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Geometry and dynamics of affine nonholonomic rolling problems

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Abstract

This thesis is concerned with nonholonomic mechanical systems with affine constraints in the velocities.

We introduce a class of examples which provide an affine generalization of the nonholonomic problem of a convex body rolling without slipping on the plane. We investigate dynamical aspects of the system such as existence of first integrals, smooth invariant measure and integrability, giving special attention to the cases in which the convex body is a dynamically balanced sphere or a body of revolution.

We provide a framework which allows us to develop a rigorous modeling of the ANAIS billiard [53] and its generalizations. The framework concerns a class of hybrid systems (in which the constraints are piecewise smooth in the velocities) and allows us to give a proof of the phenomenon observed in [53] based on general results on existence of first integrals, reversibility and symmetry of affine nonholonomic systems. We prove that analogous phenomena occur in other examples.

Introduction

This thesis concerns mechanical systems with constraints in the velocities which cannot be reduced to constraints in the configuration variables. Such systems are called *nonholonomic systems*. They arise, for instance, in mechanical systems that involve rolling contact and are relevant in robotics and other engineering applications. The main difficulty in the study of nonholonomic systems is that their equations of motion do not arise from a variational principle and, as a consequence, do not possess a Hamiltonian structure. Because of this, the current understanding of several aspects of the dynamics, like the role of continuous symmetries, the existence of an invariant measure, first integrals, reversibility, integrability and/or chaotic behavior is quite poor, especially in comparison with their Hamiltonian counterpart. These problems have been the subject of extensive investigation in recent decades. Concrete mechanical examples have shown to be quite helpful in this area, both illustrating and motivating the research.

In the nonholonomic setting, systems with affine constraints have received less attention than those with linear constraints, which has led to a less developed knowledge of the dynamics' features. In contrast with the linear case, in the affine setting, there is no general class of examples to guide or illustrate such investigations. We aim to bridge this gap by introducing and studying a broad class of examples of affine nonholonomic rolling bodies, with the idea that exhibiting a variety of dynamical phenomena could guide the development of the theoretical aspects.

After recalling the necessary background and developing in detail some of the (known) examples that we will use throughout the thesis in Part 0, we present the original results in Parts I and II. The contents of these Parts is outlined below.

Contents of Part I. In this part we introduce and study a wide class of examples of affine nonholonomic systems which generalize the classical problem of a convex body which rolls without slipping on the plane. The generalization consists in the definition of a pair of smooth vector fields, one on the plane where the rolling takes place, and another on the surface of the convex body, that specify the velocity of the contact point of the body with the plane, as illustrated in Figure 1. Special choices of these vector fields recover examples which have been previously considered in the literature, for example [54, 63, 5, 6, 45].

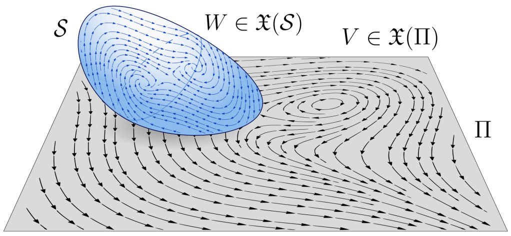


Figure 1: Graphic representation of the vector fields V on the plane Π and W on the surface $\mathcal{S}$ of the convex body.

The first results of the chapter concern the derivation of the equations of motion in all generality and their reduction in the presence of a particular class of symmetries. After this, the existence of a class of first integrals, termed *moving energies* in [31], is analyzed in terms of the symmetry properties of the underlying vector fields. The discussion then focuses on specific choices of the convex rigid body as follows:

1. The case of a dynamical balanced ball, i.e. a sphere whose center of mass coincides with its geometric center, is treated first. The resulting system is a generalization of the celebrated Chaplygin sphere, which is a classic integrable nonholonomic system [19]. We show that, regardless of the choice of vector fields, the system always possesses three independent first integrals. We then analyze the dynamics of the system for a simple particular choice of vector fields. Moreover, we show that it is integrable on the critical level sets of the integrals and instead, using an appropriate Poincare map, we give numerical evidence that the dynamics is chaotic on the generic regular level sets of the first integrals.

2. We next treat the case of a body of revolution. Assuming that the vector field on the plane vanishes and the one on the body possesses the same axial symmetry, we show that the dynamics is integrable by proving existence of a sufficiently large number of first integrals and explicitly indicating the symmetries and an invariant measure.

3. Finally, we consider the case in which the convex body is a homogeneous sphere. After proving a general result giving sufficient conditions on the vector fields on the plane and the sphere to guarantee existence of an

invariant measure, we analyze in detail the dynamics for a simple particular choice of vector fields, identifying both regular and chaotic behaviors on the phase space.

A major portion of this part's contents are contained in [22].

Contents of Part II. This part develops a geometric framework to explain a surprising phenomenon reported by Lévy-Leblond [53] regarding the motion of a homogeneous sphere that rolls without slipping on a plane with a circular uniformly rotating platform, known as the ANAIS billiard phenomenon. If the sphere is set to roll on the fixed part of the plane and towards the rotating platform, it will go outside the platform and back to the fixed part of the plane following a trajectory that is the exact prolongation of the initial one, see Figure 2.

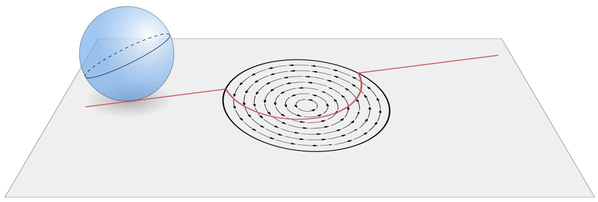


Figure 2: Graphic representation of the ANAIS billiard phenomenon.

The modeling of the system is done extending the results of Part I by allowing the vector field on the plane to be piecewise smooth. A critical issue is to propose a reasonable postulate to model the discontinuities in the velocities on the transition of the rolling regimes (the fixed part of the plane and the rotating platform). Such postulate was already tacitly assumed in [53]. We provide a justification for the postulate by taking smooth approximations of the system and applying the results about the existence of first integrals described in Part I. Specifically, the postulate assumes that the limit of the first integrals are first integrals of the limit system. This reasoning allows us to formulate a more general postulate that enables us to model a wide class of affine nonholonomic systems with piecewise smooth constraints in the velocities.

We then proceed to formulate general results about the reversibility and symmetries of nonholonomic systems with affine constraints and, in particular, of our class of examples. These results, combined with the specifics of the model described above, allow us to identify the mechanisms responsible for the phenomenon reported in [53] and to prove that it also holds for new concrete examples. This part of the thesis was inspired by Section 6 in [53]. A description of [53] and how our work is related to it is given in Appendix D.

Part 0

Preliminaries

1 Introduction to nonholonomic systems

In this chapter, we will give a brief introduction into the most important concepts in the nonholonomic systems. These arguments are well-known and can be found, for example, in [8, 56, 23].

1.1 Configuration manifold and phase space

A *configuration manifold*, denoted by Q, is a smooth n-dimensional manifold whose points label the possible positions of a mechanical system. A set of coordinates $(q^1, \ldots, q^n)$ in Q are termed *generalized coordinates*. The tangent bundle TQ of the configuration manifold Q, has induced bundle coordinates

$$(q, \dot{q}) = ((q^1, \ldots, q^n), (\dot{q}^1, \ldots, \dot{q}^n)).$$

Here, the points $q \in Q$ specify the positions and the vectors $\dot{q} \in T_qQ$ specify the velocities of the system. The space TQ is called the *velocity phase space* of the system.

1.2 Constraints

Some systems may have restrictions in positions and velocities, these are called *constraints*. Mechanical systems that are subject to constraints on their positions are known as *holonomic systems*. Instead, *nonholonomic systems* are those restricted by constraints on the system's velocities that do not arise from constraints in the position. These types of constraints are called *nonholonomic constraints*. For example, the length of the string of the simple pendulum is a holonomic constraint, while the constraint of a sphere rolling without slipping on the plane is a nonholonomic constraint.

The difference between holonomic and nonholonomic constraints can be interpreted geometrically in the following way. A curve in the system's configuration space represents a motion of the system, if the system is holonomic, then any curve in Q specifies a motion of the system. However, this is not the case in nonholonomic systems, by restricting the possible velocities of the system, we are restricting the possible tangent vectors $\dot{q} \in T_qQ$ at each point q of the configuration manifold Q. If the constraints are linear and homogeneous in the velocities, the set of possible tangent vector spans a subspace of the tangent space T_qQ and every curve that represents a motion of the system is tangent to that subspace. The set of these subspaces is in fact an important geometrical object which we define as follows.

Definition 0.1.1. Let Q be a smooth manifold. A regular *distribution* $\mathcal{D}$ on Q of rank r is a subbundle of TQ of rank r.

A rank-r distribution may be described by specifying for each $q \in Q$ a linear subspace $\mathcal{D}_q \subset T_qQ$ of dimension r, and letting $\mathcal{D} = \bigcup_{q\in Q} \mathcal{D}_q$. It follows that $\mathcal{D}$ is a smooth distribution if and only if each point $q \in Q$ has a neighborhood U on which there are smooth vector fields $X_1, \dots, X_r : U \to Q$ such that

$$\operatorname{span}\{X_1(q), \dots, X_r(q)\} = \mathcal{D}_q$$

for each $q \in U$. If a system has $n - r$ independent constraints,

$$\beta^a_j(q)\dot{q}^j = 0 \qquad \text{with } a = 1, \dots, n - r,^1 \tag{0.1.1}$$

then its constraint distribution can be written as

$$\mathcal{D} = \{(q, \dot{q}) \in TQ \,:\, (q, \dot{q}) \text{ satisfy (0.1.1)}\},$$

and each fiber satisfies,

$$\mathcal{D}_q = \{\dot{q} \in T_qQ \,:\, \beta(q)\dot{q} = 0\}, \tag{0.1.2}$$

where $\beta(q)$ denotes the $(n-r) \times n$ matrix whose a^{th} row has entries $\beta^a_j(q)$. If the constraints (0.1.1) can be integrated and expressed only in terms of the position, then they are holonomic. In this case, they define a submanifold N on Q and every point on N has tangent space $T_qN = \mathcal{D}_q$. We give the following definitions.

Definition 0.1.2. Let $\mathcal{D} \subset TQ$ be a smooth distribution. A nonempty immersed

[1] Unless otherwise stated, Einstein's convention of sum over repeated indices holds.

submanifold $N \subset M$ is called an *integral manifold of* $\mathcal{D}$ if $T_qN = \mathcal{D}_q$ at each point $q \in N$.

Definition 0.1.3. A distribution $\mathcal{D}$ on Q is said to be *integrable* if each point of Q is contained in an integral manifold.

Nonholonomic constraints are constraints in the velocities which cannot be integrated and expressed as constraints only on the positions. Therefore, nonholonomic constraints are given by nonintegrable distributions.

1.3 Affine nonholonomic constraints

In the previous section we assumed that the constraints (0.1.1) were linear homogeneous in the velocities, but we can also consider nonholonomic constraints that are nonhomogeneous or *affine* in the velocities, these can be written as

$$\beta^a_j(q)\dot{q}^j = K^a(q) \qquad \text{with } a = 1\ldots, n-r. \tag{0.1.3}$$

If the system has affine constraints, the velocities are restricted at each point $q \in Q$ to an affine, instead of linear, subspace $\mathcal{A}_q \subset T_qQ$. Hence, affine constraints determine an affine subbundle $\mathcal{A}$ of the tangent bundle TQ, which can be described by

$$\mathcal{A} = \{(q,\dot{q}) \in TQ \,:\, (q,\dot{q}) \text{ satisfy (0.1.3)}\,\}.$$

An affine distribution can be expressed as $\mathcal{A} = \mathcal{D} + Z$, where $\mathcal{D} \subset TQ$ is a linear distribution that satisfies (0.1.2) and $Z \in \mathfrak{X}(Q)$ is a vector field on the configuration manifold Q such that

$$\beta(q)Z(q) = K(q). \tag{0.1.4}$$

Each fiber of $\mathcal{A}$ satisfies

$$\mathcal{A}_q = \mathcal{D}_q + Z(q).$$

Notice that given $\mathcal{D}$ and $\mathcal{A}$, then the vector field Z is determined up to sections of $\mathcal{D}$. Clearly, if $Z(q) \in \mathcal{D}_q$ for every $q \in Q$, then $\mathcal{A} = \mathcal{D}$ is a linear distribution.

1.4 The Lagrange-D'Alembert principle

The *Lagrangian* of a system is a function $L = L(q,\dot{q}) : TQ \to \mathbb{R}$, which we will assume to be given by the kinetic minus the potential energy of the system. In coordinates,

$$L(q,\dot{q}) = T(\dot{q}) - V(q) = \frac{1}{2}\dot{q}^T A(q)\dot{q} - V(q), \tag{0.1.5}$$

where the *kinetic energy matrix* $A(q)$ is symmetric and positive definite. In systems with holonomic constraints, upon definition of an adequate configuration manifold Q, the motion of the system can be described by the *Euler-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \dots, n.$$

These equations are equivalent to a *variational principle* on a space of smooth curves with fixed endpoints. Take a curve $q : [a, b] \to Q$ that joins two fixed points in Q over a fixed time interval $[a, b]$. Then $q(t)$ is a solution of the Euler-Lagrange equations if and only if it is a stationary point of an action functional. Namely,

$$\delta \int_a^b L(q(t), \dot{q}(t))dt = 0.$$

In nonholonomic systems, the constraints give rise to a reaction force R which depends on positions and velocities, so $R = (q, \dot{q})$, and so the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = R^i(q, \dot{q}) \qquad i = 1, \dots, n, \tag{0.1.6}$$

where R^i are the components of the reaction force R of the nonholonomic constraints, and L is the Lagrangian defined in (0.1.5). To obtain an expression of R, we assume that the nonholonomic constraint satisfy a physical principle known as the *Lagrange D'Alembert principle of ideal constraints*. It states that **the reaction force annihilates any possible displacement of the system**. In other words, if $(q, \dot{q}) \in \mathcal{D}$, then $R(q, \dot{q}) \cdot \dot{q} = 0$. Under this assumption, the reaction force R is uniquely determined as a function of $(q, \dot{q}) \in \mathcal{D}$. In the following section, we show how it can be obtained explicitly.

1.4.1 Determination of the reaction force

We follow [30, 40, 1] to give the coordinate description of the equations of motion of a nonholonomic system with affine constraints. Suppose that the system has $n - r$ independent constraints that are affine on the velocities and are given by (0.1.3). These constraints determine an affine constraint distribution $\mathcal{A} = \mathcal{D} + Z$, where the linear distribution $\mathcal{D}$ is defined as (0.1.2) and $Z \in \mathfrak{X}(Q)$ satisfies (0.1.4).

From the Lagrange-D'Alembert principle, $R = R(q, \dot{q})$ must be a linear combination of the rows of $\beta(q)$. Hence $R = \beta(q)^T \lambda$ for a vector $\lambda = \lambda(q, \dot{q}) \in \mathbb{R}^{n-r}$ that we determine next. Differentiating (0.1.3) yields

$$\beta(q)\ddot{q} + \gamma(q, \dot{q}) - \xi(q, \dot{q}) = 0, \tag{0.1.7}$$

where $\gamma(q,\dot q), \xi(q,\dot q) \in \mathbb{R}^{n-r}$ have components

$$\gamma^a(q,\dot q) = \frac{\partial \beta^a_k}{\partial q^j}\dot q^j \dot q^k, \qquad \xi^a(q,\dot q) = \frac{\partial K^a}{\partial q^j}\dot q^j. \tag{0.1.8}$$

Using the expression of the Lagrangian L in (0.1.5), we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = A(q)\ddot q + \eta(q,\dot q) + V'(q)$$

where the components of the vectors $\eta(q,\dot q)$, $V'(q) \in \mathbb{R}^n$ are

$$\eta_i(q,\dot q) = \left(\frac{\partial A_{ij}}{\partial q^k}(q) - \frac{1}{2}\frac{\partial A_{jk}}{\partial q^i}(q)\right)\dot q^j \dot q^k, \qquad V'_i(q) = \frac{\partial V}{\partial q^i}(q). \tag{0.1.9}$$

Therefore (0.1.6) becomes

$$A(q)\ddot q + \eta(q,\dot q) + V'(q) = \beta(q)^T\lambda$$

Multiplying both sides of the equation by $\beta(q)A^{-1}(q)$ and using (0.1.7) yields

$$(\beta(q)A^{-1}(q)\beta(q)^T)\lambda = \beta(q)A^{-1}(q)\eta(q,\dot q) + \beta(q)A^{-1}(q)V'(q) - \gamma(q,\dot q) + \xi(q,\dot q).$$

The matrix $\beta(q)A^{-1}(q)\beta(q)^T$ is invertible, since β has full rank. Therefore,

$$\begin{aligned}\lambda(q,\dot q) =& (\beta(q)A^{-1}(q)\beta(q)^T)^{-1}(\beta(q)A^{-1}(q)\eta(q,\dot q) \\ &+ \beta(q)A^{-1}(q)V'(q) - \gamma(q,\dot q) + \xi(q,\dot q)).\end{aligned}$$

Introducing the $(n-r)\times(n-r)$ matrix

$$\Sigma(q) = (\beta(q)A^{-1}(q)\beta(q)^T)^{-1},$$

we obtain the following explicit expression for $R(q,\dot q)$:

$$R(q,\dot q) = \beta(q)^T\Sigma(q)(\beta(q)A^{-1}(q)(\eta(q,\dot q) + V'(q)) - \gamma(q,\dot q) + \xi(q,\dot q)). \tag{0.1.10}$$

The equations of motion become

$$A(q)\ddot q + \eta(q,\dot q) + V'(q) = R(q,\dot q), \tag{0.1.11}$$

with $R(q,\dot q)$ given by (0.1.10), $\eta(q,\dot q)$ by (0.1.9), and $\gamma(q,\dot q)$ and $\xi(q,\dot q)$ by (0.1.8). By construction, the constraint functions

$$\tilde\phi^a(q,\dot q) = \beta^a_i(q)\dot q^i - K^a(q)$$

are first integrals of (0.1.11). The restriction of (0.1.11) to the invariant manifold determined by the constraints (0.1.3) gives our desired equations of motion

1.4.2 Invariance and reversibility properties of the equations of motion

Consider a nonholonomic system on a configuration manifold Q with Lagrangian L given by (0.1.5) and constraint distribution $\mathcal{A} = \mathcal{D} + Z \subset TQ$ with $\mathcal{D}$ a linear distribution and $Z \in \mathfrak{X}(Q)$ a vector field. As explained above, the system has an equation of motion given by the restriction to $\mathcal{A}$ of (0.1.11). Now let $\Psi : Q \to Q$ be a diffeomorphism on the configuration manifold Q and let us denote by $\overline{\Psi} : TQ \to TQ$ its tangent lift defined as

$$\overline{\Psi}(q, \dot{q}) = (\Psi(q), \Psi'(q)\dot{q}),$$

with $\Psi' = \frac{\partial \Psi}{\partial q}$. Suppose that the Lagrangian L is invariant under $\overline{\Psi}$ and the linear distribution $\mathcal{D}$ is invariant under Ψ, i.e.

$$L \circ \overline{\Psi} = L \qquad \text{and} \qquad T_q\Psi(\mathcal{D}_q) = \mathcal{D}_{\Psi(q)} \text{ for all } q \in Q.$$

Suppose, moreover, that the vector field Z satisfies

$$\Psi_* Z = sZ, \qquad \text{with } s = \pm 1.$$

In other words, Z is Ψ-invariant if $s = 1$ and Ψ-reversible (see Chapter 3) if $s = -1$. The following result, taken from [40], whose proof is given in Appendix A, shows that if $s = 1$ then Ψ maps solutions into solutions of the nonholonomic dynamics, and instead that the nonholonomic dynamics is $\overline{\Psi}$-reversible if $s = -1$.

Theorem 0.1.4. *Let $\Psi : Q \to Q$ be a diffeomorphism such that its tangent lift $\overline{\Psi}$ preserves the Lagrangian L and the linear distribution $\mathcal{D}$. Assume that $\Psi_* Z = sZ$ for $s = \pm 1$. Suppose that $t \mapsto q_t$ is a solution of the equations of motion* (0.1.11) *satisfying* (0.1.3) *and let $\tilde{q}_t = \Psi(q_t)$. Then,*

1. *for $s = 1$, the curve $\tilde{q}_t$ is a solution of* (0.1.11) *satisfying* (0.1.3).
2. *for $s = -1$ the curve $\hat{q}_t = \tilde{q}_{-t}$ is a solution of* (0.1.11) *satisfying* (0.1.3).

2 Nonholonomic systems with linear constraints

There are important differences between holonomic and nonholonomic systems. We have mentioned that equations of motion of nonholonomic systems do not come from a variational principle in the strict sense [1]. The fact that nonholonomic systems are nonvariational implies that they do not fit within the Hamiltonian setting. This has consequences in the way nonholonomic systems behave in comparison to their holonomic counterpart. For instance, in nonholonomic systems, symmetries do not always lead to conservation laws as with the classical Noether theorem, therefore, even though there are some nonholonomic generalizations of Noether's theorem [51, 2, 8, 34, 32], the relationship between symmetries and first integrals is more complicated. Moreover, unlike in the holonomic setting, volume may not be preserved in the phase space of nonholonomic systems. This may lead to asymptotic stability in some cases [10].

Some important examples of nonholonomic systems with linear constraints regard rolling bodies. In the following section, we will restrict our attention to the sphere rolling on a fixed plane, but other examples concern the motion of a body rolling on a sphere [12] or the motion of a sphere rolling on a surface [13].

2.1 The sphere rolling on a fixed plane

The sphere rolling without slipping on a fixed plane is one of the most classical examples of a nonholonomic system. The most general case is that of an unbalanced and nonsymmetric sphere, usually known as *Chaplygin's top*, considered in [11, 12]. This system is generally chaotic, but in the presence of additional

[1]The equations of motion of nonholonomic systems are obtained from the condition of minimum of action with additional restrictions imposed on the variations.

symmetries it may be integrable. For instance, the case of an axially-symmetric sphere with displaced center of mass, studied by Routh [59] and Chaplygin [19], the case of a balanced sphere, studied by Chaplygin [20], and the case of the homogeneous sphere, considered in [56], are well-known integrable cases. Here we will present the most general system and then focus on the particular integrable cases. The description of these systems, their first integrals and invariant measures can be found in [12], where one can also find more historical references.

2.1.1 Description of the system

For this part, we follow the approach in [40]. Consider the motion of a sphere of radius r and mass m rolling without slipping on a horizontal plane. We fix a spatial frame $\Sigma_s = \{O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the horizontal plane Π contains the origin O and is spanned by the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$. We fix a body frame $\Sigma_b = \{C; \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ whose origin is the center of mass C of the sphere and such that the vectors $\boldsymbol{E}_i$ are aligned with its principal axes of inertia. We denote the inertia tensor of the sphere with respect to our choice of body axis by

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

In the general case, the three moments of inertia of the sphere are different, i.e. $I_1 \neq I_2 \neq I_3$.

A configuration of the system is defined by the position of the sphere on the plane and the orientation of the sphere. Therefore, the configuration manifold of the system is $\mathbb{R}^2 \times \mathrm{SO}(3)$ and a configuration is specified by the pair $(\boldsymbol{x}, B)$, where $\boldsymbol{x} = (x_1, x_2, 0) \in \mathbb{R}^3$ is the vector from the origin O to the point of contact, and $B \in \mathrm{SO}(3)$ is the attitude matrix that determines the orientation of the sphere (B is the change of basis matrix between $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{E}_i\}$ of $\mathbb{R}^3$).

Now we define the following points and vectors, illustrated in Fig. 2.1

- C' is the geometric center of the sphere
- C is the center of mass of the sphere
- O is the origin of the space frame
- P is the contact point
- $\boldsymbol{u} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OC}$, connecting the origin of the spatial frame and the center of mass, with respect to the spatial frame Σ_s
- $\boldsymbol{x} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OP}$, connecting the origin of the spatial frame and the contact point, with respect to the spatial frame Σ_s

- $\boldsymbol{\rho} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{CP}$, connecting the center of mass and the contact point, with respect to the body frame Σ_b
- $\boldsymbol{\gamma} = B^{-1}\boldsymbol{e}_3 \in \mathbb{R}^3$ are the coordinates of the unitary vector normal to the plane at the contact point P, with respect to the body frame Σ_b

Notice that the following relations are satisfied

$$\boldsymbol{\rho} = -r\boldsymbol{\gamma} - l\boldsymbol{E}_3, \tag{0.2.1}$$

where $l > 0$ denotes the distance between the geometric center of the sphere C' and its center of mass C, and

$$\boldsymbol{x} = \boldsymbol{u} + B\boldsymbol{\rho}. \tag{0.2.2}$$

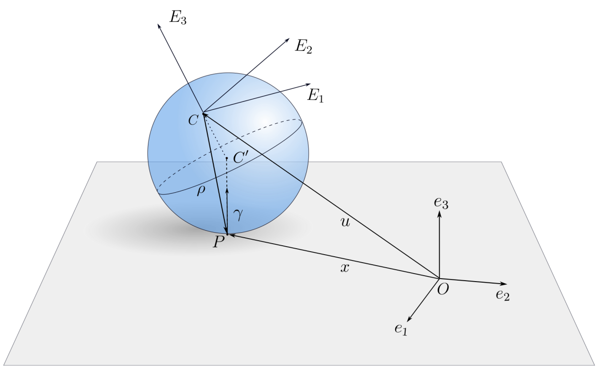


Figure 2.1

Moreover, we denote by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ the coordinates of the angular velocity vector with respect to the spatial frame Σ_s and by $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ the coordinates of the angular velocity vector with respect to the body frame Σ_b, so they satisfy

$$\boldsymbol{\Omega} = B^{-1}\boldsymbol{\omega}.$$

The space and body coordinate representations of the angular velocity $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are respectively defined by the right and left trivializations

$$\dot{B}B^{-1} = \hat{\boldsymbol{\omega}}, \qquad \text{and} \qquad B^{-1}\dot{B} = \hat{\boldsymbol{\Omega}},$$

where for $\boldsymbol{a} \in \mathbb{R}^3$, the notation $\hat{\boldsymbol{a}}$ denotes the unique 3×3 skew-symmetric real matrix such that $\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{a}}\boldsymbol{b}$ for all $\boldsymbol{b} \in \mathbb{R}^3$, i.e.

$$\hat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \hat{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

It is convenient to embed the configuration manifold to $\mathbb{R}^3 \times \mathrm{SO}(3)$ as the 5-dimensional manifold

$$Q = \{(\boldsymbol{u}, B) \in \mathbb{R}^3 \times \mathrm{SO}(3) \,:\, (0.2.2) \text{ is satisfied}\}.$$

The tangent bundle of the configuration manifold is the 10-dimensional manifold described by $TQ = \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}^3$, where we identify $T\mathrm{SO}(3)$ with $\mathrm{SO}(3) \times \mathbb{R}^3$ by considering the left trivialization of the Lie algebra of $\mathrm{SO}(3)$. Therefore, a point on TQ is given by $(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega})$.

The Lagrangian of the system $L : TQ \to \mathbb{R}$ is given by

$$L(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{m}{2}\|\dot{\boldsymbol{u}}\|^2 + mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle, \tag{0.2.3}$$

where g denotes the gravitational constant. To obtain the constraint of rolling without slipping, we consider a material point on the sphere $\mathfrak{Q} \in \mathbb{R}^3$ with constant coordinates in the body frame Σ_b. Its coordinates in the space frame Σ_s are given by

$$\boldsymbol{q}(t) = \boldsymbol{u}(t) + B(t)\mathfrak{Q},$$

differentiating with respect to time we get

$$\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{u}}(t) + \dot{B}(t)\mathfrak{Q}.$$

The no slipping constraint imposes that the velocity at the contact point is zero, so considering the point of contact $\mathfrak{Q} = \boldsymbol{\rho}$, and enforcing $\dot{\boldsymbol{q}}(t) = 0$ gives

$$\dot{\boldsymbol{u}} = -\dot{B}\boldsymbol{\rho} = -BB^{-1}\dot{B}\boldsymbol{\rho} = -B\hat{\boldsymbol{\Omega}}\boldsymbol{\rho} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}).$$

Thus, the constraint of rolling without slipping is given by

$$\dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}). \tag{0.2.4}$$

Notice that the third coordinate of (0.2.4) is given by

$$\dot{u}_3 = \langle B(\boldsymbol{\rho} \times \boldsymbol{\Omega}), \boldsymbol{e}_3\rangle = \langle (-r\boldsymbol{\gamma} - l\boldsymbol{E}_3) \times \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle = -l\langle \boldsymbol{E}_3 \times \boldsymbol{\Omega}, \boldsymbol{\gamma}\rangle = l\langle \dot{\gamma}, \boldsymbol{E}_3\rangle = l\dot{\gamma}_3,$$

which can be obtained by differentiating the holonomic constraint of the sphere staying on the plane,

$$u_3 = -\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle = r + l\gamma_3. \tag{0.2.5}$$

Therefore, (0.2.4) describes two independent nonholonomic constraints (which are linear in the velocities) and one holonomic constraint. These determine an

8-dimensional phase space described by the linear rank 3 constraint distribution $\mathcal{D} \subset TQ$, defined as

$$\mathcal{D} = \{(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) \in TQ \,:\, (0.2.4) \text{ and } (0.2.5) \text{ are satisfied}\}. \tag{0.2.6}$$

2.1.2 Equations of motion

In this section we will obtain the equations of motion of the system using the Lagrange-D'Alembert principle introduced in Section 1.4. We introduce the vector $\boldsymbol{M} \in \mathbb{R}^3$ which is written in the body coordinates Σ_b and is defined as

$$\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}). \tag{0.2.7}$$

The vector $\boldsymbol{M}$ is the angular momentum of the body about its contact point. Notice that from equation (0.2.7) we can obtain an expression for $\boldsymbol{\Omega}$ in terms of $\boldsymbol{M}$ and $\boldsymbol{\gamma}$,

$$\boldsymbol{\Omega}(\boldsymbol{M}, \gamma) = A(\gamma)\left(\boldsymbol{M} - \frac{m\langle \boldsymbol{M}, A(\gamma)\boldsymbol{\rho}\rangle}{1 - m\langle \boldsymbol{\rho}, A(\gamma)\boldsymbol{\rho}\rangle}\boldsymbol{\rho}\right), \tag{0.2.8}$$

where $A = (\mathbb{I} + m\|\boldsymbol{\rho}\|^2 \mathrm{id})^{-1}$, where id denotes the 3×3 identity matrix.

Proposition 0.2.1. *[12] The equations of motion of the system are given by*

$$\begin{aligned} \dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega} + m\dot{\boldsymbol{\rho}} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) + mg\boldsymbol{\rho} \times \boldsymbol{\gamma} \\ \dot{\boldsymbol{u}} &= B(\boldsymbol{\rho} \times \boldsymbol{\Omega}), \\ \dot{B} &= B\hat{\boldsymbol{\Omega}}, \end{aligned} \tag{0.2.9}$$

where $\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho})$.

Proof. In the absence of constraints, the equations of motion are

$$m\ddot{\boldsymbol{u}} = -mg\boldsymbol{e}_3, \qquad \mathbb{I}\dot{\boldsymbol{\Omega}} = (\mathbb{I}\boldsymbol{\Omega}) \times \boldsymbol{\Omega}.$$

Adding the reaction forces of the constraints we get

$$m\ddot{\boldsymbol{u}} = -mg\boldsymbol{e}_3 + R_1, \qquad \mathbb{I}\dot{\boldsymbol{\Omega}} = (\mathbb{I}\boldsymbol{\Omega}) \times \boldsymbol{\Omega} + R_2, \tag{0.2.10}$$

for some R_1, R_2. The Lagrange D'Alembert principle says that if $(\dot{\boldsymbol{u}}, \boldsymbol{\Omega})$ satisfy (0.2.4), then

$$\langle R_1, \dot{\boldsymbol{u}}\rangle + \langle R_2, \boldsymbol{\Omega}\rangle = 0.$$

In this case,

$$\langle R_1, B(\boldsymbol{\rho} \times \boldsymbol{\Omega})\rangle + \langle R_2, \boldsymbol{\Omega}\rangle = \langle (B^{-1}R_1) \times \boldsymbol{\rho} + R_2, \boldsymbol{\Omega}\rangle = 0.$$

Since it is valid for all $\mathbf{\Omega}$, this gives us an expression for R_2

$$R_2 = -(B^{-1}R_1) \times \boldsymbol{\rho}.$$

On the other hand, differentiating the constraints (0.2.4) gives

$$\ddot{\boldsymbol{u}} = \dot{B}(\boldsymbol{\rho} \times \mathbf{\Omega}) + B(\dot{\boldsymbol{\rho}} \times \mathbf{\Omega}) + B(\boldsymbol{\rho} \times \dot{\mathbf{\Omega}}),$$

so, substituting in equation (0.2.10) we get an expression for R_1,

$$R_1 = m\ddot{\boldsymbol{u}} - mg\boldsymbol{e}_3 = m(\dot{B}(\boldsymbol{\rho} \times \mathbf{\Omega}) + B(\dot{\boldsymbol{\rho}} \times \mathbf{\Omega}) + B(\boldsymbol{\rho} \times \dot{\mathbf{\Omega}}) - g\boldsymbol{e}_3).$$

We may now use the expressions obtained for the reaction forces by using the Lagrange-D'Alembert principle to obtain the equations of motion,

$$\begin{aligned}\mathbb{I}\dot{\mathbf{\Omega}} &= (\mathbb{I}\mathbf{\Omega}) \times \mathbf{\Omega} - (B^{-1}R_1) \times \boldsymbol{\rho} \\ &= (\mathbb{I}\mathbf{\Omega}) \times \mathbf{\Omega} + m\boldsymbol{\rho} \times (\mathbf{\Omega} \times (\boldsymbol{\rho} \times \mathbf{\Omega})) + m\boldsymbol{\rho} \times (\dot{\boldsymbol{\rho}} \times \mathbf{\Omega}) + m\boldsymbol{\rho} \times (\boldsymbol{\rho} \times \dot{\mathbf{\Omega}}) - mg\boldsymbol{\rho} \times \boldsymbol{\gamma}.\end{aligned}$$

Using the definition of the vector $\boldsymbol{M}$ introduced in (0.2.7) we obtain the first equation of (0.2.9). This equation is complemented with the constraints (0.2.4) and with $\dot{B} = B\hat{\mathbf{\Omega}}$. □

2.1.3 Symmetries and first integrals

The system has an $\mathrm{SE}(2)$-symmetry that corresponds to translations and rotations of the plane Π. We denote an element $(R_\theta, \boldsymbol{a}) \in \mathrm{SE}(2)$ with

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{a} = (a_1, a_2, 0),$$

and consider the action Ψ of $\mathrm{SE}(2)$ on Q,

$$\Psi((R_\theta, \boldsymbol{a}), (\boldsymbol{u}, B)) = (R_\theta \boldsymbol{u} + \boldsymbol{a}, R_\theta B).$$

The tangent lift of this action $\overline{\Psi} : \mathrm{SE}(2) \times TQ \to TQ$ is given by

$$\overline{\Psi}((R_\theta, \boldsymbol{a}), (\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \mathbf{\Omega})) = (R_\theta \boldsymbol{u} + \boldsymbol{a}, R_\theta B, R_\theta \dot{\boldsymbol{u}}, \mathbf{\Omega}).$$

Notice that u_3, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$ and $\boldsymbol{\Omega}$ are invariant under this action. We can check that the Lagrangian L given in (0.2.3) is invariant under this action, i.e. $L \circ \overline{\Psi} = L$,

$$L(\overline{\Psi}(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega})) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{1}{2}\|R_\theta \dot{\boldsymbol{u}}\|^2 + mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle = L(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}).$$

Moreover, the distribution $\mathcal{D}$ given by (0.2.6) is also Ψ-invariant, i.e. $\overline{\Psi}(\mathcal{D}_q) = \mathcal{D}_{\Psi(q)}$, since equations (0.2.4) and (0.2.5) are invariant under Ψ. Therefore, the system may be reduced by this symmetry. First, we notice from expression (0.2.8) that $\boldsymbol{\Omega}$ and does not depend on $\boldsymbol{u}$, so we can separate the constraints (0.2.4) from the equations of motion (0.2.9). Moreover, the third equation in (0.2.9) is equivalent to

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega},$$

and we obtain a decoupled system for $(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2$. The reduced phase space $\mathcal{D}/\mathrm{SE}(2)$ is a 5-dimensional manifold diffeomorphic to $\mathbb{R}^3 \times S^2$ and may be parametrized by $\boldsymbol{M} \in \mathbb{R}^3$ and $\boldsymbol{\gamma} \in S^2$.

Proposition 0.2.2. *The* $\mathrm{SE}(2)$*-reduced equations of motion are*

$$\begin{aligned} \dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega} + m\dot{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) + mg\boldsymbol{\rho} \times \boldsymbol{\gamma} \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \end{aligned} \tag{0.2.11}$$

where $\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho})$.

Notice that the reduced system has the geometric first integral $\|\boldsymbol{\gamma}\|^2 = 1$. Moreover, it can be checked that the energy, given by

$$E = \frac{1}{2}\langle \boldsymbol{M}, \boldsymbol{\Omega}\rangle - mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle,$$

is a first integral of the system.

2.1.4 Dynamics

According to the Euler-Jacobi theorem [2], for the system to be integrable, we would need two additional first integrals and a smooth invariant measure. In the general case, without additional symmetries, there is no invariant measure and no additional first integrals; the dynamics of the system is generally chaotic. However, there are some known integrable cases, which are cases of the general system with some additional symmetries.

2.1.5 Integrable cases

Routh's sphere

The case of the axially symmetric sphere was studied by Routh in 1884, [59] and by Chaplygin in [19] and was proven to be integrable. Suppose that the sphere is axially symmetric, i.e. $I_1 = I_2$ and with its center of mass $C \neq C'$ aligned with the axis $\boldsymbol{E}_3$. Because of the additional symmetry on the sphere, the system has an $\mathrm{SO}(2)$-symmetry, which corresponds to rotations of the body frame Σ_b about the axis $\boldsymbol{E}_3$.

In this case, the equations of motion remain as in (0.2.11). Besides the geometric integral $\|\boldsymbol{\gamma}\|^2 = 1$ and the energy integral E, it can be checked that the system has the additional first integrals

$$K_1 = \Omega_2\sqrt{I_1 I_3 + I_1 m r^2(1-\gamma_3^2) + I_3 m(r\gamma_3 + l)^2}, \qquad \text{and} \qquad K_2 = \langle \boldsymbol{M}, \boldsymbol{\rho}\rangle,$$

and smooth invariant measure $\mu d\boldsymbol{M} d\boldsymbol{\gamma}$ with

$$\mu = \frac{1}{\sqrt{I_1 I_3 \langle \boldsymbol{\rho}, \mathbb{I}\boldsymbol{\rho}\rangle}}.$$

Hence, in view of the Euler-Jacobi theorem [2], this system is integrable.

The Chaplygin sphere

Another example of an integrable system is that of a dynamically balanced sphere, i.e. a sphere with different moments of inertia in which the center of mass coincides with the geometric center. This system is known as the *Chaplygin sphere* and was proven to be integrable by Chaplygin in 1903, [20]. Other references of the Chaplygin sphere are [8, 49, 26]. If the geometric center C coincides with the center of mass C', then the relations (0.2.1) and (0.2.2) respectively become

$$\boldsymbol{\rho} = -r\boldsymbol{\gamma}, \qquad \text{and} \qquad \boldsymbol{x} = \boldsymbol{u} - r\boldsymbol{e}_3.$$

The reduced equations of motion (0.2.11) therefore take the form

$$\begin{aligned} \dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \end{aligned} \tag{0.2.12}$$

with $\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + mr^2\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma})$. From the first equation in (0.2.12), we have

$$\frac{d}{dt}B\boldsymbol{M} = \dot{B}\boldsymbol{M} + B\dot{\boldsymbol{M}} = BB^{-1}\dot{B}\boldsymbol{M} + B(\boldsymbol{M} \times \boldsymbol{\Omega}) = B(\boldsymbol{\Omega} \times \boldsymbol{M}) + B(\boldsymbol{M} \times \boldsymbol{\Omega}) = 0.$$

So the vector $\boldsymbol{M}$ is constant with respect to the space frame Σ_s. Therefore, the system has the additional first integrals

$$\|\boldsymbol{M}\|, \qquad \langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle.$$

Moreover, it has a smooth invariant measure $\mu d\boldsymbol{M} d\boldsymbol{\gamma}$, where

$$\mu = \frac{1}{\sqrt{1 - mr^2 \langle \boldsymbol{\gamma}, A\boldsymbol{\gamma} \rangle}},$$

with the matrix $A = (\mathbb{I} + mr^2 \mathrm{id})^{-1}$.

The homogeneous sphere

A particular case of those two integrable cases, is the homogeneous sphere, a dynamically balanced sphere with equal moments of inertia, $I = I_1 = I_2 = I_3$. The reduced equations of motion of the system are given by (0.2.12) with $\boldsymbol{M} = I\boldsymbol{\Omega} + mr^2 \boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma})$, and the constraints (0.2.4) take the form

$$\dot{\boldsymbol{x}} = -rB(\boldsymbol{\gamma} \times \boldsymbol{\Omega}).$$

The homogeneity of the sphere gives rise to an additional symmetry of the system by $\mathrm{SO}(3)$, corresponding to the rotations of the sphere. This symmetry can be reduced by working in the space reference frame Σ_s and decoupling the equations of motion. Considering the vector $\boldsymbol{m} = B\boldsymbol{M}$, given by

$$\boldsymbol{m} = I\boldsymbol{\omega} + mr^2 \boldsymbol{e}_3 \times (\boldsymbol{\omega} \times \boldsymbol{e}_3),$$

where $\hat{\boldsymbol{\omega}} = \dot{B}B^{-1}$, the first equation of (0.2.12) becomes

$$\dot{\boldsymbol{m}} = 0.$$

So the vector $\boldsymbol{m}$ is constant and therefore, its coordinates,

$$m_1 = (I + mr^2)\omega_1, \qquad m_2 = (I + mr^2)\omega_2, \qquad m_3 = I\omega_3$$

are first integrals of the system, which imply that the vector $\boldsymbol{\omega}$ is constant. The constraints may be rewritten using space coordinates as

$$\dot{\boldsymbol{x}} = r\boldsymbol{\omega} \times \boldsymbol{e}_3,$$

which in coordinates give

$$\dot{x}_1 = r\omega_2, \qquad \dot{x}_2 = -r\omega_1.$$

Since $\boldsymbol{\omega}$ is constant, from these last equations we know that the trajectory of the sphere on a plane follows a straight line.

3 Nonholonomic systems with affine constraints

The main examples of nonholonomic systems with affine constraints consist of rigid bodies rolling on moving surfaces. For example, in Section 3.2 we will discuss the homogeneous sphere rolling on a rotating plane; but also, more generally, one can consider the Chaplygin sphere [29] or convex body [5] rolling on a rotating plane or a sphere rolling on a rotating surface of revolution [24, 35]. Other examples of nonholonomic systems with affine constraints can be found in [54, 63, 6, 45].

In nonholonomic systems with linear constraints, the energy is preserved. However, nonholonomic systems with affine constraints need not conserve the energy. In [31], it was noticed that in nonholonomic systems with affine constraints, even if the energy is not conserved, one may find a modification of the energy, called the *moving energy*, which is preserved. We introduce the main concepts and results from [31] in the next section.

3.1 Moving energy

The idea is that given a nonholonomic system with affine constraints, under some assumptions, through a general mechanism, one can find a modification of the energy, which is conserved. These assumptions require that the affine terms of the constraints correspond to the infinitesimal generator of a continuous symmetry of the Lagrangian. To state this more precisely, we introduce some notation following [31].

