UNIVERSITÀ DEGLI STUDI DI PADOVA

Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica “Tullio Levi-Civita”

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE

CURRICOLO: MATEMATICA

CICLO XXXIII

On birational sheets and deformation theory of some Poisson varieties

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On birational sheets and deformation theory for Poisson varieties

PhD Thesis of Filippo Ambrosio
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Academic Year 2019-2020

ABSTRACT

This thesis is divided in two parts. The first one concerns geometric and representation-theoretic aspects of several subvarieties defined in complex connected reductive algebraic groups. Sheets for the adjoint action on a reductive Lie algebra $\mathfrak{g}$ were parameterized by Borho and Kraft in 1979; in 2012 Carnovale and Esposito studied sheets for the conjugacy action of a reductive group G on itself. A notion of birational sheets of $\mathfrak{g}$ first appeared in a preprint by Losev in 2016: we define analogous objects for a group and we study their features. Under the assumption that the derived subgroup of G is simply-connected, we develop a local approach to reduce the study of geometry of (closures of) Jordan classes and (birational) sheets to the study of analogous objects in subalgebras of $\mathfrak{g}$. We also obtain results about the representation theory of rings of regular functions on spherical orbits in $\mathfrak{g}$ (respectively of conjugacy classes of G), proving a special case of a conjecture of Losev.

The second part is devoted to the study of commutative and non-commutative deformation theory for conical symplectic singularities. It is well known that every conic symplectic singularity admits a universal Poisson deformation and a universal filtered quantization, thanks to the work of Namikawa (2010, 2011) and Losev (2016). We reorganize their work in categorical terms and we obtain that every such variety admits a universal equivariant Poisson deformation and universal equivariant filtered quantization with respect to any group acting on it by $\mathbb{C}^{\times}$-equivariant Poisson automorphisms. Our investigation moves from very classical problems in Lie Theory, for example the deformations of nilpotent Slodowy slices in simple Lie algebras, whose study was first treated on Slodowy's notes of 1980 and then pursued and generalized by Lehn, Namikawa and Sorger in 2012. We classify nilpotent Slodowy slices whose universal filtered quantization coincides with the associated finite W-algebra. Finally, we focus our attention on subregular slices in some non-simply-laced Lie algebras: for B_n with $n \geq 2$, C_{2m} with $m \geq 1$ and F_4 we prove that the finite W-algebra is the universal equivariant filtered quantization with respect to the Dynkin graph automorphisms coming from the unfolding of the Dynkin diagram.

RIASSUNTO

Questa tesi è suddivisa in due parti. La prima riguarda determinate sottovarietà di un gruppo algebrico complesso riduttivo connesso, trattandone aspetti geometrici e di teoria delle rappresentazioni. Le falde (*sheets*) di un'algebra di Lie riduttiva $\mathfrak{g}$ per l'azione aggiunta sono state parametrizzate da Borho e Kraft nel 1979; nel 2012, Carnovale ed Esposito hanno studiato le falde di un gruppo G per l'azione di coniugio. La nozione di falda birazionale (*birational sheet*) di $\mathfrak{g}$ è stata introdotta per la prima volta in una prepubblicazione di Losev del 2016: in questa tesi definiamo oggetti analoghi per il gruppo G e ne studiamo alcune proprietà. Nell'ipotesi in cui il sottogruppo derivato di G sia semplicemente connesso, sviluppiamo un metodo locale per ricondurre lo studio della geometria di (chiusure di) classi di Jordan e di falde (birazionali) allo studio di corrispettivi oggetti in determinate sottoalgebre di $\mathfrak{g}$. Inoltre, otteniamo alcuni risultati legati alla teoria delle rappresentazioni degli anelli delle funzioni regolari sulle orbite sferiche di $\mathfrak{g}$ e sulle classi di coniugio sferiche di G, dimostrando un caso particolare di una congettura proposta da Losev.

La seconda parte tratta alcune deformazioni (commutative e non) di varietà dette singolarità simplettiche coniche: tali varietà ammettono una deformazione di Poisson universale ed una quantizzazione filtrata universale, grazie ai lavori di Namikawa (2010, 2011) e Losev (2016). Riformulando i loro risultati in termini categorici, dimostriamo che ogni singolarità simplettica conica X ammette una deformazione di Poisson universale ed una quantizzazione filtrata universale, equivarianti rispetto ad ogni gruppo che agisce su X per automorfismi di Poisson compatibili con la contrazione indotta dalla struttura conica. La nostra indagine trae spunto da un problema classico in teoria di Lie: lo studio della sottovarietà nilpotente in una sezione trasversa di Slodowy (*Slodowy slice*) nelle algebre di Lie semplici, analizzato per la prima volta nelle note di Slodowy (1980) ed in seguito sfociato nella generalizzazione data da Lehn, Namikawa e Sorger (2012). Nella tesi, classifichiamo le sottovarietà nilpotenti delle sezioni trasverse la cui quantizzazione filtrata universale coincide con la W-algebra finita associata. Infine, ci focalizziamo sulle sezioni subregolari per $\mathfrak{g}$ di tipo B_n con $n \geq 2$, C_{2m} con $m \geq 1$ e F_4: in tali casi la W-algebra finita è la quantizzazione filtrata universale equivariante rispetto al gruppo degli automorfismi del diagramma di Dynkin ottenuto da quello di $\mathfrak{g}$ con un'operazione di dispiegamento (*unfolding*).

INTRODUCTION

The action of a complex connected reductive algebraic group G on an algebraic variety X can be studied by gathering orbits in finitely many families to deduce properties shared by orbits in the same collection. One way of grouping orbits together is to form sheets, i.e., maximal irreducibile subsets of X consisting of equidimensional orbits. In [13], Borho and Kraft studied sheets for the adjoint action of a semisimple connected group G on its Lie algebra $\mathfrak{g}$: the authors considered non-nilpotent orbits as deformations of equidimensional nilpotent ones to compare the G-module structure of their ring of regular functions. In the same paper, sheets and their closures were described set-theoretically as unions of decomposition classes (also called Jordan classes). Decomposition classes form a partition of $\mathfrak{g}$ into finitely-many, irreducible, smooth, G-stable, locally closed subvarieties grouping elements having similar Jordan decomposition. Their geometry was studied in [12, 21, 43, 88].

Every sheet $\mathfrak{S}$ of $\mathfrak{g}$ contains a dense Jordan class $\mathfrak{J}$, hence $\mathfrak{S}$ can be realized as the regular locus of the closure of $\mathfrak{J}$ and described in terms of Lusztig-Spaltenstein induction, as in [12]. Every sheet contains a unique nilpotent orbit $\mathfrak{O}$, in this sense the non-nilpotent orbits in the sheet "deform" $\mathfrak{O}$. The picture is very clear for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$: in this case any two distinct sheets have empty intersection, hence sheets are parameterized by nilpotent orbits of $\mathfrak{g}$, equivalently by partitions of n. This does not hold in general: for example, all simple non-simply-laced Lie algebras present two sheets of subregular elements which meet non-trivially at the subregular nilpotent orbit. For $\mathfrak{g}$ simple and classical, all sheets are smooth (see [55]), but this does not extend to exceptional Lie algebras (the list of smooth sheets is to appear in [26]).

Sheets are objects with an intrinsic geometric interest, but representation theorists look for a deeper understanding of sheets also for other reasons. A natural question is to consider the ring of regular functions $\mathbb{C}[\mathfrak{O}]$ as the orbit $\mathfrak{O}$ varies in a sheet and ask whether the G-modules $\mathbb{C}[\mathfrak{O}]$ are isomorphic. For G simple and adjoint acting via the adjoint action on its Lie algebra $\mathfrak{g}$, some answers were already obtained in [13]: let e be a nilpotent element of $\mathfrak{g}$, if the centralizer of e in G is connected and the closure of the orbit $\mathfrak{O}$ of e is normal, then $\mathbb{C}[\mathfrak{O}]$ is isomorphic as a G-module to $\mathbb{C}[\mathfrak{O}']$ for all $\mathfrak{O}'$ "deforming" $\mathfrak{O}$. In particular, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the G-module structure of $\mathbb{C}[\mathfrak{O}]$ is preserved along sheets, but this fails in general. We recall the counterexample contained in [13]:

for $\mathfrak{g}$ simple of type G_2 and B_2 there are two subregular sheets and the G-module structure of $\mathbb{C}[\mathfrak{O}]$ is preserved along one of them, but not along the other one. Finally, sheets in $\mathfrak{g}$ play a role in the representation theory of finite W-algebras, as shown by Premet and Topley in [87] and in the description of primitive ideals in enveloping algebras, as illustrated in [99] by Vogan.

The ideas for the topics developed in this thesis are strongly influenced by the recent work of Losev [67]: for this reason, we briefly sum up its content. Losev defines finitely-many irreducible subvarieties of $\mathfrak{g}$, called birational sheets. Following ideas already contained in [14], and further developed by Namikawa and Fu (see [45,76]), Losev defines birational sheets in terms of Lusztig-Spaltenstein induction. Induction is defined starting from a pair $(\mathfrak{l}, \mathfrak{O}^L)$ consisting of a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ and a nilpotent orbit $\mathfrak{O}^L$ in $\mathfrak{l}$. For each element ζ in the centre $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$, one can build a certain map, known as the generalized Springer map; its image is the closure of a G-adjoint orbit in $\mathfrak{g}$. When the generalized Springer map is birational, the dense orbit in the image is said to be birationally induced. Birational sheets are unions of birationally induced orbits from pairs $(\mathfrak{l}, \mathfrak{O}^L)$ which are in some sense "minimal" (namely, $\mathfrak{O}^L$ is birationally rigid, i.e., it cannot be birationally induced from a proper Levi subalgebra of $\mathfrak{l}$).

Although the definition of a birational sheet of $\mathfrak{g}$ is less intuitive than that of a sheet, the objects defined by Losev are better behaved from the geometric and representation-theoretic point of view. In particular:

Theorem (Losev [67, Theorem 4.4]). *Let $\mathfrak{g}$ be a reductive Lie algebra. The following hold.*

(i) Birational sheets form a partition of $\mathfrak{g}$.

(ii) Birational sheets are unibranch and their normalization is smooth.

(iii) A geometric quotient for the action of G on a birational sheet exists. This quotient can be explicitly calculated, and it is a unibranch variety with smooth normalization.

Moreover, birational sheets coincide with sheets in $\mathfrak{sl}_n(\mathbb{C})$ and, for $\mathfrak{g}$ simple and classical, all birational sheets are smooth, see [67, Remark 4.10]. We observe that, in contrast to sheets, not all birational sheets contain a nilpotent orbit. On the other hand, while sheets fail fulfilling this property, the G-module structure of $\mathbb{C}[\mathfrak{O}]$ is preserved along the same birational sheets, see [67, Remark 4.11]. In the same Remark, Losev gives hope for an intrinsic characterization of birational sheets in proposing the following

Conjecture (4.9). Let $\mathfrak{g}$ be a reductive Lie algebra. If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are orbits of $\mathfrak{g}$ such that $\mathbb{C}[\mathfrak{O}_1]$ and $\mathbb{C}[\mathfrak{O}_2]$ have isomorphic G-module structure, then they lie in the same birational sheet.

The work in Losev's paper [67] is formulated in the more general context of conical symplectic singularities: these are an interesting subclass of normal Poisson varieties. Examples of conical symplectic singularities arising from Lie Theory are: (finite coverings of) the normalization of the closure of a nilpotent orbit $\mathfrak{O}$ in $\mathfrak{g}$ (i.e., $\operatorname{Spec}\mathbb{C}[\mathfrak{O}]$), the Slodowy slice to a nilpotent element in $\mathfrak{g}$ and its intersection with the nilpotent cone, called nilpotent Slodowy variety.

If X is a conical symplectic singularity and $A = \mathbb{C}[X]$, then A can be seen as the central fibre of a flat family of graded commutative Poisson algebras, called Poisson deformations of

A. In [77, 78], Namikawa proved the existence of a universal Poisson deformation of a conical symplectic singularity X; he introduced a space $\mathfrak{P}_X$ and a finite group W_X acting on $\mathfrak{P}_X$ as a crystallographic group. The quotient $\mathfrak{P}_X/W_X$ parameterizes the Poisson deformations of X up to isomorphism, in the sense that every Poisson deformation of X can be obtained from the universal one as the fibre product with a point of $\mathfrak{P}_X/W_X$.

There exist also non-commutative deformations of A, called filtered quantizations. These are flat families of filtered algebras with central fibre $\mathcal{A}$ whose associated graded $\operatorname{gr}\mathcal{A}$ is isomorphic to A. In [67], Losev describes filtered quantizations of a conical symplectic singularity: similarly to Poisson deformations, they are parameterized, up to isomorphism, by the points in the quotient $\mathfrak{P}_X/W_X$ defined by Namikawa.

As the title of the manuscript suggests, our exposition deals with two subjects. The first one is the group analogue of birational sheets: we propose a possible definition of such objects and analyse some of their algebraic, geometric and representation theoretic features (Chapters 1, 2, 3). This part ends with an application to the spherical subvariety of G (Chapter 4).

The second part is devoted to the theory of Poisson deformations and filtered quantizations of nilpotent Slodowy slices (Chapter 5). We translate results of Namikawa and Losev in a very general, categorical language. We get a complete classification of the cases in which the finite W-algebra associated to a nilpotent orbit $\mathfrak{O}$ is the universal filtered quantization of the nilpotent Slodowy slice at $\mathfrak{O}$. One of the few exceptions is given by the subregular nilpotent slice in $\mathfrak{g}$ simple non-simply-laced: we conclude with some results on this case.

Despite being treated separately in the exposition, these two topics have points of connection: it is clear that both parts are deeply inspired by [67]. We conclude our overview of Losev's article by describing his construction suggesting an Orbit method for $\mathfrak{g}$ semisimple, where many of the objects we study in this thesis come into play.

To each adjoint orbit $\mathfrak{O}$, Losev assigns an algebra $\mathcal{A}_{\mathfrak{O}}$ in the following manner. Consider the pair $(\mathfrak{l}, \mathfrak{O}^L)$ attached to the unique birational sheet containing $\mathfrak{O}$ and let $s \in \mathfrak{z}(\mathfrak{l})$ be a semisimple part in $\mathfrak{O}$. If $\mathfrak{O}'$ is the nilpotent orbit induced from $(\mathfrak{l}, \mathfrak{O}^L)$, then $\mathcal{A}_{\mathfrak{O}}$ is a filtered quantization of a suitable finite covering of $X' = \operatorname{Spec}\mathbb{C}[\mathfrak{O}']$. Namely, $\mathcal{A}_{\mathfrak{O}}$ is obtained specifying the universal filtered quantization of $\mathbb{C}[X']$ to the parameter corresponding to s. The quantization $\mathcal{A}_{\mathfrak{O}}$ is endowed with a G-action and a quantum comoment map $\mathfrak{g} \to \mathcal{A}_{\mathfrak{O}}$: this yields a map $U(\mathfrak{g}) \to \mathcal{A}_{\mathfrak{O}}$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. In short, Losev's Orbit method associaties to any G-adjoint orbit $\mathfrak{O}$ a filtered $U(\mathfrak{g})$-algebra $\mathcal{A}_{\mathfrak{O}}$; denote by $I(\mathfrak{O})$ the kernel of the map $U(\mathfrak{g}) \to \mathcal{A}_{\mathfrak{O}}$. Losev's version of the Orbit method [67, Theorem 5.3] states that if two filtered quantizations $\mathcal{A}_{\mathfrak{O}_1}, \mathcal{A}_{\mathfrak{O}_2}$ are G-equivariantly isomorphic, then $\mathfrak{O}_1 = \mathfrak{O}_2$. Moreover, the map $\mathfrak{O} \mapsto I(\mathfrak{O})$ is injective for $\mathfrak{g}$ classical and Losev conjectures this is also the case for $\mathfrak{g}$ exceptional. In the case that $\mathfrak{O}$ itself is nilpotent, then $I(\mathfrak{O})$ corresponds to a one-dimensional module of the finite W-algebra associated to $\mathfrak{O}$. This leads to new conjectures in the description of the primitive ideals of $U(\mathfrak{g})$.

We now proceed to illustrate in greater detail the content of the manuscript, emphasizing its original results. The first problem addressed in our exposition is defining a group analogue of Losev's birational sheets: this is mainly contained in [1]. After fixing the notation and recollecting some general results in preliminary Chapter 0, we begin Chapter 1 with an exposition of results on induction of adjoint orbits and conjugacy classes and its interplay with birationality. Induction of unipotent classes was defined by Lusztig and Spaltenstein in [71] and it was then generalized to a non-unipotent conjugacy class in [29] readapting arguments of [12] to the group case. Following this approach and inspired by [67], for a conjugacy class $\mathcal{O}$ of G, we are led to consider two different definitions:

(a) $\mathcal{O}$ is birationally induced if it is dense in the image of a birational Springer generalized map;

(b) $\mathcal{O}$ is weakly birationally induced if the class of its unipotent part is birationally induced in the connected centralizer of its semisimple part.

As suggested by the choice of the names, (a) implies (b), but (a) is in general strictly stronger. Definition (a) is the natural extension of birational induction in the Lie algebra. On the other hand, in the group there is more control on definition (b). The most favourable situation occurs when the derived subgroup $[G, G]$ is simply-connected or when $\mathcal{O}$ is unipotent in G up to a central element: in such cases the centralizer of the semisimple part of $\mathcal{O}$ is connected and under this condition (a) is equivalent to (b). This implies also that, since the centralizer of a semisimple element of $\mathfrak{g}$ is always connected, there is no difference between birational induction and weakly birational induction for adjoint orbits in $\mathfrak{g}$.

We prove that most properties enjoyed by induction extend naturally to the case of weakly birational induction. In particular, Lemma 1.26 states a criterion which gives a sufficient condition for a unipotent conjugacy class to be birationally induced. The main result closing Chapter 1 is Theorem 1.35, which states that any conjugacy class of G is weakly birationally induced in a unique way, up to G-conjugacy, under some minimality conditions on the data needed to define induction. When $[G, G]$ is simply-connected, the adverb "weakly" can be removed.

In Chapter 2, we recall the principal facts on Jordan classes in $\mathfrak{g}$ and in G. The group analogue of Jordan classes first appeared in Lusztig's paper [69]: such objects and their closures are an essential tool in the study of the sheets for the conjugacy action of a reductive group G on itself and were thoroughly studied by Carnovale and Esposito in [29]. Jordan classes of G enjoy similar properties to decomposition classes of $\mathfrak{g}$: they are locally closed, smooth, irreducible subvarieties of G and they admit a description in terms of Lusztig-Spaltenstein induction.

With the instruments of Chapter 1, we can define the birational closure and the weakly birational closure of a Jordan class in $\mathfrak{g}$ and in G: we describe and compare these objects by means of results and examples. One of the main differences is that while the weakly birational closure of a Jordan class is a union of Jordan classes, the birational closure need not be. In the Lie algebra $\mathfrak{g}$ and for G with $[G, G]$ simply-connected there is no difference between the weakly birational closure and the birational closure of a Jordan class.

Just like sheets in G are parameterized as regular closures of certain Jordan classes, we define weakly birational sheets in G as the weakly birational closures of certain Jordan classes, thus reaching the first milestone of our project and proving an analogue of point (i) of Losev's result [67, Theorem 4.4]:

Theorem (2.25). *Weakly birational sheets form a partition of G.*

Once again, when $[G, G]$ is simply-connected, it makes sense to talk about the birational sheets of G. Every weakly birational sheet is irreducible and contained in a level set, so it is contained in a sheet of G. We conclude Chapter 2 by comparing weakly birational sheets with sheets of G from a structural point of view and studying how working in different isogeny classes of G affects weakly birational sheets.

In Chapter 3 we pave the way to prove a group analogue of (ii) in Losev's result [67, Theorem 4.4]. This aim is reached as an application of a much more general result obtained in [2]. Therein, the focus is put on the local study of (regular loci of) closures of Jordan classes in the neighbourhood of a point. The closure of a Jordan class is a union of objects of the same kind and G admits a stratification whose closed strata are given by the closures of Jordan classes: this stratification yields constructible character sheaves, see [69]. It is clear that the stratifications on $\mathfrak{g}$ and G present similarities and it is natural to expect that the geometry of Jordan classes in a group and of decomposition classes in a Lie algebra are related.

After an introductory section, recalling some notation and some preliminary results from Algebraic Geometry, we begin by proving that the exponential map identifies the Jordan stratification induced on a neighbourhood of the nilpotent cone in $\mathfrak{g}$ with the Jordan stratification induced on a neighbourhood of the unipotent variety in G, preserving closure orderings. Therefore, any closure of a Jordan class in G containing a unipotent element u is smoothly equivalent in the neighbourhood of u to the closure of a Jordan class in $\mathfrak{g}$ in the neighbourhood of the logarithm of u. Once the question has been settled for unipotent elements, under the assumption that $[G, G]$ is simply-connected, we explain how to reduce from the study of the closure $\overline{J}$ of a Jordan class J in G to the study of closures of Jordan classes in a reductive subgroup, namely:

Theorem (3.8). *Let J be a Jordan class in G with $[G, G]$ simply-connected and let $g \in \overline{J}$ with Jordan decomposition $g = rv$. Then the pointed variety $(\overline{J}, rv)$ is smoothly equivalent to $(\bigcup \overline{J}_i, v)$, where the J_i's are the Jordan classes in the centralizer of r such that $rv \in \overline{J_i}$ and $J_i \subset J$.*

At the end of Chapter 3 we collect several applications of this local analysis; we list them, reminding the reader that $[G, G]$ is assumed to be simply-connected throughout the list.

We prove that the closure of a regular Jordan class J in G is normal and Cohen-Macaulay if and only if the categorical quotient $\overline{J}//G$ is normal if and only if $\overline{J}//G$ is smooth (Theorem 3.14) by a reduction to Lie algebra case where the analogous problem was solved by [88]. Since the list of classes J for which $\overline{J}//G$ is normal is known, see [30], this gives the list of normal and Cohen-Macaulay closures of regular Jordan classes in G, see Remark 3.16.

We provide necessary and sufficient conditions for a sheet S in G to be smooth. We also show in Theorem 3.19 that if G is simple simply-connected and classical and the categorical quotient $\overline{S}//G$ is normal in codimension 1, then S is always smooth. We deduce a (non-exhaustive) list of smooth sheets for G simple, simply-connected and classical, see Remark 3.20. We also provide the list of smooth sheets when $\overline{S}//G$ is normal in codimension 1 for G exceptional and simple in Corollary 3.21.

When $G = \mathrm{SL}_n(\mathbb{C})$ we can conclude that all sheets and all Lusztig strata are smooth (Proposition 3.22). The general case is more involved and there are examples of singular and non-normal strata, for instance those containing the subregular unipotent conjugacy class when the root system is not simply-laced.

Finally, we analyse the geometry of the birational closure of a Jordan class. In this case we can use the result of Losev and obtain the analogue for the group case: birational sheets are unibranch varieties with smooth normalization and birational sheets are smooth for classical G (Theorems 3.27 and 3.28).

Chapter 4 contains research produced in [4] and deals with the aforementioned classical problem in Representation Theory: describing collections of adjoint orbits (resp. conjugacy classes) whose ring of regular functions are isomorphic as G-modules. Our research is motivated by Losev's Conjecture 4.9. We tackle this problem restricting to the context of the spherical subvarieties of $\mathfrak{g}$ and G, under the assumption that $[G, G]$ is simply-connected.

We begin Chapter 4 with an introductory part, where we briefly state the main notions and results on the G-module decomposition of rings of regular functions on orbits, and we recall the definition of spherical variety. If B is a Borel subgroup of G, the complexity of X is the codimension of a generic B-orbit in X; the variety X is spherical if it has complexity zero.

By [6, Proposition 1], the complexity of orbits as homogeneous spaces of G is constant along the sheets: it follows that the property of being spherical is preserved along sheets. A sheet S is said to be spherical if the orbits in S are spherical; research on these objects can be found in [28, 31]. Since a birational sheet is contained in a sheet, the property of being spherical is preserved along birational sheets, as well. We call spherical birational sheet any birational sheet consisting of spherical orbits. For G simple simply-connected, we classify the spherical birational sheets and observe that the union G_{sph} of all spherical conjugacy classes in G is the disjoint union of spherical birational sheets.

If $\mathcal{O}$ is a spherical conjugacy class, then $\mathbb{C}[\mathcal{O}]$ is multiplicity-free, i.e., a simple G-module occurs in $\mathbb{C}[\mathcal{O}]$ with multiplicity at most 1. Therefore, the G-module $\mathbb{C}[\mathcal{O}]$ is completely determined by its weight monoid, i.e., by the set of dominant weights λ for which the simple G-module with highest weight λ occurs in the decomposition of $\mathbb{C}[\mathcal{O}]$. In [36] the weight monoids are explicitly described for every spherical conjugacy class of G simple simply-connected. Using these results and the classification of spherical birational sheets, we prove:

Theorem (4.11)**.** *Suppose that $[G, G]$ simply-connected and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spherical conjugacy* *classes in G. Let S_1^{bir} (resp. S_2^{bir}) be the birational sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1]$ is isomorphic to $\mathbb{C}[\mathcal{O}_2]$ as a G-module if and only if $S_2^{bir} = zS_1^{bir}$ for some $z \in Z(G)$.*

From this we also deduce the validity of Losev's conjecture in the case of spherical adjoint orbits in $\mathfrak{g}$ (Theorem 4.42).

For $\mathfrak{sl}_n(\mathbb{C})$, we show that Losev's conjecture holds in all generality (Proposition 4.3) and from this we deduce that the analogous conjecture in the group case is true for $\mathrm{SL}_n(\mathbb{C})$ (Theorem 4.7).

This concludes our survey on birational sheets in reductive groups as well as the first part of the thesis.

The second part of the manuscript is Chapter 5 and contains the research produced in [3]. As noted earlier, the objects of study are Poisson deformations and filtered quantizations of nilpotent Slodowy slices.

The finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are classified by the simply-laced Dynkin diagrams. If Δ is such a diagram corresponding to a group Γ then the quotient singularity $\mathbb{C}^2/\Gamma$ is said to have type Δ. It was proven by Artin that these varieties give an exhaustive list of rational isolated surface singularities up to analytic isomorphism [5]. The classical theorem of Brieskorn [19], conjectured by Grothendieck, states that if $\mathfrak{g}$ is simple with simply-laced Dynkin diagram Δ, then the transverse slice to the subregular orbit is the $\mathbb{C}^\times$-semi-universal deformation of type Δ. This remarkable theorem was extended to the non-simply-laced types by Slodowy [91]. Let Δ_0 be a non-simply-laced diagram, let Δ be simply-laced and $\Gamma_0 \leq \mathrm{Aut}(\Delta)$ be uniquely determined by the requirement that Δ_0 is obtained by folding Δ under Γ_0 (see [32, §13], for example). Then the subregular slice in a Lie algebra of type Δ_0 is the $\mathbb{C}^\times$-semi-universal Γ_0-deformation of a singularity of type Δ.

In [65], Lehn, Namikawa and Sorger found a generalization to arbitrary nilpotent orbits. In general the nilpotent part of a Slodowy slice is not an isolated surface singularity and so there is no versal theory for deformations. The correct approach is to realize the Slodowy slice as a Poisson variety via Hamiltonian reduction, following [47]. Since the nilpotent part of the slice is a conic symplectic singularity, the afore-mentioned results of Namikawa [77, 78] show that there is a Poisson deformation which is universal. The main result of [65] gives a necessary and sufficient condition for the Slodowy slice to be the universal Poisson deformation of its nilpotent part: the exceptions are listed in Table 1.

Type of $\mathfrak{g}$	Any	B C F G	C	G
Type of $\mathcal{O}$	Regular	Subregular	Two Jordan blocks	dimension 8

Table 1

After fixing the notation and recalling some basic results on graded and filtered algebras, we begin Chapter 5 in a very general setting: in order to relate the universal Poisson deformation with the universal quantization, we reinterpret results of [67, 77, 78] in a categorical framework.

For a commutative positively graded connected Poisson algebra A, we consider the categories $\mathcal{D} = \mathcal{D}_A$ of Poisson deformations, and $\mathcal{Q} = \mathcal{Q}_A$ of quantizations of Poisson deformations of A. When these categories admit initial objects, they are equivalent to some rather elementary categories of commutative algebras. One of the key definitions is the category of deformations (or quantizations) with fixed symmetries. Suppose that Γ is a group of graded Poisson automorphisms of A. We define a Poisson Γ-deformation to be a Γ-equivariant Poisson deformation fibred over a base with trivial Γ-action, and define Γ-quantizations similarly. A universal Poisson Γ-deformation (resp. Γ-quantization) is an initial object in the category of such deformations (resp. quantizations). Our main result in this setting is the following.

Theorem (5.35)**.** *Let X be a conical symplectic singularity and Γ be a group of $\mathbb{C}^\times$-equivariant Poisson automorphisms of $A = \mathbb{C}[X]$.*

(i) There exists a universal Poisson Γ-deformation $u^\Gamma_{\mathcal{D}}$ and a universal Γ-quantization $u^\Gamma_{\mathcal{Q}}$.

(ii) $u^\Gamma_{\mathcal{Q}}$ is the unique quantization of $u^\Gamma_{\mathcal{D}}$ up to isomorphism.

As an application, we return to the Lie theoretic setting. We identify a semisimple Lie algebra $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing isomorphism. This way, we can define the nilpotent cone $\mathcal{N}^*$ of $\mathfrak{g}^*$ as the image of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ under the Killing isomorphism. For $e \in \mathcal{N}$ and its image $\chi \in \mathcal{N}^*$, we consider the Slodowy slice $\mathcal{S}_\chi$. It is known that this is a transversal slice to coadjoint G-orbits, admitting a contracting $\mathbb{C}^\times$-action. Furthermore it carries a Poisson structure via Hamiltonian reduction, see [47]. Slodowy showed in [91] that the adjoint quotient map $\mathfrak{g}^* \to \mathfrak{g}^*/\!/G$ restricts to a flat $\mathbb{C}^\times$-equivariant morphism $\mathcal{S}_\chi \to \mathfrak{g}^*/\!/G$ and it follows that the slice provides a Poisson deformation of the central fibre $\mathcal{S}_\chi \cap \mathcal{N}^*$ which we call the nilpotent Slodowy slice. On the other hand, Premet introduced a filtered quantization of the Slodowy slice known as the finite W-algebra [84]. This is a non-commutative filtered algebra $U(\mathfrak{g}, e)$ which depends only on $\mathfrak{g}$ and the orbit of e. Finite W-algebras have found numerous applications to the ordinary and modular representation theory of Lie algebras; see [86] for a detailed overview. After describing the main features of the Slodowy slice, its Poisson structure and the finite W-algebras, we prove:

Theorem (5.40)**.** *The following are equivalent:*

(i) The finite W-algebra $U(\mathfrak{g}, e)$ is the universal filtered quantization of $\mathbb{C}[\mathcal{S}_\chi \cap \mathcal{N}^*]$;

(ii) the orbit of e is not listed in Table 1.

When these equivalent conditions hold every filtered quantization of $\mathbb{C}[\mathcal{S}_\chi \cap \mathcal{N}^*]$ is isomorphic to $U(\mathfrak{g}, e)/\ker\lambda$, where λ is some central character of $U(\mathfrak{g}, e)$.

One immediate consequence of this theorem is Corollary 5.41, which states that the group of filtered automorphisms of the finite W-algebra is naturally isomorphic to the group of graded Poisson automorphisms of the nilpotent Slodowy slice.

Finally we focus on the subregular case considered by Brieskorn, Grothendieck and Slodowy [19, 91]. The non-commutative analogue of Brieskorn's theorem says that the subregular finite W-algebra attached to a simply laced Lie algebra of type Δ is the universal filtered quantization of the rational singularity of type Δ. This is a special case of Theorem 5.40. The most interesting applications arise from our non-commutative analogue of Slodowy's theorem. Let $\mathfrak{g}_0$ be a simple Lie algebra with non-simply-laced Dynkin diagram Δ_0, and let (Δ, Γ_0) be determined by Δ_0 by folding, as we described earlier. Let $e \in \mathcal{N}$ be subregular and let χ be the image of e in $\mathcal{N}^*$. With some restrictions on Δ_0 we prove the following analogue of Slodowy's theorem [91, §8.8] in the setting of universal Poisson deformations and their quantizations.

Theorem (5.47)**.** *Let $\mathfrak{g}_0$ be of type B_n, C_n or F_4, where $n \geq 2$ and n is even in type C. Let $e_0 \in \mathcal{N}_{\mathfrak{g}_0}$ be a subregular nilpotent element and χ_0 be the image of e_0 via the Killing isomorphism. Then:*

(i) the adjoint quotient $\mathcal{S}_{\chi_0} \to \mathfrak{g}_0^*//G_0$ is the universal Poisson Γ_0-deformation of $\mathbb{C}[\mathcal{S}_\chi \cap \mathcal{N}^*]$;

(ii) the subregular finite W-algebra $U(\mathfrak{g}_0, e_0)$ is the universal Γ_0-quantization of $\mathbb{C}[\mathcal{S}_\chi \cap \mathcal{N}^*]$.

We expect that the restrictions on Δ_0 are unnecessary and we conjecture that the Theorem 5.47 holds for all non-simply-laced simple Lie algebras. Theorems 5.40 and 5.47 lead to interesting surjective homomorphisms between W-algebras, which are new in the literature.

Corollary (5.48)**.** *There exists a surjective homomorphism of subregular W-algebras $U(\mathfrak{g}, e) \twoheadrightarrow U(\mathfrak{g}_0, e_0)$. When $\mathfrak{g}_0$ satisfies the hypotheses of Theorem 5.47, the kernel is generated by elements $z - \gamma \cdot z$ where $\gamma \in \Gamma_0$ and z is in the centre of $U(\mathfrak{g}, e)$.*

This result allows us to obtain new Yangian-type presentations of W-algebras. Since W-algebras are defined via quantum Hamiltonian reduction, there is no known presentation is general. This makes them difficult to work with, despite their many applications. The situation is significantly improved in type A: in this case there is an explicit isomorphism between a truncated shifted Yangian and the W-algebra [24]. This leads to an explicit presentation of the subregular finite W-algebra in type B as a quotient of a truncated shifted Yangian. This concludes Chapter 5 and the thesis.

CONTENTS

CHAPTER

ZERO

PRELIMINARY NOTATION AND DEFINITIONS

0.1 Generalities on varieties and group actions

Let G be a complex connected reductive linear algebraic group. With the notation $K \leq G$, we mean that K is a closed subgroup of G. We denote by K° its identity component, by $Z(K)$ its centre and by $[K, K]$ its derived subgroup.

If X is a K-set, we denote by X/K the set of K-orbits of elements in X. For $x \in X$, we denote the K-orbit of x by $K \cdot x$ or $\mathcal{O}_x^K$. For $x_1, x_2 \in X$ (resp. $X_1, X_2 \subset X$) we write $x_1 \sim_K x_2$ (resp. $X_1 \sim_K X_2$) to denote that the two elements (resp. the two subsets) are conjugate by K.

If $X \subset Y$ are topological spaces, we will denote by $\overline{X}^Y$ the closure of X in Y. If the ambient space is clear, we will omit the superscript Y. We denote by $\dim X$ the dimension of X and by $\operatorname{codim}_Y X = \dim X - \dim Y$ the codimension of X in Y, omitting the subscript when clear from the context.

When we want to emphasize the fact that a union of sets is disjoint, we use the symbol $\sqcup$ instead of $\cup$.

Throughout the manuscript, any variety will be intended to be a variety defined over the field of complex numbers $\mathbb{C}$ in the sense of [54, §2.5]. If X is a variety, we denote by $\mathbb{C}[X]$ the algebra of regular functions on X and by $\mathbb{C}(X)$ the function field of X.

If $f \colon X \to Y$ is a dominant rational map of varieties, the *degree* of f is defined as the degree of the function fields extension $\deg f \coloneqq [\mathbb{C}(X) \colon \mathbb{C}(Y)]$. Moreover, if f is a finite map and $y \in Y$ is a generic point, then $\left|f^{-1}(y)\right| = \deg f$, see [51, Proposition 7.16].

When $K \leq G$ acts on a variety X, the action will be intended to be regular, i.e., the map $K \times X \to X$ defined by $(k, x) \mapsto k \cdot x$ is a morphism of varieties. If $x \in X$, any K-orbit $\mathcal{O}_x^K$ is a locally closed subvariety of X and the boundary $\overline{\mathcal{O}_x^K} \setminus \mathcal{O}_x^K$ consists of K-orbits whose dimension is lower than $\dim \mathcal{O}_x^K$, see [93, Lemma 2.3.3]. Moreover, if K is connected, all K-orbits are irreducible. Recall that in this situation, K defines an action on $\mathbb{C}[X]$ via $(k \cdot f)(x) \coloneqq f(k^{-1} \cdot x)$ for all $k \in K, f \in \mathbb{C}[X], x \in X$. This action is locally finite, i.e, for each $f \in \mathbb{C}[X]$ the subspace

$\mathrm{span}_{\mathbb{C}}\{k \cdot f \mid k \in K\}$ is finite-dimensional; furthermore the representation of K on each finite-dimensional K-stable subspace is a representation as an algebraic group.

For $H \leq K$, we define the subalgebra of H-invariant regular functions as $\mathbb{C}[X]^H := \{f \in C[X] \mid h \cdot f = f \text{ for all } h \in H\}$. When X is affine and K is reductive, we write $X/\!/K := \mathrm{Spec}(\mathbb{C}[X]^K)$: this is called the *categorical quotient* of X under the action of K, it admits the structure of an affine variety and it comes with a canonical projection $\pi_X \colon X \to X/\!/K$ corresponding to the inclusion of algebras $\mathbb{C}[X]^K \to \mathbb{C}[X]$.

For any $n \in \mathbb{N}$, we define the (n-th level) subset $X_{(n)} := \{x \in X \mid \dim K \cdot x = n\} \subset X$. For $Y \subset X$, the *regular locus* of Y is $Y^{reg} := Y \cap X_{(\bar{n})}$, where $\bar{n} = \max\{n \in \mathbb{N} \mid Y \cap X_{(n)} \neq \varnothing\}$. The subset Y^{reg} is open in Y. In particular, the subsets $X_{(n)}$ are locally closed subvarieties of X, being open in $\overline{X_{(n)}} \subset \bigcup_{m \leq n} X_{(n)}$, which is a closed subset in X, see [96, Theorem 15.5.7].

Definition 0.1. Let X be a K-variety. A *sheet* of X for the action of K is an irreducible component of $X_{(n)}$ for some $n \in \mathbb{N}$ such that $X_{(n)} \neq \varnothing$.

For $Y \subset X$, the *normalizer* of Y in K is $N_K(Y) := \{k \in K \mid k \cdot y \in Y \text{ for all } y \in Y\}$. For $x \in X$, its *stabilizer* is denoted $K_x := \{k \in K \mid k \cdot x = x\}$.

0.2 Notations for adjoint and conjugacy actions

We will use fraktur characters to denote Lie algebras, so $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G and for a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$, we denote by $\mathfrak{z}(\mathfrak{k})$ its centre.

We will focus on the conjugacy (resp. the adjoint) action of G on itself (resp. on $\mathfrak{g}$). For $g \in G$ and $\eta \in \mathfrak{g}$, we define the centralizers:

$$\begin{aligned} C_G(g) &:= G_g = \{h \in G \mid hgh^{-1} = g\}; \\ C_G(\eta) &:= G_\eta = \{h \in G \mid \mathrm{Ad}(h)(\eta) = \eta\}; \\ \mathfrak{c}_{\mathfrak{g}}(g) &:= \{\eta \in \mathfrak{g} \mid \mathrm{Ad}(g)(\eta) = \eta\} = \mathrm{Lie}(C_G(g)); \\ \mathfrak{c}_{\mathfrak{g}}(\eta) &:= \mathfrak{g}_\eta = \{\eta \in \mathfrak{g} \mid [\xi, \eta] = 0\} = \mathrm{Lie}(C_G(\eta)). \end{aligned}$$

For a subset $Y \subset G$, the centralizer of Y in G is $C_G(Y) := \bigcap_{y \in Y} C_G(y)$ and the centralizer of a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ as $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{k}) := \{\xi \in \mathfrak{g} \mid [\eta, \xi] = 0 \text{ for all } \eta \in \mathfrak{k}\}$.

The conjugacy class of k in a subgroup $K \leq G$ will be denoted by $K \cdot k = \mathcal{O}_k^K$. For the K-adjoint orbit of $\xi \in \mathfrak{k}$, we use the notations $\mathrm{Ad}(K)(\xi) = \mathfrak{O}_\xi^K$. If clear from the context, subscripts or superscripts will be omitted: in particular when $K = G$.

In accordance with the notation for sets of K-orbits, the set of all K-conjugacy classes of K is denoted K/K.

We fix a maximal torus $T \leq G$ and a Borel subgroup $B \geq T$. A standard parabolic subgroup is a subgroup containing the fixed Borel B. We denote by Φ the root system of G, by Δ the base of Φ corresponding to B and by Φ^+ the subset of positive roots with respect to Δ. If Φ is irreducible, the simple roots will be denoted $\alpha_1, \ldots, \alpha_n$: we shall use the numbering and the description of the simple roots in terms of the canonical basis $(\varepsilon_1, \ldots, \varepsilon_k)$ of an appropriate $\mathbb{R}^k$ as in [16, Planches I–IX]. We denote by α_0 the opposite of the highest root in Φ and we define $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$. We denote by c_i, for $i = 1, \ldots, n$, the positive integer coefficients such that $-\alpha_0 = \sum_{i=1}^n c_i\alpha_i$ and set $c_0 \coloneqq 1$. Also, $\alpha_1^\vee, \ldots, \alpha_n^\vee$ are the simple co-roots and $\mathsf{Q}^\vee$ denotes the co-root lattice.

The one-parameter subgroup of G corresponding to $\alpha \in \Phi$ will be denoted by U_α. With this convention, the standard parabolic subgroups of G are indexed by subsets of Δ, for $\Theta \subset \Delta$, we put $P_\Theta = \langle B, U_{-\alpha} \mid \alpha \in \Theta\rangle$. We set $\mathrm{Lie}(T) = \mathfrak{h}$, $\mathrm{Lie}(B) = \mathfrak{b}$, $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$, $\mathrm{Lie}(P_\Theta) = \mathfrak{p}_\Theta$ for $\Theta \subset \Delta$.

If $s \in T$ (resp. $\sigma \in \mathfrak{h}$) we denote by Φ_s (resp. Φ_σ) the root subsystem consisting of $\alpha \in \Phi$ such that $s \in \ker\alpha$ (resp. $\sigma \in \ker\alpha$). The Weyl group of G is denoted by W. The reflection with respect to the simple root $\alpha_i \in \Delta$ is denoted by $s_i \in W$ and is called a simple reflection, the longest element of W is denoted w_0.

When we write $g_1 = su$, $g_2 = rv \in G$ or $\xi = \sigma + \nu \in \mathfrak{g}$ we implicitly assume that su (resp. rv, resp. $\sigma + \nu$) is the Jordan decomposition of g_1 (resp. g_2, resp. ξ), with s, r semisimple and u, v unipotent (σ semisimple and ν nilpotent, resp.). We agree that the elements of $\mathfrak{z}(\mathfrak{g})$ are semisimple so that, for $\zeta \in \mathfrak{z}(\mathfrak{g})$ and $\xi = \sigma + \nu \in \mathfrak{g}$, the semisimple part of $\zeta + \xi$ is $\zeta + \sigma$.

If $K \leq G$ is connected reductive and $\mathfrak{k} \coloneqq \mathrm{Lie}(K)$, we write $\mathcal{U}_K$ for the unipotent variety of K and $\mathcal{N}_\mathfrak{k}$ for the nilpotent cone of $\mathfrak{k}$; we also set $\mathcal{U} \coloneqq \mathcal{U}_G$ and $\mathcal{N} \coloneqq \mathcal{N}_\mathfrak{g}$. If $K \leq G$ is a reductive subgroup, the restriction of a Springer isomorphism $\phi_G = \phi\colon \mathcal{N} \to \mathcal{U}$ induces a a K-equivariant isomorphism $\phi_K\colon \mathcal{N}_\mathfrak{k} \to \mathcal{U}_K$. In particular, we have $C_K(\nu) = C_K(\phi_K(\nu))$ for any $\nu \in \mathcal{N}_\mathfrak{k}$ and the homogeneous spaces $\mathfrak{O}_\nu^K$ and $\mathcal{O}_{\phi_K(\nu)}^K$ are isomorphic varieties.

A partition of $n \in \mathbb{N}$, $n \geq 0$ is a sequence of non-increasing positive integers $\mathbf{d} = [d_1, \ldots, d_r] \vdash n$, where $\sum_{i=1}^n d_i = n$. If $\mathbf{d} \vdash n$, the transpose (or dual) partition is $\mathbf{d}^t = \mathbf{f}$, where $f_i = |\{j \mid d_j \geq i\}|$ for all i. We will also use the compact notation $\mathbf{d} = [e_1^{m_1}, \ldots, e_s^{m_s}]$ where $e_1 > \cdots > e_s > 0$ by grouping equal d_i's. Partitions will be used to denote nilpotent orbits in classical Lie algebras, whereas for exceptional Lie algebras we will use the Bala-Carter labeling, as in [35, §8.4].

The group G acts via the coadjoint action on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is reductive, there exists an associative symmetric G-invariant non-degenerate bilinear form on $\mathfrak{g}$ as in [15, §4, Proposition 5], thus getting a linear isomorphism of vector spaces (when $\mathfrak{g}$ is semisimple, this is known as the Killing isomorphism)

$$\kappa\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*. \tag{1}$$

We extend the notions of semisimplicity and nilpotency to elements of $\mathfrak{g}^*$ by saying that an element of $\mathfrak{g}^*$ is semisimple (resp. nilpotent) if and only if it is the image of a semisimple (resp. nilpotent) element of $\mathfrak{g}$, see [35, §1.3]. We denote by $\mathcal{N}^*$ the set of nilpotent elements of $\mathfrak{g}^*$.

As in [69, Definition 2.6], an element $g = su \in G$ is said to be *isolated* if $C_G(Z(C_G(s)^\circ)^\circ) = G$. An isolated conjugacy class is a conjugacy class consisting of isolated elements. Clearly, all unipotent conjugacy classes are isolated. For G simple, the isolated semisimple conjugacy classes are finitely many, and a complete list of representatives can be deduced from [37, Lemma 7.1].

An *isogeny* $\pi\colon K \to \overline{K}$ is a surjective group homomorphism with finite kernel; for K connected, this automatically implies $\ker\pi \leq Z(K)$.

When working with simple classical matrix groups, we adopt the following conventions. For $k \in \mathbb{N}, k \geq 1$, we denote by $M_k(\mathbb{C})$ the set of $k \times k$ square matrices with coefficients in $\mathbb{C}$ whose canonical basis is given by $\{e_{i,j} \mid 1 \leq i, j \leq k\}$, where $e_{i,j}$ is the matrix whose entries are 0 except for the entry (i, j) which is 1. Let $\mathbb{J}_k$ be the matrix in $M_k(\mathbb{C})$ whose entries on the antidiagonal are 1 and 0 elsewhere and let $\mathbb{J}'_{2k}$ be the matrix in $M_{2k}(\mathbb{C})$ defined by:

$$\mathbb{J}'_{2k} = \begin{pmatrix} 0 & \mathbb{J}_k \\ -\mathbb{J}_k & 0 \end{pmatrix}.$$

Then:

- $G = \mathrm{GL}_k(\mathbb{C})$ is the subgroup of matrices $A \in M_k(\mathbb{C})$ such that $\det A \neq 0$;
- $G = \mathrm{SL}_k(\mathbb{C})$ is the subgroup of matrices $A \in M_k(\mathbb{C})$ such that $\det A = 1$;
- $G = \mathrm{Sp}_{2k}(\mathbb{C})$ is the subgroup of matrices $A \in \mathrm{SL}_{2k}(\mathbb{C})$ such that $A^T\mathbb{J}'_{2k}A = \mathbb{J}'_{2k}$;
- $G = \mathrm{SO}_k(\mathbb{C})$ is the subgroup of matrices $A \in \mathrm{SL}_k(\mathbb{C})$ such that $A^T\mathbb{J}_kA = \mathbb{J}_k$;

Unless differently indicated, we agree that the fixed Borel B is the subgroup consisting of upper triangular matrices in G and the fixed maximal torus T is the subgroup consisting of diagonal elements in G. If π is the projection map from G to the adjoint group in the same isogeny class of G, we extend the above conventions to $\pi(G)$. We fix $\pi(B)$ as Borel subgroup, resp. $\pi(T)$ as maximal torus and we denote elements of $\pi(G)$ by a representative in G in square brackets.

0.3 Centralizers, Levi and pseudo-Levi subgroups

By a Levi subgroup of G (resp. Levi subalgebra of $\mathfrak{g}$) we mean a Levi factor of a parabolic subgroup of G (resp. parabolic subalgebra of $\mathfrak{g}$). Levi subgroups (resp. Levi subalgebras) are characterized as $C_G(Z)$ for some torus $Z \leq G$ (resp. $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{z})$ for some toral subalgebra $\mathfrak{z} \subset \mathfrak{g}$); in particular Levi subgroups are connected [93, Theorem 6.4.7].

Any Levi subgroup is G-conjugate to a privileged Levi factor of a standard parabolic subgroup, called *standard Levi subgroup*, defined by $L_\Theta \coloneqq \langle T, U_\alpha, U_{-\alpha} \mid \alpha \in \Theta\rangle$ for some $\Theta \subset \Delta$. Similarly any Levi subalgebra of $\mathfrak{g}$ is G-conjugate to a *standard Levi subalgebra* $\mathfrak{l}_\Theta \coloneqq \mathrm{Lie}(L_\Theta)$, for some $\Theta \subset \Delta$.

Fix $\sigma \in \mathfrak{h}$. Then $C_G(\sigma)^\circ = \langle T, U_\alpha \mid \alpha \in \Phi_\sigma\rangle$, see [95, Lemma 3.7] and this is a Levi subgroup of G. Moreover, under our hypothesis on the base field, $C_G(\sigma) = C_G(\sigma)^\circ$, see [95, Corollary 3.11]. Similarly, we have a decomposition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(\sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_\sigma} \mathfrak{g}_\alpha$ which is a Levi subalgebra of $\mathfrak{g}$.

For $s \in G$ semisimple, $C_G(s)^\circ$ is called a *pseudo-Levi subgroup*, following [92]. If $s \in T$, then $C_G(s)^\circ = \langle T, U_\alpha \mid \alpha \in \Phi_s\rangle$ and $C_G(s)/C_G(s)^\circ \simeq \mathrm{Stab}_W(s)/W_s$ where W_s is the subgroup of W generated by reflections with respect to roots in Φ_s, [53, §2.2]. If $[G, G]$ is simply-connected, $C_G(s)$ is connected for any semisimple element $s \in G$, see [33, Theorem 3.5.6].

If Φ is irreducible, any pseudo-Levi subgroup of G is conjugate to a *standard pseudo-Levi group* $M_\Theta := \langle T, U_\alpha, U_{-\alpha} \mid \alpha \in \Theta \rangle$ for some $\Theta \subsetneq \widetilde{\Delta}$, [92, Proposition 3], and we put $\mathfrak{m}_\Theta := \mathrm{Lie}(M_\Theta)$. It is clear that Levi subgroups are a particular type of pseudo-Levi subgroups; for Φ irreducible, a standard pseudo-Levi subgroup M_Θ is a Levi subgroup if Θ (or one of its W-conjugates) lies in Δ, see [92, §2.1].

Let $M \leq G$ be a pseudo-Levi and let $Z = Z(M)$. For $z \in Z$, we say that the connected component $Z^\circ z \subset Z$ *satisfies the regularity property* (RP) for M if

$$C_G(Z^\circ z)^\circ = M. \tag{RP}$$

We will make use of the following result:

Lemma 0.2. *Let $M \leq G$ be a pseudo-Levi subgroup, $z \in Z := Z(M)$. Then $Z^\circ z$ satisfies* (RP) *for M if and only if M is a Levi subgroup of $C_G(z)^\circ$. In particular, if $M = C_G(s)^\circ$ for a semisimple element $s \in G$, then $Z(M)^\circ s$ satisfies* (RP) *for M.*

Proof. We prove first that $C_G(Z^\circ z)^\circ = C_G(z)^\circ \cap C_G(Z^\circ) = C_{C_G(z)^\circ}(Z^\circ)$. The inclusions $C_G(Z^\circ z)^\circ \leq C_G(z)^\circ \cap C_G(Z^\circ) \leq C_G(Z^\circ z)$ are trivial and $C_G(z)^\circ \cap C_G(Z^\circ) = C_{C_G(z)^\circ}(Z^\circ)$ is connected by [11, Corollary 11.12], hence $C_G(z)^\circ \cap C_G(Z^\circ) \leq C_G(Z^\circ z)^\circ$.

Assume $Z^\circ z$ satisfies (RP) for M, then $M = C_G(Z^\circ z)^\circ = C_{C_G(z)^\circ}(Z^\circ)$, which is a Levi subgroup of $C_G(z)^\circ$ by [11, Corollary 20.4]. Conversely, if M is a Levi in $C_G(z)^\circ$, then by [11, Proposition 14.18], $M = C_{C_G(z)^\circ}(Z(M)^\circ) = C_{C_G(z)^\circ}(Z^\circ) = C_G(Z^\circ z)^\circ$. The last statement follows directly. □

Remark 0.3. Let G be simple. For $s \in T$, suppose that $M := C_G(s)^\circ$ is a standard pseudo-Levi subgroup and set $Z := Z(M)$. Observe that $Z^\circ s$ satisfies (RP) for M. We want to compute, up to G-conjugacy, the number of coclasses satisfying (RP) in Z/Z°. Let $z \in Z$ such that $Z^\circ z$ satisfies (RP) for M, then, by [92, Proposition 7] (see also [28, Theorem 4.1]), there is $w \in W$ such that $w(\Theta) = \Theta$ and $w(Z^\circ z)w^{-1} = Z^\circ \hat{z} s$ for a certain $\hat{z} \in Z(G)$. Let $W_1 = \{w \in W \mid wsw^{-1}s^{-1} \in Z^\circ Z(G)\}$, $W_2 = \{w \in W \mid wsw^{-1}s^{-1} \in Z^\circ\}$. We have an exact sequence of groups:

$$\{1\} \to W_2 \to W_1 \to \frac{Z^\circ Z(G)}{Z^\circ} \simeq \frac{Z(G)}{Z(G) \cap Z^\circ},$$

where the last map is given by $w \mapsto wsw^{-1}s^{-1}Z^\circ$. Then the number of different G-classes of pairs $(M, Z^\circ z)$ for a fixed M with $Z^\circ z$ satisfying (RP) for M is

$$d_M := \left[\frac{Z(G)}{Z(G) \cap Z^\circ} : W_1/W_2\right]. \tag{2}$$

Remark 0.4. Let $M \leq G$ be a pseudo-Levi subgroup, $z \in Z := Z(M)$. Then $Z^\circ z$ satisfies (RP) if and only if $Z^{reg} \cap Z^\circ z \neq \varnothing$ if and only if $Z = \langle Z^\circ, Z(G), z \rangle$, see [29, Remark 3.6] and [92, Theorem 7]. Moreover, Z° satisfies (RP) for M if and only if $Z^\circ z$ satisfies (RP) for M for all $z \in Z$ if and only if $Z = Z(G)Z^\circ$ if and only if M is a Levi subgroup: this follows from [74, Lemma 34].

Remark 0.5. For G connected and simple of type A all pseudo-Levi subgroups are Levi subgroups: this follows from [92, Corollary 14]. The conjugacy classes of Levi subgroups are indexed by partitions of n, as follows: for each $\mathbf{d} = [d_1, \dots, d_k] \vdash n$, we define

$$I_{\mathbf{d}} \coloneqq \{d_1, d_1 + d_2, \dots, d_1 + d_2 + \cdots + d_{k-1}\}, \qquad \Theta_{\mathbf{d}} \coloneqq \Delta \setminus \{\alpha_i \mid i \in I_{\mathbf{d}}\}, \qquad L_{\mathbf{d}} \coloneqq L_{\Theta_{\mathbf{d}}}, \quad (3)$$

where, by convention, we assume $I_{\mathbf{d}} = \varnothing$ for $\mathbf{d} = [n]$ and $I_{\mathbf{d}} = \{1, \dots, n-1\}$ for $\mathbf{d} = [1^n]$. We say that $L_{\mathbf{d}}$ is the standard Levi subgroup corresponding to the partition $\mathbf{d}$.

We will need the following results: the first one gives further information on connected components of centres of Levi subgroups, the second one computes the number of such elements for $\mathrm{SL}_n(\mathbb{C})$.

Lemma 0.6. *Let $L \leq G$ be a Levi subgroup. Then two connected components of $Z \coloneqq Z(L)$ are conjugate in G if and only if they are equal.*

Proof. Since L is a Levi subgroup, $Z = Z(G)Z^\circ$, by Remark 0.4. Let $z_1, z_2 \in Z(G)$ and suppose $gZ^\circ z_1 g^{-1} = Z^\circ z_2$ for some $g \in G$. Then $gZ^\circ g^{-1} = Z^\circ z_2 z_1^{-1}$ is a torus contained in Z, hence $Z^\circ = Z^\circ z_2 z_1^{-1}$ and $Z^\circ z_1 = Z^\circ z_2$. The converse implication is trivial. □

Lemma 0.7. *Let $L \coloneqq L_{\mathbf{d}}$ be the standard Levi subgroup of $G \coloneqq \mathrm{SL}_n(\mathbb{C})$ corresponding to $\mathbf{d} = [d_1, \dots, d_k] \vdash n$, as in (3). Then $Z(L)$ has exactly $\gcd\{d_i \mid d_i \in \mathbf{d}\}$ connected components, and no two distinct connected components are conjugate in G.*

Proof. We have:

$$Z(L) \simeq S \coloneqq \{(z_1, \dots, z_k) \in (\mathbb{C}^\times)^k \mid z_1^{d_1} \cdots z_k^{d_k} = 1\}.$$

Notice that $\mathrm{Hom}(S, \mathbb{C}^\times) \simeq X/S^\perp$, where $X = \mathrm{Hom}((\mathbb{C}^\times)^k, \mathbb{C}^\times)$ and $S^\perp = \{\chi \in X \mid \chi(S) = 1\} = \langle d_1\chi_1 + \cdots + d_k\chi_k \rangle$ with $\chi_i \in X$ the coordinate functions for $i = 1, \dots, k$. By the structure theorem for finitely generated modules over a principal ideal domain [56, §3.8], if $d = \gcd\{d_i \mid i = 1, \dots, k\}$, we have $\mathrm{Hom}(S, \mathbb{C}^\times) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}^{k-1}$, hence $Z(L)/Z(L)^\circ \simeq S/S^\circ \simeq \mathbb{Z}/d\mathbb{Z}$. The last assertion follows from Lemma 0.6. □

CHAPTER

ONE

INDUCTION AND BIRATIONALITY

The Chapter opens with Section 1.1, collecting definitions and results on varieties built with an action of G induced from a subgroup $K \leq G$. In Sections 1.2 and 1.3 we specialize to the most relevant case of induced adjoint orbits in a Lie algebra and induced conjugacy classes in a group, introducing the generalized Springer map, the main tool to realize induction. As recalled in the Introduction, we analyse in particular when the generalized Springer map is birational and how this interacts with induction. Finally, Section 1.4 is devoted to the proof of the main result of this Chapter: after fixing a minimal set of data from which one defines induction of conjugacy classes, we prove that every conjugacy class in a reductive group is birationally induced in a unique way up to G-conjugacy of the data, establishing an analogue of a theorem of Losev [67].

1.1 Action induced from a subgroup

Let $K \leq G$ and let Y be a K-set. The group K acts freely on $G \times Y$, defining the following equivalence relation: $(g_1, y_1) \sim_K (g_2, y_2)$ if there exists $k \in K$ such that $g_2 = g_1 k$ and $y_2 = k^{-1} \cdot y_1$, for $g_1, g_2 \in G$ and $y_1, y_2 \in Y$. Following terminology from [13, §7], we define the *induced* G-set $G \times^K Y := (G \times Y)/K$ and we denote by $g * y \in G \times^K Y$ the equivalence class of $(g, y) \in G \times Y$ under $\sim_K$. The group G acts on $G \times^K Y$ via $h \cdot (g * y) = (hg * y)$ for all $g, h \in G$ and $y \in Y$.

When Y is a subset of a G-set X and the K-action on Y is the restriction of the G-action on X, we have a well-defined map $\gamma \colon G \times^K Y \to X$ defined by $\gamma \colon g * x \mapsto g \cdot x$. This map is a surjection onto $G \cdot Y$: we can consider the following commutative diagram:

$$
\begin{tikzcd}
G \times Y \arrow[d, two heads] \arrow[dr, two heads] & \\
G \times^K Y \arrow[r, two heads, "\gamma"] \arrow[d, hook, "\iota"'] & G \cdot Y \\
G/K \times G \cdot Y \arrow[ur, two heads, "\mathrm{pr}_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(g, y) \arrow[d, mapsto] \arrow[dr, mapsto] & \\
g * y \arrow[r, mapsto] \arrow[d, mapsto] & g \cdot y \\
(gK, g \cdot y) \arrow[ur, mapsto] &
\end{tikzcd}
\tag{1.1}
$$

The next result sums up [13, Lemma 7.8].

