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CICLO XXXII

The Alexander polynomial of certain classes of non-symmetric line arrangements

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Riassunto

Il polinomio di Alexander di un'ipersuperficie proiettiva $V \subset \mathbb{P}^n$ è il polinomio caratteristico dell'azione di monodromia su $H^{n-1}(F, \mathbb{C})$, dove F è la fibra di Milnor di V; tranne nel caso in cui V è liscia, il suo calcolo è un problema aperto. Le ipersuperfici singolari piú studiate riguardo questo problema sono proiettivizzazioni $\overline{\mathcal{A}}$ di configurazioni centrali di iperpiani $\mathcal{A} \subset \mathbb{C}^{n+1}$, perché è possibile cercare di sfruttare la natura combinatoria di tali oggetti; senza perdita di generalitá, si puó assumere $n = 2$. In questa Tesi dimostriamo che il polinomio di Alexander di configurazioni di rette $\overline{\mathcal{A}} \subset \mathbb{P}^2$ che appartengono ad alcune classi di configurazioni di rette non simmetriche è banale: questo è un indizio a favore della validitá di una congettura proposta da Papadima e Suciu.

La Tesi è organizzata come segue. Il Capitolo 1 è una collezione di risultati noti su cui ci baseremo: la discussione delle strutture di Hodge miste sui gruppi di coomologia di varietá algebriche e il confronto tra la filtrazione polare e quella di Hodge sono di particolare importanza; anche la costruzione di iperrisoluzioni cubiche e il loro uso nel definire la coomologia di de Rham di varietá algebriche singolari sará molto utile. Il Capitolo 2 è diviso in due parti. La prima è dedicata principalmente a definire il polinomio di Alexander e a presentare una formula di Libgober che ne permette il calcolo quando V è una curva. La seconda è una panoramica di risultati noti sul problema del calcolo del polinomio di Alexander di configurazioni di rette, e si chiude con una discussione di alcuni tra gli esempi piú interessanti; cerchiamo di evidenziare come la simmetria di una configurazione di rette influisca sul suo polinomio di Alexander. Nel Capitolo 3 introduciamo alcune classi di configurazioni di rette $\overline{\mathcal{A}}$ non simmetriche e dimostriamo che i loro polinomi di Alexander sono banali. I metodi che usiamo sono sostanzialmente due: uno combina la formula di Libgober con un semplice argomento di teoria della deformazione, grazie al quale possiamo ridurci a studiare un numero finito di 'configurazioni rappresentative'; l'altro si basa sull'associare ad $\overline{\mathcal{A}}$ un threefold T fibrato in superfici su $\mathbb{P}^1$ e sullo studio della monodromia attorno ad una fibra speciale di quest'ultimo. Il punto chiave del secondo metodo è la dimostrazione dell'esistenza di un morfismo di Gysin che mette in relazione la coomologia di T con quella di una sua sezione di iperpiano S: questo risultato è di interesse indipendente, perché T ed S non soddisfano le ipotesi di solito necessarie per ottenere risultati di tipo Lefschetz.

Abstract

The Alexander polynomial of a projective hypersurface $V \subset \mathbb{P}^n$ is the characteristic polynomial of the monodromy operator acting on $H^{n-1}(F, \mathbb{C})$, where F is the Milnor fibre of V; unless V is smooth, the problem of its computation is open. The singular hypersurfaces that have drawn the most attention are projectivisations $\overline{\mathcal{A}}$ of central hyperplane arrangements $\mathcal{A} \subset \mathbb{C}^{n+1}$, as one can hope to take advantage of the combinatorial nature of such objects; one can assume without loss of generality that $n = 2$. In this Thesis we prove that the Alexander polynomials of line arrangements $\overline{\mathcal{A}} \subset \mathbb{P}^2$ belonging to some particular non-symmetric classes are trivial: this constitutes evidence in favour of the validity of a conjecture due to Papadima and Suciu.

The Thesis is organised as follows. In Chapter 1 we gather some known results on which we will build upon: the discussion of mixed Hodge structures on cohomology groups of algebraic varieties and the comparison between the polar and Hodge filtration are of particular importance; the construction of cubical hyperresolutions and their use in the definition of algebraic de Rham cohomology for singular algebraic varieties will be very useful too. Chapter 2 is divided in two parts. The first one is mainly devoted to defining the Alexander polynomial and presenting a formula by Libgober for its computation in case V is a curve. The second part is a survey of known results around the problem of determining the Alexander polynomial of a line arrangement, and closes with a discussion of some interesting examples; we try to highlight how the symmetry of the arrangement affects its Alexander polynomial. In Chapter 3 we introduce some classes of non-symmetric line arrangements $\overline{\mathcal{A}}$ and prove that their Alexander polynomials are trivial. The methods we use are essentially two: one is the combination of Libgober's formula with an easy deformation theory argument, thanks to which we can restrict ourselves to considering a finite number of 'representative arrangements'; the other relies on associating to $\overline{\mathcal{A}}$ a threefold T fibred in surfaces over $\mathbb{P}^1$ and on studying the monodromy around a special fibre of the latter. A key step of the second method is the proof of the existence of a Gysin morphism that connects the cohomology of T to that of a hyperplane section S: this result is of independent interest, as T and S do not satisfy the hypotheses usually required in order to obtain Lefschetz-type results.

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Introduction

The history of the Alexander polynomial goes back to the late 1920s: in fact, it makes its first appearance in Alexander's work [1], where it is defined as a polynomial invariant for knots. A few years later, Zariski realised that methods similar to the ones used by Alexander could be employed to study the topology of fundamental groups of complements of plane projective singular curves [71, 72]: in this way he found a connection between the irregularity of birational models of coverings of $\mathbb{P}^2$ branched over certain singular curves and the fundamental groups of the complements of the curves; in particular, he was able to exhibit two families of irreducible sextic curves with six cusps such that no member of one family can be equisingularly deformed to a member of the other [72].

After Zariski's works, the study of the fundamental groups of complements of plane projective singular curves went through a long period of stagnation, with only a few sparse results that did not generate much follow-up. It was only in the early 1980s, thanks to the then emerging field of singularity theory, that the interest in this problem was revamped. Indeed, the first explicit definition of the Alexander polynomial of a plane projective curve C appeared in Libgober's paper [42, Section 2], while in [41] the precise connection between coverings of $\mathbb{P}^2$ branched over C and the Alexander polynomial of C is worked out. This was made possible by the introduction of constants of quasi-adjunction, which are positive rational numbers associated to singularity types, and the closely related quasi-adjunction ideals $\mathcal{A}_k$:

Theorem. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d with r irreducible components, and let $k_1, \dots, k_m$ be all the constants of quasi-adjunction of C. The Alexander polynomial of C is*

$$\Delta_C(t) = (t-1)^{r-1} \prod^{dk_j \in \mathbb{Z}} [(t - e^{2\pi i k_j})(t - e^{-2\pi i k_j})]^{s(k_j)}$$

where $s(k_j) := \dim H^1(\mathbb{P}^2, \mathcal{A}_{k_j}(d-3-dk_j))$. The sum of the $s(k_j)$ is the irregularity of a resolution of singularities of a d-fold covering of $\mathbb{P}^2$ branched over C.

The values $s(k_j)$ can be interpreted as defects of linear systems of curves passing through the singularities of C, so the formula above shows a dependence between the Alexander polynomial of C and the relative position of its singularities. Indeed, the aforementioned result of Zariski can be rephrased by saying that the Alexander polynomial of an irreducible sextic with six cusps is $t^2 - t + 1$ if the cusps lie on a conic and 1 otherwise.

Soon after the works [41, 42], Randell proved in [63] that the Alexander polynomial of the curve $C = V(f)$ coincides with the characteristic polynomial of the algebraic monodromy action on $H^1(F, \mathbb{C})$, where F is the Milnor fibre of C. The latter is any smooth fibre of the smooth locally trivial fibration

$$f : \mathbb{C}^3 \setminus f^{-1}(0) \to \mathbb{C}^*$$

which is usually referred to as Milnor fibration. For this reason, in this Thesis we define the Alexander polynomial starting from the Milnor fibration.

We point out that if we write the Alexander polynomial of a curve C as

$$\Delta_C(t) = (t-1)^{r-1} q(t)$$

then it is difficult to find curves for which $q(t) \neq 1$: those for which $q(t)$ is indeed non-trivial often have a rich geometry, as Libgober's result suggests.

A class of curves that has drawn a lot of interest is that of line arrangements i.e. collections of lines in $\mathbb{P}^2$; we denote such objects by $\overline{\mathcal{A}}$, while $\mathcal{A}$ is used for their affine cones. The reason for this interest is that one may try and take advantage of the combinatorial nature of a line arrangement $\overline{\mathcal{A}}$, encoded in its intersection semilattice $L(\mathcal{A})$, to obtain information on its Alexander polynomial. Indications that such an approach could be fruitful were obtained in the 1990s: indeed, after the introduction of characteristic varieties [48] in an attempt to extend the theory of Alexander polynomials to higher dimensions, it was realised (see [43, 50]) that these varieties have a deep connection with the Orlik-Solomon algebra of $\overline{\mathcal{A}}$, which in turn depends on $L(\mathcal{A})$. The Orlik-Solomon algebra of $\overline{\mathcal{A}}$ can also be studied by means of resonance varieties, introduced in [25].

It must be mentioned that characteristic varieties are related to the cohomology of certain rank one local systems [36]. This is important, as the cohomology of local systems can be studied effectively using methods of de Rham and Hodge theory: for example, one could study the de Rham complex depending on the flat connection corresponding to a local system.

Soon after a systematic study of the Alexander polynomial of line arrangements had started, the following problem was raised [35, Problem 9A], [38, Problem 4.145]:

Problem. *Given a line arrangement $\overline{\mathcal{A}} \subset \mathbb{P}^2$, is its Alexander polynomial $\Delta_{\overline{\mathcal{A}}}$ determined by $L(\mathcal{A})$? If so, give an explicit combinatorial formula to compute it.*

As of the time of writing, it remains almost completely open. It is even still unclear which conditions a line arrangement has to satisfy in order for $q(t)$ to have positive degree; however, there is evidence that such a condition is, in some sense, symmetry. This symmetry is encoded in the combinatorial notion of *k-multinet*: this is a partition of the lines of $\overline{\mathcal{A}}$ into k classes $\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_k$ of the same cardinality such that the intersections between lines in different classes satisfy some compatibility condition.

To the best of our knowledge, all arrangements whose Alexander polynomial has non-trivial factor $q(t)$ admit a k-multinet; some of these arrangements have been known, for different reasons, for a long time: the Hesse arrangement and the Pappus arrangement, for example, admit a 4-net and a 3-net, respectively. The former is, to date, the only known non-central line arrangement admitting a 4-net.

Over the course of the years this Problem was tackled using a wide variety of techniques. On the geometric side, the main tools were defects of linear systems, logarithmic forms and mixed Hodge theory [4, 6, 13, 17, 18, 19, 44, 46]. A different approach, which can be applied to any curve, relies on establishing a connection between the Alexander polynomial of a curve and the arithmetic and geometric properties of elliptic surfaces and threefolds associated to it; it was pursued, starting from 2008, by Cogolludo-Agustin, Kloosterman, Libgober et al. [5, 38]. The topological approach can be traced back to the work of Cohen and Suciu [7, 8] on characteristic varieties of arrangements, which builds on Arapura's theory [2] of characteristic varieties of quasi-projective manifolds. Finally, combinatorial techniques allowed to find a connection between multinets on a line arrangement $\overline{\mathcal{A}}$ and complex resonance varieties of its Orlik-Solomon algebra: this connection, established in [27, 53] and further developed in [61, 69], was the key tool in may following works [10, 18, 20, 66].

A partial positive answer to the Problem above was given by Papadima and Suciu in 2017 [59]. They proved the following:

Theorem. *If $\overline{\mathcal{A}}$ is an arrangement with only double and triple points then its Alexander polynomial is*

$$\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{|\overline{\mathcal{A}}|-1}(t^2+t+1)^{\beta_3(\overline{\mathcal{A}})}$$

where $0 \leq \beta_3(\overline{\mathcal{A}}) \leq 2$ depends only on $L(\mathcal{A})$.

This result, together with the evidence gathered throughout the years, led them to formulate the following conjecture:

Conjecture. *The Alexander polynomial of a line arrangement $\overline{\mathcal{A}}$ has the form*

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\overline{\mathcal{A}}|-1}(t^2+t+1)^{\beta_3(\overline{\mathcal{A}})}[(t+1)(t^2+1)]^{\beta_2(\overline{\mathcal{A}})}$$

where $\beta_2(\overline{\mathcal{A}})$ and $\beta_3(\overline{\mathcal{A}})$ depend only on $L(\mathcal{A})$.

Recent works [52, 15, 22] have established the validity of this conjecture for all complex reflection arrangements.

In this Thesis we provide new evidence that this conjecture is true. We have focused on line arrangements that are, in a sense, dual to those for which Papadima and Suciu have obtained their theorem: indeed, their result concerns line arrangements having any number of multiple points of low multiplicity, while we focus on two classes of arrangements having exactly two points of high multiplicity. Another difference is that the methods of Papadima and Suciu are mostly topological or combinatorial, while the ones we use are much more geometric.

The Thesis is organised as follows. In Chapter 1 we recall known results which will be used throughout the rest of the Thesis: the most important ones are the content of Sections 1.2 and 1.3. In the former we illustrate the construction of cubical hyperresolutions, and explain their importance in the development of a good de Rham cohomology theory for singular algebraic varieties; in the latter we report basic facts about mixed Hodge structures on the cohomology groups of algebraic varieties. In Section 1.4 we show the construction of the polar filtration P on the cohomology groups of hypersurface complements, and state a comparison result between that and the Hodge filtration F.

In Chapter 2, which is divided in two parts, we start discussing the Alexander polynomial. In the first part we recall Milnor's fibration theorem and the main properties of the Milnor fibre, and give the definition of Alexander polynomials of hypersurfaces. We then introduce ideals and constants of quasi-adjunction, which are key tools in one of Libgober's main result (see 2.1.14); we also recall the connection between the Steenbrink spectrum of a singularity and its constants of quasi-adjunction. Lastly, in Subsection 2.1.3 we briefly discuss the approach of [5, 38] to the computation of Alexander polynomials, and show that the existence of a quasi-toric decomposition of f implies that $C = V(f)$ has Alexander polynomial with $q(t) \neq 1$.

In the second part we focus our attention on line arrangements, and try to give an overview of the methods that have been employed in their study and of the results that have been obtained. This requires us to introduce many notions from the classical theory of hyperplane arrangements, combinatorics and topology: the intersection semilattice of an arrangement and the closely related Orlik-Solomon algebra, multinets, resonance and characteristic varieties; the connection between these notions and geometry is provided by the so-called Ceva pencils of curves. The main aim of this part is to highlight the dependence between the existence of multinets on line arrangements and the Alexander polynomial of arrangements. For this reason, the last subsection of the chapter is devoted to the discussion of some interesting examples of line arrangements with non-trivial Alexander polynomials; the precise statement of the conjecture by Papadima and Suciu can also be found there.

Chapter 3 is where our results are presented. We introduce two classes of arrangements that do not admit multinets, and prove that the Alexander polynomial of arrangements belonging to such classes is trivial. The arrangements we consider have two point P_1 and P_2 of high multiplicity, with all other multiple points having multiplicity at most 3, and can contain at most one 'free line' not passing through P_1 or P_2; we denote the number of free lines by s. We have obtained results only in the cases $s = 0, 1$, as for $s \geq 2$ it becomes almost impossible to control the combinatorics of these arrangements. We now present a brief overview of the methods we used to prove our results:

$\underline{s = 1}$ First we show that the arrangements in this class fall into a finite number of deformation-equivalent classes; as the Alexander polynomial of a curve is invariant under equisingular deformation, this allows us to reduce our study to a finite number of representative arrangements. Since, up to an automorphism of $\mathbb{P}^2$, we can freely move the points P_1 and P_2 and the free line, we can use Libgober's formula (2.1.15) for the computation of Alexander polynomials, and our result follows after a long series of computations with Hilbert functions.

$\underline{s = 0}$ This case could be tackled with the same method as the case $s = 1$, but we decided to look for a more geometric one; this required an unexpectedly big amount of work, but also produced an interesting byproduct. First we associate to any arrangement in this class a threefold T, and show that its fourth primitive Betti number is $(n-1)^2 + deg(q(t))$ where n is the number of lines in the arrangement; then we take a hyperplane section S of T, and bound the dimension of $H^4(T)_{\text{prim}}$ by that of a subspace of $H^2(S)_{\text{prim}}$; finally, we show that the latter subspace has dimension $(n-1)^2$, which forces $q(t) = 1$.

In order to perform the second step we show the existence of a Gysin morphism $H^2(S) \twoheadrightarrow H^4(T)$, even though T and its hyperplane section S do not satisfy the hypotheses usually required in order to obtain Lefschetz-type results: indeed, the hyperplane that cuts S from T passes through a singular point of T, so it cannot be transversal to the strata of a Whitney stratification of T, and $T \setminus S$ is not smooth. This is very interesting, as counterexamples to the Lefschetz hyperplane theorem can usually be found when its hypotheses are not fulfilled. What allows us to obtain this result is the control over the cubical hyperresolutions of T and S, which in turn is a consequence of T and S having only ordinary multiple points as singularities; this suggests that the existence of such a Gysin morphism could not be limited to the arrangements we consider here.

CHAPTER 1

Preliminaries

Throughout this chapter, unless stated otherwise, all cohomology groups are to be understood as singular cohomology groups; when the coefficient ring is not specified, it will either be $\mathbb{C}$ or, especially when dealing with Hodge theory, $\mathbb{R}$: the context should make it so that no ambiguity arises. Similarly, all varieties and schemes we consider are over $\mathbb{C}$, with the partial exception of Section 1.2.

1.1 Cohomology of complete intersections and their complements

In this section, unless stated otherwise, V is a complete intersection of dimension n and codimension c inside $\mathbb{P}^{n+c}$.

Theorem 1.1.1 (Weak Lefschetz theorem). *(i) The pullback morphism*

$$H^k(\mathbb{P}^{n+c}, \mathbb{Z}) \to H^k(V, \mathbb{Z})$$

is an isomorphism for $k < n$ and an injective morphism with torsion-free cokernel for $k = n$.

(ii) If V is smooth then the Gysin morphism

$$H^k(V, \mathbb{Z}) \to H^{k+2c}(\mathbb{P}^{n+c}, \mathbb{Z})$$

is an isomorphism for $k > n$ and a surjective morphism for $k = n$.

A result analogous to the weak Lefschetz theorem holds for weighted complete intersections in weighted projective spaces: the proof relies on the fact that quasi-smooth weighted complete intersections and weighted projective spaces are $\mathbb{Q}$-homology manifolds, so they admit all the usual duality theorems (Poincaré duality in particular). See [16, Appendix B].

Proof. The second statement is Poincaré dual to the first one, so we only need to prove the latter, a proof of which can be found in [16, Theorem 5.2.6]. □

If $c = 1$ then V is a hypersurface, and the theorem above becomes what is known as Lefschetz hyperplane theorem; the latter can be stated in a slightly more general setting, but in order to do so we first need to recall two well-known results.

Definition 1.1.2. A *Stein variety* is a closed analytic subvariety X of some $\mathbb{C}^n$; if X is smooth, we call it a Stein manifold. In particular, all affine complex algebraic varieties are Stein.

Theorem 1.1.3 (Cartan's theorem B)**.** *If $\mathcal{F}$ is a coherent sheaf on a Stein manifold X then $H^k(X, \mathcal{F}) = 0$ for $k > 0$.*

Proof. [31, Chapter VIII, Theorem 14]. □

Corollary 1.1.4. *Let X be a Stein manifold of complex dimension n, then $H^k(X) = 0$ for $k > n$.*

Proof. The holomorphic de Rham complex of X

$$0 \to \underline{\mathbb{C}}_X \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$$

is a resolution of $\underline{\mathbb{C}}_X$ by locally free sheaves, which are in particular coherent sheaves, so by Cartan's theorem B we have $H^i(X, \Omega^j_X) = 0$ for all $i > 0$ and for all j; this means that the holomorphic de Rham complex is a $\Gamma(X, -)$-acyclic resolution of $\underline{\mathbb{C}}_X$, from which we deduce

$$H^k(X) = H^k(\Gamma(X, \Omega^\bullet_X)).$$

This implies in particular that $H^k(X) = 0$ for $k > n$. □

Theorem 1.1.5 (Lefschetz hyperplane theorem)**.** *Let X be a complex projective variety of dimension n and $Y \subset X$ be an ample divisor.*

(i) If $X \setminus Y$ is smooth then the pullback morphism

$$H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

is an isomorphism for $k \leq n - 2$ and an injection for $k = n - 1$.

(ii) If both X and Y are smooth then the Gysin morphism

$$H^k(Y, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q})$$

is an isomorphism for $k \geq n$ and a surjection for $k = n - 1$.

Proof. A proof of (i) can be found in [68, Theorem 1.23]. Here we prove (ii), but first we use the universal coefficient theorem to switch to complex coefficients. If we set $U := X \setminus Y$ then the usual Gysin long exact sequence reads

$$\cdots \to H^{k+1}(U) \to H^k(Y) \to H^{k+2}(X) \to \cdots .$$

From this sequence we deduce that it is enough to prove that $H^{k+1}(U) = 0$ for $k > \dim(X) - 1$. Now, U is an open subset of some projective space minus a hyperplane, so it is smooth and affine, and in particular a Stein manifold; since $\dim(U) = \dim(X)$, the result follows from Corollary 1.1.4. □

Remark 1.1.6. The Lefschetz hyperplane theorem eases computation of cohomology groups of smooth hypersurfaces. Let $V \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d: if we call $N := \binom{n+2}{d} - 1$ we can consider the Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, which allows us to write

$$V \simeq \nu_d(V) = \nu_d(\mathbb{P}^n) \cap H$$

for some hyperplane $H \subset \mathbb{P}^N$. If we apply the Lefschetz hyperplane theorem to $X := \nu_d(\mathbb{P}^n)$ and $Y := \nu_d(\mathbb{P}^n) \cap H$ we obtain

$$\begin{aligned} H^k(\nu_d(\mathbb{P}^n)) \to H^k(\nu_d(\mathbb{P}^n) \cap H) \quad & \text{is an isomorphism for } k \leq n-1. \\ H^k(\nu_d(\mathbb{P}^n) \cap H) \to H^{k+2}(\nu_d(\mathbb{P}^n)) \quad & \text{is an isomorphism for } k \geq n+1. \end{aligned}$$

But $H^k(\nu_d(\mathbb{P}^n) \cap H) = H^k(\nu_d(V)) \simeq H^k(V)$ and $H^k(\nu_d(\mathbb{P}^n)) \simeq H^k(\mathbb{P}^n)$; moreover we can write $H^{k+2}(\mathbb{P}^n) \simeq H^k(\mathbb{P}^n)$ (since k and $k+2$ have the same parity). From this we get

$$\begin{aligned} H^k(\mathbb{P}^n) \to H^k(V) \quad & \text{is an isomorphism for } k \leq n-1. \\ H^k(V) \to H^k(\mathbb{P}^n) \quad & \text{is an isomorphism for } k \geq n+1. \end{aligned}$$

Thus the only interesting cohomology group of V is $H^n(V)$.

Theorem 1.1.7 (Barth's theorem)**.** *Assume the singular locus of V has dimension m; the map*

$$H^k(\mathbb{P}^{n+c}, \mathbb{Z}) \to H^k(V, \mathbb{Z}) \qquad \text{for } n+m+2 \leq k \leq 2n$$

is an isomorphism, given by multiplication by deg(V).

Proof. [16, Theorem 5.2.11]. □

Lemma 1.1.8. *The pullback map $H^k(\mathbb{P}^{n+c}) \to H^k(V)$ is injective for $0 \leq k \leq 2n$.*

Proof. [16, Lemma 5.2.17]. □

Definition 1.1.9. The cokernel of the pullback map $H^k(\mathbb{P}^{n+c}) \to H^k(V)$ is called the k-th *primitive cohomology* group of V, and it is denoted by $H^k(V)_{\text{prim}}$.

We conclude this section with a result on the cohomology of the complement of a projective hypersurface:

Proposition 1.1.10. *Let V be a hypersurface with r irreducible components of degrees $d_1, \ldots, d_r$, and denote by U its complement in $\mathbb{P}^{n+1}$; then*

$$H^1(U, \mathbb{Z}) \simeq \mathbb{Z}^{r-1} \oplus \mathbb{Z}/gcd(d_1, \ldots, d_r)\mathbb{Z}.$$

Proof. [16, Proposition 4.1.3]. □

1.2 Cubical hyperresolutions and de Rham cohomology

In this section we denote the subset $\{0, \ldots, m\}$ by $[m]$; moreover, when we write n we always mean $n \in \mathbb{Z}_{\geq 0}$.

Definition 1.2.1.

1. The *n-simplicial category* is the category Δ_n with objects the sets $[m]$ for $0 \leq m \leq n$ and with morphisms the non-decreasing maps $[m] \to [m']$; if we only allow strictly increasing maps as morphisms, we speak of *n-semisimplicial category* $\triangle_n$.

2. The *n-cubical category* is the category $\square_n$ with objects the subsets of $[n-1]$ and with $\mathrm{Hom}(I, J)$ consisting of a single element if $I \subset J$ and empty otherwise.

3. If $\mathcal{C}$ is any category, an *n-semisimplicial $\mathcal{C}$-object* is a contravariant functor $K_\bullet : \triangle_n \to \mathcal{C}$, and morphisms between such objects are morphisms of the corresponding functors; similarly, we can define *n-cubical $\mathcal{C}$-objects* $K_\square$ and morphisms thereof. If we consider covariant functors, we obtain the notion of *n-cosemisimplicial $\mathcal{C}$-object* $K^\bullet$ and *n-cocubical $\mathcal{C}$-object* $K^\square$.

By the above definitions, it follows that an n-semisimplicial $\mathcal{C}$-object consists of $n+1$ objects $K_m := K_\bullet[m]$ for $[m] =\in \triangle_n$; if $\alpha : [m] \to [m']$ is a morphism in $\triangle_n$, there exists a corresponding morphism $d_\alpha : K_{m'} \to K_m$. Similarly, an n-cubical $\mathcal{C}$-object consists of 2^n objects $K_I := K_\square(I)$ for $I \in \square_n$; if $I \subset J$ there exists a corresponding morphism $d_{IJ} : K_J \to K_I$.

Remark 1.2.2. For any $m \in \mathbb{Z}_{\geq 0}$ and $i = 0, \ldots, m$ we denote by $\delta_i : [m-1] \to [m]$ the *i-th face map*, i.e. the only strictly increasing map whose image does not contain i; for $i < j$, face maps satisfy the relation $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$. Clearly, if $Q^\bullet$ is an n-cosemisimplicial $\mathcal{C}$-object, the face maps δ_i give maps $d_i := Q^\bullet(\delta_i) : Q^{m-1} \to Q^m$ satisfying the relation

$$d_j \circ d_i = d_i \circ d_{j-1} \tag{1.2.1}$$

for $i < j$. If $\mathcal{C}$ is abelian, we can set

$$\sigma^m := \sum_{i=0}^{m} (-1)^i d_i : Q^m \to Q^{m+1}$$

and thus, thanks to (1.2.1), we obtain a complex

$$CQ^\bullet := \{Q^0 \xrightarrow{\sigma^0} Q^1 \xrightarrow{\sigma^1} \dots \xrightarrow{\sigma^{n-1}} Q^n \to 0\}.$$

This construction will allow us to define the cohomology of an n-semisimplicial topological space $X_\bullet$ with values in a sheaf $\mathcal{F}^\bullet$ on $X_\bullet$.

Definition 1.2.3. If S is any object in $\mathcal{C}$, the *constant n-semisimplicial $\mathcal{C}$-object S* is the contravariant functor $S_\bullet : \triangle_n \to \mathcal{C}$ such that $S_m = S$ for all $[m] \in \triangle_n$, with all morphisms $S_{m'} \to S_m$ given by the identity of S. An *augmentation* of an n-semisimplicial $\mathcal{C}$-object $K_\bullet$ to S is a morphism of n-semisimplicial $\mathcal{C}$-objects $K_\bullet \to S$. If we replace $\triangle_n$ by $\square_n$, we obtain *constant n-cubical $\mathcal{C}$ objects* and augmentations thereof.

The next observations will be useful in what follows:

Remark 1.2.4.
1. If $X_\square$ is an n-cubical $\mathcal{C}$-object we can associate to it the augmented n-cubical $\mathcal{C}$-object $\epsilon : X_\square \to X_\emptyset$; sometimes we will call this augmentation the *natural augmentation.*
2. Any $(n+1)$-cubical $\mathcal{C}$-object $X_\square$ can be considered as a morphism $Y_\square \to Z_\square$ of n-cubical $\mathcal{C}$-objects by setting $Z_I := X_I$ and $Y_I := X_{I\cup\{n\}}$ for $I \in \square_n$; in particular, a 1-cubical $\mathcal{C}$-object is the datum of two objects $X, Y \in \mathcal{C}$ and a morphism $f : X \to Y$ between them.
3. To any $(n+1)$-cubical $\mathcal{C}$-object $X_\square$ we can associate functorially an augmented n-semisimplicial $\mathcal{C}$-object $\epsilon : X_\bullet \to Y$ with $Y := X_\emptyset$. We set:

$$X_k := \coprod_{|I|=k+1} X_I \quad \text{for } k = 0, \dots, n.$$

Let $\beta : [s] \to [r]$ be a strictly increasing map (in particular $r \geq s$). If $I \in \square_{n+1}$ has cardinality $r+1$ we can write it as $I = \{i_0, \dots, i_r\}$ with $i_0 < \dots < i_r$; the set $J := \beta(I) := \{i_{\beta(0)}, \dots, i_{\beta(s)}\}$ is contained in I, so we have a morphism $d_{JI} : X_I \to X_J$. We can now define the morphism

$$d_\beta : X_r \to X_s \quad \text{s.t. } (d_\beta)_{|X_I} = d_{\beta(I)I}.$$

Since for any $I \subset [n]$ we have a morphism $d_{\emptyset I} : X_I \to Y$ we obtain the desired augmentation by setting $\epsilon_{|X_I} = d_{\emptyset I}$.

Definition 1.2.5. The category *TopSh* has objects the pairs $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf on X, and as morphisms the pairs $(f, f^\#) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ where $f : X \to Y$ is a continuous function and $f^\# : \mathcal{G} \to f_*\mathcal{F}$ is a morphism of sheaves on Y. A *sheaf on an n-semisimplicial space* (resp. *sheaf on an n-cubical space*) is just an n-semisimplicial (resp. n-cubical) *TopSh*-object. In a similar manner, we can define complexes of sheaves and resolution of sheaves on n-semisimplicial or n-cubical spaces.

Fix an n-semisimplicial space $X_\bullet$, i.e. an n-semisimplicial Top-object, and consider a sheaf of abelian groups $\mathcal{F}^\bullet$ on $X_\bullet$: the Godement complexes $\mathcal{C}^\bullet_{Gdm}(\mathcal{F}^m)$ give injective resolutions of each $\mathcal{F}^m$, and fit together to give an injective resolution of $\mathcal{F}^\bullet$. This allows us to define the cohomology of an n-semisimplicial space with values in $\mathcal{F}^\bullet$: namely, the abelian groups

$$F^{p,q} := \Gamma(X_q, \mathcal{C}^p_{Gdm}(\mathcal{F}^q)) \tag{1.2.2}$$

are the entries of a double complex, with differentials in the p-direction coming from the Godement resolutions and differentials in the q-direction given by the differentials of the complex $CF^{p,\bullet}$ (recall Remark 1.2.2); we define

$$H^k(X_\bullet, \mathcal{F}^\bullet) := H^k(s(F^{\bullet,\bullet})) \tag{1.2.3}$$

where $s(F^{\bullet,\bullet})$ is the simple complex associated to $F^{\bullet,\bullet}$.

Remark 1.2.6. If Y is a constant n-semisimplicial space, any sheaf on Y will be denoted by $\mathcal{F}$ and not by $\mathcal{F}^\bullet$; likewise, the cohomology groups of Y with values in $\mathcal{F}$ will be denoted by $H^k(Y, \mathcal{F})$.

Suppose now that $\epsilon : X_\bullet \to Y$ is an augmented n-semisimplicial space and $\mathcal{F}^\bullet$ is a sheaf on $X_\bullet$. The sheaves $\epsilon_* \mathcal{C}^p_{Gdm}(\mathcal{F}^q)$ form a double complex of sheaves on Y, whose associated simple complex gives

$$R\epsilon_* \mathcal{F}^\bullet := s[\epsilon_* \mathcal{C}^\bullet_{Gdm}(\mathcal{F}^\bullet)]. \tag{1.2.4}$$

One can prove that the hypercohomology of the latter complex coincides with the cohomology of $X_\bullet$ with values in $\mathcal{F}^\bullet$, i.e.

$$\mathbb{H}^k(Y, R\epsilon_* \mathcal{F}^\bullet) = H^k(X_\bullet, \mathcal{F}^\bullet) \quad \text{for any } k. \tag{1.2.5}$$

Recall now that if $f : X \to Y$ is a continuous map between topological spaces and $\mathcal{G}^\bullet$ is a complex of sheaves on Y, we have a natural adjunction morphism in $D_+(Sh(Y))$:

$$\mathcal{G}^\bullet \to Rf_* f^{-1} \mathcal{G}^\bullet. \tag{1.2.6}$$

Definition 1.2.7. [9, Definition 5.3.2] An augmented n-semisimplicial space $\epsilon : X_\bullet \to Y$ is *of cohomological descent* if for any sheaf of abelian groups $\mathcal{F}$ on Y the natural adjunction morphism

$$\mathcal{F} \to R\epsilon_* \epsilon^{-1} \mathcal{F} \tag{1.2.7}$$

is an isomorphism.

If $X_\square$ is an $(n+1)$-cubical space and $\mathcal{F}^\square$ is a sheaf on X, by Remark 1.2.4(iii) to this data we can associate an augmented n-semisimplicial space $\epsilon : X_\bullet \to X_\emptyset$ and a sheaf $\mathcal{F}^\bullet$ on it. We set

$$C^\bullet(X_\square, \mathcal{F}^\square) := \mathrm{Cone}^\bullet[\mathcal{F}^\emptyset \to R\epsilon_* \epsilon^{-1} \mathcal{F}^\emptyset]. \tag{1.2.8}$$

From now on, we will take for $\mathcal{C}$ the category whose objects are reduced separated schemes of finite type over $\mathbb{C}$, which we will simply call varieties, and whose morphisms are morphisms of schemes; this is not fully consistent with the existing literature, in which the term 'algebraic variety' is usually reserved for *integral* separated schemes of finite type over some field.

Definition 1.2.8.
1. An augmented n-semisimplicial variety is *of cohomological descent* if this is the case for the associated augmented n-semisimplicial space.
2. Let X be a variety. An *n-semisimplicial resolution* of X is an n-semisimplicial variety $\epsilon : X_\bullet \to X$ augmented towards X such that all maps $X_m \to X$ are proper, X_m is smooth for all m and ϵ is of cohomological descent.
3. An $(n+1)$-cubical variety is *of cohomological descent* (resp. a *cubical hyperresolution*) if the associated augmented n-semisimplicial variety is of cohomological descent (resp. an n-semisimplicial resolution).

Every variety admits an n-cubical hyperresolution for some n, which can be constructed in a standard way; before presenting a (sketch of the) proof of this fact, we need to give some definitions:

Definition 1.2.9.
1. A *proper modification* of an n-cubical variety $X_\square$ is a proper morphism of n-cubical varieties $f : \tilde{X}_\square \to X_\square$ such that there exists an open dense (n-cubical) subset $U_\square \subset X_\square$ for which f induces an isomorphism between $f^{-1}(U_\square)$ and $U_\square$; a *resolution* of $X_\square$ is a proper modification with $\tilde{X}_\square$ smooth.
2. The *discriminant* of a proper morphism $f : X_\square \to S_\square$ of n-cubical varieties is the smallest closed n-cubical subvariety $D_\square \subset S_\square$ such that f induces isomorphisms $X_I \setminus f^{-1}(D_I) \to S_I \setminus D_I$ for all $I \subset \square_n$.
3. Let $f : \tilde{X}_\square \to X_\square$ be a proper modification (resp. resolution) of an n-cubical variety with discriminant $D_\square$, and set $E_\square := f^{-1}(D_\square)$; the *discriminant square* (resp. *resolution square*) of $X_\square$ is the $(n+2)$-cubical variety

$$\begin{tikzcd} E_\square \arrow[r] \arrow[d, "f_{|E_\square}"] & \tilde{X}_\square \arrow[d, "f"] \\ D_\square \arrow[r] & X_\square \end{tikzcd}$$

where the horizontal maps are inclusions.

Theorem 1.2.10. *For any variety X of dimension n there exists an $(n+1)$-cubical hyperresolution $X_\square$ such that* $\dim(X_I) \le n - |I| + 1$ *for all* $I \subset \square_{n+1}$.

Proof. A full proof can be found in [30, Théorème I.2.15] or, in greater generality, in [62, Theorem 5.26]; here we are only interested in showing how the cubical hyperresolution is constructed. If $\pi_1 : \tilde{X} \to X$ is a resolution of X with discriminant D, we define a 2-cubical variety $X_\square^{(1)}$ by setting

$$X^{(1)}_{\emptyset} := X, \quad X^{(1)}_{\{0\}} := \tilde{X}, \quad X^{(1)}_{\{1\}} := D, \quad X^{(1)}_{\{0,1\}} := \pi_1^{-1}(D);$$

we can see this as a morphism of 1-cubical varieties $f^{(1)} : Y^{(1)}_{\square} \to Z^{(1)}_{\square}$, with Z_I smooth for $I \neq \emptyset$. Next we consider a resolution $\pi_2 : \tilde{Y}^{(1)}_{\square} \to Y^{(1)}_{\square}$ and the corresponding resolution square

$$\begin{tikzcd}
E^{(1)}_{\square} \arrow[r] \arrow[d] & \tilde{Y}^{(1)}_{\square} \arrow[d] \\
D^{(1)}_{\square} \arrow[r] & Y^{(1)}_{\square}
\end{tikzcd}$$

where $D^{(1)}_{\square}$ is the discriminant of π_2 and $E^{(1)}_{\square} := \pi_2^{-1}(D^{(1)}_{\square})$; from this we obtain

$$\begin{tikzcd}
E^{(1)}_{\square} \arrow[r] \arrow[d] & \tilde{Y}^{(1)}_{\square} \arrow[d, "\pi_2"] \arrow[dr] & \\
D^{(1)}_{\square} \arrow[r] & Y^{(1)}_{\square} \arrow[r] & Z^{(1)}_{\square}.
\end{tikzcd}$$

The outer commutative square of 1-cubical varieties can be considered as a 3-cubical variety $X^{(2)}_{\square}$ by setting, for any $I \subset [0]$,

$$X^{(2)}_{I} := Z^{(1)}_{I}, \quad X^{(2)}_{I\cup\{1\}} := \tilde{Y}^{(1)}_{I}, \quad X^{(2)}_{I\cup\{2\}} := D^{(1)}_{I}, \quad X^{(2)}_{I\cup\{1,2\}} := E^{(1)}_{I}.$$

If we repeat this process enough times, we eventually obtain the desired cubical hyperresolution of X. $\square$

Remark 1.2.11. The theorem above provides us with a standard way to construct an $(n+1)$-cubical hyperresolution of a variety X of dimension n; however, X can admit a much simpler one, as it is the case for a variety having both the discriminant D of a resolution $\pi : \tilde{X} \to X$ and its exceptional divisor E smooth: in this scenario, X admits a 2-cubical hyperresolution (given by its resolution square) regardless of its dimension.

Observe that if we take for $\mathcal{C}$ the category of n-cubical varieties and consider $X \in \mathcal{C}$, we can still apply the construction of Theorem 1.2.10 to X: at each step we obtain an m-cubical variety whose entries are n-cubical varieties. More precisely, [30, Théorème I.2.15] implies the following:

Theorem 1.2.12. *Any n-cubical variety X admits a hyperresolution by an m-cubical variety $Y = \{Y_{IJ}\}$ of n-cubical varieties such that* $\dim(Y_{IJ}) \leq \dim(X) - |I \times J| + 1$ *for any I, J.*

The following observation will be used in Chapter 3:

Remark 1.2.13. Assume $X = \{X_\emptyset, X_{\{0\}}, X_{\{1\}}, X_{\{01\}}\}$ is a 2-cubical variety and Y is an m-cubical hyperresolution of X, then Y can be thought of as a 2-cubical variety $Y' = \{Y'_\emptyset, Y'_{\{0\}}, Y'_{\{1\}}, Y'_{\{01\}}\}$ of $(m-2)$-cubical varieties; by construction, for any $I \in \square_2$ we have that Y'_I is an $(m-2)$-cubical hyperresolution of X_I.

Let $X_\square$ be an $(n+1)$-cubical variety and $\epsilon : X_\bullet \to X_\emptyset$ be the associated augmented n-semisimplicial variety: by the definition of cohomological descent and (1.2.8), we deduce that $X_\square$ (equivalently, $X_\bullet$) is of cohomological descent if and only if $C^\bullet(X_\square, \mathcal{F}^\square)$ is acyclic for any sheaf of abelian groups $\mathcal{F}^\square$ on $X_\square$.

Lemma 1.2.14. *Let $X_\square$ be an $(n+1)$-cubical variety and consider it as a morphism $f : Y_\square \to Z_\square$ of n-cubical varieties; let $\mathcal{F}^\square$ be a sheaf on $X_\square$ restricting to sheaves on $Y_\square$ and $Z_\square$ denoted again by $\mathcal{F}^\square$. We have*

$$C^\bullet(X_\square, \mathcal{F}^\square)[1] = Cone^\bullet[C^\bullet(Z_\square, \mathcal{F}^\square) \xrightarrow{C(f^\#)} Rf_*C^\bullet(Y_\square, \mathcal{F}^\square)].$$

In particular, if $C(f^\#)$ is an isomorphism then $C^\bullet(X_\square, \mathcal{F}^\square)[1]$ is acyclic and, as a consequence, $X_\square$ is of cohomological descent.

Proof. [62, Proposition 5.27]. $\square$

Corollary 1.2.15. *Let $X_\square$ be an $(n+2)$-cubical variety and consider it as a commutative square of n-cubical varieties*

$$\begin{tikzcd} Y_\square \arrow[r, "f"] \arrow[d, "a"] & Z_\square \arrow[d, "b"] \\ T_\square \arrow[r, "g"] & W_\square. \end{tikzcd}$$

Let $\mathcal{F}^\square$ be a sheaf on $X_\square$ restricting to sheaves on $Y_\square$, $Z_\square$, $T_\square$ and $W_\square$ denoted by $\mathcal{F}^\square$ too. The cone over the map of complexes

$$C^\bullet(W_\square, \mathcal{F}^\square)[1] \to Cone^\bullet[Rb_*C^\bullet(Z_\square, \mathcal{F}^\square) \oplus Rg_*C^\bullet(T_\square, \mathcal{F}^\square) \to R(g \circ a)_*C^\bullet(Y_\square, \mathcal{F}^\square)]$$

is isomorphic to $C^\bullet(X_\square, \mathcal{F}^\square)[2]$.

Proof. We can consider $X_\square$ as a morphism of $(n+1)$-cubical varieties $(a, b) : (Y_\square \xrightarrow{f} Z_\square) \to (T_\square \xrightarrow{g} W_\square)$, with these $(n+1)$-cubical varieties being morphisms of n-cubical varieties; this implies (we omit writing $\mathcal{F}^\square$)

$$\begin{aligned} C^\bullet(X)[2] &= \mathrm{Cone}^\bullet[C^\bullet(T_\square \xrightarrow{g} W_\square) \to R(a,b)_*C^\bullet(Y_\square \xrightarrow{f} Z_\square)][1] \simeq \\ &\simeq \mathrm{Cone}^\bullet[C^\bullet(T_\square \xrightarrow{g} W_\square)[1] \to R(a,b)_*C^\bullet(Y_\square \xrightarrow{f} Z_\square)[1]] = \\ &= \mathrm{Cone}^\bullet[\mathrm{Cone}^\bullet(C^\bullet(W_\square) \to Rg_*C^\bullet(T_\square)) \to R(a,b)_*\mathrm{Cone}^\bullet(C^\bullet(Z_\square) \to Rf_*C^\bullet(Y_\square))] = \\ &= \mathrm{Cone}^\bullet[\mathrm{Cone}^\bullet(C(g^\#)) \xrightarrow{(C(a^\#), C(b^\#))} R(a,b)_*\mathrm{Cone}^\bullet(C(f^\#))] \end{aligned}$$

by a double application of the previous lemma. Now we need the following technical result:

Lemma 1.2.16. *If we have a commutative square of complexes*

$$\begin{tikzcd} A^\bullet \arrow[r, "i"] \arrow[d, "\pi'"] & B^\bullet \arrow[d, "\pi"] \\ C^\bullet \arrow[r, "j"] & D^\bullet \end{tikzcd}$$

in an abelian category $\mathcal{C}$, then the cone over the morphism

$$Cone^\bullet(i) \xrightarrow{(\pi',\pi)} Cone^\bullet(j)$$

is equal to the cone over

$$A^\bullet[1] \xrightarrow{(-i,\pi')} Cone^\bullet[B^\bullet \oplus C^\bullet \xrightarrow{\pi+j} D^\bullet].$$

We apply it to the commutative square of complexes of sheaves on $X_\emptyset$

$$\begin{tikzcd} C^\bullet(W_\square, \mathcal{F}^\square) \arrow[r, "C(g^\#)"] \arrow[d, "C(b^\#)"] & Rg_*C^\bullet(T_\square, \mathcal{F}^\square) \arrow[d, "C(a^\#)"] \\ Rg_*C^\bullet(Z_\square, \mathcal{F}^\square) \arrow[r, "C(f^\#)"] & R(g\circ a)_*C^\bullet(Y_\square, \mathcal{F}^\square) \end{tikzcd}$$

and we are done. □

The brisk presentation of the theory of cubical hyperresolutions and cohomological descent we have given here is motivated by two reasons:

1. As we shall see in the next section, it allows us to associate to a resolution $\tilde{X}$ of a complex algebraic variety X a long exact sequence of cohomology mixed Hodge structures that relates the cohomology of X to that of $\tilde{X}$, of the singular locus Σ of X and of the exceptional divisor E (Proposition 1.3.9). This result will be used extensively throughout this Thesis.