Consider a nonholonomic system on a configuration manifold Q, with Lagrangian L given by (0.1.5) and affine constraints that determine an affine constraint distribution $\mathcal{A} \subset TQ$, which we express as

$$\mathcal{A} = \mathcal{D} + Z$$

with $\mathcal{D} \subset TQ$ a linear distribution and $Z \in \mathfrak{X}(Q)$ a vector field.

Now, let $\Psi : G \times Q \to Q$ be the action of a Lie group G on Q and let $\overline{\Psi} : G \times TQ \to TQ$ be its tangent lift. Consider the Lie algebra $\mathfrak{g}$ of G and take $\xi \in \mathfrak{g}$. Let us denote by $Y_\xi \in \mathfrak{X}(Q)$ the infinitesimal generator of the action of the one-parameter subgroup generated by ξ, namely,

$$Y_\xi(q) = \frac{d}{dt}\bigg|_{t=0} \Psi_{\exp(t\xi)}(q),$$

and let us denote by $J_\xi : TQ \to \mathbb{R}$ the momentum map of the one-parameter subgroup generated by ξ,

$$J_\xi := \dot{q}^T A(q) Y_\xi(q).$$

Then, the moving energy of the nonholonomic system defined by a Lagrangian L, configuration manifold Q and affine constraint distribution $\mathcal{A}$, relative to ξ is the restriction to $\mathcal{A}$ of the function

$$E_{L,\xi} = E_L - J_\xi,$$

where $E_L : TQ \to \mathbb{R}$ is the energy of L,

$$E_L(q, \dot{q}) = \frac{1}{2}\dot{q}^T A(q)\dot{q} + V(q).$$

With this notation in mind, we state the following.

Theorem 0.3.1. *[31] Consider a (time-independent) nonholonomic system with Lagrangian L, configuration manifold Q and affine constraint distribution $\mathcal{A} = \mathcal{D} + Z$. Consider the action Ψ of a Lie group G on Q and assume that*

- *L is $\overline{\Psi}$-invariant, i.e. $L \circ \overline{\Psi}_g = L$ for all $g \in G$,*
- *$\mathcal{D}$ is Ψ-invariant, i.e. $T_q\Psi_g(\mathcal{D}_q) = \mathcal{D}_{\Psi_g(q)}$ for all $g \in G$ and $q \in Q$,*
- *$\xi \in \mathfrak{g}$ is such that $Y_\xi - Z$ is a section of $\mathcal{D}$.*

Then, the moving energy

$$E_{L,\xi,\mathcal{A}} = E_L|_{\mathcal{A}} - J_\xi|_{\mathcal{A}},$$

is a (time-independent) first integral of the system.

3.2 The homogeneous sphere rolling on a rotating plane

Consider a homogeneous sphere of radius r and mass m rolling without slipping on a horizontal plane which is rotating at constant angular speed $\eta \in \mathbb{R}$, as shown in Fig. 3.1. This is a well-known example and has been considered, for example, in [27, 57, 56, 9, 21, 46, 31].

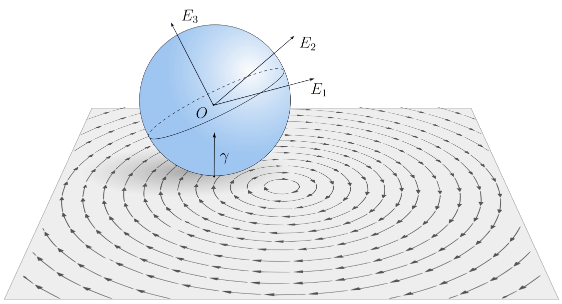


Figure 3.1: Sphere rolling on a rotating plane

We fix the space frame Σ_s in such a way that the plane is rotating about the $\boldsymbol{e}_3$ axis. The configuration manifold of the system is $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$. We use the same notation as in Section 2.1, and since the sphere is homogeneous, we have the following relations

$$\boldsymbol{u} = \boldsymbol{x} + r\boldsymbol{e}_3 \qquad \text{and} \qquad \boldsymbol{\rho} = -r\boldsymbol{\gamma}.$$

To obtain the constraints, we consider a point on the sphere $\mathcal{Q}$ with constant coordinates in the body frame Σ_b. In the space frame Σ_s its coordinates are given by

$$\boldsymbol{q}(t) = \boldsymbol{x}(t) + B(t)\mathcal{Q},$$

differentiating with respect to time we get

$$\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{x}}(t) + \dot{B}(t)\mathcal{Q}.$$

The constraint tells us that the velocity at the contact point is the velocity of the plane at that point, i.e. at the point of contact $\mathcal{Q} = -r\boldsymbol{\gamma}$, we have $\dot{\boldsymbol{q}}(t) = \eta\boldsymbol{e}_3 \times \boldsymbol{x}(t)$. Therefore,

$$\dot{\boldsymbol{x}} = r\dot{B}\boldsymbol{\gamma} + \eta\boldsymbol{e}_3 \times \boldsymbol{x} = rB\hat{\boldsymbol{\Omega}}\boldsymbol{\gamma} + \eta\boldsymbol{e}_3 \times \boldsymbol{x} = rB(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) + \eta\boldsymbol{e}_3 \times \boldsymbol{x}.$$

Then, the constraints are given by

$$\dot{\boldsymbol{x}} = rB(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) + \eta \boldsymbol{e}_3 \times \boldsymbol{x}. \tag{0.3.1}$$

Notice that, in this case, the constraints are not linear homogeneous in the velocities, but instead affine linear. Again, the third coordinate of (0.3.1) is a holonomic constraint that can be obtained by differentiating

$$x_3 = 0. \tag{0.3.2}$$

The constraint distribution is given by

$$\mathcal{A} = \{(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\Omega}) \in TQ \; : \; (0.3.1) \text{ and } (0.3.2) \text{ are satisfied}\}.$$

It is an 8-dimensional affine distribution, and it can be expressed as $\mathcal{A} = \mathcal{D} + Z$ with $\mathcal{D}$ the linear distribution and $Z \in \mathfrak{X}(Q)$ a vector field. These can be taken as

$$\begin{aligned}
&\mathcal{D} = \{(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\Omega}) \in TQ \, : \, \dot{\boldsymbol{x}} = rB(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) \text{ and } (0.3.2) \text{ are satisfied}\},\\
&Z(\boldsymbol{x}, B) = (\eta \boldsymbol{e}_3 \times \boldsymbol{x}, \boldsymbol{0}).
\end{aligned}$$

Equations of motion

Using the Lagrange-D'Alembert principle, one can get the equations of motion. First, we introduce the vector $\boldsymbol{M} \in \mathbb{R}^3$ defined by

$$\boldsymbol{M} = I\boldsymbol{\Omega} + mr^2 \boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma}) - \eta B^{-1}(\boldsymbol{x} \times \boldsymbol{e}_3). \tag{0.3.3}$$

This definition gives us an expression for $\boldsymbol{\Omega}$ in terms of $\boldsymbol{M}$, B, and $\boldsymbol{x}$

$$\boldsymbol{\Omega}(\boldsymbol{M}, B, \boldsymbol{x}) = \frac{1}{I + mr^2}\left(\boldsymbol{M} + \frac{mr^2}{I}\langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle \boldsymbol{\gamma} + mr\eta \boldsymbol{\gamma} \times ((B^{-1}\boldsymbol{x}) \times \boldsymbol{\gamma}\right)$$

Proposition 0.3.2. *The equations of motion are given by*

$$\begin{aligned}
\dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega},\\
\dot{\boldsymbol{x}} &= rB(\boldsymbol{\Omega} \times \boldsymbol{\gamma}) + \eta \boldsymbol{e}_3 \times \boldsymbol{x},\\
\dot{B} &= B\hat{\boldsymbol{\Omega}},
\end{aligned}$$

with $\boldsymbol{M} = I\boldsymbol{\Omega} + mr^2 \boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma}) - \eta B^{-1}(\boldsymbol{x} \times \boldsymbol{e}_3)$.

Proof. Following an analogous procedure to the one we used for the case of the sphere rolling on a fixed plane in Section 2.1, using the Lagrange-D'Alembert principle, we find the equations of motion. In this case, in the absence of constraints, the equations of motion are

$$m\ddot{\boldsymbol{x}} = 0, \qquad I\dot{\boldsymbol{\Omega}} = (I\boldsymbol{\Omega}) \times \boldsymbol{\Omega} = 0.$$

Adding the reaction forces of the constraints we get

$$m\ddot{\boldsymbol{x}} = R_1, \qquad I\dot{\boldsymbol{\Omega}} = R_2,$$

for some R_1, R_2. The Lagrange-D'Alembert principle says that if $(\dot{\boldsymbol{x}}, \boldsymbol{\Omega})$ satisfy $\dot{\boldsymbol{x}} = rB(\boldsymbol{\Omega} \times \boldsymbol{\gamma})$, then

$$\langle R_1, \dot{\boldsymbol{x}} \rangle + \langle R_2, \boldsymbol{\Omega} \rangle = 0.$$

In this case,

$$\langle R_1, -rB(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) \rangle + \langle R_2, \boldsymbol{\Omega} \rangle = \langle -r(B^{-1}R_1) \times \boldsymbol{\gamma} + R_2, \boldsymbol{\Omega} \rangle = 0,$$

Since it is valid for all $\boldsymbol{\Omega}$, this implies

$$R_2 = r(B^{-1}R_1) \times \boldsymbol{\gamma}.$$

Differentiating the constraints (0.3.1) gives

$$\ddot{\boldsymbol{x}} = -r\dot{B}(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) - rB(\dot{\boldsymbol{\gamma}} \times \boldsymbol{\Omega}) - rB(\boldsymbol{\gamma} \times \dot{\boldsymbol{\Omega}}) + \eta \boldsymbol{e}_3 \times \dot{\boldsymbol{x}},$$

so

$$R_1 = m\ddot{\boldsymbol{x}} = -mr(\dot{B}(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + B(\dot{\boldsymbol{\gamma}} \times \boldsymbol{\Omega}) + B(\boldsymbol{\gamma} \times \dot{\boldsymbol{\Omega}})) + m\eta \boldsymbol{e}_3 \times \dot{\boldsymbol{x}}.$$

Therefore,

$$\begin{aligned} I\dot{\boldsymbol{\Omega}} &= r(B^{-1}R_1) \times \boldsymbol{\gamma} \\ &= mr^2\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times (\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\Omega}) + (\boldsymbol{\gamma} \times \dot{\boldsymbol{\Omega}})) - mr\eta\boldsymbol{\gamma} \times B^{-1}(\boldsymbol{e}_3 \times \dot{\boldsymbol{x}}). \end{aligned}$$

Using the vector $\boldsymbol{M}$ defined by (0.3.3) it becomes

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}.$$

This equation is complemented with the constraints (0.3.1) and with equation $\dot{B} = B^{-1}\hat{\boldsymbol{\Omega}}$. □

Symmetries and first integrals

The system has an $\mathrm{SO}(3)$-symmetry corresponding to rotations of the sphere and an $\mathrm{SO}(2)$-symmetry corresponding to rotations of the plane about the $\boldsymbol{e}_3$ axis.

We do the reduction of the system by the first of these symmetries by considering the space coordinates of the vector $\boldsymbol{M}$ and using the angular velocity written on the space frame Σ_s, i.e.

$$\boldsymbol{m} = B\boldsymbol{M} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\boldsymbol{\omega} \times \boldsymbol{e}_3) - \eta\boldsymbol{x} \times \boldsymbol{e}_3, \qquad \text{and} \qquad \boldsymbol{\omega} = B\boldsymbol{\Omega}.$$

Then, the constraints (0.3.1) can be written as

$$\dot{\boldsymbol{x}} = -r\boldsymbol{e}_3 \times \boldsymbol{\omega} - \eta\boldsymbol{x} \times \boldsymbol{e}_3.$$

Since neither of these equations depend on the matrix B, we can decouple the system. The $\mathrm{SO}(3)$-reduced system can be written in terms of $\boldsymbol{m} \in \mathbb{R}^3$ and $\boldsymbol{x} \in \mathbb{R}^2$.

Proposition 0.3.3. *The* $\mathrm{SO}(3)$*-reduced equations of motion are given by*

$$\begin{aligned} \dot{\boldsymbol{m}} &= 0 \\ \dot{\boldsymbol{x}} &= -r\boldsymbol{e}_3 \times \boldsymbol{\omega} - \eta\boldsymbol{x} \times \boldsymbol{e}_3, \end{aligned} \tag{0.3.4}$$

where $\boldsymbol{m} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\boldsymbol{\omega} \times \boldsymbol{e}_3) - \eta\boldsymbol{x} \times \boldsymbol{e}_3$.

From the reduced equations of motion, it can be seen that the reduced system has first integrals

$$m_1 = (I+mr^2)\omega_1 - mr\eta x_1, \qquad m_2 = (I+mr^2)\omega_2 - mr\eta x_2, \qquad m_3 = I\omega_3. \tag{0.3.5}$$

It can be seen that the energy is not conserved. Nevertheless, as noted in [31], the moving energy,

$$E_{mov} = \frac{I}{2}\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{mr^2}{2}\|\boldsymbol{\gamma} \times \boldsymbol{\Omega}\|^2 - \frac{m}{2}\eta^2\|\boldsymbol{x}\|^2,$$

is a first integral of the system.

Dynamics

The reduced phase space $\mathcal{A}/\mathrm{SO}(3)$ has dimension 5 and the system has four first integrals, so the system is integrable and, in fact, periodic. From the first integrals (0.3.5), we may obtain an expression for $\boldsymbol{\omega}$ in terms of $\boldsymbol{x}$

$$\omega_1 = \frac{m_1 + mr\eta x_1}{I + mr^2}, \qquad \omega_2 = \frac{m_2 + mr\eta x_2}{I + mr^2}, \qquad \omega_3 = \frac{m_3}{I}.$$

Substituting in equation (0.3.1) we get

$$\dot{x}_1 = \frac{1}{I + mr^2}(rm_2 - I\eta x_2), \qquad \dot{x}_2 = \frac{1}{I + mr^2}(-rm_1 + I\eta x_1),$$

which can be rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{1}{I + mr^2} \left(\begin{pmatrix} 0 & -I\eta \\ I\eta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + r \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix} \right). \tag{0.3.6}$$

The general solution of the above linear inhomogeneous ODE is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{r}{I\eta} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

where $a, b \in \mathbb{R}$ and $\omega = \frac{I\eta}{I+mr^2}$. Therefore, the trajectories of the sphere on the plane are circles with center $\left(\frac{rm_1}{I\eta}, \frac{rm_2}{I\eta} \right)$.

Part I

Affine generalizations of the problem of the convex body rolling without slipping on the plane

1 Introduction

The role of symmetries in the reduction [47, 9, 17, 48, 16], existence of first integrals [9, 34, 30, 33, 3], invariant measures [7, 50, 62, 18, 66, 36], and integrability [37, 44, 4, 28] of nonholonomic systems with linear constraints in the velocities has been an active field of research in the last decades. Concrete examples have been very useful to illustrate, and often guide, such investigations. In this regard, the approach of Borisov, Mamaev et al. [12, 13, 14] has been very valuable. In these papers, the authors consider general rolling problems and investigate dynamical aspects as a function of the parameters entering the shape and mass distribution of the bodies, reporting a hierarchy of behaviors ranging from integrable to chaotic.

The dynamics of nonholonomic systems whose constraints are affine, instead of linear, in the velocities is much less developed. A general mechanism, arising from symmetries, which leads to existence of an energy type integral, termed *moving energy*, was only recently discovered in [31, 29] (see also [15]). On the other hand, the existence of momentum type integrals is treated in [33] but more extensive investigations remain to be done. To the best of our knowledge, general existence conditions of an invariant measure for nonholonomic systems with affine constraints are unknown.

In contrast with the linear case described above, there is no general class of examples to illustrate or guide such investigations in the affine setting. The purpose of this paper is to attempt to fill this gap by introducing a general class of examples providing affine generalizations of the classical problem of a convex body rolling without slipping on the plane. Mathematically, the systems that we propose are obtained by taking as given a vector field W on the surface of the body $\mathcal{S}$ and a vector field V on the plane Π, which determine the velocity of the contact point as illustrated in Fig 1.1. As we explain in Chapter 2, such system can be mechanically realized for specific vector fields V and W and for certain body shapes. In fact, our proposed system provides a general framework for specific examples which had been considered previously in the literature [54, 63, 5, 6, 45].

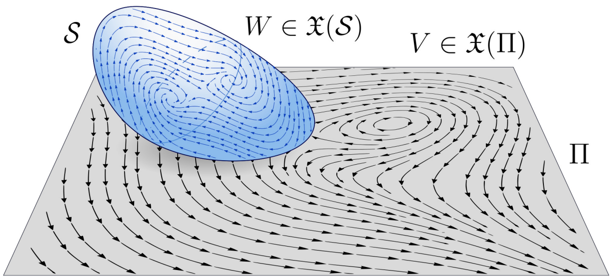


Figure 1.1: Graphic representation of the vector fields V on the plane Π and W on the surface $\mathcal{S}$ of the convex body. The nonholonomic constraint enforces the velocity of the contact point to be equal to the sum of both vectors at that point.

1.1 Structure

We begin by introducing the system in detail in Chapter 2, describing its kinematics in Section 2.1 and deriving the equations of motion for general vector fields V, W in Section 2.2. We also indicate the corresponding $\mathrm{SE}(2)$-reduction in the case where the vector field V on the plane vanishes. We then proceed to identify some special cases of existence of a preserved moving energy in Section 2.4. In Chapter 3 we focus on the case in which the convex body is a dynamically balanced sphere (i.e. a *Chaplygin sphere*) and we extend some results of [5, 6, 45] giving several dynamical contributions. Chapter 4 focuses on the case in which the convex body is a solid of revolution and we show that the system is integrable for $V = 0$ and a specific choice of W (consistent with the symmetry). Finally, in Chapter 5 we treat the case in which the convex body is a homogeneous sphere, we prove a general result on existence of an invariant measure and analyze the dynamics in detail for specific choices of V and W. We finally mention that the results of this chapter are contained in [22].

2 Description of the system

We consider the motion of a convex rigid body, with smooth surface $\mathcal{S}$, on the infinite horizontal plane $\Pi := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ subject to the following constraints:

C1. The body surface $\mathcal{S}$ and the plane Π are in contact at a unique point at all time.

C2. The velocity of the material point of the body in contact with the plane equals the sum $V_{\boldsymbol{x}} + W_{\boldsymbol{\rho}}$, where $V_{\boldsymbol{x}}, W_{\boldsymbol{\rho}}$, are prescribed horizontal vectors (i.e. tangent to Π) which respectively depend on the specific position $\boldsymbol{x} \in \Pi$ of the contact point, and on the specific material point $\boldsymbol{\rho} \in \mathcal{S}$ which is in contact with the plane.

The first condition imposes a standard holonomic constraint on the system. The second condition is a generalization of the nonholonomic constraint of rolling without slipping, which is illustrated in Fig 1.1, and is convenient to restate as:

C2′. We assume that there are two given vector fields, $V \in \mathfrak{X}(\Pi)$ and $W \in \mathfrak{X}(\mathcal{S})$, which determine the 'slipping' velocity of the contact point via the sum of their evaluations at the specific spatial point of contact $\boldsymbol{x} \in \Pi$ and the specific material point of contact $\boldsymbol{\rho} \in \mathcal{S}$.

If both vector fields V and W vanish, we recover the classical problem of rolling without slipping on the plane. On the other hand, we have the following two particular cases that are worth pointing out, can be physically realized, and will be analyzed in detail at several points of the thesis.

1. The uniformly rotating plane. If $W = 0$ and

$$V(\boldsymbol{x}) = \eta\, \boldsymbol{x} \times \boldsymbol{e}_3,$$

we recover the model for the rolling of a convex body on a plane that rotates with constant angular velocity η (see Fig 2.2a). Here $\boldsymbol{x} \in \Pi \subset \mathbb{R}^3$ is expressed with respect to a fixed spatial frame, the vector $\boldsymbol{e}_3$ is normal to Π and '$\times$' denotes the vector product in $\mathbb{R}^3$. This problem has received great attention when the convex body is a homogeneous sphere [27, 57, 56, 54, 9, 31], but also in more generality [29, 5].

2. The cat's toy mechanism. To the best of our knowledge, the case $W \neq 0$ has received very little attention. Assuming $V = 0$ for simplicity, a mechanical realization, considered recently by Bizyaev, Borisov and Mamaev [6], is obtained as follows: suppose that an arbitrary rigid body is fastened inside a spherical shell with its center of mass C located at the geometric center of the shell, and suppose that the body is set and kept in motion about an axis passing through C with constant angular speed σ, by means of some device, see Fig 2.1. If the moments of inertia tensor of the spherical shell are negligible compared to the rigid body's, and the shell is put to roll without slipping on the plane, the resulting system is modelled by our framework. Indeed, in this case the body surface $\mathcal{S}$ is a sphere and the vector field $W \in \mathfrak{X}(\mathcal{S})$ is

$$W(\boldsymbol{\rho}) = \sigma \boldsymbol{\rho} \times \boldsymbol{E}_3.$$

Here $\boldsymbol{\rho} \in \mathcal{S} \subset \mathbb{R}^3$ are coordinates on the surface of the sphere with respect to a frame centered at C and fixed in the body (so $\|\boldsymbol{\rho}\| = r$, where $r > 0$ is the radius of the shell) and $\boldsymbol{E}_3$ is the unit vector in the direction of the axis of rotation, see Fig 2.1.

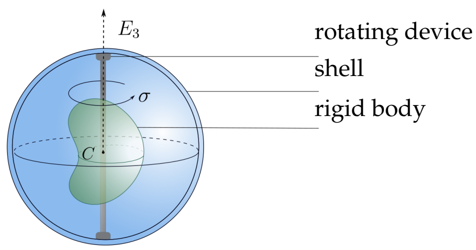
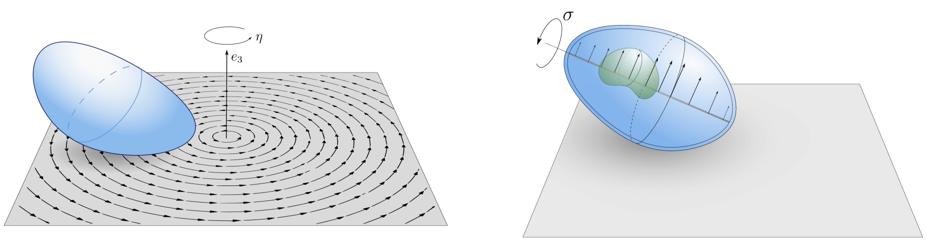


Figure 2.1: Graphic representation of the realization of the cat's toy mechanism. The center of mass C of the rigid body coincides with the geometric center of the spherical shell.

Several mechanical devices, similar to the one described above, are available in the market as toys for pets, especially cats. The idea is that the cat would amuse itself chasing the unevenly rolling spherical shell around the living room.

Inspired by this, we shall refer to the system described above as a sphere with a *cat's toy mechanism*.

A natural generalization, easily accounted for in our setup, is to assume that the shell is axially-symmetrical instead of spherical. To better align the presentation with our framework, it is convenient to think that the rigid body is steadily fastened to the shell, and it is the shell, instead of the rigid body, which is kept rotating with constant angular speed σ about its symmetry axis by means of some device, see Fig. 2.2b. We will also use the terminology "cat's toy mechanism" to refer to this case.



(a) Convex body rolling on a rotating plane. **(b)** The cat's toy mechanism.

Figure 2.2: Particular instances of our framework (see text for details).

Our motivation to consider the problem in its full generality (i.e. for arbitrary convex body and arbitrary vector fields V and W) is to illustrate dynamical phenomena that could guide the development of the theory for existence of invariant measures, existence of first integrals, integrability and chaotic behavior of mechanical systems with affine nonholonomic constraints which have received far less attention than their linear counterpart.

We mention that general possibilities for the vector field V are suggested in [53, 63] when the body is a homogeneous sphere. We also mention [45] where the authors consider the motion of a dynamically balanced ball on a vibrating plane corresponding to a non-autonomous vector field V. However, the systematic treatment of the problem that we present appears to be new.

2.1 Kinematics

We fix a spatial frame $\Sigma_s = \{O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the horizontal plane Π contains the origin O and is normal to $\boldsymbol{e}_3$. We also fix a body frame $\Sigma_b = \{C; \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ whose origin is the center of mass C of the convex body. Unless otherwise specified, we will assume that the vectors $\boldsymbol{E}_i$ are aligned with the body's principal axes of inertia.

The configuration of the body is specified by a pair $(B, \boldsymbol{x}) \in \mathrm{SO}(3) \times \mathbb{R}^3$ where $\boldsymbol{x} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OP}$ from the origin O to the contact point P (see Fig 2.3) with respect to the spatial frame Σ_s, and the attitude matrix $B \in \mathrm{SO}(3)$ determines the orientation of the body (i.e. it is the change of basis matrix between the bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{E}_i\}$ of $\mathbb{R}^3$).

The constraint C1 that the body surface $\mathcal{S}$ and the plane Π are in contact at all time at a unique point leads to the holonomic constraint

$$x_3 = 0, \tag{I.2.1}$$

so for the rest of the thesis we write

$$\boldsymbol{x} = (x_1, x_2, 0) \in \Pi \subset \mathbb{R}^3.$$

It will be convenient to think of the vector field $V \in \mathfrak{X}(\Pi)$ in constraint C2′ as the restriction to $\Pi \subset \mathbb{R}^3$ of a vector field on $\mathbb{R}^3$ which is tangent to Π. For this reason, for each $\boldsymbol{x} \in \Pi$, we will write

$$\boldsymbol{V}_s(\boldsymbol{x}) = (V_1(\boldsymbol{x}), V_2(\boldsymbol{x}), 0) \in \mathbb{R}^3, \tag{I.2.2}$$

as the coordinate expression of the vector field V with respect to the spatial frame Σ_s. In particular, for the rotating plane with constant angular velocity η about the origin O illustrated in Fig 2.2a, we have

$$\boldsymbol{V}_s(\boldsymbol{x}) = \eta \boldsymbol{e}_3 \times \boldsymbol{x}. \tag{I.2.3}$$

Similarly, it will be convenient to think of the vector field $W \in \mathfrak{X}(\mathcal{S})$ as the restriction to $\mathcal{S} \subset \mathbb{R}^3$ of a vector field on $\mathbb{R}^3$ tangent to $\mathcal{S}$. The coordinate expression for this vector field with respect to the body frame Σ_b is then given by

$$\boldsymbol{W}_b(\boldsymbol{\rho}) = (W_1(\boldsymbol{\rho}), W_2(\boldsymbol{\rho}), W_3(\boldsymbol{\rho})) \in \mathbb{R}^3, \tag{I.2.4}$$

where the tangency condition

$$\langle \boldsymbol{W}_b(\boldsymbol{\rho}), \boldsymbol{n}_b(\boldsymbol{\rho}) \rangle = 0, \tag{I.2.5}$$

holds for all $\boldsymbol{\rho} \in \mathcal{S} \subset \mathbb{R}^3$. In the above expressions, $\boldsymbol{\rho} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{CP}$, connecting the center of mass and the contact point, with respect to the body frame Σ_b (see Fig 2.3), $\boldsymbol{n}_b(\boldsymbol{\rho})$ is the outward unitary normal vector to $\mathcal{S}$ at $\boldsymbol{\rho} \in \mathcal{S}$ expressed in the body frame Σ_b, and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$. In particular, for the cat's toy mechanism described in Chapter 2 and depicted in Fig 2.2b, we have

$$\boldsymbol{W}_b(\boldsymbol{\rho}) = \sigma \boldsymbol{\rho} \times \boldsymbol{E}_3, \tag{I.2.6}$$

where the third axis of the body frame Σ_b is chosen along the direction of the shell's axis of symmetry[1].

We emphasize that the coordinate expressions for the vector fields V and W in (I.2.2) and (I.2.4) are given in distinct reference frames. V is naturally written in the space frame Σ_s whereas W is naturally written in the body frame Σ_b.

We now define a collection of vectors which will be useful to describe the system and write the equations of motion ahead. This list may provide a convenient reference for the reader to come back to when needed, so we include the definition of the vectors $\boldsymbol{x}$ and $\boldsymbol{\rho}$ given above. Some of the vectors are illustrated in Fig 2.3.

- $\boldsymbol{x} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OP}$, connecting the origin of the spatial frame and the contact point, with respect to the space frame Σ_s.
- $\boldsymbol{\rho} \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{CP}$, connecting the center of mass and the contact point, with respect to the body frame Σ_b.
- $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3$ are the *Poisson vectors*, whose components are the coordinates of the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ with respect to the body frame Σ_b. They are pairwise orthogonal unit vectors forming the rows of the attitude matrix B and given by
$$\boldsymbol{\alpha} = B^{-1}\boldsymbol{e}_1, \qquad \boldsymbol{\beta} = B^{-1}\boldsymbol{e}_2, \qquad \boldsymbol{\gamma} = B^{-1}\boldsymbol{e}_3. \tag{I.2.7}$$
- $\boldsymbol{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OC}$, connecting the origin of the spatial frame and the center of mass, with respect to the spatial frame Σ_s.
- $\boldsymbol{U} = (U_1, U_2, U_3) \in \mathbb{R}^3$ are the coordinates of the vector $\overrightarrow{OC}$ with respect to the body frame Σ_b (so $\boldsymbol{U} = B^{-1}\boldsymbol{u}$).
- $\boldsymbol{\omega} \in \mathbb{R}^3$ are the coordinates of the angular velocity vector with respect to the spatial frame Σ_s.
- $\boldsymbol{\Omega} \in \mathbb{R}^3$ are the coordinates of the angular velocity vector with respect to the body frame Σ_b (so $\boldsymbol{\Omega} = B^{-1}\boldsymbol{\omega}$).

We recall (see e.g. [55]) that the space and body coordinate representations of the angular velocity are defined by the left and right trivializations:

$$B^{-1}\dot{B} = \hat{\boldsymbol{\Omega}}, \qquad \dot{B}B^{-1} = \hat{\boldsymbol{\omega}},$$

[1] Note that, in general, this choice of third axis may be incompatible with the assumption that $\{\boldsymbol{E}_i\}$ are aligned with the principal axes of inertia.

where, for $\boldsymbol{a} \in \mathbb{R}^3$, the notation $\hat{\boldsymbol{a}}$ stands for the unique 3×3 skew-symmetric real matrix such that $\hat{\boldsymbol{a}}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ for all $\boldsymbol{b} \in \mathbb{R}^3$. It is well-known that the mapping $\hat{}: (\mathbb{R}^3, \times) \to \mathfrak{so}(3)$ is a Lie algebra isomorphism. The first of the above identities is in fact equivalent to the following well-known evolution equations for the Poisson vectors

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}. \tag{I.2.8}$$

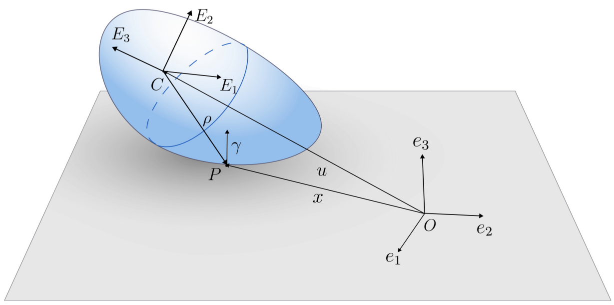


Figure 2.3: Graphic representation of the vectors $\boldsymbol{\rho}, \boldsymbol{\gamma} \in \mathbb{R}^3$ (which are written with respect to the body frame $\Sigma_b = \{C; \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$) and $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^3$ (which are written with respect to the spatial frame $\Sigma_s = \{O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$).

The relation

$$\boldsymbol{x} = \boldsymbol{u} + B\boldsymbol{\rho}, \tag{I.2.9}$$

follows from the definitions of the vectors $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{\rho}$. Taking scalar product with $\boldsymbol{e}_3$ on both sides, shows that the holonomic constraint $x_3 = 0$ may be rewritten as

$$u_3 = -\langle \boldsymbol{\rho}, \boldsymbol{\gamma} \rangle. \tag{I.2.10}$$

Following the approach of previous references [23, 12], throughout this thesis, we use the Gauss map $\boldsymbol{n}_b : \mathcal{S} \to S^2 \subset \mathbb{R}^3$ of the surface of the body to obtain a functional relation between $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$:

$$\boldsymbol{n}_b(\boldsymbol{\rho}) = -\boldsymbol{\gamma}, \qquad \boldsymbol{\rho} = \boldsymbol{n}_b^{-1}(-\boldsymbol{\gamma}). \tag{I.2.11}$$

The validity of these relations follows from our assumption that the surface $\mathcal{S}$ of the body is smooth and convex, since it guarantees that the Gauss map $\boldsymbol{n}_b$ is a diffeomorphism. Note that the tangency condition (I.2.5) implies

$$\langle \boldsymbol{W}_b(\boldsymbol{\rho}), \boldsymbol{\gamma} \rangle = 0.$$

Now, the velocity of the material point in contact with the plane, written in the space frame Σ_s, is given by $\dot{\boldsymbol{u}} + B(\boldsymbol{\Omega} \times \boldsymbol{\rho})$. Therefore, imposing C2′ leads to the nonholonomic constraint:

$$\dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}) + \boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\boldsymbol{\rho}), \tag{I.2.12}$$

where $\boldsymbol{x}$ is expressed in terms of $\boldsymbol{u}$, B and $\boldsymbol{\rho}$ by (I.2.9).

Using the kinematic condition[2] $\langle \dot{\boldsymbol{\rho}}, \boldsymbol{\gamma} \rangle = 0$, and the properties of $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ mentioned above, it is an exercise to show that the third component of (I.2.12) is the time derivative of (I.2.10). Therefore, (I.2.12) defines two independent nonholonomic constraints.

We now specify in more detail the geometry of the constraints. It is convenient to embed the configuration space Q of our problem in $\mathbb{R}^3 \times \mathrm{SO}(3)$ as the 5-dimensional submanifold

$$Q = \{(\boldsymbol{u}, B) \in \mathbb{R}^3 \times \mathrm{SO}(3) \;:\; \text{equation (I.2.10) holds}\}.$$

In the above definition of Q, and in what follows, the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ should be understood as functions of the attitude matrix B via the relations (I.2.7) and (I.2.11). The nonholonomic constraints (I.2.12) determine a rank 3 affine distribution $\mathcal{A} \subset TQ$ which is the phase space of our system and is convenient to embed inside $T(\mathbb{R}^3 \times \mathrm{SO}(3)) = T\mathbb{R}^3 \times T\mathrm{SO}(3) = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3$, where the identification $T\mathrm{SO}(3) = \mathrm{SO}(3) \times \mathbb{R}^3$ is done using the left trivialization. Specifically, we have

$$TQ = \{(\boldsymbol{u}, \dot{\boldsymbol{u}}, B, \boldsymbol{\Omega}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \;: \\ \text{(I.2.10) and the third component of (I.2.12) hold}\},$$

and

$$\mathcal{A} = \{(\boldsymbol{u}, \dot{\boldsymbol{u}}, B, \boldsymbol{\Omega}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \;:\; \text{equations (I.2.10) and (I.2.12) hold}\}.$$

As a manifold, the affine distribution $\mathcal{A}$ has dimension 8. It will be convenient to express $\mathcal{A} = \mathcal{D} + Z$ where $\mathcal{D} \subset TQ$ is the model linear distribution and $Z \in \mathfrak{X}(Q)$ is a vector field. These can be taken as

$$\mathcal{D} = \{(\boldsymbol{u}, \dot{\boldsymbol{u}}, B, \boldsymbol{\Omega}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \;:\; \dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}) \text{ and (I.2.10) holds}\}, \tag{I.2.13a}$$

$$Z(\boldsymbol{u}, B) = (\boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\boldsymbol{\rho}), \boldsymbol{0}), \tag{I.2.13b}$$

[2] Here and in what follows, $\dot{\boldsymbol{\rho}}$ is shorthand for $-D\boldsymbol{n}_b^{-1}(-\boldsymbol{\gamma})(\boldsymbol{\gamma} \times \boldsymbol{\Omega})$, which follows from (I.2.11) and (I.2.8).

where, as usual, $\boldsymbol{x}$ is expressed in terms of $\boldsymbol{u}$, B and $\boldsymbol{\rho}$ by (I.2.9).

2.2 Equations of motion

The Lagrangian $L : TQ \to \mathbb{R}$ is the sum of the kinetic energies of rotation and translation minus the gravitational potential energy. Working with the conventions of the previous section, we have

$$L(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{m}{2}\|\dot{\boldsymbol{u}}\|^2 + mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle, \tag{I.2.14}$$

where $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia tensor of the body, $m > 0$ is its total mass, and $g > 0$ is the gravitational constant.

We introduce the following vector $\boldsymbol{M} \in \mathbb{R}^3$, which is written in the body frame Σ_b, and is a generalization of the angular momentum of the body about its contact point:

$$\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + m\boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho} - B^{-1}\boldsymbol{V}_s(\boldsymbol{x}) - \boldsymbol{W}_b(\boldsymbol{\rho})), \tag{I.2.15}$$

where, according to (I.2.9), we have $\boldsymbol{x} = \boldsymbol{u} + B\boldsymbol{\rho}$. The dependence of $\boldsymbol{M}$ on the angular velocity $\boldsymbol{\Omega}$ is affine linear, depending parametrically on $\boldsymbol{u}$ and B, and may be inverted to obtain

$$\boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{u}, B) = A(\boldsymbol{\gamma})\left(\boldsymbol{M} + \boldsymbol{\zeta}(B, u) + \frac{m\langle \boldsymbol{M} + \boldsymbol{\zeta}(B, u), A(\boldsymbol{\gamma})\boldsymbol{\rho}\rangle}{1 - m\langle A(\boldsymbol{\gamma})\boldsymbol{\rho}, \boldsymbol{\rho}\rangle}\boldsymbol{\rho}\right), \tag{I.2.16}$$

where the 3×3 matrix $A(\boldsymbol{\gamma})$ and the vector $\boldsymbol{\zeta}(B, u) \in \mathbb{R}^3$ are given by

$$A(\boldsymbol{\gamma}) = (\mathbb{I} + m\|\boldsymbol{\rho}\|^2\mathrm{id})^{-1} \quad \text{and} \quad \boldsymbol{\zeta}(B, \boldsymbol{u}) = m\boldsymbol{\rho} \times (B^{-1}\boldsymbol{V}_s(\boldsymbol{x}) + \boldsymbol{W}_b(\boldsymbol{\rho})), \tag{I.2.17}$$

where id denotes the 3×3 identity matrix. To make sense of the matrix A as a function of $\boldsymbol{\gamma}$ recall that $\boldsymbol{\rho}$ is expressed as a function of $\boldsymbol{\gamma}$ by (I.2.11). On the other hand, we think of the vector $\boldsymbol{\zeta}$ as a function of $(B, \boldsymbol{u})$, since $\boldsymbol{x}$ may be expressed as a function of B and $\boldsymbol{u}$ by (I.2.9) (and $\boldsymbol{\rho}$ is a function of B through its dependence on $\boldsymbol{\gamma} = B^{-1}\boldsymbol{e}_3$). Considering that $\boldsymbol{\Omega}$ in (I.2.16) is written as a function of $(\boldsymbol{M}, \boldsymbol{u}, B)$ it would have been slightly more appropriate to write $A = A(B)$ in (I.2.16) but the notation $A = A(\boldsymbol{\gamma})$ is useful in the analysis of the equations below.

The above expression for $\boldsymbol{\Omega}$ allows us to give the following alternative parametrization of the affine distribution $\mathcal{A}$:

$$\mathcal{A} = \{(\boldsymbol{u}, \dot{\boldsymbol{u}}, B, \boldsymbol{M}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 : \\ \text{(I.2.10) and (I.2.12) hold with } \boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{u}, B)\}.$$

Proposition I.2.1. *The equations of motion of the problem are the restriction of*

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega} + m\dot{\boldsymbol{\rho}} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) + mg\boldsymbol{\rho} \times \boldsymbol{\gamma} \\ + m(B^{-1}\boldsymbol{V}_s(\boldsymbol{x}) + \boldsymbol{W}_b(\boldsymbol{\rho})) \times (\dot{\boldsymbol{\rho}} + \boldsymbol{\Omega} \times \boldsymbol{\rho}), \quad \text{(I.2.18a)}$$

$$\dot{B} = B\hat{\boldsymbol{\Omega}}, \quad \text{(I.2.18b)}$$

$$\dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}) + \boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\boldsymbol{\rho}), \quad \text{(I.2.18c)}$$

to the invariant set defined by (I.2.10) *where* $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{u}, B)$ *as in* (I.2.16)*, and, in accordance with* (I.2.9)*, we have* $\boldsymbol{x} = \boldsymbol{u} + B\boldsymbol{\rho}$.

Note that (I.2.18a) is a momentum balance equation and instead (I.2.18b) and (I.2.18c) are kinematic relations that follow from the considerations in section 2.1.

