite absent from today's Italian school context.

To answer to the first question Q1, a question about the implementation of modelling activities was inserted in the questionnaire. Only two aspects of the modelling process were considered: (i) using real contexts as starting situations for mathematical activities and (ii) presenting and working with mathematical applications. This question was split in two items of a five-Likert scale. The first item dealt with real contexts as starting points for the introduction of a new mathematical topic, while the second item dealt with mathematical applications. In the specific, we asked teachers the frequency by which they implement such activities in their classrooms: 1 = never; 2 = rarely; 3 = sometimes; 4 = often; 5 = always. Considering both the first and the second items, the total average was 3,9 (Table 8). Successively, in order to examine the presence or not of differences between primary and secondary school, data were divided between primary and secondary school teachers. The findings indicate that primary school teachers used real contexts as starting situations for modelling activities more (4,3) than secondary school teachers (3,7).

Table 8. Means of the answers to the first research question

Item	**Primary teachers**	**Secondary teachers**	**Total**
First *(real contexts)*	4,3	3,7	4
Second *(applications)*	3,8	3,8	3,8
Total	4,1	3,8	3,9

The second question Q2 was studied through one closed and one open question. The closed question asked teachers if they included or not problem-posing during their classroom activities. In Table 9 the distributions in total percentages, divided between primary and secondary teachers' answers, are reported. To have a deeper understanding of teachers' implementation of problem-posing activities respect to their school level, the row-percentages respect to Table 9 were calculated (Fig. 17). In Fig. 17 is clearly proved that problem-posing is not common in school practice, in fact more than a half of the sample, in both primary and secondary levels, did not implement problem-posing activities.

Table 9. Distributions respect to the inclusion of problem posing

		Inclusion of problem posing activities			
		Yes	No	Not answered	Total
	Primary	21,2	29,6	9,0	59,8

Teachers' school level	Secondary	18,4	19,6	2,2	40,2
	Total	39,6	49,2	11,2	100

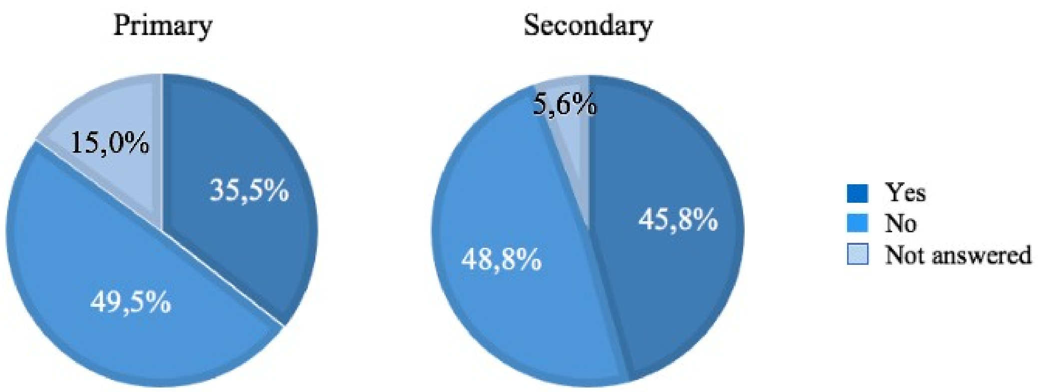


Figure 17. Percentages respect to the row of Table 9

To investigate teachers' problem-posing practices, teachers who positively answered to the closed question, i.e. who included problem-posing in their school practice, were asked to answer to an open question, in which they had to present one (or more) significant situation they used as starting point for problem-posing activities. From the coding of the answers to this open question, nine categories were identified and grouped in two families (Fig. 18). The percentages of the distributions are divided in primary (P) and secondary (S) teachers. Each teacher could report more options, so the total is higher than 100%.

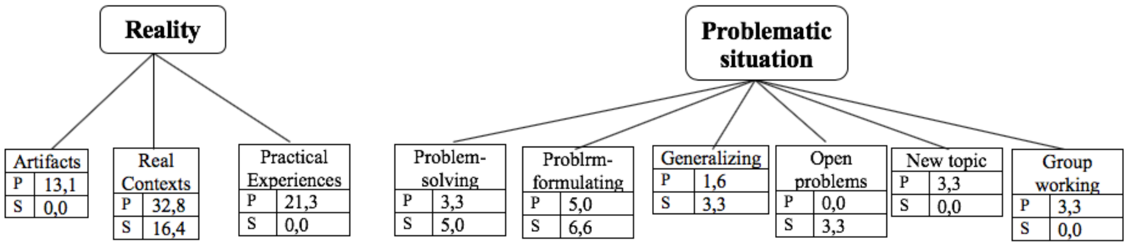


Figure 18. Families and categories of problem-posing activities

For each category the distributions divided between primary and secondary school teachers were calculated. This analysis indicates that problem-posing contexts expressed by teachers could be divided in two families: *reality* and *problematic-situations*, with distributions respectively of 83,6% and 34,7%. Note that the total is higher than 100% because teachers could express more than one situation. Moreover, the most suggested category, which is linked to the first family, was *real contexts* (49,2%). This fact remarks the importance in the choice of meaningful contexts for the implementation of problem-posing activities. Such contexts are the ones we called as realistic and rich, and that are at the basis of the modelling process. As a consequence, contexts should play a crucial role in bridging modelling and problem-posing to promote students' reasoning, critical thinking and give meaning to their mathematical activity. Data split between primary and secondary school teachers allow to have an evidence in the fact that some starting situations are used only at primary school (artefacts, practical experiences, new topic, group working), while others only at the secondary school (open problems). Instead, teachers of every level should have the possibility to face with several different contexts or tools and learn to choose which is the most appropriate in relation of the classroom and the learning process.

To have a deeper insight in the relations between the implementation of problem-posing and the use of artefacts, that in previous studies were proved to be able to foster a mindful approach towards a problem-posing attitude (Bonotto 2009), a bivariate analysis between the implementation of problem-posing and the use of artefacts was performed. Recall that ten items concerning the use of some educational tools in a five-

Likert scale (1 = never; 2 = rarely; 3 = sometimes; 4 = often; 5 = always) were inserted in the questionnaire. In the specific, teachers had to state the frequency they adopt the following educational tools during their lessons: textbooks, notes, interactive board, software, calculator, math games, audio and video supports, artefacts, library, others. Findings indicate a significant correlation between the use of artefacts and the implementation of problem-posing activities ($0{,}001<p<0{,}01$; ❑^{2}=0,13), as shown in Table 10 (the total is less than 100% because the distributions of teachers who did not answer are not reported).

Table 10. Use of artefacts and implementation of problem-posing activities

		Frequency in the use of artefacts		
		Less than sometimes	Sometimes	More than sometimes
Implementation of problem- posing activities	Yes	22,5	36,6	39,4
	No	46,5	31,8	19,3

Moreover, to obtain some information on the relations between problem-posing and problem solving, teachers were asked if they implement or not problem solving activities during their teaching. The double distribution is given in Table 11 ($p<0{,}001$; ❑^{2}=0,44). Considering the row percentages of Table 11, almost every teacher who implemented problem-posing activities implemented also problem solving ones (95,8%). The vice-versa is not true, in fact if we consider teachers who implemented problem solving, the 49,3% adopts also problem posing and the 46,5% did not.

Table 11. Implementation of problem solving and problem posing activities

	Implementation of problem solving activities

		Yes	No	Not answered	Total
Implementation of problem-posing activities	Yes	68	3	0	71
	No	64	24	0	88
	Not answered	6	14	6	20
	Total	138	41	6	179

In conclusion, we report the last question of the questionnaire, in which teachers were asked to express some suggestions they believed indispensable to improve their teaching of mathematics: *in conclusion, I ask you one (or more) suggestion you believe indispensable to improve your teaching of mathematics.*

The 71,0% of the sample answered to this question. From the coding of teachers' answers, four families (*educational strategies,* 103,4%*; math topics,* 33,3%*; school organization,* 26,1%*; teacher training,* 26,0%*)* and thirty-one categories were identified. Note that, since each teacher could express more options, the total percentage is higher than 100%. In Table 12 data whose distribution was higher than 10% are reported.

Table 12. Categories for teaching improvement

Category	**Distribution (%)**
Laboratory	32,1
Mathematics and reality	21,5
Teacher training	14,4

Students' motivation	19,1
Classroom equipment	12,0
Research in education	10,7
More hours of mathematics	12,0
Activities on problem solving	13,1

In Table 13 there are some examples of teachers' answers relative to the categories of Table 12 (only the categories with distributions higher than 10% are reported):

Table 13. Examples of teachers' answers to improve their teaching

Category	**Teachers' answers**
Laboratory	• Show the importance and use of mathematics with laboratorial activities. • Materials based on laboratorial experiences.
Mathematics and reality	• Activities based on realistic situations and daily life experiences. • Starting from concrete contexts and apply mathematics to real situations.
Teacher training	• Incrementation of training courses for both pre-service and in-service teachers. • Provide more practical professional development courses for teachers.
Students' motivation	• Enhance students' involvement, stimulate discovery and playfulness.

	• Students' motivation must be improved.
School equipment	• Improve school facilities by equipping them with laboratories or teaching tools. • Adequate classrooms (tools, software, etc.) available in all school levels.
Research in education	• More cooperation between schools and universities. • Teachers' need valid materials based on valuable teaching strategies.
More hours of Mathematic s	• More hours in the week schedule for mathematics are needed to have a deeper insight in various topics. • Renovate programs or increase the time to have more time to deepen the topics.
Activities on problem solving	• Pay more attention in the solution process of a problem and less time in calculations. • Work regularly on problem-solving activities.

From the analysis of teachers' answers, it can be deduced that teachers need more opportunities to be engaged in modelling activities. Indeed, they ask more effective experiences and practical materials based on realistic and laboratorial activities. Consequently, there is the necessity of an improvement in teachers' trainings, offering them occasions to be involved in modelling activities. We remark that teachers themselves recognized the importance in the choice of concrete and stimulating contexts for students, that we called rich and realistic. In addition to the examples reported in Table 13, other teachers' answers were directly linked to problem-posing:

- *Learn to problematize from concrete situations.*
- *Attitude to pose problems and observe.*

The request of paying more attention in problem-posing and problem solving situations, by a regular implementation of these educational strategies, supports that students' reasoning must be increased. Activities based on modelling and problem-posing, starting from meaningful contexts given for example by suitable artefacts, should represent a valuable occasion to achieve such results.

4.3 Conclusions from the questionnaire

Concerning modelling, the results from the questionnaire showed that teachers regularly included some aspects of the modelling process in their classroom activities, in terms of using real contexts as starting situations for mathematics lessons and showing real applications of mathematics. Despite this disposition, teachers expressed a need in both more materials and preparation to implement activities based on realistic contexts. This need is in line with Blum (2015), in which it is underlined the high demanding features of implementing modelling at school, and that teachers' professional development in modelling competencies (Borromeo Ferri 2018) is indispensable. Two directions seem to be important: (i) improving teachers' professional development courses, offering teachers occasions to face with modelling activities based on rich and realistic contexts, and (ii) developing prototypes of practices and textbooks based on realistic problematic contexts available for teachers of every school level. In this way teachers would have at their disposal models of modelling activities that can be adapted and implemented in their classrooms.

The analysis of the second question indicates that problem-posing is not known by teachers, and consequently not regularly implemented at school. In fact, less than a half of the participants (39,6%) adopted it during its school practice. To overcome this lack, problem-posing should become an integral part of pre-service and in-service teacher training courses, in order to give teachers opportunities to increase their knowledge, before, and their practice, after, on problem-posing. Such improvement in teachers' knowledge could help teachers to recognise intersection points between different methodologies and strategies and to adopt coherent teaching methods. Also, in our study both the relations between problem-posing and modelling and problem-posing and problem solving had been confirmed. Indeed, looking at the categories of Fig. 18, the most frequent is *real contexts,* which highlights the cooperation between modelling and problem-posing. This cooperation is natural in the choice of meaningful contexts for the implementation of both these educational strategies in order to enhance students' reasoning and critical thinking. A possible choice for that contexts is given by artefacts, whose precious contribution in problem-posing activities emerged from the bivariate analysis, in agreement with Bonotto (2013). Moreover, we remark the positive and strong relation between problem-posing and problem solving, in line with previous researches (Bonotto 2013; Kilpatrick 1987; Silver, Mamona-Downs, Leung and Kenney 1996; Silver 1994).

In conclusion, teachers' opinions about how to improve their teaching of mathematics were analyzed. The most suggested family dealt with *educational strategies*. In the specific, linked with this family there were some categories in connection with modelling and problem-posing: *laboratory, math&reality, problem solving, group work, practical experiences*. The most suggested category was *laboratory*. This means that teachers realized that a change in the way of doing mathematics is necessary. However, standard mathematics and "lab-mathematics" might not be distinguished by teachers, but activities based on modelling should become integrated in the daily mathematics activities.

The exploratory study presented has some limits. Indeed, only some aspects of modelling were considered, the ones of using real contexts as starting situations for mathematical activities and working with mathematical applications. As a consequence, a deeper understanding of teachers' effective practice of the entire modelling cycle and about their knowledge of other aspects of modelling did not emerge. Moreover, findings are based only on what teachers self-reported. As a consequence, we can only have a first overview about this issue, that was actually the aim of this exploratory study. For the future a deeper investigation in teachers' practices linked to both modelling and problem-posing would be performed through a series of interviews and classroom observations.

To conclude we remark some key points concerning what can be learned from this exploratory study for the implementation of the following design research cycles. The results from the questionnaires indicate that:

- more didactical materials are needed concerning mathematical classroom activities based on real contexts;

- problem-posing should become integrated in the school practice during mathematical lessons;

- cooperation between mathematical modelling and problem-posing is evident in the choice of meaningful contexts for mathematics lesson;

- a change is necessary in the way of teaching mathematics, making students more active in the learning process.

The results are in line with the choice of the design research methodology, because that same teachers expressed the need of innovative educational settings for their teaching. In these settings, meaningful real contexts are crucial, and as a consequence the role of the design heuristic of didactical phenomenology would become fundamental in designing instructional activities with the aim of choosing problem situations that could provide the basis for the development of the mathematical concepts or tools we want students to develop. This point represents also a possible link between modelling and problem-posing. As a consequence, problem-posing should be integrated in the school practice reinforcing the role of meaningful contexts for students and starting from such contexts to make students become familiar in posing (and solving) their own problems.

In chapter 5 and 6 the two design research cycles concerning modelling will be presented. In the specific, these cycles have the additional goal to provide teachers with design schemes and prototypes of practices in line with the aims of the University project and teachers' requests from the questionnaire. In chapter 7 and 8 the cycles concerning problem-posing will be reported. The main point is to integrate problem-posing in the school practice, and as stated in consequence of the exploratory study, a key point to pursue this goal is to reinforce the use of real contexts to make students pose their own problems. The main difficulty that could occur in this scenario is to understand which contexts should be real for students, and how different contexts can influence students' performances in problem-posing.

5. M-I Research Cycle

5.1 Introduction

The design research of this project consists in two cycles for mathematical modelling (M-I and M-II) and two cycles for problem-posing (PP-I and PP-II). This chapter addresses the first research cycle concerning mathematical modelling, M-I (Fig. 19).

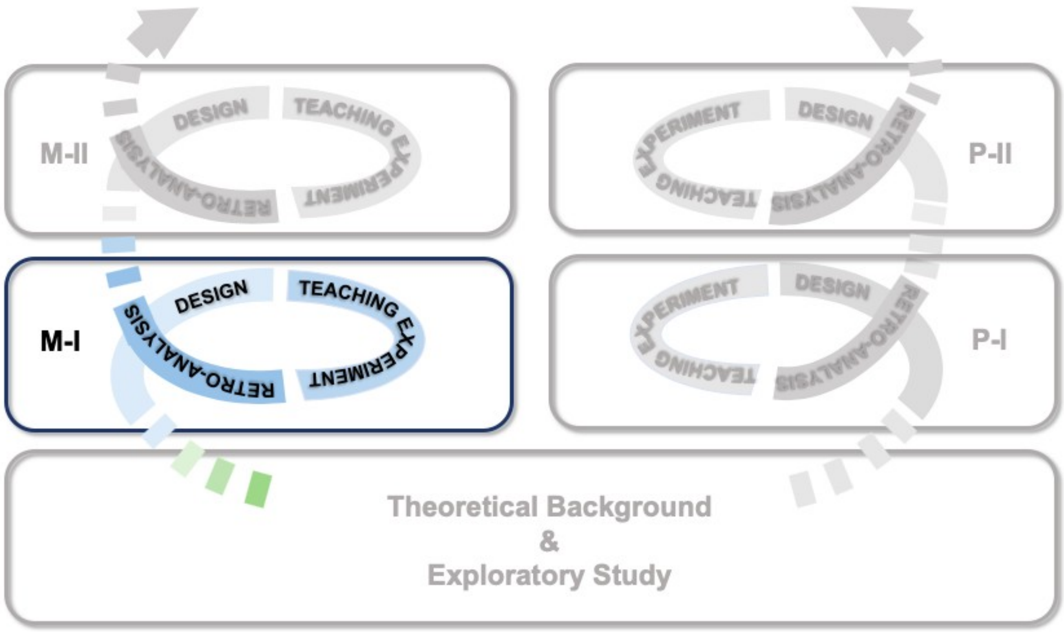


Figure 19. Research cycle M-I

The structure of the chapter is the following: in section 5.2 we introduce the context of the classroom that participated to the study, explicating the mathematical topic considered and the specific goals of this cycle in relation to the first research question. In section 5.3 the design phase is described, consisting in the development of an HLT with particular attention in designing instructional activities related to the learning goals of the HLT. HLT includes starting points and expectations and the students' hypothetical learning process. The design of key activities and materials are reported. Then the experiences during the teaching experiment are described (section 5.4). Section 5.5 is dedicated to the retrospective analysis, in which we reflect on the expectations formulated in the HLT and formulate feed-forward for the next research cycle.

5.2 Context and aim

The aim of the research cycles M-I and M-II consisted in finding possible answers to the first research question of this research project:

RQ1. How can Model Eliciting Activities promote the process of emergent modelling?

The first research cycle concerning mathematical modelling, M-I was conducted in a second-grade class (age 7) composed by nineteen students during two weeks of regular mathematics lessons. The class had never been engaged in a modelling activity before the teaching experiment. At the time of the activity, students were working on multiplication in the set of natural numbers. In particular, multiplication as iterated sum (Maffia and Mariotti 2018; Fischbein, Deri, Nello and Marino 1985) was introduced by the official mathematics teacher one week before the teaching experiment. Students were able to perform basic multiplications between numbers with one digit. As a consequence, the mathematical topic that was chosen for the modelling activity was multiplication in N. In the specific, in relation to the research question RQ1, we investigated how emergent modelling can be fostered to help students understanding some aspects of the multiplicative structure. Our hypothesis was that a modelling activity designed following a model eliciting sequence (Lesh et al. 2003) with the use if suitable artifacts could actually foster the emergent nature of modelling.

The student activities and the guidelines for the teacher, together with our intentions, were discussed beforehand with the mathematics teacher in two meetings.

The research method for the data analysis was qualitative. The aim of the data analysis was to reconstruct the classroom progress, which resulted in an empirical grounded understanding of students' reasoning during the classroom activity. In order to be able to reconstruct the learning process and verify our hypothesis, different kinds of data were collected: pre-test; transcriptions of classroom dialogs; observations of group working; students' final projects.

In the next section we present the design phase, in which the development of an HLT is described.

5.3 Design phase

In this research project the design phase of a design research cycle is characterized by the development of an HLT. In this section we first describe the starting points for the HLT and the expectations that are investigated in the following teaching experiment. Then we describe the activities and materials designed in order to foster students' cognitive development according to the goals of the HLT.

5.3.1 Starting points

Starting points for the formulation of an HLT are split in two categories. The first concerns the theoretical background specific for this research cycle and taken into consideration to design an educational setting and hypothesis about students' learning. The second deals with the classroom context, and in the specific the initial level of the students. This part could be inserted in the teaching experiment phase, but we decided to report it here because we actually considered the results of the pre-test to formulate the following learning trajectory.

Theoretical Background

In chapter 2 the background to the research was extensively described. Here we want to sum up some key points that were considered for this research cycle.

The first one deals with emergent modelling. Emergent modelling does not consist in the application of mathematical concepts to solve real problems, but actually mathematical activities are used as a vehicle for the development of mathematical concepts (Greer et al. 2007). Students, starting from a real context, begin to model their informal mathematical strategies and arrive to re-invent mathematical concepts and applications they need. These concepts and applications can be subsequently formalized in mathematical terms and generalized to other situations. To sum up, emergent modelling can be seen as a dynamic process from a *model of* students' situated informal mathematical strategies to a *model for* more formal mathematical reasoning (Gravemeijer & Doorman 1999).

Another modelling perspective considered in this research cycle is the one of model eliciting. Model eliciting activities (MEAs) are simulations of real-life problem solving situations in which students develop a model going through iterative phases of invention, refinement and revision, in which the goal is not in the application of some ready-made procedures, as in the traditional practice of word problems. In model eliciting activities, instead, students struggle to create interpretations that fit their interpretations of the starting dilemma, discuss, make sense of meaningful situations and invent, extend and refine their own mathematical constructs (Kaiser 2017). Lesh et al. (2003) developed a model development sequence whose components can easily be re-sequenced to suit the needs of researchers or teachers (Fig. 9). In this research cycle we considered three phases of such sequence: *warm-up activities*; *model construction; presentations and discussion.* Warm-up activities are usually given the day before students are expected to begin work on the model eliciting activity. Warm-up activities aimed at helping students to be confident with the context of the modelling activity and at introducing or testing eventually minimum prerequisites. The model construction is the core of the model eliciting activity, here students are engaged in performing modelling cycles to produce a model that describes the starting situation. Presentations and discussions are whole-class activities in which students make formal presentations about the results of their work.

In conclusion, to design our modelling activity, the choice of a real context followed the perspective of RME. This means that a real situation will be represented by a *realistic* and *rich* context. Realistic refers to a problem of which the problem situation is experientially real to the student (Gravemeijer & Doorman 1999), while rich (Freudenthal 1991) refers to a context that promote a structuring process as a means of organizing phenomena, physical and mathematical, and even mathematics as a whole, i.e. contexts that give more opportunities in the mathematization process.

Pre-test

Before the development of a learning trajectory, a pre-test was administered in the class were the research cycle took place. The aim was to have a picture about the starting level of the classroom concerning the mathematical topic considered for the following modelling activity: multiplication in N. Furthermore, some items were intended to be matched with post-test items in the analysis phase. 18 students participated to the pre-test. The test was composed by five problems, taken from the test *Invalsi*[2] of previous years for grade 2. The full text is presented in Appendix B. In table 14 the number of correct, incorrect and not given answers are reported. The maximum score for each item was 2, so the total score for the pre-test was 12. The mean of the classroom was 4,6 (SD=2,96) which underlines that classroom knowledge of the subject was still poor. From the analysis of the pre-test, it emerged that the notion of multiplication as iterated sum was clear to students. Indeed, in problem number 2 students had to use the

[2] INVALSI (*Istituto Nazionale per la Valutazione del Sistema Educativo di Istruzione e di Formazione*) is the acronym of the Italian National Institute for the Evaluation of the Education System, supervised by the Ministry of Education. One of the activities of the institute is to prepare and carrying out periodic and systematic tests to monitor the learning outcomes of Italian students. These tests are known as *INVALSI national tests.* In the primary school, these tests are performed in grade two (Italian and Mathematics tests) and grade five (Italian, Mathematics and English tests). Concerning mathematics, the aim is to measure the ability to solve problems, in the discipline of real-life, concerning skills in logic, interpretation of graphs, understanding of phenomena, construction of models, in relation to the mathematical curriculum described in the National Guidelines (DM 254/2012).

definition of multiplication as iterated sum, and 16 students over 18 gave a correct answer.

Table 14. Pre-test results

Problem	Max score	Number of correct answers	Number of incorrect answers	Not answered
1	2	9	8	1
2	2	16	2	0
3	2	2	11	5
4	2	5	9	4
5	2	7	5	6

However, some students did not recall the notion of multiplication and only iterated the addition (Fig. 20), while others were able to report also the definition of multiplication (Fig. 21).

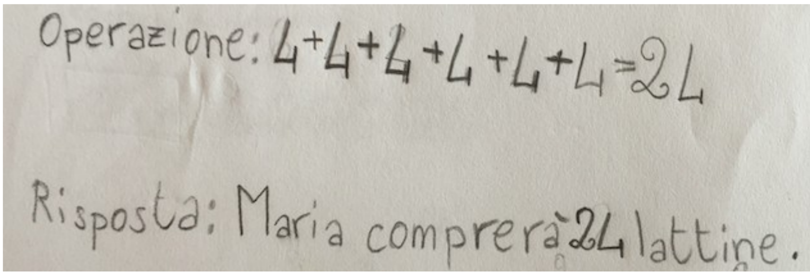


Figure 20. Example of student's answer

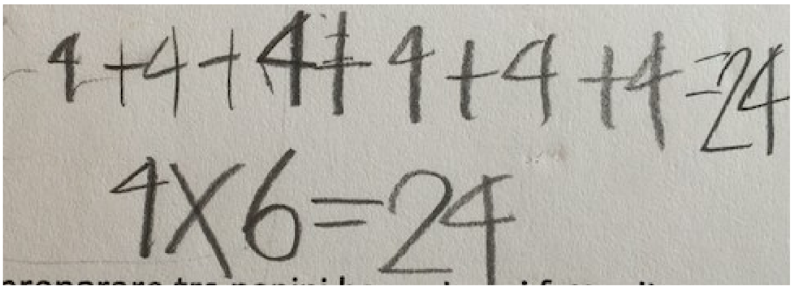


Figure 21. Example of student's answer

Concerning the two students who gave an incorrect answer to problem 2, in which students had to apply the notion of multiplication as iterated sum, the first one instead of performing a multiplication performed an addition (Fig. 22), while the second one did not actually understand the meaning of multiplying 4 by 6, because he draw four tables 4x6 (Fig. 23).

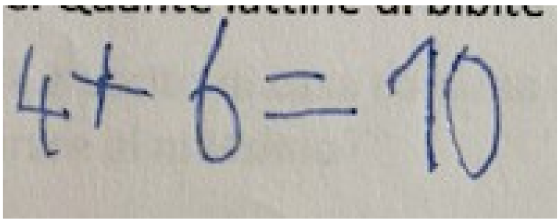


Figure 22. Example of student's incorrect answer

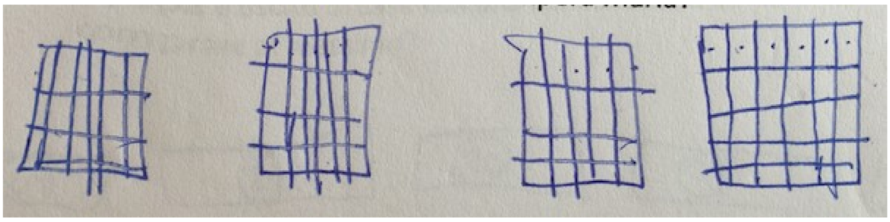


Figure 23. Example of student's incorrect answer

Regarding the other problems in the pre-test, students encountered difficulties especially in problems 3 and 4. We did not take too much into account such results, since the mathematics teachers introduced only the concept of multiplication of iterated sum one week before the implementation of the modelling activity. Nevertheless, such items would be compared with similar ones in a post-test, in order to see if the designed and implemented activity had also some positive or negative influences in solving problems related to multiplication as expected by the Italian National Evaluation System.

5.3.2 Learning goal and hypothetical learning process

As stated in section 5.2, the aim of this first research cycle consists in investigating how emergent modelling can be fostered to help students understand some aspects of the multiplicative structure. In the pre-test we saw that students at the moment of the intervention had a little knowledge on this topic, but actually they had a clear idea about the notion of multiplication as iterated sum. What we want to achieve during the teaching experiment is the re-invention of the distributivity property of multiplication over addition. As a consequence, the learning goal of the teaching experiment is represented by the distributivity property of multiplication over addition (Fig. 24).

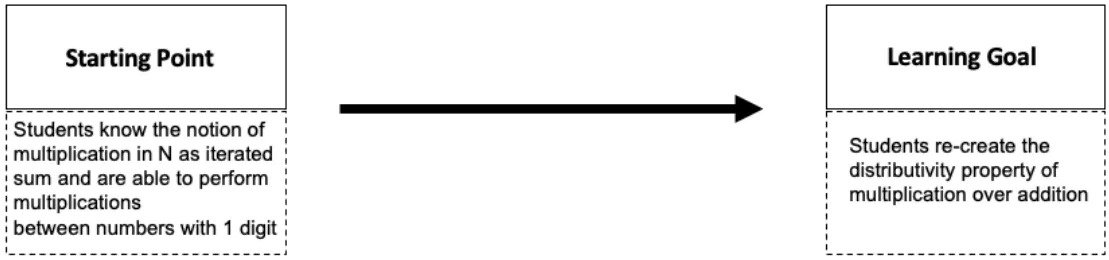


Figure 24. Learning goal of the M-I cycle

To design an HLT, together with the learning goal we need to formulate some conjectures about the learning process. Starting from the classroom level and following the heuristic of didactical phenomenology, we supposed that making students face with a problem situation in which they need a new mathematical concept to solve it could stimulate the same students in creating that concept. Moreover, we believe that not only the context, that should be rich and realistic in the perspective of RME, is important in such a re-invention process, but also the constraints given in the text of the problem could encourage or not the emergence of new mathematical concepts. In order to do that, our idea consisted in putting students face with the problem of performing multiplications between numbers with 2 digits. In Fig. 25 this problem is indicated as *critical point*: students encounter a problem situation and to solve it they need to develop a new mathematical concept or tool.

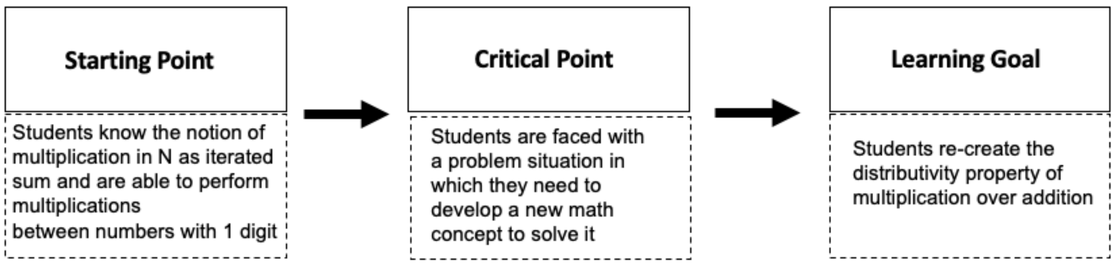


Figure 25. Hypothetical learning process of the M-I cycle

The learning process outlined in Fig. 25 is not linear. Indeed, attention should be given not only to the solution process of the given problem, but to the process of construction of a mathematical model that support such solution process. In agreement with the model eliciting approach, students develop a model going through iterative phases of invention, refinement and revision. Modelling is a process of developing representational descriptions for specific purposes in specific situations, involving iterative testing and revision cycles (Lesh and Lehrer 2003). Moreover, we think that this process could support the emergence of new mathematical knowledge, since the developed model is firstly a model that is created in a specific situation to solve a particular problem, but during the modelling activity becomes a model for a more general mathematical structure. In our case, students should start developing a model to solve a given problem, and then discover that such a model permitted them to create a general mathematical concept: the distributivity property of multiplication over addition. In the next section we present the learning activities that make those first hypothesis more concrete.