2. With this language one can define an algebraic de Rham cohomology theory for singular complex algebraic varieties, and recover some classical results that hold for smooth varieties: for example, theorems of Lefschetz type on hyperplane sections and the existence of Gysin morphisms.

The second point needs to be discussed further. One of the first thorough presentations of the theory of the de Rham cohomology (and homology) for possibly singular schemes X of finite type over a field $\mathbb{K}$ of characteristic zero was given by Hartshorne in [33]. The definition of the cohomology groups is the following:

Definition 1.2.17. Let X be a scheme as above, and assume it can be globally embedded as a closed subscheme of a scheme Y smooth over $\mathbb{K}$. Denote by $\widehat{Y|X}$ the formal completion of Y along X and by $\widehat{\Omega^\bullet_Y|X}$ the formal completion of the de Rham complex $\Omega^\bullet_Y$ of Y along X; the k-th *algebraic de Rham cohomology group* of X is

$$H^k_{DR}(X) := \mathbb{H}^k(Y\hat{\lceil}X, \Omega^\bullet_Y\hat{\lceil}X). \tag{1.2.9}$$

If V is a closed subset of X, considered as a subscheme with its induced reduced structure, the k-th algebraic de Rham cohomology group *with supports in* V is

$$H^k_{DR,V}(X) := \mathbb{H}^k(Y\hat{\lceil}V, R\Gamma_V\Omega^\bullet_Y\hat{\lceil}X) \tag{1.2.10}$$

where Γ_V denotes the restriction of sections functor.

In the same article, Hartshorne proved that this definition is well-posed (i.e. it is independent on the choice of the embedding, and it can be adapted to schemes that are not globally embeddable in a smooth scheme) and that it yields a cohomology theory with all the properties one expects: finite-dimensionality of the cohomology groups, functorial properties, duality with homology and so on. In the case $\mathbb{K} = \mathbb{C}$, he also proved a comparison theorem with the cohomology groups one can define on the analytic space X_h associated to X, namely sheaf cohomology groups and analytic de Rham cohomology groups (the latter are defined as in the algebraic case, but of course one needs to replace every object by its analytic counterpart):

Theorem 1.2.18. *Denoting by X_h the analytic space associated to X and by $H^\bullet_{DR,h}(X_h)$ the analytic de Rham cohomology groups, there exist isomorphisms*

$$H^\bullet(X_h, \underline{\mathbb{C}}) \simeq H^\bullet_{DR,h}(X_h) \simeq H^\bullet_{DR}(X). \tag{1.2.11}$$

Proof. [33, Theorem IV.1.1]. □

In particular, we obtain an isomorphism between algebraic de Rham cohomology groups of X and singular cohomology groups of X_h, as it happens in the smooth case.

In [30], a different definition of algebraic de Rham cohomology groups was given using cubical hyperresolutions:

Definition 1.2.19. Let X be a separated scheme of finite type over a field $\mathbb{K}$ of characteristic zero, and let $\epsilon : X_\square \to X$ be an $(n+1)$-cubical hyperresolution of X together with its natural augmentation; the *de Rham complex* and k-th algebraic de Rham cohomology group of X are defined as

$$DR^\bullet_X := R\epsilon_*\Omega^\bullet_{X_\bullet} \qquad H^k_{DR}(X) := \mathbb{H}^k(X, DR^\bullet_X). \tag{1.2.12}$$

If $V \subset X$ is a closed subset, the k-th algebraic de Rham cohomology group of X with supports in V is defined as

$$H^k_{DR,V}(X) := \mathbb{H}^k(V, R\Gamma_V DR^\bullet_X). \tag{1.2.13}$$

In both cases, $\epsilon : X_\bullet \to X$ is the augmented n-semisimplicial resolution of X associated to $X_\square$.

These definitions coincide with the ones given by Hartshorne in the case of an embeddable scheme X, because the complexes computing the hypercohomology are isomorphic by [30, Théorème III.1.3]. With this definition at hand the authors proved the following results, which we shall need in Chapter 3:

Lemma 1.2.20. *If X is an affine separated scheme of finite type over $\mathbb{C}$, then $H^k_{DR}(X) = 0$ for $k > \dim(X)$.*

Proof. [30, Corollaire III.3.11(i)]. □

Theorem 1.2.21. *Let X be a quasi-projective separated scheme of finite type over $\mathbb{C}$ and Y be a hyperplane section of X satisfying the following hypotheses:*

(I) There exists an augmented $(n+1)$-cubical hyperresolution $X_\square \to X$ such that $Y_\square := X_\square \times_X Y$ is an n-cubical hyperresolution of Y.

(II) For any α, there exists a closed immersion $Y_\alpha \hookrightarrow X_\alpha$ of codimension 1.

Then there exists an isomorphism

$$DR^\bullet_Y \xrightarrow{\simeq} R\Gamma_Y DR^\bullet_X[2]. \tag{1.2.14}$$

Proof. [30, Proposition III.1.20]. □

Corollary 1.2.22. *Let X be a projective separated scheme of finite type over $\mathbb{C}$ and Y be a hyperplane section of X satisfying the hypotheses of Theorem 1.2.21. There exists a Gysin morphism*

$$H^k_{DR}(Y) \to H^{k+2}_{DR}(X) \tag{1.2.15}$$

that is an isomorphism for $k > \dim(Y)$ and a surjection for $k = \dim(Y)$.

Proof. [30, Corollaire III.3.12(i)]. Call $n := \dim(Y)$ so that $\dim(X) = n+1$. The long exact sequence of the pair $(X, X \setminus Y)$ reads

$$\cdots \to H^\bullet_{DR,Y}(X) \to H^\bullet_{DR}(X) \to H^\bullet_{DR}(X \setminus Y) \to H^{\bullet+1}_{DR,Y}(X) \to \cdots .$$

Since $X \setminus Y$ is affine, by the previous lemma we have $H^k(X \setminus Y) = 0$ for $k > n+1$ and we deduce that the morphism

$$H^{k+2}_{DR,Y}(X) \to H^{k+2}_{DR}(X)$$

is an isomorphism for $k > n$ and a surjection for $k = n$. Since Theorem 1.2.21 yields isomorphisms

$$H^k_{DR}(Y) \to H^{k+2}_{DR,Y}(X)$$

we can conclude. □

Remark 1.2.23. The definition of the de Rham complex $DR^\bullet_X$ given in [30] is actually different from the one we presented here; in order to state it, and to show that the one we gave is essentially the same, we need to introduce the category $\square^*_n$: this is the full subcategory of $\square_n$ whose objects are the non-empty subsets of $[n-1]$. If we denote by $\mathcal{C}$ the category of separated schemes of finite type over a field $\mathbb{K}$ of characteristic zero, we have:

Definition 1.2.24. [30, Définition I.3.2] If $X_\square : \square_n \to \mathcal{C}$ is an n-cubical hyperresolution of $X \in \mathcal{C}$, with its natural augmentation $X_\square \to X$, the restriction of $X_\square$ to $\square_n^*$ gives a functor $X_{\square^*} : \square_n^* \to \mathcal{C}$ which has again a natural augmentation $\epsilon : X_{\square^*} \to X$ to X. The latter is an *n-cubical hyperresolution* of X.

Definition 1.2.25. [30, Définition III.1.10, Proposition III.1.12] If $X \in \mathcal{C}$ and $X_{\square^*}$ is an $(n+1)$-cubical hyperresolution of X with its natural augmentation $\epsilon : X_{\square^*} \to X$, the *de Rham complex* of X is

$$DR_X^\bullet := R\epsilon_* \Omega_{X_{\square^*}}^{\square^*}.$$

Pick now $X \in \mathcal{C}$. Let $X_\square$ be an $(n+1)$-cubical hyperresolution of X with its natural augmentation $\epsilon : X_\square \to X$, and let $\epsilon : X_\bullet \to X$ be the augmented n-semisimplicial resolution associated to it. Let $X_{\square^*}$ be the augmented $(n+1)$-cubical hyperresolution of X as in Definition 1.2.24, and denote by $\epsilon : X_{\square^*} \to X$ its augmentation. In order to show that Definitions 1.2.19 and 1.2.25 are equivalent, we need to prove that

$$R\epsilon_* \Omega_{X_\bullet}^\bullet \simeq R\epsilon_* \Omega_{X_{\square^*}}^{\square^*}.$$

But this is basically a consequence of the construction we presented in Remark 1.2.4(iii): indeed, that construction does not involve the object $X_\emptyset$ of an $(n+1)$-cubical $\mathcal{C}$-object, hence all the objects of $X_{\square^*}$ can be found in $X_\bullet$ too ('bundled together' by the coproducts); moreover, the augmentation from the objects of $X_\bullet$ to X are combinations of the augmentations from the objects of $X_{\square^*}$ to X.

1.3 Hodge theory and deformations

Definition 1.3.1. A *Hodge structure* (HS) of weight m is a pair (H, F) where:

1. H is a finite-dimensional vector space over $\mathbb{R}$.
2. F is a finite decreasing filtration, called *Hodge filtration*, on $H_\mathbb{C} := H \otimes \mathbb{C}$.
3. $H_\mathbb{C} = F^p H_\mathbb{C} \oplus \overline{F^{m-p+1} H_\mathbb{C}}$ for any $p \in \mathbb{Z}$, where the conjugation on $H_\mathbb{C}$ is induced by the complex conjugation on $\mathbb{C}$.

Given a HS of weight m, for any pair (p, q) such that $p + q = m$ we can define $H^{p,q} := F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}}$; we obtain the following equalities:

(i) $H_\mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$.

(ii) $\overline{H^{p,q}} = H^{q,p}$.

Conversely, given a finite direct sum decomposition of $H_\mathbb{C}$ by subspaces $H^{p,q}$ satisfying (i) and (ii) above, we can obtain a HS on H by defining the Hodge filtration as

$$F^p H_\mathbb{C} := \bigoplus_{s \geq p} H^{s,m-s}.$$

Given a HS of weight m, the associated *Hodge numbers* are defined, for $p+q=m$, as

$$h^{p,q} := \dim_{\mathbb{C}}(Gr_F^p H_{\mathbb{C}}) = \dim(H^{p,q}).$$

By (ii) above, we have $h^{p,q} = h^{q,p}$.

Definition 1.3.2. Let (H, F) and (H', F') be HS of the same weight m. A *morphism of HS* is an $\mathbb{R}$-linear map $\phi : H \to H'$ such that $\phi \otimes 1_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ satisfies $\phi_{\mathbb{C}}(F^p H_{\mathbb{C}}) \subseteq F'^p H'_{\mathbb{C}}$ for all p.

It is a classical result that if X is a compact Kähler manifold then $H^m(X, \mathbb{R})$ admits a HS of weight m, and the subspaces $H^{p,q}$ are given by $H^q(X, \Omega_X^p)$; in particular, for $p > \dim(X)$ we have $H^{p,q} = 0$.

The notion of HS can be generalised as follows:

Definition 1.3.3. A *mixed Hodge structure* (MHS) is a triple (H, W, F) where:

1. H is a finite dimensional vector space over $\mathbb{R}$.
2. W is a finite increasing filtration, called *weight filtration*, on H.
3. F is a finite decreasing filtration, called again Hodge filtration, on $H_{\mathbb{C}}$ with the following property: if we define $Gr_k^W H := W_k H / W_{k-1} H$ and denote again by F the filtration induced by the Hodge filtration on $(Gr_k^W H)_{\mathbb{C}} := (Gr_k^W H) \otimes \mathbb{C}$, the pair $(Gr_k^W H, F)$ is a pure HS of weight k for all k.

When the weight filtration is trivial we obtain a HS; in this case we say that the MHS is pure. The induced Hodge filtration on $(Gr_k^W H)_{\mathbb{C}}$ is given by

$$F^p (Gr_k^W)_{\mathbb{C}} := (F^p H_{\mathbb{C}} \cap W_k H_{\mathbb{C}} + W_{k-1} H_{\mathbb{C}}) / W_{k-1} H_{\mathbb{C}} \quad \text{where} \quad W_k H_{\mathbb{C}} := (W_k H) \otimes \mathbb{C}.$$

To a MHS (H, W, F) we associate the *mixed Hodge numbers*, defined as

$$h^{p,q}(H) := \dim_{\mathbb{C}}(Gr_F^p Gr_{p+q}^W H_{\mathbb{C}}).$$

Since they are the Hodge numbers of the pure HS $Gr_{p+q}^W(H_{\mathbb{C}})$ we have $h^{p,q}(H) = h^{q,p}(H)$.

Remark 1.3.4. If (V, W, F) and (U, W', F') are MHS, we can put a MHS on $\mathrm{Hom}(V, U)$ by defining the weight and Hodge filtration in the following way:

$$\begin{aligned} W_p \mathrm{Hom}(V, U) :=& \{ f : V \to U | f(W_n V) \subset W'_{n+p} U \text{ for all } n \}. \\ F^p \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, U_{\mathbb{C}}) :=& \{ f : V_{\mathbb{C}} \to U_{\mathbb{C}} | f(F^n V_{\mathbb{C}}) \subset F'^{n+p} U_{\mathbb{C}} \text{ for all } n \}. \end{aligned}$$

If the HS are actually pure of weights k and l we obtain a pure HS of weight $l-k$; in particular, if we take $U = \mathbb{R}$ and $U_{\mathbb{C}} = W^{0,0}$ we get a pure HS of weight $-k$ on the dual $V^\vee$ of V.

Definition 1.3.5. 1. Let (H, W, F) and (H', W', F') be two MHS. A *morphism of MHS* is an $\mathbb{R}$-linear map $\phi : H \to H'$ such that

$$\phi(W_k H) \subseteq W'_k H' \text{ for all } k.$$
$$\phi_{\mathbb{C}}(F^p H_{\mathbb{C}}) \subseteq F'^p H'_{\mathbb{C}} \text{ for all } p.$$

2. Given a MHS (H, W, F) and an integer $m \in \mathbb{Z}$ we can define the MHS $(H(m), W, F)$ where

$$H(m) := H,$$
$$W_k H(m) := W_{k+2m} H,$$
$$F^p H(m)_{\mathbb{C}} := F^{p+m} H_{\mathbb{C}},$$

for any $k, p \in \mathbb{Z}$.

3. A morphism of MHS of *type* (r, r) is an $\mathbb{R}$-linear map $\phi : H \to H'$ such that the induced map $\phi : H \to H'(r)$ (equivalently, the induced map $\phi : H(-r) \to H'$) is a morphism of MHS. This implies that

$$\phi(F^p H_{\mathbb{C}}) \subseteq F'^{p+r} H'_{\mathbb{C}}.$$
$$\phi(W_k H) \subseteq W'_{k+2r} H'.$$

Proposition 1.3.6. *If $\phi : H \to H'$ is a morphism of MHS, then ϕ is strictly compatible with both filtrations W and F, i.e $\phi(W_k H) = W'_k H' \cap Im(\phi)$ and $\phi_{\mathbb{C}}(F^p H_{\mathbb{C}}) = F'^p H'_{\mathbb{C}} \cap Im(\phi_{\mathbb{C}})$ for all $k, p \in \mathbb{Z}$.*

Proof. [9, Thm II.1.2.10(iii)] □

The strictness of the morphisms ϕ of MHS implies that $Ker(\phi)$, $Im(\phi)$ and $Coker(\phi)$ have canonically defined MHS. In particular, if

$$\cdots \to H_{k-1} \to H_k \to H_{k+1} \to \cdots$$

is an exact sequence of MHS, then it remains exact after taking the Gr_i^W, Gr_F^j or $Gr_F^j Gr_i^W$ parts.

Theorem 1.3.7. *For any algebraic variety X of complex dimension n there is a functorial MHS on $H^\bullet(X, \mathbb{R})$ satisfying the following properties for all $m \geq 0$:*

1. *The weight filtration W on $H^m(X, \mathbb{R})$ satisfies:*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2m} = H^m(X, \mathbb{R}).$$

Moreover, for $m \geq n$ we also have $W_{2n} = \cdots = W_{2m}$.

2. *The Hodge filtration F on $H^m(X,\mathbb{C})$ satisfies:*

$$H^m(X,\mathbb{C}) = F^0 \supseteq \cdots \supseteq F^{m+1} = 0.$$

   Moreover, for $m \geq n$ we also have $F^{n+1} = 0$.

3. *If X is smooth then $W_{m-1} = 0$ and $W_m = j^* H^m(\overline{X},\mathbb{R})$ for any compactification $j : X \hookrightarrow \overline{X}$.*

4. *If X is projective then $W_m = H^m(X,\mathbb{R})$ and $W_{m-1} = Ker(p^*)$ for any proper map $p : \tilde{X} \to X$ with $\tilde{X}$ smooth.*

Proof. All these results are contained in [9]. □

From now on, in order to simplify notations, we will omit writing the coefficients of the various cohomology groups.

MHS arise quite naturally in geometry: in fact, most of the morphisms of cohomology groups and long exact sequences of cohomology groups are actually morphisms and long exact sequences of MHS. The next result summarises some of the most important cases:

Proposition 1.3.8. *(i) Let X be a complex algebraic variety and $Y \subset X$ be any subvariety. Then there is a MHS on the relative cohomology groups $H^\bullet(X,Y)$ such that the long exact sequence of the pair (X,Y)*

$$\cdots \to H^\bullet(X,Y) \to H^\bullet(X) \to H^\bullet(Y) \to H^{\bullet+1}(X,Y) \to \cdots$$

is a long exact sequence of MHS. If Y is closed and $U := X \setminus Y$, the group $H^\bullet(X,Y)$ is usually denoted by $H^\bullet_U(X)$.

(ii) Let X be a compact complex algebraic variety, $Y \subset X$ be a closed subvariety and $U := X \setminus Y$. Then

(I) The cohomology groups with compact support $H^\bullet_c(U)$ are given a MHS via the isomorphism $H^\bullet_c(U) \to H^\bullet(X,Y)$; in particular, by point (i) we obtain the following long exact sequence of MHS:

$$\cdots \to H^\bullet_c(U) \to H^\bullet(X) \to H^\bullet(Y) \to H^{\bullet+1}_c(U) \to \cdots .$$

(II) The cup product pairings $H^i_c(U) \otimes H^j(U) \to H^{i+j}_c(U)$ are morphisms of MHS. In particular, if $d := \dim(U)$ then the pairing

$$H^i_c(U) \otimes H^{2d-i}(U) \to H^{2d}_c(U)$$

is a perfect pairing of MHS; as a consequence, we obtain the isomorphism of MHS

$$H^i_c(U) \simeq H^{2d-i}(U)^\vee(-d) \tag{1.3.1}$$

from which we deduce, for all p and k, isomorphisms

$$Gr_F^p Gr_m^W H_c^k(U) \simeq Gr_F^{d-p} Gr_{2d-m}^W H^{2d-k}(U). \tag{1.3.2}$$

Proof. [62, Sections 5.5, 6.3]. □

We now present some results on MHS and related notions that we shall use later on.

Proposition 1.3.9. *Let $f : \tilde{X} \to X$ be a proper modification of a complex algebraic variety with discriminant D. Define $E := f^{-1}(D)$ and call $g := f_{|E} : E \to D$ and $i : D \hookrightarrow X$, $j : E \hookrightarrow \tilde{X}$ the inclusions; from the discriminant square*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "g"] & \tilde{X} \arrow[d, "f"] \\ D \arrow[r, "i"] & X \end{tikzcd}$$

one obtains a long exact sequence of MHS

$$\cdots \to H^\bullet(X) \xrightarrow{(f^*,i^*)} H^\bullet(\tilde{X}) \oplus H^\bullet(D) \xrightarrow{j^*-g^*} H^\bullet(E) \to H^{\bullet+1}(X) \to \cdots$$

called the Mayer-Vietoris sequence for the discriminant square associated to f.

Proof. [62, Definition-Lemma 5.17] shows that the 2-cubical space $Y_\square$ associated to the discriminant square of a proper modification is of cohomological descent; the Proposition follows then from [62, Theorem 5.35]. □

Proposition 1.3.10. *Let X be a complex algebraic variety of dimension n with singular locus Σ. Let Z be a subvariety of dimension s of X containing Σ, and let $\pi : \tilde{X} \to X$ be a resolution of singularities of X such that $D := \pi^{-1}(Z)$ is a simple normal crossing divisor in $\tilde{X}$. Then:*

1. *For all $k \geq n + s$ we have $W_{k-1}H^k(D) = 0$.*
2. *If moreover Z is compact, then the MHS on $H^k(D)$ is pure of weight k for all $k \geq n + s$.*

Proof. [62, Theorem 6.31] There exist $s + 1$ affine open subsets of X, call them $U_0, \ldots, U_s$, whose union U covers Z; being affine, they satisfy $H^k(U_i) = 0$ for $k > n$ for all i's by Corollary 1.1.4.

We prove by induction that $H^k(U_0 \cup \cdots \cup U_t) = 0$ for $k > n + t$. The case $t = 0$ is obvious; for $t > 0$, we write the Mayer-Vietoris sequence associated to the covering $\{U_0 \cup \cdots \cup U_{t-1}, U_t\}$ of $U_0 \cup \cdots \cup U_t$:

$$\to H^{k-1}((U_0 \cup \cdots \cup U_{t-1}) \cap U_t) \to H^k(U_0 \cup \cdots \cup U_t) \to H^k(U_0 \cup \cdots \cup U_{t-1}) \oplus H^k(U_t) \to$$

We have $H^k(U_t) = 0$ for $k > n$, and $H^k(U_0 \cup \cdots \cup U_{t-1}) = 0$ for $k > n + t - 1$ by induction hypothesis. $(U_0 \cup \cdots \cup U_{t-1}) \cap U_t$ is union of the $t-1$ affine open sets $U_i \cap U_t$, so by induction hypothesis $H^{k-1}((U_0 \cup \cdots \cup U_{t-1}) \cap U_t) = 0$ for $k - 1 > n + t - 1$ i.e. for $k > n + t$. This proves our claim.

Now we define $\tilde{U} := \pi^{-1}(U)$ and consider the following resolution square:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\ Z \arrow[r] & U. \end{tikzcd}$$

The associated long exact sequence in cohomology reads:

$$\cdots \to H^k(U) \to H^k(Z) \oplus H^k(\tilde{U}) \to H^k(D) \to H^{k+1}(U) \to \cdots .$$

For $k \geq n + s$ we have $H^k(Z) = 0$, which means that the map $H^k(\tilde{U}) \to H^k(D)$ is surjective for $k = n + s$ while it is an isomorphism for $k > n + s$; since $\tilde{U}$ is smooth, this implies that for $k \geq n + s$ the group $H^k(D)$ has weights $\geq k$, and in particular $W_{k-1}H^k(D) = 0$.

If Z is compact then D is compact too, and so $H^k(D)$ has weights $\leq k$; this implies that the MHS on $H^k(D)$ is pure of weight k. $\square$

Proposition 1.3.11. *Let X be a complex algebraic variety of dimension n with singular locus Σ, and let $s := \dim(\Sigma)$; for $k > n+s$ one has $W_{k-1}H^k(X) = 0$. In particular, if X is projective then the MHS on $H^k(X)$ is pure for $k > n + s$.*

Proof. [62, Theorem 6.33] Let $\pi : \tilde{X} \to X$ be a resolution of X, and define $D := \pi^{-1}(\Sigma)$; for $k > n + s$ we have $H^k(\Sigma) = 0$ so we get the exact sequence of MHS

$$H^k(\tilde{X}) \to H^k(X) \to H^k(D).$$

As both $W_{k-1}H^k(\tilde{X})$ and $W_{k-1}H^k(D)$ are zero (the former because $\tilde{X}$ is smooth, the latter by the previous proposition) we deduce that $W_{k-1}H^k(X) = 0$. $\square$

We now present two easy consequences of Propositions 1.3.9 and 1.3.11:

Corollary 1.3.12. *If X is a complex projective surface with singular locus of dimension zero and $\tilde{X}$ is a resolution of singularities of X, then $H^i(X) \simeq H^i(\tilde{X})$ for $i = 3, 4$.*

Proof. If X is smooth there is nothing to prove. If it is not, then $H^3(X)$ and $H^4(X)$ have a pure HS by Proposition 1.3.11. If we denote by E the exceptional divisor of the resolution $\tilde{X} \to X$, the last part of the long exact sequence of MHS associated to the resolution square of X reads

$$\cdots \to H^2(E) \xrightarrow{\alpha} H^3(X) \to H^3(\tilde{X}) \to 0 \to H^4(X) \to H^4(\tilde{X}) \to 0$$

hence $H^4(X) \simeq H^4(\tilde{X})$. Since $H^3(X)$ has a pure HS of weight 3 we know that $Gr_i^W H^3(X) = 0$ for $i \neq 3$, but $Gr_3^W H^2(E) = 0$: this means α is identically zero, which implies $H^3(X) \simeq H^3(\tilde{X})$. $\square$

Corollary 1.3.13. *If X is a complex projective surface having only ADE singularities, then all its cohomology groups $H^i(X)$ admit a pure HS.*

Proof. For $i = 3, 4$ this follows from the previous corollary, while for $i = 0$ the statement is trivial. Denote by Σ the singular locus of X, by $\tilde{X}$ a resolution of X and by E the exceptional divisor; from the associated resolution square we obtain

$$\begin{aligned} 0 \to H^0(X) \to H^0(\tilde{X}) \oplus H^0(\Sigma) \to H^0(E) \xrightarrow{\sigma} H^1(X) \to H^1(\tilde{X}) \to \\ \to H^1(E) \to H^2(X) \to H^2(\tilde{X}) \to H^2(E) \xrightarrow{\alpha} H^3(X) \to \cdots . \end{aligned}$$

The exceptional divisor we obtain from the resolution of an ADE singularity consists of rational curves with intersection diagram coinciding with a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8; from this we deduce that $H^1(E) = 0$. Moreover $H^2(E)$ has pure HS by Proposition 1.3.11. As the map α is identically zero, we obtain a short exact sequence

$$0 \to H^2(X) \to H^2(\tilde{X}) \to H^2(E) \to 0$$

which implies that $H^2(X)$ has a pure HS. The map σ is identically zero too, so we find

$$0 \to H^1(X) \to H^1(\tilde{X}) \to 0$$

i.e. $H^1(X) \simeq H^1(\tilde{X})$ and in particular $H^1(X)$ has a pure HS.

□

Remark 1.3.14. The proof of the previous corollary shows that it is sufficient that X has singularities such that $H^1(E) = 0$ in order for the $H^i(X)$ to have a pure HS.

It is now natural to ask how MHS behave 'in families', i.e. what is the relation between the MHS on a variety X and the MHS of a deformation X' of it.

Definition 1.3.15. Let X be a complex algebraic variety. A *deformation* of X is given by the following data:

1. Two complex algebraic varieties $\mathcal{X}$ and B with B irreducible.
2. A flat, proper and surjective morphism $\pi : \mathcal{X} \to B$.
3. A point $b_0 \in B$ such that $X \simeq \pi^{-1}(b_0) \subset \mathcal{X}$.

This information can be encoded in the following diagram:

$$\begin{tikzcd} X \arrow[r, "\simeq"] & \pi^{-1}(b_0) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \{b_0\} \arrow[r] & B. \end{tikzcd}$$

For the Hodge numbers of a pure HS on a Kähler manifold we have the following classical result:

Proposition 1.3.16. *Let X be a Kähler manifold and $\pi : \mathcal{X} \to B$ be a deformation of $X = \pi^{-1}(b_0)$. Then for b near b_0 the fibre X_b is a Kähler manifold and $h^{p,q}(X_b) = h^{p,q}(X)$.*

Proof. [67, Proposition 9.20]. □

If X is singular, a deformation need not preserve its singularities; the deformation that do are called *equisingular*. While it is intuitively clear what 'preserving the singularities' means, the precise definition of equisingular deformation is rather involved; the interested reader can find it in any book on singularity theory, for example [28].

We are mainly interested in equisingular deformations of plane curves; we will prove in fact that:

(i) The mixed Hodge numbers of a plane curve $C \subset \mathbb{P}^2$ are invariant under equisingular deformation.

(ii) Let $C \subset \mathbb{P}^2$ be a plane curve and S be a covering of $\mathbb{P}^2$ branched along C; any equisingular deformation of C induces an equisingular deformation of S, under which the mixed Hodge numbers of S are invariant.

Thus, mixed Hodge numbers of some singular varieties behave under equisingular deformations as Hodge numbers of Kähler manifolds behave under deformations.

(i) - Curves Let C be a projective plane curve, Σ be its singular locus, $\tilde{C}$ be a resolution of C and E be the exceptional divisor. By Theorem 1.3.7 and Proposition 1.3.9 we have that:

- Both the weight and Hodge filtrations on $H^0(C)$ are trivial, so $h^{0,0}(H^0(C)) = h^0(C)$.

- On $H^1(C)$ the weight and Hodge filtrations are given by

$$0 = W_{-1}H^1(C) \subset W_0H^1(C) \subset W_1H^1(C) = H^1(C).$$
$$0 = F^2H^1(C) \subset F^1H^1(C) \subset F^0H^1(C) = H^1(C).$$

  The associated mixed Hodge numbers are thus $h^{0,0}(H^1(C))$ and $h^{0,1}(H^1(C)) = h^{1,0}(H^1(C))$.

- The MHS on $H^2(C)$ is actually pure, i.e. the weight filtration is trivial; the only mixed Hodge number is thus $h^{1,1}(H^2(C)) = h^2(C)$.

By Proposition 1.3.9, from the resolution square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & C \end{array}$$

we obtain the following long exact sequence of MHS:

$$\begin{aligned} 0 \to & H^0(C) \to H^0(\tilde{C}) \oplus H^0(\Sigma) \to H^0(E) \to H^1(C) \to \\ \to & H^1(\tilde{C}) \to 0 \to H^2(C) \to H^2(\tilde{C}) \to 0. \end{aligned}$$

From this we deduce immediately that $H^2(C) \simeq H^2(\tilde{C})$, which implies that $h^2(C)$ is invariant under equisingular deformation. Taking the graded parts Gr_i^W for $i = 0, 1$, we obtain:

- $0 \to Gr_1^W H^1(C) \to H^1(\tilde{C}) \to 0$; this implies the invariance of $h^{0,1}(H^1(C)) = h^{1,0}(H^1(C))$.
- $0 \to H^0(C) \to H^0(\tilde{C}) \oplus H^0(\Sigma) \to H^0(E) \to Gr_0^W H^1(C) \to 0$. The dimensions of the first three non-zero terms of this sequence are invariant under equisingular deformation so we deduce the invariance of $h^{0,0}(H^1(C))$.

(ii) - Branched coverings of $\mathbb{P}^2$ Let $C := V(f(x_0, x_1, x_2)) \subset \mathbb{P}^2$ be a curve of degree d, and define the surface $S := V(y^m - f) \subset \mathbb{P}(\frac{d}{m}, 1, 1, 1)$ which is an m-to-1 cover of $\mathbb{P}^2$ branched along C; the singular locus of S consists of the points $(0 : x_0 : x_1 : x_2)$ such that $(x_0 : x_1 : x_2)$ is a singular point of C, so S has only isolated singularities. We write $\Sigma := S_{sing} = \{P_1, \ldots, P_n\}$. We denote by $\tilde{S}$ a resolution of singularities of S and by E the corresponding exceptional divisor, which we can assume to be simple normal crossing.

Remark 1.3.17. A general deformation of C can be given by a flat, proper and surjective holomorphic map $\phi : \mathcal{C} \to \mathbb{C}$ where $\mathcal{C} = V(F(x_0, x_1, x_2, t)) \subset \mathbb{P}^2 \times \mathbb{C}$ is the total space, the fibre $\mathcal{C}_0 := \phi^{-1}(0)$ is isomorphic to C and ϕ is just the projection onto t. Under a general deformation the singularities of the starting variety might 'collapse' one into the other in some fibres, but to avoid this problem it is enough to remove a finite set of points Δ from the base space of the deformation; once we have removed these points from $\mathbb{C}$ and their preimages from $\mathcal{C}$, we have an equisingular deformation $\phi : \mathcal{C}' \to \mathbb{C} \setminus \Delta$.

From this we immediately obtain an equisingular deformation $\phi' : \mathcal{S} \to \mathbb{C} \setminus \Delta$ of S with total space $\mathcal{S} = V(y^m - F(x_0, x_1, x_2, t)) \subset \mathbb{P}^3 \times \mathbb{C}$, and by blowing up the singular locus of the latter total space we obtain a deformation $\phi'' : \tilde{\mathcal{S}} \to \mathbb{C} \setminus \Delta$ of $\tilde{S}$ such that any fibre $\tilde{\mathcal{S}}_t$ is a resolution of singularities of the fibre $\mathcal{S}_t$. The latter deformation gives a family of projective manifolds, for which the Hodge numbers are invariant.

Using Proposition 1.3.9, from the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ \Sigma \arrow[r] & S \end{tikzcd} \tag{1.3.3}$$

we deduce a long exact sequence of MHS:

$$\cdots \to H^\bullet(S) \to H^\bullet(\tilde{S}) \oplus H^\bullet(\Sigma) \to H^\bullet(E) \to H^{\bullet+1}(S) \to \cdots .$$

By the Lefschetz hyperplane theorem we have $H^1(S) = 0$, and clearly we have $H^3(E) = 0$ and $H^1(\Sigma) = 0$; this yields the following long exact sequence of MHS:

$$\begin{aligned} 0 &\to H^0(S) \to H^0(\Sigma) \oplus H^0(\tilde{S}) \to H^0(E) \to 0 \to H^1(\tilde{S}) \to H^1(E) \to H^2(S) \to \\ &\to H^2(\tilde{S}) \to H^2(E) \to H^3(S) \to H^3(\tilde{S}) \to 0 \to H^4(S) \to H^4(\tilde{S}) \to 0. \end{aligned}$$

By Corollary 1.3.12 we know that $H^i(S) \simeq H^i(\tilde{S})$ for $i = 3, 4$, so the mixed Hodge numbers of $H^3(S)$ and $H^4(S)$ are invariant under equisingular deformation of C. We now have to deal with the cohomology groups $H^i(S)$ for $i = 0, 1, 2$; we study them by applying Gr_2^W, Gr_1^W and Gr_0^W to the above long exact sequence of MHS.

Remark 1.3.18. *E is a simple normal crossing divisor with k smooth components of degrees $d_1, \dots, d_k$; if we deform C to C' with an equisingular deformation, the exceptional divisor E' we obtain is again simple normal crossing with k smooth components of degrees $d_1, \dots, d_k$, because it depends on data (the singularities of C') that is left untouched by equisingular deformations. In particular, all cohomology groups $H^i(E)$ and the associated mixed Hodge numbers remain invariant under equisingular deformation of C.*

- The Hodge structure on $H^2(E)$ is pure by Proposition 1.3.11, so after applying Gr_2^W we obtain

$$0 \to Gr_2^W H^2(S) \to H^2(\tilde{S}) \to H^2(E) \to 0.$$

  The Hodge numbers of $H^2(\tilde{S})$ are invariant under equisingular deformation of C, and the same goes for those of $H^2(E)$, hence the mixed Hodge numbers $h^{2,0}(H^2(S)) = h^{0,2}(H^2(S))$ and $h^{1,1}(H^2(S))$ are invariant under equisingular deformation.

- Applying Gr_1^W we obtain

$$0 \to H^1(\tilde{S}) \to Gr_1^W H^1(E) \to Gr_1^W H^2(S) \to 0$$

  which gives the invariance under equisingular deformation of C of $h^{1,0}(H^2(S)) = h^{0,1}(H^2(S))$.

- Applying Gr_0^W we obtain

$$0 \to H^0(S) \to H^0(\Sigma) \oplus H^0(\tilde{S}) \to H^0(E) \to 0 \to Gr_0^W H^2(S) \to 0$$

from which we deduce that $h^{0,0}(H^2(S)) = 0$ and that $h^{0,0}(H^0(S))$ is invariant under equisingular deformation of C.

1.4 Polar and Hodge filtrations on hypersurface complements

1.4.1 The global case

An effective approach to the study of the cohomology of a (weighted) projective hypersurface V, which was introduced first by Griffiths in his celebrated paper [29] in the projective setting, and was later extended to the weighted projective setting by Dolgachev in [23], consists in studying the cohomology of its complement. In this section we recall some results of this type, most of which can be found in [16, Chapter 6].

Fix an integer $n \geq 1$ and weights $w_0, \dots, w_n \in \mathbb{Z}_{\geq 1}$, and let $R := \mathbb{C}[x_0, \dots, x_n]$. R becomes a graded ring by setting $deg(x_0^{a_0} \cdot \ldots \cdot x_n^{a_n}) := a_0 w_0 + \cdots + a_n w_n$, and the $\mathbb{C}$-vector space Ω^p of p-forms on $\mathbb{C}^{n+1}$ becomes a graded R-module by setting $deg(x_0^{a_0} \cdot \ldots \cdot x_n^{a_n} dx_{i_1} \wedge \cdots \wedge dx_{i_p}) := a_0 w_0 + \cdots + a_n w_n + w_{i_1} + \cdots + w_{i_p}$; their homogeneous components of degree m are denoted by R_m and Ω^p_m respectively.

Assume now we are in the usual projective setting, i.e. $w_i = 1$ for all $i = 0, \dots, n$. We denote by E the Euler vector field on $\mathbb{C}^{n+1}$

$$E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$$

and by ι_E the contraction by E; this means that for any $x \in \mathbb{C}^{n+1}$, $\alpha \in \Omega^p$ and $v_1, \dots, v_{p-1} \in T_x\mathbb{C}^{n+1}$ the $(p-1)$-form $\iota_E(\alpha)$ on $\mathbb{C}^{n+1}$ is given by

$$\iota_E(\alpha)_x(v_1, \dots, v_{p-1}) := \alpha(E(x), v_1, \dots, v_{p-1}).$$

Taking advantage of the properties of $\iota_E : \Omega_p \to \Omega_{p-1}$ we obtain the following:

Proposition 1.4.1. *Let $V \subset \mathbb{P}^n$ be a hypersurface defined by a polynomial $f \in R_d$, and define $U := \mathbb{P}^n \setminus V$; any p-form w on U (for $p > 0$) can be written as*

$$w = \frac{\iota_E(\gamma)}{f^s} \tag{1.4.1}$$

for some integer $s > 0$ and $\gamma \in \Omega^{p+1}_{sd}$.

Proof. [16, Proposition 6.1.16]. □

In particular, since Ω^{n+1} is an R-module of rank 1 generated by $w_{n+1} := dx_0 \wedge \cdots \wedge dx_n$, we obtain that in the same hypotheses of the proposition above any n-form on U can be written as

$$w = \frac{h\Omega}{f^s} \qquad \text{for } h \in R_{sd-n-1} \text{ and } s > 0 \tag{1.4.2}$$

where Ω is the n-form

$$\Omega := \iota_E(w_{n+1}) = \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

From now on we will work in the weighted projective setting (to which this last result can be extended by considering a generic set of weights for the indeterminates x_i and adjusting the definition of the Euler vector field): we write $\mathbb{P} := \mathbb{P}(\underline{w})$, and consider a hypersurface $V := V(f) \subset \mathbb{P}$ of degree d and its complement $U := \mathbb{P} \setminus V$.

Definition 1.4.2. If $w \in H^0(U, \Omega^p_{\mathbb{P}})$ is a rational differential form as in (1.4.1), the minimal positive integer s for which such a representation of w exists is called the *order of the pole* of w along the hypersurface V; from now on this value will be denoted by $ord_V(w)$.

We now define the polar filtration on the de Rham complex of U, that consists of objects

$$A^m := H^0(U, \Omega^m_{\mathbb{P}}).$$

Definition 1.4.3. The *polar filtration* on $(A^\bullet, d)$ is the decreasing filtration P defined by

$$P^s A^m := \begin{cases} \{w \in A^m | ord_V(w) \leq m - s + 1\} & \text{if } m - s + 1 \geq 0. \\ 0 & \text{if } m - s + 1 < 0. \end{cases}$$

Looking back at (1.4.2), we see that there is an obvious surjective map $R_{(n-s)d-n-1} \twoheadrightarrow P^{s+1}A^n$ s.t. $h \mapsto [h\Omega/f^{n-s}]$.

Proposition 1.4.4. *The filtration P has the following properties:*

1. *P is a filtration of complexes, i.e. it is compatible with the differential:*

$$d(P^s A^m) \subset P^s A^{m+1}.$$

2. *P is decreasing and bounded above:*

$$0 = P^{n+1}A^\bullet \subset P^n A^\bullet \subset \cdots \subset P^i A^\bullet \subset P^{i-1} A^\bullet \subset \cdots .$$

3. *P is exhaustive, i.e.*

$$A^\bullet = \cup_{s \in \mathbb{Z}} P^s A^\bullet.$$

Proof. [16, Lemma 6.1.29]. □

The last two bullets of this proposition imply, by the general theory of spectral sequences, that the spectral sequence associated to the filtration P converges to $H^\bullet(U)$.

The inclusion of complexes $P^sA^\bullet \hookrightarrow A^\bullet$ induces, for every m, a morphism at the level of cohomology $H^m(P^sA^\bullet) \to H^m(A^\bullet)$; this allows us to define an induced polar filtration P on the cohomology groups $H^\bullet(U) := H^\bullet(A^\bullet)$ by

$$P^sH^m(U) := Im\{H^m(P^sA^\bullet) \to H^m(A^\bullet)\}.$$

More explicitly, we have

$$P^sH^m(U) = \begin{cases} \left\{\alpha \in H^m(A^\bullet) \,\middle|\, \begin{array}{l} \alpha \text{ has a representative } w \\ \text{s.t } ord_V(w) \le m-s+1 \end{array}\right\} & \text{if } m-s+1 \ge 0. \\ 0 & \text{if } m-s+1 < 0. \end{cases} \tag{1.4.3}$$

This filtration is clearly still decreasing. Now, since U is an algebraic variety its cohomology groups admit a natural MHS (recall Theorem 1.3.7); we are interested in comparing the polar filtration P with the Hodge filtration F on $H^\bullet(U)$. We have the following result:

Proposition 1.4.5. *$F^sH^m(U) \subset P^sH^m(U)$ for any integer s and m.*

Proof. [16, Theorem 6.1.31]. □

Corollary 1.4.6. *Any element in $H^m(U)$ can be represented by a rational form $w \in A^m$ such that $ord_V(w) \le m$.*

Proof. [16, Corollary 6.1.32]. □

As a consequence of this corollary, and of the definition of polar filtration and order of the pole of a differential form, for $m > 0$ we have the following:

$$0 = P^{m+1}H^m(U) \subseteq P^mH^m(U) \subseteq \cdots \subseteq P^2H^m(U) \subseteq P^1H^m(U) = H^m(U) \tag{1.4.4}$$

and in particular

$$H^m(U) \simeq \bigoplus_{i=1}^{m} Gr_P^iH^m(U). \tag{1.4.5}$$

We conclude this section by recalling an important result on the primitive middle cohomology groups of a (weighted) projective hypersurface. We start from the usual projective setting. Call $V := V(f) \subset \mathbb{P}^n$ and $U := \mathbb{P}^n \setminus V$, and denote by $\Omega^n_{\mathbb{P}^n}(sV)$ the sheaf of meromorphic n-forms on $\mathbb{P}^n$ having a pole of order at most s along V. For any $s = 1, \dots, n$ we can define the map

$$\phi_s : H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(sV)) \to H^n(U) \text{ s.t. } \alpha \mapsto [\alpha]$$

associating to any α its de Rham cohomology class; its image is clearly $P^{n-s+1}H^n(U)$. Assume now V has degree d; since $\Omega^n_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and $\mathcal{O}_{\mathbb{P}^n}(V) = \mathcal{O}_{\mathbb{P}^n}(d)$, we obtain a surjective map

$$\phi_s : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(sd-n-1)) \twoheadrightarrow P^{n-s+1}H^n(U)$$

which we can compose with the projection onto $P^{n-s+2}H^n(U)$ to obtain

$$\overline{\phi_s} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(sd-n-1)) \twoheadrightarrow P^{n-s+1}H^n(U)/P^{n-s+2}H^n(U).$$

Proposition 1.4.7. *Call J^f the Jacobian ideal of f in R. For $s = 1, \ldots, n$, if $w = \frac{h\Omega}{f^s}$ with $h \in J^f_{sd-n-1}$ then $w \in Ker(\overline{\phi_s})$.*

Proof. Pick an element $h := \sum_{i=0}^n h_i \frac{\partial f}{\partial x_i}$ in J^f_{sd-n-1} and call $w := \frac{h\Omega}{f^s}$ the corresponding meromorphic n-form on $\mathbb{P}^n$; if we define the $(n-1)$-form ψ as

$$\psi := \frac{1}{sf^{s-1}} \left[\sum_{i<j} (-1)^{i+j}(x_i h_j - x_j h_i) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \right]$$

a direct computation shows that

$$w + d\psi = -\frac{\Omega}{f^{s-1}} \sum_{i=0}^n \frac{\partial h_i}{\partial x_i}$$

where the right-hand side τ is a meromorphic n-form with a pole of order at most $s-1$ along V. This means that $[w] = [\tau] \in P^{n-s+2}H^n(U)$ i.e. that $\overline{\phi_s}(w) = 0$. □

From this proposition we deduce the existence of the following surjective map

$$(R/J^f)_{sd-n-1} \twoheadrightarrow P^{n-s+1}H^n(U)/P^{n-s+2}H^n(U). \tag{1.4.6}$$

If V is smooth then this map further simplifies; we have in fact:

1. The Hodge and polar filtration on $H^n(U)$ coincide, as shown by Griffiths in [29].
2. The statement in Proposition 1.4.7 becomes an if and only if.
3. We have $F^{n-s+1}H^n(U) \simeq F^{n-s}H^{n-1}(V)_{\mathrm{van}} \simeq F^{n-s}H^{n-1}(V)_{\mathrm{prim}}$ by the following arguments:
   - If we call $j : U \hookrightarrow \mathbb{P}^n$ the inclusion, we have the short exact sequence

     $$0 \to H^n(\mathbb{P}^n, \mathbb{Q})_{\mathrm{prim}} \xrightarrow{j^*} H^n(U, \mathbb{Q}) \xrightarrow{Res} H^{n-1}(X, \mathbb{Q})_{\mathrm{van}} \to 0$$

     which is compatible with the Hodge filtration. Since $H^n(\mathbb{P}^n, \mathbb{C})_{\mathrm{prim}} = 0$ and the residue is a morphism of MHS of type $(-1,-1)$, we get the first isomorphism.

