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RECOLLEMENTS FROM EXACT MODEL STRUCTURES AND HEART CONSTRUCTIONS IN TRIANGULATED CATEGORIES

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RECOLLEMENTS FROM EXACT MODEL STRUCTURES AND HEART CONSTRUCTIONS IN TRIANGULATED CATEGORIES

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ABSTRACT

We consider a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ such that $\mathcal{A}$ contains a generator of finite projective dimension. The derived category $\mathcal{D}(\mathcal{B})$ of the exact category $\mathcal{B}$ is defined as the quotient of the category $\mathrm{Ch}(\mathcal{B})$, of unbounded complexes with terms in $\mathcal{B}$, modulo the subcategory $\widetilde{\mathcal{B}}$ consisting of the acyclic complexes with terms in $\mathcal{B}$ and cycles in $\mathcal{B}$.

We prove that there are recollements

$$\begin{tikzcd} \underset{\sim}{\underline{ex\mathcal{B}}} \arrow[r, "inc"] & \mathcal{D}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

and

$$\begin{tikzcd} \underset{\sim}{\underline{ex\mathcal{B}}} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Then, we restrict our attention to the cotorsion pairs such that $\widetilde{\mathcal{B}}$ coincide with the class $ex\mathcal{B}$ of the acyclic complexes of $\mathrm{Ch}(\mathcal{G})$ with terms in $\mathcal{B}$. In this case the derived category $\mathcal{D}(\mathcal{B})$ fits into a recollement

$$\begin{tikzcd} \underset{\sim}{\underline{ex\mathcal{B}}} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{B}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

We will explore the conditions under which $ex\mathcal{B} = \widetilde{\mathcal{B}}$ and provide some examples. Symmetrically, we prove analogous results for the exact category $\mathcal{A}$.

We also introduce the notion of Nakaoka context in additive categories as couples $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$ of torsion pairs such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. We give a set of axioms for a Nakaoka context in order to ensure that the heart $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ is Abelian. Then, we inspect the properties of Nakaoka contexts in Abelian and triangulated categories. In particular, we find a bijection between the t-structures $(\mathcal{T}_1, \mathcal{F}_1[1]), (\mathcal{T}_2, \mathcal{F}_2[1])$ such that $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$ whose heart $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ is Abelian and the cohereditary torsion pairs in $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$.

RIASSUNTO

Consideriamo un cotorsion pair completo ed ereditario $(\mathcal{A}, \mathcal{B})$ in una categoria di Grothendieck $\mathcal{G}$ tale che $\mathcal{A}$ contenga un generatore di dimensione proiettiva finita. La categoria derivata $\mathcal{D}(\mathcal{B})$ della categoria esatta $\mathcal{B}$ è definita come il quoziente fra la categoria $\mathrm{Ch}(\mathcal{B})$ dei complessi illimitati a termini in $\mathcal{B}$ modulo la sottocategoria $\widetilde{\mathcal{B}}$ dei complessi aciclici a termini in $\mathcal{B}$ e cicli in $\mathcal{B}$.

Dimostriamo che esistono i recollement

$$\begin{tikzcd} \frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & \mathcal{D}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

e

$$\begin{tikzcd} \frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Successivamente restringiamo la nostra attenzione ai cotorsion pair tali che $\widetilde{\mathcal{B}}$ coincida con la classe $ex\mathcal{B}$ dei complessi aciclici di $\mathrm{Ch}(\mathcal{G})$ con termini in $\mathcal{B}$. In questo caso la categoria derivata $\mathcal{D}(\mathcal{B})$ appartiene a un recollement

$$\begin{tikzcd} \frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{B}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Studieremo le condizioni per cui $ex\mathcal{B} = \widetilde{\mathcal{B}}$ e mostreremo alcuni esempi. Simmetricamente dimostriamo risultati analoghi per la categoria esatta $\mathcal{A}$.

Inoltre, introduciamo la nozione di Nakaoka context in categorie additive come coppie di torsion pair $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ per $i = 1, 2$ tali che $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Daremo un insieme di assiomi per un Nakaoka context che garantisca l'abelianità del cuore $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$. Successivamente studieremo le proprietà dei Nakaoka context in categorie Abeliane e triangolate. In particolare troviamo una biezione tra le t-strutture $(\mathcal{T}_1, \mathcal{F}_1[1]), (\mathcal{T}_2, \mathcal{F}_2[1])$ tali che $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$ il cui cuore $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ sia abeliano e le torsion pair coereditarie in $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_{[}1]$.

Casta Diva, che inargenti
Queste sacre antiche piante,
Al noi volgi il bel sembiante,
Senza nube e senza vel!

— *Norma*, Vincenzo Bellini

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CONTENTS

INTRODUCTION

Model categories were introduced by Quillen for the first time in [40] as a formal way of dealing with the notion of homotopy in a category. The main problem is the following: we have a class of morphisms, called weak equivalences, and we want to treat them as if they were isomorphisms. Formally inverting these weak equivalences does not necessarily produce nice results, since we have to resort to some kind of calculus of fractions to construct the morphisms in the quotient category, and these methods are not guaranteed to yield a *set* of homomorphisms. This, however, happens whenever the weak equivalences are part of a model structure on a model category.

In [27] Hovey proved that there is a bijective correspondence between abelian model structures on abelian categories and certain couples of complete cotorsion pairs. Namely, if $\mathcal{Q}$, $\mathcal{W}$, and $\mathcal{R}$ are the classes of cofibrant, trivial, and fibrant objects respectively, then $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs, and conversely given three such classes forming two complete cotorsion pair, there is an abelian model structure, indicated by $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$, whose cofibrant, trivial and fibrant objects are exactly those classes.

This correspondence allows to look for models structures simply by looking for complete cotorsion pairs, for example given a ring R there is a model structure (and, in fact, many of them) on the category $\mathrm{Ch}(R)$ of unbounded cochain complexes of R-modules whose homotopy category is the derived category of R. Moreover, if several model structures on a category satisfy certain inclusions, it is possible to find a recollement between their homotopy categories.

Subsequently, the same techniques have been applied to cotorsion pairs in exact categories, showing that Hovey's correspondence holds in the case of exact model structures in exact categories. This has greatly helped the theory of localization of model structures, since it applies easily to the case of extension closed subcategories of Grothendieck categories.

The first part of this thesis follows this idea and our objective will be to construct model structures in exact subcategories of the category $\mathrm{Ch}(\mathcal{G})$ of unbounded cochain complexes on a Grothendieck category $\mathcal{G}$, in order to prove the existence of interesting recollements between their homotopy categories. More precisely, we will consider a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ and construct model structures on $\mathrm{Ch}(\mathcal{B})$ in order to find recollements involving $K(\mathcal{B})$, $D(\mathcal{B})$ together with $D(\mathcal{G})$.

The second part of this thesis is devoted to a different topic of research, although our applications will still be mainly in triangulated categories.

In [7] Beilinson, Bernstein and Deligne introduced the notion of t-structure in a triangulated category in order to build perverse sheaves. They also proved that to any t-structure is associated an abelian category, called its heart. In [33], Nakaoka proved that starting from any torsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathscr{C}$ it is possible to find an abelian heart $\underline{\mathcal{H}}$ in the quotient category $\underline{\mathscr{C}} := \mathscr{C}/\mathcal{W}$, where $\mathcal{W} = \mathcal{U}[1] \cap \mathcal{V}$. This result motivates the second part of this thesis, where we use torsion pairs in additive categories to build what we call Nakaoka contexts, and show that if they satisfy certain axioms they have an abelian heart.

We will then investigate the behaviour of Nakaoka contexts in Abelian and triangulated categories, with particular interest to the case of t-structures.

This thesis is articulated as follows:

- In Chapter 1 we introduce the notions of cotorsion pairs, model structures, and recollements, giving the statements of the main known results in the field and constructing the important cotorsion pairs that will be used later. This chapter is based on the first half of the joint work [4] with Silvana Bazzoni.

- In Chapter 2 we use the theory introduced in the previous chapter to build many new recollements in the setting of exact subcategories of the category of cochain complexes on a Grothendieck category. This chapter is based on the second half of the joint work [4].

- In Chapter 3 we introduce and study Nakaoka contexts and their hearts in the setting of additive categories, with the objective of finding an axiomatization that would guarantee the existence of an abelian heart. We examine in particular Nakaoka contexts in abelian and triangulated categories. This chapter is based on the joint work (in preparation) [43] with Manuel Saorín, Simone Virili, and Octavio Mendoza.

Next, we will give a detailed summary of each chapter of the thesis.

CHAPTER 1: COTORSION PAIRS AND MODEL STRUCTURES IN EXACT CATEGORIES

In this chapter we give a summary of the main results regarding cotorsion pairs and model structures in exact categories.

In Section 1.1 we give the relevant definitions of cotorsion pairs and model structure and state Hovey's correspondence explicitly in Theorem 1.7. Then, in Definition 1.12 we define what a recollement is and we state the main tools we will use in order to find recollements, namely Gillespie's Theorem 3.4 and Corollary 4.5 in [21] (numbered Theorem 1.14 and Corollary 1.19 respectively in this work).

First, recall the definition of injective cotorsion pair:

Definition. A complete cotorsion pair $(\mathcal{W}, \mathcal{R})$ is *injective* if $\mathcal{W}$ is thick and contains the injective objects.

If $(\mathcal{W}, \mathcal{R})$ is an injective cotorsion pair in $\mathcal{C}$, it induces a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{R})$.

Theorem (1.14)**.** *Let $\mathcal{C}$ be a weakly idempotent complete (WIC) exact category with enough injective and suppose we have three injective cotorsion pairs*

$$\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{R}_1), \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{R}_2), \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{R}_3)$$

such that $\mathcal{R}_2, \mathcal{R}_3 \subseteq \mathcal{R}_1$. If $\mathcal{W}_3 \cap \mathcal{R}_1 = \mathcal{R}_2$ (or equivalently $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$, and $\mathcal{R}_2 \subseteq \mathcal{W}_3$), then there exists a recollement

$$\begin{tikzcd} \mathcal{R}_2/\sim \arrow[r, "I"] & \mathcal{R}_1/\sim \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{C}/\mathcal{W}_3 \arrow[l, bend right, "\lambda"'] \arrow[l, bend left, "\rho"] \end{tikzcd}$$

where the functor I is simply the inclusion and Q is the quotient functor of Lemma 1.13. Moreover, λ has essential image $(\mathcal{W} \cap \mathcal{R}_1)/\sim$, ρ has essential image $\mathcal{R}_3/\sim$, and they provide an equivalence

$$\lambda\colon \mathcal{R}_3/\sim \longleftrightarrow (\mathcal{W}_2 \cap \mathcal{R}_1)/\sim\colon \rho.$$

In the case of a Frobenius category, an injective cotorsion pair is called *localizing*. If $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are both localizing cotorsion pairs in a Frobenius category, we call $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ a *localizing cotorsion triple*.

Corollary (1.19)**.** *Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a localizing cotorsion triple in a WIC Frobenius category $\mathcal{C}$. Then, there is a recollement*

$$\begin{tikzcd} \mathcal{Y}/\sim \arrow[r] & \mathcal{C}/\sim \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{C}/\mathcal{Y} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $\mathcal{C}/\sim$ is the stable category.

In Section 1.2 we give the tools needed to construct the cotorsion pairs we will use in the next chapter. We will see that starting from a completely hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category we can find several complete hereditary cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$, namely $({}^{\perp}dw\mathcal{B}, dw\mathcal{B})$, $(dw\mathcal{A}, dw\mathcal{A}^{\perp})$, $({}^{\perp}ex\mathcal{B}, ex\mathcal{B})$, $(ex\mathcal{A}, ex\mathcal{A}^{\perp})$, $(\widetilde{A}, dg\mathcal{B})$, and $(dg\mathcal{A}, \widetilde{\mathcal{B}})$ (see Notation 1.20 for the meaning of these symbols).

We conclude the chapter by stating several restriction propositions, namely 1.40, 1.41, 1.43, and 1.42. Given a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$, these results give conditions under which it is possible to restrict a cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ to one in $\mathrm{Ch}(\mathcal{B})$ (or, dually, $\mathrm{Ch}(\mathcal{A})$).

CHAPTER 2: RECOLLEMENTS FROM COTORSION PAIRS

We consider the case of a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$, sometimes requiring that $\mathcal{A}$ contains a generator of finite projective dimension. It is well known that, if $\mathcal{G}$ contains a projective generator, there is a recollement

$$\begin{tikzcd} K_{ac}(\mathcal{G}) \arrow[r] & K(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

In this chapter our objective is to find similar recollements, involving the classes $K(\mathcal{B})$ and $D(\mathcal{B})$ (and, dually, for $\mathcal{A}$). The main tools are the aforementioned [21, Theorem 3.4 and Corollary 4.5]

In Theorem 2.5 we prove that there is a recollement

$$\begin{tikzcd} \dfrac{ex\mathcal{B}}{\sim} \arrow[r] & D(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

where $D(\mathcal{B})$ is the derived category of $\mathcal{B}$ in Neeman's sense, and the homotopy category $ex\mathcal{B}/\sim$ on the left hand side is the full subcategory of $D(\mathcal{B})$ consisting of the exact complexes with terms in $\mathcal{B}$.

Then, by constructing a localizing cotorsion triple on $\mathrm{Ch}(\mathcal{B})$ of the form $(-, ex\mathcal{B}, -)$, where $ex\mathcal{B}$ is the class of exact complexes in $\mathrm{Ch}(\mathcal{G})$ with terms in $\mathcal{B}$, we use Corollary 1.19 to find in Theorem 2.11 a recollement

$$\begin{tikzcd} \frac{ex\mathcal{B}}{\sim} \arrow[r] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

It would be interesting to find cases where the previous recollement ends in the derived category $D(\mathcal{B})$. To this aim, we observe that when $ex\mathcal{B} = \widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}}$ is the class of complexes $B^\bullet \in ex\mathcal{B}$ such that $Z^n(B^\bullet) \in \mathcal{B}$ for all $n \in \mathbb{Z}$, the previous recollement becomes

$$\begin{tikzcd} \frac{\widetilde{\mathcal{B}}}{\sim} \arrow[r] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

and in Section 2.3 we describe several cases where this happens. A first example is that of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ where all objects in $\mathcal{A}$ have finite projective dimension, thus including the case of a tilting cotorsion pair, however this condition is far from necessary. In fact it was proved in [2] that the cotorsion modules satisfy $ex\mathrm{Cot} = \widetilde{\mathrm{Cot}}$.

Moreover, it is possible to find the previous recollement for cotorsion pairs that do not satisfy $ex\mathcal{B} = \widetilde{\mathcal{B}}$, e.g. the complete hereditary cotorsion pair $({}^{\perp}\mathrm{FpInj}, \mathrm{FpInj})$ generated by the class of finitely presented modules over a coherent ring, provided by Šťovíček in [44].

We also study the dual case, and find that when $ex\mathcal{A} = \widetilde{\mathcal{A}}$ we can find the dual recollement. Again, the condition $ex\mathcal{A} = \widetilde{\mathcal{A}}$ is satisfied when $\mathcal{B}$ consists of objects of finite injective dimension, this is true for example for a cotilting cotorsion pair.

Finally, one example in the case of R-modules of a cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\widetilde{\mathcal{A}} \subsetneq ex\mathcal{A}$ can be found when $\mathcal{A} = \mathrm{Flat}$ is the class of flat modules and follows by Neeman's [36].

CHAPTER 3: NAKAOKA CONTEXTS WITH ABELIAN HEARTS

In this chapter we will give a quick overview of the notion of torsion pair in an additive category. The difference with the usual notion of torsion pair is that, in lack of short exact sequences, we will resort to require the existence of pseudokernels and pseudocokernels. This will not be a problem, since our notion of torsion pair will coincide with the usual concept in Abelian categories, and t-structures in triangulated category will be (up to shift) torsion pairs.

Definition. Let $\mathcal{T}, \mathcal{F}$ be subclasses of the additive category $\mathscr{C}$, $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if:

1. $\mathcal{F} = \{X \mid \mathrm{Hom}_{\mathscr{C}}(T, X) = 0 \text{ for all } T \in \mathcal{T}\}$;
2. $\mathcal{T} = \{X \mid \mathrm{Hom}_{\mathscr{C}}(X, F) = 0 \text{ for all } F \in \mathcal{F}\}$;
3. for any $X \in \mathscr{C}$ there are two maps ε_X and λ_X

$$T_X \xrightarrow{\varepsilon_X} X \xrightarrow{\lambda_X} F^X$$

such $T_X \in \mathcal{T}$, $F^X \in \mathcal{F}$, ε_X is a pseudokernel of λ_X, and λ_X is a pseudocokernel of ε_X.

The torsion pair $(\mathcal{T}, \mathcal{F})$ is called left (resp. right) functorial if the inclusion functor $i : \mathcal{T} \to \mathscr{C}$ (resp. $j : \mathcal{F} \to \mathscr{C}$) has a right (resp. left) adjoint $t : \mathscr{C} \to \mathcal{T}$ (resp. $f : \mathscr{C} \to \mathcal{F}$).

The central notion of this chapter is the following:

Definition. A (pre-Abelian) Nakaoka context in an additive category $\mathscr{C}$ is a couple $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ of torsion pairs $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ satisfying the following axioms:

(CT.1) $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair;

(CT.2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$);

(CT.3) any $g \colon H \to H'$ in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ admits a pseudocokernel $g^C \colon H' \to C$ in $\mathcal{T}_1$, such that

$$0 \longrightarrow (C, -)_{|\mathcal{F}_2} \xrightarrow{(g^C, -)} (H', -)_{|\mathcal{F}_2} \xrightarrow{(g, -)} (H, -)_{|\mathcal{F}_2}$$

is an exact sequence in $\mathrm{Func}(\mathcal{F}_2, \mathrm{Ab})$;

(CT.3)* dual of **(CT.3)**.

Axioms **(CT.1)** and **(CT.2)** are simply setting the stage of two torsion pairs with the appropriate inclusions and functoriality. The fundamental axioms are **(CT.3)** and its dual, since they allow us to construct kernels and cokernels in the heart $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ with the following procedure: given a morphism $f \in \mathcal{H}$, take a pseudocokernel $C \in \mathcal{T}_1$ satisfying **(CT.3)**, then the kernel of f in $\mathcal{H}$ is the torsion-free part $f_2(C)$ of C with respect to $\mathcal{F}_2$. This result is stated in Theorem 3.9.

Observe that **(CT.3)** is essentially stating that there is a special pseudocokernel in $\mathcal{T}_1$, so we can regard pre-Abelian Nakaoka contexts as couples of torsion pairs with pseudokernels and pseudocokernels satisfying additional properties. In particular, since we already have a pre-Abelian heart $\mathcal{H}$, we want to give additional requirements for the pseudokernels and pseudocokernels in such a way that $\mathcal{H}$ becomes abelian.

In Theorem 3.15 we show that the abelianity of $\mathcal{H}$ is equivalent to the following axioms:

(CT.4) given a morphism $f\colon H \to H'$ in $\mathcal{H}$ that admits a pseudokernel $f^K\colon H'' \to H$ in $\mathcal{F}_2$, such that $H'' \in \mathcal{F}_1$, and the commutative diagram

$$\begin{tikzcd} & H \arrow[r, "f"] \arrow[d, "a"] & H' \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d, "\lambda_{2,T_1}"] \\ t_1F_2 \arrow[ur, dotted, "b"] \arrow[r, "\varepsilon_{1,F_2}"] & F_2 \arrow[r, "g^K"] & H' \arrow[r, "g"] & f_2T_1 \end{tikzcd}$$

where f^C is a pseudo-cokernel of f in $\mathcal{T}_1$ and g^K is a pseudo-kernel of g in $\mathcal{F}_2$, there exists a morphism $b\colon t_1F_2 \to H_1$ such that $ab = \varepsilon_{1,F_2}$;

(CT.4)* dual to **(CT.4)**.

After laying the theory, we study pre-Abelian Nakaoka contexts in special categories, namely Abelian and triangulated categories.

In the setting of Abelian categories we prove the following:

Theorem (3.18)**.** *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ Nakaoka context in an abelian category $\mathscr{A}$. Then, for $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ the following statements are equivalent:*

(a) $\mathcal{H}$ is an abelian category.

(b) The following conditions hold:

(b1) For any $f : H \to H'$ in $\mathcal{H}$, with $\mathrm{Ker}(f) \in \mathcal{F}_1$, we have that $\mathrm{Ker}(f) = 0$.

(b2) For any $f : H \to H'$ in $\mathcal{H}$, with $\mathrm{Coker}(f) \in \mathcal{T}_2$, we have that $\mathrm{Coker}(f) = 0$.

(b3) $\mathcal{H}$ is closed under kernels (resp. cokernels) of epimorphisms (resp. monomorphisms) in $\mathscr{A}$.

(c) $\mathcal{H}$ is closed under kernels and cokernels in $\mathscr{A}$.

In triangulated categories we restrict our attention to special Nakaoka contexts built from t-structures, that we call *related pairs*:

Definition (3.20)**.** Let $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_2)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ be two torsion pairs in a triangulated category $\mathcal{T}$. We will say that $\mathbb{t}$ is a *related* pair if $(\mathcal{T}_1, \mathcal{F}_1[1])$ and $(\mathcal{T}_2, \mathcal{F}_2[1])$ are t-structures and $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.

We show in Proposition 3.21 that any related pair is a pre-Abelian Nakaoka context. The reason why we restrict to related pairs is that by requiring that our torsion pairs are in fact t-structures we can compute the pseudokernels and pseudocokernels using cones and cocones. This allows us to give a much easier formulation of the axioms **(CT.4)** and **(CT.4)***, in particular Lemma 3.25 states that they are equivalent to the following:

Definition (3.22)**.** A related pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in the triangulated category $\mathcal{T}$ is *strong* if for any morphism $f : H_1 \to H_2$, in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$, and a distinguished triangle

$$V \to H_1 \xrightarrow{f} H_2 \to V[1],$$

the following conditions hold true:

(RST.1) $V \in \mathcal{F}_1$ implies $V \in \mathcal{F}_2[-1]$;

(RST.2) $V \in \mathcal{T}_2$ implies $V \in \mathcal{T}_1[1]$.

We will call such pairs *strongly related*.

We have an analogous of Theorem 3.18:

Theorem (3.30)**.** *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent.*

*(a) **(RST.1)** holds.*

(b) *For any monomorphism* $\alpha : H_1 \hookrightarrow H_2$*, in the abelian category* $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$*, with* $H_1, H_2 \in \mathcal{H}$*, we have that* $\mathrm{Coker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$*.*

(c) $\mathcal{H}$ *is closed under kernels and cokernels in the abelian category* $\mathcal{H}_1$

(d) $\mathcal{H}$ *is an abelian category.*

(e) *For any epimorphism* $H \twoheadrightarrow X$ *in* $\mathcal{H}_1$*, with* $H \in \mathcal{H}$*, we have that* $X \in \mathcal{H}$ *(i.e.* $\mathcal{H}$ *is closed under quotients in* $\mathcal{H}_1$*).*

From the Theorem above and the related proofs, it is clear that there is some relation between the heart $\mathcal{H}$ of a related pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, and the heart $\mathcal{H}_1 = \mathcal{T}_1 \cap \mathcal{F}_1[1]$ of the t-structure induced by $\mathbb{t}_1$. This relation is made clear in Theorem 3.33, where we show that there is a bijection between the following two classes:

$$RtAb(\mathcal{T}) := \{\text{related pairs } \mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2) \text{ in } \mathcal{T} \text{ s.t. } \mathcal{H}_{\mathbb{t}} \text{ is abelian}\};$$

$$\mathrm{t-}stCoh(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_1, \tau) \text{ s.t. } \bar{\mathbb{t}}_1 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1], \text{ where } \bar{\mathbb{t}}_1 = (\mathcal{T}_1, \mathcal{F}_1[1]) \end{array} \right\}.$$

1

COTORSION PAIRS AND MODEL STRUCTURES IN EXACT CATEGORIES

The notion of cotorsion pairs goes back to the seventies when it was introduced by Salce [42] in the case of abelian groups. It got an enormous impulse thanks to the discovery by Hovey [28] of the bijective correspondence between abelian model structures and cotorsion pairs in abelian categories. Many examples of cotorsion pairs and the corresponding model structures have been illustrated by Gillespie [19] who also extended the notion to the case of exact categories.

A famous example of cotorsion pair is given by the pair $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is the class of flat objects. It gave rise to the celebrated Flat Cover Conjecture by Enochs and solved in [9] in the case of module categories and in [12] for Grothendieck categories. It is particularly important in categories with no nonzero projective objects like for instance the categories of coherent sheaves.

This chapter is based on the joint work [4]. We will give an overview of the theory of cotorsion pairs and exact model structures in exact categories.

1.1 PRELIMINARIES

1.1.1 *Cotorsion pairs*

The notion of an *exact category* was introduced by Quillen in [40]. An exact category is an additive category $\mathcal{C}$ endowed with a collection Φ of kernel-cokernel pairs satisfying some axioms which allow to work with the sequences in Φ as if they were exact sequences in an abelian category. An element $E \in \Phi$ is denoted by $0 \to A \overset{i}{\to} B \overset{d}{\to} C \to 0$ and is called a *conflation or short exact sequence*. The map i is called *inflation* or *admissible monomorphism* and d is called *deflation* or *admissible epimorphism*. In an exact category pushouts (pullbacks) of inflations (deflations) exist and inflations (deflations) are stable under pushouts (pullbacks).

The axioms on conflations allow to define the Yoneda functor $\mathrm{Ext}^i_{\mathcal{C}}(M,N)$ for every pair of objects M,N in $\mathcal{C}$. For more details see [29] or [10].

We will deal with weakly idempotent complete (WIC) additive categories, that is categories such that every section has a cokernel or, equivalently, every retraction has a kernel.

Given a class $\mathcal{X}$ of objects in an exact category $\mathcal{C}$, the right orthogonal class $\mathcal{X}^{\perp}$ consists of the objects Y such that $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)=0$ for each object $X\in\mathcal{X}$. Similarly, the left orthogonal class ${}^{\perp}\mathcal{X}$ consists of the objects Y such that $\mathrm{Ext}^1_{\mathcal{C}}(Y,X)=0$ for each object $X\in\mathcal{X}$.