5.3.3 Learning activities

Which design scheme can be used to develop some learning activities to achieve the learning goal described in the previous section? Our hypothesis is to adapt the model eliciting sequence (Lesh et al. 2003) in order to translate the hypothetical learning process in concrete classroom activities. The designing scheme used for the learning activities followed the design scheme for MEAs proposed by Lesh et al. (Fig. 26). We present the activities together with the materials developed.

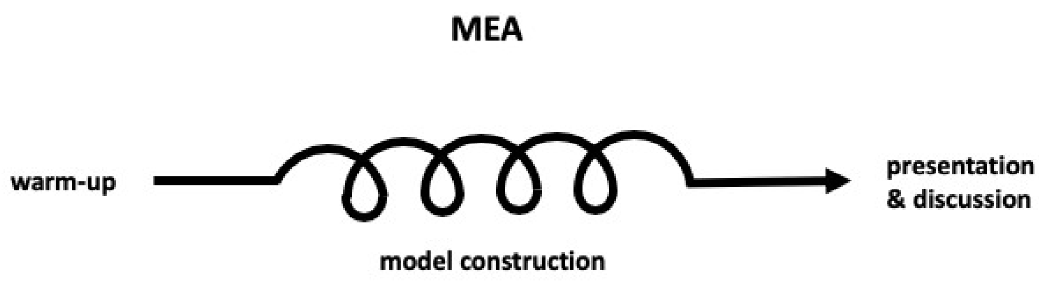


Figure 26. Design scheme for the learning activities of the M-I cycle

The first phase was represented by the warm-up phase. Recall that in a model eliciting sequence warm-up activities aimed at helping students to be confident with the context of the modelling activity and at introducing or testing eventually minimum prerequisites. In this latter connotation, also the pre-test and its results, discussed in section 5.3.1, actually are part of the warm-up phase. Next to the pre-test, the day before starting the model construction, students were engaged in a series of activities in order to make them familiar with the problem situation they would have to face. Two hours were dedicated to these activities. Each student was given the text of the modelling task together with a booklet. Since at the time of the activity the school in which the teaching experiment took place was under building renovation, we decided to choose as modelling task the following *Tiling Problem*:

The Tiling Problem

The school director decided to renovate the school. Students can design a floor tiling of their own classroom. The floor of your classroom was divided in six equal strips. Each group of students should tile a strip, using all the available types of floor tiles.

Together with the task, to each student was given a booklet which included:

- the figure of the classroom divided in six stripes (Fig. 27);
- the figure of each stripe to be tiled by a single group (Fig. 28);
- a brochure with the shapes of the available tiles (triangular, square, rectangular) to be used with their relative costs (Fig. 29).
- the task repeated in a clearer form (Fig. 30).

The brochure represented a cultural artifact that, thanks to its richness in (mathematical) meaning created a sort of hybrid space that connects mathematics and everyday contexts. Starting from the materials, students have to answer some questions dealing *The Tiling Problem*. Questions were about comprehension of the task and reasoning on the relations between different tiles and their cost (Appendix C). The aim of the questions posed to students was to start reasoning about the context of the problem. After a first individual reflection, the teacher conducts a group class discussion in order to collect ideas from students, share opinions and points out some clarifications.

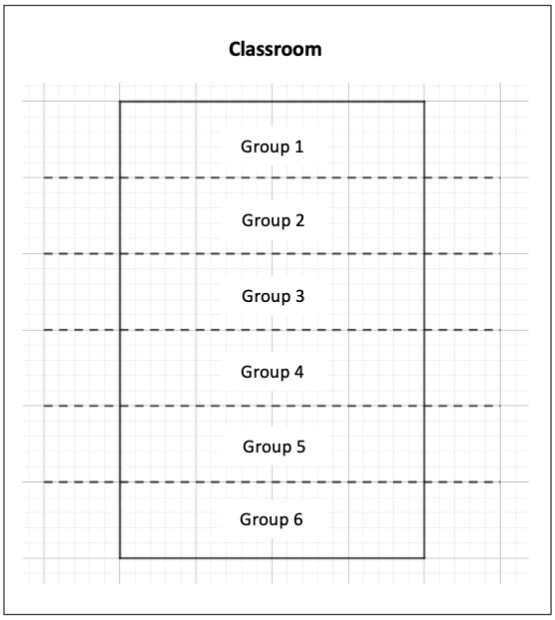


Figure 27. Classroom divided in six stripes

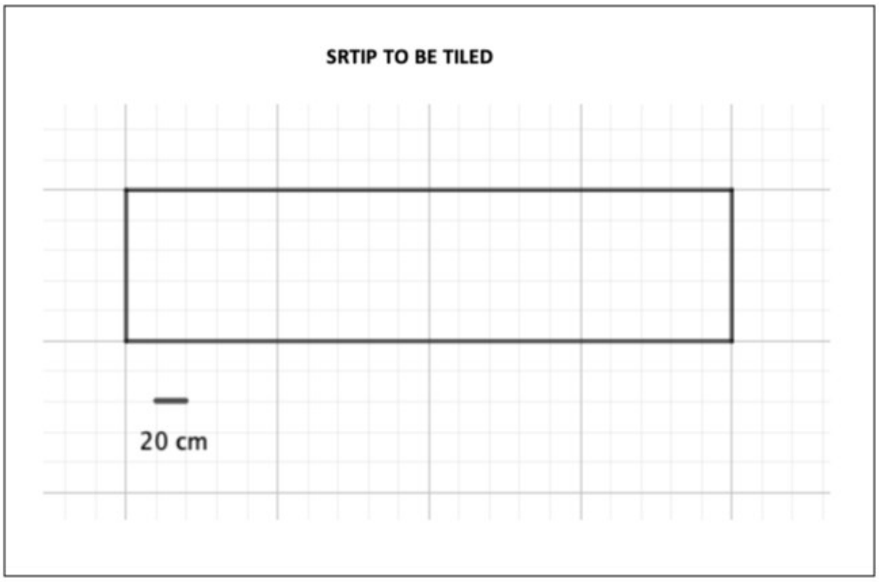
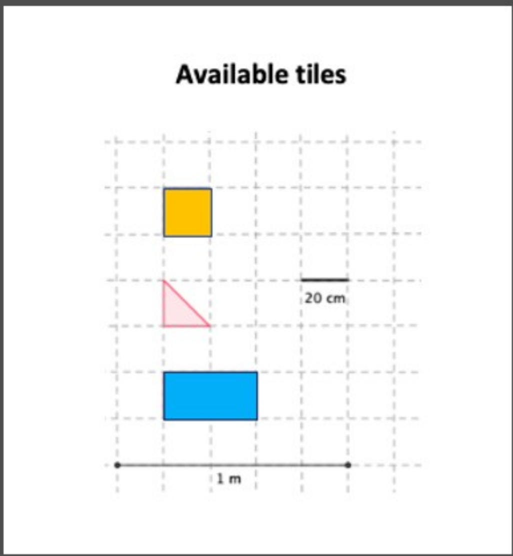


Figure 28. Stripe to be tiled by every single group



Tile	Cost
Square	6 euros
Triangular	4 euros
Rectangular	10 euros

Figure 29. Brochure with the shapes of the available tiles with their costs

Make a poster in which you:

- Design and paint the tiled floor
- Explain the steps followed to solve the problem
- Express the total cost for the designed tiling

Figure 30. Task given to students

The second phase of the MEA consists in the model construction and it covers five hours. In this phase each group has to create a poster in which design the floor tiling and explain the strategies followed to calculate its total cost. This is the core phase of a MEA. Students are expected to perform iterative cycles to build the desired model to solve the task. Two important aspects of the task are the following. The first one is the constraint of using all the available types of tiles, and the second one to cover an entire strip of the classroom. With these conditions, we suppose that students at a certain point face with the problem of performing multiplications between 2-digits numbers. Indeed, a possible solving strategy could be the following. Firstly, count how many tiles of each type (triangular, rectangular and square) have been used to cover the strip. Then, calculate the cost for each type of tiles and finally sum the obtained relative costs. In calculating the total cost associated to each type of tiles, students will encounter the problem of performing multiplications with 2-digits numbers. As a consequence, they have to find a way to perform this calculation, and we conjectured that some students to answer this problem could re-invent the distributivity property of multiplication over addition.

In the final phase of the activity (two hours), presentation and discussion, each group has to present to the classroom its project explaining the steps followed to solve the task. Each member of the group has to take part to the presentation.

5.3.4 HLT for cycle M-I

In this section we sum up the components of the designed HLT for the research cycle M-I discussed in the previous sections, namely the learning goal, the hypothetical learning process and the learning activities. Together those components define the HLT (Table 15). In the next section the M-I teaching experiment is presented.

Table 15. HLT for the M-I cycle

HLT		
Learning Goal	Hypothetical Learning Process	Learning Activities
Distributivity property of multiplication over addition	Making students face with a problem situation that need the development of new mathematical concepts to be solved could support students in	Students are engaged in a MEA in which they have to develop a model to solve the task *The Tiling Problem.* In the first phase (warm-up) all the

	achieving the learning goal. The critical point is represented by the necessity to perform multiplications with 2-digits numbers. The critical point is made explicit through a modelling task with some specific constraints. These assumptions are made explicit through the development of some learning activities.	materials are presented to the students who are engaged in a comprehension activity. During the model construction, students could face with the problem of performing multiplications with 2-digits numbers. Some students are expected to re-invent the concept of distributivity of multiplication over addition to overcome this critical point. In the end each group presents its findings to the rest of the classroom.

5.4 M-I teaching experiment

In design research the aim of the teaching experiment is to see how the developed HLT would play out in the classroom and test empirically the hypothesis that had been conjectured. We now present the teaching experiment for the M-I cycle. In Fig. 26 the designed scheme of the activities is shown. Based on such scheme, we present the implementation and the main results from the teaching experiment in its three main phases: warm-up; model construction; presentation and discussion. The activities had been carried out by the researcher with the presence of the regular mathematics teacher.

In the description of the teaching experiment, with *teacher* we refer to the researcher, who conducted the activities.

5.4.1 Warm-up

In section 5.3.3 we described the materials and setting of the warm-up phase. This phase covered 2 hours. Firstly, to each student were given the task of the modelling problem and the relative booklet. The first part of the activity is very important, since the teacher must take and keep students' attention and interest. Students have to feel motivated in doing the modelling activity. For this reason, a starting point is that all students clearly understand the task and what is required. The teacher read with students the task, the figure of the classroom and of a single strip, and reformulated together with students the request. Then, each student, before individually and then in pairs, had to answer to some questions (Appendix C). At the end, the teacher conducted a whole class discussion to check students' comprehension of the activity and clarify any doubts.

The first question asked students to explain what every group is asked to do. In Fig. 31 some students' answers are reported. Students answered in different ways to the question. Some students only focused on the fact that they have to make a project, while others specify that they actually have to renovate the floor of their classroom tiling it. Here different comprehensions occurred. Indeed, example 3 in Fig. 31 shows that a student understood that every group has to tile every strip, while the request is that every group has to tile only one strip. This request is evident in the last two examples reported. In Example 4 it is significant that the student understood that they can decide to tile the strip as they want, so they can develop a creative plan to tile their group strip. However, it does not mention about the constraint to use all the available types of tiles given in the brochure. Example 5 expresses the fact that they have to use the available types of tiles, but nothing about that they must use *all* the available types. Just in this first question different perspectives appear. As a consequence, the importance of a classroom discussion is evident. The teacher listened to students answers and pointed the attention in the fact that all the available types of tiles must be used.

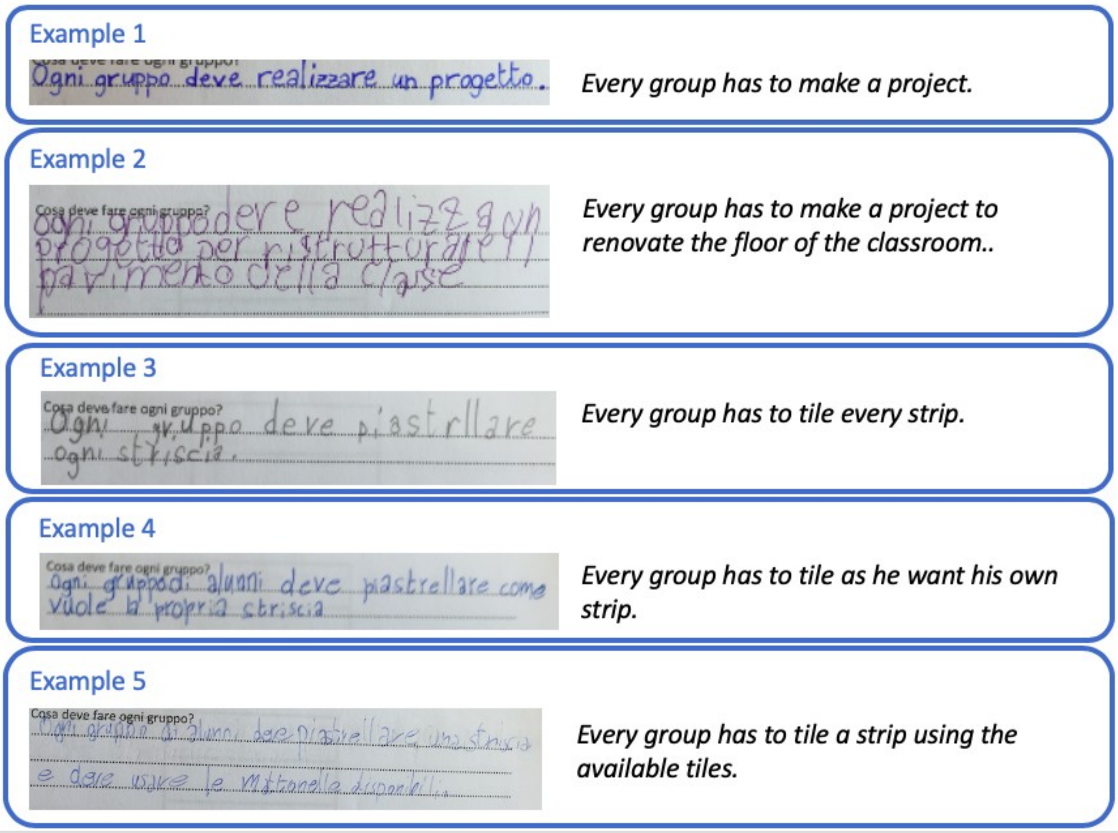


Figure 31. Students' answers

Concerning the second question, every student was able to report the correct measure of each strip using the figure with the legend. An example is given in Fig. 32.

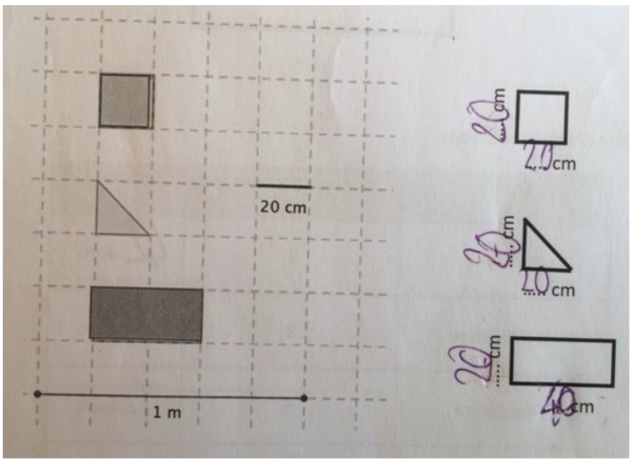


Figure 32. Student's answer

Answering the second question and using the figure of the single strip reported in students' booklet helped to answer to the third question, that consisted in calculating the measures of the sides of a single strip. We report here a dialogue in which a student explains to the teacher how she/he was able to calculate those measures (T=teacher, S1=student 1, S2= student 2):

S1: the side of the strip measures 100 cm
T: how did you obtain this result?
S1: I counted how many lines [unit of measure] … they are 5!
Then, every line is 20 cm
Twenty plus twenty plus…. = 100 cm
T: So, how many times did you sum twenty [to itself]?
S1: five times
S2: 20 x 5
S1: yes!... or 5 x 20, it's the same!

From the previous dialogue, it is evident that the student uses the definition of multiplication as iterated sum to calculate the measure of a side of the strip. Such iteration is recognized as being actually a multiplication. Moreover, the classroom reasoning highlighted also another important aspect: the commutativity property of the multiplication. This process is reconstructed in Fig. 33.

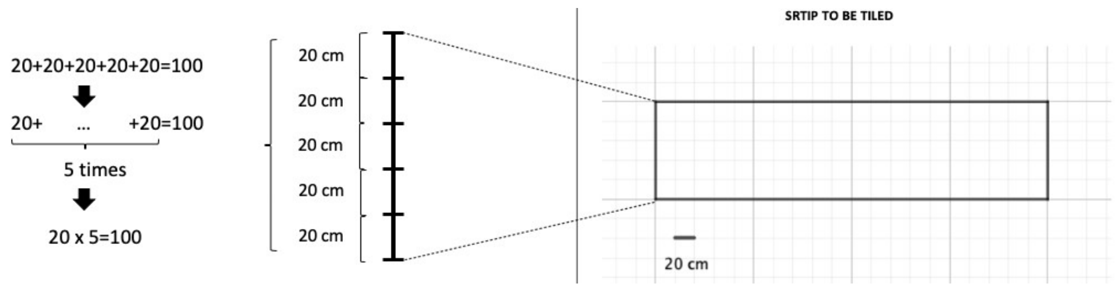


Figure 33. Process used by a student to calculate the measures of a side of the strip to be tiled

The last question consisted in filling a table about tiles and their costs. In the class discussion the teacher focused on the row highlighted in Fig. 34. In this row it is known that a person spent 12 euros buying only triangular tiles, and it is required to calculate how many tiles had been bought. A student answered in this way:

T: we spent 12 euros to buy only triangular tiles. Who help me in knowing how many
tiles had been bought?

S1: the result is 3!

T: why 3?

S1: because you had 12 euros… and… and the triangles cost 4 euros

T: the triangles?!?

S2: the tiles with triangular shape

T: ok, a triangular tile costs 4 euros

S1: so, two cost 8 euros

S2: … and three cost 12 euros

S1: 4 x 3 = 12

Shape	Number of tiles	Cost (euros)
Square	7	....
Rectangular	3	....
Triangular	6	....
Triangular	....	12
Square	....	18

Figure 34. Last question of the comprehension activity

As in the second question, students reconstructed the already known notion of multiplication as iterated sum.

In the next section we report the main results from the model construction phase.

5.4.2 Model construction

The core part of a MEA is represented by the model construction. In this phase students had to create a model to solve the task explored in the previous lesson. Students worked in groups of three. Each group had at his disposal a poster (Fig. 35) with the strip to be tiled.

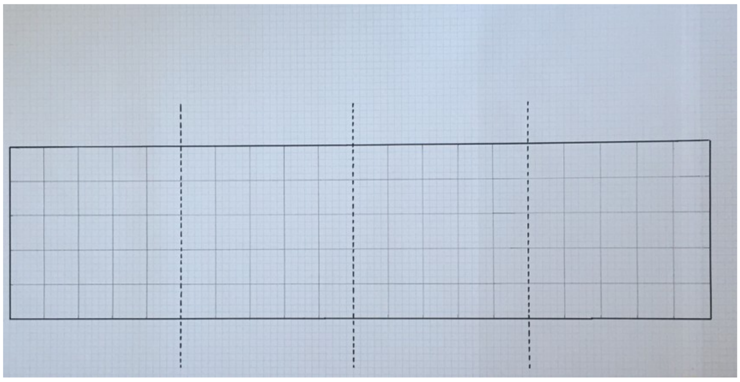


Figure 35. Poster given to students with the strip to be tiled

During the model construction phase, each group of students created a poster in which they designed the floor tiling and explained the strategies followed to calculate its total cost. In Fig. 36 there are some examples of students' group working. Students approached to the problem of tiling in different ways: who first make a project of all the tiles, who created some tiles prototypes and reported them in the poster.

Figure 36. Students working in groups to solve the tiling problem

While solving the task, all the groups developed a similar strategy to obtain the total cost. The strategy consisted in two steps. The first one consisted in counting the number of all the tiles of the same type and multiply the number obtained with the relative cost. For example, one group counted fifty square tiles, twenty-six triangular tiles and fifteen rectangular tiles. Then, the number of each type of tile was multiplied by its relative cost. In our example, students had to perform 50×6, 26×4, 15×10. This step highlights the notion of multiplication as iterated sum, already known by the students.

While performing multiplications similar to the latter one, the groups encountered the difficulty of multiplying a number with one digit and a number with two digits. Since in several groups students were not able to find a way to solve this problem, the teacher decided to reason about it in a whole class discussion. Some students suggested the strategy reported in the following dialogue (R=researcher; S1=first student; S2=second student) to calculate 6×57:

S1: I write $6 \times 57 = 57 \times 6$.

Then I divide 57 as 50 and 7…

R: Divide?

S1: Write…?

R: Decompose.

S1: Yes, I decompose 57 as 50 plus 7!

Then I calculate 50×6.

S2: That is 300!

S1: Then 6×7

S2: 42

R: Excellent, and with these number? (pointing 300 and 42)

S1: I put them together!

R: How?

S1: I compose them…

R: What does it mean?

S1: I make the sum!

After the discussion that included also other examples solved by students, each group applied the strategy suggested by their peers to perform their operations. In the group of our first example, students were able to calculate 26×4, as shown in Fig. 37.

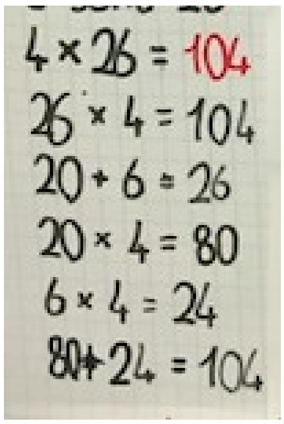


Figure 37. Students' using the distributivity property of multiplication over addition

The second step developed by students to solve *The Tiling Problem* was to sum the costs of each shape of tiles. In our example, students, having calculated $50 \times 6 = 300$, $26 \times 4 = 104$, $15 \times 10 = 150$, summed $300 + 104 + 150 = 504$, that represented the total cost in euros of their tiling design.

As hypothesized in the HLT, from their developed strategy students encountered the problem of calculating multiplications with numbers with 2-digits. Such critical point stimulated some students to re-invent the notion of distributivity of multiplication over addition.

5.4.3 Presentation and discussion

The last part of the MEA consisted in the presentation and discussion phase. This part covered two hours. Each group presented to the rest of the classroom its project. In each poster students reported the strategy they developed to solve the problem. In Fig. 38 posters created by students are reported.

Figure 38. Posters created by students

Students in their projects had been able to reproduce the strategy they developed to solve the problem. Moreover, they explained in a clear way all the calculus performed with particular attention to the distributivity property of multiplication over addition, that permitted them to calculate multiplications with numbers with 2-digits (Fig. 39).

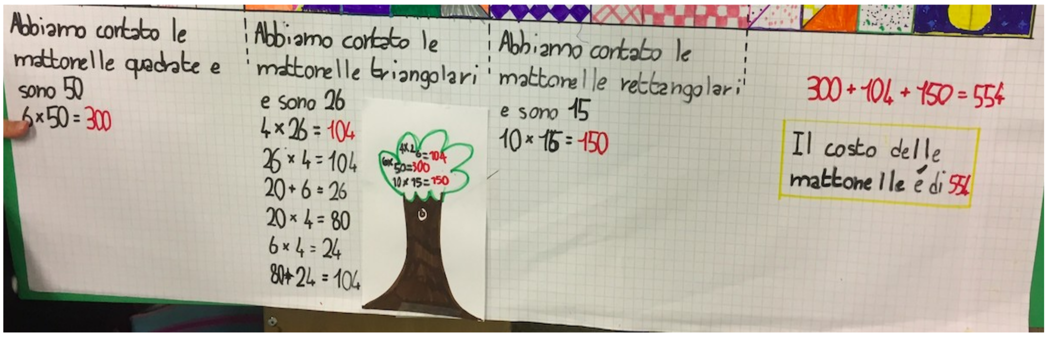


Figure 39. Students’ strategy to solve the modelling task

5.4.4 Post-test

After the conclusion of the MEA, a post-test was administered. 18 students participated to the post-test. The test was composed by four problems (the first divided in two items), comparable to the ones of the pre-test. A correspondence between pre-test and post-test problems is given in Fig. 40:

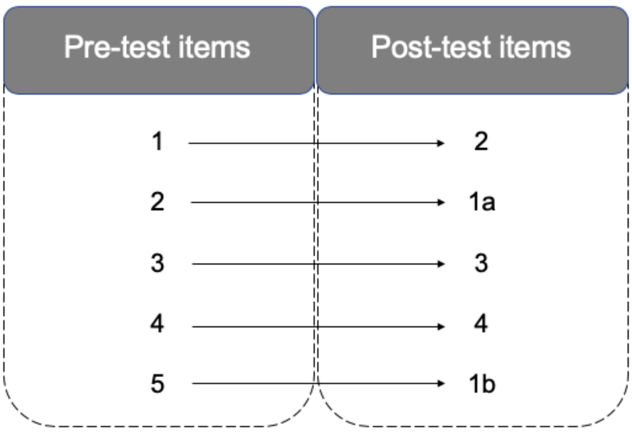


Figure 40. Correspondence between pre-test and post-test items

The full text is presented in Appendix D. Some items were matched with pre-test items, in order to see if the modelling activity had also collateral consequences in the direction of the requests from the National Evaluation System concerning multiplication (recall that this is tested through the administration of an INVALSI test, from which the problems were taken from).

In Table 16 results from the post-test are shown, concerning the numbers of correct, incorrect and not given answers for each item. In Table 17, there is a comparison between pre- and post-test results.

Table 16. Post-test results

Problem	Max score	Number of correct answers	Number of incorrect answers	Not answered
1a	2	16	2	0
1b	2	6	12	0
2	2	13	5	0
3	2	16	2	0
4	2	4	12	2

Table 17. Comparison between pre-test and post-test results

	Problem	Max score	Number of correct answers	Number of incorrect answers	Not answered
Pre-test	1	2	9	8	1

Post-test	2	2	13	5	0
Pre-test	2	2	16	2	0
Post-test	1a	2	16	2	0
Pre-test	3	2	2	11	5
Post-test	3	2	16	2	0
Pre-test	4	2	5	9	4
Post-test	4	2	4	12	2
Pre-test	5	2	7	5	6
Post-test	1b	2	6	12	0

As for the pre-test, the maximum score for each item was 2, so the total score for the post-test was 12. In the post-test the mean of the classroom was 6,25 (SD=2,25), while in the pre-test was 4,56 (SD=2,96). A Wilcoxon signed-rank test was performed making evidence of a statistically significant difference between pre- and post-test (W=10,5, $p<0.01$). The modelling activity supported students' knowledge about multiplication, as suggested by the National Curriculum. Moreover, comparing pre- and post-test results as shown in Table 17, some considerations can be done. First of all, only in the fourth item some students did not answer in the post-test, while they answered to all the other items. This can probably be attributed to a more confidence experienced by students on the topic. Item 1a for the post-test confirms students' knowledge of the notion of multiplication as iterated sum, as already seen in item 2 of the pre-test. A considerable improvement concerns item 3, with 16 correct answers respect to 2 in the pre-test (item 3).

5.5 M-I retrospective analysis

In this final section of the chapter we look back at the initial design of the HLT and compare it with the actual learning occurred and described in the teaching experiment section. This retrospective analysis could form the basis for adjusting the HLT and formulate first answers to the research questions. The section is divided in two parts: the first concerning the reflection of the teaching experiment respect to the initial HLT and the second formulating feed-forward of the M-I research cycle for the next M-II cycle.

5.5.1 Reflection on the M-I teaching experiment

The aim of this research cycle consisted in investigating how emergent modelling could be fostered to help students understanding some aspects of the multiplicative structure. In the specific, we formulated the hypothesis that a modelling activity designed following a model eliciting sequence with the use of suitable artifacts could actually foster the emergent nature of modelling. In the design phase of this research cycle we designed an HLT through the definition of its three aspects: the learning goal; the hypothetical learning process and learning activities. The learning goal was represented by the distributivity property of multiplication over addition. The hypothetical learning process consisted in putting students in a problematic situation, represented by necessity to perform 2-digits numbers multiplications, to make them re-create the notion of distributivity of multiplication over addition. Such problematic situation was defined by a task together with some constraints and instructional materials. The learning activities had been clearly presented in section 5.3.3. On the basis of the designed HLT and of the actual learning process, described in the teaching experiment phase, we can draw some reflections:

- In agreement with the process of emergent modelling, the assignment given to students stimulated them to create and work with new mathematical concepts they did not know before. In the specific, the strategy developed by students to solve the task, that consisted in grouping the tiles with the same shape and then multiply by the associated costs, showed that they were able to re-invent the mathematical concept of distributivity of multiplication respect to addiction. This is evident from different data presented in the previous section, such as: the extract of the dialogue proposed, in which students explained to the classroom their strategy to calculate 6×57; Fig. 39 in which students were able to explain and reproduce such mathematical concept. Guided by the interaction with the teacher and peers, students were able to reason and explain this property, that would be at the base of their future strategies of calculus. In this way, properties of mathematical operations become meaningful for students, because no longer mechanical rules but rooted in their experience, directly constructed by students to solve a concrete problem in a meaningful context.

- The re-invention process was possible not only thanks to the designed model eliciting sequence, but also to the use of a suitable artifact, represented by the brochure given to students (Fig. 29). Having given students the shapes of the tiles to be used and the constraint to use all of that shapes, guided them to face with the problem of performing multiplications between numbers with more than one digit, and consequently to the reformulation of the distributivity property of multiplication over addition. In this process the role of the RME heuristic of didactical phenomenology is clearly evident: it guided in choosing not only the context of the task, but also some task constraints and related materials appropriate to provide basis for the development of the mathematical concept of distributivity of multiplication respect to addition we wanted students to develop. As a consequence, model eliciting activities together with suitable artifacts could foster the emergent nature of modelling, that confirms our hypothesis. Moreover, the understanding of some aspects of the multiplicative structure in a meaningful way was enhanced. Therefore, integrating artifacts in a model eliciting sequence can actually foster emergent modelling, and in our specific situation in supporting students understanding of some aspects of the multiplicative structure.

- To achieve such results, the role of the teacher was fundamental. The teacher, indeed, encouraged students to use their own methods; stimulated students to articulate and reflect on their personal beliefs, misconceptions and informal problem-solving and modelling strategies (Bonotto 2005). Consequently, learning become a constructed understanding through a continuous interaction between teacher and students, that can be synthetized, using Freudenthal’s words, in teaching and learning as *guided reinvention*, reinforcing in this way mathematical reasoning and sense-making.

- Before the implementation of the teaching experiment, students took part to a pre-test, whose aim was to have an overview of the starting situation of the classroom concerning the topic of multiplication. The pre-test was made by items from INVALSI tests, and expressed the fact that students, despite having a poor knowledge of the topic, had a clear idea of multiplication as iterated sum. Comparing the results of the pre-test and post-test, we had also another significant collateral consequence of the modelling activity. As highlighted in section 5.4.4, there was a statistically significant increment of the classroom mean from the pre-test (4,56) to the post-test (6,25). As a consequence, the modelling activity also supported students in approaching and solving problems related to the topic of multiplication as requested by the National Evaluation System. Therefore, fostering students reasoning and critical thinking in a re-invention paradigm could help students also in achieving better learning outcomes.