Lemma 1.1. *In the above setting, the following hold.*

(i) For $y \in Y, g \in G$, we have $G_{g*y} = gK_y g^{-1}$.

(ii) There is an explicit bijection $Y/K \leftrightarrow (G \times^K Y)/G$ defined by $\mathcal{O}_y^K \mapsto \mathcal{O}_{1*y}^G = G \times^K \mathcal{O}_y^K$. Moreover, this map induces a bijection $\gamma^{-1}(\mathcal{O}_y^G)/G \leftrightarrow (\mathcal{O}_y^G \cap Y)/K$, for all $y \in Y$.

(iii) In (1.1), the map $\iota\colon (g, y) \mapsto (gK, g \cdot y)$ is injective and G-equivariant, where the G-action on the target set is given by $h \cdot (gK, y) = (hgK, h \cdot y)$, for $g, h \in G$ and $y \in Y$. □

Let X be a G-variety, let $K \leq G$ and let Y be a locally closed K-stable subvariety of X. Then $G/K \times X$ is endowed with the structure of a variety where G acts diagonally. The embedding $\iota\colon G \times^K Y \hookrightarrow G/K \times X$ identifies $G \times^K Y$ with a locally closed G-subvariety of $G/K \times X$. If Y is closed, then so is $\iota(G \times^K Y)$. We remark that $G \times^K Y$ is, in general, not affine. We have $\mathbb{C}[G \times^K Y] = \mathbb{C}[G \times Y]^K$, and, more generally, for each locally closed subset $Z \subset G/K \times X$ and its preimage $\widetilde{Z} \subset G \times X$ we have $\mathbb{C}[Z] = \mathbb{C}[\widetilde{Z}]^K$. It follows from [68, Lemma I.3] that, if $Y' \subset G \times^K Y$ is G-stable and Zariski open, resp. closed, resp. locally closed, then there exists a K-stable open, resp. closed, resp. locally closed subset $Y'' \subset Y$ such that $Y' = G \times^K Y''$.

If $K = P$ is a parabolic subgroup, then G/P is a complete variety and the projection on the second factor $\mathrm{pr}_2\colon G/P \times X \to X$ is a proper morphism (in particular, it is closed). If Y is closed in X, then $\iota(G \times^P Y)$ is closed in $G/P \times X$, so that $\mathrm{pr}_2(\iota(G \times^P Y)) = \gamma(G \times^P Y) = G \cdot Y$ is closed in X. One can prove that γ is proper. The following result is adapted from [13, Lemma 7.10].

Lemma 1.2. *In the above situation, let $x \in Y$. The following are equivalent:*

(i) $\gamma^{-1}(x)$ is finite;

(ii) $G_x^\circ \leq P$ and $\mathcal{O}_x^G \cap Y$ is a union of finitely many P-orbits.

Proof. Clearly $1 * x \in \gamma^{-1}(x)$. Observe that by G-equivariance of γ, we have

$$\left|\gamma^{-1}(x)\right| = \left|\gamma^{-1}(\mathcal{O}_x^G)/G\right| \left|\gamma^{-1}(x) \cap \mathcal{O}_{1*x}^G\right|. \tag{1.2}$$

Now, by Lemma 1.1 (d), we have $\left|\gamma^{-1}(\mathcal{O}_x^G)/G\right| = \left|(\mathcal{O}_x^G \cap Y)/P\right|$ and $\gamma^{-1}(x) \cap \mathcal{O}_{1*x}^G = \{g * x \mid g \in G_x\} \leftrightarrow G_x/G_{1*x} = G_x/P_x$ by Lemma 1.1 (a). □

We will need the following results, adapted from [57, Lemmas 8.7, 8.8].

Lemma 1.3. *In the above situation, we have:*

(i) $G \times^P Y \twoheadrightarrow G/P$ is a fibre bundle of rank $\dim Y$.

(ii) For all $x \in Y$ we have $\dim \mathcal{O}_x^G \leq \dim G/P + \dim \mathcal{O}_x^P$ with equality if and only if $G_x^\circ \leq P$.

Proof. (i) The map $p\colon G \times^P Y \twoheadrightarrow G/P$ defined by $g * x \mapsto gP$ is G-equivariant. A direct computation shows that the preimage of the coset of P (hence any fibre) is isomorphic to Y. The canonical map $p_P\colon G \to G/P$ has local sections. This means that there exists an open covering of G/P by open sets U which admit a section $s\colon U \to G$ with $p_P \circ s = \mathrm{id}_U$. For each such U, the isomorphism $U \times P \xrightarrow{\sim} p_P^{-1}(U)$ defined by $(gP, p) \mapsto s(gP)p$ has inverse $g \to (gP, s(gP)^{-1}g)$. We claim that the same open covering trivializes p, indeed for each such U we get an isomorphism:

$$\begin{aligned} p^{-1}(U) &\xrightarrow{\sim} U \times Y \\ g * x &\mapsto (gP, (s(gP)^{-1}g) \cdot x) \end{aligned}$$

with inverse $(gP, x) \mapsto s(gP) * ((g^{-1}s(gP)) \cdot x)$.

(ii) For all $x \in Y$, we have $\dim G/P_x = \dim G/P + \dim P/P_x = \dim G/P + \dim \mathcal{O}_x^P$. Since $P_x \leq G_x$, we also have $\dim G/P_x \geq \dim G/G_x = \dim \mathcal{O}_x^G$. This yields $\dim \mathcal{O}_x^G \leq \dim G/P + \dim \mathcal{O}_x^P$ for all $x \in Y$. Equality holds if and only if $\dim G_x = \dim P_x$ if and only if $G_x^\circ = P_x^\circ$, equivalently $G_x^\circ \leq P$. □

Lemma 1.4. *Let X be a G-variety, let $P \leq G$ be parabolic and let $Y \subset X$ be a closed subvariety where P acts as the restriction of the G-action. Let*

$$\gamma \colon G \times^P Y \to G \cdot Y \qquad g * x \mapsto g \cdot x. \tag{1.3}$$

Assume there exists $x \in Y$ with the following properties:

$$Y = \overline{\mathcal{O}_x^P}; \tag{P1}$$

$$G_x^\circ \leq P. \tag{P2}$$

Then the following hold:

(i) Let $\mathcal{O} := \mathcal{O}_x^G$, then $\overline{\mathcal{O}} = G \cdot Y$ and $\mathcal{O} \cap Y = \mathcal{O}_x^P$;

(ii) $\widetilde{\mathcal{O}} := \gamma^{-1}(\mathcal{O})$ is a single G-orbit, open and dense in $G \times^P Y$;

(iii) There is a natural isomorphism $G/P_x \xrightarrow{\sim} \widetilde{\mathcal{O}}$ and γ restricts to an unramified covering $\widetilde{\mathcal{O}} \to \mathcal{O}$ of degree $[G_x : P_x]$.

Proof. (i) The inclusion $\mathcal{O} \subset G \cdot Y$ is clear. Since $G \cdot Y$ is closed by properness of γ, also $\overline{\mathcal{O}} \subset G \cdot Y$. On the other hand, $Y = \overline{\mathcal{O}_x^P} \subset \overline{\mathcal{O}}$, which is G-stable, so that $G \cdot Y \subset \overline{\mathcal{O}}$. Clearly $\mathcal{O}_x^P \subset \mathcal{O} \cap Y$. We prove that $x' \in Y, x' \notin \mathcal{O}_x^P$ implies $x' \notin \mathcal{O}$. Observe that (P1) implies $x' \in Y \setminus \mathcal{O}_x^P = \overline{\mathcal{O}_x^P} \setminus \mathcal{O}_x^P$, hence $\dim \mathcal{O}_{x'}^P < \dim \mathcal{O}_x^P = \dim Y$. By Lemma 1.3 and (P2), we have $\dim \mathcal{O}_{x'}^G < \dim \mathcal{O}$ so that $x' \notin \mathcal{O}$.

(ii) Since γ is G-equivariant, we have $\widetilde{\mathcal{O}} = G \cdot \gamma^{-1}(x)$. If we show that $\gamma^{-1}(x) = G_y \cdot (1 * x)$, the first claim follows, by (1.2). Clearly, $\gamma(g * x) = x$ for all $g \in G_x$, so that $\gamma^{-1}(x) \supset G_x \cdot (1 * x)$. For the other inclusion, pick an element of $\gamma^{-1}(x)$, say $g * g^{-1} \cdot x$ with $g^{-1} \cdot x \in Y$. Then $g^{-1} \cdot x \in \mathcal{O} \cap Y$ and (a) implies there exists $p \in P$ such that $g^{-1} \cdot x = p \cdot x$. Hence $g * g^{-1} \cdot x = gp * x$ and $x = \gamma(g * g^{-1} \cdot x) = \gamma(gp * x) = gp \cdot x$, i.e., $gp \in G_x$. Denote by $\varphi_x \colon G \to \mathcal{O} \subset Y$ and $\tilde{\varphi}_x \colon G \to \widetilde{\mathcal{O}}$ the orbit maps. By Lemma 1.1 (a), $\widetilde{\mathcal{O}} \simeq G/P_x$ and $\mathcal{O} \simeq G/G_x$. Since $\dim G_x = \dim P_x$ we get $\dim \mathcal{O} = \dim \widetilde{\mathcal{O}}$. The closure of the orbit $\widetilde{\mathcal{O}}$ is an irreducible subset of $G \times^P Y$, which is irreducible as well. Since $\dim \widetilde{\mathcal{O}} = \dim G \times^P Y$, we conclude that $\widetilde{\mathcal{O}} = G \times^P Y$.

(iii) We have $\varphi_x = \gamma \circ \tilde{\varphi}_x$, hence $\mathrm{d}_1\varphi_x = \mathrm{d}_{1*x}\gamma \circ \mathrm{d}_1\tilde{\varphi}_x$. Working over $\mathbb{C}$ ensures that $G/P_y \xrightarrow{\sim} \widetilde{\mathcal{O}}$ is an isomorphism of varieties. Moreover, $\ker(\mathrm{d}_1\varphi_x) = \mathrm{Lie}(G_x)$ so that $\mathrm{d}_1\varphi_x$ is surjective and $\dim(\mathrm{T}_x\,\mathcal{O}) = \dim\mathcal{O} = \dim\widetilde{\mathcal{O}} = \dim(\mathrm{T}_{1*x}\widetilde{\mathcal{O}})$ imply $\mathrm{d}_{1*x}\gamma$ is an isomorphism. Since γ is G-equivariant, $\mathrm{d}\gamma$ is bijective at all points of $\widetilde{\mathcal{O}}$. This means that $\gamma\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an unramified covering of degree $\left|\gamma^{-1}(x)\right| = [G_x\colon P_x]$. $\square$

We give a criterion which characterizes when the map (1.3) is birational.

Corollary 1.5. *Assume we are in the setting of Lemma 1.4, let γ be as in* (1.3)*, assume* (P1) *and* (P2) *hold, and let $\mathcal{O}$ and $\widetilde{\mathcal{O}} := \gamma^{-1}(\mathcal{O})$ be as in Lemma 1.4. The following are equivalent:*

(i) γ is birational;

(ii) γ maps $\widetilde{\mathcal{O}}$ isomorphically to $\mathcal{O}$;

(iii) for all $x \in \mathcal{O}^G \cap Y$, we have $G_x = P_x$;

(iv) there exists $x \in \mathcal{O}^G \cap Y$ such that $G_x = P_x$.

Proof. Let (i) hold. By Lemma 1.4, γ induces a finite covering $\widetilde{\mathcal{O}} \to \mathcal{O}$ and these sets are open and dense. Since γ is birational and G-equivariant, it restricts to an injective map on $\widetilde{\mathcal{O}}$, i.e., $\gamma(\widetilde{\mathcal{O}}) \simeq \mathcal{O}$. The converse implication is trivial. (ii) holds if and only if the degree of γ over $\mathcal{O}$ is 1. By (1.2) this is equivalent to $[G_x\colon P_x] = 1$ for $x \in \mathcal{O} \cap Y$, that is (iii). Finally (iii) obviously implies (iv) and since $\mathcal{O} \cap Y$ is a single P-orbit, (iv) implies (iii) by P-equivariance. $\square$

1.2 Induction of adjoint orbits

Let G act on $Y = \mathfrak{g}$ via the adjoint action $\mathrm{Ad}\colon G \mapsto \mathrm{GL}(\mathfrak{g})$ and fix a Levi subgroup $L \leq G$. Choose a parabolic $P \leq G$ with Levi decomposition LU_P. Set $\mathfrak{l} := \mathrm{Lie}(L)$ and $\mathfrak{n}_P := \mathrm{Lie}(U_P)$. For $\mathfrak{O}^L \in \mathfrak{l}/L$, we take $X = \overline{\mathfrak{O}^L} + \mathfrak{n}_P \subset \mathfrak{g}$, this is a closed subvariety which is P-stable under the adjoint action. We have a well-defined map:

$$\begin{aligned} \gamma\colon G \times^P (\overline{\mathfrak{O}^L} + \mathfrak{n}_P) &\to \mathrm{Ad}(G)(\overline{\mathfrak{O}^L} + \mathfrak{n}_P) \\ g * \xi &\mapsto \mathrm{Ad}(g)(\xi) \end{aligned} \tag{1.4}$$

The image of γ is a G-stable closed subset of $\mathfrak{g}$ which projects to a point in the categorical quotient $\mathfrak{g}/\!/G$, hence it is a closed irreducible subset consisting of finitely many adjoint orbits. Irreducibility yields that $(\mathrm{Ad}(G)(\overline{\mathfrak{O}^L} + \mathfrak{n}_P))^{reg}$ is a unique adjoint orbit $\mathfrak{O} \in \mathfrak{g}/G$, called *induced* from $(\mathfrak{l}, \mathfrak{O}^L)$ and denoted $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\,\mathfrak{O}^L$, see [12]. The definition of induced conjugacy class only depends on the Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ and on $\mathfrak{O}^L$ and not on the parabolic $P \leq G$, for a proof see [12, Satz 2.6]. For any Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, we introduce the notation:

$$\begin{aligned} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\colon \mathfrak{l}/L &\to \mathfrak{g}/G \\ \mathfrak{O}^L &\mapsto \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\,\mathfrak{O}^L\,. \end{aligned}$$

Induction is transitive, i.e., if $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$, and $\mathfrak{O}^L \in \mathfrak{l}/L$, then $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\,\mathfrak{O}^L = \mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}\left(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{m}}\,\mathfrak{O}^L\right)$, see [12, §2.3]. Moreover, we have $\mathrm{codim}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\,\mathfrak{O}^L) = \mathrm{codim}_{\mathfrak{l}}\,\mathfrak{O}^L$, see [12, Satz 3.3]

A nilpotent orbit $\mathfrak{O} \in \mathcal{N}/G$ is said to be *rigid* in $\mathfrak{g}$ if it cannot be induced from a proper Levi subalgebra $\mathfrak{l} \subsetneq \mathfrak{g}$ and an orbit $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$.

Notice that $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ if and only if $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L \in \mathcal{N}/G$, in particular the restriction of $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}$ induces a function $\mathcal{N}_{\mathfrak{l}}/L \to \mathcal{N}/G$. This is the Lie algebra counterpart of a procedure called Lusztig–Spaltenstein induction of unipotent classes, appeared first in [71].

1.2.1 Reduction to the nilpotent case and birationality

Lemma 1.6. *The generalized Springer map* (1.4) *fulfils the hypotheses of Lemma 1.4 with* $X = \overline{\mathfrak{O}^L} + \mathfrak{n}_P$ *and* $Y = \mathfrak{g}$.

In particular, there is a unique open G-adjoint orbit in $G \times^P (\overline{\mathfrak{O}^L} + \mathfrak{n}_P)$ and this is a finite G-equivariant covering of the induced orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$.

Proof. Write $\mathfrak{O} := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$. Since $\dim \mathfrak{O} = \dim G - \dim L + \dim \mathfrak{O}^L$, we have $\dim \mathrm{Ad}(G)X = \dim \mathfrak{O} = \dim G - \dim P + \dim \mathfrak{O}^L + \dim U_P = \dim G \times^P X$. We claim this implies that there exists $\xi \in X$ satisfying properties (P1) and (P2). By [93, Theorem 5.1.6], γ has finite fibres on a dense open subset of $\mathrm{Ad}(G)X$. By G-equivariance, γ has finite fibres on $\mathfrak{O}$. Let $\xi \in \mathfrak{O} \cap X$. Then, by Lemma 1.2, $C_G(\xi)^\circ \leq P$ and $X \cap \mathfrak{O}$ is a union of finitely many P-orbits. Let $\mathfrak{O} \cap X = \bigsqcup_{i=1}^t \mathfrak{O}_{\xi_i}^P$ for some $t \in \mathbb{N}$ and some $\xi_i \in X$. Since $\mathfrak{O}$ is open in $G \cdot X$, the set $\mathfrak{O} \cap X$ is open and dense in X. In particular, $X = \overline{\mathfrak{O} \cap X} = \bigcup_{i=1}^t \overline{\mathfrak{O}_{\xi_i}^P}$. Irreducibility of X implies there exists $j \in \{1, \dots, t\}$ such that $X = \overline{\mathfrak{O}_{\xi_j}^P}$. Again, $\mathfrak{O} \cap X$ meets $\mathfrak{O}_{\xi_j}^P$ non-trivially, being both open and dense in X. This implies ξ_j satisfies (P1) and (P2). □

In the setting of (1.4), let $\sigma + \nu \in \mathfrak{O}^L$. Then $C_P(\sigma) = P \cap C_G(\sigma)$ is a parabolic subgroup of $C_G(\sigma)$ (it is in particular connected, since $C_G(\sigma)$ is so, see [93, Corollary 6.4.10]); moreover, $\mathfrak{c}_{\mathfrak{l}}(\sigma)$ is a Levi factor of the parabolic subalgebra $\mathfrak{c}_{\mathfrak{p}}(\sigma) := \mathrm{Lie}(C_P(\sigma))$ and we denote its nilradical by $\mathfrak{n}_{C_P(\sigma)}$. We compare the two morphisms:

$$\gamma\colon G \times^P (\overline{\mathfrak{O}_{\sigma+\nu}^L} + \mathfrak{n}_P) \to \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}_{\sigma+\nu}^L} \tag{1.5}$$

$$\gamma_\sigma\colon C_G(\sigma) \times^{C_P(\sigma)} (\overline{\mathfrak{O}_\nu^{C_L(\sigma)}} + \mathfrak{n}_{C_P(\sigma)}) \to \overline{\mathrm{Ind}_{\mathfrak{c}_{\mathfrak{l}}(\sigma)}^{\mathfrak{c}_{\mathfrak{g}}(\sigma)} \mathfrak{O}_\nu^{C_L(\sigma)}} \tag{1.6}$$

Lemma 1.7. *Let γ and γ_σ be as in* (1.5) *and* (1.6), *respectively. Then γ is birational if and only if γ_σ is birational.*

Proof. The group $C_G(\sigma)$ is connected and reductive (see §0.3) so that Lemma 1.6 also applies to (1.6). For simplicity, write $\mathfrak{O}^G := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}_{\sigma+\nu}^L$ and $\mathfrak{O}^{C_G(\sigma)} := \mathrm{Ind}_{\mathfrak{c}_{\mathfrak{l}}(\sigma)}^{\mathfrak{c}_{\mathfrak{g}}(\sigma)} \mathfrak{O}_\nu^{C_L(\sigma)}$. Suppose γ is birational and let $\nu' \in (\overline{\mathfrak{O}_\nu^{C_L(\sigma)}} + \mathfrak{n}_{C_P(\sigma)}) \cap \mathfrak{O}^{C_G(\sigma)}$. Since $\mathfrak{O}^G = \mathrm{Ad}(G)(\sigma + \mathfrak{O}_{\nu'}^{C_G(\sigma)})$, we have $\sigma + \nu' \in (\overline{\mathfrak{O}_{\sigma+\nu}^L} + \mathfrak{n}_P) \cap \mathfrak{O}^G$. The nilpotent part of $\sigma + \nu'$ is ν' because $\nu' \in C_G(\sigma)$. By Corollary 1.5 (iv), birationality of γ is equivalent to $C_G(\sigma + \nu') \leq P$. This means $C_{C_G(\sigma)}(\nu') \leq P$, equivalently $C_{C_G(\sigma)}(\nu') \leq P \cap C_G(\sigma) = C_P(\sigma)$, i.e., γ_σ is birational thanks to (iv) of Corollary 1.5. □

Arguing as in [12, Satz 2.1, 3. Fall], all generalized Springer maps can be reduced to the study of generalized Springer maps of a special type. Following [67], define the set of *induction* *data* of $\mathfrak{g}$ as the set of triples $(\mathfrak{l}, \mathfrak{O}^L, \zeta)$ where $\mathfrak{l} \subset \mathfrak{g}$ is a Levi subalgebra, $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ and $\zeta \in \mathfrak{z}(\mathfrak{l})$. Any such induction datum is associated with a generalized Springer map:

$$\begin{aligned} \gamma\colon G \times^P (\zeta + \overline{\mathfrak{O}^L} + \mathfrak{n}_P) &\to \mathrm{Ad}(G)(\zeta + \overline{\mathfrak{O}^L} + \mathfrak{n}_P) \\ g * \xi &\mapsto \mathrm{Ad}(g)(\xi) \end{aligned} \tag{1.7}$$

When γ in (1.7) is birational, $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathfrak{O}^L)$ is said to be *birationally induced* from $(\mathfrak{l}, \zeta + \mathfrak{O}^L)$ (or from the induction datum $(\mathfrak{l}, \mathfrak{O}^L, \zeta)$ with the terminology of [67]). The nilpotent orbit $\mathfrak{O}^L \subset \mathfrak{l}$ is said to be *birationally rigid* if it cannot be birationally induced from a *proper* Levi subalgebra $\mathfrak{m} \subsetneq \mathfrak{l}$. Losev calls the induction datum $(\mathfrak{l}, \mathfrak{O}^L, \zeta)$ *birationally minimal* if $\mathfrak{O}^L$ is birationally rigid and $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathfrak{O}^L)$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L, \zeta)$.

Remark 1.8. We alert the reader that Losev works in terms of coadjoint orbits of $\mathfrak{g}^*$ but it is straightforward to translate everything in terms of adjoint orbits. This can be done by fixing κ as in (1). Any Levi subalgebra $\mathfrak{l}$ admits an orthogonal complement (with respect to the fixed bilinear form inducing κ) which is stabilized by endomorphisms $\mathrm{ad}(\xi)$ with $\xi \in \mathfrak{l}$. Hence, κ allows to identify $\mathfrak{l}$ with $\mathfrak{l}^*$, $\mathfrak{z}(\mathfrak{l})$ with $(\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^*$, $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$ with $\mathfrak{n}^{\perp}$ and $\mathfrak{n}$ with $\mathfrak{p}^{\perp}$, where $\mathfrak{a}^{\perp} := \{f \in \mathfrak{g}^* \mid f(\xi) = 0 \text{ for all } \xi \in \mathfrak{a}\}$ for any vector subspace $\mathfrak{a} \subset \mathfrak{g}$. This identification restricts to a G-equivariant (resp. L-equivariant) bijection between adjoint nilpotent orbits in $\mathfrak{g}$ (resp. $\mathfrak{l}$) and coadjoint nilpotent orbits in $\mathfrak{g}^*$ (resp. $\mathfrak{l}^*$), where nilpotency for elements of $\mathfrak{g}^*$ is intended as in §0.2.

Lemma 1.9. *Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi subalgebra, let $\zeta \in \mathfrak{z}(\mathfrak{l})$ and $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$. Then the adjoint orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathfrak{O}^L) \in \mathfrak{g}/G$ is birationally induced from $(\mathfrak{l}, \zeta + \mathfrak{O}^L)$ if and only if the nilpotent orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\zeta)} \mathfrak{O}^L \in \mathcal{N}_{\mathfrak{c}_{\mathfrak{g}}(\zeta)}/C_G(\zeta)$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$.*

Proof. This follows straightforward from Lemma 1.7. □

Remark 1.10. Lemma 1.9 implies that, for the adjoint action of G on $\mathfrak{g}$, any result on the birational induction of adjoint nilpotent orbits extends to the non-nilpotent case. The following results will be therefore stated and proven in the setting of the induction of a nilpotent orbit but actually hold in the general case.

One first important fact is that birationality of a generalized Springer map as in (1.7) is independent of the chosen parabolic.

Lemma 1.11. *Let $L \leq G$ be a Levi subgroup and let $\mathfrak{l} := \mathrm{Lie}\, L$. Let $\zeta \in \mathfrak{z}(\mathfrak{l})$ and $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$. Let $P, Q \leq G$ be parabolic subgroups with Levi factor L and let $\mathrm{Lie}(P) := \mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}_P$ resp. $\mathrm{Lie}(Q) := \mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}_Q$ be the Levi decompositions for their Lie algebras. Consider the generalized Springer maps:*

$$\gamma_P\colon G \times^P (\zeta + \overline{\mathfrak{O}^L} + \mathfrak{n}_P) \to \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L}; \qquad \gamma_Q\colon G \times^Q (\zeta + \overline{\mathfrak{O}^L} + \mathfrak{n}_Q) \to \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L}.$$

Then γ_P is birational if and only if γ_Q is birational.

Proof. Let $\mathfrak{l} \ltimes \mathfrak{n}_{C_P(\zeta)}$ (resp. $\mathfrak{l} \ltimes \mathfrak{n}_{C_Q(\zeta)}$) be the Levi decomposition of $\mathfrak{c}_\mathfrak{p}(\zeta)$ (resp. of $\mathfrak{c}_\mathfrak{q}(\zeta)$) which is a parabolic subalgebra of $\mathfrak{c}_\mathfrak{g}(\zeta)$. Then we can build the generalized Springer maps:

$$\gamma_{P,\zeta} \colon C_G(\zeta) \times^{C_P(\zeta)} (\overline{\mathfrak{O}^L} + \mathfrak{n}_{C_P(\zeta)}) \to \overline{\mathrm{Ind}_\mathfrak{l}^{\mathfrak{c}_\mathfrak{g}(\zeta)} \mathfrak{O}^L};$$
$$\gamma_{Q,\zeta} \colon C_G(\zeta) \times^{C_Q(\zeta)} (\overline{\mathfrak{O}^L} + \mathfrak{n}_{C_Q(\zeta)}) \to \overline{\mathrm{Ind}_\mathfrak{l}^{\mathfrak{c}_\mathfrak{g}(\zeta)} \mathfrak{O}^L}.$$

The morphisms $\gamma_{P,\zeta}$ and $\gamma_{Q,\zeta}$ are generalized Springer maps for the nilpotent orbit $\mathrm{Ind}_\mathfrak{l}^{\mathfrak{c}_\mathfrak{g}(\zeta)} \mathfrak{O}^L \in \mathcal{N}_{\mathfrak{c}_\mathfrak{g}(\zeta)}/C_G(\zeta)$. Notice that $\deg \gamma_{P,\zeta} = \deg \gamma_{Q,\zeta}$ by [14, proof of Corollary 3.9, Remark 3.8], where a formula for the degree of the generalized Springer map for the induction of a nilpotent orbit is given only in terms of the pair $(\mathfrak{l}, \mathfrak{O}^L)$. By Lemma 1.9 γ_P (resp. γ_Q) is birational if and only if $\gamma_{P,\zeta}$ (resp. $\gamma_{Q,\zeta}$) is so if and only if $\deg \gamma_{P,\zeta} = 1$ (resp. $\deg \gamma_{Q,\zeta} = 1$) and we conclude. □

Remarks on the proof. Losev obtained, with different techniques, a stronger result: for any generalized Springer map as (1.7), the open orbit in the domain is independent of the chosen parabolic, see [67, Proposition 4.1].

The concept of birational induction first appeared in [22, 23], while in [45, 76] the focus was cast on birational induction of nilpotent adjoint orbits as conical symplectic singularities. We give the proof of a result on transitivity of birational induction.

Lemma 1.12. *Consider a Levi subgroup $M \le G$, and let L be a Levi subgroup of M; set $\mathfrak{m} := \mathrm{Lie}(M)$ and $\mathfrak{l} := \mathrm{Lie}(L)$; let $\mathfrak{O}^L \in \mathcal{N}_\mathfrak{l}/L$. Then $\mathrm{Ind}_\mathfrak{l}^\mathfrak{g} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ if and only if $\mathrm{Ind}_\mathfrak{l}^\mathfrak{m} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ and $\mathrm{Ind}_\mathfrak{m}^\mathfrak{g}(\mathrm{Ind}_\mathfrak{l}^\mathfrak{m} \mathfrak{O}^L)$ is birationally induced from $(\mathfrak{m}, \mathrm{Ind}_\mathfrak{l}^\mathfrak{m} \mathfrak{O}^L)$.*

Proof. Consider parabolic subgroups P resp. Q with Levi decompositions $P = LU_P$ resp. $Q = MU_Q$. As usual, write $\mathrm{Lie}(P) := \mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}_P$ and $\mathrm{Lie}(Q) := \mathfrak{q} = \mathfrak{m} \ltimes \mathfrak{n}_Q$ for the corresponding Levi decompositions of the corresponding Lie algebras. Then $P \cap M$ is a parabolic subgroup of M and $\mathrm{Lie}(P \cap M) = \mathfrak{p} \cap \mathfrak{m} = \mathfrak{l} \ltimes (\mathfrak{n}_P \cap \mathfrak{m})$. Write $\mathfrak{O}^M := \mathrm{Ind}_\mathfrak{l}^\mathfrak{m} \mathfrak{O}^L$ and $\mathfrak{O}^G := \mathrm{Ind}_\mathfrak{l}^\mathfrak{g} \mathfrak{O}^L = \mathrm{Ind}_\mathfrak{m}^\mathfrak{g} \mathfrak{O}^M$ with the corresponding generalized Springer maps:

$$\gamma_L^G : G \times^P (\overline{\mathfrak{O}^L} + \mathfrak{n}_P) \to \overline{\mathfrak{O}^G},$$
$$\gamma_L^M : M \times^{P\cap M} (\overline{\mathfrak{O}^L} + (\mathfrak{n}_P \cap \mathfrak{m})) \to \overline{\mathfrak{O}^M},$$
$$\gamma_M^G : G \times^Q (\overline{\mathfrak{O}^M} + \mathfrak{n}_Q) \to \overline{\mathfrak{O}^G}.$$

There exist $\sigma + \nu_1 \in \mathfrak{O}^M$ with $\sigma \in \mathfrak{O}^L$ and $\nu_1 \in \mathfrak{n}_P \cap \mathfrak{m}$. Similarly, $\mathfrak{O}^G$ has representatives of the form $\sigma + \nu_1 + \nu_2$, where $\nu_2 \in \mathfrak{n}_Q$ and $\sigma + \nu_1 \in \mathfrak{O}^M \cap (\mathfrak{O}^L + (\mathfrak{n}_P \cap \mathfrak{m}))$.

Suppose that γ_L^G is birational, then for $\sigma + \nu_1 + \nu_2 \in \mathfrak{O}^G$ as above, we have $C_G(\sigma + \nu_1 + \nu_2) \le P \le Q$, so γ_M^G is birational by Corollary 1.5. We show $C_M(\sigma + \nu_1) \le P$. Let $m \in C_M(\sigma + \nu_1)$, then $\mathrm{Ad}(m)(\sigma + \nu_1 + \nu_2) = \sigma + \nu_1 + \mathrm{Ad}(m)(\nu_2) \in (\overline{\mathfrak{O}^L} + \mathfrak{n}_P) \cap \mathfrak{O}^G = \mathfrak{O}^P_{\sigma+\nu_1+\nu_2}$, by Lemma 1.4. Hence there exists $p \in P$ such that $pm \in C_G(\sigma + \nu_1 + \nu_2) \le P$. This implies $m \in P$, i.e., γ_L^M is birational by Corollary 1.5 (iv).

For the other implication, assume γ_L^M and γ_M^G are birational. Let $\sigma + \nu_1 + \nu_2 \in \mathfrak{O}^G$ be as above and let $g \in C_G(\sigma + \nu_1 + \nu_2)$. We show that $g \in P$. Since γ_M^G is birational, then $g \in C_G(\sigma + \nu_1 + \nu_2) \leq Q$ by Corollary 1.5 (iv). So $g = mv$ with $m \in M$ and $v \in U_Q$. Then $\sigma + \nu_1 + \nu_2 = \mathrm{Ad}(mv)(\sigma + \nu_1 + \nu_2) = \mathrm{Ad}(m)(\sigma + \nu_1) + \mathrm{Ad}(m)(\tilde{\nu}_2)$, where $\tilde{\nu}_2 := \mathrm{Ad}(v)(\sigma + \nu_1 + \nu_2) - (\sigma + \nu_1) \in \mathfrak{n}_Q$. Now $\mathrm{Ad}(m)(\sigma + \nu_1) \in \mathfrak{m}$ and $\mathrm{Ad}(m)(\tilde{\nu}_2) \in \mathfrak{n}_Q$, since M stabilizes $\mathfrak{m}$ and $\mathfrak{n}_Q$. The semidirect sum decomposition $\mathfrak{m} \ltimes \mathfrak{n}_Q$ implies $\mathrm{Ad}(m)(\sigma + \nu_1) = \sigma + \nu_1$, i.e., $m \in C_M(\sigma + \nu_1)$. Since γ_L^M is also birational, we have $C_M(\sigma + \nu_1) \leq P$, by Corollary 1.5. Therefore, $g = mv \in P$, i.e., $C_G(\sigma + \nu_1 + \nu_2) \leq P$ and γ_L^G is birational, by Corollary 1.5 (iv). □

Remarks on the proof. Namikawa proved the second implication with different techniques in [76, Proof of 2.1.1].

Next result can be used to test if an orbit is birationally induced.

Lemma 1.13. *Let $\phi\colon \mathcal{N} \to \mathcal{U}$ be a Springer isomorphism and let $\pi\colon G \to \overline{G}$ be the isogeny to the adjoint group. Let $\nu \in \mathcal{N}$ and suppose $C_{\overline{G}}(\nu) = C_{\overline{G}}(\nu)^\circ$. If $\mathfrak{O}_\nu^G$ is induced from $(\mathfrak{l}, \mathfrak{O}^L)$ for a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ and some $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$, then $\mathfrak{O}_\nu^G$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$.*

Proof. Let $P \leq G$ be parabolic with Levi decomposition $P = LU$. Let f be the composition defined via the following diagram:

$$\begin{tikzcd} C_G(\nu) \arrow[r, two heads] \arrow[dr, "f"'] & C_G(\nu)/Z(G) = C_{\overline{G}}(\nu) \arrow[d, two heads] \\ & C_{\overline{G}}(\nu)/C_{\overline{G}}(\nu)^\circ \end{tikzcd}$$

We claim that: $\ker f = Z(G)C_G(\nu)^\circ$. If $g \in \ker f$, then $\pi(g) \in C_{\overline{G}}(\nu)^\circ = \pi(C_G(\nu)^\circ)$, hence $g \in Z(G)C_G(\nu)^\circ$. The other inclusion is trivial. Since by hypothesis $C_{\overline{G}}(\nu)/C_{\overline{G}}(\nu)^\circ = \{1\}$, we have $C_G(\nu) = \ker f = (\ker \pi)C_G(\nu)^\circ = Z(G)C_G(\nu)^\circ$. By Lemma 1.6, $C_G(\nu)^\circ \leq C_P(\nu) \leq C_G(\nu)$, hence $C_P(\nu) = C_G(\nu)$, because $\ker \pi = Z(G) \leq C_P(\nu)$. We conclude by Corollary 1.5 (iv). □

Remarks on the proof: the same result is contained in [45, Corollary 2.2] for G adjoint. For G simple and adjoint and $\nu \in \mathcal{N}$, the groups $C_G(\nu)/C_G(\nu)^\circ$ are known, see [35, §6.1, §8.4].

1.3 Induction of conjugacy classes

Induction can be defined also for conjugacy classes. Let G act on $Y = G$ via conjugacy, fix a Levi subgroup $L \leq G$ and choose a parabolic $P \leq G$ with Levi decomposition LU_P. For any class $\mathcal{O}^L \in L/L$, we take $X = \overline{\mathcal{O}^L}U_P \subset G$, a closed subvariety which is P-stable under the conjugacy action. Then we have a well-defined map:

$$\begin{aligned} \gamma\colon G \times^P \overline{\mathcal{O}^L}U_P &\to G \cdot (\overline{\mathcal{O}^L}U_P) \\ g * x &\mapsto gxg^{-1} \end{aligned} \tag{1.8}$$

called *generalized Springer map.*

Since the image of γ is a G-stable closed subset of G which projects to a point in the categorical quotient $G//G$, it is a closed irreducible subset consisting of finitely many conjugacy classes. Irreducibility yields that $(G \cdot X)^{reg}$ is a single conjugacy class in G, called *induced* from $(L, \mathcal{O}^L)$, and denoted $\operatorname{Ind}_L^G \mathcal{O}^L$ see [29,71]. The definition of induced conjugacy class only depends on the Levi $L \leq G$ and on $\mathcal{O}^L$ and not on the parabolic P containing L. For any Levi subgroup $L \leq G$, we introduce the notation:

$$\begin{aligned} \operatorname{Ind}_L^G \colon L/L &\to G/G \\ \mathcal{O}^L &\mapsto \operatorname{Ind}_L^G \mathcal{O}^L \,. \end{aligned}$$

We have $\operatorname{codim}_G(\operatorname{Ind}_L^G \mathcal{O}^L) = \operatorname{codim}_L \mathcal{O}^L$ (see [29, Proposition 4.6] and [71]).

Notice that $\mathcal{O}^L \in \mathcal{U}_L/L$ if and only if $\operatorname{Ind}_L^G \mathcal{O}^L \in \mathcal{U}/G$, in particular, the restriction of Ind_L^G induces a function $\mathcal{U}_L/L \to \mathcal{U}/G$, the aforementioned Lusztig–Spaltenstein induction defined in [71].

A unipotent conjugacy class $\mathcal{O}$ is *rigid* in G if it cannot be induced from a proper Levi subgroup $L \lneq G$ and a unipotent class $\mathcal{O}^L \in \mathcal{U}_L/L$.

1.3.1 Reduction to the unipotent case

We now explain how the induction of a conjugacy class in G is related to the induction of a unipotent conjugacy class in a pseudo-Levi subgroup of G. Let $L \leq G$ be a Levi subgroup and let $P \leq G$ be a parabolic with Levi decomposition $P = LU$ and let $su \in L$. It was proven in [29, Proposition 4.6] that:

$$\operatorname{Ind}_L^G \mathcal{O}_{su}^L = G \cdot (s \operatorname{Ind}_{C_L(s)^\circ}^{C_G(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ}). \tag{1.9}$$

Notice that $C_L(s)^\circ$ is a Levi subgroup in $C_G(s)^\circ$ because if $Z \coloneqq Z(L)^\circ$, then $C_L(s)^\circ = C_{G(s)^\circ}(Z)$.

Induction is transitive, i.e., if $M \leq G$ is a Levi subgroup, L is a Levi subgroup of M and $\mathcal{O}_{su}^L \in L/L$, then:

$$\begin{aligned} \operatorname{Ind}_M^G(\operatorname{Ind}_L^M \mathcal{O}_{su}^L) &= \operatorname{Ind}_M^G(M \cdot (s \operatorname{Ind}_{C_L(s)^\circ}^{C_M(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ})) = \\ &= G \cdot (s \operatorname{Ind}_{C_M(s)^\circ}^{C_G(s)^\circ}(\operatorname{Ind}_{C_L(s)^\circ}^{C_M(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ})) = \\ &= G \cdot (s \operatorname{Ind}_{C_L(s)^\circ}^{C_G(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ}) = \operatorname{Ind}_M^G \mathcal{O}_{su}^L, \end{aligned}$$

where we used transitivity for induction of unipotent classes, see [71, §1.7].

Lemma 1.14. *The generalized Springer map* (1.8) *fulfils the hypotheses of Lemma 1.4 with* $X = \overline{\mathcal{O}^L} U_P$ *and* $Y = G$.

In particular, there is a unique open orbit in $G \times^P \overline{\mathcal{O}^L} U_P$ *and this is a finite G-equivariant covering of* $\operatorname{Ind}_L^G \mathcal{O}^L$.

Proof. The argument mimics the proof of Lemma 1.6. □

In the Lie algebra setting, Lemma 1.7 allows one to reduce questions related to birationality of induction to the case of nilpotent orbits. It seems therefore natural to investigate the possibility to find a group analogue of the cited result. If $P \geq B$, for a semisimple element $s \in B$, we have that $C_B(s)^\circ = C_B(s)$ is a Borel subgroup of $C_G(s)^\circ$ and $C_P(s)^\circ$ is a parabolic subgroup of $C_G(s)^\circ$. The equality $C_P(s)^\circ = P \cap C_G(s)^\circ$ holds and $C_L(s)^\circ$ is a Levi factor of $C_P(s)^\circ$; we write $U_{C_P(s)^\circ}$ for the unipotent radical of $C_P(s)^\circ$. We compare the two morphisms:

$$\gamma \colon G \times^P \overline{\mathcal{O}_{su}^L} U \to \overline{\operatorname{Ind}_L^G \mathcal{O}_{su}^L} \tag{1.10}$$

$$\gamma_s \colon C_G(s)^\circ \times^{C_P(s)^\circ} \overline{\mathcal{O}_u^{C_L(s)^\circ}} U_{C_P(s)^\circ} \to \overline{\operatorname{Ind}_{C_L(s)^\circ}^{C_G(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ}} \tag{1.11}$$

Lemma 1.15. *Let γ and γ_s be as in* (1.10) *and* (1.11)*, respectively. Set $\mathcal{O}^G := \operatorname{Ind}_L^G \mathcal{O}_{su}^L$ and $\mathcal{O}^{C_G(s)^\circ} := \operatorname{Ind}_{C_L(s)^\circ}^{C_G(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ}$. Then:*

(i) Birationality of γ implies birationality of γ_s.

(ii) Suppose in addition that $C_G(s) = C_G(s)^\circ$. If γ_s is birational, then γ is birational. In particular, if $[G, G]$ is simply-connected, this is always the case.

Proof. Notice that Lemma 1.14 applies also to (1.11), so that we can make use of Corollary 1.5 (iv) as in the proof of Lemma 1.7.

(i) Suppose γ is birational. Let $v \in \overline{\mathcal{O}_u^{C_L(s)^\circ}} U_{C_P(s)^\circ} \cap \mathcal{O}^{C_G(s)^\circ}$. Then $sv \in \mathcal{O}^G = G\cdot(s\,\mathcal{O}^{C_G(s)^\circ})$ by (1.9) and $sv \in s\overline{\mathcal{O}_u^{C_L(s)^\circ}} U_{C_P(s)^\circ} \subset \overline{\mathcal{O}_{su}^L}\, U$, so that $sv \in \overline{\mathcal{O}_{su}^L} U \cap \mathcal{O}^G$. Since $v \in C_G(s)^\circ$, the unipotent part of sv is v. Birationality of γ yields $C_G(sv) \leq P$, so $C_{C_G(s)^\circ}(v) \leq C_{C_G(s)}(v) \leq P$. Now $C_{C_G(s)^\circ}(v) \leq C_G(s)^\circ$, hence $C_{C_G(s)^\circ}(v) \leq P \cap C_G(s)^\circ = C_P(s)^\circ$ and γ_s is birational.

(ii) We assume $C_G(s)$ connected and γ_s birational and show that γ is birational. Choose an element $sv \in s\,\mathcal{O}^{C_G(s)^\circ} \cap s\overline{\mathcal{O}^{C_L(s)^\circ}} U_{C_P(s)^\circ} \subset \mathcal{O}^G \cap \overline{\mathcal{O}_{su}^L} U_P$. Then $C_G(sv) = C_{C_G(s)^\circ}(v) \leq C_P(s)^\circ \leq P$. $\square$

The following example shows that in general, the birationality of γ_s does not imply the birationality of γ.

Example 1.16. Let $G = \mathrm{PSp}_4(\mathbb{C})$. Let $s = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in G$. Let $\Theta = \{\alpha_2\}$ and consider the standard parabolic $P := P_\Theta$ with Levi decomposition $P = LU$, where $L := L_\Theta$. Notice that $C_G(s)^\circ \lneq C_G(s)$ and that $C_L(s)^\circ = L$. Let u be in the regular unipotent class of L. Then $\operatorname{Ind}_L^{C_G(s)^\circ} \mathcal{O}_u^L = \mathcal{O}_v^{C_G(s)^\circ}$ with $v = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ while $\operatorname{Ind}_L^G \mathcal{O}_{su}^L = \mathcal{O}_{sv}$. The map γ_s as in (1.11) is birational, since $C_{C_G(s)^\circ}(v) \leq C_P(s)^\circ$. On the other hand, γ as in (1.10) is not birational, because $C_G(sv) \not\leq P$, for example $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in C_G(sv) \setminus P$.

1.3.2 Birationality for induction of conjugacy classes

In this section we discuss the definition of birational induction of a class $\mathcal{O}_{sv} \in G/G$.

Definition 1.17. Let $su \in L$ and let $\mathcal{O} = \operatorname{Ind}_L^G \mathcal{O}_{su}^L$. We say that $\mathcal{O}$ is:

(a) *birationally induced* from $(L, \mathcal{O}^L_{su})$ if the generalized Springer map

$$\gamma\colon G \times^P \overline{\mathcal{O}^L_{su}}U \to \overline{\mathrm{Ind}_L^G \mathcal{O}^L_{su}}$$

defined in (1.10) is birational;

(b) *weakly birationally induced* from $(L, \mathcal{O}^L_{su})$ if the generalized Springer map

$$\gamma_s\colon C_G(s)^\circ \times^{C_P(s)^\circ} \overline{\mathcal{O}_u^{C_L(s)^\circ}}U_{C_P(s)^\circ} \to \overline{\mathrm{Ind}_{C_L(s)^\circ}^{C_G(s)^\circ} \mathcal{O}_u^{C_L(s)^\circ}}$$

defined in (1.11) is birational.

Remark 1.18. It is a consequence of Lemma 1.15 that if $\mathcal{O}_{sv} \in G/G$ is birationally induced from $(L, \mathcal{O}^L_{su})$ then it is also weakly birationally induced from $(L, \mathcal{O}^L_{su})$. Moreover, by Lemma 1.15, the two notions coincide in the case that $[G, G]$ is simply-connected or when $\mathcal{O} = G \cdot (zu) = z\,\mathcal{O}_u$ for $z \in Z(G)$ and $u \in \mathcal{U}$ (this is the case if and only if the inducing orbit is $z\,\mathcal{O}^L$ with $\mathcal{O}^L \in \mathcal{U}_L/L$). Definition 1.17 (a) seems to be the natural way to extend Definition 1.33 to the group case. On the other hand, Definition 1.17 (b) can be dealt with more easily, allowing a straight-forward reduction to the Lie algebra setting, as we will see later.

We may drop one, or both of the elements of the pair of inducing data $(L, \mathcal{O}^L_{su})$ in the notation when they are clear from the context or they are not relevant. In particular, we will say that the class $\mathcal{O}^G \in G/G$ is (non-trivially) birationally induced (resp. weakly birationally induced) if there exist a proper Levi subgroup $L \lneq G$ and a conjugacy class $\mathcal{O}^L \in L/L$ such that $\mathcal{O}^G$ is birationally induced (resp. weakly birationally induced) from $(L, \mathcal{O}^L)$.

Definition 1.19. A unipotent conjugacy class in G is said to be *birationally rigid* if it cannot be birationally induced from a unipotent class $\mathcal{O}^L$ inside a proper Levi $L \lneq G$.

1.3.3 Springer isomorphisms and isogenies

We extend all results on birationality of induction of nilpotent orbits to the case of unipotent classes. The main instrument to perform this work is a Springer isomorphism $\phi\colon \mathcal{N} \to \mathcal{U}$. We prove that the Springer isomorphism is compatible with induction of unipotent classes (resp. of nilpotent orbits) and preserves birationality.

Lemma 1.20. *Fix a Levi subgroup $L \leq G$, set $\mathfrak{l} := \mathrm{Lie}(L)$ and let $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$.*

(i) An orbit $\mathfrak{O} \in \mathcal{N}/G$ is induced from $(\mathfrak{l}, \mathfrak{O}^L)$ if and only if $\phi(\mathfrak{O})$ is induced from $(L, \phi_L(\mathfrak{O}^L))$. In particular, rigid orbits in $\mathfrak{g}$ correspond to rigid unipotent classes in G.

(ii) An orbit $\mathfrak{O} \in \mathcal{N}/G$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ if and only if $\phi(\mathfrak{O})$ is birationally induced from $(L, \phi_L(\mathfrak{O}^L))$. In particular, birationally rigid orbits in $\mathfrak{g}$ correspond to birationally rigid classes in G.

Proof. (i) follows from definitions of induction, (ii) is a consequence of Corollary 1.5 (iii) and the fact that Springer isomorphisms preserves centralizers. □

Isogenies are compatible with induction of unipotent classes and birationality is preserved.

Lemma 1.21. *Let $\pi\colon G \to \overline{G}$ be an isogeny. Fix a Levi subgroup $L \leq G$ and let $\mathfrak{l} := \mathrm{Lie}(L)$; let $\mathcal{O}^L \in \mathcal{U}_L/L$.*

(i) A class $\mathcal{O} \in \mathcal{U}/G$ is induced from $(L, \mathcal{O}^L)$ if and only if the class $\pi(\mathcal{O})$ is induced from $(\pi(L), \pi(\mathcal{O}^L))$.

(ii) A class $\mathcal{O} \in \mathcal{U}/G$ is birationally induced from $(L, \mathcal{O}^L)$ if and only if $\pi(\mathcal{O})$ is birationally induced from $(\pi(L), \pi(\mathcal{O}^L))$.

Proof. Since $\ker \pi \subset Z(G)$ consists of semisimple elements, the isogeny π yields an identification between the unipotent varieties $\mathcal{U}_G$ and $\mathcal{U}_{\overline{G}}$. We have $\mathrm{Lie}(G) = \mathrm{Lie}(\overline{G}) = \mathfrak{g}$ and $\mathrm{Lie}(L) = \mathrm{Lie}(\pi(L)) = \mathfrak{l}$. Let $\mathfrak{O} \in \mathcal{N}/G$ be such that $\phi_G(\mathfrak{O}) = \mathcal{O}$ and $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ be such that $\phi_L(\mathfrak{O}^L) = \mathcal{O}^L$. Then $\phi_{\overline{G}}(\mathfrak{O}) = \pi(\mathcal{O})$ and $\phi_{\pi(L)}(\mathfrak{O}^L) = \pi(\mathcal{O}^L)$. By Lemma 1.20, $\mathcal{O} \in \mathcal{U}/G$ is (birationally) induced from $(L, \mathcal{O}^L)$ if and only if $\mathfrak{O}$ is (birationally) induced from $(\mathfrak{l}, \mathfrak{O}^L)$ if and only if $\pi(\mathcal{O})$ is (birationally) induced from $(\pi(L), \pi(\mathcal{O}^L))$. $\square$

Similarly to the case of nilpotent orbits, birationality of induction of unipotent classes only depends on the G-conjugacy class of the inducing pair $(L, \mathcal{O}^L)$.

Lemma 1.22. *Consider two parabolic subgroups $P, Q \leq G$ with Levi decompositions $P = LU_P, Q = LU_Q$ respectively. Let $\mathcal{O}^L \in \mathcal{U}_L/L$ and set $\mathcal{O} = \mathrm{Ind}_L^G(\mathcal{O}^L)$. Then the generalized Springer map $\gamma_P\colon G \times^P (\overline{\mathcal{O}^L} U_P) \to \overline{\mathcal{O}}$ is birational if and only if the generalized Springer map $\gamma_Q\colon G \times^Q (\overline{\mathcal{O}^L} U_Q) \to \overline{\mathcal{O}}$ is birational.*

Proof. Set $\mathfrak{p} := \mathrm{Lie}(P)$, $\mathfrak{q} := \mathrm{Lie}(Q)$ with Levi decompositions $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}_{\mathfrak{p}}$ and $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}_{\mathfrak{q}}$, respectively. Let $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ (resp. $\mathfrak{O} \in \mathcal{N}/G$) such that $\phi_L(\mathfrak{O}^L) = \mathcal{O}^L$ (resp. $\phi(\mathfrak{O}) = \mathcal{O}$). Consider the generalized Springer maps $\gamma_{\mathfrak{p}}\colon G \times^P (\overline{\mathfrak{O}^L} + \mathfrak{n}_{\mathfrak{p}}) \to \overline{\mathfrak{O}}$ and $\gamma_{\mathfrak{q}}\colon G \times^Q (\overline{\mathfrak{O}^L} + \mathfrak{n}_{\mathfrak{q}}) \to \overline{\mathfrak{O}}$. By Lemma 1.20 (ii), γ_P (resp. γ_Q) is birational if and only if $\gamma_{\mathfrak{p}}$ (resp. $\gamma_{\mathfrak{q}}$) is birational. We conclude by Lemma 1.11. $\square$

Remark 1.23. Let $\vartheta \in \mathrm{Aut}(G)$, let $L \leq G$ be a Levi subgroup and $\mathcal{O}^L \in L/L$. Then ϑ preserves parabolic subgroups, their Levi decompositions, and unipotent varieties of Levi subgroups. Thus, $\vartheta(\mathrm{Ind}_L^G \mathcal{O}^L) = \mathrm{Ind}_{\vartheta(L)}^G \vartheta(\mathcal{O}^L)$. By Corollary 1.5 (iii), birationality of unipotent induction is preserved under action by $\mathrm{Aut}(G)$ on the inducing data because ϑ maps centralizers to centralizers. In particular, for $g \in G$ and $\mathcal{O}_u^L \in \mathcal{U}_L/L$, the class $\mathcal{O}^G := \mathrm{Ind}_L^G(\mathcal{O}_u^L)$ is birationally induced from $(L, \mathcal{O}_u^L)$ if and only if $\mathcal{O}^G$ is birationally induced from $(gLg^{-1}, \mathcal{O}_{gug-1}^{gLg^{-1}})$.

Remark 1.24. By Lemma 1.21 (ii), the notion of birational rigidity for a unipotent conjugacy class does not depend on the isogeny class of the group. For a Levi subgroup $L \leq G$ and $\vartheta \in \mathrm{Aut}(G)$, a class $\mathcal{O}_u^L \in \mathcal{U}_L/L$ is birationally rigid in L if and only if $\mathcal{O}_{\vartheta(u)}^{\vartheta(L)}$ is birationally rigid in $\vartheta(L)$.

Transitivity of birational induction also holds for unipotent classes.

Proposition 1.25. *Consider a parabolic subgroup $P \leq G$ with Levi decomposition $P = LU$ and let $Q \geq P$ have Levi decomposition $Q = MV$ with $L \leq M$ and $U \geq V$. Consider $P \cap M$, a parabolic subgroup of M with Levi decomposition of $P \cap M = L(U \cap M)$. Let $\mathcal{O}^L \in \mathcal{U}_L/L$; set $\mathcal{O}^M := \mathrm{Ind}_L^M \mathcal{O}^L$ and $\mathcal{O}^G := \mathrm{Ind}_L^G \mathcal{O}^L = \mathrm{Ind}_M^G \mathcal{O}^M$. We have the generalized Springer maps:*

$$\begin{aligned} \gamma_L^G &: G \times^P \overline{\mathcal{O}^L} U \twoheadrightarrow \overline{\mathcal{O}^G}, \\ \gamma_L^M &: M \times^{P\cap M} \overline{\mathcal{O}^L}(U \cap M) \twoheadrightarrow \overline{\mathcal{O}^M}, \\ \gamma_M^G &: G \times^Q \overline{\mathcal{O}^M} V \twoheadrightarrow \overline{\mathcal{O}^G}. \end{aligned}$$

The map γ_L^G is birational if and only if the maps γ_L^M and γ_M^G are birational.

Proof. Thanks to the Springer isomorphism, one can reduce to the situation of Lemma 1.12 and conclude by Lemma 1.20. □

The sufficient condition of Lemma 1.13 can be restated as follows.

Lemma 1.26. *Let $u \in \mathcal{U}$ and let $\pi \colon G \to \overline{G}$ be the quotient to the adjoint group. Suppose $C_{\overline{G}}(\pi(u)) = C_{\overline{G}}(\pi(u))^\circ$. If $\mathcal{O}_u^G$ is induced from $(L, \mathcal{O}^L)$ for a Levi subgroup $L \leq G$ and some $\mathcal{O}^L \in \mathcal{U}_L/L$, then $\mathcal{O}_u^G$ is birationally induced from $(L, \mathcal{O}^L)$.*

Proof. This follows from Lemmas 1.20, 1.21 and 1.13. □

1.3.4 Birationally rigid unipotent classes in simple groups; examples

We conclude this section with some examples.

Example 1.27. If G is simple of type A and $\mathfrak{O} \in \mathcal{N}/G$, then $\mathfrak{O}$ is rigid if and only if it is birationally rigid if and only if $\mathfrak{O}$ is the zero orbit. By [35, Theorem 7.2.3] and adopting notation therein, if $\mathcal{O} \in \mathcal{U}/G$ corresponds to a partition $\mathbf{p}$ of n, then $\mathfrak{O} = \mathrm{Ind}_L^G\{0\}$, where $L = L_{\mathbf{p^t}}$ corresponds to the dual partition $\mathbf{p^t}$, as in (3). The induction is birational, by Lemma 1.13. Indeed, for G adjoint, we have $C_G(\nu)$ connected for all $\nu \in \mathcal{N}$, see [35, Corollary 6.1.6]. By Lemma 1.20, if G is simple of type A and $\mathcal{O} \in \mathcal{U}/G$, then $\mathcal{O}$ is rigid if and only if it is birationally rigid if and only if $\mathcal{O} = \{1\}$.

Namikawa (in [76]) and Fu (in [45]) carried out the complete list of birationally rigid nilpotent orbits for $\mathfrak{g}$ simple. For the reader's convenience and for completeness of this work, we list them: by Lemmas 1.20 and 1.21 giving this list is equivalent to giving the list of birationally rigid unipotent classes in G simple.

Namikawa dealt with birational rigidity for nilpotent orbits in simple classical Lie algebras.

If $\mathfrak{g}$ is a simple Lie algebra of type A, then the only birationally rigid orbit is the only rigid orbit, i.e., the null orbit, as already seen in Example 1.27.

Now let $\mathfrak{g}$ be a simple Lie algebra of type $\mathsf{B}, \mathsf{C}, \mathsf{D}$. Let $\mathbf{d} = [d_1, \dots, d_r]$ denote the partition corresponding to a nilpotent orbit $\mathfrak{O}$. Then $\mathfrak{O}$ is birationally rigid in $\mathfrak{g}$ if and only if $\mathbf{d}$ has *full* *members*, i.e., $1 = d_r$ and $d_i - d_{i+1} \leq 1$ for all $i = 1, \ldots, r-1$, with the exception of the case $\mathbf{d} = [2^{2n}, 1^2]$ in D_n for $n \geq 3$ odd, which is birationally induced as a Richardson orbit.

Fu worked out the exceptional cases:

- in types $\mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_6$ birationally rigid orbits coincide with rigid ones;
- in type E_7, the set of birationally rigid orbits consists of rigid orbits together with the two orbits with Bala-Carter label $A_2 + A_1$ and $A_4 + A_1$;
- in type E_8, the set of birationally rigid orbits consists of rigid orbits together with the two orbits with Bala-Carter label $A_4 + A_1$ and $A_4 + 2A_1$.

For a complete list of rigid nilpotent orbits in the exceptional cases, we refer to [73, Appendix 5.7].

Remark 1.28. Recall that an adjoint orbit in $\mathfrak{g}$ (resp. a conjugacy class in G) is said to be *characteristic* if it is stable under all Lie algebra automorphisms of $\mathfrak{g}$ (resp. under all group automorphisms of G). From [12, Lemma 3.9], we deduce that all nilpotent orbits $\mathfrak{O}$ in $\mathfrak{g}$ simple are characteristic, except for the following list:

- for $\mathfrak{g}$ of type D_4, the group of automorphisms of $\mathfrak{g}$ acts transitively on $\{\mathfrak{O}_{[4^2]}, \mathfrak{O}'_{[4^2]}, \mathfrak{O}_{[5,1^3]}\}$ and on $\{\mathfrak{O}_{[2^4]}, \mathfrak{O}'_{[2^4]}, \mathfrak{O}_{[3,1^5]}\}$.
- for $\mathfrak{g}$ of type $\mathsf{D}_{2m}, m \geq 3$, an automorphism permutes $\mathfrak{O}_{\mathbf{d}}$ and $\mathfrak{O}'_{\mathbf{d}}$ for every very even partition $\mathbf{d} \vdash 4m$.

It follows that all nilpotent birationally rigid orbits in a simple Lie algebra are characteristic, analogously for all unipotent birationally rigid classes in simple groups.