Definition 1.1. A pair of classes $(\mathcal{A},\mathcal{B})$ in an exact category $\mathcal{C}$ is called a cotorsion pair if

1. $\mathcal{A}^{\perp}=\mathcal{B}$ and ${}^{\perp}\mathcal{B}=\mathcal{A}$.

2. A cotorsion pair is *generated (cogenerated)* by a class $\mathcal{X}$ of objects if $\mathcal{B}=\mathcal{X}^{\perp}(\mathcal{A}={}^{\perp}\mathcal{X})$.

3. A cotorsion pair $(\mathcal{A},\mathcal{B})$ has *enough projectives* if every object $C\in\mathcal{C}$ has a special $\mathcal{A}$-precover, that is there is a short exact sequence $0\to B\to A\to C\to 0$ in $\mathcal{C}$ with $A\in\mathcal{A}$ and $\mathcal{B}\in B$. Dually, we say that $(\mathcal{A},\mathcal{B})$ has *enough injective* if every object $C\in\mathcal{C}$ has a special $\mathcal{B}$-preenvelope, that is there is a short exact sequence $0\to C\to B\to A\to 0$ in $\mathcal{C}$ with $A\in\mathcal{A}$ and $\mathcal{B}\in B$.

4. A cotorsion pair is *complete* when it has *enough injectives* and *enough projectives*.

5. A cotorsion pair is called *hereditary* if $\mathcal{A}$ is generating, $\mathcal{B}$ is cogenerating, and

$$\mathrm{Ext}^i_{\mathcal{C}}(A,B)=0 \text{ for all } A\in\mathcal{A},\, B\in\mathcal{B}, \text{ and } i\geq 1.$$

A class $\mathcal{C}$ of objects in an exact category is *deconstructible* and denoted by $\mathcal{F}ilt\ \mathcal{S}$, if there is a set $\mathcal{S}$ of objects such that every object of $\mathcal{C}$ is a transfinite extension of objects of $\mathcal{S}$ (for more details see [46, Definition 3.7 and 3.10]).

It is possible to prove, using the so called Small Object Argument, that any cotorsion pair $(\mathcal{A},\mathcal{B})$ generated by a set in a category of modules is complete (see [40] or [11]). The argument can be actually extended to Grothendieck categories, provided that $\mathcal{A}$ is generating. We give a precise statement in the following lemma.

Lemma 1.2. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck category such that $\mathcal{A}$ is generating. Then:*

1. *$\mathcal{A}$ is generated by a set if and only if it is deconstructible.*
2. *If the equivalent conditions in (1) hold, then $(\mathcal{A}, \mathcal{B})$ is complete.*

Proof. (1) If $\mathcal{A}$ is deconstructible, call $\mathcal{S}$ the set such that $\mathcal{A} = \mathcal{F}ilt\,\mathcal{S}$. Then, $\mathcal{F}ilt\,\mathcal{S} \subseteq {}^{\perp}(\mathcal{S}^{\perp})$ by Eklof's lemma, but ${}^{\perp}(\mathcal{S}^{\perp}) \subseteq \mathcal{A}$ so they are actually equal, i.e. $(\mathcal{A}, \mathcal{B})$ is generated by $\mathcal{S}$. Conversely, if $(\mathcal{A}, \mathcal{B})$ is generated by a set $\mathcal{S}$, it is also generated by $\mathcal{S}' = \mathcal{S} \cup \{G\}$, where $G \in \mathcal{A}$ is a generator. Then, by [46, Theorem 5.16] $\mathcal{A}$ consists of retracts of $\mathcal{F}ilt\mathcal{S}$, and by [45, Proposition 2.9(1)] it is deconstructible.

(2) [46, Theorem 5.16] actually gives a proof of this statement. □

We will mostly deal with hereditary cotorsion pairs and in order to characterize them we recall the following definition.

Definition 1.3. Let $\mathcal{C}'$ be a full subcategory of a WIC exact category $\mathcal{C}$.

1. $\mathcal{C}'$ is *thick* if it is closed under direct summands and has the 2 out of 3 property on short exact sequences.
2. $\mathcal{C}'$ is *resolving* in $\mathcal{C}$ if $A \in \mathcal{C}'$ for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ with $B, C \in \mathcal{C}'$.
3. $\mathcal{C}'$ is *coresolving* in $\mathcal{C}$ if $C \in \mathcal{C}'$ for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ with $A, B \in \mathcal{C}'$.

It can be shown that a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if $\mathcal{A}$ is resolving or, equivalently, if and only if $\mathcal{B}$ is coresolving:

Lemma 1.4. *[46, Lemma 6.17] Let $\mathcal{C}$ be a WIC exact category and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$ such that $\mathcal{A}$ is generating and $\mathcal{B}$ is cogenerating (e.g. if $(\mathcal{A}, \mathcal{B})$ is complete). Then the following are equivalent:*

1. $\mathrm{Ext}^n_{\mathcal{C}}(A, B) = 0$ *for each* $A \in \mathcal{A}$, $B \in \mathcal{B}$, *and* $n \geq 1$.
2. $\mathrm{Ext}^2_{\mathcal{C}}(A, B) = 0$ *for each* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$.
3. *$\mathcal{A}$ is resolving.*
4. *$\mathcal{B}$ is coresolving.*

1.1.2 *Model structures*

The notion of model structures on categories with finite limits and colimits was introduced by Quillen in [41]. For our purposes we refer to the book by Hovey [26] or to the survey [46].

We recall that a *model structure* on a category $\mathcal{C}$ consists of three classes of morphisms Cof, W, Fib called *cofibrations*, *weak equivalences* and *fibrations*, respectively, satisfying certain axioms.

Definition 1.5. Let Cof, W, Fib be a model structure on a category $\mathcal{C}$. An object $X \in \mathcal{C}$ is *cofibrant* if $0 \to X$ is a cofibration, *fibrant* if $X \to 0$ is a fibration and it is *trivial* if $0 \to X$ is a weak equivalence.

In particular, the class $\mathcal{W}$ of trivial objects has the 2-out-of-3 property. The *homotopy category* $\operatorname{Ho}\mathcal{C}$ is obtained by formally inverting all morphisms in W.

A tremendous impulse to the theory was given by Hovey who discovered in [28] a bijective correspondence between abelian model structures and cotorsion pairs in abelian categories. In [19] Gillespie extended the notion of model structures on exact categories and proved the analogous of Hovey's correspondence in this more general setting. We recall the basic notions and results.

Definition 1.6. An *exact model structure* on an exact category $\mathcal{C}$ is a model structure such that cofibrations (fibrations) are the inflations (deflations) with cofibrant (fibrant) cokernels (kernels).

An *abelian model structures* is an exact model structure on an abelian category considered as an exact category with the exact structure given precisely by the class of short exact sequences.

We explicitly state the aforementioned Hovey's correspondence, in the setting of WIC exact categories.

Theorem 1.7. *([28], [19]) Let $\mathcal{C}$ be a WIC exact category with an exact model structure. Let $\mathcal{Q}$ be the class of cofibrant objects, $\mathcal{R}$ the class of fibrant objects and $\mathcal{W}$ the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathcal{C}$, and $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in $\mathcal{C}$. Conversely, given three classes $\mathcal{W}, \mathcal{Q}, \mathcal{R}$ such that $\mathcal{W}$ is thick in $\mathcal{C}$, $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in $\mathcal{C}$, then there is an exact model structure on $\mathcal{C}$ where $\mathcal{Q}$ are the cofibrant objects, $\mathcal{R}$ are the fibrant objects and $\mathcal{W}$ the trivial objects.*

Given an exact model structure on $\mathcal{C}$, let $\mathcal{Q}$, $\mathcal{W}$, and $\mathcal{R}$ the three classes from the Theorem, we will follow the usual convention and indicate the model structure via $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ and call it a *Hovey triple*.

The main reason to use model structure is to be able to do homotopy theory, i.e. define good homotopy relations between maps and study the equivalence classes of morphisms modulo homotopy. The usual setting is that of a bicomplete category, however it is not necessary to assume that our category has all limits and colimits, and in fact an exact category has enough limits and colimits to be able to construct left and right homotopies from a model structure.

The following Proposition characterizes left and right homotopies in terms of cotorsion pairs.

Proposition 1.8. *[21, Proposition 2.5] Let $\mathcal{C}$ be an exact category with an exact model structure. Let $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R}$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ be the corresponding complete cotorsion pairs from Theorem 1.7.*

(1) Two maps $f, g : X \to Y$ in $\mathcal{C}$ are right homotopic *if and only if $g - f$ factors through a trivially cofibrant object, i.e. one in $\mathcal{Q} \cap \mathcal{W}$.*

(2) Two maps $f, g : X \to Y$ in $\mathcal{C}$ are left homotopic *if and only if $g - f$ factors through a trivially fibrant object, i.e. one in $\mathcal{W} \cap \mathcal{R}$.*

(3) Suppose Y is fibrant, i.e. $Y \in \mathcal{R}$. Then two maps $f, g : X \to Y$ in $\mathcal{C}$ are right homotopic if and only if $g - f$ factors through an object of $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$.

(4) Suppose X is cofibrant, i.e. $X \in \mathcal{Q}$. Then two maps $f, g : X \to Y$ in $\mathcal{C}$ are left homotopic if and only if $g - f$ factors through an object of $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$.

(5) Suppose X is cofibrant and Y is fibrant. Then two maps $f, g : X \to Y$ in $\mathcal{C}$ are homotopic if and only if $g - f$ factors through an object of $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$ if and only if $g - f$ factors through an object of $\mathcal{Q} \cap \mathcal{W}$ if and only if $g - f$ factors through an object of $\mathcal{R} \cap \mathcal{W}$.

For a WIC exact category $\mathcal{C}$, it is possible to show that the homotopy category of an exact model structure satisfies a universal property, namely that it is the triangulated localization of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of trivial objects. This was done explicitly in [21] in the context of injective model structure.

First, let's observe that the class of weak equivalence, and hence the homotopy category, depend only on the trivial objects.

Lemma 1.9. *[21, Lemma 3.1] Let $\mathcal{C}$ be a WIC exact category with an exact model structure, and let $\mathcal{W}$ denote the class of trivial objects. Then, a map f is a weak equivalence if and only if it factors as an admissible monomorphism with cokernel in $\mathcal{W}$ followed by an admissible epimorphism with kernel in $\mathcal{W}$.*

Hence, whenever we have a model structure $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ on $\mathcal{C}$ its homotopy category depends only on $\mathcal{W}$ and will be denoted as $\mathcal{C}/\mathcal{W}$, or, when it is important to remember the rest of the model structure, by $\mathrm{Ho}(\mathcal{M})$.

To be more explicit, we denote by $\gamma \colon \mathcal{C} \to \mathcal{C}/\mathcal{W}$ the canonical functor to the category $\mathcal{C}/\mathcal{W}$ where we formally inverted weak equivalences. It is a fundamental result of the theory of model categories that

$$\mathcal{C}/\mathcal{W} = \mathrm{Ho}(\mathcal{M}) \cong (\mathcal{Q} \cap \mathcal{R})/\sim$$

where $\sim$ denotes the equivalence relation in Proposition 1.8(5).

We will deal mainly with injective or projective model structure, so we will introduce them. The notions of injective and projective Hovey triples and of injective and projective cotorsion pairs have been introduced in [21] following the analogous concepts defined in [5] and [22].

Definition 1.10. Assume that a WIC exact category $\mathcal{C}$ has enough injective objects. A complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is an *injective cotorsion pair* if $\mathcal{A}$ is thick and contains the injective objects. Symmetrically, assume that $\mathcal{C}$ has enough projective objects. A complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is a *projective cotorsion pair* if $\mathcal{B}$ is thick and contains the projective objects.

Thus, an injective cotorsion pair $(\mathcal{A}, \mathcal{B})$ corresponds to the model structure $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ where all objects are cofibrant and a projective cotorsion pair $(\mathcal{A}, \mathcal{B})$ corresponds to the model structure $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ where all objects are fibrant.

Finally, we collect some important information about the localization functor γ in the following Proposition.

Proposition 1.11. *[21, Proposition 3.2] Let $\mathcal{M} = (\mathcal{W}, \mathcal{R})$ be an injective cotorsion pair in a WIC exact category $\mathcal{C}$ with enough injectives. Then:*

1. *$\mathcal{R}$ naturally inherits the structure of a Frobenius category with the projective-injective objects being precisely the injectives from $\mathcal{C}$.*

2. *The functor $\gamma\colon \mathcal{C} \to \mathcal{C}/\mathcal{W} = \mathrm{Ho}(\mathcal{M}) \cong \mathcal{R}/\sim$ is exact in the sense that it takes short exact sequences in $\mathcal{A}$ to exact triangles in $\mathrm{Ho}(\mathcal{M})$.*

3. *γ is universal among triangulated categories $\mathcal{T}$ which "kill" $\mathcal{W}$. That is, given another exact functor $F\colon \mathcal{C} \to \mathcal{T}$ with $F(\mathcal{W}) = 0$, it factors uniquely through γ.*

1.1.3 *Recollements*

A recollement is a functor diagram among triangulated categories summarizing many properties of the functors involved. They were introduced in the seminal paper [7] by Beilinson, Bernstein, and Deligne.

Definition 1.12. Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be triangulated categories. A *recollement* is a diagram of functors

$$\begin{tikzcd} \mathscr{X} \arrow[r, "i_*"] & \mathscr{Y} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{Z} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs;

(2) i_*, j_*, and $j_!$ are full embeddings;

(3) $i^! \circ j_*$ (hence, $j^* \circ i_* = 0$ and $i^* \circ j_! = 0$ too);

(4) for each $Y \in \mathscr{Y}$ there are triangles

$$i_*i^!Y \to Y \to j_*j^*Y \to i_*i^!Y[1]$$

$$j_*j^*Y \to Y \to i_*i^*Y \to j_*j^*Y[1]$$

In the case of multiple injective cotorsion pairs on a WIC exact category it is possible to define functors that give rise to a recollement.

Lemma 1.13. *[21, Lemma 3.3]*
Let $\mathcal{C}$ be a WIC exact category with enough injectives and suppose we have injective cotorsion pairs $\mathcal{M} = (\mathcal{W}, \mathcal{R})$ and $\mathcal{M}' = (\mathcal{W}', \mathcal{R}')$ with $\mathcal{R}' \subseteq \mathcal{R}$, and indicate with $\gamma_{\mathcal{W}}$ and $\gamma_{\mathcal{W}'}$ the localization functors associated with the respective model structures. Then, the quotient functor $Q : \mathcal{R}/\sim \to \mathcal{C}/\mathcal{W}'$ defined by $Q([f]) = \gamma_{\mathcal{W}'}(f)$ is well defined

We have the following *Injective Recollement Theorem*:

Theorem 1.14. *[21, Theorem 3.4] Let $\mathcal{C}$ be a WIC exact category with enough injective and suppose we have three injective cotorsion pairs*

$$\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{R}_1),\ \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{R}_2),\ \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{R}_3)$$

such that $\mathcal{R}_2, \mathcal{R}_3 \subseteq \mathcal{R}_1$. If $\mathcal{W}_3 \cap \mathcal{R}_1 = \mathcal{R}_2$ (or equivalently $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$, and $\mathcal{R}_2 \subseteq \mathcal{W}_3$), then there exists a recollement

$$\begin{tikzcd} \mathcal{R}_2/\sim \arrow[r, "I"] & \mathcal{R}_1/\sim \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{C}/\mathcal{W}_3 \arrow[l, bend right, "\lambda"'] \arrow[l, bend left, "\rho"] \end{tikzcd}$$

where the functor I is simply the inclusion and Q is the quotient functor of Lemma 1.13. Moreover, λ has essential image $(\mathcal{W} \cap \mathcal{R}_1)/\sim$, ρ has essential image $\mathcal{R}_3/\sim$, and they provide an equivalence

$$\lambda\colon \mathcal{R}_3/\sim \longleftrightarrow (\mathcal{W}_2 \cap \mathcal{R}_1)/\sim\colon \rho.$$

Injective (and projective) cotorsion pairs will turn out to be extremely powerful in the case of Frobenius categories, especially in order to find recollements as we will see at the end of this section.

Definition 1.15. ([21, Definition 4.3]) Let $\mathcal{C}$ be a WIC Frobenius category. An injective complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ is called a *localizing cotorsion pair*. If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{D})$ are injective cotorsion pairs in $\mathcal{C}$, then $(\mathcal{A}, \mathcal{B}, \mathcal{D})$ is called a *localizing cotorsion triple* in $\mathcal{C}$.

Of course, the fact that $\mathcal{C}$ is Frobenius can be used to find several descriptions of a localizing cotorsion pair. These are listed in [21, Proposition 4.2], that we state here for the reader's convenience, but first we recall the definition of syzygy and cosyzygy.

Definition 1.16. In any Frobenius category $\mathcal{C}$, the formal suspension of an object $X \in \mathcal{C}$ is an object ΣX such that there is a short exact sequence

$$0 \to X \to W \to \Sigma X \to 0$$

where W is injective. ΣX is called the cosyzygy of X and is unique up to a canonical isomorphism in the stable category.

Proposition 1.17. *[21, Proposition 4.2] Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a WIC Frobenius category $\mathcal{C}$. Then, the following are equivalent:*

1. *$(\mathcal{A}, \mathcal{B})$ is hereditary with $\mathcal{A}$ cosyzygy closed and $\mathcal{B}$ syzygy closed,*
2. *$\mathcal{A}$ is both syzygy and cosyzygy closed,*
3. *$\mathcal{B}$ is both syzygy and cosyzygy closed,*
4. *$\mathcal{A}$ is thick,*
5. *$\mathcal{B}$ is thick.*

Moreover, if $(\mathcal{A}, \mathcal{B})$ is complete then the conditions above are also equivalent to:

6. *$(\mathcal{A}, \mathcal{B})$ is an injective cotorsion pair,*
7. *$(\mathcal{A}, \mathcal{B})$ is a projective cotorsion pair.*

Remark 1.18. In any WIC Frobenius category there is a canonical cotorsion pair $\mathcal{M} = (\mathcal{W}, \mathcal{C})$, where $\mathcal{W}$ is the class of projective-injective objects. $\mathcal{M}$ is localizing, and when regarded as a projective cotorsion pair it yields the trivial model structure $(\mathcal{W}, \mathcal{C}, \mathcal{C})$. However, it can also be considered an injective cotorsion pair, yielding a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{C})$ whose homotopy category is the stable category $\mathcal{C}/\sim$ of $\mathcal{C}$.

Finally, if we have a localizing cotorsion triple in a WIC Frobenius categories we get a recollement via the following:

Corollary 1.19. *[21, Corollary 4.5] Let $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a localizing cotorsion triple in a WIC Frobenius category $\mathcal{C}$. Then, there is an equivalence of triangulated categories:*

$$\mathcal{X}/\sim \cong \mathcal{C}/\mathcal{Y} \cong \mathcal{Z}/\sim$$

where $\mathcal{X}/\sim$ and $\mathcal{Z}/\sim$ are the images of $\mathcal{X}$ and $\mathcal{Z}$ in the stable category $\mathcal{C}/\sim$.

Moreover, applying Theorem 1.14 to the injective cotorsion pairs

$$\mathcal{M}_1 = (\mathcal{W}, \mathcal{C}),\ \mathcal{M}_2 = (\mathcal{X}, \mathcal{Y}),\ \text{and}\ \mathcal{M}_3 = (\mathcal{Y}, \mathcal{X})$$

where $\mathcal{M}_1$ is the canonical localizing cotorsion pair, yields a recollement

$$\begin{tikzcd}
\mathcal{Y}/\sim \arrow[r] & \mathcal{C}/\sim \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{C}/\mathcal{Y} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}.$$

1.2 MORE ON COTORSION PAIRS

We denote by $\mathrm{Ch}(\mathcal{C})$ the category of cochain complexes X with component $X^n \in \mathcal{C}$ in degree n and with differential $d_X^n \colon X^n \to X^{n+1}$ for every $n \in \mathbb{Z}$. The morphisms in $\mathrm{Ch}(\mathcal{C})$ are the usual cochain maps. The suspension is denoted by $[-]$. If $\mathcal{C}$ is an exact category, then $\mathrm{Ch}(\mathcal{C})$ is equipped with the exact structure where the short exact sequences are the sequences which are exact in each degree. We can also consider the exact structure on $\mathrm{Ch}(\mathcal{C})$ where the short exact sequences are degreewise splitting. $\mathrm{Ext}_{dw}(X, Y)$ denotes the Yoneda group of these degreewise splitting sequences.

For every object $C \in \mathcal{C}$, $S^n(C)$ denotes the complex with entries 0 for every $i \neq n$ and with C in degree n; $D^n(C)$ denotes the complex with C in degrees n and $n+1$ and 0 elsewhere and with differential d^n being the identity on C. The homotopy category $\mathcal{K}(\mathcal{C})$ has the same objects as $\mathrm{Ch}(\mathcal{C})$ and the equivalence classes of cochain maps under the homotopy relation as morphisms.

Given two complexes X and Y, the complex $\mathcal{H}om(X, Y)$ is defined as the complex of abelian groups having

$$\prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^p, Y^{n+p})$$

in degree n and with differential $d_H(f) = d_Y \circ f - (-1)^n f \circ d_X$. The n^{th}-cohomology of $\mathcal{H}om(X, Y)$ is given by $\mathrm{Hom}_{\mathcal{K}(C)}(X, Y[n])$.

We recall the useful and important formula

$$(*) \qquad \mathrm{Ext}^1_{dw}(X, Y) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{C})}(X, Y[1]).$$

Notation 1.20. (Following Gillespie's notations) Let $\mathcal{A}$ be a class of objects in an abelian category $\mathcal{C}$. Define the following classes of cochain complexes in $\mathrm{Ch}(\mathcal{C})$:

- $dw\mathcal{A}$ is the class of all complexes $X \in \mathrm{Ch}(\mathcal{C})$ such that $X^n \in \mathcal{A}$ for all $n \in \mathbb{Z}$. $\mathrm{Ch}(\mathcal{A})$ will denote the full subcategory of $\mathrm{Ch}(\mathcal{C})$ with objects in $dw\mathcal{A}$.
- $ex\mathcal{A}$ is the class of all acyclic complexes in $dw\mathcal{A}$.
- $\tilde{\mathcal{A}}$ is the class of all complexes X in $ex\mathcal{A}$ with the cycles $Z^n(X)$ in $\mathcal{A}$ for all $n \in \mathbb{Z}$. $\mathrm{Ch}_{ac}(\mathcal{A})$ will denote the full subcategory of $\mathrm{Ch}(\mathcal{C})$ with objects in $\tilde{\mathcal{A}}$.
- If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, then:

  $dg\mathcal{A}$ is the class of all complexes $X \in dw\mathcal{A}$ such that any morphism $f : X \to Y$ with $Y \in \tilde{\mathcal{B}}$ is null homotopic. Since $\mathrm{Ext}^1_{\mathcal{C}}(A^n, B^n) = 0$ for every $n \in \mathbb{Z}$ formula $(*)$ shows that $dg\mathcal{A} = {}^{\perp}\tilde{\mathcal{B}}$.

  Similarly, $dg\mathcal{B}$ is the class of all complexes $Y \in dw\mathcal{B}$ such that any morphism $f : X \to Y$ with $X \in \tilde{\mathcal{A}}$ is null homotopic. Hence $dg\mathcal{B} = \tilde{\mathcal{A}}^{\perp}$.

Lemma 1.21. *Let $\mathcal{C}$ be an abelian category and let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of complexes in $\mathrm{Ch}(\mathcal{C})$ with the degreewise exact structure. For every $A \in \mathrm{Ch}(\mathcal{C})$ the sequence:*

$$0 \to \mathcal{H}om(A, X) \to \mathcal{H}om(A, Y) \to \mathcal{H}om(A, Z)$$

is an exact sequence of complexes in $\mathrm{Ch}(\mathbb{Z})$ and it is also right exact provided that $\mathrm{Ext}_{\mathcal{C}}(A^n, X^n) = 0$ for all $n \in \mathbb{Z}$. Dually, for every $B \in \mathrm{Ch}(\mathcal{C})$ the sequence:

$$0 \to \mathcal{H}om(Z, B) \to \mathcal{H}om(Y, B) \to \mathcal{H}om(X, B)$$

is an exact sequence of complexes in $\mathrm{Ch}(\mathbb{Z})$ and it is also right exact provided that $\mathrm{Ext}_{\mathcal{C}}(Z^n, B^n) = 0$ for all $n \in \mathbb{Z}$.

Proof. Immediate from the definition of the complex $\mathcal{H}om$. □

1.2.1 *Hereditary cotorsion pairs in Grothendieck categories*

We recall some results which will be used throughout. Their proof can be found in [45], [46], [18], [22].

Proposition 1.22. *([46, Proposition 7.13, 7.14] Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in an abelian category $\mathcal{C}$. The following hold true*

1. *A complex Y belongs to $\tilde{\mathcal{B}}$ if and only if $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^n(A), Y) = 0$ for every $n \in \mathbb{Z}$ and every $A \in \mathcal{A}$.*

2. *A complex X belongs to $\tilde{\mathcal{A}}$ if and only if $\operatorname{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, S^n(B)) = 0$ for every $n \in \mathbb{Z}$ and every $B \in \mathcal{B}$.*

3. *If $\mathcal{C}$ is a Grothendieck category and $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair, then $(\tilde{\mathcal{A}}, dg\mathcal{B})$ and $(dg\mathcal{A}, \tilde{\mathcal{B}})$ are complete hereditary cotorsion pairs in $\mathrm{Ch}(\mathcal{C})$.*

4. *$(dg\mathcal{A}, \mathcal{E}, dg\mathcal{B})$ is a model structure on $\mathrm{Ch}(\mathcal{C})$ with the acyclic complexes $\mathcal{E}$ as trivial objects. In particular, $dg\mathcal{A} \cap \mathcal{E} = \tilde{\mathcal{A}}$ and $dg\mathcal{B} \cap \mathcal{E} = \tilde{\mathcal{B}}$.*

Proof. (1) and (2) are proved in [46, Lemma 7.13]. (3) is proved in [46, Proposition 7.14]. (4) follows by (3) and by Hovey's correspondence (see Theorem 1.7). □

Proposition 1.23. *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in an Grothendieck category $\mathcal{G}$.*

1. *$(dw\mathcal{A}, dw\mathcal{A}^{\perp})$ and $({}^{\perp}dw\mathcal{B}, dw\mathcal{B})$ are cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$.*

2. *If $(\mathcal{A}, \mathcal{B})$ is generated by a set, then so is $({}^{\perp}dw\mathcal{B}, dw\mathcal{B})$.*

3. *If $(\mathcal{A}, \mathcal{B})$ is generated by a set, then so is $(dw\mathcal{A}, dw\mathcal{A}^{\perp})$.*

4. *If $\mathcal{A}$ contains a generator of $\mathcal{G}$ with finite projective dimension, then $({}^{\perp}ex\mathcal{B}, ex\mathcal{B})$ is a cotorsion pair in $\mathrm{Ch}(\mathcal{G})$. If moreover, $(\mathcal{A}, \mathcal{B})$ is generated by a set, then so is $({}^{\perp}ex\mathcal{B}, ex\mathcal{B})$.*

5. *$(ex\mathcal{A}, ex\mathcal{A}^{\perp})$ is a cotorsion pair. Moreover, if $(\mathcal{A}, \mathcal{B})$ is generated by a set, then so is $(ex\mathcal{A}, ex\mathcal{A}^{\perp})$.*

Proof. (1) is proved in [18, Proposition 3.2],

(2) is proved in [18, Proposition 4.4].