- For projective hypersurfaces the vanishing and primitive middle cohomology coincide, so we get the second isomorphism.

This means that in the smooth case the map (1.4.6) becomes an isomorphism:

$$(R/J^f)_{sd-n-1} \xrightarrow{\simeq} H^{n-s,s-1}(V)_{\text{prim}}. \tag{1.4.7}$$

In the weighted projective setting similar statements hold (although we need to put some care in the definition of a regular form on the space $\mathbb{P}(\underline{w})$: if we call W the sum of the weights w_i, we obtain

$$(R/J^f)_{sd-W} \twoheadrightarrow P^{n-s+1}H^n(U)/P^{n-s+2}H^n(U). \tag{1.4.8}$$

If V is quasi-smooth then the filtrations P and F on $H^n(U)$ coincide (this was proved by Steenbrink in [65]) and the map above becomes the following isomorphism:

$$(R/J^f)_{sd-W} \xrightarrow{\simeq} H^{n-s,s-1}(V)_{\text{prim}}. \tag{1.4.9}$$

1.4.2 The local case

We now want to describe a polar filtration in the following local setting: we assume that X is an open ball of radius $r > 0$ centred at the origin of $\mathbb{C}^n$ and Y is a hypersurface in X with equation $g = 0$ such that $\underline{0} \in Y$; we assume also that Y has a conic structure in X (this can be always achieved by taking r small enough, see [16, Theorem 1.5.1]).

We denote again by g the analytic function germ $g : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, \underline{0})$; the set $\{g^s | s \geq 0\}$ can be thought of as a multiplicative system for each stalk $\Omega^i_{\mathbb{C}^n,\underline{0}}$. We denote by $\Omega^\bullet_g$ the associated 'localised analytic de Rham complex' and define the complex $A^\bullet_\infty := H^0(X, \Omega^\bullet_{\mathbb{C}^n,\underline{0}})$; the latter can be identified with the de Rham complex of germs at the origin of $\mathbb{C}^n$ of meromorphic differential forms with poles along Y, so any element in it has a representative ω of the form

$$\omega = \frac{\beta}{g^s}$$

where β is a differential form whose coefficients are function germs in $\mathcal{O}_{\mathbb{C}^n,\underline{0}}$.

We can define a polar filtration P on $A^\bullet_\infty$ by setting

$$P^s A^m_\infty := \begin{cases} \{\omega \in A^m_\infty | ord_Y(\omega) \leq m - s + 1\} & \text{if } m - s + 1 \geq 0. \\ 0 & \text{if } m - s + 1 < 0. \end{cases}$$

This filtration has all the properties of the polar filtration we saw in the global case; in particular, it defines a spectral sequence $(E_r(Y), d_r)$ which converges to $H^\bullet(X \setminus Y)$. There is an associated polar filtration P on $H^m(X \setminus Y) = H^m(A^\bullet_\infty)$ given by:

$$P^s H^m(X \setminus Y) := Im\{H^m(P^s A^\bullet_\infty) \to H^m(A^\bullet_\infty)\}.$$

Assume from now on that $(Y, \underline{0})$ is an isolated singularity. The cohomology groups $H^m(X \setminus Y)$ also carry a Hodge filtration, but the relation between it and the polar filtration is not as simple as in the global case:

Proposition 1.4.8. *For any integer s we have*

$$F^s H^n(X \setminus Y) \subset P^s H^n(X \setminus Y).$$
$$F^s H^{n-1}(X \setminus Y) \supset P^s H^{n-1}(X \setminus Y).$$

Proof. [37]. □

We now focus our interest on a particular class of isolated singularities.

Definition 1.4.9. The singularity $(Y, \underline{0})$ is *weighted homogeneous* if there exist coordinates $y_1, \dots, y_n$ on $\mathbb{C}^n$ around the origin and weights $v_i := wt(y_i)$ such that $(Y, \underline{0})$ can be defined by a weighted homogeneous polynomial g of degree M (for some M) with respect to the weights $\mathbf{v} := (v_1, \dots, v_n)$.

Proposition 1.4.10. *$(E_r(Y), d_r)$ degenerates at E_2 if and only if the singularity $(Y, \underline{0})$ is weighted homogeneous*

Proof. [14, Corollary 3.10']. □

Remark 1.4.11. When $(Y, \underline{0})$ is weighted homogeneous the terms $E_2^{n-t,t}$ of this spectral sequence can be described easily: if we denote by $M(g)$ the Milnor algebra of g, which inherits the grading given by $\mathbf{v}$, and by v the sum of weights of $\mathbf{v}$, we have a $\mathbb{C}$-linear identification

$$E_2^{n-t,t}(Y) \simeq M(g)_{tM-v}$$

given by associating to the class of a monomial $y^\alpha \in M(g)_{tM-v}$ the class of the differential form $y^\alpha g^{-t}\omega_n$, where $\omega_n := dy_1 \wedge \dots \wedge dy_n$.

Since we have $H^n(X \setminus Y) \simeq \bigoplus_{t \in \mathbb{N}} M(g)_{tM-v}$, if $\{y^\alpha = y_1^{\alpha_1} \cdot \dots \cdot y_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in A_t\}$ is a monomial basis for $M(g)_{tM-v}$ then the differential forms

$$\omega(y^\alpha) := \frac{y^\alpha}{g^t}\omega_n$$

form a basis for $P^s H^n(X \setminus Y)$.

Since $Y - \{\underline{0}\}$ is a smooth divisor in $X - \{\underline{0}\}$ we can write the usual Gysin exact sequence, which gives in particular

$$\dots \to H^n(X \setminus \{\underline{0}\}) \to H^n(X \setminus Y) \xrightarrow{Res} H^{n-1}(Y \setminus \{\underline{0}\}) \to H^{n+1}(X \setminus \{\underline{0}\}) \to \dots .$$

$X \setminus \{\underline{0}\}$ retracts onto S^{2n-1}, so $H^k(X \setminus \{\underline{0}\}) = 0$ for $k \neq 0, 2n-1$; as a consequence, the Poincaré residue map $Res : H^n(X \setminus Y) \to H^n(Y \setminus \{\underline{0}\})$ is an isomorphism. Moreover, the assumption of Y being a cone in X means that Y is contractible, so the long exact sequence of the pair $(Y, Y \setminus \{\underline{0}\})$ gives an isomorphism $H^{n-1}(Y \setminus \{\underline{0}\}) \to H^n(Y, Y \setminus \{\underline{0}\}) := H^n_{\{\underline{0}\}}(Y)$. Putting things together, we get isomorphisms

$$H^n_{\{\underline{0}\}}(Y) \simeq H^n(X \setminus Y) \simeq \bigoplus_{t\in\mathbb{N}} M(g)_{tM-v}. \tag{1.4.10}$$

This shows that we can induce a polar filtration P on $H^n_{\{\underline{0}\}}(Y)$ too. This local cohomology group carries a natural MHS, which is actually pure of weight n when $(Y, \underline{0})$ is a weighted homogeneous singularity; in the same situation, the polar filtration P coincides with the Hodge filtration F (see [65]).

CHAPTER 2

Hyperplane arrangements and their Alexander polynomial

2.1 The Alexander polynomials of a projective hypersurface

2.1.1 Definition and basic properties

Call R the ring $\mathbb{C}[x_0, \ldots, x_n]$ and consider a polynomial $f \in R$ such that $f(\underline{0}) = 0$. For real numbers $\epsilon, \delta > 0$ we define the open ball $B_\epsilon := \{\underline{x} \in \mathbb{C}^{n+1} \text{ s.t. } |\underline{x}| < \epsilon\}$ and the punctured disk $D^*_\delta := \{x \in \mathbb{C} \text{ s.t. } 0 < |x| < \delta\}$; in [40] Lê proved that:

Theorem 2.1.1. *For any $0 < \delta \ll \epsilon$ sufficiently small, the map*

$$\psi : B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta \quad s.t. \quad \psi(\underline{x}) := f(\underline{x}) \tag{2.1.1}$$

is a smooth locally trivial fibration.

The map ψ is usually called the *Milnor fibration* associated to the polynomial f; this is due to the fact that, before the work of Lê, Milnor [54] proved the existence of a fibration equivalent to the previous one for polynomials having at worst an isolated singularity in $\underline{0}$. We are interested in the case in which f is weighted homogeneous: namely, let $\underline{w} := (w_0, \ldots, w_n)$ be a vector of non-negative integers and assign weights to the indeterminates x_i of R by $wt(x_i) := w_i$. R becomes a graded module over itself, and we can speak of its homogeneous components R_d of degree d. If $f \in R_d$, taking advantage of the Euler relation $f(t \cdot \underline{x}) = t^d f(\underline{x})$ it is easy to see that the polynomial map

$$f : \mathbb{C}^{n+1} \setminus f^{-1}(0) \to \mathbb{C}^* \tag{2.1.2}$$

is a smooth locally trivial fibration equivalent to ψ; its restriction over the unit circle S^1 is called *global affine Milnor fibration*, and the fibre $F := f^{-1}(1)$ is called *global affine Milnor fibre*; the latter is a smooth affine variety of complex dimension n so it is in particular a Stein space variety. Since we will focus only on the case in which f is weighted homogeneous, we will simply speak of Milnor fibration and Milnor fibre.

Call now $V := V(f) \subset \mathbb{P}(\underline{w})$ the weighted projective hypersurface associated to f, $U := \mathbb{P}(\underline{w}) \setminus V$ its complement and $Y := V(y^d - f) \subset \mathbb{P}(1, \underline{w})$, which is the closure of the Milnor fibre in the weighted projective space $\mathbb{P}(1, \underline{w})$; clearly

$$Y_{sing} = \{(0 : x_0 : \ldots : x_n | (x_0 : \ldots : x_n) \in V_{sing}\} \simeq V_{sing}.$$

Consider the projection map

$$\pi : Y \to \mathbb{P}(\underline{w}) \text{ s.t. } (y : x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n). \tag{2.1.3}$$

The preimage of V through π is the set $\{(0 : x_0 : \ldots : x_n)|(x_0 : \ldots : x_n) \in V\} \simeq V$, while the preimage of any point $(x_0 : \ldots : x_n) \in U$ consists of the d points $(y : x_0 : \ldots : x_n)$ with $y \neq 0$ satisfying $y^d = f(x_0, \ldots, x_n)$; this exhibits Y as a d-fold cover of $\mathbb{P}(\underline{w})$ branched along V, and F as a d-fold cover of U. In particular, F admits as a deck transformation the automorphism

$$(x_0, \ldots, x_n) \mapsto (e^{\frac{2\pi i w_0}{d}} x_0, \ldots, e^{\frac{2\pi i w_n}{d}} x_n).$$

As it happens with any fibration over S^1, to the Milnor fibration we can associate a geometric monodromy automorphism $h : F \to F$; the explicit expression of h is quite easy when f is a weighted homogeneous polynomial: we have

$$h(x_0, \ldots, x_n) = (e^{\frac{2\pi i w_0}{d}} x_0, \ldots, e^{\frac{2\pi i w_n}{d}} x_n) \tag{2.1.4}$$

i.e. the geometric monodromy coincides with the deck transformation of F we saw before. For any $i = 0, \ldots, n$, the geometric monodromy h induces an automorphism T^i on the cohomology group $H^i(F, \mathbb{C})$, which we call i-th algebraic monodromy. We can now give the following definition:

Definition 2.1.2. The i-th *Alexander polynomial* of V is the characteristic polynomial of the i-th algebraic monodromy $T^i : H^i(F, \mathbb{C}) \to H^i(F, \mathbb{C})$. We denote it by $\Delta_V^i(t)$.

As h has finite order, the same holds for the T^i's: this means the latter are diagonalisable, with (not necessarily primitive) roots of unity of order d as eigenvalues. The $\Delta_V^i(t)$ are thus products of cyclotomic polynomials $\Phi_k(t)$ with $k|d$.

Remark 2.1.3. Denote by $H^i(F, \mathbb{C})_\alpha$ the eigenspace of T^i relative to the eigenvalue α. Since F is a d-fold cover of U, and since the geometric monodromy h coincides with the generator of the group of deck transformations of F, we deduce that

$$H^i(F, \mathbb{C})_1 = H^i(F, \mathbb{C})^{T^i} \simeq H^i(F/\langle h \rangle, \mathbb{C}) = H^i(U, \mathbb{C}).$$

If V has r irreducible components, Proposition 1.1.10 implies that

$$H^1(F, \mathbb{C})_1 \simeq \mathbb{C}^{r-1}$$

so the first Alexander polynomial $\Delta_V^1(t)$ will always contain the factor $(t-1)^{r-1}$. For this reason, when $\Delta_V^1(t) = (t-1)^{r-1}$ we will say that the first Alexander polynomial is *trivial*.

We can give a different, more algebraic (but equivalent) definition of the i-th Alexander polynomial. Since T^i has order d, we can consider $H^i(F,\mathbb{Q})$ as a module over the group algebra $A := \mathbb{Q}[\mathbb{Z}/d\mathbb{Z}] \simeq \mathbb{Q}[t]/(t^d-1)$ in the following way: for $P(t) \in A$ and $\alpha \in H^i(F,\mathbb{Q})$ we set $P(t)\cdot\alpha := P(T^i)(\alpha)$. The A-module $H^i(F,\mathbb{Q})$ decomposes then into

$$H^i(F,\mathbb{Q}) = (\mathbb{Q}[t]/(t-1))^{e_1^i(V)} \oplus \bigoplus_{1<k|d} (\mathbb{Q}[t]/\Phi_k(t))^{e_k^i(V)}. \tag{2.1.5}$$

Definition 2.1.4. The i-th Alexander polynomial of V is

$$\Delta_V^i(t) := (t-1)^{e_1^i(V)} \prod_{1<k|d} \Phi_k(t)^{e_k^i(V)}.$$

Most of the Alexander polynomials of a projective hypersurface are actually identical to 1. To see this, observe first that $F = Y \cap D(y)$ and $V \simeq Y \cap Z(y)$, so we have an open immersion $j : F \hookrightarrow Y$ and a closed immersion $i : V \hookrightarrow Y$. As we have seen, the singular loci of Y and V have the same dimension m (as usual $m := -1$ if $V_{sing} = \emptyset$); this means that combining Barth's theorem 1.1.7 with the weak Lefschetz theorem 1.1.1 we obtain

$$H^i(Y,\mathbb{C}) \simeq H^i(\mathbb{P}^{n+1},\mathbb{C}) \text{ for } i<n \text{ and } n+m+2 \le i \le 2n.$$
$$H^i(V,\mathbb{C}) \simeq H^i(\mathbb{P}^{n},\mathbb{C}) \text{ for } i<n-1 \text{ and } n+m+1 \le i \le 2n-2.$$

Moreover, since F is a smooth affine hypersurface of complex dimension n by Corollary 1.1.4 we have $H^i(F,\mathbb{C}) = 0$ for $i > n$. If we combine all of this and take advantage of Poincaré duality, we obtain the following long exact sequence of cohomology groups with compact support (recall Proposition 1.3.8(ii)):

$$\begin{aligned}
&0 \to H^0(\mathbb{P}^{n+1},\mathbb{C}) \to H^0(\mathbb{P}^{n},\mathbb{C}) \to 0 \to H^2(\mathbb{P}^{n+1},\mathbb{C}) \to H^2(\mathbb{P}^{n},\mathbb{C}) \to \cdots \\
&\cdots \to 0 \to H^{n-1}(\mathbb{P}^{n+1},\mathbb{C}) \to H^{n-1}(V,\mathbb{C}) \to H^n(F,\mathbb{C}) \to H^n(Y,\mathbb{C}) \to H^n(V,\mathbb{C}) \to \cdots \\
&\cdots \to H^{n-m-1}(F,\mathbb{C}) \to H^{n+m+1}(Y,\mathbb{C}) \to H^{n+m+1}(\mathbb{P}^{n},\mathbb{C}) \to H^{n-m-2}(F,\mathbb{C}) \to \\
&\to H^{n+m+2}(\mathbb{P}^{n+1},\mathbb{C}) \to H^{n+m+2}(\mathbb{P}^{n},\mathbb{C}) \to \cdots \to 0 \to H^0(F,\mathbb{C}) \to H^{2n}(\mathbb{P}^{n+1},\mathbb{C}) \to 0.
\end{aligned}$$

As the morphisms $H^i(\mathbb{P}^{n+1},\mathbb{C}) \to H^i(\mathbb{P}^{n},\mathbb{C})$ are isomorphisms for $i = 0,\ldots,2n-2$, we find that $H^i(F,\mathbb{C})$ can be non-trivial (i.e Δ_V^i can be different from 1) only for $i = 0, n-m-1, \ldots, n$. For this reason, we shift the indices of the Alexander polynomials in the following way: for $i = 1,\ldots,m+2$ we set

$$\Delta_V^i(t) := det(I\cdot t - T^{n-m-2+i} : H^{n-m-2+i}(F,\mathbb{C}) \to H^{n-m-2+i}(F,\mathbb{C})). \tag{2.1.6}$$

An important feature of the Alexander polynomials is that they remain (almost) unchanged if we substitute the hypersurface V with a generic hyperplane section $V \cap H$; precisely, we have the following:

Theorem 2.1.5. *Let $V \subset \mathbb{P}^n$ be a hypersurface and let $m := \dim(V_{sing})$, then*

$$\Delta^k_{V\cap H} = \Delta^k_V \text{ for } k \leq m \text{ and } \Delta^{m+1}_V | \Delta^{m+1}_{V\cap H}$$

for any generic hyperplane $H \subset \mathbb{P}^n$.

Proof. [16, Theorem 4.1.24] □

Corollary 2.1.6. *Let $V \subset \mathbb{P}^n$ be a hypersurface and let $m := \dim(V_{sing}) \geq 1$, then*

$$\Delta^1_V = \Delta^1_{V\cap H_1 \cap \ldots \cap H_m}$$

for generic hyperplanes $H_1, \ldots, H_m \subset \mathbb{P}^n$.

When $m = -1$ (i.e. when V is quasi-smooth) the only interesting Alexander polynomial is Δ^1_V, and the problem of its computation has been long solved (see [3, Proposition 3.5]); on the other hand, when $m \geq 0$ the picture changes: it appears that even the Betti numbers of F (i.e. the degrees of the Alexander polynomials) are known only in some special cases (see [24, 57, 64]). This suggests that for $m \geq 0$ one should start by studying Δ^1_V: by the previous corollary, in fact, we can reduce to the case of a hypersurface $V \cap H_1 \cap \ldots \cap H_m \subset \mathbb{P}^{n-m}$ having isolated singularities. We will do just that: from now on, we will denote Δ^1_V simply by Δ_V, and we will call it *the* Alexander polynomial of V.

We conclude this section with a result that allows to 'split' the reduced cohomology groups of the Milnor fibre of a certain class of polynomials, of which we will make use in Chapter 3:

Theorem 2.1.7. *Suppose $f(x_0, \ldots, x_n)$ has an isolated singularity at the origin and $g(y_0, \ldots, y_m)$ has an arbitrary singularity at the origin. Call F, G and $F \oplus G$ the Milnor fibres of f, g and $f+g$ respectively, and denote by T^i_f, T^i_g and T^i_{f+g} the various monodromy operators on the cohomology groups. There is an isomorphism*

$$\tilde{H}^{n+k+1}(F \oplus G, \mathbb{Q}) \simeq \tilde{H}^n(F, \mathbb{Q}) \otimes \tilde{H}^k(G, \mathbb{Q}) \qquad \text{for } k = 0, \ldots, m$$

respecting the monodromy operators: $T^{n+k+1}_{f+g} = T^n_f \otimes T^k_g$.

Proof. This is a consequence of [16, Lemma 3.3.20, Corollary 3.3.21]. □

2.1.2 Constants of quasi-adjunction and a formula for Δ_V

Assume $f(x_0, \ldots, x_n) = 0$ defines a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$, and let Y_f denote the associated hypersurface germ. If ω is a non-vanishing holomorphic n-form defined on $Y_f \setminus \{\underline{0}\}$ and $\pi : \tilde{Y}_f \to Y_f$ is a resolution of Y_f, then the n-form $\pi^*(\omega)$ is a priori holomorphic on $\tilde{Y}_f \setminus \pi^{-1}(\underline{0})$ only.

Definition 2.1.8. Choose for ω the form

$$\omega = \frac{dx_1 \wedge \cdots \wedge dx_n}{\frac{\partial f}{\partial x_0}}. \tag{2.1.7}$$

The *adjoint ideal* of f is the ideal of $\mathcal{O}_{\mathbb{C}^{n+1},\underline{0}}$ formed by elements $\phi \in \mathcal{O}_{\mathbb{C}^{n+1},\underline{0}}$ such that $\pi^*(\phi\omega)$ is holomorphic on the whole $\tilde{Y}_f$ (see [56]); we denote it by Adj_f.

This local definition of the adjoint ideal can be extended to a global one, namely that of adjoint ideal of a weighted projective hypersurface V. In order to keep notations simple, we write $\mathbb{P}$ for $\mathbb{P}(w_0, \ldots, w_n)$; moreover, we denote by i the inclusion map $V \hookrightarrow \mathbb{P}$ and by w the sum of the weights w_i. Assume $\pi : \tilde{V} \to V$ is a resolution of singularities of V with exceptional locus E, and set $\mathcal{F} := \pi_*(\Omega_{\tilde{V}})(w-d)$ where $\Omega_{\tilde{V}}$ is the sheaf of holomorphic $(n-1)$-forms on $\tilde{V}$:

Definition 2.1.9. The *adjoint ideal* of V is the ideal sheaf $Adj_V := i_*\mathcal{F} \subset \mathcal{O}_{\mathbb{P}}$.

One can check that if $p \in V_{sing}$ and g_p is the associated germ of isolated hypersurface singularity (which we can assume to be at the origin of $\mathbb{C}^{n+1}$), then the stalk $Adj_{V,p}$ is exactly the adjoint ideal Adj_{g_p} of g_p.

Assume now that $f(x_0, \ldots, x_n) = 0$ defines a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$, and fix a germ of holomorphic function ϕ around the origin. For any $m \in \mathbb{N}$ set $f_m := x_{n+1}^m - f$; then $f_m = 0$ defines a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^{n+2}$. We define the function ψ_ϕ as:

$$\psi_\phi : \mathbb{N} \to \mathbb{N} \qquad \text{s.t. } m \mapsto min\{l \in \mathbb{N} | x_{n+1}^l \phi \in Adj_{f_m}\}. \tag{2.1.8}$$

We have:

Proposition 2.1.10. *For any f and ϕ as above there exists $k_{\phi,f} \in \mathbb{Q}_{\geq 0}$ such that $\psi_\phi(m) = \lfloor k_{\phi,f} \cdot m \rfloor$.*

Proof. [49, Proposition 1.7] □

Definition 2.1.11. If $k_{\phi,f} > 0$ then this number is called the *constant of quasi-adjunction* of the singularity f corresponding to the germ ϕ (or constant of quasi-adjunction of ϕ relative to the point $p \in V$ of local equation f).

In what follows we will often drop the subscript f when speaking of constants of quasi-adjunction. Moreover, we will use the expression 'constants of quasi-adjunction of a point p (a hypersurface V)' to refer to all possible constants of quasi-adjunction of function germs ϕ at the point p (at any point $p \in V_{sing}$).

Remark 2.1.12. Since we shall be interested in constants of quasi-adjunction of ordinary multiple points in the projective plane, whose local equation have the form $x^m - y^m = 0$, we recall two ways to compute the constants of quasi-adjunction of weighted homogeneous isolated singularities:

1. In [56] it was proved that if the singularity $g(x_0, \ldots, x_n, x_{n+1}) = 0$ is weighted homogeneous, then the monomial $x_0^{i_0} \cdot \ldots \cdot x_{n+1}^{i_{n+1}}$ belongs to the adjoint ideal of g if and only if the $(n+2)$-tuple $(i_0+1, \ldots, i_{n+1}+1)$ lies in the Newton polytope of g. This makes it easy to compute the constants of quasi-adjunction of a weighted homogeneous singularity $f(x_0, \ldots, x_n) = 0$. In particular, if $f = x_0^{q_0} + \cdots + x_n^{q_n}$ the Newton polytope of $g := x_{n+1}^m - f(x_0, \ldots, x_n)$ is

$$\left\{ (x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} | \sum_{i=0}^{n} \frac{mD}{q_i} x_i + Dx_{n+1} > Dm \right\} \qquad \text{where } D = \prod_{i=0}^{n} q_i.$$

In this case the minimal l such that $x_0^{i_0+1}\cdot\ldots\cdot x_n^{i_n+1}x_{n+1}^l$ belongs to the adjoint ideal of g equals $\lfloor m(1-\sum_{k=0}^n \frac{i_k+1}{q_k})\rfloor$. This means that the constant of quasi-adjunction of the monomial (function germ) $x^I := x_0^{i_0}\cdot\ldots\cdot x_n^{i_n}$ relative to a point of local equation $f = x_0^{q_0}+\cdots+x_n^{q_n}$ is

$$k_{x^I,f} = 1-\sum_{k=0}^n \frac{i_k+1}{q_k} \tag{2.1.9}$$

if this value is positive.

2. If $f(x_1,\ldots,x_{n+1})=0$ defines an isolated hypersurface singularity, the *Steenbrink spectrum* of f is the formal sum of rational numbers

$$sp(f) := \sum_{\alpha\in\mathbb{Q}} \alpha\nu(\alpha) \tag{2.1.10}$$

where $\nu(\alpha)$ is the dimension of the $e^{-2\pi i\alpha}$-eigenspace of the semisemplification of the monodromy operator acting on $Gr_F^{\lfloor n-\alpha\rfloor}H^n(F_f)$. If f is weighted homogeneous then the monodromy operator is diagonalisable, so one does not need to consider its semisemplification in order to define $\nu(\alpha)$; moreover, if f has degree d and weights w_i then

$$\nu(\alpha) = \dim M(f)_{(\alpha+1)d-w} \tag{2.1.11}$$

where $M(f)$ is the Milnor algebra of f and w is the sum of the w_i's. The spectrum is symmetric around $\frac{n-1}{2}$ and $\nu(\alpha)=0$ for $\alpha\notin(-1,n)$. The results in [51] show that when $n=2$ the constants of quasi-adjunction of f coincide with the elements of $sp(f)$ belonging to $(0,n)$.

The results illustrated above show that the constants of quasi-adjunction of the plane curve singularity $x^m-y^m=0$ are

$$\frac{1}{m},\frac{2}{m},\ldots,\frac{m-3}{m},\frac{m-2}{m}. \tag{2.1.12}$$

Definition 2.1.13. For any $k\in\mathbb{R}$, the *ideal of quasi-adjunction* $\mathcal{A}_k$ is the sheaf of ideals of $\mathcal{O}_\mathbb{P}$ s.t. for any $U\subset\mathbb{P}$:

$$H^0(U,\mathcal{A}_k) := \{\phi\in H^0(U,\mathcal{O}_\mathbb{P})|k_{\phi,f_p}<k \text{ for any } p\in U\cap V_{sing}\} \tag{2.1.13}$$

where f_p is a local equation of $V\cap U$ at the point p.

For any k the ideals of quasi-adjunction are supported on V_{sing}, hence if we define $\mathcal{Q}_k := \mathcal{O}_\mathbb{P}/\mathcal{A}_k$ and $Z_k := Supp(\mathcal{Q}_k)$ we obtain a zero-dimensional subscheme of $\mathbb{P}$; it follows that $Z_k = Spec(R_k)$ for some finite-dimensional $\mathbb{C}$-algebra R_k, so we can define the *length* of Z_k as $l(Z_k) := \dim_\mathbb{C}(R_k)$.

Now we switch back to the usual projective setting, and state the following theorem:

Theorem 2.1.14. *Let $V \subset \mathbb{P}^n$ be a projective hypersurface of degree d with isolated singularities. For any $m|d$ let X_m be the m-fold cover of $\mathbb{P}^n$ branched along V and let $\tilde{X}_m$ be a resolution of singularities of X_m. Denote by K the set of all constants of quasi-adjunction of V, and for any $\kappa \in K$ define $\delta_\kappa := \dim H^1(\mathbb{P}^n, \mathcal{A}_\kappa(d-n-1-d\kappa))$. Then*

1. *We have an equality*

$$h^{n-1,0}(\tilde{X}_m) = \sum_{\kappa \in K}^{\kappa m \in \mathbb{Z}} \delta_\kappa.$$

2. *The polynomial*

$$\Delta^{n-1,0}(t) := \prod_{\kappa \in K}^{\kappa d \in \mathbb{Z}} (t - e^{2\pi i \kappa})^{\delta_\kappa}$$

is the characteristic polynomial of the monodromy action on $H^{n-1,0}(H^{n-1}(F))$, hence $\Delta^{n-1,0}(t)\overline{\Delta^{n-1,0}(t)}|\Delta_V(t)$.

3. *If V is a curve (i.e. $n = 2$), we have the equality*

$$\Delta_V(t) = \Delta^{1,0}(t)\overline{\Delta^{1,0}(t)}(t-1)^{r-1}$$

where r is the number of irreducible components of V.

Proof. Points 1. and 2. are the content of [49, Theorem 4.1]. Here we prove 3., because in what follows we will be dealing with curves and because for the proof we will introduce a construction that will be used again in Chapter 3.

Call $C = V(f) \subset \mathbb{P}^2$ the curve, $F = V(f-1) \subset \mathbb{C}^3$ its Milnor fibre and $S = V(f - y^d) \subset \mathbb{P}^3$ the closure of F; we have a long exact sequence of MHS

$$\cdots \to H_c^\bullet(F) \to H^\bullet(S) \to H^\bullet(C) \to H_c^{\bullet+1}(F) \to \cdots$$

which gives

$$\cdots \to H^2(S) \xrightarrow{\sigma} H^2(C) \to H_c^3(F) \to H^3(S) \to 0.$$

The monodromy action on F is given by $(x_0, x_1, x_2) \mapsto \eta \cdot (x_0, x_1, x_2)$, where η is an element of the group μ_d of d-th roots of unity, and it can be extended to a μ_d-action on S by setting $\eta \cdot (y : x_0 : x_1 : x_2) := (\eta y : x_0 : x_1 : x_2)$; as $S/\mu_d \simeq \mathbb{P}^2$, we have $h^2(S)^{\mu_d} = 1$ and $h^3(S)^{\mu_d} = 0$.

Clearly $h_2(C) = r$; moreover, we notice immediately that the μ_d-action on C is trivial, hence $H^i(C)^{\mu_d} = H^i(C)$.

The fact that $C \subset S$ implies that σ is non-trivial, and the fact that C is fixed by the μ_d-action guarantees that it remains non-trivial when we consider its restriction to the μ_d-invariant parts $\sigma' : H^2(S)^{\mu_d} \to H^2(C)$. If we consider the invariant part under the μ_d-action of the previous long exact sequence, we obtain

$$0 \to Ker(\sigma') \to H^2(C) \to H^3_c(F)^{\mu_d} \to 0.$$

If $\tilde{C} \to C$ is a resolution of C, by the long exact sequence associated to this resolution we deduce that $H^2(C) \simeq H^2(\tilde{C})$; in particular, $H^2(C)$ is a pure HS of weight 2 that consists only of its $(1,1)$ part. The same is true for $Ker(\sigma')$, which is a Hodge substructure of $H^2(C)$. This implies that $H^3_c(F)_1 = H^3_c(F)^{\mu_d}$ is a pure HS of weight 2 and type $(1,1)$ too; moreover, its dimension is $r-1$.

If we look at the non-invariant part of the previous long exact sequence under the μ_d-action, we obtain

$$0 \to H^3_c(F)_{\neq 1} \to H^3(S) \to 0$$

from which we deduce that $H^3_c(F)_{\neq 1} \simeq H^3(S)$. Since the singular locus of S is zero-dimensional, by Corollary 1.3.12 we have $H^3(S) \simeq H^3(\tilde{S})$ for any resolution of singularities $\tilde{S}$ of S; this implies that $H^3_c(F)_{\neq 1}$ is a pure HS of weight 3 with parts $(1,2)$ and $(2,1)$.

What we obtained is the following:

$$\begin{aligned} H^3_c(F)_1 &= Gr^1_F Gr^W_2 H^3_c(F)_1. \\ H^3_c(F)_{\neq 1} &= Gr^1_F Gr^W_3 H^3_c(F)_{\neq 1} \oplus Gr^2_F Gr^W_3 H^3_c(F)_{\neq 1}. \end{aligned}$$

If we use the isomorphism (1.3.2), the above becomes

$$\begin{aligned} H^1(F)_1 &= Gr^1_F Gr^W_2 H^1(F)_1. \\ H^1(F)_{\neq 1} &= Gr^1_F Gr^W_1 H^1(F)_{\neq 1} \oplus Gr^0_F Gr^W_1 H^1(F)_{\neq 1}. \end{aligned}$$

This means that $H^1(F)_1$ is a pure HS of weight 2 and type $(1,1)$ and $H^1(F)_{\neq 1}$ is a pure HS of weight 1 and parts $(1,0)$ and $(0,1)$. We can thus write $H^1(F) = H^{1,1} \oplus H^{1,0} \oplus \overline{H^{1,0}}$ and

$$\Delta_C(t) = (t-1)^{r-1} P(t) \overline{P(t)}$$

where $P(t)$ is the characteristic polynomial of the monodromy action on $H^{1,0}$. But by point 2. we have $P(t) = \Delta^{1,0}(t)$, so we are done. □

Remark 2.1.15. We have the following equality:

$$\delta_\kappa = l(Z_\kappa) - h_{I_\kappa}(d - n - 1 - \kappa d) \tag{2.1.14}$$

where $I_\kappa \subset R$ denotes the homogeneous ideal corresponding to the zero-dimensional scheme $Z_\kappa \subset \mathbb{P}^n$. To prove it, we start from the short exact sequence of sheaves on $\mathbb{P}^n$

$$0 \to \mathcal{A}_\kappa \to \mathcal{O}_{\mathbb{P}^n} \to Z_\kappa \to 0.$$

Tensoring with a locally free sheaf is an exact functor, so if we tensor the above sequence by $\mathcal{O}_{\mathbb{P}^n}(d - n - 1 - \kappa d)$ we obtain

$$0 \to \mathcal{A}_\kappa(d-n-1-\kappa d) \to \mathcal{O}_{\mathbb{P}^n}(d-n-1-\kappa d) \to Z_\kappa(d-n-1-\kappa d) \to 0.$$

We have $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1-\kappa d)) = 0$; moreover, the sheaf $Z_\kappa(d-n-1-\kappa d)$ is supported on the singular locus of V i.e. on a finite set of points, so $H^0(\mathbb{P}^n, Z_\kappa(d-n-1-\kappa d)) \simeq H^0(\mathbb{P}^n, Z_\kappa)$ and from the long exact cohomology sequence of the short exact sequence above we deduce

$$H^1(\mathcal{A}_\kappa(d-n-1-\kappa d)) = Coker(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1-\kappa d)) \to H^0(Z_\kappa)).$$

Since $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1-\kappa d)) \simeq R_{d-n-1-\kappa d}$ and $H^0(Z_\kappa) \simeq R/I_\kappa$ we obtain that $H^1(\mathcal{A}_\kappa(d-n-1-\kappa d))$ is the cokernel of the projection

$$R_{d-n-1-\kappa d} \to R/I_\kappa.$$

Thus δ_κ is the difference between $\dim(S/I_\kappa) = h^0(Z_\kappa)$, which is by definition $l(Z_\kappa)$, and the Hilbert function of I_κ in degree $d-n-1-\kappa d$, i.e. the so-called defect of the linear system of hypersurfaces of degree $d-n-1-\kappa d$ whose local equations at the points of V_{sing} belong to $\mathcal{A}_\kappa$.

Since we shall be interested in the case of curves, we restate the last point of the previous theorem in a different way:

Theorem 2.1.16. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d and let $k_1, \dots, k_m$ be all the constants of quasi-adjunction of C, then*

$$\Delta_C(t) = (t-1)^{r-1} \prod^{dk_j \in \mathbb{Z}} [(t - e^{2\pi i k_j})(t - e^{-2\pi i k_j})]^{s(k_j)} \tag{2.1.15}$$

where

1. *r is the number of irreducible components of C.*
2. *$s(k_j) := \dim H^1(\mathbb{P}^2, \mathcal{A}_{k_j}(N_d(k_j)))$.*
3. *$N_d(k_j) := d - 3 - dk_j$*

As before, we have

$$s(k_j) = l(Z_{k_j}) - h_{I_{k_j}}(N_d(k_j)). \tag{2.1.16}$$

2.1.3 Δ_V and Mordell-Weil rank of abelian varieties

Formula (2.1.15) theoretically allows to compute the whole Alexander polynomial of a curve C; in order to use it, however, one needs to have information on the relative position of the relevant singular points of C (those admitting constants of quasi-adjunction), and this is often too much to ask unless one has an explicit equation for the curve. An approach that does not rely on this information is the following: one associates to C a threefold H fibred over $\mathbb{P}^2$ having C as discriminant, and relates $\Delta_C(t)$ to the Mordell-Weil rank of the Jacobian $J(H)$ of H considered as a curve over $\mathbb{C}(x,y)$; the drawback of this method, which was to be expected, is that it only provides partial information on $\Delta_C(t)$ unless C satisfies some strong hypotheses. In this subsection we illustrate some known results of this kind, which, apart from being interesting in their own regard, will allow us to introduce quasi-toric decompositions and show their influence on the Alexander polynomial of a curve.

Definition 2.1.17. Let $f \in \mathbb{C}[y_0,y_1,y_2]$ be a homogeneous polynomial of degree d. A *quasi-toric decomposition of type* (p,q,r) of f (of $C := V(f) \subset \mathbb{P}^2$) consists of co-prime homogeneous polynomials $f_1, f_2, f_3 \in \mathbb{C}[y_0,y_1,y_2]$ such that $f_1^p + f_2^q = f_3^r f$; if $f_3 = 1$, we speak of *toric decomposition of type* (p,q).

Assume $C = V(f)$ is an irreducible curve with only nodes and cusps as singularities, and associate to it the elliptic threefold (curve over $\mathbb{C}(x,y)$) H of equation $u^2 + v^3 = f(x,y,1)$; Cogolludo-Agustin and Libgober obtained the following result ([5, Theorems 1.1,1.2]):

Theorem 2.1.18. *Under the above hypotheses and notations we have:*

1. *The $\mathbb{Z}$-rank of the Mordell-Weil group of H is equal to the degree of $\Delta_C(t)$.*
2. *The set of quasi-toric decompositions of type $(2,3,6)$ of C has a group structure, and it is isomorphic to $\mathbb{Z}^{2q}$ where $\Delta_C(t) = (t^2 - t + 1)^q$.*

In particular, $\Delta_C(t)$ is non-trivial if and only if C admits a quasi-toric decomposition of type $(2,3,6)$.

Observe that quasi-toric decompositions of type $(p,q,\mathrm{lcm}(p,q))$ correspond to the $\mathbb{C}(x,y)$-rational points of the affine curve E of equation $u^p + v^q - f(x,y,1) = 0$, with toric decompositions of type (p,q) corresponding to points of E defined over $\mathbb{C}[x,y]$. Choose now weights w_0 and w_1 for u and v respectively in such a way that $h := u^p + v^q + f$ is a weighted homogeneous polynomial; call X the threefold of equation $h = 0$ in the weighted projective space $\mathbb{P} := \mathbb{P}(w_0, w_1, 1, 1, 1)$. We have the following chain of inclusions:

$$\left\{\begin{array}{l}\text{quasi-toric decompositions of} \\ C \text{ of type } (p,q,\mathrm{lcm}(p,q))\end{array}\right\} \subset J(E)(\mathbb{C}(x,y)) \subset H_4(X,\mathbb{Z})_{\mathrm{prim}}. \tag{2.1.17}$$

Call $g := u^p + v^q$, then $h = g - f$. We denote by F_h, F_g, F_f the Milnor fibres associated to h, g and f, and by T_h, T_g, T_f the various algebraic monodromy operators. If we write down the long exact sequence of cohomology groups with compact support associated to the pair $(\mathbb{P}, X)$, we obtain $H_4(X)_{\mathrm{prim}} \simeq H^5_c(\mathbb{P} \setminus X)^\vee$; by Poincaré duality we can then write $H_4(X)_{\mathrm{prim}} \simeq H^3(\mathbb{P} \setminus X)$, and since F_h is a covering space for $\mathbb{P} \setminus X$ we deduce that $H_4(X)_{\mathrm{prim}} \simeq H^3(F_h)^{T_h}$. If we combine this with Theorem 2.1.7 we can write

$$H_4(X)_{\mathrm{prim}} \supset H^1(F_g)^{T_g} \otimes H^1(F_f)^{T_f}. \tag{2.1.18}$$

Assume now that there exists a quasi-toric decomposition of f; it corresponds to a point $p \in E(\mathbb{C}(x,y))$, and by the above chain of inclusions we can associate to it an element $\zeta \in H_4(X)_{\mathrm{prim}}$ i.e. an element of $H^3(F_h)$ which is left invariant by T_h; by Theorem 2.1.7, we can write $\zeta = \sum_{i=1}^{r} \zeta_{g_i} \otimes \zeta_{f_i}$ with, for all $i = 1, \ldots, r$, $\zeta_{g_i} \in H^1(F_g)_{\alpha_i}$ and $\zeta_{f_i} \in H^1(F_f)_{\beta_i}$ where $\alpha_i + \beta_i = 1$. Now, $T_g \otimes Id$ is induced by the automorphism σ of E that multiplies u and v by the appropriate roots of unity; since $\sigma(p) \neq p$ we deduce that $(T_g \otimes Id)(\zeta) \neq \zeta$, which implies that ζ does not belong to $H^1(F_g)^{T_g} \otimes H^1(F_f)^{T_f}$. As a consequence, there exists $\zeta_f \in H^1(F_f)$ which is not invariant under T_f. We have thus proved the following:

Lemma 2.1.19. *If $C = V(f)$ admits a quasi-toric decomposition then $\Delta_C(t)$ is non-trivial.*

The approach of [5] was generalised in two ways: in [47] (see Theorem 1.2 (2)) isotrivial abelian varieties A over $\mathbb{C}(x,y)$ were considered, while in [39] the elliptic threefold $u^2 + v^3 = f(x,y,1)$ was replaced by $g(u,v) - f(s,t,1)$ with g weighted homogeneous. In particular ([39, Corollary 1.2]):

Theorem 2.1.20. *Assume $C = V(f)$ is a curve of even degree with only ADE singularities, and let e be a divisor of d. The Mordell-Weil rank of the group of $\mathbb{C}(x,y)$-valued points on the Jacobian of the general fibre of $H : u^2 = v^e + f(x,y,1)$ equals*

$$2 \sum_{j=1}^{\lfloor \frac{e-2}{2} \rfloor} ord_{t=\lambda_j} \Delta_C(t)$$

where $\lambda_j := e^{\frac{2\pi i j}{e}}$.

When $e = 3$ one obtains the same result of [5].

