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CICLO XXXI

FUNCTIONS OF BOUNDED VARIATION IN CARNOT-CARATHÉODORY SPACES

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Riassunto

Analizziamo alcune proprietà di funzioni a variazione limitata in spazi di Carnot-Carathéodory. Nel Capitolo 2 dimostriamo che esse sono approssimativamente differenziabili quasi ovunque, esaminiamo il loro insieme di discontinuità approssimata e la decomposizione della loro derivata distribuzionale. Assumendo un'ipotesi addizionale sullo spazio, che chiamiamo proprietà $\mathcal{R}$, mostriamo che quasi tutti i punti di discontinuità approssimata sono di salto e studiamo una formula per la parte di salto della derivata. Nel Capitolo 3 dimostriamo un teorema di rango uno *à la* G. Alberti per la derivata distribuzionale di funzioni vettoriali a variazione limitata in una classe di gruppi di Carnot che contiene tutti i gruppi di Heisenberg $\mathbb{H}^n$ con $n \geq 2$. Uno strumento chiave nella dimostrazione è costituito da alcune proprietà che legano le derivate orizzontali di una funzione a variazione limitata con il suo sottografico. Nel Capitolo 4 dimostriamo un risultato di compattezza per succesioni (u_j) equi-limitate in spazi metrici (X, d_j) quando lo spazio X è fissato ma la metrica può variare con j. Mostriamo inoltre un'applicazione agli spazi di Carnot-Carathéodory. I risultati del Capitolo 4 sono fondamentali per la dimostrazione di alcuni fatti contenuti nel Capitolo 2.

Abstract

We study properties of functions with bounded variation in Carnot-Carathéodory spaces. In Chapter 2 we prove their almost everywhere approximate differentiability and we examine their approximate discontinuity set and the decomposition of their distributional derivatives. Under an additional assumption on the space, called property $\mathcal{R}$, we show that almost all approximate discontinuities are of jump type and we study a representation formula for the jump part of the derivative. In Chapter 3 we prove a rank-one theorem *à la* G. Alberti for the derivatives of vector-valued maps with bounded variation in a class of Carnot groups that includes all Heisenberg groups $\mathbb{H}^n$ with $n \geq 2$. Some important tools for the proof are properties linking the horizontal derivatives of a real-valued function with bounded variation to its subgraph. In Chapter 4 we prove a compactness result for bounded sequences (u_j) of functions with bounded variation in metric spaces (X, d_j) where the space X is fixed, but the metric may vary with j. We also provide an application to Carnot-Carathéodory spaces. The results of Chapter 4 are fundamental for the proofs of some facts of Chapter 2.

Contents

Introduction

Functions of bounded variation (BV functions) play an important role in several problems of Calculus of Variation like minimal area problems and free discontinuity problems and, since their notion is closely linked to finite perimeter and rectifiable sets, they also come into use in Geometric Measure Theory. In the classical Euclidean setting, the structure of functions of bounded variation has been intensively studied. In [69, 90] BV functions have been introduced as a natural generalization of Sobolev maps while in [32] one can find BV Theory as a special case of the more recent Theory of Currents. Some properties of the distributional derivative of a BV function are described in [10, 31, 43, 44, 68, 73, 91] while the important Rank-One Theorem in the Euclidean case is proved in [1]. Most of the results about structure properties of BV functions in the Euclidean case are collected in an organic way in the book [5].

The extension of the Euclidean BV Theory to metric spaces is however much more recent. One of the milestones of Analysis on metric measure spaces is certainly [46], where Sobolev and BV functions are deeply studied and where the authors show how the validity of Poincaré-type inequalities and a doubling property of the reference measure are enough to prove fundamental results like Sobolev inequalities, Sobolev embeddings, Trudinger's inequality. The notion of BV function in metric measure spaces has been then developed in different environments like weighted Euclidean spaces (see [11]), Finsler structures (see [15]), the so-called good metric measure spaces (see [70]) and Carnot-Carathéodory spaces (see [17, 20, 23, 34, 38, 39, 36, 41] and the more recent [8, 6, 9, 19, 22, 28, 59, 66, 85]).

Carnot-Carathéodory spaces (CC spaces for short) represent one of the setting where BV functions have been most fruitfully introduced. CC spaces first naturally appeared in the Theory of hypo-elliptic operators, degenerate elliptic operators and singular integrals (see e.g. [49, 87], as well as many others) and only later on they have been object of studies from a Geometric Measure Theory point of view. The class of CC spaces is general enough to include the Euclidean spaces (as a trivial case) and all Carnot groups (or stratified groups). A Sobolev Theory in CC spaces has been systematically worked out in the literature while only partial results are known for the structure of BV functions in this setting, so far. We however point out the validity of very important results concerning BV functions in CC spaces like approximation of BV functions via

smooth maps (see [36]), coarea formulae (see [77, 36]), Sobolev-Poincaré inequalities (see [41, 35, 46]) and Isoperimetric inequalities (see [34, 41, 79]). Some of the most notable difficulties in developing analysis in this framework are the lack of a Besicovitch derivation Theorem (see e.g. [52]) and the non-existence of a group operation or a family of dilations that are compatible with the metric structure.
The goal of this Thesis is twofold. First, we extend some of the so-called *fine properties* of BV functions, that are well established in Euclidean spaces, in a setting of CC spaces (we refer the reader to [5] for a deep introduction to the Euclidean case). On a parallel line we prove a Rank-One Theorem for BV functions in a class of Carnot groups that includes all Heisenberg groups $\mathbb{H}^n$ with $n \geq 2$.
Let us now fix some notation about CC spaces. Consider an m-tuple $X = (X_1, \dots, X_m)$ of linearly independent and smooth vector fields in $\mathbb{R}^n$ satisfying Hörmander condition (named after [49]), i.e., the linear span of $X_1, \dots, X_m$ together with all their commutators computed at any point p is the whole $\mathbb{R}^n$. In this case (see [21]) for any $p, q \in \mathbb{R}^n$ there exists an *X-subunit path* γ joining them, i.e., an absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ so that, for almost every $t \in [0, T]$, one has

$$\gamma(0) = p, \quad \gamma(T) = q \quad \text{and} \quad \dot{\gamma}(t) = \sum_{i=1}^{m} h_i(t) X_i(\gamma(t)),$$

for some $h = (h_1, \dots, h_m) \in L^\infty([0, T]; \mathbb{R}^m)$ with $\|h\|_\infty \leq 1$. The map $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$d(p, q) := \inf\{T > 0 : \exists \text{ an } X\text{-subunit } \gamma : [0, T] \to \mathbb{R}^n \text{ joining } p \text{ and } q\},$$

is then a distance called *Carnot-Carathéodory distance* and the metric space $(\mathbb{R}^n, d)$ (equivalently denoted by $(\mathbb{R}^n, X)$) is said to be a *Carnot-Carathéodory space*. The metrics d and the Euclidean metric $d_e = |\cdot - \cdot|$ give the same topology but they are not metrically equivalent (see [78]). As customary in the literature, we will also assume that metric balls are bounded with respect to the Euclidean topology. Denoting by $\mathcal{L}^i(p)$ the linear span of all the commutators of $X_1, \dots, X_m$ up to order i computed at $p \in \mathbb{R}^n$, we will also assume that the dimension of $\mathcal{L}^i(p)$ is constant and equals to some integer n_i. In this case, the minimum $s \in \mathbb{N}$ such that $\mathcal{L}^s(p) = \mathbb{R}^n$ is called step of the CC space and $(\mathbb{R}^n, X)$ is said to be an *equiregular* CC space of *step s*. Equiregularity assumption will be fundamental for our purposes since by [71, 50] the Hausdorff dimension of the metric space $(\mathbb{R}^n, d)$ is given by the so-called *homogeneous dimension* $Q = \sum_{i=1}^{s} i(n_i - n_{i-1})$ and the metric measure space $(\mathbb{R}^n, d, \mathscr{L}^n)$ (where $\mathscr{L}^n$ denotes the n-dimensional Lebesgue measure) is locally Ahlfors Q-regular (see Theorem 1.2.4), i.e., for every compact set $K \subseteq \mathbb{R}^n$ there exist $C \geq 1$ and $R > 0$ such that

$$\frac{1}{C} r^Q \leq \mathscr{L}^n(B(p, r) \leq C r^Q,$$

for every $p \in K$ and for every $r \in (0, R)$. Notice that, by e.g. [82], any CC space can be lifted to an equiregular one. Despite equiregular CC spaces are not (even locally) bi-Lipschitz equivalent to any Euclidean space, a blow-up technique can still be fruitful in this framework. Indeed the metric tangent, in the Gromov-Hausdorff sense (see [45]), of an equiregular CC space at any point is a Carnot group with the same step. This fact is a consequence of the papers [13, 14, 76] and it will be heavily used throughout the Thesis. We recall that Carnot groups are connected, simply connected and nilpotent Lie groups whose Lie algebra is stratified, and we refer to [33, 75, 59, 55] for more detailed introduction to the subject. Section 1.3 below contains a brief introduction to Carnot groups.

Functions of bounded X-variation have been introduced in [41, 36]. Given an open set Ω in a CC space $(\mathbb{R}^n, X)$ and $u \in L^1(\Omega)$, we say that u has bounded X-variation ($u \in BV_X(\Omega)$) if the distributional derivative $D_X u := (D_{X_1} u, \dots, D_{X_m} u)$ is (represented) by a vector-valued Radon measure with finite total variation, i.e., for any $\varphi \in C^1_c(\Omega)$, and for any $i = 1, \dots, m$ one has

$$\int_\Omega u X_i^* \varphi \, d\mathscr{L}^n = -\int_\Omega \varphi \, d(D_{X_i} u),$$

and $|D_X u|(\Omega) < +\infty$. A measurable set E is said to have finite X-perimeter in Ω if $\chi_E \in BV_X(\Omega)$. A first goal we have in mind is to study some structural properties of the measure derivative $D_X u$, taking especially into account the decomposition $D_X u = D^a_X u + D^s_X u$ into the absolutely continuous part $D^a_X u$ and the singular part $D^s_X u$ with respect to the Lebesgue measure $\mathscr{L}^n$. To this end, as suggested by the classical theory of BV functions (see [5]), one first needs to classify, roughly speaking, the type of singularity (or regularity) that a function might have. More precisely, one needs a consistent theory that includes the notions of jump point and differentiability point in an approximate sense. This will be done in Section 2.1. Section 2.2 is then devoted to the proof of the main results about BV_X function in all equiregular CC spaces satisfying the following geometric property that we call $\mathcal{R}$: all sets of finite X-perimeter have rectifiable essential boundary. The validity of this property is crucial, non-technical and also natural since it is known to hold in all Euclidean spaces, in all Carnot groups of step 2 and in all Carnot groups of type $\star$. The importance of property $\mathcal{R}$ will be discussed into details later on, together with the definition of rectifiability. Some of the main results about fine properties of BV functions presented in Chapter 2 need some fine blow-up analysis about intrinsic regular hypersurfaces (see Section 1.5). Chapter 2 and Section 1.5 are mostly new and contained in the work of the author and his supervisor Davide Vittone [30].

Part of the analysis of singular points for BV_X functions requires some blow-up technique together with the nilpotent approximation of a CC space. Chapter 4 contains a technical but fundamental lemma (contained in [29]) that ensure compactness of equi-

bounded sequences (u_j) in BV_{X^j}, for converging smooth vector fields X^j.
The content of Chapter 3 is contained in [28] and it is devoted to the proof of a Rank-One theorem for BV functions in all Carnot groups satisfying a slightly weaker version of property $\mathcal{R}$, called w-$\mathcal{R}$, and a codimension-2 "complementability" property $\mathscr{C}_2$. The classical Rank-One Theorem, whose proof is contained in [1], states that, if u is a $\mathbb{R}^k$-valued BV function in an open set Ω and $D^s u$ is its singular part of Du with respect to $\mathscr{L}^n$, then the polar decomposition matrix $D^s u/|D^s u| : \Omega \to \mathbb{R}^{k\times n}$ has rank one $|D^s u|$-almost everywhere.

Let us analyze and discuss the content of the chapters into details.

Chapter 1 contains introductory content that will be useful in the proofs of the main results of the following chapters of this Thesis. Section 1.1 contains some covering lemmata that can be applied to CC spaces, some well-known facts of Measure Theory, a decomposition criterion for measures in product metric spaces and the definition of Hausdorff measures and pointwise densities of measures. Section 1.2 contains the definition of equiregular CC spaces, their main metric and topological properties (see Theorem 1.2.4). Subsection 1.2.1 contains a proof of Chow's Theorem (see Theorem 1.2.1). Section 1.3 includes the definition of Carnot groups, well-known facts about their structure and some examples like Heisenberg groups and the Engel group. Section 1.4 describes the tangent structure, in the Gromov-Hausdorff sense, of a CC space (see Theorem 1.4.5). Section 1.5 contains the notion of intrinsic Lipschitz and intrinsic regular hypersurfaces in the context of CC spaces. Some results of this section are due to the author and to his PhD supervisor Davide Vittone and they are contained in [30]. It is worth to mention that, by the important paper [51] we know that, already in Carnot groups, there are examples of intrinsic C^1 hypersurfaces that are (from the Euclidean point of view) fractals. However, we are able to prove some blow-up properties of such hypersurfaces in equiregular CC spaces (see Proposition 1.5.3 and Corollary 1.5.4), and to give an estimate of the Hausdorff dimension of the "transversal subset" of the intersection of two hypersurfaces (see Theorem 1.5.6). In Section 1.6 we give the definition of functions of bounded X-variation together with a list of known properties of BV_X functions in CC space: approximation by smooth maps (Theorem 1.6.3), Coarea formula (Theorem 1.6.6), Poincaré inequality (Theorem 1.6.7, see also [20, Theorem 1.2]) and Isoperimetric inequality (Theorem 1.6.8).

The aim of Chapter 2 is to establish "fine" properties of BV functions in CC spaces. A first non-trivial part of this Chapter consists in fixing the appropriate language in a consistent and robust manner. Section 2.1 is therefore devoted to the introduction of approximate notions of continuity, jump point and differentiability point for generic L^1_{loc} maps in CC spaces. The notion of approximate continuity has been already worked out in the literature (see e.g. [48, Section 2.7]) by the extension of the Lebesgue Theorem to the more general context of doubling, locally compact and separable metric measure spaces (here reported by Theorem 2.1.2). However, the definition of approximate jump triple and approximate differentiability in CC spaces (introduced in Definitions 2.1.6 and 2.1.12) are new and require some precise analysis. In the classical theory the jump set of a L^1 function u is, roughly speaking, the set of points p for which there exist $u^+(p) \neq u^-(p)$ and a unit direction $\nu_u(p)$ such that, for small $r > 0$, u is approximately equal to $u^+(p)$ on half of $B(p,r)$ and to $u^-(p)$ on the complementary half of $B(p,r)$, the two halves being separated by an hyperplane orthogonal to $\nu_u(p)$. In CC spaces this requires a certain amount of work, since there is no "linear" way to divide a ball into two "half-balls". We have to replace the notion of hyperplane orthogonal to a direction $\nu(p)$ with an equivalence class of intrinsic C^1 hypersurfaces sharing the same normal at p. To this end the local properties of intrinsic C^1 hypersurfaces proved in Section 1.5 will be of capital importance. Similarly, the classical notion of approximate differential of a L^1 map u at a point p is a linear map that, at small scales, is "almost" the incremental ratio associated with u at p. In order to define the approximate differentiability in CC spaces, we again replace the linear map with a germ of intrinsic regular hypersurfaces. Most of the results in Section 2.1 deal with well-posedness of the definitions and with Borel regularity of X-jump sets, X-differentiability sets, X-jump map $p \mapsto (u^+(p), u^-(p), \nu_u(p))$ and approximate X-gradient.

Section 2.2 contains the main results about "fine" properties of BV functions in CC spaces. An important result that holds without further assumption on the space is Theorem 1 below, and it concerns the almost everywhere approximate differentiability of BV_X functions; its classical counterpart is very well-known, see e.g. [5, Theorem 3.83].

Theorem 1. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in BV_X(\Omega; \mathbb{R}^k)$. Then u is approximately X-differentiable at $\mathscr{L}^n$-almost every point of Ω. Moreover, the approximate X-gradient of u coincides $\mathscr{L}^n$-almost everywhere with the density of $D^a_X u$ with respect to $\mathscr{L}^n$.*

The proof of Theorem 1 is based on Lemma 2.2.6, i.e., on a suitable extension to CC spaces of the inequality

$$\int_{B(p,r)} \frac{|u(q) - u(p)|}{|q - p|} \, d\mathscr{L}^n(q) \leq C \int_0^1 \frac{|Du|(B(p,tr))}{t^n} \, dt$$

valid for a classical BV function u on $\mathbb{R}^n$. Lemma 2.2.6 answers an open problem stated in [8] and it is new even in *Carnot groups*. Theorem 1 was proved in the setting of Carnot groups in [8] together with the following result, which we also extend to our more general setting. We denote by $\mathscr{H}^{Q-1}$ the Hausdorff measure of dimension $Q - 1$ and by $\mathcal{S}_u$ the set of points where a function u does not possess an approximate limit in the sense of Definition 2.1.1.

Theorem 2. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in BV_X(\Omega; \mathbb{R}^k)$. Then $\mathcal{S}_u$ is contained in a countable union of sets with finite $\mathscr{H}^{Q-1}$ measure.*

We denote by $\mathcal{J}_u \subseteq \mathcal{S}_u$ the set of X-jump points of u and by $(u^+(p), u^-(p), \nu_u(p))$ the approximate X-jump triple (see Definition 2.1.6) at a point $p \in \mathcal{J}_u$. The measures

$$D^j_X u := D^s_X u \llcorner \mathcal{J}_u, \qquad D^c_X u := D^s_X u \llcorner (\Omega \setminus \mathcal{J}_u),$$

are called, respectively, *jump part* and *Cantor part* of $D_X u$. We want to study some further properties of $D_X u$ and its decomposition

$$D_X u = D^a_X u + D^s_X u = D^a_X u + D^c_X u + D^j_X u.$$

We state some of them in the following result, which is a consequence of Theorems 2.2.20 and 2.2.4. We denote by $\mathscr{S}^{Q-1}$ the spherical Hausdorff measure of dimension $Q - 1$.

Theorem 3. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and consider an open set $\Omega \subseteq \mathbb{R}^n$, a function $u \in BV_X(\Omega; \mathbb{R}^k)$ and a Borel set $B \subseteq \Omega$. Then the following facts hold:*

(i) there exists $\lambda : \mathbb{R}^n \to (0, +\infty)$ (not depending on Ω nor u) locally bounded away from 0 such that $|D_X u| \geq \lambda |u^+ - u^-| \mathscr{S}^{Q-1} \llcorner \mathcal{J}_u$.

(ii) if $\mathscr{H}^{Q-1}(B) = 0$, then $|D_X u|(B) = 0$.

(iii) if $\mathscr{H}^{Q-1}(B) < +\infty$ and $B \cap \mathcal{S}_u = \emptyset$, then $|D_X u|(B) = 0$.

(iv) $D^a_X u = D_X u \llcorner (\Omega \setminus S)$ and $D^s_X u = D_X u \llcorner S$, where

$$S := \left\{ p \in \Omega : \lim_{r \to 0} \frac{|D_X u|(B(p,r))}{r^Q} = +\infty \right\}.$$

(v) $\mathcal{J}_u \subseteq \Theta_u$, where $\Theta_u \subseteq S$ is defined by

$$\Theta_u := \left\{ p \in \Omega : \liminf_{r \to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} > 0 \right\}.$$

However, for classical BV functions much stronger results than Theorems 1 and 3 are indeed known: some of them are proved in Section 2.2 for BV_X functions under the additional assumption that the space $(\mathbb{R}^n, X)$ satisfies the following condition.

Definition 1 (Property $\mathcal{R}$)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space with homogeneous dimension Q. We say that $(\mathbb{R}^n, X)$ satisfies the *property* $\mathcal{R}$ if, for every open set $\Omega \subseteq \mathbb{R}^n$ and every $E \subseteq \mathbb{R}^n$ with locally finite X-perimeter in Ω, the essential boundary $\partial^* E \cap \Omega$ of E in Ω is countably X-rectifiable, i.e., there exists a countable family $\{S_i : i \in \mathbb{N}\}$ of C^1_X hypersurfaces such that $\mathscr{H}^{Q-1}(\partial^* E \cap \Omega \setminus \bigcup_{i=0}^{\infty} S_i) = 0$.

We refer to Definition 1.1.21 for the essential boundary $\partial^* E$. It was proved in the fundamental paper [3] that the X-perimeter measure $|D_X\chi_E|$ of E can be represented as $\theta\mathscr{H}^{Q-1}\llcorner\partial^* E$ for a suitable positive function θ that is locally bounded away from 0, see Theorem 2.2.3.

The validity of property $\mathcal{R}$ ("rectifiability") in general equiregular CC spaces is an interesting open question even in Carnot groups (see [7] for a partial result). However, property $\mathcal{R}$ is satisfied in several interesting situations like Heisenberg groups [38], Carnot groups of step 2 [39] and Carnot groups of *type* $\star$ [66]: in particular, Theorems 4, 5 and 6 below hold is such classes. We conjecture that property $\mathcal{R}$ holds also in all CC spaces of step 2, see [6]. Building on the results of [27], we prove in Section 2.2.1 the validity of the weaker *property* $\mathcal{LR}$ ("Lipschitz rectifiability", see Definition 2.2.13) in all Carnot groups satisfying property (2.34) below; in particular, a weaker version of Theorem 4 holds in such groups (see Theorem 2.2.15).

The first result we are able to prove assuming property $\mathcal{R}$ is a refinement of Theorem 2 and, roughly speaking, it states that $\mathscr{H}^{Q-1}$-almost all singularities of a BV_X function are of jump type.

Theorem 4. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying property $\mathcal{R}$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in BV_X(\Omega;\mathbb{R}^k)$. Then $\mathcal{S}_u$ is countably X-rectifiable and $\mathscr{H}^{Q-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$.*

Assuming property $\mathcal{R}$, Theorem 3 can be refined as follows.

Theorem 5. *Under the assumption and notation of Theorem 3, assume that $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$. Then*

(i) $\mathscr{H}^{Q-1}(\Theta_u \setminus \mathcal{J}_u) = 0$ *and* $D^j_X u = D_X u\llcorner\Theta_u$;

(ii) $D^c_X u = D_X u\llcorner(S \setminus \Theta_u)$;

(iii) if $B \subseteq \Omega$ is such that $\mathscr{H}^{Q-1}\llcorner B$ is σ-finite, then $D^c_X u(B) = D^a_X u(B) = 0$.

Theorem 5 is part of Theorem 2.2.20. We also mention that, assuming property $\mathcal{R}$, one can define a *precise representative* $u^{\mathfrak{p}}$ of u (see (2.30)) and prove that the convergence of the mean values $\fint_{B(p,r)} u\, d\mathscr{L}^n$ to $u^{\mathfrak{p}}(p)$ holds for $\mathscr{H}^{Q-1}$-almost every p. See Theorem 2.2.18.

Eventually, a further natural assumption – *property* $\mathcal{D}$ ("density", see Definition 2.2.21) – concerning the local behavior of the spherical Hausdorff measure $\mathscr{S}^{Q-1}$ of C^1_X hypersurfaces allows to obtain a stronger result, Theorem 6, about the jump part $D^j_X u$. Property $\mathcal{D}$ is satisfied in Heisenberg groups, Carnot groups of step 2 and Carnot groups of type $\star$, see Subsection 2.2.1; its validity in more general settings is an interesting open problem that will be object of future investigations. Theorem 6 follows from the more general Theorem 2.2.23, which deals with a representation of the restriction of $D_X u$ to any countably X-rectifiable set R.

Theorem 6. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying properties $\mathcal{R}$ and $\mathcal{D}$; then, there exists a function $\sigma : \mathbb{R}^n \times \mathbb{S}^{m-1} \to (0, +\infty)$ such that, for every open set $\Omega \subseteq \mathbb{R}^n$ and every $u \in BV_X(\Omega; \mathbb{R}^k)$, one has*

$$D_X^j u = \sigma(\cdot, \nu_u)(u^+ - u^-) \otimes \nu_u \, \mathscr{S}^{Q-1} \llcorner \mathcal{J}_u.$$

Chapter 3 is devoted to the proof of the Rank-One Theorem in a class of Carnot groups. Its content comes from the paper [28] and it is due to the author, Annalisa Massaccesi and Davide Vittone. The Rank-one theorem represents one the most difficult results in the theory of functions with bounded variation. It states that the Radon-Nikodým derivative $\frac{D^s u}{|D^s u|}$ of $D^s u$ with respect to its total variation $|D^s u|$, which is a $|D^s u|$-measurable map from Ω to $\mathbb{R}^{d\times n}$, takes values in the space of rank-one matrices $|D^s u|$-almost everywhere in Ω.
The Rank-One Theorem was first conjectured by L. Ambrosio and E. De Giorgi in [4] and it has important applications to vectorial variational problems and systems of PDEs. It was proved by G. Alberti in [1] (see also [2, 25]): due to its complexity, Alberti's proof is generally regarded as a tour-de-force in measure theory. Two different proofs of the Rank-One Theorem were recently found. One is due to G. De Philippis and F. Rindler and it follows from a profound PDE result [26], where a rank-one property for maps with Bounded Deformation was also proved for the first time. At the same time another proof, of a geometric flavor and considerably simpler than those in [1, 26], was provided by Annalisa Massaccesi and Davide Vittone in [67].
Motivated by these results, in this chapter we consider the following natural generalization. If $\mathbb{G}$ is a Carnot group of rank m, we say that $u \in BV_{\mathbb{G}}(\Omega; \mathbb{R}^k)$ for an open set $\Omega \subseteq \mathbb{G}$, if $u \in BV_X(\Omega; \mathbb{R}^k)$ for any basis $X = (X_1, \ldots, X_m)$ of $\mathfrak{g}_1$. Upon passing in exponential coordinates, one can identify $\mathbb{G} = \mathbb{R}^n$. Consider the singular part $D^s_{\mathbb{G}} u$ of $D_{\mathbb{G}} u$ with respect to the Haar measure of the group that we can assume is $\mathscr{L}^n$. Is it true that the Radon-Nikodým derivative $\frac{D^s_{\mathbb{G}} u}{|D^s_{\mathbb{G}} u|}$ is a rank-one matrix $|D^s_{\mathbb{G}} u|$-almost everywhere?
We find two assumptions on $\mathbb{G}$, that we call properties $\mathscr{C}_2$ and w-$\mathcal{R}$ (see Definitions 3.1.3 and 3.4.1), that ensure the rank-one property for $BV_{\mathbb{G}}$ functions in $\mathbb{G}$. We will discuss later the role played by these properties in our argument. Our first main result is the following

Theorem 7. *Let $\mathbb{G}$ be a Carnot group satisfying properties $\mathscr{C}_2$ and w-$\mathcal{R}$; let $\Omega \subseteq \mathbb{G}$ be an open set and $u \in BV_{\mathbb{G},loc}(\Omega; \mathbb{R}^k)$. Then the singular part $D^s_{\mathbb{G}} u$ of $D_{\mathbb{G}} u$ is a rank-one measure, i.e., the matrix-valued function $\frac{D^s_{\mathbb{G}} u}{|D^s_{\mathbb{G}} u|}(x)$ has rank one for $|D^s_{\mathbb{G}} u|$-a.e. $x \in \Omega$.*

It is worth pointing out that Theorem 7 applies to the n-th *Heisenberg group* $\mathbb{H}^n$, provided $n \geq 2$. Heisenberg groups are defined in Example 1.3.24 and they represent some of the most simple non-trivial examples of Carnot groups. Notice also that property w-$\mathcal{R}$ is slightly weaker than property $\mathcal{R}$ used in Chapter 2. We however conjecture that property $\mathcal{R}$ and property w-$\mathcal{R}$ are indeed equivalent.

Corollary 1. *Let u be as in Theorem 7 and assume that $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$, $n \geq 2$; then, $D^s_{\mathbb{G}}u$ is a rank-one measure. More generally, the same holds if $\mathbb{G}$ is a Carnot group of step 2 satisfying property $\mathscr{C}_2$.*

Corollary 1 is an immediate consequence of Theorem 7, see Remarks 3.1.5 and 3.4.3. This basically follows from the fact that Heisenberg groups $\mathbb{H}^n$ satisfy property $\mathscr{C}_2$ if and only if $n \geq 2$ and that by [39], all step 2 Carnot groups satisfy property $\mathcal{R}$ and in particular property w-$\mathcal{R}$.
Theorem 7 does not directly follow from the outcomes of [26], see Remark 3.4.6. Its proof follows the geometric strategy devised in [67] and it is based on the relations between a (real-valued) $BV_{\mathbb{G}}$ function u in $\mathbb{G}$ and the $\mathbb{G} \times \mathbb{R}$-*perimeter* of its subgraph $E_u := \{(x,t) : t < u(x)\} \subseteq \mathbb{G} \times \mathbb{R}$. This relations can be summarized in our second main result of this Chapter.

Theorem 8. *Suppose that $\Omega \subseteq \mathbb{G}$ is open and bounded and let $u \in L^1(\Omega)$. Then u belongs to $BV_{\mathbb{G}}(\Omega)$ if and only if its subgraph E_u has finite $\mathbb{G} \times \mathbb{R}$-perimeter in $\Omega \times \mathbb{R}$.*

Actually, the proof of Theorem 7 requires much finer properties than the one stated in Theorem 8. Such properties are stated in Theorems 3.3.1 and 3.3.2 in the more general context of CC spaces. Theorem 3.3.1, from which Theorem 8 follows in a stroke, focuses on the relations between the horizontal (in $\mathbb{R}^n$) derivatives of u and the horizontal (in $\mathbb{R}^n \times \mathbb{R}$) derivatives of χ_{E_u}. Theorem 3.3.2 instead deals with the relations between the *horizontal normal* to E_u and the *polar vector* σ_u in the decomposition $D_{\mathbb{G}}u = \sigma_u |D_{\mathbb{G}}u|$, and it also deals with the relations between $D^a_{\mathbb{G}}u, D^s_{\mathbb{G}}u$ and the horizontal derivatives of χ_{E_u}. When $m = n$ and $X_i = \partial_{x_i}$ one recovers some results that belong to the folklore of Geometric Measure Theory and are scattered in the literature (see e.g. [73], [32, 4.5.9] and [43, Section 4.1.5]). We tried to collect them in a more systematic way in Section 3.3.

Property w-$\mathcal{R}$ ("weak rectifiability") intervenes in ensuring that the horizontal derivatives of χ_{E_u} are a "rectifiable" measure. A Carnot group $\mathbb{G}$ satisfies Property w-$\mathcal{R}$ (see Definition 3.4.1) if, for any open set $\Omega \subseteq \mathbb{G}$ and any $u \in BV_{\mathbb{G}}(\Omega)$, one has that the essential boundary $\partial^* E_u$ of its subgraph E_u is $\mathbb{G} \times \mathbb{R}$-rectifiable and the normal to the rectifiable set $\partial^* E_u$ coincides $\mathscr{H}^Q$-almost everywhere with the measure-theoretic horizontal normal to E_u. As already pointed out, by Theorem 8, property w-$\mathcal{R}$ is weaker than property $\mathcal{R}$ but we conjecture they are actually equivalent. Property w-$\mathcal{R}$ is a non-trivial technical obstruction one has to face when following the strategy of [67]: the rectifiability of sets with finite $\mathbb{G}$-perimeter in Carnot groups is indeed a major open problem, which has been solved only in step 2 Carnot groups (see [38, 39]) and in the class of Carnot groups *of type* $\star$ ([66]). See also [7] for a partial result in general Carnot groups.

Once the rectifiability of $\partial^* E_u$ is ensured, the proof of Theorem 7 follows rather easily from the technical Lemma 3.2.7, which is the natural counterpart of the Lemma in [67]. The latter, however, was proved by utilizing the area formula for maps between rectifiable subsets of $\mathbb{R}^n$, see e.g. [5]. A similar tool is not available in the context of Carnot groups, and this fact forces us to follow a different path. The proof of Lemma 3.2.7 is indeed achieved by a covering argument that is based on the following result.

Theorem 9. *Let $k \geq 1$ be an integer, $\mathbb{G}$ a Carnot group satisfying property $\mathscr{C}_k$ and let $\Sigma_1, \dots, \Sigma_k$ be hypersurfaces of class $C^1_{\mathbb{G}}$ with horizontal normals $\nu_1, \dots, \nu_k$. Let also $p \in \Sigma := \Sigma_1 \cap \dots \cap \Sigma_k$ be such that $\nu_1(p), \dots, \nu_k(p)$ are linearly independent. Then, there exists an open neighborhood U of p such that*

$$0 < \mathscr{H}^{Q-k}(\Sigma \cap U) < \infty.$$

In particular, the measure $\mathscr{H}^{Q-k}$ is σ-finite on the set

$$\Sigma^{\pitchfork} := \{x \in \Sigma : \nu_1(x), \dots, \nu_k(x) \text{ are linearly independent}\}.$$

By $C^1_{\mathbb{G}}$ maps (see Subsection 3.2) we mean continuous functions f for which the distributional derivative Yf is represented by a continuous function, for any $Y \in \mathfrak{g}_1$. Theorem 9 is a consequence of Theorems 3.2.3 and 3.2.5 proved, respectively, in [40] and [62]. These Theorems are collected here in Subsection 3.2.2, together with some introduction to intrinsic Lipschitz graphs. Theorem 3.2.5, in particular, states the much deeper property that the set $\Sigma^{\pitchfork}$ is locally an *intrinsic Lipschitz graph*. To this aim, one needs the intersection $T_p\Sigma_1 \cap \dots \cap T_p\Sigma_k$ of the *tangent subgroups* to Σ_i at p to admit a (necessarily commutative) complementary homogeneous subgroup that is horizontal, i.e., contained in $\exp(\mathfrak{g}_1)$. This algebraic property is guaranteed by property $\mathscr{C}_k$ ("k-codimensional complementability"), see Remark 3.1.4. We will provide a proof of Theorem 3.2.5 which does not rely on the homotopy invariance of the topological degree and is then simpler and shorter than the one in [62].
Property $\mathscr{C}_k$ might seem a restrictive one for the validity of Theorem 9. We however point out that the latter is no longer valid already when $k = 2$ and $\mathbb{G}$ is the first Heisenberg group $\mathbb{H}^1$, which does not satisfy $\mathscr{C}_2$: indeed, in this setting the measure $\mathscr{H}^{Q-2}(\Sigma^{\pitchfork})$ might be either 0 or $+\infty$ (even locally) as shown by A. Kozhevnikov [53]. See also the recent paper [63].
The fact that Theorem 9 does not apply to $\mathbb{H}^1$ (actually, to $\mathbb{H}^1 \times \mathbb{R} \times \mathbb{R}$, see the proof of Lemma 3.2.7) prevents us from proving the Rank-One Theorem for $\mathbb{G} = \mathbb{H}^1$. This does not follow from [26] either (see Remark 3.4.7) and, thus, it remains a very interesting open problem.

Chapter 4 deals with technical result about compactness for BV functions in a class of metric measure spaces. The contents of this Chapter are contained in [29] and they are due to the author and Davide Vittone. One of the milestones in the theory of functions with bounded variation is the following Rellich-Kondrachov-type theorem: given a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz regular boundary, the space $BV(\Omega)$ of functions with bounded variation in Ω compactly embeds in $L^q(\Omega)$ for any $q \in [1, \frac{n}{n-1})$. One notable consequence is the following property: if (u_j) is a sequence in $BV_{loc}(\mathbb{R}^n)$ that is locally uniformly bounded in BV, then for any $q \in [1, \frac{n}{n-1})$ there exists a subsequence (u_{j_h}) of (u_j) that converges in $L^q_{loc}(\mathbb{R}^n)$. A Rellich-Kondrachov-type result in metric measure spaces is given in [46, Theorem 8.1]: if a sequence (u_j) is bounded in some $W^{1,p}$, then a subsequence converges in some L^q.
In this chapter we study similar compactness properties for sequences (u_j) of locally uniformly bounded BV functions in metric measure spaces (M, λ, d_j) where the underlying measure space (M, λ) is fixed but the metric d_j varies with j. In our main result we prove that, if d_j converges locally uniformly to some distance d on M such that (M, λ, d) is a (locally) doubling separable metric measure space, and if the functions $u_j : X \to \mathbb{R}$ are locally uniformly (in j) bounded with respect to a BV-type norm in (M, d_j) and satisfy some local Poincaré inequality (with constant independent of j), then a subsequence of u_j converges in some $L^q_{loc}(M, \lambda)$. See Theorem 4.1.1 for a precise statement. To our knowledge, the strategy we adopt to prove Theorem 4.1.1 is novel even when the metric on M is not varying (i.e., when $d_j = d$ for any j); in particular, we are able to provide a different proof of the case $p = 1$ in [46, Theorem 8.1] for separable metric spaces.
The motivation that led us to Theorem 4.1.1 is given in Chapter 2 from an application to the study of BV functions in CC spaces. In Theorem 4.2.6 we indeed prove that, if $X^j = (X^j_1, \dots, X^j_m)$ are families of smooth vector fields in $\mathbb{R}^n$ that, as $j \to \infty$, converge in $C^\infty_{loc}(\mathbb{R}^n)$ to a family $X = (X_1, \dots, X_m)$ satisfying the Chow-Hörmander condition, and if $u_j : \mathbb{R}^n \to \mathbb{R}$ are locally uniformly bounded in $BV_{X^j,loc}$, then a subsequence u_{j_h} converges in $L^1_{loc}(\mathbb{R}^n)$ to some $u \in BV_{X,loc}(\mathbb{R}^n)$. Theorem 4.2.6 directly follows from Theorem 4.1.1 once we show that the CC distances induced by X^j converge locally uniformly to the one induced by X, and that (locally) a Poincaré inequality holds for BV_{X^j} functions with constant independent of j; these two results (Theorems 4.2.4 and 4.2.5, respectively) use in a crucial way some outcomes of the papers [18, 73].
As it is clear by the techniques used in Chapter 2, in the study of fine properties of BV_X functions in CC spaces, and in particular of their local properties, one often needs to perform a blow-up procedure around a fixed point p: as explained in Theorem 1.4.5, this produces a sequence of CC metric spaces $(\mathbb{R}^n, X^j)$ that converges to (a quotient of) a *Carnot group* structure $\mathbb{G}$. In this blow-up, the original BV_X function u_0 gives rise to a sequence (u_j) of functions in BV_{X^j} which, up to subsequences, will converge

in L^1_{loc} to a $BV_{\mathbb{G},loc}$ function u in $\mathbb{G}$. The function u will be typically a linear map, or a "jump map" taking two different values on complementary halfspaces of $\mathbb{G}$ (see Section 2.1 for a better understanding).

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Chapter 1

Preliminaries

The following chapter is devoted to the introduction of the main definitions and known results we are going to need throughout this Thesis. Section 1.1 is divided into three subsections: Subsection 1.1.1 gives some well-known notions in measure theory, Subsection 1.1.2 contains a classical result about decomposition of measures in metric spaces, Subsection 1.1.3 gives the classical covering theorems that are valid in "good" metric spaces, Subsection 1.1.4 introduces the Hausdorff measure, the Hausdorff dimension, the upper and lower k-densities of a Radon measure μ and the definition of porous sets, together with some simple (but very useful) propositions (see Propositions 1.1.18 and 1.1.19).
Section 1.2 introduces the definition of Carnot Carathéodory space. A list of some well-known (but very important) results is given in 1.2.4 while in Subsection 1.2.1 a proof of Chow's Theorem (see Theorem 1.2.1) is given. Section 1.3 is devoted to an introductory presentation of the notion of Carnot group, which will be needed especially in Chapter 3. Section 1.4 is then devoted to showing the so-called nilpotent approximation of a CC space (see Theorem 1.4.5).
Section 1.5 is devoted to the introduction of the intrinsic regular hypersurfaces. Both X-Lipschitz and C^1_X hypersurfaces are then defined and a study of "fine" properties of C^1_X hypersurfaces is worked out. Some of the results here stated are original (see Proposition 1.5.3, Corollary 1.5.4 and Theorem 1.5.6). The notion of X-rectifiable set is also given (see Definition 1.5.7).
Section 1.6 is devoted to the definition of functions of bounded X-variation and of sets of finite X-perimeter. A list of basic properties and important known results for BV functions in CC spaces is also given: smooth approximation (see Theorems 1.6.2 and 1.6.3), Coarea formula (see Theorems 1.6.5 and 1.6.6), Poincaré Inequality (see 1.6.7) and Isoperimetric inequality (see Theorem 1.6.8).

1.1 Some tools from Geometric Measure Theory in metric spaces

1.1.1 Useful facts from Measure Theory

Definition 1.1.1. Let (M, d) be a locally compact and separable metric space and let μ and μ_h $(h \in \mathbb{N})$ be $\mathbb{R}^k$-valued Radon measure on M. Then we say that μ_h weakly* converges to μ if one has

$$\lim_h \int \varphi \, d\mu_h = \int \varphi \, d\mu,$$

for every $\varphi \in C_b(M)$.

We recall that the total variation $|\mu|$ of a $\mathbb{R}^k$-valued measure $\mu = (\mu_1, \dots, \mu_k)$ is defined for Borel sets B as

$$\begin{aligned} |\mu|(B) &:= \sup \left\{ \sum_{\ell=1}^{\infty} |\mu(B_\ell)| \colon \ell \in \mathbb{N},\ B_\ell \text{ disjoint Borel subsets of } B \right\} \\ &= \sup \left\{ \int_B \varphi \cdot d\mu : \ \varphi \colon B \to \mathbb{R}^k \text{ Borel function, } |\varphi| \leq 1 \right\}. \end{aligned}$$

We recall here two important classical results: the Riesz's Representation Theorem 1.1.2 (see [80]) and the Radon-Nykodým Decomposition Theorem 1.1.3 in doubling metric measure spaces (see [84, Theorem 4.7 and Remark 4.5]).

Theorem 1.1.2. *Let M be a locally compact and separable metric space and let $L : C_b(M; \mathbb{R}^k) \to \mathbb{R}$ be an additive and bounded functional, i.e., satisfying the following conditions:*

(i) for every $u, v \in C_b(M; \mathbb{R}^k)$ one has $L(u + v) = L(u) + L(v)$;

(ii) $\|L\| := \sup\{|L(u)| \colon u \in C_b(M; \mathbb{R}^k), |u| \leq 1\} < +\infty$.

Then, there exists a unique $\mathbb{R}^k$-valued Radon measure μ on X such that

$$L(u) = \sum_{i=1}^{k} \int_X u_i \, d\mu_i,$$

for every $u \in C_b(M; \mathbb{R}^k)$. Moreover one has $\|L\| = |\mu|(M)$.

We recall that, given a metric space (M, d) and a positive Radon measure μ on M, we say that μ is doubling with respect to d if there exists $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(x, r),$$

for every $x \in M$ and for every $r > 0$.

Theorem 1.1.3. *Let M be a locally compact and separable metric space and let μ_1 and μ_2 be two positive Radon measures on M. Suppose also that μ_2 is doubling. Then the limit*

$$\frac{d\mu_1}{d\mu_2}(x) := \lim_{r\to 0} \frac{\mu_1(B(x,r))}{\mu_2(B(x,r))}$$

exists for μ_2-almost every $x \in M$ and the map $d\mu_1/d\mu_2$ is μ_2-measurable. Moreover, there exists $Z \subseteq M$ such that $\mu_2(Z) = 0$ and for any Borel set $A \subseteq M$ one has

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2}\, d\mu_2 + \mu_1^s(A),$$

where $\mu_1^s := \mu_1 \llcorner Z$.
In case also μ_1 is doubling then we may take $Z = \{d\mu_1/d\mu_2 = +\infty\}$.

The proof of Proposition 1.1.4 below can be found for instance in [5, Proposition 1.62].

Proposition 1.1.4. *Let (M,d) be a locally compact and separable metric space and let (μ_h) be a sequence of Radon measures that weakly* converges to μ. Then the following facts hold.*

(a) If $\mu_h \geq 0$ for any $h \in \mathbb{N}$, then for any lower semicontinuous function $\varphi : M \to [0,+\infty]$ one has

$$\liminf_h \int \varphi\, d\mu_h \geq \int \varphi\, d\mu,$$

and for any upper semicontinuous function $\psi : M \to [0,+\infty)$ one has

$$\limsup_h \int \psi\, d\mu_h \leq \int \psi\, d\mu.$$

(b) If the sequence of total variations $|\mu_h|$ locally weakly converges to some λ, then $\lambda \geq |\mu|$. Moreover, if $E \Subset M$ is a μ-measurable set with $\lambda(\partial E) = 0$, then $\mu_h(E) \to \mu(E)$ as $h \to +\infty$.*

1.1.2 Disintegration of measures

We here briefly describe a decomposition criterion for measures in product spaces known as *disintegration* of measure (see e.g. [5, Section 2.5]). Recall that given a σ-algebra $\mathcal{E}$ in M and a measure μ on M, we denote by $\mathcal{E}_\mu$ the smallest σ-algebra containing $\mathcal{E}$ and all the μ-negligible sets. We denote by $\mathcal{B}$ the σ-algebra of Borel sets. Actually, the careful reader will notice that all the definitions and results presented in this subsection are indeed valid in the case in which $\mathbb{R}^n$ and $\mathbb{R}^m$ with the usual Euclidean metric are replaced by two locally compact and separable metric spaces.

Definition 1.1.5. Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ be open sets, let μ be a positive Radon measure on E and let $\nu = \nu_x : E \to \mathcal{M}(F;\mathbb{R}^k)$ a map that assigns to each $x \in E$ a $\mathbb{R}^k$-valued Radon measure ν_x on F. We say that ν_x is *μ-measurable* if, for every Borel set $B \subseteq F$, the map $x \mapsto \nu_x(B)$ is μ-measurable.

Proposition 1.1.6. *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ be open sets, let μ be a positive Radon measure on E and let $\nu = \nu_x : E \to \mathcal{M}(F;\mathbb{R}^k)$. Then ν_x is μ-measurable if and only if, for any open set $A \subseteq F$, the map $x \mapsto \nu_x(A)$ is μ-measurable. Moreover, if ν_x is μ-measurable, the map*

$$x \mapsto \int_E g(x,y)\, d\nu_x(y)$$

is μ-measurable for any bounded $\mathcal{B}_\mu(E) \times \mathcal{B}(F)$-measurable function $g : E \times F \to \mathbb{R}$.

Definition 1.1.7 (Generalized product of measures)**.** Let E, F, μ and ν be as in Definition 1.1.5. Assume that for any open set $A \subseteq E$ one has

$$\int_A |\nu_x|(F)\, d\mu(x) < +\infty.$$

We say that the *generalized product* $\mu \otimes \nu_x$ of μ and ν_x is the $\mathbb{R}^k$-valued radon measure on $E \times F$ defined by

$$\mu \otimes \nu_x(B) := \int_E \left(\int_F \chi_B(x,y)\, d\nu_x(y) \right) d\mu(x),$$

for any Borel set B in $K \times F$, where K is compact in E.

Notice that Definition 1.1.7 is well-defined by Proposition 1.1.6. Moreover, the formula

$$\int_{E\times F} f(x,y)\, d(\mu \otimes \nu_x)(x,y) = \int_E \left(\int_F f(x,y)\, d\nu_x(y) \right) d\mu(x), \tag{1.1}$$

holds for any bounded Borel map $f : E \times F \to \mathbb{R}$ with $\operatorname{supt}(f) \subseteq K \times F$, for some compact $K \subseteq E$. This is a consequence of the fact that any bounded Borel map can be uniformly approximated by sequences of simple functions. Formula 1.1 still holds whenever f is $(\mu \otimes \nu_x)$-summable or, if $k = 1$ and ν_x is positive, whenever f is either positive or negative.

Theorem 1.1.8 (Disintegration of measures)**.** *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ be two open sets and let σ be a $\mathbb{R}^k$-valued Radon measure on $E \times F$. Denote by $\pi : E \times F \to E$ the canonical projection on the first factor and define $\mu := \pi_\#|\sigma|$. Assume that μ is Radon, i.e., for every compact set $K \subseteq E$ one has $|\sigma|(K \times F) < +\infty$. Then, for any $x \in E$, there exists a $\mathbb{R}^k$-valued Radon measure ν_x on F such that $x \mapsto \nu_x$ is μ-measurable and for μ-almost every $x \in E$, $|\nu_x|(F) = 1$. Moreover, for any $f \in L^1(E \times F, |\sigma|)$, we have that*

$$f(x,\cdot) \in L^1(F, |\nu_x|) \quad \text{for } \mu\text{-a.e. } x \in E, \tag{1.2}$$

$$x \longmapsto \int_F f(x,y)\, d\nu_x(y) \in L^1(E,\mu), \tag{1.3}$$

and the formula

$$\int_{E\times F} f(x,y)\,d\sigma(x,y) = \int_E \left(\int_F f(x,y)\,d\nu_x(y)\right) d\mu(x), \tag{1.4}$$

holds.

Proof. We construct ν_x by using Theorem 1.1.2. For any $g \in C_b(F)$ and for any Borel set $B \subseteq E$, we define

$$\mu_g(B) := \int_{B\times F} g(y)\,d\sigma(x,y).$$

Then μ_g is absolutely continuous with respect to μ and $\mu_g = \pi_\#(g\sigma)$. Therefore one can estimate

$$|\mu_g| \leq \pi_\#|g\sigma| \leq \|g\|_\infty \pi_\#|\sigma| = \|g\|_\infty \mu.$$

By Theorem 1.1.3, there exists $h_g \in L^\infty(E,\mu;\mathbb{R}^k)$ such that $\mu_g = h_g\mu$ and $\|h_g\|_\infty \leq \|g\|_\infty$. Since by construction $\mu_{g_1+g_2} = \mu_{g_1} + \mu_{g_2}$, one also has that $h_{g_1+g_2} = h_{g_1} + h_{g_2}$, μ-almost everywhere. Fix a countable dense subset $\mathcal{D}$ of $C_b(F)$. Then we can find a Borel set $N \subseteq E$ with $\mu(N) = 0$ and such that for any $x \in E \setminus N$ one has $h_{g_1+g_2}(x) = h_{g_1}(x) + h_{g_2}(x)$, for any $g_1, g_2 \in \mathcal{D}$. For any $x \in E \setminus N$ we can define $T_x : \mathcal{D} \to \mathbb{R}^k$, letting $T_x(g) := h_g(x)$. Then, by construction of h_g we have $|T_x(g)| \leq \|g\|_\infty$. After extending T_x on the whole $C_b(F)$, by Theorem 1.1.2, for any $x \in E \setminus N$, there exists a $\mathbb{R}^k$-valued Radon measure ν_x on F such that $|\nu_x|(F) = \|T_x\| \leq 1$ and for any $g \in C_b(F)$ one has

$$T_x(g) = \int_F g\,d\nu_x.$$

For every $x \in N$, we simply set $\nu_x = \delta_y$ for a fixed arbitrary $y \in F$. Observe now that for any $x \in E$ and any $g \in \mathcal{D}$ the map $x \mapsto T_x(g)$ is μ-measurable. By a simple approximation argument the same holds for $x \mapsto T_x(\chi_A)$, for any open set $A \subseteq E$. By Proposition 1.1.6 we get that $x \mapsto \nu_x$ is μ-measurable in the sense of Definition 1.1.5.