5.5.2 Feed-forward of the M-I

As explained in the previous section, the teaching experiment gave some precious hints to answer to the first research question, concerning how MEAs can promote the process of emergent modelling.

The model eliciting activity implemented in the first teaching experiment developed in three parts: warm-up, model construction, presentation and discussion. In particular, high attention was given to the warm-up phase, due to the low grade of students. During the model construction students in groups developed a strategy to solve the modelling task, that was explained in their final projects. After the activity a post-test was performed, in order to match some data with the pre-test and analyze students' knowledge development.

However, we realized that not a self-evaluation occurred. The model construction was characterized by several moments of classroom discussion, in which students could share their doubts. At the same time, the teacher in whole classroom discussions and group work was able to observe students' understanding. However, we noted two main problems that suggest some modifications of the instructional sequence for the next design research cycle:

- students worked always in groups. They could share their opinions, strategies, misunderstandings, but at the same time there was not enough space for individual reflection. Group work is important, but we think that also an individual work in which students have time to reflect on the modelling activity, reconstruct the entire process, clarify doubts and express their ideas is fundamental in the learning process.

- The only feedback we had from the activity was represented by the analysis of students' work during their actual learning process and of post-tests. No final considerations were taken from students and the regular mathematics teacher, who assisted the researcher during the activities. We believe that a feedback from the people who are engaged in a teaching and learning process is very important, because in one direction it may help the researcher or designer to point out weaknesses and strengths and to better plan future instructional activities; on the another direction is an opportunity for people involved to reflect deeply and freely in their learning process.

6. M-II Research Cycle

6.1 Introduction

This chapter addresses the second research cycle concerning mathematical modelling, M-II (Fig. 41).

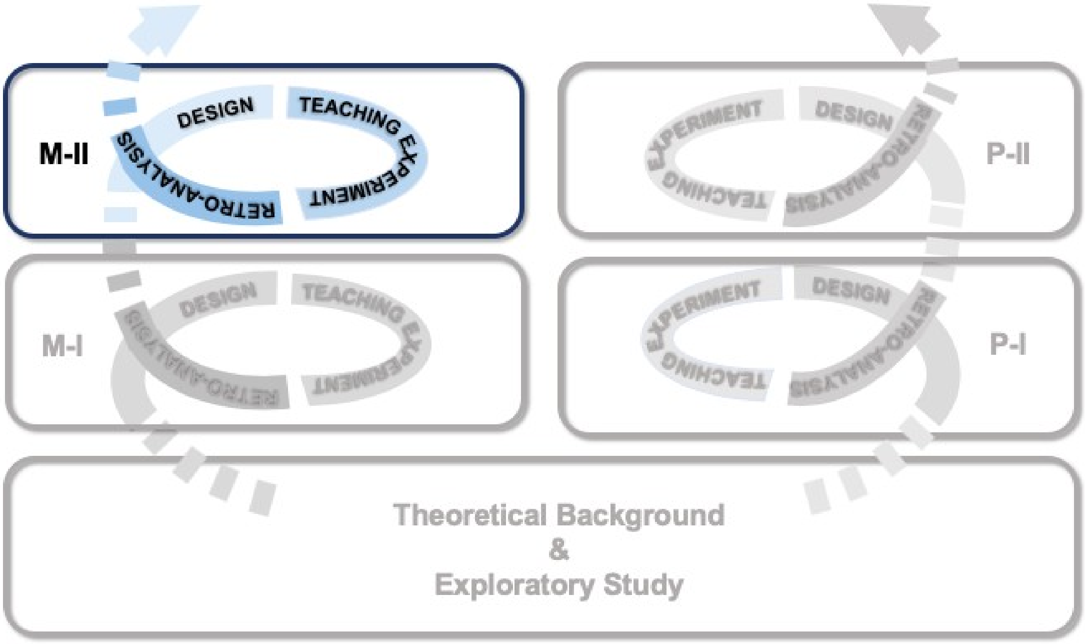


Figure 41. Research cycle M-II

The structure of the chapter is the same of the previous one: in section 6.2 we introduce the context of the classroom that participated to the study, explicating the mathematical topic considered and the specific goals of this cycle in relation to the first research question. In section 6.3 the design phase is described, consisting in the development of an HLT and with particular attention in designing instructional activities related to the learning goals of the HLT. HLT includes starting points and expectations and the students' hypothetical learning process. The design of key activities and materials are reported. Then the experiences during the teaching experiment are described in section 6.4. Section 6.5 is dedicated to the retrospective analysis.

6.2 Context and aim

The aim of the research cycles M-I and M-II consisted in finding possible answers to the first research question of this research project:

RQ1. How can Model Eliciting Activities promote the process of emergent modelling?

Starting from the retrospective analysis of the research cycle M-I, a second research cycle was designed. The research cycle M-II was conducted in a twelfth-grade class (age 17) composed by twenty-five students during two weeks of regular mathematics lessons. The class had never been engaged in a modelling activity before the teaching experiment.

In accordance with the mathematics teacher, 3D-Euclidean geometry was chosen as mathematical topic for the modelling activity. Indeed, in the Italian National Curriculum for Liceo Scientifico, at the end of the fourth year, concerning the specific learning objective of geometry, we read:

> *The study of geometry will continue with the extension to the space of some of the themes of the geometry of the plane, also in order to develop geometric intuition. In particular, the reciprocal positions of straight lines and planes in space, parallelism and perpendicularity, as well as the properties of the main geometric solids (in particular of polyhedra and rotation solids) will be studied.*

Moreover, it is suggested that students should learn to use mathematical modelling to solve problems, applying scientific results also to daily life situations.

In relation to the research question RQ1, we investigated how emergent modelling can be fostered to help students understanding some aspects of 3D-Euclidean geometry. In particular, our hypothesis was that facing students with a real problem solving situation designed following a model eliciting sequence (Lesh et al. 2003), with the use of suitable artifacts, could actually foster the emergent nature of modelling, seen as a process in which students develop mathematical concepts from informal realistic contexts.

The research method for the data analysis was mainly qualitative. The aim of the data analysis was to reconstruct the classroom progress, which resulted in an empirical grounded understanding of students' reasoning during the classroom activity. In order to be able to reconstruct the learning process and verify our hypothesis, different kinds of data were collected: pre-test; observations of group working; students' final projects; students' final individual reports; students' and teacher's feedbacks.

In the next section we present the design phase, in which the development of an HLT is described.

6.3 Design phase

In this section we first describe the starting points for the HLT and the expectations that are investigated in the following teaching experiment. Then we describe the activities and materials designed in order to foster students' cognitive development according to the goals of the HLT.

6.3.1 Starting points

Starting points for the formulation of an HLT of this second research cycle are split in two categories: (i) the classroom context, and in the specific the initial level of the students and (ii) the feed-forward formulated in the retrospective analysis of the previous research cycle M-I. Concerning the theoretical background specific for this research cycle taken into consideration to design the educational setting and hypothesis about students' learning, it is the same of the previous research cycle M-I (for more details see section 5.3.1).

Pre-test

Before the development of a learning trajectory, a pre-test was administered in the class were the research cycle took place. The aim was to have a picture about the starting level of the classroom concerning the mathematical topic considered for the modelling activity: 3D-Euclidean geometry. 25 students participated to the pre-test. The test was composed by three questions. In the specific, students were asked to give a definition of a 3D-figure, to make some examples of already known 3D-figures and to calculate the volume of a parallelepiped and of a truncated pyramid. We report the results concerning the last question of the test. The full text is presented in Appendix E. In Table 18 the number of students who were able, or not, to calculate the volume of a parallelepiped and of a truncated pyramid are presented.

Table 18. Number of students who were able or not to calculate the volume of a parallelepiped and of a truncated pyramid.

	Right calculus of the volume	Wrong calculus of the volume	Not answered
Parallelepiped	16 (64%)	7 (28%)	2 (8%)
Truncated pyramid	4 (16%)	10 (40%)	11 (44%)

Findings indicate that the 84% of students were not able to calculate the volume of a truncated pyramid. These means that before the teaching experiment the majority of students did not know or was not able to apply the equidecomposability principle to calculate the volume of an irregular solid[3] such as the truncated pyramid. Recall that equidecomposability is an equivalence relation between geometric figures, such as surfaces or solids. Two figures are equidocomposable if they can be decomposed in

[3] With irregular solid we refer to a solid that cannot be classified as a polyhedra or a rotation solid.

congruent figures. In particular, if two solids are equidecomposable then they have the same volume. The last sentence refers to what we mean by equidecomposability principle in the rest of this study. In the pre-test, in order to calculate the volume of a truncated pyramid, the equidecomposability principle could be used in the following way: the truncated pyramid can be seen as a "big" pyramid minus another "small" pyramid; calculate the volume of both the pyramids; subtract the volume of the "small" pyramid from the volume of the "big" pyramid (Fig. 42). Such principle is useful to calculate the volume, or better an approximation of the volume, of irregular solids that can be encountered in real-life. As we will see in section 6.3.3, this consideration represents the base idea to design a learning trajectory that could foster students in rediscovery such a principle from a real task.

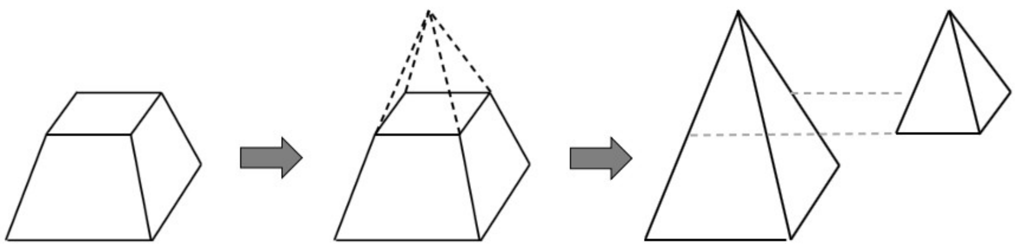


Figure 42. Equidecomposability principle to calculate the volume of a truncated pyramid

M-I feed-forward

The first research cycle dealing with mathematical modelling described in chapter 5, gave some precious hints to increase the instructional design developed for that teaching experiment. Such modifications influenced the design of the learning activities of the teaching experiment of the second research cycle M-II. In the specific, differently from the previous teaching experiment, we decided to introduce the following adjustments:

- The model eliciting sequence will remain the same in the phases of warming-up, model construction, presentation and discussion. However, after these phases, that are mainly centered in a group work setting, we decided to dedicate time also for individual activities. Firstly, students will be engaged in writing a report in which they could reflect on their work, to express doubts or misunderstandings, to reformulate in their own words the learning path. At the end of the modelling activity, we decided not to administer a post-test, but to engage students in a final personal feedback concerning the entire modelling activity.

- Since the research project has as one of its overall aim to provide teachers with methodological models and format of school practices based on mathematical modelling, in our view it was important also to have a feedback from the regular mathematics teacher who joined the researcher during the modelling activity. This feedback has a double aim: on the first hand in helping the researcher in having a direct response on the implemented project, in order to reflect in a deeper way for better future designs; on the other hand the teacher, who could express some strengthens and weaknesses of the activity, should became aware of the competencies needed from not only a disciplinary, but also a methodological perspective.

6.3.2 Learning goal and hypothetical learning process

As stated in section 6.2, the aim of this first research cycle consists in investigating how emergent modelling can be fostered to help students understand some aspects of 3D-Euclidean geometry. In the pre-test we saw that students at the moment of the intervention did not know the equidecomposability principle, or at least they were not able to apply it to calculate the volume of an irregular solid (truncated pyramid). However, this principle is the base point to calculate the volume, or better an approximation of the volume, of an irregular solid, that is the type of solids that commonly occurs in real life. As a consequence, what we want to achieve during the teaching experiment is the re-invention of the equidecomposability principle. Therefore, as shown in Fig. 43, the learning goal of the teaching experiment is represented by the equidecomposability principle.

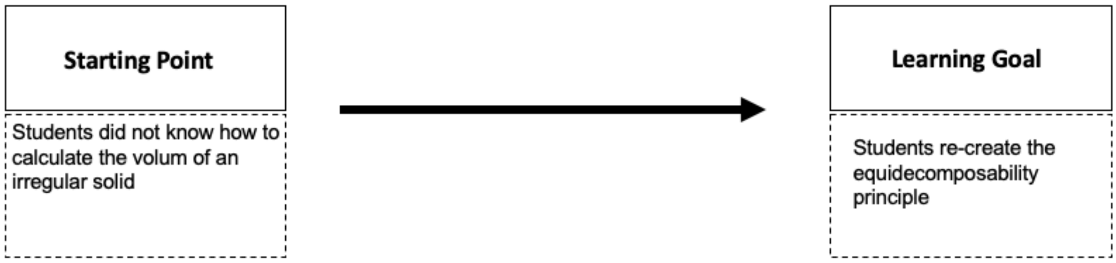


*Figure 43***.** Learning goal of the M-II cycle

To design an HLT, together with the learning goal we need to formulate some conjectures about the learning process. In analogy with the first research cycle, starting from the classroom level and following the heuristic of didactical phenomenology, we supposed that making students face with a problem situation in which they need a new mathematical concept to solve it, could stimulate the same students in creating that concept. In order to do that, our idea consisted in putting students face with a realistic problem solving situation in which at a certain point they feel the necessity to calculate the volume of an irregular realistic solid. We believe that when students deals with real objects from their world are better stimulated to develop a strategy to calculate their volume, and in the specific to re-create the idea of equidecomposability, namely decomposing a solid in regular solids that approximate it, calculate the volume of such solids, sum those volumes and obtain an approximation of the volume of the starting irregular solid. From a first exemplificatory case students should be able to generalize this principle detaching it from the starting context situation, formulating a formal mathematical concept that could be applied in other situations (Fig. 44).

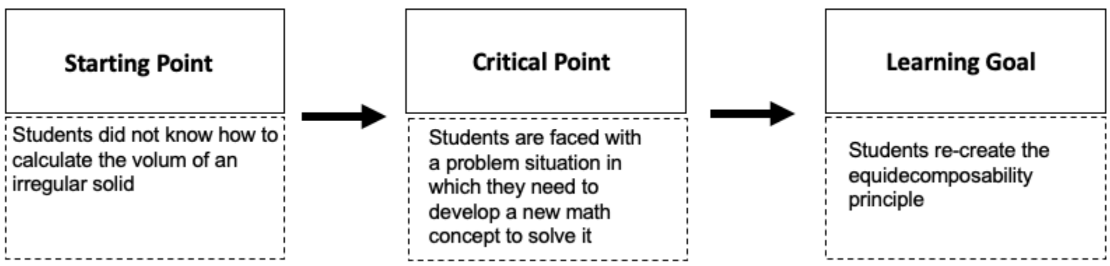


Figure 44. Hypothetical learning process of the M-II cycle

The learning process outlined in Fig. 44 is not linear. Indeed, attention should be given not only to the solution process of the given problem, but to the process of construction of a mathematical model that support such solution process. In agreement with the model eliciting approach, students develop a model going through iterative phases of invention, refinement and revision. Modelling is a process of developing representational descriptions for specific purposes in specific situations, involving iterative testing and revision cycles (Lesh and Lehrer 2003). Moreover, we think that this process could support the emergence of new mathematical knowledge, since the developed model is firstly a model that is created in a specific situation to solve a particular problem, but during the modelling activity becomes a model for a more general mathematical structure. In our case, students should start developing a model to solve a given problem, and then discover that such a model permitted them to create a general mathematical concept: the equidecomposability principle. In the next section we present the learning activities that make those first hypothesis more concrete.

6.3.3 Learning activities

The design scheme used to develop some learning activities to achieve the learning goal described in the previous section is presented in Fig. 45. Our hypothesis is to adapt the model eliciting sequence (Lesh et al. 2003) in order to translate the hypothetical learning process in concrete classroom activities. The difference with the design scheme used in the previous research cycle is that now there is an additional phase of *reflection and debriefing*, in which students reflect individually on the whole activity.

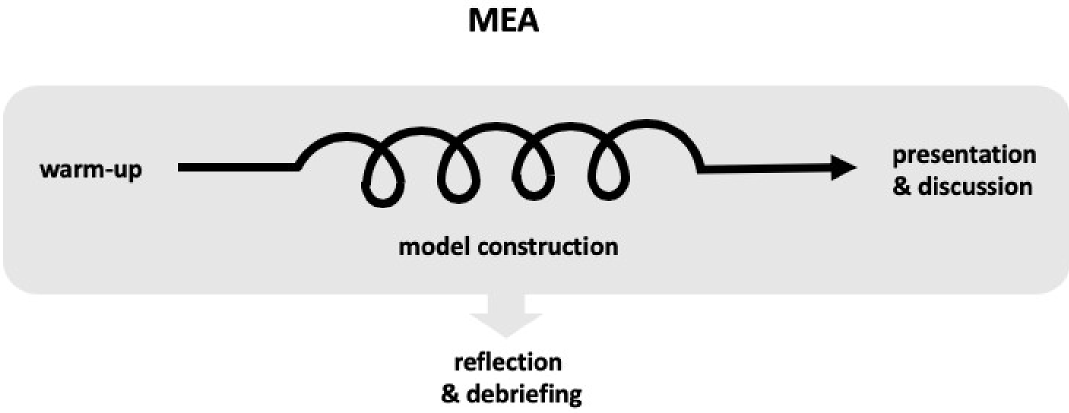


Figure 45. Design scheme for the M-II cycle

Between the pre-test and the warm-up phase, two lessons were dedicated to the introduction of some basic notions concerning 3D-Euclidean geometry. These lessons had been carried out by the mathematics teacher. The teacher made students work in groups using manipulative materials, such as bottles, straws, cans… The materials and kind of activities used by the teacher permitted to introduce some preliminary concepts useful for the following modelling activity. As a consequence, this preliminary phase can represent an extension of the warm-up phase. The notions covered were taken from the National Guidelines for Liceo Scientifico and dealt with: reciprocal positions of lines and planes in space, parallelism and perpendicularity, as well as the properties of the main geometric solids (in particular polyhedra and rotation solids). Activities and materials were designed together by the teacher and the researcher.

One week later, the first phase of the MEA started, and was represented by the warm-up phase. One lesson was dedicated to the presentation of the modelling task and to its comprehension. Students were firstly divided in groups of four. To each group was given the following modelling task (Fig. 46):

Job Summary

Company
Bevandeitalia.srl

Location
Padova, via Durer 14

Job type
Packaging department employee

Salary
19 000 euros per year

About the job
Bevandeitalia, world leader in the production and distribution of beverages, is looking for a packaging department employee. Permanent contract of 19 000 euros per year. To apply, it is necessary to present a project with your curriculum vitae attached. The project requires an estimate of the capital needed for making beverage packaging, respecting the following constraints:

- Liters to be packed of

Water	Juice	Cola
1000	500	350

- Diversification of packaging types, according to form (at least two for each drink) and materials used.

Figure 46. Modelling task given to students.

Together with the text of the problem, students are given also a brochure (Fig. 47). In the brochure, the costs of some materials useful to design some requested packaging are shown, while other information are hidden (for example, the capacity of the bottles). The brochure, together with a geometry formulary containing information about surfaces and volumes of regular solids, represented the "tool-kit" that students had at disposition to solve the assignment.

Figure 47. Brochure given to students to solve the task

During the warm-up activity, after having reviewed the materials with the researcher, students have to answer some simple comprehension questions: *what is requested for the job application? How can the brochure be useful? Which mathematical concepts or tools do you need to solve the task?* Students' answers should be discussed in a whole class discussion in order to clarify any eventual doubts.

The second phase of the MEA is dedicated to the model construction. Each group has to make a project to apply for the position as a packaging employee. The researcher gives to students some guidelines to make the project. In the specific, students are asked to report and explain clearly their reasoning, build at least two of the designed packaging and describe them from a mathematical point of view. Our hypothesis is that students would encounter the problem of calculating the volume of some irregular solids. For example, since in the brochure (Fig. 47) the information of the capacity of the bottles is removed, they have to calculate it. In so doing they need to develop a strategy, and in so doing we conjecture that they would rediscover the equidecomposability principle. At the same time, since an irregular solid is decomposed in the sum of some regular solids, students also make practice of some formulas concerning the volume of regular solids.

In the third phase each group presents its work to the rest of the classroom. In order to help students to know what will be expected from their presentations, a rubric will be offered to them. In the specific, the steps followed to solve the task must be defined in a clear way and each member of the group has to take part to the presentation. After the presentation, each group will be engaged in a Q&A session, where teachers and peers could ask questions about the presented model.

In the final phase of reflection and debriefing, an individual work is assigned to students. In the specific, every student has to write a report to the administrator of the factory in which reflect about her/his project, report strengths and weaknesses, alternative approaches and some considerations about positive and negative aspects of the whole activity. This activity should allow students to think through the activity cycle and to make their own conclusions to model a solution to the model eliciting dilemma.

6.3.4 HLT for cycle M-II

In this section we sum up the components of the designed HLT for the research cycle M-II discussed in the previous sections, namely the learning goal, the hypothetical learning process and the learning activities. Together those components define the HLT. A scheme is represented in Table 19. In the next section we present the M-II teaching experiment.

Table 19. HLT of the M-II cycle

HLT		
Learning goal	Hypothetical Learning Process	Learning Activities
Equidecomposability principle	Making students face with a problem situation that need the development of new mathematical concepts to be solved could support students in achieving the learning goal. The critical point is represented by the necessity to calculate the volume of irregular solids, represented by some packaging students have to design. The critical point is made explicit through a modelling task with some specific constraints. These assumptions are made explicit through the development of some learning activities.	Students are engaged in a MEA in which they have to develop a model to solve the task about a job employee (Fig. 46). Before the MEA the teacher introduces some basic notions about 3D-Euclidean Geometry. Materials prepare students with the modelling task. In the first phase of the MEA (warm-up) all the materials are presented to the students who are engaged in a comprehension activity. During the model construction, students could face with the problem of calculating the volume of irregular solids. Some students are expected to re-create the concept of equidecomposability. In

		the end each group presents its findings to the rest of the classroom. The last activity is represented by an individual reflection on the entire activity.

6.4 M-II teaching experiment

In this section we present results from the teaching experiment concerning the research cycle M-II.

As described in the previous section, during the modelling activity each group had to make a project to apply for the position as a packaging employee. In the specific, students were asked to report and explain clearly their reasoning and build at least two of the designed packaging and describe them from a mathematical point of view. We report now some extracts from students' projects in order to reconstruct their actual learning process. Recall that the task consisted in estimating the total cost to pack 1000 liters of water, 500 liters of juice and 250 liters of cola, using different forms and materials. All the groups were able to present in a clear way the steps followed to solve the task. However, not all the groups followed the same pattern: in Table 20 patterns of three different groups are compared.

Table 20. Steps implemented by three different groups to solve the packaging problem.

	Group 1	Group 2	Group 3
Step 1	Analysis of the request	Analysis of the request	Analysis of the request
Step 2	Calculus of capacities	Casual choice of packaging types	Choice of convenient packaging types

Step 3	Calculus of costs	Calculus of capacities	Calculus of capacities
Step 4	Cheaper choice	Calculus of costs	Calculus of costs

The first group started performing calculations to obtain the capacity of different kinds of packaging. Then a comparison between different packaging costs was made and the choice followed the cheapest. For example, to pack the 1000 liters of water, the three kinds of bottles present in the brochure were compared. For each bottle, the capacity, the total number and the total price were calculated. Then, the results were compared, and the cheapest type was chosen. We can observe that this group approximated the total number of bottles and not the capacity of each bottle (Table 21).

Table 21. First group solution steps concerning the packaging of 1000 liters of water

Type of bottle	Green	Ampolla	Giara
Capacity (l)	0,626	1,163	1,303
Total number of bottles	(1000:0,626)≈1598	(1000:1,163)≈860	(1000:1,303)≈767,5
Total cost (euros)	1278,4	1204	1152

The second group, instead, started choosing in a casual way some kinds of packaging and then calculated their relative costs. Differently from the previous group, in this case, students approximated the capacity of the packaging. For example, when calculating the volume of a “brick” for juice packaging, they obtained 0,202 liters, and the result was approximated to 0,200 liters (Fig. 48).

Tipologia	Volume contenitore (L)	Volume liquido contenuto (L)	Superficie singolo pezzo (cm2)	Prezzo singolo pezzo (€)	N° pezzi	Prezzo totale (€)
Acqua						
Modello «green»	0,59	0,5	-	0,80	1000	**800**
Modello «giara»*	1,225	1,25	-	1,5	400	**600**
Succo di frutta						
Brick	0,202	0,200	0,0215	0,213	1250	**266**
Cartone grande	1,118	1,0	0,0723	0,716	250	**178,90**
Bibita gassata						
Lattina	0,386	0,33	0,294	0,585	530	**310**
Modello «ampolla»	1,15	1,0	-	1,4	175	**245**

*Il volume calcolato risulta minore del volume di liquido contenuto perché abbiamo considerato le grandezze standard delle bottiglie e previsto un errore nell'approssimazione del volume

Figure 48. Group 2 calculations and approximations of packaging capacities and relative costs

The third group followed a sort of refinement of the second one. In this case, the choice for the packaging was not casual, but followed a "convenience" criterion. For example, when choosing the packaging for the juice, this group decided to produce two different models: a big one for domestic use and a small one for single use outside home.

When performing the calculation for the capacity of the bottles, all the teams tried to decompose the bottles as sum of regular solids, calculate the volume of each solid and then sum the volumes to obtain the capacity of the bottles. In so doing, some groups explicitly approximated the results obtained. For example, in Fig. 49, the bottle "green" was decomposed in a cone plus a pyramid. The total volume obtained was of 0,59 l, that was finally approximated to 0,5 l. Students, in order to calculate the volume of irregular 3D-figures, were able to re-invent themselves the equidecomposability principle. This means that the task stimulated students to build a mathematical concept they need.

Figure 49. Bottle “green” decomposition in a cone plus a pyramid and calculus of its total volume

Not only approximations, but also other realistic considerations were taken into account by students. For example, one group, in order to choose the best typology for the cola, made a research about different shapes and materials. In the specific, they found that

> *…the cylindrical shape allows to economize space and to be easily held. Today there are two main models of cans: a classic 11.5 cm high, and a new one introduced in 2005 that is slimmer, 14.5 cm tall. The capacity of both the models is 33cl.*

Despite the same capacity of the two models, students found different costs of production. In Fig. 50, the calculations of the total costs using the two cans models are compared, finding that the classical one is cheaper than the other.

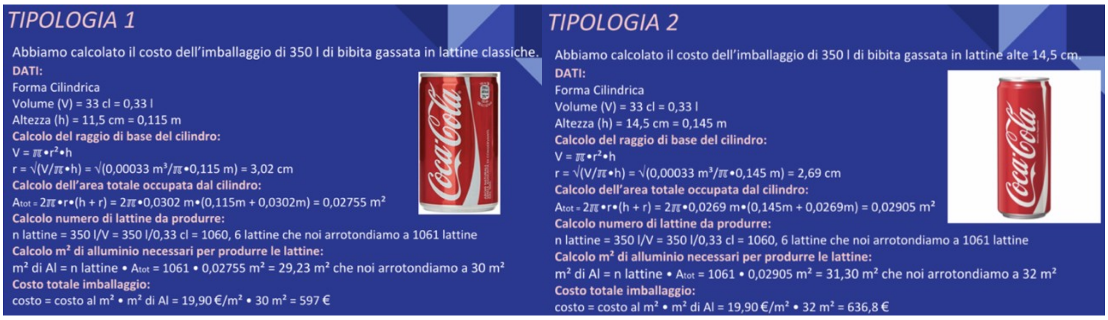


Figure 50. Comparison between the total costs of the two can models

Several groups took into account also other possible costs to be added to the packaging ones, such as: labels, transport, advertising, store…

Finally, in Fig. 51 there are some examples of students' built models.

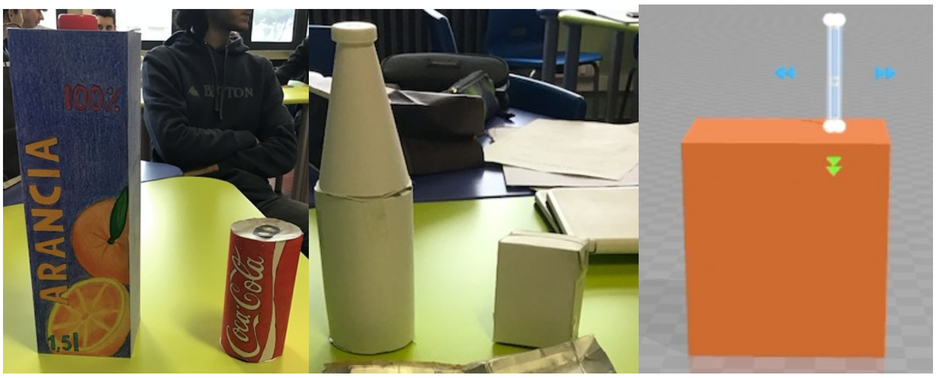


Figure 51. Students' built models. The ones on the left and in the middle were constructed in cardboard, while the one on the right is a 3D-digital construction

In conclusion, we remark the fact that during the sessions dedicated to the construction and presentation of their models, several students who did not typically engage during mathematics classes participated in an active and pro-positive way, highlighting in this way the social implications of modelling activities.

In the reflection and debriefing session, an individual work was assigned to students. In the specific, every student had to write a report to the administrator of the factory in which reflect about her/his project, report strengths and weaknesses, alternative approaches and some considerations about positive and negative aspects of the whole activity. This activity allowed students to think through the entire activity cycle and to make their own conclusions to model a solution to the model eliciting dilemma. All the students were able to present their project, specifying the steps they followed to solve the task and the results obtained. Some extracts from students' reports follow. In the first two extracts students proposed some changes to the projects of their groups. The individual activity, indeed, gave the opportunity to reflect in a deeper way on the previous group work.

E.1: *However, some changes could be made to the project, as we only considered a single type for each type of product. Instead, it would have been more appropriate to divide each product into different types of packaging. In this way the final price could have been lower, avoiding also material waste.*

E.2: *... we have not considered what we can define "chemical" aspects of drinks. First of all, the juice requires special treatments for storage. Moreover, the type of cardboard packaging we considered involves a direct contact between the drink and the glue in sealing the container, which is so harmful.*

Several students focused on similarities and differences between their project and the ones of the other groups. Not only the model construction, but also the presentation and discussion time, influenced the final individual reflection, letting every student to reflect about different possible routes to solve the modelling problem.

E.3: *Compared to the other groups we paid more attention to the additional costs (labor, advertising, etc…) but less for the calculation of the volume of the types of packaging.*

E.4*: The other groups also approximated the measurements, in fact to calculate the volume of the bottles it was necessary to divide the bottles into two solids already known to us: a cylinder and a cone. However, some groups divided the bottle into a truncated cone and a cone. All groups took into consideration the possible additional costs, even if of different natures.*

Students in their reports focused also on the new mathematical concept developed during the modelling activity and on applications of mathematical tools to solve the task. In extract 7 one student explicitly said that the activity permitted her/him to acquire new mathematical knowledge.