2.2 The case for hyperplane arrangements

2.2.1 Motivation

The computation of the Alexander polynomial of singular projective hypersurfaces $V := V(f) \subset \mathbb{P}^n$ is in general a difficult task. The class of hypersurfaces which has drawn the most interest consists of those whose defining polynomial f factors into a product of linear forms: in this situation the associated affine hypersurface (the cone over V) is a finite collection of codimension one vector subspaces of $\mathbb{C}^{n+1}$ i.e. a so-called *hyperplane arrangement*; we will denote these objects by $\mathcal{A} = \{H_1, \ldots, H_n\}$. In this situation $V = \mathbb{P}(\mathcal{A})$ is a finite union of codimension one linear subspaces of $\mathbb{P}^n$, so it is a hyperplane arrangement too; thus we will usually denote V by $\overline{\mathcal{A}}$ and write $\overline{\mathcal{A}} = \{\overline{H_1}, \ldots, \overline{H_n}\}$.

What makes this situation easier is that we may try to take advantage of the combinatorial nature of $\overline{\mathcal{A}}$, that is encoded in its intersection semilattice $L(\mathcal{A})$ (defined at the beginning of the next subsection); however, even in this setting the Alexander polynomial is not known in general.

Remark 2.2.1. Observe that if $\mathcal{A} \subset \mathbb{C}^{n+1}$ is a hyperplane arrangement then $\overline{\mathcal{A}} \subset \mathbb{P}^n$ is a projective hypersurface with singular locus of dimension $n-2$; by Corollary 2.1.6, by taking successive hyperplane sections we can reduce the computation of the Alexander polynomial to the case $n = 2$. In this case the hypersurface $\overline{\mathcal{A}} \subset \mathbb{P}^2$ is actually a line arrangement, so we will write $\overline{\mathcal{A}} = \{l_1, \ldots, l_n\}$ where l_i is the line corresponding to the (hyper)plane H_i; moreover, the Alexander polynomial is (by (2.1.6)):

$$\Delta_{\overline{\mathcal{A}}}(t) = det(I \cdot t - T^1 : H^1(F, \mathbb{C}) \to H^1(F, \mathbb{C})).$$

From now on, unless stated otherwise, we will consider only hyperplane arrangements in $\mathcal{A} \subset \mathbb{C}^3$ and the associated line arrangements $\overline{\mathcal{A}} \subset \mathbb{P}^2$.

In [35, Problem 9A] and [38, Problem 4.145], the following problem was raised:

Problem 1. *Given a line arrangement $\overline{\mathcal{A}} \subset \mathbb{P}^2$, is its Alexander polynomial $\Delta_{\overline{\mathcal{A}}}(t)$ determined by $L(\mathcal{A})$? If so, give an explicit combinatorial formula to compute it.*

In this section we gather some known results around this problem, that are interesting in their own regard and also motivate the research we have been doing.

The singular locus of $\overline{\mathcal{A}}$ consists of ordinary multiple points, which are weighted homogeneous singularities i.e. singularities for which we know the constants of quasi-adjunction: as we showed in Remark 2.1.12, if the point has order m they are $\frac{m-1-j}{m}$ for $j = 1, \ldots, m-2$. This means that if we know the position of these multiple points we can compute $\Delta_{\overline{\mathcal{A}}}(t)$ using the formula provided by Theorem 2.1.16.

Example 2.2.2. Consider a line arrangement $\overline{\mathcal{A}}$ consisting of k lines passing through the same point (which we can assume to be $(0:0:1)$); the constants of quasi-adjunction of $\overline{\mathcal{A}}$ are $c_j := \frac{k-1-j}{k}$ for $j = 1, \ldots, k-2$ and the corresponding values $N_k(c_j)$ are $j-2$. For any j the ideal associated to the scheme Z_{c_j} is $I_{c_j} = (x, y)^j$, and so we have $h_{I_{c_j}}(j-2) = \frac{j(j-1)}{2}$; on the other hand, the length of Z_{c_j} is the dimension of the vector space of polynomial function germs around $(0:0:1)$ whose constants of quasi-adjunction are bigger than or equal to c_j, so $l(Z_{c_j}) = \frac{j(j+1)}{2}$. This means that $\delta(c_j) = j$ for all $j = 1, \ldots, k-2$, so the Alexander polynomial of such an arrangement is non-trivial and is given by

$$\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{k-1} \prod_{j=1}^{k-2} [(t - e^{2\pi i c_j})(t - e^{-2\pi i c_j})]^j.$$

If we denote by η_k a primitive k-th root of unity, we can rewrite this expression as

$$\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{k-1} \prod_{j=1}^{k-2} [(t - \eta_k^{k-1-j})(t - \eta_k^{j+1})]^j.$$

Fix a $j_1 \in \{1, \ldots, k-2\}$; in order to have $k-1-j_1 = j_2+1$ we need $j_2 = k-2-j_1$, and this is always possible given the set in which j_1 varies. This means that in the above expression to any term $(t-\eta_k^h)^j$ for $j=1,\dots,k-2$ corresponds a term $(t-\eta_k^h)^{k-2-j}$; we can thus rewrite the Alexander polynomial as

$$\Delta_{\overline{\mathcal{A}}}(t)=(t-1)^{k-1}\prod_{h=1}^{k-1}(t-\eta_k^h)^{k-2}=(t-1)^{k-1}\prod_{1<d|k}\Phi_d(t)^{k-2}.$$

In view of this example, from now on we will only consider *essential* (or *non-central*) hyperplane arrangements in $\mathbb{C}^3$, i.e. arrangements $\mathcal{A}$ such that $\overline{\mathcal{A}}$ does not consist of lines that intersect in a single point.

2.2.2 Combinatorics of $\mathcal{A}$

The aim of this subsection is to introduce some combinatorial objects naturally associated to $\mathcal{A}$ (and $\overline{\mathcal{A}}$), and to illustrate their interplay. We start with the *intersection semilattice* $L(\mathcal{A})$: this is the partially ordered set of all intersection of hyperplanes in $\mathcal{A}$, usually called *flats*, ordered by reversed inclusion and ranked by codimension. Given two flats $X,Y\in L(\mathcal{A})$, their *join* is $X\vee Y:=X\cap Y$ while their *meet* is $X\wedge Y:=\cap\{Z\in L(\mathcal{A})|X\cup Y\subseteq Z\}$. If $X\in L(\mathcal{A})$ is a flat, we denote by $\mathcal{A}_X$ the subarrangement $\{H\in\mathcal{A}|X\subset H\}$ and define the *multiplicity* of X as $|\mathcal{A}_X|$. If $L_k(\mathcal{A})$ denotes the set of flats of $L(\mathcal{A})$ of rank k, we can see that there is a 1-to-1 correspondence between $L_1(\mathcal{A})$ and the lines of $\overline{\mathcal{A}}$, and between $L_2(\mathcal{A})$ and the multiple points of $\overline{\mathcal{A}}$.

Another useful object associated to $\mathcal{A}$ is its *Orlik-Solomon algebra*. This can be defined over any noetherian ring $\mathbb{K}$, but we will take for $\mathbb{K}$ a field. It is defined as follows: if $\mathcal{A}=\{H_1,\dots,H_n\}$ with $H_i=V(f_i)$ for $f_i\in\mathbb{K}[x_0,x_1,x_2]_1$, we let E be the exterior algebra over $\mathbb{K}$ generated by $e_1,\dots,e_n$ and define a degree -1 map $\partial:E_p\to E_{p-1}$ by

$$\partial(e_{i_1}\wedge\ldots\wedge e_{i_p}):=\sum_{j=1}^{p}(-1)^{j-1}e_{i_1}\wedge\ldots\wedge\widehat{e_{i_j}}.\wedge\ldots\wedge e_{i_p}$$

The Orlik-Solomon algebra $A(\mathcal{A},\mathbb{K})$ of $\mathcal{A}$ is the quotient of E by the ideal I generated by $\{\partial(e_{i_1}\wedge\ldots\wedge e_{i_p})|\{f_{i_1},\dots,f_{i_p}\}$ is a linearly dependent set$\}$; when no risk of confusion arises, we will denote it simply by A. The grading on E induces a grading $A=\oplus_pA_p$. If we denote by w_i the images of the e_i in A, we see that they form a basis for A_1, so we find an isomorphism $A_1\simeq\mathbb{K}^{|\mathcal{A}|}$.

It is clear by its definition that $A(\mathcal{A},\mathbb{K})$ only depends on $L(\mathcal{A})$ and on the choice of $\mathbb{K}$; however, $A(\mathcal{A},\mathbb{K})$ also carries geometric information: indeed, it was proved in [58] that $A(\mathcal{A},\mathbb{C})$ is isomorphic, as graded algebra, to the de Rham cohomology ring $H^*(M(\mathcal{A}),\mathbb{C})$, where $M(\mathcal{A}):=\mathbb{C}^3\setminus\mathcal{A}$. Under this isomorphism, the generator w_i of $A(\mathcal{A},\mathbb{C})$ is identified with the logarithmic one-form $d(log(\alpha_i))$, where α_i is any linear form defining H_i (for example f_i).

Remark 2.2.3. Pick a defining polynomial $f\in\mathbb{C}[x_0,x_1,x_2]_d$ for $\overline{\mathcal{A}}$: up to a change of coordinates, we can assume that $f=x_0g$ for some $g\in\mathbb{C}[x_0,x_1,x_2]_{d-1}$. If we denote by p the usual Hopf bundle map $\mathbb{C}^3\setminus\{\underline{0}\}\to\mathbb{P}^2$, we can see that its restriction

$p : \mathbb{C}^3 \setminus \{x_0 = 0\} \to \mathbb{P}^2 \setminus \mathbb{P}^1 \simeq \mathbb{C}^2$ defines a trivial $\mathbb{C}^*$-bundle; this implies that $p : M(\mathcal{A}) \to U(\overline{\mathcal{A}})$ is a trivial $\mathbb{C}^*$-bundle too, with $U(\overline{\mathcal{A}}) := \mathbb{P}^2 \setminus \overline{\mathcal{A}}$. Since, as we have already seen, the Milnor fibre $F \subset \mathbb{C}^3$ of $\overline{\mathcal{A}}$ is a d-fold cover of $U(\overline{\mathcal{A}})$ whose group of deck transformations is generated by the monodromy operator h, we can write

$$M(\mathcal{A}) \simeq U(\overline{\mathcal{A}}) \times \mathbb{C}^* \simeq (F/\langle h \rangle) \times \mathbb{C}^*.$$

This shows that $M(\mathcal{A})$ carries information on the Milnor fibre of $\overline{\mathcal{A}}$, and motivates the choice of the notation $M(\mathcal{A})$ for $\mathbb{C}^3 \setminus \mathcal{A}$.

If $a \in A_1$ then $a^2 = 0$ by the graded commutativity of A, so we can associate to a a cochain complex

$$(A, \delta_a) : \quad A_0 \xrightarrow{\delta_a} A_1 \xrightarrow{\delta_a} A_2 \xrightarrow{\delta_a} A_3 \to 0 \tag{2.2.1}$$

where $\delta_a(b) := a \wedge b$; this is called the *Aomoto-Betti complex of A relative to a.* These complexes allow us to define two other important notions: resonance varieties and Aomoto-Betti numbers. The (degree q, depth r) *resonance varieties* of A are the jumping loci for the cohomology of the Aomoto-Betti complexes, namely

$$\mathcal{R}^q_r(A, \mathbb{K}) := \{a \in A_1 |\, \dim_{\mathbb{K}} H^q((A, \mathbb{K}), \delta_a) \geq r\}. \tag{2.2.2}$$

We will be mainly interested in the degree 1 resonance variety $\mathcal{R}_r(A, \mathbb{K}) := \mathcal{R}^1_r(A, \mathbb{K})$ which, by definition, consists of $\{\underline{0}\}$ together with all $a \in A_1$ for which there exist $b_1, \ldots, b_r \in A_1$ such that $\dim_{\mathbb{K}} Span\{a, b_1, \ldots, b_r\} = r + 1$ and $ab_i = 0$. The variety $\mathcal{R}_1(A)$ is particularly well-understood, and its main properties are summarised in the following theorem (which is a collection of results in [50] and [7]):

Theorem 2.2.4. *Over a field $\mathbb{K}$ of characteristic 0, all irreducible components of $\mathcal{R}_1(\mathcal{A}, \mathbb{K})$ are Zariski-closed linear subspaces of $\mathbb{K}^n$ intersecting pairwise only at $\{\underline{0}\}$; moreover, the positive-dimensional irreducible components have dimension at least two, and the cup product map $\wedge : A_1 \times A_1 \to A_2$ vanishes identically on each such component.*

If $char(\mathbb{K}) \neq 0$ then $\mathcal{R}_1(A, \mathbb{K})$ may have irreducible components that are non-linear, or that intersect non-trivially (see [26] for some examples). An irreducible component of $\mathcal{R}^1(A, \mathbb{K})$ is called *global* if it is not contained in any coordinate hyperplane $w_i = 0$ of $A_1 \simeq \mathbb{K}^{|\mathcal{A}|}$; in this case, we say that $\mathcal{A}$ (or $\overline{\mathcal{A}}$) supports a global resonance component.

Let now σ be the element of A_1 given by $\sigma := \sum_{i=1,\ldots,|\mathcal{A}|} w_i$; the *Aomoto-Betti number* over $\mathbb{K}$ of $\overline{\mathcal{A}}$ is defined as

$$\beta_{\mathbb{K}}(\overline{\mathcal{A}}) := \dim_{\mathbb{K}} H^1((A, \mathbb{K}), \delta_\sigma) = max\{r \in \mathbb{N} | \sigma \in \mathcal{R}_r(A, \mathbb{K})\}. \tag{2.2.3}$$

It is clear from its definition that $\beta_{\mathbb{K}}(\overline{\mathcal{A}})$ depends only on $p := char(\mathbb{K})$, hence we denote it simply by $\beta_p(\overline{\mathcal{A}})$.

Recall that $H^1(F, \mathbb{Q})$ decomposes as a $\mathbb{Q}[\mathbb{Z}/d\mathbb{Z}]$-module in the following way (see (2.1.5)):

$$H^1(F,\mathbb{Q}) \simeq \mathbb{Q}^{|\overline{\mathcal{A}}|-1} \oplus \bigoplus_{1<k|d} (\mathbb{Q}[t]/\Phi_k(t))^{e_k(\overline{\mathcal{A}})}.$$

Aomoto-Betti numbers provide a bound on the $e_k(\overline{\mathcal{A}})$ when k is the power of a prime: we have in fact (see [60, Theorem 11.3]):

$$e_{p^s}(\overline{\mathcal{A}}) \leq \beta_p(\overline{\mathcal{A}}) \text{ for all } s \geq 1 \tag{2.2.4}$$

As we shall see, $\overline{\mathcal{A}}$ supports a global resonance component only if it is, in some sense, 'highly symmetric'; the precise notion that allows to detect the symmetry of $\overline{\mathcal{A}}$ is that of *multinet*:

Definition 2.2.5. Assume we have a partition $\mathcal{N}$ of $\mathcal{A}$ into $k \geq 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, a 'multiplicity function' $m : \mathcal{A} \to \mathbb{N}$ and a subset $\mathcal{X} \subset L_2(\mathcal{A})$; consider moreover the following conditions:

(i) There exists $d \in \mathbb{N}$ such that $\sum_{H\in\mathcal{A}_i} m(H) = d$ for all $i = 1, \dots, k$.

(ii) For any $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ with $i \neq j$ we have $H \cap H' \in \mathcal{X}$.

(iii) For all $l \in \mathcal{X}$ the integer $n_l := \sum_{H\in\mathcal{A}_i, l\subset H} m(H)$ does not depend on i.

(iv) For all $i = 1, \dots, k$ and any $H, H' \in \mathcal{A}_i$, there is a sequence $H = H_0, \dots, H' = H_r$ such that $H_{j-1} \cap H_j \notin \mathcal{X}$.

The couple $(\mathcal{N}, \mathcal{X})$ (sometimes just referred to as $\mathcal{N}$) is called:

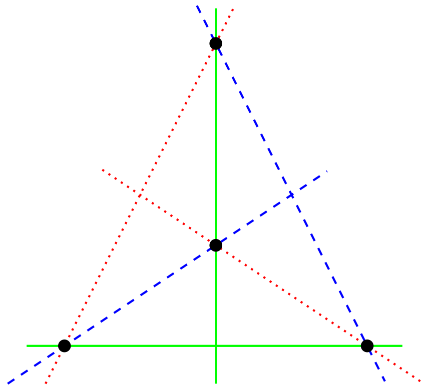
- a *weak* (k,d)*-multinet* if it satisfies (i)-(iii).
- a (k,d)*-multinet* if it satisfies (i)-(iv).
- a *reduced* (k,d)*-multinet* if it satisfies (i)-(iv) and $m(H) = 1$ for all $H \in \mathcal{A}$.
- a (k,d)*-net* if it satisfies (i)-(iv) and $n_l = 1$ for all $l \in \mathcal{X}$; if $d = 1$, the $(k,1)$-net is called a *trivial* k-net.

We call $\mathcal{A}_1, \dots, \mathcal{A}_k$ the *classes* of $\mathcal{N}$, $\mathcal{X}$ its *base locus* and d its *weight.* If $(\mathcal{N}, \mathcal{X})$ is a weak (k,d)-multinet on $\mathcal{A}$ and $l \in L_2(\mathcal{A})$, we define the *support of* l *with respect to* $\mathcal{N}$ as

$$supp_{\mathcal{N}}(l) := \{\alpha \in \{1, \dots, k\} | l \subset H \exists H \in \mathcal{A}_\alpha\}.$$

Remark 2.2.6. One can of course define in the same way weak multinets on $\overline{\mathcal{A}}$, but we chose to give the definition for $\mathcal{A}$ because it appeared first in the context of affine hyperplane arrangements; however, in what follows we will often abuse notation and speak of multinets on $\overline{\mathcal{A}}$, thanks to the correspondence between $L_1(\mathcal{A})$ and the lines of $\overline{\mathcal{A}}$ and between $L_2(\mathcal{A})$ and the multiple points of $\overline{\mathcal{A}}$.

Example 2.2.7. Consider the following construction: four points of $\mathbb{P}^2$ in general position together with all lines passing through two of them. The arrangement we obtain is the so-called A_3 arrangement: it has six lines and four triple points. It admits a $(3,2)$-net, as the figure below shows, whose base locus consists of the triple points.



Remark 2.2.8. The following are easy consequences of the above definitions:

(i) If $(\mathcal{N},\mathcal{X})$ is a weak (k,d)-multinet on $\mathcal{A}$ with multiplicity function m and $c \in \mathbb{N}$ is a natural number, we can obtain a weak (k,cd)-multinet $(\mathcal{N},\mathcal{X})$ on $\mathcal{A}$ by choosing $c \cdot m$ as multiplicity function; this means that we can choose d to be minimal, i.e. we can assume, without loss of generality, that $\gcd\{m(H)|H \in \mathcal{A}\} = 1$.

(ii) If $(\mathcal{N},\mathcal{X})$ is a weak multinet on $\mathcal{A}$ then $\mathcal{X}$ is determined by $\mathcal{N}$, as $\mathcal{X} = \{H \cap H' | H \in \mathcal{A}_i, H' \in \mathcal{A}_j, i \neq j\}$; if $(\mathcal{N},\mathcal{X})$ is a multinet on $\mathcal{A}$, $\mathcal{N}$ is determined by $\mathcal{X}$ too: if Γ is the graph with vertex set $\mathcal{A}$ and an edge connecting $H, H' \in \mathcal{A}$ if $H \cap H' \notin \mathcal{X}$, then the classes $\mathcal{A}_i$ are the connected components of Γ.

(iii) Nets are automatically reduced multinets. Indeed, let $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$: by condition (ii) of Definition 2.2.5 we have $l := H \cap H' \in \mathcal{X}$, and since $n_l = 1$ this implies $m(H) = m(H') = 1$. This means we have the following chain of inclusions

$$\{\text{weak multinets}\} \supset \{\text{multinets}\} \supset \{\text{reduced multinets}\} \supset \\ \supset \{\text{nets}\} \supset \{\text{trivial nets}\}$$

which are all strict.

(iv) $\mathcal{A}$ admits a trivial k-net $\mathcal{N} \iff \overline{\mathcal{A}}$ consists of k lines meeting in a point. Implication $\Leftarrow$ is obvious: just partition $\mathcal{A}$ into k classes containing one hyperplane each. For $\Rightarrow$, observe that $d = 1$ implies that the partition $\mathcal{N}$ consists of classes containing exactly one hyperplane each; if $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ then $l := H \cap H' \in \mathcal{X}$, and if some $H'' \in \mathcal{A}$ did not pass through l condition (iii) of Definition 2.2.5 would be violated.

(v) Let $(\mathcal{N}, \mathcal{X})$ be a weak (k,d)-multinet on $\mathcal{A}$ and let $l \in L_2(\mathcal{A})$, then clearly either $|supp_{\mathcal{N}}(l)| = 1$ (*mono-coloured* flat), or $l \in \mathcal{X}$ and so $|supp_{\mathcal{N}}(l)| = k$ (*multi-coloured* flat). If $(\mathcal{N}, \mathcal{X})$ is a reduced (k,d)-multinet, condition (iii) of Definition 2.2.5 implies that any $l \in \mathcal{X}$ has multiplicity kr_l for some r_l (i.e. any flat l in the base locus belongs to r_l hyperplanes from each class); if $(\mathcal{N}, \mathcal{X})$ is a (k,d)-net, the same condition implies that any $l \in \mathcal{X}$ has multiplicity k (i.e. any flat in the base locus belongs to exactly one hyperplane from each class).

The following is instead a much less trivial result, which summarises the results of [70],[69] and [61]:

Theorem 2.2.9. *Let $(\mathcal{N}, \mathcal{X})$ be a (k,d)-multinet on $\overline{\mathcal{A}}$. If $|\mathcal{X}| > 1$ then $k \in \{3,4\}$. In particular, the only hyperplane arrangements admitting a (k,d)-multinet with $k \geq 5$ are central hyperplane arrangements, and the multinet is actually a trivial k-net. If $|\mathcal{X}| > 1$ and $(\mathcal{N}, \mathcal{X})$ is non-reduced, then $k = 3$.*

The resonance variety $\mathcal{R}^1(A)$ is related to multinets on $\overline{\mathcal{A}}$ by the following result:

Theorem 2.2.10. *The arrangement $\overline{\mathcal{A}}$ admits a (k,d)-multinet $(\mathcal{N}, \mathcal{X}) \iff \overline{\mathcal{A}}$ supports a global resonance component of dimension $k-1$.*

A proof of this theorem, obtained by building on the results of [50], can be found in [27, Theorem 2.4, Theorem 2.5]; for a different one, the reader can look at [53, Theorem 3].

Remark 2.2.11. The methods used in the proofs of [27, Theorem 2.4, Theorem 2.5] show in particular that any weak multinet can be refined to a multinet with the same base locus.

We can give an alternative description of $\mathcal{R}^1(A)$. Let S denote $\mathbb{P}^1$ with at least 3 points removed: a map $f : M(\mathcal{A}) \to S$ is called *admissible* if it is regular, surjective and its generic fibre is connected. Arapura [2] showed that:

Theorem 2.2.12. *The correspondence $f \mapsto f^*(H^1(S, \mathbb{C}))$ gives a bijection between admissible maps (up to reparametrisation of the target S) and positive-dimensional components of $\mathcal{R}^1(A)$.*

2.2.3 Geometry of $\mathcal{A}$

Theorem 2.2.10 relates something that is purely combinatorial (multinets on $\overline{\mathcal{A}}$) with some irreducible components of an object that carries geometric information (the degree 1 depth 1 resonance variety $\mathcal{R}^1(A)$). The connection between combinatoric and geometry that the theory of hyperplane arrangements exhibits becomes even tighter when pencil of plane curves enter the picture; for the following discussion, we refer to [27, Section 3]. Pencils of plane curves are one-dimensional linear systems of plane curves, which we can think of as lines in $\mathbb{P}(\mathbb{C}[x,y,z]_d)$ for some fixed degree d: thus any two distinct plane curves (which we identify with any of their defining polynomials in $\mathbb{C}[x,y,z]_d$) define a pencil, and any pencil is uniquely determined by any two of its curves. This means that any pencil can be written as

$$aC_1 + bC_2 \qquad \text{with } (a:b) \in \mathbb{P}^1$$

an expression from which we deduce that any two of its curves meet exactly in $\mathcal{X} := C_1 \cap C_2$, called *base* of the pencil. We will always assume that $\mathcal{X}$ consists of a finite set of points, i.e. that the pencil has no fixed components.

Note that C_1 and C_2 determine a rational map

$$\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \text{ s.t. } p \mapsto (C_2(p) : -C_1(p))$$

whose indeterminacy locus is $\mathcal{X}$. The closure of the fibre of π over $(a:b)$ is the curve $aC_1 + bC_2$, and each $p \notin \mathcal{X}$ lies in exactly one of such curves.

A curve is called *completely reducible* if its defining polynomial has the form $\Pi_{i=1}^{q} \alpha_i^{m_i}$ where α_i are linear forms and $m_i \geq 1$ for all $i = 1, \dots, q$; the pencil we will be mostly interested in are those for which π has some completely reducible fibres.

Consider a pencil generated by two completely reducible curves. Let $\phi : B \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $\mathcal{X}$, then π lifts to a regular map $\pi' : B \to \mathbb{P}^1$ ([27, Lemma 3.2(i)]) whose fibres are the strict transforms of the fibres of π under the blow-up; we say the pencil is *connected* if every fibre of π' is connected.

Definition 2.2.13. A *pencil of Ceva type* (or a *Ceva pencil*) is a connected pencil of plane curves with no fixed components in which three or more fibres are completely reducible.

Now we can state the following theorem:

Theorem 2.2.14. *There is a 1-to-1 correspondence between multinets $(\mathcal{N}, \mathcal{X})$ on hyperplane arrangements $\mathcal{A} \subset \mathbb{C}^3$ and pencils of Ceva type; namely:*

(i) A pencil of Ceva type induces a multinet $\mathcal{N}$ on the hyperplane arrangement $\mathcal{A}$ consisting of the irreducible components of its completely reducible fibres $F_i = \prod_{j=1}^{i_j} \alpha_{i,j}^{m_{i,j}}$: its classes are $\mathcal{A}_i := \{H_{i,j} := V(\alpha_{i,j})\}_{j=1,\dots,i_j}$ and $m(H_{i,j}) := m_{i,j}$ for all i, j.

(ii) If $(\mathcal{N}, \mathcal{X})$ is a (k,d)-multinet on $\mathcal{A}$ let $C_i = \prod_{H \in \mathcal{A}_i} \alpha_H^{m(H)}$, where α_H is any linear form defining H; the pencil of degree d curves generated by any two of $C_1, \dots, C_k$ contains all C_i and is connected i.e. is a Ceva pencil

Proof. Points (i) and (ii) are Theorems 3.7 and 3.11 of [27] respectively. □

If $\mathcal{N}$ is a k-multinet on $\mathcal{A}$, the Ceva pencil associated to it naturally determines an admissible map $f_{\mathcal{N}} : M(\mathcal{A}) \to S$ (see [59, Section 6.2]); one can check that the component of $\mathcal{R}^1(A)$ given by $f_{\mathcal{N}}^*(H^1(S,\mathbb{C}))$ is a global component of dimension $k-1$. Conversely, by [27, Theorem 2.5] all global components of $\mathcal{R}^1(A)$ arise in this way. We can summarise these results in the following way:

Corollary 2.2.15. *The following are equivalent:*

(i) *$\mathcal{A}$ admits a (k,d)-multinet $\mathcal{N}$ for some d.*

(ii) *$\mathcal{A}$ is the set of components of $k \geq 3$ completely reducible fibres in a Ceva pencil of degree d curves, for some d.*

(iii) *There exists and admissible map $f_N : M(\mathcal{A}) \to S$.*

(iv) *$f_{\mathcal{N}}^*(H^1(S,\mathbb{C}))$ is a global resonance component of $\mathcal{R}^1(A)$ of dimension $k-1$.*

Note that the equivalence (i)$\Leftrightarrow$(ii) still holds if one restricts to reduced multinets and pencils of Ceva type with reduced completely reducible fibres, respectively.

At a first glance, it does not seem like these results have anything to do with the problem of determining the Alexander polynomial of a hyperplane arrangement $\mathcal{A}$. This is only partially true, as a result by Libgober shows: indeed, in [46, Theorem 1.2] he calls an arrangement $\overline{\mathcal{A}} \subset \mathbb{P}^2$ *composed of a reduced pencil* if there exists a pencil of plane curves $aC_1 + bC_2$ such that three of its fibres F_1, F_2, F_3 are completely reducible, reduced, and $V(F_1F_2F_3) = \overline{\mathcal{A}}$; he then proves the following:

Theorem 2.2.16. *If $\overline{\mathcal{A}}$ has only double and triple points then its Alexander polynomial has a non-trivial factor Φ_3^a if and only if $\overline{\mathcal{A}}$ is composed of a reduced pencil.*

Note that reducible pencils in the sense of Libgober are in particular reduced pencils of Ceva type, so by the above result for an arrangement $\overline{\mathcal{A}}$ with only double and triple points being composed of a reduced pencil and admitting a reduced $(3,d)$-multinet is equivalent.

The result of Libgober is a particular instance of a more general phenomenon, namely:

Lemma 2.2.17. *If $\overline{\mathcal{A}}$ is given by the components of the $k \in \{3,4\}$ completely reducible and reduced fibres in a Ceva pencil, then the Alexander polynomial of $\overline{\mathcal{A}}$ is non-trivial.*

Proof. Assume $k = 3$, then $\overline{\mathcal{A}} = V(g) = V(g_1g_2g_3)$ with $g_i \in \mathbb{C}[y_0, y_1, y_2]$ completely reducible and reduced fibres of a Ceva pencil $ah_1 + bh_2$ with $(a : b) \in \mathbb{P}^1$; up to reparametrisation of the pencil, we can assume that the g_i correspond to the values $(1:0)$, $(0:1)$ and $(1:1)$.

The equation

$$x^3 - y^3 = zw(z-w)$$

admits the solution

$$\begin{aligned} x &= (1-\eta^2)(z-\eta w) \\ y &= (1-\eta)(z-\eta^2 w) \end{aligned}$$

where η is a primitive root of unity of order three. If we substitute z with g_1 and w with g_2 we obtain $z - w = g_3$, and the corresponding polynomials $x, y \in \mathbb{C}[y_0, y_1, y_2]$ we find give a toric decomposition of type $(3,3)$ of g; by Lemma 2.1.19 $\Delta_{\overline{\mathcal{A}}}$ is non-trivial.

Assume now $k = 4$ and $\overline{\mathcal{A}} = V(g_1 g_2 g_3 g_4)$ with g_i completely reducible and reduced fibres of a Ceva pencil $ah_1 + bh_2$ with $(a : b) \in \mathbb{P}^1$ corresponding to the values $(1 : 0)$, $(0 : 1)$ and $(1 : 1)$ and $(1 : \lambda)$ for some $\lambda \in \mathbb{C}$; if $g_i \in \mathbb{C}[y_0, y_1, y_2]_d$ then $g := \prod_{i=1}^4 g_i$ belongs to $\mathbb{C}[y_0, y_1, y_2]_{4d}$.

If we give weight d to the indeterminates x_0 and x_1, the polynomial $x_0 x_1 (x_0 - x_1)(x_0 - \lambda x_1) - g(y_0, y_1, y_2)$ defines a threefold X in the weighted projective space $\mathbb{P}(d, d, 1, 1, 1)$. It can be proved, using the same 'intersection method' employed in [38, Lemma 3.8], that $D := V(x_0 - g_1, x_1 - g_2)$ defines an element in $H^{2,2}(H^4(X))$ that is linearly independent from the element given by a hyperplane section of X. This implies that

$$h^{2,2}(H^4(X)) \geq 2$$

hence by [38, Proposition 2.8] we can conclude that the Alexander polynomial of $\overline{\mathcal{A}}$ is non-trivial. □

Corollary 2.2.18. *If $\overline{\mathcal{A}}$ admits a reduced multinet then its Alexander polynomial is non-trivial.*

Thus, the existence of a reduced multinet on $\overline{\mathcal{A}}$ is a sufficient condition for the non-triviality of the Alexander polynomial of $\overline{\mathcal{A}}$; as we shall see in the last subsection of this chapter, however, this condition is not necessary.

2.2.4 Topology of $U(\overline{\mathcal{A}})$

Let X be a locally connected topological space and G be an abelian group. A *local system* of stalk G on X is a sheaf $\mathcal{G}$ on X which is locally isomorphic the the constant sheaf of stalk G, i.e. such that for any $U \subset X$ open we have $\mathcal{G}_{|U} \simeq \underline{G}_U$. If G is the vector space $\mathbb{C}^n$ we speak of local system of rank n with $\mathbb{C}$ coefficients.

It is well known (see [68, Corollary 3.10]) that for any $x \in X$ there is a bijection between the set of isomorphism classes of local systems of stalk G and the set of representations $\pi_1(X, x) \to Aut(G)$ modulo the action of $Aut(G)$ by conjugation, provided the space X is 'sufficiently nice'. In particular, local systems of rank one with $\mathbb{C}$ coefficients correspond to representations $\rho : \pi_1(X, x) \to \mathbb{C}^*$ i.e. multiplicative characters of $\pi_1(X, x)$, so they can be identified with $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$.

Given a hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_k\} \subset \mathbb{C}^3$, we will be interested in rank one local systems on $U(\overline{\mathcal{A}}) = \mathbb{P}^2 \setminus \overline{\mathcal{A}}$: if γ_i denotes the meridian around the line $\ell_i = \overline{H_i}$ of $\overline{\mathcal{A}}$, it is known that $H_1(U(\overline{\mathcal{A}}), \mathbb{Z}) \simeq \langle \gamma_1, \dots, \gamma_k | \gamma_1 + \dots + \gamma_k = 0 \rangle_{\mathbb{Z}}$ (see [16, Proposition 4.1.3]), so the rank one local systems on $U(\overline{\mathcal{A}})$ are parametrised by a $(k-1)$-dimensional torus:

$$\mathrm{Hom}(H_1(U(\mathcal{A}), \mathbb{Z}), \mathbb{C}^*) = \left\{ \lambda := (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k \middle| \prod_{i=1}^k \lambda_i = 1 \right\} \simeq (\mathbb{C}^*)^{k-1}. \quad (2.2.5)$$

Local systems provide us with a different way of describing the monodromy eigenspaces of $H^1(F, \mathbb{C})$ (see [7]): namely, for all $j = 0, \dots, k-1$ there exists a rank one local system $\mathcal{L}_j$ on $U(\overline{\mathcal{A}})$ such that

$$H^1(F,\mathbb{C})_{\lambda_j} = H^i(U(\overline{\mathcal{A}}),\mathcal{L}_j) \qquad \text{where } \lambda_j := e^{\frac{2\pi i k}{j}}. \tag{2.2.6}$$

For any $t \in (\mathbb{C}^*)^{k-1}$ we denote by $\mathbb{C}_t$ the associated local system on $U(\overline{\mathcal{A}})$. The (degree q, depth r) *characteristic variety* of $U(\overline{\mathcal{A}})$ is

$$\mathcal{V}(\mathcal{A})^q_r := \{t \in (\mathbb{C}^*)^{k-1} | \dim \mathbb{C} H^q(U(\overline{\mathcal{A}}),\mathbb{C}_t) \geq r\}. \tag{2.2.7}$$

Characteristic varieties $\mathcal{V}_r(\mathcal{A}) := \mathcal{V}^1_r(\mathcal{A})$ of degree 1 are well-understood. Indeed, it is known that $\mathcal{V}_r(\mathcal{A})$ is a finite union of translated subtori (see [2]; this is actually true in broader generality) and that all positive-dimensional components of $\mathcal{V}_r(\mathcal{A})$ pass through the identity $\mathbf{1} := (1,\dots,1)$ of $(\mathbb{C}^*)^{k-1}$ (see [43]). However, their most important feature, at least when one deals with hyperplane arrangements, is that the tangent cone of $\mathcal{V}_r(\mathcal{A})$ at $\mathbf{1}$ is exactly $\mathcal{R}_r(A)$. More specifically, the exponential homomorphism $exp : H^1(M(\mathcal{A}),\mathbb{C}) \to H^1(M(\mathcal{A}),\mathbb{C}^*)$ gives a bijection between the components of $\mathcal{R}_1(A)$ and the components of $\mathcal{V}_r(\mathcal{A})$ passing through $\mathbf{1}$ (see [7],[21]); since all positive-dimensional components P of $\mathcal{R}_1(A)$ are obtained by pullback along an admissible map $f : M(A) \to S$ by Theorem 2.2.12, each positive dimensional component of $\mathcal{V}_r(\mathcal{A})$ is of the form $exp(P) = f^*(H^1(S,\mathbb{C}^*))$.

Now, assume Y is a k-fold cyclic cover of some topological space X, then it corresponds to a surjective homomorphism $\nu : \pi_1(X) \to \mathbb{Z}_k$; if we fix an inclusion $i : \mathbb{Z}_k \to \mathbb{C}^*$ by $1 \mapsto e^{2\pi i/k}$, we obtain a character $\rho : i \circ \nu : \pi_1(X) \to \mathbb{C}^*$, and we have the following isomorphism of $\mathbb{C}[\mathbb{Z}_k]$-modules (see [66, Theorem B1]):

$$H^1(Y,\mathbb{C}) \simeq H^1(X,\mathbb{C}) \oplus \bigoplus_{1<d|k} (\mathbb{C}[t]/\Phi_d(t))^{\mathrm{depth}(\rho^{k/d})} \tag{2.2.8}$$

where $\mathrm{depth}(\rho) := \dim_{\mathbb{C}} H^1(X,\mathbb{C}_\rho) = max\{r|\rho \in \mathcal{V}_r(X)\}$. The Milnor fibre F of $\mathcal{A}$ is a k-fold cyclic covering of $U(\overline{\mathcal{A}})$, and it corresponds, by [66, Theorem 4.10], to the epimorphism $\nu : H_1(U(\overline{\mathcal{A}}),\mathbb{C}) \to \mathbb{Z}_k$ s.t. $\nu(\pi_*(a_h)) = 1 \mod k$. For any $d|k$ we can define the character $\rho_d : H_1(U(\overline{\mathcal{A}}),\mathbb{C}) \to \mathbb{C}^*$ by $\rho_d(a_h) = e^{2\pi i/d}$, and the above isomorphism becomes

$$H^1(F,\mathbb{C}) \simeq \mathbb{C}^{k-1} \oplus \bigoplus_{1<d|k} (\mathbb{C}[t]/\Phi_d(t))^{e_d(\overline{\mathcal{A}})} \tag{2.2.9}$$

where $e_d(\overline{\mathcal{A}}) := \mathrm{depth}(\rho_d)$.

We can now present a sufficient condition for the non-triviality of the Alexander polynomial of a line arrangement $\overline{\mathcal{A}}$ due to Papadima and Suciu [59, Theorem 8.3]; the same result can be obtained by combining [27, Theorem 3.11] and [20, Theorem 3.1(i)]:

Theorem 2.2.19. *Assume $\overline{\mathcal{A}}$ admits a reduced k-multinet; if $f : M(\mathcal{A}) \to S$ is the associated admissible map, the following holds:*

(i) The character ρ_k belongs to $f^*(H^1(S,\mathbb{C}^*))$, and $e_k(\overline{\mathcal{A}}) \geq k-2$.

(ii) If $k = p^s$ for some prime p, then $\rho_{p^r} \in f^*(H^1(S,\mathbb{C}^*))$ and $e_{p^r}(\overline{\mathcal{A}}) \geq k-2$ for all $1 \leq r \leq s$.

This result implies in particular Corollary 2.2.18: if we add a multiplicity assumption, it specialises to the following ([59, Theorem 1.6]):

Theorem 2.2.20. *Assume $\overline{\mathcal{A}}$ has no point of multiplicity $3r$ for $r \geq 2$, then $0 \leq e_3(\overline{\mathcal{A}}) = \beta_3(\overline{\mathcal{A}}) \leq 2$ and the following are equivalent:*

(i) *$\overline{\mathcal{A}}$ supports a reduced 3-multinet.*

(ii) *$\overline{\mathcal{A}}$ supports a 3-net.*

(iii) *$e_3(\overline{\mathcal{A}}) \neq 0$.*

As a corollary we obtain [59, Theorem 1.2]:

Corollary 2.2.21. *Let $\overline{\mathcal{A}}$ be a line arrangement with only double and triple points, then its Alexander polynomial is*

$$\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{|\overline{\mathcal{A}}|-1}(t^2+t+1)^{\beta_3(\overline{\mathcal{A}})}$$

where $0 \leq \beta_3(\overline{\mathcal{A}}) \leq 2$ depends only on $L(\mathcal{A})$.

Note that this generalises Libgober's result of Theorem 2.2.16, and gives affirmative answer to Problem 1 when $\overline{\mathcal{A}}$ has only double and triple points.

2.2.5 Examples

This last section is devoted to the discussion of some interesting line arrangements, which will highlight the role multinets have in the problem determining the Alexander polynomial.

The A_3 arrangement As we have already seen, this arrangement admits a $(3,2)$-net $\mathcal{N}$ so its Alexander polynomial is non-trivial by Corollary 2.2.18. Up to an automorphism of $\mathbb{P}^2$, we can assume that the four triple points are $\{(1:0:0),(0:1:0),(0:0:1),(1:1:1)\}$, so an equation of $\overline{\mathcal{A}}$ is given by

$$xyz(x-y)(x-z)(y-z)=0.$$

The classes of $\mathcal{N}$ are $\overline{\mathcal{A}}_1 := \{y, x-z\}$, $\overline{\mathcal{A}}_2 := \{z, x-y\}$ and $\overline{\mathcal{A}}_3 := \{x, y-z\}$; observe that in accordance with Theorem 2.2.14 the lines in each $\overline{\mathcal{A}}_i$ are the three irreducible components of the Ceva pencil given by base curves $C_1 := V(x(y-z))$ and $C_2 := V(y(x-z))$. The Alexander polynomial of this arrangement can be computed with Theorem 2.1.16: $\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^5(t^2+t+1)$.

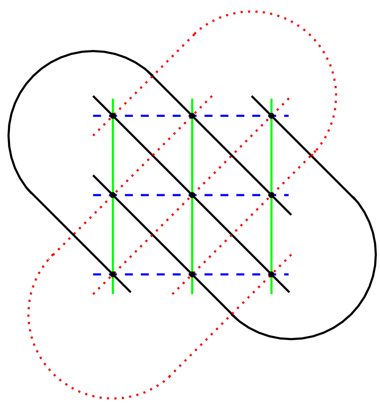
The Hesse arrangement The Hesse pencil

$$a(x^3+y^3+z^3) - bxyz \qquad \text{for } (a:b) \in \mathbb{P}^1$$

is a connected pencil with four reduced completely reducible fibres $F_1, \dots, F_4$, corresponding to the values $(a:b) = \{(1:3\eta) | \eta^3 = 1\} \cup \{(0:1)\}$; it is thus a Ceva pencil, and by Theorem 2.2.14 the line arrangement $\overline{\mathcal{A}}$ given by the irreducible components of the F_i's admits a $(4,3)$-net with classes $\overline{\mathcal{A}}_i = \{\text{irreducible components of } F_i\}$ and base locus given by the base locus of the pencil (i.e. nine points of order four). $\overline{\mathcal{A}}$ has equation

$$xyz \prod_{i,j=0}^{2} (\eta^i x + \eta^j y + z) = 0$$

and it can be pictured in the real plane in the following way:



An alternative way of obtaining this arrangement is by considering the twelve lines passing through triples of inflection points of an elliptic curve $E \subset \mathbb{P}^2$.

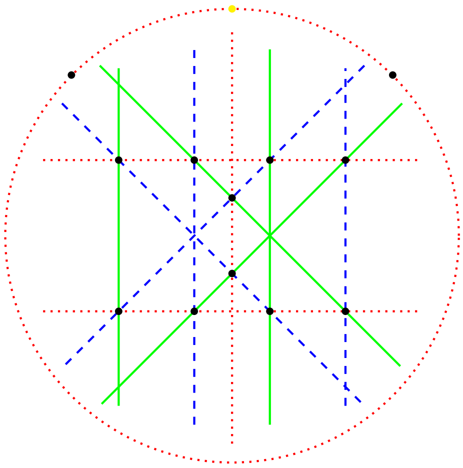
In accordance with Corollary 2.2.18 $\overline{\mathcal{A}}$ has non-trivial Alexander polynomial: indeed, $\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{11}[(t+1)(t^2+1)]^2$.

Remark 2.2.22.
1. The Hesse arrangement is the only known example of non-central line arrangement admitting a 4-net.
2. If the lines of any of the classes $\overline{\mathcal{A}}_i$ are removed from the Hesse arrangement, one obtains the Pappus arrangement of nine lines with three triple points on each line. The latter admits a 3-net, and its Alexander polynomial is $(t-1)^8(t^2+t+1)$.