(3) is proved as follows: by Lemma 1.2, $\mathcal{A}$ is deconstructible and by [45, Theorem 4.2] so is $dw\mathcal{A}$. Moreover, $dw\mathcal{A}$ contains a generator, so $(dw\mathcal{A}, dw\mathcal{A}^{\perp})$ is generated by a set by Lemma 1.2.

The first part of (4) is proved in [18, Proposition 3.3]; the second part in [18, Proposition 4.6].

The first part of (5) is again proved in [18, Proposition 3.3]; for the second part we argue as in the proof of [22, Proposition 7.3]. $ex\mathcal{A} = dw\mathcal{A} \cap \mathcal{E}$, where $\mathcal{E}$ is the class of acyclic complexes. By [45, Theorem 4.2] $\mathcal{E}$ and $dw\mathcal{A}$ are deconstructible, hence $ex\mathcal{A}$ is deconstructible by [45, Proposition 2.9]. Moreover, $ex\mathcal{A}$ contains a generator, so $(ex\mathcal{A}, ex\mathcal{A}^{\perp})$ is generated by a set by Lemma 1.2. □

Remark 1.24. If $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in an abelian category, then the complete cotorsion pairs defined in the above proposition are hereditary, too.

Lemma 1.25. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pairs in a Grothendieck category $\mathcal{G}$. Then $ex\mathcal{B} = \tilde{\mathcal{B}}$ if and only if $dw\mathcal{B} = dg\mathcal{B}$. Dually $ex\mathcal{A} = \tilde{\mathcal{A}}$ if and only if $dw\mathcal{A} = dg\mathcal{A}$*

Proof. Assume that $ex\mathcal{B} = \tilde{\mathcal{B}}$ and let $Y \in dw\mathcal{B}$. We have to show that $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, Y) = 0$ for every $X \in \tilde{\mathcal{A}}$. Equivalently we have to show that the complex $\mathcal{H}om(X, Y)$ is exact for every $X \in \tilde{\mathcal{A}}$. Since, $(\tilde{\mathcal{A}}, dg\mathcal{B})$ is a complete cotorsion pair in $\mathrm{Ch}(\mathcal{C})$ there is a short exact sequence

$$0 \to Y \to Z \to V \to 0$$

with $Z \in dg\mathcal{B}$ and $V \in \tilde{\mathcal{A}}$. Now, $\mathcal{B}$ is coresolving, hence $V \in \tilde{\mathcal{A}} \cap dw\mathcal{B} = \tilde{\mathcal{A}} \cap ex\mathcal{B}$ and the last is $\tilde{\mathcal{A}} \cap \tilde{\mathcal{B}}$ by assumption. Thus, V is contractible, hence null homotopic. By Lemma 1.21 we have a short exact sequence

$$0 \to \mathcal{H}om(X, Y) \to \mathcal{H}om(X, Z) \to \mathcal{H}om(X, V) \to 0$$

for every $X \in \tilde{\mathcal{A}}$. The second and the third nonzero terms are exact, hence also $\mathcal{H}om(X, Y)$ is exact.

Conversely, assume that $dw\mathcal{B} = dg\mathcal{B}$ and let $Y \in ex\mathcal{B}$. Then $Y \in dw\mathcal{B} \cap \mathcal{E} = dg\mathcal{B} \cap \mathcal{E}$ and by Proposition 1.22 (4), $Y \in \tilde{\mathcal{B}}$.

The dual statement is proved in similar ways. □

1.2.2 *Cotorsion pairs $(\mathcal{A}, \mathcal{B})$ satisfying $ex\mathcal{B} = \tilde{\mathcal{B}}$*

We are interested in describing cotorsion pairs $(\mathcal{A}, \mathcal{B})$ such that $ex\mathcal{B} = \tilde{\mathcal{B}}$ or $ex\mathcal{A} = \tilde{\mathcal{A}}$, since in these cases we have the following important consequences on the corresponding model structures.

Corollary 1.26. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pairs in a Grothendieck category $\mathcal{G}$. The following hold true:*

1. *If $ex\mathcal{B} = \tilde{\mathcal{B}}$, then $(dg\mathcal{A}, \mathcal{E}, dw\mathcal{B})$ is a model structure in $\mathrm{Ch}(\mathcal{G})$ for which the fibrant objects are exactly the complexes with components in $\mathcal{B}$.*

2. *If $ex\mathcal{A} = \tilde{\mathcal{A}}$, then $(dw\mathcal{A}, \mathcal{E}, dg\mathcal{B})$ is a model structure in $\mathrm{Ch}(\mathcal{G})$ for which the cofibrant objects are exactly the complexes with components in $\mathcal{A}$.*

Proof. Follows by Proposition 1.22 (4) and by Lemma 1.25. □

We say that an object M in a Grothendieck category $\mathcal{G}$ has projective dimension at most n if $\mathrm{Ext}^i_{\mathcal{G}}(M,-)$ vanishes for every $i > n$ and we denote by $\mathcal{P}_n$ the class of objects of projective dimension at most n. Analogously, M has injective dimension at most n if $\mathrm{Ext}^i_{\mathcal{G}}(-,M)$ vanishes for every $i > n$ and we denote by $\mathcal{I}_n$ the class of objects of injective dimension at most n. We denote by $\mathcal{P} = \bigcup_n \mathcal{P}_n$ the class of objects with finite projective dimension and by and $\mathcal{I} = \bigcup_n \mathcal{I}_n$ the class of objects with finite injective dimension.

Proposition 1.27. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$ and let Y be an acyclic complex with terms in $\mathcal{B}$. The following hold true:*

1. *If M is an object in $\mathcal{A}$ with finite projective dimension, then the cycles $Z^j(Y)$ of Y belong to $M^{\perp}$.*
2. *If $\mathcal{A} \subseteq \mathcal{P}$, then $Y \in \tilde{\mathcal{B}}$, hence $ex\mathcal{B} = \tilde{\mathcal{B}}$.*

   In particular, in the abelian model structure corresponding to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ by Corollary 1.26 (1), $dw\mathcal{B}$ is the class of fibrant objects.

Dually, let X be an acyclic complex with terms in $\mathcal{A}$. Then:

(3) *If N is an object in $\mathcal{B}$ with finite injective dimension, then the cycles $Z^j(X)$ of X belong to ${}^{\perp}N$.*

(4) *If $\mathcal{B} \subseteq \mathcal{I}$, then $X \in \tilde{\mathcal{A}}$, hence $ex\mathcal{A} = \tilde{\mathcal{A}}$.*

In particular, in the abelian model structure corresponding to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ by Corollary 1.26 (2), $dw\mathcal{A}$ is the class of cofibrant objects.

Proof. (1) Clearly it is enough to verify that the 0-cycle Z^0 of Y is in $M^{\perp}$. Consider the exact complex

$$\dots Y^{-n} \to \dots \to Y^{-2} \to Y^{-1} \to Z^0 \to 0.$$

If M is in $\mathcal{A}$, then $\mathrm{Ext}^j_{\mathcal{G}}(M, Y^n) = 0$ for every $n \in \mathbb{Z}$ and every $j \geq 1$. A dimension shifting argument gives $\mathrm{Ext}^i_{\mathcal{G}}(M, Z^0) \cong \mathrm{Ext}^{i+k}_{\mathcal{G}}(M, Z^{-k})$, for every $k \geq 1$. Hence by the finiteness of the projective dimension of M we conclude that $\mathrm{Ext}^i_{\mathcal{G}}(M, Z^0) = 0$ for every $i \geq 1$.

(2) The first statement follows by (1). The second statement follows by Corollary 1.26.

The proof of the dual statement is obtained by considering the acyclic complex:

$$0 \to Z^0 \to X^0 \to X^1 \to \cdots \to X^n \to \ldots$$

and using dimension shifting for the functor $\mathrm{Hom}_{\mathcal{G}}(-, N)$.

□

We consider now the particular case of a module category and we exhibit some situations in which the assumptions of the previous proposition are satisfied.

Recall that T is an n-tilting R-module if it has projective dimension at most n, $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every cardinal λ and every $i \geq 0$, and the ring R has a finite coresolution with terms in $\mathrm{Add}T$, where $\mathrm{Add}T$ denotes the class of direct summands of direct sums of copies of T. The cotorsion pair generated by T is called n-tilting cotorsion pair.

Dually, an R-module C is n-cotilting if it has injective dimension at most n, $\mathrm{Ext}^i_R(C^\lambda, C) = 0$ for every cardinal λ and every $i \geq 0$, and an injective cogenerator has a finite resolution with terms in $\mathrm{Prod}C$, where $\mathrm{Prod}C$ denotes the class of direct summands of direct products of copies of C. The cotorsion pair cogenerated by C is called n-cotilting cotorsion pair.

Proposition 1.28. *If $(\mathcal{A}, \mathcal{B})$ is an n-tilting cotorsion pair in* Mod-R*, then $ex\mathcal{B} = \tilde{\mathcal{B}}$ and $dw\mathcal{B} = dg\mathcal{B}$. Hence there is a model structure in* $\mathrm{Ch}(R)$ *in which the fibrant objects are the complexes with components in the n-tilting class $\mathcal{B}$ and the trivial objects are the acyclic complexes.*

Dually, if $(\mathcal{A}, \mathcal{B})$ is an n-cotilting cotorsion pair in Mod-R*, then $ex\mathcal{A} = \tilde{\mathcal{A}}$ and $dw\mathcal{A} = dg\mathcal{A}$. Hence there is a model structure in* $\mathrm{Ch}(R)$ *in which the cofibrant objects are the complexes with components in the n-cotilting class $\mathcal{A}$ and the trivial objects are the acyclic complexes.*

Proof. If $(\mathcal{A}, \mathcal{B})$ is a tilting (cotilting) cotorsion pair, then $\mathcal{A} \subseteq \mathcal{P}_n$ ($\mathcal{B} \subseteq \mathcal{I}_n$), by [24, Lemmas 13.10, 15.4]. Hence the conclusion follows by Proposition 1.27.

□

To exhibit other examples of cotorsion pairs $(\mathcal{A}, \mathcal{B})$ satisfying the condition $ex\mathcal{B} = \tilde{\mathcal{B}}$ we use the notion of the closure of a cotorsion pair.

Recall that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *closed* if $\mathcal{A}$ is closed under direct limits. Consider the lattice of cotorsion pairs, with respect to inclusion on the left component. Since the cotorsion pair $(\text{Mod-}R, \text{Inj})$ is closed and the meet of closed cotorsion pairs is closed (see e.g. [1] or [22, Lemma 6.1]), every cotorsion pair $(\mathcal{A}, \mathcal{B})$ is contained in a smallest closed cotorsion pair, called the closure of $(\mathcal{A}, \mathcal{B})$

Notation 1.29. Let R be a ring.

1. We denote by mod-R the class of modules M admitting a projective resolution of the form

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

with P_j finitely generated for every $j \geq 0$.

2. For every $n \geq 0$, denote by $\mathcal{P}_n(\text{mod-}R)$ the class $\mathcal{P}_n \cap \text{mod-}R$ and by $\mathcal{P}(\text{mod-}R)$ the class $\mathcal{P} \cap \text{mod-}R$.

3. The *little finitistic dimension* of R is the supremum of the projective dimension of modules in mod-R having finite projective dimension.

4. The *big projective (flat) finitistic dimension* of R is the supremum of the projective (flat) dimension of modules having finite projective (flat) dimension.

5. Denote by $(\mathcal{A}^{\omega}, \mathcal{B}_{\omega})$ the complete hereditary cotorsion pair generated by $\mathcal{P}(\text{mod-}R)$. By [1, Theorem 2.3, Corollary 2.4], its closure

$$(\mathcal{A}^{\infty}, \mathcal{B}_{\infty})$$

is a complete cotorsion pair cogenerated by the class of pure injective modules belonging to $\mathcal{B}_{\omega}$, hence it is hereditary, since cosyzygies of pure injective modules of $\mathcal{B}_{\omega}$ are in $\mathcal{B}_{\omega}$. Moreover, $\mathcal{A}^{\infty} = \varinjlim \mathcal{A}^{\omega} = \varinjlim \mathcal{P}(\text{mod-}R)$ and it is closed under pure epimorphic images.

Remark 1.30.

1. Since $\varinjlim \mathcal{P}_0(\text{mod-}R)$ is the class of flat modules, $\mathcal{A}^{\infty}$ contains all flat modules and it coincides with the class of flat modules if and only if every module in $\mathcal{P}_n(\text{mod-}R)$ is projective, i.e. if the little finitistic dimension of R is 0.

2. By part (1), $\mathcal{B}_{\infty}$ is contained in the class of cotorsion modules and it is properly contained in it whenever the little finitistic dimension of R is greater than 0.

3. Moreover, $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1(\text{mod-}R)$, hence $\mathcal{P}_1 \subseteq \mathcal{A}^{\infty}$.

4. By [3, Theorem 6.7 (vi)], if R has a classical ring of quotients Q such that Q is Von Neumann regular or has big finitistic flat dimension 0, then $\varinjlim \mathcal{P}_1$ coincides with the class $\mathcal{F}_1$ of modules of flat dimension at most 1. Hence $\mathcal{A}^\infty$ contains $\mathcal{F}_1$ and $\mathcal{B}_\infty$ is contained in the class ${\mathcal{F}_1}^\perp$ which is also called the class of weakly injective modules (see [14] and [15]). In particular, this applies to any commutative ring such that the total quotient ring is a perfect ring or a Von Neumann regular ring.

Proposition 1.31. *Let R be a (coherent) ring. The class $\mathcal{B}_\infty$ coincides with the class of injective right R-modules if and only if every module in* mod-R *(every finitely presented module) has finite projective dimension. In particular, this applies to rings with finite little finitistic dimension and thus to right semihereditary rings.*

Proof. $\mathcal{B}_\infty$ coincides with the class of injectives if and only if $\mathcal{A}^\infty = \text{Mod-}R$. If every module in mod-R has finite projective dimension, then $\mathcal{A}^\infty = \text{Mod-}R$, since $\mathcal{A}^\infty$ is closed under direct limits. Conversely, if $\mathcal{A}^\infty = \text{Mod-}R$, then every finitely presented right module X belongs to $\varinjlim \mathcal{P}(\text{mod-}R)$, hence it is a summand of a finite direct sum of modules in $\mathcal{P}(\text{mod-}R)$. Thus X has finite projective dimension and so does every module in mod-R.

The last statement follows easily. In particular, if R is right semihereditary, then every finitely presented right R-module has projective dimension at most one. □

We show now that $ex\mathcal{B}_\infty = \widetilde{\mathcal{B}_\infty}$. To this aim we apply the results proved in a recent paper [2] about periodic modules. Recall that a module M is periodic with respect to a class $\mathcal{C}$ if there exists a short exact sequence $0 \to M \to C \to M \to 0$ with $C \in \mathcal{C}$. A module M is Fp-injective if $\text{Ext}^1_R(X, M) = 0$ for every finitely presented module X.

Fact 1.32.

1. *[2, Proposition 3.8 (1)] every Fp-injective* Inj-*periodic module is injective.*

2. *[13] If $\mathcal{C}$ is a class closed under direct sums or direct products and $\mathcal{D}$ is a class closed under direct summands, then the following are equivalent:*

   (a) Every cycle of an acyclic complex with components in $\mathcal{C}$ belongs to $\mathcal{D}$.

   (b) Every $\mathcal{C}$-periodic module belongs to $\mathcal{D}$.

Proposition 1.33. *The cotorsion pair* $(\mathcal{A}^\infty, \mathcal{B}_\infty)$ *from Notation 1.29 (4) satisfies* $ex\mathcal{B}_\infty = \widetilde{\mathcal{B}_\infty}$.

Proof. Let M be a $\mathcal{B}_\infty$-periodic module. By [2, Lemma 3.4] ${}^\perp M \supseteq \mathcal{P}(\text{mod-}R)$. As mentioned in Notation 1.29 (4), the class $\mathcal{A}^\infty$ coincides with $\varinjlim \mathcal{P}(\text{mod-}R)$ and is closed under pure epimorphic images. By [2, Theorem 3.7] ${}^\perp M \supseteq \mathcal{A}^\infty$, hence $M \in \mathcal{B}_\infty$. By Fact 1.32 (2), $ex\mathcal{B}_\infty = \widetilde{\mathcal{B}_\infty}$ in $\mathrm{Ch}(R)$.

□

As a corollary we get an improvement of [44, Corollary 5.9] in the case of a module category, since $\mathcal{B}_\infty$ is in general properly contained in the class of cotorsion modules.

Corollary 1.34. *Let* Y *be an acyclic complex with injective components. Then every cycle of* Y *belongs to* $\mathcal{B}_\infty$*, hence* $Y \in \widetilde{\mathcal{B}_\infty}$.

Proof. By assumption $Y \in ex\mathcal{B}_\infty$, hence the conclusion follows by Proposition 1.33. □

The next properties will be used in Section 2.3.

Lemma 1.35. *Let* $(\mathcal{A}, \mathcal{B})$ *be a complete hereditary cotorsion pair in a Grothendieck category* $\mathcal{G}$*. Let* Inj *denote the class of injective objects of* $\mathcal{G}$*. The following hold true:*

(1) $\mathcal{B}^\perp \cap \mathcal{B} \subseteq \mathrm{Inj}$ *and* $\widetilde{\mathcal{B}}^\perp \cap dw\mathcal{B} \subseteq dw\mathrm{Inj}$.

(2) $\widetilde{\mathcal{B}}^\perp \cap dw\mathcal{B} = dw\mathrm{Inj}$ *if and only if* $\tilde{\mathcal{B}} \subseteq {}^\perp dw\mathrm{Inj}$.

Moreover, if $\mathcal{G} = \text{Mod-}R$ *and* $\mathcal{B}$ *contains the class* $\mathcal{B}_\infty$ *defined in Notation 1.29 (5) then*

(3) $\widetilde{\mathcal{B}}^\perp \cap dw\mathcal{B} = dg\mathrm{Inj}$.

(4) ${}^\perp dg\mathrm{Inj} \cap dw\mathcal{B} = ex\mathcal{B}$

Proof. (1) Let $B \in \mathcal{B}^\perp \cap \mathcal{B}$ and consider an exact sequence $0 \to B \to I \to I/B \to 0$ with $I \in \mathrm{Inj}$. Then $I/B \in \mathcal{B}$, since $\mathcal{B}$ is coresolving, hence the sequence splits and B is injective.

If $B \in \mathcal{B}$, then $D^n(B) \in \widetilde{\mathcal{B}}$ for every $n \in \mathbb{Z}$ and by [17, Lemma 3.1], $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{G})}(D^n(B), Y) \cong \mathrm{Ext}^1_{\mathcal{G}}(B, Y^n)$, for every complex Y. Thus if $Y \in \widetilde{\mathcal{B}}^\perp \cap dw\mathcal{B}$, then $Y^n \in \mathcal{B}^\perp \cap \mathcal{B}$ for every $n \in \mathbb{Z}$. By the above we conclude that $Y \in dw\mathrm{Inj}$.

(2) If $\tilde{\mathcal{B}} \subseteq {}^\perp dw\mathrm{Inj}$, then $\tilde{\mathcal{B}}^\perp \supseteq ({}^\perp dw\mathrm{Inj})^\perp = dw\mathrm{Inj}$, by [18, Proposition 4.4], hence by part (1) $\tilde{\mathcal{B}}^\perp \cap dw\mathcal{B} = dw\mathrm{Inj}$.

Conversely, if $\widetilde{\mathcal{B}}^\perp \cap dw\mathcal{B} = dw\mathrm{Inj}$, then $dw\mathrm{Inj} \subseteq \tilde{\mathcal{B}}^\perp$, hence $\tilde{\mathcal{B}} \subseteq {}^\perp(\tilde{\mathcal{B}}^\perp) \subseteq^\perp dw\mathrm{Inj}$.

(3) We show the inclusion $\widetilde{\mathcal{B}}^{\perp} \cap dw\mathcal{B} \subseteq dg\mathrm{Inj}$. Let $Y \in \widetilde{\mathcal{B}}^{\perp} \cap dw\mathcal{B}$; using the complete cotorsion pair $(\mathcal{E}, dg\mathrm{Inj})$ in $\mathrm{Ch}(R)$ we can consider a short exact sequence $(*)\quad 0 \to Y \to dgI \to E \to 0$ with $dgI \in dg\mathrm{Inj}$ and E an exact complex. By part (1) the sequence is degreewise splitting hence E^n is an injective module for every $n \in \mathbb{Z}$ which means that $E \in ex\mathrm{Inj}$. By Corollary 1.34, $ex\mathrm{Inj} \subseteq \widetilde{\mathcal{B}_\infty} \subseteq \widetilde{\mathcal{B}}$, hence the sequence $(*)$ splits showing that $Y \in dg\mathrm{Inj}$.

The other inclusion is obvious since ${}^{\perp}dg\mathrm{Inj}$ in $\mathrm{Ch}(R)$ is the class of acyclic complexes $\mathcal{E}$ and $\mathcal{E} \supseteq \widetilde{\mathcal{B}}$.

(4) Obvious, since ${}^{\perp}dg\mathrm{Inj} = \mathcal{E}$. □

Remark 1.36. If $\mathcal{G}$ has enough projective objects, then the dual of the statements in Lemma 1.35 (1) and (2) hold substituting the right orthogonal with the left orthogonal and Inj with Proj.

Points 1.35 and 1.35 of the previous Lemma can be generalized, with essentially the same proof, to a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ such that $ex\mathrm{Inj} \subseteq \widetilde{\mathcal{B}}$.

1.2.3 *Cotorsion pairs in exact categories*

We state a result valid in general for cotorsion pairs in exact categories.

Proposition 1.37. *Let $(\mathcal{A}, \mathcal{B})$ be a (hereditary) complete cotorsion pair in an exact category $\mathcal{C}$ and let $\mathcal{D}$ be an extension closed subcategory of $\mathcal{C}$ with the exact structure induced by that of $\mathcal{C}$. If $\mathcal{D}$ contains $\mathcal{A}$ and is resolving in $\mathcal{C}$ or if $\mathcal{D}$ contains $\mathcal{B}$ and is coresolving in $\mathcal{C}$, (see Definition 1.3), then $(\mathcal{A} \cap \mathcal{D},\ \mathcal{B} \cap \mathcal{D})$ is a (hereditary) complete cotorsion pair in the exact category $\mathcal{D}$.*

Proof. We prove the statement in case $\mathcal{D} \supseteq \mathcal{B}$, the other case being similar. First we show that $(\mathcal{A} \cap \mathcal{D}, \mathcal{B})$ is a cotorsion pair in $\mathcal{D}$. Clearly ${}^{\perp}\mathcal{B} = \mathcal{A} \cap \mathcal{D}$ in $\mathcal{D}$ and also $(\mathcal{A} \cap \mathcal{D})^{\perp} \supseteq \mathcal{B}$. We show that $(\mathcal{A} \cap \mathcal{D})^{\perp} = \mathcal{B}$ in $\mathcal{D}$. Let $D \in \mathcal{D}$ be such that $\mathrm{Ext}^1(X, D) = 0$ for every $X \in \mathcal{A} \cap \mathcal{D}$. Since $(\mathcal{A}, \mathcal{B})$ is complete, there is an exact sequence $0 \to D \to B \to A \to 0$ in $\mathcal{C}$, with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Since $\mathcal{D}$ is coresolving in $\mathcal{C}$ and contains $\mathcal{B}$, we have that $A \in \mathcal{D}$, hence $A \in A \cap \mathcal{D}$ showing that the exact sequence splits, thus $D \in \mathcal{B}$.

To show that $(\mathcal{A} \cap \mathcal{D}, \mathcal{B})$ is complete, let $(*)\quad 0 \to B \to A \to D \to 0$ be a special $\mathcal{A}$-precover of an object $D \in \mathcal{D}$, then $A \in \mathcal{A} \cap \mathcal{D}$, since $\mathcal{D}$ is extension closed, hence $(*)$ is a special $\mathcal{A} \cap \mathcal{D}$-precover of D. If $(**) : \quad 0 \to D \to B \to A \to 0$ is a special $\mathcal{B}$-preenvelope of $D \in \mathcal{D}$, then $A \in \mathcal{D}$ since $\mathcal{D}$ is coresolving, hence $(**)$ is special $\mathcal{B}$-preenvelope of D with respect to to $(\mathcal{A} \cap \mathcal{D}, \mathcal{B})$.

□

From now on $\mathcal{G}$ will be a Grothendieck category.

For every complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ the classes $\mathcal{A}$ and $\mathcal{B}$ are extension closed subcategories of $\mathcal{G}$, hence they inherit the exact structure from the abelian structure of $\mathcal{G}$.

Moreover, it is obvious that they are idempotent complete.

It is well known that a Grothendieck category has enough injectives. When needed we will assume that $\mathcal{G}$ has enough projectives and enough flat objects.

We will denote by Inj and Proj the classes of injective and projective objects, respectively; by Flat the class of flat objects and by Cot the class of cotorsion objects. We have the complete hereditary cotorsion pairs $(\mathrm{Proj}, \mathcal{G})$, $(\mathcal{G}, \mathrm{Inj})$ and $(\mathrm{Flat}, \mathrm{Cot})$, hence the four classes defined above are exact subcategories of $\mathcal{G}$.