Proof. As mentioned above, (I.2.18b) and (I.2.18c) are given by the definition of $\boldsymbol{\Omega}$ and the constraints (I.2.12). In order to obtain (I.2.18a), we begin by writing the equations of motion as

$$m\ddot{\boldsymbol{u}} = -mg\boldsymbol{e}_3 + \boldsymbol{R}_1, \qquad \mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} + \boldsymbol{R}_2, \quad \text{(I.2.19)}$$

where $\boldsymbol{R}_1, \boldsymbol{R}_2$ are the nonholonomic reaction forces. According to the Lagrange-d'Alembert principle,

$$\langle \boldsymbol{R}_1, \dot{\boldsymbol{u}} \rangle + \langle \boldsymbol{R}_2, \boldsymbol{\Omega} \rangle = 0$$

for all $\dot{\boldsymbol{u}}$ and $\boldsymbol{\Omega}$ satisfying the linear nonholonomic constraint specified by $\mathcal{D}$ in (I.2.13a), namely, $\dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega})$. This implies

$$\langle \boldsymbol{R}_1, B(\boldsymbol{\rho} \times \boldsymbol{\Omega}) \rangle + \langle \boldsymbol{R}_2, \boldsymbol{\Omega} \rangle = 0 \qquad \text{for all } \boldsymbol{\Omega}.$$

So we get

$$\boldsymbol{R}_2 = \boldsymbol{\rho} \times (B^{-1}\boldsymbol{R}_1). \quad \text{(I.2.20)}$$

On the other hand, differentiating the constraints (I.2.12) gives

$$\ddot{\boldsymbol{u}} = \dot{B}(\boldsymbol{\rho} \times \boldsymbol{\Omega} + \boldsymbol{W}_b(\boldsymbol{\rho})) + B(\dot{\boldsymbol{\rho}} \times \boldsymbol{\Omega} + \boldsymbol{\rho} \times \dot{\boldsymbol{\Omega}} + \boldsymbol{W}_b'(\boldsymbol{\rho})\dot{\boldsymbol{\rho}}) + \boldsymbol{V}_s'(\boldsymbol{x})\dot{\boldsymbol{x}}.$$

And from equation (I.2.19), we have $B^{-1}\boldsymbol{R}_1 = mB^{-1}\ddot{\boldsymbol{u}} + mg\boldsymbol{\gamma}$, so

$$B^{-1}\boldsymbol{R}_1 = m\boldsymbol{\Omega}\times(\boldsymbol{\rho}\times\boldsymbol{\Omega}+\boldsymbol{W}(\boldsymbol{\rho}))+m(\dot{\boldsymbol{\rho}}\times\boldsymbol{\Omega}+\boldsymbol{\rho}\times\dot{\boldsymbol{\Omega}}+\boldsymbol{W}_b'(\boldsymbol{\rho})\dot{\boldsymbol{\rho}})+B^{-1}(\boldsymbol{V}_s'(\boldsymbol{x})\dot{\boldsymbol{x}})+mg\boldsymbol{\gamma}.$$

Using this expression and (I.2.20) to express $\boldsymbol{R}_2$ and then substituting in equation (I.2.19) gives

$$\begin{aligned}\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega}\times\boldsymbol{\Omega} + m\boldsymbol{\rho}\times(\boldsymbol{\Omega}\times(\boldsymbol{\rho}\times\boldsymbol{\Omega})) + m\boldsymbol{\rho}\times(\dot{\boldsymbol{\rho}}\times\boldsymbol{\Omega}) + m\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\dot{\boldsymbol{\Omega}}) + mg\boldsymbol{\rho}\times\boldsymbol{\gamma}\\ + m\boldsymbol{\rho}\times(\boldsymbol{\Omega}\times\boldsymbol{W}_b(\boldsymbol{\rho})) + m\boldsymbol{\rho}\times(\boldsymbol{W}_b'(\boldsymbol{\rho})\dot{\boldsymbol{\rho}}) + m\boldsymbol{\rho}\times(B^{-1}\boldsymbol{V}_s'(\boldsymbol{x})\dot{\boldsymbol{x}}).\end{aligned}$$

Starting with the definition (I.2.15) of $\boldsymbol{M}$, some elementary calculations show that the above equation is equivalent to (I.2.18a). □

2.3 The case $V = 0$

If $V = 0$, the system (I.2.18) has an $\mathrm{SE}(2)$-symmetry corresponding to translations and rotations of the plane Π. Denoting elements in $\mathrm{SE}(2)$ as $(R_\theta, \boldsymbol{a})$ with

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad \boldsymbol{a} = (a_1, a_2, 0)^T,$$

and group operation

$$(R_\theta, \boldsymbol{a})(R_{\tilde{\theta}}, \tilde{\boldsymbol{a}}) = (R_{\theta+\tilde{\theta}}, R_\theta\tilde{\boldsymbol{a}} + \boldsymbol{a}),$$

then the action of $\mathrm{SE}(2)$ on Q is the restriction to Q of the following action of $\mathrm{SE}(2)$ on $\mathbb{R}^3 \times \mathrm{SO}(3)$

$$(R_\theta, \boldsymbol{a})\cdot(\boldsymbol{u}, B) = (R_\theta\boldsymbol{u} + \boldsymbol{a}, R_\theta B). \tag{I.2.21}$$

It is immediate to check that $u_3, \boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ are invariant under this action so, in view of (I.2.10), the action indeed restricts from $\mathbb{R}^3\times\mathrm{SO}(3)$ to Q. The lifted action of $\mathrm{SE}(2)$ on TQ is given by

$$(R_\theta, \boldsymbol{a})\cdot(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) = (R_\theta\boldsymbol{u} + \boldsymbol{a}, R_\theta B, R_\theta\dot{\boldsymbol{u}}, \boldsymbol{\Omega}).$$

It is not difficult to see that the Lagrangian L, given by (I.2.14), and the linear distribution $\mathcal{D}$, given by (I.2.13a), are invariant under this lifted action. If $V = 0$ then also $\mathcal{A}$ is invariant and the equations (I.2.18) may be reduced by this symmetry. The reduced phase space $\mathcal{A}/\mathrm{SE}(2)$ is diffeomorphic to $\mathbb{R}^3 \times S^2$ and may be parametrized by $\boldsymbol{M} \in \mathbb{R}^3$ and the Poisson vector $\boldsymbol{\gamma} \in S^2$. To obtain the reduced equations, note that the constraints (I.2.12) simplify to

$$\dot{\boldsymbol{u}} = B(\boldsymbol{\rho} \times \boldsymbol{\Omega}) + B\boldsymbol{W}_b(\boldsymbol{\rho}),$$

whose right-hand side is independent of $\boldsymbol{u}$. Also, the expression (I.2.16) for $\boldsymbol{\Omega}$ is independent of $\boldsymbol{u}$. Moreover, since the dependence of $\boldsymbol{\rho}$ on B is only through the Poisson vector $\boldsymbol{\gamma}$, we may write

$$\boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{\gamma}) = A(\boldsymbol{\gamma}) \left(\boldsymbol{M} + \frac{m\langle \boldsymbol{M} + m\boldsymbol{\rho} \times \boldsymbol{W}_b(\boldsymbol{\rho}), A(\boldsymbol{\gamma})\boldsymbol{\rho} \rangle}{1 - m\langle A(\boldsymbol{\gamma})\boldsymbol{\rho}, \boldsymbol{\rho} \rangle} \boldsymbol{\rho} - m\boldsymbol{\rho} \times \boldsymbol{W}_b(\boldsymbol{\rho}) \right), \tag{I.2.22}$$

which leads to a decoupled system for $(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2$. We give the reduced equations on $\mathcal{A}/\mathrm{SE}(2)$ as the following.

Proposition I.2.2. *The reduced equations on $\mathcal{A}/\mathrm{SE}(2)$ are the restriction of*

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega} + m\dot{\boldsymbol{\rho}} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) + mg\boldsymbol{\rho} \times \boldsymbol{\gamma} + m\boldsymbol{W}_b(\boldsymbol{\rho}) \times (\dot{\boldsymbol{\rho}} + \boldsymbol{\Omega} \times \boldsymbol{\rho}), \tag{I.2.23a}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \tag{I.2.23b}$$

to the invariant set $\|\boldsymbol{\gamma}\|^2 = 1$ where $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{\gamma})$ is given by (I.2.22).

2.4 Moving energy

It is well-known that nonholonomic systems with affine constraints do not in general preserve the energy. However, as first noticed in [31] (see also [15] and [29]), if the affine terms correspond to the infinitesimal generator of a continuous symmetry of the Lagrangian, then a modification of the energy, which we term *moving energy* in accordance with [31, 29], arises as a first integral. Below, we discuss some instances of existence of a preserved moving energy in our problem.

2.4.1 The case $W = 0$

As mentioned above, for a general convex body, the Lagrangian L is invariant under the lifted $\mathrm{SE}(2)$ action on TQ given by (I.2.21). If $W = 0$ and $V \in \mathfrak{X}(\Pi)$ coincides with the infinitesimal generator of the $\mathrm{SE}(2)$ action on Q, given by (I.2.21), then the system possesses a conserved moving energy. There are two possibilities for such infinitesimal generator. The first one is a steady rotation with angular frequency $\eta \in \mathbb{R}$ about a fixed point on the plane Π that can be taken as our origin O, namely

$$\boldsymbol{V}_s(\boldsymbol{x}) = \eta \boldsymbol{e}_3 \times \boldsymbol{x},$$

which is precisely the form of $\boldsymbol{V}_s$ given in (I.2.3) for the uniformly rotating plane. In this case, the conserved moving energy $E_{mov} : \mathcal{A} \to \mathbb{R}$ was found in [29] and is given by

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{m}{2}\|\boldsymbol{\rho}\times\boldsymbol{\Omega}\|^2 - mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle + \eta\langle \mathbb{I}\boldsymbol{\Omega} - m\boldsymbol{\rho}\times(\boldsymbol{\Omega}\times\boldsymbol{\rho}), \boldsymbol{\gamma}\rangle + \frac{1}{2}m\eta^2(\|\boldsymbol{\rho}\|^2 - \|\boldsymbol{u}\|^2).$$

The second possibility is that of a steady linear translation; namely,

$$V_s(\boldsymbol{x}) = \boldsymbol{v} = (v_1, v_2, 0), \tag{I.2.24}$$

for constant $v_1, v_2 \in \mathbb{R}$. In this case, following the prescription in [15, 31, 29], one computes the conserved moving energy to be

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{m}{2}\|\boldsymbol{\rho}\times\boldsymbol{\Omega}\|^2 - mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle.$$

2.4.2 The case of an axially symmetric rigid body

A further symmetry of the Lagrangian arises when the body possesses an axial symmetry, and is hence a body of revolution. Assuming that the symmetry axis is aligned with the third axis $\boldsymbol{E}_3$ of the moving frame Σ_b, then we consider the $\mathrm{SO}(2)$ action on Q given by

$$R_\phi \cdot (\boldsymbol{u}, B) = (\boldsymbol{u}, BR_\phi^{-1}), \tag{I.2.25}$$

where

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is immediate to check that under this action $\boldsymbol{\gamma}$ transforms to $R_\phi\boldsymbol{\gamma}$. Moreover, for an axisymmetric body, the Gauss map is equivariant and $\boldsymbol{\rho}$ transforms to $R_\phi\boldsymbol{\rho}$. It follows from (I.2.12) that (I.2.25) determines a well-defined $\mathrm{SO}(2)$ action on Q. The associated lifted action to TQ is

$$R_\phi \cdot (\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) = (\boldsymbol{u}, BR_\phi^{-1}, \dot{\boldsymbol{u}}, R_\phi\boldsymbol{\Omega}).$$

Our assumption that the body is axisymmetric implies $I_1 = I_2$ and it can be checked that the Lagrangian L is invariant.

Assume for simplicity that $V = 0$. If the vector field $W \in \mathfrak{X}(\mathcal{S})$ is chosen as an infinitesimal generator of the action (I.2.21), namely, if

$$\boldsymbol{W}_b(\boldsymbol{\rho}) = \sigma\boldsymbol{\rho}\times\boldsymbol{E}_3,$$

for $\sigma \in \mathbb{R}$, then $\boldsymbol{W}_b$ coincides with the expression (I.2.6) for a cat's toy mechanism. So the system under consideration corresponds to the one depicted in Fig 2.2b with the additional assumption that the internal rigid body has the same axial symmetry as the shell. This system will be studied in more detail in Chapter 4 ahead. Following the prescription in [15, 31, 29], one finds a conserved moving energy given by

$$E_{mov} = \frac{1}{2}\langle \mathbb{I}(\boldsymbol{\Omega} + \sigma \boldsymbol{E}_3), \boldsymbol{\Omega} + \sigma \boldsymbol{E}_3 \rangle + \frac{m}{2}\|\boldsymbol{\rho} \times (\boldsymbol{\Omega} + \sigma \boldsymbol{E}_3)\|^2 - mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle. \qquad \text{(I.2.26)}$$

This moving energy (I.2.26) is actually also a first integral of the system when $\boldsymbol{V}_s$ is a nonzero constant vector field (given by (I.2.24)).

Finally, we indicate that, when the axisymmetric body with a cat's toy mechanism rolls on a uniformly rotating plane (i.e.$\boldsymbol{V}_s$ is given by (I.2.3) and $\boldsymbol{W}_b$ by (I.2.6)), one may combine the $\mathrm{SE}(2)$ and $\mathrm{SO}(2)$ symmetries to derive the conserved moving energy:

$$\begin{aligned} E_{mov} =& \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \langle \mathbb{I}\boldsymbol{\Omega}, -\eta\boldsymbol{\gamma} + \sigma \boldsymbol{E}_3\rangle + \frac{m}{2}\|\boldsymbol{\rho} \times (\boldsymbol{\Omega} + \sigma \boldsymbol{E}_3)\|^2 \\ &- m\eta\langle \boldsymbol{\rho} \times (\boldsymbol{\Omega} + \sigma \boldsymbol{E}_3), \boldsymbol{\rho} \times \boldsymbol{\gamma}\rangle + \frac{m\eta^2}{2}(\|\boldsymbol{\rho}\|^2 - \|\boldsymbol{u}\|^2) - mg\langle \boldsymbol{\rho}, \boldsymbol{\gamma}\rangle. \end{aligned} \qquad \text{(I.2.27)}$$

3 A dynamically balanced sphere

Throughout this chapter we consider the special case in which the surface of the convex body is spherical, with radius $r > 0$, and the center of mass coincides with the geometric center. If both V and W vanish, we recover the classical Chaplygin ball problem [19]. Other cases previously considered for non-vanishing V, W are found in [6, 45, 5]. Here we consider the general case.

The relation (I.2.11) between $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ is

$$\boldsymbol{\rho} = -r\boldsymbol{\gamma}, \tag{I.3.1}$$

and (I.2.9) becomes

$$\boldsymbol{x} = \boldsymbol{u} - r\boldsymbol{e}_3. \tag{I.3.2}$$

In view of (I.3.1), we have $\boldsymbol{\gamma} \times \boldsymbol{\rho} = 0$ and $\dot{\boldsymbol{\rho}} = \boldsymbol{\rho} \times \boldsymbol{\Omega}$, so equation (I.2.18a) simplifies to

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}, \tag{I.3.3}$$

where in this case $\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + mr^2\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma}) + mr\boldsymbol{\gamma} \times (B^{-1}\boldsymbol{V}_s(\boldsymbol{x}) + \boldsymbol{W}_b(\boldsymbol{\rho}))$. This remarkable simplification implies that the vector $\boldsymbol{M}$, as seen in the spatial frame Σ_s is constant. As a consequence, we have.

Proposition I.3.1. *For any $V \in \mathfrak{X}(\Pi)$ and $W \in \mathfrak{X}(\mathcal{S})$, the system has first integrals*

$$\langle \boldsymbol{M}, \boldsymbol{\alpha} \rangle, \qquad \langle \boldsymbol{M}, \boldsymbol{\beta} \rangle \qquad \text{and} \qquad \langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle,$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are given by (I.2.7).

The proof is an immediate consequence of (I.3.3) and (I.2.7). The existence of these first integrals for some particular vector fields $V \in \mathfrak{X}(\Pi)$ and $W \in \mathfrak{X}(\mathcal{S})$ had been indicated in previous references [53, 63, 5, 6]. Their existence for general vector fields is actually an instance of a result which we develop on appendix B. As may be verified, the linear distribution $\mathcal{D}$ and the Lagrangian L simplify (up to the addition of a constant term in the Lagrangian that may be discarded) to

$$\mathcal{D} = \{(\boldsymbol{u}, \dot{\boldsymbol{u}}, B, \boldsymbol{\Omega}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 : \dot{\boldsymbol{u}} = -r(\boldsymbol{e}_3 \times \boldsymbol{\omega}) \text{ and (I.2.10) holds}\},$$
$$L(\boldsymbol{u}, B, \dot{\boldsymbol{u}}, \boldsymbol{\Omega}) = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega}\rangle + \frac{m}{2}\|\dot{\boldsymbol{u}}\|^2.$$

The above expressions for $\mathcal{D}$ and L do not explicitly depend on $\boldsymbol{u}$ and B. This independence is due to a very special type of symmetry: if we interpret our configuration space Q as a Lie group (isomorphic to the direct product $\mathbb{R}^2 \times \mathrm{SO}(3)$), then the distribution $\mathcal{D}$ is right invariant, and the Lagrangian L is left invariant. Therefore, the underlying linear problem is an LR system [65]. Proposition B.3 in the appendix is a robust result on the existence of first integrals of affine generalizations of LR systems which provides an explanation of the mechanism responsible for the validity of Proposition I.3.1.

Below, we consider additional aspects of the dynamics for particular choices of V and W.

3.1 The case $V = 0$

As stated in Section 2.3, when $V = 0$ the system has an SE(2)-symmetry and we can consider the reduced system. The reduced equations of motion are

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \tag{I.3.4}$$

with $\boldsymbol{M} = \mathbb{I}\boldsymbol{\Omega} + mr^2\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma}) + mr\boldsymbol{\gamma} \times \boldsymbol{W}_b(\boldsymbol{\rho})$. As a consequence of Proposition I.3.1, the reduced system (I.3.4) has first integrals

$$\|\boldsymbol{M}\|^2, \qquad \langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle \qquad \text{and} \qquad \|\boldsymbol{\gamma}\|^2 = 1. \tag{I.3.5}$$

These first integrals are insufficient to conclude integrability of (I.3.4), for instance, using the Jacobi last multiplier theorem [2] (which would require existence of an additional independent first integral and a smooth invariant measure).

Below we only consider the simplest non-zero choice of $W \in \mathfrak{X}(\mathcal{S})$, corresponding to a cat's toy mechanism (described in Chapter 2). Moreover, we will assume that the axis of rotation of the mechanism is aligned with the third principal axis of the sphere (see Fig 3.1). The corresponding form of $\boldsymbol{W}_b$ is given by (I.2.6) which in view of (I.3.1) becomes

$$\boldsymbol{W}_b(\boldsymbol{\rho}) = -r\sigma\boldsymbol{\gamma} \times \boldsymbol{E}_3.$$

For future reference we note that in the case under consideration, we may use (I.2.16) to write $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{\gamma})$ as

$$\boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{\gamma}) = A\left(\boldsymbol{M} + \boldsymbol{\zeta}(\boldsymbol{\gamma}) + \frac{mr^2\langle \boldsymbol{M} + \boldsymbol{\zeta}(\boldsymbol{\gamma}), A\boldsymbol{\gamma}\rangle}{1 - mr^2\langle A\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle}\boldsymbol{\gamma}\right), \tag{I.3.6}$$

where the matrix A is constant

$$A = (\mathbb{I} + mr^2\mathrm{id})^{-1}, \tag{I.3.7}$$

and the vector $\boldsymbol{\zeta}$ only depends on $\boldsymbol{\gamma}$ by

$$\boldsymbol{\zeta}(\boldsymbol{\gamma}) = mr^2\sigma\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{E}_3).$$

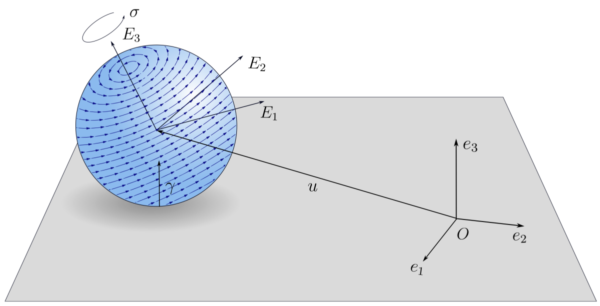


Figure 3.1: Dynamically balanced sphere with a cat's toy mechanism. It is assumed that the axis of rotation of the shell is a principal axis of inertia of the sphere.

The analysis that we present below treats separately the case in which $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ are parallel. Interestingly, in this special case the reduced dynamics is integrable (actually periodic), whereas in the general case it appears to be chaotic.

3.1.1 The case M parallel to γ

Since both $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ are body representations of vectors that are fixed in space, if they are initially parallel they will remain parallel for all time. As we prove below, the dynamics restricted to these initial conditions is integrable and in fact periodic.

It is not hard to see that those $(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2$ for which $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ are parallel are critical points of the first integrals (I.3.5). The connected components of their joint level sets are diffeomorphic to S^2 and may be parametrized by γ by putting

$$\boldsymbol{M} = \pm\|\boldsymbol{M}\|\gamma. \tag{I.3.8}$$

Writing $\lambda = \pm\|\boldsymbol{M}\|$, we may use (I.3.6) to write $\boldsymbol{\Omega}$ as a function of γ depending parametrically on λ,

$$\boldsymbol{\Omega}(\gamma;\lambda) = A\left(\lambda\gamma + \zeta(\gamma) + \frac{mr^2\langle\lambda\gamma + \zeta(\gamma), A\gamma\rangle}{1 - mr^2\langle A\gamma,\gamma\rangle}\gamma\right),$$

with $\zeta(\gamma) = mr^2\sigma\gamma \times (\gamma \times \boldsymbol{E}_3)$. The restriction of (I.3.4) to the 2-dimensional invariant submanifold determined by the condition $\boldsymbol{M} = \lambda\gamma$ is described by the equation

$$\dot{\gamma} = \gamma \times \boldsymbol{\Omega}(\gamma;\lambda). \tag{I.3.9}$$

Below we exhibit a smooth first integral and an invariant measure depending on the value of $\lambda \in \mathbb{R}$. It follows that all non-equilibrium solutions $\gamma(t)$ of (I.3.9) are periodic. Therefore, in view of (I.3.8), we also conclude that the generic solutions of (I.3.4) with the initial conditions under consideration are periodic.

Let ε be the non-dimensional number

$$\varepsilon := \frac{\|\boldsymbol{M}\|}{mr^2|\sigma|}. \tag{I.3.10}$$

If

$$\varepsilon > \frac{I_3}{I_3 + mr^2}, \tag{I.3.11}$$

then the quantity $\lambda(I_3 + mr^2) + mr^2\sigma I_3\gamma_3$ is nonzero for all $\gamma_3 \in [-1,1]$, and

$$f(\gamma) = \frac{|\lambda(I_3 + mr^2) + mr^2\sigma I_3\gamma_3|^{-\frac{mr^2}{I_3}}}{\sqrt{1 - mr^2\langle\gamma, A\gamma\rangle}}, \tag{I.3.12}$$

with A given by (I.3.7), is a smooth function of $\gamma \in S^2$ which can be checked to be a first integral of (I.3.9). Furthermore, also under the assumption (I.3.11), one can directly check that $\mu(\gamma)d\gamma$ with

$$\mu(\gamma) = |\lambda(I_3 + mr^2) + mr^2\sigma I_3\gamma_3|^{-1},$$

is an invariant measure (with smooth positive density).

If the complementary inequality of (I.3.11) holds, namely if

$$\varepsilon \le \frac{I_3}{I_3 + mr^2},$$

then f as defined by (I.3.12) is no longer a smooth function on S^2 since the expression inside the absolute value vanishes along the parallel of S^2 given by

$$\gamma_3 = -\frac{\lambda}{mr^2\sigma}\left(\frac{I_3 + mr^2}{I_3}\right) \in [-1, 1]. \tag{I.3.13}$$

Using (I.3.9) it is easy to show that this parallel is invariant. Actually, its internal dynamics is given by

$$\dot{\gamma}_1 = -\kappa\gamma_2 \qquad \dot{\gamma}_2 = \kappa\gamma_1,$$

with $\kappa = \frac{mr^2\sigma}{I_3+mr^2}$. In this case, we may use f to construct a smooth first integral $g : S^2 \to \mathbb{R}$ by

$$g(\gamma) = \begin{cases} \exp(-f(\gamma)) & \text{if } \gamma_3 \neq -\frac{\lambda}{mr^2\sigma}\left(\frac{I_3+mr^2}{I_3}\right), \\ 0 & \text{if } \gamma_3 = -\frac{\lambda}{mr^2\sigma}\left(\frac{I_3+mr^2}{I_3}\right). \end{cases}$$

By construction, the invariant parallel (I.3.13) is the zero level set of g. A smooth invariant measure in this case is given by $\nu(\boldsymbol{\gamma})d\boldsymbol{\gamma}$ where

$$\nu(\boldsymbol{\gamma}) = \begin{cases} g(\boldsymbol{\gamma})\mu(\boldsymbol{\gamma}) & \text{if } \gamma_3 \neq -\frac{\lambda}{mr^2\sigma}\left(\frac{I_3+mr^2}{I_3}\right), \\ 0 & \text{if } \gamma_3 = -\frac{\lambda}{mr^2\sigma}\left(\frac{I_3+mr^2}{I_3}\right). \end{cases}$$

We notice that the density ν is smooth and non-negative on S^2 but vanishes along the invariant parallel (I.3.13) which has measure zero. The relevance of this kind of invariant measures in nonholonomic mechanics was recently indicated in [39].

3.1.2 The general case (M and γ not parallel)

In this case, the first integrals (I.3.12) are independent and their level sets are 3-dimensional submanifolds of the phase space $\mathbb{R}^3 \times S^2$. The dynamics can be numerically investigated using a 2-dimensional Poincaré map. Below we present some numerical experiments assuming $\langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle = 0$ which lead us to conjecture that the dynamics is chaotic.

Poincaré map

We borrow techniques from [5, 12] to construct our Poincaré section. We begin by restricting the system to the four-dimensional level manifold $\mathcal{M}_4$ of the first integrals $\langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle$ and $\|\boldsymbol{\gamma}\|^2$,

$$\mathcal{M}_4 = \{(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,:\, \langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle = 0 \,\text{and}\, \|\boldsymbol{\gamma}\|^2 = 1\}.$$

In this way, we obtain a four-dimensional system with first integral $\|\boldsymbol{M}\|^2 = G^2$. To parametrize $\mathcal{M}_4$, we use the Andoyer-Deprit variables (L, G, l, g) defined by

$$M_1 = \sqrt{G^2 - L^2}\sin l, \qquad M_2 = \sqrt{G^2 - L^2}\cos l, \qquad M_3 = L$$
$$\gamma_1 = \frac{L}{G}\cos g \sin l + \sin g \cos l, \quad \gamma_2 = \frac{L}{G}\cos g \cos l - \sin g \sin l, \quad \gamma_3 = -\sqrt{1 - \frac{L^2}{G^2}},$$

where $l, g \in [0, 2\pi)$ and L, G satisfy the inequality $-1 \le \frac{L}{G} \le 1$. The system determines a three-dimensional flow on the fixed level set of the first integral $\|\boldsymbol{M}\|^2 = G^2$. We take the set $g = 0$ as a section of this flow to obtain a two-dimensional Poincaré map, which we parametrize by the variables $\left(l, \frac{L}{G}\right)$.

The Poincaré map, shown in Fig 3.2 for different values of ε (defined by (I.3.10)), resembles the Poincaré map of a non-integrable Hamiltonian system; we observe coexistence of chaotic regions and stability islands typical of KAM theory. These numerical experiments suggest that the system is non-integrable at the level $\langle \boldsymbol{M}, \gamma\rangle = 0$. We note that the experiments seem compatible with the existence of a smooth invariant measure, but we were unable to find it.

Limit cases of the dynamics

The numerical experiments in Fig 3.2 suggest that the dynamics is approximately integrable when the non-dimensional parameter ε is taken sufficiently large or small. Below we give an explanation of this phenomenon. We begin by writing

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_l + \boldsymbol{\Omega}_a, \tag{I.3.14}$$

where $\boldsymbol{\Omega}_a$ is the contribution to $\boldsymbol{\Omega}$ due to the presence of the cat's toy mechanism (i.e. if $\sigma = 0$ then $\boldsymbol{\Omega}_a = 0$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}_l$). Explicitly we have

$$\boldsymbol{\Omega}_l = A\left(\boldsymbol{M} + \frac{mr^2\langle \boldsymbol{M}, A\boldsymbol{\gamma}\rangle}{1 - mr^2\langle A\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle}\boldsymbol{\gamma}\right), \tag{I.3.15}$$

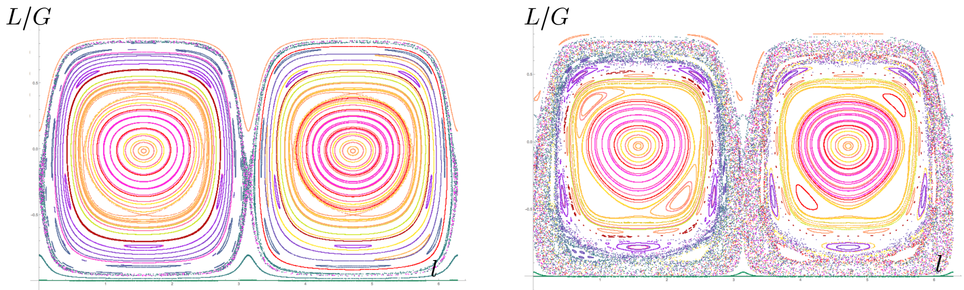
and

$$\boldsymbol{\Omega}_a = mr^2\sigma A\left(\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\boldsymbol{E}_3) + \frac{mr^2\langle\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\boldsymbol{E}_3), A\boldsymbol{\gamma}\rangle}{1 - mr^2\langle A\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle}\boldsymbol{\gamma}\right). \tag{I.3.16}$$

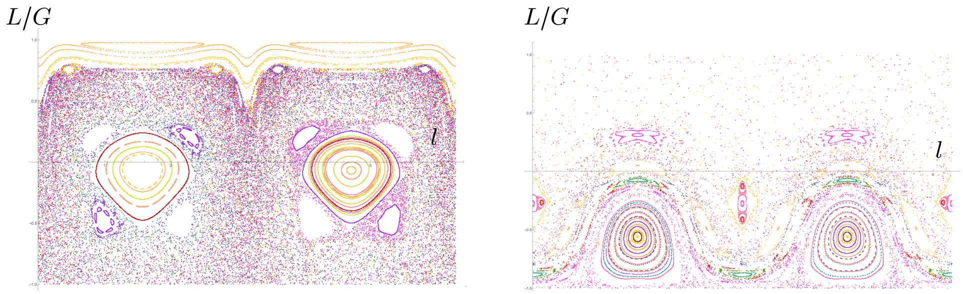
Introducing the non-dimensional time parameter $\tau = \sigma t$ the equations (I.3.4) may be written as

$$\boldsymbol{M}' = \varepsilon\left(\frac{mr^2}{\|\boldsymbol{M}\|}\boldsymbol{M}\times\boldsymbol{\Omega}_l\right) + \boldsymbol{M}\times\tilde{\boldsymbol{\Omega}}_a, \qquad \boldsymbol{\gamma}' = \varepsilon\left(\frac{mr^2}{\|\boldsymbol{M}\|}\gamma\times\boldsymbol{\Omega}_l\right) + \boldsymbol{\gamma}\times\tilde{\boldsymbol{\Omega}}_a,$$

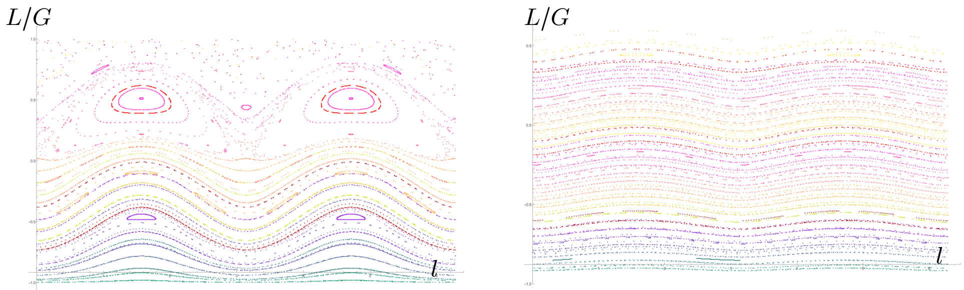
where $\tilde{\boldsymbol{\Omega}}_a := \frac{1}{\sigma}\boldsymbol{\Omega}_a$ and $' = \frac{d}{d\tau}$.



(a) $\varepsilon = 12$ **(b)** $\varepsilon = 4$



(c) $\varepsilon = 2$ **(d)** $\varepsilon = 0.4$



(e) $\varepsilon = 0.2$ **(f)** $\varepsilon = 0.04$

Figure 3.2: Poincaré map for the dynamically balanced sphere with a cat's toy mechanism for different values of ε given by (I.3.10). The system parameters were taken as $I_1 = 0.5,\ I_2 = 2.5,\ I_3 = 3,\ m = 1,\ r = 5, \sigma = 10,$ and the first integral $\langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle = 0$.

On the one hand, if $\varepsilon \gg 1$, then, neglecting the term with $\tilde{\boldsymbol{\Omega}}_a$, which encodes the effect of the cat's toy mechanism, we recover the vector field of the classical, integrable, Chaplygin sphere problem [19] multiplied by the overall factor $\frac{\varepsilon m r^2}{\|\boldsymbol{M}\|}$ which is constant along the flow.

On the other hand, if $\varepsilon \ll 1$, then, neglecting the term of order ε we obtain the equations

$$\boldsymbol{M}' = \boldsymbol{M} \times \tilde{\boldsymbol{\Omega}}_a \qquad \boldsymbol{\gamma}' = \boldsymbol{\gamma} \times \tilde{\boldsymbol{\Omega}}_a. \tag{I.3.17}$$

In addition to the first integrals (I.3.5), we now show that this system possesses an additional smooth first integral and a smooth invariant measure and is therefore integrable in virtue of Jacobi's last multiplier theorem [2]. To give the explicit form of these invariants, we proceed in analogy with the analysis in Section 3.1.1. We first observe that the set of points $(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2$ such that $\gamma_3 = 0$ is invariant. Actually, the dynamics along this set is simply harmonic. This follows from the observation that $\tilde{\boldsymbol{\Omega}}_a$ equals $-\kappa \boldsymbol{e}_3$, with $\kappa = \frac{mr^2\sigma}{I_3 + mr^2}$, when $\gamma_3 = 0$ (which can be deduced from the expression for $\boldsymbol{\Omega}_a$ in (I.3.16)). The additional smooth first integral of (I.3.17) only depends on $\boldsymbol{\gamma}$ and is given by:

$$k(\boldsymbol{\gamma}) = \begin{cases} \exp\left(\frac{|\gamma_3|^{-\frac{mr^2}{I_3}}}{\sqrt{1 - mr^2 \langle \boldsymbol{\gamma}, A\boldsymbol{\gamma} \rangle}} \right) & \text{if } \gamma_3 \neq 0, \\ 0 & \text{if } \gamma_3 = 0. \end{cases}$$

The smooth invariant measure is $\chi(\boldsymbol{\gamma}) d\boldsymbol{M} d\boldsymbol{\gamma}$ with

$$\chi(\boldsymbol{\gamma}) = \begin{cases} k(\boldsymbol{\gamma}) |\gamma_3|^{-1} & \text{if } \gamma_3 \neq 0, \\ 0 & \text{if } \gamma_3 = 0. \end{cases}$$

The density of this invariant measure is nonnegative and only vanishes along a set of measure zero and therefore also falls within the class of measures considered in [39].

3.2 The case $W = 0$

The equations of motion are

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega},$$
$$\dot{\boldsymbol{u}} = -rB(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + \boldsymbol{V}_s(\boldsymbol{x}),$$

with $\boldsymbol{x}$ given by (I.3.2) and

$$\boldsymbol{\Omega}(\boldsymbol{M}, B, \boldsymbol{u}) = A\left(\boldsymbol{M} - mr\boldsymbol{\gamma} \times B^{-1}\boldsymbol{V}_s(\boldsymbol{x}) + \frac{mr^2\langle \boldsymbol{M} - mr\boldsymbol{\gamma} \times B^{-1}\boldsymbol{V}_s(\boldsymbol{x}), A\boldsymbol{\gamma}\rangle}{1 - mr^2\langle A\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle}\gamma\right). \tag{I.3.18}$$

Under the assumption that the vector field $\boldsymbol{V}_s$ is divergence free, the system possesses an invariant measure. We state this as the following proposition whose proof is a direct calculation using equations (I.2.18c), (I.3.3) and (I.2.12).

Proposition I.3.2. *Suppose* $\operatorname{div}_{\mathbb{R}^2}\boldsymbol{V}_s = 0$. *Then*

$$\frac{1}{\sqrt{1 - mr^2\langle \boldsymbol{\gamma}, A\boldsymbol{\gamma}\rangle}} d\boldsymbol{M}\, d\boldsymbol{u}\, d\boldsymbol{\alpha}\, d\boldsymbol{\beta}\, d\boldsymbol{\gamma}$$

is an invariant measure.

The existence of this invariant measure was already known in some particular cases. In [5] it was found for $\boldsymbol{V}_s$ corresponding to the uniformly rotating plane (i.e. given by (I.2.3)) and in [45] for the non-autonomous vector field $\boldsymbol{V}_s$ corresponding to a vibrating plane.

Assuming distinct moments of inertia, I_j, and non-zero $\boldsymbol{V}_s$, we do not expect existence of additional first integrals, and we believe that the dynamics is chaotic. In fact, the papers [5] and [45] perform numerical explorations for the particular vector fields $\boldsymbol{V}_s$ mentioned above and reach this conclusion.

4 A body of revolution with a cat's toy mechanism

This chapter considers the cat's toy mechanism described in Chapter 2 and illustrated in Fig 2.2b under the additional assumption that the fastened rigid body possesses an axial symmetry along the axis of rotation of the shell. This situation puts us in the framework of Section 2.4.2. Therefore, assuming that $V = 0$ and that the axis $\boldsymbol{E}_3$ of the body frame Σ_b is aligned with the aforementioned symmetry axis, we have

$$\boldsymbol{W}_b(\boldsymbol{\rho}) = \sigma \boldsymbol{\rho} \times \boldsymbol{E}_3,$$

as in (I.2.6). In particular, the system possesses the moving energy integral (I.2.26). If $\sigma = 0$, one recovers the classical problem of a solid of revolution rolling on the plane. This problem is well-known to be integrable in virtue of the existence of two first integrals J_1, J_2 and an invariant measure found by Chaplygin [19] (see [12] for historical details).

In Section 4.2 below, we indicate that for any $\sigma \in \mathbb{R}$ the system possesses an invariant measure whose form is identical to the one found by Chaplygin in the case $\sigma = 0$. Furthermore, in Proposition I.4.2 we show that a suitable modification of J_1 and J_2 are first integrals of the system for any $\sigma \in \mathbb{R}$. The existence of these integrals, the invariant measure and the moving energy allow us to conclude that the system is integrable by Jacobi's last multiplier theorem [2].

This situation is reminiscent of the (integrable) problem of a homogeneous sphere rolling without slipping on a surface of revolution. If the surface rotates about its axis of symmetry at constant, but arbitrary angular speed, modifications of the first integrals and the invariant measure persist, and the problem remains integrable [15, 31].

4.1 Preliminaries

Given that the shell $\mathcal{S}$ is a body is of revolution, the relation (I.2.11) between $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ given by the Gauss map, may be described by (see e.g. [12, 23]):

$$\boldsymbol{\rho}(\boldsymbol{\gamma}) = -\boldsymbol{n}_b^{-1}(\boldsymbol{\gamma}) = (f_1(\gamma_3)\gamma_1, f_1(\gamma_3)\gamma_2, f_2(\gamma_3)), \tag{I.4.1}$$

where f_1, f_2 are real functions determining the shape of $\mathcal{S}$, which satisfy the differential equation

$$f_2'(\gamma_3)\gamma_3 = f_1(\gamma_3)\gamma_3 - (1-\gamma_3^2)f_1'(\gamma_3).$$

The function f_1 is strictly positive, its value being equal to a principal radius of curvature of $\mathcal{S}$ (see [38]). On the other hand, the symmetric distribution of mass of the body implies that the first two moments of inertia are equal so

$$\mathbb{I} = \mathrm{diag}(I_1, I_1, I_3).$$

From Proposition I.2.2, we have that the SE(2)-reduced equations of motion (I.2.23) are the restriction of

$$\begin{aligned} \dot{\boldsymbol{M}} &= \boldsymbol{M}\times\boldsymbol{\Omega} + m\dot{\boldsymbol{\rho}}\times(\boldsymbol{\Omega}\times\boldsymbol{\rho}) + mg\boldsymbol{\rho}\times\boldsymbol{\gamma} + m\sigma(\boldsymbol{\rho}\times\boldsymbol{E}_3)\times(\dot{\boldsymbol{\rho}}+\boldsymbol{\Omega}\times\boldsymbol{\rho}), \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma}\times\boldsymbol{\Omega}, \end{aligned} \tag{I.4.2}$$

to the invariant set $\|\boldsymbol{\gamma}\|^2 = 1$, where

$$\boldsymbol{\Omega}(\boldsymbol{M},\boldsymbol{\gamma}) = A(\boldsymbol{\gamma})\left(\boldsymbol{M} + \frac{m\langle \boldsymbol{M} + m\sigma\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\boldsymbol{E}_3), A(\boldsymbol{\gamma})\boldsymbol{\rho}\rangle}{1 - m\langle A(\boldsymbol{\gamma})\boldsymbol{\rho},\boldsymbol{\rho}\rangle}\boldsymbol{\rho} - m\sigma\boldsymbol{\rho}\times(\boldsymbol{\rho}\times\boldsymbol{E}_3)\right), \tag{I.4.3}$$

and $A(\boldsymbol{\gamma})$ is given by (I.2.17). Equations (I.4.2) have an extra SO(2)-symmetry corresponding to the rotations about the axis of symmetry of the body. This corresponds to the transformation

$$\boldsymbol{M} \mapsto R_\phi\boldsymbol{M}, \qquad \boldsymbol{\gamma}\mapsto R_\phi\boldsymbol{\gamma} \qquad \text{with} \qquad R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{I.4.4}$$

It can be checked from (I.4.1) and (I.4.3) that $\boldsymbol{\rho}$ and $\boldsymbol{\Omega}$ accordingly transform as $\boldsymbol{\rho}\mapsto R_\phi\boldsymbol{\rho}$, $\boldsymbol{\Omega}\mapsto R_\phi\boldsymbol{\Omega}$ and it is immediate to see that equations (I.4.2) are invariant.

4.2 Existence of an invariant measure

When $\sigma = 0$, the system possesses the following invariant measure found by Chaplygin [19] (see also [12]),

$$\frac{1}{\mu(\gamma_3)} d\boldsymbol{M} d\boldsymbol{\gamma}, \tag{I.4.5}$$

where

$$\begin{aligned} \mu(\gamma_3) &= \sqrt{I_1 I_3 + m\langle \boldsymbol{\rho}, \mathbb{I}\boldsymbol{\rho}\rangle} \\ &= \sqrt{I_1 I_3 + m I_1 f_1(\gamma_3)^2 (1-\gamma_3^2) + m I_3 f_2(\gamma_3)^2}. \end{aligned} \tag{I.4.6}$$

One can check that the term proportional to σ in (I.4.2) has zero divergence (with respect to $\boldsymbol{M}$) and that the terms in $\boldsymbol{\Omega}(\boldsymbol{M},\boldsymbol{\gamma})$ in (I.4.3) proportional to σ vanish when taking the divergence with respect to $\boldsymbol{M},\boldsymbol{\gamma}$. As a consequence, we have the following.

Proposition I.4.1. *The measure* (I.4.5) *is invariant by the system* (I.4.2) *for any value of* $\sigma \in \mathbb{R}$.

4.3 First integrals

A convenient approach to investigate the reduced dynamics by the $\mathrm{SO}(2)$ symmetry defined by (I.4.4) is working with coordinates on $\mathbb{R}^3 \times S^2 \ni (\boldsymbol{M},\boldsymbol{\gamma})$ that are invariant under the action. Following the approach of Borisov and Mamaev [12] for the case $\sigma = 0$, we consider the evolution of the quantities

$$K_1(\boldsymbol{M},\boldsymbol{\gamma}) = \frac{\langle \boldsymbol{M},\boldsymbol{\rho}\rangle}{f_1(\gamma_3)}, \qquad K_2(\boldsymbol{M},\boldsymbol{\gamma}) = \mu(\gamma_3)\Omega_3(\boldsymbol{M},\boldsymbol{\gamma}),$$

where $\mu(\gamma_3)$ is defined by (I.4.6) and $\Omega_3(\boldsymbol{M},\boldsymbol{\gamma})$ is the third component of $\boldsymbol{\Omega}(\boldsymbol{M},\boldsymbol{\gamma})$ given by (I.2.22). One can easily check that K_1, K_2 are $\mathrm{SO}(2)$ invariant, and a calculation shows that they satisfy the following equations

$$\begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} = \dot{\gamma}_3 \left(G(\gamma_3) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} + \sigma\, \boldsymbol{b}(\gamma_3) \right), \tag{I.4.7}$$

where the 2×2 matrix $G(\gamma_3)$ and the vector $\boldsymbol{b}(\gamma_3) \in \mathbb{R}^2$ are given by

$$\begin{aligned} G(\gamma_3) &= -\frac{1}{\mu}\begin{pmatrix} 0 & I_3\left(1-\left(\frac{f_2}{f_1}\right)'\right) \\ mf_1(f_1-f_2') & 0 \end{pmatrix}, \\ \boldsymbol{b}(\gamma_3) &= -\frac{1}{\mu}\begin{pmatrix} 0 \\ -mf_1I_1(f_1\gamma_3-(1-\gamma_3^2)f_1') \end{pmatrix}, \end{aligned} \tag{I.4.8}$$

where the dependence of f_1, f_2, f_1', f_2' and μ on γ_3 has been omitted.

