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CICLO XXXII

Weak forms of the Krull-Schmidt theorem and Prüfer rings in distinguished constructions

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Abstract

This thesis is divided in two chapters. The first one concerns direct-sum decompositions in additive categories. It is well known that if a module admits a direct-sum decomposition into indecomposable modules with local endomorphism rings, then this decomposition is essentially unique, up to isomorphism and a permutation of the direct summands. However, there are situations in which direct-sum decompositions into indecomposable modules are not essentially unique. Among these cases, particularly interesting are those in which it is possible to find some kind of regularity: direct-sum decompositions can be described via two invariants up to two permutations. Such behaviour was firstly discovered for uniserial modules by A. Facchini in 1996, and it was subsequently investigated for several other classes of modules, such as cyclically presented modules over a local ring, couniformly presented modules and kernels of morphisms between indecomposable injective modules. In this thesis, we provide examples of additive categories in which direct-sum decompositions can be classified via finitely many invariants. It is worth noting that, in our constructions, we treat cases in which the number of invariants needed to describe finite direct-sum decompositions can be arbitrarily large.

The second chapter is devoted to the study of Prüfer (commutative) rings with zero-divisors. We investigate the so-called "Prüfer-like conditions" in several constructions, most of them related to pullbacks. It is well known that fiber products provide a rich source of examples and counterexamples in Commutative Algebra, because of their ability of producing rings with certain predetermined properties. Our investigation moves from very natural settings, for example those of regular conductor squares, up to more technical constructions, such as bi-amalgamated algebras, introduced by Kabbaj, Louartiti and Tamekkante in 2017 as a generalization of that of amalgamated algebras. Our main results in the pullback framework cover several different situations studied up to now by Bakkaki and Mahdou, Boynton, Houston and Taylor. We also investigate Prüfer ring from other points of view. We introduce the notion of regular morphism and we prove that if a ring R is the homomorphic image of a Prüfer ring via a regular morphism, then R is Prüfer. Finally, we turn our attention to the ideal-theory of pre-Prüfer rings, proving a number of generalizations of some results of Boisen and Larsen.

Riassunto

Questa tesi è divisa in due capitoli. Il primo riguarda le decomposizioni in somme dirette in categorie additive. E' ben noto che se un modulo ammette una decomposizione come somma diretta di moduli con anello endomorfismi locale, allora questa decomposizione è essenzialmente unica, a meno di isomorfismo e di una permutazione degli addendi diretti. Tuttavia, ci sono situazioni in cui tale decomposizione come somma diretta di moduli indecomponibili non è unica. Tra tutte, particolarmente interessanti sono quelle classi di moduli per cui è possibile trovare una sorta di regolarità: le decomposizioni in somma diretta possono essere descritte attraverso due invarianti a meno di due permutazioni. Tale comportamento è stato individuato per la prima volta da A. Facchini nel 1996 per i moduli uniseriali, ed è successivamente stato studiato per altre classi di moduli, come i moduli ciclicamente presentati su un anello locale, i moduli couniformemente presentati e i nuclei di morfismi tra moduli iniettivi indecomponibili. In questa tesi forniamo esempi di categorie additive in cui le decomposizioni in somme dirette possono essere classificate attraverso un numero finito di invarianti. Vale la pena notare che, nelle nostre costruzioni, trattiamo casi in cui il numero di invarianti necessari per descrivere tali decomposizioni può essere arbitrariamente grande.

Il secondo capitolo è dedicato allo studio degli anelli di Prüfer (commutativi) con zero divisori. Studiamo le cosiddette "condizioni di tipo Prüfer" in diverse costruzioni, molte delle quali collegate a pullback di anelli. È noto che i prodotti fibrati forniscono una ricca fonte di esempi e controesempi in Algebra Commutativa, grazie alla loro capacità di produrre anelli con determinate proprietà. Il nostro studio va da situazioni molto naturali, come i "regular conductor squares", fino a costruzioni più tecniche, quali le algebre bi-amalgamate, introdotte da Kabbaj, Louartiti e Tamekkante nel 2017 come generalizzazione di quella delle algebre amalgamate. I nostri principali risultati nell'ambito dei pullback includono e generalizzano diverse situazioni studiate fino ad ora da Bakkaki e Mahdou, Boynton, Houston e Taylor. Studiamo gli anelli di Prüfer anche da altri punti di vista. Introduciamo la nozione di morfismo regolare e mostriamo che se un anello R è l'immagine omomorfa di un anello di Prüfer tramite un morfismo regolare, allora R è Prüfer. Infine,

rivolgiamo la nostra attenzione agli anelli pre-Prüfer, generalizzando alcuni risultati di Boisen e Larsen nel caso integro.