Let us now prove identity (1.4). For every Borel set $B \subseteq E$ and every $g \in \mathcal{D}$ one has

$$\begin{aligned}\int_{E\times F} \chi_B(x)g(y)\,d\sigma(x,y) &= \mu_g(B) = \int_B h_g(x)\,d\mu(x)\\ &= \int_B \left(\int_F g(y)\,d\nu_x(y)\right) d\mu(x) = \int_E \left(\int_F \chi_B(x)g(y)\,d\nu_x(y)\right) d\mu(x).\end{aligned}$$

By an approximation argument, the previous identity holds for all $g \in C_b(F)$ and then for all $g = \chi_A$ with $A \subseteq F$ open. Equality (1.4) holds then for all the maps $f : E \times F \to \mathbb{R}$ of the kind $f(x,y) = \chi_B(x)\chi_A(y)$ with $B \subseteq E$ Borel and $A \subseteq F$ open. This implies that (1.4) holds for all $f(x,y) = \chi_B(x,y)$ for any Borel set B in $E \times F$. In particular, if $B \subseteq E \times F$ is Borel such that $|\sigma|(B) = 0$, then $\chi_B(x,\cdot) \in L^1(F,|\nu_x|)$ and

$$\int_F \chi_B(x)g(y)\,d\nu_x(y) = 0,$$

for μ-almost every $x \in E$. Then (1.2), (1.3) and (1.4) hold for $f = \chi_B$ for $B \in \mathcal{B}_\sigma(E \times F)$. The general case follows eventually by splitting f into positive and negative part and by an approximation argument.
Let us prove that $|\nu_x|(F) = 1$ for μ-almost every $x \in E$. Define, for any $x \in E$ and for any Borel set $B \subseteq E \times F$, the set $B_x := \{y \in F : (x, y) \in B\}$. Then, taking into account (1.4), one immediately gets

$$|\sigma(B)| \leq \int_E |\nu_x|(B_x)\, d\mu(x),$$

By definition of total variation of σ, this implies

$$|\sigma|(B) \leq \int_E |\nu_x|(B_x)\, d\mu(x).$$

Hence, with the choice $B = E \times F$ and taking the definition of μ into account, one has

$$|\nu|(E \times F) \leq \int_E |\nu_x|(F)\, d\mu(x) \leq \int_E 1\, d\mu(x) = \mu(E) = |\nu|(E \times F),$$

which concludes the proof. □

Theorem 1.1.9 (Uniqueness of the disintegration). *Let $E \subseteq \mathbb{R}^n$ and $F \subseteq \mathbb{R}^m$ be two open sets and let σ be a $\mathbb{R}^k$-valued Radon measure on $E \times F$. Denote by $\pi : E \times F \to E$ the canonical projection on the first factor and define $\mu := \pi_\# |\sigma|$. Assume that μ is Radon, i.e., for every compact set $K \subseteq E$ one has $|\sigma|(K \times F) < +\infty$. Then assume $x \mapsto \nu_x$ and $x \mapsto \nu'_x$ are two μ-measurable maps satisfying conditions* (1.2) *and* (1.4) *for every bounded Borel f with compact support and are such that $x \mapsto \nu_x(F), x \mapsto \nu'_x(F) \in L^1(E, \mu)$. Then $\nu_x = \nu'_x$ for μ-almost every $x \in E$.*

Proof. Let $\mathcal{D}$ be a countable and dense set in $C_b(F)$. Then by (1.4), for any $g \in \mathcal{D}$ and any Borel set $B \Subset E$, one has

$$\int_B \left(\int_F g(y)\, d\nu_x(y) \right) d\mu(x) = \int_{B \times F} g(y)\, d\sigma(x, y) = \int_B \left(\int_F g(y)\, d\nu'_x(y) \right) d\mu(x).$$

Therefore we can find $N \subseteq E$ such that $\mu(N) = 0$ and with the property that

$$\int_F g(y)\, d\nu_x(y) = \int_F g(y)\, d\nu'_x(y),$$

for any $g \in \mathcal{D}$ and for any $x \in E \setminus N$. By density of $\mathcal{D}$ in $C_b(F)$ we can assert that $\nu_x = \nu'_x$ for μ-almost every $x \in E$. □

1.1.3 Covering Theorems

In this subsection we report the covering Theorems we are going to use throughout this Thesis.
A proof of Theorem 1.1.10 below can be found in [84, Theorem 3.3] or in [48, Theorem 1.2], while a proof of Theorem 1.1.11 can be found in [48, Theorem 1.6].

Theorem 1.1.10 (5r-Covering Lemma)**.** *Let (M,d) be a separable metric space and let $\mathcal{B}$ a family of closed balls in M such that*

$$\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < +\infty.$$

Denote by $5B$ the closed metric ball with the same center of B and radius 5 times larger than the radius of B. Then there exists a countable and pairwise disjoint subfamily $\mathcal{F} \subseteq \mathcal{B}$ such that

$$\bigcup \mathcal{B} \subseteq \bigcup_{B\in\mathcal{F}} 5B.$$

Theorem 1.1.11 (Vitali covering Lemma)**.** *Let (M,d) be a locally compact and separable metric space and let μ be a Radon measure that is doubling with respect to d. Let $A \subseteq M$ and let $\mathcal{F}$ be a family of closed balls such that for every $x \in A$*

$$\inf\{r > 0 : B(x,r) \in \mathcal{F}\} = 0.$$

Then there exists a countable family $\mathcal{G} \subseteq \mathcal{F}$ of pairwise disjoint balls such that

$$\mu\left(A \setminus \bigcup \mathcal{G}\right) = 0.$$

Actually, Theorem 1.1.11 can be strengthened to a bigger class of metric measure spaces. More precisely, let us introduce the following

Definition 1.1.12 ([84])**.** We say that a locally compact and separable metric space M satisfies the *symmetric Vitali property* with respect to a positive Radon measure μ if every family of balls $\mathcal{F}$ which covers the set $A \coloneqq \{x \in M : \exists r > 0 \text{ such that } B(x,r) \in \mathcal{F}\}$ *finely* (i.e. for all $x \in A$, $\inf\{r > 0 : B(x,r) \in \mathcal{F}\} = 0$) admits a countable and pairwise disjoint subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that

$$\mu\left(\bigcup \mathcal{F}' \setminus A\right) = 0,$$

provided $\mu(A) < +\infty$.

The importance of the symmetric Vitali property is given by Theorem 1.1.13 which generalizes Theorem 1.1.3.

Theorem 1.1.13 ([84, Theorem 4.7])**.** *Let M be a locally compact and separable metric space and let μ_1 and μ_2 be two positive Radon measures on M. Assume that M satisfies the symmetric Vitali property with respect to μ_2. Then the limit*

$$\frac{d\mu_1}{d\mu_2}(x) \coloneqq \lim_{r\to 0} \frac{\mu_1(B(x,r))}{\mu_2(B(x,r))}$$

exists for μ_2-almost every $x \in M$ and the map $d\mu_1/d\mu_2$ is μ_2-measurable. Moreover, there exists $Z \subseteq M$ such that $\mu_2(Z) = 0$ and for any Borel set $A \subseteq M$ one has

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2}\, d\mu_2 + \mu_1^s(A),$$

where $\mu_1^s := \mu_1 \llcorner Z$.
In case that M *satisfies the symmetric Vitali property with respect to* μ_1*, then we may take* $Z = \{d\mu_1/d\mu_2 = +\infty\}$.

A sufficient condition that ensures symmetric Vitali property is given in the following theorem, which is a consequence of [32, Theorem 2.8.17].

Theorem 1.1.14. *Let* M *be a locally compact and separable metric space and let* μ *be an* asymptotically doubling *positive Radon measure on* M*, i.e., such that*

$$\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < +\infty.$$

for every $x \in M$ *and every* $r > 0$*. Then* M *has the symmetric Vitali property with respect to* μ*.*

To conclude this section on covering theorems we point out that one of the main issues in the analysis of geometric properties of Carnot-Carathéodory spaces (see Section 1.2) is the lack of a Besicovitch covering Theorem. The Euclidean formulation below is contained in [5, Theorem 2.17] and its proof can be found in [16].

Theorem 1.1.15. *Let* $n \in \mathbb{N}$*. Then, there exists* $\xi \in \mathbb{N}$ *such that the following holds. For any family* $\mathcal{F}$ *of closed balls in* $\mathbb{R}^n$ *such that the set* A *of their centers is bounded, there exist* ξ *disjoint subfamilies* $\mathcal{F}_1, \dots, \mathcal{F}_\xi$ *of* $\mathcal{F}$ *such that*

$$A \subseteq \bigcup_{h=1}^{\xi} \bigcup \mathcal{F}_h.$$

In particular, any point of A *belongs to the intersection of at most* ξ *closed balls.*

Actually Theorem 1.1.15 may fail in general metric spaces and its validity depends on the metric. A counterexample to Theorem 1.1.15 in the Heisenberg group (see Section 1.3 and Example 1.3.24) endowed with the Korányi metric is given e.g. in [52, pag. 17] (see also [83, Section 4]), while a counterexample in the Heisenberg group endowed with the CC distance has been given in [81]. It is also known that, in any Carnot group of step greater than 3 endowed with the homogeneous distance, Theorem 1.1.15 is false (see [57]), while there exist homogeneous distances on the Heisenberg group for which Theorem 1.1.15 holds (see [58]).

1.1.4 Hausdorff measures and densities

We here introduce the notions of Hausdorff measure and of k-density of a measure μ and we describe their connections through Propositions 1.1.18 and 1.1.19.

Definition 1.1.16 (Hausdorff measures)**.** Let (M, d) be a metric space and $k \geq 0$. We define for any $\delta > 0$ and for any set E

$$\mathscr{H}^k_\delta(E) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{h=0}^{\infty} (\operatorname{diam} E_h)^k : E \subseteq \bigcup_{h=0}^{\infty} E_h, \operatorname{diam} E_h < \delta \right\},$$
$$\mathscr{S}^k_\delta(E) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{h=0}^{\infty} (\operatorname{diam} B_h)^k : E \subseteq \bigcup_{h=0}^{\infty} B_h, B_h \text{ balls with } \operatorname{diam} B_h < \delta \right\},$$

where $\omega_\alpha := \pi^{\alpha/2}\Gamma(1+\alpha/2)^{-1}$ and $\Gamma(t) := \int_0^{+\infty} s^{t-1}e^{-s}ds$ is the Euler Γ function. $\mathscr{H}^k_\delta$ and $\mathscr{S}^k_\delta$ are respectively called *Hausdorff premeasure* and *spherical Hausdorff premeasure* of size δ. The *Hausdorff measure* and the *spherical Hausdorff measure* of a set E are then respectively defined setting

$$\mathscr{H}^k(E) := \sup_{\delta > 0} \mathscr{H}^k_\delta(E) = \lim_{\delta \to 0} \mathscr{H}^k_\delta(E),$$
$$\mathscr{S}^k(E) := \sup_{\delta > 0} \mathscr{S}^k_\delta(E) = \lim_{\delta \to 0} \mathscr{S}^k_\delta(E).$$

It is easy to notice that for any $k \geq 0$ the following inequalities hold

$$\mathscr{H}^k \leq \mathscr{S}^k \leq 2^k \mathscr{H}^k.$$

The Hausdorff dimension of E is $\inf\{k : \mathcal{H}^k(E) = 0\} = \sup\{k : \mathcal{H}^k(E) = +\infty\}$.

Definition 1.1.17 (k-densities)**.** If (M, d, μ) is a doubling metric measure space, $k \geq 0$ and $x \in M$, we define the *upper k-density* and the *lower k-density of μ at x* respectively in the following way

$$\Theta^*_k(\mu, x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{\omega_k r^k},$$
$$\Theta_{*k}(\mu, x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{\omega_k r^k}.$$

For every Borel set $E \subseteq M$ we will also write for brevity $\Theta^*_k(E, x) := \Theta^*_k(\mathscr{H}^k \llcorner E, x)$ and $\Theta_{*k}(E, x) := \Theta_{*k}(\mathscr{H}^k \llcorner E, x)$. If $\Theta^*_k(\mu, x) = \Theta_{*k}(\mu, x)$, then the common value is denoted by $\Theta_k(\mu, x)$ and it is called *k-density of μ at x*.

The notions of k-density and of Hausdorff k-measure are linked in Propositions 1.1.18 and 1.1.19 below. The proof of Proposition 1.1.18 is an adaptation of [84, Theorem 3.2].

Proposition 1.1.18. *Let (M, d) be a locally compact and separable metric space, let μ be a positive Radon measure on M, let $E \subseteq M$ be a Borel set and let $t > 0$. Then the following facts hold.*

(i) If $\Theta^*_k(\mu, x) \geq t$ for every $x \in E$, then $\mu \geq t\mathscr{S}^k \llcorner E$.

(ii) If $\Theta^*_k(\mu, x) \le t$ for every $x \in E$, then $\mu \le 2^k t \mathscr{H}^k \llcorner E$.

In particular, for $\mathscr{H}^k$-almost every $x \in M$, we have $\Theta^*_k(\mu, x) < +\infty$.

Proof. (i) Suppose first that E is compact. Given $\delta \in (0, 1)$, take an open set with compact closure A containing E and define

$$\mathcal{F} := \left\{ \overline{B(x, r)} \subseteq A : p \in E, 0 < r < \tfrac{\delta}{2}, \mu(\overline{B(x, r)}) \ge t(1 - \delta)\omega_k r^k \right\}.$$

By Theorem 1.1.10, we get a countable sub-family $\{\overline{B(x_h, r_h)} \in \mathcal{F} : h \in \mathbb{N}\}$ of pairwise disjoint closed balls such that

$$A \subseteq \bigcup_{h=0}^{\infty} \overline{B(x_h, 5r_h)}.$$

In particular, we have

$$\mathscr{S}^k_{5\delta}(E) \le \frac{\omega_k}{2^k} \sum_{h=0}^{\infty} 5^k r_h^k \le \sum_{h=0}^{\infty} 5^k \frac{\mu(B(x_h, r_h))}{t(1 - \delta)} \le 5^k \frac{\mu(A)}{t(1 - \delta)} < +\infty.$$

By the arbitrariness of δ, we get that $\mathscr{S}^k(E) < +\infty$. Applying now Theorem 1.1.11 we get a pairwise and countable disjoint sub-family $\{\overline{B(x_h, r_h)} \in \mathcal{F} : h \in \mathbb{N}\}$ of $\mathcal{F}$ which covers $\mathscr{S}^k$-almost all E and therefore

$$\mathscr{S}^k_{2\delta}(E) \le \frac{\omega_k}{2^k} \sum_{h=0}^{\infty} 2^k r_h^k \le \sum_{h=0}^{\infty} \frac{\mu(\overline{B(x_h, r_h)})}{t(1 - \delta)} \le \frac{\mu(A)}{t(1 - \delta)}.$$

By the arbitrariness of δ and A we get the thesis in the case that E is compact. In the general case it is sufficient to notice that, in a locally compact and separable metric spaces, Radon measures are inner regular, i.e., the measure of every Borel set E can be approximated by

$$\mu(E) = \sup \{\mu(K) : K \text{ is compact }, K \subseteq E\}.$$

(ii) Suppose first that E is compact. Take $\tau > t$ and define

$$E_h := \left\{ x \in E : \frac{\mu(\overline{B(x, r)})}{\omega_k r^k} < \tau, \quad \forall r \in (0, \tfrac{1}{2h}) \right\}.$$

We have that (E_h) is an increasing sequence of sets whose union (by assumption) is E. By definition of Hausdorff measure, for every $h \in \mathbb{N}$ we can find a family $\{F_{i,h} : i \in \mathbb{N}\}$ of sets whose union covers E_h with $\operatorname{diam} F_{i,h} < 1/h$ and such that

$$\sum_{i=0}^{\infty} \frac{\omega_k}{2^k} (\operatorname{diam} F_{i,h})^k < \mathscr{H}^k_{1/h}(E_h) + \frac{1}{h}.$$

We can also suppose without loss of generality that for every $i \in \mathbb{N}$ there exists $\xi_i \in E_h \cap F_{i,h}$. Then also the family $\{\overline{B(\xi_i, 2r_i)} : i \in \mathbb{N}\}$ is a covering of E_h and

$$\mu(E_h) \leq \sum_{i=0}^{\infty} \mu(\overline{B(\xi_i, 2r_i)}) \leq \tau \sum_{i=0}^{\infty} \omega_k 2^k r_i^k < \tau 2^k \left(\mathscr{H}^k_{1/h}(E) + \frac{1}{h} \right).$$

By the arbitrariness of $\tau > t$ and $h \in \mathbb{N}$ we get the thesis in case E is compact. The general case follows as in (i). □

Corollary 1.1.19. *Let (M, d) be a locally compact and separable metric space, let μ a positive Radon measure on M, let $E \subseteq M$ a Borel set and let $f : E \to (0, +\infty)$ be a Borel map. Then the following facts hold.*

(i) If $\Theta^*_k(\mu, x) \geq f(x)$ for every $x \in E$, then $\mu \geq f \mathscr{S}^k \llcorner E$.

(ii) If $\Theta^*_k(\mu, x) \leq f(x)$ for every $x \in E$, then $\mu \leq 2^k f \mathscr{H}^k \llcorner E$.

Proof. Let $\varepsilon > 0$ and define for every $j \in \mathbb{Z}$ the set

$$E_j := \{x \in E : (1+\varepsilon)^j < f(x) \leq (1+\varepsilon)^{j+1}\}.$$

Suppose that $\Theta^*_k(\mu, x) \geq f(x)$ for every $x \in E$. Then, using (i) of Proposition 1.1.18 we get

$$\mu = \sum_{j\in\mathbb{Z}} \mu \llcorner E_j \geq \sum_{j\in\mathbb{Z}} (1+\varepsilon)^j \mathscr{S}^k \llcorner E_j \geq \sum_{j\in\mathbb{Z}} \frac{f}{1+\varepsilon} \mathscr{S}^k \llcorner E_j = \frac{f}{1+\varepsilon} \mathscr{S}^k \llcorner E,$$

which, by the arbitrariness of ε, gives (i).
If we suppose that $\Theta^*_k(\mu, x) \leq f(x)$ for every $x \in E$, using (ii) of Proposition 1.1.18 we have

$$\begin{aligned} \mu = \sum_{j\in\mathbb{Z}} \mu \llcorner E_j &\leq \sum_{j\in\mathbb{Z}} 2^k (1+\varepsilon)^{j+1} \mathscr{H}^k \llcorner E_j \\ &\leq \sum_{j\in\mathbb{Z}} 2^k (1+\varepsilon) f \mathscr{S}^k \llcorner E_j = 2^k (1+\varepsilon) f \mathscr{S}^k \llcorner E, \end{aligned}$$

which, by the arbitrariness of ε, gives (ii). □

As a consequence of the Corollary 1.1.19 we have the following remark.

Remark 1.1.20. Under the same assumptions of Corollary 1.1.19, for $\mathscr{H}^k$-almost every $x \in M$ we have $\Theta^*_k(\mu, x) < +\infty$ and for any Borel set $B \subseteq M$ the implication

$$\mu(B) = 0 \Rightarrow \Theta_k(\mu, x) = 0 \text{ for } \mathscr{H}^k\text{-a.e. } x \in B$$

holds. In particular, if $\mu = g\mathscr{H}^k \llcorner E$ we have $\Theta_k(\mu, x) = 0$ for $\mathscr{H}^k$-almost every $x \in M \setminus E$.

Definition 1.1.21. Given a metric measure space (M, d, μ), a μ-measurable set $E \subseteq M$ and $t \in [0, 1]$ we denote by E^t the set of points with μ-density t for E, i.e., all $x \in M$ satisfying

$$\lim_{r \to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = t.$$

The *essential boundary* of E is then defined by $\partial^* E := M \setminus (E^0 \cup E^1)$.

Definition 1.1.22. Let (M, d) be a metric space and let $E \subseteq M$ be a Borel set. Then E is said to be *porous* if there esist $\alpha \in (0, 1)$ and $R > 0$ such that for every $x \in M$ and every $r \in (0, R)$ there exists $y \in M$ such that $B(y, \alpha r) \subseteq B(x, r) \setminus E$.

Proposition 1.1.23. *Let let (M, d) be a locally compact and separable metric space, let μ be a doubling Radon measure on M and let $E \subseteq M$ be a porous set. Then $E^1 = \emptyset$ and in particular $\mu(E) = 0$.*

Proof. Let α and R be as in Definition 1.1.22. Suppose by contradiction there exists $x \in E^1$. For every $r \in (0, R)$ there exists $y \in M$ such that

$$B(y, \alpha r) \subseteq B(x, r) \setminus E.$$

This implies that

$$\frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \geq \frac{\mu(B(y,\alpha r))}{\mu(B(x,r))} \geq C,$$

where $C > 1$ is a suitable constant depending on α and on the doubling constant of μ. Letting $r \to 0$ and taking into account that $x \in (M \setminus E)^0$, we get a contradiction. Taking into account Lebesgue Differentiation Theorem 2.1.2 we also get $\mu(E) = 0$. □

1.2 Carnot-Carathéodory spaces

In what follows we denote by Ω an open set in $\mathbb{R}^n$ and by $X = (X_1, \dots, X_m)$ an m-tuple $(m \leq n)$ of smooth and linearly independent vector fields on $\mathbb{R}^n$, with $2 \leq m \leq n$. We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ is a X-*admissible path* joining p and q if $\gamma(0) = p$, $\gamma(T) = q$ and there exists $h = (h_1, \dots, h_m) \in L^\infty([0, T]; \mathbb{R}^m)$ such that for almost every $t \in [0, T]$ one has

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h(t) X_j(\gamma(t)). \tag{1.5}$$

For every $p, q \in \mathbb{R}^n$, we define the quantity

$$d(p, q) := \{\text{length}(\gamma) : \gamma \text{ is } X\text{-admissible curve joining } p \text{ and } q\}, \tag{1.6}$$

where we agree that $\inf \emptyset = +\infty$ and where

$$\text{length}(\gamma) := \int_0^T \|h(t)\| \, dt.$$

A sufficient condition that makes d a metric on $\mathbb{R}^n$ is given by Theorem 1.2.1, below, which is proved in [21]. A proof of Theorem 1.2.1 is given in Subsection 1.2.1.

Theorem 1.2.1 (Chow-Rashevsky)**.** *Suppose that*

$$\forall\, p \in \mathbb{R}^n \qquad \mathcal{L}ie\{X_1, \dots, X_m\}(p) = T_p\mathbb{R}^n \cong \mathbb{R}^n, \tag{1.7}$$

where $\mathcal{L}ie\{X_1, \dots, X_m\}(p)$ denotes the linear span of all the iterated commutators of the vector fields $(X_1, \dots, X_m)$ computed at p. Then d is a distance, called Carnot-Carathéodory distance *associated with X.*

We will refer to (1.7) as the *Chow-Hörmander condition.* When (1.7) holds, the metric space $(\mathbb{R}^n, d)$ is said to be a *Carnot-Carathéodory space* of *rank m* (CC space, for short). We will use the notation $(\mathbb{R}^n, X)$ to denote the metric space $(\mathbb{R}^n, d)$, where d is understood to be the Carnot-Carathéodory (CC, for short) distance associated with X. We also denote by $B(x, r)$ and $B_e(x, r)$ the metric balls of center x and radius $r > 0$ induced by the CC distance d and by the Euclidean distance d_e, respectively.

Remark 1.2.2. If the Chow-Hörmander condition holds, then for every compact set $K \subseteq \mathbb{R}^n$ there exists an integer $s(K)$ such that the following holds: for any $x \in K$, $X_1, \dots, X_m$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbb{R}^n$. This is an immediate consequence of the fact that $X_1, \dots, X_m$ and the map $A \mapsto \det(A)$ are of class C^∞.

Remark 1.2.3. Given $p, q \in \mathbb{R}^n$, denote for shortness by $\gamma_{T,h}$ the X-subunit curve in $AC([0, T]; \mathbb{R}^n)$ joining p and q and satisfying (1.5) for some $h \in L^\infty([0, T]; \mathbb{R}^m)$. The curve $\gamma_{T,h}$ is said to be X-subunit if $\sum_{j=1}^m h_j^2 \leq 1$. It is easy to observe, by a change of coordinates, that the metric d can be equivalently defined by

$$d(p, q) = \inf\{\|h\|_\infty\colon \gamma_{1,h} \text{ joins } p \text{ and } q\},$$

or by

$$d(p, q) = \inf\left\{T > 0 : \exists\, h \in L^\infty([0, T]; \mathbb{R}^m), |h| \leq 1 \text{ s.t. } \gamma_{T,h} \text{ joins } p \text{ and } q\right\}.$$

For every $p \in \mathbb{R}^n$ and for every $i \in \mathbb{N}$ we denote by $\mathcal{L}^i(p)$ the linear span of all the commutators of the vector fields $(X_1, \dots, X_m)$ up to order i computed at p. Notice that $\mathcal{L}ie\{X_1, \dots, X_m\}(p) = \bigcup_{i\in\mathbb{N}} \mathcal{L}^i(p)$. We say that $(\mathbb{R}^n, X)$ is *equiregular*, if there exist $0 = n_0 < n_1 < \cdots < n_s = n \in \mathbb{N}$ such that, for every $p \in \mathbb{R}^n$, one has $\dim \mathcal{L}^i(p) = n_i$. The natural number s is called *step* of the Carnot-Carathéodory space. In the following Theorem we resume some well known facts about the geometry of an equiregular CC space. For (i) and (iii) see [78], while for (ii) see [71].

Theorem 1.2.4. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of step s. Then the following facts hold.*

(i) For every compact set $K \subseteq \mathbb{R}^n$, there exists $M \geq 1$ such that for any $p, q \in K$

$$\frac{1}{M}\,|p-q| \leq d(p,q) \leq M|p-q|^{\frac{1}{s}}.$$

(ii) The Hausdorff dimension of the metric space $(\mathbb{R}^n, d)$ is $Q := \sum_{i=1}^{s} i(n_i - n_{i-1})$.

(iii) The metric measure space $(\mathbb{R}^n, d, \mathscr{L}^n)$ is locally Ahlfors Q-regular, i.e., for every compact set $K \subseteq \mathbb{R}^n$ there exist $R > 0$ and $C > 1$ such that

$$\frac{1}{C} r^Q \leq \mathscr{L}^n(B(p,r)) \leq C r^Q, \tag{1.8}$$

for every $p \in K$ and for every $r \in (0, R)$. In particular, the metric measure space $(\mathbb{R}^n, d, \mathscr{L}^n)$ is locally doubling.

We say that $(\mathbb{R}^n, X)$ is *geodesic* if for every $p, q \in \mathbb{R}^n$ there exists a X-admissible curve realizing the infimum in (1.6).

Proposition 1.2.5. *Let $(\mathbb{R}^n, X)$ be a geodesic equiregular CC space; then, for every $p \in \mathbb{R}^n$ and for every $r > 0$ one has $\mathscr{L}^n(\partial B(p,r)) = 0$.*

Proof. By Proposition 1.1.23 it is sufficient to prove that $\partial B(p,r)$ is a porous set. Take $q \in \partial B(p,r)$ and consider a minimizing absolutely continuous path $\gamma : [0,r] \to X$ joining p and q, i.e., such that $\gamma(0) = p$, $\gamma(r) = q$ and for every $t \in [0,r]$ one has $d(p, \gamma(t)) = t$. Consider $\varepsilon \in (0, 2r]$ and $y = \gamma(r - \frac{\varepsilon}{2})$. Then $B(y, \frac{\varepsilon}{2}) \subseteq B(q, \varepsilon)$ and obviously $B(y, \frac{\varepsilon}{2}) \cap \partial B(p,r) = \emptyset$. Then $\partial B(p,r)$ is porous with $\alpha = \frac{1}{2}$, $r_0 = 2r$. □

We assume from now on that the metric balls $B(p,r)$ are bounded with respect to the Euclidean metric in $\mathbb{R}^n$; in particular, as it has been shown in [75, Theorem 1.4.4], the CC space $(\mathbb{R}^n, X)$ is geodesic.

1.2.1 A proof of Chow's Theorem

In this Subsection we will provide a proof of Theorem 1.2.1. We will prove in particular a stronger fact, that is the Hölder-type inequality appearing in (i) of Theorem 1.2.4. We first need to introduce some notation.

Given $\alpha, \beta \in \mathbb{N}^k$ we set $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_k + \beta_k)$ and

$$|\alpha| := \alpha_1 + \cdots + \alpha_k, \qquad \alpha! := \alpha_1! \cdots \alpha_k!\,.$$

Given a vector field X in $\mathbb{R}^n$ and $k \in \mathbb{N}$ we define $(\mathrm{ad}_X)^k$ setting

$$\begin{cases} (\mathrm{ad}_X)^0 Y & := Y, \\ (\mathrm{ad}_X)^{k+1} Y & := (\mathrm{ad}_X)^k([X,Y]), \end{cases}$$

for any vector field Y in $\mathbb{R}^n$. For every $k \in \mathbb{N}$, for every $\alpha, \beta \in \mathbb{N}^k$ and for every vector fields Y, Z on $\mathbb{R}^n$ we eventually define

$$C_{\alpha\beta}(Y,Z) := \begin{cases} (\mathrm{ad}_Y)^{\alpha_1}(\mathrm{ad}_Z)^{\beta_1}\cdots(\mathrm{ad}_Y)^{\alpha_k}(\mathrm{ad}_Z)^{\beta_{k-1}}Z, & \text{if } \beta_k \neq 0, \\ (\mathrm{ad}_Y)^{\alpha_1}(\mathrm{ad}_Z)^{\beta_1}\cdots(\mathrm{ad}_Y)^{\alpha_{k-1}}Y, & \text{if } \beta_k = 0. \end{cases} \tag{1.9}$$

Theorem 1.2.6 below contains the so-called Campbell-Hausdorff formula. It is proved e.g. in [86] or [78, Appendix]. For the notion of left invariant vector field in a Lie group and of exponential map on a manifold we refer the reader to Section 1.3.

Theorem 1.2.6. *For every sufficiently small left invariant vector fields Y, Z in a Lie group M the series*

$$P(Y,Z) := \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \sum_{\alpha,\beta\in A_j} \frac{1}{\alpha!\,\beta!\,|\alpha+\beta|} C_{\alpha\beta}(Y,Z) \tag{1.10}$$

converges uniformly, where $A_j := \{(\alpha,\beta) \in \mathbb{N}^j \times \mathbb{N}^j : \alpha_i + \beta_i \geq 1$ for $i = 1,\dots,j\}$. In such a case we have $\exp(Y)\exp(Z) = \exp(P(Y,Z))$.

Notice that formula (1.10) holds also in case Y, Z are vector fields in a CC space $(\mathbb{R}^n, X)$ and Chow-Hörmander condition holds. Lemma 1.2.7 below is a consequence of Theorem 1.2.6 and its proof can be found in e.g. [78, Proposition 4.3].

Lemma 1.2.7. *Let K be a compact set in $\mathbb{R}^n$, $k \in \mathbb{N}$, let Y, Z be two vector fields in $\mathbb{R}^n$ and let*

$$P_k(Y,Z) := \sum_{j=1}^{k} \frac{(-1)^j}{j} \sum_{(\alpha,\beta)\in A_j} \frac{1}{\alpha!\,\beta!\,|\alpha+\beta|} C_{\alpha\beta}(Y,Z),$$

where A_j is defined as in Theorem 1.2.6. Then, there exists $C > 0$ such that

$$|\exp(tY)\exp(sZ)(p) - \exp(P_k(tY,sZ))(p)| \leq C(|t|^k + |s|^k)$$

for every $t, s \in \mathbb{R}$ sufficiently close to zero and every $p \in K$.

The following proof is contained in [54].

Theorem 1.2.8. *Let Ω be a connected open set in $\mathbb{R}^n$, let $K \subseteq \Omega$ be a compact set and let $X = (X_1, \dots, X_m)$ be a m-tuple of linearly independent and smooth vector fields. Assume that in K the Hörmander condition is satisfied by commutators of $X_1, \dots, X_m$ of length at most $k \in \mathbb{N}$. Then there exists $C_K > 0$ such that*

$$d(p,q) \leq C_K |p-q|^{1/k},$$

for every $p, q \in K$.

Proof. Let us start with the following consideration. Given a r-tuple $Y = (Y_1, \dots, Y_r)$ of vector fields such that $Y_j \in \{\pm X_1, \dots, \pm X_m\}$ for every $j = 1, \dots, r$, then there exists $\delta > 0$ such that the map

$$E(Y,t)(p) := \exp(tY_r)\cdots\exp(tY_1)(p)$$

is well defined for every $t \in [-\delta, \delta]$. It readily seen that $t \mapsto E(Y,t)(p)$ is X-admissible and that

$$d(p, E(Y,t)(p)) \leq |t|r. \tag{1.11}$$

For every $h \in \mathbb{N}$ and for every $\alpha \in \mathbb{N}^h$ with $1 \leq \alpha_j \leq m$ we also define the commutator (of length h) X_α setting

$$X_\alpha := [X_{\alpha_1}, [X_{\alpha_2}, \cdots [X_{\alpha_{h-1}}, X_{\alpha_h}]\cdots]].$$

By Lemma 1.2.7, for any $\alpha \in \mathbb{N}^h$, there exist $1 \leq r \leq 4^{h-1}$, a r-tuple $Y^+ = (Y_1, \dots, Y_r)$ with $Y_j^+ \in \{\pm X_1, \dots, \pm X_m\}$ and $\omega_1 \in C^1([-\delta, \delta] \times K)$ such that

$$\exp(t^h X_\alpha)(p) = E(Y^+, t)(p) + t^{h+1}\omega_1(t,p),$$

for every $p \in K$ and every $t \in [0, \delta]$. For the same reason, let Y^- be a r-tuple of vector fields (again chosen among $\pm X_1, \dots, \pm X_m$) and let $\omega_2 \in C^1([-\delta, \delta] \times K)$ be such that

$$\exp(-t^h X_\alpha)(p) = E(Y^-, t)(p) + t^{h+1}\omega_2(t,p),$$

for every $t \in [0, \delta]$ and every $p \in K$. We can therefore write

$$\exp(\tau X_\alpha)(p) = \begin{cases} E(Y^+, \tau^{1/h})(p) + \tau^{\frac{h+1}{h}}\omega_1(p), & \text{if } \tau \in [0, \delta^{1/h}] \\ E(Y^-, (-\tau)^{1/h})(p) + (-\tau)^{\frac{h+1}{h}}\omega_2(p), & \text{if } \tau \in [-\delta^{1/h}, 0], \end{cases} \tag{1.12}$$

for every $p \in K$. For any $\alpha \in \mathbb{N}^h$, we finally define

$$E_\alpha(\tau) := \begin{cases} E(Y^+, \tau^{1/h}), & \text{if } \tau \in [0, \delta^{1/h}], \\ E(Y^-, (-\tau)^{1/h}), & \text{if } \tau \in [-\delta^{1/h}, 0]. \end{cases}$$

We claim that $(\tau, p) \mapsto E_\alpha(\tau)(p)$ is of class C^1. To prove this it is enough to show that $\frac{\partial E_\alpha}{\partial \tau}(\tau)(p)$ is continuous in $\tau = 0$. For $\tau_0 > 0$, setting $t_0 = \tau_0^{1/h}$, one has

$$\begin{aligned}
\frac{\partial E_\alpha}{\partial \tau}(\tau_0)(p) &= \lim_{\tau \to \tau_0} \frac{E(Y^+, \tau^{1/h}) - E(Y^+, \tau_0^{1/h})}{\tau - \tau_0} \\
&= \lim_{t \to t_0} \frac{E(Y^+, t) - E(Y^+, t_0)}{t^h - t_0^h} \\
&= \frac{1}{h t_0^{h-1}} \frac{\partial E}{\partial t}(Y^+, t_0) \\
&= \frac{1}{h t_0^{h-1}} \frac{\partial \exp(t_0^h X_\alpha)}{\partial t} + \frac{h+1}{h} t_0 \omega_1(t,p) + \frac{t_0^2}{h}\frac{\partial}{\partial t}\omega_1(t_0, p) \\
&= X_\alpha(\exp(\tau_0 X_\alpha)(p)) + \tau_0^{1/h}\frac{h+1}{h}\omega_1(\tau_0^{1/h}, p) + \frac{\tau_0^{2/h}}{h}\frac{\partial}{\partial t}\omega_1(\tau_0^{1/h}, p),
\end{aligned}$$

where we have used (1.12). Analogously for $\tau_0 < 0$ we immediately get that $\frac{\partial E_\alpha}{\partial \tau}(\tau_0)(p)$ equals

$$X_\alpha(\exp(\tau_0 X_\alpha)(p)) + (-\tau_0)^{1/h}\frac{h+1}{h}\omega_2((-\tau_0)^{1/h}, p) + \frac{(-\tau_0)^{2/h}}{h}\frac{\partial}{\partial t}\omega_2((-\tau_0)^{1/h}, p),$$

which concludes the proof of the fact that $(t,p) \mapsto E_\alpha(t)(p)$ is C^1. Fix now $p_0 \in K$. By assumption we can find n linearly independent vector fields $X_{\alpha_1}, \dots, X_{\alpha_n}$ that are commutators of $X_1, \dots, X_m$ of length at most k. For any $t = (t_1, \dots, t_n)$ sufficiently close to 0 the map

$$F(t_1, \dots, t_n) = E_{\alpha_n}(t_n) \circ \cdots \circ E_{\alpha_1}(t_1)$$

is well defined and of class C^1. Therefore the matrix

$$dF(0) = \operatorname{col}[X_{\alpha_1}(p_0), \dots, X_{\alpha_n}(p_0)],$$

has full rank and therefore it is open. There exist $\varrho, \sigma > 0$ such that

$$B_e(p_0, \sigma) \subseteq F(B(0, \varrho),$$

and there exists $M > 0$ such that, for any $t \in \mathbb{R}^n$ with $|t| < \varrho$, one has

$$M|t| \leq |F(t) - F(0)| = |F(t) - p_0|. \tag{1.13}$$

We have then proved that, for any $p \in K$, the orbit of p given by

$$\mathcal{O}_p := \{q \in \Omega : \exists \text{ an } X\text{-admissible curve } \gamma \text{ joining } p \text{ and } q\}$$

is open. Since, by Ascoli-Arzelà's Theorem, $\mathcal{O}_p$ is also closed and since Ω is connected, then $\mathcal{O}_p = \Omega$[1]. Consider now $q \in B_e(p, \sigma)$ and let $t \in \mathbb{R}^n$ with $|t| < \varrho$ and $F(t) = q$. Then, defining $p_j = E_{\alpha_j}(t_j)(p_{j-1})$ for any $j = 1, \dots, n$, we have $p_n = q$ and, taking (1.11) into account , one has

$$\begin{aligned} d(p_0, q) &\leq \sum_{j=1}^n d(p_j, p_{j-1}) = \sum_{j=1}^n d(p_{j-1}, E_{\alpha_j}(t_j)(p_{j-1})) \\ &= \sum_{j=1}^n d(p_{j-1}, E(I_j, |t_j|^{\frac{1}{k}})(p_{j-1})) \leq C_1 \sum_{j=1}^n |t_j|^{\frac{1}{k}} \\ &\leq C_2 |t|^{\frac{1}{k}} \leq C_2 M^{-\frac{1}{k}} |F(t) - F(0)|^{\frac{1}{k}} = C_K |q - p_0|^{\frac{1}{k}}, \end{aligned}$$

where $M > 0$ comes from (1.13). This concludes the proof. □

[1] This is indeed a proof that every couple of points in Ω can be connected be a X-admissible curve, i.e., a proof of Theorem 1.2.1.

1.3 Carnot groups

Carnot groups can be seen as a remarkable subclass of CC spaces. In this section we introduce their definition and we list some theorems that will be useful in the following chapters. We start from the definition of Lie group. For an introduction to Carnot groups see e.g. [75, 59, 55].

Definition 1.3.1 (Lie group)**.** A Lie group $(\mathbb{G}, \cdot)$ is a differentiable manifold $\mathbb{G}$ endowed with a group product $\cdot$ such that the maps

$$\begin{cases} \mathbb{G} \times \mathbb{G} & \longrightarrow \mathbb{G} \\ (x, y) & \mapsto x \cdot y \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{G} & \longrightarrow \mathbb{G} \\ x & \mapsto x^{-1} \end{cases}$$

are differentiable. We will denote by 0 the neutral element of the group. Moreover for every $g \in \mathbb{G}$ we will denote by $\tau_g : \mathbb{G} \to \mathbb{G}$ the left translation map defined as

$$\tau_g(x) = g \cdot x.$$

When no confusion may arise a Lie group will be simply denoted by $\mathbb{G}$. We now recall the definition of Lie algebra.

Definition 1.3.2. A *Lie algebra* is a couple $(V, [\cdot\,, \cdot])$ such that V is a linear space on some field $\mathbb{K}$ and $[\cdot\,, \cdot]$ is a binary operation $[\cdot\,, \cdot] : V \times V \to V$ that is a *Lie bracket*, i.e., it satisfies the following properties.

(i) *Linearity.* For every $\lambda \in \mathbb{K}$ and for every $v, w, z \in V$ one has

$$[\lambda v + w, z] = \lambda[v, z] + [w, z].$$

(ii) *Anti-symmetry.* For every $v, w \in V$ we have

$$[v, w] = -[w, v].$$

(iii) *Jacobi identity.* For every $v, w, z \in V$ one has

$$[v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0.$$

Definition 1.3.3. Let $(\mathbb{G}, \cdot)$ be a Lie group and let X be a vector field on $\mathbb{G}$. X is said to be left invariant if, for every $x, g \in \mathbb{G}$ and every $f \in C^\infty(\mathbb{G})$, one has

$$(Xf)(\tau_g(x)) = X(f \circ \tau_g)(x).$$

The set $\mathfrak{g}(\mathbb{G})$ (or simply $\mathfrak{g}$, for short) denotes the vector space of all left invariant vector fields on $\mathbb{G}$.

Notice that if X and Y are two left invariant vector fields on a Lie group $\mathbb{G}$, also the Lie bracket $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(\mathbb{G}),$$

is a left invariant vector Field on $\mathbb{G}$. As a consequence, one can easily check that the couple $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra. This justifies the following

Definition 1.3.4. Let $(\mathbb{G}, \cdot)$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Define $\mathfrak{g}_1 := \mathfrak{g}$ and for every $i > 1$ we set $\mathfrak{g}_i := [\mathfrak{g}, \mathfrak{g}_{i-1}]$. We say that the Lie group $\mathbb{G}$ is nilpotent of step s ($s \in \mathbb{N}$) if $\mathfrak{g}_s \neq \{0\}$ and $\mathfrak{g}_{s+1} = \{0\}$.

We now recall the definition of exponential map on a differentiable manifold

Definition 1.3.5. Let M be a differentiable manifold, let X be a vector field on M and let $p \in M$. We define $\exp(X)(p) := \gamma(1)$ where $\gamma : [0, 1] \to M$ is the solution of

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = p. \end{cases}$$

It is well known that the exponential map around p provides a local diffeomorphism between a neighborhood of 0 in T_pM and a neighborhood of p on M. Moreover, if M is a Lie group and $X \in \mathfrak{g}$, by left invariance we have that for any $g \in M$

$$X(g) = d\tau_g X(0).$$

This gives the identity $\exp(X)(p) = p \cdot \exp(X)(0)$. Theorem 1.3.6 below gives us an important result of global diffeomorphism between the Lie group and the Lie algebra. Its proof can be found in [86].

Theorem 1.3.6. *Let* $\mathbb{G}$ *a simply connected nilpotent Lie group. Then* $\exp : \mathfrak{g} \to \mathbb{G}$ *is a diffeomorphism.*

Definition 1.3.7 (Stratified group)**.** A nilpotent Lie group $\mathbb{G}$ of step s is said to be *stratified* if there exist linear subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s, \text{ and } [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \text{ for } i = 1, \ldots, s-1.$$

Connected and simply connected stratified Lie groups are also called *Carnot groups.* For every $\lambda > 0$ we also define $\widetilde{\delta}_\lambda : \mathfrak{g} \to \mathfrak{g}$ letting $\widetilde{\delta}_\lambda(X) = \lambda^i X$ if $X \in \mathfrak{g}_i$ and then extending it by linearity on the whole $\mathfrak{g}$.

It is easy to prove the following two propositions.

Proposition 1.3.8. *Let $\mathbb{G}$ be a Carnot group. Then for every $X, Y \in \mathfrak{g}$ and for every $\lambda, \mu \in (0, +\infty)$ we have*

$$\widetilde{\delta}_\lambda([X,Y]) = [\widetilde{\delta}_\lambda(X), \widetilde{\delta}_\lambda(Y)] \quad \text{and } \widetilde{\delta}_{\lambda\mu}(X) = \widetilde{\delta}_\lambda(\widetilde{\delta}_\mu(X)).$$

Moreover $\left\{\widetilde{\delta}_\lambda : \lambda > 0\right\}$ is a family of automorphisms of $\mathfrak{g}$.

Proposition 1.3.9. *Let $(\mathbb{G}, \cdot)$ be Carnot group. For every $\lambda > 0$ define $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ letting $\delta_\lambda(x) := \exp(\widetilde{\delta}_\lambda(\exp^{-1}(x)))$. Then the following facts hold.*

(i) For every $\lambda, \mu > 0$ and every $x \in \mathbb{G}$

$$\delta_{\lambda\mu}(x) = \delta_\lambda(\delta_\mu(x)).$$

(ii) For every $\lambda > 0$ and every $x, y \in \mathbb{G}$

$$\delta_\lambda(x \cdot y) = \delta_\lambda(x) \cdot \delta_\lambda(y).$$

We define the notion of horizontal curves and their length in a Carnot group $\mathbb{G}$.

Definition 1.3.10 (Horizontal curves and horizontal length)**.** Let $\mathbb{G}$ be a Carnot group with Lie algebra decomposition given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$. An absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is said to be horizontal if, for $\mathscr{L}^1$-a.e. $t \in [0,T]$, one has $\dot{\gamma}(t) \in dL_{\gamma(t)}\mathfrak{g}_1 \cong \mathfrak{g}_1$.
Fix a scalar product $\langle \cdot, \cdot \rangle_{\mathbb{G}}$ on $\mathfrak{g}_1$. Denote by $|\cdot|_{\mathbb{G}}$ its induced norm on $\mathfrak{g}_1$ and extend it on the whole $\mathfrak{g}$ by setting $|X|_{\mathbb{G}} = +\infty$ for any $X \in \mathfrak{g} \setminus \mathfrak{g}_1$. Then the *horizontal length* of a horizontal curve γ is defined by

$$\ell_{\mathbb{G}}(\gamma) := \int_0^T |\dot{\gamma}(t)|_{\mathbb{G}} dt.$$

A proof of Theorem 1.3.11 below can be found in [86].

Theorem 1.3.11. *Let $(\mathbb{G}, \cdot)$ and $(F, *)$ be two connected and simply connected Lie groups and let $\mathfrak{g}$ and $\mathfrak{f}$ respectively be the associated Lie algebras of left invariant vector fields. Then $(\mathbb{G}, \cdot)$ is isomorphic (in the sense of Lie groups) to $(F, *)$ if and only if $\mathfrak{g}$ is isomorphic (in the sense of linear spaces) to $\mathfrak{f}$.*

The following result allows us, when dealing with Carnot groups, to always consider $\mathbb{G} = \mathbb{R}^n$ for n equal to the (topological) dimension of the manifold $\mathbb{G}$.

Proposition 1.3.12. *Let $(\mathbb{G}, \cdot)$ be a stratified Lie group of dimension n. Then there exists a group operation $*$ on $\mathbb{R}^n$ such that $(\mathbb{G}, \cdot)$ is isomorphic to $(\mathbb{R}^n, *)$.*

Proof. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and for every $x, y \in \mathbb{R}^n$ define $x * y$ by letting

$$x * y = z \quad \Leftrightarrow \quad \exp\left(\sum_{i=1}^{n} x_i X_i\right) \cdot \exp\left(\sum_{i=1}^{n} y_i X_i\right) = \exp\left(\sum_{i=1}^{n} z_i X_i\right). \tag{1.14}$$

Then it is easy to see that the Lie algebra of $(\mathbb{R}^n, *)$ is isomorphic to $\mathfrak{g}$. By Theorem 1.3.11 the thesis follows. □

Remark 1.3.13. Actually, the group law in $\mathbb{R}^n$ defined in (1.14) can always be written as

$$x * y = P(x, y) = x + y + Q(x, y), \tag{1.15}$$

where $P, Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are polynomial functions. See also 1.3.15 below for a more precise statement about P and Q.

If $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ we set $m_j = \dim(\mathfrak{g}_j)$ for $j = 1, \ldots, s$ and if i is such that $m_1 + \cdots + m_{w_{i-1}} < i \leq m_1 + \cdots + m_{w_i}$ for some $1 \leq w_i \leq s$ we say that the coordinate x_i has degree w_i. An equivalent way to define a dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with $\lambda > 0$, is then by

$$\delta_\lambda(x) := (\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_n} x_n).$$

Proposition 1.3.14 below lists some well-known properties of Carnot groups and some relations between δ_λ and the polynomial P defined in (1.14). A proof of it can be found in [75].