The structure of the system (I.4.7) allows us to apply the approach followed by Dalla Via, Fassò and Sansonetto in [24, Section 3.1] to prove the existence of first integrals. Specifically, let $Y(\gamma_3) \in \mathrm{GL}(2)$ be the solution of the (non-autonomous, linear, homogeneous) 2×2 matrix differential equation

$$\frac{dY}{d\gamma_3} = G(\gamma_3)Y, \qquad Y(0) = \mathrm{id}_2, \tag{I.4.9}$$

and $\boldsymbol{y}(\gamma_3) \in \mathbb{R}^2$ the solution of the (non-autonomous, linear, inhomogeneous) differential equation

$$\frac{d\boldsymbol{y}}{d\gamma_3} = G(\gamma_3)\boldsymbol{y} + \boldsymbol{b}(\gamma_3), \qquad \boldsymbol{y}(0) = 0. \tag{I.4.10}$$

In analogy with Proposition 2 in [24] (its second statement), we have.

Proposition I.4.2. *The two components J_1, J_2 of the map $J : \mathbb{R}^3 \times S^2 \to \mathbb{R}^2$ given by*

$$J(\boldsymbol{M}, \boldsymbol{\gamma}) = Y^{-1}(\gamma_3)\left(\begin{pmatrix} K_1(\boldsymbol{M}, \boldsymbol{\gamma}) \\ K_2(\boldsymbol{M}, \boldsymbol{\gamma}) \end{pmatrix} - \sigma \boldsymbol{y}(\gamma_3)\right)$$

are first integrals of (I.4.2).

The proof is a direct calculation relying on the definitions of $Y(\gamma_3)$, $\boldsymbol{y}(\gamma_3)$ and (I.4.7).

Remark I.4.3. Equations (I.4.7) and our observations about the invariant measure made in 4.2 resemble some aspects of the discussion in Borisov and Mamaev [12] about the gyrostatic generalization of the problem of a solid of revolution rolling without slipping on the plane. This may suggest the possibility of conjugating such problem with the one treated here via a (time-dependent) change of coordinates.

4.4 Routh's sphere with a cat's toy mechanism

As a particular example, we consider a sphere in which the center of mass O does not coincide with the geometrical center of the sphere C, but is along the axis $\boldsymbol{E}_3$, as illustrated in Fig. 4.1. In this case, the relation between $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ is given by

$$\boldsymbol{\rho} = -r\boldsymbol{\gamma} - l\boldsymbol{E}_3,$$

where l is the distance between the geometric center C of the sphere and the center of mass O. In this case, the functions f_1, f_2 are given by

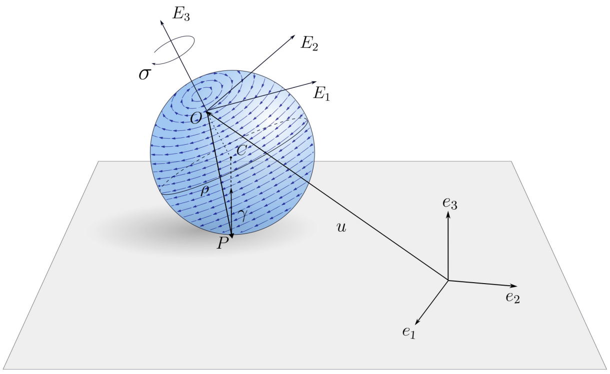


Figure 4.1: Routh's sphere with a cat's toy mechanism. It is assumed that the axis of rotation of the shell is aligned with $\boldsymbol{E}_3$.

$$f_1(\gamma_3) = -r \qquad \text{and} \qquad f_2(\gamma_3) = -r\gamma_3 - l.$$

Then the matrix $G(\gamma_3)$ and the vector $\boldsymbol{b}(\gamma_3)$ as defined in (I.4.8), in this case, become

$$G(\gamma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{b}(\gamma_3) = \frac{1}{\mu}\begin{pmatrix} 0 \\ mr^2 I_1 \gamma_3 \end{pmatrix},$$

where $\mu = \mu(\gamma_3)$ is given by (I.4.6). Taking $Y(\gamma_3) \in \mathrm{GL}(2)$ as the solution of the matrix differential equation (I.4.9) and $\boldsymbol{y}(\gamma_3) \in \mathbb{R}^2$ the solution of the differential equation (I.4.10), we have $Y = \mathrm{id}$ and

$$\boldsymbol{y}(\gamma_3) = \left(0, \frac{I_1 I_3 \sqrt{m}}{(I_1 - I_3)^{3/2}} \tan^{-1}\left(\frac{\mu'(\gamma_3)}{r\sqrt{m(I_1 - I_3)}}\right) - \frac{I_1}{I_1 - I_3}\mu(\gamma_3)\right).$$

Hence, the map $J : \mathbb{R}^3 \times S^2 \to \mathbb{R}^2$ takes the form

$$J(\boldsymbol{M}, \boldsymbol{\gamma}) = Y^{-1}\left(\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \eta \boldsymbol{y}(\gamma_3)\right) = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} - \eta \boldsymbol{y}(\gamma_3).$$

and the first integrals can be expressed in explicit form as

$$J_1 = -\frac{1}{r}\langle \boldsymbol{M}, \boldsymbol{\rho} \rangle,$$
$$J_2 - \sigma y_2(\gamma_3) = \mu\Omega_3 + \sigma\left(\frac{I_1 I_3\sqrt{m}}{(I_1 - I_3)^{3/2}}\tan^{-1}\left(\frac{\mu'}{r\sqrt{m(I_1 - I_3)}}\right) - \frac{I_1}{I_1 - I_3}\mu\right).$$

The first integral J_1 is a *Jellet integral*, [43].

5 A homogeneous sphere

We now assume that our convex body is a homogeneous sphere which puts us in the framework of Chapter 3 with the additional hypothesis of equal moments of inertia

$$I := I_1 = I_2 = I_3.$$

The equations of motion (I.2.18) may be rewritten as

$$\begin{aligned} &\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \\ &\dot{\boldsymbol{u}} = -rB(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + \boldsymbol{V}_s(\boldsymbol{u}) + B\boldsymbol{W}_b(\boldsymbol{\gamma}), \end{aligned} \tag{I.5.1}$$

where the Poisson vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are the rows of the attitude matrix $B \in \mathrm{SO}(3)$ and we have used equations (I.3.1) and (I.3.2) to write $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ as functions of $\boldsymbol{u}$ and $\boldsymbol{\gamma}$. The expression (I.2.16) for the angular velocity $\boldsymbol{\Omega}$ simplifies to

$$\boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{u}, B) = \frac{1}{I + mr^2}\left(\boldsymbol{M} - mr\boldsymbol{\gamma} \times (B^{-1}\boldsymbol{V}_s(\boldsymbol{u}) + \boldsymbol{W}_b(\boldsymbol{\gamma})) + \frac{mr^2}{I}\langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle \boldsymbol{\gamma}\right). \tag{I.5.2}$$

The following proposition gives sufficient conditions for $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ to guarantee the existence of an invariant measure whose form coincides with the one of the linear system (obtained when both $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ vanish). In the statement, $\mathrm{div}_{\mathbb{R}^2}$ and div_{S^2} denote the standard divergence of vector fields with respect to the Euclidean distance in $\mathbb{R}^2$ and the induced distance on S^2 from the ambient Euclidean metric on $\mathbb{R}^3$.

Proposition I.5.1. *Suppose that* $\mathrm{div}_{\mathbb{R}^2}\boldsymbol{V}_s(\boldsymbol{x})$ *and* $\mathrm{div}_{S^2}\boldsymbol{W}(\boldsymbol{\gamma})$ *identically vanish, then the system* (I.5.1) *possesses the invariant measure* $d\boldsymbol{M}\,d\boldsymbol{u}\,d\boldsymbol{\alpha}\,d\boldsymbol{\beta}\,d\boldsymbol{\gamma}$.

The proof follows from a direct computation and relies on the following observation:

$$\mathrm{div}_{S^2}\boldsymbol{W}(\gamma) = \mathrm{Tr}(\boldsymbol{W}'(\gamma)) - \gamma^T\boldsymbol{W}'(\gamma)\gamma, \qquad \gamma \in S^2.$$

On the left-hand side of the above relation $\boldsymbol{W}$ is a vector field on the unit sphere S^2, whereas on the right it should be interpreted as a smooth extension of $\boldsymbol{W}$ to $\mathbb{R}^3$. The formula is valid independently of the extension and may be verified using, for example, spherical coordinates on S^2.

5.1 A homogeneous sphere in space coordinates

For future reference, we mention that, when working with the homogeneous sphere, sometimes it is more convenient to work in the space frame Σ_s. This can be done by working with the space representation $\boldsymbol{\omega} = B\boldsymbol{\Omega}$ of the angular velocity and the space representation of the vector $\boldsymbol{M}$, which we will write as $\boldsymbol{m} = B\boldsymbol{M}$, and is given by

$$\boldsymbol{m} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\boldsymbol{\omega} \times \boldsymbol{e}_3) + mr\boldsymbol{e}_3 \times (\boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\gamma)). \tag{I.5.3}$$

Notice that the first equation of (I.5.1) becomes

$$\dot{\boldsymbol{m}} = 0, \tag{I.5.4}$$

and that the constraints can be written as

$$\dot{\boldsymbol{x}} = -r\boldsymbol{e}_3 \times \boldsymbol{\omega} + \boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\gamma), \tag{I.5.5}$$

where, since the sphere is dynamically balanced, we have the following relations

$$\boldsymbol{x} = \boldsymbol{u} - r\boldsymbol{e}_3 \qquad \text{and} \qquad \boldsymbol{\rho} = -r\gamma.$$

Therefore, we may write $\boldsymbol{W}_b$ as a function of γ and the constraints in terms of $\dot{\boldsymbol{x}}$. The system is completed with equation $\dot{B}B^{-1} = \hat{\boldsymbol{\omega}}$. An advantage of working in the space frame, is that we can give explicit expressions of the first integrals whose existence is guaranteed by Proposition I.3.1, which tells us that the vector $\boldsymbol{M}$ is constant in space coordinates, therefore, the coordinates of $\boldsymbol{m}$ are first integrals of the system. These are given by

$$\begin{aligned} m_1 &= (I + mr^2)\omega_1 - mr(V_2(\boldsymbol{x}) + (B\boldsymbol{W}_b(\gamma))_2), \\ m_2 &= (I + mr^2)\omega_2 + mr(V_1(\boldsymbol{x}) + (B\boldsymbol{W}_b(\gamma))_1), \\ m_3 &= I\omega_3, \end{aligned} \tag{I.5.6}$$

where $V_1(\boldsymbol{x}), V_2(\boldsymbol{x})$ are the first components of the vector field $\boldsymbol{V}_s(\boldsymbol{x})$, i.e. $\boldsymbol{V}_s(\boldsymbol{x}) = (V_1(\boldsymbol{x}), V_2(\boldsymbol{x}), 0)$, and $(B\boldsymbol{W}_b(\gamma))_1$, $(B\boldsymbol{W}_b(\gamma))_2$ are the first components of the vector defined by the product $B\boldsymbol{W}_b(\gamma)$.

Remark I.5.2. As follows from (I.5.6) the vertical component ω_3 of the angular velocity is a first integral. Therefore, the restriction of the general system to the zero level set of this integral coincides with the dynamics of the problem in the presence of an additional *rubber constraint* which forbids spinning of the sphere about the vertical axis. In particular, the integrable problem discussed in section 5.2 remains integrable if one adds a rubber (no-spin) constraint.

5.2 A homogeneous sphere with a cat's toy mechanism rolling on a fixed plane

We consider the problem of a homogeneous sphere with a cat's toy mechanism of angular speed $\sigma \in \mathbb{R}$ rolling on a fixed plane. In this case, the vector $\boldsymbol{m}$ in (I.5.3) takes the form

$$\boldsymbol{m} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\omega \times \boldsymbol{e}_3) + mr^2\sigma\boldsymbol{e}_3 \times B(\gamma \times \boldsymbol{E}_3), \tag{I.5.7}$$

and equations (I.5.4) and (I.5.5) become

$$\begin{aligned} \dot{\boldsymbol{m}} &= 0, \\ \dot{\boldsymbol{x}} &= -r\boldsymbol{e}_3 \times \boldsymbol{\omega} - r\sigma B(\gamma \times \boldsymbol{E}_3). \end{aligned} \tag{I.5.8}$$

These equations should be complemented with the evolution equations for the attitude matrix B (or the Poisson vector γ). However, due to the $\mathrm{SO}(2)$-symmetry corresponding to rotations of the body frame Σ_b about the $\boldsymbol{E}_3$ axis, it is more convenient to introduce the vector

$$\boldsymbol{\tau} = B\boldsymbol{E}_3,$$

which specifies space coordinates of the axis of rotation of the cat's toy mechanism. Its evolution is governed by the kinematic condition

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\omega} \times \boldsymbol{\tau}. \tag{I.5.9}$$

Notice that (I.5.7) and (I.5.8) may be rewritten as

$$\boldsymbol{m} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\omega \times \boldsymbol{e}_3) + mr^2\sigma\boldsymbol{e}_3 \times \boldsymbol{e}_3 \times \boldsymbol{\tau},$$

and

$$\begin{aligned} \dot{\boldsymbol{m}} &= 0, \\ \dot{\boldsymbol{x}} &= -r\boldsymbol{e}_3 \times \boldsymbol{\omega} - r\sigma(\boldsymbol{e}_3 \times \boldsymbol{\tau}). \end{aligned} \tag{I.5.10}$$

Let us now analyze the motion of the system for a fixed value of the momentum vector $\boldsymbol{m}$. It is convenient to introduce the vector

$$\overline{\boldsymbol{m}} = \boldsymbol{m} + mr^2\omega_3\boldsymbol{e}_3,$$

whose components are easily seen to be integrals of motion from (I.5.6). Notice that for a fixed value of $\overline{\boldsymbol{m}}$, one may determine $\boldsymbol{\omega}$ as a function of $\boldsymbol{\tau}$ by the relations

$$\begin{aligned} \omega_1 &= \frac{m_1}{I + mr^2} - \frac{r\sigma}{I + mr^2}\tau_2, \\ \omega_2 &= \frac{m_2}{I + mr^2} - \frac{r\sigma}{I + mr^2}\tau_2, \\ \omega_3 &= \frac{\overline{m}_3}{I + mr^2}. \end{aligned} \tag{I.5.11}$$

Therefore, for a fixed value of $\boldsymbol{m}$ (or $\overline{\boldsymbol{m}}$), the evolution of $\boldsymbol{\tau}$ reduces to the following system on S^2 (since $\|\boldsymbol{\tau}\| = 1$)

$$\dot{\boldsymbol{\tau}} = \nabla F_m(\boldsymbol{\tau}) \times \boldsymbol{\tau}, \tag{I.5.12}$$

where

$$F_m(\boldsymbol{\tau}) = m_1\tau_1 + m_2\tau_2 + \overline{m}_3\tau_3 + \frac{mr^2}{2}\sigma\tau_3^2.$$

Notice that once $\boldsymbol{\tau}$ has been determined by (I.5.12), $\boldsymbol{\omega}$ is given by (I.5.11) and the position $\boldsymbol{x}$ of the sphere on the plane can be obtained from (I.5.10) by a simple quadrature. The possibility to reduce the study of the dynamics to equation (I.5.12) is a consequence of the $\mathrm{SE}(2) \times \mathrm{SO}(2)$-symmetry of the problem.

Obviously, the system (I.5.12) has F_m as first integral and is integrable. Let us explain in more detail the mechanism leading to the integrability of the dynamics. As mentioned above, the system is $\mathrm{SE}(2) \times \mathrm{SO}(2)$ invariant, so the reduced phase space $\mathcal{A}/(\mathrm{SE}(2)\times\mathrm{SO}(2))$ is four dimensional. The moving energy E_{mov} in (I.5.25) and the third component m_3 of $\boldsymbol{m}$ are $\mathrm{SE}(2) \times \mathrm{SO}(2)$-invariant and pass to the quotient (in fact, F_m coincides, up to addition of a constant with the restriction of E_{mov} to the level sets of $\overline{\boldsymbol{m}}$). On the other hand, m_1 and m_2 are $SE(2)$-invariant and $\mathrm{SO}(2)$-equivariant with respect to the action of $\mathrm{SO}(2)$ by rotations on the $m_1 - m_2$ plane. Therefore, the invariant functions, such as $m_1^2 + m_2^2$, descend as an additional first integral to the quotient manifold

$\mathcal{A}/(\mathrm{SE}(2)\times\mathrm{SO}(2))$. The reduced dynamics on $\mathcal{A}/(\mathrm{SE}(2)\times\mathrm{SO}(2))$ constitute a system possessing 3 independent first integrals in a 4-dimensional space.

5.2.1 Analysis of the reduced dynamics

The system (I.5.12) is a Hamiltonian system on S^2 equipped with the standard area form and Hamiltonian $-F_m$. The orbits of the system (I.5.12) are contained in the intersections of the level sets of the first integral F_m (which are parabolic cylinders) and the unit sphere $\|\boldsymbol{\tau}\|^2 = 1$. Generically, these are periodic orbits. Now we will classify the equilibrium points to describe the global dynamics.

Let us first notice that we may restrict our attention to the case $m_1 = 0$ and $m_2 \neq 0$ by the SO(2)-equivariance of m_1, m_2 described above. The equilibrium points of (I.5.12) correspond to critical points of $F_m : S^2 \to \mathbb{R}$. Since S^2 is compact, F_m must reach its maximum and its minimum. Therefore, F_m has at least two critical points.

The explicit expression of (I.5.12) when $m_1 = 0$ is

$$\dot{\boldsymbol{\tau}} = \left(-\frac{m_3\tau_2 - m_2\tau_3 + mr^2\sigma\tau_2\tau_3}{I + mr^2}, \frac{(m_3 + mr^2\sigma\tau_3)\tau_1}{I + mr^2}, -\frac{m_2\tau_1}{I + mr^2}\right).$$

Therefore, critical points satisfy $\tau_1 = 0$ and

$$m_3\tau_2 - m_2\tau_3 + mr^2\sigma\tau_2\tau_3 = 0, \tag{I.5.13}$$

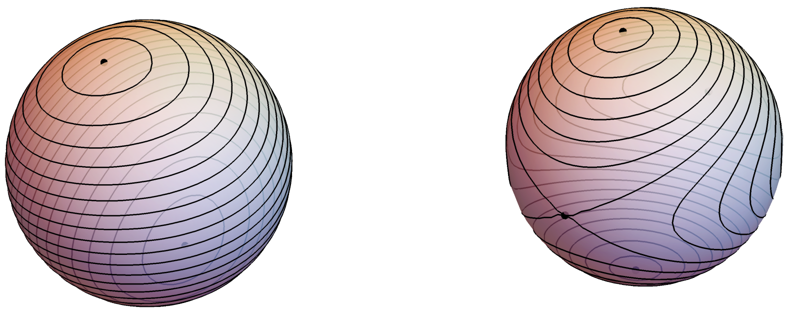
together with

$$\tau_2^2 + \tau_3^2 = 1. \tag{I.5.14}$$

So critical points constitute the intersection of the unit circle (I.5.14) and the hyperbola (I.5.13) in the plane $m_2 - \overline{m}_3$. This implies that there are at most 4 equilibrium points.

We will see that there are open subsets of the plane $m_2 - \overline{m}_3$ for which the system has 2 or 4 equilibrium points (illustrated in Fig. 5.1). The boundary of these regions corresponds to the values of $m_2, \overline{m}_3$ for which one of the solutions of (I.5.13)-(I.5.14) is a tangential point. To determine this boundary, we look for m_2 and $\overline{m}_3$ such that the normal vectors of the hyperbola (I.5.13) and the circle (I.5.14) are parallel at the point of intersection. In other words, we look for m_2 and $\overline{m}_3$ such that

$$\det\begin{pmatrix} m_3 + mr^2\sigma\tau_3 & -m_2 + mr^2\sigma\tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = 0, \tag{I.5.15}$$



(a) 2 equilibrium elliptic points.

(b) 4 equilibrium points (3 elliptic and 1 hyperbolic).

Figure 5.1: Phase space of the system (I.5.12) on S^2.

at the points that satisfy (I.5.13) and (I.5.14). Notice that (I.5.15) is equivalent to

$$m_2\tau_2 + m_3\tau_3 + mr^2\sigma(-\tau_2^2 + \tau_3^2) = 0. \tag{I.5.16}$$

Multiplying equation (I.5.13) by τ_3 and subtracting the product of equation (I.5.16) by τ_2 we obtain

$$-m_2(\tau_2^2 tau_3^2) + mr^2\sigma\tau_2^3 = 0,$$

which in view of (I.5.14) yields

$$\tau_2 = \left(\frac{m_2}{mr^2\sigma}\right)^{1/3}. \tag{I.5.17}$$

A similar procedure, multiplying (I.5.13) by τ_2 and subtracting the product of (I.5.16) by τ_3, gives

$$\tau_3 = \left(\frac{m_3}{mr^2\sigma}\right)^{1/3}. \tag{I.5.18}$$

Substituting (I.5.17) and (I.5.18) in (I.5.14), we come to the conclusion that the boundary between the regions with 2 and 4 equilibrium points in the plane $m_2 - \overline{m}_3$ is determined by the equation

$$\left(\frac{m_2}{mr^2\sigma}\right)^{2/3} + \left(\frac{\overline{m}_3}{mr^2\sigma}\right)^{2/3} = 1,$$

whose graph is shown in Figure 5.2.

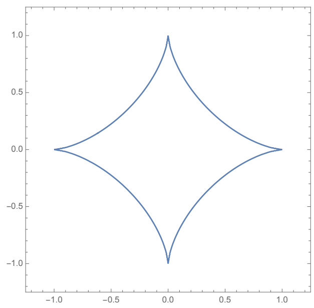


Figure 5.2: Curve on the plane $m_2 - \overline{m}_3$ separating the regions where (I.5.14) and (I.5.13) have two and four intersections.

The above observations lead to the following theorem.

Theorem I.5.3. *Consider the closed surface of revolution*

$$S = \left\{ (m_1, m_2, \overline{m}_3) \in \mathbb{R}^3 \; : \; \left(\frac{m_1^2 + m_2^2}{(mr^2\sigma)^2} \right)^{1/3} + \left(\frac{\overline{m}_3}{mr^2\sigma} \right)^{2/3} = 1 \right\},$$

depicted in Figure 5.3. If $(m_1, m_2, \overline{m}_3)$ lies inside S (i.e. $\left(\frac{m_1^2+m_2^2}{(mr^2\sigma)^2}\right)^{1/3} + \left(\frac{\overline{m}_3}{mr^2\sigma}\right)^{2/3} < 1$) then the system (I.5.12) has exactly 4 equilibrium points, 3 of which are elliptic and 1 is hyperbolic, as illustrated in Fig. 5.1b. If instead $(m_1, m_2, \overline{m}_3)$ lies outside S, (i.e. $\left(\frac{m_1^2+m_2^2}{(mr^2\sigma)^2}\right)^{1/3} + \left(\frac{\overline{m}_3}{mr^2\sigma}\right)^{2/3} > 1$), then the system (I.5.12) has exactly 2 elliptic equilibrium points, as illustrated in Fig. 5.1a.

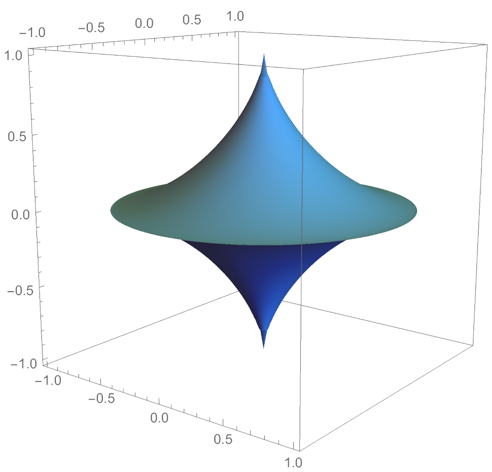


Figure 5.3: Surface S on the space $m_1, m_2, \overline{m}_3$ separating the regions in which the system (I.5.12) has 2 and 4 equilibrium points.

Remark I.5.4. Given that the dependence of $\boldsymbol{\omega}$ is affine linear, the property that $(m_1, m_2, \overline{m_3})$ lies outside S may be physically interpreted as a condition on the magnitude of angular velocity $\boldsymbol{\omega}$ to predominate over σ. In other words, the inertial motion of the sphere dominates the effect of the cat's toy mechanism.

5.3 A homogeneous sphere with a cat's toy mechanism rolling on a uniformly rotating plane

For the rest of the chapter we consider the problem of a homogeneous sphere with a cat's toy mechanism of angular speed $\sigma \in \mathbb{R}$ rolling on a uniformly rotating plane at angular velocity $\eta \in \mathbb{R}$ as depicted in Fig 5.4. The corresponding expressions for $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ are given by (I.2.3) and (I.2.6). Considering that for a spherical body $\boldsymbol{x} = \boldsymbol{u} - r\boldsymbol{e}_3$ and $\boldsymbol{\rho} = -r\boldsymbol{\gamma}$, we may write

$$\boldsymbol{V}_s(\boldsymbol{u}) = -\eta \boldsymbol{u} \times \boldsymbol{e}_3 \qquad \text{and} \qquad \boldsymbol{W}_b(\boldsymbol{\gamma}) = -r\sigma\boldsymbol{\gamma} \times \boldsymbol{E}_3.$$

Hence, equations (I.5.1) take the form

$$\begin{aligned} &\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\Omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \\ &\dot{\boldsymbol{u}} = -rB(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) - r\sigma B(\boldsymbol{\gamma} \times \boldsymbol{E}_3) - \eta \boldsymbol{u} \times \boldsymbol{e}_3, \end{aligned} \tag{I.5.19}$$

with

$$\boldsymbol{M} = I\boldsymbol{\Omega} + mr^2\boldsymbol{\gamma} \times (\boldsymbol{\Omega} \times \boldsymbol{\gamma}) - mr\boldsymbol{\gamma} \times (r\sigma\boldsymbol{\gamma} \times \boldsymbol{E}_3 + \eta B^{-1}(\boldsymbol{u} \times \boldsymbol{e}_3)).$$

The expression (I.5.2) for $\boldsymbol{\Omega}$ takes the form

$$\begin{aligned} \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{u}, B) = \frac{1}{I + mr^2} \Bigg(\boldsymbol{M} + \frac{mr^2}{I} \langle \boldsymbol{M}, \boldsymbol{\gamma} \rangle \boldsymbol{\gamma} + mr\eta\, \boldsymbol{\gamma} \times ((B^{-1}\boldsymbol{u}) \times \boldsymbol{\gamma}) \\ + mr^2 \sigma \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{E}_3) \Bigg). \end{aligned} \tag{I.5.20}$$

If $\eta = 0$ the system admits the SE(2)-symmetry described in Section 2.3 and the reduced system is integrable since it falls within the framework of Chapter 4. On the other hand, if $\sigma = 0$ we recover the classical problem of a homogeneous sphere rolling on a uniformly rotating plane, which is also well-known to be integrable. For the rest of the chapter, we analyze the dynamics for nonzero values of η and σ. We will prove that is integrable if the generalized momentum $\boldsymbol{M}$ is vertical (i.e. parallel to $\boldsymbol{\gamma}$) and exhibit numerical evidence that it is chaotic otherwise.

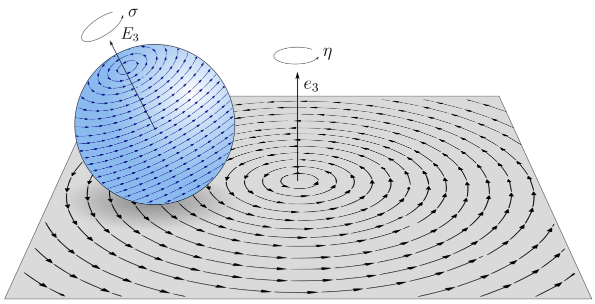


Figure 5.4: Homogeneous sphere with a cat's toy mechanism of angular speed σ rolling without slipping on a uniformly rotating plane with angular speed η.

5.3.1 Symmetries, reduction and first integrals

The system possesses two different, and commuting, $\mathrm{SO}(2)$-symmetries corresponding to rotations of the space frame Σ_s about the $\boldsymbol{e}_3$ axis and rotations of the body frame Σ_b about the $\boldsymbol{E}_3$ axis. The first of these symmetries may be reduced by working with the body frame representation $\boldsymbol{U}$ of the vector $\overrightarrow{OC}$. This vector satisfies $\boldsymbol{u} = B\boldsymbol{U}$ and, hence, the third equation in (I.5.19) yields,

$$\dot{\boldsymbol{U}} = -r(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + \boldsymbol{U} \times \boldsymbol{\Omega} - r\sigma(\boldsymbol{\gamma} \times \boldsymbol{E}_3) - \eta(\boldsymbol{U} \times \boldsymbol{\gamma}).$$

Moreover, the expression (I.5.20) implies that $\boldsymbol{\Omega}$ may be written as a function of $(\boldsymbol{M}, \boldsymbol{\gamma}, \boldsymbol{U})$ in the form

$$\begin{aligned}\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{M}, \boldsymbol{\gamma}, \boldsymbol{U}) = \frac{1}{I + mr^2}\Bigg(\boldsymbol{M} + \frac{mr^2}{I}\langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle \boldsymbol{\gamma} + mr\eta\, \boldsymbol{\gamma} \times (\boldsymbol{U} \times \boldsymbol{\gamma}) \\ + mr^2\sigma \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{E}_3)\Bigg).\end{aligned} \tag{I.5.21}$$

The expressions given above are independent of the row vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the attitude matrix B. Therefore, we may extract from (I.5.19) the following closed system for $(\boldsymbol{M}, \boldsymbol{\gamma}, \boldsymbol{U}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\begin{aligned}\dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\Omega}, \\ \dot{\boldsymbol{U}} &= -r(\boldsymbol{\gamma} \times \boldsymbol{\Omega}) + \boldsymbol{U} \times \boldsymbol{\Omega} - r\sigma(\boldsymbol{\gamma} \times \boldsymbol{E}_3) - \eta(\boldsymbol{U} \times \boldsymbol{\gamma}),\end{aligned} \tag{I.5.22}$$

with $\boldsymbol{\Omega}$ given by (I.5.21). The system possesses the geometric first integrals $\|\boldsymbol{\gamma}\|$ and $\langle \boldsymbol{U}, \boldsymbol{\gamma}\rangle$ and its restriction to the 7-dimensional manifold

$$\mathcal{M}_7 = \{(\boldsymbol{M}, \boldsymbol{\gamma}, \boldsymbol{U}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \,:\, \|\boldsymbol{\gamma}\| = 1, \quad \text{and} \quad \langle \boldsymbol{U}, \boldsymbol{\gamma}\rangle = r\}$$

defines a flow isomorphic to the reduced system on $\mathcal{A}/\mathrm{SO}(2)$. This flow has the following set of equilibrium points

$$\mathcal{M}_7^{\mathrm{eq}} = \{\boldsymbol{M} = (0,0,M_3),\ \boldsymbol{\gamma} = (0,0,\pm 1),\ \boldsymbol{U} = (0,0,\pm r)\ :\ M_3 \in \mathbb{R}\},$$

which correspond to motions where the sphere is uniformly spinning without rolling, positioned at the origin O of the plane Π and with the $\boldsymbol{E}_3$-axis of the cat's toy mechanism aligned vertically (at these configurations the vector fields $\boldsymbol{V}_s$ and $\boldsymbol{W}_b$ vanish). In what follows we shall restrict our attention to the complementary part of the phase space $\mathcal{M}_7$ which we denote by $\widetilde{\mathcal{M}}$, namely,

$$\widetilde{\mathcal{M}} = \mathcal{M}_7 \smallsetminus \mathcal{M}_7^{\mathrm{eq}}.$$

Obviously, $\widetilde{\mathcal{M}}$ is an open dense subset of $\mathcal{M}_7$ which is invariant by the dynamics.

The additional $\mathrm{SO}(2)$-symmetry described above, corresponding to rotations of the body frame about the $\boldsymbol{E}_3$-axis, results in the invariance of the manifold $\widetilde{\mathcal{M}}$ and the equations (I.5.22) under the action,

$$\boldsymbol{M} \mapsto R_\phi \boldsymbol{M}, \quad \boldsymbol{\gamma} \mapsto R_\phi \boldsymbol{\gamma}, \quad \boldsymbol{U} \mapsto R_\phi \boldsymbol{U} \quad \text{with} \quad R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{I.5.23}$$

The invariance of (I.5.22) is readily verified since, as follows from (I.5.21), the angular velocity $\boldsymbol{\Omega}$ also transforms as $\boldsymbol{\Omega} \mapsto R_\phi \boldsymbol{\Omega}$. Hence, the system may be reduced to the quotient space that we denote as

$$\mathcal{R}_6 = \widetilde{\mathcal{M}}/\mathrm{SO}(2),$$

which is a smooth 6-dimensional manifold (since the action (I.5.23) is free on $\widetilde{\mathcal{M}}$).

Additionally, the system (I.5.22) possesses the momentum first integrals

$$\|\boldsymbol{M}\|^2, \qquad \langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle, \tag{I.5.24}$$

and the preserved moving energy E_{mov} given by (I.2.27). In our case, using that $\|\boldsymbol{x}\|^2 = \|\boldsymbol{U} - r\boldsymbol{\gamma}\|^2 = \|\boldsymbol{U}\|^2 - r^2$ along $\mathcal{M}_7$, this may be written as

$$E_{mov} = \frac{I}{2}\langle \boldsymbol{\Omega} + \sigma \boldsymbol{E}_3, \boldsymbol{\Omega} + \sigma \boldsymbol{E}_3\rangle + \frac{mr^2}{2}\|\boldsymbol{\gamma} \times (\boldsymbol{\Omega} + \sigma \boldsymbol{E}_3)\|^2 - \frac{m}{2}\eta^2 \|\boldsymbol{U}\|^2. \tag{I.5.25}$$

It is easily seen that the momentum fist integrals (I.5.24), as well as the preserved moving energy (I.5.25), are invariant under the action (I.5.23) and therefore descend as first integrals of the reduced system on $\mathcal{R}_6$.

Finally, since $\mathrm{div}_{S^2}\boldsymbol{W}_b = 0$ and $\mathrm{div}_{\mathbb{R}^2}\boldsymbol{V}_s = 0$, by Proposition I.5.1, the system (I.5.19) has invariant measure $d\boldsymbol{M}d\boldsymbol{u}d\boldsymbol{\alpha}d\boldsymbol{\beta}d\boldsymbol{\gamma}$. It can be checked that $d\boldsymbol{M}d\gamma d\boldsymbol{U}$ is an invariant measure for the reduced system (I.5.22). By general results on free actions of compact groups (e.g. Lemma 3.4 in [37]), this invariant measure descends to a smooth invariant measure for the reduced system on $\mathcal{R}_6$.

Summarizing, the reduced dynamics on the 6-dimensional reduced manifold $\mathcal{R}_6$ possesses 3 first integrals (I.5.24) and (I.5.25), and a smooth invariant measure. Below, we argue that these invariants lead to the integrability of the dynamics for initial conditions with $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ parallel, and we exhibit numerical evidence indicating that the dynamics is chaotic otherwise.

5.3.2 The case M parallel to γ

It is not hard to see that initial conditions on $\widetilde{\mathcal{M}}$ with $\boldsymbol{M}$ parallel to $\boldsymbol{\gamma}$ are critical points of the momentum first integrals (I.5.24). By an argument similar to the one given in Section 3.1.1, it is seen that their level sets are 4-dimensional smooth submanifolds of $\widetilde{\mathcal{M}}$ and hence project to 3-dimensional submanifolds of the orbit space $\mathcal{R}_6$. The reduced dynamics restricted to these 3-dimensional submanifolds possesses the moving energy integral (I.5.25) and an invariant measure, and hence it is integrable by Jacobi's last multiplier theorem [2].

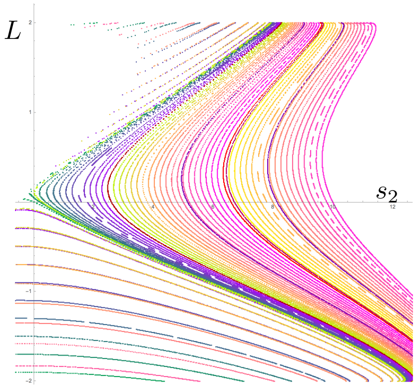
5.3.3 The general case (M and γ not parallel)

Initial conditions where $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ are not parallel are regular points of the joint map from $\widetilde{\mathcal{M}}$ to $\mathbb{R}^3$ whose components are the momentum first integrals (I.5.24) and the moving energy (I.5.25). As a consequence, their level sets are 4-dimensional submanifolds of $\widetilde{\mathcal{M}}$. These project to 3-dimensional invariant submanifolds of the orbit space $\mathcal{R}_6$ on which the dynamics can be investigated using a 2-dimensional Poincaré map.

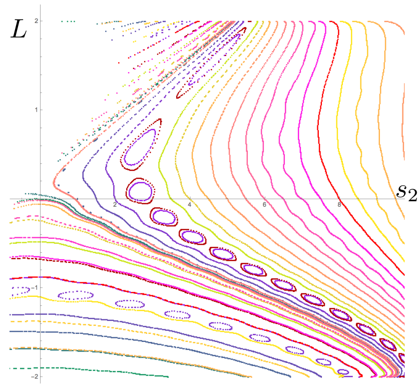
In order to construct the Poincaré map we borrow ideas of Bizyaev, Borisov, Mamaev [5] and introduce the following scalar functions on $\widetilde{\mathcal{M}}$ which are invariant under the action (I.5.23):

$$L = M_3, \qquad s_1 = U_1\gamma_1 + U_2\gamma_2, \qquad s_2 = U_1\gamma_2 - U_2\gamma_1,$$
$$G = \|\boldsymbol{M}\|, \qquad f = \langle \boldsymbol{M}, \boldsymbol{\gamma}\rangle, \qquad g = \arctan\left(\frac{G(M_2\gamma_1 - M_1\gamma_2)}{fL - G^2\gamma_3}\right).$$

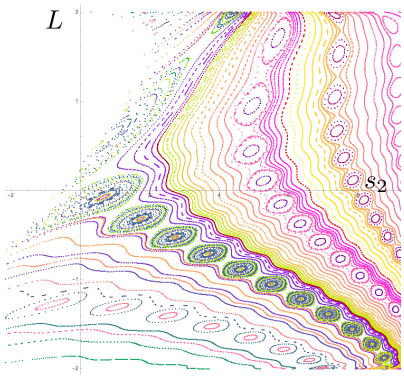
Then (L, s_1, s_2, G, f, g) are local coordinates on the reduced space $\mathcal{R}_6$ with the property that G and f are first integrals of the reduced dynamics. The explicit form of the reduced system and the moving energy integral E_{mov} in these



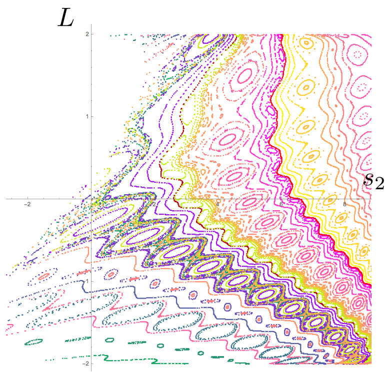
(a) $E_{mov} = -20$



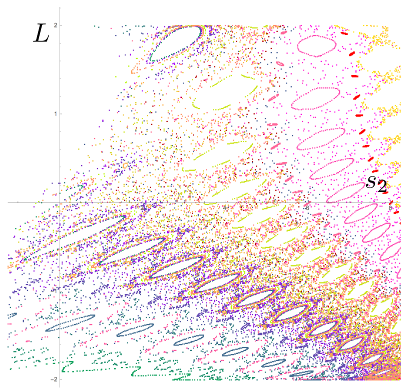
(b) $E_{mov} = -10$



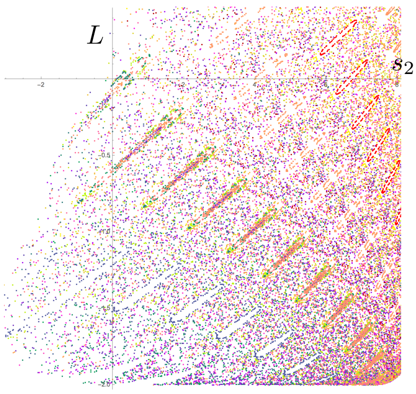
(c) $E_{mov} = -8$



(d) $E_{mov} = -7$



(e) $E_{mov} = -6$



(f) $E_{mov} = -5$

Figure 5.5: Poincaré map of the homogeneous sphere with a cat's toy mechanism rolling on a uniformly rotating plane for different values of the moving energy E_{mov}. The system parameters were taken as $I = 1$, $\eta = 1$, $\sigma = 1$, $m = 1$, $r = 2$. The momentum first integrals were fixed as $G = \|\boldsymbol{M}\| = 2$, $f = \langle \boldsymbol{M}, \gamma \rangle = 0$ and the section was defined putting $g = \frac{\pi}{4}$.

variables may be computed using the following formulae.

$$
\begin{aligned}
M_1 &= \sqrt{G^2 - L^2}\sin l, & \gamma_1 &= -\cos l \sin g + \frac{L \sin l \cos g}{G},\\
M_2 &= \sqrt{G^2 - L^2}\cos l, & \gamma_2 &= \sin l \sin g + \frac{L \cos l \cos g}{G},\\
M_3 &= L, & \gamma_3 &= -\frac{\sqrt{G^2 - L^2}\cos g}{G},
\end{aligned}
$$

and

$$
\begin{aligned}
U_1 &= \frac{G(L\cos g(s_2\cos l + s_1 \sin l) + G\sin g(-s_1\cos l + s_2 \sin l))}{L^2\cos^2 g + G^2 \sin^2 g},\\
U_2 &= \frac{G(L\cos g(s_1\cos l - s_2 \sin l) + G\sin g(s_2\cos l + s_1 \sin l))}{L^2\cos^2 g + G^2 \sin^2 g},\\
U_3 &= \frac{G s_1}{(G^2 - L^2)\cos g}.
\end{aligned}
$$

The resulting expressions for the reduced system and the moving energy E_{mov} are independent of the angle l in virtue of the $\mathrm{SO}(2)$-symmetry (I.5.23) (actually, the action (I.5.23) fixes (L, s_1, s_2, G, f, g) and shifts $l \mapsto l + \phi$).

We constructed a family of Poincaré sections (for the parameter values indicated in the caption of Fig 5.5) by setting the values of the first integrals $G = 2$, $f = 0$, fixing the value of $g = \frac{\pi}{4}$ and considering different level sets of E_{mov}. The resulting Poincaré map, projected to the plane s_2-L, is illustrated in Fig 5.5. We observe a transition from integrable to chaotic motion typical of KAM theory for Hamiltonian systems as the value of the moving energy is varied.

Part II

A theoretical framework for the ANAIS billiard

1 Introduction

The ANAIS billiard experiment [53] consists of a table with a rotating circular region and a ball which is set to roll on the table and towards the rotating part. An interesting phenomenon is observed: while the ball is rolling on the fixed part of the table it follows a straight line, but its initial rectilinear trajectory is deflected when the ball goes into the rotating part; nevertheless, the ball comes out of the circular region following the exact prolongation of the initial trajectory as illustrated in Figure 2.6 below and may be visualized in YouTube [25]. This phenomenon occurs regardless of the mass and radius of the ball, its initial velocity, and the angular velocity of the rotating disc.