Example 1.29. Let G be simple of type C and let $\mathcal{O}_u$ be the subregular unipotent class of G. Let $\Theta_1 = \{\alpha_1\}$ and $\Theta_n = \{\alpha_n\}$. Set $L_i \coloneqq L_{\Theta_i}$, then $\mathcal{O}_u$ is induced both by $(L_1, \{1\})$ and $(L_n, \{1\})$. Observe that L_1 and L_n are not conjugate in G, since α_1 and α_n have different root length, see the criterion in [92, §2.2]. Let P_1 (resp. P_n) be the standard parabolic subgroup of G with standard Levi factor L_1 (resp. L_n). A direct computation shows that $C_G(u) \leq P_1$ so that $\mathcal{O}_u$ is birationally induced from $(L_1, \{1\})$, whereas the induction $\mathrm{Ind}_{L_n}^G \{1\}$ is not birational, as $C_G(u) \not\leq P_n$. In this case Lemma 1.26 fails, as if $\pi\colon G \to \overline{G}$ denotes the isogeny to the adjoint group, $C_{\overline{G}}(\pi(u))/C_{\overline{G}}(\pi(u))^\circ \simeq \mathbb{Z}/2\mathbb{Z}$.

Example 1.30. Every rigid conjugacy class is birationally rigid. The converse is not true: if G is simple of type C_3, let $\mathcal{O}^G$ be the unipotent class indexed by the partition $[2^2, 1^2]$. This is birationally rigid, as it satisfies the full-members criterion. Nonetheless, $\mathrm{codim}_G \mathcal{O} = \dim L_\Theta = \mathrm{codim}_{L_\Theta}\{1\}$, where $\Theta = \{\alpha_2, \alpha_3\}$. Since $\mathcal{O} \in \mathcal{U}/G$ is uniquely determined by its dimension, we have $\mathcal{O} = \mathrm{Ind}_{L_\Theta}^G \{1\}$.

We have seen in Example 1.29 that there exist Levi subgroups $L_i \leq G$ and (birationally) rigid unipotent classes $\mathcal{O}^{L_i} \in \mathcal{U}_{L_i}/L_i$ such that $(L_1, \mathcal{O}^{L_1})$ and $(L_2, \mathcal{O}^{L_2})$ are not G-conjugate, but $\mathrm{Ind}_{L_1}^G \mathcal{O}^{L_1} = \mathrm{Ind}_{L_2}^G \mathcal{O}^{L_2}$. We will see in the next Section that this phenomenon does not occur if we require birationality of the induction.

1.4 Uniqueness of birational induction

In this Section we establish an explicit bijection between conjugacy classes in G and a set of data which are "minimal" with respect to weakly birational induction. The result is a group counterpart to the first assertion contained in [67, Theorem 4.4], where Losev identifies $\mathfrak{g}/G$ with the set of G-conjugacy classes of birationally minimal induction data.

Our strategy will be to start with a focus on the unipotent variety, adapting [67, Corollary 4.5] to the case of the conjugacy action of a group on itself, and then derive the more general theorem.

Definition 1.31. A *unipotent birational induction datum* is $(L, \mathcal{O}^L)$ where $L \leq G$ is a Levi subgroup, $\mathcal{O}^L \in \mathcal{U}_L/L$ is birationally rigid and $\operatorname{Ind}_L^G \mathcal{O}^L$ is birationally induced from $(L, \mathcal{O}^L)$. We denote by $\mathscr{B}(G)_u$ the set of all unipotent birational data of G.

Notice that G acts on $\mathscr{B}(G)_u$ by simultaneous conjugacy on the pairs and that $\mathscr{B}(G)_u/G$ is finite.

Lemma 1.32. *Let G be reductive and let $\mathcal{O}^G \in \mathcal{U}/G$. Then there exists, up to G-conjugacy, a unique pair $(L, \mathcal{O}^L) \in \mathscr{B}(G)_u$ such that $\mathcal{O}^G = \operatorname{Ind}_L^G \mathcal{O}^L$.*

In other words, the map

$$\begin{aligned} \mathscr{B}(G)_u/G &\longrightarrow \mathcal{U}/G \\ [(L, \mathcal{O}^L)]_\sim &\longmapsto \operatorname{Ind}_L^G \mathcal{O}^L \end{aligned}$$

is bijective.

Proof. For G reductive, $\mathcal{U} = \mathcal{U}_{[G,G]}$, hence we may assume G semisimple. Let $G = G_1 \cdots G_d$ be the decomposition into simple factors, $d \in \mathbb{N}$. The decomposition of G carries over to Levi subgroups, parabolic subgroups and unipotent conjugacy classes. For $L \leq G$ a Levi subgroup and $\mathcal{O}^L \in \mathcal{U}_L/L$, we write $L = \prod_{i=1}^d L_i$ with L_i Levi subgroup of G_i and $\mathcal{O}^L = \prod_{i=1}^d \mathcal{O}^{L_i}$ with $\mathcal{O}^{L_i} \in \mathcal{U}_{L_i}/L_i$. Observe that Lusztig–Spaltenstein induction in G is compatible with this decomposition. If $u \in \mathcal{U}$, then $u = u_1 \cdots u_d$ with u_i uniquely determined and unipotent in G_i for all $i = 1, \ldots, d$ and $C_G(u) = \prod_{i=1}^d C_{G_i}(u_i)$. The induction $\operatorname{Ind}_L^G \mathcal{O}^L$ is birational if and only if all inductions $\operatorname{Ind}_{L_i}^{G_i} \mathcal{O}^{L_i}$ are so. Also, $\mathcal{O}^L$ is birationally rigid in L if and only if each $\mathcal{O}^{L_i}$ is birationally rigid in L_i. Thus, we are reduced to proving the statement for each simple factor G_i. This follows from [67, Corollary 4.5 (i)] and the Springer isomorphism (Lemma 1.20, Remark 1.24). $\square$

We give a group analogue of Losev's definition of birationally minimal induction data.

Definition 1.33. A *weakly birational induction datum* is a triple $(M, s, \mathcal{O}^M)$ where: $M \leq G$ is a pseudo-Levi subgroup, $s \in Z(M)$ is such that $Z(M)^\circ s$ satisfies (RP), $\mathcal{O}^M \in \mathcal{U}_M/M$ is birationally rigid and $\operatorname{Ind}_M^{C_G(s)^\circ} \mathcal{O}^M$ is birationally induced from $(M, \mathcal{O}^M)$. We denote by $\mathscr{B}(G)$ the set of weakly birational induction data of G.

G acts on $\mathscr{B}(G)$ by simultaneous conjugacy on the triples.

Remark 1.34. Let $(M, s, \mathcal{O}^M) \in \mathscr{B}(G)$. Let $L(M) \coloneqq C_G(Z(M)^\circ)$ be the *Levi envelope* of M, i.e., the smallest Levi subgroup of G containing M, [29, Definition 3.7]. Notice that $C_{L(M)}(s)^\circ = C_G(s)^\circ \cap L(M) = M$, see the Proof of Lemma 0.2. Then $\operatorname{Ind}_{L(M)}^G(L(M) \cdot (s\,\mathcal{O}^M)) = G \cdot (s \operatorname{Ind}_M^{C_G(s)^\circ} \mathcal{O}^M) \coloneqq \mathcal{O}^G$ and $\mathcal{O}^G$ is weakly birationally induced from $(L(M), L(M) \cdot (s\,\mathcal{O}^M))$ in the sense of Definition 1.17 (b). For this reason, we will say that $G \cdot (s \operatorname{Ind}_M^{C_G(s)^\circ} \mathcal{O}^M)$ is *weakly birationally induced* from $(M, s, \mathcal{O}^M)$.

When $[G, G]$ is simply-connected, we will omit the adverb "weakly", i.e., we will say that $\mathscr{B}(G)$ is the set of *birational induction data* of G and that $G \cdot (s \operatorname{Ind}_M^{C_G(s)} \mathcal{O}^M)$ is *birationally induced* from $(M, s, \mathcal{O}^M) \in \mathscr{B}(G)$: this choice agrees with Remark 1.18.

Now we prove that every conjugacy class is weakly birationally induced in a unique way from a triple of birational induction data, up to conjugacy.

Theorem 1.35. *The following map is bijective:*

$$\begin{aligned} \mathscr{B}(G)/G &\to G/G \\ [(M, s, \mathcal{O}^M)]_\sim &\mapsto G \cdot (s \operatorname{Ind}_M^{C_G(s)^\circ} \mathcal{O}^M). \end{aligned}$$

In particular, if $[G, G]$ is simply-connected, every conjugacy class in G is birationally induced in a unique way from an equivalence class of birational induction data.

Proof. We prove surjectivity. Let $\mathcal{O} = \mathcal{O}_{su}^G$ with $s \in T$, $u \in K \coloneqq C_G(s)^\circ$. By Lemma 1.32, there exists, up to conjugacy in K, a unique $(L, \mathcal{O}^L) \in \mathscr{B}(K)_u$ such that $\mathcal{O}_u^K = \operatorname{Ind}_L^K(\mathcal{O}^L)$. We can assume $T \leq L$ so that L is a Levi subgroup of K with $Z(K) \leq Z(L)$. Hence, $s \in Z(L)$ and $Z(L)^\circ s$ satisfies (RP) for L, by Lemma 0.2. In particular, $L = C_G(Z(L)^\circ s)^\circ$ is a pseudo-Levi of G and $(L, s, \mathcal{O}^L) \in \mathscr{B}(G)$ satisfies $G \cdot (s \operatorname{Ind}_L^{C_G(s)^\circ} \mathcal{O}^L) = \mathcal{O}$.

Now we prove injectivity. Let $(M_1, s_1, \mathcal{O}_{u_1}^{M_1}), (M_2, s_2, \mathcal{O}_{u_2}^{M_2}) \in \mathscr{B}(G)$ and suppose that

$$G \cdot (s_1 \operatorname{Ind}_{M_1}^{C_G(s_1)^\circ} \mathcal{O}_{u_1}^{M_1}) = G \cdot (s_2 \operatorname{Ind}_{M_2}^{C_G(s_2)^\circ} \mathcal{O}_{u_2}^{M_2}) \tag{1.12}$$

where the unipotent classes $\operatorname{Ind}_{M_i}^{C_G(s_i)^\circ} \mathcal{O}_{u_i}^{M_i}$ are birationally induced from $(M_i, \mathcal{O}_{u_i}^{M_i})$ for $i = 1, 2$. We can assume that $s_1 = s_2 =: s \in T$ and set $K \coloneqq C_G(s)^\circ$. We have that (1.12) is equivalent to

$$C_G(s) \cdot (\operatorname{Ind}_{M_1}^K \mathcal{O}_{u_1}^{M_1}) = C_G(s) \cdot (\operatorname{Ind}_{M_2}^K \mathcal{O}_{u_2}^{M_2}).$$

If $v \in \operatorname{Ind}_{M_1}^K \mathcal{O}_{u_1}^{M_1}$, then there exists $g \in C_G(s)$ such that $gvg^{-1} \in \operatorname{Ind}_{M_2}^K \mathcal{O}_{u_2}^{M_2}$. Write $g = w^{-1}h$ for suitable $h \in M$ and $w \in N_G(T) \cap C_G(s)$, and up to choosing hvh^{-1} instead of v as a representative, we can assume that $g = w^{-1} \in N_G(T) \cap C_G(s)$. Therefore, we have:

$$v \in w(\operatorname{Ind}_{M_2}^K \mathcal{O}_{u_2}^{M_2})w^{-1} = \operatorname{Ind}_{wM_2w^{-1}}^{wKw^{-1}} \mathcal{O}_{wu_2w^{-1}}^{wM_2w^{-1}}.$$

Since w acts as an automorphism of K, the induction is birational (Remark 1.23) and $\mathcal{O}_{wu_2w^{-1}}^{wM_2w^{-1}}$ is birationally rigid (Remark 1.24), it follows that

$$v \in (\operatorname{Ind}_{M_1}^K \mathcal{O}_{u_1}^{M_1}) \cap (\operatorname{Ind}_{wM_2w^{-1}}^K \mathcal{O}_{wu_2w^{-1}}^{wM_2w^{-1}}).$$

By Lemma 1.32, the pairs $(M_1, \mathcal{O}_{u_1}^{M_1})$ and $(wM_2w^{-1}, \mathcal{O}_{wu_2w^{-1}}^{wM_2w^{-1}})$ are conjugate in K, so the data $(M_1, s_1, \mathcal{O}_{u_1}^{M_1})$ and $(M_2, s_2, \mathcal{O}_{u_2}^{M_2})$ are conjugate in G via $g'w^{-1}$ for some $g' \in K$.

The last statement is a consequence of Remark 1.18. $\square$

CHAPTER

TWO

JORDAN CLASSES AND BIRATIONALITY

This Chapter begins with a collections of definitions and classic results on Jordan classes in a reductive Lie algebra and in a reductive group) in Sections 2.1 and 2.2. Afterwards, we introduce the new concepts of birational closure of a Jordan class and of weakly birational closure of Jordan class: we describe these sets and we compare them in Sections 2.3 and 2.4. Finally, after recalling how sheets can be defined in terms of regular closures of Jordan classes in Section 2.5, we conclude with the original work of this Chapter. This is contained in Section 2.6, where we use results from Sections 2.4 and 1.4 to define weakly birational sheets in G and prove that they form a partition of G.

2.1 Decomposition classes of a Lie algebra

In [12, 13], decomposition classes of $\mathfrak{g}$ (*Zerlegungsklassen*) were introduced to solve the problem of the classification of sheets of $\mathfrak{g}$ for the adjoint action of G. To any element $\sigma + \nu \in \mathfrak{g}$ we can associate its *decomposition datum* $(\mathfrak{c}_{\mathfrak{g}}(\sigma), \mathfrak{O}_\nu^{C_G(\sigma)})$. Then each decomposition data is a pair consisting of a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ (see §0.3) and an adjoint orbit $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$. Denote by $\mathscr{D}(\mathfrak{g})$ the set of decomposition data of elements of $\mathfrak{g}$. Then G acts by simultaneous conjugation on $\mathscr{D}(\mathfrak{g})$. We say that two elements of $\mathfrak{g}$ are *Jordan equivalent* if their decomposition data are conjugate in G. The *decomposition class* (or *Jordan class*) of $\xi \in \mathfrak{g}$ is the set of all elements of $\mathfrak{g}$ which are Jordan equivalent to ξ and is denoted $\mathfrak{J}_{\mathfrak{g}}(\xi)$. The decomposition classes of $\mathfrak{g}$ are parameterized by $\mathscr{D}(\mathfrak{g})/G$ and the decomposition class associated to (the class of) the pair $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ is

$$\mathfrak{J}_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{O}^L) = \mathrm{Ad}(G)(\mathfrak{z}(\mathfrak{l})^{reg} + \mathfrak{O}^L). \tag{2.1}$$

In particular, for $\xi = \sigma + \nu$, we define the decomposition class of ξ as $\mathfrak{J}_{\mathfrak{g}}(\xi) := \mathfrak{J}_{\mathfrak{g}}(\mathfrak{c}_{\mathfrak{g}}(\sigma), \mathfrak{O}_\nu^{C_G(\sigma)})$. From now on, we will omit the subscript $\mathfrak{g}$ if clear from the context and simply write $\mathfrak{J}$. Denote by $\mathscr{J}(\mathfrak{g})$ the set of decomposition classes of $\mathfrak{g}$.

Let $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, then the closure and the regular closure of $\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)$ are unions of decomposition classes and admit a description in terms of induced adjoint orbits as follows:

$$\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)} = \bigcup_{\zeta \in \mathfrak{z}(\mathfrak{l})} \mathrm{Ad}(G)(\zeta + \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\zeta)} \mathfrak{O}^L}); \tag{2.2}$$

$$\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg} = \bigcup_{\zeta \in \mathfrak{z}(\mathfrak{l})} \mathrm{Ad}(G)(\zeta + \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\zeta)} \mathfrak{O}^L). \tag{2.3}$$

In particular, for any $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, we have $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg} \cap \mathcal{N} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$.

If $(\mathfrak{l}, \mathfrak{O}^L)$ and $(\mathfrak{m}, \mathfrak{O}^M) \in \mathscr{D}(\mathfrak{g})$, then $\mathfrak{J}(\mathfrak{m}, \mathfrak{O}^M) \subset \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$ if and only if $(\mathfrak{l}, \mathfrak{O}^L) \sim_G (\mathfrak{l}_0, \mathfrak{O}^{L_0})$ with $\mathfrak{l}_0 \subset \mathfrak{m}$ and $\mathrm{Ind}_{\mathfrak{l}_0}^{\mathfrak{m}} \mathfrak{O}^{L_0} = \mathfrak{O}^M$.

For $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, we have $\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L) = \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}$ if and only if $\mathfrak{z}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l})^{reg} = \mathfrak{z}(\mathfrak{g})$ (equivalently $\mathfrak{l} = \mathfrak{g}$) and $\mathfrak{O}^L$ is closed (equivalently it is the null orbit), i.e., if and only if $\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L) = \mathfrak{z}(\mathfrak{g})$.

A decomposition class $\mathfrak{J}'$ of $\mathfrak{g}$ contained in $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$ is closed therein if and only if $\overline{\mathfrak{J}'}^{reg} = \mathfrak{J}'$ and this is the case if and only if $\mathfrak{J}' = \mathfrak{z}(\mathfrak{g}) + \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$.

2.2 Jordan classes of a group

We recall the notions of Jordan classes in a reductive group, introduced in [69] and object of investigation in [29].

Definition 2.1. A *decomposition datum* of G is a triple $\tau = (M, Z(M)^\circ z, \mathcal{O}^M)$ where $M \leq G$ is a pseudo-Levi subgroup, $Z(M)^\circ z$ is a connected component of $Z(M)$ satisfying (RP) for M and $\mathcal{O}^M \in \mathcal{U}_M/M$. We denote by $\mathscr{D}(G)$ the set of decomposition data of G.

The group G acts on $\mathscr{D}(G)$ by simultaneous conjugacy on the triples with finitely many orbits. Indeed, pseudo-Levi subgroups up to conjugacy admit a description in terms of roots, thus being finitely many. Moreover, the number of connected components of an algebraic group and the number of unipotent classes of a reductive algebraic group is finite. The map assigning to any $su \in G$ its decomposition datum $(C_G(s)^\circ, Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$ is G-equivariant. Jordan classes of G are defined as the preimages of G-equivalence classes in $\mathscr{D}(G)$.

Definition 2.2. Two elements $su, rv \in G$ are *Jordan equivalent* if their decomposition data are conjugate in G. We denote by $J_G(su)$ the *Jordan class* of su, i.e., the equivalence class of all elements of G which are Jordan equivalent to su.

We have $J_G(su) = G \cdot ((Z(C_G(s)^\circ)^\circ s)^{reg} \mathcal{O}_u^{C_G(s)^\circ})$. We denote by $\mathscr{J}(G)$ the set of Jordan classes in G. The group G is partitioned into finitely many Jordan classes, which are in one-to-one correspondence with $\mathscr{D}(G)/G$. As Jordan classes only depend on the decomposition data, for $\tau = (M, Z(M)^\circ z, \mathcal{O}^M) \in \mathscr{D}(G)$, we write $J_G(\tau) \coloneqq G \cdot ((Z(M)^\circ z)^{reg} \mathcal{O}^M)$. Jordan classes are smooth irreducible locally closed subvarieties of G, they are unions of equidimensional conjugacy classes. The closure of a Jordan class is a union of Jordan classes. From now on we will omit the subscript G whenever clear from the context, and simply write J. If $\tau = (M, Z(M)^\circ z, \mathcal{O}^M) \in \mathscr{D}(G)$, we have:

$$\overline{J(\tau)} = \bigcup_{t \in Z(M)^\circ z} \overline{G \cdot (t \operatorname{Ind}_M^{C_G(t)^\circ} \mathcal{O}^M)}; \tag{2.4}$$

$$\overline{J(\tau)}^{reg} = \bigcup_{t \in Z(M)^\circ z} G \cdot (t \operatorname{Ind}_M^{C_G(t)^\circ} \mathcal{O}^M). \tag{2.5}$$

Since $\overline{J(\tau)}^{reg}$ is open in $\overline{J(\tau)}$, it is irreducible and locally closed in G.

Let $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, then $J(\tau)$ is closed in G if and only if $Z(M)^\circ s = (Z(M)^\circ s)^{reg} = Z(G)^\circ s$ and $\mathcal{O}^M$ is closed, i.e., it is the unipotent class of the identity. One can verify that this happens if and only if $u = 1$ and $M/Z(G)^\circ$ is semisimple, i.e., if and only if J consists of semisimple isolated elements.

Let $\tau_1, \tau_2 \in \mathscr{D}(G)$ such that $J(\tau_2) \subset \overline{J(\tau_1)}^{reg}$. Then $J(\tau_2)$ is closed in $\overline{J(\tau_1)}^{reg}$ if and only if $\overline{J(\tau_2)}^{reg} = J(\tau_2)$ if and only if $\tau_2 = (M, Z(M)^\circ s, \mathcal{O}^M)$ with $M/Z(G)^\circ$ semisimple, i.e., $J(\tau_2)$ consists of isolated elements (not necessarily semisimple).

We remark that, in contrast with the Lie algebra situation, not all (regular) closures of Jordan classes contain a unipotent conjugacy class, even up to a central element. In fact, $\overline{J(M, Z(M)^\circ s, \mathcal{O})}^{reg} \cap Z(G)\mathcal{U} \neq \varnothing$ if and only if $\overline{J(M, Z(M)^\circ s, \mathcal{O})} \cap Z(G) \neq \varnothing$ and the latter holds if and only if M is a Levi subgroup. Moreover, $\overline{J(M, Z(M)^\circ s, \mathcal{O})}^{reg} \cap \mathcal{U} \neq \varnothing$ if and only if $1 \in \overline{J(M, Z(M)^\circ s, \mathcal{O})}$ if and only if M is a Levi subgroup and $Z(M)^\circ s = Z(M)^\circ$, by (2.5) and [29, proof of Proposition 5.6].

Thanks to the choice of the maximal torus T, we can define a finite subset $\mathscr{T}(G) \subset \mathscr{D}(G)$ admitting an action of the Weyl group W of G which amounts to considering the G-action on $\mathscr{D}(G)$, thus leading to a simpler parameterization of $\mathscr{J}(G)$.

Proposition 2.3. *Let $\mathscr{T}(G) = \{(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G) \mid T \leq M\}$. Then the orbit set $\mathscr{T}(G)/W$ parameterizes $\mathscr{J}(G)$.*

Proof. Observe that $T \leq M$ implies $Z(M)^\circ s \subset T$. Moreover, $N_G(T)$ acts by simultaneous conjugacy on the elements of $\mathscr{T}(G)$ and T is in the kernel of this action, thus it descends to a W-action. We show that $\mathscr{D}(G)/G$ is in bijection with $\mathscr{T}(G)/W$. Any element of $\mathscr{D}(G)$ is conjugate to an element of $\mathscr{T}(G)$ because all semisimple classes admit a representative in T and connected centralizers of such elements are pseudo-Levi subgroups containing T. We conclude by showing that two representatives in $\mathscr{T}(G)$ are conjugate in G if and only if they are conjugate in $N_G(T)$. Let $(M_1, Z(M_1)^\circ s_1, \mathcal{O}^{M_1})$ and $(M_2, Z(M_2)^\circ s_2, \mathcal{O}^{M_2})$ be elements of $\mathscr{T}(G)$ and assume $(M_2, Z(M_2)^\circ s_2, \mathcal{O}^{M_2}) = g \cdot (M_1, Z(M_1)^\circ s_1, \mathcal{O}^{M_1})$ for some $g \in G$. Since all maximal tori in a pseudo-Levi subgroup are conjugate, there exists $m \in M_2$ such that $\dot{w} := mg \in N_G(T)$, and $(M_2, Z(M_2)^\circ s_2, \mathcal{O}^{M_2}) = \dot{w} \cdot (M_1, Z(M_1)^\circ s_1, \mathcal{O}^{M_1})$. ◻

2.3 Birational closure of decomposition classes

As in [67, §4], to $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ we can associate the set

$$\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L) \coloneqq \{\zeta \in \mathfrak{z}(\mathfrak{l}) \mid \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathfrak{O}^L) \text{ is birationally induced from } (\mathfrak{l}, \zeta + \mathfrak{O}^L)\} = \tag{2.6}$$

$$= \{\zeta \in \mathfrak{z}(\mathfrak{l}) \mid \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\zeta)} \mathfrak{O}^L \text{ is birationally induced from } (\mathfrak{l}, \mathfrak{O}^L)\}, \tag{2.7}$$

where the second equivalent description can be deduced from Lemma 1.9. In particular, the set $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ only depends on the pair $(\mathfrak{l}, \mathfrak{O}^L)$ and not on the parabolic subgroup of G with Levi factor L chosen for the generalized Springer map: this is stated and proven in [67, Proposition 4.2], but it can be deduced also from Lemma 1.11.

Always in [67, Proposition 4.2], it is stated and proven that $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is an open subset of $\mathfrak{z}(\mathfrak{l})$ obtainable from $\mathfrak{z}(\mathfrak{l})$ by discarding a finite number of vector spaces. We would like to give a different proof of the same result which uses more elementary arguments. To this extent, we define the set $\mathcal{L}_{\mathfrak{l}} \coloneqq \{\mathfrak{m} \supset \mathfrak{l} \mid \mathfrak{m} \text{ is a Levi subalgebra of } \mathfrak{g}\}$ and the set $\mathcal{Z}_{\mathfrak{l}} \coloneqq \{\mathfrak{z}(\mathfrak{m}) \mid \mathfrak{m} \in \mathcal{L}_{\mathfrak{l}}\}$. Inclusion of subalgebras endows the sets $\mathcal{L}_{\mathfrak{l}}$ and on $\mathcal{Z}_{\mathfrak{l}}$ with partial orders. Both sets admit maximum ($\mathfrak{g} \in \mathcal{L}_{\mathfrak{l}}$ and $\mathfrak{z}(\mathfrak{l}) \in \mathcal{Z}_{\mathfrak{l}}$) and minimum ($\mathfrak{l} \in \mathcal{L}_{\mathfrak{l}}$ and $\mathfrak{z}(\mathfrak{g}) \in \mathcal{Z}_{\mathfrak{l}}$) elements.

Lemma 2.4. *The posets $\mathcal{L}_{\mathfrak{l}}$ and $\mathcal{Z}_{\mathfrak{l}}$ are finite and anti-isomorphic.*

Proof. We can assume $\mathfrak{l}$ is standard, then there is $\sigma \in \mathfrak{h}$ such that $\mathfrak{c}_{\mathfrak{g}}(\sigma) = \mathfrak{l}$. We claim that elements of $\mathcal{L}_{\mathfrak{l}}$ can be written as centralizers in $\mathfrak{g}$ of elements in $\mathfrak{z}(\mathfrak{l})$. By [96, Theorem 29.5.7], if $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{g}$, then $\mathfrak{m} = \mathfrak{c}_{\mathfrak{g}}(\mu)$ for any $\mu \in \mathfrak{z}(\mathfrak{m})^{reg}$. We observe that if $\mathfrak{l} \subset \mathfrak{m}$, then $\mathfrak{z}(\mathfrak{m}) \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l})$, always by [96, Theorem 29.5.7]. This proves the claim and grants the finiteness of $\mathcal{L}_{\mathfrak{l}}$: if $\mathfrak{m} \in \mathcal{L}_{\mathfrak{l}}$, then $\mathfrak{m} = \mathfrak{c}_{\mathfrak{g}}(\mu)$ for some $\mu \in \mathfrak{z}(\mathfrak{l})$ and $\Phi_\mu \supset \Phi_\sigma$, but since $\Phi_\mu \subset \Phi$, we have only finitely many choices for Φ_μ. The last statement follows once more from [96, Theorem 29.5.7]: the map $\mathcal{L}_{\mathfrak{l}} \to \mathcal{Z}_{\mathfrak{l}}$ defined by $\mathfrak{l} \mapsto \mathfrak{z}(\mathfrak{l})$ and its inverse $\mathfrak{z} \mapsto \mathfrak{c}_{\mathfrak{g}}(\mathfrak{z})$ reverse inclusions. □

Lemma 2.5. *Let $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$ and let $\sigma \in \mathfrak{z}(\mathfrak{l})$.*

(i) If $\sigma \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$, then $\mu \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ for all $\mu \in \mathfrak{z}(\mathfrak{l})$ such that $\mathfrak{c}_{\mathfrak{g}}(\mu) \subset \mathfrak{c}_{\mathfrak{g}}(\sigma)$.

(ii) If $\sigma \notin \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$, then $\mu \notin \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ for all $\mu \in \mathfrak{z}(\mathfrak{l})$ such that $\mathfrak{c}_{\mathfrak{g}}(\mu) \supset \mathfrak{c}_{\mathfrak{g}}(\sigma)$.

Proof. We prove (i). Thanks to (the proof of) Lemma 2.4, we have the inclusions of Levi subalgebras $\mathfrak{l} \subset \mathfrak{c}_{\mathfrak{g}}(\mu) \subset \mathfrak{c}_{\mathfrak{g}}(\sigma)$. The induction $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\sigma)} \mathfrak{O}^L$ factorizes as $\mathrm{Ind}_{\mathfrak{c}_{\mathfrak{g}}(\mu)}^{\mathfrak{c}_{\mathfrak{g}}(\sigma)}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\mu)} \mathfrak{O}^L)$ and by Lemma 1.12, our assumption implies that $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\mu)} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$, i.e., $\mu \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$. The proof of (ii) is similar. □

Lemma 2.6. *Let $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$. Then:*

(i) The inclusion $\mathfrak{z}(\mathfrak{l})^{reg} \subset \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ always holds. In particular, $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is a non-empty subset of $\mathfrak{z}(\mathfrak{l})$.

(ii) The set $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is the complement to a finite (possibly empty) union of elements in $\mathcal{Z}_{\mathfrak{l}}$. In particular, $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is open in $\mathfrak{z}(\mathfrak{l})$.

(iii) We have $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L) = \mathfrak{z}(\mathfrak{l})$ if and only if it contains $\mathfrak{z}(\mathfrak{g})$ if and only if it contains 0.

Proof. (i) For $\zeta \in \mathfrak{z}(\mathfrak{l})^{reg}$, we have $\mathfrak{c}_{\mathfrak{g}}(\zeta) = \mathfrak{l}$. Hence $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{c}_{\mathfrak{g}}(\zeta)} \mathfrak{O}^L = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{l}} \mathfrak{O}^L = \mathfrak{O}^L$ which is trivially birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$.

(ii) By Lemmas 2.5 and 2.4, if $\mathfrak{m} \in \mathcal{L}_{\mathfrak{l}}$ is such that $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{m}} \mathfrak{O}^L$ is not birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ then $\mathfrak{z}(\mathfrak{m}) \subset (\mathfrak{z}(\mathfrak{l}) \setminus \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L))$. By Lemma 2.4, we can consider the subset

$$\mathscr{Z}'(\mathfrak{l}, \mathfrak{O}^L) := \{\mathfrak{z} \in \mathscr{Z}_{\mathfrak{l}} \mid \mathfrak{z} \subset (\mathfrak{z}(\mathfrak{l}) \setminus \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L))\}. \tag{2.8}$$

The union of vector spaces which are maximal elements in $\mathscr{Z}'(\mathfrak{l}, \mathfrak{O}^L)$ is the sought union. It is a closed subset of $\mathfrak{z}(\mathfrak{l})$, hence its complement is open.

(iii) follows directly from (ii). □

This work lays foundations for the following definition, which generalizes the concept of birational sheet defined in [67, §4.2].

Definition 2.7. For all $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$, we define the *birational closure* of the decomposition class $\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)$ as the set

$$\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir} := \bigcup_{\zeta \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathfrak{O}^L).$$

Remark 2.8. For $\mathfrak{J} \in \mathscr{J}(\mathfrak{g})$, we have $\mathfrak{J} \subset \overline{\mathfrak{J}}^{bir} \subset \overline{\mathfrak{J}}^{reg}$ by construction and Lemma 2.6 (i) and $\overline{\mathfrak{J}}^{bir}$ is G-stable, always by construction.

Proposition 2.9. *Let $\mathfrak{J} \in \mathscr{J}(\mathfrak{g})$. Then $\overline{\mathfrak{J}}^{bir}$ is obtained from $\overline{\mathfrak{J}}^{reg}$ by neglecting a finite number of regular closures of other decomposition classes of $\mathscr{J}(\mathfrak{g})$, and it is open in $\overline{\mathfrak{J}}^{reg}$. In particular, birational closures of decomposition classes are unions of decomposition classes and are irreducible locally closed subsets of $\mathfrak{g}$.*

Proof. Let $\mathfrak{J} := \mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)$ with $(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{D}(\mathfrak{g})$. We need to show that

$$\overline{\mathfrak{J}}^{reg} \setminus \overline{\mathfrak{J}}^{bir} = \bigcup_{\zeta \in \mathfrak{z}(\mathfrak{l}) \setminus \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathcal{O}^L) \tag{2.9}$$

is a union of regular closures of decomposition classes in $\mathfrak{g}$. Consider the set $\mathscr{Z}'(\mathfrak{l}, \mathfrak{O}^L)$ defined in (2.8). By Lemma 2.6 (ii), the set $\mathfrak{z}(\mathfrak{l}) \setminus \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$ is the finite union of maximal elements in $\mathscr{Z}'(\mathfrak{l}, \mathfrak{O}^L)$. We show that each $\mathfrak{z}'$ maximal in $\mathscr{Z}'(\mathfrak{l}, \mathfrak{O}^L)$ gives rise to the regular closure of a decomposition class in (2.9). Set $\mathfrak{l}' := \mathfrak{c}_{\mathfrak{g}}(\mathfrak{z}')$ and $\mathfrak{O}^{L'} := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{l}'} \mathfrak{O}^L$. Then, by transitivity of induction,

$$\bigcup_{\zeta \in \mathfrak{z}'} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\zeta + \mathcal{O}^L) = \bigcup_{\zeta \in \mathfrak{z}'} \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{l}'}(\zeta + \mathcal{O}^L)) = \bigcup_{\zeta \in \mathfrak{z}'} \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{g}}(\zeta + \mathcal{O}^{L'}) = \overline{\mathfrak{J}(\mathfrak{l}', \mathfrak{O}^{L'})}^{reg}.$$

Finally, we show that $\overline{\mathfrak{J}}^{reg} \setminus \overline{\mathfrak{J}}^{bir}$ is closed. If $\mathfrak{J} \subset \mathfrak{g}_{(n)}$, then $\overline{\mathfrak{J}}^{reg}, \overline{\mathfrak{J}}^{bir} \subset \mathfrak{g}_{(n)}$, hence also $\overline{\mathfrak{J}}^{reg} \setminus \overline{\mathfrak{J}}^{bir} \subset \mathfrak{g}_{(n)}$. Each component $\overline{\mathfrak{J}'}^{reg} \subset \overline{\mathfrak{J}}^{reg} \setminus \overline{\mathfrak{J}}^{bir}$ satisfies $\overline{\mathfrak{J}'}^{reg} = \overline{\mathfrak{J}'} \cap \mathfrak{g}_{(n)}$, i.e., it is closed in $\mathfrak{g}_{(n)}$. This implies that $\overline{\mathfrak{J}}^{reg} \setminus \overline{\mathfrak{J}}^{bir}$ is closed in $\mathfrak{g}_{(n)}$, hence it is closed in $\overline{\mathfrak{J}}^{reg}$. The last assertion follows since $\overline{\mathfrak{J}}^{reg}$ is an irreducible locally closed subset of $\mathfrak{g}$. □

2.4 Birational closures of Jordan classes

We would like to proceed in an analogous way to define the birational closure of a Jordan class in a group, but Definition 1.17 suggests that a bisection is needed. For $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, we define the sets

$$\mathrm{Bir}(Z(M)^\circ s, \mathcal{O}^M) := \{z \in Z(M)^\circ s \mid \mathrm{Ind}_{L(M)}^G(L(M)\cdot(z\,\mathcal{O}^M)) \text{ is birationally induced}\}; \quad (2.10)$$

$$\mathrm{WBir}(Z(M)^\circ s, \mathcal{O}^M) := \{z \in Z(M)^\circ s \mid \mathrm{Ind}_M^{C_G(z)^\circ}\mathcal{O}^M \text{ is birationally induced}\} = \quad (2.11)$$
$$= \{z \in Z(M)^\circ s \mid \mathrm{Ind}_{L(M)}^G(L(M)\cdot(z\,\mathcal{O}^M)) \text{ is weakly birationally induced}\}.$$

Remark 2.10. The definition in (2.11) does not depend on the choice of a parabolic subgroup, thanks to Lemma 1.22. Moreover, if $z \in (Z(M)^\circ s)^{reg}$, then $C_G(z)^\circ = M$, hence $\varnothing \neq (Z^\circ s)^{reg} \subset \mathrm{WBir}(Z(M)^\circ s, \mathcal{O}^M)$. By Remarks 1.18 and 1.34, we have the inclusion $\mathrm{Bir}(Z(M)^\circ s, \mathcal{O}^M) \subset \mathrm{WBir}(Z(M)^\circ s, \mathcal{O}^M)$. As a consequence of Lemma 1.15, when $[G,G]$ is simply-connected, we have $\mathrm{Bir}(Z(M)^\circ s, \mathcal{O}^M) = \mathrm{WBir}(Z(M)^\circ s, \mathcal{O}^M)$; on the other hand, for a general G, the inclusion can be proper as the following example shows.

Example 2.11. Consider $G = \mathrm{SL}_2(\mathbb{C})$, let $\overline{G} = \mathrm{PSL}_2(\mathbb{C})$ and let $\pi\colon G \to \overline{G}$, $\pi(g) = \bar{g}$ be the isogeny. Let us consider the torus $T \leq G$ given by diagonal elements, let $B \leq G$ be the Borel subgroup consisting of upper triangular matrices in G, let U be the unipotent radical of B and let $\overline{B} = \pi(B)$ and $\overline{U} = \pi(U)$. Let $\tau = (\overline{T}, \overline{T}, \{\bar{e}\}) \in \mathscr{D}(\overline{G})$ and let $s := \mathrm{diag}[i, -i]$. Then one can verify that $\bar{s} \in \mathrm{WBir}(\overline{T}, \{\bar{e}\}) \setminus \mathrm{Bir}(\overline{T}, \{\bar{e}\})$. The generalized Springer map (1.10) reads:

$$\gamma\colon \overline{G} \times^{\overline{B}} (\bar{s}\overline{U}) \to \overline{G}\cdot(\bar{s}\overline{U}) = \mathcal{O}_{\bar{s}}^{\overline{G}}\,.$$

This map is not birational, because $C_{\overline{G}}(\bar{s}) = N_{\overline{G}}(\overline{T}) \not\leq \overline{B}$, namely γ is a $2:1$ covering; on the other hand, $\gamma_{\bar{s}}$ as in (1.11) is trivially birational.

We now describe the structure of the set $\mathrm{WBir}(Z(M)^\circ s, \mathcal{O}^M)$ for a pseudo-Levi $M \leq G$ and $\mathcal{O}^M \in \mathcal{U}_M/M$. Thanks to Remark 2.10, when $[G,G]$ is simply-connected, the results will give a description of the set $\mathrm{Bir}(Z(M)^\circ s, \mathcal{O}^M)$. We mimic the procedure adopted in Lemmas 2.4, 2.5 and 2.6: to avoid repetitions we only focus on the discrepancies between the situation in the group and the Lie algebra.

Definition 2.12. For a pseudo-Levi subgroup $M \leq G$ and an element $s \in Z(M)$ such that $Z(M)^\circ s$ satisfies (RP) for M, we define:

$$\mathcal{Z}_{Z(M)^\circ s} := \{Z(C_G(z)^\circ)^\circ z \mid z \in Z(M)^\circ s\}. \quad (2.12)$$

Remark 2.13. We prove that the set defined in (2.12) is finite. We can assume $T \leq M$, so that $z \in Z(M) \leq T$ hence there are finitely many possibilities for the root system Φ_z of $C_G(z)^\circ$, which must satisfy $\Phi_s \subset \Phi_z \subset \Phi$, moreover the connected components of $Z(C_G(z)^\circ)$ are finitely many.

We write $\mathcal{Z}(Z(M)^\circ s) = \{Z(M_i)^\circ s_i \mid i \in I\}$ for a finite index set I and suitable $s_i \in Z^\circ s$ with $C_G(s_i)^\circ = M_i$. Thanks to Lemma 0.2, we can define a map:

$$\begin{aligned} \mathcal{Z}_{Z(M)^\circ s} &\to \{M_i \leq G \mid M \text{ is a Levi subgroup of } M_i\} \\ Z(M_i)^\circ s_i &\mapsto C_G(Z(M_i)^\circ s_i)^\circ = M_i. \end{aligned}$$

The set $\mathcal{Z}_{Z(M)^\circ s}$ is partially ordered by inclusion: if $T \leq M_1 \leq M_2$, with M_1, M_2 pseudo-Levi subgroups of G, then $Z(M_2) \leq Z(M_1)$ and $Z(M_2)^\circ z \subset Z(M_1)^\circ z$ for all $z \in Z(M_2)$. The maximum of $\mathcal{Z}_{Z(M)^\circ s}$ is $Z(M)^\circ s$ and its minimal elements are all connected components $Z(M_i)^\circ s_i$ consisting of isolated elements. The above map reverses inclusions: $Z(M_i)^\circ s_i \subset Z(M_j)^\circ s_j$ implies $M_j \leq M_i$.

Lemma 2.14. *Let $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, set $Z \coloneqq Z(M)$. Let $z \in Z^\circ s$.*

(i) If $\operatorname{Ind}_M^{C_G(z)^\circ} \mathcal{O}^M$ is birationally induced from $(M, \mathcal{O}^M)$, then $\operatorname{Ind}_M^{C_G(z')^\circ} \mathcal{O}^M$ is birationally induced from $(M, \mathcal{O}^M)$ for all $z' \in Z^\circ s$ such that $C_G(z')^\circ \leq C_G(z)^\circ$.

(ii) If $\operatorname{Ind}_M^{C_G(z)^\circ} \mathcal{O}^M$ is not birationally induced from $(M, \mathcal{O}^M)$, then $\operatorname{Ind}_M^{C_G(z')^\circ} \mathcal{O}^M$ is not birationally induced from $(M, \mathcal{O}^M)$ for all $z' \in Z^\circ s$ such that $C_G(z')^\circ \geq C_G(z)^\circ$.

Proof. The proof goes as in Lemma 2.5. □

Remark 2.15. For $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, set $Z \coloneqq Z(M)$ and define

$$\mathcal{Z}'(Z^\circ s, \mathcal{O}^M) \coloneqq \{Z(M_i)^\circ s_i \in \mathcal{Z}(Z^\circ s) \mid \operatorname{Ind}_M^{M_i} \mathcal{O}^M \text{ is not birational}\}. \tag{2.13}$$

By Remark 2.13, this set is finite and by Lemma 2.14, it is a subposet of $\mathcal{Z}_{Z^\circ s}$. Let $\hat{\mathcal{Z}}(Z^\circ s, \mathcal{O}^M)$ be the subset of maximal elements in $\mathcal{Z}'(Z^\circ s, \mathcal{O}^M)$. Observe that for all components $Z(C_G(z)^\circ)^\circ z \in \mathcal{Z}'(Z(M)^\circ s, \mathcal{O}^M)$, we have that M is a Levi subgroup in $C_G(z)^\circ$.

Now we prove that the set $\operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ is the complement in $Z^\circ s$ of the finite union (possibly empty) of shifted tori which are elements of $\mathcal{Z}'(Z^\circ s, \mathcal{O}^M)$.

Lemma 2.16. *Let $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$, set $Z \coloneqq Z(M)$.*

(i) The inclusion $(Z^\circ s)^{reg} \subset \operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ always holds. In particular, $\operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ is a non-empty subset of $Z^\circ s$.

(ii) The set $\operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ is the complement to a finite (possibly empty) union of shifted tori of G contained in $Z^\circ s$. Namely, with the notation of Remark 2.15, we have:

$$Z^\circ s \setminus \operatorname{WBir}(Z^\circ s, \mathcal{O}^M) = \bigcup_{\hat{\mathcal{Z}}(Z^\circ s, \mathcal{O}^M)} Z(M_i)^\circ s_i. \tag{2.14}$$

In particular, $\operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ is open in $Z^\circ s$.

(iii) We have $\operatorname{WBir}(Z^\circ s, \mathcal{O}^M) = Z^\circ s$ if and only if $z \in \operatorname{WBir}(Z^\circ s, \mathcal{O}^M)$ for all isolated $z \in Z^\circ s$.

Proof. The proof is similar to the one of Lemma 2.6. The main difference is in (iii), which follows from the fact that isolated elements are minimal elements of $\mathcal{Z}_{Z^\circ s}$, see Remark 2.13. □

Definition 2.17. Let $su \in G$.

(a) The *weakly birational closure* of $J(su)$ is:

$$\overline{J(su)}^{wbir} := \bigcup_{z \in \mathrm{WBir}(Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})} G \cdot \left(z \, \mathrm{Ind}_{C_G(s)^\circ}^{C_G(z)^\circ} \mathcal{O}_u^{C_G(s)^\circ} \right), \tag{2.15}$$

where $\mathrm{WBir}(Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$ is as in (2.11).

(b) We also define the set:

$$\overline{J(su)}^{bir} := \bigcup_{z \in \mathrm{Bir}(Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})} G \cdot \left(z \, \mathrm{Ind}_{C_G(s)^\circ}^{C_G(z)^\circ} \mathcal{O}_u^{C_G(s)^\circ} \right), \tag{2.16}$$

where $\mathrm{Bir}(Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$ is as in (2.10).

Remark 2.18. For $su \in G$, the sets $\overline{J(su)}^{bir}$ and $\overline{J(su)}^{wbir}$ are G-stable by construction. We have the following inclusions, by construction: $J(su) \subset \overline{J(su)}^{wbir} \subset \overline{J(su)}^{reg}$ and $\overline{J(su)}^{bir} \subset \overline{J(su)}^{wbir}$. Notice that, if $[G,G]$ is not simply-connected, we may have $J(su) \not\subset \overline{J(su)}^{bir}$, as in Example 2.11.

Lemma 2.19. *Definition 2.17 is independent of the representative of the Jordan class. In particular, for $\tau \in \mathscr{D}(G)$, the sets $\overline{J(\tau)}^{bir}$ and $\overline{J(\tau)}^{wbir}$ are well-defined and if τ and τ' represent the same class in $\mathscr{D}(G)/G$, then $\overline{J(\tau)}^{bir} = \overline{J(\tau')}^{bir}$ and $\overline{J(\tau)}^{wbir} = \overline{J(\tau')}^{wbir}$.*

Proof. We show that $J(su) = J(rv)$ implies $\overline{J(su)}^{wbir} = \overline{J(rv)}^{wbir}$. Let $s_1u_1 \in \overline{J(su)}^{wbir}$, namely we can assume that $s_1 \in Z(C_G(s)^\circ)^\circ s$ and that $\mathcal{O}_{u_1}^{C_G(s_1)^\circ} = \mathrm{Ind}_{C_G(s)^\circ}^{C_G(s_1)^\circ}(\mathcal{O}_u^{C_G(s)^\circ})$ is birationally induced from $(C_G(s)^\circ, \mathcal{O}_u^{C_G(s)^\circ})$. By hypothesis, $(C_G(s)^\circ, Z(C_G(s)^\circ)^\circ s, \mathcal{O}_u^{C_G(s)^\circ})$ and $(C_G(r)^\circ, Z(C_G(r)^\circ)^\circ r, \mathcal{O}_v^{C_G(r)^\circ})$ are conjugate by an element $g \in G$. Hence, $gs_1g^{-1} \in Z(C_G(r)^\circ)^\circ r$ and

$$\begin{aligned} gu_1g^{-1} \in g(\mathrm{Ind}_{C_G(s)^\circ}^{C_G(s_1)^\circ} \mathcal{O}_u^{C_G(s)^\circ})g^{-1} &= \mathrm{Ind}_{gC_G(s)^\circ g^{-1}}^{gC_G(s_1)^\circ g^{-1}} (g\, \mathcal{O}_u^{C_G(s)^\circ}\, g^{-1}) = \\ &= \mathrm{Ind}_{C_G(r)^\circ}^{C_G(gs_1g^{-1})^\circ} \mathcal{O}_v^{C_G(r)^\circ}, \end{aligned}$$

which is birationally induced from $(C_G(r)^\circ, \mathcal{O}_v^{C_G(r)^\circ})$ by Remark 1.23. This yields $gs_1u_1g^{-1} = (gs_1g^{-1})(gu_1g^{-1}) \in \overline{J(rv)}^{wbir}$, the proof follows from G-stability of $\overline{J(rv)}^{wbir}$. The proof for $\overline{J(su)}^{bir}$ is similar, once noticed that the Levi envelopes of conjugate pseudo-Levi subgroups are conjugate. □

We now focus on weakly birational closures of Jordan classes and state the main structural results on them.

Corollary 2.20. *Let $J_1, J_2 \in \mathscr{J}(G)$. If $J_1 \subset \overline{J_2}^{wbir}$, then $\overline{J_1}^{wbir} \subset \overline{J_2}^{wbir}$.*

Proof. This follows from Definition 2.17 and Proposition 1.25. □

Proposition 2.21. *Let $J \in \mathscr{J}(G)$. Then $\overline{J}^{wbir}$ is obtained from $\overline{J}^{reg}$ by neglecting a finite number of regular closures of other Jordan classes of G, and it is open in $\overline{J}^{reg}$. In particular, $\overline{J}^{wbir}$ is a union of Jordan classes in G and it is an irreducible locally closed subset of G.*

Proof. Let $J := J(\tau)$ with $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ and let $Z := Z(M)$. Consider the set $\hat{\mathcal{Z}}(Z^\circ s, \mathcal{O}^M)$ defined in Remark 2.15, and write $\hat{\mathcal{Z}}(Z^\circ s, \mathcal{O}^M) = \{Z(M_i)^\circ s_i \mid i \in \hat{I}\}$, for a suitable index set $\hat{I}$. By Lemma 2.16 (ii), we have $\mathrm{Bir}(Z^\circ s, \mathcal{O}^M) = Z^\circ s \setminus \bigcup_{i\in\hat{I}} Z(M_i)^\circ s_i$. Since $Z^\circ s = Z^\circ s_i$ for all $i \in \hat{I}$, Lemma 0.2 implies that $Z^\circ s_i$ satisfies (RP) for M and $C_G(s)^\circ$ is a Levi subgroup of $C_G(s_i)^\circ$. Then:

$$\overline{J}^{reg} \setminus \overline{J}^{wbir} = \bigcup_{i\in\hat{I}} \left(\bigcup_{t\in Z(C_G(s_i)^\circ)^\circ s_i} G\cdot (t\,\mathrm{Ind}_M^{C_G(t)^\circ} \mathcal{O}^M) \right).$$

Set $\mathcal{O}^{C_G(s_i)^\circ} := \mathrm{Ind}_M^{C_G(s_i)^\circ} \mathcal{O}^M$ and $\tau_i := (C_G(s_i)^\circ, Z(C_G(s_i)^\circ)^\circ s_i, \mathcal{O}^{C_G(s_i)^\circ}) \in \mathscr{D}(G)$ for all $i \in \hat{I}$. Then, by transitivity of induction:

$$\bigcup_{t\in Z(C_G(s_i)^\circ)^\circ s_i} G\cdot (t\,\mathrm{Ind}_M^{C_G(t)^\circ} \mathcal{O}^M) = \bigcup_{t\in Z(C_G(s_i)^\circ)^\circ s_i} G\cdot (t\,\mathrm{Ind}_{C_G(s_i)^\circ}^{C_G(t)^\circ} \mathcal{O}^{C_G(s_i)^\circ}) = \overline{J(\tau_i)}^{reg}.$$

For the proof of the final statement one can proceed similarly to Proposition 2.9. □

Remark 2.22. Suppose that $[G,G]$ is simply-connected. By Remark 1.18, $\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{wbir}$ for any $\tau \in \mathscr{D}(G)$. For this reason, when $[G,G]$ is simply-connected we call $\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{wbir}$ the *birational closure* of $J(\tau)$. Corollary 2.20 and Proposition 2.21 applied to the case G with $[G,G]$ simply-connected give a description of birational closures of Jordan classes.

2.5 Sheets

Since (regular) closures of Jordan classes are unions of objects of the same kind, one can put a partial order on the set of Jordan classes of $\mathfrak{g}$ by setting $\mathfrak{J}_1 \le \mathfrak{J}_2$ if and only if $\mathfrak{J}_1 \subset \overline{\mathfrak{J}_2}^{reg}$ for any $\mathfrak{J}_1, \mathfrak{J}_2 \in \mathscr{J}(\mathfrak{g})$.

Now let $n \in \mathbb{N}$ and enumerate all Jordan classes of $\mathfrak{g}$ with $\mathfrak{J}_i \subset \mathfrak{g}_{(n)}$ for $i = 1,\dots,k$. Since the Jordan classes partition $\mathfrak{g}$, we have $\mathfrak{g}_{(n)} = \bigsqcup_{i=1}^k \mathfrak{J}_i$. This implies

$$\mathfrak{g}_{(n)} \subset \bigcup_{i=1}^t \overline{\mathfrak{J}_i} \cap \mathfrak{g}_{(n)} = \bigcup_{i=1}^t \overline{\mathfrak{J}_i}^{reg},$$

where the last union consists of irreducible sets closed in $\mathfrak{g}_{(n)}$. Refine the last union to one with a minimal number of elements, in other words, select the indices $\{k_1,\dots,k_j\} \subset \{1,\dots,k\}$ such $\mathfrak{J}_{k_i}$ is a maximal element in the partial order of $\mathscr{J}(\mathfrak{g})$. Then

$$\mathfrak{g}_{(n)} \subset \bigcup_{i=1}^j \overline{\mathfrak{J}_{k_i}}^{reg},$$

where no term is redundant and the union need not be disjoint.

Suppose now $\mathfrak{S}$ is a sheet of $\mathfrak{g}$ contained in $\mathfrak{g}_{(n)}$, then by irreducibility of $\mathfrak{S}$, there exist $i \in \{k_1, \dots, k_j\}$ such that $\mathfrak{S} \subset \overline{\mathfrak{J}_i}^{reg}$ and maximality of $\mathfrak{S}$ yields equality. This proves that the map $\mathfrak{J} \mapsto \overline{\mathfrak{J}}^{reg}$ is a bijective correspondence between maximal Jordan classes in $\mathscr{J}(\mathfrak{g})$ with respect to $\leq$ and sheets of $\mathfrak{g}$.

The description of Jordan classes in terms of Lusztig–Spaltenstein induction and the transitivity of induction yield the following characterization, which traces back to [12, 13]:

Theorem (Borho–Kraft)**.** *The sheets of a reductive Lie algebra $\mathfrak{g}$ are the regular closures of Jordan classes $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$ with $\mathfrak{O}^L$ a rigid nilpotent orbit in a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$.*

If $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{reg}$ is a sheet, then it contains a unique nilpotent orbit, namely $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$ and if two sheets intersect, then they contain the same nilpotent orbit. As recalled in the Introduction, there exist sheets in simple Lie algebras which intersect non-trivially, for example the subregular sheets in simple non-simply-laced algebras, see [13, §6.6] and [12, §7.4].

Sheets are disjoint in type A, see [40]. For $\mathfrak{g}$ simple of classical type all sheets are smooth, see [55]; if $\mathfrak{g}$ is simple exceptional, there exist singular sheets, see [26] for the list of smooth ones.

The set $\mathscr{J}(G)$ is endowed with the same partial order $\leq$ on $\mathscr{J}(\mathfrak{g})$ and the same topological argument illustrated at the beginning of the Section shows that a sheet of G is the regular closure of a maximal Jordan class $J \in \mathscr{J}(G)$. Similarly, the sheets for the action of G on itself under conjugation are classified as $J = \overline{J(\tau)}^{reg}$ where $\tau = (M, Z(M)^\circ z, \mathcal{O}^M) \in \mathscr{D}(G)$ where $\mathcal{O}^M \in \mathcal{U}_M/M$ is rigid, see [29, Theorem 5.6 (a)].

In the case of a simple group G, there exist distinct sheets with non-empty intersection: if G has a non-simply-laced Dynkin diagram, then the subregular unipotent class $\mathcal{O}_{sreg}$ belongs to two sheets. This is due to the fact that $\mathcal{O}_{sreg}$ is induced in two essentially different ways, generalizing Example 1.29. Let α (resp. β) be a short (resp. long) simple root of G. Then, up to conjugation, there are two minimal standard Levi subgroups in G, i.e., $L_{\{\alpha\}}$ and $L_{\{\beta\}}$. Since $\mathcal{O}_{sreg}$ is completely determined in $\mathcal{U}$ by its dimension and $\dim L_{\{\alpha\}} = \dim L_{\{\beta\}}$, the formula of dimensions for the induced class yields $\mathrm{Ind}_{L_{\{\alpha\}}}^G \{1\} = \mathrm{Ind}_{L_{\{\beta\}}}^G \{1\} = \mathcal{O}_{sreg}$. On the other hand $(L_{\{\alpha\}}, Z(L_{\{\alpha\}})^\circ, \{1\})$ is not G-conjugate to $(L_{\{\beta\}}, Z(L_{\{\beta\}})^\circ, \{1\})$ in $\mathscr{B}\mathscr{B}(G)$. One can check that, for G simple adjoint non-simply-laced and $u \in \mathcal{O}_{sreg}$, the centralizer $C_G(u)$ is disconnected, in particular, one of the two inductions cannot be birational, by Lemma 1.26.

If two sheets intersect, then they meet at an isolated class, see [28, Proposition 3.4].

2.6 Weakly birational sheets and birational sheets

In [67, §4], Losev defines birational sheets of a reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi subalgebra and $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ be birationally rigid. In our terminology, the *birational sheet* of $\mathfrak{g}$ associated to the data $(\mathfrak{l}, \mathfrak{O}^L)$ is $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$.

As anticipated in the Introduction, not all birational sheets contain a nilpotent orbit: the birational sheet $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ contains a nilpotent orbit if and only if $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ if and only if $0 \in \mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$.

Losev defines the finite group $W(\mathfrak{l}, \mathfrak{O}^L) := \mathrm{Stab}_{N_G(L)}(\mathfrak{O}^L)/L$: it acts on $\mathfrak{z}(\mathfrak{l})$ and it stabilizes $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)$, since the latter does not depend on the parabolic subgroup containing L as a Levi factor.

We recall the main theorem by Losev on birational sheets:

Theorem (Losev [67, Theorem 4.4])**.** *Let $\mathfrak{g}$ be reductive.*

(i) Birational sheets form a partition of $\mathfrak{g}$.

(ii) Birational sheets are locally closed unibranch subvarieties of $\mathfrak{g}$ with smooth normalization.

(iii) Let $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ be a birational sheet in $\mathfrak{g}$. Then it admits a geometric quotient for the adjoint G-action. The geometric quotient $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}/G$ is unibranch and its normalization is isomorphic to $\mathrm{Bir}(\mathfrak{z}(\mathfrak{l}), \mathfrak{O}^L)/W(\mathfrak{l}, \mathfrak{O}^L)$, which is a smooth variety.

We define weakly birational sheets for the conjugation action of G on itself and prove that they partition G. We start by defining the set:

$$\mathscr{B}\mathscr{B}(G) := \{\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G) \mid \mathcal{O}^M \text{ birationally rigid}\}.$$

The group G acts on $\mathscr{B}\mathscr{B}(G)$ by simultaneous conjugacy and $\mathscr{B}\mathscr{B}(G)/G$ is finite because $\mathscr{D}(G)/G$ is so.

Definition 2.23. For $\tau \in \mathscr{B}\mathscr{B}(G)$, the *weakly birational sheet* associated to τ is: $S(\tau)^{wbir} = \overline{J(\tau)}^{wbir}$.

If $[G, G]$ is simply-connected, we write $S(\tau)^{bir} := \overline{J(\tau)}^{bir} = \overline{J(\tau)}^{wbir}$ and talk about the *birational sheets* of G.

Lemma 2.24. *Let $\tau \in \mathscr{B}\mathscr{B}(G)$. Then $S(\tau)^{wbir}$ is a G-stable irreducible locally closed subvariety of G and decomposes as a union of Jordan classes. Weakly birational sheets of G are in one-to-one correspondence with the finite set $\mathscr{B}\mathscr{B}(G)/G$.*

Proof. This follows from Definition 2.23 and Proposition 2.21. □

Theorem 2.25. *Weakly birational sheets of G form a partition of G.*

In particular, if $[G, G]$ is simply-connected, G is partitioned into its birational sheets.

Proof. We prove that $\mathcal{O}_{su} \in G/G$ belongs to a unique weakly birational sheet. By Theorem 1.35, $\mathcal{O}_{su}$ is weakly birationally induced in a unique way from $(M, s, \mathcal{O}^M) \in \mathscr{B}(G)$ up to conjugation in G. This triple uniquely determines the class of $\tau = (M, Z(M)^\circ s, \mathcal{O}^M)$ in $\mathscr{B}\mathscr{B}(G)/G$. Hence, $\mathcal{O}_{su} \subset S(\tau)^{wbir}$ and $S(\tau)^{wbir}$ is unique. □

Remark 2.26. Every weakly birational sheet, being irreducible and contained in $G_{(n)}$ for some $n \in \mathbb{N}$, is contained in a sheet.

Remark 2.27. When G is semisimple, a weakly birational sheet coincides with a single conjugacy class if and only if it is $\mathcal{O}_{su}^G$ with s isolated semisimple and $\mathcal{O}_u^{C_G(s)}$ a birationally rigid unipotent class of $C_G(s)$.