Proposition 1.3.14. *Let $(\mathbb{R}^n, \cdot)$ be a Carnot group. Then, if P is the polynomial function appearing in* (1.15)*, the following facts hold.*

(i) For every $x \in \mathbb{R}^n$ the inverse element with respect to $\cdot$ is $x^{-1} = -x$.

(ii) For every $x, y \in \mathbb{R}^n$ and for every $\lambda > 0$

$$P(\delta_\lambda(x), \delta_\lambda(y)) = \delta_\lambda P(x, y).$$

(iii) For every $x \in \mathbb{R}^n$

$$P(x, 0) = P(0, x) = 0.$$

(iv) If $(X_1, \ldots, X_n)$ is a basis of $\mathfrak{g}$ and $X_j = \sum_{i=1}^{n} a_{ij}(x)\partial_i$ for $j = 1, \ldots, n$ and for some $a_{ij} \in C^\infty(\mathbb{R}^n)$, then we have

$$a_{ij}(\delta_\lambda(x)) = \lambda^{w_i - w_j} a_{ij}(x),$$

for every $i, j = 1, \ldots, n$.

Remark 1.3.15. If $X_1, \dots, X_n$ and F are defined as in 1.4.1, then the vector fields $\widetilde{X}_1, \dots, \widetilde{X}_n$ have the structure

$$\widetilde{X}_j(x) = \partial_j + \sum_{i=n_{w_j}+1}^{n} a_{ji}(x)\partial_i,$$

where $a_{ji}(x) = a_{ji}(x_1, \dots, x_{n_{w_i}-1})$ are homogeneous polynomial of degree $w_i - w_j$.

Notice that every Carnot group $(\mathbb{R}^n, \cdot)$ with stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s$ has also a natural structure of equiregular CC space of step s. Indeed, it is sufficient to consider a basis $X = (X_1, \dots, X_m)$ of $\mathfrak{g}_1$. Directly from the definition of Carnot group we get the Hörmander condition and the equiregularity. In what follows, when dealing with a Carnot group $(\mathbb{R}^n, \cdot)$, we always denote by d the CC metric associated with $(\mathbb{R}^n, X)$ and by $B(p, r)$ a metric ball of center p and radius r. The metric space $(\mathbb{G}, d)$ has then Hausdorff dimension $Q \coloneqq \sum_{j=1}^{s} j \dim \mathfrak{g}_j$ (this is called *homogeneous dimension* of the Carnot group $(\mathbb{R}^n, \cdot)$) and it is well-known that, up to multiplicative constants, the measures $\mathscr{H}^Q$, $\mathscr{S}^Q$ and $\mathscr{L}^n$ coincide, all of them being Haar measures on $\mathbb{G}$.

Proposition 1.3.16. *Let $(\mathbb{R}^n, \cdot)$ be a Carnot group. Then, for every $x, y, g \in \mathbb{R}^n$ and every $\lambda > 0$, we have*

(i) $d(\tau_g(x), \tau_g(y)) = d(x, y)$;

(ii) $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$.

Proof. Taking into account the left invariance of the vector fields $X_1, \dots, X_m$, the proof of (i) simply follows by the fact that, if $\gamma : [0, T] \to \mathbb{R}^n$ is a subunit curve joining x and y, then $\tau_g \circ \gamma : [0, T] \to \mathbb{R}^n$ is a subunit curve joining $\tau_g(x)$ and $\tau_g(y)$.
To prove (ii) let $\gamma : [0, T] \to \mathbb{R}^n$ be a curve joining x and y such that

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} h_j(t) a_{ij}(\gamma(t)) \right) \partial_i,$$

with $|(h_1, \dots, h_m)| \leq 1$, and define $\gamma_\lambda : [0, \lambda T] \to \mathbb{R}^n$ letting $\gamma_\lambda(t) = \delta_\lambda(\gamma(\frac{t}{\lambda}))$. Then $\gamma_\lambda(0) = \delta_\lambda(x)$ and $\gamma_\lambda(\lambda T) = \delta_\lambda(y)$. By statement (iv) of Proposition 1.3.14, we have

$$\begin{aligned}
\dot{\gamma}_\lambda(t) &= \sum_{i=1}^{n} \lambda^{w_i - 1} \left(\sum_{j=1}^{m} h_j(\tfrac{t}{\lambda}) a_{ij}(\gamma(\tfrac{t}{\lambda})) \right) \partial_i \\
&= \sum_{i=1}^{n} \lambda^{w_i - 1} \left(\sum_{j=1}^{m} h_j(\tfrac{t}{\lambda}) a_{ij}(\delta_{\lambda^{-1}} \gamma_\lambda(t)) \right) \partial_i \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} h_j(\tfrac{t}{\lambda}) a_{ij}(\gamma_\lambda(t)) \right) \partial_i = \sum_{j=1}^{m} h_j(\tfrac{t}{\lambda}) X_j(\gamma_\lambda(t)).
\end{aligned}$$

Hence we get that, for every $T > 0$, if $d(x, y) \leq T$, then $d(\delta_\lambda(x), \delta_\lambda(y)) \leq \lambda T$. Then we get $d(\delta_\lambda(x), \delta_\lambda(y)) \leq \lambda d(x, y)$, for any $x, y \in \mathbb{G}$ and any $\lambda > 0$. Repeating the same argument with $\delta_\lambda(x)$, $\delta_\lambda(y)$ in place of x and y and with λ^{-1} in place of λ we also get

$$d(x, y) = d(\delta_{\lambda^{-1}}(\delta_\lambda(x)), \delta_{\lambda^{-1}}(\delta_\lambda(y))) \leq \lambda^{-1} d(\delta_\lambda(x), \delta_\lambda(y),$$

which concludes the proof. □

In a Carnot group it can be useful to define a homogeneous norm letting for every $x \in \mathbb{R}^n$

$$\|x\| := \sum_{i=1}^{n} |x|^{\frac{1}{w_i}},$$

and the corresponding boxes given by $A(x, r) := \{y \in \mathbb{R}^n : \|x^{-1} \cdot y\| \leq r\}$. We eventually set $A(r) := A(0, r)$.
A proof of the following results can be found e.g. in [75].

Proposition 1.3.17. *Let $(\mathbb{R}^n, \cdot)$ be a Carnot group. Then the following facts hold.*

(i) For every $x, g \in \mathbb{R}^n$ and every $r, \lambda > 0$ one has

$$\tau_g B(x, r) = B(\tau_g(x), r) \quad \textit{and} \quad \delta_\lambda B(x, r) = B(\delta_\lambda(x), \lambda r).$$

(ii) There exists $C > 1$ such that

$$A(x, \tfrac{1}{C} r) \subseteq B(x, r) \subseteq A(x, Cr),$$

for every $x \in \mathbb{R}^n$ and every $r > 0$.

Corollary 1.3.18. *Let $(\mathbb{R}^n, \cdot)$ be a Carnot group. Then the metric space $(\mathbb{R}^n, d)$ is geodesic, complete and locally compact.*

Proposition 1.3.19. *Let $(\mathbb{R}^n, \cdot)$ be a Carnot group, $E \subseteq \mathbb{R}^n$ a Lebesgue measurable set. Then the following facts hold.*

(i) For every $g \in \mathbb{R}^n$ one has

$$\mathscr{L}^n(\tau_g(E)) = \mathscr{L}^n(E).$$

(ii) For every $\lambda > 0$ one has

$$\mathscr{L}^n(\delta_\lambda(E)) = \lambda^Q \mathscr{L}^n(E).$$

In particular, $\mathscr{L}^n(B(x, r)) = r^Q \mathscr{L}^n(B(0, 1))$

Proof. It is sufficient to apply area formula and use the fact that $\det(d\tau_g) = 1$ and $\det(d\delta_\lambda) = \lambda^Q$. □

We here recall the notion of calibration, which is widely used in the Calculus of Variation. Proposition 1.3.21 gives a sufficient condition to find a geodesic in a Carnot group.

Definition 1.3.20. A closed 1-form ϑ on the Lie algebra $\mathfrak{g}$ of a Carnot group $\mathbb{G}$ is said to be a *calibration* on $\mathbb{G}$ if for almost every $v \in T\mathbb{G}$ one has $|\vartheta(v)| \leq |v|_{\mathbb{G}}$. We also say that a horizontal curve $\gamma : [0,T] \to \mathbb{G}$ is *calibrated by* ϑ, if ϑ is a calibration and

$$\vartheta(\dot{\gamma}(t)) = |\dot{\gamma}(t)|_{\mathbb{G}},$$

for $\mathscr{L}^1$-almost every $t \in [0,T]$.

Proposition 1.3.21. *Let $\gamma : [0,T] \to \mathbb{G}$ be a horizontal curve in a Carnot group $\mathbb{G}$ that is calibrated by ϑ. Then γ minimizes the distance between $\gamma(0)$ and $\gamma(T)$.*

Proof. Let $\omega : [0,T] \to \mathbb{G}$ be a horizontal curve joining $\gamma(0)$ and $\gamma(T)$. Since $\mathbb{G}$ is connected and simply connected, the curves $\gamma([0,T])$ and $\omega([0,T])$ are homotopic and therefore $\int_\gamma d\vartheta = \int_\omega d\vartheta$. Taking into account that γ is calibrated by ϑ, we get

$$\ell_{\mathbb{G}}(\omega) = \int_0^T |\dot{\omega}(t)|_{\mathbb{G}} dt \geq \int_\omega d\vartheta = \int_\gamma d\vartheta = \int_0^T |\dot{\gamma}(t)| dt = \ell_{\mathbb{G}}(\gamma),$$

which concludes the proof. □

Lemma 1.3.22. *Let M be a n-dimensional manifold and let $R_1, \dots, R_n, S_1, \dots, S_n$ be vector fields in M such that both $(R_1(x), \dots, R_n(x))$ and $(S_1(x), \dots, S_n(x))$ are basis for T_xM, for any $x \in M$. Let $A : M \to \mathbb{R}^{n\times n}$ be such that*

$$S_j(x) = \sum_{\ell=1}^n A_j^\ell(x) R_\ell(x),$$

for any $x \in M$ and for any $j = 1, \dots, n$, and define, for $i = 1, \dots, n$ the 1-forms R_i^*, S_i^* letting

$$R_i^*(x)(R_j(x)) = \delta_{ij},$$
$$S_i^*(x)(S_j(x)) = \delta_{ij},$$

for any $x \in M$ and for $j = 1, \dots, n$. Then $S_j^*(x) = \sum_{\ell=1}^n B_j^\ell(x) R_\ell^*(x)$, where $B = A^{-T}$.

Proof. It is enough to consider the following identity

$$\begin{aligned}\delta_{ij} = S_i^*(x)(S_j(x)) &= \sum_{\ell=1}^n B_i^\ell(x) R_\ell^*(x) \left(\sum_{k=1}^n A_j^k(x) R_k(x) \right) \\ &= \sum_{\ell=1}^n \sum_{k=1}^n B_i^\ell(x) A_j^k(x) \delta_{\ell k} = \sum_{k=1}^n B_i^k(x) A_j^k(x) = \left(B^T(x) A(x)\right)_i^j,\end{aligned}$$

to conclude the proof. □

The proof of Proposition 1.3.23 below is contained [88, Proposizione 7.4].

Proposition 1.3.23. *Let* $\mathbb{G}$ *be a Carnot group with Lie algebra* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$, *and let* $X \in \mathfrak{g}_1$. *Then the curve* $[0,1] \ni t \mapsto \exp(tX)$ *is the unique (up to reparametrizations) geodesic joining* 0 *and* $\exp(X)$. *In particular* $d(0, \exp(tX)) = |X|$.

Proof. Up to a normalization argument, we can assume without loss of generality that $|X|= 1$. Let $X =: X_1, X_2, \ldots, X_n$ be an adapted basis for $\mathfrak{g}$. We also identify $\mathbb{G}$ with $\mathbb{R}^n$ by exponential coordinates as in Definition 1.4.1 so that we can also assume (see Remark 1.3.15)

$$X_j(x) = \partial_j + \sum_{i=n_{w_j}+1}^{n} a_{ij}(x)\partial_i,$$

for every $j = 1, \ldots, n$ and $x \in \mathbb{G}$. Then, for any $x \in \mathbb{G}$, the lower triangular matrix $A(x) := \mathrm{col}(X_1, \ldots, X_n)$ is such that $A_i^i(x) = 1$ for any $i = 1, \ldots, n$. It is also clear that $X_j(x) = \sum_{\ell=1}^n A_j^\ell(x)\partial_\ell$. Define for $j = 1, \ldots, n$ the 1-form X_j^* letting

$$X_j^*(x)(X_i(x)) = \delta_{ji},$$

for any $i = 1, \ldots, n$ and any $x \in \mathbb{G}$. Then, by Lemma 1.3.22, $X_j^*(x) = \sum_{\ell=1}^n B_j^\ell(x)dx_\ell$, where $B = A^{-T}$. By the structure of A, we get that B is upper triangular and $B_i^i(x) = 1$ for any $x \in \mathbb{G}$ and any $i = 1, \ldots, n$. Then $X_1^* = dx_1$. We want to prove that X_1^* is a calibration in $\mathbb{G}$ for the curve

$$\begin{aligned} \gamma : [0,1] \to \;& \mathbb{G} \\ t \mapsto \;& \exp(tX). \end{aligned}$$

Fix $x \in \mathbb{G}$ and take $v \in T_x\mathbb{G}$. If v is not horizontal it is trivially true that $X_1^*(x)(v) \leq |v|= +\infty$. Otherwise if $v = \sum_{i=1}^m X_i(x)v_i$, then $X_1^*(x)(v) = v_1 \leq (\sum_{i=1}^m v_i^2)^{1/2}$. Notice also that $|X_1^*(x)(v)|= |v|$ only if $v = \lambda X_1(x)$ for $\lambda \in \mathbb{R}$. Since $\dot\gamma(t) = X_1(\gamma(t))$ has this form, then $X_1^*(\gamma(t))(\dot\gamma(t)) = |\dot\gamma(t)|$. Therefore X_1^* is a calibration for γ. By Proposition 1.3.21 we infer that γ is a geodesic between 0 and $\exp(X)$.
It is now enough to prove that all the geodesics joining 0 and $\exp(X)$ are a parametrization of γ. Consider a geodesic $\omega : [0,1] \to \mathbb{G}$ joining 0 and $\exp(X)$. Then γ and ω share the same extremal points and, since X_1^* is closed, we have

$$\ell_{\mathbb{G}}(\omega) = \int_0^1 |\dot\omega(t)|dt \geq \int_\omega dX_1^* = \int_\gamma dX_1^* = \int_0^1 |\dot\gamma(t)|dt = \ell_{\mathbb{G}}(\gamma) = \ell_{\mathbb{G}}(\omega).$$

Hence for $\mathscr{L}^1$-almost every $t \in [0,1]$ one has $|\dot\omega(t)|= X_1^*(\gamma(t))(\dot\gamma(t))$. This is possible only if $\dot\omega(t) = \lambda\dot\gamma(t)$ for some $\lambda \in \mathbb{R}$. □

We conclude this section presenting some notable examples of Carnot groups.

Example 1.3.24. Apart from Euclidean spaces, which are the only commutative Carnot groups, the most basic examples of Carnot groups are Heisenberg groups. Given an integer $n \geq 1$, the n-th Heisenberg group $\mathbb{H}^n$ is the $2n+1$ dimensional Carnot group of step 2 whose Lie algebra is generated by $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ and the only non-vanishing bracket relations among these generators are given by

$$[X_j, Y_j] = T \qquad \text{for any } j = 1, \dots, n.$$

The stratification of the Lie algebra is given by $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 \coloneqq \operatorname{span}\{X_j, Y_j : j = 1, \dots n\}$ and $\mathfrak{g}_2 \coloneqq \operatorname{span}\{T\}$. In exponential coordinates

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \ni (x, y, t) \longleftrightarrow \exp(x_1 X_1 + \dots + y_n Y_n + tT)$$

one has

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad T = \partial_t.$$

Example 1.3.25. The Engel group $\mathbb{E}$ is the Carnot group of step 3 and rank 2 whose Lie algebra is is generated by X_1, X_2 and the only non-vanishing bracket relations are given by

$$[X_1, X_2] =: -X_3 \qquad \text{and } [X_1, X_3] =: -X_4.$$

The stratification of the Lie Algebra is therefore given by given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ where $\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}$, $\mathfrak{g}_2 = \operatorname{span}\{X_3\}$ and $\mathfrak{g}_3 = \operatorname{span}\{X_4\}$. In exponential coordinates

$$\mathbb{R}^4 \times \mathbb{R} \ni (x_1, x_2, x_3, x_4) \longleftrightarrow \exp(x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4)$$

one has

$$\begin{aligned} X_1 &= \partial_1, \\ X_2 &= \partial_2 - x_1 \partial_3 + \frac{x_1^2}{2}\partial_4, \\ X_3 &= \partial_3 - x_1 \partial_4, \\ X_4 &= \partial_4. \end{aligned}$$

Notice that the homogeneous dimension of $\mathbb{E}$ is given by $Q = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 7$ while the topological dimension is 4.

Finally, we define the notion of Carnot group of *type* $\star$ introduced in [66]; this will be used in Section 2.2.1.

Definition 1.3.26. A Carnot group $\mathbb{G}$ is said to be of *type* $\star$ if there exists a basis $(X_1, \dots, X_m)$ of the Lie algebra $\mathfrak{g}$ such that

$$[X_j, [X_j, X_i]] = 0,$$

for every $i, j = 1, \dots, m$.

Notice that the Heisenberg group $\mathbb{H}^n$ is of type $\star$, while the Engel group is not. The group of upper triangular matrices with 1's on the diagonal is a Carnot group of type $\star$. In particular, there exist Carnot groups of type $\star$ of any dimension and arbitrarily large step.

1.4 Nilpotent approximation

In this section we describe the so-called nilpotent approximation of a CC space that is, roughly speaking, its infinitesimal structure.

Definition 1.4.1 (Adapted exponential coordinates)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $p \in \mathbb{R}^n$ be fixed; choose an open neighborhood $V \subseteq \mathbb{R}^n$ of p and smooth vector fields $Y_1, \ldots, Y_n$ such that

- $Y_i = X_i$ for any $i = 1, \ldots, m$;
- for every $q \in V$ and every $k = 1, \ldots, s$ the set $\{Y_1(q), \ldots, Y_{n_k}(q)\}$ is a basis of $\mathcal{L}^k(q)$;
- for every $i = m+1, \ldots, n$ the vector field Y_i is chosen among the iterated commutators of $X_1, \ldots, X_m$.

Then there exist a neighborhood U of 0 in $\mathbb{R}^n$ for which the map

$$\begin{aligned} F : U &\to \mathbb{R}^n \\ x &\mapsto \exp(x_1 Y_1 + \cdots + x_n Y_n)(p) \end{aligned} \tag{1.16}$$

is well defined. We say that $(x_1, \ldots, x_n)$ are *adapted exponential coordinates* around p.

The definition of F depends on p; when confusion may arise, we underline this dependence by using the notation F_p to denote (for any $x \in \mathbb{R}^n$ for which it is defined) the map $F_p(x) := \exp(x_1 Y_1 + \cdots + x_n Y_n)(p)$. When needed, we will also write $F(p, x)$ to denote $\exp(x_1 Y_1 + \cdots + x_n Y_n)(p)$; notice that, for every bounded set $V \subseteq \mathbb{R}^n$, one can find an open neighborhood U of 0 in $\mathbb{R}^n$ such that F is well defined in $V \times U$.

For every $p \in \mathbb{R}^n$ and every $j = 1, \ldots, m$, we define

$$\widetilde{X}_j := dF_p^{-1}(X_j \circ F_p).$$

It is readily seen that if X satisfies the Chow-Hörmander condition, then also $\widetilde{X}$ does and we denote by $\widetilde{d}$ the CC distance in $\mathbb{R}^n$ associated with the m-tuple of vector fields $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_m)$, and by $\widetilde{B}(0, r)$ the metric balls around the origin induced by $\widetilde{d}$. Again, when confusion may arise we shall use the notation $\widetilde{B}_p(0, r)$ to specify that the metric ball is induced by the map F_p. Since $dF_p(0)e_j = Y_j(p)$, we have $\widetilde{X}_j(0) = e_j$ for every $j = 1, \ldots, m$. Moreover, it is easy to verify that, for every $p \in \mathbb{R}^n$ and every sufficiently small $r > 0$ one has

$$d(F_p(x_1), F_p(x_2)) = \widetilde{d}(x_1, x_2),$$

for every $x_1, x_2 \in \widetilde{B}(0, r)$. Consequently $F_p(\widetilde{B}(x, r)) = B(F_p(x), r)$.

Remark 1.4.2. If we define $\mu_p := (F^{-1})_{\#}\mathscr{L}^n$, i.e. the measure such that

$$\mu_p(A) = \mathscr{L}^n (F(A)) = \int_A |\det \nabla F| \, d\mathscr{L}^n,$$

for every Borel set A in $\mathbb{R}^n$, then, it is easy to see that, whenever $0 < \varepsilon < |\det \nabla F_p(0)|$, there exists an open neighborhood U of 0 such that

$$(|\det \nabla F_p(0)| - \varepsilon) \, \mathscr{L}^n \llcorner U \leq \mu_p \llcorner U \leq (|\det \nabla F_p(0)| + \varepsilon) \, \mathscr{L}^n \llcorner U. \tag{1.17}$$

Definition 1.4.3. If $(\mathbb{R}^n, X)$ is a CC space and $Y_1, \ldots, Y_n$ are as in Definition 1.4.1 we define the j^{th} *degree of the coordinates* at p letting

$$w_j(p) := \min\{k \in \mathbb{N} : Y_j(p) \in \mathcal{L}^k(p) \setminus \mathcal{L}^{k-1}(p)\},$$

If the space $(\mathbb{R}^n, X)$ is equiregular, w_j do not depend on p and it we can define, for every $r > 0$, the *anisotropic dilation* δ_r letting

$$\begin{aligned} \delta_r : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto (x_1, \ldots, r^{w_i} x_i, \ldots, r^s x_n) \, . \end{aligned} \tag{1.18}$$

Eventually, we introduce the pseudo-norm

$$\|x\| := \sum_{j=1}^{n} |x_j|^{\frac{1}{w_j}},$$

and the sets

$$A(r) := \{y \in \mathbb{R}^n : \|y\| \leq r\} \, . \tag{1.19}$$

It is easy to prove that $\delta_r (A(1)) = A(r)$. We also say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is δ-homogeneous of degree $w \in \mathbb{N}$ if for every $p \in \mathbb{R}^n$ and every $\lambda > 0$ one has $f(\delta_\lambda p) = \lambda^w f(p)$.

By the following proposition, proved in [78], the family of balls $\{\widetilde{B}(0, r) : r \in (0, R)\}$ gives the same topology as the family $\{A(r) : r \in (0, R')\}$.

Theorem 1.4.4. *Let $K \subseteq \mathbb{R}^n$ be a compact set in an equiregular CC space $(\mathbb{R}^n, X)$ and let U be a neighborhood of 0 such that, for every $p \in K$, the map F_p is well-defined in U. Then there exists $C > 1$ such that*

$$\frac{1}{C}\|x\| \leq \widetilde{d}_p(0, x) \leq C\|x\|,$$

for every $x \in U$ and every $p \in K$.

The following Theorem is proved in [76, Theorem 2.3 and Proposition 2.5].

Theorem 1.4.5. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $p \in \mathbb{R}^n$. Then if U is a neighborhood of p and if $Y_1, \dots, Y_n$ are vector fields as in Definition 1.4.1, we define $\widetilde{Y}_i := dF_p^{-1}(Y_i \circ F_p)$ in a neighborhood V of 0 so that $F_p(V) = U$. Let $a_{ij} \in C^\infty(V)$ be such that for every $i = 1, \dots, n$ and for every $x \in \mathbb{R}^n$ one has*

$$\widetilde{Y}_i(x) = \sum_{j=1}^n a_{ij}(x)\partial_j.$$

Then, for any $i, j = 1, \dots, n$, there exist a polynomial p_{ij} and a smooth function $r_{ij} \in C^\infty(V)$ so that $a_{ij} = p_{ij} + r_{ij}$ and the following conditions hold.

(a) If $w_j \geq w_i$, then p_{ij} is δ-homogeneous of degree $w_j - w_i$.

(b) If $w_j \leq w_i$, then $p_{ij} = \delta_{ij}$ (in particular $p_{ij} = 0$ if $w_j < w_i$).

(c) $r_{ij}(0) = 0$ and $\lim_{x\to 0} \|x\|^{w_i - w_j} r_{ij}(x) = 0$.

Moreover, if we define for $i = 1, \dots, m$ and $r > 0$ the vector fields

$$\widehat{X}_i(x) := \sum_{j=1}^n p_{ij}(x)\partial_j \quad \text{and} \quad \widetilde{X}_i^r = r(d\delta_{1/r})[\widetilde{Y}_i \circ \delta_r],$$

then, for any $i = 1, \dots, m$, $\widetilde{X}_i^r$ converges to $\widehat{X}_i$ as $r \to 0$ in the C^∞_{loc}-topology and the couple $(\mathbb{R}^n, \widehat{X} := (\widehat{X}_1, \dots, \widehat{X}_m))$ is a Carnot group.

The vector fields $\widehat{X}_1, \dots, \widehat{X}_m$ introduced in Theorem 1.4.5 are known in the literature as the *nilpotent approximation* of $X_1, \dots, X_m$ at the point p. The structure $(\mathbb{R}^n, \widehat{X})$ is known as the *tangent Carnot-Carathéodory structure* of $(\mathbb{R}^n, X)$ at the point p. We shall denote by $\widehat{d}$ the Carnot-Carathéodory distance associated with $\widehat{X}$ and by $\widehat{B}$ the corresponding balls. When confusion may arise, we shall use the notation $\widehat{B}_p$ to specify the dependence on the point p. Notice that, by the fact that $(\mathbb{R}^n, \widehat{X})$ is a Carnot group, there exists $\widehat{C} > 0$ such that

$$\mathscr{L}^n(\widehat{B}(x, r)) = \widehat{C} r^Q, \tag{1.20}$$

for every $x \in \mathbb{R}^n$ and for all $r > 0$. It will be useful to notice the following

Remark 1.4.6. Let $K \subseteq \mathbb{R}^n$ be a compact set; then there exists $M \geq 1$ such that the constant $\widehat{C} = \widehat{C}_p$ appearing in (1.20) satisfies

$$\frac{1}{M} \leq \widehat{C}_p \leq M \qquad \forall\, p \in K.$$

This follows because, by Theorem 1.4.9, for any $p \in K$

$$\begin{aligned}
\widehat{C}_p &= \lim_{r\to 0} \frac{\mathscr{L}^n(\widehat{B}_p(0, r))}{r^Q} = \lim_{r\to 0} \frac{\mathscr{L}^n(\widetilde{B}_p(0, r))}{r^Q} = \lim_{r\to 0} \frac{\mathscr{L}^n(F_p^{-1} B(p, r))}{r^Q} \\
&= \frac{1}{|\det \nabla F_p(0)|} \lim_{r\to 0} \frac{\mathscr{L}^n(B(p, r))}{r^Q}
\end{aligned}$$

and one can conclude by using Theorem 1.2.4 (iii) and the smoothness of $F(p, x)$.

By [76, Remark 2.6] we also have the following

Proposition 1.4.7. *Let* $(\mathbb{R}^n, X)$ *be an equiregular CC space and* $x \in \mathbb{R}^n$. *Then we have*

$$\exp\left(x_1\widehat{X}_1 + \cdots + x_n\widehat{X}_n\right)(0) = x.$$

Corollary 1.4.8. *Let* $(\mathbb{R}^n, X)$ *be an equiregular CC space, and let* $r > 0$. *Then for every* $p \in \mathbb{R}^n$ *one has*

$$x \in \widehat{B}_p(0,r) \Longleftrightarrow -x \in \widehat{B}_p(0,r).$$

Proof. By Proposition 1.4.7 and by simple properties of Carnot groups we immediately get

$$-x = \exp\left(-\sum_{j=1}^{n} x_j\widehat{X}_j\right)(0) = \left[\exp\left(\sum_{j=1}^{n} x_j\widehat{X}_j\right)(0)\right]^{-1} = x^{-1},$$

which combined with the left invariance of $\widehat{d}$ with respect to the group operation implies

$$\widehat{d}(0,-x) = \widehat{d}(0,x^{-1}) = \widehat{d}(x\cdot 0, x\cdot x^{-1}) = \widehat{d}(x,0).$$

This concludes the proof. □

The proof of Theorem 1.4.9 below can be found in [13] or [14].

Theorem 1.4.9. *Let* $(\mathbb{R}^n, X)$ *be an equiregular CC space, and let* $\widehat{X} = (\widehat{X}_i, \ldots, \widehat{X}_m)$ *be as in Theorem 1.4.5. Then, for every* $p \in \mathbb{R}^n$ *the following holds*

$$\lim_{r\to 0}\left(\sup\left\{\frac{|\widetilde{d}(x,y) - \widehat{d}(x,y)|}{r} : x,y \in \widetilde{B}_p(0,r)\right\}\right) = 0. \tag{1.21}$$

Corollary 1.4.10. *Let* $(\mathbb{R}^n, X)$ *be an equiregular CC space and let* $p \in \mathbb{R}^n$. *For every* $\varepsilon > 0$, *there exists* $R > 0$ *such that for every* $r \in (0,R)$ *one has*

$$\widetilde{B}(0,(1-\varepsilon)r) \subseteq \widehat{B}\,(0,r) \subseteq \widetilde{B}(0,(1+\varepsilon)r).$$

Proof. By (1.21), for every $\varepsilon > 0$ there exists $R > 0$ such that, for every $r \in (0,R)$ and every $x,y \in \widetilde{B}(0,r)$, one has

$$\left|\widetilde{d}(x,y) - \widehat{d}(x,y)\right| \le \varepsilon r.$$

This completes the proof. □

1.5 Hypersurfaces of class C^1_X

This section is devoted to the study of hypersurfaces with intrinsic C^1 regularity; we work in a fixed equiregular CC space $(\mathbb{R}^n, X)$. As customary, given an open set $\Omega \subseteq \mathbb{R}^n$ we denote by $C^1_X(\Omega)$ the space of continuous functions $f : \Omega \to \mathbb{R}$ Ω such that the derivatives $X_1 f, \dots, X_m f$ are represented, in the sense of distributions, by continuous functions.

Definition 1.5.1 (C^1_X-hypersurface)**.** We say that $S \subseteq \mathbb{R}^n$ is a C^1_X*-hypersurface* (or *hypersurface of class* C^1_X) if for every $p \in S$ there exist $R > 0$ and $f \in C^1_X(B(p, R))$ such that the following facts hold

(i) $S \cap B(p, R) = \{q \in B(p, R) : f(q) = 0\}$;

(ii) $Xf(\xi) \neq 0$ on $B(p, R)$.

Moreover, for every p in S we define the *horizontal normal* $\nu_S(p) \in \mathbb{S}^{m-1}$ to S at p letting

$$\nu_S(p) := \frac{Xf(p)}{|Xf(p)|}.$$

The horizontal normal is well-defined up to a sign and, in particular, it does not depend on the choice of f: this is a consequence, for instance, of Corollary 1.5.4 below.

We also introduce the notion of intrinsic Lipschitz regularity for hypersurfaces introduced in [89]. We say that a map $f : \Omega \to \mathbb{R}$ is X-Lipschitz if it is Lipschitz with respect to the CC distance. It is well known that if f is X-Lipschitz, then $Xf = (X_1 f, \dots, X_m f)$ is in $L^\infty(\Omega)$. Vice versa, (see [37, 42]), if $f \in C(\Omega)$ and $Xf \in L^\infty(\Omega)$, then f is X-Lipschitz in any open set $\Omega' \Subset \Omega$.
Hypersurfaces with X-Lipschitz or C^1_X regularity have locally finite $(Q-1)$-dimensional Hausdorff measure, see [89].

Definition 1.5.2 (X-Lipschitz hypersurface)**.** We say that $S \subseteq \mathbb{R}^n$ is an *X-Lipschitz hypersurface* if for every $p \in S$ there exist $R > 0$ and an X-Lipschitz map $f : B(p, R) \to \mathbb{R}$ such that the following holds.

(i) $B(p, R) \cap S = \{q \in B(p, R) : f(q) = 0\}$;

(ii) there exist $C > 0$ and $1 \leq j \leq m$ such that $X_j f(q) \geq C$ for $\mathscr{L}^n$-a.e. $q \in B(p, R)$.

Given $\nu \in \mathbb{R}^m$ we define $\widetilde{L}_\nu : \mathbb{R}^n \to \mathbb{R}$ letting

$$\widetilde{L}_\nu(\xi) := \sum_{i=1}^m \nu_i \xi_i. \tag{1.22}$$

This notation will be extensively used throughout this chapter. The following proposition shows that the maps $\widetilde{L}_\nu$ provide a sort of first-order "linear" approximation for C^1_X functions.

Proposition 1.5.3. *Let $p \in \Omega$, $R > 0$ and let $f \in C^1_X(B(p,R))$. Then*

$$\lim_{r\to 0}\left(\sup\left\{\frac{|f(F_p(x)) - f(p) - \widetilde{L}_{Xf(p)}(x)|}{r} : x \in \widetilde{B}(0,r)\right\}\right) = 0.$$

Proof. It is not restrictive to assume that $f(p) = 0$. Let $r \le R$ and take $x \in \widetilde{B}(0,r)$. Denote $d = \widetilde{d}(x,0)$ and take a geodesic $\gamma \in \mathrm{Lip}([0,d];\mathbb{R}^n)$ such that $\gamma(0) = 0$, $\gamma(d) = x$ and there exists $h : [0,d] \to \mathbb{R}^m$ such that for $\mathscr{L}^1$-a.e. $t \in [0,d]$ we have

$$|h(t)| = 1 \qquad \text{and} \qquad \dot\gamma(t) = \sum_{j=1}^m h_j(t)\widetilde{X}_j(\gamma(t)).$$

Notice that $\widetilde{X}_j(0) = e_j$ and hence by Lipschitz continuity of the vector fields we find $C > 0$ such that for every $y \in \widetilde{B}(0,r)$ and for every $j = 1,\dots,m$

$$|\widetilde{X}_j(y) - e_j| \le Cr.$$

Therefore, for every $k = 1,\dots,m$

$$\begin{aligned}\left|x_k - \int_0^d h_k(t)dt\right| &= \left|\left(\int_0^d \dot\gamma(t)dt\right)_k - \int_0^d h_k(t)dt\right| \\ &= \left|\sum_{j=1}^m \int_0^d h_j(t)\left(\widetilde{X}_j(\gamma(t))\right)_k dt - \sum_{j=1}^m \int_0^d h_j(t)\,(e_j)_k\,dt\right| \\ &= \left|\sum_{j=1}^m \int_0^d h_j(t)\left(\widetilde{X}_j(\gamma(t)) - e_j\right)_k dt\right| \le mCrd \le mCr^2.\end{aligned}$$

Hence, if for every $x \in \widetilde{B}(0,r)$ we set $d := \widetilde{d}(x,0)$ and we denote by h a control associated with the geodesic γ that links 0 to x, we have

$$\lim_{r\to 0}\left(\sup\left\{\frac{1}{r}\left|x_k - \int_0^d h_k(t)dt\right| : x \in \widetilde{B}(0,r), k = 1,\dots,m\right\}\right) = 0. \tag{1.23}$$

Notice also that for every $x \in \widetilde{B}(0,r)$

$$\begin{aligned}f(F_p(x)) &= f(F_p(x)) - f(F_p(0)) = f(F_p(\gamma(d))) - f(F_p(\gamma(0))) \\ &= \int_0^d \sum_{j=1}^m X_j f(F_p(\gamma(t)))h_j(t)dt.\end{aligned}$$

Take $\varepsilon > 0$. By (1.23) and the continuity of Xf in p, we can choose $r_0 \in (0,R)$ such that for every $r \in (0,r_0)$

$$\sup\left\{\frac{1}{r}\left|(x_1,\dots,x_m) - \int_0^d h(t)dt\right| : x \in \widetilde{B}(0,r)\right\} < \frac{\varepsilon}{2|Xf(p)|}$$

and $|Xf(F_p(x)) - Xf(p)| < \varepsilon/2$ for every $x \in \widetilde{B}(0,r)$. Then, for any $r \in (0, r_0)$ and every $x \in \widetilde{B}(0,r)$, we have

$$\begin{aligned}|f(F_p(x)) - \widetilde{L}_{Xf(p)}(x)| &= \left| \int_0^d \langle h(t), Xf(F_p(\gamma(t))) \rangle \, dt - \sum_{j=1}^m X_j f(p) x_j \right| \\ &\leq \int_0^d |h(t)||Xf(F_p(\gamma(t))) - Xf(p)| dt \\ &\quad + |Xf(p)| \left| (x_1, \ldots, x_m) - \int_0^d h(t) dt \right| \\ &< d\frac{\varepsilon}{2} + |Xf(p)| \left| (x_1, \ldots, x_m) - \int_0^d h(t) dt \right| .\end{aligned}$$

The result follows by dividing both sides by r and taking into account that $d \leq r$. □

An immediate consequence of Proposition 1.5.3 is Corollary 1.5.4, where we start using the following convenient notation: given $t \in \mathbb{R}$ and a function $f : I \to \mathbb{R}$ defined on some set I, we denote by $\{f > t\}, \{f = t\}$, etc. the sets $\{x \in I : f(x) > t\}, \{x \in I : f(x) = t\}$, etc. This notation will be extensively used in this chapter.

Corollary 1.5.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in \Omega$ and $f \in C^1_X(B(p,R))$ for some $R > 0$. Suppose that $f(p) = 0$ and $|Xf(p)| = 1$. Define a C^1_X-hypersurface letting $S := \{q \in B(p,R) : f(q) = 0\}$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that*

$$F_p^{-1}(S) \cap \widetilde{B}(0,r) \subseteq \left\{ \xi \in \widetilde{B}(0,r) : -\varepsilon r \leq \widetilde{L}_{Xf(p)}(\xi) \leq \varepsilon r \right\}, \tag{1.24}$$

for every $r \in (0, r_0)$. Moreover, one has

$$\lim_{r \to 0} \frac{1}{r^Q} \mathscr{L}^n \left(\{ \xi \in \widetilde{B}(0,r) : f(F_p(\xi)) \widetilde{L}_{Xf(p)}(\xi) < 0 \} \right) = 0. \tag{1.25}$$

Proof. Fix $\varepsilon > 0$ and apply Proposition 1.5.3 to get $r_0 > 0$ such that for every $0 < r < r_0$ and for every $x \in \widetilde{B}(0,r)$ we have $|f(F_p(x)) - \widetilde{L}_{Xf(p)}(x)| \leq \varepsilon r$. Then, if we take $x \in \widetilde{B}(0,r) \cap \{\widetilde{L}_{Xf(p)} \geq 2\varepsilon r\}$, we also get

$$f(F_p(x)) \geq \varepsilon r.$$

Reasoning in the same way with the set $\{\widetilde{L}_{Xf(p)} \leq -2\varepsilon r\}$ we readily get (1.24). The previous argument shows that for any $\varepsilon > 0$ there exists $r_0 > 0$ such that for any $r \in (0, r_0)$ we have

$$\widetilde{B}(0,r) \cap \{(f \circ F_p)\widetilde{L}_{Xf(p)} \leq 0\} \subseteq \widetilde{B}(0,r) \cap \{-\varepsilon r \leq \widetilde{L}_{Xf(p)} \leq \varepsilon r\}.$$

The proof of (1.25) follows by noticing that, by Theorem 1.4.4

$$\mathscr{L}^n(\widetilde{B}(0,r) \cap \{-\varepsilon r \leq \widetilde{L}_{Xf(p)} \leq \varepsilon r\}) \leq C\varepsilon r^Q,$$

for a suitable constant C independent on r. □

Remark 1.5.5. Let $(\mathbb{R}^n, X)$ be an equiregular CC space, $p \in \mathbb{R}^n$, $R > 0$ and suppose $f_1, f_2 \in C^1_X(B(p,R))$ are such that $f_1(p) = f_2(p) = 0$ and $Xf_1(p) = Xf_2(p)$. Then one has

$$\lim_{r \to 0} \frac{1}{r^Q} \mathscr{L}^n(B(p,r) \cap \{f_1 f_2 \leq 0\}) = 0.$$

Indeed, taking into account (1.24) we observe that

$$\lim_{r \to 0} \frac{1}{r^Q} \mathscr{L}^n(B(p,r) \cap \{f_1 f_2 = 0\}) = 0.$$

On the other hand, since $\widetilde{L}_{Xf_1(p)} = \widetilde{L}_{Xf_2(p)}$ we have

$$\begin{aligned}\{\xi \in B(p,r) : f_1(\xi) f_2(\xi) < 0\} \subseteq & \left(B(p,r) \cap \{f_1 f_2 < 0\} \cap \{\widetilde{L}_{Xf_1(p)} \circ F_p^{-1} > 0\}\right) \\ & \cup \left(B(p,r) \cap \{f_1 f_2 < 0\} \cap \{\widetilde{L}_{Xf_1(p)} \circ F_p^{-1} \leq 0\}\right),\end{aligned}$$

that combined with (1.25) completes the proof.

Theorem 1.5.6. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $S_1, S_2 \subseteq \mathbb{R}^n$ be two hypersurfaces of class C^1_X. Then the set*

$$E \coloneqq \{\xi \in S_1 \cap S_2 : \nu_{S_1}(\xi) \notin \{\pm \nu_{S_2}(\xi)\}\}$$

is $\mathscr{H}^{Q-1}$-negligible.

Proof. By a localization argument and by the fact that C^1_X-hypersurfaces are σ-finite with respect to $\mathscr{H}^{Q-1}$, we can suppose without loss of generality that

$$\mathscr{H}^{Q-1}(S_1) < +\infty \quad \text{and} \quad \mathscr{S}^{Q-1}(E) < +\infty$$

and that S_1 is bounded in $\mathbb{R}^n$. If, for every $\delta > 0$, we define

$$E_\delta \coloneqq \{p \in E : |\langle \nu_{S_1}(p), \nu_{S_2}(p) \rangle| \leq 1 - \delta\},$$

then we have $E = \bigcup \{E_\delta : \delta \in (0, +\infty) \cap \mathbb{Q}\}$.
Fix $\varepsilon \in (0, 1/4)$ and define for every $R > 0$ the set $E_{\delta,R}$ of all the points p of E_δ such that the following three properties hold for every $r \leq 2R$

(a) if $C > 0$ is the constant appearing in Theorem 1.4.4, for every $x \in A(Cr)$ we have $\widehat{B}_p(x, \varepsilon r) \subseteq \widetilde{B}_p(x, 2\varepsilon r)$;

(b) for $i = 1, 2$ we have $F_p^{-1}(S_i \cap B(p, 2r)) \subseteq \{x \in \mathbb{R}^n : |\widetilde{L}_{\nu_{S_i}(p)}(x)| < \varepsilon r\}$;

(c) $\operatorname{diam} B(p,r) = \operatorname{diam} \widetilde{B}_p(0,r) \geq r$.

By Theorems 1.4.9 and 1.5.4 and the fact[2] that $\operatorname{diam} \widehat{B}_p(0,r) = 2r$ we deduce that $E_{\delta,R} \nearrow E_\delta$ as $R \to 0$.
Take now $0 < \eta < \frac{R}{2}$. Then there exist a sequence (q_h) in $\mathbb{R}^n$ and a sequence (r_h) in $(0,\eta)$ such that

$$E_{\delta,R} \subseteq \bigcup_{h=0}^{\infty} B(q_h, r_h) \quad \text{and}$$

$$\sum_{h=0}^{\infty} (r_h)^{Q-1} \leq \sum_{h=0}^{\infty} (\operatorname{diam} B(q_h, r_h))^{Q-1} \leq \mathscr{S}_\eta^{Q-1}(E_{\delta,R}) + 1.$$

We can suppose without loss of generality that for every $h \in \mathbb{N}$ there exists $p_h \in B(q_h, r_h) \cap E_{\delta,R}$. Therefore for every $h \in \mathbb{N}$ one has $B(q_h, r_h) \subseteq B(p_h, 2r_h)$ and consequently

$$E_{\delta,R} \subseteq \bigcup_{h=0}^{\infty} B(p_h, 2r_h).$$

Taking Theorem 1.4.4 into account, we can find $C > 0$ such that for every $h \in \mathbb{N}$ one has

$$F_{p_h}^{-1}(E_{\delta,R} \cap B(p_h, 2r_h)) \subseteq A_h := \left\{ x \in \mathbb{R}^n : \|x\| \leq C r_h, |\widetilde{L}_{\nu_{S_i}(p_h)}(x)| \leq \varepsilon r_h, \text{ for } i = 1, 2 \right\}.$$

We prove now that $\mathscr{L}^n(A_h) \leq C_\delta \varepsilon^2 r_h^Q$ for some $C_\delta > 0$ depending on δ. In fact, since $|\langle \nu_{S_1}(p_h), \nu_{S_2}(p_h)\rangle| \leq 1 - \delta$, we have (up to an orthogonal change of coordinates)

$$\left\{ x \in \mathbb{R}^n : |\widetilde{L}_{\nu_{S_i}(p_h)}(x)| < \varepsilon r_h \text{ for } i = 1, 2 \right\} \subseteq Q^2(0, C_\delta \varepsilon r_h) \times \mathbb{R}^{m-2},$$

where $Q^2(z,s)$ denotes the 2-dimensional cube of center z and size s. Hence

$$A_h \subseteq \left(Q^2(0, C_\delta \varepsilon r_h) \cap \left\{ x \in \mathbb{R}^m : \sum_{j=1}^m |x_j| \leq C r_h \right\} \right) \times \left\{ x \in \mathbb{R}^{n-m} : \sum_{j=m+1}^n |x_j|^{\frac{1}{d_j}} \leq C r_h \right\}$$

and consequently $\mathscr{L}^n(A_h) \leq C_\delta \varepsilon^2 r_h^Q$. For every $h \in \mathbb{N}$, combining Theorem 1.1.10 and the fact that A_h is compact, we can find $N_h \in \mathbb{N}$ and a family $\{x_{h,j} : j = 1, \ldots, N_h\}$ of points of A_h such that $\{\widehat{B}_{p_h}(x_{h,j}, \varepsilon r_h) : j = 1, \ldots, N_h\}$ covers A_h and $\{\widehat{B}_{p_h}(x_{h,j}, \varepsilon \frac{r_h}{5}) : j = 1, \ldots, N_h\}$ is pairwise disjoint. Reasoning as above, it is easy to see that

$$\mathscr{L}^n \left(\left\{ x \in \mathbb{R}^n : \widehat{d}_{p_h}(x, A_h) < \frac{\varepsilon r_h}{5} \right\} \right) \leq \widetilde{C}_\delta \varepsilon^2 r_h^Q,$$

for some $\delta > 0$. Therefore, we can estimate

$$N_h \leq \frac{\mathscr{L}^n \left(\left\{ x \in \mathbb{R}^n : \widehat{d}_{p_h}(x, A_h) < \frac{\varepsilon r_h}{5} \right\} \right)}{\mathscr{L}^n \left(\widehat{B}_{p_h}(x_{h,j}, \frac{\varepsilon r_h}{5}) \right)} \leq \widehat{C}_\delta \varepsilon^{2-Q}$$

---

[2]This is an easy consequence of the fact that, by Proposition 1.3.23, the curve $t \mapsto \exp(t\widehat{X}_1)$ is globally length minimizing.

for some $\widehat{C}_\delta > 0$ depending only on δ. By property (a) we have also $\widehat{B}_{p_h}(x_{h,j}, \varepsilon r_h) \subseteq \widetilde{B}_{p_h}(x_{h,j}, 2\varepsilon r_h)$, and hence the family

$$\left\{\widetilde{B}_{p_h}(x_{h,j}, 2\varepsilon r_h) : j = 1, \dots, N_h\right\}$$

is a covering of A_h, that is also a covering of $F_{p_h}^{-1}(E_{\delta,R} \cap B(p_h, r_h))$. Therefore, the family $\{B(F_{p_h}^{-1}(x_{h,j}), 2\varepsilon r_h) : j \in \mathbb{N}\}$ is a covering of $E_{\delta,R} \cap B(p_h, 2r_h)$. In particular, recalling that $\varepsilon \in (0, 1/4)$ we have

$$\begin{aligned}
\mathscr{S}_\eta^{Q-1}(E_{\delta,R}) &\le \mathscr{S}_{4\varepsilon\eta}^{Q-1}(E_{\delta,R}) \le \sum_{h=0}^{\infty} \mathscr{S}_{4\varepsilon\eta}^{Q-1}(E_{\delta,R} \cap B(p_h, 2r_h)) \\
&\le \sum_{h=0}^{\infty}\sum_{j=1}^{N_h} \left(\operatorname{diam} B(F_{p_h}^{-1}(x_{h,j}), 2\varepsilon r_h)\right)^{Q-1} \le \sum_{h=0}^{\infty} N_h (4\varepsilon r_h)^{Q-1} \\
&\le \sum_{h=0}^{\infty} \widehat{C}_\delta \varepsilon r_h^{Q-1} \le \widehat{C}_\delta \varepsilon(\mathscr{S}_\eta^{Q-1}(E_{\delta,R}) + 1).
\end{aligned}$$

Letting $\eta \to 0$ we get $\mathscr{S}^{Q-1}(E_{\delta,R}) \le \widehat{C}_\delta \varepsilon(\mathscr{S}^{Q-1}(E_{\delta,R}) + 1)$, which gives, letting $R \to 0$

$$\mathscr{S}^{Q-1}(E_\delta) \le \widehat{C}_\delta \varepsilon(\mathscr{S}^{Q-1}(E_\delta) + 1).$$

Letting now $\varepsilon \to 0$ we get, for any $\delta > 0$, that $\mathscr{S}^{Q-1}(E_\delta) = 0$, i.e., $\mathscr{S}^{Q-1}(E) = 0$. This concludes the proof. □

Definition 1.5.7 (X-rectifiability)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension $Q \in \mathbb{N}$ and let $R \subseteq \mathbb{R}^n$. We say that R is *countably X-rectifiable* (respectively, *countably X-Lipschitz rectifiable*) if there exists a family $\{S_h : h \in \mathbb{N}\}$ of C^1_X-hypersurfaces (resp., X-Lipschitz hypersurfaces) such that

$$\mathscr{H}^{Q-1}\left(R \setminus \bigcup_{h=0}^{\infty} S_h\right) = 0. \tag{1.26}$$

Moreover we say that R is *X-rectifiable* (resp., *X-Lipschitz rectifiable*) if R is countably X-rectifiable (resp., countably X-Lipschitz rectifiable) and $\mathscr{H}^{Q-1}(R) < +\infty$.