We first collect some well known facts.

Fact 1.38. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{G}$. The following hold true:*

1. *$\mathcal{A}$ has enough injectives and projectives: the projectives are the same as in $\mathcal{G}$ and the injectives are the objects in $\mathcal{A} \cap \mathcal{B}$.*

2. *$\mathcal{B}$ has enough injectives and projectives: the injectives are the same as in $\mathcal{G}$ and the projectives are the objects in $\mathcal{A} \cap \mathcal{B}$.*

3. *([21, Corollary 2.9] $\mathrm{Ch}(\mathcal{A})$ has enough injectives and projectives: the projectives are the same as in $\mathrm{Ch}(\mathcal{G})$ and the injectives are the contractible complexes with components in $\mathcal{A} \cap \mathcal{B}$.*

4. *([21, Corollary 2.9] $\mathrm{Ch}(\mathcal{B})$ has enough injectives and projectives: the injective are the same as in $\mathrm{Ch}(\mathcal{G})$ and the projectives are the contractible complexes with components in $\mathcal{A} \cap \mathcal{B}$.*

5. *([21, Corollary 2.8] $\mathrm{Ch}(\mathcal{A})_{dw}$ and $\mathrm{Ch}(\mathcal{B})_{dw}$ are Frobenius exact categories with the projective-injective objects being the contractible complexes with terms in $\mathcal{A}$ or $\mathcal{B}$ respectively.*

Remark 1.39. If $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$, Proposition 1.37 tells us that $(\mathcal{B}, \mathrm{Inj})$ and $(\mathcal{A} \cap \mathcal{B}, \mathcal{B})$ are complete hereditary cotorsion pairs in the exact category $\mathcal{B}$; $(\mathrm{Proj}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{A} \cap \mathcal{B})$ are complete hereditary cotorsion pairs in the exact category $\mathcal{A}$.

We conclude this chapter by giving several propositions that allow us to restrict a cotorsion pair in $\mathrm{Ch}(G)$ to the exact subcategories $\mathrm{Ch}(\mathcal{A})$ or $(\mathrm{Ch}(\mathcal{B})$ and the Frobenius categories $\mathrm{Ch}(\mathcal{A})_{dw}$ and $\mathrm{Ch}(\mathcal{B})_{dw}$.

The following Proposition 1.40 is a generalization of [21, Proposition 7.3] which was formulated for the case of the cotorsion pair $(\mathcal{F}, \mathcal{C})$ in a module category.

Moreover, in Proposition 1.41 we state a generalization of the dual of [21, Proposition 7.3].

Proposition 1.40. *Let $(\mathcal{A}, B)$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$ and let $(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ be a complete cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ with $\hat{\mathcal{A}} \subseteq dw\mathcal{A}$. Assume that $\hat{\mathcal{A}}$ is thick in the exact category $\mathrm{Ch}(\mathcal{A})$ and that it contains the contractible complexes with terms in $\mathcal{A}$. Then,*

$$\left(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cap dw\mathcal{A}\right)$$

is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{A})$. Moreover,

$$\left(\hat{\mathcal{A}}, [\hat{\mathcal{B}} \cap dw\mathcal{A}]_K\right)$$

is a localizing cotorsion pair in the Frobenius category $\mathrm{Ch}(\mathcal{A})_{dw}$, where a complex $X \in \mathrm{Ch}(\mathcal{A})$ belongs to $[\hat{\mathcal{B}} \cap dw\mathcal{A}]_K$ if and only if it is chain homotopy equivalent to a complex in $\hat{\mathcal{B}} \cap dw\mathcal{A}$.

Proof. The fact that $\left(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cap dw\mathcal{A}\right)$ is a complete cotorsion pair follows by Proposition 1.37 and it is an injective cotorsion pair by definition and by the assumptions on $\hat{\mathcal{A}}$. Moreover, $\hat{\mathcal{B}} \subseteq dw\mathcal{B}$. In fact, for every $n \in \mathbb{Z}$ and every $A \in \mathcal{A}$ the contractible complex $D^n(A)$ is in $\hat{\mathcal{A}}$, hence $\mathrm{Ext}^1_{\mathrm{Ch}}(D^n(A), B) = 0$, for every $B \in \hat{\mathcal{B}}$ and then B^n belongs to $\mathcal{B}$, by [16, Lemma 3.1]. Hence, a short exact sequence $0 \to Y \to Z \to X \to 0$ in $\mathrm{Ch}(\mathcal{A})$ with $Y \in \hat{\mathcal{B}} \cap dw\mathcal{A}$ and $X \in \hat{\mathcal{A}}$ is degreewise splitting. The second statement follows by [21, Theorem 6.3, Proposition 6.4]. □

Proposition 1.41. *Let $(\mathcal{A}, B)$ be a complete hereditary cotorsion pair in $\mathcal{G}$ and let $(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ be a complete cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ with $\hat{\mathcal{B}} \subseteq dw\mathcal{B}$. Assume that $\hat{\mathcal{B}}$ is thick in the exact category $\mathrm{Ch}(\mathcal{B})$ and contains the contractible complexes with terms in $\mathcal{B}$. Then,*

$$\left(\hat{\mathcal{A}} \cap dw\mathcal{B}, \hat{\mathcal{B}}\right)$$

is a projective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$*. Moreover,*

$$\left([\hat{\mathcal{A}} \cap dw\mathcal{B}]_K, \hat{\mathcal{B}}\right)$$

is a localizing cotorsion pair in the Frobenius category $\mathrm{Ch}(\mathcal{B})_{dw}$*, where a complex* $X \in \mathrm{Ch}(\mathcal{B})$ *belongs to* $[\hat{\mathcal{A}} \cap dw\mathcal{B}]_K$ *if and only if it is chain homotopy equivalent to a complex in* $\hat{\mathcal{A}} \cap dw\mathcal{B}$*.*

Proof. Dual of 1.40. Note that Theorem 6.3 and Proposition 6.4 in [21] have obvious dual statements for projective cotorsion pairs from which the second statement of our proposition follows. □

The next proposition is the analogue of [21, Proposition 7.2] and its dual is stated in 1.43.

Proposition 1.42. *Let* $\mathcal{G}$ *be a Grothendieck category with enough projective objects and let* $(\mathcal{P}, \mathcal{W})$ *be a projective cotorsion pair in* $\mathrm{Ch}(\mathcal{G})$ *with* $\mathcal{P} \subseteq dw\mathrm{Proj}$*. Let* $(\mathcal{A}, \mathcal{B})$ *be a complete hereditary cotorsion pair in* $\mathcal{G}$*. Then,*

$$(\mathcal{P}, \mathcal{W} \cap dw\mathcal{A})$$

is a projective cotorsion pair in $\mathrm{Ch}(\mathcal{A})$ *and*

$$\left([\mathcal{P}]_K, \mathcal{W} \cap dw\mathcal{A}\right)$$

is a localizing cotorsion pair in the Frobenius category $\mathrm{Ch}(\mathcal{A})_{dw}$*. A complex* $X \in \mathrm{Ch}(\mathcal{A})$ *is in* $[\mathcal{P}]_K$ *if and only if it is chain homotopy equivalent to a complex in* $P \in \mathcal{P}$*.*

Proof. $(\mathcal{P}, \mathcal{W} \cap dw\mathcal{A})$ is a complete cotorsion pair by Proposition 1.37 and it is automatically a projective cotorsion pair. The second statement follows by the dual of [21, Theorem 6.3, Proposition 6.4]. □

Proposition 1.43. *Let* $\mathcal{G}$ *be a Grothendieck category and let* $(\mathcal{W}, \mathcal{I})$ *be an injective cotorsion pair in* $\mathrm{Ch}(\mathcal{G})$ *with* $\mathcal{I} \subseteq dw\mathrm{Inj}$ *and let* $(\mathcal{A}, \mathcal{B})$ *be a complete hereditary cotorsion pair in* $\mathcal{G}$*. Then*

$$\left(\mathcal{W} \cap dw\mathcal{B}, \mathcal{I}\right)$$

is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$ *and*

$$\left(\mathcal{W} \cap dw\mathcal{B}, [\mathcal{I}]_K\right)$$

is a localizing cotorsion pair in the Frobenius category $\mathrm{Ch}(\mathcal{B})_{dw}$*, where a complex* $X \in \mathrm{Ch}(\mathcal{B})$ *belongs to* $[\mathcal{I}]_K$ *if and only if it is chain homotopy equivalent to a complex in* $\mathcal{I}$*.*

Proof. The first statement follows by Proposition 1.37. The second statement follows by [21, Theorem 6.3]. □

2

RECOLLEMENTS FROM COTORSION PAIRS

In [36] Neemann described the homotopy category of the projective modules as a localization of the homotopy category of flat modules and he obtained a recollement with middle term the homotopy category of flat modules. His recollement can be compared with the classical one having the homotopy category of a ring R as middle term, the derived category of R as right term and the category of acyclic complexes modulo the homotopy relation as left term.

This chapter is extracted from the joint work [4] where we exhibit many other examples of recollements of analogous type.

Our results are strongly based on the two papers [22] and [21] by Gillespie and also inspired by Becker's idea in [6] to consider triples of injective cotorsion pairs giving rise to model structures and to the corresponding recollements.

Starting from a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category, we consider triples of examples of injective and projective cotorsion pairs on the categories of unbounded complexes with components in the exact categories $\mathcal{A}$ or $\mathcal{B}$. The examples are constructed in order that the associated model structures on the categories of complexes satisfy the assumptions allowing to build the relevant recollements.

Our aim is mainly to describe the homotopy categories $K(\mathcal{B})$ and $K(\mathcal{A})$ as well as the derived categories $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A})$.

Imposing some mild conditions on a Grothendieck category $\mathcal{G}$ (which are always satisfied by module categories), Theorem 2.11 gives the recollement

$$(*)\qquad \begin{tikzcd} \frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where for every subcategory $\mathcal{C}$ of $\mathcal{G}$, $ex\mathcal{C}$ denotes the class of acyclic unbounded complexes with terms in $\mathcal{C}$. The dual is given by Theorem 2.27.

The recollement $(*)$ generalizes the recollement obtained by Krause ([31]) where the middle term is the homotopy category of the injective objects.

For a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ the term $\dfrac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}}$ is equivalent to the derived category $\mathcal{D}(\mathcal{G})$ of the Grothendieck category, essentially because $\mathrm{Ch}(\mathcal{B})$ contains the dg-injective complexes. Analogously, if $\mathcal{G}$ has enough projectives, $\dfrac{\mathrm{Ch}(\mathcal{A})}{ex\mathcal{A}}$ is equivalent to the derived category $\mathcal{D}(\mathcal{G})$ of the Grothendieck category, since $\mathrm{Ch}(\mathcal{A})$ contains the dg-projective complexes.

Concerning the derived categories $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A})$, recall that by Neeman [34] the derived category of an idempotent complete exact category $\mathcal{C}$ is defined as the quotient $\dfrac{\mathrm{Ch}(\mathcal{C})}{\widetilde{\mathcal{C}}}$, where $\widetilde{\mathcal{C}}$ denotes the class of unbounded complexes acyclic in $\mathcal{C}$, meaning that the differentials factor through short exact sequences in $\mathcal{C}$. By Theorem 2.5 we get the recollement

$$\begin{tikzcd}
\frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & \mathcal{D}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

and its dual in Theorem 2.23.

It would be important to get recollements analogous to $(*)$, but with right term $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A})$, that is the derived categories of the exact categories $\mathcal{B}$ or $\mathcal{A}$. Of course, if $\widetilde{\mathcal{B}} = ex\mathcal{B}$ or $\widetilde{\mathcal{A}} = ex\mathcal{A}$, the recollement $(*)$ degenerates into

$$(**) \quad \begin{tikzcd}
\frac{\widetilde{\mathcal{B}}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{B})}{\widetilde{\mathcal{B}}} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

(and dually for $\mathcal{A}$).

The condition $\widetilde{\mathcal{B}} = ex\mathcal{B}$ is very strong, hence it would be interesting to find other examples of $(**)$ for the case $\widetilde{\mathcal{B}} \subsetneq ex\mathcal{B}$.

The only non degenerate example of this type of which we are aware is given by the cotorsion pair $(\mathcal{A}, \mathrm{FpInj})$ over a coherent ring, where FpInj denotes the class of Fp-injective modules, that is the right Ext-orthogonal to the class of finitely presented modules. This follows by Šťovíček's results in [47] which we are able to slightly generalize in Proposition 2.20.

Symmetrically, it seems there are very few non degenerate examples of such recollements for the case $\widetilde{\mathcal{A}} \subsetneq ex\mathcal{A}$. The more important one follows by the celebrated Neeman's result in [36] and it is the case when $\mathcal{A}$ is class of flat modules. We show a slight generalization of this situation in Proposition 2.28.

From the results in Section 1.2.2 and the results in a recent paper [2] we obtain examples of cotorsion pairs $(\mathcal{A}, \mathcal{B})$ in module categories satisfying the condition $\widetilde{\mathcal{B}} = ex\mathcal{B}$. These include tilting and cotilting cotorsion pairs, the closure of the cotorsion pair generated by the compact objects of finite projective dimension and the cotorsion pair $(\mathcal{F}, \mathcal{C})$ of the flat and cotorsion modules.

2.1 PROJECTIVE COTORSION PAIRS IN THE EXACT CATEGORY $\mathrm{Ch}(\mathcal{B})$

For every complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ we look for cotorsion pairs on the exact category $\mathrm{Ch}(\mathcal{B})$ of unbounded complexes with terms in $\mathcal{B}$ in order to describe the derived category $\mathcal{D}(\mathcal{B})$ and also recollements linking it to the derived category of $\mathcal{G}$.

We start by choosing projective cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$ satisfying the assumptions of Proposition 1.41. When needed we assume some extra conditions on the Grothendieck category $\mathcal{G}$, like in example (3) below.

Example 2.1. Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$.

1. The complete hereditary cotorsion pair $(dg\mathcal{A}, \widetilde{\mathcal{B}})$ in $\mathrm{Ch}(\mathcal{G})$ satisfies the conditions in Proposition 1.41, hence we have the projective cotorsion pair:

$$\mathcal{M}_1 = \left(dg\mathcal{A} \cap dw\mathcal{B}, \widetilde{\mathcal{B}}\right)$$

in $\mathrm{Ch}(\mathcal{B})$ and the localizing cotorsion pair

$$\left([dg\mathcal{A} \cap dw\mathcal{B}]_K, \widetilde{\mathcal{B}}\right)$$

in $\mathrm{Ch}(\mathcal{B})_{dw}$.

2. The complete hereditary cotorsion pair $(\widetilde{\mathcal{A}}, dg\mathcal{B})$ in $\mathrm{Ch}(\mathcal{G})$ satisfies the conditions in Proposition 1.41, hence we have the projective cotorsion pair:

$$\mathcal{M}_2 = \left(\widetilde{\mathcal{A}} \cap dw\mathcal{B}, dg\mathcal{B}\right)$$

in $\mathrm{Ch}(\mathcal{B})$ and the localizing cotorsion pair

$$\left([\widetilde{\mathcal{A}} \cap dw\mathcal{B}]_K, dg(\mathcal{B})\right)$$

in $\mathrm{Ch}(\mathcal{B})_{dw}$.

3. If $\mathcal{A}$ contains a generator of finite projective dimension, then by Proposition 1.23 (3), $({}^{\perp}ex\mathcal{B}, ex\mathcal{B})$ is a complete hereditary cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ and it satisfies the conditions in Proposition 1.41, hence we have the projective cotorsion pair:

$$\mathcal{M}_3 = ({}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}, ex\mathcal{B})$$

in $\mathrm{Ch}(\mathcal{B})$ and the localizing cotorsion pair

$$\left([{}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}]_K, ex\mathcal{B}\right)$$

in $\mathrm{Ch}(\mathcal{B})_{dw}$.

Remark 2.2. The three examples above satisfy Proposition 1.41 since $\tilde{\mathcal{B}}$, $dg\mathcal{B}$ and $ex\mathcal{B}$ are thick in $\mathrm{Ch}(\mathcal{B})$ by Lemma 1.21 and they clearly contain the contractible complexes with terms in $\mathcal{B}$.

Theorem 2.3. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$ such that $\mathcal{A}$ contains a generator of finite projective dimension.*

The three projective cotorsion pairs in Example 2.1 satisfy the conditions of [21, Theorem 3.5], so that we get the recollement:

$$\begin{tikzcd}
\frac{\widetilde{\mathcal{A}} \cap dw\mathcal{B}}{\sim} \arrow[r, "inc"] & \frac{dg\mathcal{A} \cap dw\mathcal{B}}{\sim} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Ch}(\mathcal{B})/ex\mathcal{B} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $\sim$ denotes the homotopy relation associated to the corresponding model structure and coincides with the chain homotopy relation; moreover, inc is the inclusion and Q is the quotient functor.

Remark 2.4. In the above examples write $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i)$, for every $i = 1, 2, 3$. We have that $\mathcal{C}_i \cap \mathcal{W}_i = \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}$. Moreover, dually to [21, Proposition 3.2] $\mathcal{C}_i$ is a Frobenius category with the projective-injective objects being exactly the complexes in $\widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}$. Thus $(\widetilde{\mathcal{A}} \cap dw\mathcal{B})/\sim$ and $(dg\mathcal{A} \cap dw\mathcal{B})/\sim$ are the stable categories and they are also equivalent to the homotopy categories $K(\widetilde{\mathcal{A}} \cap dw\mathcal{B})$ and $K(dg\mathcal{A} \cap dw\mathcal{B})$. Moreover, all the three terms in the recollement are equivalent to the homotopy categories of the three model structures on $\mathrm{Ch}(\mathcal{B})$ corresponding to the projective cotorsion pairs $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. Furthermore, $\mathrm{Ch}(\mathcal{B})/ex\mathcal{B}$ is equivalent to the derived category of $\mathcal{G}$ as we will see more explicitly later in Remark 2.12.

By [34], the derived category of $\mathrm{Ch}(\mathcal{B})$ is the quotient of $\mathrm{Ch}(\mathcal{B})$ modulo the acyclic complexes in $\mathrm{Ch}(\mathcal{B})$, that is the complexes in $\widetilde{\mathcal{B}}$. Thus we need an exact model structure on $\mathrm{Ch}(\mathcal{B})$ with $\widetilde{\mathcal{B}}$ as the class of trivial objects. This is provided by Example 2.1 (1).

Theorem 2.5. *In the setting of Example 2.1 (1),*

$$\mathcal{M}_1 = \left(dg\mathcal{A} \cap dw\mathcal{B}, \widetilde{\mathcal{B}}, dw\mathcal{B}\right)$$

is an exact model structure in the category $\mathrm{Ch}(\mathcal{B})$. In particular, we can define the derived category $\mathcal{D}(\mathcal{B})$ as the quotient $\mathrm{Ch}(\mathcal{B})/\widetilde{\mathcal{B}}$.

Moreover, we have the following triangle equivalences between the derived category of $\mathrm{Ch}(\mathcal{B})$ and the homotopy category of the model structure $\mathcal{M}_1$:

$$\mathcal{D}(\mathcal{B}) = \mathrm{Ho}(\mathcal{M}_1) \cong \frac{dg\mathcal{A} \cap dw\mathcal{B}}{\widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}}$$

and in the assumptions of Theorem 2.3 there is also a recollement:

$$\begin{tikzcd}
\frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & \mathcal{D}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $ex\mathcal{B}/\sim$ is the full subcategory of $\mathcal{D}(\mathcal{B})$ consisting of exact complexes (in $\mathrm{Ch}(\mathcal{G})$).

Proof. The projective cotorsion pair $\left(dg\mathcal{A} \cap dw\mathcal{B}, \widetilde{\mathcal{B}}\right)$ in $\mathrm{Ch}(\mathcal{B})$ of Example 2.1 (1) corresponds to the exact model structure

$\left(dg\mathcal{A} \cap dw\mathcal{B}, \widetilde{\mathcal{B}}, dw\mathcal{B}\right)$. The equivalences follow from general properties of model categories (see the discussion in [21, Section 3.1]), hence the recollement follows by Theorem 2.3. □

Another way to obtain the exact model structure of Theorem 2.5 is to use results by Gillespie in [18], [20] and [23].

Theorem 2.6. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair generated by a set of objects in a Grothendieck category $\mathcal{G}$. The two complete hereditary cotorsion pairs $({}^{\perp}dw\mathcal{B}, dw\mathcal{B})$ and $(dg\mathcal{A}, \tilde{\mathcal{B}})$ in* $\mathrm{Ch}(\mathcal{G})$ *give rise to a cofibrantly generated model structure*

$$\mathcal{M} = \left(dg\mathcal{A}, \mathcal{V}, dw\mathcal{B}\right)$$

in $\mathrm{Ch}(\mathcal{G})$ *satisfying $\mathcal{V} \cap dw\mathcal{B} = \tilde{\mathcal{B}}$ and $\mathcal{V} \cap dg\mathcal{A} = {}^{\perp}dw\mathcal{B}$ whose restriction in* $\mathrm{Ch}(\mathcal{B})$ *is the exact model structure*

$$\mathcal{M}_1 = \left(dg\mathcal{A} \cap dw\mathcal{B}, \widetilde{\mathcal{B}}, dw\mathcal{B}\right)$$

of Theorem 2.5.

Moreover, if $(\mathcal{A}, \mathcal{B})$ is generated by a set of finitely presented objects then the model structure $\mathcal{M} = \left(dg\mathcal{A}, \mathcal{V}, dw\mathcal{B}\right)$ in $\mathrm{Ch}(\mathcal{G})$ *is finitely generated hence its homotopy category is compactly generated.*

Proof. By [18, Proposition 4.3 and Proposition 4.4] the cotorsion pairs $({}^{\perp}dw\mathcal{B}, dw\mathcal{B})$ and $(dg\mathcal{A}, \tilde{\mathcal{B}})$ are small and they are hereditary since $(\mathcal{A}, \mathcal{B})$ is hereditary. The existence of the model structure $\mathcal{M}$ in $\mathrm{Ch}(\mathcal{G})$ follows by [23, Theorem 1.1]. The fact that the model structure is cofibrantly generated follows by [27, Section 7.4]. The last statement follows also by [27, Section 7.4]. □

Combining Theorem 2.5 with Theorem 2.6 we obtain the following consequence:

Corollary 2.7. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *generated by a set of finitely presented modules. Then the derived category $\mathcal{D}(\mathcal{B}) \cong \mathrm{Ch}(\mathcal{B})/\tilde{\mathcal{B}}$ is compactly generated. In particular, if R is a coherent ring and $(\mathcal{A}, \mathrm{FpInj})$ is the complete cotorsion pair generated by all finitely presented modules, then $\mathcal{D}(\mathrm{FpInj})$ is compactly generated.*

Proof. Only the second statement needs a comment. If R is a coherent ring, then the complete cotorsion pair $(\mathcal{A}, \mathrm{FpInj})$ is hereditary. □

2.2 INJECTIVE COTORSION PAIRS IN THE EXACT CATEGORY $\mathrm{Ch}(\mathcal{B})$

In this section we want to investigate models for $\mathcal{K}(\mathcal{B})$ and recollements linking it to $\mathcal{D}(\mathcal{G})$, in particular we look for localizing cotorsion triple in $\mathrm{Ch}(\mathcal{B})_{dw}$ whose middle term is $ex\mathcal{B}$.

We exhibit three examples of injective cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$ satisfying the assumptions of Proposition 1.43.

When needed, we assume some extra conditions on $\mathcal{G}$ like in (2) below.

Example 2.8. Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$.

1. By Proposition 1.23 (2) we have that $({}^{\perp}dw\mathrm{Inj}, dw\mathrm{Inj})$ is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ (notice that $(\mathcal{G}, \mathrm{Inj})$ is generated by a set). Hence by Proposition 1.43 and [21, Theorem 6.3] we obtain the injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$:
$$\mathcal{N}_1 = \left({}^{\perp}dw\mathrm{Inj} \cap dw\mathcal{B}, dw\mathrm{Inj}\right)$$
and the localizing cotorsion pair
$$\left({}^{\perp}dw\mathrm{Inj} \cap dw\mathcal{B}, [dw\mathrm{Inj}]_K\right)$$
in $\mathrm{Ch}(\mathcal{B})_{dw}$.

2. If $\mathcal{G}$ has a generator of finite projective dimension, by Proposition 1.23 $({}^{\perp}ex\mathrm{Inj}, ex\mathrm{Inj})$ is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{G})$. Hence, by Proposition 1.43 and [21, Theorem 6.3] we obtain the injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$:
$$\mathcal{N}_2 = \left({}^{\perp}ex\mathrm{Inj} \cap dw\mathcal{B}, ex\mathrm{Inj}\right)$$
and the localizing cotorsion pair
$$\left({}^{\perp}ex\mathrm{Inj} \cap dw\mathcal{B}, [ex\mathrm{Inj}]_K\right)$$
in $\mathrm{Ch}(\mathcal{B})_{dw}$.

3. By Proposition 1.22 (3) $(\mathcal{E}, dg\mathrm{Inj})$ is a complete hereditary cotorsion pair in $\mathrm{Ch}(\mathcal{G})$. Hence, by Proposition 1.43 and [21, Theorem 6.3] we obtain the injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$:
$$\mathcal{N}_3 = \left(ex\mathcal{B}, dg\mathrm{Inj}\right)$$

and the localizing cotorsion pair

$$\Big(ex\mathcal{B}, [dg\mathrm{Inj}]_K\Big)$$

in $\mathrm{Ch}(\mathcal{B})_{dw}$.

In the above examples we write $\mathcal{N}_i = (\mathcal{W}_i, \mathcal{R}_i)$, for every $i = 1, 2, 3$. Then, $\mathcal{R}_2 \subseteq \mathcal{W}_3$ and $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$. In fact, by the analogous of [18, Theorem 4.7] in a Grothendieck category, ${}^{\perp}ex\mathrm{Inj} \cap \mathcal{E} = {}^{\perp}dw\mathrm{Inj}$. Thus, they satisfy the conditions of [21, Theorem 3.4] and allow to build a recollement which, as we will point out in Remark 2.10, is nothing else than Krause's recollement [31].