Example 2.28. In general, a sheet is not an union of (weakly) birational sheets. Let $G = \mathrm{Sp}_4(\mathbb{C})$, let $\Theta_i = \{\alpha_i\}$ for $i = 1, 2$ and let $L_i = L_{\Theta_i}$. Let $\mathcal{O}_{sreg}$ be the subregular unipotent class in G, then $\mathcal{O}_{sreg} = \mathrm{Ind}_{L_i}^G\{1\}$ for $i = 1, 2$, but it is birationally induced only from $(L_1, \{1\})$ (see Example 1.29). Let $\tau_i = (L_i, Z(L_i)^\circ, \{1\})$ for $i = 1, 2$, then $\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{reg}$ but $\overline{J(\tau_2)}^{reg} = \overline{J(\tau_2)}^{bir} \sqcup Z(G)\,\mathcal{O}_{sreg}$, where $\overline{J(\tau_2)}^{bir}$ is a birational sheet, but $Z(G)\,\mathcal{O}_{sreg}$ is not one.

All sheets of $\mathfrak{g}$ contain nilpotent orbits [12, §3.2], but not all birational sheets of $\mathfrak{g}$ do [67, §4]. Similarly, all sheets of G contain isolated classes, see [28, Proposition 3.1], but we give an example of a birational sheet without this property.

Example 2.29. Let $G = \mathrm{Sp}_6(\mathbb{C})$, let $s_a = \mathrm{diag}[1, a, -1, -1, a^{-1}, 1] \in T$. Fix $\bar{a} \in \mathbb{C} \setminus \{-1, 0, 1\}$ and set $\bar{s} \coloneqq s_{\bar{a}}$ and $M \coloneqq C_G(\bar{s})$. If $\tau = (M, Z(M)^\circ\bar{s}, \{1\})$, then:

$$\overline{J(\tau)}^{reg} = \bigcup_{z \in Z(M)^\circ\bar{s}} G \cdot (z\,\mathrm{Ind}_M^{C_G(z)}\{1\}) = \left(\bigcup_{a \in \mathbb{C}\setminus\{-1,0,1\}} \mathcal{O}_{s_a}\right) \sqcup \mathcal{O}_{s_1u} \sqcup \mathcal{O}_{s_{-1}v}$$

where the first member is $J(\tau)$ while $\mathcal{O}_{s_1u}$ and $\mathcal{O}_{s_{-1}v}$ are the two isolated classes of the sheet $\overline{J(\tau)}^{reg}$, indeed $C_G(s_1)$ and $C_G(s_{-1})$ are semisimple of type $\mathsf{C}_1\mathsf{C}_2$. Decompose $C_G(s_1) = K'K''$, where $K' \simeq \mathrm{Sp}_4(\mathbb{C})$ and $K'' \simeq \mathrm{Sp}_2(\mathbb{C})$ and decompose $M = M'K''$, with $M' \simeq \mathbb{C}^\times \times \mathrm{Sp}_2(\mathbb{C}) \leq K'$. Then $\mathcal{O}_u^{C_G(s_1)} = \mathrm{Ind}_M^{C_G(s_1)}\{1\} \simeq \mathcal{O}_{subreg}^{K'} \times \{1\}$, where $\{1\}$ is the trivial class in K'' and $\mathcal{O}_{subreg}^{K'} = \mathrm{Ind}_{M'}^{K'}\{1\}$ is the subregular class in $\mathcal{U}_{K'}$. By Example 2.28, the latter induction is not birational. Hence, $\mathcal{O}_u^{C_G(s_1)}$ is not birationally induced from $(M, \{1\})$ and $\mathcal{O}_{s_1u} \not\subset \overline{J(\tau)}^{bir}$. This argument can be repeated, up to reordering the decomposition into simple groups, for $C_G(s_{-1}) \simeq \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_4(\mathbb{C})$ and $v \in \mathrm{Ind}_M^{C_G(s_{-1})}\{1\}$. In particular, the birational sheet $\overline{J(\tau)}^{bir} = J(\tau)$ does not contain any isolated class.

We give a result on weakly birational sheets which are dense in sheets and characterize when the two objects coincide.

Lemma 2.30. *Let $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ with $\mathcal{O}^M \in \mathcal{U}_M/M$ rigid. Then $\overline{J(\tau)}^{wbir}$ is dense in the sheet $\overline{J(\tau)}^{reg}$ and $\overline{J(\tau)}^{reg}$ is the unique sheet of G containing $\overline{J(\tau)}^{wbir}$.*

Moreover, $\overline{J(\tau)}^{wbir} = \overline{J(\tau)}^{reg}$ if and only if all isolated classes $G \cdot (r\,\mathrm{Ind}_M^{C_G(r)^\circ}\,\mathcal{O}^M)$ with $r \in Z(M)^\circ s$ are birationally induced.

Proof. We have $\tau \in \mathscr{BB}(G)$, since $\mathcal{O}^M$ is in particular birationally rigid and $\overline{J(\tau)}^{wbir}$ is open and dense in the irreducible set $\overline{J(\tau)}^{reg}$ by Proposition 2.21. Suppose S is a sheet of G with $\overline{J(\tau)}^{wbir} \subset S$. Then the closure of $\overline{J(\tau)}^{wbir}$ equals $\overline{J(\tau)} \subset \overline{S}$, so $\overline{J(\tau)}^{reg} = S$.

The last assertion follows from Lemmas 2.14 and 2.16 (iii). ☐

Corollary 2.31. *Let G be of type A. Then all sheets are weakly birational sheets. In particular, sheets of G form a partition.*

Proof. This follows from Example 1.27 and Lemma 2.30 and Theorem 2.25. $\square$

Remark 2.32. We claim that Lusztig's strata defined in [70] are disjoint unions of weakly birational sheets. This follows from [28, Proof of Theorem 2.1]: it is proven therein that if $J \in \mathscr{J}(G)$ lies in a stratum, then $\overline{J}^{reg}$ lies in that stratum. Since $\overline{J}^{wbir} \subset \overline{J}^{reg}$, we get that strata are unions of weakly birational closures of Jordan classes. By taking maximal sets with respect to inclusion in this decomposition, we conclude our claim.

2.6.1 Interactions with isogenies and translation by central elements

Remark 2.33. Let $\tau := (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ and $z \in Z(G)$. Set $\tau_z := (M, Z(M)^\circ zs, \mathcal{O}^M)$. Then $\tau_z \in \mathscr{D}(G)$ and $J(\tau_z) = zJ(\tau)$, resp. $\overline{J(\tau_z)} = z\overline{J(\tau)}$, resp. $\overline{J(\tau_z)}^{reg} = z\overline{J(\tau)}^{reg}$. We remind the reader that, in the above situation, it may happen that $J(\tau) = zJ(\tau)$, i.e., when the triples τ and τ_z are conjugate by an element of G.

Remark 2.34. Let $\pi\colon G \to \overline{G}$ be an isogeny and let $\tau \in \mathscr{BB}(G)$. Then π induces a surjective map between the set of weakly birational sheets of G and the set of weakly birational sheets of $\overline{G}$, associating to $\overline{J_G(\tau)}^{wbir}$ the weakly birational sheet $\pi(\overline{J_G(\tau)}^{wbir}) = \overline{J_{\overline{G}}(\bar{\tau})}^{wbir}$, where $\bar{\tau}$ is the triple obtained by applying π to each entry of τ. Moreover, if $\overline{J_G(\tau_1)}^{wbir}$ and $\overline{J_G(\tau_2)}^{wbir}$ are two birational sheets of G such that $\pi(\overline{J_G(\tau_1)}^{bir}) = \pi(\overline{J_G(\tau_2)}^{bir})$, then $\overline{J_G(\tau_1)}^{bir} = z\overline{J_G(\tau_1)}^{bir}$ for some $z \in Z(G)$.

Let $\tau := (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$. We will be interested in considering the union

$$Z(G)\overline{J(\tau)}^{wbir} := \bigcup_{z \in Z(G)} z\overline{J(\tau)}^{wbir},$$

and each member of the union is $z\overline{J(\tau)}^{wbir} = \overline{J(\tau_z)}^{wbir}$, with $\tau_z := (M, Z(M)^\circ zs, \mathcal{O}^M)$. This is clear from Definition 2.17 and from the fact that $Z^\circ s$ satisfies (RP) for M if and only if $Z^\circ zs$ does so, hence $\tau \in \mathscr{D}(G)$ if and only if $\tau_z \in \mathscr{D}(G)$. Therefore, to describe the set $Z(G)\overline{J(\tau)}^{wbir}$ it is enough to describe $\overline{J(\tau)}^{wbir}$ and to count the number of birational sheets appearing in the decomposition of $Z(G)\overline{J(\tau)}^{wbir}$.

Remark 2.35. Let G be simple, let $\tau := (M, Z(M)^\circ z, \mathcal{O}^M) \in \mathscr{D}(G)$ and set $Z := Z(M)$. We have seen that the number of different G-classes of pairs $(M, Z^\circ z)$ for a fixed M, with $Z^\circ z$ satisfying (RP) for M equals the index $d_M = \left[\frac{Z(G)}{Z(G)\cap Z^\circ} : W_1/W_2\right]$, defined in Remark 2.10. If $\mathcal{O}^M$ is characteristic in M, the number of different G-classes of triples $(M, Z(M)^\circ z, \mathcal{O}^M)$ for fixed M and $\mathcal{O}^M$, with $Z^\circ z$ satisfying (RP) for M is again the index d_M.

CHAPTER

THREE

LOCAL GEOMETRY OF JORDAN CLASSES

In this Chapter we explain how the geometry of the neighbourhood of a point in the (regular, resp. weakly birational) closure of a Jordan class in G can be reduced to the study of analogous objects in suitable Lie subalgebras of $\mathfrak{g}$. After introducing some notation in Section 3.1, we first build a neighbourhood $U_{\mathcal{N}}$ of the nilpotent cone of $\mathfrak{g}$ with some particular features and we study the exponential map exp on $U_{\mathcal{N}}$. With such tools we can compare in Section 3.2 the stratification on this neighbourhood induced by Jordan classes in $\mathfrak{g}$ with the one induced by Jordan classes in G on $\exp U_{\mathcal{N}}$. In Section 3.3, we assume $[G, G]$ simply-connected and we prove the main result of the Chapter, reducing the study of the geometry of a Jordan class in G around a point to the study of several Jordan classes in a reductive subgroup around a unipotent point, where results from Section 3.2 apply. We conclude with Section 3.4, devoted to several applications.

3.1 Notation and basic results

We recall that when X and Y are algebraic varieties, the analytic closure coincides with the Zariski closure, [90, Proposition 7] and that if X is an algebraic variety and $x \in X$, then X is unibranch, normal, smooth or Cohen-Macaulay at x if and only if the corresponding analytic variety is so, [50, Exposé XII, Proposition 2.1(vi), Proposition 3.1 (vii)].

Following terminology of [52, §1.7], two pointed varieties (X, x) and (Y, y) are said to be *smoothly equivalent* if there exist a pointed variety (Z, z) and two smooth maps $\phi\colon Z \to X$ and $\psi\colon Z \to Y$ such that $\phi(z) = x$ and $\psi(z) = y$. In this case we write $(X, x) \sim_{se} (Y, y)$. Smooth equivalence is an equivalence relation on pointed varieties and it preserves the properties of being unibranch, normal, Cohen-Macaulay or smooth. By [62, Remark 2.1], if $\dim Y = \dim X + d$, then $(X, x) \sim_{se} (Y, y)$ if and only if $(X \times \mathbb{A}^d, (x, 0))$ and (Y, y) are locally analytically isomorphic. For varieties of the same dimension, smooth equivalence for (X, x) and (Y, y) and the existence of a local analytic isomorphism in a neighbourhood of x mapping x to y are equivalent properties. For any algebraic variety X, denote by X^{an} the associated analytic space.

Let X, Y be varieties and let $p\colon X \to Y$ be a surjective morphism. Then $U \subset X$ is said to be *p-saturated* if $U = p^{-1}(p(U))$. In the case of a reductive subgroup $K \leq G$ acting on an affine variety X with categorical quotient $\pi_X\colon X \to X/\!/K$, the subset $U \subset X$ is π_X-saturated if and only if it is K-stable and all $\mathcal{O}_x^K$ such that $\overline{\mathcal{O}_x^K} \cap \overline{\mathcal{O}_y^K} \neq \varnothing$ for some $y \in U$ satisfies $\mathcal{O}_x^K \subset U$ [68, §I]. In what follows, we will often make use of the categorical quotients for the adjoint action $\pi_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}/\!/G$ and for the conjugacy action $\pi_G\colon G \to G/\!/G$.

3.2 Unipotent and nilpotent elements

We start by showing how to compare the local geometry of Jordan classes containing a unipotent element in their closure with the local geometry of decomposition classes in $\mathfrak{g}$, by means of the exponential map. It is well-known that the exponential map is a G-equivariant analytic map inducing a G-equivariant analytic isomorphism $\mathcal{N} \to \mathcal{U}$, this follows from Schur's formula for the derivative of the exponential, see [89, §1.2 Theorem 5]. Since the base field is assumed to be $\mathbb{C}$, the restriction of the exponential to $\mathcal{N} \to \mathcal{U}$ is also algebraic, so it is a Springer isomorphism, see [53, §6.20].

We will work with a slight variation of the exponential map. Let $[G,G]_{sc}$ be the simply-connected cover of $[G,G]$. Denote by $\exp_{sc}\colon \mathfrak{g} \to Z(G)^\circ \times [G,G]_{sc}$ the dilation by a factor $2\pi i$ followed by the exponential map. Then if $\pi\colon Z(G)^\circ \times [G,G]_{sc} \to G$ is the isogeny, we put $\exp \coloneqq \pi \circ \exp_{sc}\colon \mathfrak{g} \to G$. In particular, when π is the identity, we have $\exp = \exp_{sc}$: for this reason, with an abuse of terminology, we will refer to exp simply as to *the exponential map* of G.

Consider G simple simply-connected, then exp restricts to a Lie group homomorphism $\mathfrak{h} \to T$ with kernel $\mathtt{Q}^\vee$, the co-root lattice of $\mathfrak{g}$. Let $\mathfrak{h}_\mathbb{R} \coloneqq \mathrm{span}_\mathbb{R}\mathtt{Q}^\vee$; we have $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, the fundamental alcove is $\mathscr{A} \coloneqq \{h \in \mathfrak{h}_\mathbb{R} \mid 0 \leq \alpha(h) \leq 1 \text{ for all } \alpha \in \widetilde{\Delta}\}$ and we define the affine hyperplanes $H_{\alpha,l} \coloneqq \{h \in \mathfrak{h} \mid \alpha(h) = l\}$ for $l \in \mathbb{Z}$ and $\alpha \in \Phi$. Let A be the interior of $W \cdot \mathscr{A} + i\mathfrak{h}_\mathbb{R}$ in $\mathfrak{h}$ and let

$$U_{sc} \coloneqq \pi_{\mathfrak{g}}^{-1}(\pi_{\mathfrak{g}}(A)). \tag{3.1}$$

It is $\pi_{\mathfrak{g}}$-saturated by construction and open by Chevalley's restriction theorem.

Lemma 3.1. *Let G be a complex connected reductive group. There exist a $\pi_{\mathfrak{g}}$-saturated analytic open neighbourhood $U_\mathcal{N}$ of $\mathcal{N}$ in $\mathfrak{g}$ and a π_G-saturated analytic open neighbourhood $U_\mathcal{U}$ of $\mathcal{U}$ in G such that the restriction to $U_\mathcal{N}$ of the exponential map gives an analytic isomorphism $\exp_{U_\mathcal{N}}\colon U_\mathcal{N} \to U_\mathcal{U}$.*

Moreover, for G simple simply-connected, one can take $U_\mathcal{N} = U_{sc}$ as in (3.1) *and $U_\mathcal{U} = \exp U_{sc}$.*

Proof. Step 1. Suppose G is a torus, then $\mathfrak{g}$ is abelian, and exp is a local analytic isomorphism at each point of $\mathfrak{g}$.

Step 2. Suppose now G simple simply-connected. We show that $\exp\colon \mathfrak{g} \to G$ is an analytic isomorphism on U_{sc}. The main result in [79, 80] (see also [49, Chapter I, Theorem 3.5]), states that the exponential map is a local analytic isomorphism at $\xi = \sigma + \nu$ if and only if the eigenvalues of $\mathrm{ad}(\xi)$ do not meet $\mathbb{Z} \setminus \{0\}$. These eigenvalues coincide with those of $\mathrm{ad}(\sigma)$, so the condition is verified if and only if, up to G-action, σ lies in $\mathfrak{h} \setminus \bigcup \{H_{\alpha,l} \mid l \in \mathbb{Z} \setminus \{0\}, \alpha \in \Phi^+\}$. As A is contained in this set, exp is a local analytic isomorphism on U_{sc}. Let $U_{\mathcal{N}} := U_{sc}$ and $U_{\mathcal{U}} := \exp U_{\mathcal{N}}$. We prove now that the restriction $\exp_{U_{\mathcal{N}}} \colon U_{\mathcal{N}} \to U_{\mathcal{U}}$ is an analytic isomorphism, i.e., that $\exp_{U_{\mathcal{N}}}$ is injective on $U_{\mathcal{N}}$. If $\exp_{U_{\mathcal{N}}}(\sigma + \nu) = \exp_{U_{\mathcal{N}}}(\sigma' + \nu')$, then $\nu = \nu'$ because exp is an isomorphism on $\mathcal{N}$ and by G-equivariance we may assume that $\exp_{U_{\mathcal{N}}}(\sigma) = \exp_{U_{\mathcal{N}}}(\sigma') \in T$, so $\sigma, \sigma' \in A$. Two elements in A cannot differ by an element in $\mathtt{Q}^\vee$ because $\mathcal{A}$ is a fundamental domain for the $\mathtt{Q}^\vee \rtimes W$-action on $\mathfrak{h}_{\mathbb{R}}$, [53, Theorem 4.8] and $\mathtt{Q}^\vee$ does not change the imaginary components of elements in $\mathfrak{h}$. Thus, $\sigma = \sigma'$ and the properties of $U_{\mathcal{U}}$ follow from those of exp and of $U_{\mathcal{N}}$.

Step 3. Now let G be as in the statement. Denote by $[G,G]_{sc}$ the simply-connected cover of $[G,G]$. Consider the isogeny $\pi \colon \widetilde{G} := Z(G)^\circ \times [G,G]_{sc} \to G$, let $\overline{\pi} \colon \widetilde{G}/\!/\widetilde{G} \to G/\!/G$ be the induced map and assume there exist a $\pi_{\mathfrak{g}}$-saturated analytic open neighbourhood $\widetilde{U}$ of $\mathcal{N}$ in $\mathfrak{g}$ such that $\exp \colon \mathfrak{g} \to \widetilde{G}$ restricts to an analytic isomorphism on $\widetilde{U}$. Let A' be an open neighbourhood of the class $[1]$ in $\widetilde{G}/\!/\widetilde{G}$ such that if $kA' \cap A' \neq \emptyset$ for some $k \in \ker \overline{\pi}$, then $k = 1$. Let $\widetilde{A} = \pi_{\widetilde{G}}^{-1}(A')$. Then $\widetilde{V} := \widetilde{A} \cap \exp(\widetilde{U})$ is a $\pi_{\widetilde{G}}$-saturated open neighbourhood of $\mathcal{U}$ in $\widetilde{G}$ and $U_{\mathcal{N}} := \widetilde{U} \cap \exp^{-1}(\widetilde{V})$ and $U_{\mathcal{U}} := \pi \circ \exp U_{\mathcal{N}} = \pi(\widetilde{V})$ are the sought neighbourhoods for $\mathfrak{g}$ and G. This allows to reduce the proof to the case in which G is a direct product of a torus and finitely many simple simply-connected groups: then it is enough to prove the statement for each factor, and this was done in steps 1 and 2. □

We describe now compatibility of the Jordan stratifications induced on neighbourhoods constructed in Lemma 3.1.

Theorem 3.2. *Let $U_{\mathcal{N}}$ be a $\pi_{\mathfrak{g}}$-saturated analytic open neighbourhood of $\mathcal{N}$ in $\mathfrak{g}$ and $U_{\mathcal{U}}$ be a π_G-saturated analytic open neighbourhood of $\mathcal{U}$ in G such that the restriction of* exp *to $U_{\mathcal{N}}$ is a G-equivariant analytic isomorphism $\exp_{U_{\mathcal{N}}} \colon U_{\mathcal{N}} \to U_{\mathcal{U}}$. Then, $\exp_{U_{\mathcal{N}}}$ identifies the stratification on $U_{\mathcal{N}}$ induced by the Jordan one in $\mathfrak{g}$ with the stratification on $U_{\mathcal{U}}$ induced by the Jordan one in G, preserving dimensions, closure orderings, orbit dimensions. More precisely, for $\tau = (M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{T}(G)$ we have $J(\tau) \cap U_{\mathcal{U}} \neq \varnothing$ if and only if M is a Levi subgroup of G and $Z(M)^\circ s = Z(M)^\circ$ and if this is the case, then*

$$J(\tau) \cap U_{\mathcal{U}} = \exp(\mathfrak{J}(\mathfrak{m}, \mathfrak{O}^M) \cap U_{\mathcal{N}})$$

where $\mathfrak{m} = \mathrm{Lie}(M)$ *and* $\exp(\mathfrak{O}^M) = \mathcal{O}^M$.

Proof. We keep notation from the proof of Lemma 3.1. Let $\mathfrak{J} := \mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L) \in \mathscr{J}(\mathfrak{g})$. Then $\overline{\mathfrak{J}} \cap \mathcal{N} \neq \varnothing$ so $\mathfrak{J} \cap U_{\mathcal{N}} \neq \varnothing$. By $\pi_{\mathfrak{g}}$-saturation of $U_{\mathcal{N}}$ we have

$$U_{\mathcal{N}} \cap (\mathfrak{z}(\mathfrak{l})^{reg} + \mathfrak{O}^L) = (U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}) + \mathfrak{O}^L .$$

If $\sigma + \nu \in (\mathfrak{z}(\mathfrak{l})^{reg} \cap U_{\mathcal{N}}) + \mathfrak{O}^L$, then by [80] we have $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(\sigma) = \mathfrak{c}_{\mathfrak{g}}(\exp \sigma)$ so $L := C_G(\exp \sigma)^\circ$ is a Levi subgroup of G and setting $\mathcal{O}^L = \exp(\mathfrak{O}^L)$ we have $\exp((\mathfrak{z}(\mathfrak{l})^{reg} \cap U_{\mathcal{N}}) + \mathfrak{O}^L) \subset U_{\mathcal{U}} \cap Z(L)\, \mathcal{O}^L$.

Observe that $\mathfrak{z}(\mathfrak{l})^{reg}$ is obtained removing finitely many vector spaces of real codimension at least 2 from a (complex) vector space, so it is connected in the analytic topology. Therefore $U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}$, $(U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}) + \mathfrak{O}^L$ and $\mathfrak{J} \cap U_{\mathcal{N}} = \mathrm{Ad}(G)((U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}) + \mathfrak{O}^L)$ are also connected. By continuity, $\exp((U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}) + \mathfrak{O}^L)$ and $\exp(U_{\mathcal{N}} \cap \mathfrak{J})$ are also connected in the analytic topology. Thus, $\exp((U_{\mathcal{N}} \cap \mathfrak{z}(\mathfrak{l})^{reg}) + \mathfrak{O}^L) \subset U_{\mathcal{U}} \cap (Z(L)^\circ s)^{reg}\, \mathcal{O}^L$ for some $s \in Z(L)$ and $\exp(U_{\mathcal{N}} \cap \mathfrak{J}) \subset J(L, Z(L)^\circ s, \mathcal{O}^L) \cap U_{\mathcal{U}}$. Observe also that $0 \in \overline{\mathfrak{J}} \cap U_{\mathcal{N}}$ so $1 \in \overline{J(L, Z(L)^\circ s, \mathcal{O}^L)} \cap U_{\mathcal{U}}$. This implies that $Z(L)^\circ s = Z(L)^\circ$.

Conversely, let $J \in \mathscr{J}(G)$ be such that $U_{\mathcal{U}} \cap J \neq \varnothing$ and let $su \in U_{\mathcal{U}} \cap J$. Set $M := C_G(s)^\circ$, by π_G-saturation of $U_{\mathcal{U}}$ we have $(Z(M)^\circ s)^{reg}\, \mathcal{O}_u^M \cap U_{\mathcal{U}} = ((Z(M)^\circ s)^{reg} \cap U_{\mathcal{U}})\, \mathcal{O}_u^M$. For any $r \in (Z(M)^\circ s)^{reg} \cap U_{\mathcal{U}}$ we have $r = \exp \rho$ for some $\rho \in U_{\mathcal{N}}$ and $\mathrm{Lie}(C_G(r)^\circ) = \mathfrak{c}_{\mathfrak{g}}(\rho) = \mathfrak{m}$. Therefore for any $rv \in ((Z(M)^\circ s)^{reg}\, \mathcal{O}_u^M) \cap U_{\mathcal{U}}$ we have $rv \in \exp(U_{\mathcal{N}} \cap \mathfrak{J}(\mathfrak{m}, \mathfrak{O}^M_{\exp^{-1}(u)})) \subseteq J(M, Z(M)^\circ, \mathcal{O}_u^M)$ so $Z(M)^\circ s = Z(M)^\circ$ and $\exp(U_{\mathcal{N}} \cap \mathfrak{J}(\mathfrak{m}, \mathfrak{O}^M_{\exp^{-1}(u)})) = U_{\mathcal{U}} \cap J(M, Z(M)^\circ, \mathcal{O}^M)$.

Finally, $\exp_{\mathcal{U}}$ is a G-equivariant analytic isomorphism, hence it preserves orbit dimensions, closure orderings, and dimensions. □

Corollary 3.3. *Let $L \leq G$ be a Levi subgroup and let $\tau = (L, Z(L)^\circ, \mathcal{O}^L) \in \mathscr{D}(G)$ and set $J := J(\tau)$. Let $\mathfrak{l} := \mathrm{Lie}(L)$ and $\mathfrak{O}^L \in \mathcal{N}_{\mathfrak{l}}/L$ such that $\exp(\mathfrak{O}^L) = \mathcal{O}^L$. Let $\mathfrak{J} := \mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)$, let $\nu \in \mathcal{N}$ and $v := \exp(\nu) \in \overline{J} \cap \mathcal{U}$. Then:*

(i) $(\overline{J}, v) \sim_{se} (\overline{\mathfrak{J}}, \nu)$.

(ii) $v \in \overline{J}^{reg}$ *if and only if* $\nu \in \overline{\mathfrak{J}}^{reg}$ *and, in this case,* $(\overline{J}^{reg}, v) \sim_{se} (\overline{\mathfrak{J}}^{reg}, \nu)$.

(iii) $v \in \overline{J}^{wbir}$ *if and only if* $\nu \in \overline{\mathfrak{J}}^{bir}$ *and, in this case,* $(\overline{J}^{wbir}, v) \sim_{se} (\overline{\mathfrak{J}}^{bir}, \nu)$.

Proof. Let $U_{\mathcal{N}}$ and $U_{\mathcal{U}}$ be neighbourhoods of $\mathcal{N}$ and $\mathcal{U}$, respectively, as in Lemma 3.1, Theorem 3.2. Then $v \in \overline{J} \cap U_{\mathcal{U}}$ and $\exp_{U_{\mathcal{N}}}$ is an analytic isomorphism mapping $\overline{\mathfrak{J}} \cap U_{\mathcal{N}}$ to $\overline{J} \cap U_{\mathcal{U}}$. This proves (i). Since exp preserves orbit dimensions, $v \in \overline{J}^{reg}$ if and only if $\nu \in \overline{\mathfrak{J}}^{reg}$ and taking regular loci at both sides of (i) yields (ii). We have $v \in \overline{J}^{wbir}$ if and only if $\mathcal{O}_v = \mathrm{Ind}_L^G \mathcal{O}^L$ is birationally induced from $(L, \mathcal{O}^L)$, if and only if $\mathfrak{O}_\nu = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{O}^L$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$, by Lemma 1.20 (ii), i.e., $\nu \in \overline{\mathfrak{J}}^{bir}$. Since $\overline{J}^{wbir}$ (resp. $\overline{\mathfrak{J}}^{bir}$) is open in $\overline{J}$ (resp. $\overline{\mathfrak{J}}$), (i) implies (iii). □

Remark on the Proof. For the unipotent element $v \in \mathcal{U}$, we have $v \in \overline{J}^{wbir}$ if and only if $v \in \overline{J}^{bir}$, by Remark 1.18, but we cannot conclude the smooth equivalence with $(\overline{J}^{bir}, v)$ because $\overline{J}^{bir}$ need not be open in $\overline{J}^{reg}$, unless $[G, G]$ is simply-connected.

Remark 3.4.

(i) The set of points $\xi \in \mathfrak{g}$ such that exp is a local analytic isomorphism at ξ is not a union of Jordan classes in general. For instance $\sigma = \mathrm{diag}[i, -i]$ and $\sigma' = \mathrm{diag}[1, -1]$ lie in the same Jordan class in $\mathfrak{sl}_2(\mathbb{C})$, and the condition on the eigenvalues in [80] holds for σ but not for σ'.

(ii) The image of exp is a union of Jordan classes in G. Indeed, $g = ru \in \exp \mathfrak{g}$ if and only if $r \in C_G(u)^\circ$, by [42]. This condition is clearly G-stable, so it is enough to show that $r \in C_G(u)^\circ$ implies $Z(C_G(r)^\circ)^\circ r \subset C_G(u)^\circ$. Now, $u \in C_G(r)^\circ$, so $Z(C_G(r)^\circ) \subset C_G(u)$. Since $Z(C_G(r))^\circ r$ is connected and contains r, we have the desired inclusion.

3.3 Reduction to unipotent elements

Throughout this Section we assume that $[G, G]$ is *simply-connected*: in particular, centralizers of semisimple elments in G are connected. To develop a local study of Jordan classes, we will use a variant of Luna's étale slice theorem. The idea is to reduce the study of the closure of a Jordan class in G in the neighbourhood of an element rv to the study of the closures of several Jordan classes in $C_G(r)$ in the neighbourhood of the unipotent part v.

We recall that a morphism of algebraic varieties $f \colon X \to Y$ is said to be *étale* at a point $x \in X$ if the differential $\mathrm{d}_x f \colon \mathrm{T}_x X \to \mathrm{T}_{f(x)} Y$ is an isomorphism.

We are indebted to Andrea Maffei for suggesting the proof of the following result.

Proposition 3.5. *Let $r \in G$ be a semisimple element and let $M = C_G(r)$. There is a Zariski open neighbourhood U of r in M such that:*

(i) U is π_M-saturated.

(ii) For any Jordan class J_M of M we have $J_M \cap U \neq \varnothing$ if and only if $r \in \overline{J_M}$.

(iii) The restriction γ_U to $G \times^M U$ of the map $\gamma \colon G \times^M M \to G$ given by $\gamma(g * x) = gxg^{-1}$ is étale.

(iv) The image $G \cdot U$ of γ_U is a π_G-saturated open neighbourhood of r in G.

Proof. Observe that $G \cdot (1 * r) = G * r$ is closed and since r is semisimple, $\mathcal{O}_r$ is closed, too. By construction, the restriction of γ to $G * r$ is injective. We claim that γ is étale at $1 * r$. We consider the map $\widetilde{\gamma} \colon G \times M \to G$ given by the conjugation action and the natural projection $p \colon G \times M \to G \times^M M$, so $\widetilde{\gamma} = \gamma \circ p$. For $m \in M$ the differential at $(1, m)$ is given by:

$$\begin{aligned} \mathrm{d}_{(1,m)}\widetilde{\gamma} \colon \mathfrak{g} \oplus \mathfrak{m} &\to \mathfrak{g} \\ (x, y) &\mapsto y - x + \mathrm{Ad}(m^{-1})x. \end{aligned}$$

For $g \in G$, let L_g be left translation in G by g. The induced map identifies $\mathfrak{g}$ with $\mathrm{T}_g G$ and $\mathfrak{m} = \mathfrak{c}_{\mathfrak{g}}(r)$ with $\mathrm{T}_g M$. This way, $\mathrm{d}_{(g,m)}\widetilde{\gamma} \colon (x, y) \mapsto \mathrm{Ad}(g)(y - x + \mathrm{Ad}(m^{-1})x)$, for $(x, y) \in \mathfrak{g} \oplus \mathfrak{m}$. Since r is semisimple, $\mathfrak{g} = \mathrm{Im}(\mathrm{Ad}(r^{-1}) - \mathrm{id}) \oplus \ker(\mathrm{Ad}(r^{-1}) - \mathrm{id})$ and $\ker(\mathrm{Ad}(r^{-1}) - \mathrm{id}) = \ker(\mathrm{id} - \mathrm{Ad}(r)) = \mathfrak{m}$ so $\mathrm{d}_{(1,r)}\widetilde{\gamma}$ is onto, yielding surjectivity of $\mathrm{d}_{1*r}\gamma$. For any pair $(g, m) \in G \times M$ the composition

$$G \times M \xrightarrow{L_g \times L_m} G \times M \xrightarrow{\ p\ } G \times^M M$$

yields an exact sequence by differentiating:

$$0 \longrightarrow N_m \longrightarrow \mathfrak{g} \oplus \mathfrak{m} \longrightarrow \mathrm{T}_{g*m}(G \times^M M) \longrightarrow 0$$

where $N_m = \{(x, x - Ad(m^{-1})(x) \mid x \in \mathfrak{m}\}$, so $\dim T_{g*m}(G \times^M M) = \dim \mathfrak{g}$ and injectivity of $\mathrm{d}_{1*r}\gamma$ follows. Therefore the hypotheses of [68, Lemme fondamental, §II.2] are satisfied for the map $\gamma \colon G \times^M M \to G$ and the point $1 * r$ and there exists an étale slice. In particular, there exists a π_M-saturated Zariski open neighbourhood U' of r in M such that the restriction of γ to $G \times^M U' \to G$ is étale with image a π_G-saturated open subset $V' = G \cdot U'$ of G.

Consider the stratification on $M//M$ with finitely many Jordan closed strata of the form $\overline{J_M}//M$, for J_M a (semisimple) Jordan class in M, and let $J_M//M$ denote the open stratum in $\overline{J_M}//M$. Let V be the union of all $J_M//M$ containing the class of r in their closure. It is open, because its complement in $M//M$ is the closed set

$$\bigcup_{[r]\notin\overline{J_M}//M} J_M//M = \bigcup_{[r]\notin\overline{J_M}//M} \overline{J_M}//M.$$

Then $U'' := \pi_M^{-1}(V)$ is a π_M-saturated open subset of M containing r and such that a Jordan class J_M in M meets U'' if and only if $r \in \overline{J}_M$. We take the π_M-saturated neighbourhood $U = U' \cap U''$. It satisfies condition (ii) and the restriction of the étale map γ to the open subset $G \times^M U$ is again étale and its image $G \cdot U$ is a π_G-saturated open neighbourhood of r in G. $\square$

Remark 3.6. With notation as above, since γ_U is étale, for any $x \in U$ we have $\dim \mathcal{O}_x^G = \dim \mathcal{O}_{\gamma(1*x)}^G = \dim \mathcal{O}_{1*x}^G$, so $\dim C_G(x) = \dim G_{1*x} = \dim C_M(x)$. Hence, $C_M(x)^\circ = C_G(x)^\circ$. Since U is π_M-saturated, if $x = su \in U$, then $s \in U$ and so $C_M(s)^\circ = C_G(s)$, see also [68, Remarque III.1.4].

Let $su \in G$, with $s \in T$ and let $J := J_G(su) \in \mathscr{J}(G)$; we have $C_G(s) = C_G(s)^\circ$. Let $rv \in \overline{J}$ with $r \in Z^\circ s \subset T$ and let $M := C_G(r) = C_G(r)^\circ$. Since $r \in Z(C_G(s))^\circ s$, we have that $C_G(s)$ is a Levi subgroup of M by Lemma 0.2. In this setting, define:

$$\mathcal{J}_{J,rv} := \{J_{M,i} \in \mathscr{J}(M) \mid J_{M,i} \subset J \text{ and } rv \in \overline{J_{M,i}}\}; \tag{3.2}$$

and let $I_{J,rv}$ be an index set for $J_{M,i} \in \mathcal{J}_{J,rv}$.

Proposition 3.7. *Let $J = J_G(\tau)$ for some $\tau \in \mathscr{T}(G)$, let $rv \in \overline{J}$ and set $M := C_G(r)$. Then*

$$(\overline{J},\, rv) \sim_{se} \left(\bigcup_{I_{J,rv}} r^{-1}\overline{J_{M,i}}, v \right) \tag{3.3}$$

where $I_{J,rv}$ is defined in (3.2).

If, in addition, $rv \in \overline{J}^{reg}$, then $rv \in \overline{J_{M,i}}^{reg}$ for all $i \in I_{J,rv}$ and

$$(\overline{J}^{reg},\, rv) \sim_{se} \left(\bigcup_{i\in I_{J,rv}} r^{-1}\overline{J_{M,i}}^{reg},\ v \right). \tag{3.4}$$

Proof. Let $\tau = (M', Z(M')^\circ s, \mathcal{O})$. Since conjugation by $g \in G$ induces the smooth equivalence $(\overline{J}, rv) \sim_{se} (\overline{J}, g \cdot (rv))$, we may assume that $r \in Z(M')^\circ s$, so $M' \subset M$. We adopt notation from Proposition 3.5 and its proof, but with γ_U viewed as a map $G \times^M U \to G \cdot U$. Let $\tilde{\gamma}_U : G \times U \to G \cdot U$ be the restriction of $\tilde{\gamma}$.

We will first show that $(\overline{J}, x) \sim_{se} (\overline{J} \cap U, x)$ for any $x \in \overline{J} \cap U$. Then, we will prove that $\overline{J} \cap U = \overline{J \cap U}^U$ and show that the irreducible components of $\overline{J \cap U}^U$ are the intersections of U with the closures of Jordan classes listed in $\mathcal{J}_{J,rv}$ as in (3.2). We will conclude the proof of

(3.3) by observing that, in order to study $(\overline{J}, x)$ we can neglect those irreducible components of $\overline{J \cap U}^U$ that do not contain x. A dimension argument will give (3.4).

We consider the following commutative diagram

$$\begin{tikzcd}
G \times U \arrow[r, "p"] & G \times^M U \arrow[r, "\gamma_U"] & G \cdot U \\
G \times (\overline{J} \cap U) \arrow[r] \arrow[u] & G \times^M (\overline{J} \cap U) \arrow[r] \arrow[u] & \overline{J} \cap G \cdot U \arrow[u]
\end{tikzcd}$$

Observe that $\tilde{\gamma}_U^{-1}(\overline{J} \cap G \cdot U)$ is a G-stable closed subset of $G \times U$, so it is of the form $G \times V$ for some V closed in U. In turn, V is the pre-image of $G \times V$ through the natural inclusion of U into $G \times U$. Therefore $\tilde{\gamma}_U^{-1}(\overline{J} \cap G \cdot U) = G \times (\overline{J} \cap U)$. This is exactly saying that the bottom composition of arrows is obtained by pulling-back $\tilde{\gamma}_U$ along the closed embedding ι. Hence, the bottom composition is also smooth and for any $x \in \overline{J} \cap U$

$$(\overline{J}, x) \sim_{se} (\overline{J} \cap G \cdot U, x) \sim_{se} (G \times (\overline{J} \cap U), (1, x)) \sim_{se} (\overline{J} \cap U, x). \tag{3.5}$$

To show $\overline{J} \cap U = \overline{J \cap U}^U$, we prove the equivalent statement $G \times^M (\overline{J} \cap U) = \overline{G \times^M (J \cap U)}^{G \times^M U}$, i.e., $\gamma_U^{-1}(\overline{J} \cap G \cdot U) = \overline{\gamma_U^{-1}(J \cap G \cdot U)}^{G \times^M U}$. By elementary topology we see that $\overline{J} \cap (G \cdot U) = \overline{J \cap G \cdot U}^{G \cdot U}$. Since γ_U is continuous, surjective and open, we have $\gamma_U^{-1}(\overline{J \cap G \cdot U}^{G \cdot U}) = \overline{\gamma_U^{-1}(J \cap G \cdot U)}^{G \times^M U}$, hence the desired equality. Thus (3.5) gives $(\overline{J}, x) \sim_{se} (\overline{J \cap U}^U, x)$ for any $x \in \overline{J} \cap U$.

We describe now the irreducible components of $\overline{J \cap U}^U$. By base-change the restriction of γ to $G \times^M (J \cap U)$ is a G-equivariant étale map to $J \cap G \cdot U \subset G_{(d)}$ for some d. Hence all G-orbits in $G \times^M (J \cap U)$ have the same dimension. By Remark 3.6 we have $J \cap U \subset M_{(d')}$ for some d'. The equivalence (3.5) implies that $(J \cap U, x) \sim_{se} (J, x)$ for any $x \in U \cap J$ and J is smooth, so the intersection $U \cap J$ is also smooth. Hence it is the union of its connected components and it is contained in the finite union of those Jordan classes in $M_{(d')}$ containing r in their closure. Let J_M be a Jordan class in M such that $J \cap U \cap J_M \neq \varnothing$. By construction of U, we have $r \in \overline{J_M}$. It follows from Remark 3.6 that if $x = tu \in J_M \cap U \cap J$, then $C_M(t)^\circ = C_G(t)$, hence $\dim Z(C_M(t)^\circ)^\circ = \dim Z(C_G(t)^\circ)^\circ = \dim Z(M')^\circ$. The proof of [29, Theorem 5.6 (e)] applied to the case of (regular closures of) arbitrary Jordan classes shows that $\dim J_M = d' + \dim Z(M')^\circ$, so all Jordan classes of M meeting $J \cap U$ have the same dimension. The same argument also shows that $(Z(C_M(t)^\circ)^\circ r)^{reg} u = (Z(C_G(t))^\circ r)^{reg} u$ and so $J_M = M \cdot ((Z(C_M(t)^\circ)^\circ r)^{reg} u) \subset G \cdot ((Z(C_G(t))^\circ r)^{reg} u) = J$. Therefore, $J_M \subset J$. Conversely, if a Jordan class $J_M \subset M$ contains r in its closure and is contained in J, then $\varnothing \neq J_M \cap U \subset J \cap U$.

Let $I_{J,r}$ be the index set defined in (3.2). Then $J \cap U = \bigcup_{i \in I_{J,r}} J_{M,i} \cap U$, and the locally closed subsets $J_{M,i} \cap U$ are finitely many, disjoint, irreducible and have all the same dimension. Hence, their closures are the irreducible components of $\overline{U \cap J}^U = \overline{J} \cap U$. Therefore, for any

$x \in \overline{U \cap J}^U$:

$$(\overline{U \cap J}^U, x) \sim_{se} (\overline{\bigcup_{i \in I_{J,r}} J_{M,i} \cap U}^U, x) \sim_{se} (\bigcup_{i \in I_{J,r}} \overline{J_{M,i} \cap U}^U, x)$$
$$\sim_{se} (\bigcup_{i \in I_{J,r}} \overline{J_{M,i}}^M \cap U, x) \sim_{se} (\bigcup_{i \in I_{J,r}} \overline{J_{M,i}}^M, x).$$

Let $I_{J,x}$ be the set of indices in $I_{J,r}$ such that $x \in \overline{J_{M,i}}$ and let U_x be a Zariski open neighbourhood of x in M such that $U_x \cap \overline{J_{M,i}} = \varnothing$ for any $i \in I_{J,r} \setminus I_{J,x}$. Then,

$$(\overline{J}, x) \sim_{se} (\bigcup_{i \in I_{J,r}} \overline{J_{M,i}}^M \cap U_x, x) \sim_{se} (\bigcup_{i \in I_{J,x}} \overline{J_{M,i}}^M \cap U_x, x) \sim_{se} (\bigcup_{i \in I_{J,x}} \overline{J_{M,i}}^M, x).$$

Taking $x = rv$ and translating by r^{-1} gives (3.3). Observe that if $rv \in \overline{J}^{reg}$, then $\mathcal{O}_{rv}^G \subset G_{(d)}$ and meets U. Since γ_U is étale, $\mathcal{O}_{rv}^M \subset M_{(d')}$ so it lies in $\overline{J}_{M,i}^{reg}$ for every $i \in I_{J,x}$. Since $\overline{J}^{reg}$ is open in $\overline{J}$ and $\bigcup_{i \in I_{J,rv}} \overline{J}_{M,i}^{reg}$ is open in the union of equidimensional closures $\bigcup_{i \in I_{J,rv}} \overline{J}_{M,i}$, equation (3.4) follows from (3.3). □

In order to provide an explicit parameterization of $I_{J,rv}$ in (3.2) in terms of data depending on J and rv, we introduce some notation. Let $\tau = (M', Z(M')^\circ s, \mathcal{O}) \in \mathscr{T}(G)$, let $rv \in \overline{J} \cap Z(M')^\circ sv$ and let $M = C_G(r)$. We set:

$$W_\tau := \mathrm{Stab}_W(\tau)$$
$$W(\tau, r) := \{w \in W \mid r \in w \cdot (Z(M')^\circ s)\}.$$

If $w \in W(\tau, r)$ then $w \cdot M' = C_G(w \cdot (Z(M')^\circ s))^\circ \subset M$ and $w \cdot M'$ is a Levi subgroup in M, [29, Lemma 4.10]. We consider then

$$W(\tau, rv) := \left\{ w \in W(\tau, r) \mid \mathcal{O}_v^M \subset \overline{\mathrm{Ind}_{w \cdot M'}^M (w \cdot \mathcal{O})} \right\}.$$

The reader should be alert that $W(\tau, r)$ and $W(\tau, rv)$ are not subgroups of W in general.

Since $W_\tau \leq \mathrm{Stab}_W(Z(M')^\circ s)$, it acts on $W(\tau, r)$ from the right. It preserves M' and $\mathcal{O}$, hence it acts also on $W(\tau, rv)$ from the right. The group $W_r := N_M(T)/T \leq W$ acts on $W(\tau, r)$ and $W(\tau, rv)$ from the left.

Theorem 3.8. *Let $J = J_G(\tau)$ for some $\tau = (M', Z(M')^\circ s, \mathcal{O}) \in \mathscr{T}(G)$, let $r \in Z(M')^\circ s$ and $M = C_G(r)$. Then*

$$(\overline{J}, r) \sim_{se} \left(\bigcup_{w \in W_r \backslash W(\tau, r)/W_\tau} \overline{J_M(w \cdot \tau)},\, r \right). \tag{3.6}$$

If $rv \in \overline{J}$ then

$$(\overline{J}, rv) \sim_{se} \left(\bigcup_{w \in W_r \backslash W(\tau, rv)/W_\tau} r^{-1}\overline{J_M(w \cdot \tau)}, v \right). \tag{3.7}$$

If $rv \in \overline{J}^{reg}$ *then*

$$(\overline{J}^{reg}, rv) \sim_{se} \left(\bigcup_{w \in W_r \backslash W(\tau, rv)/W_\tau} r^{-1}\overline{J_M(w\cdot\tau)}^{reg}, v \right). \tag{3.8}$$

Proof. We first consider the neighbourhood of r. By Proposition 3.7 it is enough to show that the right hand side of (3.6) involves precisely those Jordan classes in M that:

(1) are contained in J;
(2) contain r in their closure.

By condition (1), the latter are parameterized by W_r-orbits of triples of the form $w\cdot\tau$ for some $w \in W/W_\tau$. Condition (2) is equivalent to $r \in w\cdot(Z(M')^\circ s)$. Hence the elements w must be taken in $W(\tau, r)/W_\tau$. This gives (3.6).

Let us now consider the neighbourhood of rv. In this case we need to prove that the classes occurring in the right hand side of (3.7) are precisely those Jordan classes $J_M(M'', Z(M'')^\circ s', \mathcal{O}')$ in M that

(1) are contained in J;
(2) contain rv in their closure, that is, contain r in their closure and satisfy $\mathcal{O}_v^M \subset \overline{\mathrm{Ind}_{M''}^M \mathcal{O}'}$.

They are parameterized by W_r-orbits of triples of the form $w\cdot\tau$, where w must be taken in $W(\tau, rv)/W_\tau$, as one sees from condition (2). This gives (3.7). Equation (3.8) follows from (3.7) and (3.4). $\square$

Corollary 3.9. *Let* $J := J_G(\tau)$ *for some* $\tau \in \mathscr{T}(G)$ *and let* $rv, r'v' \in J' \subset \overline{J}$. *Then,* $(\overline{J}, rv) \sim_{se} (\overline{J}, r'v')$. *In other words, the geometry of* $\overline{J}$ *is constant along Jordan classes.*

Moreover, if $J' \subset \overline{J}^{bir}$, *then* $(\overline{J}^{bir}, rv) \sim_{se} (\overline{J}^{bir}, r'v')$.

Proof. Let $\tau = (M', Z(M')^\circ s, \mathcal{O})$. Since $(\overline{J}, x) \sim_{se} (\overline{J}, g\cdot x)$ for any $g \in G$, we may assume that $r \in Z(M')^\circ s$, $C_G(r) = C_G(r')$, $r' \in (Z(C_G(r))^\circ r)^{reg}$ and $v' = v$ so $W_{r'} = W_r$. We set $M := C_G(r)$. If $r \in w\cdot(Z(M')^\circ s)$ for some $w \in W$, then $M \supset C_G(w\cdot(Z(M')^\circ s)) = w\cdot M'$ whence $Z(M)^\circ \subset w\cdot Z(M')^\circ$, and therefore $r' \in Z(M)^\circ r \subset w\cdot(Z(M')^\circ s)$. Hence, $W(\tau, r) = W(\tau, r')$ and so $W(\tau, rv) = W(\tau, r'v)$. The first statement follows from (3.7) and left translation by $r'r^{-1} \in Z(M)^\circ$. The last statement follows from Proposition 2.21 and from the fact that $\overline{J}^{bir}$ is open in $\overline{J}$. $\square$

Corollary 3.10. *Let* $J = J_G(\tau)$, *for* $\tau = (M', Z(M')^\circ s, \mathcal{O}) \in \mathscr{T}(G)$, *let* $rv \in \overline{J} \cap Z(M')^\circ sv$ *and let* $M = C_G(r)$. *Then* $\overline{J}$ *is unibranch, respectively smooth, respectively normal, at* rv *if and only if* $|W_r\backslash W(\tau, rv)/W_\tau| = 1$ *and* $r^{-1}\overline{J_M(\tau)}$ *is so at* v.

Proof. Let U be as in the proof of Proposition 3.7. Then the irreducible components of $U \cap \overline{J}$ containing rv are precisely the subsets $\overline{J_M(w\cdot\tau)} \cap U$ for $w \in W_r\backslash W(\tau, rv)/W_\tau$. Hence, $|W_r\backslash W(\tau, rv)/W_\tau| = 1$ is a necessary condition for $\overline{J}$ being unibranch at rv, and a fortiori, normal, or smooth. In addition, if $|W_r\backslash W(\tau, rv)/W_\tau| = 1$, then $(\overline{J}, rv) \sim_{se} (r^{-1}\overline{J_M(\tau)}, v)$ and we use the properties of smooth equivalence. $\square$

Corollary 3.11. *Let $J = J_G(\tau)$, for $\tau = (M', Z(M')^\circ s, \mathcal{O}) \in \mathscr{T}(G)$, let $rv \in \overline{J} \cap Z(M)^\circ sv$ and let $M = C_G(r)$. Assume $|W_r\backslash W(\tau, rv)/W_\tau| = 1$. Then $\overline{J}$ is Cohen-Macaulay at rv if and only if $r^{-1}\overline{J_M(\tau)}$ is so at v.*

Proof. The argument is the same of Corollary 3.10. $\square$

The local study of the closure of a Jordan class $J = J_G(\tau)$ around rv simplifies drastically when $|W_r\backslash W(\tau, r)/W_\tau| = 1$ and therefore it is important to characterize when this is the case. Next result deals with this question under the assumption that $W_\tau = \mathrm{Stab}_W(Z(M')^\circ s)$, which is always satisfied when $\mathcal{O}$ is characteristic in M, e.g., when $\mathcal{O} = 1$ (semisimple Jordan classes) or when $\mathcal{O}$ is regular (regular Jordan classes), see Remark 1.28.

Lemma 3.12. *Let $J = J_G(\tau)$ for $\tau = (M', Z(M')^\circ s, \mathcal{O}) \in \mathscr{T}(G)$ and let $r \in (Z(M')^\circ s) \cap \overline{J}$. Assume that $W_\tau = \mathrm{Stab}_W(Z(M')^\circ s)$. Then $|W_r\backslash W(\tau, r)/W_\tau| = 1$ if and only if $\overline{J}//G$ is unibranch at the class $[r]$ of r.*

Proof. The isomorphism $G//G \simeq T/W$ identifies $\overline{J}//G$ with $W \cdot (Z(M')^\circ s)/W$, so we need to understand the neighbourhood of $W \cdot (Z(M')^\circ s)/W$ around $[r]$. By [8, Anhang zu K. 7, Satz 21], there is a W_r-stable analytic open neighbourhood U of r in $W \cdot (Z(M')^\circ s)$ such that U/W_r identifies with a neighbourhood of $[r]$ in $W \cdot (Z(M')^\circ s)/W$. We can choose U so that it meets only the W-translates of $Z(M')^\circ s$ containing r. Therefore

$$\begin{aligned}(W \cdot (Z(M')^\circ s)/W, [r]) &\sim_{se} (W \cdot (Z(M')^\circ s) \cap U/W_r, [r]) \\ &\sim_{se} \left(\bigcup_{w \in W(\tau,r)/W_\tau} w \cdot (Z(M')^\circ s)/W_r, [r] \right).\end{aligned}$$

Here, the group W_r acts as usual from the left. This implies that $\overline{J}//G$ is unibranch at $[r]$ if and only if $|W_r\backslash W(\tau, r)/W_\tau| = 1$. $\square$

Example 3.13. By construction $|W_r\backslash W(\tau, rv)/W_\tau| \leq |W_r\backslash W(\tau, r)/W_\tau|$ but the inequality may be strict: here is an example. Let $G = \mathrm{SL}_4(\mathbb{C})$, $M = \langle T, U_{\pm\alpha_1}\rangle$, $\tau = (M, Z(M)^\circ, 1)$. In this case $Z(M) = Z(M)^\circ$ and $W_\tau = \mathrm{Stab}_W(Z(M)) = \langle s_1, s_3\rangle$. Let $rv \in \overline{J_G(\tau)}$ with $C_G(r) = \langle T, U_{\pm\alpha_1}, U_{\pm\alpha_3}\rangle$ and $v \in \mathrm{Ind}_M^{C_G(r)}(1)$. Then $W_r = W_\tau$ and v is trivial in the component corresponding to α_1 and regular in the component corresponding to α_3. If $w \in W$ satisfies $r \in w \cdot Z(M) \neq Z(M)$, then $w \in w_0 W_\tau$. Since $w_0 \notin W_\tau$ we have $|W_r\backslash W(\tau, r)/W_\tau| = 2$. However, if w is as above, then $rv \notin \overline{J_{C_G(r)}(w \cdot \tau)}$. Indeed, if $rv' \in \overline{J_{C_G(r)}(w \cdot \tau)}$, then $v' \in \mathrm{Ind}_{w \cdot M}^{C_G(r)}(1)$ which does not contain $\mathrm{Ind}_M^{C_G(r)}(1)$. Hence $|W_r\backslash W(\tau, rv)/W_\tau| = 1$.

3.4 Applications

In this Section we assume $[G, G]$ is *simply-connected* and we apply the results from §3.2 and 3.3 to deduce geometric properties of closures of regular Jordan classes, sheets, Lusztig strata, birational closures of Jordan classes and birational sheets.

3.4.1 Closures of regular Jordan classes in G

We recall that a Jordan class $J = J_G(M, Z(M)^\circ s, \mathcal{O}^M)$ in G is called regular if $J \subset G^{reg}$, i.e., if $\mathcal{O}^M = \mathcal{O}^M_{reg}$, the regular unipotent class in M.

Theorem 3.14. *Let J be a regular Jordan class in G. Then the following statements are equivalent:*

(i) $\overline{J}$ is normal and Cohen-Macaulay.
(ii) $\overline{J}$ is normal.
(iii) $\overline{J}//G$ is normal.
(iv) $\overline{J}//G$ is smooth.

Proof. Clearly (i) implies (ii) and (ii) implies (iii), see [75, §0.2]. Also, (iii) is equivalent to (iv), by [30, Corollary 8.1]. We show that (iii) implies (i). Let $J = J_G(\tau)$ for $\tau = (M', Z(M')^\circ s, \mathcal{O}^{M'}_{reg}) \in \mathscr{T}(G)$. Recall that $\overline{J}//G = \overline{J_G(M', Z(M')^\circ s, \{1\})}//G$. Let us assume $\overline{J}//G$ is normal. Then it is everywhere unibranch and since the regular unipotent class is characteristic, Lemma 3.12 gives $|W_r \backslash W(\tau, rv)/W_\tau| = 1$ for all points $rv \in \overline{J}$. Since the locus where $\overline{J}$ is not normal (resp. not Cohen-Macaulay) is closed, [94, Tag 00RD] and the geometry of $\overline{J}$ is constant along Jordan classes by Corollary 3.9, it is enough to check the desired properties of $\overline{J}$ at points in closed Jordan classes in $\overline{J}$. These are the Jordan classes $J_G(M, Z(M)^\circ r, \{1\}) \subset \overline{J}$ with M semisimple, i.e., isolated semisimple conjugacy classes in G, see §2.2. Let $\mathcal{O}^G_r$ be such a class, with $M = C_G(r)$. By Corollaries 3.10 and 3.11 and 3.3, $\overline{J}$ is normal and Cohen-Macaulay at r if and only if $\overline{\mathfrak{J}_\mathfrak{m}(\mathfrak{m}', \mathfrak{O}^{M'}_{reg})}$ is so. By [88, Theorem B], this happens if and only if $\mathrm{Stab}_{W_r}(\mathfrak{z}(\mathfrak{m}'))$ acts on $\mathfrak{z}(\mathfrak{m}')$ as a reflection group and $\overline{\mathfrak{J}_\mathfrak{m}(\mathfrak{m}', 0)}//M$ is normal. The first condition is ensured by [30, Proposition 2.5, Lemma 8.3 (i)] applied to $\overline{\mathfrak{J}_\mathfrak{m}(\mathfrak{m}', \{0\})}//M$. The second condition is ensured by [30, Theorem 4.9]. $\square$

Remark 3.15. The fact that normality of $\overline{\mathfrak{J}_\mathfrak{m}(\mathfrak{m}', \{0\})}//M$ implies that $\mathrm{Stab}_{W_r}(\mathfrak{z}(\mathfrak{m}'))$ acts on $\mathfrak{z}(\mathfrak{m}')$ as a reflection group can also be deduced from the proof of [21, Theorem 3.1] or from the main result in [43].

Corollary 3.16. *Let G be simple and let $J = J_G(M, Z(M)^\circ s, \mathcal{O}^M_{reg})$ be a regular Jordan class. Then $\overline{J}$ is smooth if and only if M can be chosen to be either T, semisimple, or of the form M_Θ for $\varnothing \subsetneq \Theta \subsetneq \tilde{\Delta}$ as follows:*

A_n: *of type $d\mathsf{A}_h$ with $n+1 = d(h+1)$, $h \geq 1$, $d \geq 2$;*
B_n: *of type $d\mathsf{A}_h\mathsf{D}_{m_0}\mathsf{B}_{n_0}$ with $n = m_0 + n_0 + d(h+1)$ and either $m_0 \geq 2$, $n_0 \geq 0$, $h \geq 0$, or else $m_0 = 0$, $n_0 \geq 0$, $h = 0$ or odd;*
C_n: *of type $d\mathsf{A}_h\mathsf{C}_{m_0}\mathsf{C}_{n_0}$ with $m_0, n_0, h \geq 0$, $n = m_0 + n_0 + d(h+1)$;*
D_n: *of type $d\mathsf{A}_h\mathsf{D}_{m_0}\mathsf{D}_{n_0}$ with $n = m_0 + n_0 + d(h+1)$ and either $m_0, n_0 \geq 2$ and $h \geq 0$, or else $m_0 n_0 = 0$ and $h = 0$ or odd;*
E_6: *of type A_5 (there are three such subsets), $4\mathsf{A}_1$, $2\mathsf{A}_2$ (there are three such subsets);*

E_7: *of type* E_6, D_6 *(there are two such subsets),* $\mathsf{D}_5\mathsf{A}_1$ *(there are two such subsets),* $2\mathsf{A}_1\mathsf{D}_4$, $2\mathsf{A}_3$, $3\mathsf{A}_2$, $3\mathsf{A}_1\mathsf{A}_3$ *(there are two such subsets),* $5\mathsf{A}_1$, *the two subsets of type* A_5 *containing* α_2, *the subset of type* $4\mathsf{A}_1$ *which is stable under the automorphism of* $\tilde{\Delta}$, $\{\alpha_0, \alpha_2, \alpha_3\}$ *and* $\{\alpha_2, \alpha_5, \alpha_7\}$;

E_8: $\tilde{\Delta} \setminus \{\alpha_1, \alpha_3\}$, $\tilde{\Delta} \setminus \{\alpha_1, \alpha_3, \alpha_6\}$, $\tilde{\Delta} \setminus \{\alpha_4, \alpha_6, \alpha_8\}$, $\{\alpha_2, \alpha_5, \alpha_7, \alpha_0\}$ *or of type* D_7, E_7, $\mathsf{D}_6\mathsf{A}_1$, $2\mathsf{A}_3\mathsf{A}_1$, $3\mathsf{A}_2\mathsf{A}_1$, $2\mathsf{A}_1\mathsf{D}_5$, $\mathsf{D}_4\mathsf{A}_3$, $3\mathsf{A}_2$;

F_4: *of type* A_3, $\mathsf{A}_1\mathsf{B}_2$, $2\mathsf{A}_1\tilde{\mathsf{A}}_1$, B_3, C_3, $2\mathsf{A}_1$, $\tilde{\mathsf{A}}_2$;

G_2: *of type* $\tilde{\mathsf{A}}_1$.