Definition 1.5.8 (Horizontal normal)**.** Let $R \subseteq \mathbb{R}^n$ be countably X-rectifiable and let (S_h) be a family of C^1_X-hypersurfaces such that (1.26) holds. Then the *horizontal normal* $\nu_R : R \to \mathbb{S}^{m-1}$ to R is defined by

$$\nu_R(p) := \nu_{S_h}(p) \qquad \text{if } p \in R \cap S_h \setminus \bigcup_{k<h} S_k\,.$$

By Theorem 1.5.6, ν_R is well-defined $\mathscr{H}^{Q-1}$-a.e. on R, up to a sign.

1.6 Functions with bounded X-variation

In this section, Ω denotes a fixed bounded open subset of $\mathbb{R}^n$.

Definition 1.6.1. We say that $u \in L^1_{loc}(\Omega)$ is a function of *locally bounded X-variation* in Ω, and we write $u \in BV_{X,loc}(\Omega)$, if there exists a $\mathbb{R}^m$-valued Radon measure $D_X u = (D_{X_1}u, \dots, D_{X_m}u)$ in Ω such that, for every open set $A \Subset \Omega$, for every $i = 1, \dots, m$ and for every $\varphi \in C^1_c(A)$, one has

$$\int_A \varphi \, d\,(D_{X_i}u) = -\int_A u X_i^* \varphi \, d\mathscr{L}^n, \tag{1.27}$$

where X_i^* denotes the formal adjoint of X_i. If $u \in L^1(\Omega)$, we say that f has *bounded X-variation* in Ω ($u \in BV_X(\Omega)$) if, in addition, the total variation $|D_X u|$ of $D_X u$ is finite on Ω.
Moreover, we say that a measurable set $E \subseteq \mathbb{R}^n$ has *locally finite X-perimeter* (resp., *finite X-perimeter*) in Ω if $\chi_E \in BV_{X,loc}(\Omega)$ (resp., $\chi_E \in BV_X(\Omega)$). In such a case we define the *X-perimeter measure* P_X^E of E by $P_X^E := |D_X \chi_E|$. It will sometimes be useful to write $P_X(E, \cdot)$ instead of P_X^E.

As customary, we write $BV_X(\Omega;\mathbb{R}^k) := BV_X(\Omega)^k$, and similarly for $BV_{X,loc}(\Omega;\mathbb{R}^k)$. It can be useful to observe that if $u \in BV_X(\Omega;\mathbb{R}^k)$, the following inequalities hold

$$\max_{1\le i\le k} |D_X u^i|(\Omega) \le |D_X u|(\Omega) \le \sum_{i=1}^k |D_X u^i|(\Omega). \tag{1.28}$$

If $A \Subset \Omega$ is open and $u \in BV_{X,loc}(\Omega)$, one can easily prove that

$$|D_X u|(A) = \sup \left\{ \int_A u \sum_{i=1}^m X_i^*(\varphi_i)\, d\mathscr{L}^n : \varphi \in C^1_c(A;\mathbb{R}^m), |\varphi| \le 1 \right\};$$

actually, $u \in BV_X(A)$ if and only if the supremum on the right-hand-side is finite.

The following important approximation result is proved in [36, Theorem 24.2.2].

Theorem 1.6.2. *Let $u \in BV_X(\Omega;\mathbb{R}^k)$. Then there exists a sequence (u_h) in $C^\infty(\Omega;\mathbb{R}^k)$ such that*

$$\lim_h \|u_h - u\|_{L^1(\Omega;\mathbb{R}^k)} = 0 \qquad \textit{and} \qquad \lim_h |D_X u_h|(\Omega) = |D_X u|(\Omega).$$

Actually, by [36, Theorem 2.2.2], the following stronger approximation result holds.

Theorem 1.6.3. *Let $(\mathbb{R}^n, X)$ be a CC space and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then, for any $u \in BV_X(\Omega)$, there exists a sequence (u_h) in $C^\infty(\Omega) \cap BV_X(\Omega)$ such that the following convergences hold.*

$$\begin{aligned} &u_h \to u \text{ in } L^1(\Omega),\\ &|D_X u_h|(\Omega) \to |D_X u|(\Omega),\\ &|D_{X_i} u_h|(\Omega) \to |D_{X_i} u|(\Omega), \quad \forall\, i = 1, \dots, m\\ &|(D_X u_h, \mathscr{L}^n)|(\Omega) \to |(D_X u, \mathscr{L}^n)|(\Omega). \end{aligned} \tag{1.29}$$

The following easy Proposition will be useful in the sequel.

Proposition 1.6.4. *Let $\Omega, \widetilde{\Omega}$ be two open sets in $\mathbb{R}^n$ and let $G : \Omega \to \widetilde{\Omega}$ be a diffeomorphism. Let also $X_1, \dots, X_m$ be vector fields on Ω and define for every $i = 1, \dots, m$ the vector fields $Y_i := dG(X_i)$ on $\widetilde{\Omega}$. Then*

$$u \in BV_{X,loc}(\Omega) \Leftrightarrow v := u \circ G^{-1} \in BV_{Y,loc}(\widetilde{\Omega}). \tag{1.30}$$

More precisely, for every open set $U \Subset \Omega$, setting $V := G(U)$, one has, for every $u \in BV_{X,loc}(\Omega)$, that

$$m|D_X u|(U) \leq |D_Y v|(V) \leq M|D_X u|(U) \tag{1.31}$$

for $m := \inf_U |\det \nabla G|$ and $M := \sup_U |\det \nabla G|$.

Proof. We claim that, for any open set $U \Subset \Omega$ and any $u \in BV_{X,loc}(\Omega)$, one has

$$v := u \circ G^{-1} \in BV_Y(V) \qquad \text{and} \qquad |D_Y v|(V) \leq M|D_X u|(U).$$

This would be enough to conclude: indeed, the claim would imply both the $\Rightarrow$ implication in (1.30) and the second inequality in (1.31), while the $\Leftarrow$ implication in (1.30) and the first inequality in (1.31) simply follow by replacing X, U, u, G with, respectively, Y, V, v, G^{-1} and noticing that $m = (\sup_V |\det \nabla(G^{-1})|)^{-1}$.

Let us prove the claim. We first assume that $u \in C^\infty(U)$, so that also v is smooth on V. For every $\varphi \in C^1_c(V; \mathbb{R}^m)$ with $|\varphi| \leq 1$, by a change of variable we have that

$$\int_V \langle Yv, \varphi \rangle d\mathscr{L}^n = \int_U \langle Xu, |\det \nabla G| (\varphi \circ G) \rangle d\mathscr{L}^n,$$

which gives

$$|D_Y v|(V) \leq M|D_X u|(U).$$

In case $u \in BV_X(U)$ is not smooth, by Theorem 1.6.2 we can consider a sequence (u_h) in $C^\infty(U)$ that converges to u in $L^1(U)$ and such that

$$\lim_h |D_X u_h|(U) = |D_X u|(U).$$

Defining $v_h := u_h \circ G^{-1}$, we easily get that v_h converges to v in $L^1(V)$ as $h \to +\infty$. Eventually, by the lower semicontinuity of the total variation one has

$$|D_Y v|(V) \leq \liminf_h |D_Y v_h|(V) \leq M \liminf_h |D_X u_h|(U) = M|D_X u|(U),$$

which concludes the proof. □

The following Theorem links the total variation of a X-Lipschitz function to the perimeter of its sublevel-sets. Its proof can be found in [77].

Theorem 1.6.5 (Coarea Formula for X-Lipschitz functions). *Let $(\mathbb{R}^n, X)$ be a CC space, let $u : \mathbb{R}^n \to \mathbb{R}$ be a X-Lipschitz function and let $g : \mathbb{R}^n \to [0, +\infty]$ be a $\mathscr{L}^n$-measurable function. Then, if we define $E_s := \{u < s\}$, we have*

$$\int_{\mathbb{R}^n} g|Xu| d\mathscr{L}^n = \int_{-\infty}^{+\infty} \left(\int_{\{u=s\}} g d(P_X E_s) \right) ds.$$

By Theorem 1.6.2, one can easily improve Theorem 1.6.5 to Theorem 1.6.6 (see [36, Theorem 2.3.5]).

Theorem 1.6.6 (Coarea Formula for BV_X functions). *Let $(\mathbb{R}^n, X)$ be a CC space, let Ω be an open set in $\mathbb{R}^n$ and let $u \in BV_X(\Omega)$. Then, if we define $E_s := \{u < s\}$, we have*

$$|D_X u|(\Omega) = \int_{-\infty}^{+\infty} P_X(E_s, \Omega) ds.$$

The following result is essentially [20, Theorem 1.2]; note, however, that the dimension Q appearing in [20, Theorem 1.2] is slightly different from the homogeneous dimension we are considering.

Theorem 1.6.7. *Let Ω be an open subset of an equiregular CC space $(\mathbb{R}^n, X)$ of homogeneous dimension Q and let $K \subseteq \Omega$ be compact; then, there exists $C > 0$ and $R > 0$ such that, for every $p \in K$, $r \in (0, R)$ and $u \in BV_{X,loc}(\Omega)$, the inequality*

$$\left(\fint_{B(p,r)} |u - u_{p,r}|^{\frac{Q}{Q-1}} \, d\mathscr{L}^n \right)^{\frac{Q-1}{Q}} \leq \frac{C}{r^{Q-1}} |D_X u|(B(p,r)),$$

where $u_{p,r} := \fint_{B(p,r)} u \, d\mathscr{L}^n$.

Proof. The proof easily follows by [46, Theorem 5.1] on taking into account Theorem 1.2.4, [20, Theorem 1.1], [46, Corollary 9.8 and Theorem 10.3] and Theorem 1.6.2. □

An easy consequence of Theorem 1.6.7 is the following isoperimetric inequality.

Theorem 1.6.8 (Isoperimetric inequality in CC spaces). *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $K \subseteq \mathbb{R}^n$ be a compact set. Then there exist $C > 0$ and $R > 0$ such that, for every $p \in K$, $r \in (0, R)$ and every $\mathscr{L}^n$-measurable set $E \subseteq \mathbb{R}^n$, one has*

$$\min \{\mathscr{L}^n(E \cap B(p,r)), \mathscr{L}^n(B(p,r) \setminus E)\}^{\frac{Q-1}{Q}} \leq C P_X(E, B(p,r)).$$

Chapter 2

Fine properties of BV_X functions

This chapter contains the main results of [30]. Section 2.1 is devoted to the introduction of the approximate notions of continuity, X-jumps and X-differentiability for L^1_{loc} functions in an equiregular CC space. Approximate continuity is classical in locally compact, doubling and separable metric measure space (see Theorem 2.1.2) and does not require any additional work in this context. The notions of approximate X-jump and approximate X-differentiability (see Definitions 2.1.6 and 2.1.12 respectively) are instead new and require some fine results about C^1_X hypersurfaces already proved in Section 1.5. The notation $\mathcal{S}_u$, $\mathcal{J}_u$ and $\mathcal{D}_u$ is introduced to denote respectively the *singular set* of u (i.e., the set in which u is not approximately continuous), the X-jump set of u and the X-differentiability set of u. Propositions 2.1.8 and 2.1.13 are devoted to proving the well-posedness of these definition. Propositions 2.1.3, 2.1.11 and 2.1.15 deal with the Borel regularity of $\mathcal{S}_u, \mathcal{J}_u$ and $\mathcal{D}_u$. A fact widely used in this section is the nilpotent approximation of an equiregular CC space introduced with Theorem 1.4.5.

Section 2.2 contains the main results of this chapter about properties of BV functions in equiregular CC spaces. As customary in the literature, we will also assume that the CC balls are bounded with respect to the Euclidean topology. The first technical but very important result is Lemma 2.2.2 which deals with the embedding of $BV_{X,loc}(\Omega)$ into $L^{1^*}_{loc}(\Omega)$ with $1^* = \frac{Q}{Q-1}$. Although the proof might seem to follow a classical plot, we point out that a compactness result for equi-bounded sequences of BV_{X^j} functions for converging X^j is here needed. This is provided by Theorem 4.1.1 and more precisely by Theorem 4.2.6.

Theorem 2.2.9 proves that BV_X functions are approximately X-differentiable almost everywhere and this follows by the inequality proved in Lemma 2.2.6 which is new also in the framework of Carnot groups.

In the case in which the CC space satisfies property $\mathcal{R}$ (see Definition 2.2.12), then all functions of bounded X-variation u satisfy some stronger properties. Theorem 2.2.14 states that $\mathcal{J}_u$ is X-rectifiable and it coincides $\mathscr{H}^{Q-1}$-almost everywhere with

$\mathcal{S}_u$. Theorem 2.2.20 gives information on the "fine" structure of the decomposition $D_X u = D^a_X u + D^j_X u + D^c_X u$ where $D^j_X u$ and $D^c_X u$ denote respectively the X-jump part and the Cantor part of the measure derivative $D_X u$ (see Definition 2.2.8): in Theorem 2.2.20 results (a) and (b) hold without the assumption of the validity of property $\mathcal{R}$. Finally Propositions 2.1.5 and 2.2.17 then imply Theorem 2.2.18 which deals with the convergence of the mean values of a BV_X function to the so-called *precise representative*, which is $\mathscr{H}^{Q-1}$-a.e. well defined whenever the CC space satisfies property $\mathcal{R}$.
Theorem 2.2.23 instead gives a precise structure of the X-jump part of the measure derivative on a general X-rectifiable set in case the CC space satisfies both properties $\mathcal{R}$ and $\mathcal{D}$ (see 2.2.21).
Section 2.2.1 is devoted to describing some classes of CC spaces satisfying property $\mathcal{R}$, $\mathcal{LR}$ (see Definition 2.2.13) and/or $\mathcal{D}$. More specifically, a class of Carnot groups satisfying property $\mathcal{LR}$ (studied in the upcoming paper [27]) is described.

2.1 Approximate notions of continuity, X-jumps and X-differentiability

In this section we introduce the notions of approximate continuity, approximate X-jumps and approximate X-differentiability. Given a Radon measure μ, we use the notation

$$\fint_A u\,d\mu := \frac{1}{\mu(A)}\int_A u\,d\mu,$$

to denote the mean integral of a measurable function u on a μ-measurable set A with $\mu(A) > 0$. Although we are going to work in equiregular CC spaces, Definition 2.1.1 and Proposition 2.1.3 are valid in a wide class of metric measure spaces. For the reader's convenience, we are nonetheless going to show them in this higher general framework. We say that a triple (M, d, μ) is a metric measure space if (M, d) is a complete metric space and μ is a positive Radon measure on (M, d). In case (M, d) is locally compact, we also say that (M, d, μ) is locally doubling if, for every compact set $K \subseteq M$, there exist $C > 1$ and $R > 0$ such that

$$\mu(B(p, 2r)) \leq C\mu(B(p, r)),$$

for every $p \in K$ and every $r \in (0, R)$. We then often refer to equiregular CC spaces $(\mathbb{R}^n, X)$ as metric measure spaces identified with the triple $(\mathbb{R}^n, d, \mathscr{L}^n)$, where d is the CC metric associated with the m-tuple of vector fields X and $\mathscr{L}^n$ is the Lebesgue measure. Recall that, by property (iii) of Theorem 1.2.4, the measure $\mathscr{L}^n$ is locally doubling with respect to the metric d.

Definition 2.1.1 (Approximate Limit)**.** Let (M, d, μ) be a locally compact, separable and locally doubling metric measure space. Assume $\Omega \subseteq M$ is an open set, $u \in$

$L^1_{loc}(\Omega,\mu;\mathbb{R}^k)$, $z\in\mathbb{R}^k$ and $p\in\Omega$. We say that z is the *approximate limit* of u at p if

$$\lim_{r\to 0} \fint_{B(p,r)} |u(y)-z|\, d\mu(y) = 0.$$

We denote by $u^\star(p)$ the approximate limit of u at p and by $\mathcal{S}_u$ the set of points in Ω where u does not admit an approximate limit

If the approximate limit exists, it is also unique. We denote by $\mathcal{S}_u$ the subset of Ω in which u does not admit an approximate limit. In the following, if $u\in L^1(\Omega,\mu;\mathbb{R}^k)$ and $p\in\Omega\setminus\mathcal{S}_u$, we denote by $u^\star(p)$ the approximate limit of u at p.
We here state the Lebesgue's differentiation Theorem, whose proof can be found for instance in [48, Section 2.7].

Theorem 2.1.2 (Generalized Lebesgue Theorem)**.** *Let (M,d,μ) be a separable, locally compact and locally doubling metric measure space, let $\Omega\subseteq M$ be open and let $u\in L^1_{loc}(\Omega,\mu;\mathbb{R}^k)$. Then for μ-almost every $p\in\Omega$, u admits an approximate limit at p and $u(p)=u^\star(p)$.*

The proof of Proposition 2.1.3 is an easy adaptation of the Euclidean one [5, Proposition 3.64].

Proposition 2.1.3 (Properties of Approximate Limits)**.** *Let (M,d,μ) be a separable, locally compact and locally doubling metric measure space, let $\Omega\subseteq M$ be open and let $u\in L^1_{loc}(\Omega,\mu;\mathbb{R}^k)$. Suppose that $\mu(\partial B(p,r))=0$ for every $p\in X$ and for every $r>0$. Then, the following facts hold.*

(i) $\mathcal{S}_u$ is a Borel set with $\mu(\mathcal{S}_u)=0$ and $u^\star:\Omega\setminus\mathcal{S}_u\to\mathbb{R}^k$ is a Borel map.

(ii) For every $f\in\mathrm{Lip}(\mathbb{R}^k;\mathbb{R}^N)$ we have $\mathcal{S}_{f\circ u}\subseteq\mathcal{S}_u$ and for every $p\in\Omega\setminus\mathcal{S}_u$,:

$$(f\circ u)^\star(p) = f(u^\star(p));$$

Proof. (i). By the generalized Lebesgue Theorem 2.1.2 we already know that $\mu(\mathcal{S}_u)=0$ and $u(p)=u^\star(p)$ for μ-almost every $p\in\Omega$.
To prove that $\mathcal{S}_u$ is a Borel set it is sufficient to prove that

$$\Omega\setminus\mathcal{S}_u = \bigcap_{n=1}^{\infty}\bigcup_{q\in\mathbb{Q}^k}\left\{\xi\in\Omega : \limsup_{r\to 0}\fint_{B(\xi,r)}|u(y)-q|\,d\mu(y) < \frac{1}{n}\right\}. \tag{2.1}$$

Indeed, if this were true, since the function

$$\xi \longmapsto \fint_{B(\xi,r)} |u-q|\, d\mu$$

is continuous, the right-hand side of the equality (2.1) would be a Borel set.
The inclusion $\subseteq$ in (2.1) is trivial for the density of $\mathbb{Q}$ in $\mathbb{R}$. On the other hand, if p

is a point in the right-hand side of (2.1), then, for every $n \in \mathbb{N}$, we can find $q_n \in \mathbb{Q}^k$ such that $\limsup_{r\to 0} \fint_{B(p,r)} |u(y) - q_n| \, d\mu(y) < \frac{1}{n}$. We prove that (q_n) is a Cauchy sequence noticing that

$$|q_h - q_k| = \left| \fint_{B(p,r)} (q_h - q_k)\, d\mu \right| \le \fint_{B(p,r)} |u(y) - q_h| \, d\mu(y) + \fint_{B(p,r)} |u(y) - q_k| \, d\mu(y).$$

Hence there exists $z \in \mathbb{R}^k$ such that $\lim_h q_h = z$ and it is easy to see that $z = u^\star(p)$ and therefore $p \in \Omega \setminus \mathcal{S}_u$.

(ii). Let $f \in \mathrm{Lip}(\mathbb{R}^k; \mathbb{R}^N)$ and fix $p \in \Omega \setminus \mathcal{S}_u$. Then we have

$$\int_{B(p,r)} |f(u(y)) - f(u^\star(p))| \, d\mathscr{L}^n(y) \le \mathrm{Lip}(f) \int_{B(p,r)} |u(y) - u^\star(p)| \, d\mathscr{L}^n(y);$$

from which we deduce that $\mathcal{S}_{f\circ u} \subseteq \mathcal{S}_u$ and

$$(f \circ u)^\star(p) = f(u^\star(p)).$$

This concludes the proof. □

Proposition 2.1.4. *Let (M, d, μ) be a separable, locally compact and locally doubling metric measure space, let $\Omega \subseteq M$ be open and let $u \in L^1_{loc}(\Omega, \mu; \mathbb{R}^k)$. Suppose that $\mu(\partial B(p, r)) = 0$ for every $p \in M$ and for every $r > 0$. If $p \in \Omega \setminus \mathcal{S}_u$, then, for any $\varepsilon > 0$, the set*

$$E_\varepsilon := \{q \in \Omega : |u(q) - u^\star(p)| > \varepsilon\}$$

has density 0 at p. Conversely, if $u \in L^\infty_{loc}(\Omega, \mu; \mathbb{R}^k)$ and $z \in \mathbb{R}^k$ are such that, for any $\varepsilon > 0$, the set

$$E_\varepsilon := \{q \in \Omega : |u(q) - z| > \varepsilon\}$$

has density 0 at p, then $p \in \Omega \setminus \mathcal{S}_u$ and $z = u^\star(p)$.

In particular, if $p \in \Omega \setminus \mathcal{S}_u$ and $t \neq u^\star(p)$, then $p \notin \partial^*\{u > t\}$.

Proof. Suppose $p \in \Omega \setminus \mathcal{S}_u$. By Chebychev inequality we have

$$\varepsilon \frac{\mu(E_\varepsilon \cap B(p, r))}{\mu(B(p, r))} \le \fint_{B(p,r)} |u - u^\star(p)| \, d\mu,$$

which goes to 0 as $r \to 0$.

Conversely, suppose $u \in L^\infty_{loc}(\Omega, \mu; \mathbb{R}^k)$ and let z be as in the statement. Then, for any $r \in (0, 1)$, we have

$$\fint_{B(p,r)} |u - z| \, d\mathscr{L}^n \le (\|u\|_{L^\infty(B(p,1),\mu;\mathbb{R}^k)} + |z|) \frac{\mu(B(p, r) \cap E_\varepsilon))}{\mu(B(p, r))} + \varepsilon \frac{\mu(B(p, r) \setminus E_\varepsilon))}{\mu(B(p, r))}.$$

Finally, take $p \in \Omega \setminus \mathcal{S}_u$ and let $t \neq u^\star(p)$. We already know that both $\{u > u^\star(p) + \varepsilon\}$ and $\{u < u^\star(p) - \varepsilon\}$ have density 0 at p, for every $\varepsilon > 0$. If $t > u^\star(p)$, then choosing $\varepsilon = t - u^\star(p)$ we have that $\{u > t\}$ has density 0 at p. If $t < u^\star(p)$, choose $\eta > 0$ such that $\varepsilon = u^\star(p) - t - \eta > 0$ to infer that $\{u < t + \eta\}$ has density 0 at p, and consequently $\{u \ge t + \eta\}$ has density 1 at p. This implies that also $\{u > t\}$ has density 1 at p. □

We now introduce the notion of jump points in the setting of equiregular CC spaces $(\mathbb{R}^n, X)$. This requires a certain amount of work, one of the reasons being that there is no canonical way of separating a CC ball $B(p,r)$ into complementary "half-balls" $B_\nu^+(p,r), B_\nu^-(p,r)$. We will use as separating sets an arbitrary hypersurface S of class C_X^1 such that $\nu_S(p) = \nu$, and one of the issues (Remark 2.1.9 below) is proving well-posedness of our definition independently of the choice of S. For any fixed $p \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{m-1}$ and $r > 0$, we introduce the notation $B_\nu^+(p,r)$ and $B_\nu^-(p,r)$ as follows. Consider $R > 0$ and $f \in C_X^1(B(p,r))$ such that $f(p) = 0$ and $Xf(p)/|Xf(p)| = \nu$ [1]; then, for any $r \in (0,R)$, we set

$$\begin{aligned} B_\nu^+(p,r) &\coloneqq B(p,r) \cap \{f > 0\} \\ B_\nu^-(p,r) &\coloneqq B(p,r) \cap \{f < 0\}. \end{aligned}$$

These objects are well-defined only if r is small enough. Moreover, there is a clear abuse of notation, since $B_\nu^\pm(p,r)$ depend on the choice of f. However, this will not effect the validity of our results.

Before introducing the notion of approximate X-jumps we state some properties of the "half-balls" $B_\nu^\pm(p,r)$. Proposition 2.1.5 is used in the proof of Theorem 2.2.18.

Proposition 2.1.5. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then, for any $p \in \Omega$ and $\nu \in \mathbb{S}^{m-1}$.*

$$\lim_{r\to 0} \frac{\mathscr{L}^n\left(B_\nu^+(p,r)\right)}{\mathscr{L}^n\left(B(p,r)\right)} = \lim_{r\to 0} \frac{\mathscr{L}^n\left(B_\nu^-(p,r)\right)}{\mathscr{L}^n\left(B(p,r)\right)} = \frac{1}{2}$$

Proof. Let U be a neighborhood of p and let $f \in C_X^1(U)$ be such that $f(p) = 0$ and $Xf(p) = \nu$. Fix $\varepsilon \in (0,1)$. By Proposition 1.5.3 and Theorem 1.4.9 we can suppose without loss of generality that, for every small enough $r > 0$, one has $F_p(\widetilde{B}(0,r)) = B(p,r)$ and

$$\begin{aligned} F_p^{-1}(B_\nu^+(p,r)) &= \widetilde{B}(0,r) \cap \{\xi \in \mathbb{R}^n : f(F_p(\xi)) > 0\} \\ &\subseteq \widehat{B}(0,(1+\varepsilon)r) \cap \left\{\xi \in \mathbb{R}^n : \widetilde{L}_\nu(\xi) \geq -\varepsilon r\right\}. \end{aligned} \tag{2.2}$$

Analogously

$$\begin{aligned} \widehat{B}(0,(1-\varepsilon)r) \cap \left\{\xi \in \mathbb{R}^n : \widetilde{L}_\nu(\xi) \geq \varepsilon r\right\} &\subseteq \widetilde{B}(0,r) \cap \{\xi \in \mathbb{R}^n : f(F_p(\xi)) > 0\} \\ &= F_p^{-1}(B_\nu^+(p,r)). \end{aligned} \tag{2.3}$$

Applying $\delta_{1/r}$ to both sides of (2.2) and evaluating the Lebesgue measure we get

$$\begin{aligned} \frac{\mathscr{L}^n\left(F_p^{-1}(B_\nu^+(p,r))\right)}{r^Q} &= \mathscr{L}^n\left(\delta_{1/r}\left(F_p^{-1}(B_\nu^+(p,r))\right)\right) \\ &\leq \mathscr{L}^n\left(\widehat{B}(0,1+\varepsilon) \cap \left\{\xi \in \mathbb{R}^n : \widetilde{L}_\nu(\xi) \geq -\varepsilon\right\}\right). \end{aligned}$$

[1] One can consider for instance $f = \widetilde{L}_\nu \circ F_p$.

Taking the lim sup as $r \to 0$ and letting $\varepsilon \to 0$ we infer

$$\limsup_{r\to 0} \frac{\mathscr{L}^n\left(F_p^{-1}(B_\nu^+(p,r))\right)}{r^Q} \leq \mathscr{L}^n\left(\widehat{B}(0,1) \cap \left\{\xi \in \mathbb{R}^n : \widetilde{L}_\nu(\xi) \geq 0\right\}\right)$$
$$= \frac{1}{2}\mathscr{L}^n\left(\widehat{B}(0,1)\right),$$

where the last equality follows from Corollary 1.4.8. With the same argument, from (2.3) we get

$$\liminf_{r\to 0} \frac{\mathscr{L}^n\left(F_p^{-1}(B_\nu^+(p,r))\right)}{r^Q} \geq \frac{1}{2}\mathscr{L}^n\left(\widehat{B}(0,1)\right),$$

hence

$$\lim_{r\to 0} \frac{\mathscr{L}^n\left(F_p^{-1}(B_\nu^+(p,r))\right)}{r^Q} = \frac{1}{2}\mathscr{L}^n\left(\widehat{B}(0,1)\right). \tag{2.4}$$

By Corollary 1.4.10

$$\lim_{r\to 0} \frac{\mathscr{L}^n(\widetilde{B}(0,r))}{r^Q} = \lim_{r\to 0} \mathscr{L}^n(\delta_{1/r}(\widetilde{B}(0,r))) = \mathscr{L}^n(\widehat{B}(0,1)), \tag{2.5}$$

and combining (2.4) and (2.5) we get

$$\lim_{r\to 0} \frac{\mathscr{L}^n\left(F_p^{-1}\left(B_\nu^+(p,r)\right)\right)}{\mathscr{L}^n(\widetilde{B}(0,r))} = \frac{1}{2}.$$

If $c := |\det \nabla F(0)| > 0$, using (1.17) we notice that, for every $0 < \varepsilon < c$ and every sufficiently small $r > 0$, one has

$$\frac{(c-\varepsilon)\,\mathscr{L}^n\left(F_p^{-1}\left(B_\nu^+(p,r)\right)\right)}{(c+\varepsilon)\,\mathscr{L}^n(\widetilde{B}(0,r))} \leq \frac{\mathscr{L}^n\left(B_\nu^+(p,r)\right)}{\mathscr{L}^n(B(p,r))} \leq \frac{(c+\varepsilon)\mathscr{L}^n\left(F_p^{-1}\left(B_\nu^+(p,r)\right)\right)}{(c-\varepsilon)\mathscr{L}^n(\widetilde{B}(0,r))}.$$

The result follows by passing to the limit as $r \to 0$, letting $\varepsilon \to 0$ and by using a similar argument for B_ν^-. □

Definition 2.1.6 (Approximate X-jumps)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $u \in L^1_{loc}(\Omega; \mathbb{R}^k)$ and $p \in \Omega$. We say that u has an *approximate X-jump* at p if there exist $a, b \in \mathbb{R}^k$ with $a \neq b$ and $\nu \in \mathbb{S}^{m-1}$ such that

$$\lim_{r\to 0} \fint_{B_\nu^+(p,r)} |u-a|\,d\mathscr{L}^n = \lim_{r\to 0} \fint_{B_\nu^-(p,r)} |u-b|\,d\mathscr{L}^n = 0. \tag{2.6}$$

In this case we say that (a, b, ν) is an *approximate X-jump triple* of u at p. We denote by $\mathcal{J}_u$ the set of approximate X-jump points of u and by $(u^+(p), u^-(p), \nu_u(p))$ the (unique up to equivalence, see Proposition 2.1.9 below) approximate X-jump triple for u at $p \in \mathcal{J}_u$.

Remark 2.1.7. Using e.g. Proposition 2.1.5 one easily proves that $\mathcal{J}_u \subseteq \mathcal{S}_u$.

Notice that, if u has an approximate jump at p associated with (a,b,ν), then it is also associated with the triple $(b,a,-\nu)$. For this reason, it will be sometimes convenient to consider the space of approximate X-jump triples endowed with the equivalence relation $(a,b,\nu)\equiv(a',b',\nu')$ if and only if $(a,b,\nu)=(a',b',\nu')$ or $(a,b,\nu)=(b',a',-\nu')$.

The following Proposition 2.1.8 shows that the X-jump triple $(u^+(p),u^-(p),\nu_u(p))$ is unique up to equivalence, for the map $\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{S}^{m-1}\ni(a,b,\nu)\to w_{a,b,\nu}\in L^1_{loc}(\mathbb{R}^n;\mathbb{R}^k)$ defined by (2.7) below satisfies

$$w_{a,b,\nu}=w_{a',b',\nu'}\quad\iff\quad(a,b,\nu)\equiv(a',b',\nu').$$

In the theory of classical BV functions a jump point can be detected, via a blow-up procedure, in terms of L^1_{loc}-convergence to a function taking two different values on complementary half-spaces; this is the content of the next statement, which also gives an equivalent definition of approximate X-jump points.

Proposition 2.1.8. *Let $(\mathbb{R}^n,X)$ be an equiregular CC space, Ω an open set, $u\in L^1_{loc}(\Omega;\mathbb{R}^k)$, $p\in\Omega$ and let $a,b\in\mathbb{R}^k$ with $a\neq b$ and $\nu\in\mathbb{S}^{m-1}$ be fixed. Then the following statements are equivalent:*

(i) $p\in\mathcal{J}_u$ and $(u^+(p),u^-(p),\nu_u(p))\equiv(a,b,\nu)$;

(ii) if F_p denotes the map of adapted exponential coordinates around p, as $r\to0$, the functions $\widetilde{u}_r:=u\circ F_p\circ\delta_r$ converge in $L^1_{loc}(\mathbb{R}^n;\mathbb{R}^k)$ to

$$w_{a,b,\nu}(y):=\begin{cases}a & \text{if } \widetilde{L}_\nu(y)>0\\ b & \text{if } \widetilde{L}_\nu(y)<0.\end{cases}\tag{2.7}$$

Proof. We can assume without loss of generality that $k=1$.

We prove the implication (i)⇒(ii); we can assume that $(u^+(p),u^-(p),\nu_u(p))=(a,b,\nu)$ and, writing $w:=w_{a,b,\nu}$, we prove that for any fixed $R>0$ one has

$$\lim_{r\to0}\int_{\widehat{B}(0,R)}|u\circ F_p\circ\delta_r-w|\,d\mathscr{L}^n=0.$$

By a change of variables, this is equivalent to proving that

$$\lim_{r\to0}\frac{1}{r^Q}\int_{\widehat{B}(0,r)}|u\circ F_p-w|\,d\mathscr{L}^n=0.\tag{2.8}$$

Let f be the real function of class C^1_X defined on a neighborhood of p used to define, as in (2.1), the half-balls $B^\pm_\nu(p,r)$ appearing in (2.6); we set for brevity

$$\widehat{B}^+_\nu(0,r):=\widehat{B}(0,r)\cap\{\widetilde{L}_\nu>0\},\qquad \widehat{B}^-_\nu(0,r):=\widehat{B}(0,r)\cap\{\widetilde{L}_\nu<0\}$$
$$\widetilde{B}^+_\nu(0,r):=\widetilde{B}(0,r)\cap\{f\circ F_p>0\},\qquad \widetilde{B}^-_\nu(0,r):=\widetilde{B}(0,r)\cap\{f\circ F_p<0\}.$$

By Theorem 1.4.9 there exists an increasing function $\omega : (0, +\infty) \to (0, +\infty)$ such that

$$\lim_{r\to 0^+} \frac{\omega(r)}{r} = 0 \qquad \text{and} \qquad \widehat{B}(0,r) \subseteq \widetilde{B}(0, r + \omega(r))$$

for any sufficiently small r. Therefore

$$\begin{aligned}
&\frac{1}{r^Q} \int_{\widehat{B}(0,r)} |u \circ F_p - w| \, d\mathscr{L}^n \\
=&\frac{1}{r^Q} \left(\int_{\widehat{B}^+_\nu(0,r)} |u \circ F_p - a| \, d\mathscr{L}^n + \int_{\widehat{B}^-_\nu(0,r)} |u \circ F_p - b| \, d\mathscr{L}^n \right) \\
\le&\frac{1}{r^Q} \left(\int_{\widetilde{B}^+_\nu(0,r+\omega(r))} |u \circ F_p - a| \, d\mathscr{L}^n + \int_{\widehat{B}^+_\nu(0,r)\setminus \widetilde{B}^+_\nu(0,r+\omega(r))} (|u \circ F_p - b| + |a-b|) \, d\mathscr{L}^n \right. \\
&\left. + \int_{\widetilde{B}^-_\nu(0,r+\omega(r))} |u \circ F_p - b| \, d\mathscr{L}^n + \int_{\widehat{B}^-_\nu(0,r)\setminus \widetilde{B}^-_\nu(0,r+\omega(r))} (|u \circ F_p - a| + |a-b|) \, d\mathscr{L}^n \right)
\end{aligned}$$

and using $\widehat{B}^\pm_\nu(0,r) \setminus \widetilde{B}^\pm_\nu(0, r+\omega(r)) \subseteq \widetilde{B}(0, r+\omega(r)) \setminus \widetilde{B}^\pm_\nu(0, r+\omega(r)) \subseteq \overline{\widetilde{B}^\mp_\nu(0, r + \omega(r))}$

$$\begin{aligned}
\le&\frac{1}{r^Q} \left(2 \int_{\widetilde{B}^+_\nu(0,r+\omega(r))} |u \circ F_p - a| \, d\mathscr{L}^n + 2 \int_{\widetilde{B}^-_\nu(0,r+\omega(r))} |u \circ F_p - b| \, d\mathscr{L}^n \right. \\
&\left. + |a-b| \mathscr{L}^n(\widetilde{B}(0, r+\omega(r)) \cap \{(f \circ F_p)\widetilde{L}_\nu \le 0\}) \right)
\end{aligned}$$

and (2.8) follows from (2.6) and Corollary 1.5.4 taking also Theorem 1.2.4 into account.

For the converse implication one has to prove that, if (ii) holds and f is a C^1_X real function on a neighborhood of p such that $f(p) = 0$ and $Xf(p)/|Xf(p)| = \nu$, then (2.6) holds with $B^\pm_\nu(p,r)$ defined (see (2.1)) in terms of f. By Theorem 1.2.4 and a change of variables, proving (2.6) amounts to proving that

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{\widetilde{B}^+_\nu(0,r)} |u \circ F_p - a| \, d\mathscr{L}^n = \lim_{r\to 0} \frac{1}{r^Q} \int_{\widetilde{B}^-_\nu(0,r)} |u \circ F_p - b| \, d\mathscr{L}^n = 0$$

and this can be done by a boring adaptation, that we omit, of the previous argument.

□

Remark 2.1.9. The proof of Proposition 2.1.8 implicitly shows that the validity of (2.6) does not depend on the choice of the function f used in (2.1) to define $B^\pm_\nu(p,r)$.

Remark 2.1.10. Let Ω, u, z and p be as in Definition 2.1.1. Then u has approximate limit z at p if and only if, as $r \to 0$, the functions $u \circ F_p \circ \delta_r$ converge in $L^1_{loc}(\mathbb{R}^n; \mathbb{R}^k)$ to the constant function z. This is just an easy adaptation of the proof of Proposition 2.1.8 with $a = b = z$.

The proof of Proposition 2.1.11 below is standard and it is an easy adaptation of [5, Proposition 3.69].

Proposition 2.1.11. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, Ω be an open set and let $u \in L^1_{loc}(\Omega)$. Then the following facts hold:*

(i) $\mathcal{J}_u$ *is a Borel set and, up to a choice of a representative for jump triples, the function*

$$\begin{aligned}\mathcal{J}_u &\to \mathbb{R}\times\mathbb{R}\times\mathbb{S}^{m-1}\\ p &\mapsto (u^+(p), u^-(p), \nu_u(p))\end{aligned}$$

is Borel;

(ii) for every $f \in \mathrm{Lip}(\mathbb{R})$, *and* $p \in \mathcal{J}_u$ *we have*

$$p \in \mathcal{J}_{(f\circ u)} \Leftrightarrow f(u^+(p)) \neq f(u^-(p))$$

and in this case $((f\circ u)^+(p), (f\circ u)^-(p), \nu_{f\circ u}(p)) \equiv (f(u^+(p)), f(u^-(p)), \nu_u(p))$. *Otherwise* $p \notin \mathcal{S}_{(f\circ u)}$ *and* $(f\circ u)^\star(p) = f(u^+(p)) = f(u^-(p))$.

Proof. *(i)* Let $\{(a_h, b_h, \nu_h) : h \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{m-1}$. For every $h \in \mathbb{N}$, define the function $w_h : \mathbb{R}^n \to \mathbb{R}$ setting

$$w_h(y) := \begin{cases} a_h & \text{if } \widetilde{L}_{\nu_h}(y) \geq 0,\\ b_h & \text{if } \widetilde{L}_{\nu_h}(y) < 0.\end{cases}$$

Recalling notation (1.19), we first prove that

$$(\Omega \setminus \mathcal{S}_u) \cup \mathcal{J}_u = \bigcap_{\ell=1}^{\infty}\bigcup_{h=0}^{\infty}\left\{p \in \Omega : \limsup_{r\to 0} \fint_{A(r)} |u\circ F_p - w_h|\, d\mathscr{L}^n < \frac{1}{\ell}\right\}. \tag{2.9}$$

Thanks to Proposition 2.1.8 and since that the sets $A(r)$ are equivalent, for small radii, to the balls $\widehat{B}(0,r)$, the inclusion $\subseteq$ is straightforward.
To prove $\supseteq$, take $p \in \Omega$ such that for every $\ell \in \mathbb{N}\setminus\{0\}$ there exists w_{h_ℓ} such that

$$\limsup_{r\to 0} \fint_{A(r)} |u\circ F_p - w_{h_\ell}|\, d\mathscr{L}^n < \frac{1}{\ell}.$$

We prove that there exist a, b and ν such that (w_{h_ℓ}) is convergent in $L^1(A(1))$ to

$$w(y) := \begin{cases} a & \text{if } \widetilde{L}_{\nu}(y) \geq 0,\\ b & \text{if } \widetilde{L}_{\nu}(y) < 0.\end{cases}$$

Possibly passing to a subsequence, we can suppose that the sequence (ν_{h_ℓ}) converges to some ν. Define $C := \mathscr{L}^n(A(1))$ and let $\overline{k} \in \mathbb{N}$ be such that for every $h, k \geq \overline{k}$ the set

$$A^+(1) := \left\{y \in A(1) : \widetilde{L}_{\nu_h}(y) > 0 \text{ and } \widetilde{L}_{\nu_k}(y) < 0\right\}$$

is such that $\mathscr{L}^n(A^+(1)) \geq \frac{1}{4}C$. Then for such h and k, using a change of variable formula, we have

$$\begin{aligned}|a_h - a_k| &= \fint_{A^+(1)} |a_h - a_k| \, d\mathscr{L}^n \leq \frac{4}{C} \int_{A^+(1)} |w_h - w_k| \, d\mathscr{L}^n \\ &\leq \frac{4}{C} \int_{A(1)} |w_h - w_k| \, d\mathscr{L}^n = \frac{4}{Cr^Q} \int_{A(r)} |w_h - w_k| \, d\mathscr{L}^n \\ &\leq 4 \fint_{A(r)} |u \circ F_p - w_h| \, d\mathscr{L}^n + 4 \fint_{A(r)} |u \circ F_p - w_k| \, d\mathscr{L}^n.\end{aligned}$$

Passing to the lim sup as $r \to 0$ we get that (a_{h_ℓ}) is Cauchy and therefore convergent to some $a \in \mathbb{R}$. Using the same technique we also get that (b_{h_ℓ}) is convergent to some $b \in \mathbb{R}$. It is then easy to prove that w_h converges in $L^1(A(1))$ to w. Now, for sufficiently large $h \in \mathbb{N}$ and for sufficiently small $r > 0$, from

$$\int_{A(r)} |u \circ F_p \circ \delta_r - w| \, d\mathscr{L}^n \leq \int_{A(r)} |u \circ F_p \circ \delta_r - w_h| \, d\mathscr{L}^n + \int_{A(1)} |w_h - w| \, d\mathscr{L}^n,$$

we get the remaining inclusion $\supseteq$ in (2.9). Notice that the right-hand side of (2.9) is a Borel set if, for any $h \in \mathbb{N}$, and any small enough r, the function

$$p \longmapsto \fint_{A(r)} |u \circ F_p - w_h| \, d\mathscr{L}^n$$

is continuous. This is clearly true if u is of class C^∞. In the general case fix $p \in \Omega$, $r > 0$ and take $\varepsilon > 0$ and $v \in C^\infty(\Omega)$ such that

$$\|u - v\|_{L^1(B(p,C_1 r))} < \varepsilon,$$

where $C_1 > 0$ is such that $F_p(A(r)) \Subset B(p, C_1 r)$. By triangular inequality, we find

$$\begin{aligned}\fint_{A(r)} |u \circ F_p - u \circ F_q| \, d\mathscr{L}^n \leq &\fint_{A(r)} |u \circ F_p - v \circ F_p| \, d\mathscr{L}^n \\ &+ \fint_{A(r)} |v \circ F_p - v \circ F_q| \, d\mathscr{L}^n \\ &+ \fint_{A(r)} |v \circ F_q - u \circ F_q| \, d\mathscr{L}^n < C\varepsilon,\end{aligned}$$

for some $C > 0$, for every sufficiently small r and for every q sufficiently close to p; in particular, $\mathcal{J}_u$ is a Borel set.

According to Definition 2.1.6, for any $p \in \mathcal{J}_u$, we can find an X-jump triple $(u^+(p), u^-(p), \nu(p))$, and we can define the function $\phi : \mathcal{J}_u \to \mathbb{R}^m$ letting $\phi(p) := (u^+(p) - u^-(p))\nu(p)$. Since $\phi(p) \neq 0$, up to a change of sign, we can assume that $\nu(p) = \phi(p)/|\phi(p)|$. If we prove that ϕ is Borel, then also ν would be Borel. Set

$$w_p(y) := \begin{cases} u^+(p) & \text{if } \widetilde{L}_{\nu(p)}(y) > 0; \\ u^-(p) & \text{if } \widetilde{L}_{\nu(p)}(y) < 0, \end{cases}$$

and

$$\widetilde{A}(r) := \left\{ y \in \mathbb{R}^n : |(y_1, \dots, y_m)| + \sum_{j=m+1}^{n} |y_j|^{\frac{1}{w_j}} \le r \right\}.$$

Notice that the sets $\widetilde{A}(r)$ are equivalent to $A(r)$ and that $\widetilde{A}(r)$ are rotationally invariant in the first m coordinates. By Proposition 2.1.8, we have that

$$\begin{aligned}\int_{\widetilde{A}(1)} w_p \partial_i \psi \, d\mathscr{L}^n &= \lim_{\varepsilon\to 0} \int_{\widetilde{A}(1)} (u \circ F_p \circ \delta_\varepsilon) \partial_i \psi \, d\mathscr{L}^n \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^Q} \int_{\widetilde{A}(\varepsilon)} u(F_p(y)) \partial_i \psi(\delta_{\varepsilon^{-1}}(y)) \, d\mathscr{L}^n(y),\end{aligned}$$

for every $\psi \in C_c^\infty(\widetilde{A}(1))$ and for every $i = 1, \dots, n$. Hence, we get that, for every $\psi \in C_c^\infty(\widetilde{A}(1))$ and for every $i = 1, \dots, n$, the function

$$p \mapsto \int_{\widetilde{A}(1)} w_p \partial_i \psi \, d\mathscr{L}^n$$

is Borel. Fix $p \in \mathcal{J}_u$ and take a sequence (ψ_h) in $C_c^\infty(\widetilde{A}(1))$ converging to $\chi_{\widetilde{A}(1)}$. Computing the (Euclidean) measure derivative of w_p, we easily get

$$\begin{aligned}&\phi^i(p) \mathscr{H}_e^{n-1}(\widetilde{A}(1) \cap \{y \in \mathbb{R}^n : \widetilde{L}_{\nu(p)}(y) = 0\}) \\ &\quad = D^i w_p(\widetilde{A}(1)) = \lim_h \int_{\widetilde{A}(1)} \psi_h \, dD^i w_p = -\lim_h \int_{\widetilde{A}(1)} w_p \partial_i \psi_h \, d\mathscr{L}^n,\end{aligned}$$

for every $i = 1, \dots, n$. Since the quantity $\mathscr{H}_e^{n-1}(\widetilde{A}(1) \cap \{y \in \mathbb{R}^n : \widetilde{L}_{\nu(p)}(y) = 0\})$ does not depend on p, we deduce by the previous step that ϕ is a Borel function and therefore ν is Borel.
Eventually, since by Proposition 2.1.8 we have

$$u^+(p) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^Q} \int_{A(\varepsilon)} \chi_{\{\widetilde{L}_{\nu(p)} > 0\}} u \circ F_p \, d\mathscr{L}^n,$$

we complete the proof.
The proof of (ii) is completely analogous to the euclidean one. □

We are now ready to introduce the notion of approximate X-differentiability.

Definition 2.1.12 (Approximate X-differentiability)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space, $u \in L^1_{loc}(\Omega; \mathbb{R}^k)$ and $p \in \Omega \setminus \mathcal{S}_u$. We say that u is *approximately X-differentiable* at p if there exist a neighborhood U of p and $f \in C^1_X(U; \mathbb{R}^k)$ such that $f(p) = 0$ and

$$\lim_{r \to 0} \fint_{B(p,r)} \frac{|u - u^\star(p) - f|}{r} \, d\mathscr{L}^n = 0. \tag{2.10}$$

The subset of points of Ω in which u is approximately X-differentiable will be denoted by $\mathcal{D}_u$.

If f is as in Definition 2.1.12, we call $Xf(p)$ the *approximate X-gradient* of u at p. By the following Proposition approximate X-gradients are uniquely determined and therefore we denote by $D_X^{ap}u(p)$ the approximate X-gradient of u at p.