In [53], Lévy-Leblond studied the ANAIS billiard and explained that the phenomenon is due to the symmetries and time-reversibility of the system. The presence of these attributes is outstanding, since general affine nonholonomic systems do not possess them. Our work frames the explanations given by Lévy-Leblond within the general theoretic framework of affine nonholonomic systems and clarifies which are the symmetries of the Lagrangian and the constraints that are responsible for the phenomenon.

We can think of the ANAIS billiard system as a nonholonomic *hybrid system* comprised of two subsystems: one with linear constraints and one with affine constraints. Both of those subsystems fit within the framework introduced in Part I, but in order to study the system as a whole and to explain the observed phenomenon, we need to understand the transition between the regimes.

In this part of the thesis, we will develop a geometrical framework with which we can describe the ANAIS billiard system and obtain generalizations. This general framework consists on allowing the vector fields on the plane and on the surface of the sphere to be piecewise smooth, instead of smooth, which further extends the class of examples introduced in Part I. Within this framework, we will prove the aforementioned phenomenon on the trajectories of the system and prove that analogous phenomena occur for other generalizations. Moreover, we will use the ANAIS billiard as a motivation to study the role of reversibility and symmetries in this extended class of examples.

Reversibility plays an important role in mathematical physics and dynamical systems (e.g. [60, 58]) and provides useful information on the properties of the dynamics of a system. Although there has been some research regarding the role of time-reversibility in nonholonomic systems with linear constraints [41], the understanding of reversibility in nonholonomic systems with affine constraints is much less developed. We attempt to make a step forward in this direction.

1.1 Structure

We begin by giving a detailed description of the ANAIS billiard in Section 2.1 and proceed to model this example as a hybrid system by determining the change in the velocities at points where the transition between the regimes takes place in Section 2.2 and describing its geometry in Section 2.3. In Section 2.4, we introduce a class of hybrid systems and, in particular, we introduce the generalizations of the ANAIS billiard, which we will consider in the sequel. At the end of Chapter 2, in Section 2.6, we give a precise statement of the ANAIS billiard phenomenon as Theorem II.2.10 and a generalization stated as Theorem II.2.11 that are proved in Chapter 5 once the necessary tools are developed.

In Chapter 3, we focus on the role of reversibility in nonholomic systems with affine constraints and, more in particular, in the ANAIS billiard and its generalizations. In Chapter 4 we will restrict our attention to the role of symmetries in nonholonomic affine and hybrid systems and in particular, we describe the symmetries that are present in our examples. Finally, in Chapter 5 we will use the reversibility and symmetry properties of the systems to prove the ANAIS billiard phenomenon and prove that analogous phenomena occur in the generalizations.

1.2 Convention

In Part I we mainly used the left trivialization of $T\mathrm{SO}(3)$. Throughout this next part, tangent vectors to $\mathrm{SO}(3)$ will be represented in the *right* trivialization. A fundamental role will therefore be played by the vector $\boldsymbol{\omega} \in \mathbb{R}^3$ whose components are the coordinates of the angular velocity vector with respect to the *space* frame Σ_s. In other words, $\hat{\boldsymbol{\omega}} = \dot{B}B^{-1}$ where

$$\hat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Therefore, when we write

$$\boldsymbol{v} = (\dot{\boldsymbol{x}}, \boldsymbol{\omega}) \in TQ,$$

we mean $\boldsymbol{v} \in T_{(\boldsymbol{x},B)}Q$ with $\dot{\boldsymbol{x}} \in T_{\boldsymbol{x}}\mathbb{R}^2$ and $\boldsymbol{v}_B \in T_B\mathrm{SO}(3)$ such that $T_B R_B^{-1}(\boldsymbol{v}_B) = \hat{\boldsymbol{\omega}} \in \mathfrak{so}(3)$. Moreover, from now on, we will only work with the homogeneous sphere and most vectors will be expressed in the space reference frame Σ_s, more precisely, we will use the setting introduced in Section 5.1 of Part I.

2 The ANAIS billiard and its generalizations

2.1 The ANAIS billiard

The ANAIS billiard concerns the motion of a homogeneous sphere rolling on a plane with a circular platform that rotates uniformly, as illustrated in Fig. 2.1.

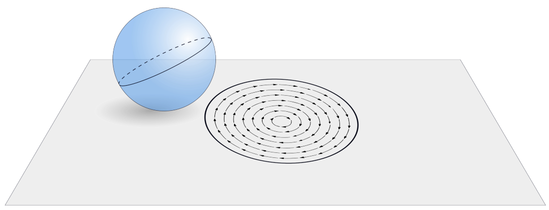


Figure 2.1: The ANAIS billiard.

The configuration manifold is given by $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ and the Lagrangian of the system is given by $L : TQ \to \mathbb{R}$

$$L(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \frac{I}{2}\|\boldsymbol{\omega}\|^2 + \frac{m}{2}\|\boldsymbol{x}\|^2, \tag{II.2.1}$$

where I is the inertia of the sphere and m is the mass of the sphere. Fixing the spatial reference frame Σ_s at the center of the rotating disc, the rolling constraints are given by

$$\dot{\boldsymbol{x}} = r(\boldsymbol{\omega} \times \boldsymbol{e}_3) + \boldsymbol{V}_s^A(\boldsymbol{x}), \tag{II.2.2}$$

where r is the radius of the sphere and $\boldsymbol{V}_s^A \in \mathfrak{X}(\Pi)$ is the piecewise smooth vector field defined by

$$\boldsymbol{V}_s^A(\boldsymbol{x}) = \begin{cases} \boldsymbol{0} & \text{if} \quad \|\boldsymbol{x}\| > R, \\ \eta e_3 \times \boldsymbol{x} & \text{if} \quad \|\boldsymbol{x}\| < R, \end{cases} \tag{II.2.3}$$

where $R > 0$ is the radius of the rotating platform and η is its uniform rotating speed.

When $\|\boldsymbol{x}\| > R$, the sphere is rolling on the fixed part of the plane, outside of the disc. If we restrict our attention to this part of the system, we recover the classical problem of the sphere rolling on the plane, which is a well-known mechanical system with linear nonholonomic constraints. As mentioned in Part 0, Section 2.1, the trajectory of the sphere on the plane is a straight line. On the other hand, when $\|\boldsymbol{x}\| < R$, the sphere is rolling on the rotating disc. Restricting our attention to this part of the system, we recover the problem of the homogeneous sphere rolling on a rotating plane, discussed in Part 0, Section 3.2 (see also [27, 57, 56, 9, 21, 31]). In this case, the system has affine nonholonomic constraints and the trajectory of the contact point on the plane traces out a circumference.

The most delicate issue in the modeling of the system is determining a reasonable transition between the two rolling regimes. This is what we do next, inspired by the results of Part I.

2.2 Modeling the ANAIS billiard as a hybrid system

The crucial point in the modeling of the ANAIS billiard is determining the velocity jump when the sphere exits the fixed part of the plane and enters the rotating disc and vice versa. In fact, the equations of motion of the system are smooth outside the submanifold N of the phase space, consisting of the configurations where the contact point of the sphere lies on the boundary of the rotating disc. As integral curves cross N, it is necessary to define a *jump* on the velocities to satisfy (II.2.2) which, due to the discontinuity of $\boldsymbol{V}_s^A$, accounts for the different nature of the constraints. The necessity to introduce this velocity jump is the reason why we use the term *hybrid* in the title of this section and in what follows. Note that the jump concerns only the velocities since the configuration of the sphere, determined by its position on the plane and its orientation, is continuous during the transition.

We begin by noticing that the ANAIS billiard system almost fits the framework of Part I (assuming that the convex body is a homogeneous sphere and setting $\boldsymbol{V}_s = \boldsymbol{V}_s^A$ and $\boldsymbol{W}_b = 0$) except for the fact that the vector field $\boldsymbol{V}_s^A$ is not smooth.

To deal with this difficulty, we approximate the ANAIS billiard system by a family of smooth affine nonholonomic systems which may be described within the framework of Part I, and converge to the ANAIS billiard in the limit.

With the above idea in mind, we define the smooth vector fields $\boldsymbol{V}_s^\varepsilon \in \mathfrak{X}(\Pi)$ parameterized by small positive $\varepsilon > 0$ given by

$$\boldsymbol{V}_s^\varepsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{0} & \text{if } \|\boldsymbol{x}\| > R + \frac{\varepsilon}{2}, \\ f_\varepsilon(\|\boldsymbol{x}\|)(\eta \boldsymbol{e}_3 \times \boldsymbol{x}) & \text{if } R - \frac{\varepsilon}{2} \le \|\boldsymbol{x}\| \le R + \frac{\varepsilon}{2}, \\ \eta \boldsymbol{e}_3 \times \boldsymbol{x} & \text{if } \|\boldsymbol{x}\| < R - \frac{\varepsilon}{2}, \end{cases}$$

where $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ is supported on the interval $(R - \frac{\varepsilon}{2}, R + \frac{\varepsilon}{2})$ and smoothly interpolates 1 and 0 as shown in figure 2.2.[1] The vector fields $\boldsymbol{V}_s^\varepsilon$ are smooth for all $\varepsilon > 0$ and pointwise (almost everywhere) converge to $\boldsymbol{V}_s^A$.

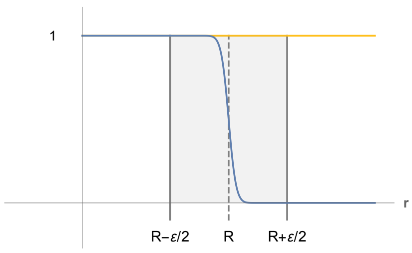


Figure 2.2: Graph of f_ε.

Now consider the problem of a homogeneous sphere of radius r rolling on the plane subject to the nonholonomic constraints

$$\dot{\boldsymbol{x}} = r(\boldsymbol{\omega} \times \boldsymbol{e}_3) + \boldsymbol{V}_s^\varepsilon(\boldsymbol{x}).$$

It is easily seen that this system is an approximation of the ANAIS billiard table in which we have introduced a transition region between the two rolling regimes given by an annulus of width ε. However, in contrast with the ANAIS billiard, due to the smoothness of $\boldsymbol{V}_s^\varepsilon$, the system falls within our standard framework and is a particular case of the systems treated in Section 5.1 of Part I. In particular, by Proposition I.3.1, it has three independent first integrals, which in view of (I.5.6) are given by

$$\begin{aligned} m_1^\varepsilon &= (I + mr^2)\omega_1 - mrV_2^\varepsilon(\boldsymbol{x}), \\ m_2^\varepsilon &= (I + mr^2)\omega_2 + mrV_1^\varepsilon(\boldsymbol{x}), \\ m_3^\varepsilon &= I\omega_3, \end{aligned} \tag{II.2.4}$$

[1] A closed form for f_ε is given in Appendix C.

where $V_1^\varepsilon(\boldsymbol{x})$ and $V_2^\varepsilon(\boldsymbol{x})$ are the first components of $\boldsymbol{V}_s^\varepsilon(\boldsymbol{x})$ (i.e. $\boldsymbol{V}_s^\varepsilon(\boldsymbol{x}) = (V_1^\varepsilon(\boldsymbol{x}), V_2^\varepsilon(\boldsymbol{x}), 0)$.

Our approach to the modeling of the transition between the two rolling regimes for the ANAIS billiard (the fixed part of the plane and the rotating disc) is that the limit of the integrals of motion (II.2.4) as $\varepsilon \to 0$ produces integrals of motion of the hybrid system. In other words, **we postulate that the jump in the velocities should be such that the following quantities remain constant:**

$$\begin{aligned} m_1^A &= (I + mr^2)\omega_1 - mrV_2^A(\boldsymbol{x}), \\ m_2^A &= (I + mr^2)\omega_2 + mrV_1^A(\boldsymbol{x}), \\ m_3^A &= I\omega_3. \end{aligned} \tag{II.2.5}$$

This condition, together with the nonholonomic constraints, uniquely determines the velocity jump.

Example: Consider the motion of the sphere that begins by rolling on the fixed part of the plane and enters the rotating disc at some time t_0. The coordinates of the point of contact at such time, $\boldsymbol{x}(t_0)$, therefore satisfy $\|\boldsymbol{x}(t_0)\| = R$. Let us indicate by t_0^- and t_0^+ the one-sided limits as time approaches t_0. At time t_0^-, the sphere is on the fixed part of the plane and at time t_0^+ it is on the rotating part. The condition that the integrals (II.2.5) remain constant through the transition implies

$$\begin{aligned} (I + mr^2)\omega_1(t_0^-) &= (I + mr^2)\omega_1(t_0^+) - mr\eta x_1(t_0), \\ (I + mr^2)\omega_2(t_0^-) &= (I + mr^2)\omega_2(t_0^+) - mr\eta x_2(t_0), \\ I\omega_3(t_0^-) &= I\omega_3(t_0^+). \end{aligned}$$

Therefore, the "jump" on the angular velocity is given by

$$\boldsymbol{\omega}(t_0^+) = \boldsymbol{\omega}(t_0^-) + \frac{mr\eta}{I + mr^2}\boldsymbol{x}(t_0),$$

(with our usual convention that the third component of $\boldsymbol{x}$ vanishes). On the other hand, using the above expressions together with the constraints (II.2.2), determines the "jump" on the linear velocity:

$$\dot{\boldsymbol{x}}(t_0^+) = \dot{\boldsymbol{x}}(t_0^-) - \eta\left(\frac{I}{I + mr^2}\right)\boldsymbol{x}(t_0) \times \boldsymbol{e}_3.$$

The above discussion and example motivate the precise statement of our postulate on the velocity jump for trajectories that change rolling regime.

Postulate II.2.1. Suppose that $q^* = (\boldsymbol{x}^*, B^*) \in Q$ is a configuration of the system such that $\|\boldsymbol{x}^*\| = R$. Assume that $(\dot{\boldsymbol{x}}^\eta, \boldsymbol{\omega}^\eta)$ and $(\dot{\boldsymbol{x}}^0, \boldsymbol{\omega}^0)$ respectively denote the velocities inside and outside the rolling platform at the instant in which the configuration is q^*. These velocities are not equal but are related by the jump condition:

$$\dot{\boldsymbol{x}}^\eta = \dot{\boldsymbol{x}}^0 - \eta\left(\frac{I}{I + mr^2}\right)\boldsymbol{x}^* \times \boldsymbol{e}_3, \qquad \boldsymbol{\omega}^\eta = \boldsymbol{\omega}^0 + \frac{mr\eta}{I + mr^2}\boldsymbol{x}^*.$$

The above postulate is tacitly assumed in [53], as we will make explicit in Appendix D. The theory developed in Part I and more precisely in Proposition I.3.1 provides a theoretical groundwork for the mathematical correctness of this approach. A discussion of the physical limitations was given in [42, 63].

Remark II.2.2. This approach can be generalized for the Chaplygin sphere using the first integrals of Proposition I.3.1. However, for more general bodies, there is no clear way to model an analogous transition.

2.3 Geometry and flow of the ANAIS billiard as a hybrid system

Let $U_0, U_\eta \subset Q$ respectively denote the configurations corresponding to the fixed and rotating parts of the plane, i.e.

$$U_0 = \{(\boldsymbol{x}, B) \in Q : \|\boldsymbol{x}\| > R\}, \qquad U_\eta = \{(\boldsymbol{x}, B) \in Q : \|\boldsymbol{x}\| < R\}, \tag{II.2.6}$$

and let $\mathcal{B}$ be the set of configurations which divides these two regions,

$$\mathcal{B} = \{(\boldsymbol{x}, B) \in Q : \|\boldsymbol{x}\| = R\}. \tag{II.2.7}$$

The phase space $\mathcal{A}^A$ of the system may be expressed as $\mathcal{A}^A = \mathcal{D} + Z^A$, where $\mathcal{D} \subset TQ$ is the linear distribution defining the constraints of rolling without slipping,

$$\mathcal{D} = \{(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) \in TQ : \dot{\boldsymbol{x}} = r\boldsymbol{\omega} \times \boldsymbol{e}_3\}, \tag{II.2.8}$$

and $Z^A \in \mathfrak{X}(Q)$ is the piecewise smooth vector field

$$Z^A(\boldsymbol{x}, B) = (\boldsymbol{V}^A(\boldsymbol{x}), \boldsymbol{0}). \tag{II.2.9}$$

Note that Z^A is smooth on U_0 and U_η but has a discontinuity along $\mathcal{B}$ corresponding to the change of regime. As a consequence, $\mathcal{A}^A$ is a piecewise smooth affine subbundle of TQ. To write things in a notation that is convenient for the generalizations of the system that we develop below, we consider the vector fields $\boldsymbol{V}^0, \boldsymbol{V}^\eta \in \mathfrak{X}(\Pi)$

$$\boldsymbol{V}^0(\boldsymbol{x}) = \boldsymbol{0}, \tag{II.2.10}$$

$$\boldsymbol{V}^\eta(\boldsymbol{x}) = \eta \boldsymbol{e}_3 \times \boldsymbol{x}, \tag{II.2.11}$$

and the vector fields $Z^0, Z^\eta \in \mathfrak{X}(Q)$ defined as

$$Z^0(\boldsymbol{x}, B) = (\boldsymbol{V}^0(\boldsymbol{x}), 0) = (\boldsymbol{0}, \boldsymbol{0}), \tag{II.2.12}$$

$$Z^\eta(\boldsymbol{x}, B) = (\boldsymbol{V}^\eta(\boldsymbol{x}), 0) = (\eta \boldsymbol{e}_3 \times \boldsymbol{x}, \boldsymbol{0}), \tag{II.2.13}$$

and define the smooth affine subbundles $\mathcal{A}^0, \mathcal{A}^\eta \subset TQ$ as

$$\mathcal{A}^0_q = \mathcal{D}_q + Z^0(q), \qquad \mathcal{A}^\eta_q = \mathcal{D}_q + Z^\eta(q).$$

With the above definitions, we may write

$$\mathcal{A}^A_q = \begin{cases} \mathcal{A}^0_q & \text{if } q \in U_0, \\ \mathcal{A}^\eta_q & \text{if } q \in U_\eta. \end{cases} \tag{II.2.14}$$

Now, since $\mathcal{A}^0$ and $\mathcal{A}^\eta$ are smooth affine subbundles, there is a well-defined non-holonomic dynamics determined by the Lagrange-D'Alembert principle. This dynamics is described by the flow of vector fields on these subbundles, which we denote as

$$X^0 \in \mathfrak{X}(\mathcal{A}^0) \qquad \text{and} \qquad X^\eta \in \mathfrak{X}(\mathcal{A}^\eta).$$

The dynamics of our system is determined by piecing together trajectories of these vector fields in a way that we make precise below. Consider the vector field $X^A \in \mathfrak{X}(\mathcal{A}^A)$ defined by

$$X^A(z) = \begin{cases} X^0(z) & \text{if } q \in U_0, \\ X^\eta(z) & \text{if } q \in U_\eta, \end{cases} \tag{II.2.15}$$

where we have written $z = (q, v) \in TQ$. Then X^A is smooth on both of the smooth components of $\mathcal{A}^A$. The dynamics of our problem is then described by the flow of X^A on each of the smooth components complemented with Postulate II.2.1 which determines the transition at the discontinuity points. In order to explain how this transition takes place, we introduce the following *jump functions*. Let

$q^* = (\boldsymbol{x}^*, B^*) \in \mathcal{B}$, inspired by Postulate II.2.1, we define

$$\begin{aligned} &\mathcal{J}_0^\eta : \mathcal{A}_{q^*}^0 \to \mathcal{A}_{q^*}^\eta, \qquad \mathcal{J}_0^\eta(\boldsymbol{x}^*, B^*, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (\dot{\boldsymbol{x}}, \boldsymbol{\omega}) + \frac{\eta}{I + mr^2}(I\boldsymbol{e}_3 \times \boldsymbol{x}^*, mr\boldsymbol{x}^*), \\ &\mathcal{J}_\eta^0 : \mathcal{A}_{q^*}^\eta \to \mathcal{A}_{q^*}^0, \qquad \mathcal{J}_\eta^0(\boldsymbol{x}^*, B^*, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (\dot{\boldsymbol{x}}, \boldsymbol{\omega}) - \frac{\eta}{I + mr^2}(I\boldsymbol{e}_3 \times \boldsymbol{x}^*, mr\boldsymbol{x}^*), \end{aligned} \tag{II.2.16}$$

where we have omitted the base point $(\boldsymbol{x}^*, B^*)$ on the tangent vectors on the right hand side of the equation.

2.3.1 Flow of the ANAIS billiard system

We define the flow $\Phi^A : \mathbb{R} \times \mathcal{A}^A \to \mathcal{A}^A$ as follows. Let Φ^0 and Φ^η respectively denote the flows of the vector fields $X^0 \in \mathfrak{X}(\mathcal{A}^0)$ and $X^\eta \in \mathfrak{X}(\mathcal{A}^\eta)$. Consider an initial condition $z_0 = (q_0, v_0) \in \mathcal{A}_{q_0}$.

1. If $z_0 = (q_0, v_0) \in \mathcal{A}_{q_0}^0$ we consider the curve $z(t) = \Phi_t^0(z_0)$ and write it as $z(t) = (q(t), v(t))$.
   (a) If $q(t) \in U_0$ for all $t \in [0, T)$ then we define $\Phi_t^A(z_0) = \Phi_t^0(z_0)$ in this interval.
   (b) If $q(t) \in U_0$ for all $t \in [0, T)$ and $q(T) \in \mathcal{B}$, we define
   $$\Phi_t^A(z_0) = \begin{cases} \Phi_t^0(z_0) & \text{if } 0 < t < T, \\ \Phi_t^\eta(\mathcal{J}_0^\eta(z(T))) & \text{for } t \in [T, T+\varepsilon], \text{ for some small } \varepsilon > 0. \end{cases}$$
2. If $z_0 = (q_0, v_0) \in \mathcal{A}_{q_0}^\eta$ we consider the curve $z(t) = \Phi_t^\eta(z_0)$ and write it as $z(t) = (q(t), v(t))$.
   (a) If $q(t) \in U_\eta$ for all $t \in [0, T)$ then we define $\Phi_t^A(z_0) = \Phi_t^\eta(z_0)$ in this interval.
   (b) If $q(t) \in U_\eta$ for all $t \in [0, T)$ and $q(T) \in \mathcal{B}$, we define
   $$\Phi_t^A(z_0) = \begin{cases} \Phi_t^\eta(z_0) & \text{if } 0 < t < T, \\ \Phi_t^0(\mathcal{J}_\eta^0(z(T))) & \text{for } t \in [T, T+\varepsilon] \text{ for some small } \varepsilon > 0. \end{cases}$$

The value of $\Phi_t^A(z_0)$ for arbitrary $t \in \mathbb{R}$ is determined from the above prescriptions requiring that the flow condition $\Phi_{t+s}^A(z_0) = \Phi_t^A(\Phi_s^A(z_0))$ holds.

Remark II.2.3. In item 1b the small number $\varepsilon > 0$ is chosen small enough so that $q(t)$ does not leave U_η. Namely, $z(t) = \Phi_t^\eta(\mathcal{J}_0^\eta(z(T)))$ satisfies $q(t) \in U_\eta$ for all $t \in (T, T+\varepsilon]$. Such ε always exist unless the trajectory in the fixed part of the plane traces out a straight line which touches the rotating platform tangentially. In such case there is no real change of regime and there is no need to introduce a jump. We have avoided mentioning this kind of trajectories in the definition of Φ^A for ease of presentation. Analogous remarks apply for item 2b.

The above prescription gives a well-defined flow $\Phi^A : \mathbb{R} \times \mathcal{A}^A \to \mathcal{A}^A$ which adequately describes the dynamics of the problem. We emphasize that Φ^A possesses the first integrals (II.2.5).

2.4 Generalizations of the ANAIS billiard

In the next sections, inspired by [53] and [63], we will consider a series of generalizations of the ANAIS billiard. These concern the motion of a homogeneous sphere of radius r on the plane subject to the nonholonomic constraints,

$$\dot{\boldsymbol{x}} = -r\boldsymbol{e}_3 \times \boldsymbol{\omega} + \boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\gamma), \tag{II.2.17}$$

where the vector fields $\boldsymbol{V}_s \in \mathfrak{X}(\Pi)$ and $\boldsymbol{W}_b \in \mathfrak{X}(\mathcal{S})$ are piecewise smooth and are allowed to have discontinuities. For simplicity, we will assume that only one of the vector fields $\boldsymbol{V}_s \in \mathfrak{X}(\Pi)$ or $\boldsymbol{W}_b \in \mathfrak{X}(\mathcal{S})$ is piecewise smooth and assume that the other one is smooth. Furthermore, we will assume that the set of discontinuities of the piecewise smooth vector field is comprised of a finite number of smooth curves, each of which separates two open sets on which the vector field is smooth. Some examples of admissible vector fields on Π and $\mathcal{S}$ are illustrated in figure 2.3.

As for the ANAIS billiard above, the modeling of such systems requires a postulate to model the velocity jumps introduced by the discontinuities on the constraints. Due to these velocity jumps, we will also use the terminology *hybrid* for these systems.

2.4.1 Planar generalizations

Let us first consider the case in which W is smooth and $\boldsymbol{V}_s \in \mathfrak{X}(\Pi)$ is piecewise smooth and is allowed to have discontinuities. Let $U_1^\Pi, \dots, U_n^\Pi \subset \Pi$ denote the n open sets in which the vector field $\boldsymbol{V}_s$ is smooth. Then, $\boldsymbol{V}_s(\boldsymbol{x})$ may be written as

$$\boldsymbol{V}_s(\boldsymbol{x}) = \begin{cases} \boldsymbol{V}^1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U_1^\Pi, \\ \vdots & \\ \boldsymbol{V}^n(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U_n^\Pi, \end{cases}$$

for certain smooth vector fields $\boldsymbol{V}^i \in \mathfrak{X}(\Pi)$, $i = 1, \dots, n$.

Suppose that U_i^Π and U_j^Π share a piece of boundary (i.e. $(\partial U_i^\Pi) \cap (\partial U_j^\Pi) \neq \varnothing$), we denote by $\mathcal{B}_{ij}^\Pi$ the curve on the plane Π that describes the boundary between U_i^Π and U_j^Π, i.e. $\mathcal{B}_{ij}^\Pi = (\partial U_i^\Pi) \cap (\partial U_j^\Pi)$.

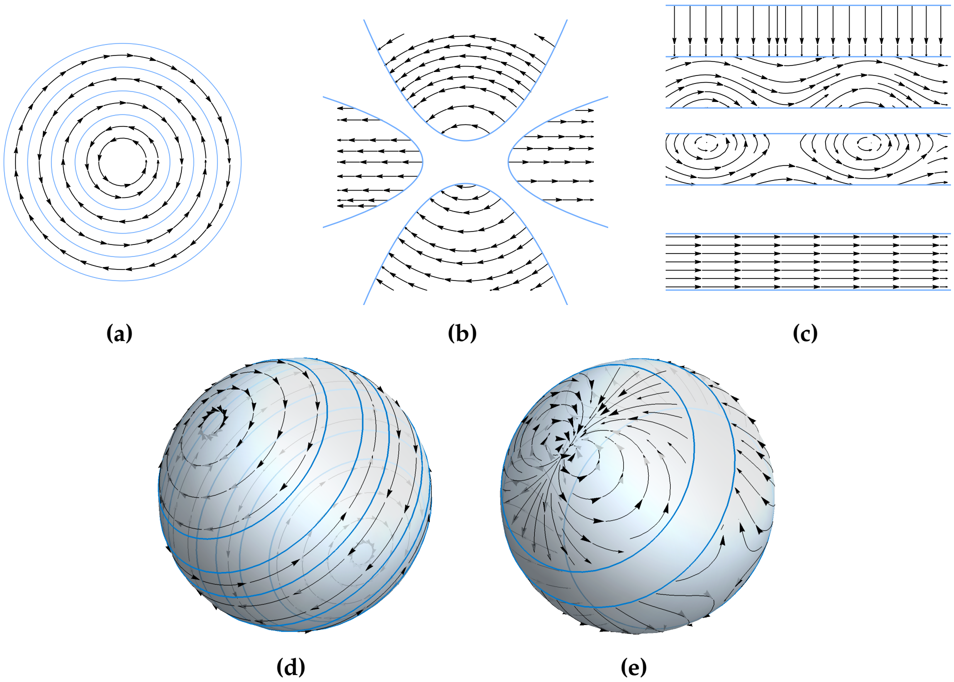


Figure 2.3: Examples of admissible vector fields on the plane Π (2.3a, 2.3b and 2.3c) and on the surface of the sphere $\mathcal{S}$ (2.3d, 2.3e).

Example: In the ANAIS billiard, the plane Π is divided in two regions $U_0^\Pi = \{\boldsymbol{x} \in \Pi : \|\boldsymbol{x}\| > R\}$ and $U_\eta^\Pi = \{\boldsymbol{x} \in \Pi : \|\boldsymbol{x}\| < R\}$ where the vector field $\boldsymbol{V}_s^A$ defined in (II.2.3) is smooth. The boundary $\mathcal{B}^\Pi \subset Q$ between these two regions is given by $\mathcal{B}^\Pi = \{\boldsymbol{x} \in \Pi : \|\boldsymbol{x}\| = R\}$.

Inspired by the discussion above for the ANAIS billiard, we formulate the following.

Postulate II.2.4. Let $q^* = (\boldsymbol{x}^*, B^*) \in Q$ be a configuration such that $\boldsymbol{x}^*$ is a point of discontinuity of the vector field $\boldsymbol{V}_s(\boldsymbol{x}) \in \mathfrak{X}(\Pi)$, i.e. $\boldsymbol{x}^* \in \mathcal{B}_{ij}^\Pi$ for some i, j. Assume that $(\dot{\boldsymbol{x}}^i, \boldsymbol{\omega}^i)$ and $(\dot{\boldsymbol{x}}^j, \boldsymbol{\omega}^j)$ respectively denote the velocities in U_i^Π and U_j^Π at the instant in which the configuration is q^*. These velocities are not equal but are related by the jump:

$$\begin{aligned}
\dot{\boldsymbol{x}}^i &= \dot{\boldsymbol{x}}^j + \frac{I}{I + mr^2}(V^i(\boldsymbol{x}^*) - V^j(\boldsymbol{x}^*)),\\
\boldsymbol{\omega}^i &= \boldsymbol{\omega}^j + \frac{mr}{I + mr^2}(V^i(\boldsymbol{x}^*) - V^j(\boldsymbol{x}^*)) \times \boldsymbol{e}_3.
\end{aligned}$$

By construction, the Postulate II.2.4 guarantees that the hybrid system possesses the first integrals (I.5.6) which we rewrite below for convenience

$$\begin{aligned} m_1 &= (I + mr^2)\omega_1 - mr(V_2(\boldsymbol{x}) + (B\boldsymbol{W}_b(\boldsymbol{\gamma}))_2), \\ m_2 &= (I + mr^2)\omega_2 + mr(V_1(\boldsymbol{x}) + (B\boldsymbol{W}_b(\boldsymbol{\gamma}))_1), \\ m_3 &= I\omega_3, \end{aligned} \tag{II.2.18}$$

where $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ denote the components of $\boldsymbol{V}_s(\boldsymbol{x})$. As a consequence of the existence of these first integrals, we have the following result stated previously in [53, 63].

Proposition II.2.5. *Consider a nonholonomic affine hybrid system on the configuration space $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ with Lagrangian (II.2.1) and constraints (II.2.17), where $W = 0$ and the vector field $\boldsymbol{V}_s$ is piecewise smooth and may have discontinuities. Suppose moreover that $\boldsymbol{V}_s(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in U_0^\Pi$ for some non-empty open set $U_0^\Pi \subset \mathbb{R}^2$. Then the linear velocity $\dot{\boldsymbol{x}}(t)$ and the angular velocity $\boldsymbol{\omega}(t)$ are constant for all t such that $\boldsymbol{x}(t) \in U_0^\Pi$.*

Proof. Considering that $W = 0$ and $\boldsymbol{V}_s(\boldsymbol{x}(t)) = 0$ if $\boldsymbol{x}(t) \in U_0^\Pi$ then the first integrals (II.2.18) take the form

$$m_1 = (I + mr^2)\omega_1(t), \qquad m_2 = (I + mr^2)\omega_2(t), \qquad m_3 = I\omega_3(t), \tag{II.2.19}$$

so the vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is constant. In view of the constraints (II.2.17), we have

$$\dot{x}_1(t) = r\omega_2, \qquad \dot{x}_2(t) = -r\omega_1, \qquad \dot{x}_3(t) = 0. \tag{II.2.20}$$

so the vector $\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(t)$ is constant as well. □

2.4.2 Spherical generalizations

We now consider the case in which the vector field on the plane V is smooth and the vector field on the sphere $\boldsymbol{W}_b(\boldsymbol{\gamma})$ is piecewise smooth and may have discontinuities. In this case, we assume that $\boldsymbol{W}_b$ is smooth in n regions $U_1^{\mathcal{S}}, \ldots, U_n^{\mathcal{S}} \subset \mathcal{S}$ of the surface of the sphere $\mathcal{S}$ and that there are no points on $\mathcal{S}$ which simultaneously belong to the boundary of three or more regions. If two regions $U_i^{\mathcal{S}}$ and $U_j^{\mathcal{S}}$ have shared boundary, then we denote by $\mathcal{B}_{ij}^{\mathcal{S}}$ the curve on the sphere that describes the boundary, i.e. $\mathcal{B}_{ij}^{\mathcal{S}} = (\partial U_i^{\mathcal{S}}) \bigcap (\partial U_j^{\mathcal{S}})$. In analogy to the previous case, the vector field $\boldsymbol{W}_b(\gamma)$ may be written as

$$\boldsymbol{W}_b(\gamma) = \begin{cases} W^1(\gamma) & \text{if } \gamma \in U_1^{\mathcal{S}}, \\ \vdots & \\ W^n(\gamma) & \text{if } \gamma \in U_n^{\mathcal{S}}, \end{cases}$$

for certain $\boldsymbol{W}^i \in \mathfrak{X}(\mathcal{S})$, $i = 1, \dots, n$. In this case, we state the following.

Postulate II.2.6. Let $q^* = (\boldsymbol{x}^*, B^*) \in Q$ be a configuration such that $\boldsymbol{\gamma}^* = (B^*)^{-1}\boldsymbol{e}_3$ is a point of discontinuity of the vector field $\boldsymbol{W}_b(\gamma) \in \mathfrak{X}(\mathcal{S})$, i.e. $\boldsymbol{\gamma}^* \in \mathcal{B}_{ij}^{\mathcal{S}}$ for some i, j. Assume that $(\dot{\boldsymbol{x}}^i, \boldsymbol{\omega}^i)$ and $(\dot{\boldsymbol{x}}^j, \boldsymbol{\omega}^j)$ respectively denote the velocities in $U_i^{\mathcal{S}}$ and $U_j^{\mathcal{S}}$ at the instant in which the configuration is q^*. These velocities are not equal but are related by the jump:

$$\begin{aligned} \dot{\boldsymbol{x}}^i &= \boldsymbol{x}^j + \frac{I}{I + mr^2} B(W^i(\gamma_0) - W^j(\gamma_0)), \\ \dot{\boldsymbol{\omega}}^i &= \boldsymbol{\omega}^j + \frac{mr}{I + mr^2} B(W^i(\gamma_0) - W^j(\gamma_0)) \times \boldsymbol{e}_3. \end{aligned}$$

In analogy to Proposition II.2.5, we have the following result whose proof is identical.

Proposition II.2.7. *Consider a nonholonomic affine hybrid system on the configuration space $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ with Lagrangian (II.2.1) and constraints (II.2.17), where $V = 0$ and the vector field $\boldsymbol{W}_b$ is piecewise smooth and may have discontinuities. Suppose moreover that $\boldsymbol{W}_b(\boldsymbol{\gamma}) = 0$ for all $\boldsymbol{\gamma} \in U_0^{\mathcal{S}}$ for some non-empty open set $U_0^{\mathcal{S}} \subset S^2$. Then the linear velocity $\dot{\boldsymbol{x}}(t)$ and the angular velocity $\boldsymbol{\omega}(t)$ are constant for all t such that $\gamma(t) \in U_0^{\mathcal{S}}$.*

2.4.3 Geometry of the generalizations

The phase space $\mathcal{A}$ of a nonholonomic system on Q with constraints (II.2.17) may be expressed as $\mathcal{A} = \mathcal{D} + Z$ where $\mathcal{D} \subset TQ$ is the linear distribution defined in (II.2.8) and $Z \in \mathfrak{X}(Q)$ is a vector field defined by

$$Z(\boldsymbol{x}, B) = (\boldsymbol{V}_s(\boldsymbol{x}) + B\boldsymbol{W}_b(\gamma), \boldsymbol{0}). \tag{II.2.21}$$

If either $\boldsymbol{V}_s(\boldsymbol{x})$ or $\boldsymbol{W}_b(\gamma)$ are piecewise smooth, instead of smooth, then the vector field Z is piecewise smooth and $\mathcal{A} \subset TQ$ defines a piecewise smooth affine subbundle. Let us denote by $U_1, \dots, U_n \subset Q$ the n regions in which Z is smooth and by $\mathcal{B}_{ij}$ the boundary between the regions U_i and U_j. Then, if $q = (\boldsymbol{x}, B)$, we may write

$$Z(q) = \begin{cases} Z^1(q) & \text{if } q \in U_1, \\ \vdots & \\ Z^n(q) & \text{if } q \in U_n, \end{cases} \tag{II.2.22}$$

for certain smooth vector fields $Z^i \in \mathfrak{X}(Q)$. Moreover, we can define the affine subbundles $\mathcal{A}^i \subset TQ$ as $\mathcal{A}^i_q = \mathcal{D}_q + Z^i(q)$ for all $i = 1, \dots, n$ and write

$$\mathcal{A}_q = \begin{cases} A^1_q & \text{if } q \in U_1, \\ \vdots & \\ A^n_q & \text{if } q \in U_n. \end{cases} \tag{II.2.23}$$

We denote by $X^i \in \mathfrak{X}(\mathcal{A}^i)$ the vector fields determined by the equations of motion of the nonholonomic system defined on $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$ with constraint distribution $\mathcal{A}^i = \mathcal{D} + Z^i$ and Lagrangian L given by (0.2.3). The dynamics of the whole system is determined by the (piecewise smooth) vector field $X \in \mathfrak{X}(\mathcal{A})$ defined for $z = (q, v) \in TQ$ by

$$X(z) = \begin{cases} X^1(z) & \text{if } q \in U_1, \\ \vdots & \\ X^n(z) & \text{if } q \in U_n. \end{cases} \tag{II.2.24}$$

together with postulates II.2.4 and II.2.6, which determine the transition at the discontinuities. We define the jump functions $\mathcal{J}^j_i : \mathcal{A}^i_q \to \mathcal{A}^j_q$ on $q \in \mathcal{B}_{ij}$ by

$$\begin{aligned} \mathcal{J}^j_i(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (\dot{\boldsymbol{x}}, \boldsymbol{\omega}) &+ \frac{1}{I + mr^2}(I(V^i(\boldsymbol{x}) - V^j(\boldsymbol{x})), mr(V^i(\boldsymbol{x}) - V^j(\boldsymbol{x})) \times \boldsymbol{e}_3) \\ &+ \frac{1}{I + mr^2}(IB(W^i(\boldsymbol{\gamma}) - W^j(\boldsymbol{\gamma})), mrB(W^i(\boldsymbol{\gamma}) - W^j(\boldsymbol{\gamma})) \times \boldsymbol{e}_3), \end{aligned} \tag{II.2.25}$$

The dynamics are determined by the flow

$$\Phi^X : \mathbb{R} \times \mathcal{A} \to \mathcal{A},$$

specified in the next subsection.

2.4.4 Flow Φ^X of the generalizations

We denote by $\Phi^i : \mathbb{R} \times \mathcal{A}^i \to \mathcal{A}^i$ the flows of the vector fields $X^i \in \mathfrak{X}(A^i)$ for $i = 1, \dots, n$, and we define the flow $\Phi^X : \mathbb{R} \times \mathcal{A} \to \mathcal{A}$ as follows. If $z_i = (q_i, v_i) \in \mathcal{A}^i$, we consider $z(t) = \Phi_t^i(z_i)$ and write it as $z(t) = (q(t), v(t))$. If $q(t) \in U_i$ for all $t \in [0, T)$ then

$$\Phi_t^X(z_i) = \Phi_t^i(z_i) \qquad \text{for all } t \in [0, T).$$

If $q(t) \in U_i$ for $t \in [0, T)$ and $q(T) \in \mathcal{B}_{ij}$, then

$$\Phi_t^X(z_i) = \begin{cases} \Phi_t^i(z_i) & \text{if } 0 < t < T, \\ \Phi_t^j(\mathcal{J}_i^j(z(T))) & \text{if } t \in [T, T+\varepsilon] \text{ for small } \varepsilon. \end{cases}$$

We consider ε to be small enough so that $\Phi_t^j(\mathcal{J}_i^j(z(T))) \in U_j$ for all $t \in (T, T+\varepsilon]$ (a remark similar to Remark II.2.3 applies for tangential trajectories). Enforcing the flow condition $\Phi_{t+s}^X = \Phi_t^X \circ \Phi_s^X$ and repeating the same prescription any time $q(t)$ changes region, we get a flow that describes the dynamics of the system.

Remark II.2.8. There is a slight ambiguity in the definition of Φ^X for the trajectories which meet the boundaries $\mathcal{B}_{ij}$ tangentially. Our convention is not to introduce a jump in this case.

We again emphasize that the flow Φ^X possesses the first integrals (II.2.18).

2.5 Examples

We introduce the main examples with which we will work throughout this part.

2.5.1 A vector field on a stripe of the plane

We consider a generalization of the ANAIS billiard which was described in [53] and is as follows: the points of a vertical strip of the plane move according to a vector field which is reversible with respect to a reflection with respect to the vertical axis, and outside this strip the plane is fixed, as illustrated in figure 2.4.

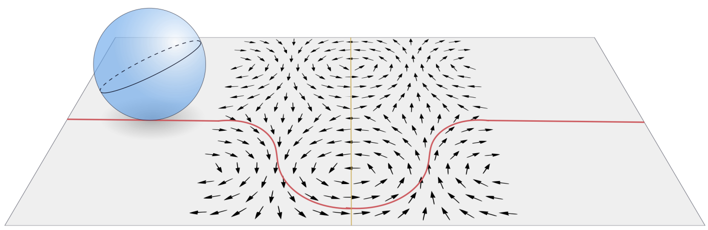


Figure 2.4: Generalization of the ANAIS billiard phenomenon with a symmetric vector field.