Theorem 2.9. *Let $\mathcal{G}$ be a Grothendieck category with a generator of finite projective dimension and let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{G}$.The three injective cotorsion pairs in Example 2.8 satisfy the conditions of [21, Theorem 3.4], so that we get the recollement:*

$$\begin{tikzcd}
\dfrac{ex\mathrm{Inj}}{\sim} \arrow[r, "inc"] & \dfrac{dw\mathrm{Inj}}{\sim} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Ch}(\mathcal{B})/ex\mathcal{B} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $\sim$ denotes the homotopy relation associated to the corresponding model structure and coincides with the chain homotopy relation; moreover, inc is the inclusion and Q is the quotient functor.

Remark 2.10.
1. Writing $\mathcal{N}_i = (\mathcal{W}_i, \mathcal{R}_i)$, for every $i = 1, 2, 3$, [21, Proposition 3.2] implies that $\mathcal{R}_i$ is a Frobenius category with the projective-injective object being exactly the injective objects in $\mathrm{Ch}(\mathcal{G})$ or, equivalently, in $\mathrm{Ch}(\mathcal{B})$. Note that, for every $i = 1, 2, 3$, $\mathcal{R}_i \cap \mathcal{W}_i$ is the class of injective objects in $\mathrm{Ch}(\mathcal{B})$. Thus, $\mathcal{R}_1/\sim$ is equivalent to the homotopy category $K(\mathrm{Inj})$ of the complexes with injective terms and $\mathcal{R}_2/\sim$ is equivalent to $K(ex\mathrm{Inj})$ the full subcategory of $K(\mathrm{Inj})$ consisting of exact complexes of injectives. Moreover, $\mathrm{Ch}(\mathcal{B})/ex\mathcal{B}$ is equivalent to the derived category $\mathcal{D}(R)$, as it will be clear from Theorem 2.11.

   That is, Theorem 2.9 is yet another instance of Krause's recollement [31], which was recovered also in [5].

2. The complexes in ${}^{\perp}dw\mathrm{Inj}$ are called coacyclic in [39] (see also [44] and [5]). By [44, Proposition 6.9] the homotopy

category of the injective cotorsion pair $\mathcal{N}_1$ is equivalent to $K(\mathrm{Inj})$ and called the coderived category of $\mathcal{G}$. Thus the central term of the above recollement is equivalent to the coderived category of $\mathcal{G}$.

Combining the above example with an example from Section 2.1 we can state the following:

Theorem 2.11. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$ such that $\mathcal{A}$ contains a generator of finite projective dimension. The triple $\left([{}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}]_K, ex\mathcal{B}, [dg\mathrm{Inj}]_K\right)$ is a localizing cotorsion triple in $\mathrm{Ch}(\mathcal{B})_{dw}$. Then, there are equivalences of triangulated categories:*

$$\frac{[{}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}]_K}{\sim} \cong \frac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}} \cong \frac{[dg\mathrm{Inj}]_K}{\sim}$$

where $\sim$ is the chain homotopy equivalence and a recollement:

$$\begin{tikzcd}
\frac{ex\mathcal{B}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}} \cong \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where the middle term is the homotopy category $K(\mathcal{B})$ of the complexes with terms in $\mathcal{B}$ modulo the chain homotopy equivalence.

Proof. By Example 2.1 (3) and Example 2.8, we have two localizing cotorsion pairs $\left([{}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}]_K, ex\mathcal{B}\right)$ and $\left(ex\mathcal{B}, [dg\mathrm{Inj}]_K\right)$ in $\mathrm{Ch}(\mathcal{B})_{dw}$. Let $\mathcal{X} = [{}^{\perp}ex\mathcal{B} \cap dw\mathcal{B}]_K$, $\mathcal{Y} = ex\mathcal{B}$ and $\mathcal{Z} = [dg\mathrm{Inj}]_K$, then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a localizing cotorsion triple in $\mathrm{Ch}(\mathcal{B})_{dw}$.

Then, the conclusion follows by [21, Corollary 4.5] . □

Remark 2.12. From the equivalence $\dfrac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}} \cong \dfrac{[dg\mathrm{Inj}]_K}{\sim}$ we see that $\dfrac{\mathrm{Ch}(\mathcal{B})}{ex\mathcal{B}}$ is equivalent to the usual derived category $\mathcal{D}(\mathcal{G})$.

2.3 WHEN IS $\tilde{\mathcal{B}}$ THE CENTRAL TERM OF A LOCALIZING COTORSION TRIPLE IN $\mathrm{Ch}(\mathcal{B})_{dw}$?

In Example 2.8 (3) we have shown that there is an injective cotorsion pair $\mathrm{Ch}(\mathcal{B})$ with $ex\mathcal{B}$ as left term and Example 2.1 (1) provides a projective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$ with right component $\widetilde{\mathcal{B}}$.

Our aim will be to find cotorsion pairs $(\mathcal{A}, \mathcal{B})$ for which there exist an injective cotorsion pair $(\widetilde{\mathcal{B}}, \mathcal{R})$ in $\mathrm{Ch}(\mathcal{B})$ with $\mathcal{R} \subseteq dw\mathrm{Inj}$ in order to obtain a localizing cotorsion triple in $\mathrm{Ch}(\mathcal{B})_{dw}$ with $\widetilde{\mathcal{B}}$ as central term.

Section 1.2.2 provides examples of cotorsion pairs $(\mathcal{A}, \mathcal{B})$ such that $ex\mathcal{B} = \widetilde{\mathcal{B}}$.

A first case appears in Proposition 1.27.

Proposition 2.13. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{G}$ with $\mathcal{A} \subseteq \mathcal{P}$ ($\mathcal{P}$ the class of objects with finite projective dimension). Then $ex\mathcal{B} = \tilde{\mathcal{B}}$, hence $(\tilde{\mathcal{B}}, dg\mathrm{Inj})$ is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$ and there is a recollement as in Theorem 2.11 with $ex\mathcal{B}$ replaced by $\tilde{\mathcal{B}}$.*

In particular, the derived category $\mathcal{D}(\mathcal{B})$ of $\mathcal{B}$ is equivalent to the usual derived category of $\mathcal{G}$.

Corollary 2.14. *Let $(\mathcal{A}, \mathcal{T})$ be an n-tilting cotorsion pair in $\mathrm{Mod}\text{-}R$. For the tilting class $\mathcal{T}$ we have a recollement:*

$$\begin{tikzcd} \dfrac{\widetilde{\mathcal{T}}}{\sim} \arrow[r, "inc"] & K(\mathcal{T}) \arrow[l, bend right=60] \arrow[l, bend left=60] \arrow[r, "Q"] & \dfrac{\mathrm{Ch}(\mathcal{T})}{\widetilde{\mathcal{T}}} \arrow[l, bend right=60] \arrow[l, bend left=60] \end{tikzcd} .$$

Proof. By Proposition 1.28 $ex\mathcal{T} = \widetilde{\mathcal{T}}$, hence the conclusion follows by Proposition 2.13. □

Proposition 2.15. *The cotorsion pair $(\mathrm{Flat}, \mathrm{Cot})$ in $\mathrm{Mod}\text{-}R$ satisfies $ex\mathrm{Cot} = \widetilde{\mathrm{Cot}}$, hence it induces a recollement:*

$$\begin{tikzcd} \dfrac{\widetilde{\mathrm{Cot}}}{\sim} \arrow[r, "inc"] & K(\mathrm{Cot}) \arrow[l, bend right=60] \arrow[l, bend left=60] \arrow[r, "Q"] & \dfrac{\mathrm{Ch}(\mathrm{Cot})}{\widetilde{\mathrm{Cot}}} \arrow[l, bend right=60] \arrow[l, bend left=60] \end{tikzcd} .$$

Proof. The fact that $ex\mathrm{Cot} = \widetilde{\mathrm{Cot}}$ in $\mathrm{Ch}(R)$ is proved in [2, Theorem 4.1 (2)]. Hence, the conclusion follows by Theorem 2.11. □

Proposition 2.16. *The cotorsion pair $(\mathcal{A}^{\infty}, \mathcal{B}_{\infty})$ from Notation 1.29 (5) satisfies $ex\mathcal{B}_{\infty} = \widetilde{\mathcal{B}_{\infty}}$, hence it induces a recollement:*

$$\begin{tikzcd} \frac{\widetilde{\mathcal{B}_\infty}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}_\infty) \arrow[l, bend right=50] \arrow[l, bend left=50] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{B}_\infty)}{\widetilde{\mathcal{B}_\infty}} \arrow[l, bend right=50] \arrow[l, bend left=50] \end{tikzcd} .$$

Proof. By Proposition 1.33 we have $ex\mathcal{B}_\infty = \widetilde{\mathcal{B}_\infty}$, hence the conclusion follows again by Theorem 2.11. $\square$

In view of Lemma 1.35 we have the following characterization.

Proposition 2.17. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in* Mod-R *with $\mathcal{B} \supseteq \mathcal{B}_\infty$.*

Then in $\mathrm{Ch}(\mathcal{B})$ there exists an injective cotorsion pair $(\tilde{\mathcal{B}}, \mathcal{R})$ with $\mathcal{R} \subseteq dw\mathrm{Inj}$ if and only if $ex\mathcal{B} = \tilde{\mathcal{B}}$.

Proof. Assume that in $\mathrm{Ch}(\mathcal{B})$ there is an injective cotorsion pair $(\tilde{\mathcal{B}}, \mathcal{R})$ with $\mathcal{R} \subseteq dw\mathrm{Inj}$. This means that $\mathcal{R} = \tilde{\mathcal{B}}^\perp \cap dw\mathcal{B}$. By Lemma 1.35 (3) and (4), $\mathcal{R} = dg\mathrm{Inj}$ and ${}^\perp\mathcal{R} \cap dw\mathcal{B} = ex\mathcal{B}$. Then, $ex\mathcal{B} = \tilde{\mathcal{B}}$.

Conversely, if $ex\mathcal{B} = \tilde{\mathcal{B}}$ then $(\tilde{\mathcal{B}}, dg\mathrm{Inj})$ is an injective cotorsion pair in $\mathrm{Ch}(\mathcal{B})$ by Example 2.8 (3). $\square$

Remark 2.18. Note that complete hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ satisfying $\mathcal{B} \supseteq \mathcal{B}_\infty$ may be abundant, since $\mathcal{B}_\infty$ may be rather small.

For instance, if the little finitistic dimension of R is finite (e.g. R is semihereditary), then $\mathcal{B}_\infty$ coincides with the class of injective modules (see Proposition 1.31).

A positive answer to the question in the title of this section is provided by Šťovíček in [44] for the cotorsion pair $(\mathcal{A}, \mathrm{FpInj})$ generated by the class of finitely presented modules over a coherent ring R. In view of Example 2.1 (1) and Example 2.8 (1), we restate Šťovíček's theorem in our notations.

Proposition 2.19. *([44, Proposition 6.11, Theorem 6.12]) Let R be a coherent ring and let $(\mathcal{A}, \mathrm{FpInj})$ be the complete hereditary cotorsion pair generated by the class of finitely presented modules. Then:*

$$^\perp dw\mathrm{Inj} \cap dw\mathrm{FpInj} = \widetilde{\mathrm{FpInj}}$$

hence $\left([dg\mathcal{A} \cap dw\mathrm{FpInj}]_K, \widetilde{\mathrm{FpInj}}, [dw\mathrm{Inj}]_K\right)$ *is a localizing cotorsion triple in $\mathrm{Ch}(\mathrm{FpInj})_{dw}$. There are equivalences:*

$$\frac{[dg\mathcal{A} \cap dw\mathrm{FpInj}]_K}{\sim} \cong \frac{\mathrm{Ch}(\mathrm{FpInj})}{\widetilde{\mathrm{FpInj}}} \cong \frac{[dw\mathrm{Inj}]_K}{\sim}$$

where $\sim$ is the chain homotopy equivalence and a recollement:

$$\begin{tikzcd}
\frac{\widetilde{\mathrm{FpInj}}}{\sim} \arrow[r, "inc"] & K(\mathrm{FpInj}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathrm{FpInj})}{\widetilde{\mathrm{FpInj}}} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd} \cong \mathcal{D}(\mathrm{FpInj}) \ .$$

We exhibit now another case of cotorsion pairs giving rise to a result analogous to Proposition 2.19

Proposition 2.20. *Let R be a coherent ring and let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in* Mod-R*. Assume that $\mathcal{B} \subseteq \mathrm{FpInj}$ and that $\mathcal{B} \subseteq \mathcal{I}_n$ (the class of modules of injective dimension at most n).*

Then, every Fp-injective $\mathcal{B}$-periodic module belongs to $\mathcal{B}$. Thus $\widetilde{\mathrm{FpInj}} \cap dw\mathcal{B} = \tilde{\mathcal{B}}$, ${}^{\perp}dw\mathrm{Inj} \cap dw\mathcal{B} = \tilde{\mathcal{B}}$ and we have a recollement:

$$\begin{tikzcd}
\frac{\widetilde{\mathcal{B}}}{\sim} \arrow[r, "inc"] & K(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{B})}{\widetilde{\mathcal{B}}} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd} \cong \mathcal{D}(\mathcal{B}) \ .$$

Proof. Let $(*) \quad 0 \to M \to B \to M \to 0$ be an exact sequence with M Fp-injective and $B \in \mathcal{B}$. Let $0 \to M \to E \to M_1 \to 0$ be an exact sequence with E injective; then, M_1 is Fp-injective. An application of the horseshoe lemma gives the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & E \oplus E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $D \in \mathcal{B}$, since $\mathcal{B}$ is coresolving. We have

$$\mathrm{inj.dim} D = \mathrm{inj.dim} B - 1,$$

hence w.l.o.g. we can assume that in our starting sequence $(*)$ B has injective dimension at most 1. Thus, in the above diagram we have that D is injective and, by Fact 1.32 (1), we conclude that M_1 is injective. The latter implies that inj.dim $M \leq 1$. Let $A \in \mathcal{A}$. Then

$$0 = \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^2_R(A, M) = 0,$$

hence $M \in \mathcal{B}$ and $\widetilde{\mathrm{FpInj}} \cap dw\mathcal{B} = \tilde{\mathcal{B}}$ by Fact 1.32 (2). Hence, the equality ${}^{\perp}dw\mathrm{Inj} \cap dw\mathcal{B} = \tilde{\mathcal{B}}$ is obtained by intersecting with $dw\mathcal{B}$ the equality ${}^{\perp}dw\mathrm{Inj} \cap dw\mathrm{FpInj} = \widetilde{\mathrm{FpInj}}$ from [44, Proposition 6.11].

The existence of a recollement as in the statement follows by the same arguments as in the proof of Proposition 2.19 applied to the cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the assumptions. □

2.4 INJECTIVE COTORSION PAIRS IN THE EXACT CATEGORY $\mathrm{Ch}(\mathcal{A})$

In this section we state results dual to the ones in Section 2.1. Their proofs are obtained by dual arguments.

Starting with a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$, we exhibit three examples of injective cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$ satisfying the assumptions of Proposition 1.40. Note that the examples below satisfy Proposition 1.40 since $\tilde{\mathcal{A}}$, $dg\mathcal{A}$ and $ex\mathcal{A}$ are thick in $\mathrm{Ch}(\mathcal{A})$ by Lemma 1.21 and they clearly contain the contractible complexes with terms in $\mathcal{A}$.

Example 2.21.

1. The complete hereditary cotorsion pair $(\widetilde{\mathcal{A}}, dg\mathcal{B})$ in $\mathrm{Ch}(\mathcal{G})$ satisfies the conditions in Proposition 1.40, hence we have the injective cotorsion pair:

$$\Delta_1 = \left(\tilde{\mathcal{A}}, dg\mathcal{B} \cap dw\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left(\tilde{\mathcal{A}}, [dg\mathcal{B} \cap dw\mathcal{A}]_K\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

2. The complete hereditary cotorsion pair $(dg\mathcal{A}, \widetilde{\mathcal{B}})$ in $\mathrm{Ch}(\mathcal{G})$ satisfies the conditions in Proposition 1.40, hence we have the injective cotorsion pair:

$$\Delta_2 = \left(dg\mathcal{A}, \tilde{\mathcal{B}} \cap dw\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left(dg\mathcal{A}, [\tilde{\mathcal{B}} \cap dw\mathcal{A}]_K\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

3. If $\mathcal{A}$ is deconstructible, then by Proposition 1.23 (5) $(ex\mathcal{A}, ex\mathcal{A}^{\perp})$ is a complete hereditary cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ and it satisfies the conditions in Proposition 1.40, hence we have the injective cotorsion pair

$$\Delta_3 = \left(ex\mathcal{A}, ex\mathcal{A}^{\perp} \cap dw\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left(ex\mathcal{A}, [ex^{\perp}\mathcal{A} \cap dw\mathcal{B}]_K\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

The three injective cotorsion pairs $\Delta_1, \Delta_2, \Delta_3$ of Example 2.21 satisfy the conditions of [21, Theorem 3.4], hence we have:

Theorem 2.22. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in a Grothendieck category $\mathcal{G}$ such that $\mathcal{A}$ is deconstructible. Then, there is a recollement:*

$$\begin{tikzcd}
\dfrac{\widetilde{\mathcal{B}} \cap dw\mathcal{A}}{\sim} \arrow[r, "inc"] & \dfrac{dg\mathcal{B} \cap dw\mathcal{A}}{\sim} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Ch}(\mathcal{A})/ex\mathcal{A} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $\sim$ denotes the homotopy relation associated to the corresponding model structure and coincides with the chain homotopy relation; moreover, inc is the inclusion and Q is the quotient functor.

Theorem 2.23. *In the setting of Example 2.21 (1),*

$$\Delta_1 = \left(dw\mathcal{A}, \widetilde{\mathcal{A}}, dg\mathcal{B} \cap dw\mathcal{A}\right)$$

is an exact model structure in the category $\mathrm{Ch}(\mathcal{A})$. In particular, we can define the derived category $\mathcal{D}(\mathcal{A})$ as the quotient $\mathrm{Ch}(\mathcal{A})/\widetilde{\mathcal{A}}$.

Moreover, we have the following triangle equivalences between the derived category of $\mathcal{D}(\mathcal{A})$ and the homotopy category of the model structure Δ_1:

$$\mathcal{D}(\mathcal{A}) = \mathrm{Ho}(\Delta_1) \cong \frac{dg\mathcal{B} \cap dw\mathcal{A}}{\widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}}$$

and in the assumptions of Theorem 2.22 there is also a recollement:

$$\begin{tikzcd}
\frac{ex\mathcal{A}}{\sim} \arrow[r, "inc"] & \mathcal{D}(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathcal{D}(\mathcal{G}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $ex\mathcal{A}/\sim$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of exact complexes (in $\mathrm{Ch}(\mathcal{G})$*).*

Proof. Dual of Theorem 2.5. □

Another way to obtain the exact model structure of Theorem 2.23 is to use results by Gillespie in [18], [20] and [23].

Theorem 2.24. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{G}$ such that $\mathcal{A}$ is deconstructible. The two cotorsion pairs $(dw\mathcal{A}, dw\mathcal{A}^{\perp})$ and $(\tilde{\mathcal{A}}, dg\mathcal{B})$ in* $\mathrm{Ch}(\mathcal{G})$ *are hereditary and complete and give rise to a cofibrantly generated model structure $\mathcal{N} = \left(dw\mathcal{A}, \mathcal{W}, dg\mathcal{B}\right)$ in* $\mathrm{Ch}(\mathcal{G})$ *satisfying $\mathcal{W} \cap dw\mathcal{A} = \tilde{\mathcal{A}}$ and $\mathcal{W} \cap dg\mathcal{B} = dw\mathcal{A}^{\perp}$ whose restriction in* $\mathrm{Ch}(\mathcal{A})$ *is the exact model structure $\mathcal{N}_1 = \left(dw\mathcal{A}, \widetilde{\mathcal{A}}, dw\mathcal{A} \cap dg\mathcal{B}\right)$ of Theorem 2.23.*

Proof. The smallness of the cotorsion pairs $(dw\mathcal{A}, dw\mathcal{A}^{\perp})$ and $(\tilde{\mathcal{A}}, dg\mathcal{B})$ follow by the fact that $dw\mathcal{A}$ and $\tilde{\mathcal{A}}$ are deconstructible in $\mathrm{Ch}(\mathcal{G})$ (see Proposition 1.23 (3) and 1.22 (3)) and they are hereditary since $(\mathcal{A}, \mathcal{B})$ is hereditary. The existence of the model structure $\mathcal{N}$ in $\mathrm{Ch}(R)$ follows by [23, Theorem 1.1]. The fact that the model structure is cofibrantly generated follows by [27, Section 7.4]. □

2.5 PROJECTIVE COTORSION PAIRS IN THE EXACT CATEGORY $\mathrm{Ch}(\mathcal{A})$

In this section we state results dual to the ones in Section 2.2.

Example 2.25. Starting with a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in a Grothendieck category $\mathcal{G}$ with enough projective objects, we exhibit three examples of projective cotorsion pairs in $\mathrm{Ch}(\mathcal{G})$ satisfying the assumptions of Proposition 1.42.

1. By Proposition 1.23 (3), $(dw\mathrm{Proj}, dw\mathrm{Proj}^{\perp})$ is a complete cotorsion pair in $\mathrm{Ch}(\mathcal{G})$, and it is a projective cotorsion

pair. By Proposition 1.42, we have the projective cotorsion pair:

$$\Gamma_1 = \left(dw\mathrm{Proj}, dw\mathrm{Proj}^{\perp} \cap dw\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left([dw\mathrm{Proj}]_K, dw\mathrm{Proj}^{\perp} \cap \mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

2. By Proposition 1.23 (6), $(ex\mathrm{Proj}, ex\mathrm{Proj}^{\perp})$ is a projective cotorsion pair in $\mathrm{Ch}(\mathcal{G})$ and by Proposition 1.42 we have the projective cotorsion pair:

$$\Gamma_2 = \left(ex\mathrm{Proj}, ex\mathrm{Proj}^{\perp} \cap dw\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left([ex\mathrm{Proj}]_K, ex\mathrm{Proj}^{\perp} \cap \mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

3. Since $(dg\mathrm{Proj}, \mathcal{E})$ is a projective cotorsion pair in $\mathrm{Ch}(\mathcal{G})$, by Proposition 1.42 we have the projective cotorsion pair:

$$\Gamma_3 = \left(dg\mathrm{Proj}, ex\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})$ and the localizing cotorsion pair

$$\left([dg\mathrm{Proj}]_K, ex\mathcal{A}\right)$$

in $\mathrm{Ch}(\mathcal{A})_{dw}$.

The above three examples $\Gamma_1, \Gamma_2, \Gamma_3$ of projective cotorsion pairs in $\mathrm{Ch}(\mathcal{A})$ satisfy the assumptions of [21, Theorem 3.5]. Hence we obtain:

Theorem 2.26. *If $\mathcal{G}$ is a Grothendieck category with enough projective objects and $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{G}$, there is a recollement*

$$\begin{tikzcd}
\frac{ex\mathrm{Proj}}{\sim} \arrow[r, "inc"] & \frac{dw\mathrm{Proj}}{\sim} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \mathrm{Ch}(\mathcal{A})/ex\mathcal{A} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

where $\sim$ denotes the homotopy relation associated to the corresponding model structure and coincides also with the chain homotopy relation; moreover, inc is the inclusion, Q is the quotient functor. In particular, the central term is the chain homotopy category $K(\mathrm{Proj})$ of the complexes with projective components and the right hand term is equivalent to the derived category of $\mathcal{G}$.

Moreover, we have:

Theorem 2.27. *If $\mathcal{G}$ is a Grothendieck category with enough projective objects and $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{G}$ such that $\mathcal{A}$ is deconstructible, the triple*

$$\Big([dg\mathrm{Proj}]_K, ex\mathcal{A}, [ex\mathcal{A}^{\perp} \cap dw\mathcal{A}]_K\Big)$$

is a localizing cotorsion triple in $\mathrm{Ch}(\mathcal{A})_{dw}$ and there are equivalences of triangulated categories:

$$\frac{[dg\mathrm{Proj}]_K}{\sim} \cong \frac{\mathrm{Ch}(\mathcal{A})}{ex\mathcal{A}} \cong \frac{[ex\mathcal{A}^{\perp} \cap dw\mathcal{A}]_K}{\sim}$$

where $\sim$ is the chain homotopy equivalence and a recollement:

$$\begin{tikzcd} \frac{ex\mathcal{A}}{\sim} \arrow[r, "inc"] & K(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\mathrm{Ch}(\mathcal{A})}{ex\mathcal{A}} \arrow[l, bend right] \arrow[l, bend left] & = \mathcal{D}(\mathcal{A})\,. \end{tikzcd}$$

Proof. By Examples 2.21 (3) and Examples 2.25 (3) we have two localizing cotorsion pairs $\Big([dg\mathrm{Proj}]_K, ex\mathcal{A}\Big)$ and $\Big(ex\mathcal{A}, [ex\mathcal{A}^{\perp} \cap dw\mathcal{A}]_K\Big)$ in $\mathrm{Ch}(\mathcal{A})_{dw}$. Thus, the statement follows by arguing as in the proof of Theorem 2.11.

□

2.6 WHEN IS $\tilde{\mathcal{A}}$ THE CENTRAL TERM OF A LOCALIZING COTORSION TRIPLE IN $\mathrm{Ch}(\mathcal{A})_{dw}$?

By Example 2.21 (1) we have shown that in $\mathrm{Ch}(\mathcal{A})$ there is an injective cotorsion pair with $\tilde{\mathcal{A}}$ as left term and Example 2.25 (3) provides a projective cotorsion pair in $\mathrm{Ch}(\mathcal{A})$ with right component $ex\mathcal{A}$. Our aim will be to find cotorsion pairs $(\mathcal{A}, \mathcal{B})$ for which there exists a projective cotorsion pair $(\mathcal{C}, \widetilde{\mathcal{A}})$ in $\mathrm{Ch}(\mathcal{A})$ with $\mathcal{C} \subseteq dw\mathrm{Proj}$ in order to obtain a localizing cotorsion triple in $\mathrm{Ch}(\mathcal{A})_{dw}$ with $\widetilde{\mathcal{A}}$ as central term.