E.5*: In the realization of this project I discovered and used the concept of equidecomposability.*

E.6*: To calculate the volume of the bottles we have decomposed them into more regular solids, we have used proportions and formulas to calculate missing data and the volumes of these solids.*

E.7: *The work was interesting and stimulating and through the study and analysis of the required packaging I was able to expand my mathematical knowledge.*

In conclusion, students did not ignore relevant, plausible and familiar aspects of reality, nor did they exclude real-world knowledge from their observation and reasoning:

E.8: *To choose the right packaging method, we have based ourselves on our personal experience: for example, water is generally found in plastic or glass bottles but not in cardboard.*

E.9: *The work of our group has focused on the fact that the products we packed should be bought by as many people as possible. This means that in addition to the usual family formats we must also think of comfortable formats in terms of use and space occupied, for this reason we tried for each type of drink to present two alternatives: a larger one and a smaller one, for all needs.*

After the conclusion of the modelling activity, students answered to some feedback questions. Firstly, students were asked to report some positive and negative aspects of the modelling activity. Twenty-three students completed the questionnaire. Students' answers concerning positive aspects of the modelling activity were grouped in eleven categories. In Table 22 these eleven categories and their distributions are reported. Students' answers concerning negative aspects of the modelling activity were grouped in seven categories. In Table 23 these seven categories and their distributions are reported.

Table 22. Categories individuated from students' answers concerning positive aspects of the modelling activity. The total is higher than 100% because each student could express more than one option

Positive aspects of the modelling activity		
Category	*(N)*	*(%)*
Group work	12	52 %
Real, concrete materials	12	52 %
Math applications to reality	4	17 %
Motivating, stimulating	3	13 %
New way to do math	3	13 %
Reasoning	2	9 %
Realistic project	2	9 %
Interaction	2	9 %
New concepts	1	4 %
Designing	1	4 %
Clear lessons	1	4 %

Table 23. Categories individuated from students' answers concerning negative aspects of the modelling activity. The total is higher than 100% because each student could express more than one option

Negative aspects of the modelling activity		
Category	*(N)*	*(%)*
Limited time to organize the exposition	8	35 %
Limited time to do exercises	8	35 %
Too slow	2	9 %

Group disorganization	2	9 %
Too simple warming activities	1	4 %
Same request for all	1	4 %
Always group work	1	4 %

Students' were also asked to express their opinion about the possibility to repeat a similar activity in the future. Fig. 52 represents a cloud with students' answers to the latter question. In particular, two main families were individuated: students who would like to repeat the activity (*Yes*) and students who would not like to repeat the activity (*No*). The family *yes* involved 21 students (92%) and was connected to six categories that consisted in the motivations given by students to implement a similar activity in the future. These categories are: *addictive and stimulating* (43%); *understanding of new concepts* (26%); *enjoyable* (13%); *group work* (13%); *innovative lessons* (13%); *real competencies* (13%). Two students (8%) did not want to repeat a similar activity. In this case, two categories were individuated: *timing inefficiency* (4%) and *not in line with the course of study* (4%).

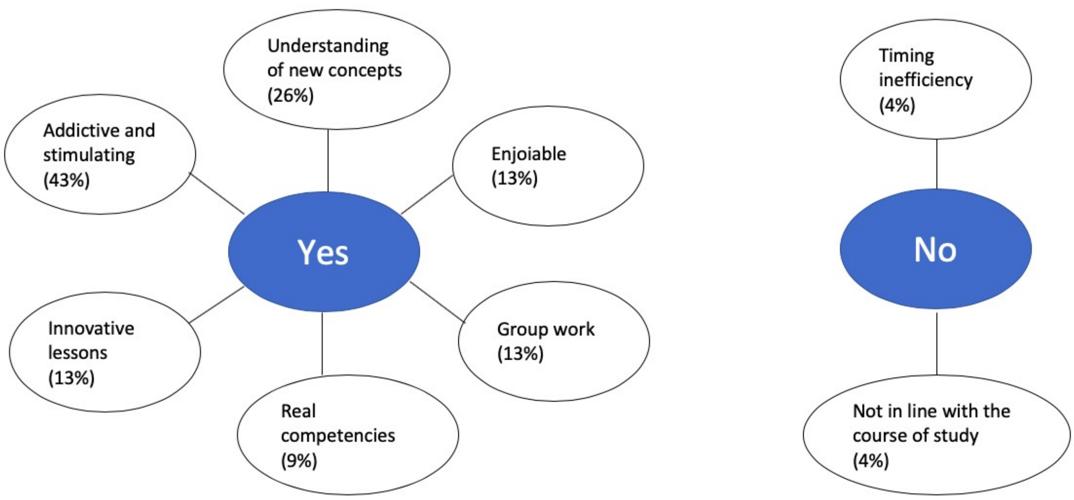


Figure 52. Students' opinion cloud about the possibility to repeat a similar modelling activity in the future

As described in section 6.3.1, at the end of the activity the mathematics teacher was asked to make a feedback of the entire activity. This feedback has a double aim: on the first hand in helping the researcher in having a direct response on the implemented project, in order to reflect in a deeper way for better future designs; on the other hand the teacher, who could express some strengthens and weaknesses of the activity, should became aware of the competencies needed from not only a disciplinary, but also a methodological perspective. We report the teacher considerations:

Strengths

The activity elicited interest in students, especially towards those who are usually not very attending classes, even just in asking questions.

I noticed that alternating theoretical explanations with group activities stimulates students to have more interest, to get less bored, to ask more questions even in their first learning phase of the notions just explained.

During the project I noticed a "team" job, and during the presentation every member of the group exposed a part of the research, so the cooperation was certainly present in each group, nobody was excluded.

Critical elements

The amount of time spent working in the classroom was reduced compared to the time it would have deserved in practice. So, students had to complete at home their work.

Some exercises assigned in the first week were too trivial, while others proved to be more complex than expected in the development phase.

Little time was spent on building the most complicated Platonic solids and identifying their properties.

I have not yet done a written test, so I have no concrete feedback to date and therefore I am not yet expressing an opinion on the effectiveness of the process.

In any case, for me having seen almost all the students working and intervening allows me to give an overall positive opinion on the activity, which I would gladly propose again.

From the analysis of the teacher's report, some points emerge. Concerning strengths of the activity, the teacher noticed that students had been stimulated and participated actively to lessons. Moreover, students who typically did not engage during mathematics classrooms showed interest and were motivated to work. Students were able to work in group with a spirit of cooperation without excluding anybody. The activity had strong social implications, fostering students' motivation, participation and a climate of inclusion. Regarding weaknesses, we point the attention in the teacher's consideration concerning the assessment. The teacher, indeed, said that she was not able to have a concrete feedback because a written evaluation had not been performed yet. This consideration highlights the fact that it is important to foster teachers' preparation not only concerning methodological issues on lesson design, but also give them some instruments to assess the process of a modelling activity. Such instruments must be accompanied by a paradigmatic change in the way of assessing students' learning, from a summative perspective to a formative one. We recognize the importance of this last consideration, that would represent a starting point for future work.

6.5 Reflection on the M-II teaching experiment

The aim of this second teaching experiment consisted in investigating how emergent modelling could be fostered to help students understanding some aspects of 3D-Euclidean geometry. In the specific, we formulated the hypothesis that a modelling activity designed following a model eliciting sequence with the use of suitable artifacts could actually foster the emergent nature of modelling. In the design phase of this research cycle we designed an HLT through the definition of its three aspects: the learning goal; the hypothetical learning process and learning activities. The learning goal was represented by the equidecomposability principle to calculate the volume of an irregular solid. The hypothetical learning process consisted in putting students in a problematic situation, represented by the necessity to calculate the volume of some irregular solids taken from a real-life task, in order to make them re-create the notion of equidecomposability. Such problematic situation was defined by a task together with some constraints and instructional materials.

On the basis of the designed HLT and of the actual learning process, described in the teaching experiment phase, we can draw some reflections:

- the first aim of the study consisted in evaluating the impact of model eliciting activities on the emergent nature of modelling. With emergent nature of modelling we mean students' attitude to create new mathematical concepts mathematizing their personal informal solving strategies. From the initial test only the 16% of students was able to calculate the volume of a truncated pyramid. This means that students before to be engaged with the modelling activity had not a clear idea about the equidecomposability principle, or at least they were not able to apply it. When solving the task, students had to calculate the capacities of some bottles given in the brochure. To do that, students had to plan a strategy to obtain the volume of these bottles. Students in all groups decided to decompose each bottle in many regular solids (cones, pyramids). Then, to obtain an approximation of the volume of the bottle considered, they calculated the volume of the regular solids by which the bottle was decomposed, and then summed these volumes. This procedure is evident in Fig. 49. In other words, since students had to calculate the volume of an irregular solid, they were able to discover, or better re-invent, the equidecomposability principle. This fact is evident also in the extract number 5 of students' individual reports, in which a student said that during the *project she/he discovered and used the concept of equidecomposability*. The assignment given to the students stimulated them to create mathematical concepts they did not know before, as confirmed also by extract 7: *the work was interesting and stimulating and through the study and analysis of the required packaging I was able to expand my mathematical knowledge*. As a consequence, we can affirm that model eliciting activities can foster emergent modelling;

- the teaching experiment M-II was characterized by a sequence of activities that followed the design scheme of Fig. 45. This scheme was an adaptation of the model development sequence proposed by Lesh et al. (2003) in the perspective of model eliciting. This development sequence represents a valid designing scheme that can be followed by teachers of every school level to implement modelling activities. We want to remark the importance of the individual reflection after the teamwork during the modelling activity. Each student should have the opportunity to reflect on her/his work, to express doubts or misunderstandings, to reformulate in her/his own words the learning path. However, the scheme itself is not sufficient to produce meaningful modelling routes. Two additional fundamental points are: the task itself and the role of the teacher;

- in our study, the task consisted in a packaging problem. Students during the warming week worked with concrete objects (bottles, straws, cans, …) that were connected to the packaging task. In this way, when students started the modelling activity were more familiar with the context of the problem. The mathematical stimulus of the context had been reinforced by the use of an artifact represented by the brochure given to students to solve the task. In particular, the brochure was reconstructed by the researcher omitting some details, in order to foster students' disposition in creating the mathematical tools and concepts they needed to solve the assignment;

- concerning the role of the teacher, this is crucial not only in designing a modelling sequence, but also during its implementation. Indeed, the teacher has to encourage students to use their own methods; stimulate students to articulate and reflect on their personal beliefs, misconceptions and informal problem solving and modelling strategies. Consequently, learning becomes a constructed understanding through a continuous interaction between teacher and students, that can be synthetized, in teaching and learning as *guided reinvention* (Freudenthal, 1991), reinforcing in this way mathematical reasoning and sense-making. In the direction of the aims of the University project, this study provides teachers with a design scheme for model eliciting activities; offers an example of implementation of a complex modelling activity; outlines the importance in the choice of an appropriate rich context problem when implementing modelling patterns; remarks the role of the teacher in a balance between the principles of guidance and invention;

- the analysis of the results from the case study that was described, shows that students understood some aspects of Euclidean geometry in a meaningful way. Students while solving the task, were able to discover the equidecomposability property to calculate the volume, or its approximation, of an irregular solid. As a consequence, geometry property were no longer mechanical rules given by the teachers to be applied to solve some numerical problems but assume meaning because rooted in students' experiential activity. In addition, during the modelling activity, students took into account several realistic considerations: approximations, historical facts, optimality choices. Significant examples are given by the extracts 8 and 9, in which students reported that they based some choices on their personal experience. This means that stimulating modelling activities can create a bridge between in-school and out-of-school mathematics, that is one of the most significant roles of mathematics teaching;

- in conclusion, we want to reflect on the results concerning students' perception about the modelling activity. Reality connotations of the task was one of the most present in students' considerations. Indeed, three categories are related to this characteristic (*real, concrete materials; math applications to reality; realistic project*). Moreover, in the cloud of Fig. 52, *realistic competencies* appears as one category for the future replication of a similar activity. We believe that students need opportunities to be engaged in activities connected to the real world. In this way, school and daily life should not be seen as separated realities but became integrated one into the other. Another category that was individuated in several students was *group work* (52%). The same category was present also in the final cloud. Despite students found significant to work in team, they also expressed the difficulty to organize the work into their team. Indeed, one of the negative aspects was *group disorganization*. We believe that working in group is a practice that needs time, and teachers should promote it starting from the first years of school. Group working, however, should not be the only modality for learning. As we have showed in our teaching experiment, also a moment for individual reflection is fundamental, in order to get a deeper understanding and involvement in the learning path. To conclude, some students (35%) expressed the fact that the time for preparing the presentation was limited. Moreover, in the final cloud one of the students said that she/he would not want to repeat a similar activity in the future because time schedule was not efficient. The reason can be attributed to the fact that this was the first modelling activity that students performed during their school lessons. As a consequence, they were not used to work in teams, to organize their group work, to choose, test and revise a solution pattern, while their usual school lessons were planned as front-lessons and applicative exercises. Instead, students need more opportunities to learn in a different way, that could enhance competencies connected to the notion of twenty-first century skills, which include creativity, decision making, critical thinking, problem solving, collaborating, communicating (Maass et al. 2019).

6.6 Conclusions from M-I and M-II research cycles

The aim of the research cycles M-I and M-II consisted in answering to the first research question, i.e. studying how MEAs can promote the process of emergent modelling. From the design research cycles, we can outline some answers to such question:

- model eliciting activities could play a central role in fostering emergent modelling. This positive result can be attributed to a combination of several factors: the choice of a realistic and rich problem, that stimulated students to elaborate formal mathematical concepts mathematizing their informal solving strategies, rooting in this way the new understandings in experientially real phenomena; the use of suitable artifacts, that presented mathematics as a means of interpreting and understanding reality and increasing the opportunities for observing mathematics outside of the school context (Bonotto 2005); the role of the teacher, who guided students in re-inventing mathematics in an active way;

- from the retrospective analysis of the M-I research cycle, we noted the necessity to engage students in a final individual reflection on the whole modelling process. This individual activity was introduced in the second research cycle in the reflection and debriefing phase. In the specific, the *Q&A session* permitted students to elaborate their solutions in a less formal manner which demonstrate understanding differently from what they showed in their prepared presentations. Then, in the *reflection and debriefing* activity students could reflect individually on the whole modelling process. Moreover, they could observe strengths and weaknesses of their projects; elaborate their own conclusions to model a solution for the model eliciting activity; analyse similarities or differences with other solution plans; change their beliefs and attitude; reinforce argumentation abilities.

- this kind of activities had also strong social implications. In fact, several students who did not typically engage during mathematics classes became active participants while solving a modelling task. As a consequence, the introduction of new socio-mathematical norms (Yackel and Cobb 1996) and the use of interactive teaching methods, promoted cooperation and inclusion. Students being active participants into the learning process were able to give meaning to new mathematical knowledge and sense to their mathematical activity.

Beside to these positive results, the implementation of such classroom activities requires very high demands on teachers, in agreement with Blum (2015). Indeed, teachers should be able to: i) see mathematics incorporated in the real world as a starting point for mathematical activities; ii) anticipate the mathematics needed for the paths that students might explore; iii) put students in familiar situations in which they clearly understand the need for mathematical constructs, integrating also their everyday knowledge; iv) provide meaningful design specs involving constraints that enable students to weed out inadequate ways of thinking. In this direction, feedbacks from the regular mathematics teacher permit to the same teacher to be aware of which competencies are needed to design and implement a modelling project. For this reason, we believe that in the future an improvement in teachers' pre-service and in-service courses are needed, in order to provide teachers with designing principles and practical materials to develop modelling activities in their classrooms.

In the final chapter of the thesis, we will reflect again on the conclusions from these first research cycles, highlighting possible future directions of research.

7. P-I Research Cycle

7.1 Introduction

The second research question of this study deals with mathematical problem-posing. Problem-posing is closely linked to mathematical modelling. A possible connection between modelling and problem-posing is given by the use of real contexts as starting situations for mathematics lessons. From the exploratory study (chapter 4) we saw that in the Italian context problem-posing is still not included in daily mathematics lessons. We believe that in order to integrate problem-posing in the teaching of mathematics the use of real contexts should be reinforced. However, how can a real context be described during problem-posing activities? Do different contexts influence differently problem-posing activities? And how? In the next two research cycles, P-I and P-II, we start investigating how different contexts for problem-posing activities can influence students' creativity and emergent problem-posing.

In the specific, this chapter addresses the first research cycle concerning problem-posing (Fig. 53), dealing with students' creativity during problem-posing activities.

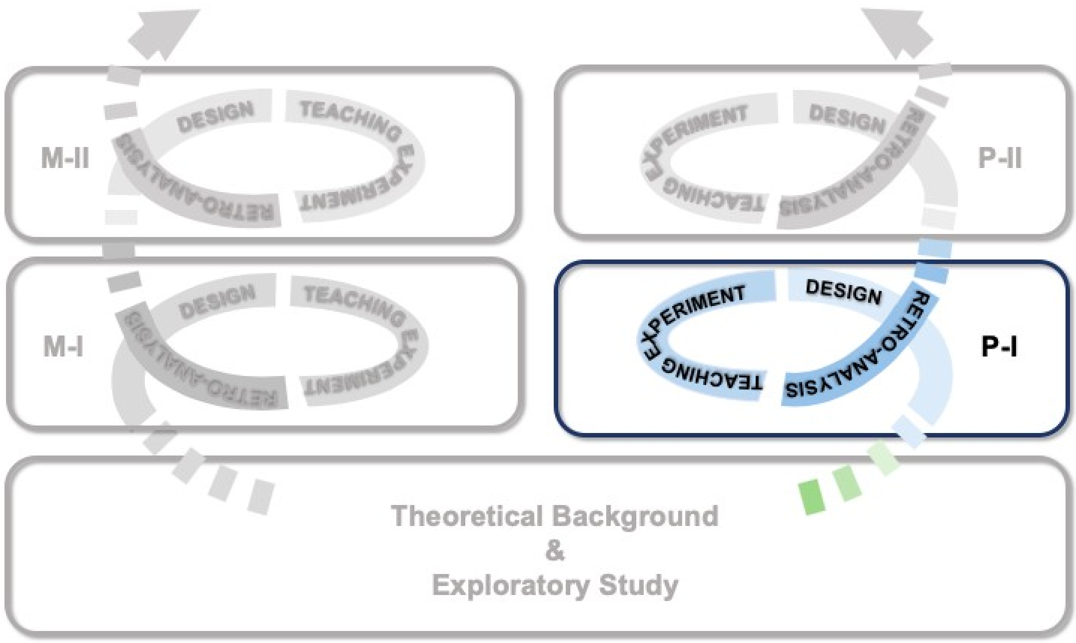


Figure 53. Research cycle P-I

The structure of the chapter is the following: in section 7.2 we introduce the classroom context for the study, explicating the mathematical topic considered and the specific goals of this cycle in relation to the second research question. In section 7.3 the design phase is described. The design phase of this research cycle is different from the ones of the previous chapters. Since in this case our aim is not to foster a particular topic, but to start investigating the role of different contexts in students' problem-posing performances in terms of creativity, we decided not to develop an HLT, but to structure the design phase explicating some conjectures together with the design of instructional activities related to that conjectures. Then the teaching experiment is described in section 7.4. Section 7.5 is dedicated to the retrospective analysis, in which we reflect on the expectations formulated in the design phase and formulate feed-forward for the next research cycle.

7.2 Context and aim

The aim of the first research cycle concerning problem-posing, P-I, consisted in studying how different contexts influence students' problem-posing performances in terms of creativity. The study was conducted in a six-grade class (age 12) composed by twenty-two students. The class had never been engaged in a problem-posing activity before the teaching experiment. At the moment of the intervention, students were working on fractions. In particular, the official mathematics teacher, who was used to teach in a traditional way, worked with students on: comparison between fractions, basic operations with fractions, fractions in the number line, word problems with fractions. As a consequence, we decided to consider fractions as mathematical topic for the problem-posing activity. In relation to the research question RQ2, we investigated how different contexts influence students' creativity in problem-posing activities. As a consequence, we focused on the sub-question:

RQ2.1. How do different contexts influence students' creativity in problem-posing?

The student activities and the guidelines for the teacher, together with our intentions, were discussed beforehand with the mathematics teacher in two meetings. The research method for the data analysis was mixed quantitative and qualitative. In section 7.3.2 we will describe in detail the method used for the data coding.

To have an overview of students' level on this topic, before the implementation of the problem-posing activity a test was performed. The test was composed by six questions (Appendix F). The maximum score for each question was 2, so the total score for the pre-test was 12. In Table 24 for each question is reported its mean. The total mean of the class was 5,2 over 12,0, which shows that students' knowledge of the subject was still poor at the moment of the intervention.

Table 24. Pre-test means

Question	Max score	Mean
1	2	0,89
2	2	1,26
3	2	0,76
4	2	0,67
5	2	1,26
6	2	0,35

Results show that students had difficulties especially in answering the last question of the test. This question consisted in solving the following problem (Fig. 54):

A cyclist decides to go from Padua to Florence by bicycle. The two cities are approximately 225 kms far away. On the first day the cyclist run 5/9 of the trip. The following morning he travelled 3/4 of the remaining kms.

a) How many kms are left to travel to Florence?
b) Express the result as a fraction of the distance between the two cities.

Figure 54. Question number 6 of the pre-test

Only 4 students over 22 were able to solve the problem. All these 4 students followed the same strategy:

1. calculate the kilometres ridden the first day performing $\frac{5}{9}\times(225)=125$;
2. calculate the left kilometres performing 225-125=100;
3. calculate the kilometres ridden the second day: $\frac{3}{4}\times(100)=75$;
4. $100-75=25$ are the left kilometres to Florence, that expressed as a fraction of the distance between the two cities is $\frac{25}{225}$.

In Fig. 55 an example of a correct student's answer is reported.

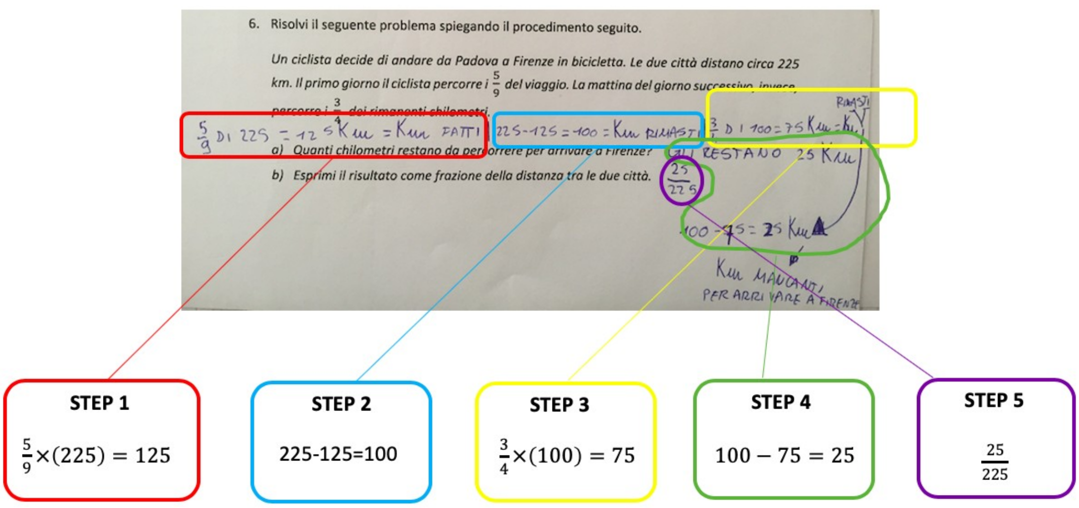


Figure 55. Example of correct answer to question 6

The most typical error, instead, consisted in the fact that students did not understand that the second day the cyclist run 3/4 of the left kilometres. As a consequence, they did not perform the second step of Fig. 55. Instead some students (8 over 22) calculate $\frac{3}{4}\times(125)$ (Fig. 56, 57, 58).

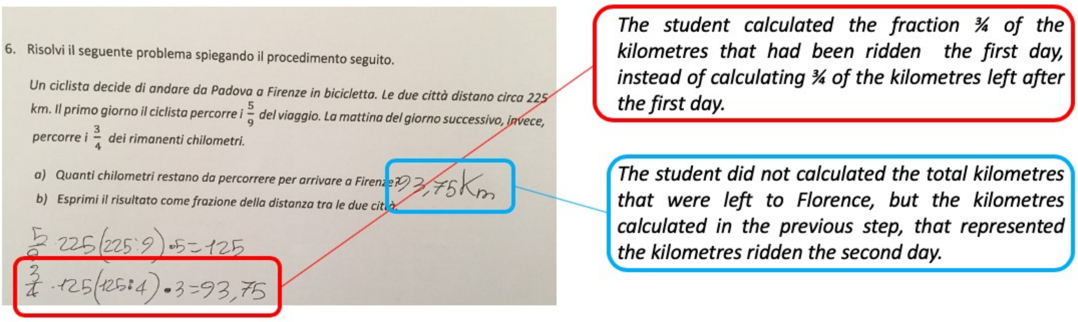


Figure 56. Example of incorrect answer to question 6

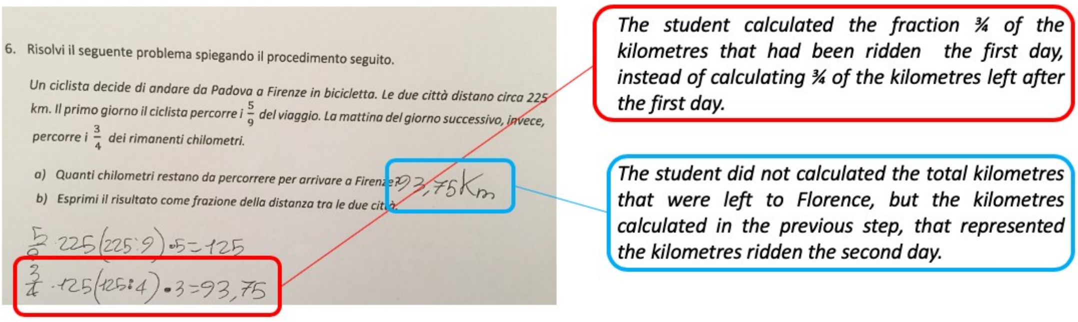


Figure 57. Example of incorrect answer to question 6

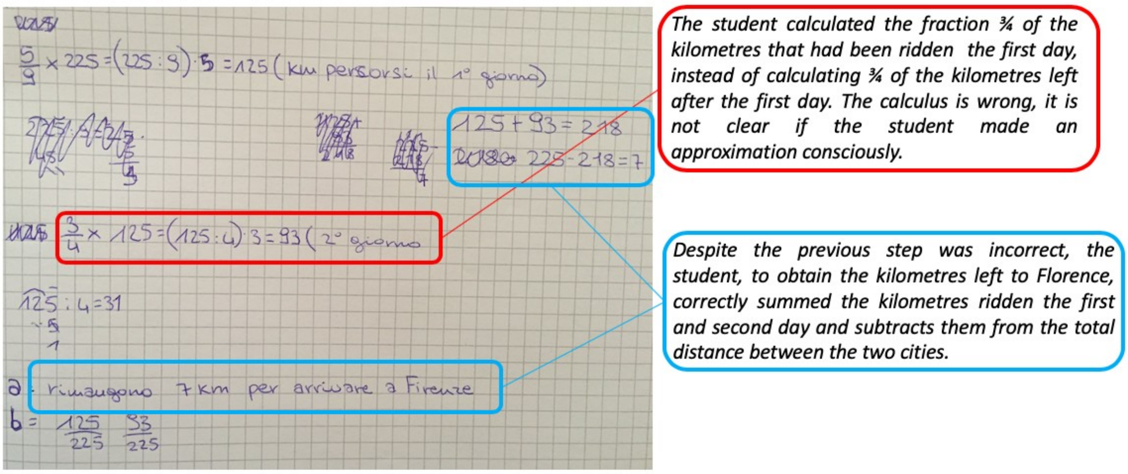


Figure 58. Example of incorrect answer to question 6

Two students had not a clear idea on how to calculate the fraction of a number. In Fig. 59 there are two examples in which students, to calculate $\frac{3}{4}\times(125)$, instead of dividing by 4 and multiply by 3, they divide by 3 and multiply by 4, exchanging numerator and denominator. The same for calculating fractions of other numbers.

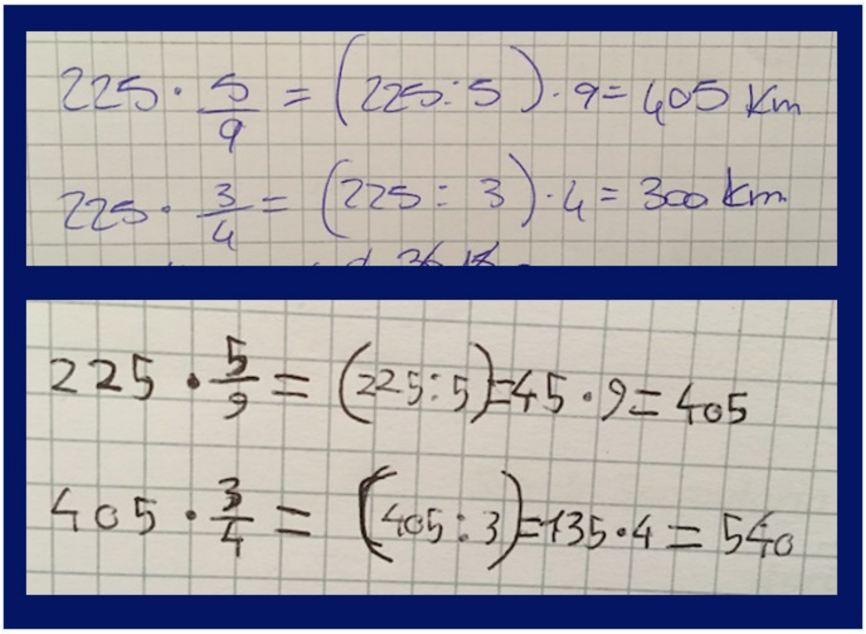


Figure 59. Incorrect calculation of the fraction of a number

Other students performed some operations with the numbers in the text of the problem, without reasoning about what they were doing (Fig. 60).

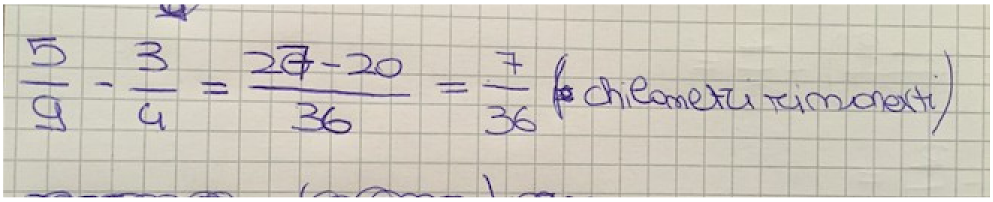


Figure 60. Students’ incorrect answer

In the next section we present the design phase, in which the development of an instructional design is described.

7.3 Design phase

In this research cycle the design phase is characterized by the development of an instructional design through the formulation of some hypothesis together with instructional activities and materials. In this section we first remark the theoretical background concerning problem-posing and the expectations that are investigated in the following teaching experiment. Then we describe the designed activities and materials.

7.3.1 Starting points

In the previous section we described the classroom context and the initial level of students. In this section we remark some key points from the literature on problem-posing, extensively reported in section 2.3.