Proof. This follows from Theorem 3.14 and the list of classes J for which $\overline{J}//G$ is normal in [30, Theorem 8.7]. □

3.4.2 Sheets

In this Subsection we apply the local description to the case of sheets. We will apply repeatedly the following argument.

Remark 3.17. Let $S = \overline{J}^{reg}$, with $J = J_G(M, Z(M)^\circ s, \mathcal{O}^M)$ be a sheet in G.

(i) The locus where S is not smooth, respectively normal, is closed. Thus, by Corollary 3.9 it is enough to check smoothness or normality of S at a point in each closed Jordan class in S. These are Jordan classes of triples $(M', Z(M')^\circ s', \mathcal{O}^{M'})$ with M' semisimple and are precisely the conjugacy classes of isolated elements contained in S, see §2.2.

(ii) The conjugacy class $w \cdot \mathcal{O}^M$ is rigid in $w \cdot M$ for any $w \in W$ and therefore (3.8) implies that S in the neighbourhood of an isolated point rv is smoothly equivalent to a union of sheets in the semisimple group $C_G(r)$ in the neighbourhood of v.

(iii) As exp is compatible with induction, it maps rigid nilpotent orbits in $\mathfrak{g}$ to rigid unipotent conjugacy classes in G. Hence, it identifies a neighbourhood of v in a sheet in $C_G(r)$ with a neighbourhood of a nilpotent element in a sheet of $\mathfrak{c}_{\mathfrak{g}}(r)$.

Theorem 3.18. *Let* Φ *be classical and let* $S = \overline{J(\tau)}^{reg}$ *be a sheet in* G. *Then* S *is smooth if and only if it is normal if and only if it is unibranch.*

Proof. One direction is immediate. Assume S is unibranch: we prove that it is smooth. Let $\tau = (M, Z(M)^\circ s, \mathcal{O}^M)$ and $\mathcal{O}^M = \exp \mathfrak{O}^M$. By Corollary 3.10 we have $|W_r \backslash W(\tau, rv)/W_\tau| = 1$ for any point $rv \in S$. Hence (3.8) and Corollary 3.3 imply that S is smooth at rv if and only if $\overline{\mathfrak{J}_{\mathfrak{c}_{\mathfrak{g}}(r)}(\mathfrak{m}, \mathfrak{O}^M)}^{reg}$ is smooth. By Remark 3.17 part 1, it suffices to prove smoothness of S at isolated classes. In this case $\mathfrak{c}_{\mathfrak{g}}(r)$ is semisimple and classical because its Dynkin diagram is a sub-diagram of the extended Dynkin diagram of $\mathfrak{g}$. In addition, $\overline{\mathfrak{J}_{\mathfrak{c}_{\mathfrak{g}}(r)}(\mathfrak{m}, \mathfrak{O}^M)}^{reg}$ is a sheet in $\mathfrak{c}_{\mathfrak{g}}(r)$ by Remark 3.17, part 2. Since all sheets in classical Lie algebras are smooth [10, 55, 83], we have the statement. □

Theorem 3.19. *Let* Φ *be classical and irreducible and let* $S = \overline{J(\tau)}^{reg}$ *be a sheet in* G. *If* $\overline{J}//G$ *is normal in codimension* 1, *then* S *is smooth.*

Proof. By Theorem 3.18 it is enough to show that S is unibranch at every isolated $rv \in S$. Let $\tau = (M, Z(M)^\circ s, \mathcal{O}^M)$. If $\overline{J(\tau)}//G$ is normal in codimension 1, then it is unibranch by [30, Lemma 8.2, Lemma 8.3]. By [31, Lemma 3.3] if G is simple and simply-connected and $\overline{J(\tau)}^{reg}$ is a sheet we always have $W_\tau = \mathrm{Stab}_W(Z(M)^\circ s)$, so Lemma 3.12 applies. □

Corollary 3.20. *Let G be simple with Φ classical and assume M is either: T, semisimple, or M_Θ for $\varnothing \subsetneq \Theta \subsetneq \tilde{\Delta}$ as follows:*

- A_n*: of type $d\mathsf{A}_h$ with $n+1 = d(h+1)$, $h \geq 1$, $d \geq 2$;*
- B_n*: of type $d\mathsf{A}_h\mathsf{D}_{m_0}\mathsf{B}_{n_0}$ with $n = m_0 + n_0 + d(h+1)$ and either $m_0 \geq 2$, $n_0 \geq 0$, $h \geq 0$, or else $m_0 = 0$, $n_0 \geq 0$, $h = 0$ or odd;*
- C_n*: of type $d\mathsf{A}_h\mathsf{C}_{m_0}\mathsf{C}_{n_0}$ with $m_0, n_0, h \geq 0$, $n = m_0 + n_0 + d(h+1)$;*
- D_n*: of type $d\mathsf{A}_h\mathsf{D}_{m_0}\mathsf{D}_{n_0}$ with $n = m_0 + n_0 + d(h+1)$ and either $m_0, n_0 \geq 2$ and $h \geq 0$, or else $m_0 n_0 = 0$ and $h = 0$ or odd.*

Then, for any $Z(M)^\circ s$ satisfying (RP) *for M and any rigid unipotent class $\mathcal{O}^M$ in M, the sheet $S = \overline{J_G(M, Z(M)^\circ s, \mathcal{O}^M)}^{reg}$ is smooth.*

Proof. The result follows from Theorem 3.19 and the list of varieties $\overline{J}//G$ that are normal in codimension 1 for G simple, [30, Proposition 8.6]. □

For G simple exceptional, the subset of sheets S in G such that quotient $\overline{S}//G$ is normal in codimension 1 can be deduced from [30, Proposition 8.6]. For each S in this subset, our previous results we can compute whether S is smooth or not: we record this in the following

Corollary 3.21. *Let G be simple with Φ exceptional. Let M be either semisimple, T, or M_Θ for $\varnothing \subsetneq \Theta \subsetneq \tilde{\Delta}$ of the following type:*

- E_6*:* A_5, D_4, $4\mathsf{A}_1$, $2\mathsf{A}_2$;
- E_7*:* E_6, D_6, $2\mathsf{A}_1\mathsf{D}_4$, $3\mathsf{A}_2$, $2\mathsf{A}_3$, $3\mathsf{A}_1\mathsf{A}_3$, $\mathsf{D}_4\mathsf{A}_1$, $5\mathsf{A}_1$, $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, D_4, $\{\alpha_0, \alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_5, \alpha_7\}$, $\{\alpha_0, \alpha_3, \alpha_5, \alpha_7\}$;
- E_8*:* $\tilde{\Delta} \setminus \{\alpha_1, \alpha_3\}$, $\tilde{\Delta} \setminus \{\alpha_1, \alpha_3, \alpha_6\}$, $\tilde{\Delta} \setminus \{\alpha_4, \alpha_6, \alpha_8\}$, $\{\alpha_2, \alpha_5, \alpha_7, \alpha_0\}$, D_7, E_7, $\mathsf{D}_6\mathsf{A}_1$, $2\mathsf{A}_3\mathsf{A}_1$, $3\mathsf{A}_2\mathsf{A}_1$, $2\mathsf{A}_1\mathsf{D}_5$, $\mathsf{D}_4\mathsf{A}_3$, D_6, E_6, $2\mathsf{A}_1\mathsf{D}_4$, $3\mathsf{A}_2$, D_4;
- F_4*:* A_3, $\mathsf{A}_1\mathsf{B}_2$, $2\mathsf{A}_1\tilde{\mathsf{A}}_1$, B_3, C_3, $2\mathsf{A}_1$, $\tilde{\mathsf{A}}_2$, B_2;
- G_2*:* $\tilde{\mathsf{A}}_1$;

Consider $(M, Z(M)^\circ s, \mathcal{O}^M) \in \mathscr{D}(G)$ with $\mathcal{O}^M$ rigid. Then the sheet $S = \overline{J_G(M, Z(M)^\circ s, \mathcal{O}^M)}^{reg}$ is smooth, with the following exceptions:

type of G	M	$Z(M)^\circ s$	$\mathcal{O}^M$
E_7	D_6	any	$[2^4,1^4]$
E_8	D_6	any	$[2^4,1^4]$
	E_7	any	$2A_1$ and $(A_1+A_3)a$
	$\mathsf{D}_6\mathsf{A}_1$	any	$[2^4,1^4]\times[1^4]$
F_4	B_2	any	$[1^5]$

Table 3.1

Proof. If $M=T$ or M is semisimple, then $S=G^{reg}$ or S coincides with a single conjugacy class and there is nothing to prove. Let $M=M_\Theta$ with Θ from the above list. We apply Remark 3.17 and we look at S in the neighbourhood of isolated elements rv. For all Θ the quotient $\overline{S}//G$ is normal in codimension 1, [30, Proposition 8.6], hence it is unibranch. In addition, [31, Lemma 3.3] ensures that $W_\tau=\mathrm{Stab}_W(Z(M)^\circ s)$ for any choice of $Z(M)^\circ s$. By Lemma 3.12, Corollary 3.10 and [55] the problem is reduced to showing that $\overline{\mathfrak{J}_{\mathfrak{m}'}(\mathfrak{m},\mathfrak{O}^M)}^{reg}$ is smooth for $\mathcal{O}^M=\exp\mathfrak{O}^M$ and any $\mathfrak{m}'=\mathfrak{c}_{\mathfrak{g}}(r)$ semisimple exceptional containing $\mathfrak{m}$. Such Lie subalgebras are conjugate to $\mathfrak{m}_{\Theta'}:=\mathrm{Lie}(M_{\Theta'})$ for some $\Theta'\subset\tilde{\Delta}$ with $|\Theta'|=|\Delta|$. Let $W_{\Theta'}$ be the Weyl group of $M_{\Theta'}$, then $\mathfrak{m}$ is $W_{\Theta'}$-conjugate to a standard Levi subalgebra therein, [29, Lemma 4.9]. However, normality in codimension 1 of $\overline{J}//G$ is equivalent to the condition $\{w\Theta\in\mathscr{P}(\Phi)\mid w\in W,\, w\Theta\subset\tilde{\Delta}\}=\{\Theta\}$, where $\mathscr{P}(\Phi)$ denotes the power set of Φ. Therefore we are left to verify smoothness of the sheets $\overline{\mathfrak{J}_{\mathfrak{m}_{\Theta'}}(\mathfrak{m}_\Theta,\mathfrak{O})}^{reg}$ for all exceptional $\Theta'\supset\Theta$ with $|\Theta'|=|\Delta|$. This is done by using the list in [26, §4] of smooth and singular sheets in simple exceptional Lie algebras on each simple component of $\mathfrak{m}_{\Theta'}$. □

3.4.3 Sheets and Lusztig strata in $\mathrm{SL}_n(\mathbb{C})$

The case in which $G=\mathrm{SL}_n(\mathbb{C})$ is particularly simple and we retrieve information on all its sheets and, as a consequence, on all Lusztig strata as defined in [70, §2], see also [70, §3.2,3.3].

Proposition 3.22. *Let $G=\mathrm{SL}_n(\mathbb{C})$. Then every sheet and Lusztig stratum in G is smooth.*

Proof. Let S be a sheet in $G=\mathrm{SL}_n(\mathbb{C})$. By Remark 3.17 (i), it suffices to prove smoothness at its isolated classes. These are all of the form zv with $z\in Z(G)$ and v unipotent, hence $(S,zv)\sim_{se}(z^{-1}S,v)\sim_{se}(\mathfrak{S},\exp^{-1}(v))$ where $\mathfrak{S}$ is a sheet in $\mathfrak{sl}_n$ by Corollary 3.3 and Remark 3.17 (iii). All sheets in $\mathfrak{sl}_n$ are smooth by [10,83]. Hence S is smooth.

We turn now to Lusztig strata. It follows from [28, §2] that their irreducible components are sheets in G. Strata of $G=\mathrm{SL}_n(\mathbb{C})$ are of the form $X_S=\bigcup_{k\in Z(G)}kS$ for $S=\overline{J_G(L,Z(L)^\circ,1)}^{reg}$ a given (birational) sheet. By Corollary 2.31, the stratum X_S can be expressed as a non-redundant, disjoint union of (birational) sheets. Hence the connected components of strata are (birational) sheets in G, which are smooth by (i), so strata are smooth as well. □

3.4.4 Birational closures of Jordan classes and birational sheets

Suppose now that $rv \in \overline{J}^{bir}$ where $J := J_G(su) \in \mathscr{J}(G)$ and set $M := C_G(r)$, $C := C_G(s)$ and $Z := Z(C)$. We will prove that the two pointed spaces $(\overline{J}^{bir}, rv)$ and $(\bigcup_{I_{J,rv}} \overline{J}_{M,i}^{wbir}, rv)$ are smoothly equivalent, where $I_{J,rv}$ is defined as in (3.2).

Lemma 3.23. *Let $su \in G$ with $s \in T$ and $J_G(su) =: J \in \mathscr{J}(G)$ and let $rv \in \overline{J}$. Then $rv \in \overline{J}^{bir}$ if and only if $rv \in \overline{J}_{M,i}^{wbir}$ for all $i \in I_{J,rv}$. In this case,*

$$(\overline{J}^{bir}, rv) \sim_{se} \left(\bigcup_{I_{J,rv}} r^{-1}\overline{J}_{M,i}^{wbir}, v \right). \tag{3.9}$$

Proof. We can assume $r \in Z^\circ s$. Denote by

$$\mathscr{D}_{J,rv} := \{(C_i, Z(C_i)^\circ z_i, \mathcal{O}^{C_i}) \in \mathscr{D}(M) \mid r \in Z(C_i)^\circ z_i \text{ and } \mathcal{O}_v^M \subset \overline{\operatorname{Ind}_{C_i}^M \mathcal{O}^{C_i}}\}.$$

If $J_{M,i} \in \mathcal{J}_{J,rv}$ as in (3.2), there exists $\tau_i \in \mathscr{D}_{J,rv}$ such that $J(\tau_i) = J_{M,i}$, by Theorem 3.8. Note that $\tau := (C, Z^\circ s, \mathcal{O}_u^C) \in \mathscr{D}_{J,rv}$. By Lemma 0.2, since $r \in Z(C_i)^\circ z_i$, we have that C_i is a Levi subgroup of M. Hence, by [29, Lemma 3.10] C_i is a pseudo-Levi of G, so that $\tau_i \in \mathscr{D}(G)$. By definition of $\mathcal{J}_{J,rv}$, we have that $J_G(\tau_i) = J$ for every $\tau_i \in \mathscr{D}_{J,rv}$. Suppose that $rv \in \overline{J}^{bir}$. Then, $rv \in \overline{J_G(\tau_i)}^{bir}$ for all $\tau_i \in \mathscr{D}_{J,rv}$, i.e.,

$$G \cdot (r\,\mathcal{O}_v^M) \subset \bigcup_{z \in \operatorname{Bir}(Z(C_i)^\circ z_i, \mathcal{O}^{C_i})} G \cdot (z \operatorname{Ind}_{C_i}^{C_G(z)} \mathcal{O}^{C_i}).$$

Since $[G,G]$ is simply-connected and $r \in Z^\circ s$, this is equivalent to $r\,\mathcal{O}_v^M = \mathcal{O}_{rv}^M$ being birationally induced (equiv. weakly birationally induced, by Remark 1.18) from $(C_i, r, \mathcal{O}^{C_i}) \in \mathscr{B}(M)$ for all $\tau_i \in \mathscr{D}_{J,rv}$, i.e., $rv \in \overline{J_M(\tau_i)}^{wbir}$, for all $\tau_i \in \mathscr{D}_{J,rv}$. For the other implication, observe that $J(\tau) \in \mathcal{J}_{J,rv}$, and if $\mathcal{O}_{rv}^M \subset \overline{J(\tau)}^{wbir}$, then $\mathcal{O}_{rv}^G = G \cdot (\mathcal{O}_{rv}^M) \subset \overline{J}^{bir}$. This proves the first assertion.

For the last part of the statement, let $rv \in \overline{J}^{bir} \subset \overline{J}^{reg}$. Observe that $r \in Z(M)$ implies $r^{-1}\overline{J}_{M,i}^{wbir} = \overline{r^{-1}J_{M,i}}^{wbir}$. By Proposition 3.7, $(\overline{J}^{reg}, rv) \sim_{se} (\bigcup_{I_{J,rv}} r^{-1}\overline{J}_{M,i}^{reg}, v)$. The set $\overline{J}^{bir}$ is open in $\overline{J}^{reg}$, so $(\overline{J}^{reg}, rv) \sim_{se} (\overline{J}^{bir}, rv)$; similarly each $\overline{J}_{M,i}^{wbir}$ is open in $\overline{J}_{M,i}^{reg}$, hence $(\bigcup_{I_{J,rv}} \overline{J}_{M,i}^{reg}, rv) \sim_{se} (\bigcup_{I_{J,rv}} \overline{J}_{M,i}^{wbir}, rv)$. Transitivity allows to conclude. □

We prove that the local study of a birational sheet in G can be reduced to the study of a neighbourhood of a unipotent element in some connected reductive subgroup of G.

Theorem 3.24. *Let $\tau = (C, Z^\circ s, \mathcal{O}^C) \in \mathscr{B}\mathscr{B}(G)$. For $rv \in \overline{J_G(\tau)}^{bir}$ with $r \in Z^\circ s$, we have*

$$\left(\overline{J_G(\tau)}^{bir}, rv\right) \sim_{se} \left(r^{-1}\overline{J_{C_G(r)}(\tau)}^{wbir}, v\right).$$

Proof. The statement follows from Lemma 3.23, provided that $|I_{J,rv}| = 1$ if $\overline{J}^{bir}$ is a birational sheet, where $I_{J,rv}$ is as in (3.2). Since $\mathcal{O}^C$ is birationally rigid in C, also $\mathcal{O}^{C_i}$ is birationally rigid in C_i for all $i \in I_{J,rv}$, as $(C, \mathcal{O}^C)$ and $(C_i, \mathcal{O}^{C_i})$ are conjugate in G (see Remark 1.24). We set $M := C_G(r)$. Thus each $\overline{J_{M,i}}^{wbir}$ in (3.9) is a weakly birational sheet of M containing rv. By Theorem 2.25, M is partitioned into its weakly birational sheets, hence the set $\mathcal{J}_{J,rv}$ as in (3.2) only contains $J_M(\tau)$. $\square$

The following result compares the local structures of a birational sheet of G and of a birational sheet in a reductive Lie subalgebra of $\mathfrak{g}$.

Theorem 3.25. *Let $\tau = (C, Z^\circ s, \mathcal{O}^C) \in \mathscr{B}\mathscr{B}(G)$ and let $rv \in \overline{J_G(\tau)}^{bir}$ with $r \in Z^\circ s$. Then*

$$(\overline{J_G(\tau)}^{bir}, rv) \sim_{se} (\overline{\mathfrak{J}_{\mathfrak{c}_\mathfrak{g}(r)}(\mathfrak{c}, \mathfrak{O}^C)}^{bir}, \nu)$$

where $\mathfrak{c} := \mathrm{Lie}(C)$, the orbit $\mathfrak{O}^C \in \mathcal{N}_\mathfrak{c}/C$ satisfies $\exp \mathfrak{O}^C = \mathcal{O}^C$, the set $\overline{\mathfrak{J}_{\mathfrak{c}_\mathfrak{g}(r)}(\mathfrak{c}, \mathfrak{O}^C)}^{bir}$ is a birational sheet in $\mathfrak{c}_\mathfrak{g}(r)$ and $\nu \in \mathcal{N}_{\mathfrak{c}_\mathfrak{g}(r)}$ is such that $\exp \nu = v$.

Proof. This follows from Theorem 3.24, Corollary 3.3 and the fact that $\mathcal{O}^C \in \mathcal{U}_C/C$ is birationally rigid if and only if $\mathfrak{O}^C \in \mathcal{N}_\mathfrak{c}/C$ is so (Lemma 1.20). $\square$

We conclude this Chapter with some applications of Theorem 3.25 and Losev's results on the geometry of birational sheets in $\mathfrak{g}$ in [67, §4]. We will first need the following fact from Algebraic Geometry.

Lemma 3.26. *Let X and Y be complex algebraic varieties with $\dim X = \dim Y + d$. Let X and Y be unibranch at $x \in X$ and at $y \in Y$, respectively. Suppose $(X, x) \sim_{se} (Y, y)$. Let $\psi_X : \widetilde{X} \to X$ and $\psi_Y : \widetilde{Y} \to Y$ be the normalizations of X and Y, respectively. Let $\tilde{x} \in \widetilde{X}$ and $\tilde{y} \in \widetilde{Y}$ with $\psi_X(\tilde{x}) = x$ and $\psi_Y(\tilde{y}) = y$, respectively. Then $(\widetilde{X}, \tilde{x}) \sim_{se} (\widetilde{Y}, \tilde{y})$.*

Proof. By assumption, (X^{an}, x) and $(Y^{an} \times \mathbb{A}^d, (y, 0))$ are locally isomorphic as analytic pointed spaces. Let $\widetilde{X^{an}}$ (resp. $\widetilde{Y^{an}}$) be the normalization of X^{an} (resp. of Y^{an}). By [63, §5, Satz 4], we have $\widetilde{X^{an}} = \widetilde{X}^{an}$ and $\widetilde{Y^{an}} = \widetilde{Y}^{an}$. Thus, $(\widetilde{X}^{an}, \tilde{x})$ is the analytic normalization of (X, x) and $(\widetilde{Y}^{an} \times \mathbb{A}^d, (\tilde{y}, 0))$ is the analytic normalization of $(Y \times \mathbb{A}^d, (y, 0))$. Hence, $(\widetilde{X}^{an}, \tilde{x})$ and $(\widetilde{Y}^{an} \times \mathbb{A}^d, (\tilde{y}, 0))$ are locally analytically isomorphic and this concludes the proof. $\square$

Theorem 3.27. *Let $\tau = (C, Z^\circ s, \mathcal{O}^C) \in \mathscr{B}\mathscr{B}(G)$. Then:*

(i) $\overline{J_G(\tau)}^{bir}$ is unibranch;

(ii) the normalization of $\overline{J_G(\tau)}^{bir}$ is smooth.

Proof. Retain notation from Theorem 3.25, and set $J := J_G(\tau)$ and $\mathfrak{J} := \mathfrak{J}_{\mathfrak{c}_\mathfrak{g}(r)}(\mathfrak{c}, \mathfrak{O}^C)$.

(i) Let $rv \in \overline{J}^{bir}$. By Theorem 3.25, we have $(\overline{J}^{bir}, rv) \sim_{se} (\overline{\mathfrak{J}}^{bir}, \nu)$. By Remark 1.24, the orbit $\mathfrak{O}^C$ is birationally rigid and the statement follows from [67, Theorem 4.4 (2)] applied to the birational sheet $\overline{\mathfrak{J}}^{bir}$ in the reductive Lie algebra $\mathfrak{c}_\mathfrak{g}(r)$.

(ii) By (i), we have that $(\overline{J}^{bir}, rv) \sim_{se} (\overline{\mathfrak{J}}^{bir}, \nu)$ are both unibranch. Let $\psi_J : \widetilde{J} \to \overline{J}^{bir}$ and $\psi_\mathfrak{J} : \widetilde{\mathfrak{J}} \to \overline{\mathfrak{J}}^{bir}$ denote the normalization maps of $\overline{J}^{bir}$ and $\overline{\mathfrak{J}}^{bir}$, respectively. Then $(\widetilde{J}, \psi_J^{-1}(rv)) \sim_{se} (\widetilde{\mathfrak{J}}, \psi_\mathfrak{J}^{-1}(\nu))$ by Lemma 3.26 and we conclude by [67, Theorem 4.4 (2)]. $\square$

Theorem 3.28. *Suppose that the simple factors of G have classical root system. Then the birational sheets of G are smooth.*

Proof. We prove that for $\tau = (C, Z^\circ s, \mathcal{O}^C) \in \mathscr{B}\mathscr{B}(G)$ and $rv \in \overline{J(\tau)}^{bir}$ with $r \in Z^\circ s$, $\overline{J(\tau)}^{bir}$ is smooth at rv. By Theorem 3.25, from which we retain notation, we have $(\overline{J(\tau)}^{bir}, rv) \sim_{se} (\overline{\mathfrak{J}_{\mathfrak{c}_\mathfrak{g}(r)}(\mathfrak{c}, \mathfrak{O}^C)}^{bir}, \nu)$. The algebra $\mathfrak{c}_\mathfrak{g}(r)$ is reductive, hence $\overline{\mathfrak{J}_{\mathfrak{c}_\mathfrak{g}(r)}(\mathfrak{c}, \mathfrak{O}^C)}^{bir}$ decomposes as a product of a vector space and some birational sheets in simple classical Lie subalgebras of $[\mathfrak{c}_\mathfrak{g}(r), \mathfrak{c}_\mathfrak{g}(r)]$, and all such objects are smooth by [67, Remark 4.10]. $\square$

CHAPTER
FOUR

RINGS OF REGULAR FUNCTIONS OF ORBITS ALONG BIRATIONAL SHEETS

An important numerical invariant attached to each adjoint orbit $\mathfrak{O} \subset \mathfrak{g}$ is given by the multiplicities with which each irreducible finite-dimensional G-module M occurs in $\mathbb{C}[\mathfrak{O}]$; the aim is to characterize subsets of $\mathfrak{g}$ consisting of orbits parameterized by the same invariant. In Section 4.1 we introduce the necessary notation and results. We then give a brief historical overview about the problem of multiplicities in Section 4.2: we recall the first formulation by Dixmier [41] for the case of $\mathfrak{sl}_n(\mathbb{C})$ and we illustrate results obtained by Borho and Kraft in [13]: for $\mathfrak{sl}_n(\mathbb{C})$, the sheets of $\mathfrak{g}$ coincide with the sets of orbits whose rings of regular functions are isomorphic as G-modules. We introduce the similar problem in the framework of conjugacy classes of a simply-connected group: for $\mathrm{SL}_n(\mathbb{C})$ we can prove that the situation is analogous to its Lie algebra. The generalization of Dixmier's problem to $\mathfrak{g}$ simple of type different from A has been addressed more recently by Losev, [67]. As already mentioned in the Introduction, Losev has proven that multiplicities of orbits are preserved along birational sheets (which coincide with sheets for $\mathfrak{sl}_n(\mathbb{C})$) and he has conjectured that this invariant characterizes birational sheets, [67]. After an exposition of Losev's contributions, we conclude with the original content of this Chapter, contained in Section 4.3. For G simple simply-connected, restricting to the subset of spherical conjugacy classes, we can prove group analogues to Losev's result and conjecture. Finally, some remarks on Losev's conjecture for the subset of spherical adjoint orbits are collected in Section 4.4.

4.1 Notations and basic results

In this Chapter, unadorned tensor products should be read as tensor products over $\mathbb{C}$. If M_1 and M_2 are G-modules, we write $M_1 \simeq_G M_2$ to denote that they are isomorphic as G-modules. The set of isomorphism classes of finite-dimensional irreducible G-modules is denoted by $\mathrm{Irr}\, G$ and its elements are parameterized by $X(T)^+$, the set of dominant weights of T with respect to Φ^+. We write $V(\lambda)$ for the irreducible G-module of highest weight λ. We denote by P the lattice of integral weights of $\mathfrak{g}$, by $\mathtt{P}^+$ the monoid of dominant weights. We denote the fundamental weights with $\omega_1, \ldots, \omega_n$ and the fundamental co-weights with $\check{\omega}_1, \ldots, \check{\omega}_n$: these are the elements $\check{\omega}_j \in \mathfrak{h}$ defined by $\alpha_i(\check{\omega}_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. For G semisimple simply-connected, it is well-known that $X(T)$ identifies with $\mathtt{P}$ and $X(T)^+$ identifies with $\mathtt{P}^+$.

By Peter-Weyl Theorem, any G-module M admits a canonical decomposition

$$M \simeq_G \bigoplus_{V \in \operatorname{Irr} G} \operatorname{Hom}_G(V, M) \otimes V \simeq_G \bigoplus_{V \in \operatorname{Irr} G} (M \otimes V^*)^G \otimes V,$$

where $\operatorname{Hom}_G(V, M)$ denotes the set of G-equivariant linear maps $V \to M$. For $V \in \operatorname{Irr} G$ and a G-module M, we define the *multiplicity of M in V* as the integer $[M : V] \coloneqq \dim \operatorname{Hom}_G(V, M)$. This yields the following decomposition for $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ if $H \leq G$:

$$\mathbb{C}[G/H] \simeq_G \bigoplus_{V \in \operatorname{Irr} G} (\mathbb{C}[G]^H \otimes V^*)^G \otimes V \simeq_G \bigoplus_{V \in \operatorname{Irr} G} (V^*)^H \otimes V \simeq_G \bigoplus_{V \in \operatorname{Irr} G} n_{V,H} V, \tag{4.1}$$

where $n_{V,H} = [\mathbb{C}[G/H] : V]$. Let X be a G-variety and let $x \in X$; since the ground field is $\mathbb{C}$, the orbit map $G \to X$ defined by $g \mapsto g \cdot x$ induces an isomorphism of varieties $G/G_x \xrightarrow{\sim} \mathcal{O}_x$ and we have $\mathbb{C}[\mathcal{O}_x] \simeq_G \mathbb{C}[G/G_x] = \mathbb{C}[G]^{G_x}$. Then (4.1) applies with $H = G_x$.

A transitive G-space X is called *spherical* if a Borel subgroup of G has a dense orbit in it. Now suppose X is a conjugacy class of G or a G-adjoint orbit of $\mathfrak{g}$: in this case X is spherical if and only if $\mathbb{C}[X]$ is *multiplicity-free*, i.e., $n_{V,G_x} \in \{0, 1\}$ for all $V \in \operatorname{Irr}(G)$, for $x \in X$, [20, 98]. We denote by $\Lambda(X)$ the monoid of dominant weights occurring in $\mathbb{C}[X]$, in particular we have $\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda(X)} V(\lambda)$. We will describe the weight monoid $\Lambda(X)$ by means of a generic element contained in it, i.e., $\sum_{i=1}^n n_i \omega_i$, where $n_i \in \mathbb{N}$ and ω_i is the i-th fundamental dominant weight. A subgroup $H \leq G$ is said to be *spherical* if the homogeneous space G/H is a spherical variety.

We denote by G_{sph} (resp. $\mathfrak{g}_{sph}$) the union of all spherical conjugacy classes in G (resp. spherical adjoint orbits in $\mathfrak{g}$). By [6, Corollary 2], the subset G_{sph} (resp. $\mathfrak{g}_{sph}$) is closed in G (resp. $\mathfrak{g}_{sph}$), in particular it is a subvariety.

4.2 The ring of regular functions as an invariant

One of the main motivations which led to the study of sheets of $\mathfrak{g}$ in [13] was Dixmier's multiplicity conjecture, left open from [41] which we now state in our notation. Let $G = \mathrm{PSL}_n(\mathbb{C})$, and let $\mathbf{d} \vdash n$. Consider the nilpotent orbit in $\mathfrak{g}$ attached to it, $\mathfrak{O}_{\mathbf{d}}$, and the standard Levi subgroup $L_{\mathbf{d}^t}$ defined in (3). For $V \in \operatorname{Irr} G$, is it true that $[\mathbb{C}[\overline{\mathfrak{O}_{\mathbf{d}}}] : V] = \dim(V^*)^{L_{\mathbf{d}^t}}$?

The answer to this question is positive: it was obtained by Borho and Kraft in [13], as a corollary of a more general result, which we sum up in the following statement, in a form adapted to our notation.

Theorem 4.1 (Borho–Kraft [13, Theorems 3.8 and 6.3])**.** *Let G be simple and adjoint. Let $\mathfrak{S}$ be a sheet of $\mathfrak{g}$ and let $\mathfrak{O}$ be the unique nilpotent orbit in $\mathfrak{S}$, let $\nu \in \mathfrak{O}$. If $\overline{\mathfrak{O}}$ is normal and $C_G(\nu)$ is connected, then $\mathbb{C}[\overline{\mathfrak{O}}] \simeq_G \mathbb{C}[\mathfrak{O}] \simeq_G \mathbb{C}[\mathfrak{O}'] \simeq_G \mathbb{C}[\overline{\mathfrak{O}'}]$ for all orbits $\mathfrak{O}'$ in $\mathfrak{S}$.*

Let us now return to the setting of $G = \mathrm{PSL}_n(\mathbb{C})$. Recall that sheets of $\mathfrak{sl}_n(\mathbb{C})$ are disjoint and are parameterized by the unique nilpotent orbit which they contain: this was first recorded in [40]. Moreover, the assumptions of Theorem 4.1 hold, since $C_G(\nu)$ is connected for all $\nu \in \mathcal{N}$ and $\overline{\mathfrak{O}}$ is normal for each $\mathfrak{O} \in \mathcal{N}/G$: normality of closures of nilpotent orbits of $\mathfrak{sl}_n(\mathbb{C})$ was established by Kraft and Procesi in [61]. Therefore, Borho and Kraft could conclude in [13, *Nachtrag bei der Korrektur*] that the G-module structure of the algebras of regular functions on adjoint orbits is preserved along sheets of $\mathfrak{sl}_n(\mathbb{C})$. In particular, Dixmier's multiplicity conjecture is true, since for every $\mathbf{d} \vdash n$, the nilpotent orbit $\mathfrak{O}_{\mathbf{d}}$ lies in the same sheet of the orbits of semisimple elements centralized by $L_{\mathbf{d}^t}$.

The natural following question is: do multiplicities "separate" distinct nilpotent orbits, hence distinct sheets, in $\mathfrak{sl}_n(\mathbb{C})$? The answer is again positive, as recorded in the following statement. We thank Eric Sommers for suggesting the use of small modules in the proof; recall that a G-module $V(\lambda) \in \operatorname{Irr} G$ is said to be *small* if twice a root never occurs as a weight.

Proposition 4.2. *Let G be simple with $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two distinct nilpotent orbits in $\mathfrak{g}$, then $\mathbb{C}[\mathfrak{O}_1] \not\simeq_G \mathbb{C}[\mathfrak{O}_2]$.*

Proof. Let $\mathbb{C}$ denote the trivial representation of a group, let $\lambda \in \mathtt{P}^+$ and let $V(\lambda)_0 = V(\lambda)^T$ be the zero weight subspace in $V(\lambda)$: then $V(\lambda)_0$ is a W-module (in general, reducible). Let $L \leq G$ be a Levi subgroup with Weyl group W_L. By [21, Proof of Corollary 1], if $V(\lambda)$ is small, we have $V(\lambda)^L = (V(\lambda)^T)^{W_L} = (V(\lambda)_0)^{W_L}$ and, by Frobenius reciprocity, $\dim(V(\lambda)_0)^{W_L} = [\operatorname{Ind}_{W_L}^W \mathbb{C} : V(\lambda)_0]$. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, for every irreducible W-module M there exists $\lambda \in \mathtt{P}^+ \cap \mathbb{Z}\Phi$ such that $V(\lambda)$ is small and $V(\lambda)_0 \simeq_W M$, see [21, example p. 389]. Let $\mathbf{d} = [d_1, \ldots, d_k] \vdash n$, then the Weyl group $W_{\mathbf{d}}$ of $L_{\mathbf{d}}$ is isomorphic to the direct product of the symmetric groups $\operatorname{Sym}(d_i)$, for $i = 1, \ldots k$. The Richardson orbit $\operatorname{Ind}_{L_{\mathbf{d}}}^G \{0\} = \mathfrak{O}_{\mathbf{d}^t}$ lies in the same sheet containing the semisimple elements centralized by $L_{\mathbf{d}}$. By Theorem 4.1 and [59, §2.1], $[\mathbb{C}[\mathfrak{O}_{\mathbf{d}^t}] : V(\lambda)] = \dim V(\lambda)^{L_{\mathbf{d}}}$ for every simple G-module $V(\lambda)$. If we denote by $V_{\mathbf{d}}$ the Specht module corresponding to $\mathbf{d}$, then:

$$\operatorname{Ind}_{W_{\mathbf{d}}}^W \mathbb{C} = V_{\mathbf{d}} \oplus \bigoplus_{\mathbf{f} > \mathbf{d}} K_{\mathbf{f}\,\mathbf{d}} V_{\mathbf{f}}$$

where the coefficients $K_{\mathbf{f}\,\mathbf{d}}$ are the Kostka numbers and $<$ is the lexicographic total order on partitions of n, see [46, Corollary 4.39]. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be different partitions of n, we may assume $\mathbf{d}_1 > \mathbf{d}_2$. Then there exists a small simple $\mathfrak{g}$-module $V(\lambda)$ such that $V_{\mathbf{d}_2} \simeq_W V(\lambda)_0$. Then $[\mathbb{C}[\mathfrak{O}_{\mathbf{d}_2}] : V(\lambda)] = 1 \neq 0 = [\mathbb{C}[\mathfrak{O}_{\mathbf{d}_1}] : V(\lambda)]$, and we conclude. □

The above arguments can be gathered in the following result, which illustrates the situation for $\mathfrak{sl}_n(\mathbb{C})$, where sheets coincide with birational sheets.

Proposition 4.3. *Let G be simple with $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then $\mathfrak{O}_1$ and $\mathfrak{O}_2 \in \mathfrak{g}/G$ are in the same (birational) sheet of $\mathfrak{g}$ if and only if $\mathbb{C}[\mathfrak{O}_1] \simeq_G \mathbb{C}[\mathfrak{O}_2]$.* □

We now consider the group case: for a connected reductive group G, what can be said on the behaviour of the G-module structure of $\mathbb{C}[\mathcal{O}]$ as the conjugacy class $\mathcal{O}$ varies along the sheets of

G? We begin with some easy observations. In [23, §3.7], it is proven that all adjoint orbits in the same Jordan class of $\mathfrak{g}$ are isomorphic as G-homogeneous spaces: in particular their rings of regular functions are isomorphic as G-modules. In the group case, one needs to take into account central elements, as the following remark shows.

Remark 4.4. Let $su \in G$ and $z \in Z(G)$. Then $\mathcal{O}_{su}$ and $\mathcal{O}_{zsu} = z\,\mathcal{O}_{su}$ are isomorphic G-homogeneous spaces: in fact, $C_G(su) = C_G(zsu)$ so that $\mathbb{C}[\mathcal{O}_{su}] \simeq_G \mathbb{C}[\mathcal{O}_{zsu}]$.

Proposition 4.5. *Let $[G,G]$ be simply-connected and let $J \in \mathscr{J}(G)$. Then for all $\mathcal{O}_1, \mathcal{O}_2 \subset Z(G)J$, the G-homogeneous spaces $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic. In particular, $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$.*

Proof. Suppose $J = J(\tau)$ for $\tau = (M, Z(M)^\circ s, \mathcal{O}_u^M) \in \mathscr{D}(G)$. Then $J = G \cdot ((Z(M)^\circ s)^{reg} u)$ and all conjugacy classes in J meet $(Z(M)^\circ s)^{reg} u$. We show that $s_1 u, s_2 u \in (Z(M)^\circ s)^{reg} u$ imply $G/C_G(s_1 u) \simeq G/C_G(s_2 u)$: indeed, $C_G(s_1 u) = C_G(s_1) \cap C_G(u) = M \cap C_G(u) = C_G(s_2 u)$. The statement follows from Remark 4.4. □

Observe that in Proposition 4.5 the hypothesis on $[G,G]$ cannot be relaxed, as illustrated by the following example.

Example 4.6. Recall the situation of Example 2.11 and retain notation from therein: $\pi\colon G = \mathrm{SL}_2(\mathbb{C}) \to \overline{G} = \mathrm{PSL}_2(\mathbb{C})$. Let $t_k = \mathrm{diag}[k, k^{-1}] \in T^{reg}$ for $k \in \mathbb{C}^\times$ and denote $\bar{t}_k := \pi(t_k)$. We have $C_{\overline{G}}(\bar{t}_k)$:

$$C_{\overline{G}}(\bar{t}_k) = \begin{cases} \overline{H} := N_{\overline{G}}(\overline{T}) & \text{if } k = \pm i; \\ \overline{T} = \overline{H}^\circ & \text{otherwise.} \end{cases}$$

Neither the $\overline{G}$-module structure nor the $\overline{G}$-homogeneous space structure of conjugacy classes is preserved along the Jordan class consisting of regular semisimple elements. In particular, we have:

k	$\mathbb{C}[\mathcal{O}_{\bar{t}_k}^{\overline{G}}]$	$\Lambda(\mathcal{O}_{\bar{t}_k}^{\overline{G}})$
$k = \pm i$	$\mathbb{C}[\overline{G}/\overline{H}]$	$4n_1\omega_1$
$k \in \mathbb{C} \setminus \{0, \pm 1, \pm i\}$	$\mathbb{C}[\overline{G}/\overline{T}]$	$2n_1\omega_1$

Table 4.1: Regular semisimple spherical classes in $\mathrm{PSL}_2(\mathbb{C})$.

Let us consider the most familiar case $G = \mathrm{SL}_n(\mathbb{C})$. Since pseudo-Levi subgroups of G are Levi subgroups (Remark 0.5), we get the following analogue of Proposition 4.3.

Theorem 4.7. *Let $G = \mathrm{SL}_n(\mathbb{C})$, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be conjugacy classes of G and let S_1 (resp. S_2) be the (birational) sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if $S_2 = zS_1$ for some $z \in Z(G)$.*

Proof. Recall that (birational) sheets in G are disjoint (see Corollary 2.31) and parameterized by G-classes of pairs $(L, Z(L)^\circ z)$, with L a Levi sugbroup of G and suitable z in $Z(G)$. For every

$g \in G$ there exists $\xi \in \mathfrak{g}$ such that $C_G(\xi) = C_G(g)$: if the sheet of G containing $\mathcal{O}_g$ corresponds to the G-class of $(L, Z(L)^\circ z)$, then the sheet of $\mathfrak{g}$ containing ξ corresponds to the G-class of $\mathrm{Lie}(L)$. Let $g_i \in \mathcal{O}_i$ and $\xi_i \in \mathfrak{g}$ such that $C_G(\xi_i) = C_G(g_i)$, and let $(L_i, Z(L_i)^\circ z_i)$ correspond to S_i for $i = 1, 2$. Then $\mathbb{C}[\mathcal{O}_i] = \mathbb{C}[\mathfrak{O}_{\xi_i}]$ for $i = 1, 2$. Therefore $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if and only if L_1 and L_2 are G-conjugate by Proposition 4.3, hence if and only if $S_2 = zS_1$ for some $z \in Z(G)$. □

As we anticipated, it is not true, in general, that the G-module structure of algebras of regular functions on adjoint orbits are preserved along sheets of any simple Lie algebra. Already Borho and Kraft came up with examples of this phenomenon in [13, §6.6]: for $\mathfrak{g}$ of type B_2 or G_2, the algebra $\mathfrak{g}$ presents two distinct subregular sheets $\mathfrak{S}_1, \mathfrak{S}_2$ such that $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \mathfrak{O}_{sreg}$, the nilpotent subregular orbit. Then the G-module structure of $\mathbb{C}[\mathfrak{O}]$ is constant as $\mathfrak{O}$ varies in $\mathfrak{S}_1$, while $\mathbb{C}[\mathfrak{O}] \not\simeq_G \mathbb{C}[\mathfrak{O}_{sreg}]$ for $\mathfrak{O}$ semisimple in $\mathfrak{S}_2$.

It was not until the recent work of Losev [67] that the problem of multiplicities of orbits of $\mathfrak{g}$ has found new and interesting developments. Considering birational sheets instead of sheets, Losev obtained the desired result:

Proposition 4.8 (Losev [67, Remark 4.10])**.** *Let $\mathfrak{g}$ be reductive. If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two orbits of $\mathfrak{g}$ lying in the same birational sheet, then $\mathbb{C}[\mathfrak{O}_1] \simeq_G \mathbb{C}[\mathfrak{O}_1]$.*

As announced in the Introduction, the result was obtained via the theory of Poisson deformations. In addition, Losev conjectured that the converse is also true:

Conjecture 4.9 (Losev [67, Remark 4.10]). Let $\mathfrak{g}$ be reductive. If $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are two orbits of $\mathfrak{g}$ satisfying $\mathbb{C}[\mathfrak{O}_1] \simeq_G \mathbb{C}[\mathfrak{O}_1]$, then they lie in the same birational sheet.

With the notion of birational sheets of a group G introduced in Chapter 2, we are motivated to investigate group analogues of Proposition 4.8 and Conjecture 4.9, restricted to the spherical subvariety $G_{sph} \subset G$: this is done in the next Section.

4.3 Spherical birational sheets

In this Section, unless otherwise stated, we assume that $[G, G]$ is *simply-connected*. In particular, centralizers of semisimple elements of G are connected. Recall that, for a spherical conjugacy class $\mathcal{O}$ the G-module structure of $\mathbb{C}[\mathcal{O}]$ is completely determined by the weight lattice $\Lambda(\mathcal{O})$. To tackle the problem, we want to make use of the complete description of the weight monoids $\Lambda(\mathcal{O})$ for $\mathcal{O}$ spherical in G simple simply-connected given in [27, 36].

The property of being spherical is preserved along sheets, as proven in [6, Proposition 1]. As in [28], we call a sheet *spherical* if it is contains a spherical orbit, equivalently if it consists of spherical orbits. Since every birational sheet is contained in a sheet, the following definition is well-posed.

Definition 4.10. Let $\tau \in \mathscr{B}\mathscr{B}(G)$. The birational sheet $\overline{J(\tau)}^{bir}$ is said to be *spherical* if one of the following equivalent properties is satisfied:

(a) all conjugacy classes $\mathcal{O} \subset \overline{J(\tau)}^{bir}$ are spherical;
(b) there exists a spherical conjugacy class $\mathcal{O} \subset \overline{J(\tau)}^{bir}$;
(c) $\overline{J(\tau)}^{bir}$ is contained in a spherical sheet.

We will prove the following result.

Theorem 4.11. *Let G be a complex connected reductive algebraic group with $[G, G]$ simply-connected. Then the spherical birational sheets form a partition of G_{sph}. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spherical conjugacy classes in G. Let $\overline{J_G(\tau_1)}^{bir}$ (resp. $\overline{J_G(\tau_2)}^{bir}$) be the birational sheet containing $\mathcal{O}_1$ (resp. $\mathcal{O}_2$). Then $\mathbb{C}[\mathcal{O}_1] \simeq_G \mathbb{C}[\mathcal{O}_2]$ if and only if $\overline{J_G(\tau_2)}^{bir} = z\overline{J_G(\tau_1)}^{bir}$ for some $z \in Z(G)$.*

Clearly, it is enough to prove Theorem 4.11 for G simple: let G be simple, we illustrate the procedure for the proof of Theorem 4.11.

The list of spherical conjugacy classes in G simple was first completely carried out in [27]; we rather adopt notation from [36], where G is assumed to be simply-connected. The fact that G_{sph} is partitioned as the union of spherical birational sheets of G is a direct consequence of Theorem 2.25 and of Definition 4.10. With this in mind, we compute the list of spherical birational sheets in G, proceeding as follows.

- For $z \in Z(G)$, we have $\mathcal{O}_z = \{z\}$, $w_z = 1$ and $\mathbb{C}[\mathcal{O}_z] = \mathbb{C}$: then $\{z\}$ is the unique sheet and the unique birational sheet containing z. Therefore, we can restrict to considering non-central spherical conjugacy classes.
- We begin the list by computing all spherical birational sheets containing semisimiple elements. Notice that such birational sheets are exactly those obtained as $\overline{J(\tau)}^{bir}$ with $\tau = (M, Z(M)^\circ s, \{1\}) \in \mathscr{B}\mathscr{B}(G)$ and M a spherical pseudo-Levi subgroup of G: this follows from Definitions 2.23, 4.10 and Remark 2.18. We deduce the list of spherical pseudo-Levi subgroups M up to G-conjugacy from [27, 36]: by inspection, we have two possibilities.
  - (i) If M is a spherical Levi subgroup, the birational sheet $\overline{J(\tau)}^{bir}$ is dense in the spherical sheet $\overline{J(\tau)}^{reg}$. Moreover, it turns out that $\overline{J(\tau)}^{reg} \setminus J(\tau)$ is a union of isolated classes. Since $J(\tau) \subset \overline{J(\tau)}^{bir}$, we can compute $\overline{J(\tau)}^{bir}$ by checking whether the isolated classes in $\overline{J(\tau)}^{reg}$ are birationally induced or not, using Corollary 1.5 or Lemma 1.26 and the list of birationally rigid unipotent classes in Section 1.3.4.
  - (ii) If M is a spherical pseudo-Levi subgroup which is not Levi, then M is semisimple and $\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{reg} = \overline{J(\tau)} = J(\tau)$ is an isolated semisimple class.
- At this point we are left with considering all spherical conjugacy classes which are not semisimple, nor birationally induced as in (i). By inspecting the lists in [27,36], we see that all such classes are spherical classes $\mathcal{O}_{su}$ with s semisimple isolated and $\mathcal{O}_u^{C_G(s)}$ birationally rigid, confronting the list in Section 1.3.4. We conclude by Remark 2.27 that these classes are spherical birational sheets.

We collect the list of spherical birational sheets in a table. In the first column we write a triple $\tau = (M, Z(M)^\circ s, \mathcal{O}_u^M) \in \mathscr{B}\mathscr{B}(G)$ corresponding to a spherical birational sheet $\overline{J(\tau)}^{bir}$, whose decomposition into conjugay classes is described in the second column. From the tables in [36] we verify that the weight monoid is constant on the classes in $\overline{J(\tau)}^{bir}$ and we describe $\Lambda(\mathcal{O})$ in the third column. In the cases when $Z(G)$ is non-trivial, we list also a fourth column indicating the number of (disjoint) birational sheets in $Z(G)\overline{J(\tau)}^{bir}$. This is produced by applying Remark 2.35 when $\mathcal{O}^M$ is characteristic in M and by direct computation otherwise (this case occurs only for M semisimple of type $\mathsf{C}_m\mathsf{C}_m$ in G of type $\mathsf{C}_{2m}, m \geq 2$, see Remark 4.22). The fact that $\Lambda(\mathcal{O})$ is independent of the orbit $\mathcal{O}$ in $\overline{J(\tau)}^{bir}$ (and hence in $Z(G)\overline{J(\tau)}^{bir}$) proves the group analogue of Proposition 4.8 (for spherical conjugacy classes in G simple simply-connected). To prove the validity of the group analogue of Conjecture 4.9 one has to check that the entries in the third column are pairwise distinct.

We set the following notation. For $i = 1, \dots, n$, recall that c_i is the coefficient of α_i in the expansion of the highest root. We put: $r_i \coloneqq \exp \frac{\check{\omega}_i}{c_i}$ for $i = 1, \dots, n$. These elements are a family of representatives of the semisimple isolated classes, [38]; moreover, $Z(G) = \{1\} \cup \{r_i \mid c_i = 1\}$. For $i = 1, \dots, n$, we set:

$$\Theta_i \coloneqq \Delta \setminus \{\alpha_i\} \qquad\qquad L_i \coloneqq L_{\Theta_i}; \tag{4.2}$$

$$\widetilde{\Theta}_i \coloneqq \widetilde{\Delta} \setminus \{\alpha_i\} \qquad\qquad M_k \coloneqq M_{\widetilde{\Theta}_i} = C_G(r_i). \tag{4.3}$$

We shall freely use the notation from in [36] to describe unipotent elements as products of elements of the root subgroups, namely, for $\alpha \in \Phi$ the root subgroup U_α will be expressed through a one-parameter additive subgroup as follows $U_\alpha = \{x_\alpha(\zeta) \mid \zeta \in \mathbb{C}\}$.

4.3.1 Type A_n

Let $G = \mathrm{SL}_{n+1}(\mathbb{C})$, for $n \geq 1$. Clearly, Theorem 4.11 holds for G: this is a consequence of Theorem 4.7 applied to the spherical subvariety G_{sph}. For the sake of completeness, we list the spherical birational sheets of G and the weight monoids of orbits contained in them. Set $m \coloneqq \left\lfloor \frac{n+1}{2} \right\rfloor$.

If $n = 1$, every conjugacy class of G is spherical and there are three (birational) sheets: $\{-1\}, \{1\}$ and G^{reg}.

Let $n \geq 2$. Consider the Levi subgroups L_i, for all $i = 1, \dots, m$. Then $[L_i, L_i]$ is of type $\mathsf{A}_{n-i}\mathsf{A}_{i-1}$, the centre $Z(L_i)$ is one-dimensional and consists of $d_i = \gcd(n+1-i, i) = \gcd(i, n+1)$ distinct connected components which are not conjugate in G. Let $\mathbf{d}_i \coloneqq [n+1-i, i]$ and let $\tau_i \coloneqq (L_i, Z(L_i)^\circ, \{1\})$, then $Z(L_i)^\circ = \exp(\mathbb{C}\check{\omega}_i)$, and $Z(G) \cap Z(L_i)^\circ$ has order $\frac{n+1}{d_i}$. We have:

$$\overline{J(\tau_i)}^{bir} = \overline{J(\tau_i)}^{reg} = \bigcup_{z \in Z(L_i)^\circ} G \cdot (z \operatorname{Ind}_{L_i}^{C_G(z)}\{1\}) = \bigcup_{\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_i)} \sqcup \bigcup_{z \in Z(G) \cap Z(L_i)^\circ} z\, \mathcal{O}_{\mathbf{d}_i^t},$$

by [35, Theorem 7.2.3]. It follows that $Z(G)\overline{J(\tau_i)}^{bir}$ is the disjoint union of d_i birational sheets.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_\ell, Z(L_\ell)^\circ, \{1\})$ $\ell = 1\dots, m-1$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_\ell)} \sqcup$ $\sqcup (Z(G)\cap Z(L_i)^\circ)\,\mathcal{O}_{\mathbf{d}^t_\ell}$	$\sum_{k=1}^{\ell} n_k(\omega_k+\omega_{n-k+1})$	$\gcd(\ell, n+1)$
$(L_m, Z(L_m), \{1\})$ $n=2m$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_m)} \sqcup Z(G)\,\mathcal{O}_{\mathbf{d}^t_m}$	$\sum_{k=1}^{m} n_k(\omega_k+\omega_{n-k+1})$	1
$(L_m, Z(L_m)^\circ, \{1\})$ $n+1=2m$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_m)} \sqcup \pm\,\mathcal{O}_{\mathbf{d}^t_m}$	$\sum_{k=1}^{m-1} n_k(\omega_k+\omega_{n-k+1})+$ $+2n_m\omega_m$	m

Table 4.2: Type $\mathsf{A}_n, n\geq 1, m=\lfloor \frac{n+1}{2}\rfloor$.

4.3.2 Type $\mathsf{B}_n, n\geq 3$

Set $\hat{z} \coloneqq \alpha_n^\vee(-1)$, so that $Z(G)=\langle\hat{z}\rangle\simeq\mathbb{Z}_2$.

The following result holds indepedently of the parity of n.

Lemma 4.12. *Consider $\tau_1 \coloneqq (L_1, Z(L_1), \{1\}) \in \mathscr{BB}(G)$. Then the spherical sheet $S_1 \coloneqq \overline{J_G(\tau_1)}^{reg} = \overline{J_G(\tau_1)}^{bir}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{d}}$, with $\mathbf{d}=[3,1^{2n-2}]$.*

Proof. The group L_1 is maximal in G of type $\mathsf{T}_1\mathsf{B}_{n-1}$ and its centre is connected:

$$Z(L_1)=\{\alpha_1^\vee(x^2)\dots\alpha_{n-1}^\vee(x^2)\alpha_n^\vee(x)\mid x\in\mathbb{C}^\times\}.$$

We have

$$S_1=\bigcup_{z\in Z(L_1)} G\cdot(z\,\mathrm{Ind}_{L_1}^{C_G(z)}\{1\}) = G\cdot(Z(L_1)^{reg})\sqcup Z(G)\,\mathcal{O}_{\mathbf{d}},$$

where $\mathcal{O}_{\mathbf{d}}=\mathrm{Ind}_{L_1}^G\{1\}$ with $\mathbf{d}\coloneqq[3,1^{2n-2}]\vdash 2n+1$ is the only isolated class in S_1 up to central elements. By Lemma 2.30, $S_1=\overline{J_G(\tau_1)}^{bir}$ if and only if $\mathcal{O}_{\mathbf{d}}$ is birationally induced from $(L_1,\{1\})$. This is true by Corollary 1.5: we have $u=x_{\varepsilon_1}(1)\in\mathcal{O}_{\mathbf{d}}$, where ε_1 is the highest short root of G; a direct computation shows $C_G(u)\leq P_{\Theta_1}$. Therefore

$$S_1=\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}}\mathcal{O}_{\exp(\zeta\check{\omega}_1)}\sqcup\mathcal{O}_{\mathbf{d}}\sqcup\hat{z}\,\mathcal{O}_{\mathbf{d}}$$

and $Z(G)\overline{J(\tau_1)}^{bir}=\overline{J(\tau_1)}^{bir}$. □

Type $\mathsf{B}_n, n=2m+1, m\geq 1$

Let $\beta_k=\varepsilon_{2k-1}+\varepsilon_{2k}$ and $u_k\coloneqq\prod_{i=1}^k x_{\beta_i}(1)$ for $k=1,\dots,m$.

Lemma 4.13. *Consider $\tau_n \coloneqq (L_n, Z(L_n), \{1\}) \in \mathscr{BB}(G)$. Then the spherical sheet $S_n \coloneqq \overline{J_G(\tau_n)}^{reg} = \overline{J_G(\tau_n)}^{bir}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{f}}$, where $\mathbf{f}=[3,2^{n-1}]$.*

Proof. Let $n\in\mathbb{N}, n\geq 3$, we have $L_n<M_n<G$ and $r_n^2=\hat{z}$. Observe that:

$$Z(L_n)=\left\{\prod_{i=1}^{n-1}\alpha_i^\vee(x_1^i)\alpha_n^\vee(x_n)\mid x_1,x_n\in\mathbb{C}^\times, x_n^2=x_1^n\right\}$$

is connected. Then:

$$\begin{aligned} S_n &= \bigcup_{z\in Z(L_n)} G\cdot (z\operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = \\ &= G\cdot (Z(L_n)^{reg}) \sqcup G\cdot (Z(G)r_n \operatorname{Ind}_{L_n}^{M_n}\{1\}) \sqcup Z(G)\operatorname{Ind}_{L_n}^{G}\{1\}, \end{aligned}$$

where the last two members in the union contain the isolated classes in S_n. Observe that M_n is simple of type D_n. By Lemma 2.30 to prove that $S_n = \overline{J_G(\tau_n)}^{bir}$, it is enough to prove that $\operatorname{Ind}_{L_n}^{M_n}\{1\}$ and $\mathcal{O} \coloneqq \operatorname{Ind}_{L_n}^{G}\{1\}$ are birationally induced from $(L_n,\{1\})$. We show that $G\cdot (r_n \operatorname{Ind}_{L_n}^{M_n}\{1\})$ is birationally induced from $(L_n,\{1\})$. Observe that $\mathcal{O}_{u_m}^{M_n} = \operatorname{Ind}_{L_n}^{M_n}\{1\}$, the unipotent class in M_n corresponding to the partition $[2^{2n-1},1^2]$. This class satisfies the criterion in Lemma 1.26. The unipotent class $\mathcal{O}$ in G corresponds to the partition $\mathbf{f}$, hence by Lemma 1.26, it is birationally induced from $(L_n,\{1\})$. Thus S_n is a birational sheet in G. Finally, we observe that $r_n \sim_W \hat{z}r_n = r_n^{-1}$ and $\mathcal{O}_{u_m}^{M_n}$ is characteristic in the simple group M_n, so that $G\cdot (Z(G)r_n \operatorname{Ind}_{L_n}^{M_n}\{1\})$ is a single class, hence

$$S_n = \bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \mathcal{O}_{r_n u_m} \sqcup \mathcal{O}_{\mathbf{f}} \sqcup \hat{z}\,\mathcal{O}_{\mathbf{f}},$$

and $Z(G)S_n = S_n$. $\square$

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2,\dots,n$, the pseudo-Levi subgroup M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{B}_{n-\ell}$, we have:

(i) $r_\ell^2 = 1$ and $Z(M_\ell) = \langle r_\ell\rangle \times Z(G)$, for ℓ even;
(ii) $r_\ell^2 = \hat{z}$ and $Z(M_\ell) = \langle r_\ell\rangle$, for ℓ odd.

In any case, $Z(M_\ell)^{reg} = \{r_\ell, \hat{z}r_\ell\}$ and $r_\ell \sim_W \hat{z}r_\ell$ via the reflection corresponding to the root ε_1. Then $\mathcal{O}_{r_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\,\mathcal{O}_{r_\ell} = \mathcal{O}_{r_\ell}$.