The most famous example of this situation is provided by the cotorsion pair $(\text{Flat}, \text{Cot})$. In fact, by [36, Theorem 8.6] $dw(\text{Proj})^{\perp} \cap dw\text{Flat} = \widetilde{\text{Flat}}$. Hence, as noted by Gillespie in [21], Example 2.25 (1) provides the wanted example and induces Neeman's recollement, that is the recollement as in Theorem 2.27:

$$(a) \quad \begin{tikzcd} \frac{\widetilde{\text{Flat}}}{\sim} \arrow[r, "inc"] & K(\text{Flat}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\text{Ch}(\text{Flat})}{\widetilde{\text{Flat}}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Another case of cotorsion pairs giving rise to a result analogous to the previous one is provided by the following:

Proposition 2.28. *Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in* $\text{Mod-}R$*. Assume that $\mathcal{A} \subseteq \text{Flat}$ and that $\mathcal{A} \subseteq \mathcal{P}_n$, where $\mathcal{P}_n$ is the class of modules of projective dimension at most n. Then, every flat $\mathcal{A}$-periodic module belongs to $\mathcal{A}$. Thus $\widetilde{\text{Flat}} \cap dw\mathcal{A} = \tilde{\mathcal{A}}$, $(dw\text{Proj})^{\perp} \cap dw\mathcal{A} = \tilde{\mathcal{A}}$ and there is a recollement:*

$$\begin{tikzcd} \frac{\widetilde{\mathcal{A}}}{\sim} \arrow[r, "inc"] & K(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \frac{\text{Ch}(\mathcal{A})}{\widetilde{\mathcal{A}}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. Let $0 \to F \to A \to F \to 0$ be an exact sequence with F flat and $A \in \mathcal{A}$. Let $0 \to F_1 \to P \to F \to 0$ be an exact sequence with P projective; then F_1 is flat. An application of the horseshoe lemma gives the following commutative diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & F_1 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P \arrow[r] \arrow[d] & P \oplus P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where $Q \in \mathcal{A}$, since $\mathcal{A}$ is resolving. We have $\text{p.dim } Q = \text{p.dim } A - 1$, hence w.l.o.g. we can assume that in our starting sequence $0 \to F \to A \to F \to 0$ A has projective dimension at most 1. Thus, in the above diagram we have that Q is projective and by [8], F_1 is projective. The latter implies that p.dim $F \leq 1$. Let $B \in \mathcal{B}$. Then

$$0 = \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(F, B) \to \mathrm{Ext}^2_R(F, B) = 0,$$

hence $F \in \mathcal{A}$ and $\widetilde{\mathrm{Flat}} \cap dw\mathcal{A} = \tilde{\mathcal{A}}$ by Fact 1.32 (2). Now the equality $(dw\mathrm{Proj})^{\perp} \cap dw\mathcal{A} = \tilde{\mathcal{A}}$ is obtained by intersecting with $dw\mathcal{A}$ the equality $dw(\mathrm{Proj})^{\perp} \cap dw\mathrm{Flat} = \widetilde{\mathrm{Flat}}$ from [36, Theorem 8.6].

The arguments used above to obtain the recollement (a) for the cotorsion pair $(\mathrm{Flat}, \mathrm{Cot})$ can be repeated for the case of the cotorsion pair $(\mathcal{A}, \mathcal{B})$ in our assumption to obtain the stated recollement. □

Remark 2.29. The above proposition applies, for example, to the case of the cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by the localizations $\{R[s^{-1}] \mid s \in R\}$ of a commutative ring R. The class $\mathcal{A}$ was introduced by Positselski in [38] and called the class of very flat modules. Clearly $\mathcal{A} \subsetneq \mathrm{Flat} \cap \mathcal{P}_1$.

Another situation is provided by Proposition 1.27:

Proposition 2.30. *Let $\mathcal{G}$ be a Grothendieck category with enough projective objects. Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathcal{G}$ with $\mathcal{B} \subseteq \mathcal{I}$ ($\mathcal{I}$ the class of objects with finite injective dimension). Then $ex\mathcal{A} = \tilde{\mathcal{A}}$, hence $(dg\mathrm{Proj}, \tilde{\mathcal{A}})$ is a projective cotorsion pair in $\mathrm{Ch}(\mathcal{A})$ and there is a recollement as in Theorem 2.27 with $ex\mathcal{A}$ replaced by $\tilde{\mathcal{A}}$. In particular, the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is equivalent to the usual derived category of $\mathcal{G}$.*

Corollary 2.31. *Let $(\mathcal{C}, \mathcal{B})$ be an n-cotilting cotorsion pair in* Mod-R. *For the cotilting class $\mathcal{C}$ we have a recollement:*

$$\begin{tikzcd} \dfrac{\widetilde{\mathcal{C}}}{\sim} \arrow[r, "inc"] & K(\mathcal{C}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "Q"] & \dfrac{\mathrm{Ch}(\mathcal{C})}{\widetilde{\mathcal{C}}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. By Proposition 1.28, $ex\mathcal{C} = \widetilde{\mathcal{C}}$, hence the conclusion follows by Proposition 2.30. □

3 NAKAOKA CONTEXTS WITH ABELIAN HEARTS

Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure in a triangulated category $\mathscr{C}$. It is well known (see [7]) that its heart $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category.

A different result but in the same spirit arises for a cluster tilting subcategory $\mathscr{T}$ of a triangulated category $\mathscr{C}$. It was showed by Koenig and Zhu in [30] that in this case $\mathscr{C}/\mathscr{T}$ carries an abelian structure.

In both cases the construction of the abelian structure, i.e. of the kernels and cokernels, relies heavily on the triangulated structure of $\mathscr{C}$. In [32], Nakaoka proved that starting from any torsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathscr{C}$ it is possible to construct an abelian heart $\underline{\mathcal{H}}$ in the quotient category $\underline{\mathscr{C}} = \mathscr{C}/\mathcal{W}$, where $\mathcal{W} = \mathcal{U}[1] \cap \mathcal{V}$. This result recovers both cases above giving a unified way of constructing kernels and cokernels in the heart.

Keeping this setting in mind, we will provide a set of axioms for a pair of torsion pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ in a (sufficiently nice) additive category in such a way that $\mathcal{H} = \mathcal{T}_1 \cap \mathcal{F}_2$ becomes an abelian category.

Since we don't have a triangulated structure to rely on, the axioms will impose some requirements on $\mathbb{t}_1$ and $\mathbb{t}_2$ to compensate. Moreover, we will require that idempotents split in $\mathscr{C}$.

In Section 3.1 we will introduce the notion of torsion pairs in additive categories and show some elementary properties of left (and right) functorial torsion pairs. These will coincide with the usual notion of torsion pairs in the case of Abelian categories. Moreover, any t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ induces a torsion pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ in this sense.

In Section 3.2 we will use these as building bricks to define Nakaoka contexts, i.e. couples of torsion pairs $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, $\mathbb{t}_1$ is left functorial and $\mathbb{t}_2$ is right functorial. In particular, we will be interested in pre-Abelian Nakaoka contexts, namely those whose heart $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ has all kernels and cokernels. Finally, we will give conditions for a pre-Abelian Nakaoka context that will ensure the abelianity of $\mathcal{H}$.

In Section 3.3 we will study Nakaoka contexts in Abelian and triangulated categories.

A Nakaoka context in an Abelian category $\mathscr{A}$ is simply a couple of torsion pairs $(\mathbb{t}_1, \mathbb{t}_2)$ such that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, since their functoriality is automatic. Furthermore, any Nakaoka context is pre-Abelian, and actually its heart is an integral category. In this setting, it is possible to show that the heart $\mathcal{H}$ of a Nakaoka context is abelian if and only if it is closed under kernels and cokernels in $\mathscr{A}$ (Theorem 3.18).

In the case of triangulated categories we will restrict our attention to a special class of Nakaoka contexts arising from t-structures that we will call related pairs. The important property of these Nakaoka contexts is that they satisfy the additional inclusion $\mathcal{T}_1[1] \subseteq \mathcal{T}_2$. We will show that any related pair is a pre-Abelian Nakaoka context, and that strongly related pairs have an abelian heart (Theorem 3.23).

Finally, starting with a related pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, we will use Polishchuk correspondence to see that its heart $\mathcal{H}$ is a torsion free class in $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$ (i.e. the heart of the t-structure $(\mathcal{T}_1, \mathcal{F}_1[1])$). From this, we will prove that $\mathcal{H}$ is abelian if and only if it is closed under quotients in $\mathcal{H}_1$ (Theorem 3.30) and that there is a bijection between related pairs with abelian heart and t-structures whose heart contains a cohereditary torsion pair (Theorem 3.33).

The results in this chapter are part of a work in preparation by the author in conjunction with Manuel Saorín, Simone Virili and Octavio Mendoza.

3.1 TORSION PAIRS

Let $\mathscr{C}$ be an additive category and $\mathcal{A} \subseteq \mathscr{C}$ a class of objects. We introduce the following notation:

- $\mathcal{A}^{\perp} = \{X \in \mathscr{C} \,|\, \mathrm{Hom}_{\mathscr{C}}(\mathcal{A}, X) = 0\}$
- ${}^{\perp}\mathcal{A} = \{X \in \mathscr{C} \,|\, \mathrm{Hom}_{\mathscr{C}}(X, \mathcal{A}) = 0\}$.

Given a map $\varphi : X \to Y$, recall that a *pseudocokernel* of φ is a map $\psi : Y \to Z$ such that $\psi\varphi = 0$ and any other $\psi' : Y \to Z$ satisfying $\psi'\varphi = 0$ factors (not necessarily uniquely) through ψ. There an obvious dual notion of *pseudokernel*.

Recall that a full subcategory $\mathcal{T}$ of $\mathscr{C}$ is *coreflective* (resp. *reflective*) if the inclusion functor $i : \mathcal{T} \to \mathscr{C}$ has a right (resp. left) adjoint t.

Definition 3.1. A pair $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathscr{C}$ is a *torsion pair* if:

1. $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$,
2. given $X \in \mathscr{C}$ there is a pseudokernel-pseudocokernel sequence

$$T_X \xrightarrow{\varepsilon_X} X \xrightarrow{\lambda_X} F^X$$

   where $T_X \in \mathcal{T}$ and $F^X \in \mathcal{F}$.

Moreover, a torsion pair $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ is called *left* (resp. *right*) *functorial* if $\mathcal{T}$ (resp. $\mathcal{F}$) is a coreflective (resp. reflective) subcategory of $\mathscr{C}$. If $\mathbb{t}$ is both left and right functorial it is called just *functorial*.

Remark. Our definition of a torsion pair in additive categories requires the existence of a pseudokernel-pseudocokernel sequence, while the usual definition of a torsion pair in abelian categories requires a short exact sequence instead. However, as we'll show in the following lemma, the two definitions actually coincide in abelian categories.

Lemma 3.2. *Let $\mathscr{C}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ be a torsion pair according to Definition 3.1. Then, for any $X \in \mathscr{C}$ there is a short exact sequence*

$$0 \longrightarrow T_X \longrightarrow X \longrightarrow F^X \longrightarrow 0$$

with $T_X \in \mathcal{T}$ and $F^X \in \mathcal{F}$.

Proof. Consider a pseudokernel-pseudocokernel sequence $T_X \xrightarrow{\varepsilon_X} X \xrightarrow{\lambda^X} F^X$ with $T_X \in \mathcal{T}$ and $F^X \in \mathcal{F}$, and let $K = \mathrm{Ker}(\lambda^X)$. Then, we have the following diagram:

$$\begin{tikzcd}
 & K \arrow[d, tail, "k"] \arrow[dl, dashed, "v"] & \\
T_X \arrow[ur, dashed, "u"] \arrow[r, "\varepsilon_X"'] & X \arrow[r, "\lambda^X"] & F^X
\end{tikzcd}$$

where u and v are given by the (pseudo)kernel properties of K and T_X respectively, i.e. $k \circ u = \varepsilon_X$ and $\varepsilon_X \circ v = k$. Combining the two we get that $k \circ u \circ v = \varepsilon_X \circ v = k$, so $u \circ v = 1_K$ (since k is mono). Hence, u is a section and $K \underset{\oplus}{<} T_X$.

This implies that $\mathrm{Hom}_{\mathscr{C}}(K, F) = 0$ for all $F \in \mathcal{F}$, i.e. $K \in \mathcal{T}$. Moreover, λ^X is a pseudocokernel of k. Thus, for any $X \in \mathscr{C}$ there is a short exact sequence $0 \to K \xrightarrow{k} X \xrightarrow{\lambda^X} F^X$ with $K \in \mathcal{T}$ and $F^X \in \mathcal{F}$ such that λ^X is a pseudocokernel of k.

To conclude the proof, consider the cokernel of k and use a dual argument to get the desired short exact sequence. □

It is a classical result that torsion pairs in Abelian categories are automatically functorial. Similarly, if $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure in a triangulated category, $(\mathcal{T}^{\leq 0}[1], \mathcal{T}^{\geq 0})$ is an example of functorial torsion pair.

The following lemma gives basic properties of left functorial torsion pairs. The proof is omitted, since it is a straightforward application of the definitions. Of course, the dual statement holds for right functorial torsion pairs.

Lemma 3.3. *Let $\mathbb{t} = (\mathcal{T}, \mathcal{F})$ be a left functorial torsion pair in $\mathscr{C}$. Then*

(a) for any $M \in \mathscr{C}$, $T' \in \mathcal{T}$ and $\alpha \in \mathrm{Hom}_{\mathscr{C}}(T', M)$ there is a unique $\alpha' \in \mathrm{Hom}_{\mathscr{C}}(T', t(M))$ such that $\varepsilon_M \circ \alpha' = \alpha$;

(b) for a morphism $g : T_1 \to T_2$ in $\mathcal{T}$, any pseudocokernel $g^C : T_2 \to C$ in $\mathcal{T}$ is also a pseudocokernel of g in $\mathscr{C}$.

3.2 NAKAOKA CONTEXTS IN ADDITIVE CATEGORIES

In this section we will introduce the notion of a Nakaoka context in an additive category and provide axioms that will guarantee the existence of kernels and cokernels in the heart and later ensure its abelianity.

3.2.1 *Nakaoka contexts and the heart construction*

Definition 3.4. A *Nakaoka context* is a pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ of torsion pairs in $\mathscr{C}$, satisfying the following axioms:

(CT.1) $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are respectively a left functorial and a right functorial torsion pair;

(CT.2) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ (equiv. $\mathcal{F}_1 \subseteq \mathcal{F}_2$).

Notation. If $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a Nakaoka context, we will always use the notation $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$ and the coreflection and reflection will be indicated as:

$$(i_1, t_1)\colon \mathcal{T}_1 \underset{t_1}{\overset{i_1}{\rightleftarrows}} \mathscr{C} \qquad \text{and} \qquad (f_2, j_2)\colon \mathcal{F}_2 \underset{j_2}{\overset{f_2}{\leftrightarrows}} \mathscr{C}.$$

The counit of the adjunction (i_1, t_1) will be denoted by $\varepsilon_1\colon t_1 i_1 \to \mathrm{id}_{\mathcal{T}_1}$, while the unit of (f_2, j_2) will be denoted by $\lambda_2\colon \mathrm{id}_{\mathcal{F}_2} \to f_2 j_2$.

Definition 3.5. The *heart* of a Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$.

In the following lemma we explicitly state two general observations about Nakaoka contexts.

Lemma 3.6. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a Nakaoka context. Then, the followings hold true:*

(a) $\mathcal{F}_1 \cap \mathcal{T}_2 = 0$;

(b) $f_2(\mathcal{T}_1) \subseteq \mathcal{H}$ *and* $t_1(\mathcal{F}_2) \subseteq \mathcal{H}$.

Proof. (a) Since $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we have $\mathcal{F}_1 \cap \mathcal{T}_2 \subseteq \mathcal{F}_1 \cap \mathcal{T}_1 = 0$.
(b) Let $T_2 \in \mathcal{T}_2$ and $F_2 \in \mathcal{F}_2$. Then,

$$\mathrm{Hom}_{\mathscr{C}}(T_2, t_1(F_2)) \cong \mathrm{Hom}_{\mathscr{C}}(T_2, F_2) = 0.$$

Hence, $t_1(F_2) \in \mathcal{T}_2^{\perp} = \mathcal{F}_2$. An analogous proof shows that $f_2(\mathcal{T}_1) \subseteq \mathcal{T}_1$. □

In the following lemma we introduce a technical condition under which we can easily construct kernels of morphisms in the heart of a given Nakaoka context.

Lemma 3.7. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a Nakaoka context and let $f\colon H \to H'$ be a morphism in the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$. If there is a morphism $f^K\colon K \to H$, with $K \in \mathcal{F}_2$, such that the following sequence is exact in* $\mathrm{Func}(\mathcal{T}_1, \mathrm{Ab})$

$$(*) \qquad 0 \to (-, K) \restriction_{\mathcal{T}_1} \to (-, H) \restriction_{\mathcal{T}_1} \to (-, H') \restriction_{\mathcal{T}_1},$$

then the composition $f^K \circ \varepsilon_{1,K}\colon t_1 K \to K \to H$ is a kernel for f in $\mathcal{H}$.

Proof. Consider the exact sequence in $(*)$ and notice that it gives, by restriction of the functors, an exact sequence of the form:

$$0 \to (-, K) \restriction_{\mathcal{H}} \to (-, H) \restriction_{\mathcal{H}} \to (-, H') \restriction_{\mathcal{H}} .$$

The map $\varepsilon_{1,K}\colon t_1K \to K$, induces a natural isomorphism $(-,t_1K)\restriction_{\mathcal{H}} \to (-,K)\restriction_{\mathcal{H}}$, so we get an exact sequence

$$0 \longrightarrow (-,t_1K)\restriction_{\mathcal{H}} \xrightarrow{k\circ\varepsilon_{1,K}\circ -} (-,H)\restriction_{\mathcal{H}} \longrightarrow (-,H')\restriction_{\mathcal{H}} .$$

To conclude one notices that, since $t_1K \in \mathcal{H}$ by Lemma 3.6, the above exact sequence means exactly that $f^K \circ \varepsilon_{1,K}$ is a kernel of f. □

3.2.2 *Pre-abelian Nakaoka contexts*

Recall that an additive category is *pre-Abelian* if any morphism has a kernel and a cokernel. In view of Lemma 3.7, it is natural to introduce the following definition:

Definition 3.8. A Nakaoka context is said to be *pre-Abelian* if it satisfies the following axioms:

(CT.3) any $g\colon H \to H'$ in $\mathcal{H}(\subseteq \mathcal{T}_1)$ admits a pseudocokernel $g^C\colon H' \to C$ in $\mathcal{T}_1$, such that

$$0 \longrightarrow (C,-)_{|\mathcal{F}_2} \xrightarrow{(g^C,-)} (H',-)_{|\mathcal{F}_2} \xrightarrow{(g,-)} (H,-)_{|\mathcal{F}_2}$$

is an exact sequence in $\mathrm{Func}(\mathcal{F}_2, \mathrm{Ab})$.

(CT.3*) Dual of **(CT.3)**.

Theorem 3.9. *For a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is a pre-Abelian category.*

Proof. This is a consequence of the axioms, Lemma 3.7 and its dual. □

In the following proposition we give a characterization of those morphisms that are monomorphisms in the heart. For this, remember that, in a pre-Abelian category, a morphism is mono if and only if its kernel is trivial.

Proposition 3.10. *The following are equivalent for a morphism $f\colon H \to H'$ in the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ of a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$:*

(a) f is a monomorphism (in $\mathcal{H}$);

(b) there is a pseudokernel $f^K\colon K \to H$ of f in $\mathcal{F}_2$ such that $K \in \mathcal{F}_1$.

Proof. For any morphism $f\colon H \to H'$ in $\mathcal{H}$, by the axiom (CT.3*), we can consider a diagram as follows

$$\begin{tikzcd} F_2 \arrow[r, "f^K"] & H \arrow[r, "f"] & H' \\ t_1F_2 \arrow[u, "\varepsilon_{1,F_2}"] \arrow[ur, "\bar{f}^K"'] & & \end{tikzcd}$$

where $F_2 \in \mathcal{F}_2$ is a pseudo-kernel of f in $\mathcal{F}_2$ and, by Lemma 3.7, $t_1F_2 \to H$ is the kernel of f in $\mathcal{H}$.

(a)⇒(b). Since f is a monomorphism in $\mathcal{H}$, its kernel is trivial, that is, $t_1F_2 = 0$, i.e. $F_2 \in \mathcal{F}_1$.

(b)⇒(a). If $f^K\colon K \to H$ is a pseudokernel of f in $\mathcal{F}_2$ such that $K \in \mathcal{F}_1$, then the kernel $0 = t_1K \to H$ of f in $\mathcal{H}$ is trivial, that is, f is a monomorphism. □

3.2.3 *Integral Nakaoka contexts*

Our final objective is to find conditions that make $\mathcal{H}_{\mathbb{t}}$ into an Abelian category. For a pre-Abelian category, being Abelian amounts to saying that every mono is a kernel and every epi is a cokernel. However, we can consider a middle step before abelianity, namely *integral categories*.

Definition 3.11. A pre-Abelian category is called *integral* if pullbacks of epis are epi and pushout of monos are mono.

An integral category is Abelian if any morphism that is both epi and mono is an isomorphism. In this section and the next we will introduce axioms that guarantee the integrality and abelianity, respectively, of $\mathcal{H}$.

Definition 3.12. A pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is said to be *integral* provided the following axioms hold:

(CT.4.1) Given two morphisms $f\colon H \to H'$ and $g\colon K \to H'$ in $\mathcal{H}$ such that f has a pseudo-cokernel $f^C\colon H' \to C$ with $C \in \mathcal{T}_2$, we can construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, dotted, "x_K"] \arrow[d, dotted, "x_H"'] & K \arrow[r, "x_K^C"] \arrow[d, "g"] & C' \\ H \arrow[r, "f"'] & H' \arrow[r, "f^C"'] & C \end{tikzcd}$$

where $x_K^C\colon K \to C$ is a pseudo-cokernel of x_K, $X \in \mathcal{H}$ and $C' \in \mathcal{T}_2$.

(CT.4.1*) Dual to (CT.4.1).

Theorem 3.13. *The following are equivalent for a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$:*

(a) $\mathbb{t}$ is integral;

(b) the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an integral pre-Abelian category.

Proof. (a)⇒(b). Consider a pullback diagram in $\mathcal{H}$:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "PB"] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

and suppose that f is an epi in $\mathcal{H}$. By (the dual of) Proposition 3.10, we can find a pseudo-cokernel $f^C \colon D \to T_2$ of f, with $T_2 \in \mathcal{T}_2$. By the axiom (CT.4.1), we obtain a commutative diagram as follows

$$\begin{tikzcd} X \arrow[r, dotted, "x_B"] \arrow[d, dotted, "x_C"'] & B \arrow[r, "f''"] \arrow[d, "g"] & T_2' \\ C \arrow[r, "f"'] & D \arrow[r, "f^C"'] & T_2 \end{tikzcd}$$

where $f'' \colon B \to T_2$ is a pseudo-cokernel of x_B, and $X \in \mathcal{H}$. Since the square we started with is a pullback, there is a unique morphism $\alpha \colon X \to A$, making the following diagram commute:

$$\begin{tikzcd} X \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "x_B"] \arrow[ddr, bend right, "x_C"'] & & \\ & A \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "PB"] & B \arrow[d, "g"] \\ & C \arrow[r, "f"'] & D \end{tikzcd}$$

Now, since x_B is a morphism in $\mathcal{H}$ that admits a pseudo-cokernel (in $\mathcal{T}_1$) which belongs to $\mathcal{T}_2$, x_B is an epimorphism in $\mathcal{H}$ by Proposition 3.10. Furthermore, since $x_B = f' \circ \alpha$, also f' is an epi. One can prove dually that pushouts of monomorphisms are mono, so $\mathcal{H}$ is integral.

(b)⇒(a). Suppose that $\mathcal{H}$ is an integral category and let us show how this implies the axiom (CT.4.1), one can verify (CT.4.1*) dually. Given two morphisms $f\colon H \to H'$ and $g\colon K \to H'$ in $\mathcal{H}$ such that f has a pseudo-cokernel $f^C\colon H' \to C$ with $C \in \mathcal{T}_2$, we can take the following pullback diagram in $\mathcal{H}$

$$\begin{tikzcd} X \arrow[r, "x_K"] \arrow[d, "x_H"'] \arrow[dr, phantom, "PB"] & K \arrow[d, "g"] \\ H \arrow[r, "f"'] & H' \end{tikzcd}$$

Now, by the integrality of $\mathcal{H}$, x_K is an epimorphism in $\mathcal{H}$ and so, by Proposition 3.10, it admits a pseudo-cokernel $K \to C'$ (in $\mathcal{T}_1$) such that $C' \in \mathcal{T}_1$. □

3.2.4 *Abelian Nakaoka contexts*

Definition 3.14. An integral Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in $\mathscr{C}$ is said to be *Abelian* provided the following axioms hold:

(CT.4.2) A morphism $f\colon H \to H'$ in $\mathcal{H}$ is an isomorphism if and only if it fits in a sequence:

$$K \xrightarrow{f^K} H \xrightarrow{f} H' \xrightarrow{f^C} C$$

where f^K is a pseudo-kernel of f in $\mathcal{F}_2$, f^C is a pseudo-cokernel of f in $\mathcal{T}_1$, $K \in \mathcal{F}_1$ and $C \in \mathcal{T}_2$;

(CT.4.2*) dual to (CT.4.2).

Theorem 3.15. *The following are equivalent for a pre-Abelian Nakaoka context $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$:*

(a) the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an Abelian category;

(b) $\mathbb{t}$ is Abelian (that is, it satisfies (CT.4.1), (CT.4.2) and their duals);

(c) $\mathbb{t}$ satisfies the following axioms:

(CT.4) given a morphism $f\colon H \to H'$ in $\mathcal{H}$ that admits a pseudo-kernel $f^K\colon H'' \to H$ in $\mathcal{F}_2$, such that $H'' \in \mathcal{F}_1$, and the commutative diagram

$$\begin{tikzcd} & H \arrow[r, "f"] \arrow[d, "a"] & H' \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d, "\lambda_{2,T_1}"] \\ t_1F_2 \arrow[ur, dotted, "b"] \arrow[r, "\varepsilon_{1,F_2}"] & F_2 \arrow[r, "g^K"] & H' \arrow[r, "g"] & f_2T_1 \end{tikzcd}$$

where f^C is a pseudo-cokernel of f in $\mathcal{T}_1$ and g^K is a pseudo-kernel of g in $\mathcal{F}_2$, there exists a morphism $b\colon t_1F_2 \to H_1$ such that $ab = \varepsilon_{1,F_2}$;

(CT.4∗) dual to (CT.4).