A particular simplicial arrangement The arrangement $\overline{\mathcal{A}}$ of twelve lines given by

$$xz(x \pm 2z)(x \pm 4z)(y \pm z)(y + x \pm 3z)(y - x \pm 3z) = 0$$

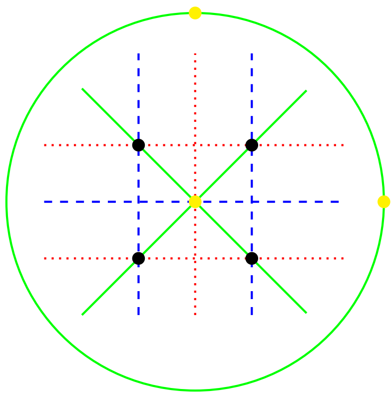
admits a reduced $(3,4)$-multinet $\mathcal{N}$ which is not a net, whose classes are $\overline{\mathcal{A}}_1 := \{x, z, y+z, y-z\}$ (red), $\overline{\mathcal{A}}_2 := \{x+4z, x+2z, y-x+3z, y+x-3z\}$ (blue) and $\overline{\mathcal{A}}_3 := \{x-4z, x-2z, y-x-3z, y+x+3z\}$ (green); the line $z=0$ is portrayed at infinity. The base locus of $\mathcal{N}$ consists of one point of order six (in $(0:1:0)$) and twelve triple points: $\{(\pm 4 : \pm 1 : 1), (\pm 2 : \pm 1 : 1), (1 : \pm 1 : 0), (0 : \pm 3 : 1)\}$. As predicted by Corollary 2.2.18 the Alexander polynomial of $\overline{\mathcal{A}}$ is non-trivial: in fact, it is $(t-1)^{11}(t^2+t+1)$.



The B_3 arrangement This is the arrangement of equation

$$xyz(x-y)(x-z)(y-z)(x+y)(x+z)(y+z) = 0.$$

It admits a non-reduced $(3,4)$-multinet $\mathcal{N}$ with classes $\overline{\mathcal{A}}_1 := \{x, y-z, y+z\}$ (red), $\overline{\mathcal{A}}_2 := \{y, x-z, x+z\}$ (blue), $\overline{\mathcal{A}}_3 := \{z, x-y, x+y\}$ (green) and multiplicities 2 for the lines x, y, z and 1 for the other lines; the line $z = 0$ is portrayed at infinity. Its base locus is given by three points of order four in $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ and four points of order three in $(1:1:1)$, $(1:0:-1)$, $(1:-1:0)$ and $(1:-1:1)\}$. The Alexander polynomial of this arrangement is trivial, so the hypotheses of Corollary 2.2.18 cannot be weakened.



Full monomial arrangements Consider the family of arrangements $\{\overline{\mathcal{A}}_m\}_{m\geq 1}$ defined by equations

$$Q_m := (x^m - y^m)(x^m - z^m)(y^m - z^m) = 0.$$

Each $\overline{\mathcal{A}}_m$ admits a $(3,m)$-net, with classes consisting of the irreducible factors of each factor of Q_m: there are three mono-coloured points of multiplicity m and m^2 multi-coloured triple points. Direct computations via [59, Lemma 3.1] show that if $3 \nmid m$ then $\beta_3(\overline{\mathcal{A}}_m) = 1$ while $3|m$ gives $\beta_3(\overline{\mathcal{A}}_m) = 2$. In the former case Theorem 2.2.20 implies that $e_3(\overline{\mathcal{A}}_m) = 1$, while in the latter we need to distinguish two cases:

- If $m = 3$ we can invoke again Theorem 2.2.20 to conclude that $e_3(\overline{\mathcal{A}}_3) = 2$.
- If $m = 3d$ with $d > 1$, the multiplicity assumption of Theorem 2.2.20 no longer holds, but it is still possible to prove that $e_3(\overline{\mathcal{A}}_{3d}) = 2$.

This example shows that $e_3(\overline{\mathcal{A}})$ can indeed take all values between 0 and 2.

Remark 2.2.23.
1. The arrangement $\overline{\mathcal{A}}_3$ can be obtained as the dual of the nine inflection point of an elliptic curve $E \subset (\mathbb{P}^2)^\vee$, see [4, Remarks 3.2(i)] and [43, Example 3]; in particular, it is the dual of the Hesse arrangement.
2. It can actually be shown (see [52]) that $e_p(\overline{\mathcal{A}}_m) = \beta_p(\overline{\mathcal{A}}_m)$ for all primes p and $m \geq 1$.

Monomial arrangements There exist arrangements that in spite of admitting only a non-reduced multinet still have non-trivial Alexander polynomial: consider for example the family of arrangements $\{\overline{\mathcal{A}}_m\}_{m\geq 1}$ of equation

$$Q_m := xyz(x^m - y^m)(x^m - z^m)(y^m - z^m) = 0.$$

For any m, $\overline{\mathcal{A}}_m$ admits a $(3,2m)$-multinet $(\mathcal{N}, \mathcal{X})$ with classes

$$\begin{aligned}\overline{\mathcal{A}}_1 &= \{x, \text{factors of } (y^m - z^m)\}.\\ \overline{\mathcal{A}}_2 &= \{y, \text{factors of } (x^m - z^m)\}.\\ \overline{\mathcal{A}}_3 &= \{z, \text{factors of } (x^m - y^m)\}.\end{aligned}$$

The lines x, y and z have multiplicity m, while all other lines have multiplicity 1 (in particular, this multinet is reduced only for $m = 1$); $\mathcal{X}$ consists of the three points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ of multiplicity $m+2$ and of other m^2 triple points.

If $m = 1$ we obtain the A_3 arrangement, which we have already studied. For the other values of m we have two possibilities:

- If $m \not\equiv 1 \bmod 3$ then the Alexander polynomial of $\overline{\mathcal{A}}_m$ has the form $(t-1)^{3m-2}(t^2+t+1)^{e_3}$; but it can be proven that $\beta_3 = 0$, so from the modular bound (2.2.4) we deduce that $\overline{\mathcal{A}}_m$ has trivial Alexander polynomial.
- If $m = 3d+1$ with $d > 0$ it can be proven that $\overline{\mathcal{A}}_m$ admits no reduced multinets but it still satisfies $e_3(\overline{\mathcal{A}}_m) \geq 1$ i.e. it has non-trivial Alexander polynomial; in particular, Corollary 2.2.18 is not an if and only if. Observe that in this case the multiplicities of the points in the base locus have greatest common divisor equal to 3.

For a thorough discussion of this example, the reader can consult [59, Example 8.11].

The results and examples presented so far led to the formulation of the following conjecture:

Conjecture 1. *Let $\overline{\mathcal{A}}$ be an essential line arrangement, then $e_{p^s}(\overline{\mathcal{A}}) = 0$ for all primes p and integers $s \geq 1$ with two possible exceptions:*

$$e_2(\overline{\mathcal{A}}) = e_4(\overline{\mathcal{A}}) = \beta_2(\overline{\mathcal{A}}), \qquad e_3(\overline{\mathcal{A}}) = \beta_3(\overline{\mathcal{A}}). \tag{2.2.10}$$

Moreover, if $e_k(\overline{\mathcal{A}}) = 0$ for all divisors k of $|\overline{\mathcal{A}}|$ that are not prime powers, then the Alexander polynomial of $\overline{\mathcal{A}}$ is

$$\Delta_{\overline{\mathcal{A}}}(t) = (t-1)^{|\overline{\mathcal{A}}|-1}(t^2+t+1)^{\beta_3(\overline{\mathcal{A}})}[(t+1)(t^2+1)]^{\beta_2(\overline{\mathcal{A}})}. \tag{2.2.11}$$

The validity of this conjecture in the strong form (2.2.11) would give an affirmative answer to Problem 1.

We conclude the discussion of these examples with two remarks:

1. To the best of our knowledge, there are no line arrangements with non-trivial Alexander polynomial that do not support multinets.

2. As we have already observed with the B_3 arrangement, the existence of a multinet on $\overline{\mathcal{A}}$ is not a sufficient condition for the non-triviality of the Alexander polynomial of $\overline{\mathcal{A}}$; however, the last example shows that there are indeed arrangements with non-trivial Alexander polynomial that only admit non-reduced multinets ($\overline{\mathcal{A}}_m$ with $m = 3d+1$ and $d > 1$). Observe that in the former case the greatest common divisor of the multiplicities of the points in the base locus is 1, while in the latter case this value is 3; it is interesting to notice that this value is always greater than 1 when $\overline{\mathcal{A}}$ admits a reduced multinet (recall Remark 2.2.8(v)) which is a sufficient condition for the non triviality of $\Delta_{\overline{\mathcal{A}}}$.

CHAPTER 3

Arrangements with two points of high order

As we have seen in Chapter 2, it seems that a line arrangement $\overline{\mathcal{A}}$ needs to have some symmetry properties (i.e. needs to admit at least a multinet) in order for its Alexander polynomial to be non-trivial; the existence of a multinet on $\overline{\mathcal{A}}$, in turn, imposes restrictions on the multiplicities of the singular points of $\overline{\mathcal{A}}$. Motivated by this, we will focus on a class of line arrangements that do not admit multinets. Namely, we will consider line arrangements $\overline{\mathcal{A}}$ of n lines having the following properties:

1. $n \geq 7$.
2. $\overline{\mathcal{A}}$ has two multiple points P_1 and P_2 with $ord(P_1), ord(P_2) \geq 3$.
3. If P is a multiple point of $\overline{\mathcal{A}}$ different from P_1 and P_2, then $ord(P) \leq 3$.
4. If P is a multiple point of $\overline{\mathcal{A}}$ with $ord(P) = 3$, then at least one of the lines through P passes through P_1 or P_2.

We impose condition 1. in order to rule out the arrangement A_3, as it satisfies 2.-4. and it admits a 3-net. One should think of these arrangements as possessing two 'anchor points' P_1 and P_2 of high order.

The complexity of such arrangements depends on the number of lines not passing through P_1 or P_2, to which we shall refer as to *free lines*; we will denote their number by s. For example, if $s = 0$ then P_1 and P_2 are the only points of $\overline{\mathcal{A}}$ of order greater than or equal to three; if $s = 1$, the arrangement can have triple points different from P_1 and P_2, but they all belong to the free line; if $s \geq 2$, the combinatorics of the triple points becomes much more complicated.

In this chapter we will consider arrangements satisfying 1.-4. with $s \in \{0, 1\}$.

3.1 $s = 1$ with common line

We denote the free line by ℓ. Without loss of generality, we assume that P_1 has order p, P_2 has order q and $p \geq q$. We assume moreover that P_1 and P_2 lie on a same line ℓ_c of the arrangement, so that $p + q = n$; this implies that $p \leq n - 3$, so we can write $n - 3 \geq p \geq q \geq 3$. Note that arrangements of this type can have up to $q - 1$ triple points different from P_1 and P_2, all of them lying on ℓ.

We begin our study by showing that arrangements of this type are indeed non-symmetric:

Lemma 3.1.1. *An arrangement $\overline{\mathcal{A}}$ of this type does not support weak (α, β)-multinets.*

Proof. Since any weak multinet can be refined to a multinet (recall Remark 2.2.11), it is enough to prove that $\overline{\mathcal{A}}$ cannot support a multinet; moreover, since $\overline{\mathcal{A}}$ is non-central, by Theorem 2.2.9 it is enough to prove that it cannot support (k, d)-multinets for $k \in \{3, 4\}$. If $\overline{\mathcal{A}}$ supported a $(4, d)$-multinet $(\mathcal{N}, \mathcal{X})$ with classes $\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_4$, then all points of $\mathcal{X}$ would have multiplicity at least 4 (recall that for any $p \in \mathcal{X}$ there exists $l_i \in \overline{\mathcal{A}}_i$ such that $p \in l_i$); now we have to distinguish two cases:

- P_2 has order 3. In this case $\mathcal{X} = \{P_1\}$ and P_1 is the only multi-coloured point of $\overline{\mathcal{A}}$. Assume $\ell \in \overline{\mathcal{A}}_1$: since P_1 is multi-coloured, there is a line ℓ' through P_1 such that $\ell' \in \overline{\mathcal{A}}_j$ with $i \neq j$. This means the point $P := \ell \cap \ell'$ belongs to $\mathcal{X}$, which is impossible because $ord(P) \leq 3$.

- P_2 has order 4 or bigger. The cases $\mathcal{X} = \{P_1\}$ and $\mathcal{X} = \{P_2\}$ are impossible by the same reasoning as above, so it remains to rule out the possibility that $\mathcal{X} = \{P_1, P_2\}$. Again, since P_1 and P_2 are multi-coloured we can find $\ell_1 \in \overline{\mathcal{A}}_1$ through P_1 and $\ell_2 \in \overline{\mathcal{A}}_2$ through P_2; this means $P := \ell_1 \cap \ell_2 \in \mathcal{X}$, which is again impossible because $ord(P) \leq 3$.

Assume now $\overline{\mathcal{A}}$ supports a $(3, d)$-multinet with classes $\overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2, \overline{\mathcal{A}}_3$. Assume $P_1, P_2 \in \mathcal{X}$, then they are multi-coloured and we can find ℓ_1 through P_1 and ℓ_2 through P_2 such that $\ell_1 \in \overline{\mathcal{A}}_1$, $\ell_2 \in \overline{\mathcal{A}}_2$ and $\ell_c \in \overline{\mathcal{A}}_3$; this means that $P := \ell_1 \cap \ell_2$ belongs to $\mathcal{X}$. If $\overline{\mathcal{A}}$ has no triple points other than (possibly) P_1 and P_2 this is impossible, otherwise the only way this can happen is by having $\ell \in \overline{\mathcal{A}}_3$ and $P \in \ell$. Now, if other lines belonging to $\overline{\mathcal{A}}_2$ or $\overline{\mathcal{A}}_3$ passed through P_2 then their intersection with ℓ_1 would be in $\mathcal{X}$, but this is impossible because that intersection point has multiplicity two; we deduce that all the $q - 2$ lines through P_2 different from ℓ_2 and ℓ_c must belong to $\overline{\mathcal{A}}_1$ and, by symmetry, all the $p - 2$ lines through P_1 different from ℓ_1 and ℓ_c must belong to $\overline{\mathcal{A}}_2$. These groups of lines intersect in $(q - 2)(p - 2)$ multi-coloured points, but ℓ can pass through at most $q - 2$ of them (since it already passes through P); this would imply that there are multi-coloured points of multiplicity two, impossible.

Assume $P_1 \in \mathcal{X}$ and $P_2 \notin \mathcal{X}$, then P_2 is mono-coloured and we can assume that all lines through it belong to $\overline{\mathcal{A}}_2$. Let now ℓ_1 be a line through P_1 different from ℓ_c and belonging to $\overline{\mathcal{A}}_1$ (or to $\overline{\mathcal{A}}_3$): its intersection with the lines through P_2 gives $q - 1$ points that must belong to the base locus, but since they lie on the same line this is impossible (the free line ℓ can only pass through one of them). By symmetry, $P_2 \in \mathcal{X}$ and $P_1 \notin \mathcal{X}$ is impossible too.

Thus the only possibility that remains is that $\mathcal{X}$ contains only triple points P different from P_1 and P_2. But if $P \in \mathcal{X}$ is such a triple point we can find ℓ_1 through P_1 and P belonging to $\overline{\mathcal{A}}_1$ and ℓ_2 through P_2 and P belonging to $\overline{\mathcal{A}}_2$; regardless of which class ℓ_c belongs to, we obtain that at least one between P_1 and P_2 belongs to $\mathcal{X}$, and we have shown that this is impossible. $\square$

Now we state a result that allows to obtain information on the Alexander polynomial of a curve from the first Betti number of a smooth projective surface associated to it; it is an immediate consequence of points 1. and 3. of Theorem 2.1.14, but since we focus on the case of curves we present a proof for the convenience of the reader.

Theorem 3.1.2. *Let $C := V(f(x_0, x_1, x_2)) \subset \mathbb{P}^2$ be a reduced curve of degree d; for any $l|d$ the l-fold cover of $\mathbb{P}^2$ branched along C is the surface S_l of equation $y^l = f$ in the weighted projective space $\mathbb{P}(\frac{d}{l}, 1, 1, 1)$. If we call $\tilde{S}_l$ a resolution of S_l and write the non-trivial part of the Alexander polynomial of C as $\prod_{1<k|d} \Phi_k^{\alpha_k}$, we have*

$$2q(\tilde{S}_l) = b_1(\tilde{S}_l) = \sum_{1<k|l} deg(\Phi_k)\alpha_k. \tag{3.1.1}$$

This theorem tells us that the value $h^1(\tilde{S}_l)$ gives information on the degree of the product of the factors of the Alexander polynomial of C that vanish on the $\frac{l}{k}$-th root of unity $e^{2\pi i \frac{k}{l}}$. In particular, if $q(\tilde{S}_d) = 0$ then the Alexander polynomial of C is trivial.

Note that the right-hand side of (3.1.1) is always even, which is no surprise: Φ_k has even degree for $k \geq 3$, and Φ_2 can only appear in the Alexander polynomial with an even α_2 by formula (2.1.15).

Proof. If we denote by F the Milnor fibre of C then S_l is the closure of F in $\mathbb{P}(\frac{d}{l}, 1, 1, 1)$, thus $F = S_l \cap D(y)$ and $C = S_l \cap Z(y)$; in particular, the singular locus of S_l consists of a finite number of points. By Proposition 1.3.8(ii) we have a long exact sequence of MHS

$$\cdots \to H_c^\bullet(F) \to H^\bullet(S_l) \to H^\bullet(C) \to H_c^{\bullet+1}(F) \to \dots$$

which gives

$$\cdots \to H^2(S_l) \to H^2(C) \to H_c^3(F) \to H^3(S_l) \to 0$$

The monodromy action on F is given by $(x_0, x_1, x_2) \mapsto \eta \cdot (x_0, x_1, x_2)$, where η is an element of the group μ_l of l-th roots of unity, and it can be extended to a μ_l-action on S_l by setting $\eta \cdot (y : x_0 : x_1 : x_2) := (\eta^{-\frac{d}{l}} y : x_0 : x_1 : x_2)$; as $S_l/\mu_l \simeq \mathbb{P}^2$, we have $h^2(S_l)^{\mu_l} = 1$ and $h^3(S_l)^{\mu_l} = 0$.

Calling r the number of irreducible components of C, we have $b_2(C) = r$; moreover, we notice immediately that the μ_l-action on C is trivial, so $H^i(C)^{\mu_l} = H^i(C)$.

The fact that $C \subset S_l$ implies that the morphism $\sigma : H^2(S_l) \to H^2(C)$ is non-trivial, and the fact that C is fixed by the μ_l-action guarantees that it remains non-trivial when we consider its restriction to the μ_l-invariant parts $\sigma' : H^2(S_l)^{\mu_l} \to H^2(C)$. If we consider the invariant part under the μ_l-action of the previous long exact sequence, we obtain then $0 \to H^2(C)/Im(\sigma') \to H_c^3(F)^{\mu_l} \to 0$; in particular $H_c^3(F)_1 = H_c^3(F)^{\mu_l}$ has dimension $r-1$. On the other hand, $H_c^3(F)$ is Poincaré dual to $H^1(F) = H^1(F)_1 \oplus H^1(F)_{\neq 1}$, and we know that $H^1(F)_1$ has dimension $r-1$ too; in particular $H_c^3(F)_1 \simeq H^1(F)_1$ and $H_c^3(F)_{\neq 1} \simeq H^1(F)_{\neq 1}$.

More importantly, looking at the non-invariant part of the previous long exact sequence under the μ_l-action we obtain $0 \to H^3_c(F)_{\neq 1} \to H^3(S_l) \to 0$, from which we deduce $H^1(F)_{\neq 1} = H^3(S_l)$. Since the singular locus of S_l is zero-dimensional, we have $H^3(S_l) \simeq H^3(\tilde{S}_l)$ by Corollary 1.3.12.

From this fact and Hodge symmetry we deduce $\dim H^1(F)_{\neq 1} = h^3(S_l) = h^3(\tilde{S}_l) = h^1(\tilde{S}_l) = 2q(\tilde{S}_l)$; since $\dim H^1(F)_{\neq 1}$ is, by definition, equal to $\sum_{1<k|l} deg(\Phi_k)\alpha_k$, the proof is complete. □

One may ask what happens to the Alexander polynomial when we deform the curve C. For some types of deformation, namely equisingular ones, the answer is very simple: nothing, as the following well-known result shows:

Corollary 3.1.3. *The Alexander polynomial of a curve $C \subset \mathbb{P}^2$ is invariant under equisingular deformation.*

Proof. As we noticed in Remark 1.3.17, an equisingular deformation of C induces an equisingular deformation of S_l (for any l) and a deformation of the projective manifold $\tilde{S}_l$; since the Hodge numbers of a family of projective manifolds are constant, we have proved our claim. □

Remark 3.1.4. For any deformation $\mathcal{X} \to \mathbb{C}$ with $\mathcal{X} \subset \mathbb{P}^N$ for some N, the dimension, degree and arithmetic genus of the fibres X_t are independent of t (see [32, Corollary III.9.10]); this implies that all fibres of any equisingular deformation $\phi : \mathcal{C} \to \mathbb{C} \setminus \Delta$ of a line arrangement C are still line arrangements. In order to see this, set

- $C_t := \phi^{-1}(t)$ for the fibres of ϕ; in particular, there exists t_0 in $\mathbb{C} \setminus \Delta$ such that $C \simeq C_{t_0}$ (we can assume without loss of generality that $0 \notin \Delta$ and $t_0 = 0$, so that $C \simeq C_0$).
- $\tilde{\phi} : \tilde{\mathcal{X}} \to \mathbb{C} \setminus \Delta$ for the deformation of $\tilde{C} \simeq \tilde{C}_0$ obtained by ϕ after resolving the singular locus of $\mathcal{X}$, and $\tilde{C}_t := \tilde{\phi}^{-1}(t)$ for its fibres.

Since for any plane curve C we have $h^2(C) = h^2(\tilde{C})$, if we call d the degree of our line arrangement $C \simeq C_0$ and pick any $t \in \mathbb{C} \setminus \Delta$ we can write

$$d = deg(C_0) = h^2(C_0) = h^2(\tilde{C}_0) = h^2(\tilde{C}_t) = h^2(C_t)$$

where we have used the invariance of the Hodge numbers for families of smooth manifolds. Since the corollary cited above gives $d = deg(C_t)$ we get $h^2(C_t) = deg(C_t)$, which implies that C_t is a line arrangement.

We say that $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}'$ are *ED-equivalent* if one can be obtained from the other by an equisingular deformation; by the previous corollary and remark, the study of this class of arrangements reduces to two steps:

(a) We partition the set of arrangements into ED-equivalence classes, and we choose a suitable representative for each class.

(b) We study the Alexander polynomial of each representative.

Up to an isomorphism of $\mathbb{P}^2$ we can assume that $P_1 = (0:0:1)$, $P_2 = (0:1:0)$ and ℓ has equation $y - z = 0$. If an arrangement has t triple points, they can only lie on ℓ, and the only effect of an equisingular deformation is to move them along that line; this means we have an ED-equivalence class X_t for every possible number of triple points, i.e. $t = 0, \dots, q-1$. As representative of X_0 we choose any arrangement $\overline{\mathcal{A}}_0$ without triple points; as representative of X_t with $t = 1, \dots, q-1$ we choose any arrangement $\overline{\mathcal{A}}_t$ whose triple points lie in $(j:1:1)$ for $j = 1, \dots, t$. This takes care of step (a).

For step (b) observe that since we know the position of the points of $\overline{\mathcal{A}}_t$ of multiplicity three or more, the Alexander polynomials $\Delta_{\overline{\mathcal{A}}_t}$ can be computed using formula (2.1.15); while the computation we need to carry out are fairly easy, for the sake of clarity it is better to establish some notations and lemmas before moving on.

We call $R := \mathbb{C}[x, y, z]$ and $A := (x, y)$, $B := (x, z)$ the ideals of P_1 and P_2. We denote the triple points of $\overline{\mathcal{A}}_t$, whose ideals are $I_j := (y - z, x - jy)$ for $j \leq t$, by T_j ; the intersection between r of the I_j will be denoted by II_r. Now we recall some well-known results from algebra.

Proposition 3.1.5. *If $I, J \subset R$ are monomial ideals, say $I = (m_1, \dots, m_r)$ and $J = (n_1, \dots, n_s)$, then $I \cap J = (lcm(m_i, n_j) | i = 1, \dots, r, j = 1, \dots, s)$; in particular, for any $a \in \mathbb{Z}_{\geq 0}$ we have $(I \cap J)^a = I^a \cap J^a$.*

Proposition 3.1.6. *Let $f, g, h \in R$ such that $gcd(f, h) = gcd(g, h) = 1$, then $(f, g) \cap (f, h) = (f, \frac{gh}{gcd(g,h)})$.*

Proof. The inclusion $\supset$ is obvious. For the other inclusion, pick a polynomial $Q \in (f, g) \cap (f, h)$ and use the division algorithm to write it as $Q = f\bar{Q} + gp_1 = f\bar{Q} + hp_2$, so that $hp_2 = gp_1$; this gives both $h|gp_1$ i.e. $\frac{h}{gcd(g,h)}|p_1$ and $g|hp_2$ i.e. $\frac{g}{gcd(g,h)}|p_2$ so we are done. $\square$

Lemma 3.1.7. *For any two ideals $J_1, J_2 \subset R$ there exists a short exact sequence*

$$0 \to R/J_1 \cap J_2 \xrightarrow{\rho} R/J_1 \oplus R/J_2 \xrightarrow{\pi} R/(J_1 + J_2) \to 0 \tag{3.1.2}$$

where $\rho([f]) := ([f], [f])$ and $\pi([f], [g]) := [f - g]$.

Proof. We begin by showing that ρ and π are well-defined.

Pick $[f] \in R/J_1 \cap J_2$; if $[f] = [f']$ in $R/J_1 \cap J_2$ for some $f' \neq f$ then $f - f' \in J_1 \cap J_2$, and we get $\rho([f]) - \rho([f']) = ([f - f'], [f - f']) = ([0], [0])$. This proves ρ is well-defined.

Pick now $[f] \in R/J_1$ and $[g] \in R/J_2$; if $[f] = [f']$ in R/J_1 for some $f \neq f'$ and $[g] = [g']$ in R/J_2 for some $g \neq g'$ then $f - f' \in J_1$ and $g - g' \in J_2$. This means that $\pi([f], [g]) - \pi([f'], [g']) = [f - f' - (g - g')] = [0]$ because $f - f' - (g - g') \in J_1 + J_2$; this proves π is well-defined.

ρ is injective because $\rho([f]) = ([0], [0])$ if and only if $[f] = [0]$ in both R/J_1 and R/J_2 i.e if and only if $f \in J_1$ and $f \in J_2$. π is surjective because any element $[f] \in R/(J_1 + J_2)$ can be written as $\pi([f], [0])$. The inclusion $Im(\rho) \subseteq Ker(\pi)$ is obvious; we only need to show that $Ker(\pi) \subseteq Im(\rho)$ in order to conclude.

Pick $([f], [g]) \in Ker(\pi)$, then $[f - g] = [0]$ in $R/(J_1 + J_2)$ i.e. $f - g \in J_1 + J_2$; this means there exist $h_1 \in J_1$ and $h_2 \in J_2$ such that $f - g = h_1 + h_2$, and this allows us to write $f - h_1 = g + h_2$. Now, clearly $[f] = [f - h_1]$ in R/J_1 and $[g] = [g + h_2]$ in R/J_2, so we can write $([f], [g]) = ([f - h_1], [g + h_2])$; but since $f - h_1 = g + h_2$, we can actually write $([f], [g]) = ([f - h_1], [g + h_2]) = ([g + h_2], [g + h_2]) = \rho([g + h_2])$. This proves the lemma. □

Corollary 3.1.8. *If J_1 and J_2 are homogeneous ideals, the previous short exact sequence remains exact after taking the homogeneous parts of any fixed degree; in particular, for any $m \in \mathbb{Z}$ one has*

$$h_{J_1 \cap J_2}(m) = h_{J_1}(m) + h_{J_2}(m) - h_{J_1 + J_2}(m). \tag{3.1.3}$$

In order to use formula (2.1.15) we need to understand which constants of quasi-adjunction of the $\overline{\mathcal{A}}_i$ can actually contribute to the Alexander polynomial $\Delta_{\overline{\mathcal{A}}_i}$, and whether they are relative to one or more multiple points of $\overline{\mathcal{A}}_i$; this requires us to take into consideration the divisibility relations between 3, p, q and n. Since $p+q = n$, any integer d dividing two of p, q and n actually divides all three of them, so $\gcd(p, q) = \gcd(p, n) = \gcd(q, n) = \gcd(p, q, n)$; we denote this integer by d, and write $p = dp'$, $q = dq'$ and $n = dn'$.

By Remark 2.1.12, the constants of quasi-adjunction relative to P_1 are $\frac{j}{p}$ for $j = 1, \dots, p-2$; however, $\frac{j}{p}$ can only contribute to the Alexander polynomial if $n\frac{j}{p} \in \mathbb{Z}_{>0}$ i.e. $n'\frac{j}{p'} \in \mathbb{Z}_{>0}$. Since $(p', n') = 1$, this means that the only constants of quasi-adjunction $\frac{j}{p}$ we need to consider have the form $\frac{jp'}{p} = \frac{j}{d}$ for $j = 1, \dots, d-1$ (unless $d = p$, in which case $j = 1, \dots, d-2$). Likewise, the only constants of quasi-adjunction relative to P_2 that can contribute to the Alexander polynomial have the form $\frac{jq'}{q} = \frac{j}{d}$ for $j = 1, \dots, d-1$ (unless $d = q$, in which case $j = 1, \dots, d-2$).

If $d < q$ then all the constants of quasi-adjunction $\frac{j}{d}$ with $j = 1, \dots, d-1$ we need to consider are relative to both P_1 and P_2. If $d = q$ then the constants of quasi-adjunction relative to P_2 that we need to consider are $\frac{j}{q}$ with $j = 1, \dots, q-2$ but we have to distinguish two cases:

1. If $p > q$ we have to consider the constant of quasi-adjunction $\frac{q-1}{q}$ too, and that one is only relative to the point P_1.

2. If $p = q$, all the constants of quasi-adjunction we have to consider are $\frac{j}{q}$ with $j = 1, \dots, q-2$ and they are relative to both P_1 and P_2.

Lastly, the constant of quasi-adjunction $\frac{1}{3}$ relative to the triple points has to be considered if and only if $3|n$. In order to maintain the exposition as organised as possible, we separate the cases $p = q$ and $p \neq q$ and start from the former.

3.1.1 $p = q$

In this scenario we have $p = q = d$ and $n = 2d$, so $3|n$ if and only if $3|d$. We will show the following:

Theorem 3.1.9. *The Alexander polynomial of each $\overline{\mathcal{A}}_t$ is trivial.*

We separate cases again, depending on whether 3 divides n or not, and show that Theorem 3.1.9 holds in each of them.

$3 \nmid d$

The constants of quasi-adjunction we need to consider are $c_j := \frac{d-1-j}{d}$ for $j = 1, \dots, d-2$. For each of them we need to compute the length of the associated scheme Z_{c_j} and the Hilbert function in degree $N_{2d}(c_j) = 2d - 3 - 2dc_j = 2j - 1$ of the associated ideal I_{c_j}. If $c_j > \frac{1}{3}$ we have $I_{c_j} = A^j \cap B^j = (A \cap B)^j$, while for $c_j \leq \frac{1}{3}$ we have $I_{c_j} = II_t \cap A^j \cap B^j = II_t \cap (A \cap B)^j$.

Lemma 3.1.10. *Consider $a, b \in \mathbb{Z}_{>0}$ such that $a \geq b$; a minimal system of generators for the ideal $A^a \cap B^b$ is given by monomials $x^{a-i}y^i$ for $i = 0, \dots, a-b$ and $x^{b-i}y^{a-b+i}z^i$ for $i = 1, \dots, b$.*

Proof. Since A and B are monomial ideals, certainly $A^a \cap B^b$ is generated by monomials $lcm(x^{a-i}y^i, x^{b-h}z^h)$ for $i = 0, \dots, a$ and $h = 0, \dots, b$; we will extract our desired minimal system of generators from this one.

Assume that $i = 0, \dots, a-b$: we have $lcm(x^{a-i}y^i, x^{b-h}z^h) = x^{a-i}y^iz^h$ for any $h = 0, \dots, b$ but clearly all these monomials are multiples of $x^{a-i}y^i$; this gives our first group of generators.

Consider now terms in A^a like $x^{b-i}y^{a+i-b}$ for $i = 1, \dots, b$: we have

$$lcm(x^{b-i}y^{a+i-b}, x^{b-h}z^h) = \begin{cases} x^{b-i}y^{a+i-b}z^h & \text{if } h \geq i. \\ x^{b-h}y^{a+i-b}z^h & \text{otherwise.} \end{cases}$$

For a fixed i, the monomials we get for $h \geq i$ are all multiples of $x^{b-i}y^{a+i-b}z^i$, and this gives our second group of generators; in order to conclude, we need to prove that for any fixed $i \in \{1, \dots, b\}$ the terms $x^{b-h}y^{a+i-b}z^h$ with $h < i$ are multiples of generators from the first or second group. Fix an $i \in \{1, \dots, b\}$. If $h = 0$ then we get x^by^{a+i-b} which is multiple of x^by^{a-b}; if $0 < h < i$ the monomial $x^{b-h}y^{a+i-b}z^h$ is multiple of $x^{b-h}y^{a+h-b}z^h$, and the latter monomial is in the second group of generators since we can write it as $x^{b-i}y^{a+i-b}z^h$ for $h = i$. □

Lemma 3.1.11. $h_{I_{c_j}}(m) = j(j+1)$ *for any* $m \geq j$

Proof. By Lemma 3.1.10 we have $I_{c_j} = (x^a(yz)^b | a, b \geq 0, a + b = j)$ so a non-zero monomial $x^ay^bz^c$ in $(R/I_{c_j})_m$ needs to satisfy $a = j - h$ for some $h \in \{1, \dots, j\}$ because $x^j \in I_{c_j}$; moreover, since $x^{j-h}y^hz^h \in I_{c_j}$, it needs to have $b < h$ or $c < h$ (or both), so we can write b (or c) as $h - l$ for some $l \in \{1, \dots, h\}$, and this forces c (or b) to be $m - (j - h + h - l) = m - j + l$. Since $h = 1, \dots, j$ and $l = 1, \dots, h$, we obtain $h_{I_{c_j}}(m) = \sum_{h=1}^{j} 2(\sum_{l=1}^{h} 1) = 2\sum_{h=1}^{j} h = 2\frac{j(j+1)}{2} = j(j+1)$. □

Corollary 3.1.12. *The constants of quasi-adjunction c_j s.t. $c_j > \frac{1}{3}$ do not contribute to $\Delta_{\mathcal{A}_t}$ for all t.*

Proof. The local ring of the scheme Z_{c_j} at both P_1 and P_2 consists of all the germs of holomorphic functions whose constant of quasi-adjunction at P_1 (or P_2) is bigger than or equal to c_j, so it is the vector subspace of R generated by the monomials x^ay^b for $a, b \geq 0$ and $a + b \leq j$, while the local ring of Z_{c_j} at the T_i is trivial because triple points admit only $\frac{1}{3}$ as constants of quasi-adjunction for constant function germs. This means that for any $c_j > \frac{1}{3}$ we have $l(Z_{c_j}) = 2(\sum_{h=1}^{j} h) = 2\frac{j(j+1)}{2} = j(j+1)$. Since $2j-1 \geq j$ (because $j \geq 1$), by Lemma 3.1.11 we get $h_{I_{c_j}}(2j-1) = j(j+1)$ too and we are done. $\square$

Now we need to study the contribution of the constants of quasi-adjunction $c_j \leq \frac{1}{3}$: in this situation we have $j \geq \lceil \frac{2d-3}{3} \rceil$ and $I_{c_j} = II_t \cap A^j \cap B^j$.

Remark 3.1.13. Before moving on, note that it must be $t < 2j$. If $t \geq 2j$ in fact we would get $t \geq 2\lceil \frac{2d-3}{3} \rceil = 2\lceil d-1-\frac{d}{3} \rceil \geq 2(d-1) = 2d-2$; but $2d-2 > d-1$ as long as $d > 1$, which is true in our scenario. Since the $\overline{\mathcal{A}}_i$ can have at most $d-1$ triple points, we would get a contradiction.

Lemma 3.1.14. *For any $r \geq 2$ we have $R/(II_{r-1} + I_r) \simeq \mathbb{C}[y]/(y^{r-1})$. In particular $h_{II_{r-1}+I_r}(m)$ is 1 for $m < r-1$ and 0 otherwise.*

Proof. By Proposition 3.1.6 we have $II_{r-1} = (y-z, \prod_{i=1}^{r-1}(x-iy))$ so $II_{r-1}+I_r = (y-z, \prod_{i=1}^{r-1}(x-iy), x-ry)$ and we get $R/(II_{r-1}+I_r) \simeq \mathbb{C}[y]/(\prod_{i=1}^{r-1}(ry-iy)) = \mathbb{C}[y]/(y^{r-1})$. The last assertion is obvious. $\square$

Proposition 3.1.15. *For any $m \in \mathbb{Z}_{\geq 0}$ and any $r \geq 2$ we have*

$$h_{II_r}(m) = r - \sum_{i=2}^{r} \dim(\mathbb{C}[y]/(y^{i-1}))_m. \tag{3.1.4}$$

In particular, for $m \geq r-1$ we have $h_{II_r}(m) = r$.

Proof. We proceed by induction on r. When $r = 2$ the proposition follows immediately from the previous lemma and the short exact sequence

$$0 \to R/II_2 \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0$$

since $R/I_i \simeq \mathbb{C}[y]$ for any i. Now we assume the proposition holds true for $r-1$. From the short exact sequence

$$0 \to R/II_r \to R/II_{r-1} \oplus R/I_r \to R/(II_{r-1} + I_r) \to 0$$

we obtain, using the induction hypothesis and the previous lemma, that

$$\begin{aligned} h_{II_r}(m) &= h_{II_{r-1}}(m) + h_{I_r}(m) - h_{II_{r-1}+I_r}(m) = \\ &= r - 1 - \sum_{i=2}^{r-1} \dim(\mathbb{C}[y]/(y^{i-1}))_m + h_{I_r}(m) - \dim(\mathbb{C}[y]/(y^{r-1}))_m \\ &= r - \sum_{i=2}^{r} \dim(\mathbb{C}[y]/(y^{i-1}))_m. \end{aligned}$$

$\square$

Lemma 3.1.16. *Assume $a, b \in \mathbb{Z}_{>0}$, then $h_{II_t+A^a \cap B^b}(m) = 0$ for any $m \geq a+b-1$.*

Proof. We can assume without loss of generality that $a \geq b$. Thanks to Lemma 3.1.10 we can write

$$R/(II_t + A^a \cap B^b) \simeq \mathbb{C}[x,y]/I \subset \mathbb{C}[x,y]/I'$$

where I is generated by the homogeneous degree t polynomial $\prod_{i=1}^{t}(x - iy)$ and by monomials $x^{a-h}y^h$ for $h = 0, \ldots, a-b$ and $x^{b-h}y^{a-b+2h}$ for $h = 1, \ldots, b$ while I' is only generated by the latter two sets of monomials; it is clearly sufficient to prove that $h_{I'}(m) = 0$ for $m \geq a + b - 1$.

Monomials $x^{a-h}y^{m-a+h}$ with $h = 0, \ldots, a-b$ belong to I' if and only if $m-a+h \geq h$ i.e. if and only if $m \geq a$, and this holds since $b \geq 1$ and $m \geq a + b - 1$. Similarly, monomials $x^{b-h}y^{m-b+h}$ with $h = 1, \ldots, b$ belong to I' if and only if $m-b+h \geq a-b+2h$ i.e. if and only if $m \geq a - h$; but the maximum possible value for $a - h$ when $h \in \{1, \ldots, b\}$ is $a - 1$, and by hypothesis we have $m \geq a + b - 1 \geq a - 1$. □

Corollary 3.1.17. *The constants of quasi-adjunction c_j s.t. $c_j \leq \frac{1}{3}$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. Arguing as in Corollary 3.1.12 we can say that the sum of the dimensions of the local rings of Z_{c_j} at the points P_1 and P_2 is $j(j+1)$; however, since $c_j \leq \frac{1}{3}$ the local ring of Z_{c_j} at the T_i has dimension one. This means that for any $c_j \leq \frac{1}{3}$ and for all t we have $l(Z_{c_j}) = j(j+1) + t$.

Now we need to compute $h_{I_{c_j}}(2j-1)$: we do it using the short exact sequence

$$0 \to R/I_{c_j} \to R/II_t \oplus R/(A^j \cap B^j) \to R/(II_t + A^j \cap B^j) \to 0.$$

Since by Remark 3.1.13 we have $2j - 1 \geq t$, Proposition 3.1.15 and Lemmas 3.1.10 and 3.1.16 allow us to conclude that $h_{I_{c_j}}(2j-1) = j(j+1) + t$ too, so we are done. □

Corollaries 3.1.12 and 3.1.17 together imply Theorem 3.1.9.

$3|d$

If $3|d$ we write $d = 3d'$ (with $d' > 1$ since $2d = n \geq 7$) and notice that only two things are different from the case $3 \nmid d$. First, the constant of quasi-adjunction $\frac{1}{3}$ is now one of the c_j and it could give a non-trivial factor $\Phi_3^{e_3}$ in the Alexander polynomial; second, the inequality in Remark 3.1.13 becomes much simpler: we need to prove that we cannot have $t \geq 2j$ for $j \geq 2d' - 1$, and in this situation we have $2j \geq 4d' - 2 = d + d' - 2 > d - 1$ since $d' > 1$ so that is indeed impossible. All other computations go through without any change, so we can conclude that Theorem 3.1.9 holds in this case too.

3.1.2 $p \neq q$

We write $n = d(q' + p')$. If $d < q$ then the constants of quasi-adjunction we have to consider are $\frac{d-j}{d}$ with $j = 1, \ldots, d-1$, and they are all relative to both P_1 and P_2. If $d = q$ the constants of quasi-adjunction we have to consider are $\frac{q-j}{q}$ with $j = 1, \ldots, q-1$; they are all relative to both P_1 and P_2 save for $\frac{q-1}{q}$ which is only relative to P_1. If $3 \nmid n$ we know that the constant of quasi-adjunction $\frac{1}{3}$ does not contribute to the Alexander polynomial; if $3|n$ the constant of quasi-adjunction $\frac{1}{3}$ might contribute to the Alexander polynomial, but we have to distinguish two cases:

- If $3|d$ then $\frac{1}{3}$ is one of the $\frac{d-j}{d}$: indeed, $\frac{d-j}{d} = \frac{1}{3}$ for $j = \frac{2}{3}d$, and $\frac{2}{3}d \leq d-1$ if and only if $d \geq 3$; since $3|d$, this holds.
- If $3 \nmid d$ then we have to study $\frac{1}{3}$ separately.

Before starting with the computations, we prove two easy lemmas:

Lemma 3.1.18. *Let $a, b \in \mathbb{Z}_{\geq 0}$, then $h_{A^a+B^b}(m) = 0$ for any $m \geq a+b-1$.*

Proof. We can assume, without loss of generality, that $a \geq b$. In order for $x^{c_1}y^{m-c_1-c_2}z^{c_2}$ to be non-zero in $(R/(A^a+B^b))_m$ it must be $c_1 = b-j$ for some $j \in \{1, \dots, b\}$ and $c_2 = j-h$ for some $h \in \{1, \dots, j\}$ (since $x^{b-j}z^j \in B^b$ for $j = 0, \dots, b$); this implies that $m - c_1 - c_2 = m - b + h$. The monomial $x^{b-j}y^{a-(b-j)}$ belongs to A^a, so we need to have $m - b + h < a - (b-j)$ i.e. $m < a + j - h$; the biggest value $j - h$ can take is $b-1$, but by hypothesis we have $m \geq a+b-1$ so the lemma is proved. $\square$

Lemma 3.1.19.

$$h_{A^a}(m) = \begin{cases} \frac{a(a+1)}{2} & \text{if } m \geq a. \\ \frac{(m+1)(m+2)}{2} & \text{otherwise.} \end{cases} \qquad h_{B^b}(m) = \begin{cases} \frac{b(b+1)}{2} & \text{if } m \geq b. \\ \frac{(m+1)(m+2)}{2} & \text{otherwise.} \end{cases}$$

Proof. It is clearly enough to prove the lemma for A^a. If $m < a$ then we need to count all monomials $x^{c_1}y^{c_2}z^{c_1+c_2+c_3}$ with $c_1+c_2+c_3 = m$, and they are $\binom{m+3-1}{2} = \frac{(m+1)(m+2)}{2}$. If $m \geq a$, in order for $x^{c_1}y^{c_2}z^{m-c_1-c_2}$ to be non-zero in $(R/A^a)_m$ it must be $c_1 = a-j$ for some $j \in \{1, \dots, a\}$ and $c_2 = j-h$ for some $h \in \{1, \dots, j\}$; the number of these monomials is $\sum_{j=1}^{a}\sum_{h=1}^{j} 1 = \frac{a(a+1)}{2}$. $\square$

Using these lemmas, we will prove that

Theorem 3.1.20. *The Alexander polynomial of each $\overline{\mathcal{A}}_t$ is trivial.*

As before, we distinguish various cases and show that Theorem 3.1.20 holds in each of them.

$d < q$ and $3 \nmid n$

In this case triple points cannot contribute to the Alexander polynomial with a term $\Phi_3^{e_3}$, but we still need to consider them when studying constants of quasi-adjunction that are less than or equal to $\frac{1}{3}$.