We proceed with the non-semisimple spherical conjugacy classes: for $\ell = n$, the group M_n is simple of type D_n contains the birationally rigid unipotent class $\mathcal{O}_{u_k}^{M_n}$ corresponding to the partition $[2^{2k},1^{2(n-2k)}]$, for $k = 1,\dots,m-1$. Hence $\mathcal{O}_{r_n u_k}$ is a (birational) sheet consisting of an isolated class. Since $r_n \sim_W r_n^{-1}$ (by s_n) and $\mathcal{O}_{u_k}^{M_n}$ is characteristic in M_n we have $Z(G)\,\mathcal{O}_{r_n u_k} = \mathcal{O}_{r_n u_k}$ by Remark 2.35. The remaining spherical conjugacy classes in G are unipotent birationally rigid, up to central elements.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup$ $\sqcup Z(G)\,\mathcal{O}_{[3,1^{2n-2}]}$	$2n_1\omega_1 + n_2\omega_2$	1
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup$ $\sqcup\,\mathcal{O}_{\sigma_n u_m} \sqcup Z(G)\,\mathcal{O}_{[3,2^{n-1}]}$	$\sum_{i=1}^{n-1} n_i\omega_i + 2n_n\omega_n$	1
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = 2,\dots,m$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	1
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = m+2,\dots,n$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$	1
$(M_{m+1}, \{r_{m+1}\}, \{1\})$	$\mathcal{O}_{r_{m+1}}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(M_n, \{r_n\}, \mathcal{O}_{u_\ell}^{M_n})$ $\ell = 1,\dots,m-1$	$\mathcal{O}_{r_n u_\ell}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell},1^{2n+1-4\ell}]})$ $\ell = 1,\dots,m$	$\mathcal{O}_{[2^{2\ell},1^{2n+1-4\ell}]}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(G, \{1\}, \mathcal{O}_{[3,2^{2(\ell-1)},1^{2n+2-4\ell}]})$ $\ell = 2,\dots,m$	$\mathcal{O}_{[3,2^{2(\ell-1)},1^{2n+2-4\ell}]}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2

Table 4.3: Type B_n, $n = 2m+1$, $m \geq 1$.

Proposition 4.14. *Theorem 4.11 holds for G of type B_{2m+1}, for $m \geq 1$.*

Proof. From [36, Table 13, 14, 15] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.3 are pairwise distinct. □

Type $\mathsf{B}_n, n = 2m, m \geq 2$

Let $\beta_k = \varepsilon_{2k-1} + \varepsilon_{2k}$ and $u_k := \prod_{i=1}^{k} x_{\beta_i}(1)$ for $k = 1,\dots,m$.

Lemma 4.15. *Let $\tau_n = (L_n, Z(L_n)^\circ, \{1\})$, set $\mathbf{f} := [3, 2^{2(m-1)}, 1^2]$ and let $S_n := \overline{J_G(\tau_n)}^{reg}$. Then $S_n = \overline{J_G(\tau_n)}^{bir} \sqcup \mathcal{O}_{\mathbf{f}}$, where $\mathcal{O}_{\mathbf{f}}$ is a birationally rigid unipotent class in G. Similarly, the sheet $\hat{z}S_n$ decomposes as the union of two birational sheets $\hat{z}\overline{J_G(\tau_n)}^{bir} \sqcup \hat{z}\,\mathcal{O}_{\mathbf{f}}$.*

Proof. We have $L_n < M_n < G$ and $Z(L_n) = \{\alpha_1^\vee(x)\alpha_2^\vee(x^2)\cdots\alpha_{n-1}^\vee(x^{n-1})\alpha_n^\vee(\pm x^m) \mid x \in \mathbb{C}^\times\} = Z(L_n)^\circ \cup Z(L_n)^\circ\hat{z}$; observe that $r_n^2 = 1$. By Lemma 0.6, Z° and $Z^\circ\hat{z}$ are not conjugate in G.

Then

$$S_n = \bigcup_{z\in Z(L_n)^\circ} G\cdot(z\,\mathrm{Ind}_{L_n}^{C_G(z)}\{1\}) = G\cdot(Z(L_n))^{reg} \sqcup G\cdot(r_n\,\mathrm{Ind}_{L_n}^{M_n}\{1\}) \sqcup \mathrm{Ind}_{L_n}^{G}\{1\},$$

where the last two members in the union are the isolated classes in S_n. We show that $G\cdot(r_n\,\mathrm{Ind}_{L_n}^{M_n}\{1\})$ is birationally induced from $(L_n,\{1\})$. We have M_n is simple of type D_n and $\mathrm{Ind}_{L_n}^{M_n}\{1\} = \mathcal{O}_{u_m}^{M_n}$ is the one of the two unipotent class in M_n corresponding to $[2^n]$, which fulfils the criterion in Lemma 1.26. We have $\mathrm{Ind}_{L_n}^{G}\{1\} = \mathcal{O}_{\mathbf{f}}$ with $\mathbf{f} = [3, 2^{2(m-1)}, 1^2] \vdash 2n+1$

a full-member partition (see §1.3.4), hence $\mathcal{O}_{\mathbf{d}}$ is not birationally induced from $(L_n, \{1\})$ and it forms a single birational sheet in G. Thus

$$\overline{J_G(\tau_n)}^{bir} = G \cdot ((Z(L_n)^\circ)^{reg}) \sqcup G \cdot (r_n \operatorname{Ind}_{L_n}^{M_n}\{1\}) = \bigcup_{\zeta \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_n)} \sqcup \mathcal{O}_{r_n u_m}.$$

and $Z(G)\overline{J_G(\tau_n)}^{bir} = \overline{J_G(\tau_n)}^{bir} \sqcup \hat{z}\overline{J_G(\tau_n)}^{bir}$. The description of $\hat{z}S_n$ follows easily. $\square$

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2, \dots, n$, the pseudo-Levi subgroup M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{B}_{n-\ell}$ and we have:

(i) $r_\ell^2 = 1$ and $Z(M_\ell) = \langle r_\ell \rangle \times Z(G)$ for even ℓ;

(ii) $r_\ell^2 = \hat{z}$ and $Z(M_\ell) = \langle r_\ell \rangle$ for odd ℓ.

In any case, $Z(M_\ell)^{reg} = \{r_\ell, \hat{z}r_\ell\}$ and $r_\ell \sim_W \hat{z}r_\ell$ via the reflection with respect to the root ε_1. Then $\mathcal{O}_{r_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\,\mathcal{O}_{r_\ell} = \mathcal{O}_{r_\ell}$.

We are now left with the non-semisimple spherical conjugacy classes. For $\ell = n$, the subgroup M_n is simple of type D_n and contains the birationally rigid unipotent classes $\mathcal{O}_{u_k}^{M_n}$, corresponding to the partition $[2^{2k}, 1^{2(n-2k)}]$ in $\mathrm{SO}_{2n}(\mathbb{C})$, for $k = 1, \dots, m-1$. Hence, $\mathcal{O}_{r_\ell u_k}$ is a (birational) sheet consisting of an isolated class. Moreover, $r_n \sim_W \hat{z}r_n$ and $\mathcal{O}_{u_k}^{M_n}$ is characteristic in the simple group M_n, hence by Remark 2.35 we have $Z(G)\,\mathcal{O}_{r_\ell u_k} = \mathcal{O}_{r_\ell u_k}$, for $k = 1, \dots, m-1$. Up to central elements, the remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup$ $\sqcup Z(G)\,\mathcal{O}_{[3,1^{2n-2}]}$	$2n_1\omega_1 + n_2\omega_2$	1
$(L_n, Z(L_n)^\circ, \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \mathcal{O}_{\sigma_n u_m}$	$\sum_{i=1}^{n-1} n_i\omega_i + 2n_n\omega_n$	2
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	1
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = m+1, \dots, n$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$	1
$(M_m, \{r_m\}, \{1\})$	$\mathcal{O}_{r_m}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(M_n), \{r_n\}, \mathcal{O}_{u_\ell}^{M_n})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{r_n u_\ell}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell},1^{2n+1-4\ell}]})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{[2^{2\ell},1^{2n+1-4\ell}]}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(G, \{1\}, \mathcal{O}_{[2^{2m},1]})$	$\mathcal{O}_{[2^{2m},1]}$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_n\omega_n$	2
$(G, \{1\}, \mathcal{O}_{[3,2^{2(\ell-1)},1^{2n+2-4\ell}]})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{[3,2^{2(\ell-1)},1^{2n+2-4\ell}]}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2
$(G, \{1\}, \mathcal{O}_{[3,2^{2(m-1)},1^2]})$	$\mathcal{O}_{[3,2^{2(m-1)},1^2]}$	$\sum_{i=1}^{n} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1}, n_n \in 2\mathbb{N}$	2

Table 4.4: Type B_n, $n = 2m$, $m \geq 2$.

Proposition 4.16. *Theorem 4.11 holds for G of type B_{2m}, for $m \geq 2$.*

Proof. From [36, Table 10, 11, 12] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.4 are pairwise distinct. $\square$

4.3.3 Type $\mathsf{C}_n, n \geq 2$

We have $Z(G) = \langle \hat{z} \rangle$ with $\hat{z} = \prod_{j\ odd} \alpha_j^\vee(-1)$. We set $m := \lfloor \frac{n}{2} \rfloor$ and $\beta_k = 2\varepsilon_k$ for $k = 1, \ldots, m$.

Type C_2

Lemma 4.17. *Let $G = \mathrm{Sp}_4(\mathbb{C})$. For $i = 1, 2$, consider $\tau_i := (L_i, Z(L_i)^\circ, \{1\}) \in \mathscr{B}\mathscr{B}(G)$ and set $S_i := \overline{J(\tau_i)}^{reg}$. Then:*

(i) The spherical sheet $S_2 = \overline{J(\tau_2)}^{bir}$ is birational.

(ii) The spherical sheet S_1 (resp. $\hat{z}S_1$) decomposes as the union of two birational sheets $\overline{J(\tau_1)}^{bir} \sqcup \mathcal{O}_{[2^2]}$ (resp. $\hat{z}\overline{J(\tau_1)}^{bir} \sqcup \hat{z}\,\mathcal{O}_{[2^2]}$).

Proof. For (i), observe that $Z(L_2) = \{\alpha_1^\vee(x)\alpha_2^\vee(x^2) \mid x \in \mathbb{C}^\times\}$ is connected. Then $S_2 = \bigcup_{z\in Z(L_2)} G \cdot (z \operatorname{Ind}_{L_2}^{C_G(z)}\{1\})$ contains two isolated classes: $\operatorname{Ind}_{L_2}^{G}\{1\}$ and $\hat{z}\operatorname{Ind}_{L_2}^{G}\{1\}$. We have $\operatorname{Ind}_{L_2}^{G}\{1\} = \mathcal{O}_{[2^2]} \in \mathcal{U}/G$, and $u := x_{\beta_1}(1)x_{\beta_2}(1) \in \mathcal{O}_{[2^2]}$ satisfies $C_G(u) \leq P_{\Theta_2}$, so that $\mathcal{O}_{[2^2]}$ is birationally induced from $(L_2, \{1\})$ and S_2 is a birational sheet, by Corollary 1.5.

For (ii), notice that $Z(L_1) = Z(L_1)^\circ \sqcup Z(L_1)^\circ \hat{z}$. We have

$$S_1 = \bigcup_{z\in Z(L_1)^\circ} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)}\{1\}) = G \cdot ((Z(L_1)^\circ)^{reg}) \sqcup G \cdot (r_1 \operatorname{Ind}_{L_2}^{M_1}\{1\}) \sqcup \operatorname{Ind}_{L_1}^{G}\{1\}.$$

Observe that $M_1 \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, so the mixed class $G \cdot (r_1 \operatorname{Ind}_{L_1}^{M_1}\{1\}) = \mathcal{O}_{r_1 x_{\beta_1}(1)}$ is birationally induced by Lemma 1.26. The unipotent class $\mathcal{O}_{[2^2]} = \operatorname{Ind}_{L_1}^{G}\{1\}$ is not birationally induced from $(L_1, \{1\})$, so that

$$\overline{J(\tau_1)}^{bir} = \bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{r_1 x_{\beta_1}(1)}$$

and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir} \sqcup \hat{z}\overline{J(\tau_1)}^{bir}$. $\square$

There is one more spherical pseudo-Levi subgroup M_1 giving rise to the (birational) sheet $\mathcal{O}_{r_1}$. Note that $r_1 \sim_W \hat{z}r_1$, hence $Z(G)\,\mathcal{O}_{r_1} = \mathcal{O}_{r_1}$.

Up to central elements, there is one more spherical conjugacy class, it is unipotent and corresponds to the partition $[2, 1^2]$: this is birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_2, Z(L_2), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_2)} \sqcup Z(G)\,\mathcal{O}_{[2^2]}$	$2n_1\omega_1 + 2n_2\omega_2$	1
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{r_1 x_{\beta_1}(1)}$	$2n_1\omega_1 + n_2\omega_2$	2
$(C_1, \{r_1\}, \{1\})$	$\mathcal{O}_{r_1}$	$n_2\omega_2$	1
$(G, \{1\}, \mathcal{O}_{[2,1^2]})$	$\mathcal{O}_{[2,1^2]}$	$2n_1\omega_1$	2

Table 4.5: Type C_2.

Proposition 4.18. *Theorem 4.11 holds for G of type C_2.*

Proof. From [36, Table 3, 4, 5] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.5 are pairwise distinct. □

Remark 4.19. The subregular unipotent class $\mathcal{O}_{[2^2]}$ lies in both the sheets S_1 and S_2. This agrees with what is stated in [13, §6(c)]: $\mathcal{O}_{[2^2]}$ can be deformed in semisimple classes of both types $\mathcal{O}_{\exp(\zeta\check{\omega}_1)}$ and $\mathcal{O}_{\exp(\zeta\check{\omega}_2)}$, but in general the multiplicities of the weights can decrease. Indeed, for $\zeta \in \mathbb{C}\setminus\frac{1}{2}\mathbb{Z}$, we have $\Lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \langle 2\omega_1, \omega_2\rangle > \langle 2\omega_1, 2\omega_2\rangle = \Lambda(\mathcal{O}_{[2^2]}) = \Lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_2)})$.

Type C_n, $n \geq 3$

Lemma 4.20. *Let $\tau_1 := (L_1, Z(L_1)^\circ, \{1\}) \in \mathscr{B}\mathscr{B}(G)$; then the spherical sheet $S_1 := \overline{J_G(\tau_1)}^{reg}$ decomposes as $S_1 = \overline{J_G(\tau_1)}^{bir} \sqcup \mathcal{O}_{\mathbf{d}}$, where $\mathcal{O}_{\mathbf{d}}$ is the unipotent birationally rigid class with $\mathbf{d} = [2^2, 1^{2(n-1)}]$. Similarly, the spherical sheet $\hat{z}S_1$ decomposes as $\hat{z}\overline{J_G(\tau_1)}^{bir} \sqcup \hat{z}\,\mathcal{O}_{\mathbf{d}}$.*

Proof. We have $L_1 < M_1 < G$ and

$$Z(L_1) = \left\{ \prod_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \alpha^\vee_{2j+1}(x_1) \prod_{j=1}^{m} \alpha^\vee_{2j}(x_2) \mid x_1, x_2 \in \mathbb{C}^\times, x_1^2 = x_2^2 \right\},$$

so that $Z(L_1) = Z(L_1)^\circ \sqcup Z(L_1)^\circ \hat{z}$ and the two connected components are not conjugate in G by Lemma 0.6. We have:

$$S_1 = \bigcup_{z\in Z(L_1)^\circ} G\cdot(z\,\mathrm{Ind}_{L_1}^{C_G(z)}\{1\}) = G\cdot((Z(L_1)^\circ)^{reg}) \sqcup G\cdot(r_1\,\mathrm{Ind}_{L_1}^{M_1}\{1\}) \sqcup \mathrm{Ind}_{L_1}^{G}\{1\}.$$

We check whether the two isolated classes in S_1 are birationally induced. The mixed class $G\cdot(\hat{z}s_1\,\mathrm{Ind}_{L_1}^{C_G(s_1)}\{1\})$ is birationally induced, as $[L_1, L_1] \simeq \mathrm{Sp}_{2n-2}(\mathbb{C})$ and $M_1 \simeq \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. The unipotent class $\mathcal{O}_{\mathbf{d}} = \mathrm{Ind}_{L_1}^{G}\{1\}$ is not birationally induced from $(L_1, \{1\})$, indeed it is birationally rigid, see §1.3.4, and it coincides with a whole birational sheet. Hence

$$\overline{J(\tau_1)}^{bir} = \bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{r_1 x_{\beta_1}(1)}$$

and $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir} \sqcup \hat{z}\overline{J(\tau_1)}^{bir}$. □

Lemma 4.21. *Let $\tau_n = (L_n, Z(L_n), \{1\}) \in \mathscr{B}\mathscr{B}(G)$; then the spherical sheet $S_n := \overline{J_G(\tau_n)}^{reg}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{f}}$, with $\mathbf{f} = [2^n]$.*

Proof. Observe that $Z(L_n) = \left\{ \prod_{i=1}^{n} \alpha_i^\vee(x^i) \mid x \in \mathbb{C}^\times \right\}$ is connected. We have:

$$S_n = \bigcup_{z \in Z(L_n)} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = G \cdot (Z(L_n)^{reg}) \sqcup \hat{z} \operatorname{Ind}_{L_n}^G\{1\} \sqcup \operatorname{Ind}_{L_n}^G\{1\}.$$

The isolated classes in S_n are $\operatorname{Ind}_{L_n}^G\{1\}$ and $\hat{z}\operatorname{Ind}_{L_n}^G\{1\}$, where $\operatorname{Ind}_{L_n}^G\{1\} = \mathcal{O}_{\mathbf{f}}^G$, with $\mathbf{f} = [2^n]$. A representative for $\mathcal{O}_{\mathbf{f}}$ is $u = \prod_{i=1}^{n} x_{\beta_i}(1)$, and $C_G(u) \le P_{\Theta_n}$ implies $\mathcal{O}_{\mathbf{f}}$ is birationally induced from $(L_n, \{1\})$, by Corollary 1.5. Hence

$$\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{reg} = \bigcup_{\zeta \in \mathbb{C} \setminus \mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_n)} \sqcup \mathcal{O}_{\mathbf{f}} \sqcup \hat{z}\, \mathcal{O}_{\mathbf{f}},$$

and $Z(G)\overline{J(\tau)}^{bir} = \overline{J(\tau)}^{bir}$. □

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 1, \dots, m$, the pseudo-Levi M_ℓ is maximal of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ and $Z(C_G(r_\ell)) = \langle r_\ell \rangle \times Z(G)$. Then, for $(M_\ell, \{r_\ell\}, \{1\})$ we get $\mathcal{O}_{r_\ell}$, a (birational) sheet consisting of an isolated class. We have $Z(G)\,\mathcal{O}_{r_\ell} = \mathcal{O}_{r_\ell} \sqcup \hat{z}\, \mathcal{O}_{r_\ell}$ except when $n = 2m$, $\ell = m$, in which case $r_m \sim_G \hat{z} r_m$ and $Z(G)\,\mathcal{O}_{r_m} = \mathcal{O}_{r_m}$.

We are left with non-semisimple spherical conjugacy classes.

(i) For $\ell = 2, \dots, p$, the pseudo-Levi M_ℓ of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ contains the birationally rigid unipotent class $\mathcal{O}_{x_{\beta_1}(1)}^{M_\ell}$ of the form $[2, 1^{2\ell-2}] \times \{1\}$. Then $\mathcal{O}_{r_\ell x_{\beta_1}(1)}$ is a (birational) sheet consisting of an isolated class.

(ii) For $\ell = 1, \dots, m$, the pseudo-Levi $C_G(r_\ell)$ of type $\mathsf{C}_\ell\mathsf{C}_{n-\ell}$ has the birationally rigid unipotent class $\mathcal{O}_{x_{\alpha_n}(1)}^{M_\ell}$ of the form $\{1\} \times [2, 1^{2(n-\ell)-2}]$. Then $\mathcal{O}_{r_\ell x_{\alpha_n}(1)}$ is a (birational) sheet consisting of an isolated class.

In both cases, we have that $Z(G)\,\mathcal{O}_{r_\ell x_{\beta_1}(1)} = \mathcal{O}_{r_\ell x_{\beta_1}(1)} \sqcup \hat{z}\, \mathcal{O}_{r_\ell x_{\beta_1}(1)}$ and $Z(G)\,\mathcal{O}_{r_\ell x_{\alpha_n}(1)} = \mathcal{O}_{r_\ell x_{\alpha_n}(1)} \sqcup \hat{z}\, \mathcal{O}_{r_\ell x_{\alpha_n}(1)}$. The only case which needs inspection is $n = 2m$, $\ell = m$: we have $r_m \sim_W \hat{z} r_m$, but $r_m x_{\beta_1} \not\sim_G \hat{z} r_m x_{\beta_1}$.

Remark 4.22. This is an example of $(M, Z^\circ s_1, \mathcal{O}^M)$, $(M, Z^\circ s_2, \mathcal{O}^M)$ in $\mathscr{B}\mathscr{B}(G)$ with $(M, Z^\circ s_1)$, $(M, Z^\circ s_2)$ G-conjugate, but $(M, Z^\circ s_1, \mathcal{O}^M)$, $(M, Z^\circ s_2, \mathcal{O}^M)$ not G-conjugate: in this case the (birationally) rigid class $\mathcal{O}^M$ is not characteristic in the semisimple pseudo-Levi subgroup M.

Up to central elements, the remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup Z(G)\,\mathcal{O}_{[2^n]}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\frac{1}{2}\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{r_1 x_{\beta_1}(1)}$	$2n_1\omega_1 + n_2\omega_2$	2
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	2
$(M_m, \{r_m\}, \{1\})$ if $n = 2m+1$	$\mathcal{O}_{r_m}$	$\sum_{i=1}^{m} n_{2i}\omega_{2i}$	2
$(M_m, \{r_m\}, \{1\})$ if $n = 2m$			1
$(M_m, \{r_m\}, \mathcal{O}^{M_m}_{\{1\}\times[2,1^{2(n-m)-2}]})$	$\mathcal{O}_{r_m \alpha_n(1)}$	$\sum_{i=1}^{n} n_i\omega_i \mid \sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} n_{2i-1} \in 2\mathbb{N}$	2
$(M_\ell, \{r_\ell\}, \mathcal{O}^{M_\ell}_{\{1\}\times[2,1^{2(n-\ell)-2}]})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{r_\ell \alpha_n(1)}$	$\sum_{i=1}^{2\ell+1} n_i\omega_i \mid \sum_{i=1}^{\ell+1} n_{2i-1} \in 2\mathbb{N}$	2
$(M_\ell, \{r_\ell\}, \mathcal{O}^{M_\ell}_{[2,1^{2\ell-2}]\times\{1\}})$ $\ell = 2, \dots, m$	$\mathcal{O}_{r_\ell x_{\beta_1}(1)}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	2
$(G, \{1\}, \mathcal{O}_{[2^\ell,1^{2n-2\ell}]})$ $\ell = 1, \dots, n-1$	$\mathcal{O}_{[2^\ell,1^{2n-2\ell}]}$	$\sum_{i=1}^{\ell} 2n_i\omega_i$	2

Table 4.6: Type $\mathsf{C}_n, n \geq 3, m = \lfloor \frac{n}{2} \rfloor$.

Proposition 4.23. *Theorem 4.11 holds for G of type C_n, $n \geq 3$.*

Proof. From [36, Table 3, 4, 5] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. Assume $\Lambda(\mathcal{O}_1) = \Lambda(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset Z(G)\overline{J(\tau_1)}^{bir}, \mathcal{O}_2 \subset Z(G)\overline{J(\tau_2)}^{bir}$. This is possible if and only if $n = 2m$ for $m \in \mathbb{N}, m \geq 1$, and

$$\tau_1 = (M_m, \{r_m\}, \mathcal{O}^{M_m}_{\{1\}\times[2,1^{n-2}]}) = (M_m, \{r_m\}, \mathcal{O}^{M_m}_{x_{\alpha_n(1)}})$$

$$\tau_2 = (M_m, \{r_m\}, \mathcal{O}^{M_m}_{[2,1^{n-2}]\times\{1\}}) = (M_m, \{r_m\}, \mathcal{O}^{M_m}_{x_{\beta_1(1)}})$$

However in this case $r_m \sim_G \hat{z} r_m$, and also $r_m x_{\alpha_n}(1) \sim_G \hat{z} r_m x_{\beta_1}(1)$. Therefore, the triples τ_1 and $(M_m, \{\hat{z} r_m\}, \mathcal{O}^{M_m}_{x_{\beta_1(1)}})$ are G-conjugate and $\overline{J(\tau_1)}^{bir} = \hat{z}\overline{J(\tau_2)}^{bir}$. □

4.3.4 Type $\mathsf{D}_n, n \geq 4$

Let ϑ denote the graph automorphism of G which swaps α_{n-1} and α_n. We observe that $r_1 = \alpha^\vee_{n-1}(-1)\alpha^\vee_n(-1) \in Z(G)$.

Type $\mathsf{D}_n, n = 2m, m \geq 2$

For G of type $\mathsf{D}_{2m}, m \geq 2$ we have that r_{n-1} and r_n are involutions and $r_n r_{n-1} = r_1$; in particular,

$$\prod_{j=0}^{m-1} \alpha^\vee_{2j+1}(-1) = \begin{cases} r_n & m \text{ even} \\ r_{n-1} & m \text{ odd} \end{cases}$$

hence $Z(G) = \langle r_1, r_n \rangle = \langle r_1, r_{n-1} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 4.24. *Let $\tau_1 \coloneqq (L_1, Z(L_1)^\circ, \{1\}) \in \mathscr{B}\mathscr{B}(G)$. Then the spherical sheet $S_1 \coloneqq \overline{J(\tau_1)}^{reg}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{d}}$ where $\mathbf{d} = [3, 1^{2n-3}]$. Similarly, the spherical sheet $r_n S_1$ is birational.*

Proof. Remark that:

$$Z(L_1) = \left\{ \prod_{j=0}^{m-2} \alpha_{2j+1}^\vee(\pm x^2) \prod_{j=1}^{m-1} \alpha_{2j}^\vee(x^2) \alpha_{2m-1}^\vee(\pm x) \alpha_{2m}^\vee(x) \mid x \in \mathbb{C}^\times \right\},$$

so that $Z(L_1) = Z(L_1)^\circ \sqcup Z(L_1)^\circ r_n$. We have:

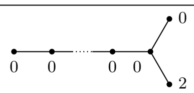
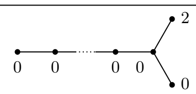
$$S_1 = \bigcup_{z \in Z(L_1)^\circ} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)}\{1\}) = G \cdot ((Z(L_1)^\circ)^{reg}) \sqcup \operatorname{Ind}_{L_1}^G\{1\} \sqcup r_1 \operatorname{Ind}_{L_1}^G\{1\},$$

where the last two members of the union are the only isolated classes in S_1. We have $\operatorname{Ind}_{L_1}^G\{1\} = \mathcal{O}_{\mathbf{d}}$, where $\mathbf{d} = [3, 1^{2n-3}]$. By Lemma 1.26, the class $\mathcal{O}_{\mathbf{d}}$ is birationally induced from $(L_1, \{1\})$, hence $\overline{J_G(\tau_1)}^{bir} = S_1$ by Lemma 2.30. Therefore,

$$\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{reg} = \bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_1)} \sqcup \mathcal{O}_{\mathbf{d}} \sqcup r_1 \mathcal{O}_{\mathbf{d}}$$

and $Z(G)S_1 = S_1 \sqcup r_n S_1$ ∎

For the following result set some additional notation: we consider the very even partition $[2^n]$, it is well-known that there exist two unipotent conjugacy classes in G corresponding to this partition, and they are swapped by the automorphism ϑ. We set:

$\mathcal{O}_{[2^n]} = \operatorname{Ind}_{L_n}^G\{1\}$ 	$\mathcal{O}'_{[2^n]} = \operatorname{Ind}_{L_{n-1}}^G\{1\}$ 

Lemma 4.25. *Let $\tau_n \coloneqq (L_n, Z(L_n)^\circ, \{1\})$ and $\tau_{n-1} \coloneqq (L_{n-1}, Z(L_{n-1})^\circ, \{1\}) \in \mathscr{B}\mathscr{B}(G)$. The following spherical sheets of G are birational:*

(i) $S_n \coloneqq \overline{J_G(\tau_n)}^{reg}$ and $r_1 S_n$.

(ii) $S_{n-1} \coloneqq \vartheta(S_n) = \overline{J_G(\tau_{n-1})}^{reg}$ and $r_1 S_{n-1}$.

Proof. For (i), we have $Z(L_n) = \{\alpha_1^\vee(x)\alpha_2^\vee(x^2) \cdots \alpha_{n-2}^\vee(x^{n-2})\alpha_{n-1}^\vee(\pm x^m)\alpha_n^\vee(\pm x^{m-1}) \mid x \in \mathbb{C}^\times\} = Z(L_n)^\circ \sqcup Z(L_n)^\circ r_1$. By Lemma 0.6, $Z(L_n)^\circ$ and $Z(L_n)^\circ r_1$ are not conjugate in G, hence we have two distinct sheets S_n and $r_1 S_n$, where $S_n = \overline{J_G(\tau_n)}^{reg}$. We prove that $S_n = \overline{J_G(\tau_n)}^{bir}$. We have

$$S_n = \bigcup_{z \in Z(L_n)^\circ} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = G \cdot ((Z(L_n)^\circ)^{reg}) \sqcup \operatorname{Ind}_{L_n}^G\{1\} \sqcup r_n \operatorname{Ind}_{L_n}^G\{1\}.$$

The two isolated classes in S_n are $\operatorname{Ind}_{L_n}^G\{1\} = \mathcal{O}_{[2^n]}$ and $r_n \mathcal{O}_{[2^n]}$. We conclude by Lemma 1.26. Therefore

$$S_n = \bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta \check{\omega}_n)} \sqcup \mathcal{O}_{[2^n]} \sqcup r_n \mathcal{O}_{[2^n]}$$

is a spherical birational sheet and $Z(G)S_n = S_n \sqcup r_1 S_n$. By applying the automorphism ϑ we get

$$S_{n-1} = \bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})} \sqcup \mathcal{O}'_{[2^n]} \sqcup r_{n-1}\,\mathcal{O}'_{[2^n]}$$

and $Z(G)S_{n-1} = S_{n-1} \sqcup r_1 S_{n-1}$, proving (ii). □

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2, \dots, m$, the pseudo-Levi subgroup M_ℓ is maximal of type $\mathsf{D}_\ell\mathsf{D}_{n-\ell}$.

(i) For ℓ odd we have $r_\ell^2 = r_1$ and $Z(M_\ell) = \langle r_\ell\rangle \times \langle r_n\rangle$;

(ii) For ℓ even we have $r_\ell^2 = 1$ and $Z(M_\ell) = \langle r_\ell\rangle \times Z(G)$.

In both cases, $Z(M_\ell)^{reg} = \{r_\ell, r_1r_\ell, r_{n-1}r_\ell, r_nr_\ell\}$. Observe that for $\ell = 2, \dots, m-2$, we have $r_\ell \sim_W r_1r_\ell \not\sim_W r_{n-1}r_\ell \sim_W r_nr_\ell$, whereas for $\ell = m$, all elements of $Z(M_\ell)^{reg}$ are W-conjugate. To sum up, for all $\ell = 2, \dots, m$ we have that $\mathcal{O}_{r_\ell}$ is a (birational) sheet consisting of an isolated class. Moreover, for $\ell = 2, \dots, m-1$ we have $Z(G)\,\mathcal{O}_{r_\ell} = \mathcal{O}_{r_\ell} \sqcup r_n\,\mathcal{O}_{r_\ell}$, whereas $Z(G)\,\mathcal{O}_{r_m} = \mathcal{O}_{r_m}$.

Up to central elements, the remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \sqcup\,\mathcal{O}_{[3,1^{2n-3}]} \sqcup r_1\,\mathcal{O}_{[3,1^{2n-3}]}$	$2n_1\omega_1 + n_2\omega_2$	2
$(L_n, Z(L_n)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \sqcup\,\mathcal{O}_{[2^n]} \sqcup r_n\,\mathcal{O}_{[2^n]}$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_n\omega_n$	2
$(L_{n-1}, Z(L_{n-1})^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})} \sqcup \sqcup\,\mathcal{O}'_{[2^n]} \sqcup r_{n-1}\,\mathcal{O}'_{[2^n]}$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + 2n_{n-1}\omega_{n-1}$	2
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	2
$(M_m, \{r_m\}, \{1\})$	$\mathcal{O}_{r_m}$	$\sum_{i=1}^{n} 2n_i\omega_i$	1
$(G, \{1\}, \mathcal{O}_{[2^{2\ell},1^{2n-4\ell}]})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{[2^{2\ell},1^{2n-4\ell}]}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	4
$(G, \{1\}, \mathcal{O}_{[3,2^{2(\ell-1)},1^{2n-4\ell+1}]})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{[3,2^{2(\ell-1)},1^{2n-4\ell+1}]}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	4
$(G, \{1\}, \mathcal{O}_{[3,2^{2(m-1)},1]})$	$\mathcal{O}_{[3,2^{2(m-1)},1]}$	$\sum_{i=1}^{n} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1}, n_{n-1}+n_n \in 2\mathbb{N}$	4

Table 4.7: Type $\mathsf{D}_n, n = 2m, m \geq 2$.

Proposition 4.26. *Theorem 4.11 holds for G of type D_{2m}, with $m \geq 2$.*

Proof. From [36, Table 6, 7] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.7 are pairwise distinct. □

Type $\mathsf{D}_n, n = 2m+1, m \geq 2$

For G simple simply-connected of type $\mathsf{D}_{2m+1}, m \geq 2$, we have

$$r_n = r_{n-1}^{-1} = \begin{cases} \prod\limits_{j=0}^{m-1} \alpha_{2j+1}^\vee(-1)\alpha_{n-1}^\vee(-i)\alpha_n^\vee(i) & m \text{ even} \\ \prod\limits_{j=0}^{m-1} \alpha_{2j+1}^\vee(-1)\alpha_{n-1}^\vee(i)\alpha_n^\vee(-i) & m \text{ odd} \end{cases}$$

has order 4 and $r_n^2 = r_1$, hence $Z(G) = \langle r_n \rangle = \langle r_{n-1} \rangle \simeq \mathbb{Z}_4$.

Lemma 4.27. *Let $\tau_1 := (L_1, Z(L_1)^\circ, \{1\}) \in \mathscr{BB}(G)$. Then the spherical sheet $S_1 := \overline{J(\tau_1)}^{reg}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{d}}$ with $\mathbf{d} = [3, 1^{2n-3}]$. Similarly, the spherical sheet $r_n S_1$ is birational.*

Proof. Remark that:

$$Z(L_1) = \left\{ \prod_{j=0}^{m-1} \alpha_{2j+1}^\vee(x^2) \prod_{j=1}^{m-1} \alpha_{2j}^\vee(\pm x^2)\alpha_{2m}^\vee(\pm x)\alpha_{2m+1}^\vee(x) \mid x \in \mathbb{C}^\times \right\},$$

so that $Z(L_1) = Z(L_1)^\circ \cup Z(L_1)^\circ r_n$. We have:

$$S_1 = \bigcup_{z \in Z(L_1)^\circ} G \cdot (z \operatorname{Ind}_{L_1}^{C_G(z)}\{1\}) = G \cdot ((Z(L_1)^\circ)^{reg}) \sqcup \operatorname{Ind}_{L_1}^G\{1\} \sqcup r_1 \operatorname{Ind}_{L_1}^G\{1\},$$

where the last two members of the union are the only isolated classes in S_1. We have $\operatorname{Ind}_{L_1}^G\{1\} = \mathcal{O}_{\mathbf{d}}$, where $\mathbf{d} = [3, 1^{2n-3}]$. By Lemma 1.26, the class $\mathcal{O}_{\mathbf{d}}$ is birationally induced from $(L_1, \{1\})$, hence $\overline{J_G(\tau_1)}^{bir} = S_1$ by Lemma 2.30. Therefore

$$\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{reg} = \bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup \mathcal{O}_{\mathbf{d}} \sqcup r_1 \mathcal{O}_{\mathbf{d}}$$

and $Z(G)S_1 = S_1 \sqcup r_n S_1$ $\square$

We remark that w_0 acts as $-\vartheta$, so that in this case $L_n \sim_G \vartheta(L_n) = L_{n-1}$.

Lemma 4.28. *Let $\tau_n := (L_n, Z(L_n), \{1\}) \in \mathscr{BB}(G)$. The spherical sheet $S_n := \overline{J_G(\tau_n)}^{reg}$ is birational and it contains the unipotent class $\mathcal{O}_{\mathbf{f}}$ with $\mathbf{f} = [2^{n-1}, 1^2]$.*

Proof. Remark that $Z(L_n) = \{\alpha_1^\vee(x)\alpha_2^\vee(x^2)\cdots\alpha_{n-2}^\vee(x^{n-2})\alpha_{n-1}^\vee(x_{n-1})\alpha_n^\vee(x_n) \mid x \in \mathbb{C}^\times, x_{n-1}^2 = x^{n-2}, x_{n-1}x_n = x^{n-1}\}$ is connected. We have

$$S_n = \bigcup_{z \in Z(L_n)} G \cdot (z \operatorname{Ind}_{L_n}^{C_G(z)}\{1\}) = G \cdot (Z(L_n)^{reg}) \cup Z(G) \operatorname{Ind}_{L_n}^G\{1\}.$$

Let $\mathbf{f} = [2^{n-1}, 1^2]$, then $\mathcal{O}_{\mathbf{f}} = \operatorname{Ind}_{L_n}^G\{1\}$, and by Lemma 2.30, $\overline{J_G(\tau_n)}^{reg} = \overline{J_G(\tau_n)}^{bir}$ if and only if $\mathcal{O}_{\mathbf{f}}$ is birationally induced from $(L_n, \{1\})$. This is true by Lemma 1.26, hence

$$S_n = \overline{J(\tau_n)}^{bir} = \bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup \mathcal{O}_{\mathbf{f}} \sqcup r_1 \mathcal{O}_{\mathbf{f}} \sqcup r_{n-1} \mathcal{O}_{\mathbf{f}} \sqcup r_n \mathcal{O}_{\mathbf{f}},$$

and $Z(G)S_n = S_n$. $\square$

Proposition 4.29. *Theorem 4.11 holds for G simple simply-connected of type D_{2m+1}, with $m \geq 2$.*

We consider the remaining spherical pseudo-Levi subgroups. For $\ell = 2, \dots, m$, the pseudo-Levi subgroup M_ℓ is maximal of type $\mathsf{D}_\ell \mathsf{D}_{n-\ell}$.

(i) For ℓ even we have $r_\ell^2 = 1$ and $Z(M_\ell) = \langle r_\ell \rangle \times Z(G)$.

(ii) For ℓ odd we have $r_\ell^2 = r_1$ and $Z(M_\ell) = \langle r_\ell, z_n \rangle$.

In both cases $Z(M_\ell)^{reg} = \{r_\ell, r_1 r_\ell, r_{n-1} r_\ell, r_n r_\ell\}$ and $r_\ell \sim_W r_1 r_\ell \not\sim_W r_{n-1} r_\ell \sim_W r_n r_\ell$. Then $\mathcal{O}_{r_\ell}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\,\mathcal{O}_{r_\ell} = \mathcal{O}_{r_\ell} \sqcup r_n\, \mathcal{O}_{r_\ell}$ for all $\ell = 2, \dots, m$.

Up to central elements, the remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1)^\circ, \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup$ $\sqcup\, \mathcal{O}_{[3,1^{2n-3}]} \sqcup r_1\, \mathcal{O}_{[3,1^{2n-3}]}$	$2n_1\omega_1 + n_2\omega_2$	2
$(L_n, Z(L_n), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \sqcup$ $\sqcup Z(G)\, \mathcal{O}_{[2^{n-1},1^2]}$	$\sum_{i=1}^{m-1} n_{2i}\omega_{2i} + n_{n-1}(\omega_{n-1}+\omega_n)$	1
$(M_\ell, \{r_\ell\}, \{1\})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{r_\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$	2
$(M_m, \{r_m\}, \{1\}$	$\mathcal{O}_{r_m}$	$\sum_{i=1}^{n-2} 2n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n)$	2
$(G, \{1\}, \mathcal{O}_{[2^{2\ell},1^{2n-4\ell}]})$ $\ell = 1, \dots, m-1$	$\mathcal{O}_{[2^{2\ell},1^{2n-4\ell}]}$	$\sum_{i=1}^{\ell} n_{2i}\omega_{2i}$	4
$(G, \{1\}, \mathcal{O}_{[3,2^{2(\ell-1)},1^{2n-4\ell+1}]})$ $\ell = 2, \dots, m-1$	$\mathcal{O}_{[3,2^{2(\ell-1)},1^{2n-4\ell+1}]}$	$\sum_{i=1}^{2\ell} n_i\omega_i \mid \sum_{i=1}^{\ell} n_{2i-1} \in 2\mathbb{N}$	4
$(G, \{1\}, \mathcal{O}_{[3,2^{2(m-1)},1^3]})$	$\mathcal{O}_{[3,2^{2(m-1)},1^3]}$	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n) \mid$ $\sum_{i=1}^{m} n_{2i-1} \in 2\mathbb{N}$	4

Table 4.8: Type $\mathsf{D}_n, n = 2m+1, m \geq 2$.

Proposition 4.30. *Theorem 4.11 holds for G of type D_{2m+1}, with $m \geq 2$.*

Proof. From [36, Table 8, 9] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.8 are pairwise distinct. $\square$

4.3.5 Type E_6

Let G be of type E_6. We have $Z(G) = \langle \hat{z} \rangle$, with $\hat{z} = \alpha_1^\vee(x)\alpha_3^\vee(x^{-1})\alpha_5^\vee(x)\alpha_6^\vee(x^{-1})$ where x is a primitive third root of 1.

Lemma 4.31. *Let $\tau_1 = (L_1, Z(L_1), \{1\}) \in \mathscr{BB}(G)$. Then the spherical sheet $S_1 := \overline{J(\tau_1)}^{reg}$ is a birational sheet containing the unipotent class $2A_1$.*

Proof. L_1 is maximal of type $\mathsf{D}_5\mathsf{T}_1$ and $Z(L_1)$ is connected, since

$$Z(L_1) = \{\alpha_1^\vee(y^2)\alpha_2^\vee(x_2)\alpha_3^\vee(x_3)\alpha_4^\vee(y^3)\alpha_5^\vee(y^2)\alpha_6^\vee(y) \mid x_1, x_2 \in \mathbb{C}^\times, y = x_2^{-1}x_3, y^3 = x_2^2\}.$$

We have

$$S_1 = \bigcup_{z\in Z(L_1)} G\cdot(z\,\mathrm{Ind}_{L_1}^G\{1\}) = \bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z(G)\,\mathcal{O}_{2A_1}\,.$$

By Lemma 2.30, $S_1 = \overline{J(\tau_1)}^{bir}$ if and only if the isolated class $\mathcal{O}_{2A_1}$ is birationally induced from $(L_1, \{1\})$; we conclude by Lemma 1.26. Moreover, it is clear that $Z(G)\overline{J(\tau_1)}^{bir} = \overline{J(\tau_1)}^{bir}$. □

There is one more spherical pseudo-Levi subgroup, i.e., M_2 of type $\mathsf{A}_1\mathsf{A}_5$ with $Z(M_2) = \langle r_2\rangle \times Z(G)$. Observe that r_2 is an involution, we have $Z(M_2)^{reg} = \{r_2, \hat{z}r_2, \hat{z}^2 r_2\}$ and no pair of elements in this set is W-conjugate. Hence, the isolated class $\mathcal{O}_{r_2}$ is a (birational) sheet and $Z(G)\,\mathcal{O}_{r_2} = \mathcal{O}_{r_2} \sqcup \hat{z}\,\mathcal{O}_{r_2} \sqcup \hat{z}^2\,\mathcal{O}_{r_2}$. Up to central elements, the remaining spherical conjugacy classes in G are birationally rigid unipotent conjugacy classes in G.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_1, Z(L_1), \{1\})$	$\bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_1)} \sqcup Z(G)\,2A_1$	$n_1(\omega_1+\omega_6)+n_2\omega_2$	1
$(M_2, \{r_2\}, \{1\})$	$\mathcal{O}_{r_2}$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+2n_2\omega_2+2n_4\omega_4$	3
$(G, \{1\}, A_1)$	A_1	$n_2\omega_2$	3
$(G, \{1\}, 3A_1)$	$3A_1$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+n_2\omega_2+n_4\omega_4$	3

Table 4.9: Type E_6.

Proposition 4.32. *Theorem 4.11 holds for G of type E_6.*

Proof. From [36, Table 16, 17] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.9 are pairwise distinct. □

4.3.6 Type E_7

Let G be of type E_7. We have $Z(G) = \langle\hat{z}\rangle$, where $\hat{z} = \alpha_2^\vee(-1)\alpha_5^\vee(-1)\alpha_7^\vee(-1)$.

Lemma 4.33. *Let $\tau_7 = (L_7, Z(L_7), \{1\}) \in \mathscr{BB}(G)$. Then the spherical sheet $S_7 := \overline{J(\tau_7)}^{reg}$ is a birational sheet containing the unipotent class $(3A_1)''$.*

Proof. L_7 is maximal of type $\mathsf{E}_6\mathsf{T}_1$ and $Z(L_7)$ is connected, since

$$Z(L_7) = \{\alpha_1^\vee(x_1)\alpha_2^\vee(x_2)\alpha_3^\vee(x_1^2)\alpha_4^\vee(x_1^3)\alpha_5^\vee(x_1x_2)\alpha_6^\vee(x_1^2)\alpha_7^\vee(x_2) \mid x_1, x_2 \in \mathbb{C}^\times, x_1^3 = x_2^2\}.$$

We have

$$S = \bigcup_{z\in Z(L_7)} G\cdot(z\,\mathrm{Ind}_{L_7}^G\{1\}) = \bigcup_{\zeta\in\mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \sqcup Z(G)\,\mathcal{O}_{3A_1''}\,.$$

By Lemma 2.30, to check that $\overline{J(\tau)}^{reg} = \overline{J(\tau)}^{bir}$, it is enough to check that the isolated class $\mathcal{O}_{3A_1''}$ is birationally induced from $(L_7, \{1\})$: this is true by Lemma 1.26. Moreover, we have $Z(G)\overline{J(\tau_7)}^{bir} = \overline{J(\tau_7)}^{bir}$. □

We consider the remaining spherical pseudo-Levi subgroups:

(i) The pseudo-Levi subgroup M_2 is maximal of type A_7. We have $r_2^2 = \hat{z}$ and $Z(M_2) = \langle r_2 \rangle$; moreover, r_2 and $\hat{z}r_2 = r_2^{-1}$ are conjugate via the longest element $w_0 \in W$. Then $\mathcal{O}_{r_2}$ is a (birational) sheet consisting of an isolated class, and $Z(G)\,\mathcal{O}_{r_2} = \mathcal{O}_{r_2}$.

(ii) The pseudo-Levi subgroup M_1 is maximal of type $\mathsf{D}_6\mathsf{A}_1$. We have $r_1^2 = 1$ and $Z(M_1) = \langle \hat{z}, r_1 \rangle$; moreover, r_1 and $\hat{z}r_1$ are not G-conjugate (in fact G has 2 classes of non-central involutions: $\mathcal{O}_{r_1}$ and $\mathcal{O}_{\hat{z}r_1}$). Then $\mathcal{O}_{r_1}$ is a (birational) sheet consisting of an isolated class, and $Z(G) = \mathcal{O}_{r_1} \sqcup \hat{z}\,\mathcal{O}_{r_1}$.

Up to central elements, the remaining spherical conjugacy classes are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$	d
$(L_7, Z(L_7), \{1\})$	$\bigcup_{\zeta \in \mathbb{C}\setminus\mathbb{Z}} \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \sqcup Z(G)\ (3A_1)''$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$	1
$(M_1, \{r_1\}, \{1\})$	$\mathcal{O}_{r_1}$	$2n_1\omega_1 + 2n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	2
$(M_2, \{r_2\}, \{1\})$	$\mathcal{O}_{r_2}$	$\sum_{i=1}^{7} 2n_i\omega_i$	1
$(G, \{1\}, A_1)$	A_1	$n_1\omega_1$	2
$(G, \{1\}, 2A_1)$	$2A_1$	$n_1\omega_1 + n_6\omega_6$	2
$(G, \{1\}, (3A_1)')$	$(3A_1)'$	$n_1\omega_1 + n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	2
$(G, \{1\}, 4A_1)$	$4A_1$	$\sum_{i=1}^{7} n_i\omega_i \mid n_2 + n_5 + n_7 \in 2\mathbb{N}$	2

Table 4.10: Type E_7.

Proposition 4.34. *Theorem 4.11 holds for G of type E_7.*

Proof. From [36, Table 18, 19] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.10 are pairwise distinct. □

4.3.7 Type E_8

Let G be of type E_8. There are no spherical proper Levi subgroups.

We list the spherical pseudo-Levi subgroups.

(i) The pseudo-Levi subgroup M_8 is maximal of type $\mathsf{A}_1\mathsf{E}_7$. We have $r_8^2 = 1$ and $Z(M_8) = \langle r_8 \rangle$. Then $\mathcal{O}_{r_8}$ is a (birational) sheet consisting of an isolated class.

(ii) The pseudo-Levi subgroup M_1 is maximal of type D_8. We have $r_1^2 = 1$ and $Z(M_1) = \langle r_1 \rangle$ Then $\mathcal{O}_{r_1}$ is a (birational) sheet consisting of an isolated class.

The remaining spherical conjugacy classes in G are birationally rigid unipotent.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_8, \{r_8\}, \{1\})$	$\mathcal{O}_{r_8}$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 + 2n_8\omega_8$
$(M_1, \{r_1\}, \{1\})$	$\mathcal{O}_{r_1}$	$\sum_{i=1}^{8} 2n_i\omega_i$
$(G, \{1\}, A_1)$	A_1	$n_8\omega_8$
$(G, \{1\}, 2A_1)$	$2A_1$	$n_1\omega_1 + n_8\omega_8$
$(G, \{1\}, 3A_1)$	$3A_1$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7 + n_8\omega_8$
$(G, \{1\}, 4A_1)$	$4A_1$	$\sum_{i=1}^{8} n_i\omega_i$

Table 4.11: Type E_8.

Proposition 4.35. *Theorem 4.11 holds for G of type E_8.*

Proof. From [36, Table 20, 21] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.11 are pairwise distinct. □

4.3.8 Type F_4

Let G be of type F_4. There are no spherical proper Levi subgroups. We list the spherical pseudo-Levi subgroups.

(i) The pseudo-Levi subgroup M_1 is maximal of type $\mathsf{A}_1\mathsf{C}_3$. We have $r_1^2 = 1$ and $Z(M_1) = \langle r_1 \rangle$. Then $\mathcal{O}_{r_1}$ is a (birational) sheet consisting of an isolated class.

(ii) The pseudo-Levi subgroup M_4 is maximal of type B_4. We have $r_4^2 = 1$ and $Z(M_4) = \langle r_4 \rangle$. Then $\mathcal{O}_{r_4}$ is a (birational) sheet consisting of an isolated class.

We proceed with non-semisimple spherical conjugacy classes: the pseudo-Levi subgroup M_4 contains the birationally rigid unipotent class $\mathcal{O}^{M_4}_{x_{\beta_1}(1)}$, where $\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. In particular $\mathcal{O}^{M_4}_{x_{\beta_1}(1)}$ corresponds to the partition $[2^2, 1^5]$ in $\mathrm{SO}_9(\mathbb{C})$ and is a (birational) sheet consisting of an isolated class. The remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_4, \{r_4\}, \{1\})$	$\mathcal{O}_{r_4}$	$n_4\omega_4$
$(M_1, \{r_1\}, \{1\})$	$\mathcal{O}_{r_1}$	$\sum_{i=1}^{4} 2n_i\omega_i$
$(M_4, \{r_4\}, \mathcal{O}^{M_4}_{[2^2,1^5]})$	$\mathcal{O}_{r_4 x_{\beta_1}(1)}$	$\sum_{i=1}^{4} n_i\omega_i$
$(G, \{1\}, A_1)$	A_1	$n_1\omega_1$
$(G, \{1\}, \widetilde{A}_1)$	$\widetilde{A}_1$	$n_1\omega_1 + 2n_4\omega_4$
$(G, \{1\}, A_1 + \widetilde{A}_1)$	$A_1 + \widetilde{A}_1$	$n_1\omega_1 + n_2\omega_2 + 2n_3\omega_3 + 2n_4\omega_4$

Table 4.12: Type F_4.

Proposition 4.36. *Theorem 4.11 holds for G of type F_4.*

Proof. From [36, Table 22, 23, 24] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.12 are pairwise distinct. ☐

4.3.9 Type G_2

Let G be of type G_2. There are no spherical proper Levi subgroups. We list the spherical pseudo-Levi subgroups.

(i) The pseudo-Levi subgroup M_2 is maximal of type $\mathsf{A}_1\tilde{\mathsf{A}}_1$. We have $r_2^2 = 1$ and $Z(M_2) = \langle r_2 \rangle$. Then $\mathcal{O}_{r_2}$ is a (birational) sheet consisting of an isolated class.

(ii) The pseudo-Levi subgroup M_1 is maximal of type A_2. We have $r_1^3 = 1$ and $Z(M_1) = \langle r_1 \rangle$; moreover, r_1 and r_1^{-1} are G-conjugate. Then $\mathcal{O}_{r_1}$ is a (birational) sheet consisting of an isolated class.

The remaining spherical conjugacy classes in G are unipotent birationally rigid.

τ	$\overline{J(\tau)}^{bir}$	$\Lambda(\mathcal{O})$
$(M_2, \{r_2\}, \{1\})$	$\mathcal{O}_{r_2}$	$2n_1\omega_1 + 2n_2\omega_2$
$(M_1, \{r_1\}, \{1\})$	$\mathcal{O}_{r_1}$	$n_1\omega_1$
$(G, \{1\}, A_1)$	A_1	$n_2\omega_2$
$(G, \{1\}, \widetilde{A}_1)$	$\widetilde{A}_1$	$n_1\omega_1 + 2n_2\omega_2$

Table 4.13: Type G_2.

Proposition 4.37. *Theorem 4.11 holds for G of type G_2.*

Proof. From [36, Table 25, 26] the weight monoid is preserved along classes in $Z(G)\overline{J(\tau)}^{bir}$ for every spherical birational sheet $\overline{J(\tau)}^{bir}$. The entries in the third column of Table 4.13 are pairwise distinct. ☐

The proof of Theorem 4.11 is complete. □

Remark 4.38. The classification implies that the birationally rigid unipotent conjugacy class $\mathcal{O}^M$ appearing in the decomposition datum $\tau = (M, Z(M)^\circ z, \mathcal{O}^M)$ is in fact rigid, with two exceptions:

(i) for G of type C_n, $n \geq 3$, let $\tau = (G, \{1\}, [2^2, 1^{2n-4}])$. Then $\overline{J(\tau)}^{bir}$ is contained only in the (spherical) sheet $\overline{J(L_1, Z(L_1)^\circ, \{1\})}^{reg}$;

(ii) for G of type B_{2m}, $m \geq 2$, let $\tau = (G, \{1\}, [3, 2^{2(m-1)}, 1^2])$. Then $\overline{J(\tau)}^{bir}$ is contained only in the (spherical) sheet $\overline{J(L_n, Z(L_n)^\circ, \{1\})}^{reg}$.

In the other cases $\overline{J(\tau)}^{bir}$ is contained only in the sheet $\overline{J(\tau)}^{reg}$: in particular every spherical birational sheet is contained in a unique sheet.

4.3.10 Characterization via horospherical contractions

We conclude this Section with another characterization of spherical birational sheets up to central elements: in order to do this, we recollect some facts on spherical conjugacy classes from [27, 28, 36].

From the Bruhat decomposition $G = \bigcup_{w \in W} BwB$, it follows that for every conjugacy class $\mathcal{O}$ of G there exists a unique $w_{\mathcal{O}} \in W$ such that $\mathcal{O} \cap Bw_{\mathcal{O}}B$ is dense in $\mathcal{O}$. Similarly, for S a sheet in G, there is a unique $w_S \in W$ such that $S \cap Bw_S B$ is dense in S. By [28, Proposition 5.3] if S is a spherical sheet, then for every conjugacy class $\mathcal{O}$ lying in S we have $w_{\mathcal{O}} = w_S$. For a birational sheet $\overline{J(\tau)}^{bir}$ we may define w_τ as the unique element of W such that $\overline{J(\tau)}^{bir} \cap Bw_\tau B$ is dense in $\overline{J(\tau)}^{bir}$. We have $w_{\mathcal{O}} = w_\tau$ for all classes $\mathcal{O} \subset \overline{J(\tau)}^{bir}$ and $w_\tau = w_S$ for all sheets S containing $\overline{J(\tau)}^{bir}$.

If H is a spherical subgroup of G, by [20, Theorem 1], there exists a flat deformation of G/H to a homogeneous spherical space G/H_0, where H_0 contains a maximal unipotent subgroup of G: such an homogeneous space is called *horospherical*, and H_0 a *horospherical contraction* of H, see also [97]. If G/H is isomorphic to a conjugacy class, then $\mathbb{C}[G/H] \simeq_G \mathbb{C}[G/H_0]$, see [36, Theorem 3.15].

Proposition 4.39. *Let G be a complex connected reductive algebraic group with $[G, G]$ simply-connected. Let $x_1, x_2 \in G_{sph}$. Then $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2}$ are contained in the same birational sheet up to a central element if and only if $C_G(x_1)$ and $C_G(x_2)$ have the same horospherical contraction.*

Proof. Let $x \in G_{sph}$ and $H = C_G(x)$. We recall the description of the horospherical contraction H_0 of H containing U from [36, Corollary 3.8]. Let w be the unique element in W such that $\mathcal{O}_x \cap BwB$ is dense in $\mathcal{O}_x$. Up to G-conjugacy we can assume that $x \in wB$, so the dense B-orbit in $\mathcal{O}_x$ is $B \cdot x = \mathcal{O}_x^B$. Then $P := \{g \in G \mid g \cdot \mathcal{O}_x^B = \mathcal{O}_x^B\}$ is a parabolic subgroup containing B, i.e., it is standard. Let $\Theta \subset \Delta$ be such that $P = P_\Theta$ and let w_Θ denote the longest element of the Weyl group of the standard Levi subgroup L_Θ. One has $H_0 = \langle U^-, U_{w_\Theta}, T_x \rangle$, where, $w := w_0 w_\Theta$, $U_{w_\Theta} := U \cap L_\Theta$, $T_x := T \cap C_G(x)$.

We may assume that x_i lies in the dense B-orbit $\mathcal{O}^B_{x_i} \subset Bw_iB$ for $i = 1, 2$. We have seen that $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2}$ lie in the same birational sheet up to a central element if and only if $\Lambda(\mathcal{O}_{x_1}) = \Lambda(\mathcal{O}_{x_2})$. The last equality is equivalent to $w_1 = w_2$ and $T_{x_1} = T_{x_2}$ by [36, Lemma 3.9 and Theorem 3.23]. $\square$

4.4 Remarks for Lie algebras

Let G be simple with $[G, G]$ simply-connected, having described the spherical birational sheets in G, from the tables in Section 4.3 we can build a method to classify all spherical birational sheets in $\mathfrak{g}$. Consider the subset of spherical birational sheets of G defined by:

$$\mathscr{L}(G_{sph}) := \{\overline{J(\tau)}^{bir} \subset G_{sph} \mid \tau = (L, Z(L)^\circ, \mathcal{O}^L) \in \mathscr{B}\mathscr{B}(G)\}. \tag{4.4}$$

Observe that L is in particular a Levi subgroup of G.

Proposition 4.40. *Let G be simple with $[G, G]$ simply-connected and consider $\mathscr{L}(G_{sph})$ as in (4.4). To $\overline{J(\tau)}^{bir} \in \mathscr{L}(G_{sph})$ we attach the birational sheet $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ in $\mathfrak{g}$, where $\mathfrak{l} := \mathrm{Lie}(L)$ and $\exp \mathfrak{O}^L = \mathcal{O}^L$. Then:*

(i) Either $\tau = (L, Z(L)^\circ, \{1\}) \in \mathscr{B}\mathscr{B}(G)$ with L a spherical proper Levi subgroup of G or $\tau = (G, \{1\}, \mathcal{O}^G) \in \mathscr{B}\mathscr{B}(G)$ with $\mathcal{O}^G$ spherical unipotent birationally rigid.

(ii) The birational sheet $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ is spherical in $\mathfrak{g}$.

(iii) All elements in $\mathfrak{g}_{sph}$ are contained in a unique birational sheet in $\mathfrak{g}$ of this form.

In particular, spherical birational sheets of $\mathfrak{g}$ are in bijective correspondence with $\mathscr{L}(G_{sph})$.

Proof. Part (i) follows by inspection of the tables in Section 4.3 and the definition of $\mathscr{L}(G_{sph})$.