- In this study problem-posing is seen as the process by which students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems (Stoyanova and Ellerton 1996). These concrete situations considered as starting point for the practice of problem-posing could be divided in three categories (Stoyanova & Ellerton 1996): free situations, semi-structured situations and structured situations. In this study we will focus only on semi-structured situations, that we recall are situations in which students are provided with an open situation and are invited to explore its structure and to complete it using their personal previous mathematical experience.

- Cultural artefacts represent a precious tool to provide students with meaningful contexts, and especially when engaged in problem-posing activities (Bonotto 2013). Thanks to its complexity and richness in mathematical meaning, an artefact lives in both the world of symbols and the real one, creating a sort of hybrid space that connects mathematics and everyday contexts. A re-mathematization process is thereby favoured, wherein students are invited to unpack from artefacts the mathematics that has been hidden in them (Bonotto 2013).

- An aspect that is investigated in this research cycle is the relation between problem-posing and creativity. Problem-posing, in fact, is a form of creative activity that can operate within tasks involving rich situations (Freudenthal 1991), using real-life artefacts and human interactions (English 2009). Creativity is directly linked to the mathematical activity of problem-posing, being the act of creating mathematical problems in specific contexts (Bonotto and Dal Santo 2015). To encourage the creative process in school mathematics we will use semi-structured situations (Stoyanova and Ellerton 1996) as starting contexts for problem-posing activities. In particular the use of cultural artefacts can help creating such situations. Several studies used problem posing and problem solving to promote and assess creativity (Xie and Masingila 2018; Bonotto and Dal Santo 2015; Bonotto 2013; Yuan and Sriraman 2010; Sriraman 2009; Leung 1997; Silver 1997, Leung and Silver 1997), proving that an inquiry-oriented mathematics instruction, including problem-posing activities, could assist students to develop more creative approaches to mathematics. However, given the value of problem-posing activities as opportunities for measuring students' creativity, or other mathematical learning outcomes, it is mandatory to develop and validate suitable problem-posing instruments, understanding which kind of problem-posing tasks best reveal students' creativity and their mathematical understandings (Cai, Hwang, Jiang and Silber 2015). In the next section we will design some teaching activities, through tasks and artefacts, that may foster students' creativity.

- We remark that through problem-posing students can actively construct meaning in both the natural and simulated worlds in classrooms. Moreover, teachers and students might create knowledge together in a variety of contexts and generate and address critical questions about the knowledge they produce. In this direction, we believe that problem-posing is a valuable educational strategy to enhance a guided re-invention approach to mathematics education.

- As starting contexts for problem-posing activities we will consider different types of real contexts. Such real contexts will be defined using the frameworks of RME and Palm (2006), described in section 2.4.

The aim of this research cycle is to start investigating how different contexts might influence students' creativity in problem-posing activities. Our idea is that semi-structured problem-posing activities in which real contexts are used as starting situations could foster students' creativity. However, it is not clear if different contexts influence in the same way students' creativity, and which characteristics such contexts should have. As a consequence, we decided to consider different real contexts from different theoretical perspectives (RME and Palm (2006)), and study how they could influence students' creativity when engaged in semi-structured problem-posing activities.

In the next section we describe the instructional activities and related materials developed for the implementation of the teaching experiment. Moreover, the data analysis scheme used for the data analysis is described.

7.3.2 Instructional activities

Since the aim of the study was to investigate the impact of different contexts in students' creativity in problem-posing, we decided to split the problem-posing activity in two sessions. In each session of forty minutes students had to pose at least three problems dealing with fractions from a given context. In the specific, we decided to consider two contexts that stressed the contrast *realistic-rich/feasibility* between RME and Palm's frameworks for real contexts. In the specific, the context considered for the first problem-posing session consisted in a number line with some rational numbers, while the context for the second session consisted in an advertising leaflet containing discounts for mobile phones (Fig. 61, 62). The first context (number line) is closer to the perspective of RME. Indeed, it can be considered as a realistic and rich context, since significant for students, who previously worked on it with their teacher, and rich in mathematical stimulus. The second context, instead, is closer to Palm's framework, since it consists in a real leaflet, and so represents an event that can occur in real life.

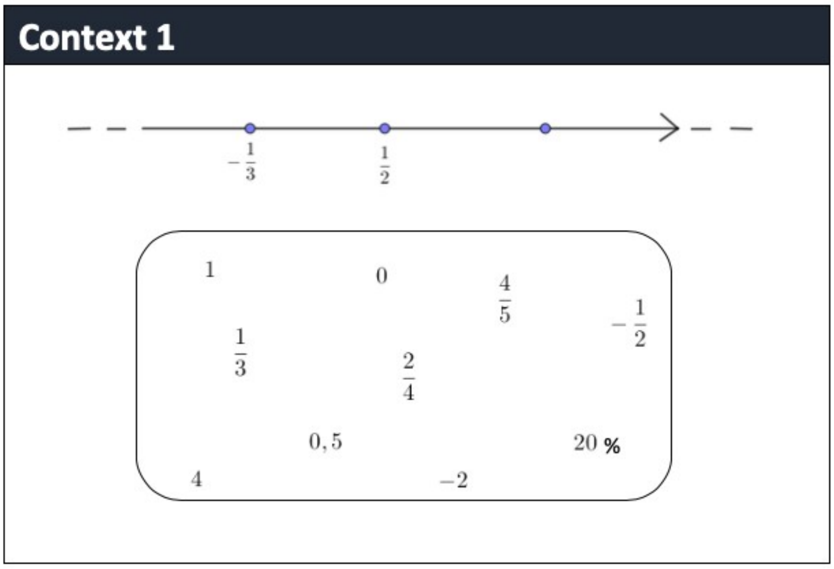


Figure 61. First context for the problem-posing activity: numbers line

Figure 62. Second context for the problem-posing activity: mobile phones leaflet

After the problem-posing sessions, students had been engaged in a problem-solving activity. Some problems were chosen by the teacher from the ones posed by students. The full text of the problem-solving activity is given in Appendix G.

Data coding

A summary of the data coding scheme used in this study is provided in Fig. 63. The first phase of the data coding consisted in a variation of the model proposed by Leung and Silver (1997). Students' problem-posing responses were firstly categorized as *problems* or *statements*. Then, problems were classified as *mathematical* or *non-mathematical problems*. Each mathematical problem was analysed in two directions. First, a mathematical problem was classified as *context related*, i.e. set in its starting context (respectively the number line for the first session and the leaflet for the second session), or as *not context related*. Second, mathematical problems were divided between *solvable* and *not solvable*. Problems were considered to be not solvable if they lacked sufficient information or if they posed a goal that was incompatible with the given information. The last phase of the data coding involved examining the creativity of the posed problems that had been previously classified as solvable.

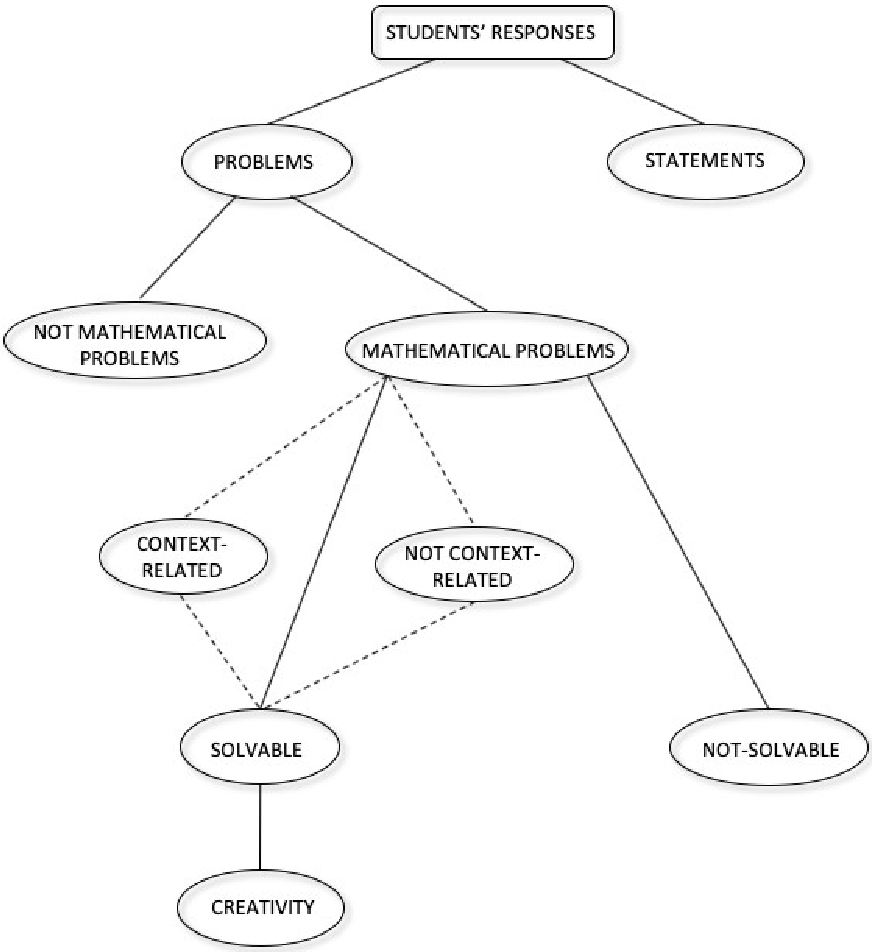


Figure 63. Data coding scheme to analyse students' problem-posing responses

Creativity was considered as follows. Starting from the rubric for the evaluation of teachers' creativity in problem-posing proposed by Xie and Masingila (2017) (Table 5), we developed the following analytic scheme to calculate the creativity of students' responses in a problem-posing activity in which students had to pose three problems from a given context. We call P_i^n the i-th problem posed by the n-th student, so in our case $i=1,2,3;n>0$. For every n, we consider the first posed problem, namely P_1^n, and we start comparing it with the second posed problem P_2^n:

$$\left(P\text{¿¿}2^n, P_1^n\right) = \begin{cases} 0, if\ P_2^n \wedge P_1^n\ are\ comparable \\ \text{¿}+1, if\ P_2^n \wedge P_1^n\ are\ somewhat\ different\ \text{¿} \\ \text{¿}+2, if\ P_2^n \wedge P_1^n\ are\ completely\ different \end{cases} \quad \text{(i)}$$

Then we consider the third posed problem and compare it with both the second and the first one:

$$(P¿¿3^n, P_2^n) = \begin{cases} 0, if\ P_3^n \wedge P_2^n are\ comparable \\ ¿+1, if\ P_3^n \wedge P_2^n are\ somewhat\ different\ ¿ \\ ¿+2, if\ P_3^n \wedge P_2^n are\ completely\ different \end{cases} \quad \text{(ii)}$$

$$(P¿¿3^n, P_1^n) = \begin{cases} -2, if\ P_3^n \wedge P_1^n are\ comparable \\ ¿-1, if\ P_3^n \wedge P_1^n are\ somewhat\ different\ ¿ \\ ¿0, if\ P_3^n \wedge P_1^n are\ completely\ different \end{cases} \quad \text{(iii)}$$

In conclusion, we calculate the sum $c := (P_2^n, P_1^n) + (P_3^n, P_2^n) + (P¿¿3^n, P_1^n) \in \{-2,-1,0,1,2,3,4\}¿$. At the end, to each student is associated a level of creativity in accordance to Table 25.

Table 25. Student’s level of creativity associated to the value of c

c	**level of creativity**
≤ 0	Low (L)
$1 \vee 2$	Medium (M)
$¿2$	High (H)

In the case that one student posed only one problem (*d=1*), we define $c=0$. If a student posed only two problems, we calculate *c* according to (i).

The scheme can easily be extended to activities in which students could pose a number of problems greater than three. In this case, in fact, the level of creativity of each student is associated to the quantity

$$c := \sum_{j=2}^{d} \left(\sum_{k=1}^{j-1} \left(P_j^n, P_k^n \right) \right)$$

with

$$\left(P\text{¿¿}\, j^n, P_k^n\right) \in \begin{cases} \{0,+1,+2\}, \wedge\, j-k=1 \\ \text{¿}\{-2,-1,0\}, \wedge\, j-k>1 \end{cases} \text{¿}$$

according to (i) and (iii), where d is the maximal number of problems that can be posed by a single student, and P_i^n is the i-th problem posed by the n-th student, $i=1,\ldots,d$.We can observe that in general there are $d-1$ pairs $\left(P_j^n, P_k^n\right)$ with $j-k=1$, and $(\sum_{m=1}^{d-2} m\text{¿}$ pairs $\left(P_j^n, P_k^n\right)$ with $i-j>1$. As a consequence, the quantity c lives in $B \coloneqq \{z \in Z \mid h<z<l\}$, where $h=-2 \cdot \sum_{m=1}^{d-2} m,$ and $l=2\cdot(d-1)$.

7.4 P-I teaching experiment

In design research the aim of the teaching experiment is to see how the developed instructional activities would play out in the classroom and test empirically the hypothesis that had been conjectured. We now present the main results from the teaching experiment for the P-I cycle. The results are presented in three subsections: the first dealing with students' problem-posing responses including *mathematical/not mathematical* problems, *context/not context related* problems, *solvable/not solvable* problems; the second dealing with students' creativity; the third reports results from the final problem-solving activity. In the specific, we focused on how the two contexts used as starting situations for the problem-posing sessions might have influenced students' responses in terms of creativity.

7.4.1 Problem-posing responses

Students responses had been firstly analysed using the data coding scheme reported in Fig. 63. The results split between the two contexts are reported in Table 26. All the students' responses had been classified as problems, so no statement occurred. Students posed totally 122 problems, of which the 95% were mathematical problems. The number of mathematical problems was comparable between the two contexts, respectively the 97% of the posed problems for the number line and the 92% of the posed problems for the leaflet. The main difference between the two contexts dealt with context/not context related problems ($p<0.001$; $V=0.81$). In the case of the number line, the 92% of the mathematical problems were problems that did not refer to the number line. Instead, for the leaflet there was an opposite behaviour, since the 100% of the mathematical problems were context related, which means that they referred to the leaflet itself. In Table 27 some examples of context/not context related problems are reported. In conclusion, not a significant difference occurred between the two contexts in terms of solvable and not solvable posed problems ($p<0.05$; $V=0.19$). In fact, for the number line the 98% of mathematical problems were solvable, and for the leaflet the 90% of the mathematical problems were solvable. Further examples of *not context related problems* from the first problem-posing session are reported in Table 28.

Table 26. Students' responses in terms of *mathematical/not mathematical* problems, *context/not context* related problems, *solvable/not solvable* problems. Results are split between the two contexts used for the problem-posing sessions. All the responses were classified as *problems*

		Math Problems	Not Math Problems		Context Related	Not Context Related	Solvable	Not Solvable
Cont. 1 (number line)	count	64	2	count	5	59	63	1
	% within problems	97	3	% within math problems	8	92	98	2
Cont. 2 (leaflet)	count	58	5	count	58	0	52	6
	% within problems	92	8	% within math problems	100	0	90	10
Total	count	122	7	count	57	65	115	7
	% within problems	95	5	% within math problems	47	53	94	6
					($p<0.001$; $V=0.81$)		($p<0.05$; $V=0.19$)	

Table 27. Examples of context and not context related problems for both the contexts.

Context 1 (numbers line)	context related	Marco is doing his math homework, he has to place the following numbers on the number line: 4, 0, 2/4. Help Marco by finding their value and placing them on the numbers line
	not context related	Luisa is reading a book of 350 pages. If she has already read the 4/5 of the book, how many pages will she have to read to finish it?
Context 2 (leaflet)	context related	A phone is sold in a shop for the price of 299 euros. It is discounted by 40% and Tommaso paid 1/4 of the cost. How much does he still have to pay by counting the discount and the advance payment?
	not context related	No problem was not context related

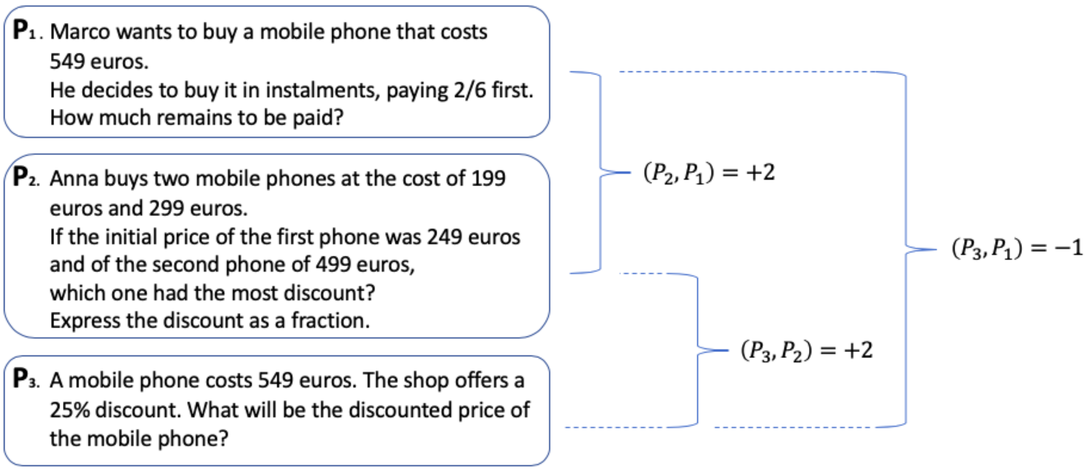
Table 28. Examples of *not context related* problems from the first context

P.1	Giada would like to buy a toy that costs 10 euros, but she has only the 4/5 of 10 euros. How much more money does she need to buy the toy?
P.2	Luca is playing football. One side of the football field measures 25m, and the other its 4/5. Which is its measure?
P.3	Marco and Anna collected 40 shells on the beach. Anna collected 2/4 of those collected by Marco. How many shells did Marco collect?
P.4	Matilde has 30 euros and Gianna has 2/4 of 30 euros. How much money does Gianna have? Who has more money?
P.5	Luigi has 27 marbles. Luca has 1/3 of those who Luigi has. How many marbles does Luca have? Which is the total number of marbles?
P.6	Laura has 70 euros. Giovanni has the 2/4 of Anna’s money who has the 4/5 of Laura’s. How much money does Giovanni have? And Anna?
P.7	Sara has 125 euros. In a shop there is a bicycle that costs 135 euros, but it is

	discounted by 20%. Will Sara be able to buy that bicycle?
P.8	Marco has to cover 6 kilometers to go to school, and Giulia 1/3 of Marcos's. How many kilometers does Giulia have to cover to go to school?
P.9	Umberto buys a copybook that costs 20% more than the older one, that costs 1,99 euros. How much does the new copybook cost?
P.10	Anna is reading a book of 420 pages. She has already read the 4/5 of the total. How many pages has she already read? How many pages are left to finish the book?

7.4.2 Creativity

The second phase of the data analysis consisted in classifying students' creativity. In each of the two problem-posing sessions, for each student we considered her/his posed problems that had been classified as solvable problems in the previous phase, and we applied the scheme proposed in the section 7.3.2 to associate a level of creativity to each student. An example of the process is given in Fig. 64. Students' results, split between the two problem-posing sessions, are reported in Table 29.



$$c = (P_2, P_1) + (P_3, P_1) + (P_3, P_2) = 3 \ \rightarrow \text{level of creativity: } H$$

Figure 64. Example to calculate student's the level of creativity

Concerning the first context (number line), the 45% of students had a low level of creativity, the 50% of students a medium level of creativity and the 5% of students a high level of creativity. Concerning the second context (leaflet), the 45% of students had a low level of creativity, the 45% of students a medium level of creativity and the 10% of students a high level of creativity. As clearly shown in Fig. 65, students' problem-posing responses in terms of creativity are comparable between the two contexts. This is supported also by a Wilcoxon test, that indicated that students had not a significant difference in terms of creativity between the two contexts (z=-0.1; p=0.9).

Table 29. Students' level of creativity distributions split between the contexts used in the two problem-posing sessions

	Context 1 (number line)				Context 2 (leaflet)			
	Count	%	Mean	SD	Count	%	Mean	SD
Low creativity level	10	45	- 0,6	1,0	10	45	-0,8	1,0
Medium creativity level	11	50	1,3	0,5	10	45	1,2	0,4
High creativity level	1	5	3	0,0	2	10	3,5	0,7
Total	22	100	0,5	1,5	22	100	0,5	1,3

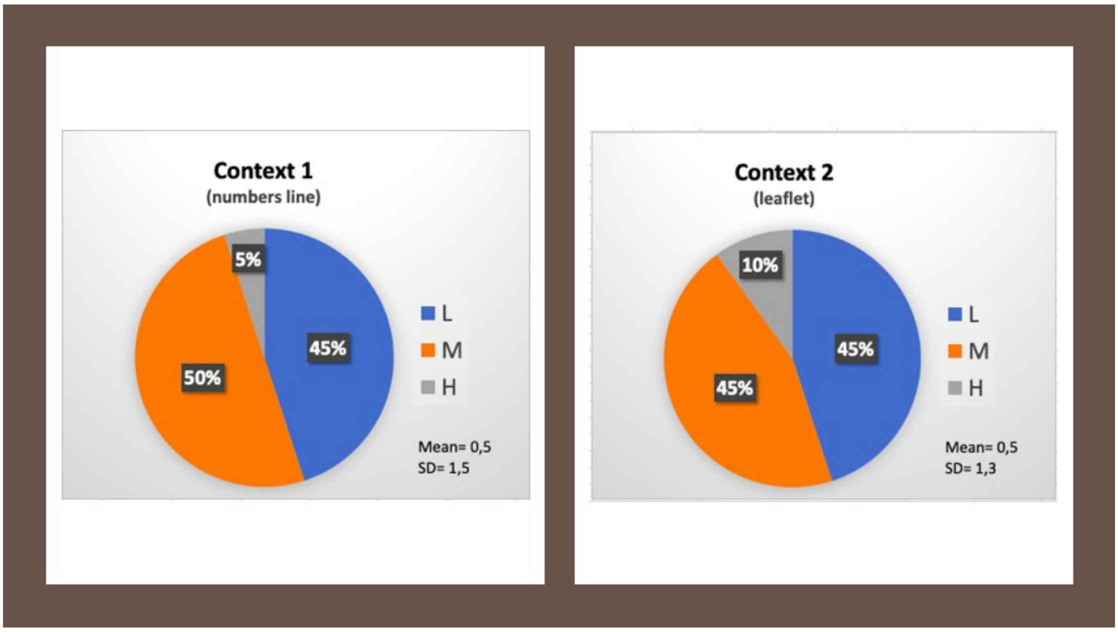


Figure 65. Students' level of creativity distributions split between the contexts used in the two problem-posing sessions. Not a significant difference in terms of creativity between the two contexts (z=-0.1; p=0.9) was observed

7.4.3 Problem-solving

The lesson after the problem-posing sessions students had been engaged in a problem-solving session. 10 problems were chosen from the ones posed by students (Appendix G). We focus here on two of that problems: number 2 and number 5.

The problem number 2 was the following:

Sara has 125 euros. A bicycle costs 135 euros, but it is discounted by 20%. Will Sara be able to buy the bicycle?

Figure 66. Problem number 2 of the problem posing activity

This problem was chosen because students posed a problem with the concept of percentage, that was never treated by the teacher. However, students used the percentages present in the second context and posed a problem with it. 12 students over

23 answered correctly to the question, while the rest of the classroom did not answer. In Fig. 67 there are three examples of students who answered to the question. The first student calculated the cost of the discounted bicycle and then subtracts it from the money that Sara has, obtaining also how much money Sara has after having bought the bicycle. It is interesting that this student to calculate the cost of the discounted bicycle, divided the initial cost by 100 hundred, that corresponds to 1% of the cost of the bicycle. In the second example the student was able to stress that 20% is actually the fraction 20/100. In the last example, despite the student performed incorrectly the final subtraction, she/he expressed both the percentage as a fraction and the final Sara's rest.

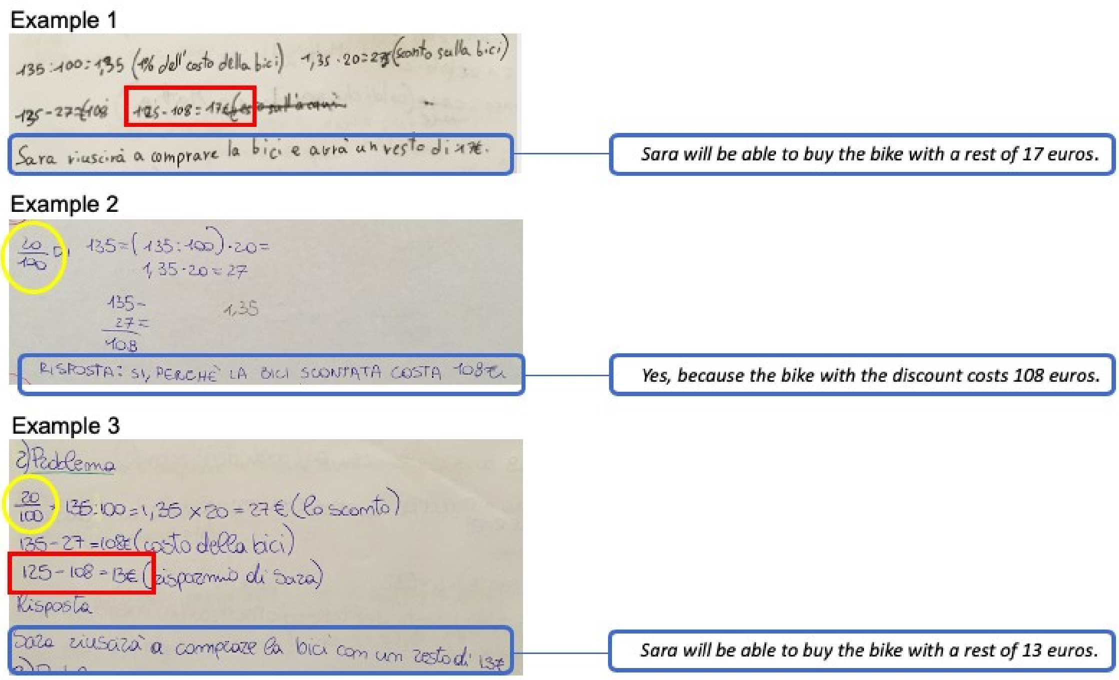


Figure 67. Students' answers to the second problem of the problem solving activity

Problem number 5 was the following:

> Valentina was running in a pedestrian park to stretch her legs a little. Knowing that the route is 10 km long and that she has already walked 1/3 of it, how many km has she already covered? At a certain point Valentina realized she has no longer her house keys in her pocket. She went back 1/2 and found them. How many km from the start of the route did she start running again? How many km has she covered so far?

Figure 68. Problem number 5 of the problem posing activity

This problem was very similar to the problem of the question number 6 of the pre-test, that was the one with the lowest number of correct answers. However, in this case several students were able to give a correct answer. In Fig. 69 there are two examples of students' answers. Both the students, to calculate the total kilometers run by Valentina, adopted the following strategy:

1. calculate 1/3 of the total route, that represents the first part run by Valentina;
2. calculate 1/2 of the previous result, in order to calculate how many kilometers, she run back to look for her keys;
3. sum the two previous results to obtain the total kilometers run.

In performing the first point, the first student made an approximation of the result. Indeed, 1/3 of 10 is 3,333…, and the students approximated it as 3,5. The second student, instead, took as value 3,33.

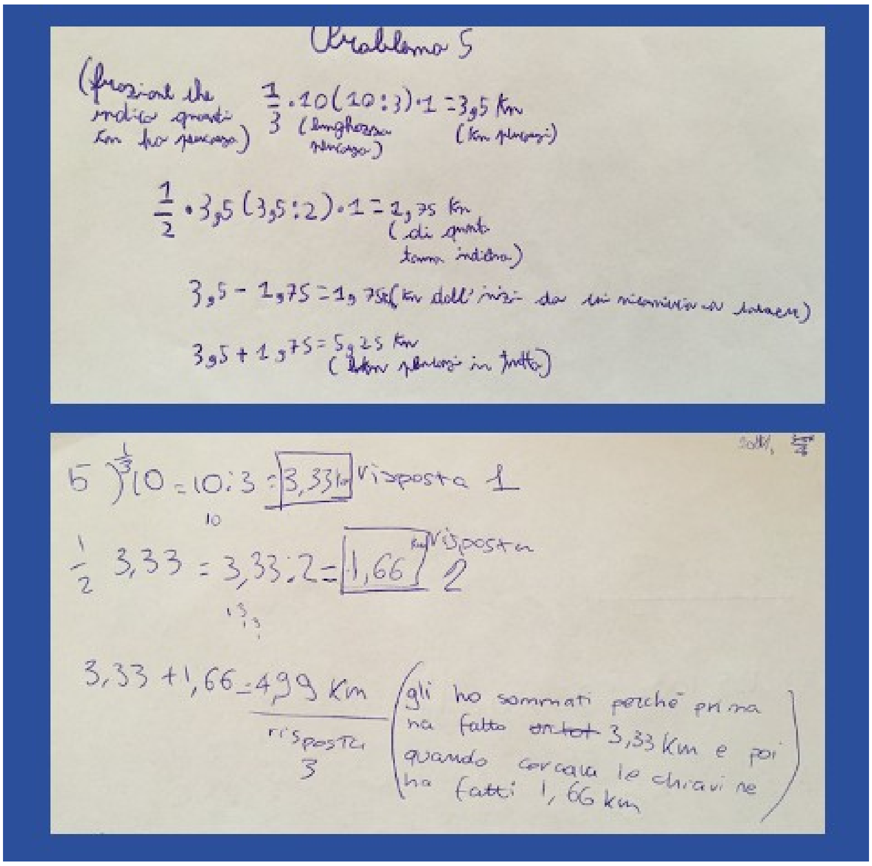


Figure 69. Students' answers to the fifth problem of the problem solving activity

7.5 P-I retrospective analysis

In this final section we look back at our initial instructional design and compared it with the results from the teaching experiment. This retrospective analysis could form the basis to formulate first answers to our second research question, to adjust the instructional design for the next research cycle and to formulate additional hypothesis. The section is divided in two parts: the first concerning the reflection of the teaching experiment and the second formulating feed-forward of the P-I research cycle for the next P-II cycle.

7.5.1 Reflection on the P-I teaching experiment

The aim of this first teaching experiment consisted in investigating how different contexts influence students' problem-posing performances in terms of creativity. We decided to consider fractions as mathematical topic for the problem-posing activity. In relation to the research question RQ2.1, we investigated how different contexts influence students' creativity in problem-posing activities. In the specific, our idea was that semi-structured problem-posing activities in which real contexts are used as starting situations could foster students' creativity. However, it was not clear how different contexts influence students' creativity, and which characteristics such contexts should have. As a consequence, we decided to consider different real contexts from different theoretical perspectives. In the specific, two contexts were chosen: a number line and a mobile phones leaflet. The first context can be seen as a rich and realistic context in the perspective of RME, that comes from the world of mathematics. The second context, instead, can be seen as an artifact (Bonotto 2013), and represents a real context in Palm's framework, since it denotes a context that could really happen in real life. In order to evaluate the influence of each of these contexts in students' problem-posing performances, two problem-posing sessions had been implemented. In each session students had to pose at least three mathematical problems (dealing with fractions) from the given contexts, respectively the number line and the mobile phones leaflet.