The system is described by setting $W = 0$ and considering the sets $U_0^\Pi, U_1^\Pi \subset \Pi$ as

$$U_0^\Pi = \{\boldsymbol{x} \in \Pi : |\langle \boldsymbol{x}, \boldsymbol{e}_1\rangle| > A\}, \qquad U_1^\Pi = \{\boldsymbol{x} \in \Pi : |\langle \boldsymbol{x}, \boldsymbol{e}_1\rangle| < A\},$$

with $A > 0$. Then, we denote by

$$\mathcal{B}_{01}^\Pi = \{\boldsymbol{x} \in \Pi : |\boldsymbol{x}| = A\},$$

the set where the transition between the regimes occurs. We define the vector field $\boldsymbol{V}_s^G \in \mathfrak{X}(\Pi)$ as

$$\boldsymbol{V}_s^G(\boldsymbol{x}) = \begin{cases} \boldsymbol{V}^0(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U_0^\Pi, \\ \boldsymbol{V}^1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U_1^\Pi, \end{cases} \tag{II.2.26}$$

with $\boldsymbol{V}^0(\boldsymbol{x}) = \boldsymbol{0}$ and we assume that the vector field $\boldsymbol{V}^1$ on the plane satisfies

$$\Sigma \boldsymbol{V}^1(\boldsymbol{x}) = \boldsymbol{V}^1(-\Sigma \boldsymbol{x}), \qquad \text{with } \Sigma \text{ as in (II.3.5)}. \tag{II.2.27}$$

Remark II.2.9. Notice that, in coordinates, the property (II.2.27) gives

$$\begin{pmatrix} V_1^1(x_1,x_2,0) \\ -V_2^1(x_1,x_2,0) \\ 0 \end{pmatrix} = \Sigma \begin{pmatrix} V_1^1(x_1,x_2,0) \\ V_2^1(x_1,x_2,0) \\ 0 \end{pmatrix} = \begin{pmatrix} V_1^1(-\Sigma(x_1,x_2,0)) \\ V_2^1(-\Sigma(x_1,x_2,0)) \\ 0 \end{pmatrix} = \begin{pmatrix} V_1^1(-x_1,x_2,0) \\ V_2^1(-x_1,x_2,0) \\ 0 \end{pmatrix},$$

and in particular

$$V_2^1(0,x_2,0) = -V_2^1(0,x_2,0) = 0 \qquad \text{for every } x_2 \in \mathbb{R}.$$

Now, the sets $U_0, U_1, \mathcal{B}_{10} \subset Q$ are defined as

$$U_0 = U_0^\Pi \times \mathrm{SO}(3) \subset Q, \qquad U_1 = U_1^\Pi \times \mathrm{SO}(3) \qquad \text{and} \qquad \mathcal{B}_{01} = \mathcal{B}_{01}^\Pi \times \mathrm{SO}(3).$$

We define the smooth vector fields $Z^0, Z^1 \in \mathfrak{X}(Q)$ as

$$\begin{aligned} Z^0(\boldsymbol{x}, B) &= (\boldsymbol{0}, \boldsymbol{0}), \\ Z^1(\boldsymbol{x}, B) &= (\boldsymbol{V}^1(\boldsymbol{x}), \boldsymbol{0}), \end{aligned} \tag{II.2.28}$$

and the piecewise smooth vector field $Z^G \in \mathfrak{X}(Q)$ as

$$Z^G(q) = \begin{cases} Z^0(q) & \text{if } q \in U_0, \\ Z^1(q) & \text{if } q \in U_1. \end{cases}$$

Following the notation introduced in Section 2.4.3, we will denote by $\mathcal{A}^0, \mathcal{A}^1$ the affine distributions defined by $\mathcal{A}^0 = \mathcal{D} + Z^0$ and $\mathcal{A}^1 = \mathcal{D} + Z^1$. Moreover, the piecewise smooth affine distribution $\mathcal{A}^G$ is defined as

$$A_q^G = \begin{cases} A_q^0 & \text{if } q \in U_0, \\ A_q^1 & \text{if } q \in U_1. \end{cases}$$

We denote by $X^0 \in \mathfrak{X}(\mathcal{A}^0)$ and $X^1 \in \mathfrak{X}(\mathcal{A}^1)$ the vector fields defined by the equations of motion of the systems determined by Q, L and $\mathcal{A}^0$ or $\mathcal{A}^1$; while $X^G \in \mathfrak{X}(\mathcal{A}^G)$ is the vector field defined by

$$X^G(q,v) = \begin{cases} X^0(q,v) & \text{if } q \in U_0, \\ X^1(q,v) & \text{if } q \in U_1. \end{cases}$$

Finally, Φ^0 and Φ^1 respectively denote the flows of X^0 and X^1, while Φ^G is the flow defined by piecing together Φ^0 and Φ^1 (with the jump functions $\mathcal{J}_0^1, \mathcal{J}_1^0$) as defined in Section 2.4.4 for this specific example.

2.5.2 The Janus sphere

As the main example of the spherical generalizations, we will consider a sphere with a particular kind of cat's toy mechanism (described in Part I, Chapter 2) in which only one part of the shell is rotating along the $\boldsymbol{E}_3$ axis and the rest of the shell is fixed. We assume that the curve between the fixed and the rotating part of the sphere is a circular section which is perpendicular to the rotation axis at a constant height $a \in (-r, r)$ with respect to the center of the sphere, see Figure 2.5. We will call this example *Janus sphere* after the Roman god Janus[2].

Figure 2.5: The Janus sphere.

This system can be described within our framework by setting $V = 0$ and

[2]we are thankful to Jair Koiller for suggesting this name.

letting $\boldsymbol{W}_b^C \in \mathfrak{X}(\mathcal{S})$ be the piecewise smooth vector field defined by

$$\boldsymbol{W}_b^C(\boldsymbol{\gamma}) = \begin{cases} \boldsymbol{W}^0(\boldsymbol{\gamma}) & \text{if } \boldsymbol{\gamma} \in U_0^{\mathcal{S}}, \\ \boldsymbol{W}^\sigma(\boldsymbol{\gamma}) & \text{if } \boldsymbol{\gamma} \in U_\sigma^{\mathcal{S}}, \end{cases}$$

with

$$\boldsymbol{W}^0(\boldsymbol{\gamma}) = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{W}^\sigma(\boldsymbol{\gamma}) = -r\sigma(\gamma_3)(\boldsymbol{\gamma} \times \boldsymbol{E}_3), \tag{II.2.29}$$

where $\sigma \in \mathbb{R}$ the angular velocity of the rotating part of the shell, and the sets $U_0^{\mathcal{S}}, U_\sigma^{\mathcal{S}} \subset \mathcal{S}$ are defined as

$$U_0^{\mathcal{S}} = \{\boldsymbol{\gamma} \in S^2 \,:\, \gamma_3 < a\}, \qquad U_\sigma^{\mathcal{S}} = \{\boldsymbol{\gamma} \in S^2 \,:\, \gamma_3 > a\}, \tag{II.2.30}$$

where a is the height of the transition. The boundary $\mathcal{B}_{0\sigma} \subset \mathcal{S}$ between $U_0^{\mathcal{S}}$ and $U_1^{\mathcal{S}}$ is described by

$$\mathcal{B}_{0\sigma} = \{\boldsymbol{\gamma} \in S^2 \,:\, \gamma_3 = a\}.$$

For future reference, we write the explicit form of the constraints:

$$\dot{\boldsymbol{x}} = \begin{cases} -r\boldsymbol{e}_3 \times \boldsymbol{\omega} - r\sigma(\boldsymbol{\gamma} \times \boldsymbol{E}_3) & \text{if } \gamma_3 > a, \\ -r\boldsymbol{e}_3 \times \boldsymbol{\omega} & \text{if } \gamma_3 < a. \end{cases} \tag{II.2.31}$$

Now, we denote by $U_0, U_\sigma \mathcal{B}_{0\sigma} \subset Q$ the sets

$$U_0 = \mathbb{R}^2 \times U_1^{\mathcal{S}} \subset Q, \qquad U_\sigma = \mathbb{R}^2 \times U_\sigma^{\mathcal{S}} \subset Q \qquad \text{and} \qquad \mathcal{B}_{0\sigma} = \mathbb{R}^2 \times \mathcal{B}_{0\sigma}^{\mathcal{S}}.$$

In this case, the vector smooth fields $Z^0, Z^\sigma \in \mathfrak{X}(Q)$ are given by

$$\begin{aligned} Z^0(\boldsymbol{x}, B) &= (\mathbf{0}, \mathbf{0}), \\ Z^\sigma(\boldsymbol{x}, B) &= (B\boldsymbol{W}^\sigma(\boldsymbol{\gamma}), \mathbf{0}), \end{aligned} \tag{II.2.32}$$

and the piecewise vector field $Z^C \in \mathfrak{X}(Q)$ by

$$Z^C = \begin{cases} Z^0(\boldsymbol{x}, B) & \text{if } q \in U_0^{\mathcal{S}}, \\ Z^\sigma(\boldsymbol{x}, B) & \text{if } q \in U_1^{\mathcal{S}}. \end{cases} \tag{II.2.33}$$

Following the notation introduced in Section 2.4.3, we will denote the corresponding affine distributions $\mathcal{A}^i$, vector fields X^i and flows Φ^i with indices $0, \sigma, C$ where the flow Φ^C is defined as in Section 2.4.4 with the jump functions $\mathcal{J}_i^j$ with $i, j = 0, \sigma$ defined as in (II.2.25) for this specific example.

2.6 The ANAIS billiard phenomenon and its generalizations

The ANAIS billiard phenomenon, illustrated in Fig 2.6 (for a video see [25]), was observed at the Nice Exploratory, and has been considered in [53, 42, 64, 63]. Below, we state it as a theorem that we will prove in Chapter 5.

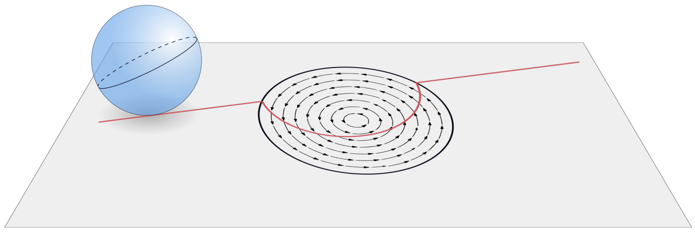


Figure 2.6: The ANAIS billiard phenomenon.

Theorem II.2.10. *[53] Consider the ANAIS billiard system described in section 2.1 above. Suppose that the sphere is set to roll on the fixed part of the plane and goes into the rotating disc. When the sphere goes out of the disc and back to the fixed part of the plane, it will follow a trajectory that is the exact prolongation of the initial one.*

The proof of Theorem II.2.10 that we present formalizes some aspects of the discussion of Lévy-Leblond [53]. Furthermore, it will allow us to show that similar rolling phenomena hold for appropriate generalizations of the system. In particular, our approach will allow us to prove that a similar phenomenon occurs for the rolling Janus sphere, as stated in the following theorem and illustrated in figure 2.7.

Theorem II.2.11. *Suppose that the Janus sphere, described in Section 2.4.2 above, is set to roll on the table in such a way that the fixed part of the sphere is in contact with the plane and that at some instant the moving part of the shell goes into contact with the plane. If the fixed part of the sphere comes back into contact with the plane at a later time*[3]*, it will follow the exact prolongation of the initial trajectory.*

Notice that for Theorems II.2.10 and II.2.11, Proposition II.2.5 already proves that the trajectories of the ball before and after it crosses the rotating parts of the system are along parallel lines. The conclusion that the path of the sphere

[3] the proof of Theorem II.2.11 given in Section 5.2.2 below shows, in particular, that the condition that the fixed part of the sphere comes into contact with the plane at a later time is verified by generic solutions of the system.

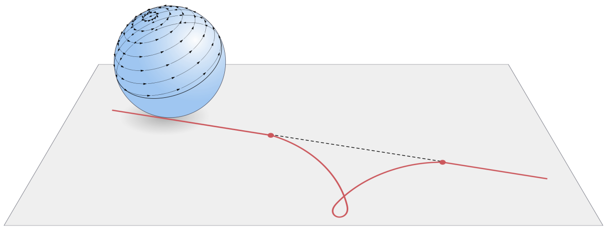


Figure 2.7: The ANAIS billiard phenomenon for the Janus sphere.

is the exact prolongation follows from further symmetry considerations of the problem that will be analyzed in the next chapters. A fundamental role will be played by the reversibility of the system with respect to suitable involutions of the phase space.

Remark II.2.12. In the spirit of Remark I.5.2, we observe that since ω_3 is a constant of motion, the phenomena in the statements of Theorems II.2.10 and II.2.11 remain valid in the presence of an additional rubber (i.e. no-spin) constraint.

3 Reversibility

3.1 Reversible systems

We recall the notions of reversible systems that we will need ahead. Some of the contents of this section may be found, for instance, in [41, 52, 58]. We begin by recalling that an *involution* κ on a manifold M is a diffeomorphism $\kappa : M \to M$ such that $\kappa^2 = \mathrm{id}_M$ and proceed to define a reversible system.

Definition II.3.1. Let $\kappa : M \to M$ be an involution on the manifold M. A complete vector field $X \in \mathfrak{X}(M)$ is called *reversible with respect to* κ or κ*-reversible* if its flow $\Phi : \mathbb{R} \times M \to M$ satisfies

$$\kappa \circ \Phi_t(m) = \Phi_{-t} \circ \kappa(m) = \Phi_t^{-1} \circ \kappa(m) \quad \text{for all } t \in \mathbb{R} \text{ and } m \in M. \tag{II.3.1}$$

In this case, κ is called a *reversing symmetry*.

The infinitesimal version of the above condition is

$$\kappa_* X = -X.$$

Identity (II.3.1) implies one of the most important properties of reversible systems: if $\Phi_t(m)$ is a solution of the system, then $\kappa(\Phi_{-t}(\kappa(m)))$ is also a solution of the system. Moreover, from identity (II.3.1) we may see that the involution κ maps an orbit of X to another orbit of X, reversing the time direction. In particular, if the orbit goes through a point $m \in M$ such that $\kappa(m) = m$, then it maps the orbit to itself, reversing time parametrization. In fact, the fixed point set of an involution κ will play an important part in the study of reversible systems;

we will denote it by

$$\mathrm{Fix}(\kappa) := \{m \in M \,:\, \kappa(m) = m\}.$$

Definition II.3.2. An integral curve $t \mapsto z(t)$ of the vector field $X \in \mathfrak{X}(M)$ is *symmetric with respect to an involution* κ on M if there exists an $s \in \mathbb{R}$ such that for every $t \in \mathbb{R}$,

$$z(s+t) = \kappa(z(s-t)).$$

Proposition II.3.3. *Let $X \in \mathfrak{X}(M)$ be a reversible vector field with respect to the involution κ and let $z(t)$ be an integral curve of X. Then $z(t)$ is symmetric with respect to κ if and only if there exists $s \in \mathbb{R}$ such that $z(s) \in \mathrm{Fix}(\kappa)$.*

Proof. Suppose $z(s) = m \in \mathrm{Fix}(\kappa)$. Then $z(s+t) = \Phi_t(m)$ and $z(s-t) = \Phi_{-t}(m) = \Phi_{-t}(\kappa(m))$. Using (II.3.1) we have $\kappa(z(s+t)) = z(s-t)$ which is equivalent to $z(s+t) = \kappa(z(s-t))$. Conversely, if $z(s+t) = \kappa(z(s-t))$, for all t then evaluation at $t = 0$ gives $z(s) = \kappa(z(s))$. □

Finally, we mention a property of symmetric integral curves which can be immediately proved.

Lemma II.3.4. *Let $X \in \mathfrak{X}(M)$ be a vector field reversible by an involution $\kappa : M \to M$ and let $f : M \to \mathbb{R}$ be a κ-invariant function. If $z(t)$ is a symmetric integral curve of X such that for $z(s) \in \mathrm{Fix}(\kappa)$ for $s \in \mathbb{R}$, then*

$$f(z(s-t)) = f(z(s+t)).$$

3.1.1 Reversible systems with symmetries

Suppose now that there is a Lie group action $\Psi : H \times M \to M$ of the Lie group H on the phase manifold M. Suppose that the complete vector field $X \in \mathfrak{X}(M)$ is reversible with respect to the involution $\kappa : M \to M$ and, moreover, H-invariant. This means that, apart from (II.3.1), the flow Φ of X satisfies

$$\Psi_h \circ \Phi_t = \Phi_t \circ \Psi_h, \qquad \text{for all } h \in H, t \in \mathbb{R}. \tag{II.3.2}$$

Let us fix $h \in H$ and consider the map

$$\kappa_h := \Psi_h^{-1} \circ \kappa \circ \Psi_h : M \to M. \tag{II.3.3}$$

It is straightforward to check that $\kappa_h^2 = \mathrm{id}_M$ so it is an involution. Moreover, as the following proposition shows, X is also κ_h-reversible.

Proposition II.3.5. *Under the hypothesis given above, X is κ_h-reversible for all $h \in H$. Moreover,*

$$\mathrm{Fix}(\kappa_h) = \Psi_{h^{-1}}(\mathrm{Fix}(\kappa)). \tag{II.3.4}$$

Proof. Take $h \in H$, $m \in M$ and $\kappa_h : M \to M$ as in (II.3.3). Then, by κ-reversibility and Ψ-invariance of Φ,

$$\begin{aligned}\kappa_h \circ \Phi_t(m) &= \Psi_h^{-1} \circ \kappa \circ \Psi_h \circ \Phi_t(m)\\ &= \Psi_h^{-1} \circ \kappa \circ \Phi_t(\Psi_h(m))\\ &= \Psi_h^{-1} \circ \Phi_t(\kappa \circ \Psi_h(m))\\ &= \Phi_{-t} \circ \Psi_h^{-1} \circ \kappa \circ \Psi_h(m) = \Phi_{-t} \circ \kappa_h(m).\end{aligned}$$

Therefore, Φ is reversible with respect to κ_h. To prove (II.3.4), note that $m \in \mathrm{Fix}(\kappa_h)$ if and only if

$$\kappa_h(m) = \Psi_{h^{-1}} \circ \kappa \circ \Psi_h(m) = m,$$

if and only if $\kappa \circ \Psi_h(m) = \Psi_h(m)$ if and only if $\Psi_h(m) \in \mathrm{Fix}(\kappa)$ if and only if $m \in \Psi_{h^{-1}}(\mathrm{Fix}(\kappa))$. □

3.2 Reversibility of affine nonholonomic systems

It is well known that nonholonomic systems with linear constraints are time-reversible, but the presence of affine nonholonomic constraints usually destroys time-reversibility. However, in the following theorem we will see that, under some conditions, affine nonholonomic systems may be reversible as well.

Theorem II.3.6. *Consider a nonholonomic system described by a configuration manifold Q, a Lagrangian $L : TQ \to \mathbb{R}$ and an affine constraint distribution $\mathcal{A}$, which can be expressed as $\mathcal{A} = \mathcal{D} + Z$, with $\mathcal{D} \subset TQ$ the linear constraint distribution and $Z \in \mathfrak{X}(Q)$. Let $\lambda : Q \to Q$ be an involution on Q such that the Lagrangian L and the linear distribution $\mathcal{D}$ are λ-invariant (i.e. $L \circ T\lambda = L$ and $T\lambda(\mathcal{D}) = \mathcal{D}$). If $Z \in \mathfrak{X}(Q)$ is reversible with respect to λ (i.e. $\lambda_* Z = -Z$), then $\kappa := \iota \circ T\lambda : TQ \to TQ$, with $\iota(q, v) = (q, -v)$ for $(q, v) \in TQ$, satisfies*

1. *$\kappa(\mathcal{A}) = \mathcal{A}$,*
2. *$\kappa|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is an involution,*
3. *$\kappa_* X^{(Q,\mathcal{A},L)} = -X^{(Q,\mathcal{A},L)}$,*

where $X^{(Q,\mathcal{A},L)} \in \mathfrak{X}(\mathcal{A})$ denotes the vector field defined by the equations of motion of the nonholonomic system defined by Q, $\mathcal{A}$ and L.

Proof. To prove item 1, we notice that κ is a vector bundle homomorphism. Considering that $\mathcal{A}_q = D_q + Z(q)$, we have

$$\begin{aligned}\kappa(\mathcal{A}_q) &= \kappa(\mathcal{D}_q) + \kappa(Z(q)) = -T\lambda(\mathcal{D}_{\lambda(q)}) - T\lambda(Z(\lambda(q))) \\ &= -\mathcal{D}_{\lambda(q)} + Z(\lambda(q)) = \mathcal{D}_{\lambda(q)} + Z(\lambda(q)) = \mathcal{A}_{\lambda(q)},\end{aligned}$$

where we have used the λ-invariance of $\mathcal{D}$, the property $\lambda_* Z = -Z$ and the fact that $\mathcal{D}$ is a linear distribution.

To prove item 2, we first write an explicit expression for κ,

$$\kappa(q, v) = \iota \circ T\lambda(q, v) = \iota(\lambda(q), T_q\lambda(v)) = (\lambda(q), -T_q\lambda(v)).$$

Now, we see that since $\lambda^2 = \mathrm{id}_Q$ and $(T_q\lambda)^2 = \mathrm{id}_{T_qQ}$ are involutions,

$$\kappa^2(q, v) = (\lambda^2(q), (-T_q\lambda)^2(v)) = (q, v).$$

Finally, to prove item 3 we use Theorem 0.1.4 which is proved in Appendix A. We notice that λ satisfies the hypothesis of Theorem 0.1.4 for $s = -1$. So the vector field $X^{Q,\mathcal{A},L} \in \mathfrak{X}(\mathcal{A})$ is reversible with respect to κ. □

3.3 A useful involution

In what follows, we will use Theorem II.3.6 to derive reversibility properties of the ANAIS billiard and the class of hybrid systems described in Section 5.2 with a specific choice of involution. Let $\lambda : Q \to Q$, with $Q = \mathbb{R}^2 \times SO(3)$ be given by

$$\lambda : (\boldsymbol{x}, B) \mapsto (-\Sigma\boldsymbol{x}, \Sigma B \Sigma), \quad \text{where} \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(3). \tag{II.3.5}$$

Note that $\lambda^2 = \mathrm{id}_{\mathrm{Q}}$, so it is an involution on the configuration space Q. For future reference, we note that the fixed points of λ are

$$\mathrm{Fix}(\lambda) = \{(\boldsymbol{x}, B) \in Q \,:\, x_1 = 0 \text{ and } \Sigma B = B\Sigma\}. \tag{II.3.6}$$

The tangent map of λ is given by

$$\begin{aligned} T\lambda : & TQ \to TQ \\ & (\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) \mapsto (-\Sigma\boldsymbol{x}, \Sigma B\Sigma, -\Sigma\dot{\boldsymbol{x}}, \Sigma\boldsymbol{\omega}).\end{aligned}$$

With the above expression it is a routine calculation to check that the following proposition holds.

Proposition II.3.7. *The Lagrangian L given by* (II.2.1) *and the linear distribution $\mathcal{D}$ given by* (II.2.8) *are λ-invariant, namely $L = L \circ T\lambda$ and $T\lambda(\mathcal{D}_q) = \mathcal{D}_{\lambda(q)}$.*

In view of Theorem II.3.6, if $Z \in \mathfrak{X}(Q)$ is reversible by λ (i.e. $\lambda_* Z = -Z$) then, denoting $\mathcal{A} = \mathcal{D} + Z$, the vector field $X = X^{(Q,\mathcal{A},L)} \in \mathcal{A}$ is reversible by $\kappa = \iota \circ T\lambda : TQ \to TQ$, which in our case is given by

$$\kappa(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (-\Sigma\boldsymbol{x}, \Sigma B \Sigma, \Sigma\dot{\boldsymbol{x}}, -\Sigma\boldsymbol{\omega}). \tag{II.3.7}$$

Because of this, it is useful to give conditions on the vector fields $\boldsymbol{V}_s \in \mathfrak{X}(\Pi)$ and $\boldsymbol{W}_b \in \mathfrak{X}(\mathcal{S})$ which guarantee that Z is λ-reversible.

Proposition II.3.8. *Suppose that $Z \in \mathfrak{X}(Q)$ is given by $Z(\boldsymbol{x}, B) = (\boldsymbol{V}_s(\boldsymbol{x}), 0)$. Then $\lambda_* Z = -Z$ if and only if*

$$\Sigma \boldsymbol{V}_s(\boldsymbol{x}) = \boldsymbol{V}_s(-\Sigma\boldsymbol{x}). \tag{II.3.8}$$

Similarly, if $Z(\boldsymbol{x}, B) = (B\boldsymbol{W}_b(\gamma), 0)$, then $\lambda_* Z = -Z$ if and only if

$$\Sigma B \boldsymbol{W}_b(\gamma) = B\boldsymbol{W}_b(-\Sigma\gamma).$$

Proof. We begin by proving the first case of the proposition. On the one hand,

$$\lambda_* Z(\lambda(q)) = T_q\lambda(Z(q)) = T_q\lambda(\boldsymbol{V}_s(\boldsymbol{x}), \boldsymbol{0}) = (-\Sigma\boldsymbol{V}_s(\boldsymbol{x}), \boldsymbol{0}).$$

On the other hand,

$$Z(\lambda(q)) = (\boldsymbol{V}_s(-\Sigma\boldsymbol{x}), \boldsymbol{0}).$$

Therefore, $\lambda_* Z(\lambda(q)) = -Z(\lambda(q))$ if and only if (II.3.8) is satisfied. The second part of the proof is done analogously. First, we notice that,

$$\lambda_* Z(\lambda(q)) = T_q\lambda(Z(q)) = T_q\lambda(B\boldsymbol{W}_b(\gamma), \boldsymbol{0}) = (-\Sigma B\boldsymbol{W}_b(\gamma), \boldsymbol{0}).$$

On the other hand, noticing that $\boldsymbol{\gamma} = B^{-1}\boldsymbol{e}_3$ and $\Sigma B^{-1}\Sigma\boldsymbol{e}_3 = -\Sigma\boldsymbol{\gamma}$, we have

$$Z(\lambda(q)) = (B\boldsymbol{W}_b(-\Sigma\gamma), \boldsymbol{0}).$$

□

Finally, for future reference we indicate that the fixed point set of the involution κ is given by

$$\operatorname{Fix}(\kappa) = \{(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) \,:\, x_1 = 0,\ \Sigma B = B\Sigma,\ \dot{x}_2 = 0 \text{ and } \omega_1 = 0\}. \tag{II.3.9}$$

3.4 Reversibility of the ANAIS billiard

We now proceed to argue that the ANAIS billiard is reversible with respect to the involution κ given by (II.3.7). We begin by observing that the vector fields $\boldsymbol{V}^0$ and $\boldsymbol{V}^\eta$ defined by (II.2.10) satisfy the condition (II.3.8). As a consequence, the corresponding vector fields X^0, X^η (defined by the equations of motion on $\mathcal{A}^0 = \mathcal{D} + Z^0$ and $\mathcal{A}^\eta = \mathcal{D} + Z^\eta$ with Z^0 and Z^η given by (II.2.12)) are κ-reversible. Considering that the flow Φ^A of the ANAIS billiard system is obtained by piecing together the integral curves of X^0 and X^η in accordance with Postulate II.2.1, we still need to check that our prescription for the velocity jumps is compatible with κ (in a certain way). The crucial ingredients are contained in the following proposition.

Proposition II.3.9. *The involutions $\lambda : Q \to Q$ and $\kappa : TQ \to TQ$ respectively defined by* (II.3.5) *and* (II.3.7) *satisfy*

1. $\lambda(U_0) = U_0$ *and* $\lambda(U_\eta) = U_\eta$*,*
2. $\lambda(\mathcal{B}) = \mathcal{B}$*,*
3. $\kappa \circ \mathcal{J}_0^\eta = \mathcal{J}_0^\eta \circ \kappa$ *and* $\kappa \circ \mathcal{J}_\eta^0 = \mathcal{J}_\eta^0 \circ \kappa$*,*

where $U_0, U_\eta \subset Q$ are defined by (II.2.6)*, $\mathcal{B}$ is defined by* (II.2.7) *and the jump functions $\mathcal{J}_0^\eta$ and $\mathcal{J}_\eta^0$ by* (II.2.16)*.*

Proof. It is immediate to check that items 1 and 2 hold. To prove 3, let $q^* = (\boldsymbol{x}^*, B^*) \in \mathcal{B}$ (so $\|\boldsymbol{x}^*\| = R$), and a velocity vector $(\dot{\boldsymbol{x}}, \boldsymbol{\omega}) \in \mathcal{A}^0_{q^*}$. On the one hand, using (II.2.16) and (II.3.7), we have

$$\begin{aligned}\kappa \circ \mathcal{J}_0^\eta(\dot{\boldsymbol{x}}, \boldsymbol{\omega}) &= \kappa\left((\dot{\boldsymbol{x}}, \boldsymbol{\omega}) + \frac{\eta}{I + mr^2}(I\boldsymbol{e}_3 \times \boldsymbol{x}^*, mr\boldsymbol{x}^*)\right) \\ &= \left((\Sigma\dot{\boldsymbol{x}}, -\Sigma\boldsymbol{\omega}) + \frac{\eta}{I + mr^2}(I\Sigma(\boldsymbol{e}_3 \times \boldsymbol{x}^*), -mr\Sigma\boldsymbol{x}^*)\right)\end{aligned}$$

On the other hand, considering that $\kappa(\dot{\boldsymbol{x}}, \boldsymbol{\omega})$ is a tangent vector at $\lambda(q^*) = (-\Sigma\boldsymbol{x}^*, \Sigma B^*\Sigma) \in \mathcal{B}$, the prescription for $\mathcal{J}_0^\eta$ in (II.2.16) gives

$$\begin{aligned}\mathcal{J}_0^\eta \circ \kappa(\dot{\boldsymbol{x}}, \boldsymbol{\omega}) &= \mathcal{J}_0^\eta(\Sigma\dot{\boldsymbol{x}}, -\Sigma\boldsymbol{\omega}) \\ &= \left((\Sigma\dot{\boldsymbol{x}}, -\Sigma\boldsymbol{\omega}) + \frac{\eta}{I + mr^2}(I\boldsymbol{e}_3 \times (-\Sigma\boldsymbol{x}^*), -mr\Sigma\boldsymbol{x}^*)\right).\end{aligned}$$

Using that $\Sigma\boldsymbol{e}_3 = -\boldsymbol{e}_3$, we get the desired equality. The one for $\mathcal{J}_\eta^0$ can be proved analogously. ☐

We are now ready to present the main result of the section.

Theorem II.3.10. *The flow Φ^A of the ANAIS billiard system defined in Subsection 2.3.1 is κ-reversible. Namely,*

$$\kappa(\Phi^A_t(z)) = \Phi^A_{-t}(\kappa(z)) \qquad \textit{for all } t \in \mathbb{R} \textit{ and } z \in \mathcal{A}^A. \tag{II.3.10}$$

Proof. We know that the flows Φ^0_t and Φ^η_t are κ-reversible, so if $\Phi^A_t(z) = (q(t), v(t))$ is such that $q(t)$ stays in U_0 or U_η for all t, then (II.3.10) holds. To check that (II.3.10) holds for all t, allowing for transitions between the regions, we proceed as follows. Consider $z_0 = (q_0, v_0) \in \mathcal{A}^0_{q_0}$ and the solution with initial condition z_0, $z(t) = (q(t), v(t)) = \Phi^A_t(z_0)$. Suppose that $q(t) \in U_0$ for all $t \in [0, T)$ and $q(T) \in \mathcal{B}$. Then, for times $t \in [0, T+\varepsilon)$ and a small $\varepsilon > 0$ we have

$$\Phi^A_t(z_0) = \begin{cases} \Phi^0_t(z_0) & \text{if } 0 \le t < T \\ \Phi^\eta_t(\mathcal{J}^\eta_0(z(T))) & \text{if } T \le t \le T+\varepsilon. \end{cases} \tag{II.3.11}$$

On the other hand, since $\lambda(q_0) \in U_0$, then $\kappa(z_0) \in \mathcal{A}^0_{\lambda(q_0)}$ and since Φ^0 is κ-reversible and $\kappa(\mathcal{B}) = \mathcal{B}$, then $\tilde{z}(t) = (\tilde{q}(t), \tilde{v}(t)) = \Phi^A_{-t}(\kappa(z_0))$ is a trajectory such that $\tilde{q}(t) \in U_0$ for all $t \in [0, T)$ and $\tilde{q}(T) \in \mathcal{B}$. Therefore,

$$\Phi^A_{-t}(\kappa(z_0)) = \begin{cases} \Phi^0_{-t}(\kappa(z_0)) & \text{if } 0 \le t < T \\ \Phi^\eta_{-t}(\mathcal{J}^\eta_0(\kappa(z(T))) & \text{if } T \le t \le T+\varepsilon. \end{cases}$$

The identity (II.3.10) holds for $t \in [0, T)$ by κ reversibility of $\Phi^0_t(z_0)$. On the other hand, by item 3 of Proposition II.3.9 and κ-reversibility of Φ^η, we have

$$\begin{aligned} \Phi^\eta_{-t}(\mathcal{J}^\eta_0(\kappa(z(T))) &= \Phi^\eta_{-t}(\kappa(\mathcal{J}^\eta_0(z(T)))) \\ &= \kappa(\Phi^\eta_t(\mathcal{J}^\eta_0(z(T)))). \end{aligned} \tag{II.3.12}$$

Combining (II.3.11) and (II.3.12) shows that (II.3.10) holds for $t \in [T, t+\varepsilon)$.

An analogous argument is valid if $z_0 = (q_0, v_0)$ is such that $q_0 \in U_\eta$ and $\Phi^A_t(z_0) = (q(t), v(t))$ satisfies that $q(T) \in U_0$ for some T. ☐

3.5 Reversibility of hybrid systems

Consider a hybrid system like the ones described in Section 2.4. We briefly recall the notation introduced in Section 2.4.3. The vector field $Z \in \mathfrak{X}(Q)$ is a piecewise smooth vector field defined by (II.2.22) in terms of the smooth vector fields $Z^i \in \mathfrak{X}(Q)$, $i = 1, \ldots, n$, so Z coincides with Z^i on the open set $U^i \subset Q$, and therefore Z is smooth on these open sets. The boundaries of these sets are assumed to separate at most two sets U_i, U_j and the boundary between the regions U_i, U_j is denoted by $\mathcal{B}_{ij} \subset Q$. The constraints of the system determine a piecewise smooth affine distribution $\mathcal{A}$ which coincides with the smooth distribution $\mathcal{A}^i$ on each U_i. Moreover, $\mathfrak{J}_i^j : \mathcal{A}^i \to \mathcal{A}^j$ are the jump functions defined in (II.2.25) which model the velocity discontinuities as the configuration crosses $\mathcal{B}_{ij} \subset Q$. Finally, $X \in \mathfrak{X}(\mathcal{A})$ is the vector field defined by the equations of motion of the system with piecewise smooth affine distribution $\mathcal{A}$ complemented with the transition at the discontinuity points determined by the jump functions $\mathfrak{J}_i^j$ and the flow of X is defined in Section 2.4.4. Inspired by the ideas of the previous section, we have the following.

Proposition II.3.11. *Let $\lambda : Q \to Q$ and $\kappa : TQ \to TQ$ be respectively defined by* (II.3.5) *and* (II.3.7)*. Suppose that the following conditions are satisfied for all $i, j = 1, \ldots, n$:*

1. $\lambda_* Z^i = -Z^i$,
2. $\lambda(U_i) = U_i$,
3. $\lambda(\mathcal{B}_{ij}) = \mathcal{B}_{ij}$,
4. $\kappa \circ \mathfrak{J}_i^j = \mathfrak{J}_i^j \circ \kappa$,

Then the flow Φ^X of the vector field X as defined in Section 2.4.4 is reversible by κ.

Proof. The first condition together with Theorem 3.2 and Proposition II.3.7 guarantees that the vector fields X^i are κ-reversible so their flows, Φ^i satisfy (II.3.1). Hypotheses 2-4 in the statement of the proposition guarantee that the discontinuities of the vector field Z are compatible with κ. More precisely, an analogous procedure as the one done in the proof of Theorem II.3.10 shows that the flow Φ^X as defined in section 2.4.4 satisfies the reversibility condition $\kappa(\Phi_t^X(z)) = \Phi_t^X(\kappa(z))$ for trajectories crossing the boundaries $\mathcal{B}_{ij}$. □

3.5.1 Planar generalization

As a first illustration of the setting described above, consider the system described in Section 2.5.1. From Proposition II.3.7 we know that the Lagrangian L and the distribution $\mathcal{D}$ are λ-invariant. Given that the vector field $\boldsymbol{V}^1$ satisfies

$$\Sigma \boldsymbol{V}^1(\boldsymbol{x}) = \boldsymbol{V}^1(-\Sigma \boldsymbol{x}), \qquad \text{with } \Sigma \text{ as in (II.3.5)}$$

from Proposition II.3.8, we get $\lambda_* Z^1 = -Z^1$ and therefore, by Theorem II.3.6 the vector field X^1 is κ-reversible. It is easy to check that all the other conditions of Proposition II.3.11 hold, and so the flow of the system Φ^G defined following the prescription in Section 2.4.4 is reversible by κ.

3.5.2 The Janus sphere

We now consider the Janus sphere described in Section 2.5.2 and illustrated in Figure 2.5.

We notice that the vector field $\boldsymbol{W}^\sigma$ (defined in (II.2.29)) given by

$$\boldsymbol{W}^\sigma(\boldsymbol{\gamma}) = -r\sigma(\gamma_3)(\boldsymbol{\gamma} \times \boldsymbol{E}_3)$$

satisfies the property

$$\Sigma B \boldsymbol{W}^\sigma(\boldsymbol{\gamma}) = B \boldsymbol{W}^\sigma(-\Sigma \boldsymbol{\gamma}), \qquad \text{with } \Sigma \text{ as in (II.3.5).} \tag{II.3.13}$$

As in the previous example, from Proposition II.3.7 we get that L and $\mathcal{D}$ are λ-invariant and from Proposition II.3.8, we get $\lambda_* Z^\sigma = -Z^\sigma$. Therefore, by Theorem II.3.6, we know that the vector field X^σ is κ-reversible. It can be checked that all the other conditions of Proposition II.3.11 hold, so the flow Φ^C of the system defined by following the prescription in Section 2.4.4 is reversible by κ.

4 Symmetries

4.1 Symmetries of affine nonholonomic systems

It is well known that, for nonholonomic systems with linear constraints, if the Lagrangian and the distribution are invariant under the tangent lift of the action of a Lie group, then the system is also invariant under the tangent lift of this action. Under reasonable assumptions on the affine terms, we see that this is also the case for nonholonomic systems with affine constraints.

Proposition II.4.1. *Consider the nonholonomic system determined by a configuration manifold Q, a Lagrangian $L : TQ \to \mathbb{R}$, and an affine constraint distribution $\mathcal{A}$, which can be expressed as $\mathcal{A} = \mathcal{D} + Z$, with $\mathcal{D}$ a linear distribution and $Z \in \mathfrak{X}(Q)$ a vector field. Let H be a Lie group that acts on Q by a Lie group action $\Psi : H \times Q \to Q$ with tangent lift $\overline{\Psi} : H \times TQ \to TQ$. Suppose that*

- *The Lagrangian is Ψ-invariant, i.e. $L \circ \overline{\Psi}_h = L$ for all $h \in H$,*
- *The distribution $\mathcal{D}$ is Ψ-invariant, i.e. $\mathcal{D}_{\Psi_h(q)} = T_q\Psi_h(\mathcal{D}_q)$ for all $h \in H$ and $q \in Q$,*
- *The vector field Z is Ψ-invariant, i.e $\overline{\Psi}(Z(q)) = Z(\Psi(q))$ for all $q \in Q$.*

Then the flow Φ_t of X satisfies

$$\overline{\Psi}_h \circ \Phi_t = \Phi_t \circ \overline{\Psi}_h \qquad \text{for all } t \in \mathbb{R} \text{ and all } h \in H. \tag{II.4.1}$$

Proof. Again we use Theorem 0.1.4 (proved in Appendix A). In this case, Ψ_h satisfies the hypothesis of Theorem 0.1.4 for all $h \in H$ and $s = 1$. Therefore, the vector field $X^{Q,\mathcal{A},L} \in \mathfrak{X}(\mathcal{A})$ is invariant with respect to $\overline{\Psi}_h$ and (II.4.1) is satisfied. □

4.2 Symmetries of the ANAIS billiard

The ANAIS billiard system has an $\mathrm{SO}(3)$-symmetry corresponding to rotations of the reference frame on the sphere Σ_b and an $\mathrm{SO}(2)$-symmetry corresponding to rotations of the space reference frame Σ_s. Let us consider the Lie group $H^A = \mathrm{SO}(3) \times \mathrm{SO}(2)$ and let us denote an element $h \in H^A$ as $h = (k, R_\theta) \in H^A$, with $k \in \mathrm{SO}(3)$ and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{II.4.2}$$

Now, let us consider the Lie group action of H^A on Q, $\Psi^A : H^A \times Q \to Q$ defined as

$$\Psi_h^A(\boldsymbol{x}, B) = (R_\theta \boldsymbol{x}, R_\theta B k^{-1}), \tag{II.4.3}$$

with tangent lift $\overline{\Psi}^A : H^A \times TQ \to TQ$,

$$\overline{\Psi}^A(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (R_\theta \boldsymbol{x}, R_\theta B k^{-1}, R_\theta \dot{\boldsymbol{x}}, R_\theta \boldsymbol{\omega}). \tag{II.4.4}$$

As we will see in the following proposition, the ANAIS billiard is invariant under $\overline{\Psi}$.

Proposition II.4.2. *The flow Φ^A is $\overline{\Psi}^A$-invariant. Namely,*

$$\overline{\Psi}_h^A \circ \Phi_t^A = \Phi_t^A \circ \overline{\Psi}_h^A \qquad \textit{for all } h \in H^A \textit{ and all } t \in \mathbb{R}.$$

Proof. Clearly the Lagrangian L as defined in (II.2.1) is $\overline{\Psi}^A$-invariant, and the linear distribution $\mathcal{D}$ defined in (II.2.8) is Ψ^A-invariant. Now, considering Z^η as in (II.2.12), taking $q = (\boldsymbol{x}, B) \in Q$ and $h = (k, R_\theta) \in H^A$, we have

$$\overline{\Psi}_h^A(Z^\eta(q)) = \overline{\Psi}_h^A(\boldsymbol{V}_s^\eta(\boldsymbol{x}), \boldsymbol{0}) = (R_\theta \boldsymbol{V}_s^\eta(\boldsymbol{x}), \boldsymbol{0}) = (\boldsymbol{V}_s^\eta(R_\theta \boldsymbol{x}), \boldsymbol{0}) = Z^\eta(\Psi_h^A(q)),$$

by the invariance of $\boldsymbol{V}_s^\eta$ under rotations R_θ. So Z^η is Ψ^A-invariant. Clearly, Z^0 is also Ψ^A-invariant. Therefore, the flows Φ^0 and Φ^η are $\overline{\Psi}^A$-invariant. Moreover, it can easily be checked that the sets $U_0, U_\eta \subset Q$ and the boundary $\mathcal{B}$ are Ψ^A-invariant, i.e. $\Psi_h^A(U_0) = U_0$, $\Psi_h^A(U_\eta) = U_\eta$ and $\Psi_h^A(\mathcal{B}) = \mathcal{B}$. It can be directly checked that the flow defined in section 2.3.1 is $\overline{\Psi}^A$-invariant, since if $q \in \mathcal{B}$, the jump functions $\mathcal{J}_0^\eta : \mathcal{A}_q^0 \to \mathcal{A}_q^\eta$ and $\mathcal{J}_\eta^0 : \mathcal{A}_1^\eta \to \mathcal{A}_q^0$ satisfy

$$\mathcal{J}_0^\eta \circ \overline{\Psi}_h^A = \overline{\Psi}_h^A \circ \mathcal{J}_0^\eta \qquad \text{and} \qquad \mathcal{J}_\eta^0 \circ \overline{\Psi}_h^A = \overline{\Psi}_h^A \circ \mathcal{J}_\eta^0.$$

Therefore, the flow Φ^A defined in Section 2.3.1 is $\overline{\Psi}^A$-invariant. □

4.3 Symmetries of hybrid systems

Let us now consider a hybrid system like the ones introduced in Section 2.4. Namely, a nonholonomic system determined by a configuration manifold $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$, a Lagrangian L given by (II.2.1) and by a piecewise smooth affine constraint distribution $\mathcal{A}$. We briefly recall the notation used there: the piecewise smooth affine distribution $\mathcal{A}$ is expressed as $\mathcal{D}+Z$, where $\mathcal{D}$ is a linear distribution and $Z \in \mathfrak{X}(Q)$ is a piecewise smooth vector field which is defined in such a way that it coincides with some smooth vector fields $Z^i \in \mathfrak{X}(Q)$ in the open sets $U_i \subset Q$; the boundary between the sets U_i, U_j is described by $\mathcal{B}_{ij} \subset Q$ and the jump in the velocities at the discontinuities is given by the jump functions $\mathcal{J}_i^j : \mathcal{A}^i \to \mathcal{A}^j$ defined in (II.2.25); the vector fields $X^i \in \mathfrak{X}(\mathcal{A}^i)$ are defined by the equations of motion of the system determined by Q, L and $\mathcal{A}^i$ and the vector field $X \in \mathfrak{X}(\mathcal{A})$ coincides with each X^i when $q \in U_i$. Using the transition determined by $\mathcal{J}_i^j$, the flow Φ_t of X is defined in section 2.4.4. We have the following.