To prove (ii), recall that the spherical property is preserved along birational sheets: indeed, by [6, Proposition 1], the subvariety $\mathfrak{g}_{sph}$ consisting of spherical ajoint orbits is a union of sheets and every birational sheet is contained in a sheet. Hence, it is enough to prove that each $\overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ obtained as in the statement contains a spherical element. Let $\xi \in \mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L) \subset \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$. Then, by (i), we have two possibilities. In one case, $(\mathfrak{l}, \mathfrak{O}^L) = (\mathfrak{l}, \{0\})$ with $\mathfrak{l} = \mathrm{Lie}(L)$ a proper Levi subalgebra, then we may assume $\xi \in \mathfrak{z}(\mathfrak{l})^{reg}$ and $\mathfrak{c}_\mathfrak{g}(\xi) = \mathfrak{l}$ implies $C_G(\xi) = L$, which is a spherical Levi subgroup. In the other case, $(\mathfrak{l}, \mathfrak{O}^L) = (\mathfrak{g}, \mathfrak{O}^G)$ with $\mathfrak{O}^G$ nilpotent birationally rigid. Then $\xi \in \mathfrak{O}^G$ and $C_G(\xi) = C_G(\exp \xi)$. This is a spherical subgroup, as $\exp \xi \in \mathcal{O}^G$.

We prove (iii): birational sheets are disjoint by [67, Theorem 4.4], hence $\mathfrak{g}_{sph}$ is the disjoint union of the spherical birational sheets of $\mathfrak{g}$. To complete the proof, for $\xi \in \mathfrak{g}_{sph}$, we build $\overline{J(\tau)^{bir}} \in \mathscr{L}(G_{sph})$ such that $\xi \in \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ with the notation in the statement. Let $\mathfrak{J}$ be the decomposition class of $\mathfrak{g}$ containing ξ, in particular $\mathfrak{J} \subset \mathfrak{g}_{sph}$, by [23, §3.7]. Then, $\mathfrak{J} \cap U_\mathcal{N} \neq \varnothing$ for a neighbourhood $U_\mathcal{N}$ of $\mathcal{N}$ defined as in Theorem 3.2. Let $\xi = \sigma + \nu \in \mathfrak{J} \cap U_\mathcal{N}$, set $g := \exp \xi$. The properties of $\mathcal{U}_\mathcal{N}$ imply $C_G(g) = C_G(\xi)$, hence $\mathcal{O}_g \subset G_{sph}$. Moreover, $g \in J(\tau)$ such that $J(\tau) \cap \mathcal{U}_U \neq \varnothing$ by Theorem 3.2. By Theorem 4.11, the class $\mathcal{O}_g$ is contained in a unique $Z(G)\overline{J(\tau')}^{bir}$ where $\overline{J(\tau')}^{bir}$ is a spherical birational sheet in G. Moreover, $J(\tau') \cap U_\mathcal{U} \neq \varnothing$ by construction, hence $\tau' = (L, Z(L)^\circ, \mathcal{O}^L)$ with L a Levi subgroup of G, by Theorem 3.2. We prove that $\mathfrak{O}_\xi \subset \overline{\mathfrak{J}(\mathfrak{l}, \mathfrak{O}^L)}^{bir}$ where $\mathfrak{l} := \mathrm{Lie}(L)$ and $\exp \mathfrak{O}^L = \mathcal{O}^L$. Let $g = su$ with $s := \exp \sigma \in Z(L)^\circ$ and $u := \exp \nu$. Since $U_{\mathcal{U}}$ is π_G-saturated, $s \in Z(L)^\circ \cap U_{\mathcal{U}}$. Observe that $C_G(s) \supset L$ so that $\mathrm{Lie}(C_G(s)) = \mathfrak{c}_\mathfrak{g}(s) = \mathfrak{c}_\mathfrak{g}(\sigma) \supset \mathfrak{l}$. Moreover, $\mathcal{O}_u^{C_G(s)}$ is birationally induced from $(L, \mathcal{O}^L)$: this implies that $\mathfrak{O}_\nu^{C_G(s)}$ is birationally induced from $(\mathfrak{l}, \mathfrak{O}^L)$ where $\exp \nu = u$ and $\exp \mathfrak{O}^L = \mathcal{O}^L$ by Lemma 1.20 (ii). $\square$

The spherical birational sheets in $\mathfrak{g}$ are:

(i) $\overline{\mathfrak{J}(\mathfrak{l}, \{0\})}^{bir}$ with $\mathfrak{l} = \mathrm{Lie}(L)$ where L is a proper Levi spherical subgroup of G.

(ii) $\overline{\mathfrak{J}(\mathfrak{g}, \mathfrak{O})}^{bir} = \mathfrak{O}$ a nilpotent spherical birationally rigid orbit in $\mathfrak{g}$.

Since $\mathfrak{l}$ as in (i) is in particular a maximal Levi subalgebra of $\mathfrak{g}$, we have $\mathfrak{z}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l})^{reg} \sqcup \{0\}$ and this yields two possibilities:

(i) if $\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}\{0\}$ is birationally induced from $(\mathfrak{l}, \{0\})$, the spherical birational sheet $\overline{\mathfrak{J}(\mathfrak{l}, \{0\})}^{bir}$ coincides with the sheet $\overline{\mathfrak{J}(\mathfrak{l}, \{0\})}^{reg} = \mathrm{Ad}(G)(\mathfrak{z}(\mathfrak{l})^{reg}) \sqcup \mathrm{Ind}_\mathfrak{l}^\mathfrak{g}\{0\}$;

(ii) if $\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}\{0\}$ is not birationally induced from $(\mathfrak{z}(\mathfrak{l}), \{0\})$, then $\overline{\mathfrak{J}(\mathfrak{l}, \{0\})}^{bir} = \mathrm{Ad}(G)(\mathfrak{z}(\mathfrak{l})^{reg})$ and the nilpotent orbit $\mathrm{Ind}_\mathfrak{l}^\mathfrak{g}\{0\}$ is either birationally rigid or it is contained in another spherical birational sheet.

We prove Losev's Conjecture 4.9 restricted to the spherical subvariety $\mathfrak{g}_{sph}$.

Proposition 4.41. *Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be spherical adjoint orbits of $\mathfrak{g}$ and suppose $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$. Then $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are contained in the same birational sheet of $\mathfrak{g}$.*

Proof. We can assume G simple simply-connected. Let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be adjoint orbits in $\mathfrak{g}_{sph}$ such that $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$. For $i = 1, 2$, let $\mathfrak{J}_i$ be the decomposition class in $\mathfrak{g}$ containing $\mathfrak{O}_i$, in particular $\mathfrak{J}_i \subset \mathfrak{g}_{sph}$. Then pick a neighbourhood $U_\mathcal{N}$ of $\mathcal{N}$ defined as in Theorem 3.2 such that $\mathfrak{J}_i \cap U_\mathcal{N} \neq \varnothing$ for $i = 1, 2$. Let $\xi_i = \sigma_i + \nu_i \in \mathfrak{J}_i \cap U_\mathcal{N}$, set $x_i := \exp \xi_i$ for $i = 1, 2$. The properties of $\mathcal{U}_\mathcal{N}$ imply $C_G(x_i) = C_G(\xi_i)$, hence $\mathcal{O}_{x_i} \subset G_{sph}$ and $\Lambda(\mathcal{O}_{x_i}) = \Lambda(\mathfrak{O}_{\xi_i})$ for $i = 1, 2$; transitivity yields $\Lambda(\mathcal{O}_{x_1}) = \Lambda(\mathcal{O}_{x_2})$. Theorem 4.11 and Proposition 4.41 imply that $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2} \subset Z(G)\overline{J(\tau)}^{bir}$ for a unique $\overline{J(\tau)}^{bir} \in \mathscr{L}(G_{sph})$. We conclude that $\mathfrak{O}_1$ and $\mathfrak{O}_2$ lie in the same birational sheet of $\mathfrak{g}$, by Proposition 4.40. $\square$

Putting together Losev's Proposition 4.8 restricted to $\mathfrak{g}_{sph}$ and Proposition 4.41, we get:

Theorem 4.42. *Let $\mathfrak{g}$ be reductive and let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be spherical adjoint orbits of $\mathfrak{g}$. Then $\Lambda(\mathfrak{O}_1) = \Lambda(\mathfrak{O}_2)$ if and only if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are contained in the same birational sheet of $\mathfrak{g}$.* $\square$

CHAPTER

FIVE

UNIVERSAL FILTERED QUANTIZATIONS OF NILPOTENT SLODOWY VARIETIES

As announced in the Introduction, this Chapter is self-contained: it deals with commutative and non-commutative deformation theory of some varieties called conical symplectic singularities, which are very common in Lie Theory. As usual, we begin with some notation and basic results, contained in Section 5.1. We proceed with Section 5.2, where we introduce universal Poisson deformations and filtered quantizations of Poisson algebras in a categorical language. After recalling the definition of a conical symplectic singularity, in Section 5.3 we focus on a particular example of such varieties, the nilpotent Slodowy slice. To any nilpotent orbit $\mathfrak{O}$ one can attach, on one hand, the algebra of regular function on its nilpotent Slodowy slice, on the other hand, a finite W-algebra: we explain when the latter is the universal filtered quantization of the former. We conclude with a deeper analysis of the case of nilpotent subregular orbits in Section 5.4.

5.1 Notations and preliminaries

We have tried to adopt the same notation used in the rest of the work (for Lie algebras, adjoint and coadjoint actions, etc.) whenever possible, but some repetitions were unavoidable. This phenomenon will mostly affect lower case Greek letters, which will lose their connotation with the setting of Lie Theory adopted up to now. Unless explicitly specified, the letters $\alpha, \beta, \gamma, \lambda, \xi, \pi, \phi, \varphi$ will exclusively denote maps and no more imply the meaning in the former chapter. The symbol ω will denote a 2-form. Moreover, elements of Lie algebras will be denoted by lower case Latin letters: $e, f, h, \dots$. In any case, we will define all symbols when we introduce them for the first time.

For $\mathfrak{g}$ a semisimple Lie algebra, the letter ρ will denote the half-sum of positive roots (equivalently the sum of all fundamental weights). Recall that the Weyl group acts on $\mathfrak{h}^*$ via the *dot action* (also called the affine action) in the following manner: $w \bullet x = w(x + \rho) - \rho$ for all $w \in W, x \in \mathfrak{h}^*$. When we write $W_\bullet$ instead of W we mean that the dot action is being considered rather than the usual one.

The universal enveloping algebra of $\mathfrak{g}$ is denoted by $U(\mathfrak{g})$ and we write $Z(\mathfrak{g})$ for the centre of $U(\mathfrak{g})$.

As in the previous Chapter, unadorned tensor products should be read as tensors over $\mathbb{C}$. Every algebra is assumed to be finitely generated over $\mathbb{C}$. If A, B, C are algebras and $\phi\colon A \to C$, $\psi\colon B \to C$ are homomorphisms then we define $\phi\overline{\otimes}\psi\colon A \otimes B \to C$ to be the composition of $\phi \otimes \psi$ with multiplication $C \otimes C \to C$. We use the same notation when the tensor product is taken over an arbitrary ring, provided the resulting map is well-defined.

A vector space V is filtered if it comes with a filtration of finite-dimensional vector subspaces $V_0 \subset V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots$ satisfying $V = \bigcup_{i\geq 0} V_i$. A vector space V is graded if there exist finite-dimensional vector subspaces $V_0, V_1, \ldots, V_i, V_{i+1}, \ldots$ satisfying $V = \bigoplus_{i\geq 0} V_i$; the subspace V_i is the component of degree i. If $V = \bigcup_{i\geq 0} V_i$ is a filtered vector space, the associated graded space is $\operatorname{gr} V = \bigoplus_{i\geq 0} V_i/V_{i-1}$, where we take $V_{-1} = 0$ by convention.

For the purposes of our work, we will assume implicitly that all filtrations and gradings satisfy the condition $\dim V_0 = 1$.

As usual, if V is a vector space, we denote by $S(V)$ the symmetric algebra on V: it comes with the natural grading $S(V) = \bigoplus_{i\geq 0} S^i(V)$, where $S^i(V)$ is the vector subspace of i-th symmetric powers of V. We put $S^{>1}(V) := \oplus_{i>1} S^i(V)$. When V is finite-dimensional, it is well-known that the symmetric algebra $S(V)$ identifies with the $\mathbb{C}[V^*]$, the algebra of regular functions on the dual space V^*. In our exposition we will need the following result.

Lemma 5.1. *Let $V = \bigoplus_{i=1}^n V_i$ and $U = \bigoplus_{i=1}^n U_i$ be finite-dimensional graded vector spaces, with V_i and U_i in degree i.*[1] *Suppose that $\tau\colon S(V) \to S(U)$ is a graded algebra homomorphism. Then τ is surjective if and only if its linear term $\mathrm{d}_0\tau\colon V \to U$ is surjective.*

Proof. The gradings on V and U induce gradings on the symmetric algebras $S(V) = \bigoplus_{k\geq 0} S(V)_k$ and $S(U) = \bigoplus_{k\geq 0} S(U)_k$. For each $i = 1, \ldots, n$, we have $S(U)_i = U_i \oplus S^{>1}(U)_i$, where $S^{>1}(U)_i = S(U)_i \cap S^{>1}(U)$; an analogous description holds for $S(V)$. For each $i = 1, \ldots, n$, we write $\tau_i = \tau|_{V_i}$ and observe that $\tau_i = \mathrm{d}_0\tau_i + \bar{\tau}_i\colon V_i \to S(U)_i$ where $\mathrm{d}_0\tau_i\colon V_i \to U_i$ and $\bar{\tau}_i\colon V_i \to S^{>1}(U)_i$ are linear maps. The linear term of τ is $\mathrm{d}_0\tau = \bigoplus_{i=1}^n \mathrm{d}_0\tau_i$.

Suppose τ surjects, so for $u \in U_i$ there exists $p \in S(V)_i$ such that $\tau(p) = u$. Since $\tau(S^{>1}(V)) \subset S^{>1}(U)$, we see that, if $p = p^1 + p^{>1} \in V_i \oplus S^{>1}(V)_i$, then $\tau(p) = \tau(p^1) = u$. It follows that $\mathrm{d}_0\tau(p) = u$, hence $\mathrm{d}_0\tau$ surjects. Now suppose that $\mathrm{d}_0\tau$ surjects and that $u \in U_i$ with $\mathrm{d}_0\tau(v) = u$ for some $v \in V$. Then $\tau_i(v) = u + \bar{\tau}_i(v)$ and an inductive argument shows that $\bar{\tau}_i(v)$ lies in the image of τ. Hence U lies in the image, which proves that τ is surjective. □

A graded module $M = \bigoplus_{i\geq 0} M_i$ over a graded algebra A is *free graded* if it has a basis $\{m_j\}_{j\in J}$ consisting of homogeneous elements (i.e., $\{m_j\}_{j\in J} \subset \bigcup_{i\geq 0} M_i$).

We say that a filtered map $V \to W$ of filtered vector spaces $\phi\colon V \to W$ is *strictly filtered* if $\phi(V_i) = W_i \cap \phi(V)$. The importance of this definition is that gr is an exact functor from

[1] Note that we do not insist that $V_i \neq 0$ or $U_i \neq 0$ for $1 \leq i \leq n$.

the category of filtered vector spaces with strict morphisms to the category of graded vector spaces [72, Proposition 7.6.13] so that, for instance, a strictly filtered embedding induces an embedding of associated graded vector spaces. All filtered morphisms of vector spaces in this work are assumed to be strictly filtered.

When V is a graded vector space we may regard it as filtered in the usual manner, and identify V with $\operatorname{gr} V$ via the obvious splitting. Note that every graded map of graded vector spaces is a strictly filtered map. We shall often need to consider a map $\phi\colon V \to W$ from a graded space to a filtered space, and we call such a map strictly filtered if it is so when regarded as a map of filtered spaces. Finally, for $v \in V_i \setminus V_{i-1}$ we write $\bar{v} = v + V_{i-1} \in \operatorname{gr} V$ for the top graded component of v.

Lemma 5.2. *Let $A = \bigoplus_{i\geq 0} A_i$ be a finitely generated graded algebra with $A_0 = \mathbb{C}$ and let $M = \bigoplus_{j\geq 0} M_j$ be a graded A-module. Then M is flat if and only if M is a free graded module.*

Proof. This follows directly from [34, Lemma 2.2]. □

Lemma 5.3. *Suppose that B is a commutative filtered algebra and that C and A are commutative filtered B-algebras such that the natural maps $B \to A$ and $B \to C$ are strictly filtered. Assume in addition that $\operatorname{gr} A$ is $\operatorname{gr} B$-flat. There is a natural isomorphism*

$$\operatorname{gr} A \otimes_{\operatorname{gr} B} \operatorname{gr} C \xrightarrow{\sim} \operatorname{gr}(A \otimes_B C)$$

Proof. Since $\operatorname{gr} A$ is flat it is free graded by Lemma 5.2. Hence, A is a free object in the category of filtered B-modules [81, Lemma 5.1, 3°]. By [81, Lemma 8.2] the natural homomorphism $\varphi\colon \operatorname{gr} A \otimes_{\operatorname{gr} B} \operatorname{gr} C \twoheadrightarrow \operatorname{gr}(A \otimes_B C)$ defined on homogeneous elements by $\varphi(\bar{a} \otimes \bar{c}) = \overline{a \otimes c}$ is an isomorphism. A direct verification shows that it is also an algebra homomorphism. □

Fix $n \in \mathbb{N}$. A Poisson algebra is a commutative algebra $\mathcal{A}$ equipped with a Lie bracket $\{\cdot,\cdot\}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in \mathcal{A}$. The *Poisson centre* of $\mathcal{A}$ is $PZ(\mathcal{A}) \coloneqq \{z \in \mathcal{A} \mid \{a, z\} = 0 \text{ for all } a \in \mathcal{A}\}$: for a Poisson $\mathbb{C}$-algebra $\mathcal{A}$, we always have $\mathbb{C} \subset PZ(\mathcal{A})$. We say that a graded (resp. filtered) Poisson algebra $\mathcal{A} = \bigoplus_{i\geq 0} \mathcal{A}_i$ (resp. $\mathcal{A} = \bigcup_{i\geq 0} \mathcal{A}_i$) has Poisson bracket in degree $-n$ if $\{a, b\} \in \mathcal{A}_{i+j-n}$ for $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$. If $\mathcal{A}$ is a filtered Poisson algebra with Poisson bracket in degree $-n$ then $\operatorname{gr} \mathcal{A}$ is a graded Poisson algebra with Poisson bracket in degree $-n$. Similarly we say that a filtered (non-commutative) associative algebra $\mathcal{A} = \bigcup_{i\geq 0} \mathcal{A}_i$ has bracket in degree $-n$ if the commutator $[a, b] = ab - ba \in \mathcal{A}_{i+j-n}$ whenever $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_j$. Such filtered algebras have the property that $\operatorname{gr} \mathcal{A}$ is a graded Poisson algebra with Poisson bracket in degree $-n$ via the formula

$$\{a + \mathcal{A}_{i-1}, b + \mathcal{A}_{j-1}\} \coloneqq [a, b] + \mathcal{A}_{i+j-n-1} \tag{5.1}$$

whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$. Similarly, filtered homomorphisms between filtered algebras with bracket in degree $-n$ induce graded homomorphisms between Poisson algebras with Poisson bracket in degree $-n$. These observations can be upgraded to a well-known categorical statement.

Lemma 5.4. *The associated graded construction defines a functor from the category of filtered algebras with bracket in degree $-n$ to the category of graded Poisson algebras with Poisson bracket in degree $-n$.* $\square$

5.2 Universal Poisson deformations and filtered quantizations

5.2.1 The category of Poisson deformations

For the rest of the Section we keep fixed $n \in \mathbb{N}$ and a positively graded, finitely generated, commutative, integral Poisson algebra $A = \bigoplus_{i \geq 0} A_i$ with Poisson bracket in degree $-n$. Denote by $\mathrm{PAut}(A)$ the group of graded Poisson automorphisms of A. Recall that all graded algebras are assumed to be connected graded, i.e., their component of degree zero is one-dimensional, hence isomorphic to the ground field $\mathbb{C}$. When B is a commutative graded algebra we write B_+ for the unique graded maximal ideal, and $\mathbb{C}_+$ for the corresponding quotient $B/B_+ \simeq \mathbb{C}$. Furthermore, when we say that $\mathcal{A}$ is a graded Poisson B-algebra we insist that the map $B \to \mathcal{A}$ is a graded homomorphism whose image is Poisson central.

Definition 5.5. A *Poisson deformation of* A is a triple $(\mathcal{A}, B, \iota)$ where:

(i) B is a positively graded commutative algebra;

(ii) $\mathcal{A}$ is a graded Poisson B-algebra in degree $-n$, flat as a B-module;

(iii) $\iota\colon \mathcal{A} \otimes_B \mathbb{C}_+ \to A$ is a graded isomorphism of Poisson algebras.

We refer to B as the *base* of the deformation.

Example 5.6. Let $\mathcal{A}$ be a filtered Poisson algebra with bracket in degree $-n$, and let $\iota\colon\ \mathrm{gr}\,\mathcal{A} \to A$ be an isomorphism of Poisson algebras. Define the graded algebra $\mathcal{A}' \coloneqq \bigoplus_{i \geq 0} \mathcal{A}_i t^i$. It inherits a Poisson algebra structure with bracket in degree $-n$ via the rule $\{a_i t^i, a_j t^j\} \coloneqq \{a_i, a_j\} t^{i+j-n}$, for each $a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j$. Let $B \coloneqq \mathbb{C}[t]$ and let ι' be the isomorphism $\mathcal{A}' \otimes_B \mathbb{C}_+ \xrightarrow{\sim} \mathcal{A}'/\mathcal{A}'t \xrightarrow{\sim} \mathrm{gr}\,\mathcal{A} \xrightarrow{\sim} A$. Then $(\mathcal{A}', B, \iota')$ is a Poisson deformation of A and the pair $(\mathcal{A}, \iota)$ is called a *filtered Poisson deformation of* A.

Definition 5.7. A *morphism* $(\mathcal{A}_1, B_1, \iota_1) \to (\mathcal{A}_2, B_2, \iota_2)$ *of Poisson deformations of* A is a pair $\phi = (\phi_1, \phi_2)$ such that:

(i) $\phi_2 \colon B_1 \to B_2$ is a graded algebra homomorphism;

(ii) $\phi_1 \colon \mathcal{A}_1 \to \mathcal{A}_2$ is a graded Poisson algebra homomorphism such that the following diagrams commute

$$\begin{tikzcd} B_1 \arrow[r, "\phi_2"] \arrow[d] & B_2 \arrow[d] \\ \mathcal{A}_1 \arrow[r, "\phi_1"] & \mathcal{A}_2 \end{tikzcd} \tag{5.2}$$

$$\begin{tikzcd}
 & A & \\
\mathcal{A}_1 \otimes_{B_1} \mathbb{C}_+ \arrow[ur, "\iota_1"] \arrow[rr, "\phi_1\otimes\mathrm{id}"] & & \mathcal{A}_2 \otimes_{B_2} \mathbb{C}_+ \arrow[ul, "\iota_2"']
\end{tikzcd} \tag{5.3}$$

We say that ϕ is an isomorphism if both ϕ_1, ϕ_2 are isomorphisms. Write $\mathcal{D} = \mathcal{D}_A$ for *the category of Poisson deformations of* A.

Example 5.8. Let G be semisimple simply-connected. The Lie bracket on $\mathfrak{g}$ endows the coordinate ring $\mathbb{C}[\mathfrak{g}^*]$ with the structure of a graded Poisson algebra. Its Poisson centre coincides with $\mathbb{C}[\mathfrak{g}^*]^G$ which is isomorphic to $\mathbb{C}[\mathfrak{h}^*]^W = \mathbb{C}[\mathfrak{h}^*/W]$ by Chevalley's restriction theorem. One may choose the isomorphism to ensure that it is an isomorphism of graded algebras. The coordinate ring $\mathbb{C}[\mathcal{N}^*]$ of the nullcone is graded via the contracting $\mathbb{C}^\times$-action on $\mathfrak{g}^*$ and a famous theorem of Kostant [57, Proposition 7.13] says that the vanishing ideal of $\mathcal{N}^*$ in $\mathbb{C}[\mathfrak{g}^*]$ is generated by $\mathbb{C}[\mathfrak{g}^*]^G_+$. Hence $\mathbb{C}[\mathcal{N}^*]$ is a positively graded Poisson algebra and there is an isomorphism $\iota \colon \mathbb{C}[\mathfrak{g}^*] \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}_+ \to \mathbb{C}[\mathcal{N}^*]$. Another result of Kostant, [59, Theorem 0.2] implies that $\mathbb{C}[\mathfrak{g}^*]$ is a free $\mathbb{C}[\mathfrak{g}^*]^G$-module and so $(\mathbb{C}[\mathfrak{g}^*], \mathbb{C}[\mathfrak{h}/W], \iota)$ is a Poisson deformation of $\mathbb{C}[\mathcal{N}^*]$.

5.2.2 The category of quantizations of Poisson deformations

We continue to fix $n \in \mathbb{N}$ and A, and we remind the reader that all filtered maps in this Chapter are assumed to be strictly filtered. Our goal is to define a category similar to $\mathcal{D}$ whose objects are the quantizations of A.

Definition 5.9. Recall that for a filtered algebra of degree $-n$, the associated graded algebra carries a Poisson structure via (5.1). A *filtered quantization of* A is a pair $(\mathcal{A}, \iota)$ consisting of a filtered algebra of degree $-n$, and an isomorphism $\iota\colon \operatorname{gr}\mathcal{A} \to A$ of Poisson algebras.

If B is a positively graded commutative algebra and $\mathcal{A}$ is a filtered B-algebra, then we always assume that the natural map $B \to \mathcal{A}$ is strictly filtered. By the introductory remarks, we may identify B with $\operatorname{gr} B$ and regard $\operatorname{gr}\mathcal{A}$ as a $\operatorname{gr} B$-algebra. In this way, $\operatorname{gr}\mathcal{A}$ is also a B-algebra.

Definition 5.10. A *quantization (of a Poisson deformation) of* A is a triple $(\mathcal{A}, B, \iota)$ where:
(i) B is a positively graded commutative algebra;
(ii) $\mathcal{A}$ is a filtered B-algebra of degree $-n$, flat as a B-module;
(iii) $(\operatorname{gr}\mathcal{A}, B, \iota)$ is a Poisson deformation of A.
Once again, we call B the *base* of the quantization.

Definition 5.11. A *morphism* $(\mathcal{A}_1, B_1, \iota_1) \to (\mathcal{A}_2, B_2, \iota_2)$ *of quantizations of* A is a pair $\phi = (\phi_1, \phi_2)$ such that:
(i) $\phi_2 \colon B_1 \to B_2$ is a filtered algebra homomorphism;
(ii) $\phi_1 \colon \mathcal{A}_1 \to \mathcal{A}_2$ is a filtered homomorphism such that (5.2) commutes;
(iii) $\operatorname{gr}\phi = (\operatorname{gr}\phi_1, \operatorname{gr}\phi_2)$ is a morphism $(\operatorname{gr}\mathcal{A}_1, B_1, \iota_1) \to (\operatorname{gr}\mathcal{A}_2, B_2, \iota_2)$ of Poisson deformations where we view $\operatorname{gr}\phi_1$ as a map $\operatorname{gr}\mathcal{A}_1 \otimes_{B_1} B_2 \to \operatorname{gr}\mathcal{A}_2$ via Lemma 5.3.

We say that ϕ is an isomorphism if both ϕ_1, ϕ_2 are so, and we write $\mathcal{Q} = \mathcal{Q}_A$ for *the category of Poisson deformations of* A.

Example 5.12. Retain notation from Example 5.8. Consider the enveloping algebra $U(\mathfrak{g})$ which is filtered of degree -1 with $\operatorname{gr} U(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$. We choose a grading on the centre $Z(\mathfrak{g})$ such that the inclusion $Z(\mathfrak{g}) \to U(\mathfrak{g})$ is strictly filtered. Furthermore $\operatorname{gr} Z(\mathfrak{g}) \simeq Z(\mathfrak{g})$ identifies with $\mathbb{C}[\mathfrak{g}^*]^G \subseteq \mathbb{C}[\mathfrak{g}^*]$ as a subalgebra of $\operatorname{gr} U(\mathfrak{g})$. Thanks to the Harish–Chandra restriction theorem we know that there is a graded algebra isomorphism $Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^{W_\bullet} = \mathbb{C}[\mathfrak{h}^*/W_\bullet] \simeq \mathbb{C}[\mathfrak{h}^*]^W$, see §5.3.2 for a more detailed account. In virtue of Example 5.8 we may regard $(U(\mathfrak{g}), \mathbb{C}[\mathfrak{h}^*/W_\bullet], \iota)$ as a filtered quantization of $\mathbb{C}[\mathcal{N}^*]$. The construction depends on a choice of grading on $Z(\mathfrak{g})$, which is the same as fixing a choice of strictly filtered isomorphism $\mathbb{C}[\mathfrak{h}^*]^W \to Z(\mathfrak{g})$ and it is easily seen that these various choices of isomorphism lead to isomorphic quantizations of $\mathcal{N}^*$.

5.2.3 Properties of morphisms

Throughout the Section we suppose for $i = 1, 2$ that $d_i = (\mathcal{A}_i, B_i, \iota_i)$ are Poisson deformations with algebra maps $\pi_i \colon B_i \to \mathcal{A}_i$. The next lemma suggests an alternative, equivalent definition of a morphism.

Lemma 5.13. *Let $\phi_2 \colon B_1 \to B_2$ be a homomorphism, so that $\pi_2 \circ \phi_2$ endows $\mathcal{A}_2$ with the structure of a B_1-algebra. Let $\varphi_{B_2} \colon \mathcal{A}_1 \otimes_{B_1} B_2 \otimes_{B_2} \mathbb{C}_+ \xrightarrow{\sim} \mathcal{A}_1 \otimes_{B_1} \mathbb{C}_+$ be the natural isomorphism and let $\eta_{\mathcal{A}_1} \colon \mathcal{A}_1 \to \mathcal{A}_1 \otimes_{B_1} B_2$ be the natural map $a \mapsto a \otimes 1$. Then:*

(i) $d = (\mathcal{A}_1 \otimes_{B_1} B_2, B_2, \iota_1 \circ \varphi_{B_2})$ is a Poisson deformation of A.

(ii) The assignment $\phi_1 \mapsto \phi_1 \overline{\otimes} \pi_2$ establishes a (natural) vector space isomorphism

$$\alpha \colon \operatorname{Hom}_{B_1}(\mathcal{A}_1, \mathcal{A}_2) \xrightarrow{\sim} \operatorname{Hom}_{B_2}(\mathcal{A}_1 \otimes_{B_1} B_2, \mathcal{A}_2) \tag{5.4}$$

whose inverse β is given by $\psi \mapsto \psi \circ \eta_{\mathcal{A}_1}$.

(iii) The pair $\phi = (\phi_1, \phi_2) \colon d_1 \to d_2$ is a morphism of Poisson deformations if and only if $(\alpha(\phi_1), \mathrm{id}) \colon d \to d_2$ is so.

Proof. (i) follows directly from the definitions and [17, I.2.7, Corollary 2], whilst (ii) is [18, Chapter II, §4, Proposition 1 (a)].

We prove (iii). The isomorphisms α and β preserve gradings and Poisson structures. By (ii) the pair ϕ satisfies (5.2) if and only if $\alpha(\phi_1)$ is B_2-linear, i.e., if $(\alpha(\phi_1), \mathrm{id})$ satisfies (5.2). Equivalence of (5.3) for ϕ and $(\alpha(\phi_1), \mathrm{id})$ is a consequence of the equality $\phi_1 \overline{\otimes} \pi_2 = (\phi_1 \otimes \mathrm{id}) \circ \varphi_{B_2}$. $\square$

Remark 5.14. A statement analogous to Lemma 5.13 holds if we replace d_1, d_2 with quantizations and φ_{B_2} by $\operatorname{gr} \varphi_{B_2}$ in part (i). The proof is similar.

Lemma 5.15. *Any morphism of deformations of the form $\psi = (\psi_1, \mathrm{id}) \colon (\mathcal{A}_1, B, \iota_1) \to (\mathcal{A}_2, B, \iota_2)$ is an isomorphism of deformations. In particular, the morphisms ψ as in Lemma 5.13 (iii) are* *always isomorphisms. The same is true for morphisms of quantizations* $\phi = (\phi_1, \phi_2)$ *satisfying* $\operatorname{gr}\phi_2 = \mathrm{id}$.

Proof. We start by considering Poisson deformations. Since $\mathcal{A}_1$ is B-flat, it is free over B by Lemma 5.2. Let $V_1 \subseteq \mathcal{A}_1$ be a graded subspace such that the multiplication map $B \otimes V_1 \to \mathcal{A}_1$ is an isomorphism. Let V_2 be the image of V_1 in $\mathcal{A}_2$. We have the following commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, hook] \arrow[d] & \mathcal{A}_1 \arrow[r, two heads] \arrow[d] & \mathcal{A}_1 \otimes_B \mathbb{C}_+ \arrow[d, "(\iota_2)^{-1}\circ\iota_1"] \\ V_2 \arrow[r, hook] & \mathcal{A}_2 \arrow[r, two heads] & \mathcal{A}_2 \otimes_B \mathbb{C}_+ \end{tikzcd} \tag{5.5}$$

Since $\mathcal{A}_1 = V_1 \oplus B_+\mathcal{A}_1$ the horizontal composition along the top row of (5.5) is an isomorphism. At the same time, $(\iota_2)^{-1} \circ \iota_1$ is an isomorphism by (5.3). It follows that the map $V_1 \to V_2$ is injective, whilst it is surjective by definition. We deduce that the horizontal composition along the bottom row of (5.5) is an isomorphism, which implies that $\mathcal{A}_2 = V_2 \oplus B_+\mathcal{A}_2$ as graded vector spaces. Since $\mathcal{A}_2$ is B-flat it is B-free (Lemma 5.2) and $B \otimes V_2 \to \mathcal{A}_2$ is an isomorphism. A $\mathbb{C}$-basis for V is a basis of the free B-module $\mathcal{A}_1$ and ψ_1 sends it to a basis of the free B-module $\mathcal{A}_2$ bijectively, which completes the proof of the first claim.

The statement for morphisms of quantizations follows immediately because ϕ_1 is an isomorphism and all filtered pieces are finite-dimensional. $\square$

Observe that Lemmas 5.13 (iii) and Lemma 5.15 imply that (5.2) is a pushout diagram, in analogy to the requirements on morphisms of deformations of varieties in [91, §2.1].

Corollary 5.16. *Let* $\phi = (\phi_1, \phi_2)\colon (\mathcal{A}_1, B_1, \iota_1) \to (\mathcal{A}_2, B_2, \iota_2)$ *be a morphism in* $\mathcal{D}$ *or* $\mathcal{Q}$*. Then* ϕ *is surjective if and only if* ϕ_2 *is surjective.*

Proof. By Lemma 5.13(ii) we see that ϕ_1 factors as $\phi_1\overline{\otimes}\pi_2 \circ \eta_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{A}_1 \otimes_{B_1} B_2 \to \mathcal{A}_2$. When ϕ_2 is surjective $\eta_{\mathcal{A}_1}$ is surjective, whilst Lemma 5.15 implies that $\phi_1\overline{\otimes}\pi_2$ is an isomorphism, which proves the claim. $\square$

We record another important fact which follows from the definitions, using Lemma 5.4.

Lemma 5.17. *The associated graded construction defines a functor* $\operatorname{gr}\colon \mathcal{Q}_A \to \mathcal{D}_A$*: it is defined on objects by* $(\mathcal{A}, B, \iota) \mapsto (\operatorname{gr}\mathcal{A}, B, \iota)$ *and on morphisms by* $(\phi_1, \phi_2) \mapsto (\operatorname{gr}\phi_1, \operatorname{gr}\phi_2)$ *.* $\square$

5.2.4 Universal deformations and quantizations

Keep fixed $n \in \mathbb{N}$.

Definition 5.18. Let A be a graded Poisson algebra of degree $-n$ and write $\mathcal{D} = \mathcal{D}_A$, $\mathcal{Q} = \mathcal{Q}_A$.

(a) A *universal Poisson deformation* of A is an initial object in $\mathcal{D}$, denoted $u_{\mathcal{D}} := (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota_{\mathcal{D}})$. If $d \in \mathcal{D}$ then we write $\phi^d = (\phi_1^d, \phi_2^d)$ for the unique morphism $u_{\mathcal{D}} \to d$;

(b) A *universal quantization of* A is an initial object in $\mathcal{Q}$, denoted $u_{\mathcal{Q}} := (\mathcal{U}_{\mathcal{Q}}, B_{\mathcal{Q}}, \iota_{\mathcal{Q}})$. If $q \in \mathcal{Q}$ then we write $\phi^q = (\phi_1^q, \phi_2^q)$ for the unique morphism $u_{\mathcal{Q}} \to q$.

As usual the universal objects enjoy the following uniqueness property.

Lemma 5.19. *If an initial object exists in $\mathcal{D}$ or $\mathcal{Q}$ then it is determined upto a unique isomorphism.* $\square$

5.2.5 Interactions between universal deformations and quantizations

Let $\mathcal{C}$ be the category of finitely-generated, commutative graded algebras, a subcategory of all commutative algebras. For a given $B \in \mathcal{C}$ we consider the category $\mathcal{C}_B^{\mathrm{gr}}$ of commutative graded B-algebras, whose objects are (C, ψ) where $C \in \mathcal{C}$ and $\psi\colon B \to C$ is a graded homomorphism. The morphisms $(C_1, \psi_1) \to (C_2, \psi_2)$ in $\mathcal{C}_B^{\mathrm{gr}}$ are the commutative triangles consisting of graded homomorphisms

$$\begin{tikzcd} & B \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ C_1 \arrow[rr, "\psi_0"] & & C_2. \end{tikzcd} \tag{5.6}$$

We also consider the category $\mathcal{C}_B^{\mathrm{filt}}$ whose objects are pairs (C, ψ) where $C \in \mathcal{C}$ is a graded algebra, $\psi\colon B \to C$ is a strictly filtered homomorphism, and morphisms are given by commutative triangles (5.6) consisting of strictly filtered homomorphisms. The first claim of the following lemma is an easy exercise, whilst the second follows from the fact that a strictly filtered map is an isomorphism if and only if the associated graded map is an isomorphism.

Lemma 5.20. *Let $B \in \mathcal{C}$ be a finitely generated, commutative graded algebra.*

(i) Both $\mathcal{C}_B^{\mathrm{gr}}$ and $\mathcal{C}_B^{\mathrm{filt}}$ admit initial objects, given by pairs (B, ψ) with ψ bijective.

(ii) Let $(B, \psi) \in \mathcal{C}_B^{\mathrm{filt}}$. Then (B, ψ) is initial in $\mathcal{C}_B^{\mathrm{filt}}$ if and only if $(B, \mathrm{gr}\,\psi)$ is initial in $\mathcal{C}_B^{\mathrm{gr}}$.

We retain the notation introduced throughout this Section as well as the notation from Lemma 5.13. Suppose that initial objects in $\mathcal{D} = \mathcal{D}_A$ and $\mathcal{Q} = \mathcal{Q}_A$ exist and fix representatives $u_{\mathcal{D}} = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota_{\mathcal{D}})$ and $u_{\mathcal{Q}} = (\mathcal{U}_{\mathcal{Q}}, B_{\mathcal{Q}}, \iota_{\mathcal{Q}})$ in their isomorphism classes.

Proposition 5.21. *When the universal deformation $u_{\mathcal{D}}$ exists, there is an equivalence of categories $\mathcal{F}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{C}_{B_{\mathcal{D}}}^{\mathrm{gr}}$ defined on objects by $d := (\mathcal{A}, B, \iota) \mapsto (B, \phi_2^d)$.*

Proof. Throughout this proof we use the notation $d_i = (\mathcal{A}_i, B_i, \iota_i)$ with $i = 1, 2$ to denote a pair of deformations, and we set $u^{d_i} = (\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} B_i, B_i, \iota_{\mathcal{D}} \circ \varphi_{B_i})$. By Lemma 5.13, $u^{d_i} \in \mathcal{D}$. Also write $\eta_{\mathcal{A}}$ for the natural map from a B-algebra $\mathcal{A}$ to $\mathcal{A} \otimes_B C$ for a B-module C, and $\varphi_B\colon \mathcal{A}_1 \otimes_{B_1} B_2 \otimes_{B_2} \mathbb{C}_+ \xrightarrow{\sim} \mathcal{A}_1 \otimes_{B_1} \mathbb{C}_+$ for a natural isomorphism as in Lemma 5.13. Finally, we write $\pi_i\colon B_i \to \mathcal{A}_i$ for the algebra maps.

Let $\phi = (\phi_1, \phi_2)$ be a morphism $d_1 \to d_2$. Uniqueness of the morphisms ϕ^{d_i} for $i = 1, 2$ ensures commutativity of the triangle

$$\begin{tikzcd} & u_{\mathcal{D}} \arrow[dl, "\phi^{d_1}"'] \arrow[dr, "\phi^{d_2}"] & \\ d_1 \arrow[rr, "\phi"] & & d_2 \end{tikzcd} \tag{5.7}$$

so we define $\mathcal{F}_{\mathcal{D}}$ on morphisms by mapping ϕ to the triangle

$$\begin{tikzcd} & B_{\mathcal{D}} \arrow[dl, "\phi_2^{d_1}"'] \arrow[dr, "\phi_2^{d_2}"] & \\ B_1 \arrow[rr, "\phi_2"] & & B_2. \end{tikzcd} \tag{5.8}$$

It is straightforward to verify that $\mathcal{F}_{\mathcal{D}}$ so defined is a functor. We show that it is an equivalence.

Step 1: $\mathcal{F}_{\mathcal{D}}$ is full. Let $\mathcal{F}_{\mathcal{D}}(d_1) \to \mathcal{F}_{\mathcal{D}}(d_2)$ be a morphism, i.e., a triangle as in (5.6). We need to construct a morphism $\phi\colon d_1 \to d_2$ such that $\mathcal{F}_{\mathcal{D}}(\phi)$ is (5.8). Recall from Lemma 5.15 that $(\phi_1^{d_i} \overline{\otimes} \operatorname{id}, \operatorname{id})\colon u^{d_i} \to d_i$ is an isomorphism. Therefore the composition

$$(\phi_1^{d_2} \overline{\otimes} \operatorname{id}, \operatorname{id}) \circ (\operatorname{id} \otimes \phi_2, \phi_2) \circ ((\phi_1^{d_1} \overline{\otimes} \operatorname{id})^{-1}, \operatorname{id})\colon d_1 \xrightarrow{\sim} u^{d_1} \to u^{d_2} \xrightarrow{\sim} d_2$$

is the sought morphism, where ϕ_2 is an in (5.8).

Step 2: $\mathcal{F}_{\mathcal{D}}$ is faithful. Let $\phi\colon d_1 \to d_2$ be a morphism. We show that we can recover ϕ from $\mathcal{F}_{\mathcal{D}}(\phi)$, i.e., that we can recover ϕ_1 from ϕ_2 and data depending only on d_1 and d_2. By Lemma 5.13(ii) it is enough to recover $\alpha(\phi_1) = \phi_1 \overline{\otimes} \pi_2$. Iterated application of the same lemma gives a morphism

$$\psi\colon (\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} B_1 \otimes_{B_1} B_2, B_2, \iota_{\mathcal{D}} \circ \varphi_{B_1 \otimes_{B_1} B_2}) \xrightarrow{\sim} (\mathcal{A}_1 \otimes_{B_1} B_2, B_2, \iota_1 \circ \varphi_{B_2}) \xrightarrow{\sim} d_2$$

where $\psi = (\psi_1, \psi_2) = ((\phi_1 \overline{\otimes} \pi_2) \circ (\phi_1^{d_1} \overline{\otimes} \pi_1 \otimes \operatorname{id}), \operatorname{id})$. Its factors are isomorphisms by Lemma 5.15. By B_2-linearity of the first component ψ_1, it is enough to recover ψ_1 on elements $x \otimes 1 \otimes 1$, with $x \in \mathcal{U}_{\mathcal{D}}$. i.e., to recover $\psi_1 \circ \eta_{\mathcal{U}_{\mathcal{D}}}$.

It is not hard to verify that $(\eta_{\mathcal{U}_{\mathcal{D}}}, \phi_2^{d_1})\colon u_{\mathcal{D}} \to (\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} B_1 \otimes_{B_1} B_2, B_2, \iota_{\mathcal{D}} \circ \varphi_{B_1 \otimes_{B_1} B_2})$ is a morphism in $\mathcal{D}$. Uniqueness of the morphism $u_{\mathcal{D}} \to d_2$ gives $\psi \circ (\eta_{\mathcal{U}_{\mathcal{D}}}, \phi_2^{d_1}) = \phi^{d_2}$, and so we get $\psi_1 \circ \eta_{\mathcal{U}_{\mathcal{D}}} = \phi_1^{d_2}$.

Step 3: $\mathcal{F}_{\mathcal{D}}$ is essentially surjective. If (B, ϕ) is any object in $\mathcal{C}^{\mathrm{gr}}_{B_{\mathcal{D}}}$ then we can consider the Poisson deformation $d := (\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} B, B, \iota_{\mathcal{D}} \circ \varphi_B)$ and the morphism $\phi^d = (\eta_{\mathcal{U}_{\mathcal{D}}}, \phi)$. Then $\mathcal{F}_{\mathcal{D}}(d) = (B, \phi)$.

It can be verified that the functor $\mathcal{G}_{\mathcal{D}}\colon \mathcal{C}^{\mathrm{gr}}_{B_{\mathcal{D}}} \to \mathcal{D}$ sending the object (C, ψ) to $(\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} C, C, \iota_{\mathcal{D}} \circ \varphi_C)$ and a triangle (5.6) to the morphism $(\operatorname{id} \otimes \psi_0, \psi_0)$ is a quasi-inverse for $\mathcal{F}_{\mathcal{D}}$. $\square$

The same recipe gives a functor $\mathcal{F}_{\mathcal{Q}}\colon \mathcal{Q} \to \mathcal{C}^{\mathrm{filt}}_{B_{\mathcal{Q}}}$ and the proof of the following statement is almost identical to Proposition 5.21, so we omit it.

Proposition 5.22. *When $u_{\mathcal{Q}}$ exists $\mathcal{F}_{\mathcal{Q}}$ is an equivalence of categories.*

Theorem 5.23. *Suppose that $\mathcal{D}$ and $\mathcal{Q}$ admit initial objects $u_{\mathcal{D}} = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota_{\mathcal{D}})$ and $u_{\mathcal{Q}} = (\mathcal{U}_{\mathcal{Q}}, B_{\mathcal{Q}}, \iota_{\mathcal{Q}})$, respectively. Suppose furthermore that:*

(1) there is an isomorphism $\xi\colon B_{\mathcal{Q}} \simeq B_{\mathcal{D}}$ of graded algebras;

(2) $\operatorname{gr} \phi_2^q = \phi_2^{\operatorname{gr} q} \circ \xi$ for every $q \in \mathcal{Q}$.

Then $\operatorname{gr} q \simeq u_{\mathcal{D}}$ *if and only if* $q \simeq u_{\mathcal{Q}}$ *for any* $q \in \mathcal{Q}$.

Proof. Assumption (1) implies that there is a natural associated graded functor $\operatorname{gr}\colon \mathcal{C}^{\text{filt}}_{B_{\mathcal{Q}}} \to \mathcal{C}^{\text{gr}}_{B_{\mathcal{D}}}$, whilst (2) states that we have a natural isomorphism $\operatorname{gr} \circ \mathcal{F}_{\mathcal{Q}} \simeq \mathcal{F}_{\mathcal{D}} \circ \operatorname{gr}\colon \mathcal{Q} \to \mathcal{C}^{\text{gr}}_{B_{\mathcal{D}}}$. Write $q = (\mathcal{A}, B, \iota)$. It follows from Proposition 5.21 that $\operatorname{gr} q \simeq u_{\mathcal{D}}$ if and only if $\mathcal{F}_{\mathcal{D}}(\operatorname{gr} q) = (B, \phi_2^{\operatorname{gr} q})$ is initial in $\mathcal{C}^{\text{gr}}_{B_{\mathcal{D}}}$. Similarly, by Proposition 5.22, $q \simeq u_{\mathcal{Q}}$ if and only if $\mathcal{F}_{\mathcal{Q}}(q) = (B, \phi_2^q)$ is initial in $\mathcal{C}^{\text{filt}}_{B_{\mathcal{Q}}}$. By Lemma 5.20 and the identification ξ these two conditions are equivalent under our assumptions. □

5.2.6 Poisson Γ-deformations

Keep fixed $n \in \mathbb{N}$ and A as before, and now fix a group Γ. If B is a graded commutative algebra with an action of Γ by graded automorphisms, we define the homogeneous ideal $I(\Gamma)$ as the ideal in B generated by $\{b - \gamma \cdot b \mid b \in B, \gamma \in \Gamma\}$ and we write $B_\Gamma := B/I(\Gamma)$. If C is a (graded) algebra equipped with trivial Γ-action then every (graded) Γ-equivariant map $B \to C$ factors through B_Γ. If C is graded and $B \to C$ preserves the gradings then the induced map $B_\Gamma \to C$ also preserves the grading.

Recall that $\operatorname{PAut}(A)$ is the group of graded Poisson automorphisms, and suppose $\Gamma \leq \operatorname{PAut}(A)$.

Definition 5.24. A Γ*-equivariant Poisson deformation of* A is a Poisson deformation $(\mathcal{A}, B, \iota)$ such that:

(i) Γ acts on B by graded automorphisms;
(ii) Γ acts on $\mathcal{A}$ by graded Poisson automorphisms;
(iii) $B \to \mathcal{A}$ is Γ-equivariant;
(vi) $\iota : \mathcal{A} \otimes_B \mathbb{C}_+ \to A$ is Γ-equivariant.

A *morphism of* Γ*-equivariant Poisson deformations of* A is a morphism $\phi = (\phi_1, \phi_2)$ such that both ϕ_1 and ϕ_2 are Γ-equivariant.

Definition 5.25. When the Γ-action on B is trivial we call $(\mathcal{A}, B, \iota)$ a *Poisson* Γ*-deformation of* A. Write $\mathcal{D}^\Gamma$ for the category of Poisson Γ-deformations of A together with their Γ-equivariant morphisms. A *universal Poisson* Γ*-deformation* of A is an initial object in $\mathcal{D}^\Gamma$.

If $(\mathcal{A}, B, \iota)$ is a Poisson Γ-deformation of A then Γ also acts by automorphisms on the fibres of the map $\operatorname{Spec} \mathcal{A} \to \operatorname{Spec} B$. More generally, when $(\mathcal{A}, B, \iota)$ is a Γ-equivariant Poisson deformation Γ acts on the fibres over the set of the Γ-fixed points $(\operatorname{Spec} B)^\Gamma$.

If $u = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota_{\mathcal{D}}) \in \mathcal{D}$ is a universal deformation and $\gamma \in \operatorname{PAut}(A)$ then we can define another universal deformation by

$$ {}^\gamma u = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \gamma^{-1} \circ \iota_{\mathcal{D}}) \tag{5.9} $$

By the universal property of u there is a unique morphism

$$ \phi^\gamma = (\phi_1^\gamma, \phi_2^\gamma) : u \to {}^\gamma u \tag{5.10} $$

Proposition 5.26. *Suppose that A admits a universal Poisson deformation $u = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota_{\mathcal{D}}) \in \mathcal{D}$ and Γ is a subgroup of $\mathrm{PAut}(A)$. Then:*

(i) The map $\gamma \mapsto \phi_1^\gamma$ defines an isomorphism $\mathrm{PAut}(A) \xrightarrow{\sim} \mathrm{PAut}(\mathcal{U}_{\mathcal{D}})$;

(ii) u admits a unique Γ-equivariant structure;

(iii) A admits a universal Poisson Γ-deformation u^Γ determined upto a unique isomorphism:

$$u^\Gamma \coloneqq (\mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} (B_{\mathcal{D}})_\Gamma, (B_{\mathcal{D}})_\Gamma, \iota_{\mathcal{D}} \circ \varphi_{(B_{\mathcal{D}})_\Gamma}).$$

Proof. For the sake of simplicity we write $u = (\mathcal{U}, B, \iota)$. If $\gamma \in \mathrm{PAut}(A)$ then clearly ${}^\gamma u$ is an object in $\mathcal{D}$. For $\gamma, \delta \in \mathrm{PAut}(A)$, we consider ${}^\gamma u$ and ${}^{\gamma\delta} u$. It is not hard to verify that the pair $(\phi_1^\gamma, \phi_2^\gamma)$ also defines a morphism ${}^\delta u \to {}^{\gamma\delta} u$, where (5.3) follows from the same diagram for $u \mapsto {}^\gamma u$ as follows:

$$\begin{tikzcd}
 & A & \\
 & A \arrow[u, "\delta^{-1}"] & \\
\mathcal{U} \otimes_B \mathbb{C}_+ \arrow[ur, "\iota"] \arrow[rr, "\phi_1^\gamma \otimes 1"] & & \mathcal{U} \otimes_B \mathbb{C}_+ \arrow[ul, "\gamma^{-1} \circ \iota"']
\end{tikzcd} \tag{5.11}$$

Uniqueness of the morphism $u \to {}^{\gamma\delta} u$ gives $(\phi_1^\gamma, \phi_2^\gamma) \circ (\phi_1^\delta, \phi_2^\delta) = (\phi_1^{\gamma\delta}, \phi_2^{\gamma\delta})$ so $\mathrm{PAut}(A) \to \mathrm{PAut}(\mathcal{U})$ is a homomorphism. The map $\mathrm{PAut}(A) \to \mathrm{PAut}(\mathcal{U})$ is surjective because for $\phi_1 \in \mathrm{PAut}(\mathcal{U})$ the map $\gamma \coloneqq \iota \circ (\phi_1 \otimes \mathrm{id}) \circ \iota^{-1} \colon A \to A$ lies in $\mathrm{PAut}(A)$ and satisfies $\phi_1^\gamma = \phi_1$. Furthermore, if $\phi_1^\gamma = \mathrm{id}$ then the commutative diagram (5.3) for ϕ^γ would give $\iota = \gamma^{-1} \circ \iota$, implying $\gamma = \mathrm{id}$. Hence $\mathrm{PAut}(A) \to \mathrm{PAut}(\mathcal{U})$ is an isomorphism as claimed and (i) is proven.

Now suppose $\Gamma \leq \mathrm{PAut}(A)$. We claim that the Γ-action on u via $\gamma \mapsto \phi^\gamma$ equips u with a Γ-equivariant structure. The Γ-equivariance of $B \to \mathcal{U}$ may be deduced from (5.2) for ϕ^γ and the Γ-equivariance of ι follows directly from diagram (5.3) for ϕ^γ. Now we claim that this is the unique Γ-equivariant structure for Γ. Suppose that u is equipped with another Γ-equivariant structure, and the induced Γ-action on A coincides with the one we have chosen. Now $\gamma \in \Gamma$ defines a pair of maps ${}^\gamma\phi = ({}^\gamma\phi_1, {}^\gamma\phi_2) \in \mathrm{PAut}(\mathcal{U}) \times \mathrm{Aut}(B)$. In fact the Γ-equivariance of ι implies that ${}^\gamma\phi \colon u \to {}^\gamma u$ is a morphism in $\mathcal{D}$, and so ${}^\gamma\phi = \phi^\gamma$ by uniqueness. This shows that there is a unique Γ-equivariant structure extending the action of Γ on A, settling (ii).

We now prove (iii). It is easy to see that u^Γ is an object in $\mathcal{D}^\Gamma$. Let $d \in \mathcal{D}^\Gamma$ be an object with base B_0. Then there is a unique morphism $\phi^d = (\phi_1^d, \phi_2^d) \colon u \to d$ in $\mathcal{D}$. Since B_0 has trivial Γ-action the map $\phi_2^d \colon B \to B_0$ factors through the quotient $B \to B_\Gamma$. Thanks to Proposition 5.21 we see that ϕ^d factors through $u \to u^\Gamma$. We deduce that there exists a morphism of Poisson deformations $\psi = (\psi_1, \psi_2) \colon u^\Gamma \to d$. The uniqueness of ψ follows quickly from the universal property of u, using the fact that $\phi_1^{u^\Gamma}$ and $\phi_2^{u^\Gamma}$ are both surjective, hence epimorphic. Since ψ_1, ψ_2 are each obtained by factorising a Γ-equivariant map over another such map, it follows that ψ is a Γ-equivariant morphism, which completes the proof. $\square$

In the case of a conical symplectic singularity X the relation between the automorphism group of the graded Poisson algebra $\mathbb{C}[X]$ and isomorphisms of its filtered quantizations is discussed in [67, §3.7].

5.2.7 Γquantizations

Let A be a graded Poisson algebra with Poisson brackets in degree $-n$, and let $\Gamma \leq \mathrm{PAut}(A)$. Our purpose here is to record a version of Proposition 5.26 for the category $\mathcal{Q} = \mathcal{Q}_A$.

Definition 5.27. A Γ-*equivariant quantization* of A is a quantization $(\mathcal{A}, B, \iota)$ such that:

(i) Γ acts on B by filtered automorphisms;

(ii) Γ acts on $\mathcal{A}$ by filtered automorphisms;

(iii) $B \to \mathcal{A}$ is Γ-equivariant;

(vi) $(\mathrm{gr}\,\mathcal{A}, B, \iota)$ is a Γ-equivariant Poisson deformation (in view of (i), (ii), (iii) this is equivalent to asking that $\iota : \mathrm{gr}\,\mathcal{A} \otimes_B \mathbb{C}_+ \to A$ is Γ-equivariant where Γ acts on $(\mathrm{gr}\,\mathcal{A}, B)$ via the associated graded action).

A Γ-*equivariant morphism of quantizations* is a morphism $\phi = (\phi_1, \phi_2)$ between two Γ-equivariant quantizations such that ϕ_1 and ϕ_2 are both Γ-equivariant.

Definition 5.28. A Γ-*quantization* is a Γ-equivariant quantization $(\mathcal{A}, B, \iota)$ such that Γ acts trivially on the base B. The category of Γ-quantizations together with the Γ-equivariant morphisms is denoted $\mathcal{Q}^\Gamma$. A *universal* Γ-*quantization* is an initial object in $\mathcal{Q}^\Gamma$.

If $u = (\mathcal{U}_\mathcal{Q}, B_\mathcal{Q}, \iota_\mathcal{Q}) \in \mathcal{Q}$ is a universal quantization and $\gamma \in \mathrm{PAut}(A)$ then we can define ${}^\gamma u$ and $\phi^\gamma \colon u \to {}^\gamma u$ mimicking the constructions in (5.9) and (5.10).

Proposition 5.29. *Suppose that A admits a universal quantization $u = (\mathcal{U}_\mathcal{Q}, B_\mathcal{Q}, \iota_\mathcal{Q}) \in \mathcal{Q}$. Then:*

(i) the map $\gamma \mapsto \phi_1^\gamma$ defines an isomorphism from $\mathrm{PAut}(A)$ to filtered automorphisms of $\mathcal{U}_\mathcal{Q}$;

(ii) u admits a unique Γ-equivariant structure;

(iii) A admits a universal Γ-quantization determined upto unique isomorphism by

$$u^\Gamma := (\mathcal{U}_\mathcal{Q} \otimes_{B_\mathcal{Q}} (B_\mathcal{Q})_\Gamma, (B_\mathcal{Q})_\Gamma, \iota_\mathcal{Q} \circ \varphi_{(B_\mathcal{Q})_\Gamma}).$$

Proof. The proof is almost identical to the proof of Proposition 5.26 and so we only highlight the differences in the current argument. Write $u = (\mathcal{U}, B, \iota)$ for simplicity. Once again we consider ${}^\gamma u$ and $\phi^\gamma \colon u \to {}^\gamma u$. The maps ϕ_1^γ and ϕ_2^γ are filtered automorphisms by definition. Therefore one can show that $\gamma \to \phi^\gamma$ gives homomorphisms from $\mathrm{PAut}(A)$ to the groups $\mathrm{FAut}(\mathcal{U})$ and $\mathrm{FAut}(B)$ of filtered automorphisms of $\mathcal{U}$ and B, respectively. If $\phi_1 \in \mathrm{FAut}(\mathcal{U})$ then $\phi_1 = \phi_1^{\iota \circ (\mathrm{gr}\,\phi_1 \otimes \mathrm{id}) \circ \iota^{-1}}$ and so $\mathrm{PAut}(A) \to \mathrm{FAut}(\mathcal{U})$ is surjective, and it is injective thanks to (5.3) for $\mathrm{gr}\,\phi_1^\gamma$.

Now ϕ^γ equips u with a Γ-equivariant structure, where (5.3) for $\mathrm{gr}\,\phi^\gamma$ ensures equivariance of ι. Uniqueness of the structure follows from universal property of u. The remaining claims are checked in the same manner as the proof of Proposition 5.26, making use of Proposition 5.22 instead of Proposition 5.21 □

In particular, in the presence of universal objects, the categories $\mathcal{D}^\Gamma$ and $\mathcal{Q}^\Gamma$ have a rather elementary structure, as recorded by the following result.