Proposition 2.1.13 (Uniqueness of approximate X-differential)**.** *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, $u \in L^1_{loc}(\Omega;\mathbb{R}^k)$ and $p \in \Omega$. Let $R > 0$ and let $f_1, f_2 \in C^1_X(B(p,R);\mathbb{R}^k)$. Suppose formula* (2.10) *holds for both $f = f_1$ and for $f = f_2$. Then $p \in \mathcal{D}_u$, $f_1(p) = f_2(p) = 0$ and $Xf_1(p) = Xf_2(p)$. Conversely, if $f_1(p) = f_2(p) = 0$ and $Xf_1(p) = Xf_2(p)$, then formula* (2.10) *holds for $f = f_1$ if and only if it holds for $f = f_2$.*

Proof. It is not restrictive to assume $k = 1$. Define for $i = 1, 2$ the functions $L_i \coloneqq \widetilde{L}_{Xf_i(p)}$. Suppose first that both f_1, f_2 satisfy (2.10). Fix $\varepsilon > 0$ and, by Proposition 1.5.3, choose $r > 0$ such that

$$\frac{|f_i(F(x)) - L_i(x)|}{\varrho} < \frac{\varepsilon}{2},$$

for every $\varrho \in (0, r)$ and $x \in \widetilde{B}(0,\varrho)$. Then, for such values of ϱ, we have

$$\begin{aligned}
\fint_{\widetilde{B}(0,\varrho)} \frac{|L_1 - L_2|}{\varrho}\, d\mathscr{L}^n &\le \fint_{\widetilde{B}(0,\varrho)} \frac{|f_1 \circ F_p - f_2 \circ F_p|}{\varrho}\, d\mathscr{L}^n + \varepsilon \\
&\le C \fint_{B(p,\varrho)} \frac{|f_1 - f_2|}{\varrho}\, d\mathscr{L}^n + \varepsilon \\
&\le C \fint_{B(p,\varrho)} \frac{|u - u^\star(p) - f_1| + |u - u^\star(p) - f_2|}{\varrho}\, d\mathscr{L}^n + \varepsilon.
\end{aligned}$$

It follows that

$$\lim_{\varrho\to 0} \fint_{\widetilde{B}(0,\varrho)} \frac{|L_1 - L_2|}{\varrho}\, d\mathscr{L}^n = 0. \tag{2.11}$$

If $Xf_1(p) \neq Xf_2(p)$, by Theorem 1.4.4 one would get, for some $C, C_1 > 0$

$$\begin{aligned}
\fint_{\widetilde{B}(0,\varrho)} |L_1 - L_2|\, d\mathscr{L}^n &= \frac{1}{\mathscr{L}^n(\widetilde{B}(0,\varrho))} \int_{\widetilde{B}(0,\varrho)} |L_1 - L_2|\, d\mathscr{L}^n \\
&\ge \frac{1}{\mathscr{L}^n\left(A(C_1\varrho)\right)} \int_{A\left(\frac{\varrho}{C_1}\right)} |L_1 - L_2|\, d\mathscr{L}^n \\
&= C\frac{\varrho^{Q+1}}{\varrho^Q} = C\varrho,
\end{aligned}$$

that contradicts (2.11). This proves the first part of the statement

Suppose now that $Xf_1(p) = Xf_2(p)$ and that f_1 satisfies (2.10). Then we have $L_1 = L_2$ and

$$\begin{aligned}
&\fint_{B(p,\varrho)} \frac{|u - u^\star(p) - f_2|}{\varrho}\, d\mathscr{L}^n \\
&\le \fint_{B(p,\varrho)} \frac{|f_1 - L_1 \circ F_p^{-1}| + |u(y) - u^\star(p) - f_1| + |f_2 - L_2 \circ F_p^{-1}|}{\varrho}\, d\mathscr{L}^n.
\end{aligned}$$

By Proposition 1.5.3, this completes the proof. □

As for approximate X-jump points and approximate continuity points, also approximate X-differentiability points can be detected by a blow-up procedure.

Proposition 2.1.14. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, Ω be an open subset of $\mathbb{R}^n$, $u \in L^1_{loc}(\Omega; \mathbb{R}^k)$ and let $p \in \Omega \setminus \mathcal{S}_u$. Then u is approximate X-differentiable at p if and only if there exists $z = (z_1, \dots, z_k) \in \mathbb{R}^{k\times m}$ such that, as $r \to 0$, the functions*

$$\frac{u \circ F_p \circ \delta_r - u^\star(p)}{r}$$

converge in $L^1_{loc}(\mathbb{R}^n; \mathbb{R}^k)$ to $(\widetilde{L}_{z_1}, \dots, \widetilde{L}_{z_k})$. In this case we have $D^{ap}_X u(p) = z$.

Proof. Assume first that $p \in \mathcal{D}_u$ and let $z := D^{ap}_X u(p)$. Given $R > 0$, by Corollary 1.4.10 one has

$$\begin{aligned}\int_{\widehat{B}(0,R)} \left| \frac{u \circ F_p \circ \delta_\varepsilon - u^\star(p)}{\varepsilon} - \widetilde{L}_z \right| d\mathscr{L}^n &= \frac{1}{\varepsilon^Q} \int_{\widehat{B}(0,\varepsilon R)} \left| \frac{u \circ F_p - u^\star(p) - \widetilde{L}_z}{\varepsilon} \right| d\mathscr{L}^n \\ &\le C \fint_{\widetilde{B}(0,2\varepsilon R)} \left| \frac{u \circ F_p - u^\star(p) - \widetilde{L}_z}{2\varepsilon R} \right| d\mathscr{L}^n,\end{aligned}$$

which proves the first implication $\Rightarrow$. Conversely, for any $R > 0$ and any small enough $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\varepsilon^Q} \int_{\widetilde{B}(0,\varepsilon R)} \left| \frac{u \circ F_p - u^\star(p) - \widetilde{L}_z}{\varepsilon} \right| d\mathscr{L}^n &\le \frac{1}{\varepsilon^Q} \int_{\widehat{B}(2\varepsilon R)} \left| \frac{u \circ F_p - u^\star(p) - \widetilde{L}_z}{\varepsilon} \right| d\mathscr{L}^n \\ &= \int_{\widehat{B}(0,2R)} \left| \frac{u \circ F_p \circ \delta_\varepsilon - u^\star(p)}{\varepsilon} - \widetilde{L}_z \right| d\mathscr{L}^n(y),\end{aligned}$$

which concludes the proof. $\square$

The proofs of the following two Propositions are standard and follows closely [5, Proposition 3.71] and [5, Proposition 3.73], respectively.

Proposition 2.1.15 (Properties of approximate differentiability points)**.** *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, Ω be an open set in $\mathbb{R}^n$ and let $u \in L^1_{loc}(\Omega; \mathbb{R}^k)$. Then $\mathcal{D}_u$ is a Borel set and the map $D^{ap}_X u : \mathcal{D}_u \to \mathbb{R}^{m\times k}$ is a Borel map.*

Proof. Consider a dense subset $\{z_i : i \in \mathbb{N}\}$ of $\mathbb{R}^{m\times k}$. Reasoning as in Proposition 2.1.11 one can prove that

$$\mathcal{D}_u = \bigcap_{h=1}^{\infty} \bigcup_{i=0}^{\infty} \left\{ p \in \Omega \setminus \mathcal{S}_u : \limsup_{\varrho\to 0} \frac{1}{\varrho^{Q+1}} \int_{A(\varrho)} \left| u \circ F_p - u^\star(p) - \widetilde{L}_{z_i} \right| d\mathscr{L}^n < \frac{1}{h} \right\},$$

which implies that $\mathcal{D}_u$ is a Borel set.
We now prove that $D^{ap}_X u$ is Borel. Using Theorem 1.4.4, for any $p \in \mathcal{D}_u$ one has

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{Q+1}} \int_{\delta_\varepsilon P} \left| u \circ F_p - u^\star(p) - \widetilde{L}_{D^{ap}_X u(p)} \right| d\mathscr{L}^n = 0,$$

where for every n-tuple of positive real numbers $(\ell_1, \dots, \ell_n)$

$$P = P(\ell_1, \dots, \ell_n) \coloneqq \{\xi \in \mathbb{R}^n : \forall j = 1, \dots, n\ 0 \le \xi_j^{1/w_j} \le \ell_j\}$$

is the anisotropic box with axis that are parallel to the coordinate ones $(e_1, \dots, e_n)$. By a change of variable formula we get

$$\frac{1}{\mathscr{L}^n(P)} \int_P \widetilde{L}_{D_X^{ap}u(p)}\, d\mathscr{L}^n = \frac{1}{\mathscr{L}^n(P)} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{Q+1}} \int_{\delta_\varepsilon P} (u \circ F_p - u^\star(p))\, d\mathscr{L}^n.$$

From this we deduce that, for any n-tuple $(\ell_1, \dots, \ell_n)$ the function

$$p \longmapsto \frac{1}{\mathscr{L}^n(P)} \int_P \widetilde{L}_{D_X^{ap}u(p)}\, d\mathscr{L}^n \tag{2.12}$$

is Borel. Now, for every $i = 1, \dots, m$ and for every $h \in \mathbb{N} \setminus \{0\}$, define the rectangles $P_h^i \coloneqq P(1/h, \dots, 1/h, 1, 1/h, \dots, 1/h)$. A simple computation shows that

$$\lim_h \frac{1}{\mathscr{L}^n(P_h^i)} \int_{P_h^i} \widetilde{L}_{D_X^{ap}u(p)}\, d\mathscr{L}^n = \tfrac{1}{2} \left(D_X^{ap}u(p)\right)_i.$$

This completes the proof. □

Proposition 2.1.16 (Locality). *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, Ω an open set in $\mathbb{R}^n$ and $u, v \in L^1_{loc}(\Omega; \mathbb{R}^k)$. Suppose $p \in \Omega$ is of density one for the set $\{q \in \Omega : u(q) = v(q)\}$. Then the following facts hold.*

(a) If $p \in \Omega \setminus (\mathcal{S}_u \cup \mathcal{S}_v)$, then $u^\star(p) = v^\star(p)$.

(b) If $p \in \mathcal{J}_u \cap \mathcal{J}_v$, then $(u^+(p), u^-(p), \nu_u(p)) \equiv (v^+(p), v^-(p), \nu_v(p))$.

(c) If $p \in \mathcal{D}_u \cap \mathcal{D}_v$ then $D_X^{ap}u(p) = D_X^{ap}v(p)$.

Proof. To prove (a), let $p \in \Omega \setminus (\mathcal{S}_u \cup \mathcal{S}_v)$. By Remark 2.1.10, the functions $\widetilde{u}_\varepsilon \coloneqq u \circ F_p \circ \delta_\varepsilon$ and $\widetilde{v}_\varepsilon \coloneqq u \circ F_p \circ \delta_\varepsilon$ converge respectively to $u^\star(p)$ and $v^\star(p)$ in $L^1_{loc}(\mathbb{R}^n; \mathbb{R}^k)$, as $\varepsilon \to 0$. In particular, as $\varepsilon \to 0$, the families $(\widetilde{u}_\varepsilon)$ and $(\widetilde{v}_\varepsilon)$ converge in measure in $\widehat{B}(0, R)$ to $u^\star(p)$ and $v^\star(p)$, respectively. By a change of variable formula we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathscr{L}^n &\left(\left\{\xi \in \widehat{B}(0, R) : \widetilde{v}_\varepsilon(\xi) \ne \widetilde{u}_\varepsilon(\xi)\right\}\right) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^Q} \mathscr{L}^n \left(\left\{\xi \in \widehat{B}(0, \varepsilon R) : u(F_p(\xi)) \ne v(F_p(\xi))\right\}\right) = 0,\end{aligned}$$

which tells us that $(\widetilde{u}_\varepsilon)$ and $(\widetilde{v}_\varepsilon)$ must have the same measure limit and so $u^\star(p) = v^\star(p)$. To prove (b), let $p \in \mathcal{J}_u \cap \mathcal{J}_v$. By using Proposition 2.1.8 and the same argument used in (a) we easily get that the functions

$$U(y) \coloneqq \begin{cases} u^+(p) & \text{if } \widetilde{L}_{\nu_u(p)}(y) > 0; \\ u^-(p) & \text{if } \widetilde{L}_{\nu_u(p)}(y) < 0, \end{cases}$$

and

$$V(y) := \begin{cases} v^+(p) & \text{if } \widetilde{L}_{\nu_v(p)}(y) > 0; \\ v^-(p) & \text{if } \widetilde{L}_{\nu_v(p)}(y) < 0, \end{cases}$$

coincide for $\mathscr{L}^n$-almost every $y \in \widehat{B}(0,R)$. Therefore one has

$$(u^+(p), u^-(p), \nu_u(p)) \equiv (v^+(p), v^-(p), \nu_v(p)).$$

To prove (c), let $p \in \mathcal{D}_u \cap \mathcal{D}_v$. By (a) we already know that $u^\star(p) = v^\star(p)$. It is also clear that

$$\frac{u(F_p(\delta_\varepsilon(y))) - u^\star(p)}{\varepsilon} \neq \frac{v(F_p(\delta_\varepsilon(y))) - v^\star(p)}{\varepsilon}$$

if and only if $u(F_p(\delta_\varepsilon(y))) \neq v(F_p(\delta_\varepsilon(y)))$. The thesis follows by Proposition 2.1.14 and by an argument that is similar to part (a) of the proof. □

2.2 Fine properties of BV functions in CC spaces

Recall Definition 1.1.16 for the notion of Hausdorff measure. We denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure built with respect to the CC metric. We denote by $\mathscr{H}_e^k$ the k-dimensional Hausdorff measure built with respect to the Euclidean metric.

Lemma 2.2.1. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let (E_h) be a sequence of measurable sets in Ω such that*

$$\lim_h \mathscr{L}^n(E_h) = 0 \quad \textit{and} \quad \lim_h P_X(E_h; \Omega) = 0.$$

Then, for every $\alpha \in (0,1)$, we have

$$\mathscr{H}^{Q-1}\left(\bigcap_{h=1}^{\infty} \left\{p \in \Omega : \limsup_{r \to 0} \frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))} \geq \alpha\right\}\right) = 0.$$

Proof. Denote by E_h^α the set

$$\left\{q \in \Omega : \limsup_{r \to 0} \frac{\mathscr{L}^n(E_h \cap B(q,r))}{\mathscr{L}^n(B(q,r))} \geq \alpha\right\},$$

and suppose without loss of generality that $\mathscr{L}^n(E_h) > 0$ for every $h \in \mathbb{N}$.
Let $K \Subset \Omega$. By Theorem 1.2.4 there exist $C > 1$ and $R > 0$ such that, for every $q \in K$ and every $0 < r < 2R$, we have

$$\frac{1}{C} r^Q \leq \mathscr{L}^n(B(q,r)) \leq C r^Q. \tag{2.13}$$

On the other hand, for any sufficiently large $h \in \mathbb{N}$, we have

$$\left(\frac{2C\mathscr{L}^n(E_h)}{\alpha}\right) < R^Q.$$

Fix now $p \in E_h^\alpha \cap K$ and define $\delta_h := \left(\frac{4C\mathscr{L}^n(E_h)}{\alpha}\right)^{\frac{1}{Q}}$. Then we have

$$\frac{\mathscr{L}^n(E_h \cap B(p,\delta_h))}{\mathscr{L}^n(B(p,\delta_h))} \leq \frac{C\mathscr{L}^n(E_h)}{\delta_h^Q} = \frac{\alpha}{4}.$$

On the other hand, by definition of E_h^α we can find arbitrarily small radii $r > 0$ such that

$$\frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))} \geq \frac{\alpha}{2}.$$

Taking into account Proposition 1.2.5, a continuity argument allows us to find $0 < \varrho \leq \delta_h$ such that

$$\mathscr{L}^n(E_h \cap B(x,\varrho)) = \frac{\alpha}{2}\mathscr{L}^n(B(x,\varrho)). \tag{2.14}$$

By the $5r$-covering Lemma 1.1.10, we can find a family $\{B(p_j,\varrho_j) : j \in \mathbb{N}\}$ of pairwise disjoint balls in Ω such that, for every $j \in \mathbb{N}$,

$$\begin{aligned} &p_j \in E_h^\alpha \cap K, \\ &\mathscr{L}^n(E_h \cap B(p_j,\varrho_j)) = \frac{\alpha}{2}\mathscr{L}^n(B(p_j,\varrho_j)), \\ &E_h^\alpha \cap K \subseteq \bigcup_{j=0}^\infty B(p_j, 5\varrho_j). \end{aligned} \tag{2.15}$$

Since $\mathscr{L}^n(E_h)$ is finite, by Theorem 1.6.8 we get $M > 0$ such that

$$\frac{\alpha}{2C}\varrho_j^Q \leq \frac{\alpha}{2}\mathscr{L}^n(B(p_j,\varrho_j)) = \mathscr{L}^n(E_h \cap B(p_j,\varrho_j)) \leq \Big(M\, P_X(E_h; B(p_j,\varrho_j))\Big)^{\frac{Q}{Q-1}}.$$

Therefore we have that

$$\varrho_j^{Q-1} \leq M\left(\frac{2C}{\alpha}\right)^{\frac{Q-1}{Q}} P_X(E_h; B(p_j,\varrho_j)),$$

for every $j \in \mathbb{N}$. Finally

$$\begin{aligned} \mathscr{H}^{Q-1}_{10\delta_h}\left(K \cap \bigcap_{i=0}^\infty E_i^\alpha\right) &\leq \mathscr{H}^{Q-1}_{10\delta_h}(K \cap E_h^\alpha) \overset{(2.15)}{\leq} \omega_{Q-1}5^{Q-1}\sum_{j=0}^\infty \varrho_j^{Q-1} \\ &\leq \omega_{Q-1}5^{Q-1}M\left(\frac{2C}{\alpha}\right)^{\frac{Q-1}{Q}} \sum_{j=0}^\infty P_X(E_h; B(p_j,\varrho_j)) \\ &\leq \omega_{Q-1}5^{Q-1}M\left(\frac{2C}{\alpha}\right)^{\frac{Q-1}{Q}} P_X(E_h;\Omega). \end{aligned}$$

Taking the limit for $h \to \infty$ we get

$$\mathscr{H}^{Q-1}\left(K \cap \bigcap_{i=0}^\infty E_i^\alpha\right) = 0,$$

which, by the arbitrariness of K, completes the proof. □

Before passing to the next result, we introduce some notation that we are going to use frequently in what follows. Let $p \in \mathbb{R}^n$ be fixed and let F_p denote exponential coordinates as in (1.16), for a fixed choice of a basis $Y_1, \dots, Y_n$ as in (1.16). Given $r > 0$ and $i \in \{1, \dots, m\}$, define

$$\widetilde{X}_i^r := r(d\delta_{r^{-1}})[\widetilde{X}_i \circ \delta_r]. \tag{2.16}$$

By Theorem 1.4.5, we know that $\widetilde{X}_i^r$ converges to $\widehat{X}_i$ in C^∞_{loc}, for every $i = 1, \dots, m$. Moreover, if $\widetilde{d}_r, \widetilde{B}_r(\xi, \varrho)$ denote, respectively, distance and balls with respect to the metric induced by the vector fields $(\widetilde{X}_1^r, \dots, \widetilde{X}_m^r)$, it is easy to see that the function $\delta_r : (\mathbb{R}^n, \widetilde{X}^r) \to (\mathbb{R}^n, \widetilde{X})$ satisfies

$$\widetilde{d}_r(\xi, \eta) = \frac{1}{r}\widetilde{d}(\delta_r\xi, \delta_r\eta).$$

By Theorem 1.4.9, the convergence

$$\lim_{r\to 0} \widetilde{B}_r(0, \varrho) = \widehat{B}(0, \varrho) \tag{2.17}$$

holds in the Gromov-Hausdorff sense, $\widehat{B}(0, \varrho)$ denoting a ball in the tangent Carnot group at p (recall Theorem 1.4.5). Moreover, given $u \in BV_{X,loc}(\mathbb{R}^n)$ we set

$$\widetilde{u} := u \circ F_p \qquad \text{and} \qquad \widetilde{u}_r := \widetilde{u} \circ \delta_r; \tag{2.18}$$

notice that

$$|D_{\widetilde{X}^r}\widetilde{u}_r|(\widetilde{B}_r(0, \varrho)) = r^{1-Q}|D_{\widetilde{X}}\widetilde{u}|(\widetilde{B}(0, r\varrho)).$$

We implicitly assume from now on in this chapter that the CC balls are bounded with respect to the Euclidean metric. This natural hypothesis will guarantee the use of Theorem 4.2.6.

Lemma 2.2.2. *Let $u \in BV_X(\Omega)$. Then*

$$\mathscr{H}^{Q-1}\left(\left\{p \in \Omega : \limsup_{r\to 0} \fint_{B(p,r)} |u|^{\frac{Q}{Q-1}}\, d\mathscr{L}^n = +\infty\right\}\right) = 0.$$

Proof. Possibly taking $|u|$ instead of u, we can suppose that $u \geq 0$; we also assume without loss of generality that Ω is bounded in $\mathbb{R}^n$. Define the set

$$D = \left\{p \in \Omega : \limsup_{r\to 0} \frac{|D_X u|(B(p, r))}{r^{Q-1}} = +\infty\right\}.$$

By Proposition 1.1.18 we have that $\mathscr{H}^{Q-1}(D) = 0$. For every $h \in \mathbb{N}$ we can find $t_h \in (h, h + 1)$ such that

$$P_X(\{u > t_h\}, \Omega) \leq \int_h^{h+1} P_X(\{u > t\}, \Omega)\, dt.$$

Define $E_h := \{u > t_h\}$. Since $u \in L^1(\Omega)$ we have that $\lim_h \mathscr{L}^n(E_h) = 0$ and applying the Coarea Formula of Theorem 1.6.6 we get

$$\sum_{h=0}^{\infty} P_X(E_h, \Omega) \leq \int_0^{+\infty} P_X(\{u > t\}, \Omega)\, dt = |D_X u|(\Omega) < +\infty,$$

and therefore $\lim_h P_X(E_h, \Omega) = 0$. We are in a position to apply Lemma 2.2.1. Defining for every $h \in \mathbb{N}$

$$F_h = \left\{ p \in \Omega : \limsup_{r \to 0} \frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))} \geq \alpha \right\},$$

where $\alpha > 0$ will be chosen later depending on Ω only, we have that $\mathscr{H}^{Q-1}\left(\bigcap_{h=0}^{\infty} F_h\right) = 0$. It is then sufficient to prove the inclusion

$$L := \left\{ p \in \Omega : \limsup_{r \to 0} \fint_{B(p,r)} |u|^{\frac{Q}{Q-1}} d\mathscr{L}^n = +\infty \right\} \subseteq D \cup \bigcap_{h=0}^{\infty} F_h. \tag{2.19}$$

To this aim, we fix $p \notin D \cup \bigcap_{h=0}^{\infty} F_h$ and we prove that $p \notin L$. Define $u_{p,r} := \fint_{B(p,r)} u d\mathscr{L}^n$. Applying Theorem 1.6.7, we get $C > 0$ and $R > 0$ such that

$$\fint_{B(q,r)} |u(y) - u_{q,r}|^{\frac{Q}{Q-1}} d\mathscr{L}^n(y) \leq C \left(\frac{|D_X u|(B(q,r))}{r^{Q-1}} \right)^{\frac{Q}{Q-1}}, \tag{2.20}$$

for every $q \in \Omega$ and all $0 < r < R$. It is then enough to prove that $\limsup_{r \to 0} u_{p,r} < +\infty$: in this case, in fact, the previous inequality and the definition of D would imply that $p \notin L$.

Suppose by contradiction that there exists a sequence (r_j) such that $\lim_j r_j = 0$ and $\lim_j u_{p,r_j} = +\infty$. Define $\widetilde{u}, \widetilde{u}_{r_j}$ as in (2.18) (with $r = r_j$) and $\widetilde{v}_j := \widetilde{u}_{r_j} - u_{p,r_j}$; set also

$$\widetilde{X}_i^j := \widetilde{X}_i^{r_j} \qquad \text{and} \qquad \widetilde{X}^j := (\widetilde{X}_1^j, \ldots, \widetilde{X}_m^j).$$

Since $p \notin D$, for any $\varrho > 0$ the sequence $r_j^{1-Q} |D_X u|(B(p, \varrho r_j))$ is uniformly bounded with respect to $j \in \mathbb{N}$; by Proposition 1.6.4, the same is true for the sequence

$$|D_{\widetilde{X}^j} \widetilde{v}_j|(\widetilde{B}_j(0, \varrho)) = r_j^{1-Q} |D_{\widetilde{X}} \widetilde{u}|(\widetilde{B}(0, \varrho r_j)),$$

where $\widetilde{B}_j(0, \varrho) := \widetilde{B}_{r_j}(0, \varrho)$, according to the notation introduced before (2.18). Taking also (2.17) into account, this proves that, for any compact set $K \subseteq \mathbb{R}^n$, the sequence $|D_{\widetilde{X}^j} \widetilde{v}_j|(K)$ is bounded; by (2.20), also $\|\widetilde{v}_j\|_{L^1(K)}$ is bounded.

Taking Theorem 1.4.5 into account, by Theorem 4.2.6, we can find $w \in L^1(\widehat{B}(0,1))$ such that (possibly extracting a subsequence)

$$\lim_j \|\widetilde{v}_j - w\|_{L^1(\widehat{B}(0,1))} = 0.$$

Consequently, for almost every $q \in \widehat{B}(0,1)$, we have

$$\lim_j u(F_p(\delta_{r_j} q)) = +\infty,$$

and then, for every $h \in \mathbb{N}$,

$$\begin{aligned}
\mathscr{L}^n(\widehat{B}(0,1)) &= \lim_j \mathscr{L}^n(\{q \in \widetilde{B}_j(0,1) : u(F_p(\delta_{r_j} q)) > t_h\}) \\
&= \lim_j \frac{\mathscr{L}^n(\{q \in \widetilde{B}(0,r_j) : u(F_p(q)) > t_h\})}{r_j^Q} \\
&= \lim_j \frac{1}{r_j^Q} \int_{B(p,r_j)\cap E_h} |\det \nabla F_p^{-1}| \, d\mathscr{L}^n \\
&\le |\det \nabla F_p^{-1}(p)| \limsup_{r\to 0} \frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))} \, \frac{\mathscr{L}^n(B(p,r))}{r^Q} \\
&\le \frac{C}{|\det \nabla F_p(0)|} \limsup_{r\to 0} \frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))}
\end{aligned}$$

where $C > 0$ is given by Theorem 1.2.4 with $K = \overline{\Omega}$. Notice that $\mathscr{L}^n(\widehat{B}(0,1))$ depends on p. Using (2.17) we obtain

$$\begin{aligned}
\limsup_{r\to 0} \frac{\mathscr{L}^n(E_h \cap B(p,r))}{\mathscr{L}^n(B(p,r))} &\ge \frac{|\det \nabla F_p(0)|}{C} \mathscr{L}^n(\widehat{B}(0,1)) \\
&= \frac{|\det \nabla F_p(0)|}{C} \lim_{r\to 0} \mathscr{L}^n(\widetilde{B}_r(0,1)) \\
&= \frac{|\det \nabla F_p(0)|}{C} \lim_{r\to 0} \frac{1}{r^Q} \mathscr{L}^n(\widetilde{B}(0,r)) \\
&= \frac{|\det \nabla F_p(0)|}{C} \lim_{r\to 0} \frac{1}{r^Q} \int_{B(p,r)} |\det \nabla F_p^{-1}| \, d\mathscr{L}^n \\
&\ge \frac{1}{C} \liminf_{r\to 0} \frac{\mathscr{L}^n(B(p,r))}{r^Q} \ \ge \ \frac{1}{C^2}.
\end{aligned}$$

This proves that $p \in \bigcap_{h=0}^\infty F_h$ for $\alpha := 1/C^2$, a contradiction. □

The following result is proved in [3] and it will be of capital importance throughout this Chapter. Recall Definition 1.1.21 for the notion of essential boundary $\partial^* E$ of a measurable set E. Observe also that, in the context of CC spaces, the reference measure is $\mathscr{L}^n$ and $\partial^* E$ is equivalently defined as the set of points $p \in \mathbb{R}^n$ such that

$$\liminf_{r\to 0} \frac{\mathscr{L}^n(E \cap B(p,r))}{\mathscr{L}^n(B(p,r))} < 1 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mathscr{L}^n(E \cap B(p,r))}{\mathscr{L}^n(B(p,r))} > 0.$$

Theorem 2.2.3. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q; let $E \subseteq \mathbb{R}^n$ be a set with finite X-perimeter in an open set $\Omega \subseteq \mathbb{R}^n$. Then*

$$P_X^E \llcorner \Omega = \eta \mathscr{H}^{Q-1} \llcorner (\Omega \cap \partial^* E) \tag{2.21}$$

for a suitable positive function η that is locally bounded away from zero. Moreover

$$\limsup_{r\to 0} \frac{P_X^E(B(p,2r))}{P_X^E(B(p,r))} < \infty \qquad \text{for } P_X^E\text{-a.e. } p \in \Omega \cap \partial^* E.$$

Theorem 2.2.4. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q. Then there exists $\lambda : \mathbb{R}^n \to (0, +\infty)$ locally bounded away from 0 such that, for every open set $\Omega \subseteq \mathbb{R}^n$ and any $u \in BV_X(\Omega; \mathbb{R}^k)$ one has*

$$|D_X u| \geq \lambda |u^+ - u^-| \mathscr{S}^{Q-1} \llcorner \mathcal{J}_u.$$

Moreover, for any Borel set $B \subseteq \Omega$ the following implications hold:

$$\mathscr{H}^{Q-1}(B) = 0 \quad \Rightarrow \quad |D_X u|(B) = 0; \tag{2.22}$$

$$\mathscr{H}^{Q-1}(B) < +\infty \quad \textit{and} \quad B \cap \mathcal{S}_u = \emptyset \quad \Rightarrow \quad |D_X u|(B) = 0. \tag{2.23}$$

Proof. Take $p \in \mathcal{J}_u$. By Proposition 2.1.8 the sequence $\widetilde{u}_r := u \circ F_p \circ \delta_r$ converges in $L^1(\widehat{B}(0,1))$ as $r \to 0$ to the function

$$w_p(y) := \begin{cases} u^+(p) & \text{if } \langle \overline{\nu}(p), y \rangle \geq 0 \\ u^-(p) & \text{if } \langle \overline{\nu}(p), y \rangle < 0. \end{cases}$$

Defining $\widetilde{X}_i^r$ as in (2.16) and using Propositions 4.2.7 and 1.6.4 we obtain for any positive ε that

$$\begin{aligned} \liminf_{r\to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} &\geq |\det \nabla F_p(0)| \liminf_{r\to 0} |D_{\widetilde{X}^r} \widetilde{u}_r|(\widetilde{B}_r(0,1)) \\ &\geq |\det \nabla F_p(0)| \liminf_{r\to 0} |D_{\widetilde{X}^r} \widetilde{u}_r|(\widehat{B}(0,1-\varepsilon)) \\ &\geq |\det \nabla F_p(0)| \, |D_{\widehat{X}} w_p|(\widehat{B}(0,1-\varepsilon)), \end{aligned}$$

whence

$$\begin{aligned} \liminf_{r\to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} &\geq |\det \nabla F_p(0)| \, |D_{\widehat{X}} w_p|(\widehat{B}(0,1)) \\ &\geq |\det \nabla F_p(0)| \, |u^+(p) - u^-(p)| \mathscr{H}_e^{n-1}(\overline{\nu}^\perp \cap \widehat{B}(0,1)) \end{aligned} \tag{2.24}$$

Using the Ball-Box Theorem (see, for instance, the version given in [72, equation (1.1)]) one can easily see that, for any $p \in \mathbb{R}^n$, there exist $c > 0$ and a neighborhood U of p such that the function $\lambda(q) := |\det \nabla F_q(0)| \mathscr{H}_e^{n-1}(\overline{\nu}^\perp \cap \widehat{B}_q(0,1))$ is such that $\lambda \geq c$ on U. By Corollary 1.1.19, this proves the first part of the statement.

By Theorem 2.2.3, the implication (2.22) is trivially true in case $k = 1$ and $u = \chi_E$ for some $E \subseteq \mathbb{R}^n$ with finite X-perimeter. If $k = 1$ and $u \in BV_X(\Omega)$, we define $E_s := \{u > s\}$ and we apply Theorem 1.6.6 (and, again, Theorem 2.2.3) to get

$$|D_X u|(B) = \int_{-\infty}^{+\infty} P_X(E_s; B) \, ds = \int_{-\infty}^{+\infty} \left(\int_{B \cap \partial^* E_s} \eta_s \, d\mathscr{H}^{Q-1} \right) ds$$

for suitable positive functions η_s. This allows to infer (2.22). In the general case $k \geq 1$, it is sufficient to recall inequality (1.28).

In order to prove (2.23) we take a Borel subset B of Ω and $u \in BV_X(\Omega)$ such that $B \cap \mathcal{S}_u = \emptyset$. If $k = 1$, by Theorem 1.6.6 we obtain again

$$\begin{aligned} |D_X u|(B) &= \int_{-\infty}^{+\infty} \left(\int_{B \cap \partial^* E_s} \eta_s \, d\mathscr{H}^{Q-1} \right) ds \\ &= \int_B \int_{\mathbb{R}} \eta_s(p) \chi_{\partial^* E_s}(p) \, ds \, d\mathscr{H}^{Q-1}(p) = 0, \end{aligned}$$

the last equality following from Proposition 2.1.4. In the case $u \in BV_X(\Omega; \mathbb{R}^k)$, $k \geq 2$, it is sufficient to notice that $B \cap \mathcal{S}_u = \emptyset$ implies $B \cap \mathcal{S}_{u^j} = \emptyset$ for every $j = 1, \dots, k$. Using inequality (1.28) we can complete the proof. □

Let us recall once more the notation $u_{p,r} := \fint_{B(p,r)} u \, d\mathscr{L}^n$.

Lemma 2.2.5. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q and let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set. Then there exist $C = C(\Omega) > 0$ and $R = R(\Omega) > 0$ such that, for every $p \in \Omega$, every $u \in BV_X(\Omega)$ and every $0 < r < \min\{R, \frac{1}{2} d(p, \partial\Omega)\}$, one has*

$$|u_{p,2r} - u_{p,r}| \leq C r^{1-Q} |D_X u|(B(p, 2r)).$$

Proof. We use Theorems 1.2.4 and 1.6.7 to estimate

$$\begin{aligned} |u_{p,2r} - u_{p,r}| = &\left| \fint_{B(p,r)} (u - u_{p,2r}) \, d\mathscr{L}^n \right| \leq C \fint_{B(p,2r)} |u - u_{p,2r}| \, d\mathscr{L}^n \\ \leq &C \left(\fint_{B(p,2r)} |u - u_{p,2r}|^{\frac{Q}{Q-1}} \, d\mathscr{L}^n \right)^{\frac{Q-1}{Q}} \leq C r^{1-Q} |D_X u|(B(p, 2r)). \end{aligned}$$

□

Lemma 2.2.6. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q and let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set. Then there exist $C = C(\Omega) > 0$ and $R = R(\Omega) > 0$ such that the following holds: for every $p \in \Omega$, $u \in BV_X(\Omega)$ and $0 < r < \min\{R, \frac{1}{2} d(p, \partial\Omega)\}$ with $p \notin \mathcal{S}_u$, one has*

$$\int_{B(p,r)} \frac{|u(q) - u^\star(p)|}{d(p,q)} \, d\mathscr{L}^n(q) \leq C \left(|D_X u|(B(p,r)) + \int_0^1 \frac{|D_X u|(B(p,tr))}{t^Q} \, dt \right).$$

In particular we have also

$$\int_{B(p,r)} \frac{|u(q) - u^\star(p)|}{d(p,q)} \, d\mathscr{L}^n(q) \leq C \int_0^2 \frac{|D_X u|(B(p,tr))}{t^Q} \, dt.$$

Proof. Let u, p, r be as in the statement; denote for shortness $u_i := u_{p,2^{-i}r}$, $i \in \mathbb{N}$. Since $u_i \to u^\star(p)$ as $i \to \infty$ we estimate

$$\begin{aligned}
&\int_{B(p,r)} \frac{|u(q) - u^\star(p)|}{d(p,q)}\, d\mathscr{L}^n(q)\\
\le&\sum_{i=1}^{\infty} \int_{B(p,2^{-i+1}r)\setminus B(p,2^{-i}r)} \frac{|u(q) - u^\star(p)|}{2^{-i}r}\, d\mathscr{L}^n(q)\\
\le&\sum_{i=1}^{\infty} \frac{2^i}{r} \int_{B(p,2^{-i+1}r)\setminus B(p,2^{-i}r)} \left(|u(q) - u_{i-1}| + \sum_{j=i-1}^{\infty} |u_j - u_{j+1}| \right) d\mathscr{L}^n(q)
\end{aligned}$$

and use Lemma 2.2.5 and Theorem 1.6.7 to get

$$\begin{aligned}
\le&C\sum_{i=1}^{\infty} \frac{2^i}{r} \left(2^{-i}r|D_Xu|(B(p,2^{1-i}r)) + \sum_{j=i-1}^{\infty} \left(2^{1-i}r\right)^Q \left(2^{-(j+1)}r\right)^{1-Q} |D_Xu|(B(p,2^{-j}r)) \right)\\
\le&C\sum_{i=1}^{\infty} \left(|D_Xu|(B(p,2^{1-i}r)) + \sum_{j=i-1}^{\infty} 2^{(j-i+1)(Q-1)}|D_Xu|(B(p,2^{-j}r)) \right)\\
=&C\sum_{k=0}^{\infty} \left(1 + 1 + 2^{Q-1} + (2^{Q-1})^2 + \cdots + (2^{Q-1})^k\right)|D_Xu|(B(p,2^{-k}r))\\
\le&C\sum_{k=0}^{\infty} \frac{2^{(k+1)(Q-1)} - 1}{2^{Q-1} - 1}|D_Xu|(B(p,2^{-k}r)).
\end{aligned}$$

Since $Q \ge 2$ we have $2^{Q-1} - 1 \ge \frac{2^{Q-1}}{2}$ and hence

$$\begin{aligned}
\int_{B(p,r)} \frac{|u(q) - u^\star(p)|}{d(p,q)}\, d\mathscr{L}^n(q) &\le C\sum_{k=0}^{\infty} 2^{k(Q-1)}|D_Xu|(B(p,2^{-k}r))\\
&= C\left(|D_Xu|(B(p,r)) + \sum_{k=1}^{\infty} 2^{k(Q-1)}|D_Xu|(B(p,2^{-k}r)) \right)\\
&= C\left(|D_Xu|(B(p,r)) + \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{1-k}} 2^{kQ}|D_Xu|(B(p,2^{-k}r))\, dt \right)\\
&\le C\left(|D_Xu|(B(p,r)) + \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{1-k}} \frac{|D_Xu|(B(p,tr))}{t^Q}\, dt \right)\\
&= C\left(|D_Xu|(B(p,r)) + \int_0^1 \frac{|D_Xu|(B(p,tr))}{t^Q}\, dt \right),
\end{aligned}$$

which completes the proof. □

Definition 2.2.7 (Absolutely continuous and singular parts)**.** Let $u \in BV_X(\Omega;\mathbb{R}^k)$. We denote by $D^a_X u$ and $D^s_X u$, respectively, the absolutely continuous and singular part of D_Xu with respect to $\mathscr{L}^n$.

Definition 2.2.8 (Jump and Cantor parts)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $u \in BV_X(\Omega; \mathbb{R}^k)$. The measures

$$D_X^j u := D_X^s u \llcorner \mathcal{J}_u, \qquad D_X^c u := D_X^s u \llcorner (\Omega \setminus \mathcal{J}_u),$$

are called, respectively, *jump part* of the measure derivative of u and *Cantor part* of the measure derivative of u.

Theorem 2.2.9. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in BV_X(\Omega; \mathbb{R}^k)$. Then u is approximately X-differentiable at $\mathscr{L}^n$-almost every point of Ω. Moreover the approximate differential $D_X^{ap} u$ coincides $\mathscr{L}^n$-almost everywhere with the density of the absolutely continuous part of the distributional derivative $D_X u$ with respect to $\mathscr{L}^n$.*

Proof. We can assume without loss of generality that $k = 1$. Suppose $D_X u = v\mathscr{L}^n + D_X^s u$ is the Radon-Nykodým decomposition of the measure $D_X u$ with respect to $\mathscr{L}^n$. By the Radon-Nykodým Theorem in doubling metric spaces (see Theorem 1.1.3), at $\mathscr{L}^n$-almost every $p \in \Omega$ we have

$$\lim_{r \to 0} \frac{D_X^s u(B(p,r))}{r^Q} = 0. \tag{2.25}$$

It is sufficient to prove that, for every $p \in \Omega \setminus (\mathcal{S}_u \cup \mathcal{S}_v)$ for which (2.25) holds, u is approximately X-differentiable at p with $D_X^{ap} u(p) = v^\star(p)$.

Let $R > 0$ and $f \in C^1(B(p,R))$ be such that $f(p) = 0$ and $Xf(p) = v^\star(p)$ and define

$$w(q) := u(q) - u^\star(p) - f(q).$$

Then $w \in BV(B(p,R))$, $p \in B(p,R) \setminus \mathcal{S}_w$ and $w^\star(p) = 0$. We are in a position to apply Lemma 2.2.6 to the function w and get $C > 0$ so that, for small enough r,

$$\begin{aligned} \frac{1}{r^Q} \int_{B(p,r)} \frac{|u(q) - u^\star(p) - f(q)|}{d(p,q)} \, d\mathscr{L}^n(q) &\leq \frac{C}{r^Q} \int_0^2 \frac{|D_X w|(B(p,tr))}{t^Q} \, dt \\ &\leq C \sup_{t \in (0,2)} \frac{|D_X w|(B(p,tr))}{(tr)^Q}. \end{aligned}$$

It is then enough to show that $\lim_{r \to 0} r^{-Q} |D_X w|(B(p,r)) = 0$. Taking into account that $D_X w = (v - Xf)\mathscr{L}^n + D_X^s u$ and (2.25), it suffices to check that

$$\lim_{r \to 0} \frac{1}{r^Q} \int_{B(p,r)} |v - Xf| \, d\mathscr{L}^n = 0,$$

which follows by Theorem 2.1.2 and the inequality $|v - Xf| \leq |v - v^\star(p)| + |v^\star(p) - Xf|$. □

One important fact about BV function is about the existence of a trace operator depending on a sufficiently smooth boundary. The following theorem is a consequence of some results contained in [89]. For the purpose, we introduce the notation

$$B_f^{\pm}(p,r) := \{q \in B(p,r) : \pm f > 0\},$$

for $p \in \mathbb{R}^n$, $r > 0$ and a function f.

Theorem 2.2.10. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $f \in C_X^1(\Omega)$ be such that $Xf \neq 0$ on Ω; let S be the C_X^1-hypersurface $S := \Omega \cap \{f = 0\}$. Then, for any open set $U \Subset \Omega$, we have $\mathscr{H}^{Q-1}(S \cap U) < \infty$. Moreover, there exist two linear operators $T^+, T^- : BV_{X,loc}(\Omega) \to L^1_{loc}(S, \mathscr{H}^{Q-1})$ such that, for any $u \in BV_{X,loc}(\Omega)$, one has*

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{B_f^+(p,r)} |u - T^+u(p)|\, d\mathscr{L}^n = \lim_{r\to 0} \frac{1}{r^Q} \int_{B_f^-(p,r)} |u - T^-u(p)|\, d\mathscr{L}^n = 0,$$

for $\mathscr{H}^{Q-1}$-a.e. $p \in S$. In particular,

$$T^{\pm}u(p) = \lim_{r\to 0} \frac{1}{r^Q} \int_{B_f^{\pm}(p,r)} u \, d\mathscr{L}^n,$$

for $\mathscr{H}^{Q-1}$-a.e. $p \in S$.

Proposition 2.2.11. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $R \subseteq \Omega$ be a countably X-rectifiable set. Then, for every $u \in BV_X(\Omega; \mathbb{R}^k)$ and for $\mathscr{H}^{Q-1}$-almost every $p \in R$, there exists a couple $(u^+(p), u^-(p)) \in \mathbb{R}^k \times \mathbb{R}^k$ such that*

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{\Omega\cap B^+_{\nu_R(p)}(p,r)} |u - u^+(p)|\, d\mathscr{L}^n = \lim_{r\to 0} \frac{1}{r^Q} \int_{\Omega\cap B^-_{\nu_R(p)}(p,r)} |u - u^-(p)|\, d\mathscr{L}^n = 0. \quad (2.26)$$

Proof. Without loss of generality we can assume $k = 1$. Let $u \in BV_X(\Omega)$ be fixed. By definition of countable X-rectifiability we can find a family $\{S_i : i \in \mathbb{N}\}$ of C_X^1-hypersurfaces in $\mathbb{R}^n$ such that

$$\mathscr{H}^{Q-1}\left(R \setminus \bigcup_{i=0}^{\infty} S_i\right) = 0.$$

For every $i \in \mathbb{N}$ we can write, at least locally, $S_i = \{f_i = 0\}$ and we can suppose that $Xf_i \neq 0$ on S_i. Formula (2.26) easily follows (with $u^{\pm}(p) = T^{\pm}u(p)$ and $\nu(p) = \nu_R(p)$) from Theorem 2.2.10 for $\mathscr{H}^{Q-1}$-a.e. $p \in R$ such that $\#\{i \in \mathbb{N} : p \in S_i\} = 1$. It is then enough to show that, for any fixed couple $i, j \in \mathbb{N}$ with $i \neq j$, the following holds: for $\mathscr{H}^{Q-1}$-almost every point $p \in S_i \cap S_j$, the equivalence

$$(T_i^+u(p), T_i^-u(p), \nu_{S_i}(p)) \equiv (T_j^+u(p), T_j^-u(p), \nu_{S_j}(p)) \quad (2.27)$$

holds. Here, $T_i^\pm, T_j^\pm$ are the trace operators provided by Theorem 2.2.10 with $f = f_i, f_j$.

Fix a point $p \in S_i \cap S_j$ where $\nu_{S_i}(p) = \pm\nu_{S_j}(p)$; recall that this fact occurs at $\mathscr{H}^{Q-1}$-a.e. $p \in S_i \cap S_j$. Assume that $\nu_{S_i}(p) = \nu_{S_j}(p)$, i.e., $\frac{Xf_i(p)}{|Xf_i(p)|} = \frac{Xf_j(p)}{|Xf_j(p)|}$; by Theorem 2.2.10 we have, for $\mathscr{H}^{Q-1}$-a.e. such p, that

$$\begin{aligned}
|T_i^\pm(p) - T_j^\pm(p)| &= \lim_{r\to 0} \frac{1}{r^Q} \left| \int_{\{\pm f_i>0\}\cap B(p,r)} u \, d\mathscr{L}^n - \int_{\{\pm f_j>0\}\cap B(p,r)} u \, d\mathscr{L}^n \right| \\
&\le \lim_{r\to 0} \frac{1}{r^Q} \int_{\{f_i f_j \le 0\}\cap B(p,r)} |u| \, d\mathscr{L}^n \\
&\le \lim_{r\to 0} \frac{1}{r^Q} \mathscr{L}^n(\{f_i f_j \le 0\} \cap B(p,r))^{1/Q} \left(\int_{B(p,r)} |u|^{\frac{Q}{Q-1}} d\mathscr{L}^n \right)^{\frac{Q-1}{Q}} .
\end{aligned}$$

By Remark 1.5.5, we have

$$\lim_{r\to 0} \frac{1}{r^Q} \mathscr{L}^n(\{f_i f_j \le 0\} \cap B(p,r)) = 0,$$

while by Lemma 2.2.2 we also have that for $\mathscr{H}^{Q-1}$-almost every $p \in \Omega$

$$\limsup_{r\to 0} \frac{1}{r^Q} \int_{B(p,r)} |u|^{\frac{Q}{Q-1}} d\mathscr{L}^n < +\infty.$$

This proves that $T_i^\pm(p) = T_j^\pm(p)$ for $\mathscr{H}^{Q-1}$-a.e. $p \in S_i \cap S_j$ such that $\nu_{S_i}(p) = \nu_{S_j}(p)$. A similar argument shows that $T_i^\pm(p) = T_j^\mp(p)$ holds for $\mathscr{H}^{Q-1}$-a.e. $p \in S_i \cap S_j$ with $\nu_{S_i}(p) = -\nu_{S_j}(p)$. This proves (2.27) and concludes the proof. $\square$

The results below show how some assumptions on the regularity of the essential boundary of sets with finite perimeter can induce some regularity of the sets $\mathcal{S}_u$, whenever $u \in BV_X(\Omega; \mathbb{R}^k)$.

Definition 2.2.12 (Property $\mathcal{R}$)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space with homogeneous dimension $Q \in \mathbb{N}$. We say that $(\mathbb{R}^n, X)$ satisfies *property* $\mathcal{R}$ if, for every open set $\Omega \subseteq \mathbb{R}^n$ and every $E \subseteq \mathbb{R}^n$ with locally finite X-perimeter in Ω, the essential boundary $\partial^* E \cap \Omega$ is countably X-rectifiable.

Definition 2.2.13 (Property $\mathcal{LR}$)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space with homogeneous dimension $Q \in \mathbb{N}$. We say that $(\mathbb{R}^n, X)$ satisfies *property* $\mathcal{LR}$ if, for every open set $\Omega \subseteq \mathbb{R}^n$ and every $E \subseteq \mathbb{R}^n$ with locally finite X-perimeter in Ω, the essential boundary $\partial^* E \cap \Omega$ is countably X-Lipschitz rectifiable.