Proof. (a)⇔(b). This is almost tautological since, by Theorem 3.13, (CT.4.1) is equivalent to saying that $\mathcal{H}$ is integral while, by Proposition 3.10, (CT.4.2) is equivalent to saying that, in $\mathcal{H}$, morphisms that are both epi and mono are isomorphisms; to conclude it is enough to notice that Abelian categories are exactly those pre-Abelian categories which are integral and where morphisms that are both epi and mono are isomorphisms.

(a)⇒(c). Let $f\colon H \to H'$ be a morphism in $\mathcal{H}$ that admits a pseudo-kernel $f^K\colon K \to H$ in $\mathcal{F}_2$, such that $K \in \mathcal{F}_1$, and the commutative diagram

$$\begin{tikzcd}
 & H \arrow[r, "f"] \arrow[d, "a"] \arrow[dl, dotted, "\beta"'] & H' \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d, "\lambda_{2,T_1}"] \\
t_1F_2 \arrow[r, "\varepsilon_{1,F_2}"] & F_2 \arrow[r, "g^K"] & H' \arrow[r, "g"] & f_2T_1
\end{tikzcd}$$

where f^C is a pseudo-cokernel of f in $\mathcal{T}_1$ and g^K is a pseudo-kernel of g in $\mathcal{F}_2$. Since f is a monomorphism in $\mathcal{H}$ by Proposition 3.10 (a), and $\mathcal{H}$ is Abelian, we have that $f = \mathrm{Ker}_{\mathcal{H}}\,\mathrm{Coker}_{\mathcal{H}}(f)$. On the other hand, by Lemma 3.3 (a), there is $\beta\colon H_1 \to t_1F_2$ such that $\varepsilon_{1,F_2}\beta = a$ completing the diagram above. Since $f = \mathrm{Ker}\,\mathrm{Coker}(f)$, it follows that β is an isomorphism and (CT.4) follows by setting $b := \beta^{-1}$. (CT.4*) can be verified by a dual argument.

(c)⇒(a). By Theorem 3.9, $\mathcal{H}$ is pre-Abelian. To prove that $\mathcal{H}$ is Abelian, we just need to show that any monomorphism (resp. epimorphism) is a kernel (resp. cokernel). Hence, let $f\colon H \to H'$ be a monomorphism in $\mathcal{H}$; by Proposition 3.10 we know that f admits a pseudo-kernel in $\mathcal{F}_2$ that belongs in $\mathcal{F}_1$ and thus, by (CT.4), we get a commutative diagram as follows

$$\begin{tikzcd}
 & H \arrow[r, "f"] \arrow[d, "a"] & H' \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d, "\lambda_{2,T_1}"] \\
t_1F_2 \arrow[r, "\varepsilon_{1,F_2}"] \arrow[ur, "b"] & F_2 \arrow[r, "g^K"] & H' \arrow[r, "g"] & f_2T_1
\end{tikzcd}$$

where f^C is a pseudo-cokernel of f in $\mathcal{T}_1$ and g^K is a pseudo-kernel of g in $\mathcal{F}_2$. Let $\alpha\colon H \to H'$ be a morphism such that

$g\alpha = 0$. Since F_2 is a pseudo-kernel of g, there is a morphism $\alpha' \colon H \to F_2$ such that $g^K\alpha' = \alpha$. By Lemma 3.3, α' factors through ε_{1,F_2}, as in the following diagram:

$$\begin{tikzcd}
 & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & T_1 \arrow[d, "\lambda_{2,T_1}"] \\
t_1F_2 \arrow[ur, "b"] \arrow[r, "\varepsilon_{1,F_2}"] & F_2 \arrow[r, "g^K"] & H_2 \arrow[r, "g"] & f_2T_1 \\
 & H. \arrow[ul, dotted, "\alpha''"] \arrow[u, dotted, "\alpha'"] \arrow[ur, "\alpha"] & &
\end{tikzcd}$$

By setting $\alpha''' := b\alpha'' \colon H \to H$, we get

$$f\alpha''' = g^K ab\alpha'' = g^K\varepsilon_{1,F_2}\alpha'' = g^K\alpha' = \alpha.$$

Thus, any morphism $\alpha \colon H \to H'$ such that $g\alpha = 0$ factors through f. □

3.3 NAKAOKA CONTEXTS IN SPECIAL CATEGORIES

Now we will apply the results of the previous section to some well behaved categories, namely Abelian and triangulated. In the first case we will show that any Nakaoka context is pre-Abelian and show necessary and sufficient conditions for the abelianity of the heart.

For triangulated categories we will restrict to Nakaoka contexts built from t-structures and investigate when their heart is abelian.

3.3.1 *Nakaoka contexts in Abelian categories*

Let's consider the case $\mathscr{C} = \mathscr{A}$ of an Abelian category with two torsion pairs $\mathbb{t}_i = (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$.

Lemma 3.16. *For an Abelian category $\mathscr{A}$, any Nakaoka context is integral.*

Proof. Let $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we need to show that **(CT.1)**, **(CT.3)** and **(CT.3*)** hold.

(CT.1) It is well known that any torsion pair in an abelian category is functorial.

(CT.3) Let $g : T_1 \to T_1'$ be a morphism in $\mathcal{T}_1$. Consider the cokernel morphism of g in $\mathscr{A}$

$$\mathrm{Coker}_{\mathscr{A}}(T_1 \xrightarrow{g} T_1') = (T_1' \xrightarrow{c_g} \mathrm{Coker}(g)).$$

Since $\mathcal{T}_1$ is closed under quotient objects, we get that $\mathrm{Coker}(g) \in \mathcal{T}_1$. Therefore, we can choose $c_g : T_1' \to \mathrm{Coker}(g)$ as $g^C : T_1' \to \mathrm{PCok}_{\mathscr{A}}(g)$.

(CT.3*) Analogous to the previous.

Let us now verify the axiom (CT.4.1). Indeed, consider two morphisms $f \colon H \to H'$ and $g \colon K \to H'$ in $\mathcal{H}$ such that f has a pseudo-cokernel $f^C \colon H' \to C$ with $C \in \mathcal{T}_2$ and consider the following pullback diagram in $\mathscr{A}$

$$\begin{tikzcd}
X \arrow[r, dotted, "x_K"] \arrow[d, dotted, "x_H"'] \arrow[dr, phantom, "P.B."] & K \arrow[r, "x_K^C"] \arrow[d, "g"] & C \arrow[d, equal] \\
H \arrow[r, "f"'] & H' \arrow[r, "f^C"'] & C
\end{tikzcd}$$

where $x_K^C \colon K \to C$ is a pseudo-cokernel of x_K. Clearly $X \leq H \oplus K$, so that $X \in \mathcal{F}_2$.

$$\begin{tikzcd}
t_1X \arrow[drr, dotted] \arrow[ddr, dotted] & & & \\
& X \arrow[r, dotted, "x_K"] \arrow[d, dotted, "x_H"'] \arrow[dr, phantom, "P.B."] & K \arrow[r, "x_K^C"] \arrow[d, "g"] & C \arrow[d, equal] \\
& H \arrow[r, "f"'] & H' \arrow[r, "f^C"'] & C
\end{tikzcd}$$

□

Corollary 3.17. *Let* $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ *Nakaoka context in* $\mathscr{A}$. *Then, for* $f : H_1 \to H_2$ *in* $\mathcal{H}$, *the following statements hold:*

(a) the cokernel of f *in* $\mathcal{H}$ *is the composition of the morphisms*

$$H_2 \xrightarrow{c_f} \mathrm{Coker}(f) \xrightarrow{\lambda_{2,\mathrm{Coker}(f)}} f_2(\mathrm{Coker}(f));$$

(b) the kernel of f *in* $\mathcal{H}$ *is the composition of the morphisms*

$$t_1(\mathrm{Ker}(f)) \xrightarrow{\varepsilon_{1,\mathrm{Ker}(f)}} \mathrm{Ker}(f) \xrightarrow{k_f} H_1;$$

(c) *f is an epimorphism in $\mathcal{H}$ if and only if* $\mathrm{Coker}(f) \in \mathcal{T}_2$*;*

(d) *f is a monomorphism in $\mathcal{H}$ if and only if* $\mathrm{Ker}(f) \in \mathcal{F}_1$.

Proof. (a) and (b) follow from the proof of Lemmas 3.16 and 3.7.

(c)

$\Leftarrow$ is trivial.

$\Rightarrow$ By 3.10(b) there exists $f^C : H_2 \to T_2$, where

$$T_2 = \mathrm{Coker}_{T_1}(f) \in \mathcal{T}_2.$$

Then, we have

$$\begin{tikzcd} H_2 \arrow[rr, "f^C"] \arrow[dr, "c_f"'] & & T_2 \arrow[dl, dashed, shift left, "v"] \\ & \mathrm{Coker}(f) \arrow[ur, dashed, shift left, "u"] & \end{tikzcd} \quad \text{such that} \quad \begin{cases} uf^C = c_f, \\ vc_f = f^C. \end{cases}$$

Hence, $uvc_f = c_f$, but c_f is epi, therefore $uv = 1$. Hence, $\mathrm{Coker}(f)$ is a direct summand of $T_2 \in \mathcal{T}_2$ so $\mathrm{Coker}(f) \in \mathcal{T}_2$.

(d) Similar to the previous proof. □

Theorem 3.18. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ Nakaoka context in an abelian category $\mathscr{A}$. Then, for $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$ the following statements are equivalent:*

(a) *$\mathcal{H}$ is an abelian category.*

(b) *The following conditions hold:*

(b1) *For any $f : H \to H'$ in $\mathcal{H}$, with* $\mathrm{Ker}(f) \in \mathcal{F}_1$*, we have that* $\mathrm{Ker}(f) = 0$.

(b2) *For any $f : H \to H'$ in $\mathcal{H}$, with* $\mathrm{Coker}(f) \in \mathcal{T}_2$*, we have that* $\mathrm{Coker}(f) = 0$.

(b3) *$\mathcal{H}$ is closed under kernels (resp. cokernels) of epimorphisms (resp. monomorphisms) in $\mathscr{A}$.*

(c) *$\mathcal{H}$ is closed under kernels and cokernels in $\mathscr{A}$.*

Proof. $((a) \Rightarrow (b1), (b2))$ Assume that $\mathcal{H}$ is an abelian category. By 3.17(d), $(b1)$ holds if and only if any monomorphism in $\mathcal{H}$ is a monomorphism in $\mathscr{A}$.

Observe that for any $f : H \to H'$, we have that

$$\mathrm{Ker}_{\mathcal{H}}(f) \xrightarrow{\tilde{f}^H} H$$

is a monomorphism in $\mathscr{A}$. Indeed, by 3.17(b), we know that $\tilde{f}^H$ is the composition of

$$t_1(\mathrm{Ker}(f)) \xrightarrow{\varepsilon_{1,\mathrm{Ker}(f)}} \mathrm{Ker}(f) \xrightarrow{k_f} H;$$

and, since $\varepsilon_{1,\mathrm{Ker}(f)}$ and k_f are monomorphism in $\mathscr{A}$, so is $\tilde{f}^H$.

Furthermore, if $f : H \to H'$ is a monomorphism in $\mathcal{H}$, then $f = \mathrm{Ker}_{\mathcal{H}}(\mathrm{Coker}_{\mathcal{H}}(f))$ since $\mathcal{H}$ is abelian. Thus, f is a monomorphism in $\mathscr{A}$.

A dual argument can be used to prove $(b2)$.

$((a) \Rightarrow (b3))$ Let $f : H \to H'$ in $\mathcal{H}$ be a monomorphism in $\mathscr{A}$. We want to prove that $\mathrm{Coker}(f) \in \mathcal{H}$.

Observe that the diagram

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "f"] \arrow[d, "\alpha"] & H' \arrow[r, "c_f"] \arrow[d, equal] & \mathrm{Coker}(f) \arrow[r] \arrow[d, "\lambda_2"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(g) \arrow[r, "k_g"] & H' \arrow[r, "g:=\lambda_2 c_f"] & f_2(\mathrm{Coker}(f)) \arrow[r] & 0
\end{tikzcd} \tag{3.1}$$

is commutative and has exact rows. Then, by Snake's lemma we get that $\mathrm{Ker}(\alpha) = 0$ and

$$\mathrm{Coker}(\alpha) \cong \mathrm{Ker}(\lambda_2) = t_2(\mathrm{Coker}(f)) \in \mathcal{T}_2.$$

Since $\mathcal{H}$ is abelian, we know that $f = \mathrm{Ker}_{\mathcal{H}}(\mathrm{Coker}_{\mathcal{H}}(f))$. Therefore, by 3.17 and (3.1), we get that the dashed morphism in the following commutative diagram exists

$$\begin{tikzcd}
 & & H \arrow[r, "f"] \arrow[dll, dashed, "\exists\varphi"'] & H' \arrow[d, equal] \\
t_1(\mathrm{Ker}(g)) \arrow[rr, "\varepsilon_{1,\mathrm{Ker}(g)}"] & & \mathrm{Ker}(g) \arrow[r, "k_g"] & H'
\end{tikzcd}$$

and it is an isomorphism. But, $k_g\varepsilon_{1,\mathrm{Ker}(g)}\varphi = f = k_g\alpha$ and k_g is mono, hence $\varepsilon_{1,\mathrm{Ker}(g)}\varphi = \alpha$. Therefore,

$$\mathrm{Coker}(\alpha) \cong \mathrm{Coker}(\varepsilon_{1,\mathrm{Ker}(g)}) = \frac{\mathrm{Ker}(g)}{t_1(\mathrm{Ker}(g))} = f_1(\mathrm{Ker}(g)) \in \mathcal{F}_1.$$

Thus, $\mathrm{Coker}(\alpha) \in \mathcal{T}_2 \cap \mathcal{F}_1 = 0$ and so α is an isomorphism. By (3.1) is follows that $\mathrm{Coker}(f) \cong f_2(\mathrm{Coker}(f)) \in \mathcal{F}_2$ and, using the fact that T_1 is closed under quotients, we conclude that $\mathrm{Coker}(f) \in \mathcal{T}_1 \cap \mathcal{F}_2 = \mathcal{H}$. A dual argument shows that $\mathcal{H}$ is closed under kernels of epimorphism in $\mathscr{A}$.

$((b) \Rightarrow (a))$ We already know that $\mathcal{H}$ is pre-Abelian. On order to prove that $\mathcal{H}$ is abelian we need to show that any monomorphism (resp. epimorphism) in $\mathcal{H}$ is a kernel (resp. cokernel).

Let $f : H \to H'$ be a monomorphism in $\mathcal{H}$. By 3.17 and $(b1)$, it follows that f is a monomorphism in $\mathscr{A}$. Thus, we have a short exact sequence in $\mathscr{A}$

$$0 \longrightarrow H \xrightarrow{f} H' \xrightarrow{c_f} \mathrm{Coker}(f) \longrightarrow 0.$$

Furthermore, from $(b3)$ we know that $\mathrm{Coker}(f) \in \mathcal{H}$. Then, $\mathrm{Coker}(f) = \mathrm{Coker}_{\mathcal{H}}(f)$ and $\mathrm{Ker}(c_f) = \mathrm{Ker}_{\mathcal{H}}(c_f)$. Therefore, $f = \mathrm{Ker}(\mathrm{Coker}(f)) = \mathrm{Ker}_{\mathcal{H}}(\mathrm{Coker}_{\mathcal{H}}(f))$.

$((a), (b) \Rightarrow (c))$ Let $f : H \to H'$ be a morphism in $\mathcal{H}$. We want to show that $\mathrm{Ker}(f) \in \mathcal{H}$. Consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow H \longrightarrow \mathrm{Im}(f) \longrightarrow 0,$$

by $(b3)$ it is enough to show that $\mathrm{Im}(f) \in \mathcal{H}$. Since $\mathcal{H}$ is abelian, we have the canonical factorization of f in $\mathcal{H}$

$$\begin{tikzcd}
H \arrow[rr, "f"] \arrow[dr, two heads, "f'"'] & & H' \\
& \mathrm{Im}_{\mathcal{H}}(f) \arrow[ur, hook, "f''"'] &
\end{tikzcd}$$

where f' is an epi and f'' is a mono (in $\mathcal{H}$). Then, by $(b1)$ and $(b2)$ we have that f' and f'' are respectively an epi and a mono in $\mathscr{A}$. Therefore, $\mathrm{Im}(f) \cong \mathrm{Im}_{\mathcal{H}}(f) \in \mathcal{H}$, and by the exact sequence

$$0 \longrightarrow \mathrm{Im}(f) \longrightarrow H' \longrightarrow \mathrm{Coker}(f) \longrightarrow 0,$$

and $(b3)$ we conclude that $\mathrm{Coker}(f) \in \mathcal{H}$.

$(c) \Rightarrow (a)$ is clear.

□

3.3.2 *Nakaoka contexts in triangulated categories*

Let $\mathscr{C} = \mathcal{T}$ be a triangulated category on which idempotents split. We start by recalling the definition of a t-structure in $\mathcal{T}$.

Definition 3.19. A pair $(\mathcal{A}, \mathcal{B})$ of full subcategories of $\mathcal{T}$ is a t-structure in $\mathcal{T}$ if

(a) $\mathcal{A} = {}^{\perp}\mathcal{B}[-1]$ and $\mathcal{B} = \mathcal{A}[1]^{\perp}$,

(b) for any $X \in \mathcal{T}$ there is a distinguished triangle

$$U_X \longrightarrow X \longrightarrow V^X \longrightarrow A_X[1]$$

with $U_X \in \mathcal{A}$ and $V^X \in \mathcal{B}[-1]$.

(c) $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark. It is well known that any t-structure $(\mathcal{A}, \mathcal{B})$ in $\mathcal{T}$ gives a functorial torsion pair $\mathbb{t} = (\mathcal{A}, \mathcal{B}[-1])$ and $\mathcal{B}[-1] \subseteq \mathcal{B}$. Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions and direct summands. Note that the t-structure $(\mathcal{A}, \mathcal{B})$ depends only on $\mathcal{A}$, since $\mathcal{B} = \mathcal{A}^{\perp}[1]$.

Definition 3.20. Let $\mathbb{t}_1 = (\mathcal{T}_1, \mathcal{F}_2)$ and $\mathbb{t}_2 = (\mathcal{T}_2, \mathcal{F}_2)$ be two torsion pairs in a triangulated category $\mathcal{T}$. We will say that $\mathbb{t}$ is a *related* pair if $(\mathcal{T}_1, \mathcal{F}_1[1])$ and $(\mathcal{T}_2, \mathcal{F}_2[1])$ are t-structures and $\mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proposition 3.21. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in $\mathcal{T}$. Then*

(a) $\mathbb{t}$ is a pre-Abelian Nakaoka context in $\mathcal{T}$;

(b) the heart $\mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$ is a pre-Abelian category.

Proof. Assuming (a), (b) is a consequence of Theorem 3.9. Therefore, it is enough to prove (a), i.e. to show that axioms **(CT.3)** and **(CT.3*)** hold, since **(CT.1)** and **(CT.2)** are essentially among the hypothesis.

(CT.3) Let $g : H \to H'$ in $\mathcal{H}$ and complete it to a triangle:

$$H \xrightarrow{g} H' \xrightarrow{g^C} C \longrightarrow H[1].$$

Since $\mathcal{H} \subseteq \mathcal{T}_1$, we have that $H[1] \in \mathcal{T}_1[1] \subseteq \mathcal{T}_1$. Moreover, since $\mathcal{T}_1$ is closed under extensions, C belongs to $\mathcal{T}_1$. Hence, $g^C : H' \to C$ is a pseudocokernel of g in $\mathcal{T}_1$.

Take $F \in \mathcal{F}_2$. Applying the functor $(-, F) := \mathrm{Hom}(-, F)$ to the triangle above yields an exact sequence of abelian groups

$$(H[1], F) \longrightarrow (C, F) \longrightarrow (H', F) \longrightarrow (H, F).$$

Observe that $\mathcal{F}_2 = \mathcal{T}_2^{\perp} \subseteq \mathcal{T}_1^{\perp}[1]$, i.e. $\mathrm{Hom}(H[1], F) = 0$. This concludes the proof.

(CT.3*) Dual to the previous.

□

Definition 3.22. A related pair $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ in the triangulated category $\mathcal{T}$ is *strong* if for any morphism $f : H_1 \to H_2$, in $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$, and a distinguished triangle

$$V \to H_1 \xrightarrow{f} H_2 \to V[1],$$

the following conditions hold true:

(RST.1) $V \in \mathcal{F}_1$ implies $V \in \mathcal{F}_2[-1]$;

(RST.2) $V \in \mathcal{T}_2$ implies $V \in \mathcal{T}_1[1]$.

We will call such pairs *strongly related*.

Theorem 3.23. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a strongly related pair in the triangulated category $\mathcal{T}$. Then, the heart $\mathcal{H} = \mathcal{H}_{\mathbb{t}}$ is an abelian category.*

Proof. By Proposition 3.21, $\mathbb{t}$ is a pre-Abelian Nakaoka context. Therefore, by Proposition 3.21 it is enough to check that **(CT.4)** and **(CT.4∗)** hold.

(CT.4) Consider a map $f\colon H \to H'$ in $\mathcal{H}$ that admits a pseudo-kernel $f^K\colon H'' \to H$ in $\mathcal{F}_2$ such that $H'' \in \mathcal{F}_1$ as in the statement of **(CT.4)** and consider the commutative diagram

$$\begin{tikzcd}
H'' \arrow[r] \arrow[d, "\lambda_2[-1]"] & H \arrow[r, "f"] \arrow[d, "a"] & H' \arrow[r, "f^C"] \arrow[d, equal] & H''[1] \arrow[d, "\lambda_2"] \\
f_2(H''[1])[-1] \arrow[r] & F_2 \arrow[r] & H' \arrow[r] & f_2(H''[1])
\end{tikzcd}$$

whose rows are distinguished triangles. By construction H'' belongs to $\mathcal{F}_1$, hence it belongs to $\mathcal{F}_2[-1]$ by (RST1) and so $H''[1]$ belongs to $\mathcal{F}_2$. Thus, λ_2 is an iso and as a consequence so is a. By setting $b := a^{-1}\varepsilon_1\colon t_1(F_2) \to H$, we see that $\mathbb{t}$ satisfies **(CT.4)**.

(CT.4∗) Dual to the previous.

□

In the next example we use the previous theorem to recover the classical result, that is the heart of any t-structure in a triangulated category is abelian.

Example 3.24. Let $(\mathcal{A}, \mathcal{B})$ be a t-structure in $\mathcal{T}$. Consider $\mathbb{t}_1 := (\mathcal{A}, \mathcal{B}[-1])$ and $\mathbb{t}_2 := (\mathcal{A}[1], \mathcal{B})$. It is not hard to see that $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a strongly related torsion pair in $\mathcal{T}$. In this case, by 3.23, we get that $\mathcal{H} = \mathcal{A} \cap \mathcal{B}$ is an abelian category (recovering the classical result found in [7]).

Lemma 3.25. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in a triangulated category $\mathcal{T}$. Then,* ***(RST.1)*** *is equivalent to* ***(CT.4)*** *and dually* ***(RST.2)*** *is equivalent to* ***(CT.4∗)****.*

Proof. **(RST.1)⇒(CT.4)** was proved in Theorem 3.23.

Conversely, assume that **(CT.4)** holds. Consider the solid part of the diagram

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "{\lambda[-1]}"] & H_1 \arrow[r, "f"] \arrow[d, "\alpha"] & H_2 \arrow[r, "f^C"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[d, "\lambda"] \\
f_2(\mathrm{Cone}(f))[-1] \arrow[r, dashed] & F_2 \arrow[r] & H_2 \arrow[r] & f_2(\mathrm{Cone}(f)) \\
& t_1(F_2) \arrow[u, "\varepsilon"] \arrow[uu, dotted, bend left, "\beta"] & &
\end{tikzcd}$$

with $\mathrm{Cone}(f)[-1] \in \mathcal{F}_1$. Neeman [35, Lemma 1.4.3] guarantees that α can be taken so that the square on the left is a pullback. Axiom **(CT.4)** gives the existence of $\beta : t_1(F_2) \to H_1$ such that $\alpha \circ \beta = \varepsilon$.

Since t_1 is a functor, there is also a morphism

$$t_1(\alpha) : t_1(H_1) = H_1 \to t_1(F_2)$$

such that $\varepsilon \circ t_1(\alpha) = \alpha$, hence $\varepsilon \circ t_1(\alpha) \circ \beta = \varepsilon$. By the functoriality of the torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$, this means that $t_1(\alpha) \circ \beta = 1_{t_1(F_2)}$. Then, β is a section.

Hence, we can write $t_1(\alpha) : H_1 \to t_1(F_2)$ as

$$t_1(\alpha) : t_1(F_2) \oplus H_1' \xrightarrow{(\ast\ 0)} t_1(F_2)$$

for some $H_1' \underset{\oplus}{<} H_1$ such that α vanishes on H_1'. If we consider the solid part of the diagram

$$\begin{tikzcd}
& H_1' \arrow[dl, dashed] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[dr, "0"] & & \\
\mathrm{Cone}(t_1(\alpha))[-1] \arrow[r] & H_1 \arrow[r, "t_1(\alpha)"] & t_1(F_2) \arrow[r, dashed, "+"] &
\end{tikzcd}$$

we can construct the dashed arrow, and the fact that the triangle commutes means that $H_1' \underset{\oplus}{<} \mathrm{Cone}(t_1(\alpha))[-1]$.

Observe that $\mathrm{Cone}(\alpha) = \mathrm{Cone}(\lambda)[-1]$, since the square

$$\begin{tikzcd}
\mathrm{Cone}(f)[-1] \arrow[r, "f^K"] \arrow[d, "\lambda[-1]"] & H_1 \arrow[d, "\alpha"] \\
f_2(\mathrm{Cone}(f))[-1] \arrow[r] & F_2
\end{tikzcd}$$

is a pullback. Moreover,

$$\mathrm{Cone}(\lambda)[-1] = (t_2(\mathrm{Cone}(f))[1])[-1] = t_2(\mathrm{Cone}(f)).$$

Hence, $\mathrm{Cone}(\alpha) \in \mathcal{T}_2$ and $f_1(\mathrm{Cone}(\alpha)) = 0$, that is, $\mathrm{Cone}(\alpha) \in \mathcal{T}_1$, and since there is a distinguished triangle

$$H_1 \xrightarrow{\alpha} F_2 \to \mathrm{Cone}(\alpha) \xrightarrow{+}$$

with $H_1, \mathrm{Cone}(\alpha) \in \mathcal{T}_1$ it follows that $F_2 \in \mathcal{T}_1$. Hence, $t_1(F_2) \cong F_2$.