Proposition 5.30. *Suppose that A admits a universal deformation $u_{\mathcal{D}}$ and a universal quantization $u_{\mathcal{Q}}$.*

(i) The functor $\mathcal{F}_{\mathcal{D}}$ restricts to an equivalence $\mathcal{D}^\Gamma \to \mathcal{C}^{\mathrm{gr}}_{(B_{\mathcal{D}})_\Gamma}$.

(ii) The functor $\mathcal{F}_{\mathcal{Q}}$ restricts to an equivalence $\mathcal{Q}^\Gamma \to \mathcal{C}^{\mathrm{filt}}_{(B_{\mathcal{Q}})_\Gamma}$.

Proof. This follows from Propositions 5.21 and 5.22 along with Propositions 5.26 and 5.29. $\square$

Theorem 5.31. *Suppose that the hypotheses of Theorem 5.23 are satisfied. Let $u^\Gamma_{\mathcal{D}}$ and $u^\Gamma_{\mathcal{Q}}$ be a universal Poisson Γ-deformation and Γ-quantization, respectively. Then for $q \in \mathcal{Q}$ we have $\operatorname{gr} q \simeq u^\Gamma_{\mathcal{D}}$ if and only if $q \simeq u^\Gamma_{\mathcal{Q}}$.*

Proof. Write $q = (\mathcal{A}, B, \iota)$ and let $\psi_{\mathcal{D}} \colon B_{\mathcal{D}} \to (B_{\mathcal{D}})_\Gamma$ and $\psi_{\mathcal{Q}} \colon B_{\mathcal{Q}} \to (B_{\mathcal{Q}})_\Gamma$ be the natural projections. It follows from Proposition 5.21 that $\operatorname{gr} q \simeq u^\Gamma_{\mathcal{D}}$ if and only if $\mathcal{F}_{\mathcal{D}}(\operatorname{gr} q) = (B, \phi_2^{\operatorname{gr} q})$ is isomorphic to $((B_{\mathcal{D}})_\Gamma, \psi_{\mathcal{D}})$ in $\mathcal{C}^{\mathrm{gr}}_{B_{\mathcal{D}}}$. By Proposition 5.22 we have $q \simeq u^\Gamma_{\mathcal{Q}}$ if and only if $\mathcal{F}_{\mathcal{Q}}(q) = (B, \phi_2^q)$ is isomorphic to $((B_{\mathcal{Q}})_\Gamma, \psi_{\mathcal{Q}})$ in $\mathcal{C}^{\mathrm{filt}}_{B_{\mathcal{Q}}}$. The uniqueness condition in Proposition 5.26 ensures that the associated graded Γ-action on $B_{\mathcal{Q}}$ is precisely the Γ-action on $B_{\mathcal{D}}$. This completes the proof. $\square$

5.2.8 Conical symplectic singularities and their deformations

In this Section we apply all of the above results to an important class of Poisson varieties, known as conical symplectic singularities. Let X be a normal algebraic variety such that its smooth locus X^{sm} carries a symplectic form, ω. Since X is normal, the form ω gives rise to the Poisson bracket on $\mathbb{C}[X]$ so X becomes a Poisson algebraic variety. Following [7] we say that X is a *symplectic singularity* if there is a projective resolution of singularities $\varrho \colon \widetilde{X} \to X$ such that the pull-back $\varrho^*\omega$ on $\varrho^{-1}(X^{sm})$ extends to a regular (possibly degenerate) 2-form on $\widetilde{X}$. This property does not depend on the chosen resolution, see [58, §2.1]. An affine Poisson variety X is said to be *conical* if $A = \mathbb{C}[X]$ is a positively graded Poisson algebra in degree $-n$ for some $n \in \mathbb{N}$. Geometrically, this means that X is endowed with a contracting $\mathbb{C}^\times$-action.

A resolution of singularities $\varrho \colon \widetilde{X} \to X$ is called a *symplectic resolution* if $\varrho^*\omega$ is a symplectic form on $\widetilde{X}$.

Example 5.32. Since their introduction, symplectic singularities have become of central interest in Lie Theory: we give here two of the most significant examples, extracted from [9, §1].

(i) Let (V, ω) be a complex symplectic vector space and let $\Gamma \leq \mathrm{Sp}(V)$ be a finite group. Then the quotient variety $V/\Gamma = \operatorname{Spec} \mathbb{C}[V]^\Gamma$ is a conical symplectic singularity by [7, Proposition 2.4]. In particular, for $V = \mathbb{C}^2$, we have $\mathrm{Sp}(V) \simeq \mathrm{SL}_2(\mathbb{C})$. As recalled in the Introduction, in this case the possible groups Γ are parameterized, up to conjugation, by the simply-laced Dynkin diagrams, see [91, §6.1] and the varieties $\mathbb{C}^2/\Gamma$ classify the rational isolated surface singularities, up to analytic isomorphism.

(ii) For $\mathfrak{g}$ simple, let $e \in \mathcal{N}$ with adjoint orbit $\mathfrak{O}$ and consider the bilinear map on $\mathfrak{g}$: $\omega(x, y) \coloneqq \kappa(e)([x, y])$ for all $x, y \in \mathfrak{g}$. Then ω induces a symplectic form (the Kirillov-Kostant-Souriau form) on the quotient $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(e)$, which identifies with the tangent space $\mathrm{T}_e\, \mathfrak{O}$. The Killing isomorphism induces a symplectic structure on the coadjoint orbit $\mathfrak{O}^* \coloneqq \kappa(\mathfrak{O}) \subset \mathcal{N}^*$. Let $X = \operatorname{Spec} \mathbb{C}[\mathfrak{O}^*]$: the boundary of $\mathfrak{O}^*$ has codimension at least 2 in its closure, hence X equals the normalization of the closure of $\mathfrak{O}^*$, see [57, §8.3]. By the work of Panyushev [82], any such X is a conical symplectic singularity.

The following theorem combines results of Losev and Namikawa; see [67, 77, 78].

Theorem 5.33 (Losev, Namikawa)**.** *Let X be a conical symplectic singularity and set $A \coloneqq \mathbb{C}[X]$. Then the categories $\mathcal{Q} = \mathcal{Q}_A$ and $\mathcal{D} = \mathcal{D}_A$ of filtered quantizations and Poisson deformations of X admit initial objects satisfying the hypotheses of Theorem 5.23.*

Proof. Thanks to [67, Proposition 2.12] there is a Poisson deformation $u_{\mathcal{D}} = (\mathcal{U}_{\mathcal{D}}, B_{\mathcal{D}}, \iota)$ such that for any $d \coloneqq (\mathcal{A}, B, \iota) \in \mathcal{D}$ there is a unique pair of homomorphisms (ψ, ϕ) with $\phi \colon B_{\mathcal{D}} \to B$ and $\psi \colon \mathcal{U}_{\mathcal{D}} \otimes_{B_{\mathcal{D}}} B \to \mathcal{A}$ a $B_{\mathcal{D}}$-linear isomorphism. Thanks to Lemma 5.13 this is equivalent to the existence of a unique morphism $u_{\mathcal{D}} \to d$ of Poisson deformations. By the same reasoning, [67, Proposition 3.5] implies that $\mathcal{Q}$ admits an initial object with base isomorphic (actually, equal) to $B_{\mathcal{D}}$ as a graded algebra. Furthermore condition (1) of the universal property in [67, Proposition 3.5] states that $\operatorname{gr} \phi^q = \phi^{\operatorname{gr} q}$ for all $q \in \mathcal{Q}$. □

Corollary 5.34. *Let X be a conical symplectic singularity with Γ a group of $\mathbb{C}^\times$-equivariant Poisson automorphisms. Then $\mathbb{C}[X]$ admits a universal Poisson Γ-deformation $u_{\mathcal{D}}^{\Gamma}$ and a universal Γ-quantization $u_{\mathcal{Q}}^{\Gamma}$. Moreover, for $q = (\mathcal{A}, B, \iota) \in \mathcal{Q}$ we have $q \simeq u_{\mathcal{Q}}^{\Gamma}$ if and only if $\operatorname{gr} q \simeq u_{\mathcal{D}}^{\Gamma}$.*

Proof. By Propositions 5.26 and 5.29 there exist a universal Poisson Γ-deformation $u_{\mathcal{D}}^{\Gamma}$ and a universal Γ-quantization $u_{\mathcal{Q}}^{\Gamma}$. Applying Theorem 5.31 we conclude. □

We now have all the tools to prove one of our main results.

Theorem 5.35. *Let X be a conic symplectic singularity and Γ a group of $\mathbb{C}^\times$-equivariant Poisson automorphisms of $A = \mathbb{C}[X]$.*

(i) There exists a universal Poisson Γ-deformation $u_{\mathcal{D}}^{\Gamma}$ and a universal Γ-quantization $u_{\mathcal{Q}}^{\Gamma}$.
(ii) $u_{\mathcal{Q}}^{\Gamma}$ is the unique quantization of $u_{\mathcal{D}}^{\Gamma}$ up to isomorphism.

Proof. This follows from Corollary 5.34 and Propositions 5.26 and 5.29. □

5.3 Nilpotent Slodowy slices and their quantizations

Throughout this Section we assume G simple and simply-connected. We fix $e \in \mathcal{N}$ and embed it in an $\mathfrak{sl}_2$-triple (e, h, f); we also set $\chi \coloneqq \kappa(e)$. We choose $\mathfrak{h}$ containing h, and we set $\lambda \colon \mathbb{C}^\times \to G$ a cocharacter with $\mathrm{d}_1\lambda(t) = th$. Finally, we denote by $\mathcal{S}_\chi = \chi + \kappa(\mathfrak{g}^f) \subseteq \mathfrak{g}^*$ the Slodowy slice.

5.3.1 Poisson structures on Slodowy slices

We begin by explaining how $\mathcal{S}_\chi$ is naturally equipped with a conical Poisson structure. This structure can be understood in two different ways: either as the transverse Poisson structure to $\mathfrak{g}^*$ at χ as in [39, §2.3], or alternatively via Poisson reduction similar to [47], as we now explain. Along the way, we describe the Kazhdan grading on $\mathbb{C}[\mathfrak{g}^*]$ and $\mathbb{C}[\mathcal{S}_\chi]$.

The torus $\mathrm{Ad}^* \lambda(\mathbb{C}^\times) \leq \mathrm{GL}(\mathfrak{g}^*)$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}^*$: for $i \in \mathbb{Z}$, we set $\mathfrak{g}^*(i) = \{\eta \in \mathfrak{g}^* \mid \mathrm{Ad}^* \lambda(t)\eta = t^i \eta\}$. Using the representation theory of $\mathfrak{sl}_2$, we have $\kappa(\mathfrak{g}^f) \subseteq \bigoplus_{i \leq 0} \mathfrak{g}^*(i)$. Consider the cocharacter:

$$\begin{aligned} \lambda_K \colon \mathbb{C}^\times &\to \mathrm{GL}(\mathfrak{g}^*) \\ t &\mapsto t^{-2} \mathrm{Ad}^* \lambda(t). \end{aligned}$$

This is a linear action on $\mathfrak{g}^*$: it is clear that λ_K shifts the $\mathbb{Z}$-grading on $\mathfrak{g}^*$ relative to λ by -2. In particular, χ is fixed and $\kappa(\mathfrak{g}^f)$ is stabilized by λ_K, hence λ_K stabilizes the slice $\mathcal{S}_\chi$, it acts with negative weights on it, contracting it to χ.

The map λ_K induces an action of $\mathbb{C}^\times$ on the algebra $\mathbb{C}[\mathfrak{g}^*]$ in the usual manner: set

$$\lambda^\#(t)v(\xi) = v(\lambda_K^{-1}(t)(\xi)) = v(t^2 \mathrm{Ad}^* \lambda(t^{-1})(\xi)),$$

for $t \in \mathbb{C}^\times, v \in \mathbb{C}[\mathfrak{g}^*]$ and $\xi \in \mathfrak{g}^*$. This action endows the Poisson algebra $\mathbb{C}[\mathfrak{g}^*]$ with the so-called *Kazhdan grading*: $\mathbb{C}[\mathfrak{g}^*] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\mathfrak{g}^*](i)$, where

$$\mathbb{C}[\mathfrak{g}^*](i) = \{v \in \mathbb{C}[\mathfrak{g}^*] \mid \lambda^\#(t)(v) = t^i v, \forall t \in \mathbb{C}^\times\},\ i \in \mathbb{Z}.$$

Via the identification with the symmetric algebra $\mathbb{C}[\mathfrak{g}^*] \simeq S(\mathfrak{g}) = \bigoplus_{j \geq 0} S^j(\mathfrak{g})$, we get another description for the Kazhdan grading. For each component define $S^j(\mathfrak{g})(k) = \{x \in S^j(\mathfrak{g}) \mid [h, x] = kx\}$, then we have that $S(\mathfrak{g})(i)$ is spanned by all $S^j(\mathfrak{g})(k)$ such that $k + 2j = i$: it follows that $\mathbb{C}[\mathfrak{g}^*]$ with the Kazhdan grading is a Poisson algebra with Poisson brackets in degree -2.

Since $\mathcal{S}_\chi \subset \mathfrak{g}^*$ is stabilized by $\lambda^\#$, through the restriction map $\mathbb{C}[\mathfrak{g}^*] \twoheadrightarrow \mathbb{C}[\mathcal{S}_\chi]$ we get a grading on $\mathbb{C}[\mathcal{S}_\chi]$, which is also referred to as Kazhdan grading: it is non-negative.

We now proceed to describe algebraic Poisson reduction, see [64] for more detail. Thanks to the representation theory of $\mathfrak{sl}_2$ we have an isomorphism $\mathrm{ad}(e) \colon \mathfrak{g}(-1) \xrightarrow{\sim} \mathfrak{g}(1)$. This implies that the skew-symmetric bilinear form

$$\begin{aligned} \omega \colon \mathfrak{g}(-1) \times \mathfrak{g}(-1) &\to \mathbb{C} \\ (x, y) &\mapsto \chi[x, y] \end{aligned}$$

is non-degenerate. We pick an isotropic subspace $\ell \subseteq \mathfrak{g}(-1)$. We let $\ell^{\perp_\omega} = \{x \in \mathfrak{g}(-1) \mid \omega(x, y) = 0\ \forall\, y \in \ell\}$ and set

$$\mathfrak{m}_\ell = \ell \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i), \qquad \mathfrak{n}_\ell = \ell^{\perp_\omega} \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i).$$

Then $\mathfrak{m}_\ell \subset \mathfrak{n}_\ell$ are nilpotent Lie subalgebras of $\mathfrak{g}$ and χ vanishes on $[\mathfrak{m}_\ell, \mathfrak{n}_\ell]$. Let $N_\ell \leq G$ be the connected unipotent algebraic subgroup with $\mathfrak{n}_\ell = \mathrm{Lie}(N_\ell)$, then N_ℓ acts by Poisson automorphisms on $\mathfrak{g}^*$ and $\mathfrak{m}_\ell^*$, and the restriction map $\mu_\ell \colon \mathfrak{g}^* \to \mathfrak{m}_\ell^*$ is N_ℓ-equivariant. Thanks to [47, Lemma 2.1] the coadjoint action gives an isomorphism of affine varieties

$$N_\ell \times \mathcal{S}_\chi \xrightarrow{\sim} \mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell}) = \chi + \mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{m}_\ell) \subset \mathfrak{g}^*, \tag{5.12}$$

where $\mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{m}_\ell) \coloneqq \{\xi \in \mathfrak{g}^* \mid \xi(m) = 0 \text{ for all } m \in \mathfrak{m}_\ell\}$. Therefore N_ℓ acts freely on the fibre $\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})$ and the slice $\mathcal{S}_\chi$ parameterises N_ℓ-orbits in $\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})$. It follows that there is a natural isomorphism of Kazhdan graded algebras

$$\mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)} = \mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{N_\ell} \xrightarrow{\sim} \mathbb{C}[\mathcal{S}_\chi]. \tag{5.13}$$

We define the shift of $\mathfrak{m}_\ell$ as

$$\mathfrak{m}_{\ell,\chi} \coloneqq \{x - \chi(x) \mid x \in \mathfrak{m}_\ell\}. \tag{5.14}$$

Then it can be checked that $I_\chi \coloneqq \mathbb{C}[\mathfrak{g}^*]\mathfrak{m}_{\ell,\chi}$ is the defining ideal of the subvariety $\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell}) \subset \mathfrak{g}^*$, in other words $\mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})] = \mathbb{C}[\mathfrak{g}^*]/I_\chi$.

If $f + I_\chi$ lies in the subalgebra $\mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)}$, a short calculation shows that $\{f, I_\chi\} \subseteq I_\chi$. Therefore there is a natural Kazhdan graded Poisson structure of degree -2 on $\mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)}$ given by $\{f + I_\chi, g + I_\chi\} \coloneqq \{f, g\} + I_\chi$ for $f + I_\chi, g + I_\chi \in \mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)}$. Finally, this Poisson structure is transferred from $\mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)}$ to $\mathbb{C}[\mathcal{S}_\chi]$ via the isomorphism (5.13).

In the special case where ℓ is a Lagrangian subspace of $\mathfrak{g}(-1)$ we have $\mathfrak{m}_\ell = \mathfrak{n}_\ell$ and μ_ℓ is actually a moment map for the action of N_ℓ on $\mathfrak{g}^*$ [47, §3.2], thus the action is Hamiltonian.

Since $\mathfrak{m}_\ell \supset \mathfrak{m}_0$, we have $\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell}) \hookrightarrow \mu_0^{-1}(\chi|_{\mathfrak{m}_0})$ and so $\mathbb{C}[\mu_0^{-1}(\chi|_{\mathfrak{m}_0})] \twoheadrightarrow \mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]$. Using the fact that $\mathcal{S}_\chi \subseteq \mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})$, along with (5.12) we see that $\mathbb{C}[\mu_0^{-1}(\chi|_{\mathfrak{m}_0})]^{\mathrm{ad}(\mathfrak{n}_0)} \hookrightarrow \mathbb{C}[\mu_\ell^{-1}(\chi|_{\mathfrak{m}_\ell})]^{\mathrm{ad}(\mathfrak{n}_\ell)}$ which is an isomorphism of Poisson algebras because both algebras are isomorphic to $\mathbb{C}[\mathcal{S}_\chi]$ as Kazhdan graded algebras, by (5.13). Hence the Poisson structure which we have placed on $\mathcal{S}_\chi$ does not depend on ℓ.

5.3.2 Finite W-algebras

We are ready to introduce finite W-algebras associated to $\mathfrak{g}$: these objects arise as a generalization of the universal enveloping algebra $U(\mathfrak{g})$. We start by explaining their construction and some of their main properties. In the following part, we will prove that these algebras arise as filtered quantizations of algebras of certain conical symplectic singularities which are subvarieties of $\mathfrak{g}^*$.

For $\mathfrak{m}_{\ell,\chi}$ as in (5.14), we consider the left $U(\mathfrak{g})$-ideal $J_\chi \coloneqq U(\mathfrak{g})\mathfrak{m}_{\ell,\chi}$. Since N_ℓ preserves $\mathfrak{m}_\ell$ and χ vanishes on $[\mathfrak{n}_\ell, \mathfrak{m}_\ell]$, then N_ℓ preserves J_χ and the N_ℓ-invariants in the left $U(\mathfrak{g})$-module $Q \coloneqq U(\mathfrak{g})/J_\chi$ inherit an algebra structure from $U(\mathfrak{g})$. The algebra $U(\mathfrak{g}, e) \coloneqq Q^{\mathrm{ad}(\mathfrak{n}_\ell)}$ is known as the *finite W-algebra*. Gan and Ginzburg proved in [47] that the definition is independent of the chosen isotropic subspace $\ell \subset \mathfrak{g}(-1)$. Moreover, if e' is G-conjugate to e, then $U(\mathfrak{g}, e) \simeq U(\mathfrak{g}, e')$;

in particular, if we fix $\mathfrak{g}$ simple, there are only finitely many isomorphism classes of finite W-algebras, indexed by the nilpotent orbits of $\mathfrak{g}$.

We remark that for $e = 0$, the construction for $U(\mathfrak{g}, 0)$ becomes trivial and we get $U(\mathfrak{g}, 0) = U(\mathfrak{g})$: this suggests the possibility to think of (isomorphism classes of) finite W-algebras as a finite family of associative algebras related to $\mathfrak{g}$ and generalizing the universal enveloping algebra.

At the other end, one can consider e regular nilpotent: the W-algebras relative to such elements were first defined by Kostant in [60]. Choosing e as the sum of the root vectors relative to simple roots, the construction yields $U(\mathfrak{g}, e) = U(\mathfrak{b})^{\mathrm{ad}(\mathfrak{n}_-)}$, where $\mathfrak{n}_-$ is the negative nilpotent subalgebra with $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_-$.

The algebra $U(\mathfrak{g})$ is naturally endowed with the standard PBW filtration: the subspace $F_i^{\mathrm{st}} U(\mathfrak{g}) = \mathrm{span}\{x_1 \cdots x_j \mid j \leq i,\, x_1, \ldots, x_j \in \mathfrak{g}\}$. For $j \in \mathbb{Z}$, let $U(j) := \{u \in U(\mathfrak{g}) \mid [h, u] = ju\}$. The *Kazhdan filtration* on $U(\mathfrak{g})$ is defined as follows: $F_i U(\mathfrak{g}) := \sum_{2j+k \leq i} F_j^{\mathrm{st}} U(\mathfrak{g}) \cap U(k)$. We warn the reader that $F_i U(\mathfrak{g}) \neq 0$ for all $i \in \mathbb{Z}$, contrary to the conventions of the rest of this Chapter. In particular $\mathfrak{g}(i) \subset F_{i+2} U(\mathfrak{g})$; one can check that the commutator decreases the degree by -2 and that $\mathrm{gr}\, U(\mathfrak{g})$ is commutative with the induced grading. This descends to a non-negative filtration on both Q and $U(\mathfrak{g}, e)$, known as the *Kazhdan filtration*, as well. The associated graded algebra is $\mathrm{gr}\, U(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ with the Kazhdan grading and under this isomorphism we have an identification $\mathrm{gr}\, J_\chi = I_\chi$. Moreover, $\mathrm{gr}\, U(\mathfrak{g}, e)$ is commutative, so we deduce that $\mathrm{gr}\, U(\mathfrak{g}, e)$ is equipped with a Poisson structure as in (5.1).

By [47, Proposition 5.2] the natural inclusion $\mathrm{gr}\, U(\mathfrak{g}, e) \subseteq (\mathbb{C}[\mathfrak{g}^*]/I_\chi)^{\mathrm{ad}(\mathfrak{n}_\ell)} \simeq \mathbb{C}[\mathcal{S}_\chi]$ is an equality, and it is not hard to check that the Poisson structure on $\mathrm{gr}\, U(\mathfrak{g}, e)$ arising from the noncommutative multiplication coincides with the structure coming from Poisson reduction of $\mathbb{C}[\mathfrak{g}^*]$. Thus $U(\mathfrak{g}, e)$ can be viewed as a quantum analogue of the Kazhdan graded Poisson algebra $\mathbb{C}[\mathcal{S}_\chi]$.

This procedure generalizes the well-known isomorphism $\mathrm{gr}\, U(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ from Example 5.12, which is recovered in the case $e = 0$, where $U(\mathfrak{g}, 0) = U(\mathfrak{g})$ and the Slodowy slice coincides with the whole space $\mathfrak{g}^*$. At the other extremum, for $e \in \mathcal{N}$ regular, we recover the isomorphism $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}^*]^G$, since in this case $\mathbb{C}[\mathcal{S}_\chi] \simeq \mathbb{C}[\mathfrak{h}/W]$ and $U(\mathfrak{g}, e) \simeq Z(\mathfrak{g})$, as described by Kostant in [60].

Actually, for a fixed simple $\mathfrak{g}$, the centre is an invariant of the family of finite W-algebras $U(\mathfrak{g}, e)$ as e varies in $\mathcal{N}$. We have chosen our maximal toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ so that $h \in \mathfrak{h}$. Therefore $\mathbb{C}[\mathfrak{h}^*] \hookrightarrow \mathbb{C}[\mathfrak{g}^*]$ is a Kazhdan graded subalgebra with $\mathfrak{h} \subseteq \mathbb{C}[\mathfrak{h}^*]$ concentrated in degree 2, and the Weyl group W acts on $\mathbb{C}[\mathfrak{h}^*]$ by graded automorphisms. Recall that ρ denotes the half sum of positive roots of $\mathfrak{g}$, and the ρ-shifted invariants are denoted $\mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$. The Poisson centre of $\mathbb{C}[\mathfrak{g}^*]$ is $PZ\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[\mathfrak{g}^*]^G$ and the centre of $U(\mathfrak{g})$ is $Z(\mathfrak{g}) = U(\mathfrak{g})^G$. These algebras are well-understood by the Chevalley restriction theorem and the Harish-Chandra restriction theorem. Consider the natural projection maps

$$\begin{array}{rcccccc} PZ\mathbb{C}[\mathfrak{g}^*] & = & \mathbb{C}[\mathfrak{g}^*]^G & \to & PZ\mathbb{C}[\mathcal{S}_\chi] & & \\ Z(\mathfrak{g}) & = & U(\mathfrak{g})^G & \to & Z(\mathfrak{g}, e) & := & ZU(\mathfrak{g}, e) \end{array} \tag{5.15}$$

Lemma 5.36 (Premet [85, Footnote 1])**.** *The maps* (5.15) *are isomorphisms.* ☐

We have the following commutative diagram

$$\begin{tikzcd} \mathbb{C}[\mathfrak{g}^*] \arrow[d, "\mathrm{res}"] & \mathbb{C}[\mathfrak{g}^*]^G \arrow[l, hook'] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathbb{C}[\mathfrak{h}^*]^W \arrow[r, "\simeq"] & \mathbb{C}[\mathfrak{h}^*]^{W_\bullet} & U(\mathfrak{g})^G \arrow[l, "\simeq"'] \arrow[r, hook] \arrow[d, "\simeq"] & U(\mathfrak{g}) \arrow[d, "\mathrm{pr}"] \\ \mathbb{C}[\mathcal{S}_\chi] & PZ\mathbb{C}[\mathcal{S}_\chi] \arrow[l, hook'] \arrow[rrr, "\simeq"] & & & Z(\mathfrak{g}, e) \arrow[r, hook] & Q. \end{tikzcd} \tag{5.16}$$

The restriction map $\mathbb{C}[\mathfrak{g}^*]^G \to \mathbb{C}[\mathfrak{h}^*]^W$ is an isomorphism by Chevalley's restriction theorem, $U(\mathfrak{g})^G \to \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is the Harish-Chandra isomorphism and the isomorphism $\mathbb{C}[\mathfrak{h}^*]^W \to \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is the shift map $x \mapsto x - \rho(x)$ which maps invariants to ρ-shifted invariants. The isomorphism $PZ\mathbb{C}[\mathcal{S}_\chi] \to Z(\mathfrak{g}, e)$ is the unique map making the diagram commute. Every algebra on the left half of (5.16) is Kazhdan graded. Furthermore, if we consider the grading on $\mathbb{C}[\mathfrak{h}^*]$ with $\{x - \rho(x) \mid x \in \mathfrak{h}\}$ in degree 2, then $\mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is a graded subalgebra and $\mathbb{C}[\mathfrak{h}^*]^W \to \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is a graded homomorphism. Furthermore the isomorphism $\mathbb{C}[\mathfrak{h}^*]^{W_\bullet} \to Z(\mathfrak{g}, e)$ is strict for the Kazhdan filtration.

5.3.3 The universal deformation of a nilpotent Slodowy slice

Now consider the Poisson subvariety $\mathcal{S}_{\chi,\mathcal{N}} := \mathcal{S}_\chi \cap \mathcal{N}^*$, known as the *nilpotent Slodowy variety.*

Lemma 5.37. *$\mathcal{S}_{\chi,\mathcal{N}}$ is a conical symplectic singularity.*

Proof. Thanks to [84, §5, Theorem] the fibres of the restriction of the adjoint quotient map $\pi_{\mathfrak{g}} \colon \mathcal{S}_\chi \to \mathfrak{h}/W$ are irreducible normal complete intersections; moreover, for $x \in \mathfrak{h}/W$, we have $\pi_{\mathfrak{g}}^{-1}(x)^{sm} = \pi_{\mathfrak{g}}^{-1}(x) \cap \mathfrak{g}^{reg}$. In particular, the nullfibre $\mathcal{S}_{\chi,\mathcal{N}}$ is irreducible and normal and $\mathcal{S}_{\chi,\mathcal{N}}^{sm} = \mathcal{S}_\chi \cap \kappa(\mathfrak{O}_{reg})$. The slice $\mathcal{S}_\chi$ inherits a Poisson structure via Hamiltonian reduction as in Section 5.3.1, and on $\mathcal{S}_{\chi,\mathcal{N}}^{sm}$ this Poisson structure agrees with the Kostant–Kirillov–Souriau one descending from $\kappa(\mathfrak{O}_{reg})$, [47, §3.2]. It follows from [48, Proposition 2.1.2] that $\mathcal{S}_{\chi,\mathcal{N}}$ admits a symplectic resolution. The Kazhdan grading on $\mathbb{C}[\mathcal{S}_\chi]$ endows $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ with a positive grading with Poisson brackets in degree -2, thus proving the result. ☐

From now on we identify $PZ\mathbb{C}[\mathcal{S}_\chi] = \mathbb{C}[\mathfrak{h}^*/W]$ and $Z(\mathfrak{g}, e) = \mathbb{C}[\mathfrak{h}^*/W_\bullet]$ as Kazhdan graded algebras, via (5.16). Since the scheme-theoretic fibres of the adjoint quotient map $\mathcal{S}_\chi \to \mathfrak{h}^*/W$ are reduced [84, Theorem 5.4(ii)], it follows from Kostant's theorem [57, Proposition 7.13] that $\mathbb{C}[\mathcal{S}_\chi] \otimes_{\mathbb{C}[\mathfrak{h}^*/W]} \mathbb{C}_+ \simeq \mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ as graded Poisson algebras. For the rest of the Section we pick a graded Poisson isomorphism

$$\iota \colon \mathbb{C}[\mathcal{S}_\chi] \otimes_{\mathbb{C}[\mathfrak{h}^*/W]} \mathbb{C}_+ \to \mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}].$$

Recall that, by Lemma 5.17, there is a grading functor gr from quantizations to Poisson deformations. By [91, Corollary 7.4.1], the adjoint quotient map $\mathcal{S}_\chi \to \mathfrak{h}^*/W$ is flat, which completes the proof of the next result.

Lemma 5.38. *The following hold:*

(i) $(\mathbb{C}[\mathcal{S}_\chi], \mathbb{C}[\mathfrak{h}^*/W], \iota)$ *is a Poisson deformation of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$*;*

(ii) $(U(\mathfrak{g}, e), \mathbb{C}[\mathfrak{h}^*/W_\bullet], \iota)$ *is a quantization of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$*;*

(iii) $(\mathbb{C}[\mathcal{S}_\chi], \mathbb{C}[\mathfrak{h}^*/W], \iota)$ *is the associated graded deformation of* $(U(\mathfrak{g}, e), \mathbb{C}[\mathfrak{h}^*/W_\bullet], \iota)$. □

Combining Theorem 5.33 and Lemma 5.37 we see that $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ always admits a universal Poisson deformation and a universal filtered quantization, and these are fibred over the same base space. It is natural to wonder under what circumstances the objects in (i) and (ii) Lemma 5.38 are universal in their respective categories. As far as Poisson deformations are concerned, the question was answered comprehensively by Lehn–Namikawa–Sorger, by means of the following result.

Theorem 5.39 (Lehn–Namikawa–Sorger [65, Theorems 1.2 and 1.3])**.** *Let* $\mathfrak{g}$ *be a simple Lie algebra and* $e \in \mathcal{N}$ *with* $\mathfrak{O} := \mathfrak{O}_e^G$*. Then* $(\mathbb{C}[\mathcal{S}_\chi], \mathbb{C}[\mathfrak{h}^*/W], \iota)$ *is the universal Poisson deformation of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ *if and only if* $(\mathfrak{g}, \mathfrak{O})$ *does not occur in Table 5.1.*

Type of $\mathfrak{g}$	Any	B C F G	C	G
Type of $\mathfrak{O}$	Regular	Subregular	Two Jordan blocks	dimension 8

Table 5.1

In [65] the authors actually classified the nilpotent orbits for which the adjoint quotient $\mathcal{S}_\chi \to \mathfrak{h}^*/W$ is the formally universal Poisson deformation. It is explained by Namikawa in [78, §5] that when the underlying affine Poisson variety is conical a formally universal deformation can be globalised, leading to a universal Poisson deformation in the sense of Definition 5.18 (see also [67, §2.4]). The regular Slodowy slice is not discussed explicitly in [65], however it is a classical theorem of Kostant [60] that $\mathcal{S}_\chi \to \mathfrak{h}^*/W$ is an isomorphism for χ regular, and so the Poisson structure is trivial in these cases by Lemma 5.36.

The following is one of our main results.

Theorem 5.40. *The following are equivalent:*

(i) $(U(\mathfrak{g}, e), \mathbb{C}[\mathfrak{h}^*/W_\bullet], \iota)$ *is the universal filtered quantization of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$*;*

(ii) the orbit of e *is not listed in Table 5.1.*

Proof. This follows from Corollary 5.34 for $\Gamma = \{1\}$, Lemmas 5.37 and 5.38 and Theorem 5.39. □

Write FAut for filtered automorphisms and PAut for graded Poisson automorphisms. Suppose we choose an element $\mathrm{FAut}\, U(\mathfrak{g}, e)$. Taking the associated graded automorphism and then restricting the scalars to $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ defines a group homomorphism

$$\mathrm{FAut}\, U(\mathfrak{g}, e) \longrightarrow \mathrm{PAut}(\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]), \qquad \phi \mapsto \iota \circ (\mathrm{gr}\, \phi \otimes \mathrm{id}) \circ \iota^{-1} \tag{5.17}$$

Corollary 5.41. *If the orbit of e does not appear in the above table then* (5.17) *is an isomorphism.*

Proof. Combine Proposition 5.29, Theorem 5.33 and Lemma 5.37. $\square$

Remark 5.42. This universal property leads to exceptional isomorphisms with other interesting algebras arising in Representation Theory. In particular, [67, Proposition 3.13] shows that the universal quantization of a simple surface singularity is given by (the Namikawa–Weyl group invariants in) the rational Cherednik algebra for the Weyl group of the same Dynkin type. By the work of Brieskorn and Slodowy we know that these surface singularities are isomorphic to subregular nilpotent Slodowy slices for simply-laced Lie algebras. Hence the subregular simply-laced finite W-algebras are isomorphic to the corresponding spherical symplectic reflection algebras. This observation also follows from Losev's Theorems 5.3.1 and 6.2.2 of [66].

5.4 Deformations in the subregular case

On top of this we assume henceforth that $e \in \mathfrak{g}$ is a subregular nilpotent element; we retain the notation established at the beginning of Section 5.3.

5.4.1 The subregular slice and the automorphism group

Consider the subgroup $C = \mathrm{Stab}_{\mathrm{Aut}(\mathfrak{g})}(\{e,h,f\}) \leq \mathrm{Aut}(\mathfrak{g})$ consisting of automorphisms fixing the $\mathfrak{sl}_2$-triple (e,h,f). Its structure is described in [91, §7.5]. The action of C on $\mathfrak{g}$ descends to an action on $\mathcal{S}_\chi$.

Lemma 5.43. *C acts on $\mathbb{C}[\mathcal{S}_\chi]$ by graded Poisson automorphisms.*

Proof. Recall the notation $\ell, N_\ell, \mu, I_\chi$ from Section 5.3.1 and set $\ell = 0$ and $\mu \coloneqq \mu_0$. Since the Poisson structure on $\mathbb{C}[\mathcal{S}_\chi]$ is defined via the graded isomorphism (5.13) it will suffice to show that C acts by Poisson automorphisms on $\mathbb{C}[\mu^{-1}(\chi|_{\mathfrak{m}_0})]^{N_0}$. Since C preserves the graded pieces of $\mathfrak{g}$, it stabilises both $\mathfrak{m}_0$ and $\mathfrak{n}_0$, and furthermore acts on $\mathbb{C}[\mathfrak{g}^*]$ by automorphisms which preserve the Kazhdan grading. The defining ideal I_χ of $\mu^{-1}(\chi|_{\mathfrak{m}_0})$ in $\mathbb{C}[\mathfrak{g}^*]$ is generated by the Kazhdan graded vector space $\{x - \chi(x) \mid x \in \mathfrak{m}_0\}$ and so C acts by graded automorphisms on $\mathbb{C}[\mu^{-1}(\chi|_{\mathfrak{m}_0})]^{\mathrm{ad}(\mathfrak{n}_0)}$. Since N_0 is connected and unipotent the latter algebra coincides with $\mathbb{C}[\mu^{-1}(\chi|_{\mathfrak{m}_0})]^{N_0}$. To see that the C-action on $\mathbb{C}[\mu^{-1}(\chi|_{\mathfrak{m}_0})]^{N_0}$ is Poisson it suffices to recall that $\{f + I_\chi, g + I_\chi\} \coloneqq \{f, g\} + I_\chi$ for $f + I_\chi, g + I_\chi \in \mathbb{C}[\mu^{-1}(\chi|_{\mathfrak{m}_0})]^{N_0}$. $\square$

5.4.2 The equivariant universal deformation of a subregular nilpotent Slodowy slice

Assume now $\mathfrak{g}_0$ is not simply-laced and choose a simple Lie algebra $\mathfrak{g}$ by determining the Dynkin type as follows:

$$\begin{array}{cl} \mathsf{A}_{2n-1} & \text{if } \mathfrak{g}_0 \text{ is of type } \mathsf{B}_n \\ \mathsf{D}_{n+1} & \text{if } \mathfrak{g}_0 \text{ is of type } \mathsf{C}_n \\ \mathsf{E}_6 & \text{if } \mathfrak{g}_0 \text{ is of type } \mathsf{F}_4 \\ \mathsf{D}_4 & \text{if } \mathfrak{g}_0 \text{ is of type } \mathsf{G}_2. \end{array} \tag{5.18}$$

In this Section we consider the subregular Slodowy slice in $\mathfrak{g}_0^*$ and so we use notation $e_0, \chi_0, \mathcal{S}_{\chi_0}$ to mirror the notation for $\mathfrak{g}$. The nilpotent subregular Slodowy slice for $\mathfrak{g}_0$ is denoted $\mathcal{S}_{\chi_0,\mathcal{N}_0}$. The following result appeared in [44, Lemma 2.23], we include here another proof for the reader's convenience.

Lemma 5.44. *The Poisson varieties $\mathcal{S}_{\chi,\mathcal{N}}$ and $\mathcal{S}_{\chi_0,\mathcal{N}_0}$ are $\mathbb{C}^\times$-isomorphic.*

Proof. It follows from the proofs of [91, Theorem 8.4 & 8.7] that $\mathcal{S}_{\chi,\mathcal{N}}$ and $\mathcal{S}_{\chi_0,\mathcal{N}_0}$ are both $\mathbb{C}^\times$-isomorphic to a simple surface singularity, say $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}] \simeq \mathbb{C}[x,y]^\Gamma \simeq \mathbb{C}[\mathcal{S}_{\chi_0,\mathcal{N}_0}]$ where Γ is a finite subgroup of SL_2. Let $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_0$ denote the Poisson structures on $\mathbb{C}^2/\Gamma$ transported from $\mathcal{S}_{\chi,\mathcal{N}}$ and $\mathcal{S}_{\chi_0,\mathcal{N}_0}$ respectively. Applying the argument in the final paragraph of the proof of [64, Proposition 9.24] we see that $\{\cdot,\cdot\} = c\{\cdot,\cdot\}_0$ for some $c \in \mathbb{C}^\times$. Now apply [64, Remark 6.19] to complete the proof. $\square$

Let Γ_0 be the finite subgroup of C defined in [91, p. 143]. It is isomorphic to the group $\mathrm{Aut}(\Delta)$ of Dynkin diagram automorphism of $\mathfrak{g}$ for all pairs $(\mathfrak{g}, \mathfrak{g}_0)$ except for $(\mathsf{D}_4, \mathsf{C}_3)$ in which case Γ_0 is isomorphic to a subgroup of order 2. In all cases the composition $\Gamma_0 \hookrightarrow C \hookrightarrow \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\Delta)$ is injective and its image is the subgroup of $\mathrm{Aut}(\Delta)$ realizing the Dynkin diagram Δ_0 of $\mathfrak{g}_0$ as a folding of Δ.

By [91, §8.7, Remark 3] there is a morphism of deformations of the algebraic variety $\mathcal{S}_{\chi_0,\mathcal{N}_0} \simeq \mathcal{S}_{\chi,\mathcal{N}}$ where the right hand side is Γ_0-equivariant and the vertical arrows are the adjoint quotient maps:

$$\begin{tikzcd} \mathcal{S}_{\chi_0} \arrow[r, "i"] \arrow[d, "\delta_0"'] & \mathcal{S}_\chi \arrow[d, "\delta"] \\ \mathfrak{h}_0^*/W_0 \arrow[r, "j"] & \mathfrak{h}^*/W. \end{tikzcd} \tag{5.19}$$

By [91, §8.8, Remark 4] the maps (i,j) induce isomorphisms of varieties

$$\mathfrak{h}_0^*/W_0 \simeq (\mathfrak{h}^*/W)^{\Gamma_0}, \qquad \mathcal{S}_{\chi_0} \simeq \mathcal{S}_\chi \times_{\mathfrak{h}^*/W} \mathfrak{h}_0^*/W_0 \tag{5.20}$$

where $(\mathfrak{h}^*/W)^{\Gamma_0} \subseteq \mathfrak{h}^*/W$ is the subscheme of Γ_0-fixed points.

Example 5.45. Assume $\mathfrak{g}_0$ is of type B_n, and $\mathfrak{g}$ is of type A_{2n-1}. Then Γ_0 is a cyclic group of order 2 and $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x_1, \ldots, x_{2n}]/(x_1 + \cdots + x_{2n})$. The only non-trivial element γ in Γ_0 maps x_i to $-x_{2n+1-i}$. Furthermore

$$\mathbb{C}[\mathfrak{h}^*/W] = (\mathbb{C}[x_1, \ldots, x_{2n}]/(x_1 + \cdots + x_{2n}))^{\mathrm{Sym}(2n)} = \mathbb{C}[e_2, e_3, \ldots, e_{2n}]$$

where e_j is the j-th elementary symmetric polynomial. Thus $\gamma e_j = e_j$ for j even and $\gamma e_j = -e_j$ for j odd and the kernel of the natural projection $\mathbb{C}[\mathfrak{h}^*/W] \to \mathbb{C}[(\mathfrak{h}^*/W)^{\Gamma_0}]$ is generated by all e_{2r+1} for $r = 1, \ldots, n-1$.

For each piece of notation at the beginning of Section 5.3 we introduce the same notation for $\mathfrak{g}_0$. For example, $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a maximal toral subalgebra and W_0 is the corresponding Weyl group. Applying the remarks of Section 5.3.3 we see that we may fix graded isomorphisms

$$\iota_0 \colon \mathbb{C}[\mathcal{S}_{\chi_0}] \otimes_{\mathbb{C}[\mathfrak{h}_0^*/W_0]} \mathbb{C}_+ \to \mathbb{C}[\mathcal{S}_{\chi_0, \mathcal{N}_0}]; \qquad \iota \colon \mathbb{C}[\mathcal{S}_\chi] \otimes_{\mathbb{C}[\mathfrak{h}^*/W]} \mathbb{C}_+ \to \mathbb{C}[\mathcal{S}_{\chi, \mathcal{N}}].$$

such that ι is Γ_0-equivariant. Since Γ_0 acts on $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ by graded Poisson automorphisms we can consider the universal Γ_0-deformation of $\mathcal{S}_{\chi,\mathcal{N}}$. We are ready to prove the main results of this Section.

Proposition 5.46. *Let* $(\mathfrak{g}, \mathfrak{g}_0)$ *be as in* (5.18). *Set:*

$$\begin{aligned} u &\coloneqq (\mathbb{C}[\mathcal{S}_\chi], \mathbb{C}[\mathfrak{h}^*/W], \iota), & u_0 &\coloneqq (\mathbb{C}[\mathcal{S}_{\chi_0}], \mathbb{C}[\mathfrak{h}_0^*/W_0], \iota_0); \\ q &\coloneqq (U(\mathfrak{g}, e), Z(\mathfrak{g}, e), \iota), & q_0 &\coloneqq (U(\mathfrak{g}_0, e_0), Z(\mathfrak{g}_0, e_0), \iota_0). \end{aligned}$$

Then the unique morphism $u \to u_0$ (resp. $q \to q_0$) is a surjective morphism of Poisson deformations (resp. of filtered quantizations) of $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}] \simeq \mathbb{C}[\mathcal{S}_{\chi_0,\mathcal{N}_0}]$.

Proof. First, we prove the statement relative to Poisson deformations. By Theorem 5.39, the triple u is the universal Poisson deformation of $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$. By Lemmas 5.38 and 5.44 the triple u_0 is a Poisson deformation of $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$. Hence there is a unique morphism of Poisson deformations $\phi = (\phi_1, \phi_2) \colon u \to u_0$. By $\mathbb{C}^\times$-semi-universality of $\mathbb{C}[\mathcal{S}_\chi]$, see [91, §2.5 and Theorem 8.7], the differentials at zero are equal for j as in (5.19) and the morphism $\mathfrak{h}_0^*/W_0 \to \mathfrak{h}^*/W$ corresponding to ϕ_2. Algebraically this means that $\mathrm{d}_0 j^* = \mathrm{d}_0 \phi_2$, in the notation of Lemma 5.1. Since j is a closed inclusion of affine varieties, j^* and $\mathrm{d}_0 j^*$ are surjective, and so we deduce from Lemma 5.1 that ϕ_2 is surjective. We conclude by Corollary 5.16.

As far as filtered quantizations are concerned, by Theorem 5.40 we have that q is the universal quantization and $\operatorname{gr} q$ the universal Poisson deformation, so there is a unique morphism $\phi^{q_0} \colon q \to q_0$ in $\mathcal{Q}$ and a unique morphism $\phi^{\operatorname{gr} q_0} \colon \operatorname{gr} q \to \operatorname{gr} q_0$ is $\mathcal{D}$. Part (i) implies that $\phi_2^{\operatorname{gr} q_0}$ is surjective. Theorem 5.33 implies that $\operatorname{gr} \phi_2^{q_0} = \phi_2^{\operatorname{gr} q_0}$ and we conclude that $\phi_2^{q_0}$ is also surjective. Now we apply Corollary 5.16 to see that $\phi_1^{q_0}$ is surjective. $\square$

With some restrictions on the Dynkin type, we can prove the following result.

Theorem 5.47. *Let $\mathfrak{g}_0$ be of type B_n, C_n or F_4, where $n \geq 2$ and n is even in type C. Then:*

(i) $(\mathbb{C}[\mathcal{S}_{\chi_0}], \mathbb{C}[\mathfrak{h}_0^*/W_0], \iota_0)$ *is isomorphic to the universal Poisson* Γ_0*-deformation of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$*;*

(ii) $(U(\mathfrak{g}_0, e_0), Z(\mathfrak{g}_0, e_0), \iota_0)$ *is isomorphic to the universal* Γ_0*-quantization of* $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$.

Proof. Retain notation from Proposition 5.46. By Proposition 5.26, the universal Poisson deformation $u = (\mathbb{C}[\mathcal{S}_\chi], \mathbb{C}[\mathfrak{h}^*/W], \iota)$ of $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ admits a unique Γ_0-equivariant structure. Proposition 5.26 implies that the universal Γ_0-deformation of $\mathbb{C}[\mathcal{S}_{\chi,\mathcal{N}}]$ is $u^{\Gamma_0} = (\mathbb{C}[\mathcal{S}_\chi] \otimes_{\mathbb{C}[\mathfrak{h}^*/W]} \mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0}, \mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0}, \iota)$. There exists a unique morphism of Poisson deformations $\psi \colon u \to u^{\Gamma_0}$, we write $\psi = (\psi_1, \psi_2)$ and we claim that $\ker(\phi_2) = \ker(\psi_2)$ when $\mathfrak{g}_0$ has one of the Dynkin types listed in the statement. The map ψ_2 is the natural projection $\mathbb{C}[\mathfrak{h}^*/W] \to \mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0}$, so its kernel is generated by $f - \gamma \cdot f$ where $f \in \mathbb{C}[\mathfrak{h}^*/W]$ and $\gamma \in \Gamma_0$, however we will obtain a different description of the kernel.

If r and s denote the ranks of $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively then we write $d_i, i = 1, \ldots, r$ and $d_i^0, i = 1, \ldots, s$ for the Kazhdan graded degrees of the elementary homogeneous generators of $\mathbb{C}[\mathfrak{h}^*/W]$ and $\mathbb{C}[\mathfrak{h}_0^*/W_0]$ respectively. These degrees are listed in [91, p. 112], and they coincide with the total degrees doubled, viewed as polynomials on $\mathfrak{h}^*$ or $\mathfrak{h}_0^*$. Let $I_0, I_2 \subseteq \{1, ..., r\}$ be the two complementary sets consisting of indexes i such that $d_i \equiv 0 \,(\mathrm{mod}\, 4)$ or $d_i \equiv 2 \,(\mathrm{mod}\, 4)$, respectively. Thanks to our restrictions on the Dynkin label of $\mathfrak{g}_0$ the set $\{d_i \mid i \in I_0\}$ coincides with the collection of all degrees of homogeneous generators of $\mathbb{C}[\mathfrak{h}_0^*]^{W_0}$, whilst $\dim \mathfrak{h}_0 = |I_0|$. Therefore the Kazhdan grading on $\mathbb{C}[\mathfrak{h}_0^*/W_0]$ has degree concentrated in $4\mathbb{Z}$. Since ϕ_2 is graded, the generators of degree d_i with $i \in I_2$ are mapped to zero. Since ϕ_2 is surjective by Proposition 5.46, the generators with degrees d_i where $i \in I_0$ are sent to algebraically independent elements. It follows that $\ker(\phi_2) = (e_i \mid i \in I_2)$.

It is explained in [32, §13] (see also [91, Remark 8.8.4]) that $\mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0} \simeq \mathbb{C}[\mathfrak{h}_0^*/W_0]$ as algebras graded by total degree and, equivalently, by Kazhdan degree. Since ψ_2 is a surjection we can apply the argument of the previous paragraph verbatim to deduce that $\ker(\psi_2) = (e_i \mid i \in I_2)$.

Since $\ker(\phi_2) = \ker(\psi_2)$ we can define a graded isomorphism $\sigma : \mathbb{C}[\mathfrak{h}_0^*/W_0] \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0}$ by setting $\sigma(\phi_2(f)) \coloneqq \psi_2(f)$ where $f \in \mathbb{C}[\mathfrak{h}^*/W]$. We have a commutative triangle of homomorphisms in $\mathcal{C}^{\mathrm{gr}}_{\mathbb{C}[\mathfrak{h}^*/W]}$

$$\begin{tikzcd} & \mathbb{C}[\mathfrak{h}^*/W] \arrow[dl, "\phi_2"'] \arrow[dr, "\psi_2"] & \\ \mathbb{C}[\mathfrak{h}_0^*/W_0] \arrow[rr, "\sigma"] & & \mathbb{C}[\mathfrak{h}^*/W]_{\Gamma_0}. \end{tikzcd} \tag{5.21}$$

Thanks to Proposition 5.21 we see that σ corresponds to an isomorphism of Poisson deformations $(\mathbb{C}[\mathcal{S}_{\chi_0}], \mathbb{C}[\mathfrak{h}_0^*/W_0], \iota_0) \xrightarrow{\sim} u^{\Gamma_0}$, as required. Part (ii) follows from Corollary 5.34. □

We have the following consequence.

Corollary 5.48. *Under the assumptions of Theorem 5.47 the kernel of the map* $U(\mathfrak{g}, e) \twoheadrightarrow U(\mathfrak{g}_0, e_0)$ *is generated by* $\{z - \gamma \cdot z \mid \gamma \in \Gamma_0, z \in Z(\mathfrak{g}, e)\}$.

Proof. This follows directly from Propositions 5.29, 5.46 and Theorem 5.47. $\square$

We conjecture that Theorem 5.47 and Corollary 5.48 hold in general, without the restrictions on Dynkin type.

5.4.3 A presentation for the subregular W-algebra of type B

In this Section we let $G_0 = \mathrm{SO}_{2n+1}$ and $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$. Let $e_0 \in \mathfrak{g}_0$ be a subregular nilpotent element of $\mathfrak{g}_0$ and $\chi_0 \in \mathfrak{g}_0^*$ the corresponding element with respect to the Killing identification. Our purpose here is to give a presentation of the finite W-algebra $U(\mathfrak{g}_0, e_0)$ as a quotient of a shifted Yangian.

By Corollary 5.48 we can express $U(\mathfrak{so}_{2n+1}, e)$ as a quotient of $U(\mathfrak{sl}_{2n}, e)$, whilst [24] allows us to express $U(\mathfrak{gl}_{2n}, e)$ as a truncated shifted Yangian. In order to tie these threads together we record the following observation which follows straight from the definitions.

Lemma 5.49. *The centre of $\mathfrak{gl}_n$ maps to a central element of $U(\mathfrak{gl}_n, e)$ and the quotient by that element is isomorphic to $U(\mathfrak{sl}_n, e)$.* $\square$

In [24] the *shifted Yangian associated to $\mathfrak{gl}_n$* is introduced in full generality, however in this work we only require a special case: we define the shifted Yangian $Y_2(\sigma)$ to be the algebra with generators

$$\{D_1^{(r)}, D_2^{(r)} \mid r > 0\} \cup \{E^{(r)} \mid r > 2n-2\} \cup \{F^{(r)} \mid r > 0\} \tag{5.22}$$

and relations (2.4)-(2.9) from [24] with $n = 2$. Our generators $E^{(r)}$ and $F^{(r)}$ are denoted $E_1^{(r)}$ and $F_1^{(r)}$ in [24] and our definition above corresponds to the shift matrix $\sigma = (s_{i,j})_{1\le i,j\le 2}$ with $s_{1,2} = 2n-2$ and $s_{i,j} = 0$ otherwise. We gather the diagonal generators $D_i^{(r)}$ into power series by setting $D_i(u) := \sum_{r\ge 0} D_i^{(r)} u^{-r} \in Y_2(\sigma)[[u^{-1}]]$ where $D_i^{(0)} := 1$ and consider the series

$$\begin{aligned} Z(u) &= u^{2n} + \sum_{r>0} Z^{(r)} u^{2n-r} \\ &:= u(u-1)^{2n-1} D_1(u) D_2(u-1) \in u^{2n} Y_2(\sigma)[[u^{-1}]]. \end{aligned} \tag{5.23}$$

Lemma 5.50. *The elements $\{Z^{(r)} \mid r > 0\}$ are algebraically independent generators of the centre of $Y_2(\sigma)$. Furthermore for $r = 1, ..., 2n$ we have*

$$Z^{(r)} = \sum_{s=0}^{r} \binom{2n-1}{2n-1-s} (-1)^{2n-s} \sum_{t=0}^{s} D_1^{(t)} \mathring{D}_2{}^{(s-t)} \tag{5.24}$$

where $\mathring{D}_2{}^{(r)} := \sum_{s=0}^{r} \binom{r-1}{r-s} D_2^{(s)}$ and $\mathring{D}_2{}^{(-1)} := 0$.

Proof. The first claim follows from [25, Theorem 2.6] in view of the fact that $u^{-2n+1}(u-1)^{2n-1}$ is invertible in $\mathbb{C}[[u^{-1}]]$. We proceed to prove formula (5.24). Using the binomial theorem we have $(u-1)^{-s} = \sum_{r\ge s} \binom{r-1}{r-s} u^{-r}$. It follows that

$$D_2(u-1) = \sum_{r\ge 0} (u-1)^{-r} D_2^{(r)} = \sum_{r\ge 0} u^{-r} \sum_{s=0}^{r} D_2^{(s)} \binom{r-1}{r-s} = \sum_{r\ge 0} u^{-r} \mathring{D}_2{}^{(r)}. \tag{5.25}$$

If we define $C(u) = \sum_{r\geq 0} C^{(r)} u^{-r} := D_1(u) D_2(u-1)$ then we have

$$C(u) = \sum_{r,s\geq 0} D_1^{(r)} \overset{\circ}{D}_2{}^{(s)} u^{-r-s} = \sum_{r\geq 0} \sum_{s=0}^{r} u^{-r} D_1^{(s)} \overset{\circ}{D}_2{}^{(r-s)}. \tag{5.26}$$

At the same time we have

$$u(u-1)^{2n-1} = \sum_{i=1}^{2n} \binom{2n-1}{i-1} (-u)^i. \tag{5.27}$$

Finally if we have a polynomial $f(u) = \sum_{i=0}^{m} f_i u^i$ and a power series $A(u) = \sum_{r\geq 0} A_r u^{-r}$ then for $r = 0, ..., m$ the u^{m-r} coefficient of $f(u)A(u)$ is $\sum_{s=0}^{r} f_{m-s} A_s$. Since $\binom{2n-1}{-1} = 0$ we can combine this last statement together with (5.26) and (5.27) we arrive at the proof of (5.24). $\square$

Theorem 5.51. *There is a surjective algebra homomorphism*

$$Y_2(\sigma) \twoheadrightarrow U(\mathfrak{g}_0, e_0)$$

with kernel generated by

$$\{D_1^{(r)} \mid r > 1\} \cup \{Z^{(2r-1)} \mid r = 1, ..., n\}.$$

Proof. Let e be a subregular nilpotent element of $\mathfrak{sl}_{2n} \subseteq \mathfrak{gl}_{2n}$. The main result of [24] implies that there is a surjective homomorphism $Y_2(\sigma) \twoheadrightarrow U(\mathfrak{gl}_{2n}, e)$ with kernel generated by $\{D_1^{(r)} \mid r > 1\}$. It follows from [25, Lemma 3.7] that the image of the element $Z^{(1)}$ in (5.23) under the map $Y_2(\sigma) \to U(\mathfrak{gl}_{2n}, e)$ lies in the image of $\mathfrak{z}(\mathfrak{gl}_{2n}) \to Z(\mathfrak{gl}_{2n}) \to U(\mathfrak{gl}_{2n}, e)$. Together with Lemma 5.49 this implies that $U(\mathfrak{sl}_{2n}, e)$ is naturally isomorphic to the quotient of $U(\mathfrak{gl}_{2n}, e)$ by $Z^{(1)}$. Finally by Example 5.45 and Corollary 5.48 there is a surjective algebra homomorphism $U(\mathfrak{sl}_{2n}, e) \twoheadrightarrow U(\mathfrak{g}_0, e_0)$ and the kernel is generated by the image of the elementary symmetric polynomials $\{e_{2r+1} \mid r = 1, ..., n-1\}$ under the isomorphism $\mathbb{C}[\mathfrak{h}^*/W] \to Z(\mathfrak{sl}_{2n}, e)$ discussed in (5.16). Here we use $(\mathfrak{h}, W)$ to denote a torus and Weyl group for $\mathfrak{sl}_{2n}$. To complete the proof of the current Theorem it suffices to show, for $r = 1, ..., n-1$, that the image of e_{2r+1} under $\mathbb{C}[\mathfrak{h}^*/W] \to Z(\mathfrak{sl}_{2n}, e)$ is equal to the image of $Z^{(2r+1)}$ under $Y_2(\sigma) \to U(\mathfrak{gl}_{2n}, e) \to U(\mathfrak{sl}_{2n}, e)$. Once again this follows from [25, Lemma 3.7]. $\square$

ACKNOWLEDGEMENTS

First, I would like to thank Giovanna Carnovale for her support and patience in guiding me through my research. I am very grateful to Francesco Esposito, as well. I think Giovanna and Francesco have done a great job as supervisors: they have always given me more than I asked for, but most importantly, they could inspire and help me through the most frustrating moments of my PhD studies.

Secondly, I am highly indebted to my collaborators Mauro Costantini and Lewis Topley: I have learnt a lot from them, and I hope that they enjoyed working together at least as much as I did. I would also like to thank Bart Van Steirteghem for letting me visit him in Erlangen and for interesting mathematical discussions with him. Mauro, Lewis and Bart could also give me very useful advice for my future and my career: this was very kind of them.

Thirdly, my gratitude goes to the referees of this PhD thesis: Gwyn Bellamy and Eric Sommers, who carefully read the manuscript and gave a very positive feedback along with important suggestions, improving the exposition.

As far as the results in this thesis are concerned, I have to thank George Lusztig for his email which has led to Remark 2.32, Andrea Maffei for his help in the proof of Theorem 3.8 and Eric Sommers for his help in the proof of Proposition 4.3.

My parents Marisa and Franco deserve a special place on this page, as they have never prevented me from pursuing what I wanted to do, even without completely understanding it.

To my closest friends Alessandro, Angela, Beatrice, Ilaria and Valeria: your friendship is very important to me, thanks for being there through happy and sad times. We will find an occasion to celebrate together this achievement. I wish you all obtain what you are fighting for, and that many more reasons to celebrate are to come!

To my former flatmate and PhD colleague Angelina: even if we were not very lucky with the landlord and the house, I would go through it all again – the fun outweighed all the rest!

To all my fellow PhD colleagues: thanks for mutual support in the most difficult moments and for good laughs when we needed some. Whether we stay in academia or not, I hope that we will all remember our PhD as a fulfilling journey.

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