Theorem 2.2.14. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u \in BV_X(\Omega; \mathbb{R}^k)$. Then $\mathcal{S}_u$ is contained in a countable union of sets with finite $\mathscr{H}^{Q-1}$ measure. Moreover, if $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$, then $\mathcal{S}_u$ is countably X-rectifiable and $\mathscr{H}^{Q-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$.*

Proof. Since $\mathcal{S}_u = \cup_{\alpha=1}^k \mathcal{S}_{u^\alpha}$, it is not restrictive to suppose $k = 1$. By the Coarea Formula we get a countable and dense set $D \subseteq \mathbb{R}$ such that for every $t \in D$ the level set $\{u > t\}$ has finite X-perimeter. We first prove that

$$\mathcal{S}_u \setminus L \subseteq \bigcup_{t\in D} \partial^*\{u > t\} \tag{2.28}$$

where, as in Theorem 2.2.2, L denotes the $\mathscr{H}^{Q-1}$-negligible set

$$L := \left\{ p \in \Omega : \limsup_{r\to 0} \fint_{B(p,r)} |u|^{\frac{Q}{Q-1}} d\mathscr{L}^n = +\infty \right\}.$$

For this purpose, take $p \notin L$ and suppose that $p \notin \bigcup_{t\in D} \partial^*\{u > t\}$; we will prove that $p \notin \mathcal{S}_u$. By definition, p is either a point of density 1 or a point of density 0 in $\{u > t\}$, for every $t \in D$. Notice that for every $t \in D \cap (0, +\infty)$ one has

$$\frac{\mathscr{L}^n\left(\{u > t\} \cap B(p, r)\right)}{\mathscr{L}^n(B(p, r))} \le \frac{1}{t} \fint_{B(p,r)} |u|\, d\mathscr{L}^n \le \frac{1}{t} \left(\fint_{B(p,r)} |u|^{\frac{Q}{Q-1}} d\mathscr{L}^n \right)^{\frac{Q-1}{Q}}$$

and therefore, if $t \in D \cap (0, +\infty)$ is large enough, p is a point of density 0 for $\{u > t\}$. Analogously, if $t \in D \cap (-\infty, 0)$ and $-t$ is large enough, p is a point of density 1 for $\{u > t\}$. Hence we can find a real number

$$z = z(p) := \sup \{t \in D : \{u > t\} \text{ has density 1 at } p\}.$$

By the density of D in $\mathbb{R}$ we get that for every $t > z$, $\{u > t\}$ has density 0 at p and for every $t < z$, $\{u > t\}$ has density 1 at p.
We prove now that z is the approximate limit of u at p. To this end define $E_\varepsilon := \{|u - z| > \varepsilon\}$ and estimate

$$\begin{aligned}
\frac{1}{r^Q} \int_{B(p,r)} |u - z|\, d\mathscr{L}^n &\le \varepsilon C + \frac{1}{r^Q} \int_{E_\varepsilon \cap B(p,r)} |u - z|\, d\mathscr{L}^n \\
&\le \varepsilon C + \frac{1}{r^Q} \left(\mathscr{L}^n(E_\varepsilon \cap B(p, r))\right)^{1/Q} \left(\int_{B(p,r)} |u - z|^{\frac{Q}{Q-1}}\, d\mathscr{L}^n \right)^{\frac{Q-1}{Q}} \\
&= \varepsilon C + \left(\frac{\mathscr{L}^n(E_\varepsilon \cap B(p, r))}{r^Q} \right)^{1/Q} \left(\frac{1}{r^Q} \int_{B(p,r)} |u - z|^{\frac{Q}{Q-1}}\, d\mathscr{L}^n \right)^{\frac{Q-1}{Q}}.
\end{aligned}$$

Since both $\{u > z + \varepsilon\}$ and $\{u < z - \varepsilon\}$ have density 0 at p, one has

$$\lim_{r\to 0} \frac{\mathscr{L}^n(E_\varepsilon \cap B(p, r))}{r^Q} = 0$$

and, since $p \notin L$, we get

$$\limsup_{r\to 0} \frac{1}{r^Q} \int_{B(p,r)} |u - z|\, d\mathscr{L}^n \le C\varepsilon,$$

from which we deduce that $p \notin \mathcal{S}_u$, as desired.

Assume now $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$. Then, (2.28) together with the fact that $\mathscr{H}^{Q-1}(L) = 0$, imply that $\mathcal{S}_u$ is countably X-rectifiable. It remains to prove that $\mathscr{H}^{Q-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$. Let $\nu = \nu_{\mathcal{S}_u}$ be the horizontal normal to $\mathcal{S}_u$. By Proposition 2.2.11, for $\mathscr{H}^{Q-1}$-almost every $p \in \mathcal{S}_u$, there exist $u^+(p)$ and $u^-(p)$ in $\mathbb{R}^k$ such that

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{B^+_{\nu(p)}(p,r)} |u - u^+(p)| \, d\mathscr{L}^n = 0$$

and

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{B^-_{\nu(p)}(p,r)} |u - u^-(p)| \, d\mathscr{L}^n = 0.$$

According to Definition 2.1.6, we are equivalently saying that the approximate jump triple $(u^+(p), u^-(p), \nu(p))$ exists for $\mathscr{H}^{Q-1}$-almost every $p \in \mathcal{S}_u$. This concludes the proof. □

The proof of Theorem 2.2.14 can be easily extended in order to prove the following result.

Theorem 2.2.15. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying property $\mathcal{LR}$ and let $u \in BV_X(\Omega; \mathbb{R}^k)$. Then $\mathcal{S}_u$ is countably X-Lipschitz rectifiable.*

Remark 2.2.16. Notice that combining Theorem 2.2.14 and Proposition 2.2.11, we have that, whenever $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$, the set $\mathcal{J}_u$ is rectifiable and therefore, for $\mathscr{H}^{Q-1}$-almost every $p \in \mathcal{J}_u$, one has that $(u^+_{\mathcal{J}_u}, u^-_{\mathcal{J}_u}, \nu_{\mathcal{J}_u})$ is an approximate X-jump triple for u at p.

As for classical BV functions (see e.g. [5, pag. 177]), the (approximate) convergence of $u \in BV_X$ to $u^\star(p)$ at points $p \notin \mathcal{S}_u$ can be improved in a L^{1^*}-sense, as we now state.

Proposition 2.2.17. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space, $\Omega \subseteq \mathbb{R}^n$ an open set and let $u \in BV_X(\Omega)$. Then*

$$\lim_{r\to 0} ⨍_{B(p,r)} |u - u^\star(p)|^{\frac{Q}{Q-1}} d\mathscr{L}^n = 0 \qquad \text{for } \mathscr{H}^{Q-1}\text{-a.e. } p \in \Omega \setminus \mathcal{S}_u.$$

Proof. We first prove that

$$\lim_{r\to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} = 0 \qquad \text{for } \mathscr{H}^{Q-1}\text{-a.e. } p \in \Omega \setminus \mathcal{S}_u. \tag{2.29}$$

Let $t > 0$ be fixed and consider the set

$$E_t := \left\{ p \in \Omega \setminus \mathcal{S}_u : \limsup_{r\to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} > t \right\}.$$

By Theorem 1.1.18 one has $\mathscr{H}^{Q-1}(E_t) < +\infty$ and then, by Theorem 2.2.4, we have $|D_X u|(E_t) = 0$ and again Proposition 1.1.18 gives $\mathscr{H}^{Q-1}(E_t) = 0$. Since this is true for all positive t, formula (2.29) immediately follows.

Combining Theorem 1.6.7 and (2.29) we immediately get that

$$\lim_{r\to 0} \fint_{B(p,r)} |u - u_{p,r}|^{\frac{Q}{Q-1}} d\mathscr{L}^n = 0,$$

for $\mathscr{H}^{Q-1}$-a.e. $p \in \Omega$. The conclusion then follows by

$$|u - u^\star(p)|^{\frac{Q}{Q-1}} \le 2^{\frac{1}{Q-1}} \left(|u_{p,r} - u^\star(p)|^{\frac{Q}{Q-1}} + |u - u_{p,r}|^{\frac{Q}{Q-1}} \right),$$

together with $u^\star(p) = \lim_{r\to 0} u_{p,r}$. □

When $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$, $\Omega \subseteq \mathbb{R}^n$ is open and $u \in BV_X(\Omega;\mathbb{R}^k)$, by Theorem 2.2.14 the *precise representative* $u^{\mathfrak{p}}$

$$u^{\mathfrak{p}}(p) := \begin{cases} u^\star(p) & \text{if } p \in \Omega \setminus \mathcal{S}_u, \\ \dfrac{u^+(p) + u^-(p)}{2} & \text{if } p \in \mathcal{J}_u \end{cases} \tag{2.30}$$

is defined $\mathscr{H}^{Q-1}$-a.e. on Ω. We have the following result.

Theorem 2.2.18. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying property $\mathcal{R}$, $\Omega \subseteq \mathbb{R}^n$ an open set and let $u \in BV_X(\Omega;\mathbb{R}^k)$. Then*

$$\lim_{r\to 0} \fint_{B(p,r)} u \, d\mathscr{L}^n = u^{\mathfrak{p}}(p) \qquad \text{for } \mathscr{H}^{Q-1}\text{-a.e. } p \in \Omega.$$

Proof. The statement easily follows for $\mathscr{H}^{Q-1}$-a.e. $p \in \Omega \setminus \mathcal{S}_u$ by Proposition 2.2.17. By Theorem 2.2.14 it suffices to prove the statement for all $p \in \mathcal{J}_u$, which directly follows from Proposition 2.1.5 and Definition 2.1.6. □

Remark 2.2.19. When $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$, then $D^c_X u = D^s_X u \llcorner (\Omega \setminus \mathcal{S}_u)$: to see this, it is enough to combine Theorems 2.2.4 and 2.2.14.

We now want to study the properties of the decomposition $D_X u = D^a_X u + D^c_X u + D^j_X u$.

Theorem 2.2.20 (Properties of Cantor part and jump part)**.** *Let $u \in BV_X(\Omega;\mathbb{R}^k)$. Then the following facts hold:*

(a) $D^a_X u = D_X u \llcorner (\Omega \setminus S)$ and $D^s_X u = D_X u \llcorner S$, where

$$S := \left\{ p \in \Omega : \lim_{r\to 0} \frac{|D_X u|(B(p,r))}{r^Q} = +\infty \right\}.$$

Moreover, if $E \subseteq \mathbb{R}^k$ is such that $\mathscr{H}^1_e(E) = 0$, then $D^{ap}_X u = 0$ $\mathscr{L}^n$-a.e. in $(u^\star)^{-1}(E)$.

(b) *Let* $\Theta_u \subseteq S$ *be defined by*

$$\Theta_u := \left\{ p \in \Omega : L(p) := \liminf_{r \to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} > 0 \right\}.$$

Then $\mathscr{J}_u \subseteq \Theta_u$.

Moreover, if $(\mathbb{R}^n, X)$ *satisfies property* $\mathcal{R}$*, then*

(c) $\mathscr{H}^{Q-1}(\Theta_u \setminus \mathscr{J}_u) = 0$ *and* $D^j_X u = D_X u \llcorner \Theta_u$*. More generally, for every Borel set* Σ *containing* $\mathscr{J}_u$ *and* σ*-finite with respect to* $\mathscr{H}^{Q-1}$*, we have* $D^j_X u = D_X u \llcorner \Sigma$.

(d) $D^c_X u = D_X u \llcorner (S \setminus \Theta_u)$.

(e) *if* $B \subseteq \Omega$ *is such that either* $\mathscr{H}^{Q-1} \llcorner B$ *is* σ*-finite or* $B = (u^\star)^{-1}(E)$ *for some* $\mathscr{H}^1_e$*-negligible set* $E \subseteq \mathbb{R}^k$*, then* $D^c_X u(B) = 0$.

Proof. In order to prove the first part of statement (a) it is sufficient to apply Radon-Nykodým Theorem in doubling metric spaces (see e.g. [84, Theorem 4.7 and Remark 4.5]). Concerning the second part, assume first that $k = 1$ and let $B := (u^\star)^{-1}(E)$. By Proposition 2.1.4, for any $t \notin E$ we have $B \cap \partial^*\{u > t\} = \emptyset$. By Theorems 1.6.6 and 2.2.3 we obtain

$$|D_X u|(B) = \int_{\mathbb{R}} P_X(\{u > t\} \cap B)\, dt = 0 = \int_{\mathbb{R}\setminus E} \int_{\partial^*\{u>t\}\cap B} \theta_t \, d\mathscr{H}^{Q-1}\, dt = 0,$$

where θ_t denote suitable positive functions. When $k \geq 1$ and $i = 1, \dots, k$ we set $E_i := \{t \in \mathbb{R} : t = z_i \text{ for some } z \in E\}$; the set E_i is such that $\mathscr{L}^1(E_i) = 0$ and by (1.28)

$$|D_X u|(B) \leq \sum_{i=1}^k |D_X u^i|(B) \leq \sum_{i=1}^k |D_X u^i|(((u^i)^\star)^{-1}(E_i)) = 0.$$

We then conclude by Theorem 2.2.9.

By (2.24) in the proof of Theorem 2.2.4 we have $\mathscr{J}_u \subseteq \Theta_u$, and statement (b) follows.

We now prove (c). Applying Proposition 1.1.18 we get that for every $h \in \mathbb{N} \setminus \{0\}$

$$|D_X u| \llcorner \{L \geq \tfrac{1}{h}\} \geq \frac{1}{h} \omega_{Q-1} \mathscr{H}^{Q-1} \llcorner \{L \geq \tfrac{1}{h}\}, \tag{2.31}$$

where L is defined in statement (b). In particular, $\mathscr{H}^{Q-1}\left(\{L \geq \tfrac{1}{h}\}\right) < +\infty$. By (2.23)

$$|D_X u|\big(\{L \geq \tfrac{1}{h}\} \setminus \mathcal{S}_u\big) = 0$$

and consequently (by (2.31)) also $\mathscr{H}^{Q-1}(\{L \geq \tfrac{1}{h}\} \setminus \mathcal{S}_u) = 0$. Since $\{L \geq \tfrac{1}{h}\} \nearrow \Theta_u$, on passing to the limit for $h \to +\infty$ we get $\mathscr{H}^{Q-1}(\Theta_u \setminus \mathcal{S}_u) = 0$. Taking Theorem 2.2.14 into account, we conclude that $\mathscr{H}^{Q-1}(\Theta_u \setminus \mathscr{J}_u) = 0$.

Let now Σ be as in statement (c). Then, taking into account Theorem 2.2.4 and the fact that $\mathscr{H}^{Q-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$, we have

$$\begin{aligned} D_X u \llcorner \Sigma &= D_X u \llcorner \mathcal{J}_u + D_X u \llcorner (\Sigma \setminus \mathcal{J}_u) \\ &= D_X^j u + D_X u \llcorner (\Sigma \setminus \mathcal{S}_u) + D_X u \llcorner (\Sigma \cap \mathcal{S}_u \setminus \mathcal{J}_u) \\ &= D_X^j u + D_X u \llcorner (\Sigma \setminus \mathcal{S}_u). \end{aligned}$$

Since Σ is σ-finite with respect to $\mathscr{H}^{Q-1}$, using (2.23) we get that $D_X u \llcorner (\Sigma \setminus \mathcal{S}_u) = 0$, and so $D_X u \llcorner \Sigma = D_X^j u$.

Statement (d) follows from (a), (b), (c) and the decomposition $D_X u = D_X^a u + D_X^c u + D_X^j u$, which immediately give that $D_X^c u = D_X u \llcorner (S \setminus \Theta_u)$.

We prove (e) in case $\mathscr{H}^{Q-1} \llcorner B$ is σ-finite; we can assume (see e.g. [5, Theorem 1.43]) that B is a Borel set. Using Theorems 2.2.4 and 2.2.14 we get that $|D_X u|(B \setminus \mathcal{J}_u) = 0$, which gives $(D_X^a u + D_X^c u) \llcorner B = 0$.
Concerning the second part of statement (e), suppose first that $k = 1$ and let $B = (u^\star)^{-1}(E)$ with $\mathscr{L}^1(E) = 0$. By Proposition 2.1.4 we know that $\partial^*\{u > t\} \cap B = \emptyset$ for every $t \notin E$. Applying the Coarea Formula of Theorem 1.6.6 we get

$$|D_X u|(B) = \int_E \int_{\partial^*\{u>t\}\cap B} \theta_t \, d\mathscr{H}^{Q-1} dt = 0$$

for suitable positive functions θ_t. In the general case $k \geq 2$ define for every $i = 1, \ldots, k$ the sets $E_i := \pi_i(E)$, where π_i denotes the canonical projection $\pi_i(x_1, \ldots, x_k) = x_i$. Noticing that $\mathscr{L}^1(E_i) \leq \mathscr{H}_e^1(E) = 0$, we can use (1.28) to estimate

$$|D_X u|((u^\star)^{-1}(E)) \leq \sum_{i=1}^{k} |D_X u^i|((u^\star)^{-1}(E)) \leq \sum_{i=1}^{k} |D_X u^i|(((u^i)^\star)^{-1}(E_i)) = 0,$$

and conclude the proof. □

The problem of studying "intrinsic" measures of submanifolds of a CC space goes back to M. Gromov [45, 0.6.b]: the interested reader might consult [60, 64, 65, 77] and the references therein. Since we do not intend to dwell on such questions, we follow a different ("axiomatic") path; this is based on the following definition, where we choose to work with the spherical Hausdorff measure $\mathscr{S}^{Q-1}$, rather than the standard one, because the results mentioned above (as well as [38, 39]) suggest $\mathscr{S}^{Q-1}$ to be more natural than the standard measure $\mathscr{H}^{Q-1}$.

Definition 2.2.21 (Property $\mathcal{D}$)**.** Let $(\mathbb{R}^n, X)$ be an equiregular CC space with homogeneous dimension $Q \in \mathbb{N}$. We say that $(\mathbb{R}^n, X)$ satisfies *property* $\mathcal{D}$ if there exists a function $\zeta : \mathbb{R} \times \mathbb{S}^{m-1} \to (0, +\infty)$ such that, for every C_X^1-hypersurface $S \subseteq \mathbb{R}^n$ and every $p \in S$, one has

$$\lim_{r\to 0} \frac{\mathscr{S}^{Q-1}(S \cap B(p, r))}{r^{Q-1}} = \zeta(p, \nu_S(p)).$$

Remark 2.2.22. If $(\mathbb{R}^n, X)$ is an equiregular CC space satisfying *property* $\mathcal{D}$ and $R \subseteq \mathbb{R}^n$ is X-rectifiable, then we have

$$\lim_{r\to 0} \frac{\mathscr{S}^{Q-1}(R \cap B(p,r))}{r^{Q-1}} = \zeta(p, \nu_R(p)) \qquad \text{for } \mathscr{S}^{Q-1}\text{-a.e. } p \in R,$$

where ζ is as in Definition 2.2.21.

Let us prove this fact. Let $S_i, i \in \mathbb{N}$, be a family of C^1_X-hypersurfaces such that $\mathscr{S}^{Q-1}(R \setminus \cup_{i\in\mathbb{N}} S_i) = 0$; it is enough to show that, for any fixed $i \in \mathbb{N}$, we have

$$\lim_{r\to 0} \frac{\mathscr{S}^{Q-1}(R \cap B(p,r))}{r^{Q-1}} = \zeta(p, \nu_R(p)) \qquad \text{for } \mathscr{S}^{Q-1}\text{-a.e. } p \in R \cap S_i.$$

Setting $R \Delta S_i := (R \setminus S_i) \cup (S_i \setminus R)$, by Remark 1.1.20 (applied with $\mu := \mathscr{S}^{Q-1} \llcorner (R\Delta S_i))$ we obtain

$$\lim_{r\to 0} \frac{\mathscr{S}^{Q-1}((R\Delta S_i) \cap B(p,r))}{r^{Q-1}} = 0 \qquad \text{for } \mathscr{S}^{Q-1}\text{-a.e. } p \in R \cap S_i,$$

which gives for $\mathscr{S}^{Q-1}$-a.e. $p \in R \cap S_i$

$$\lim_{r\to 0} \frac{\mathscr{S}^{Q-1}(R \cap B(p,r))}{r^{Q-1}} = \lim_{r\to 0} \frac{\mathscr{S}^{Q-1}(S_i \cap B(p,r))}{r^{Q-1}} = \zeta(p, \nu_{S_i}(p)) = \zeta(p, \nu_R(p))$$

as desired.

Assuming properties $\mathcal{R}$ and $\mathcal{D}$ we are able to prove the following result, where we use the notation u^+_R, u^-_R of Proposition 2.2.11.

Theorem 2.2.23. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying properties $\mathcal{R}$ and $\mathcal{D}$; then, there exists a function $\sigma : \mathbb{R}^n \times \mathbb{S}^{m-1} \to (0, +\infty)$ such that the following holds. For every open set $\Omega \subseteq \mathbb{R}^n$, $u \in BV_X(\Omega; \mathbb{R}^k)$ and every countably X-rectifiable set $R \subseteq \mathbb{R}^n$ one has*

$$D_X u \llcorner R = \sigma(\cdot, \nu_R)(u^+_R - u^-_R) \otimes \nu_R \, \mathscr{S}^{Q-1} \llcorner R.$$

In particular, $D^j_X u = \sigma(\cdot, \nu_u)(u^+ - u^-) \otimes \nu_u \, \mathscr{S}^{Q-1} \llcorner \mathcal{J}_u$.

Proof. We can assume without loss of generality that $k = 1$ and $\mathscr{S}^{Q-1}(R) < +\infty$. By Theorems 2.2.14 and 2.2.4 we can also assume that $R \subseteq \mathcal{J}_u$. Given $p \in \mathbb{R}^n$, we work in adapted exponential coordinates F_p around p and we define

$$\sigma(p, \nu) := \frac{|\det \nabla F_p(0)| \mathscr{H}^{n-1}_e(\overline{\nu}^\perp \cap \widehat{B}_p(0,1))}{\zeta(p,\nu)},$$

where ζ is as in Definition 2.2.21 and, as in the proof of Theorem 2.2.4, $\mathscr{H}^{n-1}_e$ denotes the Euclidean Hausdorff measure in $\mathbb{R}^n$.

Let $\mu_R := D_X u \llcorner R$; by Theorem 2.2.4 we have $\mu_R \ll \mathscr{S}^{Q-1} \llcorner R$. By Remark 2.2.22 and Theorem 1.1.13, it is enough to prove that for $\mathscr{S}^{Q-1}$-a.e. $p \in R$

$$\lim_{r\to 0} \frac{\mu_R(B(p,r))}{\mathscr{S}^{Q-1}(R\cap B(p,r))} = \sigma(p,\nu_R(p))(u_R^+(p) - u_R^-(p))\nu_R(p);$$

notice that the limit above exists $\mathscr{S}^{Q-1}$-almost everywhere. Taking into account Remark 2.2.22 and the fact that (by Remark 1.1.20)

$$\lim_{r\to 0} \frac{|D_X u - \mu_R|(B(p,r))}{r^{Q-1}} = 0 \qquad \text{for } \mathscr{S}^{Q-1}\text{-a.e. } p \in R,$$

it suffices to prove that, for $\mathscr{S}^{Q-1}$-a.e. $p \in R$, there exists a sequence $r_i \to 0$ such that

$$\lim_{i\to +\infty} \frac{D_X u(B(p,r_i))}{r_i^{Q-1}} = |\det \nabla F_p(0)| \mathscr{H}_e^{n-1}(\overline{\nu}^\perp \cap \widehat{B}_p(0,1))(u_R^+(p) - u_R^-(p))\nu_R(p).$$

We prove that such a sequence exists at all points where $\limsup_{r\to 0} \frac{|D_X u|(B(p,r))}{r^{Q-1}} < \infty$, which holds for $\mathscr{S}^{Q-1}$-a.e. $p \in R$ due to Remark 1.1.20.

Let then such a $p \in R$ be fixed; since $R \subseteq \mathcal{J}_u$, the functions $\widetilde{u}_r := u \circ F_p \circ \delta_r$ converge in $L^1_{loc}(\mathbb{R}^n)$ to

$$w_p(y) := \begin{cases} u^+(p) & \text{if } \widetilde{L}_{\nu_R(p)}(y) \ge 0 \\ u^-(p) & \text{if } \widetilde{L}_{\nu_R(p)}(y) < 0, \end{cases}$$

where we used the fact that $\nu_R = \nu_{\mathcal{J}_u} = \nu_u$ $\mathscr{S}^{Q-1}$-a.e. on R. Let $\widetilde{u} := u \circ F_p$; since (recall notation (2.16)) $|D_{\widetilde{X}^r}\widetilde{u}_r|(\widetilde{B}_r(0,\varrho)) = |D_{\widetilde{X}}\widetilde{u}|(\widetilde{B}(0,r\varrho))/r^{Q-1}$ is bounded as $r \to 0$ for any positive ϱ, by Remark 4.2.8 the sequence $D_{\widetilde{X}^r}\widetilde{u}_r$ weakly* converges in $\mathbb{R}^n$ to $D_{\widehat{X}} w_p$ as $r \to 0$. Let s_i be an infinitesimal sequence such that $|D_{\widetilde{X}^{s_i}}\widetilde{u}_{s_i}|$ weakly* to some measure λ in $\mathbb{R}^n$; let $\varrho \in (0,1)$ be such that $\lambda(\partial\widehat{B}_p(0,\varrho)) = 0$ (which holds for all except at most countably many ϱ) and define $r_i := \varrho s_i$. Proposition 1.6.4 gives

$$\begin{aligned}\lim_{i\to\infty} \frac{D_X u(B(p,r_i))}{r_i^{Q-1}} =& |\det \nabla F_p(0)| \lim_{i\to\infty} \frac{D_{\widetilde{X}}\widetilde{u}(\widetilde{B}(0,r_i))}{r_i^{Q-1}} \\ =& |\det \nabla F_p(0)| \lim_{i\to\infty} \frac{D_{\widetilde{X}^{s_i}}\widetilde{u}^{s_i}(\widetilde{B}_{s_i}(0,\varrho))}{\varrho^{Q-1}}.\end{aligned}$$

We prove in a moment that

$$\lim_{i\to\infty} \frac{D_{\widetilde{X}^{s_i}}\widetilde{u}^{s_i}(\widetilde{B}_{s_i}(0,\varrho))}{\varrho^{Q-1}} = \frac{D_{\widehat{X}} w_p(\widehat{B}_p(0,\varrho))}{\varrho^{Q-1}}; \tag{2.32}$$

assuming this to be true, we have

$$\begin{aligned}\lim_{i\to\infty} \frac{D_X u(B(p,r_i))}{r_i^{Q-1}} =& |\det \nabla F_p(0)| \ \frac{D_{\widehat{X}} w_p(\widehat{B}_p(0,\varrho))}{\varrho^{Q-1}} \\ =& |\det \nabla F_p(0)| \mathscr{H}_e^{n-1}(\overline{\nu}^\perp \cap \widehat{B}_p(0,1))(u_R^+(p) - u_R^-(p))\nu_R(p).\end{aligned}$$

and the proof would be concluded.

Let us prove (2.32). Defining

$$\mu_i := D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i} \llcorner \widetilde{B}_{s_i}(0, \varrho), \qquad \mu := D_{\widehat{X}} w_p \llcorner \widehat{B}_p(0, \varrho)$$

and taking into account part (b) of Proposition 1.1.4, it suffices to show that

$$\mu_i \overset{*}{\rightharpoonup} \mu \qquad \text{and} \qquad |\mu_i| \overset{*}{\rightharpoonup} \lambda \llcorner \widehat{B}_p(0, \varrho). \tag{2.33}$$

Concerning the first statement in (2.33), fix a test function $\varphi \in C_c(\mathbb{R}^n)$; then

$$\begin{aligned}
&\lim_{i\to\infty} \int \varphi \, d\mu_i = \lim_{i\to\infty} \int_{\widetilde{B}_{s_i}(0,\varrho)} \varphi \, dD_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i} \\
&= \lim_{i\to\infty} \int_{\widehat{B}_p(0,\varrho)} \varphi \, dD_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i} + \int_{\widetilde{B}_{s_i}(0,\varrho)\setminus \widehat{B}_p(0,\varrho)} \varphi \, dD_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i} - \int_{\widehat{B}_p(0,\varrho)\setminus \widetilde{B}_{s_i}(0,\varrho)} \varphi \, dD_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i} \\
&= \lim_{i\to\infty} \int_{\widehat{B}_p(0,\varrho)} \varphi \, dD_{\widehat{X}} w_p,
\end{aligned}$$

where the last equality follows from the weak* convergence of $D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i}$ to $D_{\widehat{X}} w_p$ and the fact that (denoting by Δ the symmetric difference of sets)

$$\lim_{i\to\infty} |D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i}|(\widetilde{B}_{s_i}(0, \varrho) \Delta \widehat{B}_p(0, \varrho)) = 0$$

that, in turn, can be proved as follows. For any $\varepsilon > 0$ there exists $\delta \in (0, \varrho)$ such that

$$\lambda(\overline{\widehat{B}_p(0, \varrho + \delta)} \setminus \widehat{B}_p(0, \varrho - \delta)) < \varepsilon;$$

by Theorem 1.4.9 we obtain

$$\begin{aligned}
\limsup_{i\to\infty} |D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i}|(\widetilde{B}_{s_i}(0, \varrho) \Delta \widehat{B}_p(0, \varrho)) &\le \limsup_{i\to\infty} |D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i}|(\overline{\widehat{B}_p(0, \varrho + \delta)} \setminus \widehat{B}_p(0, \varrho - \delta)) \\
&\le \lambda(\overline{\widehat{B}_p(0, \varrho + \delta)} \setminus \widehat{B}_p(0, \varrho - \delta)) < \varepsilon,
\end{aligned}$$

where we used part (a) of Proposition 1.1.4.

The first statement in (2.33) is proved; the second one can be easily proved by the very same argument taking into account that $|\mu_i| = |D_{\widetilde{X}^{s_i}} \widetilde{u}^{s_i}| \llcorner \widetilde{B}_{s_i}(0, \varrho)$. □

2.2.1 An application to some classes of Carnot groups

Some of the main results of this chapter rely on properties $\mathcal{R}, \mathcal{LR}$ or $\mathcal{D}$; in this section we show how they can be in some meaningful CC spaces and, in particular, in some large classes of Carnot groups.

We start by introducing the X-reduced boundary $\mathcal{F}_X E$ of a set E with finite X-perimeter and its measure-theoretic horizontal inner normal. Recall that the reduced boundary was the object originally considered by E. De Giorgi in the seminal paper [24] about the rectifiability of sets with finite (Euclidean) perimeter in $\mathbb{R}^n$.

Definition 2.2.24. If $(\mathbb{R}^n, X)$ is a CC space and E is a set of locally finite X-perimeter, then by Riesz representation theorem there exists a P_X^E-measurable function $\nu_E : \mathbb{R}^n \to \mathbb{S}^{m-1}$ such that

$$D_X \chi_E = \nu_E P_X^E.$$

We call ν_E the *measure-theoretic horizontal inner normal* to E.

Definition 2.2.25 (Reduced boundary)**.** Let $E \subseteq \mathbb{R}^n$ be a set with locally finite X-perimeter. The *X-reduced boundary* $\mathcal{F}_X E$ of E is the set of points $p \in \mathbb{R}^n$ such that $P_X(E, B(p,r)) > 0$ for any $r > 0$ and

$$\widetilde{\nu}_E(p) := \lim_{r \to 0} \frac{D_X \chi_E(B(p,r))}{|D_X \chi_E|(B(p,r))}$$

exists with $|\widetilde{\nu}_E(p)| = 1$.

For sets with finite (Euclidean) perimeter in $\mathbb{R}^n$ the symmetric difference between the essential boundary and the reduced one is $\mathscr{H}_e^{n-1}$-negligible, see e.g. [5, Theorem 3.61]. In our setting we have the following result, which is a known consequence of Theorem 2.2.3, see e.g. [38, Theorem 7.3] for the *Heisenberg group* case and [39, Lemma 2.26] for step 2 Carnot groups. Notice that the proof of Theorem 2.2.26 below also shows that $\nu_E = \widetilde{\nu}_E$ a.e. on $\mathcal{F}_X E$.

Theorem 2.2.26. *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q and let $E \subseteq \mathbb{R}^n$ be a set of locally finite X-perimeter. Then $\mathscr{H}^{Q-1}(\partial^* E \setminus \mathcal{F}_X E) = 0$.*

Proof. By Theorem 2.2.3 we have $D_X \chi_E = \theta \nu_E \mathscr{H}^{Q-1} \llcorner \partial^* E$ for a suitable positive function θ. Therefore it is enough to prove that, for $\mathscr{H}^{Q-1}$-almost every $p \in \partial^* E$, one has

$$\lim_{r \to 0} \frac{D_X \chi_E(B(p,r))}{|D_X \chi_E|(B(p,r))} = \nu_E(p).$$

This fact directly follows from [32, Theorem 2.9.8] taking into account Theorem 2.2.3 and [32, Theorem 2.8.17]. □

The papers [38, 39, 66] prove the countable X-rectifiability of the reduced boundary of sets with locally finite X-perimeter in, respectively, Heisenberg groups, Carnot groups of step 2, and Carnot groups *of type* $\star$. These results, in conjunction with Theorem 2.2.26, show that property $\mathcal{R}$ is satisfied in these settings.

Actually, Theorem 2.2.26 and the results about blow-up and representation of the X-perimeter available in Heisenberg groups ([38, Theorems 4.1 and 7.1]), step 2 Carnot groups ([39, Theorems 3.1 and 3.9]) and Carnot groups of type $\star$ [66, Theorems 4.12 and 4.13] imply that also property $\mathcal{D}$ is satisfied in these settings.

Using also the left-invariance of the structure we can conclude what follows.

Theorem 2.2.27. *Heisenberg groups, Carnot groups of step 2 and Carnot groups of type $\star$ satisfy properties $\mathcal{R}$ and $\mathcal{D}$. In particular, Theorems 2.2.14, 5, 6 and 2.2.18 hold in these settings.*

Moreover, the function $\sigma(p,\nu)$ appearing in 6 and 2.2.18 does not depend on the point $p \in \mathbb{R}^n$.

In [27] a class of Carnot groups $\mathbb{G}$ satisfying the following assumption

$$\text{there exists at least one direction } X \text{ in the first layer of the stratified Lie algebra of } \mathbb{G} \text{ such that } t \mapsto \exp(tX) \text{ is not an abnormal curve} \tag{2.34}$$

is considered (see e.g. [74] for the notion of *abnormal curve*). This class includes, for instance, the *Engel group*, which is the simplest example where the rectifiability problem for sets with finite X-perimeter is open. One of the main results of [27] is the following one: for any set E with finite X-perimeter in a Carnot group $\mathbb{G}$ satisfying (2.34), the reduced boundary $\mathcal{F}_X E$ is countably X-Lipschitz rectifiable. Together with Theorem 2.2.26, this gives the following result.

Theorem 2.2.28. *The property $\mathcal{LR}$ is satisfied in all Carnot groups $\mathbb{G}$ such that* (2.34) *holds; in particular, Theorem 2.2.15 holds in such groups.*

For the reader's convenience, we here introduce the notion of *end-point map* and of *abnormal curve* in a Lie group and we show that Condition (2.34) is purely algebraic.

Definition 2.2.29. Let $\mathbb{G}$ be a Lie group and let $V \subseteq \mathfrak{g}$ be a linear subspace of its Lie algebra $\mathfrak{g}$ identified with $T_0\mathbb{G}$ and let $u \in L^2([0,1];V)$. We denote by γ_u the (unique) solution of the following ODE

$$\begin{cases} \dot\gamma(t) &= \left(dL_{\gamma(t)}\right)_0 u(t), \\ \gamma(0) &= 0. \end{cases} \tag{2.35}$$

Vice versa, if γ is a solution of (2.35) for some $u \in L^2([0,1];V)$, then we set $u_\gamma := u$. We define the *end-point map* $\mathrm{End} : L^2([0,1];V) \to \mathbb{G}$ letting $\mathrm{End}(u) = \gamma_u(1)$.

The proof of Proposition 2.2.30 below can be found in [74, Proposition 5.2.5] (for the proof of (2.2.30)) and in [56, Proposition 2.3] (for the proof of 2.37). Recall that $\mathrm{Ad}_g := \left(dL_g \circ dR_{g^{-1}}\right)_0$ denotes the *adjoint map* associated with $g \in \mathbb{G}$.

Proposition 2.2.30. *Let $\mathbb{G}$ be a Lie group and let $V \subseteq \mathfrak{g}$ be a linear subspace of its Lie algebra $\mathfrak{g}$. The end-point map* End *is smooth and its differential is given by*

$$d(\mathrm{End}(u))(v) = \left(dR_{\gamma_u(1)}\right)_0 \int_0^1 \mathrm{Ad}_{\gamma(t)} v(t)\, dt, \tag{2.36}$$

for any $v \in L^2([0,1];V)$. In particular, the image of the differential is given by

$$\mathrm{Im}(d\,\mathrm{End}(u)) = \left(dR_{\gamma(1)}\right)_0 \mathrm{span}\{\mathrm{Ad}_{\gamma(t)} V : t \in [0,1]\}. \tag{2.37}$$

Definition 2.2.31. Let $\mathbb{G}$ be a Lie group and let $V \subseteq \mathfrak{g}$ be a linear subspace of its Lie algebra $\mathfrak{g}$. An absolutely continuous curve $\gamma : [0,1] \to \mathbb{G}$ is said to be abnormal if $\mathrm{Im}(d\,\mathrm{End}(u_\gamma)) \neq T_{\gamma(1)}\mathbb{G}$.

Remark 2.2.32. Combining Definition 2.2.31 and identity (2.37) it is readily seen that, in a Carnot group $\mathbb{G}$ of step s, condition (2.34) is equivalent to

$$\left(dR_{\gamma(1)}\right)_0 \mathrm{span}\{\mathrm{Ad}_{\gamma(t)}V : t \in [0,1]\} = T_{\gamma(1)}\mathbb{G},$$

for $V = \mathfrak{g}_1$, and $\gamma(t) := \exp(tX)$ for $X \in \mathfrak{g}_1$. Since $dR_{\gamma(1)}$ is a diffeomorphism we just need to compute the dimension of $\mathrm{span}\{\mathrm{Ad}_{\exp(tX)}\mathfrak{g}_1 : t \in [0,1]\} =: W$.
Recalling that $\mathrm{Ad}_{\exp(tX)}Y = e^{\mathrm{ad}_X}Y$, $\mathrm{ad}_{\mathrm{X}}Y = [X,Y]$ and that

$$e^{\mathrm{ad}_X} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathrm{ad}_{\mathrm{X}}\right)^k,$$

one gets the formula

$$\mathrm{Ad}_{\exp(tX)}Y = Y + t[X,Y] + \frac{t^2}{2}[X,[X,Y]] + \frac{t^3}{6}[X,[X,[X,Y]]] + ... \tag{2.38}$$

for any $X, Y \in \mathfrak{g}$ and $t \in \mathbb{R}$. Evaluating (2.38) in $t = 0$ one has that $\mathfrak{g}_1 \subseteq W$. Since W is a linear space and $\mathfrak{g}_1 \subseteq W$ one gets that also $[X,Y] \in W$ for any $Y \in \mathfrak{g}_1$, i.e., $[X,\mathfrak{g}_1] = \mathrm{ad}_X\mathfrak{g}_1 \subseteq W$. Reasoning in the same way one gets $(\mathrm{ad}_X)^k\mathfrak{g}_1 \subseteq W$ for any $k = 1, \dots, s$. This implies

$$\mathrm{span}\{\mathrm{Ad}_{\exp(tX)}\mathfrak{g}_1 : t \in [0,1]\} = \bigoplus_{k=0}^{s} (\mathrm{ad}_X)^k\mathfrak{g}_1.$$

Therefore Condition (2.34) is equivalent to say that there exists $X \in \mathfrak{g}_1$ such that $\bigoplus_{k=0}^{s} (\mathrm{ad}_X)^k\mathfrak{g}_1 = \mathfrak{g}$.

Chapter 3

The Rank-One Theorem in a class of Carnot groups

The current Chapter is devoted to the proof of the Rank-One Theorem for BV functions in a class of Carnot groups satisfying properties $\mathscr{C}_2$ and w-$\mathcal{R}$ (see Definitions 3.1.3 and 3.4.1 below). The results of this chapter are contained in [28]. The Rank-One Theorem is stated and proved in Section 3.4 (see Theorem 3.4.5) and it is a consequence of the results proved in Section 3.3 and Lemma 3.2.7.
Section 3.3 deals with the relations between the total variation of a function of bounded X-variation and the $X^\dagger$-perimeter of its subgraph, whenever $X = (X_1, \ldots, X_m)$ is a family of smooth and linearly independent vector fields in $\mathbb{R}^n$ and $X^\dagger$ is the corresponding $(m+1)$-tuple of vector fields in $\mathbb{R}^{n+1}$ defined according to (3.10): Theorem 3.3.1 has as first consequence that a function $u \in L^1$ has bounded X-variation in an open set Ω if and only if its subgraph χ_{E_u} has bounded $X^\dagger$-perimeter in $\Omega \times \mathbb{R}$; Theorem 3.3.2 instead deals with the relations between the measure-theoretic horizontal inner normal to the subgraph E_u of $u \in BV_X$ and the polar vector of $D_X u$.
Lemma 3.2.7 is proved in Section 3.2 and it is a consequence of Theorem 3.2.6 which gives an estimate on the Hausdorff dimension of the transversal subset of the intersection of k regular hypersurfaces assuming the Carnot group satisfies the algebraic property $\mathscr{C}_k$. Section 3.2 introduces the notation about intrinsic regular hypersurfaces in Carnot groups (see Subsection 3.2.1, these results are contained also in Section 1.5 in the more general context of CC spaces), and the notion of intrinsic Lipschitz graphs in Carnot groups (see Subsection 3.2.2). The most notable result of Subsection 3.2.2 is represented by Theorem 3.2.5 which is proved in [62] and it guarantees that, in a Carnot group of rank m, the "transverse" subset of the intersection of k intrinsic regular hypersurfaces is locally an intrinsic Lipschitz graph whenever $k \le m$. The proof of this result is here given by a more simple argument based on the extension Lemma 3.2.4 for C^1 regular maps.

3.1 Preliminaries

In this chapter, $\mathbb{G}$ will denote a Carnot group of rank m, step s, Lie algebra $\mathfrak{g}$ and Ω will be an open set in $\mathbb{G}$. Notice that by Theorem 1.3.12 we will assume that $\mathbb{G} = \mathbb{R}^n$ by means of exponential coordinates

$$F(x_1, \ldots, x_n) = \exp\left(x_1 X_1 + \cdots + x_n X_n\right).$$

Given a Carnot group $\mathbb{G}$ we will frequently deal with products like $\mathbb{G} \times \mathbb{R}^N$. This is the Carnot group with algebra $\mathfrak{g} \times \mathbb{R}^N$ with product defined by $[(X,t),(Y,s)] = ([X,Y],0)$ for any $X, Y \in \mathfrak{g}$, $t, s \in \mathbb{R}^N$ and whose stratification is given by $(\mathfrak{g}_1 \times \mathbb{R}^N) \oplus (\mathfrak{g}_2 \times \{0\}) \oplus \ldots \oplus (\mathfrak{g}_s \times \{0\})$. Throughout this chapter, given a Borel set $E \subseteq \mathbb{G}$ and $r > 0$, we denote by E_r the open neighborhood of E of size r given by

$$E_r := \{p \in \mathbb{G} : d(p, E) < r\}.$$

Since Carnot groups are special cases of equiregular CC spaces we say that $u \in BV_{\mathbb{G}}(\Omega)$ if $u \in BV_X(\Omega)$ for any basis $X = (X_1, \ldots, X_m)$ of $\mathfrak{g}_1$. This is a well-posed definition by the fact that, if $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ are two basis of $\mathfrak{g}_1$ and $u \in L^1_{loc}(\mathbb{G})$, then $u \in BV_{X,loc}(\mathbb{G})$ if and only if $BV_{Y,loc}(\mathbb{G})$ (see also Proposition 1.6.4).

Definition 3.1.1. We say that $\mathbb{W} \subseteq \mathbb{G}$ is a *vertical plane of codimension* k, $1 \leq k \leq m$, if there exists a linear subspace $\mathfrak{w} \subseteq \mathfrak{g}_1$ of dimension $m-k$ such that $\mathbb{W} = \exp(\mathfrak{w} \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_s)$.

Notice that a vertical plane $\mathbb{W}$ is a homogeneous subgroup (i.e. $\delta_r \mathbb{W} = \mathbb{W}$ for any $r > 0$) of $\mathbb{G}$ with topological dimension $(n-k)$ and Hausdorff dimension $Q-k$. It is also easy to see that intersections of vertical planes is again a vertical plane (of possibly higher codimension). The following simple Lemma will be used in the proof of Lemma 3.2.7.

Lemma 3.1.2. *Let $\mathbb{W} \subseteq \mathbb{G}$ be a vertical plane of codimension k and let $x \in \mathbb{W}$, $r > 0$ and $\varepsilon \in (0,1)$ be fixed. Then, the set $\mathbb{W} \cap B(x,r)$ can be covered by a family of balls $\{B(y_\ell, \varepsilon r)\}_{\ell \in L}$ of radius εr with cardinality $\#L \leq (4/\varepsilon)^{Q-k}$.*

Proof. By dilation and translation invariance, it is not restrictive to assume that $x = 0$ and $r = 1$. Let $\{y_\ell : \ell \in L\}$ be a maximal family of points of $\mathbb{W} \cap B(0,1)$ such that the balls $B(y_\ell, \varepsilon/2)$ are pairwise disjoint; working by contradiction, it can be easily seen that the family $\{B(y_\ell, \varepsilon) : \ell \in L\}$ covers $\mathbb{W} \cap B(0,1)$. The measure $\mathscr{H}^{Q-k}$ is locally finite on $\mathbb{W}$ (see e.g. [61, 65, 64]), left-invariant and it is $(Q-k)$-homogeneous with respect to dilations. In particular, setting $M := \mathscr{H}^{Q-k}(\mathbb{W} \cap B(0,1))$, we have

$$\left(\frac{\varepsilon}{2}\right)^{Q-k} M \, \#L = \sum_{\ell \in L} \mathscr{H}^{Q-k}(\mathbb{W} \cap B(y_\ell, \varepsilon/2)) \leq \mathscr{H}^{Q-k}(\mathbb{W} \cap B(0,2)) = 2^{Q-k} M,$$

which proves the claim. □

Definition 3.1.3. Let $\mathbb{G}$ be a Carnot group with rank m and let $1 \le k \le m$ be an integer. We say that $\mathbb{G}$ satisfies *property* $\mathscr{C}_k$ if the first layer $\mathfrak{g}_1$ of its Lie algebra has the following condition: for any linear subspace $\mathfrak{w}$ of $\mathfrak{g}_1$ of codimension k there exists a commutative complementary subspace in $\mathfrak{g}_1$, i.e., a k-dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $[\mathfrak{h}, \mathfrak{h}] = 0$ and $\mathfrak{g}_1 = \mathfrak{w} \oplus \mathfrak{h}$.

Remark 3.1.4. According to Definition 3.1.1, a Carnot group satisfies property $\mathscr{C}_k$ if and only if, for any vertical plane $\mathbb{W}$ in $\mathbb{G}$, there exists a complementary homogeneous subgroup $\mathbb{H}$ that is horizontal, i.e., such that $\mathbb{H} \subseteq \exp(\mathfrak{g}_1)$. Notice also that, in this case, $\mathbb{H}$ is necessarily commutative.

Remark 3.1.5. The Heisenberg group $\mathbb{H}^n$ satisfies property $\mathscr{C}_k$ if and only if $1 \le k \le n$.

All Carnot groups satisfy property $\mathscr{C}_1$. *Free* Carnot groups (see [47]) satisfy property $\mathscr{C}_k$ if and only if $k = 1$.

A Carnot group $\mathbb{G}$ of rank m satisfies property $\mathscr{C}_m$ if and only if $\mathbb{G}$ is abelian (i.e., $\mathbb{G} \equiv \mathbb{R}^m$).

Remark 3.1.6. It is an easy exercise to show that, if $k \ge 2$ and $\mathbb{G}$ satisfies property $\mathscr{C}_k$, then $\mathbb{G}$ satisfies property $\mathscr{C}_h$ for any $1 \le h \le k$.

Lemma 3.1.7. *Let $N \ge 1$ be an integer. Then, a Carnot group $\mathbb{G}$ has the property $\mathscr{C}_k$ if and only if $\mathbb{G} \times \mathbb{R}^N$ has the property $\mathscr{C}_k$.*

Proof. It is clearly enough to prove the statement for $N = 1$.

Assume first that $\mathbb{G}$ has the property $\mathscr{C}_k$ and let $\mathfrak{w}$ be a k-codimensional subspace of the first layer $\mathfrak{g}_1 \times \mathbb{R}$ of the Lie algebra of $\mathbb{G} \times \mathbb{R}$. We have two cases according to the dimension of $\mathfrak{w}' := \mathfrak{w} \cap (\mathfrak{g}_1 \times \{0\})$:

- if $\dim \mathfrak{w}' = m - k$, by using property $\mathscr{C}_k$ of $\mathbb{G}$ one can find a k-dimensional commutative subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 \times \{0\} = \mathfrak{w}' \oplus (\mathfrak{h} \times \{0\})$. In particular, $\mathfrak{g}_1 \times \mathbb{R} = \mathfrak{w} \oplus (\mathfrak{h} \times \{0\})$;

- if $\dim \mathfrak{w}' = m + 1 - k$, then $\mathfrak{w} = \mathfrak{w}' \subseteq \mathfrak{g}_1 \times \{0\}$ and, by Remark 3.1.6, one can find a $(k-1)$-dimensional commutative subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 \times \{0\} = \mathfrak{w} \oplus (\mathfrak{h} \times \{0\})$. In particular, $\mathfrak{g}_1 \times \mathbb{R} = \mathfrak{w} \oplus (\mathfrak{h} \times \mathbb{R})$.

In both cases we have found a commutative complementary subspace of $\mathfrak{w}$.

Assume now that $\mathbb{G} \times \mathbb{R}$ satisfies property $\mathscr{C}_k$ and let $\mathfrak{w}$ be a k-codimensional linear subspace of $\mathfrak{g}_1$. Then $\mathfrak{w} \times \mathbb{R}$ is a k-codimensional linear subspace of $\mathfrak{g}_1 \times \mathbb{R}$, hence it admits a k-dimensional commutative complementary subspace $\mathfrak{h}$ in $\mathfrak{g}_1 \times \mathbb{R}$. Denoting by $\pi : \mathfrak{g}_1 \times \mathbb{R} \to \mathfrak{g}_1$ the canonical projection, it is readily noticed that $\pi(\mathfrak{h})$ is a k-dimensional commutative subspace of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 = \mathfrak{w} \oplus \pi(\mathfrak{h})$. This concludes the proof. □

3.2 Intrinsic hypersurfaces and graphs

3.2.1 Intrinsic regular hypersurfaces

We now introduce some notation about intrinsic regular maps and hypersurfaces in Carnot groups, taking into account Section 1.5. For the purpose, fix an orthonormal basis $(X_1, \dots, X_m)$ in $\mathfrak{g}_1$. We say that a continuous real function f on an open set $\Omega \subseteq \mathbb{G}$ is *of class* $C^1_{\mathbb{G}}$ if, for any $Y \in \mathfrak{g}_1$, the *horizontal derivative* Yf, in the sense of distributions, is represented by a continuous map in Ω. In this case we write $f \in C^1_{\mathbb{G}}(\Omega)$ and we set $\nabla_{\mathbb{G}} f \coloneqq (X_1 f, \dots, X_m f)$.