The constants of quasi-adjunction we need to consider are $c_j := \frac{d-j}{d}$ for $j = 1, \dots, d-1$, and they are all relative to both P_1 and P_2; we have $N_n(c_j) = p'j + q'j - 3$. Since $\frac{d-j}{d} = \frac{p-p'j}{p} = \frac{p-1-(p'j-1)}{p}$ and $\frac{d-j}{d} = \frac{q-q'j}{q} = \frac{q-1-(q'j-1)}{q}$ we have

1. If $c_j > \frac{1}{3}$ then $I_{c_j} = A^{p'j-1} \cap B^{q'j-1}$.

2. If $c_j \leq \frac{1}{3}$ then $I_{c_j} = II_t \cap A^{p'j-1} \cap B^{q'j-1}$.

Lemma 3.1.21. *The constants of quasi-adjunction c_j s.t. $c_j > \frac{1}{3}$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. Since $(p'j-1)+(q'j-1)-1 = p'j+q'j-3$, we can use Lemma 3.1.18 to conclude that $h_{A^{q'j-1}+B^{p'j-1}}(q'j+p'j-3) = 0$; from the short exact sequence

$$0 \to R/I_{c_j} \to R/A^{p'j-1} \oplus R/B^{q'j-1} \to R/(A^{p'j-1}+B^{q'j-1}) \to 0$$

we deduce that $h_{I_{c_j}}(p'j+q'j-3) = h_{A^{p'j-1}}(p'j+q'j-3)+h_{B^{q'j-1}}(p'j+q'j-3)$. Now, if $q'j \geq 2$ then $p'j+q'j-3 \geq p'j-1$ and by Lemma 3.1.19 we get $h_{A^{p'j-1}}(p'j+q'j-3) = \frac{p'j(p'j-1)}{2}$; if $j = q' = 1$ then $p'j+q'j-3 = p'-2$ and $p'j-1 = p'-1$, so by Lemma 3.1.19 we get again $h_{A^{p'j-1}}(p'j+q'j-3) = \frac{p'j(p'j-1)}{2}$. Since a similar argument works for the ideal $B^{q'j-1}$ too, we can conclude that

$$h_{I_{c_j}}(p'j+q'j-3) = \frac{p'j(p'j-1)}{2} + \frac{q'j(q'j-1)}{2}.$$

The local ring at P_1 of the scheme Z_{c_j} contains all germs of holomorphic function whose constant of quasi-adjunction is bigger than or equal to c_j; since $\frac{q-j}{q} = \frac{p-1+(p'j-1)}{p}$, it is the vector subspace of R generated by the monomials $x^a y^b$ for $a, b \geq 0$ and $a+b = h$ for all $h \leq p'j-1$, so it has dimension $\frac{p'j(p'j-1)}{2}$. Similarly, the local ring at P_2 of Z_{c_j} has dimension $\frac{q'j(q'j-1)}{2}$. The local ring of Z_{c_j} at the T_i is trivial instead, since the only constant of quasi-adjunction of a triple point is $\frac{1}{3} > c_j$. This implies $l(Z_{c_j}) = \frac{p'j(p'j-1)}{2} + \frac{q'j(q'j-1)}{2} = h_{I_{c_j}}(p'j+q'j-3)$ so we are done. $\square$

Now we need to study the c_j such that $c_j \leq \frac{1}{3}$, which give $I_{c_j} = II_t \cap A^{p'j-1} \cap B^{q'j-1}$.

Remark 3.1.22. In order to have $c_j \leq \frac{1}{3}$ we need $j \geq \frac{2d}{3}$, but since $3 \nmid d$ and j must be an integer we can actually write $j \geq \frac{2d+1}{3}$. This means that $p'j+q'j-3 \geq \frac{2d+1}{3}(p'+q')-3 = \frac{2p+2q+p'+q'}{3}-3 \geq \frac{2(q+2)+2q+p'+q'}{3}-3 = \frac{4q+p'+q'+4}{3}-3$; the right-hand side is bigger than or equal to $q-1$ if and only if $q+p'+q' \geq 2$, which is clearly true in our situation.

Lemma 3.1.23. *The constants of quasi-adjunction c_j s.t. $c_j \leq \frac{1}{3}$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. Using Lemma 3.1.16 we can conclude that $h_{II_t+A^{p'j-1}\cap B^{q'j-1}}(p'j+q'j-3) = 0$, while using Lemma 3.1.18 we can write $h_{A^{p'j-1}\cap B^{q'j-1}}(p'j+q'j-3) = h_{A^{p'j-1}}(p'j+q'j-3)+h_{B^{q'j-1}}(p'j+q'j-3)$. Combining the two things, we get

$$h_{I_{c_j}}(p'j+q'j-3) = h_{II_t}(p'j+q'j-3)+h_{A^{p'j-1}}(p'j+q'j-3)+h_{B^{q'j-1}}(p'j+q'j-3).$$

Remark 3.1.22 together with Proposition 3.1.15 gives

$$h_{I_{c_j}}(p'j+q'j-3) = t + \frac{p'j(p'j-1)}{2} + \frac{q'j(q'j-1)}{2}.$$

Arguing as in Lemma 3.1.21 we can say that the sum of the dimensions of the local rings of Z_{c_j} at the points P_1 and P_2 is $\frac{p'j(p'j-1)}{2} + \frac{q'j(q'j-1)}{2}$; however, since $c_j \leq \frac{1}{3}$ the local ring of Z_{c_j} at the T_i has dimension one. This means that for any $c_j \leq \frac{1}{3}$ and for all t we have $l(Z_{c_j}) = \frac{p'j(p'j-1)}{2} + \frac{q'j(q'j-1)}{2} + t$. Since this value coincides with $l(Z_{c_j})$ we are done. $\square$

Lemmas 3.1.21 and 3.1.23 imply Theorem 3.1.20.

$d < q$ **and** $3|n$

The triple points T_i can contribute to the Alexander polynomial with the constant of quasi-adjunction $\frac{1}{3}$. If $3|d$ the constant of quasi-adjunction $\frac{1}{3}$ is one of the c_j, so we can assume $3 \nmid d$. The computations for the constants of quasi-adjunction c_j are the same as before, so they do not contribute to the $\Delta_{\overline{\mathcal{A}}_t}$ by Lemmas 3.1.21 and 3.1.23; we only need to study $\frac{1}{3}$. If we write $n = 3n'$ we obtain $N_n(\frac{1}{3}) = 2n' - 3$.

In order for $\frac{1}{3}$ to be greater than or equal to a constant of quasi-adjunction $\frac{p-1-j}{p}$ of P_1 (respectively, a constant of quasi-adjunction $\frac{q-1-h}{q}$ of P_2), we need $j \geq \frac{2p-3}{3}$ (respectively, $h \geq \frac{2q-3}{3}$); we call $t_p := \frac{2p-3}{3}$ and $t_q := \frac{2q-3}{3}$. Since the indices j and h have to be integers, and neither t_p nor t_q is an integer, we actually need $j \geq \lceil t_p \rceil$ and $h \geq \lceil t_q \rceil$; this means that $I_{\frac{1}{3}} = II_t \cap A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1}$. We need to compute $l(Z_{\frac{1}{3}}) - h_{I_{\frac{1}{3}}}(2n' - 3)$.

Lemma 3.1.24. *The constant of quasi-adjunction $\frac{1}{3}$ does not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. Since $t_p + t_q = 2n' - 2 \in \mathbb{Z}$ we have $t_p + t_q = \lceil t_p + t_q \rceil$ and $\lceil t_p + t_q \rceil + 1 = \lceil t_p \rceil + \lceil t_q \rceil$; this implies that $2n' - 3 - (\lceil t_p \rceil + \lceil t_q \rceil - 3) = 2n' - \lceil t_p \rceil - \lceil t_q \rceil = 2n' - 1 - \lceil t_p + t_q \rceil = 2n' - 1 - t_p - t_q = 2n' - 1 - (2n' - 2) = 1 \geq 0$, so by Lemma 3.1.18 we conclude that $h_{A^{\lceil t_p \rceil - 1} + B^{\lceil t_q \rceil - 1}}(2n' - 3) = 0$.

We have $2n' - 3 \geq t_p$ if and only if $q \geq 3$, which is true under our hypotheses; since $t_p \notin \mathbb{Z}$ implies that $t_p \geq \lceil t_p \rceil - 1$, we can conclude that $2n' - 3 \geq \lceil t_p \rceil - 1$. With the same argument we can prove that $2n' - 3 \geq \lceil t_q \rceil - 1$. In particular, from the usual short exact sequence

$$0 \to R/A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1} \to R/A^{\lceil t_p \rceil - 1} \oplus R/B^{\lceil t_q \rceil - 1} \to R/(A^{\lceil t_p \rceil - 1} + B^{\lceil t_q \rceil - 1}) \to 0$$

and Lemma 3.1.19 we deduce that

$$h_{A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1}}(2n' - 3) = \frac{\lceil t_p \rceil (\lceil t_p \rceil - 1)}{2} + \frac{\lceil t_q \rceil (\lceil t_q \rceil - 1)}{2}.$$

Now we need to use the short exact sequence

$$0 \to R/I_{\frac{1}{3}} \to R/II_t \oplus R/A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1} \to R/(II_t + A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1}) \to 0.$$

Since $2n' - 3 = t_p + t_q - 1 = \lceil t_p + t_q \rceil - 1 = \lceil t_p \rceil + \lceil t_q \rceil - 2$, using Lemma 3.1.16 we can conclude that $h_{II_t + A^{\lceil t_p \rceil - 1} \cap B^{\lceil t_q \rceil - 1}}(2n' - 3) = 0$.

Now, we have $q-1 \leq \lfloor \frac{n}{2} \rfloor - 1 = n'-1+\lfloor \frac{n'}{2} \rfloor$, and $2n'-3 \geq n'-1+\lfloor \frac{n'}{2} \rfloor$ if and only if $n'-2 \geq \lfloor \frac{n'}{2} \rfloor$. If n' is even this is true if and only if $n' \geq 4$, which holds under our hypotheses ($n'=2$ would give $n=6$, contradiction); if n' is odd then $\lfloor \frac{n'}{2} \rfloor = \frac{n'-1}{2}$, so that inequality holds if and only if $n' \geq 3$ (which is the case under our hypotheses, since $n \geq 7$ implies $n' \geq 3$). In any case we can conclude that $2n'-3 \geq q-1 \geq t$, so using Proposition 3.1.15 and what we found above we obtain

$$h_{I_{\frac{1}{3}}}(2n'-3) = h_{II_t}(2n'-3)+h_{A^{\lceil t_p \rceil -1} \cap B^{\lceil t_q \rceil -1}}(2n'-3) = t+\frac{\lceil t_p \rceil(\lceil t_p \rceil - 1)}{2}+\frac{\lceil t_q \rceil(\lceil t_q \rceil - 1)}{2}.$$

The local ring at P_1 of the scheme $Z_{\frac{1}{3}}$ contains all germs of holomorphic function whose constant of quasi-adjunction is bigger than or equal to $\frac{1}{3}$; by the relation we saw above between $\frac{1}{3}$ and the constants of quasi-adjunction of P_1, this is the vector subspace of R generated by the monomials $x^a y^b$ for $a,b \geq 0$ and $a+b=h$ for all $h \leq \lceil t_p \rceil - 1$, so it has dimension $\frac{\lceil t_p \rceil(\lceil t_p \rceil - 1)}{2}$. Similarly, the local ring at P_2 of $Z_{\frac{1}{3}}$ has dimension $\frac{\lceil t_q \rceil(\lceil t_q \rceil - 1)}{2}$. The local ring of $Z_{\frac{1}{3}}$ at any triple point is simply $\mathbb{C}$, since constants are the only functions whose constant of quasi-adjunction around a triple point is greater than or equal to $\frac{1}{3}$.

As $l(Z_{\frac{1}{3}}) - h_{I_{\frac{1}{3}}}(2n'-3) = 0$ we are done. □

Lemmas 3.1.24, 3.1.21 and 3.1.23 imply Theorem 3.1.20.

$d=q$ and $3 \nmid d$

In this case the triple points cannot contribute to the Alexander polynomial with a term $\Phi_3^{e_3}$, but we still need to take them into account when studying constants of quasi-adjunction which are less than or equal to $\frac{1}{3}$. The constants of quasi-adjunction we need to consider are the $c_j := \frac{q-j}{q}$ for $j=1,\dots,q-1$, so $N_n(c_j) = p'j+j-3$. Since $\frac{q-j}{q} = \frac{p-p'j}{p} = \frac{p-1-(p'j-1)}{p}$ and $\frac{q-j}{q} = \frac{q-1-(j-1)}{q}$ we obtain:

1. If $c_j > \frac{1}{3}$ then $I_{c_j} = A^{p'j-1} \cap B^{j-1}$.
2. If $c_j \leq \frac{1}{3}$ then $I_{c_j} = II_t \cap A^{p'j-1} \cap B^{j-1}$.

Lemma 3.1.25. *The constants of quasi-adjunction c_j s.t. $c_j > \frac{1}{3}$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. First we study c_1. Since $q \geq \lfloor \frac{n}{2} \rfloor \geq 3$ we have $c_1 > \frac{1}{3}$, so $I_{c_1} = A^{p'-1}$ and $l(Z_{c_1}) = \frac{p'(p'-1)}{2}$ (arguing as usual); since $h_{A^{p'-1}}(p'-2) = \frac{p'(p'-1)}{2}$ by Lemma 3.1.19, we can conclude that c_1 does not contribute to the Alexander polynomial.

Assume now that $j \geq 2$. Since $(p'j-1)+(j-1)-1 = p'j+j-3$, we can use Lemma 3.1.18 to conclude that $h_{A^{p'j-1}+B^{j-1}}(p'j+j-3) = 0$; from the short exact sequence

$$0 \to R/I_{c_j} \to R/A^{p'j-1} \oplus R/B^{j-1} \to R/(A^{p'j-1}+B^{j-1}) \to 0$$

we deduce that $h_{I_{c_j}}(p'j+j-3) = h_{A^{p'j-1}}(p'j+j-3) + h_{B^{j-1}}(p'j+j-3)$. Since we are in the case $j \geq 2$, applying Lemma 3.1.19 we obtain

$$h_{I_{c_j}}(p'j+j-3) = \frac{p'j(p'j-1)}{2} + \frac{j(j-1)}{2}.$$

Arguing as in Lemma 3.1.21 we find $l(Z_{c_j}) = \frac{p'j(p'j-1)}{2} + \frac{j(j-1)}{2} = h_{I_{c_j}}(p'j+j-3)$so we are done. □

Now we study the $c_j \geq \frac{1}{3}$, for which $I_{c_j} = II_t \cap A^{p'j-1} \cap B^{j-1}$.

Remark 3.1.26. In order to have $c_j \leq \frac{1}{3}$ we need $j \geq \frac{2q}{3}$, but since $3 \nmid q$ and j must be an integer we can actually write $j \geq \frac{2q+1}{3}$. This means that $p'j+j-3 \geq \frac{2q+1}{3}p' + \frac{2q+1}{3} - 3 = \frac{2p+2q+p'+1}{3} - 3 \geq \frac{2(q+2)+2q+p'+1}{3} - 3 = \frac{4q+p'+5}{3} - 3$; the right-hand side is bigger than or equal to $q-1$ if and only if $q+p' \geq 1$, which is clearly true in our situation.

Lemma 3.1.27. *The constants of quasi-adjunction c_j s.t. $c_j \leq \frac{1}{3}$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$ for all t.*

Proof. Using Lemma 3.1.16 we can conclude that $h_{II_t + A^{p'j-1} \cap B^{j-1}}(p'j+j-3) = 0$, while using Lemma 3.1.18 we can write $h_{A^{p'j-1} \cap B^{j-1}}(p'j+j-3) = h_{A^{p'j-1}}(p'j+j-3) + h_{B^{j-1}}(p'j+j-3)$. Combining the two things, we get

$$h_{I_{c_j}}(p'j+j-3) = h_{II_t}(p'j+j-3) + h_{A^{p'j-1}}(p'j+j-3) + h_{B^{j-1}}(p'j+j-3).$$

Remark 3.1.26 together with Proposition 3.1.15 gives

$$h_{I_{c_j}}(p'j+j-3) = t + \frac{p'j(p'j-1)}{2} + \frac{j(j-1)}{2}.$$

Arguing as in Lemma 3.1.23 we find that this value is exactly $l(Z_{c_j})$, so we are done. □

Lemmas 3.1.25 and 3.1.27 imply Theorem 3.1.20.

$d = q$ and $3|d$

The triple points can contribute to the Alexander polynomial with a factor $\Phi_3^{e_3}$, so we also need to consider the constant of quasi-adjunction $\frac{1}{3}$. By Lemmas 3.1.25 and 3.1.27 we know that the constants of quasi-adjunction $c_j = \frac{q-j}{q}$ for $j = 1, \ldots, q-1$ do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$. If $3|q$ the constant of quasi-adjunction $\frac{1}{3}$ is one of the c_j, so we have nothing to do; if $3 \nmid q$ then we do have to study the constant of quasi-adjunction $\frac{1}{3}$, but the computations we need to do are the same we did in Lemma 3.1.24.

In any case Theorem 3.1.20 is still true.

3.2 $s = 1$ without common line

Now we consider arrangements in which no line passes through both P_1 and P_2; we assume that P_1 has order p and P_2 has order q, and we still denote by ℓ the free line. The argument we use is the same as before: the ED-equivalence classes of such arrangements are finite, as we have one for each number of triple points these arrangements can have (i.e. classes $\overline{\mathcal{A}}_t$ for $t = 0\ldots, m := min\{p, q\}$), and the Alexander polynomial of each $\overline{\mathcal{A}}_t$ can be determined using formula (2.1.15). The results we obtain are also the same as before:

Theorem 3.2.1. *The Alexander polynomial of each $\overline{\mathcal{A}}_t$ is trivial.*

From this we conclude that the Alexander polynomial of any arrangement of this type is trivial using Corollary 3.1.3; but first, we show that these arrangements are indeed non-symmetric:

Lemma 3.2.2. *An arrangement $\overline{\mathcal{A}}$ of this type does not support weak (α, β)-multinets.*

Proof. Any line r through P_1 meets the lines $l_1, \ldots, l_q$ through P_2 in at least $q - 1$ double points, which cannot belong to the base locus of a weak multinet; this means r and the l_i must belong to the same class, say $\overline{\mathcal{A}}_1$. Since this argument holds for any line through P_1, we deduce that $\overline{\mathcal{A}} - \overline{\mathcal{A}}_1 = \{\ell\}$ which is impossible because weak multinets must have at least three classes. ☐

As we did before we call $d_p := (p, n)$, $d_q = (q, n)$ and write $p = p'd_p$, $q = q'd_q$, $n = d_p n_p = d_q n_q$; we obtain that if $d_p > 1$ (resp. $d_q > 1$) the constants of quasi-adjunction we have to consider are:

$$\text{Relative to } P_1\text{: } cp_j := \frac{d_p - j}{d_p} \qquad \text{with } j = \begin{cases} 1, \ldots, d_p - 1 & \text{if } d_p \neq p. \\ 2, \ldots, p - 1 & \text{if } d_p = p. \end{cases} \tag{3.2.1}$$

$$\text{Relative to } P_2\text{: } cq_h := \frac{d_q - h}{d_q} \qquad \text{with } h = \begin{cases} 1, \ldots, d_q - 1 & \text{if } d_q \neq q. \\ 2, \ldots, q - 1 & \text{if } d_q = q. \end{cases} \tag{3.2.2}$$

$$\text{Relative to triple points: } \frac{1}{3} \text{ (if and only if } 3|n).$$

This situation however is more complicated than before: since there is no common line, we have $p + q = n - 1$, which means in particular that the equality $d_p = d_q$ no longer holds; in fact, we actually have $(d_p, d_q) = 1$. This means that no constant of quasi-adjunction cp_j will ever coincide with a constant of quasi-adjunction cq_h.

3.2.1 Only one of d_p and d_q is greater than 1

Without loss of generality we can assume $d_p > 1$ and $d_q = 1$. For any constant of quasi-adjunction c we study the difference $l(Z_c) - h_{I_c}(N_n(c))$ where $N_n(c) = n - 3 - nc$.

We have equalities

$$cp_j = \frac{d_p - j}{d_p} = \frac{p-1-(p'j-1)}{p} = \frac{q-1-(j\frac{q}{d_p}-1)}{q}.$$

Moreover, in order to have $cp_j \leq \frac{1}{3}$ (resp. $cq_h \leq \frac{1}{3}$) we need $j \geq t_p := \frac{2p-3}{3}$ (resp. $h \geq t_q := \frac{2q-3}{3}$). This implies the following:

1. For $cp_j > \frac{1}{3}$ and $j\frac{q}{d_p} < 2$ we have $I_{cp_j} = A^{p'j-1}$.
2. For $cp_j > \frac{1}{3}$ and $j\frac{q}{d_p} \geq 2$ we have $I_{cp_j} = A^{p'j-1} \cap B^{\lfloor j\frac{q}{d_p}\rfloor -1}$.
3. For $cp_j \leq \frac{1}{3}$ and $j\frac{q}{d_p} \geq 2$ we have $I_{cp_j} = A^{p'j-1} \cap B^{\lfloor j\frac{q}{d_p}\rfloor -1} \cap II_t$.
4. (To be considered if and only if $3|n$ and $3 \nmid d_p$) $I_{\frac{1}{3}} = A^{\lceil t_p\rceil -1} \cap B^{\lceil t_q\rceil -1} \cap II_t$.

Remark 3.2.3. It cannot happen that $cp_j \leq \frac{1}{3}$ and $j\frac{q}{d_p} < 2$: indeed, $cp_j \leq \frac{1}{3}$ if and only if $j \geq \frac{2}{3}d_p$ which implies $j\frac{q}{d_p} \geq \frac{2q}{3} \geq 2$ because $q \geq 3$.

Lemma 3.2.4. *None of the constants of quasi-adjunction satisfying one of 1.-3. contributes to* $\Delta_{\overline{\mathcal{A}}_t}$.

Proof. We proceed case-by-case:

Case 1. $N_n(cp_j) = n_pj - 3$ so $N_n(cp_j) \geq p'j-1$ if and only if $j(n_p - p') \geq 2$. If $n_p - p' \geq 2$ this is true. $n_p - p' = 0$ cannot happen, as it would give $n = p$. If $n_p - p' = 1$ then $N_n(cp_j) = p'j + j - 3$: $j = 1$ gives $N_n(cp_j) = p' - 2$ and $p'j - 1 = p' - 1$, while $j \geq 2$ guarantees $N_n(cp_j) \geq p'j - 1$. In any case we can conclude by Lemma 3.1.19 that $h_{I_{cp_j}}(N_n(cp_j)) = \frac{p'j(p'j-1)}{2}$. $l(Z_{cp_j})$ can be computed by the same reasoning we used so far, and turns out to be $\frac{p'j(p'j-1)}{2}$. Hence these constants of quasi-adjunction do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$.

Case 2. $N_n(cp_j) \geq (p'j-1) + (\lfloor j\frac{q}{d_p}\rfloor - 1) - 1$ if and only if $j(n_p - p') \geq \lfloor j\frac{q}{d_p}\rfloor$, and it is enough to prove that $j(n_p - p') \geq j\frac{q}{d_p}$ i.e. $n_p - p' - \frac{q}{d_p} \geq 0$; but this is true, since that difference is $\frac{1}{d_p}$. By Lemmas 3.1.18 and 3.1.7 we conclude that $h_{I_{cp_j}}(N_n(cp_j)) = h_{A^{p'j-1}}(N_n(cp_j)) + h_{B^{\lfloor j\frac{q}{d_p}\rfloor -1}}(N_n(cp_j))$.

Proceeding as in Case 1. we can show that $h_{A^{p'j-1}}(N_n(cp_j)) = \frac{p'j(p'j-1)}{2}$.

$N_n(cp_j) \geq \lfloor j\frac{q}{d_p}\rfloor - 1$ if and only if $n_pj - \lfloor j\frac{q}{d_p}\rfloor \geq 2$, and it is enough to prove that $n_pj - j\frac{q}{d_p} \geq 2$ i.e $j\frac{p+1}{d_p} \geq 2$. If $d_p = p$ then $j \geq 2$ and $\frac{p+1}{d_p} = \frac{p+1}{p} > 1$, so that holds; if $d_p < p$ we must have $d_p \leq \frac{p}{2}$ so $\frac{p+1}{d_p} \geq \frac{2}{p}(p+1) \geq 2$ and we are done. In any case we can conclude, by Lemma 3.1.19, that $h_{B^{\lfloor j\frac{q}{d_p}\rfloor -1}}(N_n(cp_j)) = \frac{\lfloor j\frac{q}{d_p}\rfloor(\lfloor j\frac{q}{d_p}\rfloor-1)}{2}$.

We have obtained

$$h_{I_{cp_j}}(N_n(cp_j)) = \frac{p'j(p'j-1)}{2} + \frac{\lfloor j\frac{q}{d_p}\rfloor(\lfloor j\frac{q}{d_p}\rfloor - 1)}{2}$$

and since this value coincides with $l(Z_{cp_j})$ we can conclude that these constants of quasi-adjunction do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$.

Case 3. As before, we have $N_n(cp_j) \geq (p'j-1) + (\lfloor j\frac{q}{d_p}\rfloor - 1) - 1$, so by Lemmas 3.1.7, 3.1.16 and 3.1.18 we conclude that $h_{I_{cp_j}}(N_n(cp_j)) = h_{A^{p'j-1}}(N_n(cp_j)) + h_{B^{\lfloor j\frac{q}{d_p}\rfloor -1}}(N_n(cp_j)) + h_{II_t}(N_n(cp_j))$.

Proceeding as in Case 1. we can show that $h_{A^{p'j-1}}(N_n(cp_j)) = \frac{p'j(p'j-1)}{2}$, while proceeding as in Case 2. we can show that $h_{B^{\lfloor j\frac{q}{d_p}\rfloor -1}}(N_n(cp_j)) = \frac{\lfloor j\frac{q}{d_p}\rfloor(\lfloor j\frac{q}{d_p}\rfloor-1)}{2}$.

$N_n(cp_j) \geq t-1$ if and only if $n_pj - t \geq 2$, and it is enough to prove that $n_pj - m \geq 2$; observe that $cp_j \leq \frac{1}{3}$ implies $j \geq \frac{2}{3}d_p$, so $n_pj - m \geq \frac{2}{3}n - m = \frac{2}{3}(p+q+1) - m$. Since the right-hand side of the last inequality is symmetric in p and q, we can assume without loss of generality that $p < q$ i.e. $m = p$; we obtain $n_pj - m \geq \frac{2}{3}(q+1) - \frac{1}{3}p > \frac{q}{3} + \frac{2}{3}$. Since $q \geq 3$ and $n_pj - m$ is an integer we are done. By Proposition 3.1.15, we have $h_{II_t}(N_n(cp_j)) = t$.

We have obtained

$$h_{I_{cp_j}}(N_n(cp_j)) = \frac{p'j(p'j-1)}{2} + \frac{\lfloor j\frac{q}{d_p}\rfloor(\lfloor j\frac{q}{d_p}\rfloor - 1)}{2} + t$$

and since this value coincides with $l(Z_{cp_j})$ we can conclude that these constants of quasi-adjunction do not contribute to $\Delta_{\overline{\mathcal{A}}_t}$. □

Now, if $3 \nmid n$ the cp_j are the only constants of quasi-adjunction we have to consider, so Lemma 3.2.4 implies immediately Theorem 3.2.1. If $3|n$ we have to consider the constant of quasi-adjunction $\frac{1}{3}$ too, but we need to distinguish two cases: if $3|d_p$ then $\frac{1}{3}$ is one of the cp_j, so we actually have already taken care of it, while if $3 \nmid d_p$ we do have to study the constant of quasi-adjunction $\frac{1}{3}$.

Lemma 3.2.5. *The constant of quasi-adjunction $\frac{1}{3}$ does not contribute to $\Delta_{\overline{\mathcal{A}}_t}$.*

Proof. If we write $n = 3n'$ we find $N_n(\frac{1}{3}) = 2n' - 3$. In order to prove that $2n' - 3 \geq (\lceil t_p\rceil - 1) + (\lceil t_q\rceil - 1) - 1$ it is enough to prove that $2n' \geq t_p + t_q + 2$; the right-hand side is $\frac{2p-3}{3} + \frac{2q-3}{3} + 2 = \frac{2(n-1)}{3} = 2n' - \frac{2}{3}$, so it is indeed smaller than $2n'$. By Lemmas 3.1.7, 3.1.16 and 3.1.18 we deduce that $h_{I_{\frac{1}{3}}}(2n'-3) = h_{A^{\lceil t_p\rceil-1}}(2n'-3) + h_{B^{\lceil t_q\rceil-1}}(2n'-3) + h_{II_t}(2n'-3)$.

Now, $2n' - 3 \geq t_p$ if and only if $p \geq 3$, which is true under our hypotheses; as $t_p \geq \lceil t_p\rceil - 1$, this implies $2n' - 3 \geq \lceil t_p\rceil - 1$ (and the same goes for t_q). $2n' - 3 \geq m - 1$ if and only if $\frac{2p+2q+2}{3} - 2 - m \geq 0$, and we can assume without loss of generality that $m = q$; in this case, the left-hand side is $\frac{2p-q-4}{3} \geq \frac{p-4}{3}$. $p = 3$ cannot happen, as it would force $q = 3$ and $n = 7$, so that value is indeed greater than or equal to zero.

By Lemma 3.1.19 and Proposition 3.1.15, we find

$$h_{I_{\frac{1}{3}}}(2n'-3) = \frac{\lceil t_p\rceil(\lceil t_p\rceil - 1)}{2} + \frac{\lceil t_q\rceil(\lceil t_q\rceil - 1)}{2} + t$$

and since this value coincides with $l(Z_{\frac{1}{3}})$ we can conclude that the constant of quasi-adjunction $\frac{1}{3}$ does not contribute to $\Delta_{\overline{\mathcal{A}}_t}$. □

Hence Theorem 3.2.1 holds when $3|n$ too.

3.2.2 $d_p, d_q > 1$

We certainly have to consider the constants of quasi-adjunction cp_j in (3.2.1) and cq_h in (3.2.2), and only these if $3 \nmid n$. If $3|n$ we have to consider the constant of quasi-adjunction $\frac{1}{3}$ too; however, if $3|d_p$ (resp. $3|d_q$) then $\frac{1}{3}$ is one of the cp_j (resp. cq_h), so we have nothing to do, while if $3 \nmid d_p, d_q$ we do have to study the constant of quasi-adjunction $\frac{1}{3}$. As before, for any constant of quasi-adjunction c we study the difference $l(Z_c) - h_{I_c}(N_n(c))$ where $N_n(c) = n - 3 - nc$.

We have equalities

$$cp_j = \frac{d_p - j}{d_p} = \frac{p - 1 - (p'j - 1)}{p} = \frac{q - 1 - (j\frac{q}{d_p} - 1)}{q}.$$
$$cp_h = \frac{d_q - h}{d_q} = \frac{q - 1 - (q'h - 1)}{q} = \frac{p - 1 - (h\frac{p}{d_q} - 1)}{p}.$$

and in order to have $cp_j \leq \frac{1}{3}$ (resp. $cq_h \leq \frac{1}{3}$) we need $j \geq t_p := \frac{2p-3}{3}$ (resp. $h \geq t_q := \frac{2q-3}{3}$). This implies the following:

1. For $cp_j > \frac{1}{3}$ and $j\frac{q}{d_p} < 2$ we have $I_{cp_j} = A^{p'j-1}$.
2. For $cp_j > \frac{1}{3}$ and $j\frac{q}{d_p} \geq 2$ we have $I_{cp_j} = A^{p'j-1} \cap B^{\lfloor j\frac{q}{d_p}\rfloor - 1}$.
3. For $cp_j \leq \frac{1}{3}$ and $j\frac{q}{d_p} \geq 2$ we have $I_{cp_j} = A^{p'j-1} \cap B^{\lfloor j\frac{q}{d_p}\rfloor - 1} \cap II_t$.
4. (To be considered if and only if $3|n$ and $3 \nmid d_p, d_q$) $I_{\frac{1}{3}} = A^{\lceil t_p\rceil - 1} \cap B^{\lceil t_q\rceil - 1} \cap II_t$.
5. For $cq_h > \frac{1}{3}$ and $h\frac{p}{d_q} < 2$ we have $I_{cq_h} = B^{q'h-1}$.
6. For $cq_h > \frac{1}{3}$ and $h\frac{p}{d_q} \geq 2$ we have $I_{cq_h} = B^{q'h-1} \cap A^{\lfloor h\frac{p}{d_q}\rfloor - 1}$.
7. For $cq_h \leq \frac{1}{3}$ and $h\frac{p}{d_q} \geq 2$ we have $I_{cq_h} = B^{q'h-1} \cap A^{\lfloor h\frac{p}{d_q}\rfloor - 1} \cap II_t$.

Again, it cannot happen that $cp_j \leq \frac{1}{3}$ and $j\frac{q}{d_p} < 2$: indeed, $cp_j \leq \frac{1}{3}$ if and only if $j \geq \frac{2}{3}d_p$ which implies $j\frac{q}{d_p} \geq \frac{2q}{3} \geq 2$ because $q \geq 3$. Likewise, we cannot have $cq_h \leq \frac{1}{3}$ and $h\frac{p}{d_q} < 2$.

Now we should do computations analogous to those we did to prove Lemmas 3.2.4 and 3.2.5. In the cases 1. – 4. the computations are actually exactly the same of the previous Lemmas; in cases 5. – 7., we just need to switch the roles of p (resp. A) and q (resp. B). The result is again that none of the constants of quasi-adjunction listed above contributes to $\Delta_{\overline{\mathcal{A}}_t}$, which means Theorem 3.2.1 holds under these hypotheses too.

3.3 $s = 0$

We consider arrangements $\overline{\mathcal{A}}$ with no free lines in which any line passes through only one of P_1 or P_2. Without loss of generality, we may assume that $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$ with $p := ord(P_1) \geq q := ord(P_2)$; we deduce that $n - 3 \geq p \geq q \geq 3$ and $p + q = n$. Arrangements of this type cannot support weak multinets, because double points must be mono-coloured, and this would force all the lines of $\overline{\mathcal{A}}$ to belong to the same class. We will prove that

Theorem 3.3.1. *The Alexander polynomial of arrangements of this type is trivial.*

While this could be done via direct computation using formula (2.1.15), like we did in Sections 3.1 and 3.2 of this chapter, we will resort to a more geometric argument, which can be summarised as follows:

(i) We call $g := y^n + z^n$ and we associate to $\overline{\mathcal{A}} = V(f)$ the threefold $T := V(g - f) \subset \mathbb{P}^4$; we show that from the latter we obtain a fibration $\psi : T' \to \mathbb{P}^1$ with a surface $S \subset \mathbb{P}^3$ as generic fibre. We explicitly compute the geometric monodromy of ψ around a pole of the fibration, which we denote by ϕ; the action of the algebraic monodromy T^ϕ on $H^\bullet(S)$ clearly extends to $H^\bullet(\mathbb{P}^3 \setminus S)$.

(ii) We prove the existence of a surjective Gysin morphism $H^2_{DR}(S) \twoheadrightarrow H^4_{DR}(T)$, which yields by Theorem 1.2.18 a surjective Gysin morphism $\gamma : H^2(S) \twoheadrightarrow H^4(T)$ too; using the global invariant cycle theorem we prove that $\gamma(H^2(S)) = \gamma(H^2(S)^{T^\phi})$, and then we show that everything restricts to the primitive cohomology groups: this gives

$$H^2(S)^{T^\phi}_{\text{prim}} \twoheadrightarrow H^4(T)_{\text{prim}}. \tag{3.3.1}$$

(iii) As $H^2(S)_{\text{prim}}$ is isomorphic to $H^3(\mathbb{P}^3 \setminus S) \simeq \bigoplus_{i=1}^3 Gr^i_P H^3(\mathbb{P}^3 \setminus S)$, where P denotes the polar filtration (recall (1.4.4)), and the map (1.4.6) is compatible with T^ϕ, we can bound the dimension of $H^2(S)^{T^\phi}_{\text{prim}}$ and, in turn, of $H^4(T)_{\text{prim}}$; moreover, the eigenspaces of $H^1(F_g, \mathbb{C})$ under the action of the algebraic monodromy T_g can be explicitly computed. Since $H^4(T, \mathbb{C})_{\text{prim}} \simeq H^3(F_{g-f}, \mathbb{C})^{T_{g-f}}$, we can use Theorem 2.1.7 to deduce information on the eigenspaces of $H^1(F_f, \mathbb{C})$ under the algebraic monodromy T_f (i.e. on the Alexander polynomial of $\overline{\mathcal{A}}$). This allows us to conclude.

3.3.1 Part (i) - The fibred threefold T' and its monodromy

A polynomial $f \in \mathbb{C}[x_0, x_1, x_2]$ describing an arrangement of this type can be written as

$$f = \prod_{i=1}^{p}(x_0 - \lambda_i x_1) \prod_{i=1}^{q}(x_0 - \mu_i x_2) \qquad \lambda_i \neq 0, \mu_i \neq 0.$$

Any hyperplane $V(\alpha x_1 - \beta x_0) \subset \mathbb{P}^4$ cuts a surface from T; if we assume $\alpha \neq 0$ and call $s := \beta/\alpha$ then this surface, which we denote by S_s, is a hypersurface of $\mathbb{P}^3$ defined by the polynomial

$$f_s := y^n + z^n - h(s)x_0^p \prod_{i=1}^{q}(x_0 - \mu_i x_2) \qquad \text{where } h(s) := \prod_{i=1}^{p}(1 - \lambda_i s).$$

If $\alpha = 0$ we denote the corresponding surface by S_∞, whose defining polynomial as hypersurface of $\mathbb{P}^3$ is

$$f_\infty := y^n + z^n - (-1)^n \left(\prod_{i=1}^{p} \lambda_i \prod_{i=1}^{q} \mu_i\right) x_1^p x_2^q.$$

If we call B the blow-up of $\mathbb{P}^2$ at P_1 and set $T' := T \times_{\mathbb{P}^2} B$, we can write

$$\begin{aligned}
B &= \{(t_0 : t_1 : t_2) \times (\alpha : \beta) \text{ s.t. } t_0\beta = t_1\alpha\} \subset \mathbb{P}^2 \times \mathbb{P}^1,\\
T' &= \{(y : z : x_0 : x_1 : x_2) \times (t_0 : t_1 : t_2) \times (\alpha : \beta) \text{ s.t. } (x_0 : x_1 : x_2) = (t_0 : t_1 : t_2),\\
&\qquad t_0\beta = t_1\alpha, y^n + z^n - f(x_0, x_1, x_2) = 0\} =\\
&\quad = \{(y : z : x_0 : x_1 : x_2) \times (x_0 : x_1 : x_2) \times (\alpha : \beta) \text{ s.t.}\\
&\qquad x_0\beta = x_1\alpha, y^n + z^n - f(x_0, x_1, x_2) = 0\} \simeq\\
&\quad \simeq \{(y : z : x_0 : x_1 : x_2) \times (\alpha : \beta) \text{ s.t. } x_0\beta = x_1\alpha, y^n + z^n - f(x_0, x_1, x_2) = 0\},
\end{aligned}$$

and we can write the following diagram

$$\begin{tikzcd}
T \arrow[r, dashed, "pr"] & \mathbb{P}^2 \\
T' \arrow[u] \arrow[r] \arrow[dr, "\psi"'] & B \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
& \mathbb{P}^1
\end{tikzcd}$$

where pr is the rational map given by $(y : z : x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2)$, π_i is the projection from B onto $\mathbb{P}^i$, ψ is given by $(y : z : x_0 : x_1 : x_2) \times (\alpha : \beta) \mapsto (\alpha : \beta)$ and the maps from T' are the projections. $\psi : T' \to \mathbb{P}^1$ is the fibration we want: we have in fact

$$\psi^{-1}(1 : s) = \{(y : z : x_0 : sx_0 : x_2) \times (1 : s) | y^n + z^n - h(s)x_0^p \prod_{i=1}^{q}(x_0 - \mu_i x_2)\} \simeq S_s.$$

$$\psi^{-1}(0 : 1) = \{(y : z : 0 : x_1 : x_2) \times (0 : 1) | y^n + z^n - (-1)^n \left(\prod_{i=1}^{p} \lambda_i \prod_{i=1}^{q} \mu_i\right) x_1^p x_2^q\} \simeq S_\infty.$$

Now we examine the singular loci of T and of the various S_s; this discussion will come into play in Part (ii). We have

$$\begin{aligned} T_{sing} &= \{(0:0:a:b:c) | (a:b:c) \text{ is a multiple point of } \overline{\mathcal{A}}\}. \\ S_{s\,sing} &= \begin{cases} \{(0:0:0:0:1)\} & \text{if } h(s) \neq 0. \\ L_s := \{(0:0:a:as:b)\} & \text{if } h(s) = 0. \end{cases} \\ S_{\infty\,sing} &= \{(0:0:0:0:1), (0:0:0:1:0)\}. \end{aligned}$$

In particular the point $(0:0:0:0:1)$ is a singular point of both T and any surface S cut out from it by hyperplanes $V(\alpha x_0 - \beta x_1)$.

$\overline{\mathcal{A}}$ has nodes at the points $(1 : \frac{1}{\lambda_j} : \frac{1}{\mu_h})$. If we fix $j_0 \in [1,p]$, $h_0 \in [1,q]$ and perform the change of coordinates $x_1 \mapsto x_1 + \frac{1}{\lambda_{j_0}} x_0$, $x_2 \mapsto x_2 + \frac{1}{\mu_{h_0}} x_0$ we can rewrite f as

$$x_1 x_2 \lambda_{j_0} \mu_{h_0} \prod_{j \neq j_0} \left[\left(1 - \frac{\lambda_j}{\lambda_{j_0}}\right) x_0 - x_1 \right] \prod_{h \neq h_0} \left[\left(1 - \frac{\mu_h}{\mu_{h_0}}\right) x_0 - x_2 \right].$$

If we restrict to the local chart $x_0 \neq 0$ and introduce affine coordinates $v := \lambda_{j_0} x_1 \prod_{j \neq j_0} [(1 - \frac{\lambda_j}{\lambda_{j_0}}) - x_1]$ and $w := \mu_{h_0} x_2 \prod_{h \neq h_0} [(1 - \frac{\mu_h}{\mu_{h_0}}) - x_2]$, we can write the local equation of $\overline{\mathcal{A}}$ around $(1 : \frac{1}{\lambda_j} : \frac{1}{\mu_h})$ as vw. This means that the singularities of T at the points $P_{j,h} := (0:0:1: \frac{1}{\lambda_j} : \frac{1}{\mu_h})$ are topologically equivalent to $y^n + z^n - v^2 - w^2 = 0$.

$\overline{\mathcal{A}}$ has a point of order p at $P_1 = (0:0:1)$. If we restrict to the chart $\{x_2 \neq 0\}$ and change coordinates by $v := x_0 \sqrt[p]{\prod_{h=1}^q (x_0 - \mu_h)}$, $w := x_1 \sqrt[p]{\prod_{h=1}^q (x_0 - \mu_h)}$ we can write the local equation of $\overline{\mathcal{A}}$ around P_1 as $\prod_{j=1}^p (v - \lambda_j w)$. This means that the singularity of T at the point $P_p := (0:0:0:0:1)$ is topologically equivalent to $y^n + v^n - v^p - w^p = 0$. In a similar fashion, we can see that the singularity of T at the point $P_q := (0:0:0:1:0)$ is topologically equivalent to $y^n + v^n - v^q - w^q = 0$.

Now we deal with S_∞. If we restrict to the affine chart $\{x_2 \neq 0\}$ and change coordinates by introducing $v := \sqrt[p]{(-1)^n (\prod_{i=1}^p \lambda_i)(\prod_{i=1}^q \mu_i)} x_1$ we see that the point $(0:0:0:0:1)$ has local equation $y^n + z^n - v^p = 0$, while if we restrict to the affine chart $\{x_1 \neq 0\}$ and change coordinates by introducing $v := \sqrt[q]{(-1)^n (\prod_{i=1}^p \lambda_i)(\prod_{i=1}^q \mu_i)} x_2$ we see that the point $(0:0:0:1:0)$ has local equation $y^n + z^n - v^q = 0$.

As for the S_s, assume first that $h(s) \neq 0$. In the affine chart $\{x_2 \neq 0\}$ the surface S_s is given by $y^n + z^n - h(s) x_0^p \prod_{i=1}^q (x_0 - \mu_i) = 0$; since for $x_0 = 0$ we have $\prod_{i=1}^q (x_0 - \mu_i) \neq 0$, the coordinate change $v := x_0 \sqrt[p]{h(s) \prod_{i=1}^q (x_0 - \mu_i)}$ is holomorphic, and turns the previous equation into $y^n + z^n - v^p = 0$.

The surfaces S_s with $h(s) = 0$ are given by $y^n + z^n = 0$, so they consist of n planes containing the line L_s (so they are not even normal).