The first part of the data analysis consisted in evaluating differences and/or similarities in the two problem-posing sessions in terms of the quality of the posed problems, including: *problems/statements*; *mathematical/not mathematical problems*; *context/not context related problems*; *solvable/not solvable problems*. Results (Table 26) show that the only significant difference between the two problem-posing sessions was in terms of *context/not context related problems*. In the specific, the 92% of mathematical problems posed starting from the first context, the number line, had been classified as not context related. This means that students did not consider the number line as a context wherein setting in their problems. The majority of students, instead, used the numbers given in such first context in posing problems that were not related to the number line context. Such problems were connected to their real-world experience and were posed similarly to world problems they found in their previous mathematical experience, from textbooks or teacher (Table 28). From the second context, instead, the 90% of students' mathematical problems were context related. Students recognised the leaflet as a familiar context, because close to their experience, being a context that possibly occur in their real life. In this way, students were able to pose problems that were set in that context. As a consequence, despite the number line was used by the teacher in previous mathematics lesson, it was not really a significant context for students, lacking in a possible real-life occurrence, and so it was not realistic, not experientially meaningful for them. Moreover, we noticed that students tended to use numbers present in the number line to pose problems linked to their experience. The meaningfulness of the context in terms of real occurrence did not affect problems solvability. The majority of students' mathematical problems, in fact, in both cases was made by solvable problems (respectively the 98% for the numbers line and the 90% for the leaflet).

The second part of the analysis consisted in studying students' creativity in the two problem-posing sessions. In both the sessions, students' creativity was approximately equally distributed between low and medium-high level of creativity. Moreover, not a significant difference occurred between the use of the number line and the leaflet. When posing their problems starting from the number line, students created a realistic setting linked to the number line context, posing not context related problems, but problems linked to their school and real-life experience. As a consequence, we cannot say that a most meaningful context for students, such as the mobile phone leaflet, promoted students' creativity more than a less meaningful one, such as the number line. In fact, since students created personal realistic settings when posing problems from the number line context, the significance of the context itself did not influence their creativity process, and both the contexts stimulated in the same way students' creativity (Table 27). However, the results indicate that a fundamental factor that influenced students' problem-posing performances, and especially in terms of creativity, was the significance given by students to the context. Indeed, when the context was not experientially meaningful for a student, as in the case of the number line, she/he tried to associate a new meaning to the context, using some elements from it and re-creating a new more meaningful context in which setting her/his problems. In our study, this process of context free re-construction was possible because no constraint was explicitly maid during the problem-posing activity concerning the fact that students had to refer to the number line when posing problems, but the only one constraint was to pose at least three problems (dealing with fraction) *from* that context, and not *in* that context.

7.5.2 Feed-forward of the P-I

The analysis of the teaching experiment permitted to formulate feed-forward of the P-I research cycle for the next P-II cycle.

In section 7.4.1, 7.4.2, 7.4.3 we presented some results from the two problem-posing sessions and the problem solving one. During the problem-posing session, students, starting from given contexts, formulated some mathematical problems that took into account some concepts and notions that have not already been introduced by the teacher during the previous lessons. For example, several students posed problems concerning percentages. During the problem solving activity, some students were able to solve that problems, and also to interpret percentages as fractions. Such students probably already had this notion from their experience outside the school context and had been able to apply it to solve a mathematical problem. This fact may suggest that problems posed by students starting from real contexts should be used to introduce and work with new mathematical objects. However, the previous are very superficial considerations, that lead to some questions that need further investigation, such as:

- how can problem-posing be used to introduce new mathematical concepts for students?

- do different contexts in semi-structured problem-posing situations play a different role in fostering the development and understanding of new mathematical concepts?

- which characteristics should such contexts have to promote students' re-invention of mathematical concepts or tools?

We realized that despite several studies focused on students' or teachers' responses in problem-posing activities, no research is present concerning the implementation of problem-posing activities to enhance students' mathematical knowledge. Consequently, starting from the observations and suggestions of the P-I cycle just described, we believe indispensable to start investigating how different contexts could promote or not emergent problem-posing, and this will be exactly the focus of the next research cycle P-II.

8. P-II Research Cycle

8.1 Introduction

Starting from the reflections of the previous research cycle, a second cycle concerning mathematical problem-posing had been designed. This chapter addresses this second research cycle concerning problem-posing, P-II (Fig. 70).

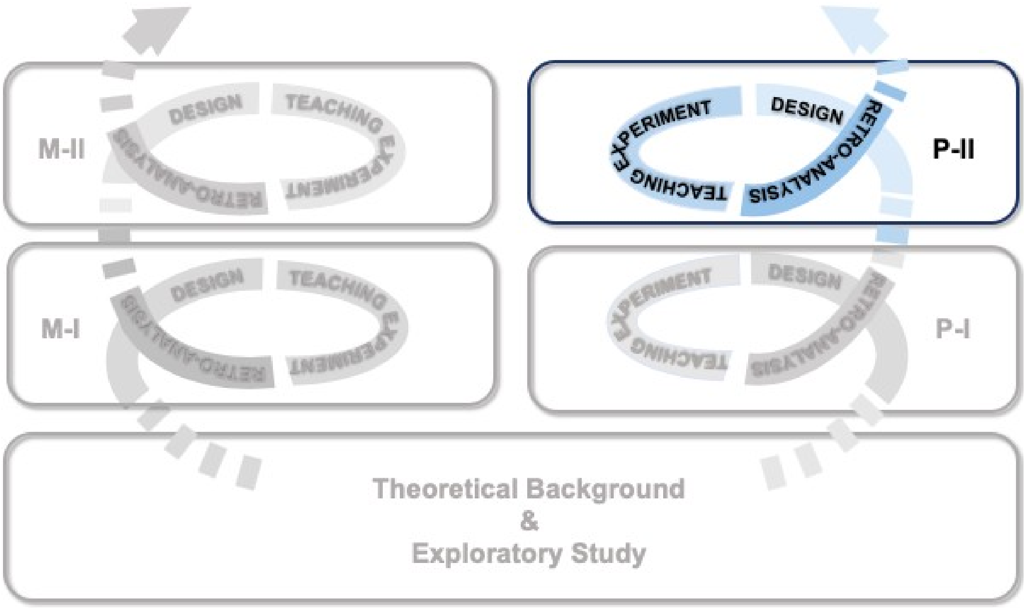


Figure 70. Research cycle P-II

The structure of the chapter is the following: in section 8.2 we introduce the context of the classroom that participated to the study, explicating the mathematical topic considered and the specific goals of this cycle in relation to the second research question. In section 8.3 the design phase is described, consisting in the development of an HLT and with particular attention in designing instructional activities related to the learning goals of the HLT. HLT includes starting points and expectations and students' hypothetical learning process. The design of key activities and materials are reported. Then the experiences during the teaching experiment are described (section 8.4). Section 8.5 is dedicated to the retrospective analysis, in which we reflect on the expectations formulated in the HLT and formulate some conclusions for the second research question.

8.2 Context and aim

The aim of the research cycles P-I and P-II consisted in finding possible answers to the second research question of this research project:

RQ2. How do different contexts influence the process of problem-posing?

Concerning the second research question, we focused on two aspects of the problem-posing process, namely its relations with creativity and emergent problem-posing. The research cycle P-I, described in the previous chapter, dealt with students' creativity focusing on the research sub-question RQ2.1, while this chapter will focus on emergent problem-posing, answering to:

RQ2.2. How do different contexts influence emergent problem-posing?

Starting from the retrospective analysis of the research cycle P-I, a second research cycle was designed. The research cycle P-II was conducted in a fourth-grade class (age 9) composed by twenty-five students. The classroom involved in the study had never been engaged in problem-posing activities before the study. The activity was implemented by the author with the presence of the official mathematics teacher. At the moment of the intervention, students were working on decimal numbers. In particular, the official mathematics teacher, who was used to teach through a traditional method, worked with students on: decimal numbers and equivalent fractions; comparison between decimal numbers; decimal numbers in the number line. No operations between decimal numbers were introduced by the mathematics teacher. In this situation, addition between decimal numbers was chosen as mathematical topic for the problem-posing activity.

In relation to the research question RQ2.2, we investigated how different contexts could promote or not emergent problem-posing. We believe that semi-structured situations could support the process of emergent problem-posing, i.e. the process in which problem-posing is a way for the emergence of students' formal ways of knowing.

In the next section we present the design phase, in which the development of an HLT is described.

8.3 Design phase

In this section we first describe the starting points for the HLT and the expectations that are investigated in the following teaching experiment. Then we describe the activities and materials designed in order to foster students' cognitive development according to the goals of the HLT.

8.3.1 Starting points

Starting points for the formulation of an HLT of this second research cycle on problem-posing are split in three categories. The first concerns the theoretical background specific for this research cycle and taken into consideration to design an educational setting and hypothesis about students' learning. The second deals with the classroom context, and in the specific the initial level of the students. The third is represented by the feed-forward formulated in the retrospective analysis of the previous research cycle P-I.

Theoretical Background

In addition to the theoretical background of the previous research cycle P-I, we recall here some considerations concerning emergent problem-posing. This notion was introduced starting from the considerations that when generating a problem, students do not always take into account possible solving strategies related to that problem, instead often they are not able to solve the problems they have posed. In this situation, problems posed by students that require new mathematical knowledge for their solution can be used as a vehicle to introduce new mathematical concepts. Moreover, these new concepts assume meaning for students, because rooted in their personal experience and for the specific purpose of solving the problems posed by themselves. As a consequence, new mathematical knowledge should be not only introduced, but also re-invented (Freudenthal 1991) by students. This aspect of problem-posing is called emergent problem-posing, highlighting its aim to support the emergence of formal mathematical ways of knowing.

Pre-test

Before the development of a learning trajectory, a pre-test was administered in the class were the research cycle took place. The aim was to have a picture about the starting level of the classroom concerning the mathematical topic considered for the teaching experiment: decimal numbers. 25 students participated to the pre-test. The test was composed by five questions. The full text is presented in Appendix H. The maximum score for every question was 2, so the maximum score was 10. In Table 30 we report the mean of students' answers to each question. The total mean of the classroom was 7,2.

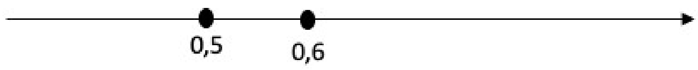
Table 30. Mean of students' answers to the pre-test

Question	Q1	Q2	Q3	Q4	Q5	Total
Mean	1,6	1,7	1,5	1,0	1,4	7,2

Findings indicate that students had a good knowledge of the subject at the moment of the teaching experiment. We remark in particular one fact that emerged from the pre-test, dealing with the last question (Fig. 71).

5. **Compare the numbers and put them in the numbers line, as shown in the example:**

0,5 ; 0,6 ; 0,46 ; 0,721 ; 0,72



Example: $0{,}5 = \frac{5}{10}$

$0{,}6 = \frac{6}{10}$

$\frac{5}{10} < \frac{6}{10}$

Figure 71. Last question of the pre-test

In this case all the students had the same score and answered in the same way. Students, indeed, were able to perform the comparisons between the last three numbers in the text of the problem (0,46; 0,721; 0,72), however they did not compare them with the numbers given in the example. Consequently, they put in a correct progression the last numbers, but not in relation to the given ones. Students were able to compare decimal numbers recurring to their notation as fractions but did not have a complete confidence with the number line (Fig. 72).

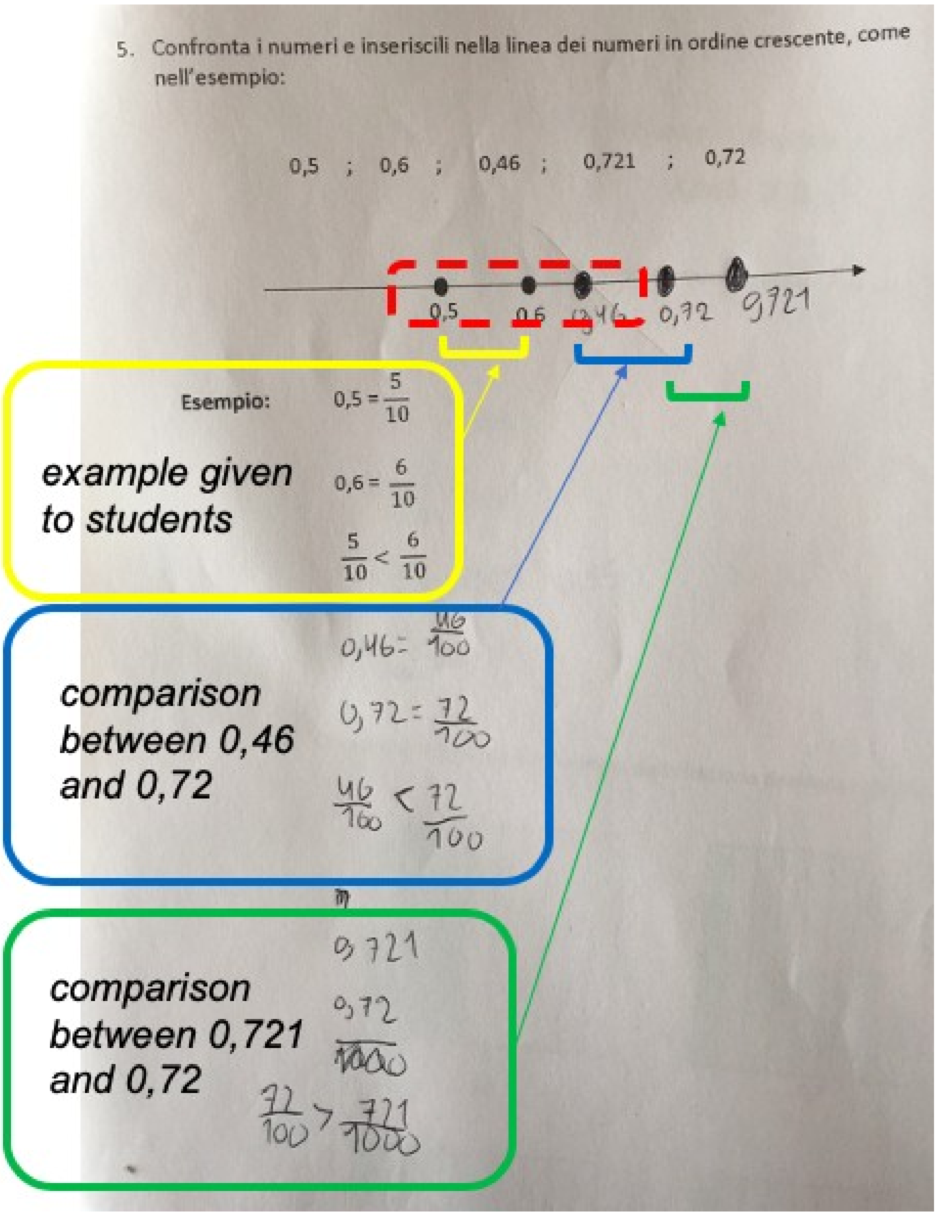


Figure 72. Example of a student's answer to the last question of the pre-test

P-II feed-forward

The research cycle P-I described in chapter 7, gave some suggestions for the instructional design of the next research cycle P-II. In the specific, from the results of the previous research cycle, we saw that students, starting from given contexts, formulated some mathematical problems that take into account concepts and notions that have not already been introduced by the teacher during the previous lessons. As a consequence, the problem-posing activity gave the opportunity to work and reflect on new concepts, objects, tools, starting from students posed problems. This fact may suggest that problems posed by students starting from real contexts should be used to introduce and work with new mathematical objects. Starting from these observations and suggestions, in this second research cycle we want to start investigating how different contexts could promote or not emergent problem-posing.

8.3.2 Learning goal and hypothetical learning process

As stated in section 8.2, the aim of this first research cycle consists in investigating how emergent problem-posing can be fostered to help students understand some aspects of decimal numbers. In the pre-test we saw that students at the moment of the intervention had a good knowledge of the topic. The regular mathematics teacher had not already introduced and worked with operations between decimal numbers. As a consequence, what we want to achieve during the teaching experiment is the re-invention of the algorithm (or more algorithms) to calculate additions between decimal numbers. Therefore, as shown in Fig. 73, the learning goal of the teaching experiment is represented by the addition between decimal numbers.

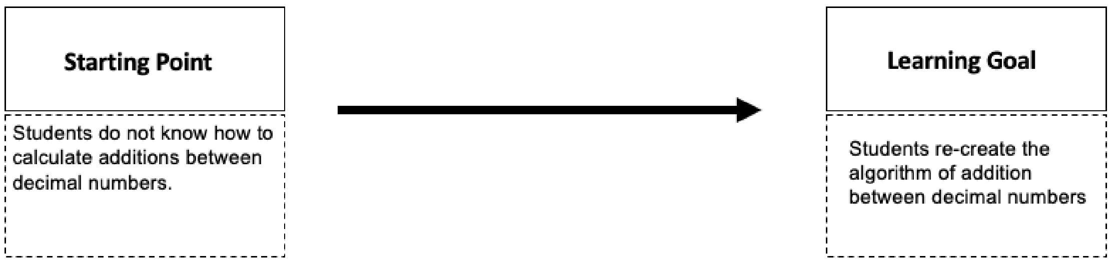


Figure 73. Learning goal of the P-II cycle

To design an HLT, together with the learning goal we need to formulate some conjectures about the learning process. In analogy with the research cycle M-I and M-II, starting from the classroom initial level and following the heuristic of the didactical phenomenology, we supposed that making students face with a problem situation in which they need a new mathematical concept to solve it could stimulate the same students in creating that concept. Differently from the modelling activities described in chapters 5 and 6, in this case students are not given a task, but they have to pose problems that could stimulate the described process. As a consequence, our idea consists in putting students face with problem-posing situations that could stimulate them to pose problems dealing with decimal numbers and that could bring to the need of introducing addition between decimal numbers. Then, focusing on that problems posed by students which need to develop a strategy to perform addition between decimal numbers, the teacher can foster students' creation of one or more algorithms in a guided re-invention (Freudenthal 1991) way. At the same time, we also want to know which characteristics should have a context for a problem-posing activity to foster this process of emergent problem-posing, i.e. of concept re-creation. Therefore, students would not be engaged in only one problem-posing activity, but in more problem-posing sessions that start from different contexts. A priory we believe that both these contexts could foster students' emergent problems in the same way.

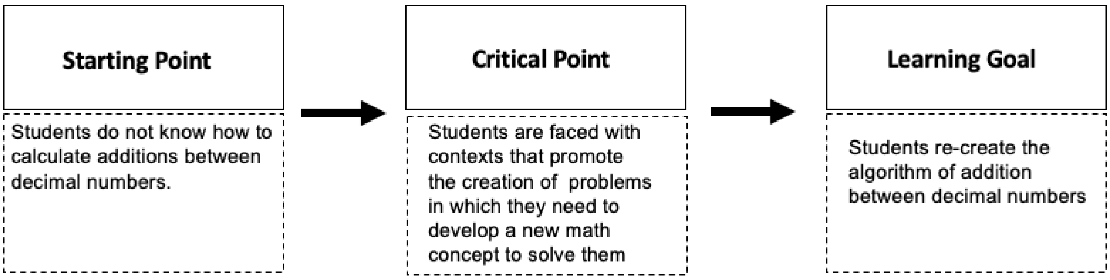


Figure 74. Hypothetical learning process of the P-II cycle

The learning process outlined in Fig. 74 is not linear. Indeed, attention should be given not only to the creation of problems and their solution, but to the process of construction of concepts that support such solution process.

8.3.3 Learning activities

Starting from the hypothetical learning process and the learning goal outlined in the previous sections, some learning activities had been designed. The first one consists in a problem-posing activity. Since the aim of the study is to investigate the impact of different contexts in students' problem-posing abilities in terms of emergent problem-posing, we decided to split the lesson dedicated to the problem-posing activity in two sessions. In each session of forty minutes students had to pose at least three problems dealing with decimal numbers from a given context. In the first session the context was represented by a number line with decimal numbers (Fig. 75), while in the second session the context was represented by a survey statistic about people practicing sports (Fig. 76). Similarly to the cycle P-I, we decided to consider two different kinds of contexts that highlight one of the main differences between RME and Palm's frameworks for real contexts: *realistic-rich* vs *feasibility*. We believe that such contexts could stimulate students in posing problems that, to be solved, need the necessity to develop a strategy to perform additions between decimal numbers. A priory we do not know which of the two contexts should support better emergent problem-posing. Our aim is not to compare RME and Palm's frameworks, but to start understanding which features a context should have to support students in creating mathematical concepts starting from their own informal mathematical strategies, represented also by posing problems from a realistic situation and solving problems created by themselves.

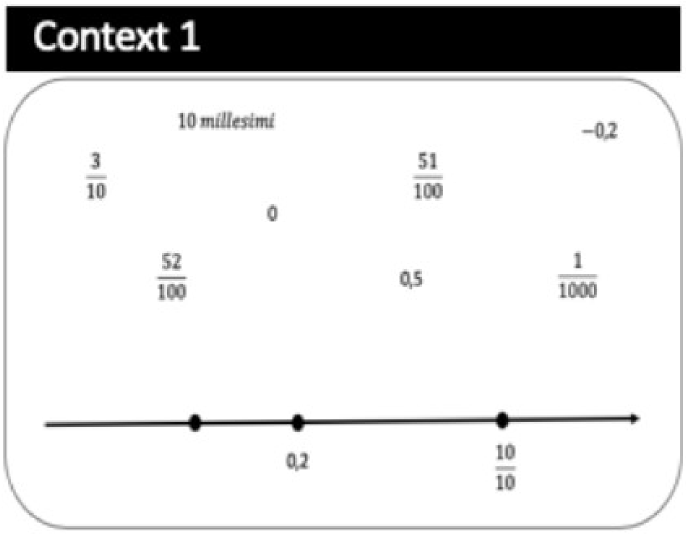


Figure 75. Context of the first problem-posing session: number line

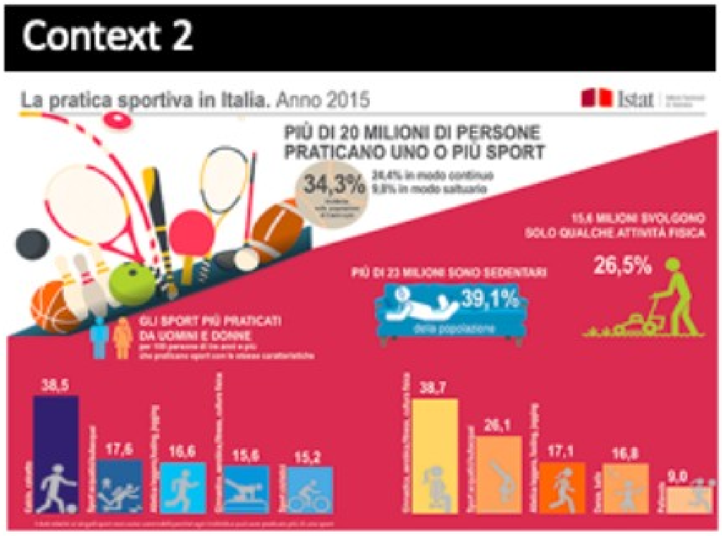


Figure 76. Context of the second problem-posing session: survey sport statistic

The lesson after the problem-posing sessions, students have to solve some problems chosen by the teacher from the ones posed in the previous problem-posing activity. The teacher should stimulate students' re-creation of the algorithm of addition between decimal numbers.

8.3.4 HLT for cycle P-II

In this section we sum up the components of the designed HLT for the research cycle P-II discussed in the previous sections, namely the learning goal, the hypothetical learning process and the learning activities. Together those components define the HLT. A scheme is represented in Table 31.

Table 31. HLT of the P-II cycle

HLT		
Learning goal	Hypothetical Learning Process	Learning Activities
Addition between decimal numbers	Making students face with contexts that stimulate the	Students are engaged in two problem-posing

	creation of problems in which, in order to solve them, they need to develop new mathematical concepts or strategy. The critical point is represented by the necessity to calculate addition between decimal numbers.	sessions in which they have to pose at least three problems dealing with decimal numbers. The contexts chosen reflect the RME and Palm's frameworks for real problems, respectively a number line and a sport survey statistic. After the problem-posing activity, students are asked to solve some problems that had been chosen by their teacher from the ones posed during the problem-posing activity. Such problems together with a classroom discussion coordinated by the teacher, should support the emergence of new mathematical knowledge in a guided re-invention perspective.

8.4 P-II teaching experiment

In this section we present results from the teaching experiment concerning the research cycle P-II.

As described in the previous section, during the problem-posing activity each student had to pose at least three problems from two given contexts: a number line and a sport survey.

For the analysis of the posed problems, we modified the scheme proposed by Leung and Silver (1997), introducing the new category *emergent problems*. The difference between *solvable problems* and *emergent problems* is that for the firsts, students know the mathematics needed to solve them, while the seconds refer to problems that require new mathematical concepts to be solved. The data analysis scheme is reported in Fig. 77.

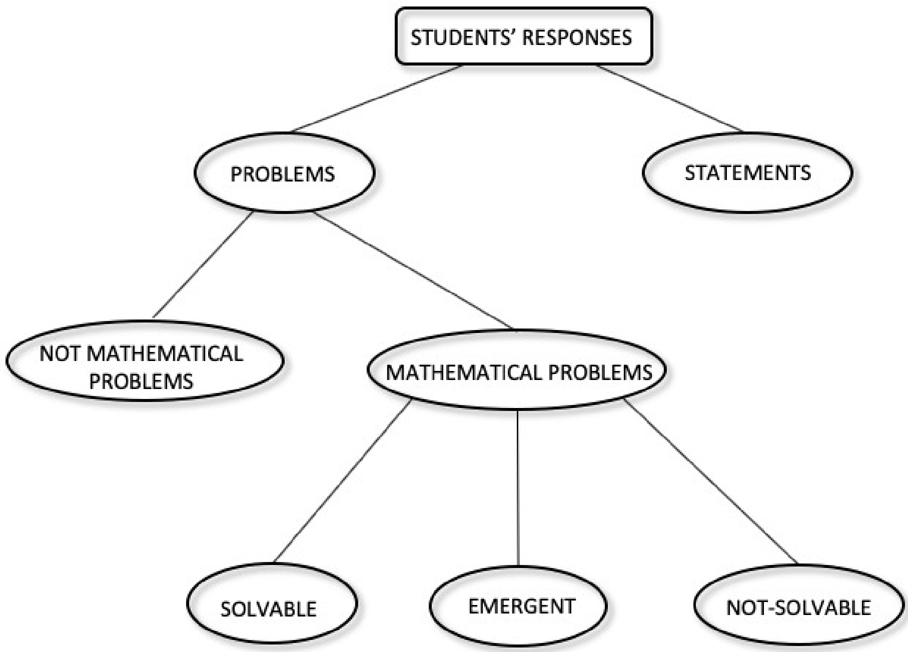


Figure 77. Data analysis scheme of students' responses in terms of emergent problems

Students posed totally 229 problems (95% of students' responses) and 11 statements (5% of students' responses). Concerning the quality of the posed problems, including *mathematical problems/not mathematical problems* and *solvable/emergent/not solvable problems*, the results split between the two contexts used as starting points for the problem-posing sessions, respectively the number line and the sport survey, are reported in Table 32.

Table 32. Students’ responses in terms of *mathematical/not mathematical* problems, *solvable/emergent/not solvable* problems. Results are split between the two contexts used for the problem-posing sessions.

		Math. Problems	Not Math Problems		Solvable	Emergent	Not Solvable
Cont. 1 (numbers line)	count	91	36	count	65	7	19
	% within problems	72	28	% within math problems	71	8	21
Cont. 2 (sport survey)	count	92	10	count	13	55	24
	% within problems	90	10	% within math problems	14	60	26
Total	count	122	7	count	78	62	43
	% within problems	95	5	% within math problems	43	34	23
		($p<0.05$; V=0.15)				($p<0.001$; V=0.63)	

The first (also statistically significant) difference between the two contexts appears in terms of *mathematical problems* ($p<0.05$; V=0.15). Indeed, for the first context (number line), the 72% of problems had been classified as *mathematical problems* and the 28% of the problems as *not mathematical problems*, while for the second context (sport survey), the 90% of problems had been classified as *mathematical problems* and the 10% of the problems as not *mathematical problems*.

The second difference occurs considering *solvable* and *emergent problems* ($p<0{,}001$; V=0,63). Despite the total number of mathematical problems that can be solved (*solvable* plus *emergent*) is comparable for the two contexts (79% for the number line one and 74% for the sport survey), from the number line the 71% of *mathematical problems* had been classified as *solvable* and the 8% as *emergent*, while for the sport survey the 14% of *mathematical problems* had been classified as *solvable* and the 60% as *emergent*.

In the lesson that followed the problem-posing activity, students solved some of the problems they posed. In particular, the author chose some emergent problems dealing with addition between decimal numbers (Appendix I). Students had to solve such problems in pairs and explain their solving strategies. One of these problems consisted in calculating the total number in percentages of people who practice swimming, that consisted in performing 17,6+26,1. Some students in order to perform the calculation, transformed the decimal numbers in fractions and then summed the results:

$$17{,}6+26{,}1=\frac{176}{10}+\frac{261}{10}=\frac{437}{10}=43{,}7$$

In another case, that consisted in performing 14,5+26,7, students summed separately units and tenths and then summed together the obtained results. In this case, as shown in Fig. 78, students had to reason about the meaning of obtaining 12 tenths in a decimal system. The author decided to stimulate all the class to reason about this point, and students were able to understand their previous error and correctly perform the calculation paying attention to the positional notation. From this classroom discussion, other students were able to reinvent the algorithm for the calculus in column (Fig. 79).

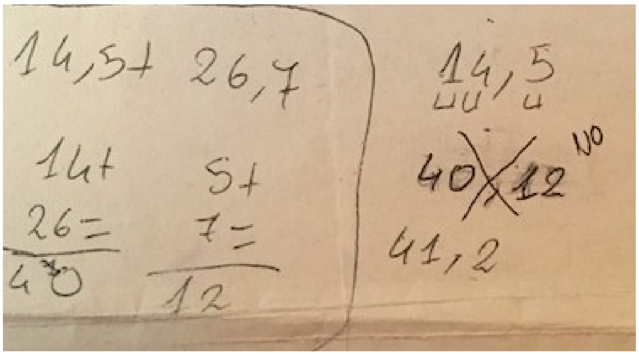


Figure 78. Students' solving strategy to perform the addition 14,5+26,7

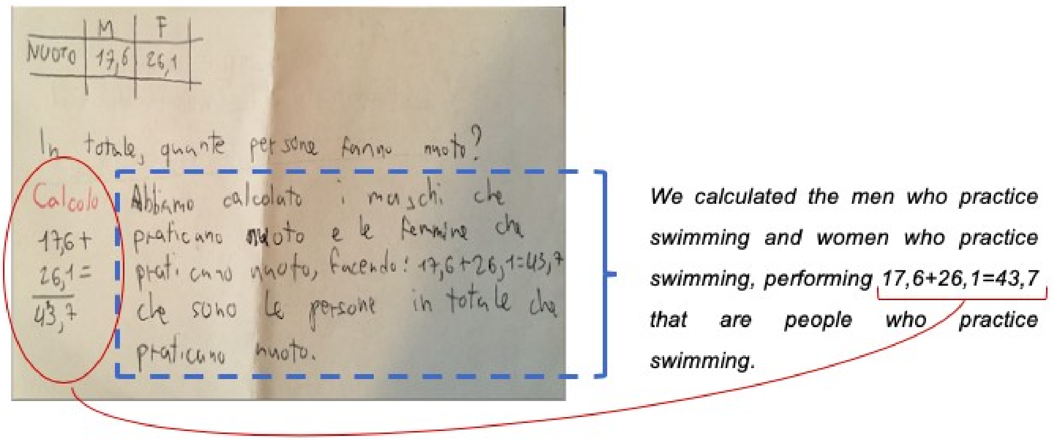


Figure 79. Students' solving strategy to perform the addition 17,6+26,1

8.5 Reflection on the P-II teaching experiment

The aim of this teaching experiment consisted in investigating how different contexts could promote or not emergent problem-posing. In the specific, we formulated the hypothesis that semi-structured situations could support the process of emergent problem-posing, i.e. the process in which problem-posing is a way for the emergence of students' formal ways of knowing. The mathematical topic that was chosen for the activity was given by decimal numbers, and in particular we studied how emergent problem-posing could be fostered to help students in understanding addition between decimal numbers.