A set $S \subseteq \mathbb{G}$ is a $C^1_{\mathbb{G}}$ *hypersurface* if, for any $p \in S$, there exist an open neighborhood U of p and $f \in C^1_{\mathbb{G}}(U)$ such that

$$S \cap U = \{y \in U : f(y) = 0\} \quad \text{and} \quad \nabla_{\mathbb{G}} f \neq 0 \text{ on } U.$$

In this case, we define the *horizontal normal* to S at p as $\nu_S(p) \coloneqq \frac{\nabla_{\mathbb{G}} f(p)}{|\nabla_{\mathbb{G}} f(p)|} \in \mathbb{S}^{m-1}$. The normal $\nu_S(p) = ((\nu_S(p))_1, \dots, (\nu_S(p))_m)$ is defined up to sign and it can be identified with a horizontal vector at p by

$$\nu_S(p) = (\nu_S(p))_1 X_1(p) + \cdots + (\nu_S(p))_m X_m(p).$$

We also recall that a $C^1_{\mathbb{G}}$-hypersurface has locally finite $\mathscr{H}^{Q-1}$-measure, see e.g. [89].[1]

Given $p \in S$, the hyperplane $\nu_S(p)^\perp$ in $\mathfrak{g}$ is a Lie subalgebra. The associated subgroup $T_pS \coloneqq \exp(\nu_S(p)^\perp)$ is called *tangent subgroup* to S at p. T_pS is an example of vertical plane of codimension 1.
Restating Corollary 1.5.4 in this context, we can say that

$$\forall p \in S,\ \forall \varepsilon > 0,\ \exists R > 0 :\ (p^{-1}S) \cap B(0,r) \subseteq (T_pS)_{\varepsilon r} \cap B(0,r),\ \forall r \in (0,R). \tag{3.1}$$

Notice also that

$$T_pS = \exp(\{X \in \mathfrak{g}_1 : Xf(p) = 0\} \oplus \mathfrak{g}_2 \dots \oplus \mathfrak{g}_s);$$

in particular, while $\nu_S(p)$ depends on the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, the subgroup T_pS is intrinsic.

3.2.2 Intrinsic Lipschitz graphs

The aim of this section is proving Theorem 3.2.5, due to V. Magnani [62], for which we will need the preparatory Lemma 3.2.4. Actually, its use could be avoided by utilizing a local version of Theorem 3.2.3 which, even though not explicitly stated there, would easily follow adapting the techniques of [40]. We note however that Lemma 3.2.4, and

[1] Actually, this also follows from Theorem 3.2.6 with $k = 1$.

(3.2) in particular, provide also a proof of (3.1).
To introduce the notion of intrinsic Lipschitz graphs we follow [40]. Let $\mathbb{W}, \mathbb{H}$ be homogeneous complementary subgroups of $\mathbb{G}$, i.e., such that $\mathbb{W} \cap \mathbb{H} = \{0\}$ and $\mathbb{G} = \mathbb{W}\mathbb{H}$. In particular, for any $x \in \mathbb{G}$ there exist unique $x_\mathbb{W} \in \mathbb{W}$ and $x_\mathbb{H} \in \mathbb{H}$ such that $x = x_\mathbb{W} x_\mathbb{H}$. Recall (see e.g. [40, Remark 2.3]) that any homogeneous subgroup $\mathbb{W}$ is *stratified*, that is, its Lie algebra $\mathfrak{w}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{w} = \mathfrak{w}_1 \oplus \ldots \oplus \mathfrak{w}_s$ where $\mathfrak{w}_i = \mathfrak{w} \cap \mathfrak{g}_i$. Moreover, the metric (Hausdorff) dimension of $\mathbb{W}$ is $Q_\mathbb{W} := \sum_{i=1}^s i \dim \mathfrak{w}_i$.

The *intrinsic graph* of a function $\phi : \mathbb{W} \to \mathbb{H}$ is defined by

$$\mathrm{gr}\ \phi := \{w\phi(w) : w \in \mathbb{W}\}.$$

We introduce the homogeneous cones $C_{\mathbb{W},\mathbb{H}}(x, \alpha)$ of center $x \in \mathbb{G}$ and aperture $\alpha > 0$ as

$$C_{\mathbb{W},\mathbb{H}}(x, \alpha) := xC_{\mathbb{W},\mathbb{H}}(0, \alpha) \quad \text{where} \quad C_{\mathbb{W},\mathbb{H}}(0, \alpha) := \{y \in \mathbb{G} : \|x_\mathbb{W}\| \leq \alpha \|x_\mathbb{H}\|\}.$$

Definition 3.2.1. A function $\phi : \mathbb{W} \to \mathbb{H}$ is *intrinsic Lipschitz* if there exists $\alpha > 0$ such that

$$\forall\, x \in \mathrm{gr}\ \phi \qquad \mathrm{gr}\ \phi \cap C_{\mathbb{W},\mathbb{H}}(x, \alpha) = \{x\}.$$

We say that $S \subseteq \mathbb{G}$ is an *intrinsic Lipschitz graph* if there exists an intrinsic Lipschitz map $\phi : \mathbb{W} \to \mathbb{H}$ such that $S = \mathrm{gr}\ \phi$.

Remark 3.2.2. A function $\phi : \mathbb{W} \to \mathbb{H}$ is intrinsic Lipschitz if and only if there exists $\beta > 0$ such that for any $x \in \mathrm{gr}\ \phi$

$$\mathrm{gr}\ \phi \cap D(x, \mathbb{H}, \beta) = \{x\},$$

where the homogeneous cone $D(x, \mathbb{H}, \beta)$ is defined by

$$D(x, \mathbb{H}, \beta) := xD(\mathbb{H}, \beta) \quad \text{and} \quad D(\mathbb{H}, \beta) := \bigcup_{h \in \mathbb{H}} \overline{B(h, \beta d(h, 0))}.$$

Indeed, it is enough to observe that, for any $\alpha > 0$ and $\beta > 0$, there exist $\beta_\alpha > 0$ and $\alpha_\beta > 0$ such that

$$C_{\mathbb{W},\mathbb{H}}(0, \alpha) \supset D(\mathbb{H}, \beta_\alpha) \quad \text{and} \quad D(\mathbb{H}, \beta) \supset C_{\mathbb{W},\mathbb{H}}(0, \alpha_\beta).$$

This, in turn, is a consequence of a homogeneity argument based on the following fact: if $S := \{x \in \mathbb{G} : \|x\| = 1\}$ and

$$A_\alpha := S \cap \mathrm{int}(C_{\mathbb{W},\mathbb{H}}(0, \alpha)), \qquad B_\beta := S \cap \mathrm{int}(D(\mathbb{H}, \beta)),$$

then $\{A_\alpha\}_{\alpha>0}$ and $\{B_\beta\}_{\beta>0}$ are monotone families of (relatively) open subsets of S such that the intersection

$$\bigcap_{\alpha>0} A_\alpha = \bigcap_{\beta>0} B_\beta = \mathbb{H} \cap S$$

is a compact set.

A key tool in the proof of the rank-one Theorem 3.4.5 is Lemma 3.2.7 which, in turn, uses Theorem 3.2.6 below. We denote by $\pi : \mathbb{G}\times\mathbb{R} \to \mathbb{G}$ the canonical projection $\pi(x,t) = x$.

The following result will be used in the proof of Theorem 3.2.6.

Theorem 3.2.3 ([40, Theorem 3.9]). *Let $\mathbb{W}, \mathbb{H}$ be homogeneous complementary subgroups of $\mathbb{G}$, let $\phi : \mathbb{W} \to \mathbb{H}$ be intrinsic Lipschitz and let $\alpha > 0$ be as in Definition 3.2.1. Then there exists a positive $C = C(\mathbb{W}, \mathbb{H}, \alpha)$ such that*

$$\frac{1}{C} r^{Q_\mathbb{W}} \le \mathcal{H}^{Q_\mathbb{W}}(\mathrm{gr}\ \phi \cap B(x,r)) \le C r^{Q_\mathbb{W}} \qquad \forall\, x \in \mathrm{gr}\ \phi, r > 0.$$

Lemma 3.2.4. *Let $\Omega \subseteq \mathbb{G}$ be open, $f \in C^1_\mathbb{G}(\Omega)$, $p \in \Omega$ and let $A := \nabla_\mathbb{G} f(p)$. Then, for any $\varepsilon > 0$ there exist an open set $U \subseteq \Omega$ with $p \in U$ and a function $g \in C^1_\mathbb{G}(\mathbb{G})$ such that*

(i) $g = f$ *on* U*;*

(ii) $|\nabla_\mathbb{G} g - A| < \varepsilon$ *on* $\mathbb{G}$.

Proof. Without loss of generality we can assume that $p = 0$ and identify $\mathbb{G} = \mathbb{R}^n$ by means of exponential coordinates. We preliminarily fix a smooth function $\chi : \mathbb{G} \to [0,1]$ such that $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ on $\mathbb{G} \setminus B(0,2)$. For any $r > 0$, the functions $\chi_r := \chi \circ \delta_{1/r}$ satisfy

$$0 \le \chi_r \le 1, \quad \chi \equiv 1 \text{ on } B(0,r), \quad \chi \equiv 0 \text{ on } \mathbb{G}\setminus B(0,2r), \quad |\nabla_\mathbb{G}\chi_r| \le \frac{C}{r}$$

for some positive C independent of r.

Let $\varepsilon > 0$. By Proposition 1.5.3, we can fix $r > 0$ such that $|\nabla_\mathbb{G} f - A| < \varepsilon$ on $B(0,2r)$ and for every $\xi \in B(0,2r)$

$$|f(\xi) - \widetilde{L}_A(\xi)| < 2\varepsilon r. \tag{3.2}$$

We now define $g := \chi_r f + (1-\chi_r)\widetilde{L}_A$; statement (i) is readily checked, while for (ii)

$$\begin{aligned} |\nabla_\mathbb{G} g - A| &= |\chi_r \nabla_\mathbb{G} f + (1-\chi_r)A + (f - \widetilde{L}_A)\nabla_\mathbb{G}\chi_r - A| \\ &\le \chi_r |\nabla_\mathbb{G} f - A| + |f - \widetilde{L}_A||\nabla_\mathbb{G}\chi_r| \\ &\le \varepsilon + 2C\varepsilon. \end{aligned}$$

The proof is then accomplished. □

We can now prove the main result of this section. Since property $\mathscr{C}_1$ holds in any Carnot group, when $k = 1$ Theorem 3.2.5 states in particular that hypersurfaces of class $C^1_\mathbb{G}$ in a Carnot group $\mathbb{G}$ are locally intrinsic Lipschitz graphs of codimension 1.

Theorem 3.2.5 ([62, Theorem 1.4]). *Let $\mathbb{G}$ be a Carnot group of rank m and let $\Sigma_1, \dots, \Sigma_k$, $k \leq m$, be hypersurfaces of class $C^1_{\mathbb{G}}$ with horizontal normals $\nu_1, \dots, \nu_k$; let $p \in \Sigma := \Sigma_1 \cap \ldots \cap \Sigma_k$ be such that $\nu_1(p), \dots, \nu_k(p)$ are linearly independent. Consider the vertical plane $\mathbb{W} := T_x\Sigma_1 \cap \ldots \cap T_x\Sigma_k$ of codimension k and assume that there exists a complementary homogeneous horizontal subgroup $\mathbb{H}$ such that $\mathbb{G} = \mathbb{W}\mathbb{H}$. Then, there exists an open neighborhood U of p and an intrinsic Lipschitz $\phi : \mathbb{W} \to \mathbb{H}$ such that*

$$\Sigma \cap U = \mathrm{gr}\ \phi \cap U.$$

Proof. We work in exponential coordinates associated with an adapted basis $X_1, \dots, X_n$ of $\mathfrak{g}$ such that

$$\mathbb{H} = \exp(\mathrm{span}\,\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp((\mathrm{span}\,\{X_{k+1}, \dots, X_s\}) \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_s).$$

By definition we can find an open neighborhood U of x and $f = (f_1, \dots, f_k) \in C^1_{\mathbb{G}}(U, \mathbb{R}^k)$ such that $\Sigma \cap U = \{q \in U : f(q) = 0\} \cap U$ and the $m \times k$ matrix-valued function $\nabla_{\mathbb{G}} f$ has rank k in U. Actually, by our choice of the basis, the $k \times k$ minor $M := (X_1 f(p), \dots, X_k f(p))$ has rank k.

Let $\varepsilon > 0$, to be fixed later and only depending on M. By Lemma 3.2.4, possibly restricting U we can assume that f is defined on the whole $\mathbb{G}$, that $f \in C^1_{\mathbb{G}}(\mathbb{G}; \mathbb{R}^k)$ and $|\nabla_{\mathbb{G}} f - \nabla_{\mathbb{G}} f(p)| < \varepsilon$; in particular,

$$|(X_1 f, \dots, X_k f) - M| < \varepsilon \quad \text{on } \mathbb{G}.$$

It will be enough to prove that the level set $R := \{q \in \mathbb{G} : f(q) = 0\}$ is an intrinsic Lipschitz graph. We divide the proof of this claim into two steps.

Step 1: R is the intrinsic graph of some $\phi : \mathbb{W} \to \mathbb{H}$. It is enough to show that, for any $w \in \mathbb{W}$, there exists a unique $h \in \mathbb{H}$ such that $f(wh) = 0$; this will allow to define the map $\phi(w) := h$.

The map $(h_1, \dots, h_k) \longleftrightarrow \exp(h_1 X_1 + \cdots + h_k X_k)$ is a group isomorphism between $\mathbb{H}$ and $\mathbb{R}^k$. Upon identifying $\mathbb{H}$ and $\mathbb{R}^k$ in this way, for any $w \in \mathbb{W}$ we can consider $f_w : \mathbb{R}^k \to \mathbb{R}^k$ defined by $f_w(h) := f(wh)$. This map is of class C^1 and

$$\nabla f_w(h) = (X_1 f(wh), \dots, X_k f(wh)).$$

We have $|\nabla f_w - M| < \varepsilon$ which, if ε is small enough, implies that f_w is a C^1 diffeomorphism of $\mathbb{R}^k$: see e.g. the argument in [32, 3.1.1][2]. This concludes the proof of Step 1; we also notice that, possibly reducing ε, there exists $c > 0$ such that (see again [32, 3.1.1])

$$|f(wh_1) - f(wh_2)| = |f_w(h_1) - f_w(h_2)| \geq c|h_1 - h_2|, \quad \forall\, h_1, h_2 \in \mathbb{R}^k. \tag{3.3}$$

[2]The careful reader will notice that the argument in [32, 3.1.1] works also when the parameter δ introduced therein is $+\infty$.

Step 2: ϕ is intrinsic Lipschitz. By Remark 3.2.2 it is enough to prove that, for any $x \in \mathbb{G}$, one has

$$\text{gr } \phi \cap D(x, \mathbb{H}, \beta) = \{x\}$$

for a suitable $\beta > 0$ that will be chosen in a moment.

Let then $x \in \text{gr } \phi$ be fixed; consider $x' \in D(x, \mathbb{H}, \beta)$, so that $x' = xy$ for some $y \in D(\mathbb{H}, \beta)$. By definition, there exists $h \in \mathbb{H}$ such that

$$d(0, h^{-1}y) = d(h, y) \leq \beta d(h, 0).$$

Denoting by L the Lipschitz constant of f, and using (3.3), we deduce that

$$\begin{aligned} |f(x')| &= |f(xhh^{-1}y) - f(x)| \\ &\geq |f(xh) - f(x)| - |f(xhh^{-1}y) - f(xh)| \geq c\|h\| - Ld(h, y) \geq (\widetilde{c} - \beta L) d(0, h) \end{aligned}$$

for some $\widetilde{c} > 0$. In particular, if β is small enough, one can have $f(x') = 0$ only if $h = 0$, which immediately gives $x' = x$. This concludes the proof. □

3.2.3 Hypersurfaces vs. Lipschitz graphs

Theorem 3.2.6. *Let $k \geq 1$ be an integer, $\mathbb{G}$ a Carnot group satisfying property $\mathscr{C}_k$ and let $\Sigma_1, \ldots, \Sigma_k$ be $C^1_{\mathbb{G}}$-hypersurfaces with horizontal normals $\nu_1, \ldots, \nu_k$. Let also $p \in \Sigma := \Sigma_1 \cap \ldots \cap \Sigma_k$ be such that $\nu_1(p), \ldots, \nu_k(p)$ are linearly independent. Then, there exists an open neighborhood U of p such that*

$$0 < \mathscr{H}^{Q-k}(\Sigma \cap U) < \infty.$$

In particular, the measure $\mathscr{H}^{Q-k}$ is σ-finite on the set

$$\Sigma^{\pitchfork} := \{x \in \Sigma : \nu_1(x), \ldots, \nu_k(x) \text{ are linearly independent}\}.$$

Proof. By property $\mathscr{C}_k$ and Remark 3.1.4, the vertical plane $\mathbb{W} := T_p\Sigma_1 \cap \ldots \cap T_p\Sigma_k$ admits a complementary horizontal homogeneous subgroup $\mathbb{H}$. One can then easily conclude using Theorems 3.2.3 and 3.2.5. □

Lemma 3.2.7. *Let $\mathbb{G}$ be a Carnot group satisfying property $\mathscr{C}_2$. Let Σ_1, Σ_2 be $C^1_{\mathbb{G}}$ hypersurfaces in $\mathbb{G} \times \mathbb{R}$ with unit normals $\nu_{\Sigma_1}, \nu_{\Sigma_2}$. Then, the set*

$$R := \left\{ p \in \Sigma_1 : \exists\, q \in \Sigma_2 \text{ such that } \begin{array}{l} \pi(q) = \pi(p), \\ (\nu_{\Sigma_1}(p))_{m+1} = (\nu_{\Sigma_2}(q))_{m+1} = 0, \\ \nu_{\Sigma_1}(p) \neq \pm \nu_{\Sigma_2}(q) \end{array} \right\}$$

is $\mathscr{H}^Q$-negligible.

Proof. Let us consider the distances $d_{\mathbb{G}\times\mathbb{R}}$ and $d_{\mathbb{G}\times\mathbb{R}\times\mathbb{R}}$ on (respectively) $\mathbb{G}\times\mathbb{R}$ and $\mathbb{G}\times\mathbb{R}\times\mathbb{R}$ defined by

$$\begin{aligned} d_{\mathbb{G}\times\mathbb{R}}((x,t),(x',t')) &:= d(x,x') + |t-t'| && \forall\, x,x'\in\mathbb{G}, t,t'\in\mathbb{R},\\ d_{\mathbb{G}\times\mathbb{R}\times\mathbb{R}}((x,t,s),(x',t',s')) &:= d(x,x') + |t-t'|+|s-s'| && \forall\, x,x'\in\mathbb{G}, t,t',s,s'\in\mathbb{R}, \end{aligned}$$

where d is the Carnot-Carathéodory distance on $\mathbb{G}$. Such distances are left-invariant and homogeneous, hence they are equivalent to the Carnot-Carathéodory distances on (respectively) $\mathbb{G}\times\mathbb{R}$ and $\mathbb{G}\times\mathbb{R}\times\mathbb{R}$; in particular, it is enough to prove the statement when the Hausdorff measure $\mathscr{H}^Q$ is the one induced by $d_{\mathbb{G}\times\mathbb{R}}$ on $\mathbb{G}\times\mathbb{R}$. We will use the same notation $B(a,r)$ for balls of radius $r>0$ in either $\mathbb{G}, \mathbb{G}\times\mathbb{R}$ or $\mathbb{G}\times\mathbb{R}\times\mathbb{R}$, according to which group the center a belongs to.

The sets

$$\begin{aligned} \widetilde\Sigma_1 &:= \{(x,t,s)\in\mathbb{G}\times\mathbb{R}\times\mathbb{R} : (x,t)\in\Sigma_1, s\in\mathbb{R}\}\\ \widetilde\Sigma_2 &:= \{(x,t,s)\in\mathbb{G}\times\mathbb{R}\times\mathbb{R} : (x,s)\in\Sigma_2, t\in\mathbb{R}\} \end{aligned}$$

are clearly $C^1_{\mathbb{G}}$-hypersurfaces in $\mathbb{G}\times\mathbb{R}\times\mathbb{R}$ and, moreover,

$$\begin{aligned} \nu_{\widetilde\Sigma_1}(x,t,s) &= ((\nu_{\Sigma_1}(x,t))_1,\dots,(\nu_{\Sigma_1}(x,t))_m, (\nu_{\Sigma_1}(x,t))_{m+1},\ 0\ )\\ \nu_{\widetilde\Sigma_2}(x,t,s) &= ((\nu_{\Sigma_2}(x,s))_1,\dots,(\nu_{\Sigma_2}(x,s))_m,\ 0\ ,(\nu_{\Sigma_2}(x,s))_{m+1}). \end{aligned}$$

Let us define

$$\begin{aligned} \widetilde R :=&\{P\in\widetilde\Sigma_1\cap\widetilde\Sigma_2 : (\nu_{\widetilde\Sigma_1}(P))_{m+1} = (\nu_{\widetilde\Sigma_2}(P))_{m+2} = 0,\ \nu_{\widetilde\Sigma_1}(P)\neq\pm\nu_{\widetilde\Sigma_2}(P)\}\\ =&\{(x,t,s)\in\widetilde\Sigma_1\cap\widetilde\Sigma_2 : (\nu_{\Sigma_1}(x,t))_{m+1} = (\nu_{\Sigma_2}(x,s))_{m+1} = 0,\ \nu_{\Sigma_1}(x,t)\neq\pm\nu_{\Sigma_2}(x,s)\}. \end{aligned}$$

By construction we have $\widetilde\pi(\widetilde R) = R$, where $\widetilde\pi : \mathbb{G}\times\mathbb{R}\times\mathbb{R}\to\mathbb{G}\times\mathbb{R}$ is the group homomorphism defined by $\widetilde\pi(x,t,s) := (x,t)$; moreover the measure $\mathscr{H}^Q\llcorner\widetilde R$ is σ-finite by Theorem 3.2.6 (notice that we are also using Lemma 3.1.7). We are going to show that $\mathscr{H}^Q(\widetilde\pi(T)) = 0$ for any fixed $T\subseteq\widetilde R$ such that $\mathscr{S}^Q(T)<\infty$; this will be clearly enough to conclude.

For any $P\in T$ and $i=1,2$, the tangent space $T_P\widetilde\Sigma_i$ equals $\mathbb{W}_i\times\mathbb{R}\times\mathbb{R}$ for a suitable vertical hyperplane $\mathbb{W}_i$ of $\mathbb{G}$. In particular, setting $\mathbb{W}=\mathbb{W}(P):=\mathbb{W}_1\cap\mathbb{W}_2$, we have by (3.1) that, for any $P\in T$ and any $\varepsilon\in(0,1)$, there exists $r_0=r_0(\varepsilon,P)>0$ such that

$$\begin{aligned} (P^{-1}T)\cap B(0,r) \subseteq&(\mathbb{W}\times\mathbb{R}\times\mathbb{R})_{\varepsilon r}\cap B(0,r)\\ =&(\mathbb{W}_{\varepsilon r}\times\mathbb{R}\times\mathbb{R})\cap B(0,r), \qquad \text{for any } r\in(0,r_0). \end{aligned} \tag{3.4}$$

Notice also that $\mathbb{W}$ is a vertical plane of codimension 2 in $\mathbb{G}$. Let $\varepsilon>0$ be fixed and for any $j\in\mathbb{N}\setminus\{0\}$ define

$$T_j := \{P\in T : r_0(\varepsilon,P)\geq \tfrac{1}{j}\}$$

Since $T_j \nearrow T$, the proof will be accomplished by showing that for any fixed j

$$\mathscr{H}^Q(\widetilde{\pi}(T_j)) < C\varepsilon, \tag{3.5}$$

where $C > 0$ is a constant that will be determined in the sequel.

Let us prove (3.5). Fix $\delta \in (0, 1/j)$; since $\mathscr{H}^Q(T_j) \leq \mathscr{H}^Q(T) < +\infty$, one can find a (countable or finite) family $\{B(Q_i, r_i/2) : i \in I\}$ of balls in $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$ such that $0 < r_i < \delta$,

$$T_j \subseteq \bigcup_{i\in I} B(Q_i, r_i/2) \qquad \text{and} \qquad \sum_{i\in I}(r_i/2)^Q \leq \sum_{i\in I}(\text{diam } B(Q_i, r_i/2))^Q \leq C_1$$

where $C_1 := \mathscr{H}^Q(T) + 1$. We can also assume that $T_j \cap B(Q_i, r_i/2)$ is non-empty for any i. Choosing $P_i \in T_j \cap B(Q_i, r_i/2)$, for any i the balls $B(P_i, r_i)$ have then the following properties:

$$P_i \in T_j, \quad 0 < r_i < \delta, \quad T_j \subseteq \bigcup_{i\in I} B(P_i, r_i) \quad \text{and} \quad \sum_{i\in I} r_i^Q \leq 2^Q C_1. \tag{3.6}$$

Setting $\mathbb{W}_i := \mathbb{W}(P_i)$, by (3.4) we have

$$\begin{aligned}(P_i^{-1}T_j) \cap B(0, r_i) \subseteq &((\mathbb{W}_i)_{\varepsilon r_i} \times \mathbb{R} \times \mathbb{R}) \cap B(0, r_i) \\ =&((\mathbb{W}_i)_{\varepsilon r_i} \cap B(0, r_i)) \times (-r_i, r_i) \times (-r_i, r_i).\end{aligned} \tag{3.7}$$

By Lemma 3.1.2, for any i we can find a family of balls $\{B(y_{i,\ell}, \varepsilon r_i) : \ell \in L_i\}$ such that, for any $\ell \in L_i$ and any $y_{i,\ell} \in \mathbb{W}_i$, we have

$$\#L_i \leq (8/\varepsilon)^{Q-2} \quad \text{and} \quad \mathbb{W}_i \cap B(0, 2r_i) \subseteq \bigcup_{\ell\in L_i} B(y_{i,\ell}, \varepsilon r_i).$$

In particular

$$(\mathbb{W}_i)_{\varepsilon r_i} \cap B(0, r_i) \subseteq (\mathbb{W}_i \cap B(0, r_i + \varepsilon r_i))_{\varepsilon r_i} \subseteq \bigcup_{\ell\in L_i} B(y_{i,\ell}, 2\varepsilon r_i). \tag{3.8}$$

Let us also fix points $\{\tau_k\}_{k\in K_i} \subseteq (-r_i, r_i)$ such that $\#K_i \leq 2\varepsilon^{-1}$ and

$$(-r_i, r_i) \subseteq \bigcup_{k\in K_i} (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i). \tag{3.9}$$

By (3.7), (3.8) and (3.9) we get

$$(P_i^{-1}T_j) \cap B(0, r_i) \subseteq \bigcup_{\substack{\ell\in L_i \\ k,h\in K_i}} B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i).$$

For any $\ell \in L_i$ and $k, h, h' \in K_i$ one has

$$\begin{aligned}&\widetilde{\pi}(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i)) \\ =&\widetilde{\pi}(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_{h'} - 2\varepsilon r_i, \tau_{h'} + 2\varepsilon r_i)) \\ =&B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \\ \subseteq&B((y_{i,\ell}, \tau_k), 4\varepsilon r_i),\end{aligned}$$

which, using (3.6), implies that

$$\begin{aligned}
\widetilde{\pi}(T_j) \subseteq & \bigcup_i \widetilde{\pi}(T_j \cap B(P_i, r_i)) \\
\subseteq & \bigcup_i \bigcup_{\substack{\ell \in L_i \\ k,h \in K_i}} \widetilde{\pi}(P_i(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i))) \\
\subseteq & \bigcup_i \bigcup_{\substack{\ell \in L_i \\ k \in K_i}} \widetilde{\pi}(P_i) B((y_{i,\ell}, \tau_k), 4\varepsilon r_i) \\
= & \bigcup_i \bigcup_{\substack{\ell \in L_i \\ k \in K_i}} B(p_{i\ell k}, 4\varepsilon r_i),
\end{aligned}$$

where $p_{i\ell k} := \widetilde{\pi}(P_i)(y_{i,\ell}, \tau_k) \in \mathbb{G} \times \mathbb{R}$. Using again (3.6) we obtain that

$$\mathscr{H}^Q_{2\varepsilon\delta}(T_j) \le \sum_{i \in I} \# L_i \; \# K_i \; (8\varepsilon r_i)^Q \le \varepsilon \sum_{i \in I} 2^{6Q-5} r_i^Q \le 2^{7Q-5} C_1 \varepsilon,$$

which, by the arbitrariness of $\delta \in (0, 1/j)$, gives claim (3.5). □

3.3 BV_X functions and their subgraphs

Given a system $X = (X_1, \dots, X_m)$ of linearly independent and smooth vector fields in $\mathbb{R}^n$, we introduce the family $X^\dagger = (X_1^\dagger, \dots, X_{m+1}^\dagger)$ of linearly independent vector fields in $\mathbb{R}^{n+1}$ defined for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ by

$$\begin{aligned}
&X_i^\dagger(x,t) := (X_i(x), 0) \in \mathbb{R}^{n+1} \equiv \mathbb{R}^n \times \mathbb{R} \qquad \text{if } i = 1, \dots, m \\
&X_{m+1}^\dagger(x,t) := \partial_t.
\end{aligned} \tag{3.10}$$

The aim of this section is the study of the relations occurring between a function $u \in BV_X(\Omega)$ and the $X^\dagger$-perimeter of its *subgraph*

$$E_u := \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\} \subseteq \Omega \times \mathbb{R},$$

where Ω is an open set in $\mathbb{R}^n$.

The following result is the natural generalization of some classical facts about Euclidean functions of bounded variation, see e.g. [43, Section 4.1.5]. We denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the canonical projection $\pi(x,t) := x$ and by $\pi_\#$ the associated push-forward of measures.

Theorem 3.3.1. *Suppose Ω is bounded in $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. Then u belongs to $BV_X(\Omega)$ if and only if its subgraph E_u has finite $X^\dagger$-perimeter in $\Omega \times \mathbb{R}$.*
Moreover, writing $D'_{X^\dagger}\chi_{E_u} := (D_{X_1^\dagger}\chi_{E_u}, \dots, D_{X_m^\dagger}\chi_{E_u})$, then the following statements hold.

(i) $\pi_\# D_{X_i^\dagger}\chi_{E_u} = D_{X_i} u$ for any $i = 1, \dots, m$;

(ii) $\pi_{\#}\partial_t \chi_{E_u} = -\mathscr{L}^n$;

(iii) $\pi_{\#}|D_{X_i^\dagger}\chi_{E_u}|= |D_{X_i}u|$ *for any* $i = 1,\dots,m$;

(iv) $\pi_{\#}|\partial_t\chi_{E_u}|= \mathscr{L}^n$;

(v) $\pi_{\#}|D'_{X^\dagger}\chi_{E_u}|= |D_X u|$.

(vi) $\pi_{\#}|D_{X^\dagger}\chi_{E_u}|= |(D_X u, -\mathscr{L}^n)|$.

Proof. Suppose first that $\chi_{E_u} \in BV_{X^\dagger}(\Omega\times\mathbb{R})$. We fix a sequence (g_h) in $C_c^\infty(\mathbb{R})$ such that g_h is even, $g_h \equiv 1$ on $[0, h]$, $g_h \equiv 0$ on $[h + 1, +\infty)$ and $\int_{\mathbb{R}} g_h(t)dt = 2h + 1$. Let $\varphi \in C_c^1(\Omega;\mathbb{R}^m)$ with $|\varphi|\le 1$ be fixed. By the Dominated Convergence Theorem, we have that

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \varphi(x)\cdot d(D'_{X^\dagger}\chi_{E_u})(x,t) &= \lim_{h\to+\infty}\int_{\Omega\times\mathbb{R}} g_h(t)\varphi(x)\cdot d(D'_{X^\dagger}\chi_{E_u})(x,t)\\ &= -\lim_{h\to+\infty}\int_{\Omega\times\mathbb{R}} \chi_{E_u}(x,t)g_h(t)\mathrm{div}\varphi(x)\, d\mathscr{L}^{n+1}(x,t)\\ &= -\lim_{h\to+\infty}\int_{\Omega}\left(\int_{-\infty}^{u(x)} g_h(t)\,dt\right)\mathrm{div}\varphi(x)\, d\mathscr{L}^{n}(x).\end{aligned}$$

For every $z\in\mathbb{R}$ and every $h\in\mathbb{N}$ we have

$$\int_{-\infty}^{z} g_h(t)\,dt \le |z|+h+\frac{1}{2} \quad\text{and}\quad \lim_{h\to+\infty}\left(\int_{-\infty}^{z} g_h(t)\,dt - h - \frac{1}{2}\right) = z;$$

using the fact that $\int_\Omega \mathrm{div}\varphi(x)d\mathscr{L}^n(x) = 0$, by the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \varphi(x)\cdot d(D'_{X^\dagger}\chi_{E_u})(x,t) &= -\lim_{h\to+\infty}\int_{\Omega}\left(\int_{-\infty}^{u(x)} g_h(t)\,dt - h - \frac{1}{2}\right)\mathrm{div}\varphi(x)\, d\mathscr{L}^{n}(x)\\ &= -\int_\Omega u(x)\mathrm{div}\varphi(x)\, d\mathscr{L}^n(x)\\ &= \int_\Omega \varphi(x)\cdot d(D_X u)(x).\end{aligned}\tag{3.11}$$

In particular, $u \in BV_X(\Omega)$ and, for any open set $A\subseteq\Omega$,

$$\begin{aligned}&|D_X u|(A) \le |D'_{X^\dagger}\chi_{E_u}|(A\times\mathbb{R}),\\ &|D_{X_i}u|(A) \le |D_{X_i^\dagger}\chi_{E_u}|(A\times\mathbb{R}) \quad \text{for any } i = 1,\dots,m.\end{aligned}\tag{3.12}$$

Before passing to the reverse implication we observe two facts. First, for any $\varphi\in C_c^1(\Omega)$, one has

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \varphi(x)\, d\left(\partial_t \chi_{E_u}\right)(x,t) &= \lim_{h\to+\infty} \int_{\Omega\times\mathbb{R}} \varphi(x) g_h(t)\, d\left(\partial_t \chi_{E_u}\right)(x,t) \\ &= -\lim_{h\to+\infty} \int_{\Omega\times\mathbb{R}} \varphi(x) g_h'(t) \chi_{E_u}(x,t)\, d\mathscr{L}^{n+1}(x,t) \\ &= -\lim_{h\to+\infty} \int_{\Omega} \varphi(x) \left(\int_{-\infty}^{u(x)} g_h'(t)\, dt \right) d\mathscr{L}^n(x) \\ &= -\lim_{h\to+\infty} \int_{\Omega} \varphi(x) g_h(u(x))\, d\mathscr{L}^n(x) \\ &= -\int_\Omega \varphi\, d\mathscr{L}^n\end{aligned} \tag{3.13}$$

whence, for any open set $A \subseteq \Omega$,

$$\mathscr{L}^n(A) \le |\partial_t \chi_{E_u}|(A\times\mathbb{R}). \tag{3.14}$$

Second, if $\varphi \in C^1_c(\Omega;\mathbb{R}^{m+1})$, one has by (3.11) and (3.13)

$$\int_{\Omega\times\mathbb{R}} \varphi(x)\cdot d(D_{X^\dagger}\chi_{E_u})(x,t) = \int_\Omega \varphi(x)\cdot d(D_X u, -\mathscr{L}^n)(x),$$

which gives for any open set $A\subseteq\Omega$

$$|(D_X u, -\mathscr{L}^n)|(A) \le |D_{X^\dagger}\chi_{E_u}|(A\times\mathbb{R}). \tag{3.15}$$

Suppose now that $u \in BV_X(\Omega)$. Let $A \subseteq \Omega$ be open and let $\varphi \in C^1_c(A\times\mathbb{R})$ and $i = 1,\dots,m$ be fixed. Let (u_h) be a sequence in $C^\infty(A)\cap BV_X(A)$ satisfying (1.29) (with A in place of Ω); then

$$\begin{aligned}&\int_{A\times\mathbb{R}} \varphi\, d(D_{X_i^\dagger}\chi_{E_{u_h}}) \\ &= -\int_{A\times\mathbb{R}} \chi_{E_{u_h}}(x,t)[(X_i^\dagger)^*\varphi](x,t)\, d\mathscr{L}^{n+1}(x,t) \\ &= -\int_A \left(\int_{-\infty}^{u_h(x)} \sum_{j=1}^n \partial_{x_j}\left(a_{ij}(x)\varphi(x,t)\right) dt \right) d\mathscr{L}^n(x) \\ &= -\int_A \left(\sum_{j=1}^n \partial_{x_j} \int_{-\infty}^{u_h(x)} a_{ij}(x)\varphi(x,t)\, dt - \sum_{j=1}^n a_{ij}(x)\varphi(x,u_h(x))\partial_{x_j}u_h(x) \right) d\mathscr{L}^n(x) \\ &= \int_A \varphi(x,u_h(x)) X_i u_h(x)\, d\mathscr{L}^n(x),\end{aligned} \tag{3.16}$$

where we used the fact that $x \mapsto a_{ij}(x)\int_{-\infty}^{u_h(x)} \varphi(x,t)\, dt$ is in $C^1_c(A)$. In a similar way

$$\begin{aligned}\int_{A\times\mathbb{R}} \varphi\, d(\partial_t\chi_{E_{u_h}}) &= -\int_A \left(\int_{-\infty}^{u_h(x)} \partial_t\varphi(x,t) dt \right) d\mathscr{L}^n(x) \\ &= -\int_A \varphi(x,u_h(x))\, d\mathscr{L}^n(x).\end{aligned} \tag{3.17}$$

Formulas (3.16) and (3.17) imply that for any $\varphi \in C^1_c(A \times \mathbb{R}; \mathbb{R}^{m+1})$

$$\int_{A\times\mathbb{R}} \varphi \cdot d(D_{X^\dagger}\chi_{E_{u_h}}) = \int_A \varphi(x, u_h(x)) \cdot d(D_X u_h, -\mathscr{L}^n)(x).$$

Since $\chi_{E_{u_h}} \to \chi_{E_u}$ in $L^1(A \times \mathbb{R})$, we obtain

$$\begin{aligned} |D_{X^\dagger}\chi_{E_u}|(A \times \mathbb{R}) \le &\liminf_{h\to+\infty} |D_{X^\dagger}\chi_{E_{u_h}}|(A \times \mathbb{R}) \le \lim_{h\to+\infty} |(D_X u_h, -\mathscr{L}^n)|(A) \\ =&|(D_X u, -\mathscr{L}^n)|(A) < +\infty, \end{aligned} \tag{3.18}$$

which proves that $\chi_{E_u} \in BV_{X^\dagger}(\Omega \times \mathbb{R})$, as desired. Notice that, using the lower semicontinuity in a similar way, one also gets

$$\begin{aligned} &|D'_{X^\dagger}\chi_{E_u}|(A \times \mathbb{R}) \le |D_X u|(A) \\ &|D_{X_i^\dagger}\chi_{E_u}|(A \times \mathbb{R}) \le |D_{X_i} u|(A) \quad \text{for any } i = 1, \dots, m \\ &|\partial_t \chi_{E_u}|(A \times \mathbb{R}) \le \mathscr{L}^n(A) < +\infty. \end{aligned} \tag{3.19}$$

Eventually, statements (i) and (ii) follow from (3.11) and (3.13), while statements (iii)–(vi) are consequences of formulas (3.12), (3.14), (3.15), (3.18) and (3.19). □

For $u \in BV_{X,loc}(\Omega)$ we recall the decomposition of its distributional derivatives $D_X u = D^a_X u + D^s_X u$ as introduced in Chapter 2. We also write $D^a_X u = Xu\,\mathscr{L}^n$ for some function $Xu \in L^1_{loc}(\Omega; \mathbb{R}^{k\times m})$.

We will also consider the polar decomposition $D_X u = \sigma_u |D_X u|$, where $\sigma_u : \Omega \to \mathbb{S}^{m-1}$ is a $|D_X u|$-measurable function. In case $u = \chi_E$ is the characteristic function of a set $E \subseteq \Omega \times \mathbb{R}$ of locally finite $X^\dagger$-perimeter in $\Omega \times \mathbb{R}$, we write $D_{X^\dagger}\chi_E = \nu_E |D_{X^\dagger}\chi_E|$ for some Borel function $\nu_E = ((\nu_E)_1, \dots, (\nu_E)_{m+1})$ called measure-theoretic *horizontal inner normal* to E.

The following result is basically a consequence of Theorem 3.3.1.

Theorem 3.3.2. *Let $u \in BV_X(\Omega)$ and define*

$$\begin{aligned} S &\coloneqq \{(x, t) \in \Omega \times \mathbb{R} : (\nu_{E_u})_{m+1}(x, t) = 0\} \\ T &\coloneqq \{(x, t) \in \Omega \times \mathbb{R} : (\nu_{E_u})_{m+1}(x, t) \neq 0\}. \end{aligned}$$

Then, the following identities hold

$$\nu_{E_u}(x, t) = (\sigma_u(x), 0) \quad \textit{for } |D_{X^\dagger}\chi_{E_u}|\textit{-a.e. } (x, t) \in S; \tag{3.20}$$

$$\nu_{E_u}(x, t) = \frac{(Xu(x), -1)}{\sqrt{1 + |Xu(x)|^2}} \quad \textit{for } |D_{X^\dagger}\chi_{E_u}|\textit{-a.e. } (x, t) \in T; \tag{3.21}$$

$$\pi_\#(D_{X^\dagger}\chi_{E_u} \llcorner S) = (D^s_X u, 0); \tag{3.22}$$

$$\pi_\#(D_{X^\dagger}\chi_{E_u} \llcorner T) = (D^a_X u, -\mathscr{L}^n). \tag{3.23}$$

Proof. Thanks to Theorem 3.3.1 (vi), we can disintegrate the measure $|D_{X^\dagger}\chi_{E_u}|$ with respect to $|(D_X u, -\mathscr{L}^n)|$ (see Theorem 1.1.8): for every $x \in \Omega$, there exists a probability measure μ_x on $\mathbb{R}$ such that for every Borel function $g \in L^1(\Omega \times \mathbb{R}, |D_{X^\dagger}\chi_{E_u}|)$

$$\int_{\Omega\times\mathbb{R}} g(x,t)\, d|D_{X^\dagger}\chi_{E_u}|(x,t) = \int_\Omega \left(\int_\mathbb{R} g(x,t)\, d\mu_x(t)\right) d|(D_X u, -\mathscr{L}^n)|(x).$$

It follows that, for any Borel function $\varphi : \Omega \to \mathbb{R}$, one has

$$\begin{aligned}\int_\Omega \varphi(x)\, d(D_X u, -\mathscr{L}^n)(x) &= \int_\Omega \varphi(x)\, d\pi_\#(\nu_{E_u}|D_{X^\dagger}\chi_{E_u}|)(x)\\ &= \int_{\Omega\times\mathbb{R}} \varphi(x)\nu_{E_u}(x,t)\, d|D_{X^\dagger}\chi_{E_u}|(x,t)\\ &= \int_\Omega \varphi(x)\left(\int_\mathbb{R} \nu_{E_u}(x,t)\, d\mu_x(u)\right) d|(D_X u, -\mathscr{L}^n)|(x).\end{aligned} \tag{3.24}$$

Since $D^a_X u$ and $D^s_X u$ are mutually singular we have

$$|(D_X u, -\mathscr{L}^n)| = |(D^a_X u, -\mathscr{L}^n)| + |(D^s_X u, 0)| = \sqrt{1+|Xu|^2}\mathscr{L}^n + |D^s_X u|$$

and (3.24) gives

$$\begin{aligned}&\int_\Omega \varphi\, d\Big((Xu,-1)\mathscr{L}^n + (\sigma_u, 0)|D^s_X u|\Big)\\ &= \int_\Omega \varphi(x)\left(\int_\mathbb{R} \nu_{E_u}(x,t)\, d\mu_x(t)\right) d\left(\sqrt{1+|Xu|^2}\mathscr{L}^n + |D^s_X u|\right)(x).\end{aligned}$$

Denote by I a subset of Ω such that $\mathscr{L}^n(I) = 0$ and $|D^s_X u|(\Omega \setminus I) = 0$. Considering Borel test functions φ such that $\varphi = 0$ in $\Omega \setminus I$, we deduce that for $|D^s_X u|$-a.e. $x \in I$ one has

$$(\sigma_u(x), 0) = \int_\mathbb{R} \nu_{E_u}(x,t)\, d\mu_x(t).$$

Taking on both sides the scalar product with $(\sigma_u(x), 0)$ we get

$$\left\langle (\sigma_u(x), 0), \int_\mathbb{R} \nu_{E_u}(x,t)\, d\mu_x(t)\right\rangle = 1,$$

and, since $\mu_x(\mathbb{R}) = 1$ and (for $|(D_X u, -\mathscr{L}^n)|$-a.e. $x \in \Omega$) $|\nu_{E_u}(x,t)| = 1$ for μ_x-a.e. t, we deduce that

$$\nu_{E_u}(x,t) = (\sigma_u(x), 0) \qquad \text{for } |D^s_X u|\text{-a.e. } x \in I \text{ and } \mu_x\text{-a.e. } t \in \mathbb{R},$$

i.e.,

$$\nu_{E_u}(x,t) = (\sigma_u(x), 0) \qquad \text{for } |D_{X^\dagger}\chi_{E_u}|\text{-a.e. } (x,t) \in I \times \mathbb{R}. \tag{3.25}$$

Taking into account again (3.3) and letting φ be such that $\varphi = 0$ on I, we instead obtain

$$\begin{aligned}&\int_\Omega \varphi\, \frac{(Xu,-1)}{\sqrt{1+|Xu|^2}}\sqrt{1+|Xu|^2}\, d\mathscr{L}^n\\ &= \int_\Omega \varphi(x)\left(\int_\mathbb{R} \nu_{E_u}(x,t)\, d\mu_x(t)\right)\sqrt{1+|Xu(x)|^2}\, d\mathscr{L}^n(x).\end{aligned}$$

Consequently, for $\mathscr{L}^n$-a.e. $x \in \Omega \setminus I$, we have

$$\int_{\mathbb{R}} \nu_{E_u}(x,t)\, d\mu_x(t) = \frac{(Xu(x),-1)}{\sqrt{1+|Xu(x)|^2}}.$$

Reasoning as before, we deduce that

$$\nu_{E_u}(x,t) = \frac{(Xu(x),-1)}{\sqrt{1+|Xu(x)|^2}} \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \setminus I \text{ and } \mu_x\text{-a.e. } t \in \mathbb{R},$$

or equivalently

$$\nu_{E_u}(x,t) = \frac{(Xu(x),-1)}{\sqrt{1+|Xu(x)|^2}} \quad \text{for } |D_{X^\dagger}\chi_{E_u}|\text{-a.e. } (x,t) \in (\Omega \setminus I) \times \mathbb{R}. \tag{3.26}$$

Formula (3.25) implies that $|D_{X^\dagger}\chi_{E_u}|$-a.e. $(x,t) \in I \times \mathbb{R}$ belongs to S and that $|D_{X^\dagger}\chi_{E_u}|$-a.e. $(x,t) \in T$ belongs to $(\Omega \setminus I) \times \mathbb{R}$. Similarly, (3.26) says that $|D_{X^\dagger}\chi_{E_u}|$-a.e. $(x,t) \in (\Omega \setminus I) \times \mathbb{R}$ belongs to T and that $|D_{X^\dagger}\chi_{E_u}|$-a.e. $(x,t) \in S$ belongs to $I \times \mathbb{R}$. Since S and T are disjoint, this is enough to conclude (3.20) and (3.21). Statement (3.22) now easily follows because

$$\pi_\#(D_{X^\dagger}\chi_{E_u} \llcorner S) = \pi_\#(\nu_{E_u}|D_{X^\dagger}\chi_{E_u}| \llcorner (I \times \mathbb{R})) = (\sigma_u, 0)|(D_X u, -\mathscr{L}^n)| \llcorner I = (D_X^s u, 0)$$

Similarly, one has

$$\begin{aligned} \pi_\#(D_{X^\dagger}\chi_{E_u} \llcorner T) =& \pi_\#(\nu_{E_u}|D_{X^\dagger}\chi_{E_u}| \llcorner ((\Omega \setminus I) \times \mathbb{R})) \\ =& \frac{(Xu,-1)}{\sqrt{1+|Xu|^2}}|(D_X u, -\mathscr{L}^n)| \llcorner (\Omega \setminus I) = (Xu,-1)\mathscr{L}^n, \end{aligned}$$

which gives (3.23). □

3.4 The rank-one theorem in a class of Carnot groups

Definition 3.4.1. We say that a Carnot group $\mathbb{G}$ satisfies property w-$\mathcal{R}$ if for any bounded open set $\Omega \subseteq \mathbb{G}$ and any $u \in BV_{\mathbb{G}}(\Omega)$, the set $\partial^* E_u$ is $(\mathbb{G} \times \mathbb{R})$-rectifiable set and the identity

$$\nu_{E_u} = \nu_{\partial^* E_u},$$

holds $\mathscr{H}^Q$-a.e. on $\partial^* E_u$, where $\nu_{\partial^* E_u}$ is the normal of the rectifiable set $\partial^* E_u$ while ν_{E_u} is the measure-theoretic horizontal inner normal to E_u.

Notice that, by Theorem 2.2.3, we have that the measure derivative $D_{\mathbb{G}\times\mathbb{R}}\chi_{E_u}$ of the characteristic function of the subgraph E_u of u can be represented as

$$D_{\mathbb{G}\times\mathbb{R}}\chi_{E_u} = \nu_{\partial^* E_u}\theta \mathscr{H}^Q \llcorner \partial^* E_u, \tag{3.27}$$

for some positive density $\theta \in L^1(\partial^* E_u, \mathscr{H}^Q)$.
Notice also that, in Definition 3.4.1, the measure $D_{\mathbb{G}\times\mathbb{R}}\chi_{E_u}$ has finite total variation by Theorem 3.3.1.

Remark 3.4.2. In view of Theorem 3.3.1, if $\mathbb{G}$ satisfies property $\mathcal{R}$, then $\mathbb{G}$ satisfies property w-$\mathcal{R}$. We however conjecture that property w-$\mathcal{R}$ is indeed equivalent to property $\mathcal{R}$.