We can then write $F_2 \underset{\oplus}{<} H_1$ and consider the commutative diagram

$$\begin{tikzcd}
H_1 \cong H_1' \oplus F_2 \arrow[r, "(f' \; \tilde{f})"] \arrow[d, "(0\;1)"] & H_2 \arrow[d, equal] \\
F_2 \arrow[r] & H_2
\end{tikzcd}$$

so $f' = 0$. Hence, the inclusion $\binom{1}{0} : H_1' \to H_1' \oplus F_2$ can be lifted to $\mathrm{Cone}(f)[-1]$ and $H_1' \underset{\oplus}{<} \mathrm{Cone}(f)[-1]$.

Since $\mathrm{Cone}(f)[-1] \in \mathcal{F}_1$, so does H_1'. Similarly, $H_1' \in \mathcal{T}_1$ because $H_1 \in \mathcal{T}_1$. Hence, $H_1' = 0$ and $\alpha : H_1 \to F_2$ is an iso. The same follows for λ. Therefore, $\mathrm{Cone}(f) \in \mathcal{F}_2$ which proves **(CT.4)**. □

Lemma 3.26. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a strongly related pair in $\mathcal{T}$. Then:*

1. *For any $H, H' \in \mathcal{H}_{\mathbb{t}}$, $(H, H'[-1]) = 0$,*
2. *$A \to B \to C$ is a short exact sequence in $\mathcal{H}_{\mathbb{t}}$ if and only if $A \to B \to C \xrightarrow{+}$ is a triangle in $\mathcal{T}$.*

Proof. 1 follows observing that

$$\mathcal{H}[-1] = \mathcal{T}_1[-1] \cap \mathcal{F}_2[-1] \subseteq \mathcal{F}_2[-1] \subseteq (\mathcal{F}_1[1])[-1] = \mathcal{F}_1.$$

To prove 2 first consider a short exact sequence $A \xrightarrow{f} B \to C$ in $\mathcal{H}$, we want to prove that it is a triangle. Consider the triangle

$$\mathrm{Cone}(f)[-1] \to A \xrightarrow{f} B \to \mathrm{Cone}(f),$$

since f is mono, its pseudocokernel $\mathrm{Cone}(f)[-1]$ belongs to $\mathcal{F}_1$, so $\mathrm{Cone}(f) \in \mathcal{F}_2[-1]$ by **(RST.1)**, hence $\mathrm{Cone}(f) \in \mathcal{F}_2$. This implies that $\mathrm{Cone}(f)$ is the cokernel of f and so $A \xrightarrow{f} B \to C \xrightarrow{+}$ is a triangle.

Conversely, assume that $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{+}$ is a triangle. Then, $\mathrm{Cone}(f) \cong C \in \mathcal{H} = \mathcal{T}_1 \cap \mathcal{F}_2$ and so

$$\mathrm{Cone}(f)[-1] \in \mathcal{F}_2[-1] \subseteq \mathcal{F}_1.$$

Hence, f is mono. A dual argument shows that g is epi. Therefore, $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence in $\mathcal{H}$. □

Example 3.27. Let R be any (associative with 1) ring. Consider the triangulated category $\mathcal{T} := \mathcal{D}(R)$. The derived category $\mathcal{D}(R)$ has the so called natural t-structure $(\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq}(R))$ where

$$\mathcal{D}^{\leq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(X) = 0 \text{ for } i > 0\},$$
$$\mathcal{D}^{\geq 0}(R) := \{X \in \mathcal{D}(R) \mid H^i(x) = 0 \text{ for } i < 0\}.$$

For any ideal $I \trianglelefteq R$, we have the TTF-triple $(\mathcal{C}_I, \mathcal{T}_I, \mathcal{F}_I)$ associated to I, where

$$\mathcal{C}_I := \{M \in \text{Mod-}R \mid IM = M\},$$
$$\mathcal{T}_I := \{M \in \text{Mod-}R \mid IM = 0\} \cong \text{Mod-}\frac{R}{I},$$
$$\mathcal{F}_I := \{M \in \text{Mod-}R \mid Ix = 0 \text{ and } x \in M \Rightarrow x = 0\}.$$

Consider the t-structure by Happel, Reiten, and Smalø's [25]

$$(\mathcal{D}^{\leq 0}_{t_I}(R), \mathcal{D}^{\geq 0}_{t_I}(R))$$

associated to the torsion pair $t_I = (\mathcal{C}_I, \mathcal{T}_I)$, where

$$\mathcal{D}^{\leq 0}_{t_I}(R) := \{X \in \mathcal{D}^{\leq 0}(R) \mid H^0(X) \in \mathcal{C}_I\},$$
$$\mathcal{D}^{\geq 0}_{t_I}(R) := \{X \in \mathcal{D}^{\geq 0}(R) \mid H^0(X) \in \mathcal{T}_I\}.$$

It can be seen that $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ where $\mathbb{t}_1 := (\mathcal{D}^{\leq 0}(R), \mathcal{D}^{\geq 1}(R))$ and $\mathbb{t}_2 := (\mathcal{D}^{\leq 0}_{t_I}(R), \mathcal{D}^{\geq 1}_{t_I}(R))$, is a strongly related pair in $\mathcal{T} = \mathcal{D}(R)$.

3.3.3 *Polishchuk correspondence*

We recall the following bijection given by A. Polishchuk in [37], where for a t-structure $(\mathcal{T}_1, \mathcal{F}_1[1])$ in $\mathcal{T}$, we indicate its heart by $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$.

Proposition 3.28 (Polishchuk's bijection)**.** *Let $(\mathcal{T}_1, \mathcal{F}_1[1])$ be a t-structure in a triangulated category. Then we have a bijection*

$$\left\{ \begin{array}{c} \textit{torsion pairs in} \\ \mathcal{H}_1 = \mathcal{T}_1 \cap \mathcal{F}_1[1] \end{array} \right\} \overset{\mathrm{Pol}_{\mathcal{H}_1}}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{t-structures } (\mathcal{T}_2, \mathcal{F}_2) \\ \textit{in } \mathcal{D} \textit{ satisfying} \\ \mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1 \end{array} \right\}$$

$$(\mathcal{X}, \mathcal{Y}) \longmapsto (\mathcal{T}_2, \mathcal{F}_2[1])$$

$$(\mathcal{T}_2 \cap \mathcal{H}_1, \mathcal{F}_2 \cap \mathcal{H}_1) \longleftarrow\!\shortmid (\mathcal{T}_2, \mathcal{F}_2[1])$$

where

$$\mathcal{T}_2 = \{X \in \mathcal{T}_1 \mid H_1^0(X) \in \mathcal{X}\}$$
$$\mathcal{F}_2 = \{Y \in \mathcal{F}_1 \mid H_1^0(Y) \in \mathcal{Y}\}.$$

Remark. (1) Note that $\mathrm{Pol}_{\mathcal{H}_1}^{-1}(\mathcal{T}_2, \mathcal{U}_2^{\perp}[1]) = (\mathcal{T}_2 \cap \mathcal{F}_1[1], \mathcal{H})$, where $\mathcal{H} := \mathcal{T}_1 \cap \mathcal{F}_2$.

(2) By (1), it follows that $\mathcal{H}$ is a torsion free class in the abelian category $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$.

Lemma 3.29. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in a triangulated category $\mathcal{T}$. Let $\mathcal{H} := \mathcal{H}_{\mathbb{t}} = \mathcal{T}_1 \cap \mathcal{F}_2$ and $\mathcal{H}_1 = \mathcal{T} \cap \mathcal{F}_1[1]$. Observe that $\mathcal{H}_1$ is abelian since it is the heart of a t-structure and $\mathcal{H}_{\mathbb{t}} \subseteq \mathcal{H}_1$. Then, $\mathcal{H}_{\mathbb{t}}$ is closed under kernels taken in $\mathcal{H}_1$.*

Proof. Let $V \xrightarrow{f^K} H_1 \xrightarrow{f} H_2 \xrightarrow{f^C} V[1]$ be a distinguished triangle in $\mathcal{T}$, with $H_1, H_2 \in \mathcal{H}$. We recall that $\mathcal{H}$ is a pre-Abelian category, by Proposition 3.21(a).

Consider the kernels $\mathrm{Ker}_{\mathcal{H}_1}(f) \to H_1$ and $\mathrm{Ker}_{\mathcal{H}}(f) \to H_1$ of f in $\mathcal{H}_1$ and $\mathcal{H}$ respectively and draw the diagram (in $\mathcal{T}$)

$$\begin{tikzcd} t_1(V) = \mathrm{Ker}_{\mathcal{H}}(f) \arrow[d] & \mathrm{Ker}_{\mathcal{H}_1}(f) \arrow[l, dashed, "\exists!"'] \arrow[d] \arrow[dl, dashed, "\exists!"] & & \\ V \arrow[r] & H_1 \arrow[r, "f"] & H_2 \arrow[r] & V[1] \end{tikzcd}$$

whose solid part is commutative and where the row is a distinguished triangle. The arrow $\mathrm{Ker}_{\mathcal{H}_1}(f) \to V$ exists and is unique by **(CT.3*)** and $\mathrm{Ker}_{\mathcal{H}_1}(f) \to \mathrm{Ker}_{\mathcal{H}}(f)$ exists and is unique by the functoriality of the t-structure $(\mathcal{T}_1, \mathcal{F}_1[1])$. Moreover, $\mathrm{Ker}_{\mathcal{H}_1}(f) \to H_1$ is obviously a mono in $\mathcal{H}_1$, so $\mathrm{Ker}_{\mathcal{H}_1}(f) \to \mathrm{Ker}_{\mathcal{H}}$ is a mono in $\mathcal{H}_1$. Since $\mathcal{H}$ is a torsion free class in $\mathcal{H}_1$, this means that $\mathrm{Ker}_{\mathcal{H}_1}(f) \in \mathcal{H}$. In particular, $\mathrm{Ker}_{\mathcal{H}_1}(f) \to H_1$ is the kernel of f in $\mathcal{H}$. This also proves that monomorphisms in $\mathcal{H}$ are mono in $\mathcal{H}_1$ too.

□

Theorem 3.30. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent.*

(a) ***(RST.1)*** *holds, i.e. for any distinguished triangle*

$$V \to H_1 \xrightarrow{f} H_2 \to V[1],$$

with f a morphism in $\mathcal{H} = \mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$, we have that

$$V \in \mathcal{F}_1 \Rightarrow V[1] \in \mathcal{F}_2.$$

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1]$, with $H_1, H_2 \in \mathcal{H}$, we have that $\mathrm{Coker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$.

(c) $\mathcal{H}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_1$

(d) $\mathcal{H}$ is an abelian category.

(e) For any epimorphism $H \twoheadrightarrow X$ in $\mathcal{H}_1$, with $H \in \mathcal{H}$, we have that $X \in \mathcal{H}$ (i.e. $\mathcal{H}$ is closed under quotients in $\mathcal{H}_1$).

Proof. (a)⇒(b). Let f be a monomorphism in $\mathcal{H}_1$. By Lemma 3.29 $0 = \mathrm{Ker}_{\mathcal{H}_1}(f) = \mathrm{Ker}_{\mathcal{H}}(f)$, so $V \in \mathcal{F}_1$ (by 3.10). By (a), it follows that $V[1] \in \mathcal{F}_2$. In the following diagram the solid part is commutative:

$$\begin{tikzcd}
 & & \mathrm{Coker}_{\mathcal{H}_1}(f) \arrow[dd, dashed, bend left, "\exists s"] \\
H_1 \arrow[r, "f"] & H_2 \arrow[ur, "\pi"] \arrow[r, "f^C"] \arrow[dr, "t"'] & V[1] \arrow[u, dashed, "\exists r"'] \arrow[d, "\lambda_2"] \\
 & & f_2(V[1]).
\end{tikzcd}$$

By (the dual of) Lemma 3.7, $\mathrm{Coker}_{\mathcal{H}}(f) = \lambda_2 \circ f^C$.

Since $V[1] \in \mathcal{F}_2$, λ_2 is an isomorphism. Since $\pi f = 0$ and $V[1]$ is pseudocokernel of f, there is a map r such that $rf^C = \pi$. Similarly, the map s exists by the cokernel property of $\mathrm{Coker}_{\mathcal{H}_1}(f)$ in $\mathcal{H}_1$.

Define $\bar{s} := r\lambda_2^{-1}$. We have $\bar{s}s\pi = \bar{s}t = \pi$. Since π is an epimorphism in $\mathcal{H}_1$, we have that $\bar{s}s = 1$, hence

$$\mathrm{Coker}_{\mathcal{H}_1}(f) \overset{s}{\hookrightarrow} f_2(V[1]) \in \mathcal{H}.$$

So $\mathrm{Coker}_{\mathcal{H}_1}(f) \in \mathcal{H}$, since $\mathcal{H}$ is a torsion free class in $\mathcal{H}_1$.

(b)$\Rightarrow$(c). It is enough to show that $\mathcal{H}$ is closed under cokernels in $\mathcal{H}_1$. Let $H_1 \xrightarrow{f} H_2$ in $\mathcal{H}$. Then, $\mathrm{Im}_{\mathcal{H}_1}(f) \hookrightarrow H_2$ and so $\mathrm{Im}_{\mathcal{H}_1}(f) \in \mathcal{H}$, since $\mathcal{H}$ is torsion free in $\mathcal{H}_1$. Therefore, $\mathrm{Coker}_{\mathcal{H}_1}(f) \in \mathcal{H}$.

(c)$\Rightarrow$(d). Trivial.

(d)$\Rightarrow$(a). Let $V \xrightarrow{f^K} H_1 \xrightarrow{f} H_2 \xrightarrow{f^C} V[1]$ be a distinguished triangle with $V \in \mathcal{F}_1$. To prove that $V[1] \in \mathcal{F}_2$ it is enough to show that the arrow λ_2 in the approximation triangle

$$t_2(V[1]) \xrightarrow{\varepsilon_2} V[1] \xrightarrow{\lambda_2} f_2(V[1]) \xrightarrow{\mu_2} t_2(V[1])[1]$$

is an isomorphism. To see that, apply the octaedral axiom to $H_2 \xrightarrow{f^C} V[1] \xrightarrow{f_2} (V[1])$ to get the following:

$$
\begin{tikzcd}
H_2 \arrow[rr, "f^C"] \arrow[drrr, "\lambda_2 \circ f^c"'] & & V[1] \arrow[rr, "-f^K[1]"] \arrow[dr, "\lambda_2"] & & H_1[1] \arrow[rr, "-f[1]"] \arrow[dd, dashed, "\alpha"] & & H_2[1] \arrow[ddd, equals] \\
& & & f_2(V[1]) \arrow[dr] \arrow[dddrr, "\mu_2"'] & & & \\
& & & & T \arrow[drr] \arrow[ddr, dashed, "\gamma"] & & \\
& & & & & & H_2[1] \arrow[dd, "f^C[1]"] \\
& & & & & t_2(V[1])[1] \arrow[dr, "-\varepsilon_2[1]"] \arrow[ddr, dashed] & \\
& & & & & & V[2] \arrow[d, "-f^K[2]"] \\
& & & & & & H_1[2]
\end{tikzcd}
$$

From this diagram, by setting $a := -\alpha[-1]$, $b := -\beta[-1]$ and $c = -\gamma[-1]$ we get the following commutative diagram:

$$
\begin{tikzcd}
H_2[-1] \arrow[r, "-f^C[-1]"] \arrow[d, equals] & V \arrow[r, "f^K"] \arrow[d, "-\lambda_2[-1]"] & H_1 \arrow[r, "f"] \arrow[d, "a"] & H_2 \arrow[d, equals] \\
H_2[-1] \arrow[r] & f_2(V[1])[-1] \arrow[r] & T[-1] \arrow[r, "b"] \arrow[d, "c"] & H_2 \arrow[d, "f^C"] \\
& & t_2(V[1]) \arrow[r, "\varepsilon_2"] \arrow[d] & V[1] \arrow[d, "f^K[1]"] \\
& & H_1[1] \arrow[r, equals] & H_1[1]
\end{tikzcd}
$$

where $T[-1] \in \mathcal{H}$ since $\mathcal{H}$ is closed under extensions and the column $T[-1]$ belongs to is a distinguished triangle.

We claim that $a : H_1 \to T[-1]$ is an isomorphism. Indeed $V \in \mathcal{F}_1$, hence f is a mono in $\mathcal{H}$. Moreover, $f = ba$ and so a too is a mono. The cokernel of a is $T[-1] \xrightarrow{\lambda_2 c} f_2(t_2(V[1]))$, which is 0 since $f_2(t_2(V[1])) = 0$. Hence a is an isomorphism, since it is both mono and epi an the abelian category $\mathcal{H}$. Thus, $\lambda_2[-1]$ too is an isomorphism.

(c)⇒(e). Let $H \xrightarrow{\alpha} X$ be an epimorphism in $\mathcal{H}_1$ with $H \in \mathcal{H}$. Then

$$0 \longrightarrow \operatorname{Ker}_{\mathcal{H}_1}(\alpha) \longrightarrow H \xrightarrow{\alpha} X \longrightarrow 0$$

is an exact sequence in $\mathcal{H}_1$.

By Lemma 3.26 there is a distinguished triangle in $\mathcal{T}$

$$\operatorname{Ker}_{\mathcal{H}_1}(\alpha) \to H \to X \to \operatorname{Ker}_{\mathcal{H}_1}(\alpha)[1].$$

Let $T \in \mathcal{T}_2$, by applying $\operatorname{Hom}(T, -)$ to the above triangle we find that $\operatorname{Ker}_{\mathcal{H}_1}(\alpha) \in \mathcal{F}_2$, that is $\operatorname{Ker}_{\mathcal{H}_1}(\alpha) \in \mathcal{H}$. Together with (c), this implies that $X \in \mathcal{H}$.

(e)⇒(c). Follows from Lemma 3.29.

□

Let $t = (\mathcal{A}, \mathcal{B})$ be a pair of full subcategories of the triangulated category $\mathcal{T}$. We will use the following notation:

$$t[1] := (\mathcal{A}[1], \mathcal{B}[1]),$$
$$\bar{t} := (\mathcal{A}, \mathcal{B}[1]).$$

Note that $\bar{t}$ is a t-structure in $\mathcal{T}$ if and only if t is a torsion pair $\mathcal{T}$ such that $\mathcal{A}[1] \subseteq \mathcal{A}$.

Remark. Consider $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$, where $\mathbb{t}_i := (\mathcal{T}_i, \mathcal{F}_i)$ for $i = 1, 2$. We have

1. $\mathcal{H}_{\mathbb{t}} := \mathcal{T}_1 \cap \mathcal{F}_2$, $\mathcal{H}_i := \mathcal{T}_i \cap \mathcal{F}_i[1]$,
2. $\mathbb{t}' := (\mathbb{t}_2, \mathbb{t}_1[1])$

Note that

3. $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related pair in $\mathcal{T}$

$$\Leftrightarrow \mathcal{T}_1[1] \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$$
$$\Leftrightarrow \mathcal{T}_2[1] \subseteq \mathcal{T}_1[1] \subseteq \mathcal{T}_2$$
$$\Leftrightarrow \mathbb{t}' = (\mathbb{t}_2, \mathbb{t}_1[1]) \text{ is a related pair in } \mathcal{T}.$$

4. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ is a related pair in $\mathcal{T}$. In this case, we have

$$\mathcal{H}_{\mathbb{t}} = \mathcal{T}_1 \cap \mathcal{F}_2, \ \mathcal{H}_{\mathbb{t}'} = \mathcal{T}_2 \cap \mathcal{F}_1[1],$$
$$\operatorname{Pol}^{-1}_{\mathcal{H}_1}(\bar{\mathbb{t}}_2) = \operatorname{Pol}^{-1}_{\mathcal{H}_1}(\mathcal{T}_2, \mathcal{F}_2[1]) = (\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}}),$$
$$\operatorname{Pol}^{-1}_{\mathcal{H}_2}(\bar{\mathbb{t}}_1[1]) = \operatorname{Pol}^{-1}_{\mathcal{H}_1}(\mathcal{T}_1[1], \mathcal{F}_1[2]) = (\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'}).$$

Thus, $(\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$ is a torsion pair in the abelian category $\mathcal{H}_1$, $(\mathcal{H}_{\mathbb{t}}[1], \mathcal{H}_{\mathbb{t}'})$ is a torsion pair in the abelian category $\mathcal{H}_2$.

Corollary 3.31. *Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2)$ be a related pair in a triangulated category $\mathcal{T}$. Then, the following statements are equivalent:*

(a) For any distinguished triangle $V \to H_1 \xrightarrow{f} H_2 \xrightarrow{+}$, with f a morphism in $\mathcal{H}_{\mathbb{t}'} = \mathcal{T}_2 \cap \mathcal{F}_1[1]$, we have that $V \in \mathcal{F}_2$ implies $V \in \mathcal{F}_1$.

(b) For any monomorphism $\alpha : H_1 \hookrightarrow H_2$, in the abelian category $\mathcal{H}_2 := \mathcal{T}_2 \cap \mathcal{F}_2[1]$, with $H_1, H_2 \in \mathcal{H}_{\mathbb{t}'}$, we have that $\mathrm{Coker}_{\mathcal{H}_2}(\alpha) \in \mathcal{H}_{\mathbb{t}'}$.

(c) $\mathcal{H}_{\mathbb{t}'}$ is closed under kernels and cokernels in the abelian category $\mathcal{H}_2$.

(d) $\mathcal{H}_{\mathbb{t}'}$ is an abelian category.

(e) $\mathcal{H}_{\mathbb{t}'}$ is closed under quotients in $\mathcal{H}_2$.

We recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is cohereditary if the class $\mathcal{F}$ is closed under quotients in $\mathcal{A}$.

Definition 3.32. For a triangulated category $\mathcal{T}$, we consider the following classes:

$$RtAb(\mathcal{T}) := \{\text{related pairs } \mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2) \text{ in } \mathcal{T} \text{ s.t. } \mathcal{H}_{\mathbb{t}} \text{ is abelian}\};$$

$$\mathrm{t}{-}stCoh(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_1, \tau) \text{ s.t. } \bar{\mathbb{t}}_1 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_1 := \mathcal{T}_1 \cap \mathcal{F}_1[1] \end{array} \right\};$$

$$RtAb'(\mathcal{T}) := \{\text{related pairs } \mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2) \text{ in } \mathcal{T} \text{ s.t. } \mathcal{H}_{\mathbb{t}'} \text{ is abelian}\};$$

$$\mathrm{t}{-}stCoh'(\mathcal{T}) := \left\{ \begin{array}{c} \text{pairs } (\bar{\mathbb{t}}_2, \tau) \text{ s.t. } \bar{\mathbb{t}}_2 \text{ is a t-structure in } \mathcal{T} \text{ and } \tau \text{ is a} \\ \text{cohereditary torsion pair in the abelian category} \\ \mathcal{H}_2 := \mathcal{T}_2 \cap \mathcal{F}_2[1] \end{array} \right\}.$$

Theorem 3.33. *For a triangulated category $\mathcal{T}$, the following statements hold true.*

(a) There is a bijective correspondence

$$\begin{array}{ccc} RtAb(\mathcal{T}) & \overset{\alpha}{\longleftrightarrow} & \mathrm{t} - stCoh(\mathcal{T}) \\ \mathbb{t} & \longmapsto & (\bar{\mathbb{t}}_1, \mathrm{Pol}^{-1}_{\mathcal{H}_1}(\bar{\mathbb{t}}_2)) \\ (\mathbb{t}_1, \mathbb{t}_2) & \longleftarrow\!\shortmid & (\bar{\mathbb{t}}_1, \tau) \end{array}$$

where $\bar{\mathbb{t}}_2 = \mathrm{Pol}_{\mathcal{H}_1}(\tau)$.

(b) *There is a bijective correspondence*

$$RtAb'(\mathcal{T}) \overset{\alpha'}{\longleftrightarrow} \mathrm{t}-stCoh'(\mathcal{T})$$

$$\mathbb{t} \longmapsto (\bar{\mathbb{t}}_2, \mathrm{Pol}^{-1}_{\mathcal{H}_2}(\bar{\mathbb{t}}_1[1]))$$

$$(\mathbb{t}_1, \mathbb{t}_2) \longleftarrow\!\shortmid (\bar{\mathbb{t}}_2, \tau)$$

where $\bar{\mathbb{t}}_1 = \mathrm{Pol}_{\mathcal{H}_2}(\tau)[-1]$.

Proof. **(a)**. Let $\mathbb{t} = (\mathbb{t}_1, \mathbb{t}_2) \in RtAb(\tau)$. Since

$$\mathrm{Pol}^{-1}_{\mathcal{H}_1}(\bar{\mathbb{t}}_2) = (\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$$

and $\mathcal{H}_{\mathbb{t}}$ is Abelian, we get from Theorem 3.30 that $(\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$ is a cohereditary torsion pair in $\mathcal{H}_1$. Hence, the correspondence α is well defined. Let $(\bar{\mathbb{t}}_1, \tau) \in \mathrm{t}-stCoh(\mathcal{T})$ and

$$\bar{\mathbb{t}}_2 := \mathrm{Pol}_{\mathcal{H}_1}(\tau) = (\mathcal{T}_2, \mathcal{F}_2[1]).$$

Hence, $\tau = \mathrm{Pol}^{-1}_{\mathcal{H}_1}(\mathcal{T}_2, \mathcal{F}_2[1]) = (\mathcal{H}_{\mathbb{t}'}, \mathcal{H}_{\mathbb{t}})$. Therefore, $\mathcal{H}_{\mathbb{t}}$ is closed under quotients in $\mathcal{H}_1$ and thus $\mathcal{H}_{\mathbb{t}}$ is Abelian by Theorem 3.30. This proves that β is well defined.

The proof that β and α are mutually inverse is straightforward.

(b). Dual. $\square$

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