In the design phase of this research cycle we designed an HLT through the definition of its three aspects: the learning goal; the hypothetical learning process and learning activities. The learning goal was represented by addition between decimal numbers. The hypothetical learning process consisted in making students experiencing a problem-posing activity, in which a prominent role was given by the contexts from which students had to pose their problems dealing with decimal numbers. Such contexts could stimulate students in formulating problems that need new mathematical knowledge for their solution. On the basis of the designed HLT and of the actual learning process, described in the teaching experiment phase, we can make some reflections.

As said before, the aim of the study consisted in evaluating how different contexts could impact emergent problem-posing in semi-structured situations (Stoyanova and Ellerton 1997). Two contexts were chosen: a number line and a sport survey. Similarly to the previous research cycle (chapter 7), the first context can be seen as a rich and realistic context in the perspective of RME, that comes from the world of mathematics. The second context, instead, can be seen as an artifact (Bonotto 2013), and represents a real context in Palm's framework, since it represents a context that could really happen in real life. From the data analysis two main aspects emerged.

The first aspect consists in the different impact of the two contexts in students' responses, and in particular in the difference that occurred in terms of *solvable* and *emergent problems*. Table 32, indeed, shows that students' problem-posing responses in terms of *emergent mathematical problems* were completely different. From the first context the 8% of *mathematical problems* were *emergent* ones, while from the second only the 60%. This means that the second context, the sport survey, offered more opportunities in fostering the emergent nature of problem-posing. Moreover, a shift between *emergent* and *solvable mathematical problems* from the first to the second context occurred. One reason for such a significant difference can be attributed to the fact that the first context was probably more abstract and less experientially significant for students, and consequently they paid more attention to the solving aspect of the problems they posed. At the same time, since the second context was richer in real data, students were stimulated to pose problems in a freeway. Moreover, starting from the first context the 28% of problems were *not mathematical problems*, while for the second context the 10%. This means that a meaningful context for students fostered not only emergent problem-posing, but also permitted students to focus on the mathematical sense and meaning of the problems they posed.

The second aspect we want to focus on is that emergent problem-posing can promote prospective learning (Freudenthal 1991). Indeed, in this study the fact that students had to solve a problem created by themselves, stimulated them to develop a mathematical strategy to solve a new mathematical task: performing addition between decimal numbers. Emergent problem-posing, in analogy with emergent modelling, encouraged students in developing mathematical algorithms and procedures starting from their informal mathematical strategies. As a consequence, problems posed by students supported the same students in creating new mathematical knowledge, fostering in this sense their prospective learning in a re-invention process. Moreover, students, while solving the problems created by themselves, were able to develop more than only one strategy, fact that contrasts the conviction that there is only one possible correct way to solve a problem. We remark the fact that the kind of contexts used to implement problem-posing activities was fundamental. Indeed in this study is clearly proved that some contexts, that are experientially meaningful for students, such as the sport survey, can foster in a deeper and significant way emergent problem-posing, while other contexts, such as the number line in our example, did not give the same opportunity to increase students' knowledge. As a consequence, different contexts actually have a different potential in enhancing emergent problem-posing, and such a potential seems to be connected with the significance given by students to the context.

8.6 Conclusions from P-I and P-II research cycles

The aim of the research cycles P-I and P-II consisted in answering to the second research question, i.e. studying how different contexts can influence the process of problem-posing. In the specific, we focused on two aspects of the problem-posing process, namely its relations with creativity and emergent problem-posing. As a consequence, from the second research questions, two more specific sub-questions emerged:

RQ2.1. How do different contexts influence students' creativity in problem-posing?

RQ2.2. How do different contexts influence emergent problem-posing?

The research cycle P-I focused on the research question RQ2.1, while the research cycle P-II focused on the research question RQ2.2. From these design research cycles some conclusions can be drawn.

The aim of the research cycles P-I and P-II consisted in studying how different contexts influence students' problem-posing abilities in terms of creativity and emergent problem-posing. To answer to our research questions, the results of two teaching experiments had been reported, in which students had been engaged in semi-structured (Stoyanova and Ellerton 1997) problem-posing activities that started from different contexts. In both the teaching experiments, the chosen contexts had a main difference: one context was taken from the mathematical world (number line) and the other was represented by an artifact (mobile phones leaflet and sport survey statistic) (Bonotto 2013).

Concerning students' creativity, the first research cycle showed that there was not a significant difference between the two contexts used, i.e. the mobile phone leaflet and the number line. However, significant differences occurred in terms of *context* and *not context related* problems. Indeed, students when had to pose problems from the number line, constructed a new setting for their problems. This fact has two main implications. The first one consists in the fact that actually the number line was not a realistic context for students. In fact, even if they previously worked with it in mathematics classrooms, they did not recognise it as a meaningful context, and the posed problems had been set in other real contexts chosen by the same students. The second point, that is linked to the first one, is that it is fundamental that the context should be experientially significant for students. When the context is not significant, students attach to it a new meaning, not working with the given context but building a new one. In conclusion, answering to the research question RQ2.1, if the context is meaningful for students there is no evidence in differences in terms of creativity.

Concerning emergent problem-posing, instead, the number line and the sport survey produced completely different results. Indeed, the sport survey offered more opportunities in fostering the emergent nature of problem-posing, and as a consequence also students' prospective learning. This means that the kind of context used to implement problem-posing activities is fundamental. Indeed, in our case we proved that a suitable artifact can foster in a deeper and more significant way emergent problem-posing, while other contexts, such as the number line in our example, did not give the same opportunity to increase students' knowledge.

We want to stress the fact that the aim of these cycles was to start investigating the impact of different contexts on some aspects of problem-posing (creativity and emergent problem-posing), and not to compare the frameworks of RME and Palm about real contexts. What emerged from the reported studies, is that artifacts, that were closer to Palm's framework since their component of feasibility, were more familiar to students who felt them as more realistic. This fact tells that this context is not more valid in general, but, since it was more significant for students, it offered them more opportunities in terms of emergent problems. Therefore, the key point is that the context must be meaningful for students, and this characteristic is actually asked by both the proposed frameworks. However, from this analysis another consideration appears: it is very difficult to make students familiar with contexts that come from mathematics. We believe that this could be a great challenge for the future, since such contexts offer significant starting points also for vertical mathematization (Freudenthal 1991; Treffers 1987).

9. Conclusions and Discussion

9.1 Introduction

This research aimed at designing a re-invention process (Freudenthal 1991) to integrate mathematical modelling in the regular school practice in the Italian context. In the specific, we investigated how this process should be implemented and used to help students give sense to their mathematical activity. The main research question of this project was:

> *How, and to what extent, can mathematical modelling be integrated in the teaching and learning of mathematics in a guided re-invention paradigm?*

In order to answer to our research question, we decided to adopt the perspective of Realistic Mathematics Education, in order to design a learning trajectory that brings students to invent their mathematical principles in a modelling environment. To design such a re-invention process, our choices had been the design heuristics of didactical phenomenology and emergent modelling, and the use of problem-posing in relation to such heuristics.

The main research question was split in two more specific questions. The first research question dealt with the design of such modelling activities with the focus in the promotion of students' creation of new mathematical concepts or strategies they need to solve a real problem. In the specific, we investigated how activities designed following the MEA principles could foster the emergent nature of modelling.:

RQ1. How can Model Eliciting Activities promote the process of emergent modelling?

The second research question consisted in start investigating the impact in the use of different contexts during problem-posing activities in terms of students' creativity and emergent problems:

RQ2. How do different contexts influence the process of problem-posing?

Concerning the second research question, we focused on two aspects of the problem-posing process, namely its relations with creativity and emergent problem-posing. As a consequence, from the second research questions, two more specific sub-questions emerged:

RQ2.1. How do different contexts influence students' creativity in problem-posing?

RQ2.2. How do different contexts influence emergent problem-posing?

To be able to answer to the research questions we had to create an instructional environment with which it could be possible to study how and to what extent the suggested processes could be fostered. An instructional sequence was therefore necessary to answer to the research questions, and consequently a research design that allows for revising theories, hypothesis and instructional activities was needed. Furthermore, new teaching materials that support new types of learning must be developed, making the design process an integrated part of the research. In order to answer to our research questions, the research methodology adopted was the one of *design research*, in which planning and creating innovative educational settings and analyzing teaching and learning processes is given a central role.

The answers to the first research question are described in section 9.2, while the second research question is answered in section 9.3. Section 9.4 is dedicated to the discussion of other important factors of the research project, and to some conclusions in relation to the global purpose of the Univerisity project that this research is part of. We conclude in section 9.5 with some recommendations for teaching and future research.

9.2 Answers to the first research question

In order to answer to the first research question, we implemented two design research cycles: M-I and M-II. Each cycle was made by three phases: a design phase, a teaching experiment and a retrospective analysis. Concerning the design phase, this was explicated through the development of a Hypothetical Learning Trajectory, that consisted in defining a learning goal, some learning activities and a hypothetical learning process.

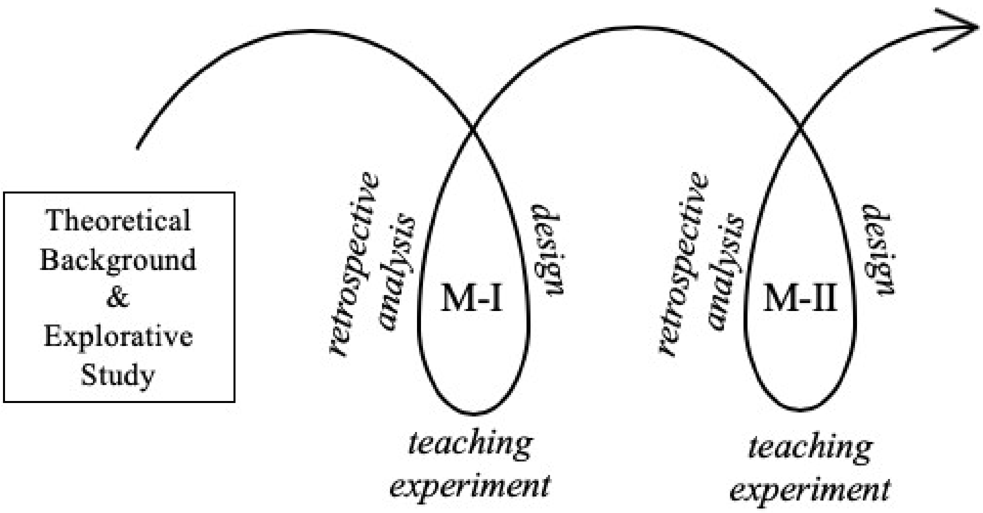


Figure 80. Research cycles M-I and M-II

Before the implementation of the research cycles M-I and M-II, a questionnaire for mathematics teachers was developed. This questionnaire, together with the theoretical background, constituted the preparation phase of the design research. The aim of the questionnaire was to know more about teachers' knowledge and practice of mathematical modelling and problem-posing in the Italian context, in order to identify possible starting points for the research cycles. The questionnaire had an exploratory character. Concerning modelling, its aim was to know if teachers include in their school practice some aspects of the modelling process. The results from the questionnaire show that teachers regularly include some aspects of modelling in their classroom activities, such as in using real contexts as starting situations for mathematics lessons and showing real applications of mathematics. Despite this disposition, teachers expressed a need in both more materials and preparation to implement activities based on realistic contexts, in line with Blum (2015). As a consequence, from the questionnaire we understood that teachers must be given more opportunities to face with modelling activities, and moreover prototypes of practices based on realistic problematic contexts available for teachers of every school level need to be developed. In this way teachers would have at their disposal models of modelling activities that can be adapted and implemented in their classrooms. The results are in line with the choice of the design research methodology, because the same teachers expressed the need of innovative educational settings for their teaching. In such settings, meaningful real contexts are crucial, and as a consequence the role of the design heuristic of didactical phenomenology would become fundamental in designing instructional activities with the aim of choosing problem situations that could provide the basis for the development of the mathematical concepts or tools we wanted students to develop.

In chapter 5 and 6 the two design research cycles M-I and M-II have been presented. Such cycles had the additional goal to provide teachers with design schemes and prototypes of practices in line with the University project and teachers' requests from the questionnaire.

The first research cycle M-I was conducted in a second-grade class (age 7). Its focus consisted in studying how emergent modelling should be fostered to help students understanding some aspects of the multiplicative structure, and in particular the distributivity property of multiplication over addition. Our hypothesis was that a modelling activity designed following a model eliciting sequence (Lesh et al. 2003) with the use of suitable artifacts could actually foster the emergent nature of modelling. The design phase, the teaching experiment and the retrospective analysis of this research cycle are described in sections 5.3, 5.4, 5.5.

The second research cycle M-II was conducted in a twelfth-grade class (age 17). In relation to the research question RQ1, we investigated how emergent modelling can be fostered to help students understanding some aspects of 3D-Euclidean geometry. In particular, our hypothesis was that facing students with a real problem solving situation designed following a model eliciting sequence (Lesh et al. 2003), with the use if suitable artifacts, could actually foster the emergent nature of modelling, seen as a process in which students develop mathematical concepts from informal realistic contexts. The design phase, the teaching experiment and the retrospective analysis of this research cycle are described in sections 6.3, 6.4, 6.5.

In both the research cycles the aim of the data analysis was to reconstruct the classroom progress, which resulted in an empirical grounded understanding of students' reasoning during the classroom activity. In order to be able to reconstruct the learning process and verify our hypothesis, different kinds of data were collected: pre-test; transcriptions of classroom dialogs; observations of group working; students' final projects, students' individual reflections and feedbacks.

The results from the research cycles M-I and M-II, given by the analysis of our preliminary hypothesis respect to the actual learning process occurred in the teaching experiments, permitted to formulate answers to the first research question RQ1.

In agreement with the process of emergent modelling, the assignments given to students stimulated them to create and work with new mathematical concepts they did not know before. In the first teaching experiment, students were asked to design a floor tiling of their classroom and to specify the cost for such design. The strategy developed by students to solve the task, that consisted in grouping the tiles with the same shape and then multiply by the associated costs, permitted them to re-invent the mathematical concept of distributivity of multiplication respect to addiction. This is evident from different data such as classroom dialogues and students' final projects, in which students were able to explain and reproduce such mathematical concept. In the second teaching experiment, students had to make a project dealing with packaging (Fig. 46). In particular, students had to calculate the capacities of some packaging seen as irregular solids. Students decomposed such packaging in many regular solids, and to obtain an approximation of their volume, they calculated the volume of the regular solids by which the packaging was decomposed, and then summed these volumes. In other words, since students had to calculate the volume of an irregular solid, they were able to re-invent the equidecomposability principle. This fact is evident in students' projects (Fig. 49) and in their final individual reflections. Guided by the interaction with the teacher and peers, students in both the teaching experiments were able to reason and explain new mathematical concepts or properties. Such re-invented concepts become meaningful for students, because no longer mechanical rules but rooted in their experience and directly constructed by themselves to solve a concrete problem in a meaningful context.

The re-invention process was possible not only thanks to the designed model eliciting sequence, but also to the use of suitable artifacts, that permitted to foster students' disposition in creating mathematical tools and concepts needed to solve the assignments. Moreover, the way in which the problem or these artifacts are presented to students is fundamental, since some constraints, omission of data or other information could stimulate deeply such re-invention process. As a consequence, the role of the RME heuristic of didactical phenomenology is clearly evident: it guides in choosing not only the contexts of the task, but also some task constraints and related materials appropriate to provide basis for the development of mathematical concepts (in our research cycles distributivity of multiplication respect to addition, equidecomposability principle) we wanted students to develop.

Another important aspect from these research cycles is the importance of moments for personal reflection, in particular during group work. Indeed, the retrospective analysis of the M-I research cycle highlighted the necessity to engage students in a final individual reflection on the whole modelling process. This individual activity was introduced in the second research cycle in the *reflection and debriefing phase*, wherein students had the possibility to reflect individually on the whole modelling process, observing strengths and weaknesses of their projects; elaborating their own conclusions to model a solution for the model eliciting activity; analysing similarities or differences with other solution plans; changing their beliefs and attitude; reinforcing argumentation abilities.

The modelling activities had also strong social implications. Several students who did not typically engage during mathematics classes became active participants while solving the task. As a consequence, the introduction of new socio-mathematical norms (Yackel and Cobb 1996) and the use of interactive teaching methods, could promote cooperation and inclusion. Students being active participants to the learning process are able to give meaning to new mathematical knowledge and sense to their mathematical activity.

In conclusion, answering to the first research question of this project, we can affirm that model eliciting activities together with suitable artifacts could foster the emergent nature of modelling, that confirms our hypothesis. As explained before, this result can be attributed to a combination of several factors: the choice of realistic and rich problems that stimulate students to elaborate formal mathematical concepts mathematizing their informal solving strategies, rooting in this way the new understandings in experientially real phenomena; the use of suitable artifacts, that present mathematics as a means of interpreting and understanding reality and increasing the opportunities for observing mathematics outside of the school context (Bonotto 2005); the role of the teacher, who guides students in re-inventing mathematics in an active way.

9.3 Answers to the second research question

In order to answer to the second research question, we implemented two design research cycles: P-I and P-II. Each cycle was made by three phases: a design phase, a teaching experiment and a retrospective analysis. Concerning the design phase, in the research cycle P-I this was explicated through the development of an instructional design that consisted in the formulation of some hypothesis together with instructional activities and materials, while in the research cycle P-II through the development of a Hypothetical Learning Trajectory, that consisted in defining a learning goal, some learning activities and a hypothetical learning process.

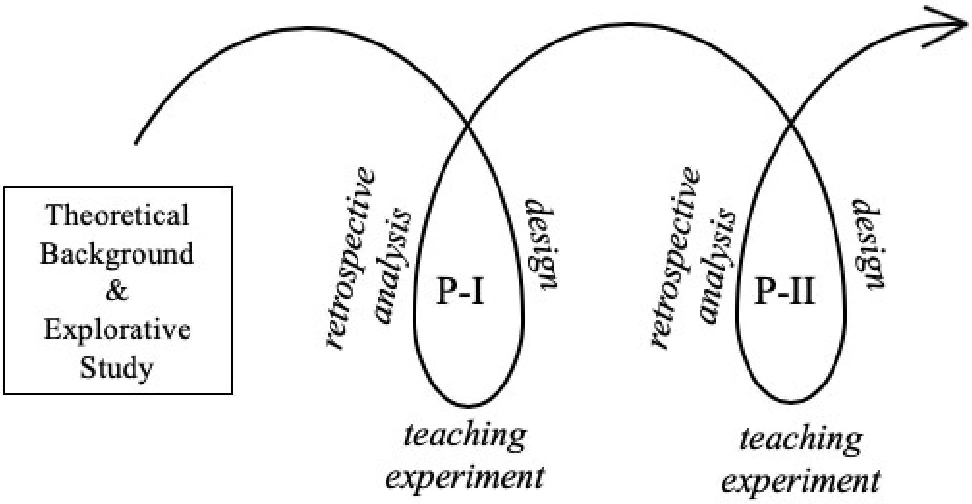


Figure 81. Research cycles P-I and P-II

Before the implementation of the research cycles P-I and P-II, a questionnaire for mathematics teachers was developed. This questionnaire, together with the theoretical background, constituted the preparation phase of the design research. The aim of the questionnaire was to know more about teachers' knowledge and practice of mathematical modelling and problem-posing in the Italian context, in order to identify possible starting points for the research cycles. The questionnaire had an exploratory character.

Concerning problem-posing, it emerged that this educational strategy is not known by teachers, and consequently not regularly implemented at school. In fact, less than a half of the participants (39,6%) adopted it during its school practice. Problem-posing, instead, should become an integral part of pre-service and in-service teacher training courses, in order to give teachers opportunities to increase their knowledge, before, and their practice, after, on problem-posing. Such improvement in teachers' knowledge could help teachers to recognise intersection points between different methodologies and strategies and to adopt coherent teaching methods.

In chapter 7 and 8 the cycles concerning problem-posing had been reported. The main point was to integrate problem-posing in the school practice, and as stated in consequence of the exploratory study, a key point to pursue this goal is to reinforce the use of real contexts to make students pose their own problems. The main difficulty that could occur in this scenario is to know what a real context could be for students, and how different contexts can affect students' performances in problem-posing. In this direction, we investigated such influence in terms of creativity and emergent problem-posing.

The first research cycle P-I was conducted in a sixth-grade class (age 12). It aimed at studying how different contexts influence students' problem-posing performances in terms of creativity, focusing in this way on the sub-question RQ2.1. Our idea was that semi-structured problem-posing activities in which real contexts are used as starting situations could foster students' creativity. However, it was not clear how different contexts influence students' creativity, and which characteristics such contexts should have. As a consequence, we decided to consider different real contexts from different theoretical perspectives. The design phase, the teaching experiment and the retrospective analysis of this research cycle are described in sections 7.3, 7.4, 7.5.

The second research cycle P-II was conducted in a fourth-grade class (age 9). In relation to the research question RQ2.2, we investigated how different contexts could promote or not emergent problem-posing. Our hypothesis was that semi-structured situations could support the process of emergent problem-posing, i.e. the process in which problem-posing is a way for the emergence of students' formal ways of knowing. However, it was not clear a priori how different contexts could influence the emergent nature of problem-posing. The design phase, the teaching experiment and the retrospective analysis of this research cycle are described in sections 8.3, 8.4, 8.5.

The results from the research cycles P-I and P-II permitted to formulate answers to the second research question RQ2. In particular, the influence of different contexts was analysed in terms of creativity in the research cycle P-I, that aimed at answering to the sub-question RQ2.1, and in terms of emergent problem-posing in the research cycle P-II, that aimed at answering to the sub-question RQ2.2.

Concerning creativity, we investigated how different contexts influence students' creativity in semi-structured problem-posing activities. Real contexts from different theoretical perspectives were chosen: a number line, that can be seen as a rich and realistic context in the perspective of RME, and a mobile phones leaflet, that can be seen as an artifact (Bonotto 2013), representing a real context in Palm's framework. Not a significant difference occurred between the use of the two contexts. Indeed, both the contexts stimulated in the same way students' creativity (Table 27). However, the results indicate that a fundamental factor that influenced students' problem-posing performances, and especially in terms of creativity, is the significance given by students to the context. Indeed, when the context is not experientially meaningful for a student, as in the case of the number line, she/he associates a new meaning to the context, using some elements from it and re-creating a new more meaningful context in which setting her/his problems. In our study, this process of context free re-construction was possible because no constraint was explicitly maid during the problem-posing activity concerning the fact that students had to refer to the number line when posing problems, but the only constraint was to pose at least three problems *from* that context, and not *in* that context. However, significant differences occurred in terms of *context* and *not context related* problems. Indeed, students when had to pose problems from the number line, constructed a new setting for their problems. This fact has two main implications. The first one consists in the fact that actually the number line was not a realistic context for students. In fact, even if they previously worked with it in mathematics classrooms, they did not recognise it as a meaningful context, and their posed problems had been set in other real contexts chosen by students. The second point, that is linked to the first one, is that it is fundamental that the context should be experientially significant for students. When the context is not significant, students attach to it a new meaning, not working with the given context but building a new one. In conclusion, answering to the research question RQ2.1, if the context is meaningful for students there is no evidence in differences in terms of creativity.

During the problem-posing sessions of the research cycle P-I, students, starting from given contexts, formulated some mathematical problems that take into account some concepts and notions that have not already been introduced by the teacher during previous lessons. Starting from these suggestions, we believed indispensable to start investigating how different contexts could promote or not emergent problem-posing, that was investigated in the following research cycle P-II.

The aim of the research cycle P-II consisted in evaluating how different contexts could impact emergent problem-posing in semi-structured situations (Stoyanova and Ellerton 1997). As in the previous cycle, two contexts were chosen: a number line, that can be seen as a rich and realistic context in the perspective of RME, and a sport survey, that can be seen as an artifact (Bonotto 2013), representing a real context in Palm's framework. In terms of emergent problem-posing, the two contexts had a different impact on students' responses. Indeed, students' problem-posing responses in terms of *emergent mathematical problems* were completely different (Table 32), indeed the sport survey offered more opportunities in fostering the emergent nature of problem-posing. Moreover, a shift between *emergent* and *solvable mathematical problems* from the first to the second context occurred. One reason for such a significant difference can be attributed to the fact that the first context was probably more abstract and less experientially significant for students, and consequently they paid more attention to the solving aspect of the problems they posed. At the same time, since the second context was richer in real data, students were stimulated to pose problems in a freeway. Moreover, emergent problem-posing actually promoted prospective learning (Freudenthal 1991). The fact that students had to solve a problem created by themselves, stimulated them to develop a mathematical strategy to solve a new mathematical task: performing addition between decimal numbers. Emergent problem-posing, in analogy with emergent modelling, encouraged students in developing mathematical algorithms and procedures starting from their informal mathematical strategies. As a consequence, problems posed by students supported the same students in creating new mathematical knowledge, fostering in this sense their prospective learning in a re-invention process. Moreover, students, while solving the problems created by themselves, were able to develop more than only one strategy, fact that contrasts the conviction that there is only one possible correct way to solve a problem. We remark the fact that the kind of contexts used to implement problem-posing activities was fundamental. Indeed in this study is clearly proved that some contexts, that are experientially meaningful for students, such as the sport survey, can foster in a deeper and more significant way emergent problem-posing, while other contexts, such as the number line in our example, did not give the same opportunities to increase students' knowledge. As a consequence, different contexts actually have a different potential in enhancing emergent problem-posing, and such a potential seems to be connected with the significance given by students to the context.

9.4 Discussion

In the previous section we presented and discussed answers to the research questions of the study. In this section, we will focus on other aspects related to this project. Firstly, we will discuss the results in relation to the general purpose of the University project that this research is part of. Then, we report some considerations concerning two components that played a fundamental role in achieving the results: the RME heuristics of didactical phenomenology and guided reinvention.

Overall aim of the project

The present research project found its roots in the need of a paradigmatic change in the didactics of mathematics that aims to build a bridge between reality, in which intuition plays a fundamental role, and school life, in which exercise and memorization continue to play an important role. In the specific we looked at the Italian context, where, despite some experiences of innovation and reflection on the curriculum, teaching strategies and learning environments, still persists a resistance to abandoning traditional teaching models of transmission type.

As stated in section 1.2, this study is part of a larger project of the University of Padova, concerning teachers' professional development. Concerning mathematics education, its overall purpose is to provide mathematics teachers with methodological models and format of school practices based on mathematical modelling. This purpose was outlined in the following points:

- implementing some teaching experiments wherein connecting mathematics and daily-life experiences;
- start developing prototypes of significant didactics practices based on mathematical modelling ready to be transferred and implemented in different concrete school contexts;
- developing specific models for professional development courses based on mathematical modelling for mathematics teachers of every school level.

The results and conclusions from the design research cycles outlined in the previous section, have also important implications in relation to the purposes of such University project. The methodology of design research, indeed, permitted to work in direct contact with schools in collaborations with mathematics teachers. The research cycles had been implemented in schools of different levels, from primary to secondary. Each research cycle has provided a teaching experiment, wherein one of the main points consisted in reducing the gap between mathematics in- and out- the school context. In the teaching experiments concerning mathematical modelling, the methodological approach of model eliciting was presented. In the specific, the activities had been designed following specific methodological principles, that can be used and/or adapted by mathematics teachers to design modelling activities in other contexts. Concerning problem-posing, two analytic schemes were presented, in order to evaluate students' problem-posing performances in terms of creativity and emergent problems. Moreover, in all the research project the design phase had a prominent position, designing activities, materials and their implementations were extensively described, in order to furnish teachers with operative examples of didactics practices on modelling and problem-posing, ready to be transferred and implemented in different concrete school contexts. Concerning the last point of the University project, namely the development of models for professional courses based on mathematical modelling, in November 2019 we started a teacher professional development course on mathematical modelling. The course was structured in five meetings each of 2 hours. The aim of the course consisted in improving teachers' mathematical modelling competencies (Borromeo Ferri 2014): *theoretical competence; task competence; instruction competence; diagnostic competence* (Table 2). However, due to the pandemic circumstances of COVID-19, it was not possible to complete the last part of the teachers' professional development course, regarding the implementation of teachers' designed modelling activities and their presentations and reflections, before the end of this project. As a consequence, not enough data had been collected to analyze the efficacy of such training. The conclusion of the course is postponed to the next year.

Didactical phenomenology

The goal of this research was to design a re-invention process (Freudenthal 1991) to integrate mathematical modelling in the regular school practice in the Italian context. In the specific, we wanted to investigate how this process can be implemented and used to help students give sense to their mathematical activity.

In section 9.2, 9.3 we answered to the research questions of the study. To achieve such results two fundamental tools have been represented by the RME heuristics of didactical phenomenology and guided reinvention.

Indeed, when designing and implementing activities based on mathematical modelling and problem-posing, a central role should be given to the contexts for mathematical problems. These contexts must be experientially significant for students and able to evoke new mathematical concepts or strategies for their solution. In this direction the challenge is to find phenomena that beg to be organized by the concepts that are to be taught (Freudenthal 1983). As a consequence, the role of didactical phenomenology is crucial, since it guides teachers to find problem situations that could provide the basis for the development of the mathematical concepts or tools they want students to develop. Such problem situations could lead to solutions that are first specific for that situation but can be generalized to other problem situations.

Guided reinvention

The meaning of guided reinvention is that students should experience the learning of mathematics as a process similar to the one by which mathematics was invented (Gravemeijer 1994). Students, mathematizing their mathematical activity can reinvent mathematics. This mathematization process is possible thanks to the guidance of the teacher and the instructional design.

In this research project, the role of the teacher, represented by the researcher that designed and conducted the teaching activities, was fundamental. Indeed, the teacher encouraged students to use their own methods; stimulated students to articulate and reflect on their personal beliefs, misconceptions and informal problem-solving and modelling strategies (Bonotto 2005). Consequently, learning become a constructed understanding through a continuous interaction between teacher and students, that can be synthetized, using Freudenthal’s words, in teaching and learning as *guided reinvention*, reinforcing in this way mathematical reasoning and sense-making.

9.5 Recommendations

9.5.1 Recommendations for teaching

This final section contains recommendations concerning instruction theories, educational practices and future research on mathematical modelling.

This research contributes to a local instruction theory on the teaching and learning of modelling and problem-posing, through the descriptions of shifts in students' reasoning, specific activities, tasks development, the design of the sequences of the activities, the role of didactical phenomenology and guided reinvention, the role of the teacher, and analytic schemes.