Remark 3.4.3. If $\mathbb{G}$ is a Carnot group of step 2, then $\mathbb{G}$ satisfies property w-$\mathcal{R}$: this follows from the fact that $\mathbb{G}\times\mathbb{R}$ is also a step 2 Carnot group and that the rectifiability theorem holds in any step 2 Carnot group, see [39].

Remark 3.4.4. If (3.27) holds, then

$$|D_{\mathbb{G}\times\mathbb{R}}\chi_{E_u}|=\theta\mathscr{H}^Q \llcorner \partial^* E_u.$$

Theorem 3.4.5. *Let $\mathbb{G}$ be a Carnot group satisfying properties $\mathscr{C}_2$ and w-$\mathcal{R}$; let $\Omega\subseteq\mathbb{G}$ be an open set and $u\in BV_{\mathbb{G},loc}(\Omega;\mathbb{R}^k)$. Then the singular part $D^s_{\mathbb{G}}u$ of $D_{\mathbb{G}}u$ is a rank-one measure, i.e., the matrix-valued function $\frac{D^s_{\mathbb{G}}u}{|D^s_{\mathbb{G}}u|}(x)$ has rank one for $|D^s_{\mathbb{G}}u|$-a.e. $x\in\Omega$.*

Proof. Without loss of generality, one can assume that $u=(u_1,\dots,u_k)\in BV_{\mathbb{G}}(\Omega;\mathbb{R}^k)$ and that Ω is bounded. For any $i=1,\dots,k$ we write $D^s_{\mathbb{G}}u^i=\sigma_i|D^s_{\mathbb{G}}u_i|$ for a $|D^s_{\mathbb{G}}u_i|$-measurable map $\sigma_i:\Omega\to\mathbb{S}^{m-1}$; notice that, using the notation of Section 3.3, the equality $\sigma_i=\sigma_{u_i}$ holds $|D^s_{\mathbb{G}}u_i|$-a.e. We also let $E_i:=\{(x,t)\in\Omega\times\mathbb{R}: t<u^i(x)\}$ be the subgraph of u^i, that has finite $\mathbb{G}$-perimeter in $\Omega\times\mathbb{R}$ by Theorem 3.3.1. Denoting by ∂^*E_i the essential boundary of E_i and writing $\nu_i=\nu_{E_i}$ for the measure-theoretic horizontal inner normal to E_i, we have, by Theorem 3.3.2 and Remark 3.4.4, that

$$|D^s_{\mathbb{G}}u_i|=\pi_\#(\theta_i\mathscr{H}^Q\llcorner S_i)\quad\text{for some positive }\theta_i\in L^1(\partial^*E_i,\mathscr{H}^Q),$$

where $S_i:=\big\{p\in\partial^*E_i:(\nu_i(p))_{m+1}=0\big\}$ and $\pi_\#$ denotes push-forward of measures through the projection π defined by $\mathbb{G}\times\mathbb{R}\ni(x,t)\mapsto x\in\mathbb{G}$. By rectifiability, we can assume that ∂^*E_i is contained in a union $\cup_{\ell\in\mathbb{N}}\Sigma^i_\ell$ of $C^1_{\mathbb{G}}$ hypersurfaces Σ^i_ℓ in $\mathbb{G}\times\mathbb{R}$.

Using Theorem 3.3.2, Remark 3.4.4 and Lemma 3.2.7, the following properties hold for $\mathscr{H}^Q$-a.e. $p\in S_1\cup\ldots\cup S_k$:

$$\text{if } p\in S_i,\text{ then } \nu_i(p)=(\sigma_i(\pi(p)),0) \tag{3.28}$$

$$\text{if } p\in\Sigma^i_\ell,\ \text{ then } \nu_i(p)=\pm\nu_{\Sigma^i_\ell}(p) \tag{3.29}$$

$$\text{if } p\in\Sigma^i_\ell \text{ and } \exists\, q\in S_j\cap\Sigma^j_k\cap\pi^{-1}(\pi(p)),\ \text{ then } \nu_{\Sigma^i_\ell}(p)=\pm\nu_{\Sigma^j_k}(q). \tag{3.30}$$

Up to modifying each S_i on a $\mathscr{H}^Q$-negligible set and each σ_i on a $|D^s_{\mathbb{G}}u^i|$-negligible set, we can assume that (3.28), (3.29) and (3.30) hold for any $p\in S_1\cup\ldots\cup S_k$ and that, for any $i=1,\dots,k$, $\sigma_i=0$ on $\Omega\setminus\pi(S_i)$.

Since $D^s_{\mathbb{G}}u=(\sigma_1|D^s_{\mathbb{G}}u^1|,\dots,\sigma_k|D^s_{\mathbb{G}}u^k|)$ and $|D^s_{\mathbb{G}}u|$ is concentrated on $\pi(S_1)\cup\ldots\cup\pi(S_k)$, it is enough to prove that the matrix-valued function $(\sigma_1,\dots,\sigma_k)$ has rank 1 on $\pi(S_1)\cup\ldots\cup\pi(S_k)$. This will follow if we prove that the implication

$$i,j\in\{1,\dots,k\},\ i\neq j,\ x\in\pi(S_i)\implies\sigma_j(x)\in\{0,\sigma_i(x),-\sigma_i(x)\},$$

holds. If i, j, x are as above and $x \notin \pi(S_j)$, then $\sigma_j(x) = 0$. Otherwise, $x \in \pi(S_i) \cap \pi(S_j)$, i.e., there exist $p \in S_i$ and $\ell \in \mathbb{N}$ such that $\pi(p) = x$ and $\sigma_i(x) = \pm\nu_{\Sigma^i_\ell}(p)$ and there exist $q \in S_j$ and $N \in \mathbb{N}$ such that $\pi(q) = x$ and $\sigma_j(x) = \pm\nu_{\Sigma^j_N}(p)$. By (3.30), we obtain $\sigma_j(x) = \pm\sigma_i(x)$, as wished. □

Remark 3.4.6. As an easy consequence of Remarks 3.1.5 and 3.4.3, Theorem 3.4.5 holds for the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ provided $n \geq 2$. This result does not directly follow from [26], as we now briefly explain using the notation of Example 1.3.24 and restricting for simplicity to $n = 2$, the general case $n \geq 2$ being a straightforward generalization.

Let $u \in BV_{\mathbb{G}}(\Omega; \mathbb{R}^k)$ for some open set $\Omega \subseteq \mathbb{H}^2$. It can be easily seen that the matrix-valued measure $(\mu_1, \mu_2, \mu_3, \mu_4) \coloneqq D_{\mathbb{G}}u = (X_1u, X_2u, Y_1u, Y_2u)$ satisfies the equations

$$\mathscr{A}\mu \coloneqq \begin{pmatrix} X_1\mu_2 - X_2\mu_1 \\ Y_1\mu_4 - Y_2\mu_3 \\ X_1\mu_4 - Y_2\mu_1 \\ Y_1\mu_2 - X_2\mu_3 \\ X_1\mu_3 - Y_1\mu_1 + Y_2\mu_2 - X_2\mu_4 \end{pmatrix} = 0$$

in the sense of distributions. Write the first-order differential operator $\mathscr{A}$ (the *horizontal curl* in $\mathbb{H}^2$, see [12, Example 3.12]) in the form

$$\mathscr{A} = A_1\partial_{x_1} + A_2\partial_{x_2} + A_3\partial_{y_1} + A_4\partial_{y_2} + A_5\partial_t$$

for suitable $A_j = A_j(x, y, t)$ and consider the *wave cone* $\Lambda_{\mathscr{A}}(x, y, t)$ (see [26]) associated with $\mathscr{A}$

$$\Lambda_{\mathscr{A}}(x, y, t) \coloneqq \bigcup_{q \in \mathbb{R}^5 \setminus \{0\}} \ker \mathbb{A}_{x,y,t}(q), \qquad \text{where } \mathbb{A}_{x,y,t}(q) \coloneqq 2\pi i \sum_{j=1}^{5} A_j(x, y, t) q_j.$$

One can readily check that

$$\mathbb{A}_{x,y,t}(q) = 0 \quad \text{for } q \coloneqq (-2y, 2x, 1) \in \mathbb{R}^5 \setminus \{0\},$$

i.e., the wave cone $\Lambda_{\mathscr{A}}(x, y, t)$ is the full space for any $(x, y, t) \in \mathbb{H}^2$. In particular, [26, Theorem 1.1] gives no information on the polar decomposition of $D^s_{\mathbb{G}}u$.

Remark 3.4.7. The rank-one property for BV functions in the first Heisenberg group remains a very interesting open question, since it does not follow either from Theorem 3.4.5 (because property $\mathscr{C}_2$ fails for $\mathbb{H}^1$) or from [26, Theorem 1.1], as we now explain.

Let $u \in BV_{\mathbb{G}}(\Omega; \mathbb{R}^k)$ for some open set $\Omega \subseteq \mathbb{G} \coloneqq \mathbb{H}^1$; we use again the notation of Example 1.3.24 and we set $p = (x, y, t) \in \mathbb{H}^1 \equiv \mathbb{R}^3$. One can check that $(\mu_1, \mu_2) \coloneqq D_{\mathbb{G}}u = (Xu, Yu)$ satisfies

$$\mathscr{A}\mu \coloneqq \begin{pmatrix} YX\mu_1 - 2XY\mu_1 + XX\mu_2 \\ YY\mu_1 - 2YX\mu_2 + XY\mu_2, \end{pmatrix} = 0$$

in the sense of distributions. Now $\mathscr{A}$ (the horizontal curl in $\mathbb{H}^1$, see [12, Example 3.11]) is a second-order differential operator that one can write as

$$\mathscr{A} = \sum_{|\alpha|=2} A_\alpha(p)\partial^\alpha,$$

where $\alpha \in \mathbb{N}^3$ is a multi-index and $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3}$. As before, one can define the wave cone

$$\Lambda_{\mathscr{A}}(p) = \bigcup_{q\in\mathbb{R}^3\setminus\{0\}} \ker \mathbb{A}_p(q), \qquad \text{where } \mathbb{A}_p(q) = (2\pi i)^2 \sum_{|\alpha|=2} A_\alpha(p)q^\alpha.$$

Again, one has

$$\mathbb{A}_p(q) = 0 \qquad \text{for } q := (-2y, 2x, 1) \in \mathbb{R}^3 \setminus \{0\}$$

and the wave cone $\Lambda_{\mathscr{A}}(x, y, t)$ is the full space. More precisely, this follows from the computations below. First, we observe that

$$\begin{aligned} XY &= \partial_{xy} - 2x\partial_{xt} + 2y\partial_{yt} - 4xy\partial_{tt} - 2\partial_t = \\ &= \partial^{(1,1,0)} - 2x\partial^{(1,0,1)} + 2y\partial^{(0,1,1)} - 4xy\partial^{(0,0,2)} - 2\partial^{(0,0,1)}; \end{aligned}$$

$$\begin{aligned} YX &= \partial_{xy} - 2x\partial_{xt} + 2y\partial_{yt} - 4xy\partial_{tt} + 2\partial_t = \\ &= \partial^{(1,1,0)} - 2x\partial^{(1,0,1)} + 2y\partial^{(0,1,1)} - 4xy\partial^{(0,0,2)} + 2\partial^{(0,0,1)}; \end{aligned}$$

$$XX = \partial_{xx} + 4y\partial_{xt} + 4y^2\partial_{tt} = \partial^{(2,0,0)} + 4y\partial^{(1,0,1)} + 4y^2\partial^{(0,0,2)};$$

$$YY = \partial_{yy} - 4x\partial_{yt} + 4x^2\partial_{tt} = \partial^{(0,2,0)} - 4x\partial^{(0,1,1)} + 4x^2\partial^{(0,0,2)}.$$

Then, the matrices A_α's are given by

$$\begin{aligned} A_{(1,1,0)}(x, y, t) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A_{(0,1,1)}(x, y, t) &= \begin{pmatrix} -2y & 0 \\ -4x & -2y \end{pmatrix} \\ A_{(1,0,1)}(x, y, t) &= \begin{pmatrix} 2x & 4y \\ 0 & 2x \end{pmatrix} \\ A_{(2,0,0)}(x, y, t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ A_{(0,2,0)}(x, y, t) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ A_{(0,0,2)}(x, y, t) &= \begin{pmatrix} 4xy & 4y^2 \\ 4x^2 & 4xy \end{pmatrix}. \end{aligned}$$

It follows that for every $(\xi, \eta, \tau) \neq (0,0,0)$ one has

$$\begin{aligned}
\mathbb{A}^2_{(x,y,t)}(\xi,\eta,\tau) &= -4\pi^2 \big(\xi\eta A_{(1,1,0)}(x,y,t) + \eta\tau A_{(0,1,1)}(x,y,t) + \xi\tau A_{(1,0,1)}(x,y,t) \\
&\qquad + \xi^2 A_{(2,0,0)}(x,y,t) + \eta^2 A_{(0,2,0)}(x,y,t) + \tau^2 A(0,0,2)\big) \\
&= -4\pi^2 \begin{pmatrix} -\xi\eta - 2y\eta\tau + 2x\xi\tau + 4xy\tau^2 & 4y\xi\tau + \xi^2 + 4y^2\tau^2 \\ -4x\eta\tau + \eta^2 + 4x^2\tau^2 & -\xi\eta - 2y\eta\tau + 2x\xi\tau + 4xy\tau^2 \end{pmatrix} \\
&= -4\pi^2 \begin{pmatrix} -\xi\eta - 2y\eta\tau + 2x\xi\tau + 4xy\tau^2 & (\xi + 2y\tau)^2 \\ (\eta - 2x\tau)^2 & -\xi\eta - 2y\eta\tau + 2x\xi\tau + 4xy\tau^2 \end{pmatrix},
\end{aligned}$$

and hence, with the choice $\tau = 1$, $\xi = -2y$ and $\eta = 2x$, we get

$$\mathbb{A}^2_{(x,y,t)}(\xi,\eta,\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

as claimed.

Chapter 4

A compactness result for BV functions in metric spaces

The following chapter deals with a compactness criterion for equibounded sequences (u_j) in metric measure spaces when the underlying metric varies with $j \in \mathbb{N}$. The results of this chapter are contained in [29]. Section 4.1 is devoted to the proof of the main Theorem 4.1.1. The proof follows basically by combining a Poincaré inequality and an approximation scheme of functions in terms of their mean values on balls.
Section 4.2 has the goal of showing an application of Theorem 4.1.1 to the case of equiregular CC spaces $(\mathbb{R}^n, X)$ with bounded (in the Euclidean metric) metric balls. The first part of this section is devoted to proving that the sequence of CC metric (d_j) built with respect to moving vector fields $X^j = (X_1^j, \dots, X_m^j)$ converges uniformly to the reference CC distance d if the frame X^j converges to X smoothly enough (see Theorem 4.2.4). The proof of Theorem 4.2.4 requires some preparatory lemmata (see Lemmata 4.2.2 and 4.2.3) and a uniform ball-box inequality (see Theorem 4.2.1) coming from the application of the results of [18, 73]. The main result of this section (see Theorem 4.2.6) then follows by taking into account the uniform Poincaré inequality given by Theorem 4.2.5.

4.1 The main result

In the statement of the following theorem the locality is to be understood with respect to the topology induced by d. Also, all the compacts sets considered are compact with respect to the topology induced by d.

Theorem 4.1.1. *Let M be a set, $q \geq 1$, $\delta > 0$ and let d, d_j $(j \in \mathbb{N})$ be metrics on M such that (M, d) is locally compact and separable. Let λ, μ_j $(j \in \mathbb{N})$ be Radon measures on M and consider a sequence (u_j) in $L^q_{loc}(M, \lambda)$. Suppose that the following assumptions hold.*

(i) The sequence (d_j) *converges to* d *in* $L^\infty_{loc}(M \times M, \lambda)$*.*

(ii) (M, d, λ) *is a locally doubling metric measure space, i.e., for any compact set* $K \subseteq M$ *there exist* $C_D \geq 1$ *and* $R_D > 0$ *such that*

$$\forall\, x \in K,\ \forall r \in (0, R_D) \qquad \lambda(B(x, 2r)) \leq C_D \lambda(B(x, r)).$$

(iii) For every compact set $K \subseteq M$ *there exist* $C_P, R_P > 0$ *and* $\alpha \geq 1$ *such that*

$$\forall x \in K,\ \forall j \in \mathbb{N},\ \forall r \in (0, R_P) \qquad \|u_j - u_j(B^j)\|_{L^q(B^j,\lambda)} \leq C_P\, r^\delta \mu_j(\alpha B^j),$$

where $B^j := B^j(x, r)$ *denotes a ball in* (M, d_j)*,* $\alpha B^j := B^j(x, \alpha r)$ *and* $u_j(B^j) := \fint_{B^j} u_j d\lambda$*.*

(iv) For every compact set $K \subseteq M$ *there exists* $M_K > 0$ *such that*

$$\forall j \in \mathbb{N} \qquad \|u_j\|_{L^1(K,\lambda)} + \mu_j(K) \leq M_K.$$

Then there exist $u \in L^q_{loc}(M; \lambda)$ *and a subsequence* (u_{j_h}) *of* (u_j) *such that* (u_{j_h}) *converges to* u *in* $L^q_{loc}(M; \lambda)$ *as* $h \to +\infty$*.*

Concerning the classical Euclidean case when $(M, d_j, \lambda) = (M, d, \lambda) = (\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$, we invite the reader to compare the assumption in (iii) with the well-known Poincaré inequality

$$\|u - u(B_r)\|_{L^q(B_r)} \leq C r^\delta |Du|(B_r), \qquad \forall\, q \in [1, \tfrac{n}{n-1}) \text{ with } \delta := \tfrac{n}{q} + 1 - n > 0$$

valid for for any BV function u on any ball $B_r \subseteq \mathbb{R}^n$ of radius r and where $u(B_r)$ denotes the mean value $\mathscr{L}^n(B_r)^{-1} \int_{B_r} u \, d\mathscr{L}^n$ of u in B_r, $C > 0$ is a geometric constant, and $|Du|$ denotes the total variation measure associated with u (i.e., the total variation of the distributional derivatives of u).

Proof. Let $K \subseteq M$ be a fixed compact set and let $\varepsilon > 0$. We first prove that there exists a subsequence (u_{j_h}) such that

$$\limsup_{h,k \to +\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} \leq 2C_0 \varepsilon, \tag{4.1}$$

for some $C_0 > 0$ depending on K only.

Consider an open set $U_1 \subseteq M$ such that $K \subseteq U_1$, $\overline{U}_1$ is compact and

$$\lambda(U_1 \setminus K) \leq \frac{1}{4C_D^{\beta+3}} \lambda(K), \tag{4.2}$$

where β is an integer such that $2^\beta > 2\alpha$ and α is given by condition (iii). By the $5r$-covering Theorem (see Theorem 1.1.10) we can find a family $\{B(x_\ell, r_\ell) : \ell \in \mathbb{N}\}$ of pairwise disjoint balls such that $x_\ell \in K$, $0 < r_\ell < \min\{\varepsilon^{1/\delta}, R_D/4, 2\alpha R_P\}$, $\overline{B(x_\ell, 5r_\ell)} \subseteq U_1$ and

$$K \subseteq \bigcup_{\ell=0}^{\infty} \overline{B(x_\ell, 5r_\ell)}.$$

Denote for shortness $B_\ell \coloneqq B(x_\ell, r_\ell)$; then

$$\lambda(K) \leq \sum_{\ell=0}^{\infty} \lambda(5\overline{B}_\ell) \leq \sum_{\ell=0}^{\infty} \lambda(8B_\ell) \leq C_D^{\beta+3} \sum_{\ell=0}^{\infty} \lambda(\tfrac{1}{2^\beta} B_\ell) = C_D^{\beta+3} \lambda\left(\bigcup_{\ell=0}^{\infty} \tfrac{1}{2^\beta} B_\ell\right).$$

Hence we can choose $L \in \mathbb{N}$ such that

$$\lambda\left(\bigcup_{\ell=0}^{L} \tfrac{1}{2^\beta} B_\ell\right) \geq \frac{1}{2C_D^{\beta+3}} \lambda(K).$$

Taking into account (4.2), we easily get that $A_1 \coloneqq K \cap \bigcup_{\ell=0}^{L} \frac{1}{2^\beta} B_\ell$ satisfies

$$\lambda(A_1) \geq \frac{1}{4C_D^{\beta+3}} \lambda(K).$$

For $j \in \mathbb{N}$ and $\ell = 1, \dots, L$ set for shortness $B_\ell^j \coloneqq B^j(x_\ell, r_\ell)$. By assumption (i), there exists $J \in \mathbb{N}$ such that for every $j \geq J$, and for every $\ell = 0, \dots, L$

$$\tfrac{1}{2^\beta} B_\ell \subseteq \tfrac{1}{2\alpha} B_\ell^j \qquad \text{and} \qquad \tfrac{1}{2} B_\ell^j \subseteq B_\ell. \tag{4.3}$$

Hence for every $j \geq J$ one has

$$\left|u_j\left(\tfrac{1}{2\alpha} B_\ell^j\right)\right| \leq \lambda\left(\tfrac{1}{2\alpha} B_\ell^j\right)^{-1} \|u_j\|_{L^1(U_1;\lambda)} \leq M_{\overline{U}_1} \max\{\lambda\left(\tfrac{1}{2^\beta} B_\ell\right)^{-1} : \ell = 0, \dots, L\} < +\infty.$$

By Bolzano-Weierstrass Theorem we get an increasing function $\nu_1 : \mathbb{N} \to \mathbb{N}$ such that

$$\text{the sequence } \left(u_{\nu_1(j)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(j)}\right)\right)_j \text{ is convergent for every } \ell = 0, \dots, L. \tag{4.4}$$

Then

$$\begin{aligned}
&\limsup_{h,k\to+\infty} \|u_{\nu_1(h)} - u_{\nu_1(k)}\|_{L^q(A_1;\lambda)} \\
&\leq \limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \Bigg(\left\|u_{\nu_1(h)} - u_{\nu_1(h)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)}\right)\right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \\
&\qquad + \left\|u_{\nu_1(k)} - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \\
&\qquad + \left\|u_{\nu_1(h)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)}\right) - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2^\beta} B_\ell;\lambda\right)} \Bigg)
\end{aligned}$$

and, using (4.3) and (4.4),

$$
\begin{aligned}
&\leq \limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \left(\left\| u_{\nu_1(h)} - u_{\nu_1(h)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(h)}\right)\right\|_{L^q\left(\frac{1}{2\alpha}B_\ell^{\nu_1(h)};\lambda\right)} \right. \\
&\qquad\qquad \left. + \left\| u_{\nu_1(k)} - u_{\nu_1(k)}\left(\tfrac{1}{2\alpha} B_\ell^{\nu_1(k)}\right)\right\|_{L^q\left(\frac{1}{2\alpha}B_\ell^{\nu_1(k)};\lambda\right)} \right) \\
&\leq \limsup_{h,k\to+\infty} \sum_{\ell=0}^{L} \frac{C_P\, r_\ell^\delta}{(2\alpha)^\delta} \left(\mu_{\nu_1(h)}\left(\tfrac{1}{2} B_\ell^{\nu_1(h)}\right) + \mu_{\nu_1(k)}\left(\tfrac{1}{2} B_\ell^{\nu_1(k)}\right)\right) \\
&\leq \limsup_{h,k\to+\infty} \frac{C_P\,\varepsilon}{(2\alpha)^\delta} \left(\mu_{\nu_1(h)}\left(\overline{U}_1\right) + \mu_{\nu_1(k)}\left(\overline{U}_1\right)\right) \leq C_0\varepsilon,
\end{aligned}
$$

where C_0 depends only on U_1 and thus only on K.
We proved that there exist $A_1 \subseteq K$ and a subsequence $(u_{\nu_1(h)})$ of (u_j) such that

$$
\begin{aligned}
&\lambda(K \setminus A_1) \leq \left(1 - \frac{1}{4C_D^{\beta+3}}\right)\lambda(K), \\
&\limsup_{h,k\to+\infty} \|u_{\nu(h)} - u_{\nu(k)}\|_{L^q(A_1;\lambda)} \leq C_0\varepsilon.
\end{aligned}
$$

Since the set $K_2 = K \setminus A_1$ is compact we can repeat the same argument on K_2, with $\frac{\varepsilon}{2}$ in place of ε, and paying attention to choose an open set $U_2 \subseteq U_1$ so that C_0 can be left unchanged. By a recursive argument, for every $j \in \mathbb{N}$ we get pairwise disjoint sets $A_j \subseteq K$ and subsequences $(u_{\nu_j(h)})$ such that for every $j \geq 1$

(a) $(u_{\nu_{j+1}(h)})$ is a subsequence of $(u_{\nu_j(h)})$;

(b) $\lambda\left(K \setminus \bigcup_{i=1}^j A_i\right) \leq \left(1 - \frac{1}{4C_D^{\beta+3}}\right)^j \lambda(K)$;

(c) $\limsup_{h,k\to+\infty} \|u_{\nu_j(h)} - u_{\nu_j(k)}\|_{L^q(A_j;\lambda)} \leq C_0 2^{1-j}\varepsilon$.

Inequality (b) immediately implies that $\lambda\left(K \setminus \bigcup_{i=1}^\infty A_i\right) = 0$. Working on the diagonal subsequence $(u_{\nu_h(h)})$, we can conclude that

$$
\begin{aligned}
\limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(K;\lambda)} &= \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q\left(\bigcup_{i=1}^\infty A_i;\lambda\right)} \\
&\leq \sum_{i=1}^{\infty} \limsup_{h,k\to+\infty} \|u_{\nu_h(h)} - u_{\nu_k(k)}\|_{L^q(A_i;\lambda)} \leq 2C_0\varepsilon.
\end{aligned} \tag{4.5}
$$

This proves (4.1).

Let us write for simplicity (u_h) instead of $(u_{\nu_h(h)})$. We now prove that, for every compact set $K \subseteq M$, there exists a subsequence (u_{j_h}) of (u_h) such that

$$
\lim_{h,k\to+\infty} \|u_{j_h} - u_{j_k}\|_{L^q(K;\lambda)} = 0. \tag{4.6}
$$

By (4.5), for every $i \in \mathbb{N}$, we can recursively build a subsequence $(u_{\nu_{i+1}(h)})$ of $(u_{\nu_i(h)})$ such that

$$\limsup_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K;\lambda)} \leq \tfrac{2}{i+1} C_0.$$

Then the diagonal sequence $(u_{\nu_h(h)})$ satisfies (4.6).

Eventually, take a sequence (K_j) of compact sets such that $K_j \subseteq \mathrm{int}(K_{j+1})$ and $\bigcup_{j\in\mathbb{N}} K_j = M$. By (4.6), for every $i \in \mathbb{N}$ we can recursively build a subsequence $(u_{\nu_i(h)})$ such that $(u_{\nu_{i+1}(h)})$ is a subsequence of $(u_{\nu_i(h)})$ and

$$\lim_{h,k\to+\infty} \|u_{\nu_i(h)} - u_{\nu_i(k)}\|_{L^q(K_i;\lambda)} = 0.$$

The diagonal subsequence $(u_{\nu_h(h)})$ will then converge to some u in $L^q_{loc}(M;\lambda)$. This concludes the proof. □

Remark 4.1.2. The careful reader will easily notice that Theorem 4.1.1 holds also when assumption (iii) is replaced by the following weaker one:

(iii') For every compact set $K \subseteq M$ there exist $R_P > 0, \alpha \geq 1$ and a function $f : (0,+\infty) \to (0,+\infty)$ such that $\lim_{r\to 0} f(r) = 0$ and

$$\|u_j - u_j(B^j)\|_{L^q(B^j)} \leq f(r)\, \mu_j(\alpha B^j), \qquad \forall x \in K,\ \forall j \in \mathbb{N},\ \forall r \in (0, R_P).$$

4.2 An application to Carnot-Carathéodory spaces

Let $(\mathbb{R}^n, X)$ be a CC space, let Ω be an open set in $\mathbb{R}^n$ and assume that the metric balls are bounded with respect to the Euclidean metric. This implies that the space $(\mathbb{R}^n, X)$ is geodesic, as it has been shown in [75, Theorem 1.4.4].

For $j \in \mathbb{N}$ let $X^j = (X^j_1, \ldots, X^j_m)$ be a family of linearly independent vector fields such that, for every fixed $i = 1, \ldots, m$, X^j_i converges to X_i in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to +\infty$. We denote by d_j, $j \in \mathbb{N}$, the CC distance associated with X^j. If $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \leq 1$, $T > 0$ and $x \in \mathbb{R}^n$, it is convenient to define $\gamma_{h,x}, \gamma^j_{h,x} : [0,T] \to \mathbb{R}^n$ as the AC curves such that $\gamma_{h,x}(0) = \gamma^j_{h,x}(0) = x$ and for almost every $t \in [0,T]$

$$\dot\gamma_{h,x}(t) = \sum_{i=1}^m h_i(t) X_i(\gamma_{h,x}(t)), \qquad \dot\gamma^j_{h,x}(t) = \sum_{i=1}^m h_i(t) X^j_i(\gamma^j_{h,x}(t)).$$

With this notation, an equivalent definition of the CC distance is

$$d(x,y) = \inf\{\|h\|_{L^\infty(0,1)} : h \in L^\infty([0,1];\mathbb{R}^m) \text{ and } \gamma_{h,x}(1) = y\}. \tag{4.7}$$

The boundedness of metric balls implies that, for every $T > 0$ and $h \in L^\infty([0,T];\mathbb{R}^m)$, the curve $\gamma_{h,x}$ is well-defined on $[0,T]$.

As already observed in Remark 1.2.2, if the Chow-Hörmander condition holds, then for every compact set $K \subseteq \mathbb{R}^n$ there exists an integer $s(K)$ such that the following holds:

for any $x \in K$, $X_1, \dots, X_m$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbb{R}^n$. In an analogous way the following fact holds: for any compact set $K \subseteq \mathbb{R}^n$ there exists $J \in \mathbb{N}$ such that, for any $x \in K$ and $j \geq J$, the vector fields $X_1^j, \dots, X_m^j$ and their commutators up to order $s(K)$ computed at x span the whole $\mathbb{R}^n$. The following theorem gives a sort of quantitative version of some of the celebrated results of [78]. The proof of Theorem 4.2.1 follows fairly easily from [18, 73] (see in particular [18, Proposition 5.8 and Claim 3.3]).

Theorem 4.2.1. *For every compact set $K \subseteq \mathbb{R}^n$ there exist $J_0 \in \mathbb{N}$ and $C_K > 0$ such that for every $x, y \in K$ and $j \geq J_0$*

$$\frac{1}{C_K}|x-y| \leq d(x,y) \leq C_K|x-y|^{1/s(K)}$$
$$\frac{1}{C_K}|x-y| \leq d_j(x,y) \leq C_K|x-y|^{1/s(K)}.$$

We aim at proving that the sequence of distances d_j converges to d locally uniformly; we need some preparatory lemmata.

Lemma 4.2.2. *Let K be a compact set in $\mathbb{R}^n$. Then for every $T > 0$, there exist $J_1 = J_1(K,T) \in \mathbb{N}$ and $R = R(K,T) > 0$ such that for every $x \in K$, $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \leq 1$ and any $j \geq J_1$ the following hold:*

(a) the curve $\gamma^j_{h,x}$ is well defined on $[0,T]$;

(b) $\gamma^j_{h,x}([0,T]) \subseteq B_e(0,R)$.

Proof. Define first

$$K' := \{\gamma_{h,x}(T) : x \in K, h \in L^\infty([0,T];\mathbb{R}^m), \|h\| \leq 1\} = \bigcup_{x \in K} \overline{B(x,T)}.$$

Since metric balls are bounded, also K' is bounded. We can therefore find $R > 0$ such that $K' \subseteq B_e(0,R)$ and $d_e(K', \mathbb{R}^n \setminus B_e(0,R)) > 1$. Choose $J_1 \in \mathbb{N}$ such that for every $j \geq J_1$

$$T\left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i|\right) e^{mCT} \leq \frac{1}{2},$$

where $C > 0$ will be determined later. Let $h \in L^\infty([0,T];\mathbb{R}^m)$ and $j \geq J_1$ be fixed; define

$$t_j := \sup\{t > 0 : \gamma^j_{h,x} \text{ is well-defined on } [0,t] \text{ and } \gamma^j_{h,x}([0,t]) \subseteq B_e(0,R)\}$$

and suppose by contradiction that $t_j < T$. Then $\gamma^j_{h,x}(t_j) \in \partial B_e(0,R)$ and for every

$\tau < t_j$ one has

$$\begin{aligned}\left|\gamma^j_{h,x}(\tau) - \gamma_{h,x}(\tau)\right| &\leq \int_0^\tau \sum_{i=1}^m \left|h_i(s)X_i^j(\gamma^j_{h,x}(s)) - h_i(s)X_i(\gamma_{h,x}(s))\right| ds \\ &\leq \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma^j_{h,x}(s)) - X_i^j(\gamma_{h,x}(s))\right| ds \\ &\quad + \int_0^\tau \sum_{i=1}^m \left|X_i^j(\gamma_{h,x}(s)) - X_i(\gamma_{h,x}(s))\right| ds.\end{aligned}$$

Notice that, since X_i^j is converging to X_i locally in C^1, and since $\gamma^j_{h,x}(s), \gamma_{h,x}(s) \in B_e(0,R)$, the Lipschitz constants

$$c_i^j := \sup_{x,y\in B_e(0,R)} \frac{|X_i^j(x) - X_i^j(y)|}{|x-y|}$$

are converging to the Lipschitz constant $c_i := \sup_{x,y\in B_e(0,R)} \frac{|X_i(x)-X_i(y)|}{|x-y|}$. Therefore we can choose $C > 0$ such that $c_i^j, c_i \leq C$ for any $j \in \mathbb{N}$ and $i = 1, \dots, m$, which gives

$$\left|\gamma^j_{h,x}(\tau) - \gamma_{h,x}(\tau)\right| \leq \int_0^\tau \left(mC\left|\gamma^j_{h,x}(s) - \gamma_{h,x}(s)\right| + \sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i| \right) ds.$$

We can therefore apply Grönwall's Lemma to get

$$\left|\gamma^j_{h,x}(t_j) - \gamma_{h,x}(t_j)\right| \leq t_j \left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i| \right) e^{mCt_j} \leq \frac{1}{2}.$$

Notice that $\gamma_{h,x}(t_j) \in K'$ and $\gamma^j_{h,x}(t_j) \in \partial B_e(0,R)$: this contradicts the definition of R, giving $t_j = T$. The lemma is proved. □

Lemma 4.2.3. *Fix $\varepsilon \in (0,1)$ and a compact set K in $\mathbb{R}^n$. Then, for every $T > 0$ there exists $J_2 = J_2(K,T,\varepsilon) \in \mathbb{N}$ such that for every $x \in K$, $j \geq J_2$, $h \in L^\infty([0,T];\mathbb{R}^m)$ with $\|h\| \leq 1$ and $t \in [0,T]$ one has*

$$|\gamma^j_{h,x}(t) - \gamma_{h,x}(t)| \leq \varepsilon$$

Proof. Let $J_1 = J_1(K,T)$ and $R = R(K,T)$ be given by Lemma 4.2.2 and let $C > 0$ be the constant appearing in its proof. We can reason as in Lemma 4.2.2 above and use Grönwall's Lemma to get, for any x, j, h, t as in the statement, that

$$\left|\gamma^j_{h,x}(t) - \gamma_{h,x}(t)\right| \leq t \left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i| \right) e^{mCt}.$$

The proof is then accomplished by choosing $J_2 \geq J_1$ sufficiently large to have

$$T \left(\sum_{i=1}^m \sup_{B_e(0,R)} |X_i^j - X_i| \right) e^{mCT} < \varepsilon.$$

□

Clearly, J_2 can be chosen to be increasing in T, i.e., $J_2(K,T_1,\varepsilon) \le J_2(K,T_2,\varepsilon)$ whenever $0 < T_1 \le T_2$.

Theorem 4.2.4. *Let $X = (X_1,\dots,X_m)$ and $X^j = (X_1^j,\dots,X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1,\dots,m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Then the sequence (d_j) converges to d in $L^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ as $j \to +\infty$.*

Proof. Let $K \subseteq \mathbb{R}^n$ be a fixed compact set.

We first prove that for every $\varepsilon \in (0,1)$ there exists $J_3 = J_3(K,\varepsilon) \in \mathbb{N}$ such that for every $x, y \in K$ and $j \ge J_3$ one has

$$d_j(x,y) \le d(x,y) + \varepsilon.$$

Consider $x, y \in K$; by the existence of geodesics, there exists $h \in L^\infty([0,1];\mathbb{R}^m)$ such that $\|h\|_{L^\infty} = d(x,y)$ and $\gamma_{h,x}(1) = y$. We set $y_j := \gamma^j_{h,x}(1)$ and consider J_0 and $C_K > 0$ as given by Theorem 4.2.1. Define $J_3 := \max\{J_0, J_2(K, \mathrm{diam}_d K, (\varepsilon/C_K)^{s(K)})\}$. Then, by Lemma 4.2.3, for $j \ge J_3$ we have

$$|y_j - y| = |\gamma^j_{h,x}(1) - \gamma_{h,x}(1)| \le \left(\frac{\varepsilon}{C_K}\right)^{s(K)}.$$

By Theorem 4.2.1 we deduce that $d_j(y_j, y) \le \varepsilon$; in particular, for any $j \ge J_3$ one has

$$d_j(x,y) \le d_j(x,y_j) + d_j(y_j,y) \le d(x,y) + \varepsilon, \tag{4.8}$$

as claimed. Notice also that $\sup_{j \ge J_3} \mathrm{diam}_{d_j} K \le \mathrm{diam}_d K + 1 =: L$ is finite.

We now prove that for any $x, y \in K$ and $\varepsilon \in (0,1)$ there exists $J_4 = J_4(K,x,y,\varepsilon) \in \mathbb{N}$ such that for every $j \ge J_4$

$$d(x,y) \le d_j(x,y) + \varepsilon. \tag{4.9}$$

For every $j \ge J_3$ let $h^j \in L^\infty([0,1];\mathbb{R}^m)$ be such that

$$\gamma^j_{h^j,x}(1) = y \qquad \text{and} \qquad \|h^j\|_{L^\infty} = d_j(x,y) \le L.$$

The sequence $(h^j)_j$ is bounded in L^∞ and therefore there exists a subsequence (h^{j_ℓ}) and $h \in L^\infty([0,1];\mathbb{R}^m)$ such that

$$h^{j_\ell} \overset{*}{\rightharpoonup} h \text{ in } L^\infty \qquad \text{and} \qquad \lim_{\ell\to\infty} \|h^{j_\ell}\|_{L^\infty} = \liminf_{j\to\infty} \|h^j\|_{L^\infty} = \liminf_{j\to\infty} d_j(x,y).$$

Denoting by $\gamma^{j_\ell} := \gamma^{j_\ell}_{h^{j_\ell},x}$ and considering $R = R(K,L) > 0$ as given by Lemma 4.2.2, one has $\gamma^{j_\ell}([0,1]) \subseteq B_e(0,R)$. Since X_i^j are converging to X_i uniformly in C^∞ $(i = 1,\dots,m)$, such vector fields are equibounded on $B_e(0,R)$. By Ascoli-Arzelà

Theorem, up to a further subsequence, there exists a curve $\gamma \in AC([0,1],\mathbb{R}^n)$ such that γ^{j_ℓ} uniformly converges to γ in $[0,1]$ as $\ell \to \infty$. For every $t \in [0,1]$ one has

$$\gamma^{j_\ell}(t) = x + \int_0^t \sum_{i=1}^m h_i^{j_\ell}(s) X_i^{j_\ell}(\gamma^{j_\ell}(s))\, ds$$

and, taking into account that $X_i^{j_\ell} \circ \gamma^{j_\ell} \to X_i \circ \gamma$ uniformly in $[0,1]$ and that $h^j \overset{*}{\rightharpoonup} h$ in L^∞, by letting $\ell \to \infty$ one gets

$$\gamma(t) = x + \int_0^t \sum_{i=1}^m h_i(s) X_i(\gamma(s))\, ds.$$

In particular $\gamma = \gamma_{h,x}$, $\gamma(1) = y$ and

$$d(x,y) \le \|h\|_{L^\infty} \le \liminf_{\ell\to\infty} \|h_{j_\ell}\|_{L^\infty} = \liminf_{j\to\infty} d_j(x,y),$$

which proves (4.9).

By the compactness of K we can find $x_1,\dots,x_k \in K$ such that $K \subseteq \bigcup_{\ell=1}^k B(x_\ell,\varepsilon)$. Using Theorem 4.2.1 and (4.9) we can find $\widetilde{C} = \widetilde{C}(K) > 0$ and $J_5 = J_5(K,\varepsilon) \in \mathbb{N}$ such that for $j \ge J_5$

$$\begin{aligned} &B(x_\ell,\varepsilon) \subseteq B^j(x_\ell, \widetilde{C}\varepsilon^{1/s(K)}) && \forall\, \ell = 1,\dots,k \\ &d(x_{\ell_1},x_{\ell_2}) \le d_j(x_{\ell_1},x_{\ell_2}) + \varepsilon && \forall\, \ell_1,\ell_2 = 1,\dots,k. \end{aligned}$$

For every $x,y \in K$ we can find $x_{\ell_1}, x_{\ell_2} \in K$ (with $1 \le \ell_1,\ell_2 \le k$) such that $x \in B(x_{\ell_1},\varepsilon)$ and $y \in B(x_{\ell_2},\varepsilon)$, hence for $j \ge J_5$ we have

$$\begin{aligned} d(x,y) &\le d(x,x_{\ell_1}) + d(x_{\ell_1},x_{\ell_2}) + d(y,x_{\ell_2}) \\ &\le \varepsilon + d_j(x_{\ell_1},x_{\ell_2}) + \varepsilon + \varepsilon \\ &\le d_j(x_{\ell_1},x) + d_j(x,y) + d_j(y,x_{\ell_2}) + 3\varepsilon \\ &= d_j(x,y) + 3\varepsilon + 2\widetilde{C}\varepsilon^{1/s(K)}, \end{aligned}$$

which, combined with (4.8), concludes the proof. □

Theorem 4.2.5 below gives a uniform Poincaré inequality when the moving vector fields are converging. The proof follows directly from [18, Theorem 7.2 and considerations above].

Theorem 4.2.5. *Let $X = (X_1,\dots,X_m)$ and $X^j = (X_1^j,\dots,X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1,\dots,m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Then, for every compact set $K \subseteq \mathbb{R}^n$ there exist $C_P > 1$, $\alpha \ge 1$, $R_P > 0$ and $J \in \mathbb{N}$ such that for every $j \ge J$, $u \in BV_{X^j,loc}(\mathbb{R}^n)$, $x \in K$ and $r \in (0,R_P)$ one has*

$$\int_{B^j} \left|u - u(B^j)\right| d\mathscr{L}^n \le C_P\, r\, |D_{X^j} u|(\alpha B^j), \tag{4.10}$$

where $B^j := B^j(x,r)$ and $u(B^j) = ⨍_{B^j} u\, d\mathscr{L}^n$.

We can then state our main application. See [46, Section 8] for more references about compactness results for Sobolev or BV functions in CC spaces.

Theorem 4.2.6. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that X satisfies the Chow-Hörmander condition and its CC balls are bounded in $\mathbb{R}^n$; assume that, for every $i = 1, \dots, m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Let $u_j \in BV_{X^j,loc}(\mathbb{R}^n)$ be a sequence of functions that is locally uniformly bounded in BV, i.e., such that for any compact set $K \subseteq \mathbb{R}^n$ there exists $M > 0$ such that for any $j \in \mathbb{N}$ one has*

$$\|u_j\|_{L^1(K)} + |D_{X^j} u_j|(K) \le M.$$

Then, there exist $u \in BV_{X,loc}(\mathbb{R}^n)$ and a subsequence (u_{j_h}) of (u_j) such that $u_{j_h} \to u$ in $L^1_{loc}(\mathbb{R}^n)$ as $h \to \infty$. Moreover, for any bounded open set $\Omega \subseteq \mathbb{R}^n$, the semicontinuity inequality

$$|D_X u|(\Omega) \le \liminf_{j\to\infty} |D_{X^j} u_j|(\Omega)$$

holds.

Proof. We use Theorem 4.1.1 with $X = \mathbb{R}^n$, $\lambda = \mathscr{L}^n$, $\delta = q = 1$, $\mu_j := |D_{X^j} u|$ and d, d_j the CC distances associated with X, X^j respectively. Assumption (i) follows from Theorem 4.2.4, while the local doubling property (ii) of d is a well-known fact (see e.g. [78]). The validity of (iii) (with $\delta = q = 1$) follows from Theorem 4.2.5, while (iv) is satisfied by assumption.

Theorem 4.1.1 ensures that, up to subsequences, u_j converges to some u in $L^1_{loc}(\mathbb{R}^n)$; we need to show that $u \in BV_{X,loc}(\mathbb{R}^n)$. To this aim, for any $i = 1, \dots, m$, we denote by X_i^* the formal adjoint to X_i and write

$$X_i(x) = \sum_{k=1}^n a_{i,k}(x)\partial_k \qquad \text{and} \qquad X_i^j(x) = \sum_{k=1}^n a_{i,k}^j(x)\partial_k,$$

for suitable smooth functions $a_{i,k}, a^j_{i,k}$. Then, for any bounded open set $\Omega \subseteq \mathbb{R}^n$, any test function $\varphi \in C^1_c(\Omega)$ and any $i = 1, \dots, m$, we have

$$\begin{aligned}\int_\Omega u\, X_i^* \varphi \, d\mathscr{L}^n = \int_\Omega u \sum_{k=1}^n \partial_k(a_{i,k}\varphi)\, d\mathscr{L}^n &= \lim_{j\to\infty} \int_\Omega u_j \sum_{k=1}^n \partial_k(a^j_{i,k}\varphi)\, d\mathscr{L}^n \\ &= -\lim_{j\to\infty} \int_\Omega \varphi \, dD_{X_i^j} u_j \le \|\varphi\|_{L^\infty(\Omega)} \liminf_{j\to\infty} |D^j_X u_j|(\Omega) < +\infty.\end{aligned}$$

This proves that $u \in BV_{X,loc}(\mathbb{R}^n)$ as well as the semicontinuity of the total variation. The proof is accomplished. □

Proposition 4.2.7. *Let $X = (X_1, \dots, X_m)$ and $X^j = (X_1^j, \dots, X_m^j)$, $j \in \mathbb{N}$, be m-tuples of linearly independent smooth vector fields on $\mathbb{R}^n$ such that, for every $i =$*

$1,\dots,m$, $X_i^j \to X_i$ in $C^\infty_{loc}(\mathbb{R}^n)$ as $j\to\infty$. Let (u_j) be a sequence converging in $L^1_{loc}(\mathbb{R}^n)$ to some u; then, for any open bounded set $\Omega\subseteq\mathbb{R}^n$ one has

$$|D_X u|(\Omega) \le \liminf_{j\to\infty} |D_{X^j} u_j|(\Omega)$$

Proof. For any $i=1,\dots,m$ and any $j\in\mathbb{N}$ we write

$$X_i(x) = \sum_{k=1}^n a_{i,k}(x)\partial_k \qquad \text{and} \qquad X_i^j(x) = \sum_{k=1}^n a^j_{i,k}(x)\partial_k,$$

for suitable smooth functions $a_{i,k}, a^j_{i,k}$. Then, for any test function $\varphi\in C^1_c(\Omega;\mathbb{R}^m)$, we have

$$\begin{aligned}\int_\Omega u\mathrm{div}_X\varphi\, d\mathscr{L}^n &= \int_\Omega u\sum_{i=1}^m\sum_{k=1}^n \partial_k(a_{i,k}\varphi_i)\, d\mathscr{L}^n \ = \ \lim_{j\to\infty}\int_\Omega u_j \sum_{i=1}^m\sum_{k=1}^n \partial_k(a^j_{i,k}\varphi_i)\, d\mathscr{L}^n \\ &= \lim_{j\to\infty}\int_\Omega u_j \mathrm{div}_{X^j}\varphi\, d\mathscr{L}^n \ \le \ \|\varphi\|_{L^\infty(\Omega)}\liminf_{j\to\infty}|D_{X^j}u_j|(\Omega).\end{aligned}$$

The proof is accomplished. □

Remark 4.2.8. Let X, X^j, u_j, u be as in Proposition 4.2.7 and assume that $|D_{X^j}u^j|$ are locally uniformly bounded in $\mathbb{R}^n$, i.e., for any compact set $K\subseteq\mathbb{R}^n$ there exists $C_K<\infty$ such that $|D_{X^j}u^j|(K)<C_K$ for all j. Then $D_{X^j}u^j$ weakly* converges to D_Xu in $\mathbb{R}^n$.

Indeed, one can reason as in Proposition 4.2.7 to show that for any test function $\varphi\in C^1_c(\mathbb{R}^n)$ and any $i=1,\dots,m$

$$\lim_{j\to\infty}\int \varphi\, dD_{X_i^j}u^j = \int\varphi\, dD_{X_i}u$$

and the density of C^1_c in C_c allows to conclude.

Remark 4.2.9. We conjecture that, when the CC space $(\mathbb{R}^n, X)$ is *equiregular*, the convergence $u_{j_h}\to u$ in Theorem 4.2.6 holds in L^q_{loc} for any $q\in[1,\frac{Q}{Q-1})$, where Q is the Hausdorff dimension of $(\mathbb{R}^n,X)$. This would easily follow in case the Poincaré inequality (4.10) could be strengthened to

$$\|u-u(B^j)\|_{L^q(B^j)}\le C_P\, r^\delta\, |D_{X^j}u|(\alpha B^j)$$

for some $\delta>0$ (arguably, $\delta=\frac{Q}{q}+1-Q$). The key point would be proving that the constant C_P can be chosen independent of j but, as far as we know, no investigation in this direction has been attempted in the literature, so far.

Remark 4.2.10. Theorems 4.2.4, 4.2.5 and 4.2.6 hold also under a slightly weaker assumption: it is indeed enough that, for any compact set $K\subseteq\mathbb{R}^n$, the convergence $X_i^j\to X_i$ holds in $C^k(K)$ for a suitable $k=k(K)$ (actually, k depends only on $s(K)$) that one could explicitly compute. See [18, 73] for more details.

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