Proposition II.4.3. *Let H be a Lie group and let it act by $\Psi : H \times Q \to Q$ on the configuration manifold Q. Suppose that the following are satisfied*

1. $L \circ \overline{\Psi}_h = L$ *for all* $h \in H$,
2. $\overline{\Psi}_h(\mathcal{D}_q) = \mathcal{D}_{\Psi(q)}$ *for all* $h \in H$ *and* $q \in Q$,
3. $\overline{\Psi}(Z(q)) = Z(\Psi(q))$ *for all* $q \in Q$,
4. $\Psi_h(U_i) = U_i$ *for all* $h \in H$ *and* $i = 1, \dots, n$,
5. $\Psi_h(\mathcal{B}_{ij}) = (\mathcal{B}_{ij})$ *for all* $h \in H$ *and* $i, j = 1, \dots, n$,
6. $\overline{\Psi}_h \circ \mathcal{J}_i^j = \mathcal{J}_i^j \circ \overline{\Psi}_h$ *for all* $h \in H$ *and* $i, j = 1, \dots, n$.

Then the flow Φ_t of the vector field X as defined in section 2.4.4 satisfies

$$\overline{\Psi}_h \circ \Phi_t = \Phi_t \circ \overline{\Psi}_h \qquad \textit{for all } h \in H.$$

Proof. The first three conditions, together with Proposition II.4.1, imply that the vector fields $X^i \in \mathfrak{X}(\mathcal{A}^i)$ are $\overline{\Psi}_h$ invariant for all $h \in H$. Conditions 4-6 imply that the discontinuities of the vector field Z are compatible with the action Ψ. It can be checked directly from the definition of the flow Φ of X that Φ is $\overline{\Psi}_h$-invariant for all $h \in H$. □

4.3.1 Planar generalization

Consider the system described in Section 2.5.1 and illustrated in Figure 2.4. In this system, we no longer have the $\mathrm{SO}(2)$-symmetry corresponding to rotations of the plane that we had in the case of the ANAIS billiard. Nonetheless, we still have the $\mathrm{SO}(3)$-symmetry corresponding to rotations of the sphere. Therefore, we may consider the Lie group $H^G = \mathrm{SO}(3)$ and the action $\Psi^G : H^G \times Q \to Q$ on Q by right multiplication given by

$$\Psi_h^G(\boldsymbol{x}, B) = (\boldsymbol{x}, Bh), \tag{II.4.5}$$

and with tangent lift $\overline{\Psi}^G : H \times TQ \to TQ$ given by

$$\overline{\Psi}_h^G(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = (\boldsymbol{x}, Bh, \dot{\boldsymbol{x}}, \boldsymbol{\omega}).$$

It can be checked that the Lagrangian L given by (II.2.1), the distribution $\mathcal{D}$ given by (II.2.8) and the vector fields Z^1, Z^0 as defined in (II.2.28) are $\overline{\Psi}^G$-invariant. Therefore, the vector fields X^0, X^1 are $\overline{\Psi}^G$-invariant. It is clear that the sets $U_0, U_1 \subset Q$ and $\mathcal{B}_{01} \subset Q$ are all Ψ^G-invariant and that the jump functions defined in (II.2.25) applied to this example are $\overline{\Psi}^G$-invariant. Therefore, from Proposition II.4.3 we get that the flow Φ^G defined as in section 2.4.4 is Ψ^G- invariant. Namely,

$$\overline{\Psi}_h^G \circ \Phi_t^G = \Phi_t^G \circ \overline{\Psi}_h^G \qquad \text{for all } t \in \mathbb{R} \text{ and } h \in H^G$$

Moreover, in Section 3.5.1 we saw that Φ^G is reversible with respect to κ, so, by Proposition II.3.5, Φ^G is reversible with respect to κ_h for all $h \in H^G$.

4.3.2 The Janus sphere

Now consider the system described in Section 2.5.2. In this case, the system has an $\mathrm{SE}(2)$-symmetry corresponding to rotations and translations of the space frame Σ_s and an $\mathrm{SO}(2)$ symmetry corresponding to rotations about the $\boldsymbol{E}_3$ axis of the body frame Σ_b. We consider the Lie group $H^C = \mathrm{SE}(2) \times \mathrm{SO}(2)$, and describe its elements by $h = (\boldsymbol{a}, R_\theta, R_\phi^{-1}) \in H^G$ where

$$\boldsymbol{a} = (a_1, a_2, 0), \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we consider the Lie group action $\Psi^C : H^C \times Q \to Q$ of H^C on Q defined as

$$\Psi_h(\boldsymbol{x}, B) = (R_\theta \boldsymbol{x} + \boldsymbol{a}, R_\theta B R_\phi^{-1}),$$

and its tangent lift $\overline{\Psi}^C : H^C \times TQ \to TQ$ given by

$$\overline{\Psi}^C_{(\boldsymbol{a},R_\theta,R_\phi^{-1})}(\boldsymbol{x},B,\dot{\boldsymbol{x}},\boldsymbol{\omega}) = (R_\theta\boldsymbol{x}+\boldsymbol{a},R_\theta B R_\phi^{-1},R_\theta\dot{\boldsymbol{x}},R_\theta\boldsymbol{\omega}). \qquad \text{(II.4.6)}$$

It is clear that the Lagrangian L given by (II.2.1) and the linear distribution $\mathcal{D}$ given by (II.2.8) are Ψ^C-invariant. Now we check that the vector field Z^C given by (II.2.33) is also Ψ^C-invariant, i.e. $\overline{\Psi}^C_h(Z(q)) = Z(\Psi^C_h(q))$. It is clear that $Z^0 = (\mathbf{0},\mathbf{0})$ (given in (II.2.32)) is Ψ^C-invariant. To check that Z^σ, given by (II.2.32) with $\boldsymbol{W}^\sigma(\boldsymbol{\gamma}) = -r\sigma(\boldsymbol{\gamma}\times\boldsymbol{E}_3)$ is also Ψ^C-invariant, we write

$$\boldsymbol{W}^\sigma(B) = -r\sigma(B^{-1}\boldsymbol{e}_3)\times\boldsymbol{E}_3$$

and notice that, taking $h = (\boldsymbol{a},R_\theta,R_\phi^{-1})$,

$$\begin{aligned}
Z^\sigma(\Psi^C_h(q)) &= (R_\theta B R_\phi^{-1}\boldsymbol{W}^\sigma(R_\theta B R_\phi^{-1}),\mathbf{0})\\
&= (-r\sigma R_\theta B R_\phi^{-1}((R_\phi B^{-1}R_\theta^{-1}\boldsymbol{e}_3)\times\boldsymbol{E}_3),\mathbf{0})\\
&= (-r\sigma R_\theta B((B^{-1}\boldsymbol{e}_3)\times\boldsymbol{E}_3),\mathbf{0})\\
&= (R_\theta B\boldsymbol{W}^\sigma,\mathbf{0}) = \overline{\Psi}^C_h(Z^\sigma(q)).
\end{aligned}$$

Therefore, the flows Φ^0 and Φ^σ of X^0 and X^σ are $\overline{\Psi}^C$-invariant. It is easy to see that the sets U^0, U^σ and $\mathcal{B}_{0\sigma}$ are Ψ^C-invariant. Therefore, by means of Proposition II.4.3, the flow Φ^C is $\overline{\Psi}^C$-invariant. Moreover, since Φ^C is reversible with respect to κ (see Section 3.5.2), by means of Proposition II.3.5, we know that the flow Φ^C is reversible with respect to κ_h for all $h \in H^C$.

5 The ANAIS billiard phenomenon and its generalizations

5.1 The ANAIS billiard phenomenon

In what follows, we will see how the symmetry and reversibility properties of the system allow us to prove the phenomenon observed in the ANAIS billiard experiment, stated in Theorem II.2.10. As previously mentioned, from Proposition II.2.5, we know that the velocity vector $\dot{\boldsymbol{x}}$ when the sphere enters the rotating disc is the same as when it goes out of it and therefore, the exit trajectory must be parallel to the initial one. We will show that the exit trajectory is not only parallel to the initial one, but it is the exact prolongation of it. To prove Theorem II.2.10, we begin by proving the following propositions. We recall from Chapter 2 that the flow Φ^A of the system possesses the first integrals

$$\begin{aligned} m_1^A &= (I + mr^2)\omega_1 - mrV_2^A(\boldsymbol{x}), \\ m_2^A &= (I + mr^2)\omega_2 + mrV_1^A(\boldsymbol{x}), \\ m_3^A &= I\omega_3. \end{aligned} \tag{II.5.1}$$

Proposition II.5.1. *If the sphere is set to roll on the fixed part of the plane following a trajectory which is parallel to $\boldsymbol{e}_1$, then it will reach the set $\{x_1 = 0\}$ at some point.*

Proof. Let $z(t) = (q(t), v(t))$ be an integral curve of the system such that for some $t_0 \in \mathbb{R}$ the sphere is outside the rotating disc, i.e. $q(t) \in U_0$ and $\|\boldsymbol{x}(t_0)\| > R$, and the linear velocity is parallel to $\boldsymbol{e}_1$, i.e. $\langle \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = 0$. By means of the constraints (II.2.2) we therefore have that $\langle \boldsymbol{\omega}(t_0), \boldsymbol{e}_1 \rangle = 0$ and thus, $m_1^A = m_1^A(t_0) = 0$. If the sphere does not go inside the disc, then its trajectory on the plane is a straight line parallel to $\boldsymbol{e}_1$, so it must cross $\{x_1 = 0\}$ at some point. Now assume that the sphere goes inside the rotating disc, i.e. $q(t) \in U_\eta$ for $t \in (t_i, t_f)$ for some

$t_i, t_f \in \mathbb{R}$ and let us consider the vectors $\tilde{\boldsymbol{\omega}} = (\omega_1, \omega_2, 0)$ and $\tilde{\boldsymbol{m}}^A = (m_1^A, m_2^A, 0)$. From the expression of the first integrals (II.5.1) when $\boldsymbol{V}^A = 0$, we can obtain an expression for $\tilde{\boldsymbol{\omega}}(t)$,

$$\tilde{\boldsymbol{\omega}}(t) = \frac{1}{I + mr^2}(\tilde{\boldsymbol{m}}^A + mr\eta\boldsymbol{x}(t)),$$

for $t \in (t_i, t_f)$. Substituting this expression in the constraints (II.2.2), we get that the trajectory of the sphere on the plane, while it is inside the rotating disc, is described by the solutions of

$$\dot{\boldsymbol{x}}(t) = \alpha\boldsymbol{x}(t) \times \boldsymbol{e}_3 + \beta\tilde{\boldsymbol{m}}^A \times \boldsymbol{e}_3, \tag{II.5.2}$$

for some nonzero constants $\alpha, \beta \in \mathbb{R}$ (which depend on the system parameters) and $t \in (t_i, t_f)$. Given that $m_1 = 0$, equation (II.5.2) takes the form

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \beta \begin{pmatrix} m_2^A \\ 0 \end{pmatrix}.$$

Therefore, the solutions of (II.5.2) describe circles with center at $-\frac{\beta}{\alpha}(0, m_2)$ (see Part 0, Section 3.2), and hence the trajectories must pass through $\{x_1 = 0\}$ at some point. □

Proposition II.5.2. *If $z(t) = (q(t), v(t))$ is an integral curve of X^A such that:*

- *there exists $t_0 \in \mathbb{R}$ such that $q(t_0) \in U_0$ and $\langle \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = 0$,*
- *there exists $s \in \mathbb{R}$ such that $q(s) \in \mathrm{Fix}(\lambda)$,*

then $z(s) \in \mathrm{Fix}(\kappa)$.

Proof. Notice that the fixed point set of κ given in (II.3.9) can be written as

$$\mathrm{Fix}(\kappa) = \{(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) \,:\, (\boldsymbol{x}, B) \in \mathrm{Fix}(\lambda),\, \dot{\boldsymbol{x}} = \Sigma\dot{\boldsymbol{x}},\, \boldsymbol{\omega} = -\Sigma\boldsymbol{\omega}\,\}.$$

Since $q(s) \in \mathrm{Fix}(\lambda)$, we only need to check that $\dot{\boldsymbol{x}}(s) = \Sigma\dot{\boldsymbol{x}}(s)$ and $\boldsymbol{\omega}(s) = -\Sigma\boldsymbol{\omega}(s)$. From $\langle \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = 0$, through the equations (II.2.2) and (II.2.5), we get $m_1^A = m_1^A(t_0) = m^A(s) = 0$. Therefore $\langle \omega(s), \boldsymbol{e}_1 \rangle = 0$ and, in view of (II.2.2), this implies $\langle \dot{\boldsymbol{x}}(s), \boldsymbol{e}_2 \rangle = 0$. Hence $z(s) = (\boldsymbol{x}(s), B(s), \dot{\boldsymbol{x}}(s), \boldsymbol{\omega}(s)) \in \mathrm{Fix}(\kappa)$. □

We now notice that the set of fixed points $\mathrm{Fix}(\lambda)$ given in (II.3.6) can be expressed as

$$\mathrm{Fix}(\lambda) = \{(\boldsymbol{x}, B) \in Q \,:\, x_1 = 0,\, B \in C(\Sigma)\,\},$$

where $C(\Sigma)$ denotes the centralizer of Σ on $\mathrm{SO}(3)$. Proposition II.5.2 tells us that if the orientation of the sphere happens to be in the centralizer of Σ when it gets to the set $\{x_1 = 0\}$, then, by κ-reversibility of the system, the integral curve $z(t)$ is symmetric with respect to κ. Since, as discussed in section 4.2, the system is invariant under rotations of the body frame Σ_b, we can remove the condition on the orientation B of the sphere. Moreover, since it is invariant under rotations of the space frame Σ_s, we can remove the condition on the initial linear velocity being parallel to $\boldsymbol{e}_1$. As in section 4.2, we consider the Lie group $H^A = \mathrm{SO}(2) \times \mathrm{SO}(3)$ and its Lie group action $\Psi^A : H^A \times Q \to Q$ given by (II.4.3). Since X^A is $\overline{\Psi}^A$-invariant, if we consider an integral curve $z(t) = (q(t), v(t)) = \Phi_t^A(z_0)$ for some $z_0 \in \mathcal{A}^A$, then the curve

$$\overline{\Psi}^A(z(t)) = \overline{\Psi}^A(\Phi_t^A(z_0)) = \Phi_t^A(\overline{\Psi}^A(z_0)),$$

is also an integral curve of the system. With this in mind, we may generalize the previous propositions to the following.

Proposition II.5.3. *Let $z(t) = (q(t), v(t))$ be an integral curve of the system, and suppose that for some $t_0 \in \mathbb{R}$ it satisfies $q(t_0) \in U_0$. There exists $h = (R_\theta, k) \in H^A$ and $s \in \mathbb{R}$ such that $z(s) \in \mathrm{Fix}(\kappa_h)$.*

Proof. Suppose that $z(t) = (q(t), v(t))$ is an integral curve of the system such that for some $t_0 \in \mathbb{R}$ the sphere is on the fixed part of the plane, i.e. $q(t_0) \in U_0$, and the linear velocity is given by $\dot{\boldsymbol{x}}(t_0)$. Take $\theta \in [0, 2\pi)$ such that $\dot{\boldsymbol{x}}(t_0)$ is parallel to the vector $(\cos\theta, -\sin\theta, 0)$. Then, considering R_θ as in (II.4.2),

$$\langle R_\theta \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = \langle \dot{\boldsymbol{x}}(t_0), R_\theta^{-1} \boldsymbol{e}_2 \rangle = \langle \dot{\boldsymbol{x}}(t_0), (\sin\theta, \cos\theta, 0) \rangle = 0.$$

Considering $\overline{\Psi}^A_{(R_\theta, \mathrm{id})}(z(t))$, by proposition II.5.1 there is an $s \in \mathbb{R}$ such that

$$\langle R_\theta \boldsymbol{x}(s), \boldsymbol{e}_1 \rangle = \langle \boldsymbol{x}(s), R_\theta^{-1} \boldsymbol{e}_1 \rangle = \langle \boldsymbol{x}(s), (\cos\theta, -\sin\theta, 0) \rangle = 0.$$

Take $k \in \mathrm{SO}(3)$ such that $R_\theta B(s) k^{-1} \in C(\Sigma)$ (for example $k = R_\theta B(s)$) and $h = (R_\theta, k) \in H^A$. Then $\Psi_h^G(q(s)) = (R_\theta \boldsymbol{x}(s), R_\theta B(s) k^{-1}) \in \mathrm{Fix}(\lambda)$. Consider the integral curve

$$\overline{\Psi}_h^A(z(t)) = (R_\theta \boldsymbol{x}(t), R_\theta B(t) k^{-1}, R_\theta \dot{\boldsymbol{x}}(t), R_\theta \boldsymbol{\omega}(t)),$$

it is a curve that satisfies the hypothesis of proposition II.5.2, therefore $\overline{\Psi}_h^A(z(s)) \in \mathrm{Fix}(\kappa)$, or equivalently,

$$z(s) \in \overline{\Psi}^A_{h^{-1}}(\mathrm{Fix}(\kappa)) = \mathrm{Fix}(\kappa_h).$$

Since X^A is κ_h-reversible, then $z(t)$ is symmetric with respect to κ_h. □

Now we are able to give a formal proof of the ANAIS billiard phenomenon.

Proof of Theorem II.2.10

Proof. Consider an integral curve $z(t) = (q(t), v(t))$ of X^A and suppose that at time $t_0 \in \mathbb{R}$, the sphere is outside the rotating disc, i.e. $q(t_0) \in U_0$, and that the linear velocity at that point is given by $\dot{\boldsymbol{x}}(t_0)$. By proposition II.5.3, we know that there exist $h \in H^A$ and $s \in \mathbb{R}$ such that $z(s) \in \text{Fix}(\kappa_h)$ and $z(t)$ is symmetric with respect to κ_h . Therefore,

$$z(s-t) = \kappa_h(z(s+t)).$$

Now, we notice that the function $n : TQ \to \mathbb{R}$ defined as

$$n(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \|\boldsymbol{x}\|^2$$

is $\overline{\Psi}^A$-invariant and κ-invariant. Therefore, it is κ_h-invariant, so by lemma II.3.4, we have

$$n(z(s-t)) = n(z(s+t)).$$

Taking $t_i = s - t^*$ with $t^* \in \mathbb{R}$ such that $n(z(s-t^*)) = R$ as the time of entrance, then $t_f = s+t^*$ satisfies $n(z(t_f)) = R$ and denotes the exit time. From proposition II.2.5 we know that the linear velocity just before entering the rotating disc $\dot{\boldsymbol{x}}(t_i)$ is parallel to the linear velocity just after going out of the rotating disc $\dot{\boldsymbol{x}}(t_f)$, and we know that they are parallel to $R_\theta^{-1}\boldsymbol{e}_1$. Now we notice that the function $a_\theta : TQ \to \mathbb{R}$ defined as

$$a_\theta(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \langle R_\theta \boldsymbol{x}, \boldsymbol{e}_2 \rangle, \tag{II.5.3}$$

gives us the height with respect to $R_\theta^{-1}\boldsymbol{e}_1$ of the position of the sphere on the plane and is κ_h-invariant, since

$$\begin{aligned} a_\theta \circ \kappa_h(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) &= \langle R_\theta(-R_\theta^{-1}\Sigma R_\theta \boldsymbol{x}), \boldsymbol{e}_2 \rangle = \langle -\Sigma R_\theta \boldsymbol{x}, \boldsymbol{e}_2 \rangle \\ &= \langle R_\theta \boldsymbol{x}, -\Sigma \boldsymbol{e}_2 \rangle = \langle R_\theta \boldsymbol{x}, \boldsymbol{e}_2 \rangle = a_\theta(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}). \end{aligned}$$

Therefore, by means of lemma II.3.4, we have

$$a_\theta(t_i) = a_\theta(t_f).$$

Which means that the exit trajectory is the exact prolongation of the initial one. □

5.2 Generalizations of the ANAIS billiard phenomenon

Consider a hybrid generalization of the ANAIS billiard like the ones described in section 2.4. Following the notation introduced there, suppose that the vector field $Z \in \mathfrak{X}(Q)$ vanishes in some set $U_0 \subset Q$. Moreover, suppose that the flow Φ^X described in section 2.4.4 is reversible with respect to κ defined in (II.3.7). Then we have the following.

Proposition II.5.4. *Let $z(t) = (q(t), v(t))$, with $q(t) = (\boldsymbol{x}(t), B(t)) \in Q$, be an integral curve of X and suppose that:*

- *there exists $t_0 \in \mathbb{R}$ such that $q(t_0) \in U_0$ and $\langle \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = 0$,*
- *there exists $s \in \mathbb{R}$ such that $q(s) \in \operatorname{Fix}(\lambda)$.*

Then $z(s) \in \operatorname{Fix}(\kappa)$.

The proof is analogous to the one of proposition II.5.2 using the constraints (II.2.17).

As in the case of the ANAIS billiard, we can use the symmetries of the system to generalize the result. Let H be a Lie group that acts on Q by an action $\Psi : H \times Q \to Q$ and consider its tangent lift $\overline{\Psi}_h : TQ \to TQ$. Suppose that the flow Φ of the system is invariant under $\overline{\Psi}$ and consider $z_0 \in \mathcal{A}$ and an integral curve $z(t) = (q(t), v(t)) = \Phi_t(z_0)$ of X. Let us consider the curve $z_h(t) = \overline{\Psi}_h(z(t))$. By $\overline{\Psi}$-invariance of Φ,

$$z_h(t) = \overline{\Psi}_h(\Phi_t(z_0)) = \Phi_t(\overline{\Psi}_h(z_0))$$

is also an integral curve of X. If $z(t) = (q(t), v(t))$, with $q(t) = (\boldsymbol{x}(t), B(t))$ and $v(t) = (\dot{\boldsymbol{x}}(t), \boldsymbol{\omega}(t))$, we will denote $z_h(t) = (q_h(t), v_h(t))$ with

$$q_h(t) = (\boldsymbol{x}_h(t), B_h(t)) \qquad \text{and} \qquad v_h(t) = (\dot{\boldsymbol{x}}_h(t), \boldsymbol{\omega}_h(t)).$$

With this notation, we have the following.

Proposition II.5.5. *Consider $z(t) = (q(t), z(t))$ an integral curve of X. Suppose that the flow of X is invariant under the lift of the action of a Lie group H on Q and suppose that there is some $h \in H$ such that*

- *there exists $t_0 \in \mathbb{R}$ such that $q_h(t_0) \in U_0$ and $\langle \dot{\boldsymbol{x}}_h(t_0), \boldsymbol{e}_2 \rangle = 0$,*
- *there exists $s \in \mathbb{R}$ such that $q_h(s) \in \operatorname{Fix}\lambda$.*

Then $z(t)$ is symmetric with respect to the involution $\kappa_h : TQ \to TQ$ defined by (II.3.3).

Proof. If $z_h(t)$ satisfies the conditions, then by proposition II.5.4, we know that $z_h(t) = \overline{\Psi}_h(z(t)) \in \operatorname{Fix}(\kappa)$, or equivalently, $z(t) \in \Psi_{h^{-1}}(\operatorname{Fix}(\kappa))$. By proposition II.3.5 we have $z(t) \in \operatorname{Fix}(\kappa_h)$. □

5.2.1 Planar generalization

As a first example, we consider the system described in Section 2.5.1 and considered in Sections 3.5.1 and 4.3.1 and, using the above results, we prove an analogous to the ANAIS billiard phenomenon illustrated in figure 2.4 and stated below. In this case, since we do not have the $\mathrm{SO}(2)$-symmetry on the plane, we do not get the phenomenon for any curve that crosses the strip but just for the ones that enter perpendicularly to the symmetry axis and that reach the symmetry axis at some point.

Corollary II.5.6. *Consider the system described in section 2.5.1. If the sphere enters the strip of the plane in which $V \neq 0$ following a trajectory perpendicular to $\{x_1 = 0\}$, and arrives to $\{x_1 = 0\}$ at some point, then it goes out of the strip following a trajectory which is the exact prolongation of the initial one.*

Proof. Let $z(t) = (q(t), v(t))$ be an integral curve of X^G and assume that at some time $t_0 \in \mathbb{R}$, we have

$$q(t_0) \in U_0 \quad \text{and} \quad \langle \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = 0. \tag{II.5.4}$$

Moreover, we assume that the sphere reaches $\{x_1 = 0\}$ at some point, so there is $s \in \mathbb{R}$ such that $x_1(s) = 0$. Consider the Lie group $H^G = \mathrm{SO}(3)$ and its Lie group action Ψ^G in Q given by (II.4.5) and chose $h \in H^G = \mathrm{SO}(3)$ such that $B(s)h \in C(\Sigma)$ (for example $h = B(s)^{-1}$). Then, since

$$x_1(s) = 0 \quad \text{and} \quad B(s)h \in C(\Sigma),$$

we have $q_h(s) = (\boldsymbol{x}(s), B(s)h) \in \mathrm{Fix}(\lambda)$. This fact, together with (II.5.4) (and noticing that Ψ^G does not act on the first coordinate of q) allow us to use proposition II.5.5. Hence, $z(t)$ is symmetric with respect to the involution $\kappa_h : TQ \to TQ$ and $z(s) \in \mathrm{Fix}(\kappa_h)$, so

$$z(s - t) = \kappa_h(z(s + t)).$$

Now, we notice that the functions $a_1, a_2 : TQ \to TQ$ defined as

$$a_1(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = |\langle \boldsymbol{x}, \boldsymbol{e}_1 \rangle|, \qquad a_2(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \langle \boldsymbol{x}, \boldsymbol{e}_2 \rangle \tag{II.5.5}$$

are κ-invariant and $\overline{\Psi}^G$-invariant, therefore, they are κ_h-invariant. We can take $t_i = s - t^*$ for $t^* \in \mathbb{R}$ such that $a_1(z(t_i)) = A$ as the time of entrance. Then, in view of lemma II.3.4, $t_f = s + t^*$ satisfies $a_2(z(t_f)) = A$ and denotes the time at which the sphere exits the moving strip. From proposition II.2.5, we know that $\dot{\boldsymbol{x}}(t_i) = \dot{\boldsymbol{x}}(t_f)$ and from our hypothesis we know that both $\dot{\boldsymbol{x}}(t_i)$ and $\dot{\boldsymbol{x}}(t_f)$ are parallel to $\boldsymbol{e}_1$. Using lemma II.3.4, we get

$$a_2(z(t_i)) = a_2(z(t_f))$$

which implies that the exit trajectory is the exact prolongation of the initial one. □

5.2.2 The Janus sphere

Now we will prove that the Janus sphere described in Section 2.5.2 (and considered in Sections 3.5.2 and 4.3.2) exhibits the same phenomenon as the ANAIS billiard, as stated in Theorem II.2.11. To ease the notation, let us consider the vector

$$\boldsymbol{\tau} = B\boldsymbol{E}_3 = (\tau_1, \tau_2, \tau_3) \in S^2.$$

Considering $\boldsymbol{W}^\sigma(\gamma)$ defined in (II.2.29), we can define the vector field $\tilde{\boldsymbol{W}}^\sigma$ by

$$\tilde{\boldsymbol{W}}^\sigma(\boldsymbol{\tau}) = B\boldsymbol{W}^\sigma(\gamma) = \sigma\boldsymbol{e}_3 \times \boldsymbol{\tau}.$$

Then, we can write the equations of the system as

$$\begin{aligned} \dot{\boldsymbol{m}} &= 0, \qquad \dot{B}B^{-1} = \hat{\boldsymbol{\omega}} \\ \dot{\boldsymbol{x}} &= -r\boldsymbol{e}_3 \times \boldsymbol{\omega} + \tilde{\boldsymbol{W}}_b(\boldsymbol{\tau}), \end{aligned} \tag{II.5.6}$$

with

$$\boldsymbol{m} = I\boldsymbol{\omega} + mr^2\boldsymbol{e}_3 \times (\boldsymbol{\omega} \times \boldsymbol{e}_3) + mr\boldsymbol{e}_3 \times \tilde{\boldsymbol{W}}_b(\boldsymbol{\tau}), \tag{II.5.7}$$

where

$$\tilde{\boldsymbol{W}}_b(\tau) = \begin{cases} \boldsymbol{W}^0(\boldsymbol{\tau}) & \text{if } \tau_3 < a, \\ \tilde{\boldsymbol{W}}^\sigma(\boldsymbol{\tau}) & \text{if } \tau_3 > a, \end{cases}$$

with $\boldsymbol{W}^0(\boldsymbol{\tau}) = 0$. Finally, we notice that the third component of equation $\dot{B}B^{-1} = \hat{\boldsymbol{\omega}}$ is equivalent to

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\omega} \times \boldsymbol{\tau} \tag{II.5.8}$$

Now we prove the following.

Proposition II.5.7. *Consider the set $\mathcal{E}$ of solutions, $(\boldsymbol{x}(t), \boldsymbol{\tau}(t), \boldsymbol{\omega}(t))$ of (II.5.6) satisfying the following properties:*

1. *the fixed part of the sphere is in contact with the plane at time t_0 and at such instant the vector $\dot{\boldsymbol{x}}(t_0)$ is parallel to $\boldsymbol{e}_1$;*

2. *the rotating part of the shell goes into contact with the plane at a later time $t_i > t_0$.*

Then the generic solutions in $\mathcal{E}$, reach the set $\{\langle B\boldsymbol{E}_3, \boldsymbol{e}_1\rangle = 0\}$ at some time $s > t_i$. Moreover, for all time $t_i < t \le s$, the moving part of the shell is in contact with the plane.

Proof. Let $z(t) = (q(t), v(t))$ be an element of $\mathcal{E}$ (so $\gamma_3(t_0) < a$). In view of the constraints (II.2.31), the condition that $\dot{\boldsymbol{x}}(t_0)$ is parallel to $\boldsymbol{e}_1$ implies $\langle \boldsymbol{\omega}(t_0), \boldsymbol{e}_1\rangle = 0$ and hence, by (II.2.19), $m_1 = 0$.

Suppose now that the moving part of the shell is in contact with the plane during the time interval (t_i, t_f). From the expression for $\boldsymbol{m}$ given by (II.5.7), we can express $\boldsymbol{\omega}(t)$ as

$$\boldsymbol{\omega}(t) = \frac{1}{I + mr^2}(\overline{\boldsymbol{m}} + mr\sigma\boldsymbol{e_3} \times (\boldsymbol{e}_3 \times \boldsymbol{\tau}(t))),$$

for $t \in (t_i, t_f)$, where the (constant) vector $\overline{\boldsymbol{m}} = \boldsymbol{m} + mr^2\omega_3\boldsymbol{e}_3$. Now observe that, since $m_1 = 0$, the right-hand side of the above equation may be rewritten as

$$\frac{1}{I + mr^2}(\overline{\boldsymbol{m}} + mr\sigma\boldsymbol{e_3} \times (\boldsymbol{e}_3 \times \boldsymbol{\tau}(t))) = \nabla F(\boldsymbol{\tau}),$$

where

$$F(\boldsymbol{\tau}) = m_2\tau_2 + m_3\tau_3 + \frac{1}{2}mr\sigma\tau_3^2.$$

It follows from (II.5.8) that when the moving part of the shell is in contact with the plane, the vector τ evolves according to the equation

$$\dot{\boldsymbol{\tau}} = \boldsymbol{\tau} \times \nabla F(\boldsymbol{\tau}). \tag{II.5.9}$$

The last component of (II.5.9) is

$$\dot{\tau}_3 = m_2\tau_1. \tag{II.5.10}$$

Now, the condition that the moving part of the shell enters in contact with the plane at time t_i implies $\dot{\tau}_3(t_i) > 0$. Hence, the above equation implies that $m_2 \ne 0$ and $\tau_1(t_i) \ne 0$. If the fixed part of the shell comes into touch at a later time t_f then necessarily $\dot{\tau}_3(t_f) < 0$ which implies that the sign of $\tau_1(t_f)$ is the opposite of $\tau_1(t_i)$. In particular, this shows that a necessary condition for the fixed part of the sphere to go back into contact with the plane is that $\tau_1(s) = 0$ for some $s < t_f$. This shows that if such s exists, then the moving part of the shell is in contact with the plane for all time $t_i < t \le s$ as stated.

Now we prove the existence of the time instant s such that $\tau_1(s) = 0$. The main observation is that the system (II.5.9) is a Hamiltonian system on the unit sphere $\|\boldsymbol{\tau}\|^2 = 1$ with Hamiltonian function F. Since F is independent of τ_1, the integral curves of (II.5.9) are symmetric with respect to the reflection $(\tau_1, \tau_2, \tau_3) \mapsto$

$(-\tau_1, \tau_2, \tau_3)$. In particular, all equilibrium points have $\tau_1 = 0$ (which actually follows directly from (II.5.10)). On the other hand, the generic trajectories, corresponding to regular values of F, are periodic orbits which, by the symmetry described above, cross the set where $\tau_1 = 0$. The only trajectories which may not cross this set are the invariant manifolds of hyperbolic equilibrium points of (II.5.9) corresponding to (exceptional) critical values of F. □

Lemma II.5.8. *Given $B \in \mathrm{SO}(3)$ such that $\langle B\boldsymbol{E}_3, \boldsymbol{e}_1\rangle = 0$, there exists $\phi \in [0, 2\pi)$ such that*

$$BR_\phi \in C(\Sigma), \qquad \textit{with } R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $C(\Sigma)$ denotes the centralizer in $\mathrm{SO}(3)$ of $\Sigma = \mathrm{diag}(1, -1, -1)$.

Proof. Since $B \in \mathrm{SO}(3)$, it can be written as the product of three rotation matrices $B = R_{\psi_1} R_{\psi_2} R_{\psi_3}$, where

$$R_{\psi_1} = \begin{pmatrix} \cos\psi_1 & -\sin\psi_1 & 0 \\ \sin\psi_1 & \cos\psi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{\psi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_2 & -\sin\psi_2 \\ 0 & \sin\psi_2 & \cos\psi_2 \end{pmatrix},$$

$$R_{\psi_3} = \begin{pmatrix} \cos\psi_3 & -\sin\psi_3 & 0 \\ \sin\psi_3 & \cos\psi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\langle B\boldsymbol{E}_3, \boldsymbol{e}_1\rangle = 0$, then $\sin\psi_1 \sin\psi_2 = 0$. We have the following cases:

- $\sin\psi_1 = 0$: then $B = DR_{\psi_2}R_{\psi_3}$ with D diagonal. Take $\phi = -\psi_3$ so $BR_\phi = DR_{\psi_2} \in C(\Sigma)$,
- $\psi_2 = 0$: then $B = R_{\psi_1+\psi_3}$. Take $\phi = -\psi_1 - \psi_2$ so $BR_\phi = R_{\psi_1+\psi_3+\phi} = \mathrm{id} \in \mathrm{C}(\Sigma)$,
- $\psi_2 = \pi$: then $B = R_{\psi_1}\Sigma R_{\psi_3}$. Take $\phi = \psi_1 - \psi_3$ so $BR_\phi = \Sigma \in C(\Sigma)$.

□

We are now ready to give a proof of Theorem II.2.11 which states that a phenomenon analogous to the ANAIS billiard also holds for the Janus sphere.

Proof of Theorem II.2.11.

Proof. Let $z(t) = (q(t), v(t))$ be a solution such that the fixed part of the sphere is in contact with the plane at $t = t_0$ and the rotating part of the sphere comes into contact with the plane at a later time $t = t_i > t_0$. Throughout the proof, we will write $q(t) = (\boldsymbol{x}(t), B(t))$.

We begin by observing that, by the symmetry considerations of Section 4.3.2, the curve $z_h(t) = (q_h(t), v_h(t)) := \overline{\Psi}^C_h(z(t))$ is a solution of the system for any $h \in H^C = \mathrm{SE}(2) \times \mathrm{SO}(2)$ possessing the same properties. Indeed, this follows since the vector $\boldsymbol{\gamma}$ transforms according to the action of $\overline{\Psi}^C_{(\boldsymbol{a},R_\theta,R_\phi)}$ by

$$\boldsymbol{\gamma} \mapsto R_\phi \boldsymbol{\gamma},$$

and so the coordinate γ_3, which is responsible for the transition between the regimes, is invariant.

Assume that at time t_0 the linear velocity $\dot{\boldsymbol{x}}(t_0)$ is parallel to the vector $(\cos\theta, -\sin\theta, 0)$, for some $\theta \in [0, 2\pi)$. Then

$$\langle R_\theta \dot{\boldsymbol{x}}(t_0), \boldsymbol{e}_2 \rangle = \langle \dot{\boldsymbol{x}}(t_0), R_\theta^{-1} \boldsymbol{e}_2 \rangle = \langle \dot{\boldsymbol{x}}(t_0), (\sin\theta, \cos\theta, 0) \rangle = 0.$$

Hence, taking $h_1 = (\boldsymbol{0}, R_\theta, \mathrm{id}) \in H^C$ we obtain the solution

$$t \mapsto z_{h_1}(t) = (q_{h_1}(t), v_{h_1}(t))$$

satisfying $\langle \dot{\boldsymbol{x}}_{h_1}(t_0), \boldsymbol{e}_2 \rangle = 0$. In other words, $t \mapsto z_{h_1}(t)$ satisfies the hypotheses of Proposition II.5.7. Assuming that $t \mapsto z_{h_1}(t)$ is generic, we conclude the existence of a time $s \in \mathbb{R}$, $s > t_i$, such that

$$\langle R_\theta B(s) \boldsymbol{E}_3, \boldsymbol{e}_1 \rangle = \langle R_\theta \boldsymbol{\tau}(s), \boldsymbol{e}_1 \rangle = 0,$$

where, as usual, we have written $\boldsymbol{\tau} = B\boldsymbol{E}_3$.

Now, in view of Lemma II.5.8, there exists $\phi \in [0, 2\pi)$ such that $R_\theta B(s) R_\phi^{-1} \in C(\Sigma)$. Taking $\boldsymbol{a} = -R_\theta \boldsymbol{x}(s)$ and considering $h_2 = (\boldsymbol{a}, R_\theta, R_\phi) \in H^C$, we can build the solution

$$t \mapsto \overline{\Psi}^C_{h_2}(z(t)) = z_{h_2}(t) = (q_{h_2}(t), v_{h_2}(t)),$$

which satisfies $q_{h_2}(s) \in \mathrm{Fix}(\lambda)$ and also $\langle \dot{\boldsymbol{x}}_{h_2}(t_0), \boldsymbol{e}_2 \rangle = 0$. Therefore, by Proposition II.5.5, we conclude that $z(s) \in \mathrm{Fix}(\kappa_{h_2})$ and $z(t)$ is symmetric with respect to κ_{h_2}.

Now consider the function $g : TQ \to \mathbb{R}$ defined by

$$g(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \langle B^{-1} \boldsymbol{e_3}, \boldsymbol{E}_3 \rangle = \gamma_3.$$

As mentioned above, it is $\overline{\Psi}^C$-invariant. A direct calculation using (II.3.7) shows that g is also κ-invariant and therefore, it is κ_h-invariant for all $h \in H^C$, and in particular κ_{h_2}-invariant. Therefore, by Lemma II.3.4, we have

$$g(z(s-t)) = g(z(s+t)),$$

for all $t \in \mathbb{R}$. Evaluating at the time value $t = t^* := s - t_i$, we get

$$g(z(t_i)) = g(z(s+t^*)).$$

But $g(z(t_i)) = a$ since t_i is the instant at which the rotating part of the shell comes into contact with the plane. This shows that $t_f := s + t^*$ is the instant at which the fixed part of the sphere comes back into contact with the plane.

Let us now recall the function $a_\theta : TQ \to \mathbb{R}$ defined by (II.5.3), which gives the projection of $\boldsymbol{x}$ to the line generated by $R_\theta^{-1}\boldsymbol{e}_2$ (to which the trajectory is perpendicular before the transition)

$$a_\theta(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) = \langle \boldsymbol{x}, R_\theta^{-1}\boldsymbol{e}_2\rangle.$$

Below we prove that a_θ is κ_{h_2}-invariant, first notice that κ_{h_2} maps $\boldsymbol{x} \mapsto -R_\theta^{-1}\Sigma R_\theta \boldsymbol{x}$, so

$$\begin{aligned} a_\theta \circ \kappa_{h_2}(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}) &= \langle -R_\theta^{-1}\Sigma R_\theta \boldsymbol{x}, R_\theta^{-1}\boldsymbol{e}_2\rangle = \langle \boldsymbol{x}, -R_\theta^{-1}\Sigma \boldsymbol{e}_2\rangle \\ &= \langle \boldsymbol{x}, R_\theta^{-1}\boldsymbol{e}_2\rangle = a_\theta(\boldsymbol{x}, B, \dot{\boldsymbol{x}}, \boldsymbol{\omega}). \end{aligned}$$

Therefore, again by Lemma II.3.4, we have

$$a_\theta(\boldsymbol{x}(t_i)) = a_\theta(\boldsymbol{x}(t_f)).$$

Since the linear velocities at the exit and the entrance time are both parallel to R_θ^{-1} (by Proposition II.2.5) and both the exit and entrance heights with respect to the line generated by $R_\theta^{-1}\boldsymbol{e}_1$ are equal, we conclude that the exit trajectory is the exact prolongation of the initial one. □

Further research

We now list a series of topics of further research. Some of them were proposed by A. Kilin and J. Koiller who acted as reviewers of this thesis.

- In all examples considered in detail in the thesis, the vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ have divergence zero. Moreover, Propositions I.3.2 and I.5.1 give some general results on the existence of an invariant measure for nondivergent fields $\boldsymbol{V}$, $\boldsymbol{W}$. As suggested by Kilin, it would be interesting to provide a physical interpretation of the constraints when the vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ have nonzero divergence, and to analyze the dynamics in this case.
- As suggested by Kilin (also by D. Barilari in the Ph.D. seminar of the University of Padova), one may generalize the setting described in Part I incorporating more nonholonomic constraints. For instance, considering an additional rubber (no-spin) constraint. As mentioned in remark I.5.2, for the case of the homogeneous sphere, the dynamics of the system with the additional rubber constraint coincides with the restriction of the system to the zero level set of the first integral ω_3. The influence of a rubber constraint for more general bodies remains to be investigated.
- The central observation for the development of Part II of this thesis was to recognize the mechanism, which exploits the existence of a robust class of first integrals, to deal with the discontinuity of the constraints of the ANAIS billiard. Such mechanism applies to more general systems, specifically to those which fall within the framework of Appendix B.2. Particular cases are the motion of a Chaplygin sphere with discontinous vector fields V and W, its multi-dimensional generalizations, and the rubber-constraint version of these problems. An interesting question, posed by Koiller, is to propose a reasonable transition recipe for more general nonholonomic systems with discontinuous constraints. Some simple examples needing a

new recipe are the motion of an unbalanced sphere on the ANAIS table or the motion of a sphere with a rubber hemisphere on the plane.