Assume s_1 and s_2 are not roots of $h(s)$, then we can find a diffeomorphism $S_{s_1} \to S_{s_2}$. Pick in fact $(y : z : x_0 : s_1 x_0 : x_2) \in S_{s_1}$, which satisfies $y^n + z^n - h(s_1) x_0^p \prod_{i=1}^q (x_0 - \mu_i x_2) = 0$: we can find $(\alpha y : \beta z : x_0 : s_2 x_0 : x_2) \in S_{s_2}$ for simple values of α and β. Namely, in order to have $(\alpha y : \beta z : x_0 s_2 x_0 : x_2) \in S_{s_2}$ the equation $\alpha^n y^n + \beta^n z^n - h(s_2) x_0^p \prod_{i=1}^q (x_0 - \mu_i x_2) = 0$ must be satisfied; as $x_0^p \prod_{i=1}^q (x_0 - \mu_i x_2) = \frac{y^n + z^n}{h(s_1)}$, we need to find α and β satisfying

$$\alpha^n y^n + \beta^n z^n - h(s_2)\frac{y^n + z^n}{h(s_1)} = 0 \Longleftrightarrow y^n \left(\alpha^n - \frac{h(s_2)}{h(s_1)}\right) = z^n \left(\beta^n - \frac{h(s_2)}{h(s_1)}\right)$$

and this gives $\alpha^n = \beta^n = \frac{h(s_2)}{h(s_1)} =: \gamma$. The inverse diffeomorphism $S_{s_2} \to S_{s_1}$ is clearly given by $(y : z : x_0 : s_2x_0, x_2) \mapsto (\alpha^{-1}y : \beta^{-1}z : x_0 : s_1x_0 : x_2)$.

If we call $\Delta := \{(0 : 1)\} \cup \{(1 : s)|h(s) = 0\}$ we obtain a locally trivial fibration $\psi' : T' \setminus \psi^{-1}(\Delta) \to \mathbb{P}^1 \setminus \Delta$, with the S_s with $h(s) \neq 0$ as generic fibres. We now compute the monodromy of ψ around one of the special fibres, i.e. the S_s with $s \in \Delta$:

Lemma 3.3.2. *If S_s is any generic fibre of ψ, the geometric monodromy around a special fibre of ψ is given by*

$$S_s \to S_s \text{ s.t. } (y : z : x_0 : sx_0 : x_2) \mapsto (\eta_n y : \eta_n z : x_0 : sx_0 : x_2) \tag{3.3.2}$$

where η_n is an n-th primitive root of unity.

Proof. Assume the special fibre we are considering is $S_{\frac{1}{\lambda_1}}$. Consider a loop $s(t) = \frac{1}{\lambda_1} + re^{2\pi it}$ around $\frac{1}{\lambda_1}$: by the above discussion, the diffeomorphism between $S_{s(0)}$ and $S_{s(t)}$ is governed by

$$\gamma_t := \frac{h(s(t))}{h(s(0))} = e^{2\pi it} \prod_{i=2}^{p} \frac{\lambda_1 - re^{2\pi it}\lambda_1\lambda_i - \lambda_i}{\lambda_1 - r\lambda_1\lambda_i - \lambda_i}$$

We can choose branch cuts for the n-th root function in such a way that, for r small enough, the loop $s(t)$ remains in a zone of the complex plane in which the n-th root is a single-valued function. The only indeterminacy lies then in the term $e^{2\pi it}$; since we look for automorphisms $\phi_t : S_{s(0)} \to S_{s(t)}$ giving the identity for $t = 0$, we deduce that the monodromy action ϕ on $S_{s(0)}$ is given by $y \mapsto \eta_n y$, $z \mapsto \eta_n z$, and this clearly holds for any S_s with $h(s) \neq 0$. $\square$

3.3.2 Part (ii) - The Gysin morphism

We can rephrase Corollary 1.2.22 in the following way:

Theorem 3.3.3. *Assume X is a quasi-projective separated scheme of finite type over $\mathbb{C}$ and Y is a hyperplane section of X satisfying the following hypotheses:*

(I) There exists an augmented n-cubical hyperresolution $X_\square \to X$ such that $Y_\square := X_\square \times_X Y$ is an n-cubical hyperresolution of Y.

(II) For any α, there exists a closed immersion $Y_\alpha \hookrightarrow X_\alpha$ of codimension 1.

Then there exists a map $H^k_{DR}(Y) \to H^{k+2}_{DR}(X)$ that is an isomorphism for $k > \dim(Y)$ and a surjection for $k = \dim(Y)$.

We want to apply this corollary our situation, so the first step is to find a cubical hyperresolution of T and to check that its section by the hyperplane H defining S is a cubical hyperresolution of S.

Property (I) - Resolution of T and S

The singular points of T not belonging to S are the $P_{j,h}$ and P_q, whose local equations are $y^n + z^n - v^k - w^k$ with $k = 2, q$ respectively; P_p is a singular point of both T and S, with local equation $y^n + z^n - v^p - w^p$. For this reason we will show how to resolve the singularity at the origin of the affine threefold $T := V(y^n + z^n - v^k - w^k) \subset \mathbb{C}^4$ and of its hyperplane section $S := T \cap V(w) \simeq V(y^n + z^n - v^k) \subset \mathbb{C}^3$; recall that by our hypotheses on the arrangements we have $k \leq n - 3$. The blow-up of $\mathbb{C}^4$ at the origin is by definition

$$X_1 := Bl_{\underline{0}}\mathbb{C}^4 = \left\{ (y,z,v,w) \times [a:b:c:d] \middle| \begin{array}{c} yb = za, yc = va, yd = wa \\ zc = vb, zd = wb, vd = wc \end{array} \right\} \subset \mathbb{C}^4 \times \mathbb{P}^3.$$

X_1 can be covered by the four affine charts $X_1^a := X_1 \cap (\mathbb{C}^4 \times D_+(a))$, $X_1^b := X_1 \cap (\mathbb{C}^4 \times D_+(b))$, $X_1^c := X_1 \cap (\mathbb{C}^4 \times D_+(c))$ and $X_1^d := X_1 \cap (\mathbb{C}^4 \times D_+(d))$, all of which are isomorphic to $\mathbb{C}^4$. The blow-up map $\pi : X_1 \to \mathbb{C}^4$ can be easily read when restricted to these affine charts; we have in fact:

$$\begin{array}{lccc}
\pi_1^a : X_1^a \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y,p,q,r) \mapsto (y, py, qy, ry) & \text{with} & p := \frac{b}{a}, \quad q := \frac{c}{a}, \quad r := \frac{d}{a}. \\
\pi_1^b : X_1^b \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (s,z,t,u) \mapsto (sz, z, tz, uz) & \text{with} & s := \frac{a}{b}, \quad t := \frac{c}{b}, \quad u := \frac{d}{b}. \\
\pi_1^c : X_1^c \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (i,j,v,l) \mapsto (iv, jv, v, lv) & \text{with} & i := \frac{a}{c}, \quad j := \frac{b}{c}, \quad l := \frac{d}{c}. \\
\pi_1^d : X_1^d \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (m,k,o,w) \mapsto (mw, kw, ow, w) & \text{with} & m := \frac{a}{d}, \quad k := \frac{b}{d}, \quad o := \frac{c}{d}.
\end{array}$$

We will denote the intersections of the strict transforms T_1 and S_1 (and of the exceptional divisors E_1 and F_1) with the various charts by the appropriate apexes. We obtain:

(a) $T_1^a = V(y^{n-k} + y^{n-k}p^n - q^k - r^k)$, which is singular along $L^a = \{(0,p,0,0)\}$, and $E_1^a = \{(0,p,q,r) | q^k + r^k = 0\} = \{(0,p,\eta r, r) | \eta^k = -1\}$ i.e. k planes containing L^a; $S_1^a = \{(y,p,q,0) | y^{n-k} + y^{n-k}p^n - q^k = 0\} = T_1^a \cap V(r)$, which is singular along L^a too, and $F_1^a = \{(0,p,q,0) | q^k = 0\} = E_1^a \cap V(r) = L^a$.

(b) $T_1^b = V(z^{n-k}s^n + z^{n-k} - t^k - u^k)$, which is singular along $L^b = \{(s,0,0,0)\}$, and $E_1^b = \{(s,0,t,u) | t^k + u^k = 0\} = \{(s,0,\eta u, u) | \eta^k = -1\}$ i.e. k planes containing L^b; $S_1^b = \{(s,z,t,0) | z^{n-k}s^n + z^{n-k} - t^k = 0\} = T_1^b \cap V(u)$, which is singular along L^b too, and $F_1^b = \{(s,z,t,0) | t^k = 0\} = E_1^b \cap V(u) = L^b$.

(c) $T_1^c = V(v^{n-k}i^n + v^{n-k}j^n - 1 - l^k)$, which is smooth, and $E_1^c = \{(i,j,0,l) | l^k = -1\} = \{(i,j,0,\eta) | \eta^k = -1\}$ i.e. k disjoint planes; $S_1^c = \{(i,j,v,0) | v^{n-k}i^n + v^{n-k}j^n + 1 = 0\} = T_1^c \cap V(l)$, which is smooth, while $F_1^c = \emptyset = E_1^c \cap V(l)$.

(d) $T_1^d = V(w^{n-k}m^n + w^{n-k}k^n - o^k - 1)$, which is smooth, and $E_1^d = \{(m,k,o,0) | o^k = -1\} = \{(m,k,\eta,0) | \eta^k = -1\}$ i.e. k disjoint planes; in this chart $S_1^d = F_1^d = \emptyset$.

Thus, after blowing up $\underline{0}$ we obtain a threefold T_1 which is singular along a line L not meeting X_1^c, X_1^d, and whose exceptional divisor E_1 consists of k planes containing

L. Moreover, there is a hyperplane H passing through L and not meeting X_1^d that cuts S_1 from T_1 and F_1 from E_1. Now we have to blow up the line L. To ease computations, we will blow up L^b in the chart X_1^b (as the situation in X_1^a is analogous), but first we set $h_1 := n-k$ and rewrite

$$\begin{aligned} T_1 = V(y^n z^{h_1} + z^{h_1} - v^k - w^k), \quad & S_1 = V(w, y^n z^{h_1} + z^{h_1} - v^k). \\ E_1 = \{(y,0,\eta w, w) | \eta^k = -1\}, \quad & L = V(z,v,w). \end{aligned}$$

By definition, the blow-up of $\mathbb{C}^4$ along L is

$$X_2 := Bl_L \mathbb{C}^4 = \{(y,z,v,w) \times [a:b:c] | zb = va, zc = wa, vc = wb\} \subset \mathbb{C}^4 \times \mathbb{P}^2.$$

X_2 can be covered by charts X_2^a, X_2^b and X_2^c, and the expression of the blow-up map $\pi_2 : X_2 \to \mathbb{C}^4$ when restricted to these charts is:

$$\begin{array}{llll} \pi_2^a : X_2^a \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y,z,p,q) \mapsto (y,z,pz,qz) & \text{with} & p := \frac{b}{a}, \quad q := \frac{c}{a}. \\ \pi_2^b : X_2^b \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y,r,v,s) \mapsto (y,rv,v,sv) & \text{with} & r := \frac{a}{b}, \quad s := \frac{c}{b}. \\ \pi_2^c : X_2^c \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y,t,u,w) \mapsto (y,tw,uw,w) & \text{with} & t := \frac{a}{c}, \quad u := \frac{b}{c}. \end{array}$$

Observe that now we have to keep track not only of the exceptional divisors $E_2 \subset T_2$ and $F_2 \subset S_2$, but also of the strict transform $st_L(E_1)$ of E_1 (only that of E_1 because F_1 is the center of the blow-up). We need to distinguish three cases.

$\underline{k < h_1}$ With the same convention on notations as before, we obtain:

(a) $T_2^a = V(y^n z^{h_1-k} + z^{h_1-k} - p^k - q^k)$ whose singular locus is contained in $L^a := (y:0:0:0)$ and $E_2^a = \{(y,0,p,q) | p^k + q^k = 0\} = \{(y,0,\eta q, q) | \eta^k = -1\}$ i.e. k planes containing L^a; $S_2^a = \{(y,z,p,0) | y^n z^{h_1-h_2} + z^{h_1-h_2} - p^{h_2} = 0\} = T_2^a \cap V(q)$, whose singular locus is contained in L^a too, and $F_2^a = \{(y,0,p,0) | p^{h_2} = 0\} = E_2^a \cap V(q) = L^a$; $st_L(E_1)^a = \emptyset$.

(b) $T_2^b = V(y^n r^{h_1} v^{h_1-k} + r^{h_1} v^{h_1-k} - 1 - s^k)$ smooth and $E_2^b = \{(y,r,0,s) | s^k = -1\} = \{(y,r,0,\eta) | \eta^k = -1\}$ i.e. k disjoint planes; $st_L(E_1)^b = \{(y,0,v,\eta) | \eta^k = -1\}$ i.e. k disjoint planes, and $st_L(E_1)^b \cap E_2^b = \{(y,0,0,\eta) | \eta^k = -1\}$ i.e. k disjoint lines. $S_2^b = \{(y,r,v,0) | y^n r^{h_1} v^{h_1-h_2} + r^{h_1} v^{h_1-h_2} - 1 = 0\} = T_2^b \cap V(s)$ smooth and $F_2^b = \emptyset = E_2^b \cap V(s)$.

(c) $T_2^c = V(y^n t^{h_1} w^{h_1-k} + t^{h_1} w^{h_1-k} - u^k - 1)$ smooth and $E_2^c = \{(y,t,u,0) | u^k = -1\} = \{(y,t,\eta,0) | \eta^k = -1\}$ i.e. k disjoint planes; $st_L(E_1)^c = \{(y,0,\eta,w) | \eta^k = -1\}$ i.e. k disjoint planes and $st_L(E_1)^c \cap E_2^c = \{(y,0,\eta,0) | \eta^k = -1\}$ i.e. k disjoint lines. We have $S_2^c = F_2^c = \emptyset$.

Thus, after the blow we obtain a threefold T_2 whose singular locus is contained in a line L not meeting X_2^b and X_2^c, with $E_2 = \cup_{i=1}^k Z_i^{(1)}$ and $st_L(E_1) = \cup_{j=1}^k Z_j^{(0)}$ not meeting X_2^a. S_2 is singular with $S_{2sing} \subset L$ too; moreover, $F_2 = L$. The $Z_i^{(1)}$ meet in L, while the $Z_j^{(0)}$ are disjoint. We have

$$Z_i^{(1)} \cap Z_j^{(0)} = \begin{cases} \text{a line} & \text{if } i = j. \\ \emptyset & \text{if } i \neq j. \end{cases}$$

If we set $h_1 := h_1 - k$, we can see that $T_2^a \subset X_2^a$ is in the same situation as $T_1^b \subset X_1^b$, the only difference being that now each $Z_i^{(1)}$ making up E_2^a meets the plane $Z_i^{(0)} \subset st_L(E_1)$ in a line; since the $Z_i^{(0)}$ do not meet L, they remain untouched under the blow-up of L.

$\underline{k = h_1}$ With the same convention on notations as before, we obtain:

(a) $T_2^a = V(y^n + 1 - p^k - q^k)$ and $E_2^a = \{(y, 0, p, q) | y^n + 1 - p^k - q^k = 0\}$ are smooth; $S_2^a = \{(y, z, p, 0) | y^n + 1 - p^k = 0\} = T_2^a \cap V(q)$ and $F_2^a = \{(y, 0, p, 0) | y^n + 1 - p^k = 0\}$ are smooth; $st_L(E_1)^a = \emptyset$.

(b) $T_2^b = V(y^n r^k + r^k - 1 - s^k)$ and $E_2^b = \{(y, r, 0, s) | y^n r^k + r^k - 1 - s^k = 0\}$ are smooth; $S_2^b = \{(y, r, v, 0) | y^n r^k + r^k - 1 = 0\} = T_2^b \cap V(s)$ and $F_2^b = \{(y, r, 0, 0) | y^n r^k + r^k - 1 = 0\} = E_2^b \cap V(s)$ are smooth; $st_L(E_1)^b = \{(y, 0, v, \eta) | \eta^k = -1\}$ i.e. k disjoint planes, and $st_L(E_1)^b \cap E_2^b = \{(y, 0, 0, \eta) | \eta^k = -1\}$ i.e. k disjoint lines.

(c) $T_2^c = V(y^n t^k + t^k - u^k - 1)$ and $E_2^c = \{(y, t, u, 0) | y^n t^k + t^k - u^k - 1 = 0\}$ are smooth, while $S_2^c = F_2^c = \emptyset$; $st_L(E_1)^c = \{(y, 0, \eta, w) | \eta^k = -1\}$ i.e. k disjoint planes and $st_L(E_1)^c \cap E_2^c = \{(y, 0, \eta, 0) | \eta^k = -1\}$ i.e. k disjoint lines.

T_2 is smooth, and its 'total exceptional divisor' $D_T := E_2 \cup st_L(E_1)$ consists of a smooth surface and k disjoint planes $Z_1^{(0)}, \ldots, Z_k^{(0)}$ not meeting X_2^a, with each $Z_i^{(0)}$ intersecting E_2 in a line. Moreover, there is a hyperplane H meeting neither X_2^c nor $st_L(E_1)$ (the latter fact can be read in X_2^b) that cuts S_2 (which is smooth) from T_2; the 'total exceptional divisor' of S_2 is $D_S := F_2$ i.e. a smooth curve which is cut from E_2 by H. We have thus obtained a resolution of both T and S.

If we get to this point after having gone through the step $k < h_1$ for s times, the only difference is that the k planes $Z_i^{(0)}$ are replaced by a 'string' of planes $Z_i^{(0)} \cup Z_i^{(1)} \cup \cdots \cup Z_i^{(s)}$ with

$$Z_i^{(t_1)} \cap Z_j^{(t_2)} = \begin{cases} \text{a line} & \text{if } i = j \text{ and } t_1 = t_2 \pm 1. \\ \emptyset & \text{otherwise.} \end{cases}$$

$$Z_i^{(t)} \cap E_2 = \begin{cases} \text{a line} & \text{if } t = 0. \\ \emptyset & \text{otherwise.} \end{cases}$$

$\underline{k > h_1}$ With the same notations as before, we obtain

(a) $T_2^a = V(y^n + 1 - p^k z^{k-h_1} - q^k z^{k-h_1})$ smooth and $E_2^a = \{(\sigma, 0, p, q) | \sigma^n = -1\}$ i.e. n disjoint planes, $S_2^a = \{(y, z, p, 0) | y^n + 1 - p^k z^{k-h_1} = 0\} = T_2^a \cap V(q)$ smooth and $F_2^a = \{(\sigma, 0, p, 0) | \sigma^n = -1\} = E_2 \cap V(q)$ i.e. n disjoint lines; $st_L(E_1)^a = \emptyset$.

(b) $T_2^b = V(y^n r^{h_1} + r^{h_1} - v^{k-h_1} - s^k v^{k-h_1})$ whose singular locus is contained in $H^b := (y, 0, 0, s)$ and $E_2^b = H^b \cup \{(\sigma, r, 0, s)|\sigma^n = -1\}$ i.e. the plane H^b together with n disjoint planes meeting H^b in the n lines $\{(\sigma, 0, 0, s)|\sigma^n = -1\}$. $S_2^b = \{(y, r, v, 0)|y^n r^{h_1} + r^{h_1} - v^{k-h_1} = 0\} = T_2^b \cap V(s)$ whose singular locus is contained in $L^b := \{(y, 0, 0, 0)\} = H^b \cap V(s)$ and $F_2^b = L^b \cup \{(\sigma, r, 0, 0)|\sigma^n = -1\} = E_2^b \cap V(s)$ i.e. the line L^b together with n disjoint lines L_i meeting L^b in $\{(\sigma, 0, 0, 0)|\sigma^n = -1\}$. $st_L(E_1)^b = \{(y, 0, v, \eta)|\eta^k = -1\}$ with $st_L(E_1)^b \cap H^b = \{(y, 0, 0, \eta)|\eta^k = -1\}$ (k disjoint lines) and $st_L(E_1)^b \cap E_2^b = \{(\sigma, 0, 0, \eta)|\sigma^n = -1, \eta^k = -1\}$ (nk points belonging to H^b).

(c) $T_2^c = V(y^n t^{h_1} + t^{h_1} - u^k w^{k-h_1} - w^{k-h_1})$ whose singular locus is contained in $H^c := \{(y, 0, u, 0)\}$ and $E_2^c = H_c \cup \{(\sigma, t, u, 0)|\sigma^n = -1\}$ i.e. the plane H^c together with n disjoint planes meeting H^c in the n lines $\{(\sigma, 0, u, 0)|\sigma^n = -1\}$. $S_2^c = F_2^c = \emptyset$. $st_L(E_1)^c = \{(y, 0, \eta, w)|\eta^k = -1\}$ with $st_L(E_1)^c \cap H^c = \{(y, 0, \eta, 0)|\eta^k = -1\}$ (k disjoint lines) and $st_L(E_1)^c \cap E_2^c = \{(\sigma, 0, \eta, 0)|\sigma^n = -1, \eta^k = -1\}$ (nk points belonging to H^c).

Thus T_2 is singular with $T_{2sing} \subset H$, where H is a plane not meeting X_2^a. The exceptional divisor generated by the two blow-ups consists of the plane H, n disjoint planes $Y_1, \dots, Y_n$ and k disjoint planes $Z_1^{(0)}, \dots, Z_k^{(0)}$ not meeting X_2^a such that $st_L(E_1) = \cup_{i=1}^k Z_i^{(0)}$. $H \cap Y_i$ gives a line L_i, $H \cap Z_j^{(0)}$ gives a line R_j. S_2 is cut from T_2 by a hyperplane H' (not meeting X_2^c) that cuts H in the line L containing S_{2sing}; H' also cuts F_2 from E_2.

Again, if we get to this point after having gone through the step $k < h_1$ for s times, the only difference is that the k planes $Z_i^{(0)}$ are replaced by a 'string' of planes $Z_i^{(0)} \cup Z_i^{(1)} \cup \dots \cup Z_i^{(s)}$ with

$$\begin{aligned} Z_i^{(t_1)} \cap Z_j^{(t_2)} &= \begin{cases} \text{a line} & \text{if } i = j \text{ and } t_1 = t_2 \pm 1. \\ \emptyset & \text{otherwise.} \end{cases} \\ Z_i^{(t)} \cap H &= \begin{cases} \text{the line } R_i & \text{if } t = 0. \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Next we blow up the plane H. We will blow up H^b in the chart X_2^b, which is the chart that meets all exceptional divisors generated so far.

Remark 3.3.4. By the discussion above, the only planes $Z_i^{(t)}$ affected by the blow-up of H are the $Z_i^{(0)}$ i.e. those that make up $st_L(E_1)$, hence we can assume without loss of generality that $s = 0$.

Before starting the computations, we set $h_2 := k - h_1$ and rewrite

$$\begin{aligned} & T_2 = V(y^n z^{h_1} + z^{h_1} - v^{h_2} - v^{h_2} w^k), \quad S_2 = V(w, y^n z^{h_1} + z^{h_1} - v^{h_2}). \\ H = V(z, v), \quad & E_2 = H \cup \{(\sigma, z, 0, w)|\sigma^n = -1\}, \quad st_L(E_1) = \{(y, 0, v, \eta)|\eta^k = -1\}. \\ & L = V(z, v, w), \quad F_2 = L \cup \{(\sigma, z, 0, 0)|\sigma^n = -1\} = E_2 \cap V(w). \end{aligned}$$

By definition, the blow-up of $\mathbb{C}^4$ at H is

$$X_3 := Bl_H\mathbb{C}^4 = \{(y, z, v, w) \times [a : b] | zb = va\} \subset \mathbb{C}^4 \times \mathbb{P}^1$$

and it can be covered by charts X_3^a and X_3^b; the expression of the blow-up map $\pi_3 : X_3 \to \mathbb{C}^4$ when restricted to these charts is:

$$\begin{array}{llll} \pi_3^a : X_3^a \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y, z, r, w) \mapsto (y, z, rz, w) & \text{with} & r := \frac{b}{a}. \\ \pi_3^b : X_3^b \simeq \mathbb{C}^4 \to \mathbb{C}^4 \text{ s.t.} & (y, s, v, w) \mapsto (y, sv, v, w) & \text{with} & s := \frac{a}{b}. \end{array}$$

At this point we need to distinguish three cases, again.

$\underline{h_2 < h_1}$ We obtain

(a) $T_3^a = V(y^n z^{h_1-h_2} + z^{h_1-h_2} - r^{h_2} - r^{h_2}w^k)$ whose singular locus is contained in $H^a := \{(y, 0, 0, w)\}$ and $E_3^a = \{(y, 0, r, \eta) | \eta^k = -1\}$. $S_3^a = V(w, y^n z^{h_1-h_2} + z^{h_1-h_2} - r^{h_2}) = T_3^a \cap V(w)$ with singular locus contained in $L^a := \{(y, 0, 0, 0)\} = H^a \cap V(w)$ and $F_3^a = L^a = E_3^a \cap V(w)$. $st_H(E_2)^a = \{(\sigma, z, 0, w) | \sigma^n = -1\}$, $st_H(F_2)^a = \{(\sigma, z, 0, 0) | \sigma^n = -1\} = st_H(E_2)^a \cap V(w)$ and $st_H(st_L(E_1))^a = \emptyset$.

(b) $T_3^b = V(y^n s^{h_1} v^{h_1-h_2} + s^{h_1} v^{h_1-h_2} - 1 - w^k)$ smooth and $E_3^b = \{(y, s, 0, \eta) | \eta^k = -1\}$. $S_3^b = V(w, y^n s^{h_1} v^{h_1-h_2} + s^{h_1} v^{h_1-h_2} - 1) = T_3^b \cap V(w)$ smooth and $F_3^b = E_3^b \cap V(w) = \emptyset$. $st_H(st_L(E^1))^b = \{(y, 0, v, \eta) | \eta^k = -1\}$ and $st_H(E_2)^b = st_H(F_2)^b = \emptyset$.

The exceptional divisors of T_3 we have obtained are:

- A plane H not meeting X_3^b.
- k disjoint planes $Z_1^{(0)}, \dots, Z_k^{(0)} \subset E_3$ meeting both X_3^a and X_3^b and intersecting H in k lines R_i not meeting X_3^b.
- n disjoint planes $Y_1^{(0)}, \dots, Y_n^{(0)} \subset st_H(E_2)$ not meeting X_3^b and intersecting H in n lines L_i (not meeting X_3^b).
- k disjoint planes $Z_1^{(1)}, \dots, Z_k^{(1)} \subset st_H(st_L(E_1))$ not meeting X_3^a with $Z_i^{(1)}$ intersecting $Z_j^{(0)}$ in a line if and only if $i = j$ (clearly these lines too do not meet X_3^a).

For S_3 we have

- A line $L = H \cap V(w)$ not meeting X_3^b.
- n disjoint lines $K_i^{(0)} := Y_i^{(0)} \cap V(w)$ such that $\cup_{i=1}^n K_i^{(0)} = st_H(F_2) = st_H(E_2) \cap V(w)$; the $K_i^{(0)}$ do not meet X_3^b and intersect L in n points.

Observe that the $Z_i^{(1)}$ do not intersect H, hence they are not affected by blow ups with center H.

<u>$h_2 > h_1$</u> We obtain

(a) $T_3^a = V(y^n+1-r^{h_2}z^{h_2-h_1}-r^{h_2}z^{h_2-h_1}w^k)$ smooth and $E_3^a = \{(\sigma,0,r,w)|\sigma^n = -1\}$. $S_3^a = V(w, y^n+1-r^{h_2}z^{h_2-h_1}) = T_3^a \cap V(w)$ smooth and $F_3^a = \{(\sigma,0,r,0)|\sigma^n = -1\} = E_3^a \cap V(w)$. $st_H(E_2)^a = \{(\sigma,z,0,w)|\sigma^n = -1\}$, $st_H(F_2)^a = \{(\sigma,z,0,0)|\sigma^n = -1\} = st_H(E_2)^a \cap V(w)$ and $st_H(st_L(E_1))^a = \emptyset$.

(b) $T_3^b = V(y^n s^{h_1} + s^{h_1} - v^{h_2-h_1} - v^{h_2-h_1}w^k)$ whose singular locus is contained in $H^b := \{(y,0,0,w)\}$ and $E_3^b = H^b \cup \{(\sigma,s,0,w)|\sigma^n = -1\}$. $S_3^b = V(w, y^n s^{h_1} + s^{h_1} - v^{h_2-h_1}) = T_3^b \cap V(w)$ singular in $L^b := \{(y,0,0,0)\} = H^b \cap V(w)$ and $F_3^b = L^b \cup \{(\sigma,s,0,0)|\sigma^n = -1\} = E_3^b \cap V(w)$. $st_H(E_2)^b = st_H(F_2)^b = \emptyset$ and $st_H(st_L(E_1))^b = \{(y,0,v,\eta)|\eta^k = -1\}$.

The exceptional divisors of T_3 we have obtained so far are:

- A plane H not meeting X_3^a.
- n disjoint planes $Y_1^{(0)},\dots,Y_n^{(0)} \subset E_3$ meeting both X_3^a and X_3^b and intersecting H in n lines L_i not meeting X_3^a.
- n disjoint planes $Y_1^{(1)},\dots,Y_n^{(1)} \subset st_H(E_2)$ not meeting X_3^b with $Y_i^{(0)}$ intersecting $Y_j^{(1)}$ in a line if and only if $i = j$ (clearly these lines too do not meet X_3^b).
- k disjoint planes $Z_1^{(0)},\dots,Z_k^{(0)} \subset st_H(st_L(E_1))$ not meeting X_3^a and intersecting H in k lines R_i not meeting X_3^a.

Fr S_3 we have

- A line $L = H \cap V(w)$ not meeting X_3^a.
- n disjoint lines $K_i^{(0)} := Y_i^{(0)} \cap V(w)$ such that $\cup_{i=1}^n K_i^{(0)} = F_3 = E_3 \cap V(w)$; the $K_i^{(0)}$ intersect L in n points.
- n disjoint lines $K_i^{(1)} := Y_i^{(1)} \cap V(w)$ such that $\cup_{i=1}^n K_i^{(1)} = st_H(F_2) = st_h(E_2) \cap V(w)$; the $K_i^{(0)}$ do not meet X_3^b, and $K_i^{(0)}$ intersects $K_j^{(1)}$ in a point if and only if $i = j$.

Observe that the $Y_i^{(1)}$ do not intersect H, hence they are not affected by blow ups with center H.

<u>$h_2 = h_1$</u> If we set $h := h_1 = h_2$ we obtain

(a) $T_3^a = V(y^n+1-r^h-r^hw^k)$ and $E_3^a = \{(y,0,r,w)|y^n+1-r^h-r^hw^k = 0\}$ are smooth. $S_3^a = V(w, y^n+1-r^h) = T_3^a \cap V(w)$ and $F_3^a = \{(y,0,r,0)|y^n+1-r^h = 0\} = E_3^a \cap V(w)$ are smooth. $st_H(E_2)^a = \{(\sigma,z,0,w)|\sigma^n = -1\}$ (this was to be expected, as in X_2^b the plane H is a hyperplane of $V(v)$ with $\{(\sigma,z,0,w)|\sigma^n = -1\} \subset V(v)$), $st_H(F_2)^a = \{(\sigma,z,0,0)|\sigma^n = -1\} = st_H(E_2)^a \cap V(w)$ and $st_H(st_L(E_1))^a = \emptyset$.

(b) $T_3^b = V(y^n s^h + s^h - 1 - w^k)$ and $E_3^b = \{(y,s,0,w) | y^n s^h + s^h - 1 - w^k = 0\}$ are smooth. $S_3^b = V(w, y^n s^h + s^h - 1) = T_3^b \cap V(w)$ and $F_3^b = \{(y,s,0,0) | y^n s^h + s^h - 1 = 0\} = E_3^b \cap V(w)$ are smooth. $st_H(E_2)^b = st_H(F_2)^b = \emptyset$ and $st_H(st_L(E_1))^b = \{(y,0,v,\eta) | \eta^k = -1\}$.

Thus T_3 is smooth with exceptional divisors:

- The smooth surface E_3 meeting both X_3^a and X_3^b.
- n disjoint planes $Y_i^{(0)}$ not meeting X_3^b with $Y_1^{(0)} \cup \cdots \cup Y_n^{(0)} = st_H(E_2)$; each $Y_i^{(0)}$ intersects E_3 in the line L_i.
- k disjoint planes $Z_i^{(0)}$ not meeting X_3^a with $Z_1^{(0)} \cup \cdots \cup Z_k^{(0)} = st_H(st_L(E_1))$; each $Z_i^{(0)}$ intersects E_3 in the line R_i.

S_3 is smooth too, with exceptional divisors:

- The smooth curve $F_3 = E_3 \cap V(w)$ meeting both X_3^a and X_3^b.
- n disjoint lines $K_i^{(0)} := Y_i^{(0)} \cap V(w)$ not meeting X_3^b with $K_1^{(0)} \cup \cdots \cup K_n^{(0)} = st_H(F_2) = st_H(E_2) \cap V(w)$; each $K_i^{(0)}$ intersects F_3 in a point.

Conclusion At each step of the resolution one of the h_i decreases, so we are guaranteed that this procedure terminates at either the step $h_1 = k$ or the step $h_1 = h_2$ with $\tilde{T}$ and $\tilde{S} = \tilde{T} \cap V(w)$ smooth; if we have performed the steps $k < h_1$, $h_2 < h_1$ and $h_2 > h_1$ respectively s, r and u times we end up with the following divisors:

- For $\tilde{T}$:
  - A smooth surface E.
  - Planes $Z_i^{(t)}$ with $i = 1, \ldots, k$ and $t = 0, \ldots, s+r$ such that

$$\begin{aligned} Z_i^{(t_1)} \cap Z_j^{(t_2)} &= \begin{cases} \text{a line} & \text{if } i = j \text{ and } t_1 = t_2 \pm 1. \\ \emptyset & \text{otherwise.} \end{cases} \\ Z_i^{(t)} \cap E &= \begin{cases} \text{the line } R_i & \text{if } t = 0. \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

  - Planes $Y_i^{(t)}$ with $i = 1, \ldots, n$ and $t = 0, \ldots, u$ such that

$$\begin{aligned} Y_i^{(t_1)} \cap Y_j^{(t_2)} &= \begin{cases} \text{a line} & \text{if } i = j \text{ and } t_1 = t_2 \pm 1. \\ \emptyset & \text{otherwise.} \end{cases} \\ Y_i^{(t)} \cap E &= \begin{cases} \text{the line } L_i & \text{if } t = 0. \\ \emptyset & \text{otherwise.} \end{cases} \\ Y_i^{(t_1)} \cap Z_j^{(t_2)} &= \emptyset. \end{aligned}$$

- For $\tilde{S}$:
  - A smooth curve $F = E \cap V(w)$.
  - Lines $K_i^{(t)} = Y_i^{(t)} \cap V(w)$ with $i = 1, \dots, n$ and $t = 0, \dots, u$ such that

$$\begin{aligned} K_i^{(t_1)} \cap K_j^{(t_2)} &= \begin{cases} \text{a point} & \text{if } i = j \text{ and } t_1 = t_2 \pm 1. \\ \emptyset & \text{otherwise.} \end{cases} \\ K_i^{(t)} \cap F &= \begin{cases} \text{a point} & \text{if } t = 0. \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Property (I) - The cubical hyperresolutions

Assume T has d points of type $P_{j,h}$; if we denote by Σ_T and Σ_S the singular loci of T and S respectively, then $\Sigma_T = \{P_p, P_q\} \cup \{P_1, \dots, P_d\}$ and $\Sigma_S = \{P_p\}$.

$\underline{p = q = n/2}$ In this case n is even, and we write $n = 2n'$. The exceptional divisors we have generated on $\tilde{T}$ in order to resolve Σ_T are:

- (Resolution of P_p) A smooth surface E_p and p disjoint planes Y_i meeting E_p in the lines L_i.
- (Resolution of P_q) A smooth surface E_q and q disjoint planes Z_j meeting E_q in the lines R_j.
- (Resolution of each P_k) A smooth surface E_2^k and planes $W_{h,l}^k$ for $h = 1, 2$ and $l = 1, \dots, n' - 1$ such that the $W_{h,1}^k$ meet E_2^k in lines $V_{h,1}^k$, the $W_{h,l}^k$ meet the $W_{h,l-1}^k$ in lines $V_{h,l}^k$ and $V_{1,l_1}^k \cap V_{2,l_2}^k = \emptyset$.

We call D_T the union of all the divisors above. In order to construct a cubical hyperresolution of T we start with the following resolution square, where $\tilde{T}$ denotes the resolution of T we found:

$$\begin{array}{ccc} D_T & \longrightarrow & \tilde{T} \\ \downarrow & & \downarrow \\ \Sigma_T & \longrightarrow & T \end{array}$$

Since D_T is not smooth, we proceed to resolve the 1-cubical variety $(D_T \to \Sigma_T)$. The easiest way to do so is by separating its irreducible components, so we set

$$D'_T := \left(E_p \coprod \left(\coprod_{i=1}^{p} Y_i \right) \right) \coprod \left(E_q \coprod \left(\coprod_{j=1}^{q} Z_j \right) \right) \coprod \left(\coprod_{k=1}^{d} \left(E_2^k \coprod \left(\coprod_{l=1,\dots,n'-1}^{h=1,2} W_{h,l}^k \right) \right) \right).$$

The discriminant of the map $(D'_T \to \Sigma_T) \to (D_T \to \Sigma_T)$ of 1-cubical varieties is given by $(K_T \to \Sigma_T)$, where

$$K_T = \left(\bigcup_{i=1}^{p} L_i \right) \bigcup \left(\bigcup_{j=1}^{q} R_j \right) \bigcup \left(\bigcup_{k=1}^{d} \left(\bigcup_{l=1,\dots,n'-1}^{h=1,2} V_{h,l}^{k} \right) \right).$$

In order to obtain a resolution square of $(D_T \to \Sigma_T)$, we set

$$K'_T := \left(\coprod_{i=1}^{p} L_i^0 \coprod L_i^1 \right) \coprod \left(\coprod_{j=1}^{q} R_j^0 \coprod R_j^1 \right) \coprod \left(\coprod_{k=1}^{d} \left(\coprod_{l=1,\dots,n'-1}^{h=1,2} V_{h,l}^{k,0} \coprod V_{h,l}^{k,1} \right) \right)$$

where the apexes distinguish between the variety being thought as belonging to one or the other of the irreducible components of D_T in which it is contained. We can now complete the square with $(K'_T \to \Sigma_T)$, obtaining

$$\begin{tikzcd}
(K'_T \to \Sigma_T) \arrow[r, "\tau"] \arrow[d, "\sigma"] & (D'_T \to \Sigma_T) \arrow[d] \\
(K_T \to \Sigma_T) \arrow[r, "\gamma"] & (D_T \to \Sigma_T)
\end{tikzcd}$$

The maps γ and σ are simply inclusions, with σ sending the L_i^0's into E_p and each L_i^1 into the plane Y_i and $\tau(L_i^0) = \tau(L_i^1) = L_i$ (the same goes for all other lines in K'_T). The picture we have now is the following:

$$\begin{tikzcd}
 & K'_T \arrow[rr] \arrow[dl] \arrow[ddd] & & D'_T \arrow[dl] \arrow[ddd] \arrow[dddrr, dashed] & & & & K'_T \arrow[rr] \arrow[dl] \arrow[ddd] & & D'_T \arrow[dl] \arrow[ddd] \\
\Sigma_T \arrow[rr] \arrow[ddd] & & \Sigma_T \arrow[ddd] \arrow[dddrr, dashed] & & & & \Sigma_T \arrow[rr] \arrow[ddd] & & \Sigma_T \arrow[ddd] & \\
 & & & & {} \arrow[rr, shift left] \arrow[rr, shift right] & & {} & & & \\
 & K_T \arrow[rr] \arrow[dl] & & D_T \arrow[rr] \arrow[dl] & & \tilde{T} \arrow[dl] & & K_T \arrow[rr] \arrow[dl] & & \tilde{T} \arrow[dl] \\
\Sigma_T \arrow[rr] & & \Sigma_T \arrow[rr] & & T & & \Sigma_T \arrow[rr] & & T &
\end{tikzcd}$$

and if we contract the diagram using the dashed maps we obtain our desired cubical hyperresolution of T.

For S the situation is much simpler: its only singular point is P_p, and its resolution generates on $\tilde{S}$ a smooth curve F as exceptional divisor. If we cut all terms of the cubical hyperresolution of T above by the hyperplane H determining S, we obtain

$$\begin{tikzcd}
& \emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & F \arrow[dl] \arrow[dd] \\
\Sigma_S \arrow[rr, crossing over] \arrow[dd] & & \Sigma_S & \\
& \emptyset \arrow[rr] \arrow[dl] & & \tilde{S} \arrow[dl] \\
\Sigma_S \arrow[rr] & & S \arrow[from=2-3, crossing over] &
\end{tikzcd}$$

which is a cubical hyperresolution of S. Note, however, that Property (II) of 1.2.21 is not satisfied: for example, the closed immersions $\Sigma_S \hookrightarrow \Sigma_T$ have codimension zero.

__General case__ The only difference with the previous case is that the resolution of P_p generates a string of s_p groups of k planes with pairwise intersection in lines (resp. a string of u_p groups of n planes with pairwise intersection in lines) where s_p (resp. u_p) is the number of times the steps $k < h_1$ and $h_2 < h_1$ (resp. $h_2 > h_1$) are executed during the resolution; the same goes of course for the resolution of P_q. Moreover, on $\tilde{S}$ now we have also a string of u_p groups of n lines with pairwise intersection. If we define D_S, K_S and K'_S in the same way we did in the previous case, we obtain the hyperresolutions

$$\begin{tikzcd}
& K'_T \arrow[rr] \arrow[dl] \arrow[dd] & & D'_T \arrow[dl] \arrow[dd] \\
\Sigma_T \arrow[rr, crossing over] \arrow[dd] & & \Sigma_T & \\
& K_T \arrow[rr] \arrow[dl] & & \tilde{T} \arrow[dl] \\
\Sigma_T \arrow[rr] & & T \arrow[from=2-3, crossing over] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& K'_S \arrow[rr] \arrow[dl] \arrow[dd] & & D'_S \arrow[dl] \arrow[dd] \\
\Sigma_S \arrow[rr, crossing over] \arrow[dd] & & \Sigma_S & \\
& K_S \arrow[rr] \arrow[dl] & & \tilde{S} \arrow[dl] \\
\Sigma_S \arrow[rr] & & S \arrow[from=2-3, crossing over] &
\end{tikzcd}$$

where the one of S is the section of the one of T by the hyperplane H. Property (II) of 1.2.21 is not satisfied in this situation either: the closed immersions $\Sigma_S \hookrightarrow \Sigma_T$ still have codimension zero.

Property (II) - A workaround

We want to find a surjective 'Gysin morphism'

$$\gamma : H^2(S) \twoheadrightarrow H^4(T). \tag{3.3.3}$$

Our strategy is to first obtain a Gysin morphism $H^2_{DR}(S) \hookrightarrow H^4_{DR}(T)$ at the level of algebraic de Rham cohomology and then use Theorem 1.2.18 to find the one in singular cohomology. After all, while it is true that hypothesis (II) of Theorem 1.2.21 does not hold in our situation, the only problem lies in codimension zero closed immersions $\Sigma_S \hookrightarrow \Sigma_T$ between zero-dimensional varieties: it is thus reasonable to hope that the failure of property (II) will only prevent us from finding the Gysin morphisms $H^k_{DR}(S) \to H^{k+2}_{DR}(T)$ for small values of k.

A resolution square for S is the 2-cubical variety

$$\begin{tikzcd} D_S \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ \Sigma_S \arrow[r] & S. \end{tikzcd} \tag{3.3.4}$$

By Theorem 1.2.12 and Remark 1.2.13 we can find an m-cubical hyperresolution $Y_\square$ of (3.3.4) which, as 2-cubical variety of $(m-2)$-cubical hyperresolutions, can be written as

$$\begin{tikzcd} D_{S\square} \arrow[r, "f"] \arrow[d, "a"] & \tilde{S}_\square \arrow[d, "b"] \\ \Sigma_{S\square} \arrow[r, "g"] & S_\square. \end{tikzcd} \tag{3.3.5}$$

Being a hyperresolution, $Y_\square$ is in particular of cohomological descent: hence, if $\underline{\mathbb{C}}_{Y_\square}$ denotes the constant sheaf on $Y_\square$ then $C^\bullet(Y_\square, \underline{\mathbb{C}}_{Y_\square})$ is acyclic, and the same is true of $C^\bullet(Y_\square, \underline{\mathbb{C}}_{Y_\square})[2]$; by Corollary 1.2.15 we deduce the existence of an isomorphism

$$C^\bullet(S_\square, \underline{\mathbb{C}}_{S_\square}) \xrightarrow{\simeq} \mathrm{Cone}^\bullet[Rb_*C^\bullet(\tilde{S}_\square, \underline{\mathbb{C}}_{\tilde{S}_\square}) \oplus Rg_*C^\bullet(\Sigma_{S\square}, \underline{\mathbb{C}}_{\Sigma_{S\square}}) \xrightarrow{(C(a^\#),C(b^\#))}$$
$$\xrightarrow{(C(a^\#),C(b^\#))} R(g\circ a)_*C^\bullet(D_{S\square}, \underline{\mathbb{C}}_{D_{S\square}})][-1].$$

If we shift by -1 the short exact sequence of the cone over the morphism $(C(a^\#), C(b^\#))$ we obtain

$$0 \to R(g\circ a)_*C^\bullet(D_{S\square}, \underline{\mathbb{C}}_{D_{S\square}})[-1] \to \mathrm{Cone}^\bullet[Rb_*C^\bullet(\tilde{S}_\square, \underline{\mathbb{C}}_{\tilde{S}_\square}) \oplus Rg_*C^\bullet(\Sigma_{S\square}, \underline{\mathbb{C}}_{\Sigma_{S\square}}) \to$$
$$\xrightarrow{(C(a^\#),C(b^\#))} R(g\circ a)_*C^\bullet(D_{S\square}, \underline{\mathbb{C}}_{D_{S\square}})][-1] \to Rb_*C^\bullet(\tilde{S}_\square, \underline{\mathbb{C}}_{\tilde{S}_\square}) \oplus Rg_*C^\bullet(\Sigma_{S\square}, \underline{\mathbb{C}}_{\Sigma_{S\square}}) \to 0$$

so using the isomorphism above we get the short exact sequence

$$0 \to R(g\circ a)_*C^\bullet(D_{S\square}, \underline{\mathbb{C}}_{D_{S\square}})[-1] \to C^\bullet(S_\square, \underline{\mathbb{C}}_{S_\square}) \to$$
$$\to Rb_*C^\bullet(\tilde{S}_\square, \underline{\mathbb{C}}_{\tilde{S}_\square}) \oplus Rg_*C^\bullet(\Sigma_{S\square}, \underline{\mathbb{C}}_{\Sigma_{S\square}}) \to 0.$$

Now, since the $(m-2)$-cubical hyperresolution $\epsilon : S_\square \to S$ is of cohomological descent $C^\bullet(S_\square, \underline{\mathbb{C}}_{S_\square})$ is acyclic; hence, if we denote by $S_\bullet$ the $(m-3)$-semisimplicial space associated to $S_\square$ we can write the following isomorphism in $D_+(Sh(S))$:

$$\underline{\mathbb{C}}_S \xrightarrow{\simeq} R\epsilon_* \underline{\mathbb{C}}_{S_\bullet}.$$

Since all elements of the $(m-3)$-semisimplicial variety $S_\bullet$ are smooth, in $D_+(Sh(S_\bullet))$ we have an isomorphism $\underline{\mathbb{C}}_{S_\bullet} \xrightarrow{\simeq} \Omega^\bullet_{S_\bullet}$, so we can substitute $R\epsilon_*\underline{\mathbb{C}}_{S_\bullet}$ with $R\epsilon_*\Omega^\bullet_{S_\bullet}$; the same can of course be done with the other three $(m-2)$-cubical hyperresolutions in $Y_\square$.