The design research approach resulted in an empirically based contribution to a local instruction theory. However, experimenting in a settled school program limited the possibilities for our research and the teaching experiments were confined to a series of mathematics lessons. Moreover, teachers were not able to guide the students during the lessons, that instead had been carried on by the researcher. We recommend future experiments where mathematics teachers, after having received a professional training on such educational strategies, implement these instructional sequences in their regular lessons. These experiments are needed for the constitution of a robust local instruction theory for the learning and teaching of modelling and problem-posing.

The local instruction theory, including the instructional activities, offered teachers a framework of reference for planning their lessons and their practical teaching (Gravemeijer 2004a). Further research is needed to investigate which description of the instructional sequence, together with the underlying theory, can indeed be used as a means of support for teachers and for other parties who influence the course of affairs in education (De Lange et al. 2001).

In order to fill the gap between in- and out-of-school mathematical competencies, we recommend the positive role of emergent modelling and problem-posing activities. Designing model eliciting activities could actually increase students' mathematical knowledge. Students indeed can re-create mathematical concepts, tools, strategies, starting from their informal solving strategies. In this process the importance of the context of the problems and the role of the teacher must be taken into account. Contexts should be experientially meaningful for students, in order to stimulate them to root their new mathematical knowledge in their personal experience. Such contexts are central also in problem-posing activities. In particular, they enhance both students' creativity and also the emergence of new mathematical concepts starting from students' posed problems. Students are able to develop concepts and instrumental competencies through activities based on real situations. These modelling activities should be valued and discussed seriously by the class. Current assessment practices appeared to focus both teacher and students on algorithmic skills. Instead, more emphasis should be put on assessment which addresses modelling competencies through open-ended investigations (Goldin, 2003; De Haan & Wijers, 2000; van den Heuvel-Panhuizen, 1996; De Lange, 1987, 1999).

9.5.2 Recommendations for future research

This study has provided some insights into the constraints and possibilities for the integration of modelling and problem-posing in mathematics lessons. Future research is recommended especially in the following points:

- in accordance with Blum (2015), mathematical modelling is high demanding for teachers. Indeed, mathematics teachers should become able of recognizing the mathematics incorporated in daily life. This requires knowing how to integrate pedagogical-didactic and disciplinary knowledge together, paying attention also to the particular school context and the cultural environment in which operating. Indeed, teachers should be able to: i) see mathematics incorporated in the real world as a starting point for mathematical activities (Bonotto 2005); ii) anticipate the mathematics needed for the paths that students might explore; iii) put students in familiar situations in which they clearly understand the need for mathematical constructs, integrating also their everyday knowledge; iv) provide meaningful design specs involving constraints that enable students to weed out inadequate ways of thinking. In order to provide teachers with designing principles and practical materials to develop modelling activities in their classrooms (and to increase their knowledge of problem-posing), in the future we believe that improvements in teachers' pre-service and in-service courses are needed, in particular: i) changing the type of activities with more realistic problem situations; ii) improving teachers' knowledge of some teaching strategies, such as problem-posing, that could be adequately chosen by teachers for the teaching of specific mathematical topics; iii) connecting mathematics and classroom teaching creating prototypes of practices based on modelling and problem-posing available for teachers;

- in this project we started investigating the impact of different contexts on some aspects of problem-posing, i.e. creativity and emergent problem-posing. What emerged from the research cycles is that artifacts, closer to Palm's framework since their component of feasibility, were more familiar to students who felt them as more realistic, respect to contexts from the world of mathematics. This fact tells that artefacts are not more valid in general, but, since they were more significant for students, they offered them more opportunities in terms of emergent problems. Therefore, the key point is that the context must be meaningful for students. However, from this analysis appears very difficult to make students familiar with contexts that come from mathematics. We believe that this could be a great challenge, since such (mathematical) contexts can offer significant base points also for vertical mathematization (Freudenthal 1991; Treffers 1987);

- in the explorative study, only some aspects of the modelling process were taken into account. A deeper understanding of teachers' effective practice of the entire modelling process and about their knowledge of other aspects of modelling is needed. For the future a deeper investigation of teachers' practices linked to both modelling and problem-posing would be performed through a series of interviews and classroom observations;

- modelling and problem-posing are closely related. However, further investigation is needed in: i) analyzing specificities and common points of each strategy; ii) developing designing principles to integrate problem-posing in the modelling process; iii) providing empirical evidence from the classroom in order to support the integration between mathematical modelling and problem-posing

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Appendix

A

QUESTIONNAIRE
Educational practices in the teaching of Mathematics

The following questionnaire is part of a project at the University of Padova, Italy. The questionnaire is made by three sections and it deals with the educational practices of Mathematics teachers. 20 minutes are needed to complete it.

We remind you that the questionnaire is anonymous, and the data collected will be used only for this research in respect of the privacy.

Thank you for your collaboration

ANAGRAFIC SECTION

1. Year of birth:

2. Nationality:

3. Gender: M F I don't want to say

4. Higher Graduation:

5. School level in which you are teaching:

   Primary

   Secondary

6. How many years are you been teaching in this level?

7. In which town are you teaching?

EDUCATIONAL PRACTICES

8. During your teaching activity, you adopt the following strategies:

	Never	*Rarely*	*Sometimes*	*Often*	*Always*
Lectures	1	2	3	4	5
Individual work	1	2	3	4	5
Group work	1	2	3	4	5
Guided lessons	1	2	3	4	5
Support activity	1	2	3	4	5
Laboratories	1	2	3	4	5
Other………..	1	2	3	4	5

9. During your teaching activity, you adopt the following tools:

	Never	*Rarely*	*Sometimes*	*Often*	*Always*
Textbooks	1	2	3	4	5
Notes	1	2	3	4	5
Interactive board	1	2	3	4	5
Software	1	2	3	4	5
Calcolator	1	2	3	4	5
Math games	1	2	3	4	5

Audio and video tools	1	2	3	4	5
Artefacts	1	2	3	4	5
Library	1	2	3	4	5
Other……….	1	2	3	4	5

10. Based on your teaching experience, you perform the following activities:

	Never	***Rarely***	***Sometimes***	***Often***	***Always***
I use starting real contexts for mathematical lessons	1	2	3	4	5
I show and work with some applications of mathematics	1	2	3	4	5

11. Do you include *problem-solving* activities during your teaching? Yes No
If yes, describe a significant example.

____
____
____
____

12. Do you implement *problem-posing* activities during your teaching? Yes No
If yes, describe a significant example.

____
____
____
____

TEACHING DIFFICULTIES

13. Based on your experience, express the level of difficulty you have found teaching the following topics:

	No one	*Just a few*	***Enough***	***Many***	***A lot***
Arithmetic and Algebra	1	2	3	4	5
Euclidean geometry	1	2	3	4	5
Analitic geometry	1	2	3	4	5
Functions	1	2	3	4	5
Probability and Statistics	1	2	3	4	5
Logic	1	2	3	4	5

14. In conclusion I ask you one (or more) suggestions you believe indispensable to improve your teaching of mathematics.

B

Scuola Primaria – Peschiera del Garda

Risolvi i seguenti problemi. Ricordati di spiegare il procedimento che hai seguito.
[*solve the following problems*, *explaining your solving strategy*]

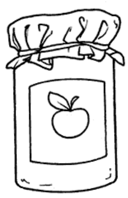
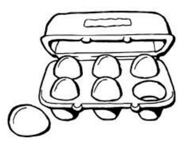
1. Un giornalino costa 3 euro e 50 centesimi. Questi sono i soldi che hanno tre bambini:
   [*A newspaper costs 3 euros and 50 cents. Children have the following money*:]

Uno dei bambini non ha abbastanza soldi per comprare il giornalino. Chi?
[*One of the children does not have enough money to buy the newspaper. Who is*?]

2. Maria va al supermercato per comprare 4 confezioni di bibite. Ogni confezione contiene 6 lattine. Quante lattine di bibite compera Maria?
   [*Maria goes to the supermarket to buy 5 packs of drinks. Each pack is made by 6 cans. How many cans does Maria buy?]*

3. Un barista per preparare tre panini ha usato sei fette di pane, tre fette di pomodoro, una mozzarella. Per fare sei panini ha bisogno di quanto pane, pomodoro e mozzarella?
   [*A bartender used six slices of bread*, *three slices of tomato*, *one mozzarella to make three sandwiches. To make six sandwiches*, *how much bread*, *tomato and mozzarella does he need?]*

4. La nonna decide di preparare quattro crostate. Decide di usare:
   [*the Grandmother wants to bake four pies*. *She uses:]*

	Un vasetto di marmellata ogni due crostate [*a jar of jam every two pies*]
	Tre uova per ogni crostata [*three eggs for each pie*]

Quanti vasetti di marmellata e quante uova dovranno essere usati in tutto?
[*How many jars of jam and eggs does she totally need?]*

5. Il papà è uscito di casa con una banconota da 20 euro. Quante riviste da 6 euro può comprare al massimo?
   [*Dad left the house with 20 euros. If a magazine costs 6 euros, how many magazines can he buy at most?]*

C

1. Explain the task in your own words.

………

………

………

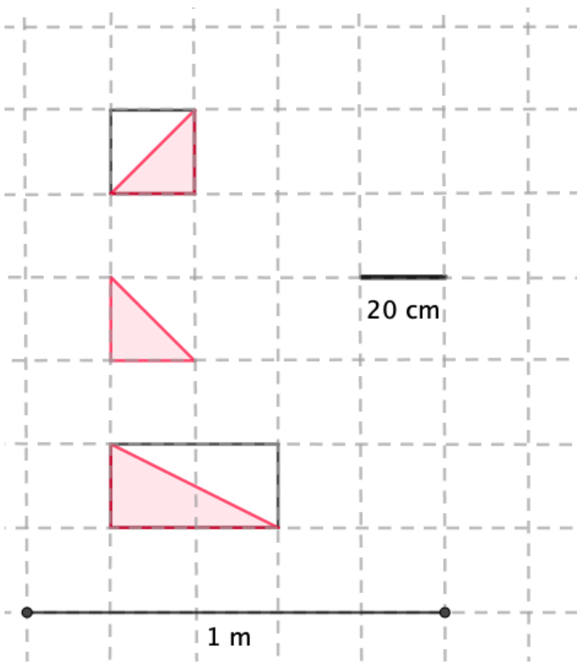
………

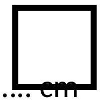
………

………

2. Write the measures of the tiles:

..... cm





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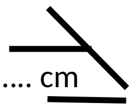


..... cm

..... cm

..... cm

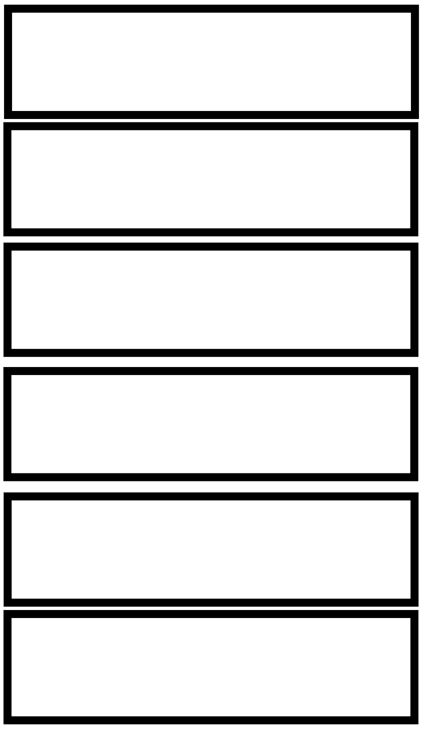
..... cm

3. How many centimeters do the sides of the strip measure?

..... cm

4. How many centimeters do the sides of the classroom measure?

..... cm



..... cm

5. Fill the table:

Shape	Number of tiles	Cost (euros)
Square	7	....
Rectangular	3	....
Triangular	6	....
Triangular	....	12

Square	....	18

D

Scuola Primaria – Peschiera del Garda

Risolvi i seguenti problemi. Ricordati di spiegare il procedimento che hai seguito.
[*solve the following problems, explaining your solving strategy*]

1. Un pacchetto di figurine costa 3 euro.

   Quanto costano tre pacchetti di figurine?

   [*a pack of stickers costs 3 euros. How much are three packs of stickers?*]

   ……

   ……

   Simone è uscito con una banconota da 20 euro.

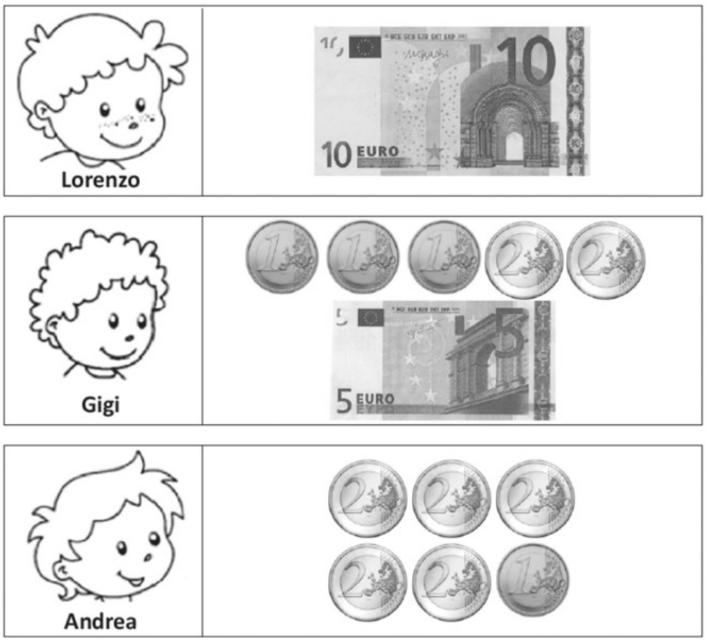
   Quanti pacchetti di figurine può comprare?

   [*Simone left with 20 euros. How many packs of stickers can he buy?*]

   ……

   ……

2. Una tavoletta di cioccolata costa quattro euro. Questi sono i soldi che hanno tre bambini:
   [*A bar of chocolate costs four euros. This is the money that children have:*]



Chi potrà acquistare tre tavolette di cioccolata?

[*Who can buy three bars of chocolate?]*

……

……

……

3. Guarda la figura qui sotto.
   [*Look at the picture*]

Quanti tappi è lunga la penna?

[*How many caps is the pen long?]*

…….

…….

Ogni tappo misura 2 cm. Quanti centimetri è lunga la penna?

[*Every cap measures 2cm. How many centimeters is long the pen?]*

…….

……

4. I venti alunni di una classe vogliono preparare una macedonia di fragole e banane per tutta la classe. Decidono di usare:

   [*A class is made by 20 students, who want to prepare a fruit salad for all the classroom. They use: a banana every 4 students; 3 strawberries for every student]]*

	**1 banana ogni 4 alunni**
	**3 fragole per ogni alunno**

Quante fragole e banane dovranno usare in tutto?

[*How many strawberries and bananas do they totally need?]*

……

……

……

……

E

Liceo Scientifico Curiel - Padova

1. Quale definizione daresti di solido?
[What may you define as a solid?]

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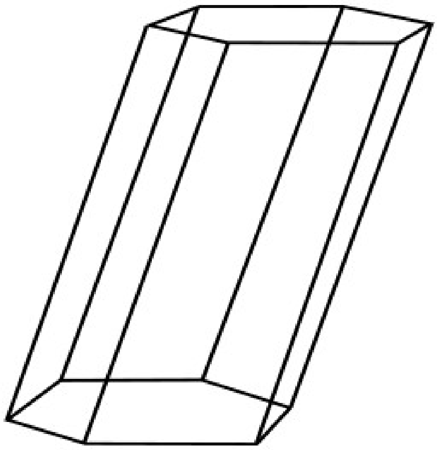
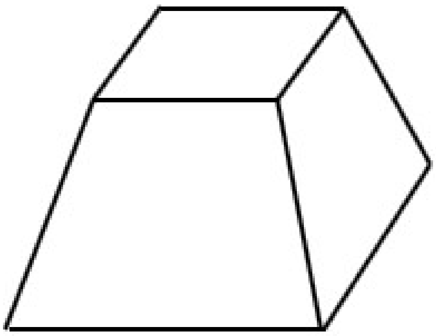
2. Conosci alcuni solidi? Sì No
[Do you know some solid?]

Se sì, disegna i solidi che conosci e scrivine il nome.
[If so, paint solids you know and write their names]

3. Come calcoleresti il volume dei seguenti solidi?

[how do you calculate the volume of the following solids?]





F

Scuola Secondaria – Peschiera del Garda

1. Rispondi alle seguenti domande:
   [Answer to the following questions]

   a) Calcola i $\frac{6}{11}$ del numero 88.
   [perform 6/11 of 88]

   b) Ho speso $\frac{5}{8}$ di 96 euro. Quanti soldi ho ora?
   [I spent 5/8 of 96 euros. How much money do I still have?]

2. Semplifica ai minimi termini le seguenti frazioni:
   [*reduce the following fractions]*

   c) $\frac{33}{11}$ = ……

   a) $\frac{54}{99}$ = ……

   d) $\frac{275}{325}$= ……

   b) $\frac{750}{10}$= ……

3. Senza eseguire operazioni, inserisci le frazioni sulla linea dei numeri:
   [without performing any operations, put the fractions on the numbers line]

$-\frac{1}{3}$; 1 ; $\frac{1}{2}$; $\frac{4}{5}$; $-2; 0{,}5$

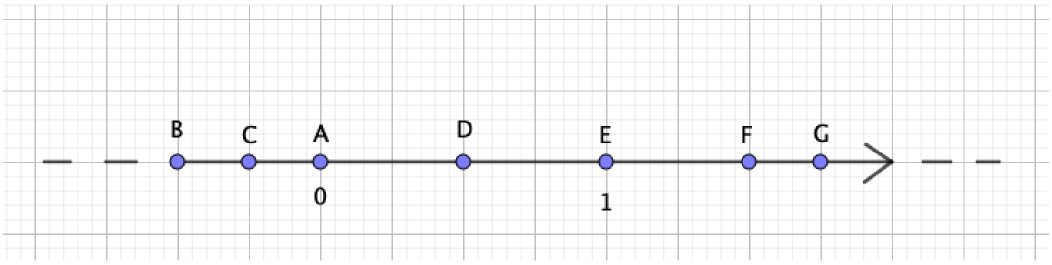
— — • — —
0

4. Risolvi le seguenti operazioni:
   [solve the following expressions]

   a) $\left(\frac{1}{5}-\frac{3}{10}\right)+\left(\frac{4}{3}+\frac{7}{5}\right)-1$

   b) $\left(\frac{2}{4}+\frac{1}{4}\right)\left(\frac{4}{3}-\frac{1}{2}\right)$

5. Ogni lettera sulla linea dei numeri corrisponde ad un numero. Ad esempio, la lettera A corrisponde al numero 0.
   [each letter on the numbers line corresponds to a number. For example, A corresponds to 0]



   a) Quale lettera corrisponde al numero $0{,}5$?
      [which letter corresponds to 0,5?]

   b) Quale lettera corrisponde al numero $\frac{6}{6}$?
      [which letter corresponds to 6/6?]

   c) Quale lettera corrisponde al numero $\frac{-1}{2}$?
      [which letter corresponds to -1/2]

   d) Quale lettera corrisponde al numero $\frac{3}{2}$?
      [*which letter corresponds to 3/2?]*

6. Risolvi il seguente problema spiegando il procedimento seguito.
   [solve the following problem, explaining your strategy (Fig. 54)]

   Un ciclista decide di andare da Padova a Firenze in bicicletta. Le due città distano circa 225 km. Il primo giorno il ciclista percorre i $\frac{5}{9}$ *del viaggio. La mattina del giorno successivo, invece, percorre i* $\frac{3}{4}$ *dei rimanenti chilometri.*

   a) *Quanti chilometri restano da percorrere per arrivare a Firenze?*
   b) *Esprimi il risultato come frazione della distanza tra le due città.*

G

Scuola Secondaria "Felice Chiarle"

Risolvi i seguenti problemi. Ricorda di spiegare il procedimento seguito.

1. Marco deve fare i compiti, posizionando sulla linea dei numeri i seguenti numeri: $4, 0, \frac{2}{4}$. Aiutalo trovando il loro valore e posizionandoli sulla linea dei numeri.

2. Sara ha 125 euro. Una bici costa 135 euro, però è scontata del 20%. Sara riuscirà a comprare la bici?
   [for the translation see Fig. 66]

3. Divido una pizza in 5 fette per i miei amici. A fine giornata mi avanza una fetta. Quante fette avranno mangiato i miei amici?

4. Luisa legge un libro di 350 pagine. Se ha letto i $\frac{4}{5}$ del libro, quante pagine dovrà leggere per finirlo? Se le rimangono tre giorni per finire il libro, quante pagine dovrà leggere al giorno?

5. Valentina è andata a correre in un parco pedonale per sgranchirsi un po' le gambe. Sapendo che il percorso è lungo 10km e che lei ne ha percorsi $\frac{1}{3}$, quanti km ha già percorso?
   Ad un certo punto Valentina si accorge di non avere più le chiavi di casa in tasca. Torna indietro di $\frac{1}{2}$ di quello che già aveva percorso e le ritrova. A quanti km dall'inizio del percorso ricomincia a correre? Quanti km ha percorso fino ad adesso?
   [for the translation see Fig. 68]

6. Mattia va a comprare un telefono della samsung da 249 euro e ha solamente i $\frac{3}{5}$ di quei soldi. Quanti soldi mancano a Mattia?

7. Un cliente compra un cellulare a 549 euro con lo sconto del 50%. Il cliente paga il cellulare a rate. Quante rate deve pagare se ha già pagato $\frac{1}{3}$ del telefono?

8. Michela ha un telefono che ha comprato a 399 euro. Ha visto che in un negozio il suo telefono costava i $\frac{4}{3}$ di quello che l'ha pagato. Se l'avesse comprato in negozio, avrebbe speso più o meno di quanto l'ha pagato? Quanto?

9. Gianluca prende uno smartphone che costa 499 euro. Adesso il telefono ha lo sconto del 200%. Quanti soldi costa adesso il telefono?

10. Giorgia vuole comprare un telefono a rate. Anche Luca, che paga rate da 13 euro. Ogni rata di Giorgia costa i $\frac{3}{9}$ della rata di Luca. Quanto costa una rata di Giorgia? Dopo quanto finisce di pagare il telefono se la rata è settimanale?

H

Scuola Primaria – Peschiera del Garda

1. Scrivi la frazione decimale come numero decimale, e viceversa:
   [write the fraction as a decimal number and viceversa]

   a) $\frac{3}{10}$= 0,3

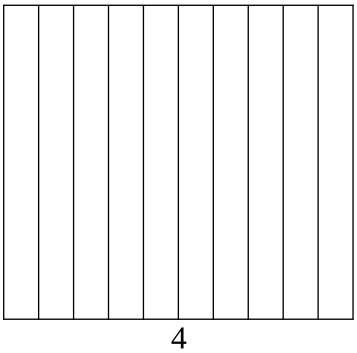
   b) $\frac{34}{100}$= .....

   c) $\frac{\cdots}{\cdots}$= 0,123

   d) $\frac{15}{1000}$= ....

   e) $\frac{\cdots}{\cdots}$= 5,67

2. Colora sulla figura la parte indicata dalla frazione decimale o dal numero decimale:
   [color on the figure the part represented by the fraction or by the decimal number]



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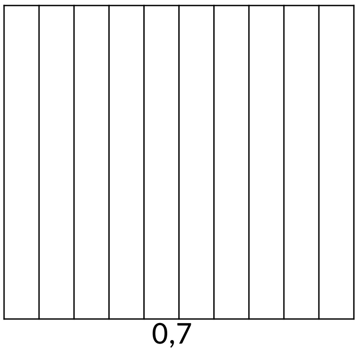
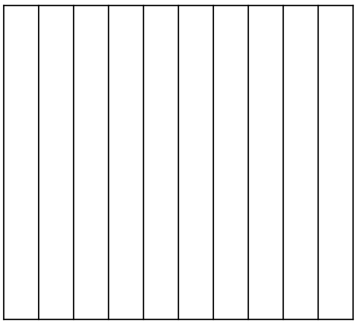
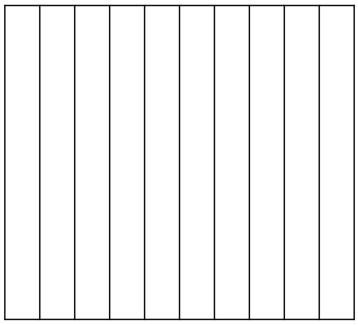


0,7



$\frac{6}{10}$



$\frac{1}{2}$

3. Completa la tabella come nell'esempio:
[fill the table as in the example]

A parole	Frazione decimale	Numero decimale
40 centesimi	$\frac{40}{100}$	0,40
3 decimi		
		$\frac{57}{100}$
	$\frac{98}{10}$	
5 millesimi		

4. Completa la tabella come nell’esempio:
   [fill the table as in the example]

Soldi	Valore in euro	Frazione decimale
	0,60 euro	$\frac{60}{100}$
		
		
	2,40 euro	
		$\frac{55}{100}$

5. Confronta i numeri e inseriscili nella linea dei numeri in ordine crescente, come nell’esempio:
   [compare the numbers and put them in the numbers line, as shown in the example]

0,5 ; 0,6 ; 0,46 ; 0,721 ; 0,72

● 0,5 ● 0,6

Esempio: $0{,}5 = \frac{5}{10}$

$0{,}6 = \frac{6}{10}$

$\frac{5}{10} < \frac{6}{10}$

I

Scuola Primaria "Dante Alighieri"

Risolvi i seguenti problemi. Ricorda di spiegare il procedimento.
[solve the following problems, explaining your solving strategy]

1. Sommando 24,4 e 9,8 che numero decimale ottengo? Si può trasformare in frazione? Se sì, qual è la frazione?
   [summing 24,4 and 9,8, which decimal number do I obtain? Can I write it as a fraction? If so, which fraction?]
   ……
   ……
   ……
   ……
   ……
2. In Italia le donne che praticano nuoto sono il 26,1 % e gli uomini il 17,6 %. Qual è la differenza di percentuali tra donne e uomini che fanno nuoto?
   [In Italy, women who swim are the 26,1 % and men the 17,6 %. Which is the difference between women and men who swim?]
   ……
   ……
   ……
   ……
   ……

3. Gli uomini che fanno corsa sono il 16,6 %. Le donne che fanno corsa sono in percentuale $\frac{1}{4}$ in più degli uomini. Quante sono in percentuale le donne che fanno corsa?

[Men who run are the 16,6 %, while the percentage of women ¼ more than the percentage of men. How much are in percentage the women who run?]

……

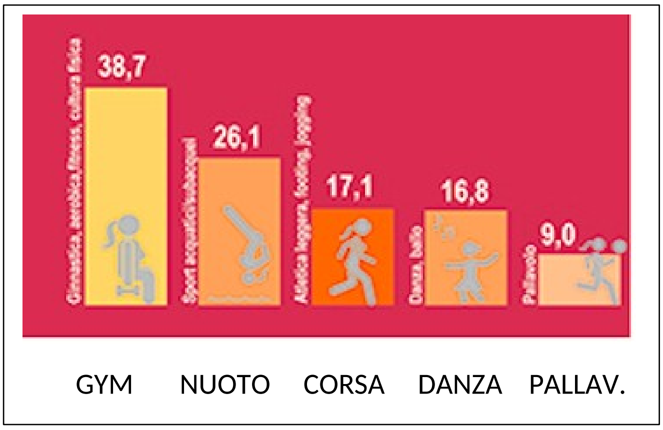
……

……

……

Se guardi la figura, cosa puoi dire sul risultato di prima?

[*If you look at the picture, what can you say about the previous result?]*



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4. Il papà ha comprato una scatola di cioccolatini. Il giorno dopo ne trovati i $\frac{3}{10}$. Quanti ne sono stati mangiati?

   [Dad bought a box of chocolates. The following day Dad finds 3/10 of them. How many chocolates had been eaten?]

   ……

   ……

   ……

   ……

   ……

5. Ad una maratona ci sono 100 partecipanti. I $\frac{51}{100}$ dei partecipanti sono riusciti a finire la maratona. Quanti non la hanno finita?

   [There are 100 participants in a marathon. The 51/100 of the participants completed the marathon. How many did not complete it?]

   ……

   ……

   ……

   ……

……

6. Nella classe di mio fratello ci sono 27 alunni. I $\frac{10}{10}$ degli alunni sono italiani. Tra gli stranieri 2 sono francesi. Quanti stranieri non francesi ci sono?
[In my brothers' classroom there 27 students. 10/10 of them are Italian. Among the foreigners 2 are French. How many non-French foreigners are there?]
……
……
……
……
……

7. Inserisci sulla linea dei numeri i seguenti numeri: $\frac{52}{100}$; $\frac{51}{100}$; 0,5 ; - 0,2 .

[Put in the numbers line the following numbers]

Summary

This thesis finds its roots in the necessity of a paradigmatic change in the didactics of mathematics that aims to build a bridge between reality and school life, as desired by national and international curricula. A valid educational strategy to reduce school and extra-school mathematical competences is that of *mathematical modeling*, seen not only as a process of solving real problems, but as a possibility to achieve a process of mathematization and reflection on mathematics that leads to the construction of new mathematical concepts and tools. The goal of this research is to design a re-invention process (Freudenthal 1991) to integrate mathematical modelling in the regular school practice in the Italian context. The main question of this project is: *How, and to what extent, can mathematical modelling be integrated in the teaching and learning of mathematics in a guided re-invention paradigm?* The underlying theory is the one of Realistic Mathematics Education, wherein learning occurs through experience, the experience of mathematizing experientially real situations, extending day-to-day reasoning to acquire new mathematical knowledge. As a consequence, a central role is be covered by contexts for mathematical problems that must be experientially significant for students and able to evoke new mathematical concepts or strategies for their solution. Therefore, a learning trajectory that brings students to invent their mathematical principles in a modelling environment need to be designed. To design such a re-invention process, our choices are the design heuristics of *didactical phenomenology* and *emergent modelling*, and the use of problem-posing in relation to such heuristics. Our main research question is split in two more specific questions. The first deals with the design of modelling activities with the focus in the promotion of students' reinvention of new mathematical concepts they need to solve a real problem. In the specific, we investigated how activities designed following the principles of

Model Eliciting Activities could foster the emergent nature of modelling. The second research question consists in start investigating the impact in the use of different contexts during problem-posing activities in terms of students' creativity and emergent problems. To be able to answer the research questions we had the necessity to create an instructional environment with which it could be possible to study how and to what extent the suggested processes could be fostered. An instructional sequence was therefore needed to answer to the research questions. A research design that allows for revising theories, hypothesis and instructional activities was needed. Furthermore, new teaching materials that support new types of learning must be developed, making the design process an integrated part of the research. As a consequence, a research approach that consists in planning and creating innovative educational settings and analyzing teaching and learning processes was followed, namely that of *design research*. The structure of the thesis is the following. Starting from the theoretical background and an exploratory study represented by a questionnaire for mathematics teachers, that aimed at investigating teachers' school practices in relations to mathematical modelling and problem-posing, two cycles of design research are reported, respectively for mathematical modelling (M-I and M-II) and problem-posing (P-I and P-II), in order to answer the research questions. Each cycle is made by three main parts: a design phase, a teaching experiment and a retrospective analysis. The final part of the thesis concerns the discussion of the results from the design research cycles, in order to find answers to the research questions and to suggest some recommendations for future research and teaching.