- The properties of the ANAIS billiard may be abstracted as follows:
  - The configuration manifold Q is a Lie group.
  - The Lagrangian L is biinvariant.
  - The constraints are given by an affine subbundle $\mathcal{A} = \mathcal{D} + Z$, with Z a discontinuous vector field on Q, and the linear distribution $\mathcal{D}$ is right invariant (i.e. the system $(Q, L, \mathcal{D})$ is LR and RR).
  - There is an action of a Lie group H on Q which leaves L, $\mathcal{D}$ and Z invariant.
  - There is an involution $\lambda : Q \to Q$, which is also a Lie group automorphism that leaves L and $\mathcal{D}$ invariant and satisfies $\lambda_* Z = -Z$.

It would be interesting to prove analogous results to the ANAIS billiard phenomenon in this degree of generality.

Appendix

A Invariance of affine nonholonomic systems

This part is taken from [40]. Since it is not publicly available, we reproduce its contents here. We will briefly recall the notation introduced in Part 0, Section 1.4 and prove Theorem 0.1.4.

We assume that $q = (q^1, \dots, q^n)$ are any set of coordinates on a manifold Q that specify the configuration of the system and that the motion of the system is subjected to $n-k$ independent constraints that are affine linear and homogeneous on the velocities, say

$$\beta^a_k(q)\dot{q}^k = K^a(q), \qquad a = 1, \dots, n-r, \tag{A.1}$$

where the vectors $\beta^a(q)$, $a = 1, \dots, n-r$ are linearly independent. The constraints define a regular, rank r, affine distribution $\mathcal{A} = \mathcal{D} + Z$ on Q. Here the linear distribution $\mathcal{D}$ is defined as

$$\mathcal{D}_q = \{\dot{q} \in T_qQ : \beta(q)\dot{q} = 0\} \tag{A.2}$$

and $Z \in \mathfrak{X}(Q)$ satisfies

$$\beta(q)Z(q) = K(q). \tag{A.3}$$

Then the equations of motion can be written as the restriction to (A.1) of

$$A(q)\ddot{q} + \eta(q,\dot{q}) + V'(q) = R(q,\dot{q}), \tag{A.4}$$

where the components of the vectors $\eta(q,\dot{q})$, $V'(q) \in \mathbb{R}^n$ are

$$\eta_i(q,\dot{q}) = \left(\frac{\partial A_{ij}}{\partial q^k}(q) - \frac{1}{2}\frac{\partial A_{jk}}{\partial q^i}(q)\right)\dot{q}^j\dot{q}^k, \qquad V'_i(q) = \frac{\partial V}{\partial q^i}(q), \tag{A.5}$$

and the reaction force $R(q,\dot q)$ is given by

$$R(q,\dot q) = \beta(q)^T\Sigma(q)(\beta(q)A^{-1}(q)(\eta(q,\dot q)+V'(q)) - \gamma(q,\dot q)+\xi(q,\dot q)), \tag{A.6}$$

where $\gamma(q,\dot q),\xi(q,\dot q)\in\mathbb{R}^{n-r}$ have components

$$\gamma^a(q,\dot q) = \frac{\partial\beta^a_k}{\partial q^j}\dot q^j\dot q^k, \qquad \xi^a(q,\dot q) = \frac{\partial K^a}{\partial q^j}\dot q^j. \tag{A.7}$$

Suppose that $\Psi : Q\to Q$ is a diffeomorphism whose tangent lift preserves the Lagrangian and the linear distribution $\mathcal{D}$. In coordinates, $q\in Q\subset\mathbb{R}^n$ this implies that

$$\Psi'(q)^T A(\Psi(q))\Psi'(q) = A(q), \tag{A.8}$$

$$V(\Psi(q)) = V(q), \tag{A.9}$$

and

$$\ker((\beta(\Psi(q))\Psi'(q)) = \ker(\beta(q)), \tag{A.10}$$

for all q. The condition (A.10) implies the existence of an invertible $(n-k)\times(n-k)$ matrix $\mathcal{Q}(q)$[1] such that

$$\beta(q) = \mathcal{Q}(q)\beta(\Psi(q))\Psi'(q). \tag{A.11}$$

In what follows we denote the components of the matrix $\mathcal{Q}(q)$ by $\mathcal{Q}^b_a(q)$, $a,b = 1,\ldots,n-k$. We also denote its inverse matrix by $\mathcal{Q}(q)^{-1} = \mathcal{P}(q)$ with components $\mathcal{P}^b_a(q)$ (so $\mathcal{P}^a_c(q)\mathcal{Q}^c_b(q) = \delta^a_b$).

Proposition A.1. *Consider an arbitrary curve $t\mapsto q_t$ on Q and denote $\tilde q_t = \Psi(q_t)$. Under the invariance conditions stated above, the following relations hold:*

$$\begin{aligned}
\Psi'(q_t)^T A(\tilde q_t)\ddot{\tilde q}_t &= A(q_t)\ddot q_t + \Psi'(q_t)^T\Gamma_t,\\
\Psi'(q_t)^T\eta(\tilde q_t,\dot{\tilde q}_t) &= \eta(q_t,\dot q_t) - \Psi'(q_t)^T\Gamma_t,\\
\Psi'(q_t)^T V'(\tilde q_t) &= V'(q_t),\\
\mathcal{Q}(q_t)\gamma(\tilde q_t,\dot{\tilde q}_t) &= \gamma(q_t,\dot q_t) - (T_1)_t - (T_2)_t,
\end{aligned} \tag{A.12}$$

[1] do not confuse matrix $\mathcal{Q}$ with configuration space Q

where $\Gamma_t \in \mathbb{R}^n$ *and* $(T_1)_t, (T_2)_t \in \mathbb{R}^{n-k}$ *have components*

$$\begin{aligned}(\Gamma_t)_i &= A_{ij}(\tilde{q}_t)\frac{\partial^2 \Psi^j}{\partial q^k \partial q^m}(q_t)\dot{q}_t^k \dot{q}_t^m,\\ (T_1)_t^a &= \frac{\partial \mathcal{Q}_b^a}{\partial q^j}(q_t)\mathcal{P}_c^b(q_t)\beta_i^c(q_t)\dot{q}_t^i \dot{q}_t^j,\\ (T_2)_t^a &= \mathcal{Q}_b^a(q_t)\beta_j^b(\tilde{q}_t)\frac{\partial^2 \Psi^j}{\partial q^k \partial q^m}(q_t)\dot{q}_t^k \dot{q}_t^m.\end{aligned} \tag{A.13}$$

Proof. The proof follows from long but elementary calculations in components using the chain rule. Below we give all the details, omitting the subindex t everywhere.

Differentiating $\tilde{q}^j = \Psi^j(q)$ with respect to time gives

$$\dot{\tilde{q}}^j = \frac{\partial \Psi^j}{\partial q^i}(q)\dot{q}^i \qquad \text{and} \qquad \ddot{\tilde{q}}^j = \frac{\partial \Psi^j}{\partial q^i}(q)\ddot{q}^i + \frac{\partial^2 \Psi^j}{\partial q^i \partial q^k}(q)\dot{q}^i \dot{q}^k. \tag{A.14}$$

Therefore,

$$\begin{aligned}(\Psi'(q)^T A(\tilde{q})\ddot{\tilde{q}})_l &= \frac{\partial \Psi^m}{\partial q^l}(q)A(\tilde{q})_{mj}\ddot{\tilde{q}}^j\\ &= \frac{\partial \Psi^m}{\partial q^l}(q)A(\tilde{q})_{mj}\frac{\partial \Psi^j}{\partial q^i}(q)\ddot{q}^i + \frac{\partial \Psi^m}{\partial q^l}(q)A(\tilde{q})_{mj}\frac{\partial^2 \Psi^j}{\partial q^i \partial q^k}(q)\dot{q}^i \dot{q}^k\\ &= A(q)_{il}\ddot{q}^i + \frac{\partial \Psi^m}{\partial q^l}(q)\Gamma_m\\ &= (A(q)\ddot{q} + \Psi'(q)^T\Gamma)_l,\end{aligned}$$

where in the penultimate equality we have used the definition of Γ given in (A.13) and the invariance condition

$$\frac{\partial \Psi^m}{\partial q^l}(q)A(\tilde{q})_{mj}\frac{\partial \Psi^j}{\partial q^i} = A(q)_{li}, \tag{A.15}$$

which is the coordinate form of (A.8). This proves the validity of the first equality in (A.12).

We now show that the second equality in (A.12) holds. Starting from the definition of η in (A.5) and using the expression for $\dot{\tilde{q}}^j$ in (A.14) we get

$$
\begin{aligned}
(\Psi'(q)^T\eta(\tilde{q},\dot{\tilde{q}}))_l &= \frac{\partial\Psi^i}{\partial q^l}(q)\eta_i(\tilde{q},\dot{\tilde{q}}) \\
&= \frac{\partial\Psi^i}{\partial q^l}(q)\left(\frac{\partial A_{ij}}{\partial q^r}(\tilde{q}) - \frac{1}{2}\frac{\partial A_{jr}}{\partial q^i}(\tilde{q})\right)\frac{\partial\Psi^j}{\partial q^m}(q)\frac{\partial\Psi^r}{\partial q^k}(q)\dot{q}^m\dot{q}^k \\
&= \frac{\partial A_{ij}}{\partial q^r}(\tilde{q})\frac{\partial\Psi^i}{\partial q^l}(q)\frac{\partial\Psi^j}{\partial q^m}(q)\frac{\partial\Psi^r}{\partial q^k}(q)\dot{q}^m\dot{q}^k \\
&\quad - \frac{1}{2}\frac{\partial A_{jr}}{\partial q^i}(\tilde{q})\frac{\partial\Psi^i}{\partial q^l}(q)\frac{\partial\Psi^j}{\partial q^m}(q)\frac{\partial\Psi^r}{\partial q^k}(q)\dot{q}^m\dot{q}^k.
\end{aligned}
\tag{A.16}
$$

On the other hand, differentiating the invariance condition (A.15) (with different choices of indices) and using the chain rule yields,

$$
\begin{aligned}
\frac{\partial A_{kl}}{\partial q^m}(q) = \frac{\partial A_{ij}}{\partial q^r}(\tilde{q})\frac{\partial\Psi^r}{\partial q^m}(q)\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^l}(q) + A_{ij}(\tilde{q})\frac{\partial^2\Psi^i}{\partial q^m\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^l}(q) \\
+ A_{ij}(\tilde{q})\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial^2\Psi^j}{\partial q^m\partial q^l}(q),
\end{aligned}
\tag{A.17}
$$

and also,

$$
\begin{aligned}
\frac{\partial A_{km}}{\partial q^l}(q) = \frac{\partial A_{ij}}{\partial q^r}(\tilde{q})\frac{\partial\Psi^r}{\partial q^l}(q)\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^m}(q) + A_{ij}(\tilde{q})\frac{\partial^2\Psi^i}{\partial q^l\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^m}(q) \\
+ A_{ij}(\tilde{q})\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial^2\Psi^j}{\partial q^l\partial q^m}(q).
\end{aligned}
\tag{A.18}
$$

Now, by the definition (A.5) of η, we have

$$
\eta_l(q,\dot{q}) = \left(\frac{\partial A_{kl}}{\partial q^m}(q) - \frac{1}{2}\frac{\partial A_{km}}{\partial q^l}(q)\right)\dot{q}^m\dot{q}^k. \tag{A.19}
$$

Inserting the expressions (A.17) and (A.18) into (A.19), and exploiting the symmetry of A_{ij} on the indices i,j, and the equality of the mixed second derivatives, leads to a cancellation of three terms, and leads to the expression

$$
\begin{aligned}
\eta_l(q,\dot{q}) = A_{ij}(\tilde{q})\frac{\partial^2\Psi^i}{\partial q^k\partial q^m}(q)\frac{\partial\Psi^j}{\partial q^l}(q)\dot{q}^k\dot{q}^m + \frac{\partial A_{ij}}{\partial q^r}(\tilde{q})\frac{\partial\Psi^r}{\partial q^m}(q)\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^l}(q)\dot{q}^k\dot{q}^m \\
- \frac{1}{2}\frac{\partial A_{ij}}{\partial q^r}(\tilde{q})\frac{\partial\Psi^r}{\partial q^l}(q)\frac{\partial\Psi^i}{\partial q^k}(q)\frac{\partial\Psi^j}{\partial q^m}(q)\dot{q}^k\dot{q}^m.
\end{aligned}
$$

Taking into account (A.16), exploiting the interchangeability of certain subindices and recalling the definition (A.13) of Γ, this implies

$$\begin{aligned}\eta_l(q,\dot q) &= (\Psi'(q)^T\eta(\tilde q,\dot{\tilde q}))_l + \frac{\partial\Psi^j}{\partial q^l}(q)A_{ij}(\tilde q)\frac{\partial^2\Psi^i}{\partial q^k\partial q^m}(q)\dot q^k\dot q^m\\ &= (\Psi'(q)^T\eta(\tilde q,\dot{\tilde q}))_l + \frac{\partial\Psi^j}{\partial q^l}(q)\Gamma_j\\ &= (\Psi'(q)^T\eta(\tilde q,\dot{\tilde q}) + \Psi'(q)^T\Gamma)_l\end{aligned}$$

which shows that the second equality in (A.12) also holds.

The proof of the third identity in (A.12) follows at once by differentiating (A.9) with respect to q^j. Finally, to prove the fourth identity in (A.12), we start by using the definition (A.7) of γ together with the expressions (A.14) for $\dot{\tilde q}^j$ to obtain

$$\begin{aligned}(\mathcal{Q}(q)\gamma(\tilde q,\dot{\tilde q}))^a &= \mathcal{Q}^a_b(q)\gamma^b(\tilde q,\dot{\tilde q})\\ &= \mathcal{Q}^a_b(q)\frac{\partial\beta^b_i}{\partial q^j}(\tilde q)\frac{\partial\Psi^i}{\partial q^l}(q)\frac{\partial\Psi^j}{\partial q^m}(q)\dot q^l\dot q^m.\end{aligned}\tag{A.20}$$

On the other hand, the coordinate form of the invariance condition (A.11) is given by

$$\beta^a_i(q) = \mathcal{Q}^a_b(q)\beta^b_k(\tilde q)\frac{\partial\Psi^k}{\partial q^i}(q),\tag{A.21}$$

or, equivalently, by

$$\mathcal{P}^b_a(q)\beta^a_i(q) = \beta^b_k(\tilde q)\frac{\partial\Psi^k}{\partial q^i}(q).\tag{A.22}$$

Differentiating (A.21) with respect to q^j gives

$$\begin{aligned}\frac{\partial\beta^a_i}{\partial q^j}(q) &= \frac{\partial\mathcal{Q}^a_b}{\partial q^j}(q)\beta^b_k(\tilde q)\frac{\partial\Psi^k}{\partial q^i}(q) + \mathcal{Q}^a_b(q)\frac{\partial\beta^b_k}{\partial q^r}(\tilde q)\frac{\partial\Psi^r}{\partial q^j}(q)\frac{\partial\Psi^k}{\partial q^i}(q)+\\ &\qquad \mathcal{Q}^a_b(q)\beta^b_k(\tilde q)\frac{\partial^2\Psi^k}{\partial q^j\partial q^i}(q)\\ &= \frac{\partial\mathcal{Q}^a_b}{\partial q^j}(q)\mathcal{P}^b_c(q)\beta^c_i(q) + \mathcal{Q}^a_b(q)\frac{\partial\beta^b_k}{\partial q^r}(\tilde q)\frac{\partial\Psi^r}{\partial q^j}(q)\frac{\partial\Psi^k}{\partial q^i}(q)\\ &\qquad + \mathcal{Q}^a_b(q)\beta^b_k(\tilde q)\frac{\partial^2\Psi^k}{\partial q^j\partial q^i}(q),\end{aligned}\tag{A.23}$$

where we have used (A.22) in the second identity. Inserting (A.23) into the definition (A.7) of γ and using (A.20) yields

$$\begin{aligned}\gamma^a(q,\dot q) &= \frac{\partial \mathcal{Q}^a_b}{\partial q^j}(q)\mathcal{P}^b_c(q)\beta^c_i(q)\dot q^i\dot q^j + \mathcal{Q}^a_b(q)\frac{\partial \beta^b_k}{\partial q^r}(\tilde q)\frac{\partial \Psi^r}{\partial q^j}(q)\frac{\partial \Psi^k}{\partial q^i}(q)\dot q^i\dot q^j \\ &\quad + \mathcal{Q}^a_b(q)\beta^b_k(\tilde q)\frac{\partial^2\Psi^k}{\partial q^j\partial q^i}(q)\dot q^i\dot q^j \\ &= (T_1 + \mathcal{Q}(q)\gamma(\tilde q,\dot{\tilde q}) + T_2)^a,\end{aligned} \tag{A.24}$$

where the second identity follows from the definitions (A.13) of T_1 and T_2 and the expression (A.20) for $(\mathcal{Q}(q)\gamma(\tilde q,\dot{\tilde q}))^a$. The fourth identity in (A.12) is clearly equivalent to (A.24).

□

Now make an additional assumption which is that the vector field $Z \in \mathfrak{X}(Q)$ is invariant or reversible with respect to Ψ. We treat both cases simultaneously by assuming

$$\Psi_* Z = sZ, \qquad \text{where } s = \pm 1.$$

In coordinates, this means that

$$Z(\Psi(q)) = s\Psi'(q)Z(q).$$

Using (A.3) and (A.11) this implies

$$K(q) = s\mathcal{Q}(q)K(\Psi(q)) \qquad \text{for all } q. \tag{A.25}$$

Proposition A.2. *Consider an arbitrary curve $t \mapsto q_t$ on Q and denote $\tilde q_t = \Psi(q_t)$. The condition (A.25) implies*

$$\mathcal{Q}(q_t)\xi(\tilde q_t,\dot{\tilde q}_t) = s\xi(q_t,\dot q_t) - U_t, \tag{A.26}$$

where $U_t \in \mathbb{R}^{n-k}$ has components

$$U^a_t = \frac{\partial \mathcal{Q}^a_b}{\partial q^j}(q_t)\mathcal{P}^b_c(q_t)K^c(q_t)\dot q^j_t. \tag{A.27}$$

Proof. The proof follows from chain rule calculations similar to the ones performed in the proof of Proposition (A.1) starting from the component version of (A.25),

$$K^a(q) = s\mathcal{Q}^a_b(q)K^b(\tilde q).$$

The details are left to the reader. □

Theorem A.3. *Assume the above invariance conditions, i.e. $L \circ T\Psi = L$, $\Psi_*\mathcal{D} = \mathcal{D}$*

and $\Psi_* Z = sZ$ with $s = \pm 1$. Suppose that $t \mapsto q_t$ is a solution of the equations of motion (A.4) *satisfying* (A.1) *and let $\tilde{q}_t = \Psi(q_t)$. Then,*

1. *for $s = 1$, the curve $\tilde{q}_t$ is a solution of* (A.4) *satisfying* (A.1).
2. *for $s = -1$ the curve $\hat{q}_t = \tilde{q}_{-t}$ is a solution of* (A.4) *satisfying* (A.1).

Proof. First note that in view of (A.12) we have

$$A(\tilde{q}_t)\ddot{\tilde{q}}_t + \eta(\tilde{q}_t, \dot{\tilde{q}}_t) + V'(\tilde{q}_t) = \Psi'(q_t)^{-T}(A(q_t)\ddot{q}_t + \eta(q_t, \dot{q}_t) + V'(q_t)). \tag{A.28}$$

On the other hand, using (A.8) and (A.11), it is easy to show that

$$\begin{aligned} \beta(\tilde{q}_t) &= \Psi'(q_t)^{-T}\beta(q_t)^T P(q_t)^T, \\ \Sigma(\tilde{q}_t) &= \mathcal{Q}(q_t)^T \Sigma(q_t)\mathcal{Q}(q_t), \\ \beta(\tilde{q}_t)A(\tilde{q}_t)^{-1} &= P(q_t)\beta(q_t)A(q_t)^{-1}\Psi'(q_t)^T. \end{aligned}$$

Inserting the above relations into the explicit expression (A.6) for R yields

$$\begin{aligned} R(\tilde{q}_t, \dot{\tilde{q}}_t) =& \Psi'(q_t)^{-T}\beta(q_t)^T\Sigma(q_t)\left(\beta(q_t)A(q_t)^{-1}(\Psi'(q_t)^T\eta(\tilde{q}_t, \dot{\tilde{q}}_t) + \Psi'(q_t)^T V'(\tilde{q}_t)) + \right. \\ & \left. + \mathcal{Q}(q_t)\xi(\tilde{q}_t, \dot{\tilde{q}}_t) - \mathcal{Q}(q_t)\gamma(\tilde{q}_t, \dot{\tilde{q}}_t)\right). \end{aligned}$$

Using (A.12) and (A.26) we may rewrite the above expression as

$$\begin{aligned} R(\tilde{q}_t, \dot{\tilde{q}}_t) =& \Psi'(q_t)^{-T}\beta(q_t)^T\Sigma(q_t)\left(\beta(q_t)A(q_t)^{-1}(\eta(q_t, \dot{q}_t) + V'(q_t)) - \gamma(q_t, \dot{q}_t) + \right. \\ & \left. s\xi(q_t, \dot{q}_t)\right) + \Psi'(q_t)^{-T}\beta(q_t)^T\Sigma(q_t)\mu_t, \end{aligned} \tag{A.29}$$

where

$$\mu_t = -\beta(q_t)A(q_t)\Gamma_t + (T_1)_t + (T_2)_t - U_t.$$

Now note that, from (A.13) and (A.27), the components of $(T_1)_t - U_t$ are:

$$((T_1)_t - U_t)^a = \frac{\partial \mathcal{Q}^a_b}{\partial q^j}(q_t)\mathcal{P}^b_c(q_t)(\beta^c_i(q_t)\dot{q}^i_t - K^c(q_t))\dot{q}^j_t.$$

Therefore, our hypothesis that q_t satisfies the constraints (A.1) implies that $(T_1)_t - U_t = 0$. On the other hand, using (A.11) one checks that $-\beta(q_t)A(q_t)\Gamma_t + (T_2)_t = 0$. Therefore, the term μ_t in (A.29) vanishes and we have

$$\begin{aligned} R(\tilde{q}_t, \dot{\tilde{q}}_t) = & \Psi'(q_t)^{-T}\beta(q_t)^T\Sigma(q_t)\left(\beta(q_t)A(q_t)^{-1}(\eta(q_t, \dot{q}_t) + V'(q_t)) \right. \\ & \left. -\gamma(q_t, \dot{q}_t) + s\xi(q_t, \dot{q}_t)\right). \end{aligned} \tag{A.30}$$

If $s = 1$ we deduce from (A.30) that $R(\tilde{q}_t, \dot{\tilde{q}}_t) = \Psi'(q_t)^{-T} R(q_t, \dot{q}_t)$. This relation, combined with (A.28) and our hypothesis that q_t is a solution of (A.4) proves that $\tilde{q}_t$ is also a solution of (A.4).

If $s = -1$ we instead deduce from (A.30) that

$$\begin{aligned} R(\tilde{q}_t, \dot{\tilde{q}}_t) = \Psi'(q_t)^{-T} \beta(q_t)^T \Sigma(q_t) \big(\beta(q_t) A(q_t)^{-1} (\eta(q_t, \dot{q}_t) + V'(q_t)) \\ -\gamma(q_t, \dot{q}_t) - \xi(q_t, \dot{q}_t) \big) . \end{aligned} \tag{A.31}$$

Considering that $\hat{q}_t = \tilde{q}_{-t}$ and $\dot{\hat{q}}_t = -\dot{\tilde{q}}_{-t}$, we have

$$\begin{aligned} R(\hat{q}_t, \dot{\hat{q}}_t) &= R(\tilde{q}_{-t}, -\dot{\tilde{q}}_{-t}) \\ &= \Psi'(q_{-t})^{-T} \beta(q_{-t})^T \Sigma(q_{-t}) \big(\beta(q_{-t}) A(q_{-t})^{-1} (\eta(q_{-t}, -\dot{q}_{-t}) + V'(q_{-t})) \\ &\quad -\gamma(q_{-t}, -\dot{q}_{-t}) - \xi(q_{-t}, -\dot{q}_{-t}) \big) . \end{aligned}$$

Given that η and γ are homogeneous of degree 2 and ξ is homogeneous of degree 1 in the velocities, we have

$$\eta(q_{-t}, -\dot{q}_{-t}) = \eta(q_{-t}, \dot{q}_{-t}), \quad \gamma(q_{-t}, -\dot{q}_{-t}) = \gamma(q_{-t}, \dot{q}_{-t}), \quad \xi(q_{-t}, -\dot{q}_{-t}) = -\xi(q_{-t}, \dot{q}_{-t}).$$

This implies that

$$R(\hat{q}_t, \dot{\hat{q}}_t) = \Psi'(q_{-t})^{-T} R(q_{-t}, \dot{q}_{-t}). \tag{A.32}$$

Considering that $\ddot{\hat{q}}_t = \ddot{\tilde{q}}_{-t}$, and using again the degree 2 homogeneity of η, we have

$$\begin{aligned} A(\hat{q}_t)\ddot{\hat{q}}_t + \eta(\hat{q}_t, \dot{\hat{q}}_t) + V'(\hat{q}_t) &= A(\tilde{q}_{-t})\ddot{\tilde{q}}_{-t} + \eta(\tilde{q}_{-t}, \dot{\tilde{q}}_{-t}) + V'(\tilde{q}_{-t}) \\ &= \Psi'(q_{-t})^{-T} (A(q_{-t})\ddot{q}_{-t} + \eta(q_{-t}, \dot{q}_{-t}) + V'(q_{-t})). \end{aligned}$$

which in view of (A.28) shows that

$$A(\hat{q}_t)\ddot{\hat{q}}_t + \eta(\hat{q}_t, \dot{\hat{q}}_t) + V'(\hat{q}_t) = \Psi'(q_{-t})^{-T} (A(q_{-t})\ddot{q}_{-t} + \eta(q_{-t}, \dot{q}_{-t}) + V'(q_{-t})).$$

Equations (A.32) and (A) show that $\hat{q}_t$ is a solution of (A.4) if $s = -1$.

Now, combining the invariance relations (A.11) and (A.25), and recalling that the matrix Ω is invertible, it is not hard to show that $\tilde{q}_t$ satisfies

$$\beta(\tilde{q}_t)\dot{\tilde{q}}_t = sK(\tilde{q}_t).$$

This shows that $\tilde{q}_t$ satisfies the constraints (A.1) if $s = 1$ and, considering again that $\dot{\hat{q}}_t = -\dot{\tilde{q}}_{-t}$, that $\hat{q}_t$ satisfies the constraints (A.1) if $s = -1$. □

B Existence of first integrals of affine LR systems

We provide a general framework explaining the existence of the first integrals in proposition I.3.1. Throughout this appendix we assume that we are given a configuration manifold Q, a Lagrangian $L : TQ \to \mathbb{R}$ and a (linear) constraint distribution $\mathcal{D} \subset TQ$ specifying some nonholonomic constraints. We are interested in determining conditions for the existence of first integrals for the nonholonomic system with Lagrangian L and affine nonholonomic constraints described by the affine distribution $\mathcal{A} = \mathcal{D} + Z$ where Z is a given vector field on Q.

B.1 Affine nonholonomic Noether's theorem

Let $\Psi : G \times Q \to Q$ be an action of the Lie group G on Q and $\overline{\Psi} : G \times TQ \to TQ$ be the lifted action. Let ξ_Q be the infinitesimal generator of the Ψ-action on Q corresponding to an element ξ of the Lie algebra $\mathfrak{g}$ of G (i.e. $\xi_Q(q) = \frac{d}{dt}\big|_{t=0} \exp(\xi t) \cdot q \in T_qQ$). Finally, let $J_\xi : TQ \to \mathbb{R}$ be the momentum component in the direction of ξ, namely,

$$J_\xi(q, \dot{q}) = \left\langle \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \, , \, \xi_Q(q) \right\rangle .$$

The following well-known result is sometimes referred to as nonholonomic Noether's theorem [2, 8, 34].

Proposition B.1. *If the Lagrangian L is invariant under the lifted action $\overline{\Psi}$ and $\xi \in \mathfrak{g}$ is such that $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$ (i.e. ξ is a* horizontal symmetry*), then $J_\xi|_{\mathcal{D}}$ is a first integral of the nonholonomic system $(L, Q, \mathcal{D})$.*

This result admits the following immediate generalization to the affine case, and is a particular instance of proposition 2 in [33].

Proposition B.2. *Let $Z \in \mathfrak{X}(Q)$ be any vector field and consider the affine distribution $\mathcal{A} = \mathcal{D} + Z \subset TQ$. Under the same hypothesis of proposition B.1, the function $J_\xi|_\mathcal{A}$ is a first integral of the nonholonomic system determined by L and $\mathcal{A}$.*

The key observation to connect this result to proposition 2 in [33] is that the vector field ξ_Q is annihilated by the reaction force by the assumption that $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$.

B.2 Affine LR systems

Now suppose that $Q = G$ is a Lie group and the action Ψ of the previous section is left multiplication. The invariance of L under the lifted action $\overline{\Psi}$ is usually called *left invariance*. In addition, we assume that the distribution $\mathcal{D}$ is right invariant (i.e. $\mathcal{D}_{gh} = T_g R_h(\mathcal{D}_g)$ for all $g, h \in G$, where $R_h : G \to G$ is right multiplication by h). These systems where introduced by Veselov and Veselova [65] and are termed *LR systems*.

By right invariance, we have $\mathcal{D}_g = T_e(\mathfrak{d})$ for all $g \in G$ where $\mathfrak{d}$ is the value of $\mathcal{D}$ at the identity $e \in G$, namely $\mathfrak{d} = \mathcal{D}_e \subset \mathfrak{g}$. Non-integrability of $\mathcal{D}$ is equivalent to the condition that $\mathfrak{d}$ is not a subalgebra of $\mathfrak{g}$. A direct consequence of Proposition B.2 is the following.

Proposition B.3. *Let $\xi \in \mathfrak{d}$, then $J_\xi|_\mathcal{A}$ is a first integral of the nonholonomic system determined by L and $\mathcal{A} = \mathcal{D} + Z$ where Z is any vector field on G.*

Proof. It is easily seen that such ξ is a horizontal symmetry. Indeed,

$$\xi_G(g) = \left.\frac{d}{dt}\right|_{t=0} L_{\exp(\xi t)} g = \left.\frac{d}{dt}\right|_{t=0} R_g(\exp \xi t) = T_e R_g(\xi) \in \mathcal{D}_g,$$

where $L_{\exp(\xi t)}$ is left multiplication by $\exp(\xi t)$ (there should be no risk of confusion with the Lagrangian function, also denoted by L). □

C A closed form for the transition function f_ε

We first consider the bump function $h : \mathbb{R} \to \mathbb{R}$ defined as:

$$h(r) = \begin{cases} e^{-1/r^2} & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases}.$$

It is a C^∞-function with support $(0, \infty)$ satisfying $0 < h(r) < 1$ for $r > 0$ and $\lim_{r\to\infty} h(r) = 1$, see Fig. C.1.

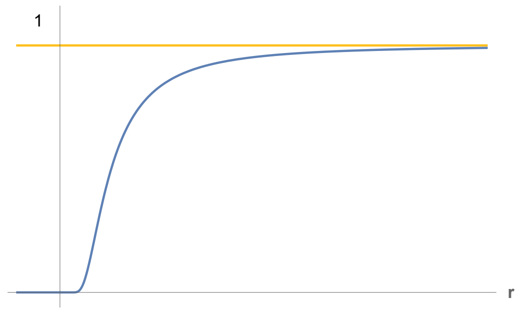


Figure C.1: $h(r)$

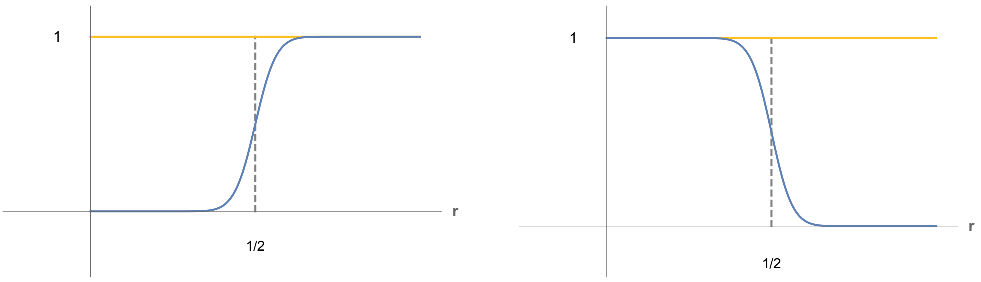
Now we define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(r) = \frac{h(r)}{h(r) + h(1-r)}.$$

Since the denominator is greater than 0 for all $r \in \mathbb{R}$, it is a differentiable function and $g(r) = 0$ for $r \leq 0$ and $g(r) = 1$ for $r \geq 1$. Moreover, g provides a smooth transition from 0 to 1, see Fig. C.2a. We now define the function $k : \mathbb{R} \to \mathbb{R}$ as

$$k(r) = g(1 - r),$$

which instead provides a smooth transition from 1 to 0, see Fig. C.2b.



(a) $g(r)$ **(b)** $k(r)$

Figure C.2

Finally, we define $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ as

$$f_\varepsilon(r) = k\left(\frac{r - (R - \varepsilon/2)}{\varepsilon}\right).$$

Then f_ε is a smooth function with support $(R - \frac{\varepsilon}{2}, R + \frac{\varepsilon}{2})$ that provides a smooth transition from 1 to 0 in an interval of width ε centered at R, see Fig. C.3.

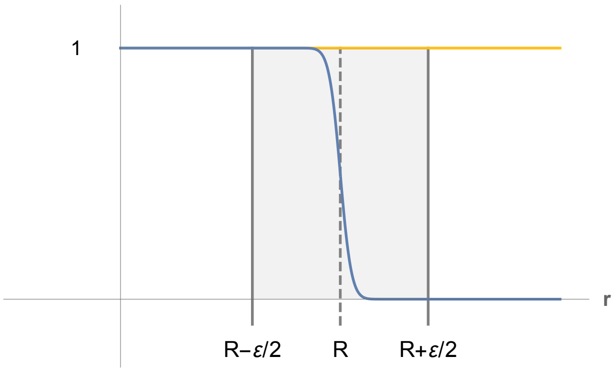


Figure C.3: $f_\varepsilon(r)$

D The work of Lévy-Leblond

In this appendix, we will give a summary of the original reasoning in [53] and that was then elaborated in [63] aiming at framing the contributions of this thesis. All vectors appearing below are written with respect to the inertial frame Σ_s and we follow our notation.

Lévy-Leblond proves the ANAIS billiard phenomenon by considering the problem of the motion of a sphere rolling without slipping on a "horizontal substrate characterized by any vector field" and using Newtonian mechanics. We consider a sphere (of radius r, mass m and moment of inertia I) that is rolling without slipping on a horizontal substrate characterized by a vector field $\boldsymbol{V}(\boldsymbol{x})$[1], where, as usual, $\boldsymbol{x} = (x_1, x_2, 0) \in \mathbb{R}^3$ is the contact point of the sphere on the plane. We also write $\boldsymbol{V}(\boldsymbol{x}) = (V_1(\boldsymbol{x}), V_2(\boldsymbol{x}), 0) \in \mathbb{R}^3$.

Consider the reaction force $\boldsymbol{F}$ which is exerted at the contact point. Newton's equations of motion are

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}$$

for the displacement and

$$I\dot{\boldsymbol{\omega}} = \boldsymbol{F} \times (r\boldsymbol{e}_3)$$

for the rotation of the sphere. These two equations imply that the components of

$$\boldsymbol{m} = I\boldsymbol{\omega} - mr\dot{\boldsymbol{x}} \times \boldsymbol{e}_3, \tag{D.1}$$

are integrals of motion.

[1]actually, part of the discussion by Lévy-Leblond in [53] considers non-autonomous vector fields $\boldsymbol{V}(\boldsymbol{x}, t)$.

On the other hand, the no-slip rolling condition is given by

$$\dot{\boldsymbol{x}} = -r\boldsymbol{e}_3 \times \boldsymbol{\omega} + \boldsymbol{V}(\boldsymbol{x}), \tag{D.2}$$

Combining (D.1) and (D.2), one can easily check that we recover the expression of $\boldsymbol{m}$ given in (I.5.3) for $\boldsymbol{W}_b = 0$.

We may eliminate $\boldsymbol{\omega}$ combining (D.1) and (D.2) to obtain

$$\dot{\boldsymbol{x}} = \frac{I}{I + mr^2}\boldsymbol{V}(\boldsymbol{x}) + \frac{r}{I + mr^2}\boldsymbol{m} \times \boldsymbol{e}_3, \tag{D.3}$$

This first order differential equation for $\boldsymbol{x}$ determines the trajectory of the contact point of the sphere on the plane and will be used to arrive to desired conclusions.

The argument given by Lévy-Leblond is based on the assumption that $\boldsymbol{m}$ given by (D.1) is constant along the motion, even if $\boldsymbol{V}$ is a discontinuous vector field. As a consequence, it is assumed that (D.3) is valid for discontinuous $\boldsymbol{V}$. Lévy-Leblond does mention that it is possible to justify this assumption from a physical point of view but does not provide any details. In this regard, we mention the work of Ivanov [42] which delves precisely into this issue. Our discussion in Section 2.2 of Part II gives a mathematical justification for the validity of this postulate.

D.1 The rotating plane

Consider the vector field $\boldsymbol{V}(\boldsymbol{x}) = \eta\boldsymbol{e}_3 \times \boldsymbol{x}$. Equation (D.3) takes the form

$$\dot{\boldsymbol{x}} = \frac{I}{I + mr^2}(\eta\boldsymbol{e}_3 \times \boldsymbol{x} + r\boldsymbol{m} \times \boldsymbol{e}_3), \tag{D.4}$$

which is equivalent to (0.3.6). As shown at the end of Section 3.2 of Part 0, the trajectories of the contact point on the plane are circles with center

$$\boldsymbol{C} = \frac{r}{I\eta}\boldsymbol{e}_3 \times (\boldsymbol{m} \times \boldsymbol{e}_3). \tag{D.5}$$

If at time $t = \tilde{t}$ the sphere has position $\tilde{\boldsymbol{x}}$ and velocity $\tilde{\boldsymbol{v}}$, then, eliminating $\boldsymbol{m}$ from (D.4) and (D.5), we get that the center of the circle is given by

$$\boldsymbol{C} = \frac{I + mr^2}{I\eta}\boldsymbol{e}_3 \times \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{x}}. \tag{D.6}$$

D.2 The rotating disk

Now consider the vector field

$$\boldsymbol{V}(\boldsymbol{x}) = \begin{cases} \eta \boldsymbol{e}_3 \times \boldsymbol{x} & \text{if } \|\boldsymbol{x}\| < R, \\ 0 & \text{if } \|\boldsymbol{x}\| > R. \end{cases}$$

Equation (D.3) states that the velocity vector is constant $\boldsymbol{v}_0$ whenever the sphere is outside the rotating disk (i.e. whenever $\|\boldsymbol{x}_0\| > R$). Therefore, after the sphere passes through the rotating disk, it regains its initial velocity in magnitude and direction. Note that this is a particular instance of our more general Proposition II.2.5.

Let us now explain how Lévy-Leblond explains the phenomenon. Suppose that the sphere is rolling in the fixed part of the plane with linear velocity $\boldsymbol{v}_0$ and angular velocity $\boldsymbol{\omega}_0$. These two remain constant until the sphere arrives to the rotating circular platform and due to the no-slip constraints satisfy

$$\boldsymbol{v}_0 = -r\boldsymbol{e}_3 \times \boldsymbol{\omega}_0. \tag{D.7}$$

Now suppose that the sphere reaches the boundary of the rotating disc at $\boldsymbol{x} = \boldsymbol{x}_0$ (so $\|\boldsymbol{x}_0\| = R$). At this point the ball experiences a discontinuity in the linear and angular velocities which become $\boldsymbol{v}_1$, $\boldsymbol{\omega}_1$. The no-slip nonholonomic constraints imply

$$\boldsymbol{v}_1 = -r\boldsymbol{e}_3 \times \boldsymbol{\omega}_1 + \eta \boldsymbol{e}_3 \times \boldsymbol{x}_0. \tag{D.8}$$

As mentioned above, the fundamental assumption is that the discontinuity in the velocities is such that $\boldsymbol{m}$ remains constant. In view of (D.1) we get

$$I\boldsymbol{\omega}_0 - mr\boldsymbol{v}_0 \times \boldsymbol{e}_3 = I\boldsymbol{\omega}_1 - mr\boldsymbol{v}_1 \times \boldsymbol{e}_3, \tag{D.9}$$

A simple calculation which combines (D.7), (D.8) and (D.9) yields

$$\boldsymbol{v}_1 = \boldsymbol{v}_0 + \frac{I\eta}{I + mr^2}\boldsymbol{e}_3 \times \boldsymbol{x}_0. \tag{D.10}$$

Now, as mentioned in Section D.1, when the sphere is inside the circular platform, the contact point traces a circumference with center $\boldsymbol{C}$ determined by (D.6). Putting $\tilde{\boldsymbol{v}} = \boldsymbol{v}_1$ in (D.6) with $\boldsymbol{v}_1$ given by (D.10) gives

$$\boldsymbol{C} = \frac{I + mr^2}{I\eta}\boldsymbol{e}_3 \times \boldsymbol{v}_0.$$

This implies that the center of the circumference lies on the line Δ passing through the origin and perpendicular to $\boldsymbol{v}_0$. In other words, the piece of the circumference traced by the contact point while in the inside of the platform is a curve which is symmetric with respect to the line Δ which passes through the origin and is perpendicular to the straight line path followed by the sphere before entering the disc, see Fig. D.1. In particular, the exit point corresponds to the reflection of the entrance point with respect to Δ. This conclusion, together with the observation that the linear velocity is the same in direction and magnitude before and after entering the platform, proves that the exit trajectory is the exact prolongation of the initial one.

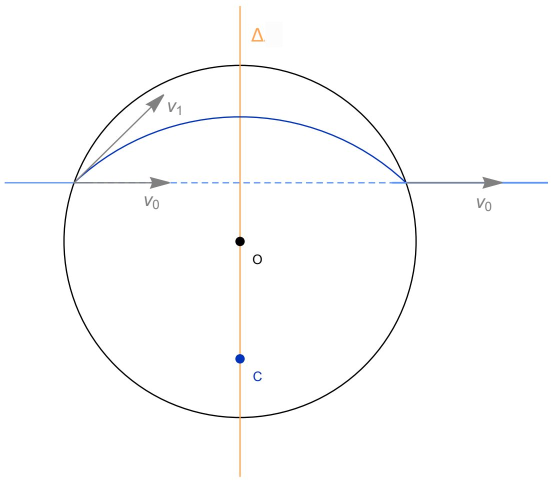


Figure D.1

D.3 Generalization

In Section 6 of [53], Lévy-Leblond explains how the argument given above only relies on the symmetries of the problem and may therefore be formulated in a broader context which allows for more general vector fields $\boldsymbol{V}$. His symmetry arguments (which were apparently proposed by Christian Vanneste) are applicable to a family of vector fields $\boldsymbol{V}$ possessing specific symmetric properties and rely on the reversibility of the dynamics. Our development in Part II was greatly inspired by this discussion. Our work formalizes and extends the ideas presented in this section. In particular, in Chapter 3 we give general results on the reversibility of affine nonholonomic systems which explain the reversibility of the given problem. On the other hand, our work in Chapter 4 gives a formal treatment of the continuous symmetries of the problem which is necessary to rigorously prove the phenomenon (actually, the argument given in Section 6 of [53] overlooks the orientation of the sphere throughout the motion, which may be justified exploiting the $\mathrm{SO}(3)$-symmetry as in Chapter 4).

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