In this way we obtain a short exact sequence of objects of $D_+(Sh(S))$

$$0 \to R(g \circ a)_* DR^\bullet_{D_S}[-1] \to DR^\bullet_S \to Rb_* DR^\bullet_{\tilde{S}} \oplus Rg_* DR^\bullet_{\Sigma_S} \to 0 \tag{3.3.6}$$

which yields the long exact sequence of algebraic de Rham cohomology groups

$$\cdots \to H^\bullet_{DR}(\Sigma_S) \oplus H^\bullet_{DR}(\tilde{S}) \to H^\bullet_{DR}(D_S) \to H^{\bullet+1}_{DR}(S) \to \cdots . \tag{3.3.7}$$

We want to apply a similar argument to T. A resolution square for T is the 2-cubical variety

$$\begin{tikzcd} D_T \arrow[r] \arrow[d] & \tilde{T} \arrow[d] \\ \Sigma_T \arrow[r] & T \end{tikzcd} \tag{3.3.8}$$

and as before we can associate to it an m'-cubical hyperresolution $X_\square$ which, as a 2-cubical variety of $(m'-2)$-cubical hyperresolutions, can be written as:

$$\begin{tikzcd} D_{T\square} \arrow[r, "f"] \arrow[d, "a"] & \tilde{T}_\square \arrow[d, "a"] \\ \Sigma_{T\square} \arrow[r, "g"] & T_\square. \end{tikzcd} \tag{3.3.9}$$

We observe the following:

(a) irreducible components of D_T and D_S and intersections thereof are smooth, and each irreducible component of D_S is an hyperplane section of an irreducible component of D_T.

(b) Σ_T and Σ_S are smooth, with the latter being a hyperplane section of the former. The same goes for $\tilde{T}$ and $\tilde{S}$.

These facts imply that considering in each entry of $X_\square$ the corresponding hyperplane section yields precisely $Y_\square$, so there is a natural closed immersion $Y_\square \hookrightarrow X_\square$; hence we can consider the restriction of sections functor $\Gamma_{Y_\square} : Sh(X_\square) \to Sh(X_\square)$.

From this, passing to semisimplicial objects, we deduce the existence of the restriction of sections functor $\Gamma_{Y_\bullet} : Sh(X_\bullet) \to Sh(X_\bullet)$ and of its total derived functor $R\Gamma_{Y_\bullet} : D_+(Sh(X_\bullet)) \to D_+(Sh(X_\bullet))$. The same reasoning applies to all the entries of (3.3.5) and (3.3.9), and yields the restriction of sections functors $\Gamma_{S_\bullet}$, $\Gamma_{\tilde{S}_\bullet}$, $\Gamma_{\Sigma_{S_\bullet}}$ and $\Gamma_{D_{S_\bullet}}$ (plus the corresponding total derived functors).

Now we apply the same argument as before to the complex of sheaves on $Y_\square$ given by $R\Gamma_{Y_\square}\underline{\mathbb{C}}_{X_\square}$.

Remark 3.3.5. We have the following commutative diagram of functors:

$$\begin{tikzcd} Sh(T_\bullet) \arrow[r, "\Gamma_{S_\bullet}"] \arrow[d, "\epsilon_*"] & Sh(T_\bullet) \arrow[d, "\epsilon_*"] \\ Sh(T) \arrow[r, "\Gamma_S"] & Sh(T). \end{tikzcd}$$

From this we deduce the equality of the total derived functors $R(\epsilon_* \circ \Gamma_{S_\bullet}) = R(\Gamma_S \circ \epsilon_*)$. But pushforwards preserve injective objects, and the same holds for $\Gamma_{S_\bullet}$ because $S_\bullet$ is closed in $T_\bullet$; since injective objects are adapted to any functor, we obtain isomorphisms

$$R\epsilon_* \circ R\Gamma_{S_\bullet} \simeq R(\epsilon_* \circ \Gamma_{S_\bullet}) = R(\Gamma_S \circ \epsilon_*) \simeq R\Gamma_S \circ R\epsilon_*. \tag{3.3.10}$$

Of course this commutativity holds for all the restriction of sections functors previously listed.

As before we have isomorphisms

$$R\Gamma_S\underline{\mathbb{C}}_T \simeq R\Gamma_S R\epsilon_*\underline{\mathbb{C}}_{T_\bullet} \simeq R\Gamma_S R\epsilon_*\Omega^\bullet_{T_\bullet} = R\Gamma_S DR^\bullet_T$$

(that have counterparts for all the $(m'-2)$-cubical hyperresolution $X_\square$ is composed of). Thus we obtain the short exact sequence of objects of $D_+(Sh(T))$

$$0 \to R(g\circ a)_* R\Gamma_{D_{S_\bullet}} DR^\bullet_{D_{T_\bullet}}[-1] \to \Gamma_{S_\bullet} DR^\bullet_{T_\bullet} \to Rb_*\Gamma_{\tilde{S}} DR^\bullet_{\tilde{T}} \oplus Rg_*\Gamma_{\Sigma_S} DR^\bullet_{\Sigma_T} \to 0 \tag{3.3.11}$$

which yields the long exact sequence of algebraic de Rham cohomology groups with supports

$$\cdots \to H^\bullet_{DR,\Sigma_S}(\Sigma_T) \oplus H^\bullet_{DR,\tilde{S}}(\tilde{T}) \to H^\bullet_{DR,D_S}(D_T) \to H^{\bullet+1}_{DR,S}(T) \to \cdots. \tag{3.3.12}$$

Now we need to find a way to compare the long exact sequences (3.3.7) and (3.3.12); in order to do this, we will consider some particular hyperresolutions of the entries of (3.3.4) and (3.3.8).

We start from (3.3.4), and we assume first that $p \neq q$. Σ_S and $\tilde{S}$ are smooth, so they are already 0-cubical hyperresolutions. D_S instead is not smooth, and it fits in the resolution square

$$\begin{tikzcd} K'_S \arrow[r] \arrow[d] & D'_S \arrow[d] \\ K_S \arrow[r] & D_S \end{tikzcd}$$

which is a 2-cubical variety $D_{S\square}$; since discriminant squares are of cohomological descent (see [62, Lemma-Definition 5.17]), this is a 2-cubical hyperresolution of D_S.

Similarly, Σ_T and $\tilde{T}$ are smooth while D_T fits in the resolution square (i.e. 2-cubical hyperresolution) $D_{T\square}$ given by

$$\begin{tikzcd} K'_T \arrow[r] \arrow[d] & D'_T \arrow[d] \\ K_T \arrow[r] & D_T. \end{tikzcd}$$

Hence we can write closed immersions $\Sigma_S \hookrightarrow \Sigma_T$ of codimension zero, $\tilde{S} \hookrightarrow \tilde{T}$ of codimension one and $D_{S\square} \hookrightarrow D_{T\square}$, with the latter giving a codimension one closed immersion if restricted to any irreducible component of any entry of $D_{S\square}$.

If we switch to 2-semisimplicial varieties, we can write closed immersions

$$\begin{aligned} D_{S_\bullet} &\hookrightarrow D_{T_\bullet} \\ \tilde{S} &\hookrightarrow \tilde{T} \qquad \text{of codimension 1,} \\ \Sigma_S &\hookrightarrow \Sigma_T \qquad \text{of codimension 0.} \end{aligned}$$

By [33, Lemma 3.1] the corresponding trace maps

$$\begin{aligned} &\Omega^\bullet_{D_{S_\bullet}} \to R\Gamma_{D_{S_\bullet}} \Omega^\bullet_{D_{T_\bullet}}[2] \\ &\Omega^\bullet_{\tilde{S}} \to R\Gamma_{\tilde{S}} \Omega^\bullet_{\tilde{T}}[2] \\ &\Omega^\bullet_{\Sigma_S} \to R\Gamma_{\Sigma_S} \Omega^\bullet_{\Sigma_T} \end{aligned}$$

are isomorphisms (the reader can find the explicit construction of the trace maps in [34, Chapter VI, Section 4.2]). These extend to isomorphisms of the associated de Rham complexes $DR^\bullet$, because the latter do not depend on the particular choice of a hyperresolution (see [30, Proposition III.1.12(i)]), and yield isomorphisms of algebraic de Rham cohomology groups

$$\begin{aligned} &H^\bullet_{DR}(D_S) \xrightarrow{\simeq} H^{\bullet+2}_{DR,D_S}(D_T) \\ &H^\bullet_{DR}(\tilde{S}) \xrightarrow{\simeq} H^{\bullet+2}_{DR,\tilde{S}}(\tilde{T}) \\ &H^\bullet_{DR}(\Sigma_S) \xrightarrow{\simeq} H^\bullet_{DR,\Sigma_S}(\Sigma_T). \end{aligned} \tag{3.3.13}$$

When $p = q$ we have to do the same, but this time D_S is already smooth, so the cubical hyperresolution $D_{S\square}$ we construct is somewhat artificial. For D_T we consider the same 2-cubical hyperresolution as in the case $p \neq q$. For D_S, we first obtain a 1-cubical variety by considering the identity morphism $D_S \to D_S$; then we choose points $Q_0 \in D_S$ and $Q \in K_T$, with the latter that belongs to the intersection $U \cap W$ of irreducible components of D_T. The 2-cubical variety

$$\begin{array}{ccc} \{Q\} & \xrightarrow{a} & D_S \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle id} \\ \{Q\} & \xrightarrow{c} & D_S \end{array}$$

where a and c send everything to Q_0 is a discriminant square for D_S, so it is of cohomological descent and since all its entries are smooth we conclude that it is a 2-cubical hyperresolution $D_{S\square}$ of D_S. In this way, we obtain a closed immersion $D_{S\square} \hookrightarrow D_{T\square}$ with the same properties as in the case $p \neq q$ and, reasoning as before, morphisms of cohomology groups like in 3.3.13.

At this point we can write the following diagram

$$\begin{array}{ccccccccc} H^1_{DR}(\tilde{S}) & \xrightarrow{\alpha} & H^1_{DR}(D_S) & \xrightarrow{\beta} & H^2_{DR}(S) & \xrightarrow{\delta} & H^2_{DR}(\tilde{S}) & \xrightarrow{\sigma} & H^2_{DR}(D_S) \\ \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} & & & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\ H^3_{DR,\tilde{S}}(\tilde{T}) & \xrightarrow{\alpha'} & H^3_{DR,D_S}(D_T) & \xrightarrow{\beta'} & H^4_{DR,S}(T) & \xrightarrow{\delta'} & H^4_{DR,\tilde{S}}(\tilde{T}) & \xrightarrow{\sigma'} & H^4_{DR,D_S}(D_T). \end{array} \tag{3.3.14}$$

The two squares are commutative. Indeed, by construction the trace maps are functorial so the same holds for the isomorphisms of cohomology groups they yield, which are the vertical maps of this diagram; as the horizontal maps are obtained from resolution squares of S and T they are functorial too, and this gives the commutativity of the squares. This implies in particular that $Ker(\alpha) \simeq Ker(\alpha')$ (hence $Im(\alpha) \simeq Im(\alpha')$ too) and $Ker(\sigma) \simeq Ker(\sigma')$, from which we deduce isomorphisms

$$\begin{aligned} Ker(\delta) = Im(\beta) \simeq H^1_{DR}(D_S)/Ker(\beta) = H^1_{DR}(D_S)/Im(\alpha) \simeq \\ \simeq H^3_{DR,D_S}(D_T)/Im(\alpha') = H^3_{DR,D_S}(D_T)/Ker(\beta') \simeq Im(\beta') = Ker(\delta') \end{aligned}$$
$$H^2_{DR}(S)/Ker(\delta) \simeq Im(\delta) = Ker(\sigma) \simeq Ker(\sigma') = Im(\delta') \simeq H^4_{DR,S}(T)/Ker(\delta').$$

From this we obtain the existence of an isomorphism

$$H^2_{DR}(S) \simeq Ker(\delta) \oplus H^2_{DR}(S)/Ker(\delta) \simeq Ker(\delta') \oplus H^4_{DR,S}(T)/Ker(\delta') \simeq H^4_{DR,S}(T).$$

We will choose a particular isomorphism $\theta : H^2_{DR}(S) \to H^4_{DR,S}(T)$. Namely:

1. We send any basis of $Ker(\delta)$ to any basis of $Ker(\delta')$ (no actual choice here).

2. Call h the isomorphism $H^2_{DR}(\tilde{S}) \to H^4_{DR,\tilde{S}}(\tilde{T})$. If $x \in H^2_{DR}(S)/Ker(\delta)$ then $\delta(x) \in Ker(\sigma)$ and $h(\delta(x)) \in Ker(\sigma')$; this means there exists $y \in H^4_{DR,S}(T)/Ker(\delta')$ s.t. $\delta'(y) = h(\delta(x))$. We set $\theta(x) := y$, so that in particular $\delta' \circ \theta = h \circ \delta$. This defines an isomorphism between $H^2_{DR}(S)/Ker(\delta)$ and $H^4_{DR,S}(T)/Ker(\delta')$.

With this choice of θ, we can rewrite diagram (3.3.14) as

$$\begin{tikzcd} H^1_{DR}(\tilde{S}) \arrow[r, "\alpha"] \arrow[d, "\simeq"] & H^1_{DR}(D_S) \arrow[r, "\beta"] \arrow[d, "\simeq"] & H^2_{DR}(S) \arrow[r, "\delta"] \arrow[d, "\theta"] & H^2_{DR}(\tilde{S}) \arrow[r, "\sigma"] \arrow[d, "\simeq"] & H^2_{DR}(D_S) \arrow[d, "\simeq"] \\ H^3_{DR,\tilde{S}}(\tilde{T}) \arrow[r, "\alpha'"] & H^3_{DR,D_S}(D_T) \arrow[r, "\beta'"] & H^4_{DR,S}(T) \arrow[r, "\delta'"] & H^4_{DR,\tilde{S}}(\tilde{T}) \arrow[r, "\sigma'"] & H^4_{DR,D_S}(D_T). \end{tikzcd} \tag{3.3.15}$$

Observe that all squares of this diagram, with the exception of the second from the left, are commutative.

Now, $T \setminus S$ is affine so $H^k_{DR}(T \setminus S) = 0$ for $k \geq 4$ by Lemma 1.2.20; writing down the long exact sequence of algebraic de Rham cohomology groups associated to the pair $(T, T \setminus S)$, we find

$$\cdots \to H^3_{DR}(T \setminus S) \to H^4_{DR,S}(T) \to H^4_{DR}(T) \to 0$$

so there is a surjective morphism $H^4_{DR,S}(T) \twoheadrightarrow H^4_{DR}(T)$. If we pre-compose it with θ we obtain $H^2_{DR}(S) \twoheadrightarrow H^4_{DR}(T)$, so by Theorem 1.2.18 we obtain the desired surjective morphism $\gamma : H^2(S) \twoheadrightarrow H^4(T)$ as in (3.3.3).

Now we need to study how the induced monodromy action T^ϕ on $H^2(S)$ interacts with the Gysin morphism we just found; in order to do this, we will make use of the global invariant cycle theorem:

Theorem 3.3.6. *Let $\tau : X \to Y$ be a morphism between smooth projective varieties such that the general fibre is smooth and connected; set $B := \{y \in Y | X_y := \tau^{-1}(y) \text{ is singular}\}$, call $F := X_y$ for some $y \notin B$ and denote by i the closed immersion $F \hookrightarrow X$. The image of the restriction map*

$$i^* : H^k(X, \mathbb{Q}) \to H^k(F, \mathbb{Q})$$

is the invariant part of $H^k(F, \mathbb{Q})$ under the monodromy action $\pi_1(Y, y) \to Aut(H^k(F, \mathbb{Q}))$.

Proof. [68, Theorem 4.24]. □

Corollary 3.3.7. *The image of the Gysin morphism $H^k(F, \mathbb{Q}) \to H^{k+2}(K, \mathbb{Q})$ does not change if we restrict it to the invariant part of $H^k(F, \mathbb{Q})$ under the monodromy action.*

Proposition 3.3.8. *We have $\gamma(H^2(S)) = \gamma(H^2(S)^{T^\phi})$, so there is a surjective morphism*

$$H^2(S)^{T^\phi} \twoheadrightarrow H^4(T). \tag{3.3.16}$$

Proof. First, we denote by ϕ' the extension of the monodromy action from the fibration $T' \to \mathbb{P}^1$ to $\tilde{T} \to \mathbb{P}^1$, and by $T^{\phi'}$ the induced automorphism of the cohomology groups of $\tilde{T}$ and of the generic fibre $\tilde{S}$. If we denote by $\tilde{\gamma}$ the usual Gysin morphism $H^2(\tilde{S}) \to H^4(\tilde{T})$ then by the global invariant cycle theorem we have

$$\tilde{\gamma}(H^2(\tilde{S})) = \tilde{\gamma}(H^2(\tilde{S})^{T^{\phi'}}).$$

From the resolution square of T we obtain the exact sequences of MHS

$$\cdots \to H^3(D_T) \to H^4(T) \to H^4(\tilde{T}) \to \cdots ;$$

since the Hodge structure on $H^4(T)$ is pure by Proposition 1.3.11 and $H^3(D_T)$ has weights up to 3 by Theorem 1.3.7, we deduce that $H^4(T) \to H^4(\tilde{T})$ is injective.

Now we observe that the diagram

$$\begin{tikzcd} H^2(S) \arrow[r, two heads, "\gamma"] \arrow[d] & H^4(T) \arrow[d, hook] \\ H^2(\tilde{S}) \arrow[r, two heads, "\tilde{\gamma}"] & H^4(\tilde{T}) \end{tikzcd} \tag{3.3.17}$$

is commutative. This can be read off the following diagram (there is a slight abuse of notation: we have switched to singular cohomology, but we maintain the names we gave to morphisms in the algebraic setting):

$$\begin{tikzcd} H^2(S) \arrow[r, "\theta"] \arrow[d, "\delta"] & H^4_S(T) \arrow[r, two heads] \arrow[d, "\delta'"] & H^4(T) \arrow[d, hook] \\ H^2(\tilde{S}) \arrow[r, "\simeq"] & H^4_{\tilde{S}}(\tilde{T}) \arrow[r, two heads] & H^4(\tilde{T}). \end{tikzcd}$$

The left square is commutative, because it is simply the equivalent, in singular cohomology, of the third square of diagram (3.3.15); the right square is commutative too, because the vertical maps are pullbacks and the horizontal maps come from the long exact sequences of the pairs $(T, T \setminus S)$ and $(\tilde{T}, \tilde{T} \setminus \tilde{S})$ respectively, which are functorial. Since the compositions of the maps on the top and on the bottom give exactly the Gysin morphisms γ and γ', we obtain the commutativity of diagram 3.3.17.

The pullback morphism $H^2(S) \to H^2(\tilde{S})$ maps the subspace $V \subset H^2(S)$ which is not T^ϕ-invariant to the subspace $\tilde{V} \subset H^2(\tilde{S})$ which is not $T^{\phi'}$-invariant, and the latter is sent to zero by $\tilde{\gamma}$ by the global invariant cycle theorem; since the diagram (3.3.17) is commutative and $H^4(T) \to H^4(\tilde{T})$ is injective, we deduce that $\gamma(V) = 0$. ☐

The commutativity of (3.3.17) actually allows us to further refine this result. Since $H^2(\tilde{S})$ is a pure HS of weight 2, the kernel of $H^2(S) \to H^2(\tilde{S})$ contains $W_1 H^2(S)$; this, together with the injectivity of $H^4(T) \to H^4(\tilde{T})$ and the commutativity of (3.3.17), implies that $W_1H^2(S) \subset Ker(\gamma)$. The same holds true if we restrict to the invariant part of $H^2(S)$ under the action of T^ϕ, which proves that

$$\gamma(H^2(S)^{T^\phi}) = \gamma(W_2H^2(S)^{T^\phi}). \tag{3.3.18}$$

Remark 3.3.9. The Gysin morphism (3.3.3) restricts to primitive cohomology groups yielding a surjective map $H^2(S)_{\mathrm{prim}} \twoheadrightarrow H^4(T)_{\mathrm{prim}}$. This can be seen in the following way. Assume H_0 is the hyperplane of $\mathbb{P}^4$ that cuts S from T, and choose another hyperplane H of $\mathbb{P}^4$ such that $H \cap T_{sing} = \emptyset$; we can find a resolution of singularities $\pi_T : \tilde{T} \to T$ such that $\tilde{S} := \tilde{T} \cap \pi_T^{-1}(H_0)$ is smooth. If we call $\pi_S : \tilde{S} \to S$ the restriction of π_T to $\tilde{S}$, we can write functorial morphisms

$$\begin{aligned} \pi_S^* : H^2(S) \to H^2(\tilde{S}) &\quad \text{(pullback)} \\ \pi_T^* : H^4(T) \to H^4(\tilde{T}) &\quad \text{(pullback)} \\ \tilde{\gamma} : H^2(\tilde{S}) \twoheadrightarrow H^4(\tilde{T}) &\quad \text{(Gysin).} \end{aligned}$$

Since $\pi_T^{-1}(H) \simeq H$ and $\pi_S^{-1}(H_0 \cap H) \simeq H_0 \cap H$ we deduce that $\tilde{\gamma}([\pi_S^{-1}(H_0 \cap H)]) = [\pi_T^{-1}(H)]$; moreover, the functoriality of the pullback maps implies that $\pi_T^*([H]) = [\pi_T^{-1}(H)]$ and $\pi_S^*(H_0 \cap H) = [\pi_S^{-1}(H_0 \cap H)]$.

The commutativity of (3.3.17) now implies that $\gamma([H_0 \cap H])$ can be written as $[H] + Ker(\pi_T^*)$, but since π_T^* is injective it must be $\gamma([H_0 \cap H]) = [H]$; this proves our claim.

In particular, reasoning as before we obtain a surjective morphism

$$\gamma : H^2(S)^{T^\phi}_{\mathrm{prim}} \twoheadrightarrow H^4(T)_{\mathrm{prim}} \tag{3.3.19}$$

satisfying

$$\gamma(H^2(S)^{T^\phi}_{\mathrm{prim}}) = \gamma(W_2H^2(S)^{T^\phi}_{\mathrm{prim}}). \tag{3.3.20}$$

3.3.3 Part (iii) - Final computations

If we call $U := \mathbb{P}^4 \setminus T$ then from the long exact sequence of MHS associated to the pair $(\mathbb{P}^4, T)$ we deduce

$$\cdots \to H^4(\mathbb{P}^4) \to H^4(T) \to H^5_c(U) \to 0.$$

By using Poincaré duality and the isomorphism of homology and cohomology we obtain the isomorphism $H^5_c(U) \simeq H^3(U)^\vee$; since the map $H^4(\mathbb{P}^4) \to H^4(T)$ is injective we obtain $H^4(T)_{\mathrm{prim}} \simeq H^3(U)^\vee$. This implies in particular that $\dim H^4(T)_{\mathrm{prim}} = \dim H^3(F_{g-f})^{T_{g-f}}$.

If $S \subset \mathbb{P}^3$ is any of the surfaces cut out from T by the hyperplanes $V(ax_0 - bx_1)$ and we call $U' := \mathbb{P}^3 \setminus S$, then from the long exact sequence of MHS associated to the pair $(\mathbb{P}^3, S)$ we deduce

$$\cdots \to H^2(\mathbb{P}^3) \to H^2(S) \to H^3_c(U') \to 0$$

and we obtain $H^2(S)_{\mathrm{prim}} \simeq H^3_c(U')$ at the level of vector spaces. Since we will need to study in detail the MHS on $H^2(S)_{\mathrm{prim}}$ we write the Poincaré duality isomorphism at the level of MHS: by (1.3.1) we have

$$H^2(S)_{\mathrm{prim}} \simeq H^3(U')^\vee(-3). \tag{3.3.21}$$

In order to simplify notations we call $V := H^2(S)_{\mathrm{prim}}$. The isomorphism above implies the following equality of mixed Hodge numbers:

$$h^{p,q}(V) = h^{3-p,3-q}(H^3(U')). \tag{3.3.22}$$

V is a mixed Hodge substructure of $H^2(S)$, so it has weights ≤ 2 and its Hodge filtration can be written as

$$0 = F^3 V \subset F^2 V \subset F^1 V \subset F^0 V = V$$

while for $H^3(U')$ we have

$$0 = F^4 H^3(U') \subset F^3 H^3(U') \subset F^1 H^3(U') \subset F^0 H^3(U') = H^3(U').$$

On $H^3(U')$ we also have the polar filtration (recall 1.4.4):

$$0 = P^4 H^3(U') \subset \cdots \subset P^1 H^3(U') = H^3(U').$$

Since the action of T^ϕ is compatible with all these filtrations, from (3.3.22), the inclusion $F^k H^3(U') \subseteq P^k H^3(U')$ given by Proposition 1.4.5 and the symmetry of mixed Hodge numbers we deduce

$$\begin{aligned} h^{2,0}(V^{T^\phi}) + h^{1,0}(V^{T^\phi}) + h^{0,0}(V^{T^\phi}) &\leq \dim P^3 H^3(U')^{T^\phi} \\ h^{2,0}(V^{T^\phi}) + 2h^{1,0}(V^{T^\phi}) + h^{0,0}(V^{T^\phi}) + h^{1,1}(V^{T^\phi}) &\leq \dim P^2 H^3(U')^{T^\phi}. \end{aligned} \tag{3.3.23}$$

We call now $R := \mathbb{C}[y, z, x_0, x_2]$, $f_S \in R$ the polynomial defining S and $J_{f_S} \subset R$ the associated Jacobian ideal; in this case the map (1.4.6) reads, for $t = 1, 2, 3$, as

$$(R/J_{f_S})_{tn-4} \twoheadrightarrow Gr_P^{4-t} H^3(U') = P^{4-t} H^3(U') / P^{5-t} H^3(U'). \tag{3.3.24}$$

Any class in $P^k H^3(U')$ has a representative of the form

$$\omega_h := \frac{h\Omega}{f_S^k} \qquad \text{with } h \in R_{kn-4}$$

(where $\Omega = y dz \wedge dx_0 \wedge dx_2 - z dy \wedge dx_0 \wedge dx_2 + x_0 dy \wedge dz \wedge dx_2 - x_2 dy \wedge dz \wedge dx_0$), and T^ϕ acts on it by multiplying y and z by η_n; this means that if $h(y, z, x_0, x_2)$ is an element of $(R/J_{f_S})_{kn-4}$ such that

$$h(y, z, x_0, x_2) y z x_0 x_2 = h(\eta_n y, \eta_n z, x_0, x_2) \eta_n^2 y z x_0 x_2 \tag{3.3.25}$$

then the cohomology class $[\omega_h] \in H^3(U')$ is fixed by T^ϕ. If we denote by $((R/J_{f_S})_{tn-4})^{T^\phi}$ the elements of $(R/J_{f_S})_{tn-4}$ satisfying condition (3.3.25), from (3.3.24) we deduce

$$(R/J_{f_S})^{T^\phi}_{tn-4} \twoheadrightarrow Gr_P^{4-t}H^3(U')^{T^\phi} = P^{4-t}H^3(U')^{T^\phi}/P^{5-t}H^3(U')^{T^\phi} \quad \text{for } t = 1, 2, 3. \tag{3.3.26}$$

Let us compute the dimensions of the $(R/J_{f_S})^{T^\phi}_{tn-4}$: a monomial $y^a z^b x_0^c x_2^d$ satisfies condition (3.3.25) if and only if

$$e^{\frac{2\pi i(a+1)}{n}} e^{\frac{2\pi i(b+1)}{n}} = 1 \Longleftrightarrow a + b = kn - 2 \ \exists k \in \mathbb{Z}. \tag{3.3.27}$$

Since J_{f_S} contains y^{n-1} and z^{n-1}, a monomial $y^a z^b x_0^c x_2^d \in (R/J_{f_S})_{tn-4}$ can satisfy (3.3.27) only for $k = 1$; this implies in particular that $(R/J_{f_S})^{T^\phi}_{n-4} = 0$. From this we deduce that

$$Gr^3_P H^3(U')^{T^\phi} = P^3 H^3(U')^{T^\phi} = 0$$

which implies $Gr^2_P H^3(U')^{T^\phi} = P^2 H^3(U')^{T^\phi}$; by (3.3.23) we obtain

$$\dim V^{T^\phi} = h^{1,1}(V)^{T^\phi} \leq \dim Gr^2_P H^3(U')^{T^\phi}.$$

Since there are $n-1$ choices of non-negative $a, b < n-1$ that give $a + b = n - 2$, we have $(n-1)^2$ monomials in $(R/J_{f_S})_{2n-4}$ satisfying condition (3.3.25); this gives

$$\dim V^{T^\phi} \leq (n-1)^2. \tag{3.3.28}$$

Now we compute the dimension of $H^1(F_{y^n+z^n})$ by studying the Steenbrink spectra of the homogeneous isolated singularities of $\mathbb{C}$ given by $y^n = 0$ and $z^n = 0$. Recall (Remark 2.1.12) that if $h(y_1, \dots, y_{m+1}) = 0$ is an isolated weighted homogeneous singularity of degree d and weights w_i, the Steenbrink spectrum of h is the formal sum

$$sp(h) := \sum_{\alpha \in \mathbb{Q}} \alpha \nu(\alpha) \qquad \text{with } \nu(\alpha) = \dim M(h)_{(\alpha+1)d-w} \tag{3.3.29}$$

and that $\nu(\alpha)$ is also the dimension of the $e^{-2\pi i\alpha}$-eigenspace of the monodromy operator acting on $Gr_F^{\lfloor m-\alpha \rfloor} H^m(F_h)$.

For y^n we have $d = n$, $w = 1$ and $M(y^n) = \mathbb{C} \oplus \mathbb{C}y \oplus \cdots \oplus \mathbb{C}y^{n-2}$, so the non-zero parts of $M(y^n)$ have weights $0, \dots, n-2$ and dimension 1. In order to have $(\alpha+1)n-1 = j$ for $j \in [0, n-2]$ we need $\alpha = \frac{j+1-n}{n}$, which implies $sp(y^n) = \sum_{j=0}^{n-2} (\frac{j+1-n}{n})$; this means the monodromy operator on $H^0(F_{y^n}, \mathbb{C})$ has $n-1$ eigenspaces of dimension 1 with eigenvalues η_n^a for $a \in [1, n-1]$ (and the same goes for $H(F_{z^n})$).

By Theorem 2.1.7 we deduce that $H^1(F_{y^n+z^n})$ has dimension $(n-1)^2$ and it is the direct sum of monodromy eigenspaces with eigenvalues η_n^{a+b} for $a, b \in [1, n-1]$. The equality $\eta_n^{a+b} = \eta_n^k$ is satisfied by $n-2$ choices of the couple (a, b) for $k \neq 0$, while for $k = 0$ the choices are $n-1$: this means that in $H^1(F_{y^n+z^n})$ the fixed part under the monodromy action has dimension $n-1$, while all the other $n-1$ eigenspaces have dimension $n-2$.

Theorem 2.1.7 also allows us to write

$$H^3(F_{g-f})^{T_{g-f}} = \bigoplus_{0 \leq \alpha < 1} H^1(F_g)_{1-\alpha} \otimes H^1(F_f)_\alpha$$

where the subscript α indicates the eigenspace relative to $e^{2\pi i\alpha}$. If we denote by ϵ_i the dimension of $H^1(F_f)_{\eta_n^i}$ then $\epsilon_0 = n-1$, so we can write the dimension of the right-hand side as

$$(n-1)^2 + \sum_{i=1}^{n-1}(n-2)\epsilon_i = (n-1)^2 + (n-2)\sum_{i=1}^{n-1}\epsilon_i$$

From the surjective Gysin morphism (3.3.19) and from (3.3.28) we deduce that

$$\dim H^3(F_{g-f})^{T_{g-f}} \leq (n-1)^2$$

so $\epsilon_i = 0$ for all $i \neq 0$, which means exactly that the Alexander polynomial of the arrangements we consider is trivial.

This concludes the proof of Theorem 3.3.1.

Remark 3.3.10. While at the beginning of this section we assumed that $\overline{\mathcal{A}}$ does not contain the line connecting P_1 and P_2, the proof we have given can be applied to that case too.

Bibliography

[1] Alexander J.W. - Topological invariants of knots and links. Trans. Amer. Math. Soc. **30** (1928), no. 2, 275-306.

[2] Arapura D. - Geometry of cohomology support loci for local systems I. J. Alg. Geom. **6** (1997), no. 3, 563-597.

[3] Brieskorn E. - Die Monodromie der isolierten Singularitäten von Hyperflächen. Manuscr. Math **2** (1970), 103-161.

[4] Budur N., Dimca A., Saito M. - First Milnor cohomology of hyperplane arrangements. Topology of algebraic varieties and singularities, Contemp. Math. **538** (2011), 279-292, Amer. Math. Soc., Providence, RI.

[5] Cogolludo-Agustin J-I., Libgober A. - Mordell-Weil groups of elliptic threefolds and the Alexander module of pane curves. J. Reine Angew. Math **697** (2014), 15-55.

[6] Cohen D., Dimca A., Orlik P. - Nonresonance conditions for arrangements. Ann. Inst. Fourier (Grenoble) **53** (2003), no. 6, 1883-1896.

[7] Cohen D., Suciu A.I. - Characteristic varieties of arrangements. Math. Proc. Cambridge Phil. Soc. **127** (1999), no. 1, 33-53.

[8] Cohen D., Suciu A.I. - On Milnor fibrations of arrangements. J. London Math. Soc. (2) **51** (1995), no. 1, 105-119.

[9] Deligne P. - Théorie de Hodge II, III. Publ. Math. IHES **40** (1971), 5-58 and **44** (1974), 5-77.

[10] Denham G., Suciu A.I. - Multinets, parallel connections, and Milnor fibrations of arrangements. Proc. London Math. Soc. **108** (2014), no. 6, 1435-1470.

[11] Dimca A. - Betti numbers of hypersurfaces and defects of linear systems. Duke Math. J. **60** (1990), 285-298.

[12] Dimca A. - Hyperplane Arrangements, an introduction. Universitext, Springer-Verlag, New York, 2017.

[13] Dimca A. - Monodromy of triple point line arrangements. Singularities in Geometry and Topology (2011), 71-80, Adv. Stud. Pure Math., vol. 66, Math. Soc. Japan, Tokyo (2015).

[14] Dimca A. - On the Milnor fibrations of weighted homogeneous polynomials. Compositio Math. **76** (1990), 19-47.

[15] Dimca A. - On the Milnor monodromy of the irreducible complex reflection arrangements. arXiv:1606.04048.

[16] Dimca A. - Singularities and topology of hypersurfaces. Universitext, Springer-Verlag, New York, 1992.

[17] Dimca A. - Tate properties, polynomial-count varieties, and monodromy of hyperplane arrangements. Nagoya Math. J. **206** (2012), 75-97.

[18] Dimca A., Ibadula D., Măcinic A. - Pencil type line arrangements of low degree: classification and monodromy. Ann. Sc. Norm. Super. Pisa Cl. Sci. **15** (2016), 249-267.

[19] Dimca A., Lehrer G. - On the cohomology of the Milnor fibre of a hyperplane arrangement with symmetry. Configuration Spaces: Geometry, Topology and Representation Theory, 319-360, Springer INDAM Series, vol. 14, Springer, Cham (2016).

[20] Dimca A., Papadima S. - Finite Galois covers, cohomology jump loci, formality properties, and multinets. Ann. Sc. Norm. Super. Pisa Cl. Sci. **10** (2011), no. 2, 253-268.

[21] Dimca A., Papadima S., Suciu A.I. - Topology and geometry of cohomology jump loci. Duke Math. Journal **148** (2009), no. 3, 405-457.

[22] Dimca A., Sticlaru G. - On the Milnor monodromy of the exceptional reflection arrangement of type G_{31}. arXiv:1606.06615.

[23] Dolgachev I. - Weighted projective varieties. Group Actions and Vector Fields, Proceedings 1981, Lecture Notes in Mathematics 956, Springer-Verlag, Berlin (1982), 34-71.

[24] Esnault H. - Fibre de Milnor d'un cône sur une courbe plane singulière. Invent. Math. **68** (1982), 477-496.

[25] Falk M. - Arrangements and cohomology. Ann. Combin. **1** (1997), 135-157.

[26] Falk M. - Resonance varieties over fields of positive characteristic. Int. Math. Res. Not. **2007** (2007), no.3, article ID rnm009, 25 pages.

[27] Falk M., Yuzvinsky S. - Multinets, resonance varieties, and pencils of plane curves. Compositio Math. **143** (2007), no. 4, 1069-1088.

[28] Greuel G.M., Lossen C., Shustin E. - Introduction to singularities and deformations. Springer-Verlag, New York, 2007.

[29] Griffiths P. - On the periods of certain rational integrals I, II. Ann. of Math. **90** (1969), 460-541.

[30] Guillén, F., Navarro Aznar, V., Pascual Gainza, P., Puerta, F. - Hyperrésolutions cubiques et descente cohomologique. Lecture Notes in Mathematics, vol. 1335, Springer-Verlag, Berlin, 1988, Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982.

[31] Gunning R.C., Rossi H. - Analytic functions of several complex variables. Prentice-Hall, Englewood Cliffs, N.J., 1965.

[32] Hartshorne R. - Algebraic geometry. Springer-Verlag, New York, 1997.

[33] Hartshorne R. - On the de Rham cohomology of algebraic varieties. Publ. Math. I.H.E.S., **45** (1976), 5-99.

[34] Hartshorne R. - Residues and duality. Lecture Notes in Math. Springer, **20**, 1966.

[35] Hauser H., Randell R. - Report on the problem session, in: Singularities (Iowa City, IA, 1986), 119-134, Contemp. Math., vol. 90, Amer. Math. Soc., Providence, RI, 1989.

[36] Hironaka E. - Alexander stratifications of character varieties. Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 555-583.

[37] Karpishpan, Y. - Pole order filtration on the cohomology of algebraic links. Compositio Math. **78** (1991), 213-226.

[38] Kirby R. - Problems in low-dimensional topology, in: Geometric Topology (Athens, GA, 1993), 35-473, AMS/IP Stud. Adv. Math., vol. 2.2., Amer. Math. Soc., Providence, RI, 1997.

[39] Kloosterman R. - Mordell-Weil lattices and toric decompositions of plane curves. Mathematische Annalen **367**, 755-783, 2017.

[40] Lê Dung Trang - Some remarks on relative monodromy. Real and complex singularities (Oslo 1976), Sijthoff and Noordhoff, Amsterdam (1977), 397-403.

[41] Libgober A. - Alexander invariants of plane algebraic curves. Singularities, Part 2 (Arcata, Calif., 1981), 135-143, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983.

[42] Libgober A. - Alexander polynomial of plane algebraic curves and cyclic multiple planes. Duke Math. J. **49** (1982), no. 4, 833-851.

[43] Libgober A. - Characteristic varieties of algebraic curves. Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), Kluwer Acad. Publ., Dordrecht (2001), 215-254.

[44] Libgober A. - Eigenvalues for the monodromy of the Milnor fibers of arrangements. Trends in singularities, 141-150, Trends Math., Birkhäuser, Basel (2002).

[45] Libgober A. - Lectures on topology of complements and fundamental groups. Singularity theory, 71137, World Sci. Publ., Hackensack, NJ, 2007.

[46] Libgober A. - On combinatorial invariance of the cohomology of Milnor fiber of arrangements and Catalan equation over function fields. Arrangements of hyperplanes (Sapporo 2009), 175-187, Adv. Stud. Pure Math., vol. 62, Math. Soc. Japan, Tokyo, 2012.

[47] Libgober A. - On Mordell-Weil group of isotrivial abelian varieties over function fields. Math. Ann. **357** (2013), 605-629.

[48] Libgober A. - On the homology of finite abelian coverings. Topology Appl. **43** (1992), no. 2, 157-166.

[49] Libgober A. - Position of singularities of hypersurfaces and the topology of their complements. J. Math. Sci. **82** (1996), 3194-3210. Algebraic Geometry, 5.

[50] Libgober A., Yuzvinsky S. - Cohomology of the Orlik-Solomon algebras and local systems. Compositio Math. **21** (2000), no. 3, 337-361.

[51] Loeser F., Vaquié M. - Le polynôme d'Alexander d'une courbe plane projective. Topology **29** (1990), no. 2, 163-173.

[52] Măcinic A., Papadima S., Popescu R. - Modular equalities for complex reflection arrangements. Doc. Math. **22** (2017), 135-150.

[53] Marco-Buzunariz M.A. - A description of the resonance variety of a line combinatorics via combinatorial pencils. Graphs Combin. **25** (2009), no. 4, 469-488.

[54] Milnor J. - Singular points of complex hypersurfaces. Annals of Math. Studies, vol. 61, Princeton Univ. Press, Princeton, NJ, 1968.

[55] Milnor J., Orlik P. - Isolated singularities defined by weighted homogeneous polynomials. Topology **9** (1970), 385-393.

[56] Merle M., Tessier B. - Conditions d'adjonction, d'après Du Val. Lect. Notes Math. **777** (1976), 229-295.

[57] Oka M. - On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials. Topology 12 (1973), 19-32.

[58] Orlik P., Solomon L. - Combinatorics and topology of complements of hyperplanes. Invent. Math. **56** (1980), no. 2, 167-189.

[59] Papadima S., Suciu A.I. - The Milnor fibration of a hyperplane arrangement: from modular resonance to algebraic monodromy. Proc. of the London Math. Soc **114**, no. 6 (2017), 961-1004.

[60] Papadima S., Suciu A.I. - The spectral sequence of an equivariant chain complex and homology with local coefficients. Trans. Amer. Math. Soc. **362** (2010) no.5, 2685-2721.

[61] Pereira J., Yuzvinsky S. - Completely reducible hypersurfaces in a pencil. Adv. Math. **219** (2008), no. 2, 672-688.

[62] Peters C.A.M., Steenbrink J. - Mixed Hodge Structures. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008.

[63] Randell R. - Milnor fibers and Alexander polynomials of plane curves. Singularities, Part 2 (Arcata, Calif., 1981), 415-419, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983.

[64] Siersma D. - Quasihomogeneous singularities with transversal type A_1. Contemporary Mathematics 90, Amer. Math. Soc. (1989), 261-294.

[65] Steenbrink J. - Intersection form for quasi-homogeneous singularities. Compositio Math. **34** (1977), 211-223.

[66] Suciu A.I. - Hyperplane arrangements and Milnor fibrations. Ann. Fac. Sci. Toulouse Math. **23** (2014), no. 2, 417-481.

[67] Voisin C. - Hodge theory and complex algebraic geometry I. Cambridge University Press, **76** (2002).

[68] Voisin C. - Hodge theory and complex algebraic geometry II. Cambridge University Press, **77** (2003).

[69] Yuzvinsky S. - A new bound on the number of special fibres in a pencil of curves. Proc. Amer. Math. Soc. **137** (2009), no. 5, 1641-1648.

[70] Yuzvinsky S. - Realization of finite abelian groups by nets in $\mathbb{P}^2$. Compositio Math. **140** (2004), no. 6, 1614-1624.

[71] O. Zariski - On linear connection index of the algebraic surfaces $z^n = f(x, y)$. Proc. National Acad. Sci. USA, vol. 15, (1929).

[72] O. Zariski - On the irregularity of cyclic multiple planes. Ann. Mat. **32** (1931), 485-511.

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