Contents

Introduction

This thesis is divided in two chapters and its content is based on the research done during my PhD under the supervision of Professor Alberto Facchini, as part of different collaborations with him, Dott. Susan El-Deken and Dott. Carmelo Antonio Finocchiaro. This thesis contains results both in module and category theory and in commutative ring theory.

Chapter 1

The first chapter is devoted to the so-called "weak forms of the Krull-Schmidt Theorem" in additive categories and it is based on joint works with A. Facchini and S.F. El-Deken [19, 20, 21].
In algebra, structures - for instance objects of a category - some times decompose into indecomposable ones. The study of this kind of phenomena has its roots in the well known result of G. Frobenius and L. Stickelberger (1879 [51]), which states that every finite abelian group is isomorphic to a direct sum of primary cyclic groups whose orders are uniquely-determined powers of primes. In 1909, J.H. Wedderburn [82] tried to extend this result to non-abelian groups, showing that any two direct product decompositions of a finite group $G = H_1 \times \cdots \times H_r \cong K_1 \times \cdots \times K_t$ into indecomposable factors are isomorphic, that is, $r = t$ and there exist an automorphism $\varphi : G \to G$ and a permutation σ of $\{1, 2, \dots, r\}$ such that $\varphi(H_i) = K_{\sigma(i)}$ for every $i = 1, 2, \dots, r$. Nevertheless, his proof had some gaps. Two years later, in 1911, R. Remak [89] proved Weddenburn's result, also showing that the automorphism can be chosen to be central. Then, W. Krull (1925 [77]) and O. Schmidt (1929 [92]) transferred these results to the case of modules that are both Artinian and Noetherian. The classical Krull-Schmidt Theorem for modules states that any module of finite composition length decomposes as a direct sum of indecomposable modules in an essentially unique way up to isomorphism and a permutation of the indecomposable direct summands. This result was later extended to the case of arbitrary direct sums of modules with a local endomorphism ring by G. Azumaya (1950 [5]). Nowadays the name "Krull-Schmidt" is given to any theorem concerning uniqueness of direct-sum decompositions into indecomposable direct summands. This is a very classical topic that has a crucial relevance in the study of algebraic

structures. Of course, there are cases in which direct-sum decompositions are not essentially unique. It is worth mentioning that Krull already knew that the Krull-Schmidt Theorem does not hold for arbitrary Noetherian modules, which means that the ascending chain condition does not suffices to have uniqueness of direct-sum decompositions. In light of this fact, a question that naturally arises is if the Krull-Schmidt Theorem holds for the class of Artinian modules. This problem was originally posed by Krull himself in 1932 but the answer was found only sixty years later, in 1995, when Facchini, Herbera, Levy and Vámos showed that the Krull-Schmidt Theorem fails for Artinian modules [40].

In the last three decades, new interesting examples in which direct-sum decompositions are not unique made their appearance. The starting point of these examples can be dated back to 1975, when Warfield proved that every finitely presented module over a serial ring is a finite direct sum of uniserial modules and posed a problem similar to that of Krull, essentially asking whether the Krull-Schimdt Theorem holds for finite direct sums of uniserial modules (in particular, for finitely presented modules over serial rings). The negative answer to Warfield's question was given by A. Facchini in 1996 [34]. Even though the lack of uniqueness, this situation displays some kind of regularity: direct-sum decompositions can be classified via two invariants and a weak version of the Krull-Schmidt Theorem can be proved. Such behaviour has subsequently been discovered and studied for several classes of modules, including biuniform modules (Facchini, 1996 [34]), cyclically presented modules over local rings (Amini, Amini and Facchini, 2008, [3]), couniformly presented modules (Facchini and Girardi, 2010 [37]), kernels of morphisms between indecomposable injective modules (Facchini, Ecevit and Koşan, 2010 [36]). But what do we mean by "weak version of the Krull-Schmidt Theorem"? In order to answer this question and try to better explain what we mean, let $\mathcal{C}$ be any full subcategory of Mod-R (for a fixed ring R) whose class of objects consists of one of those just mentioned above (for instance, the class of biuniform modules). Then, it is possible to find two invariants under isomorphisms on $\mathcal{C}$, say $\sim_a$ and $\sim_b$, such that the behaviour of direct-sums of objects of $\mathcal{C}$ can be described as follows: let $M_1, \dots, M_r, N_1, \dots, N_t$ be $r + t$ non-zero objects in $\mathcal{C}$; then $M_1 \oplus \cdots \oplus M_r \cong N_1 \oplus \cdots \oplus N_t$ if and only if $r = t$ and there exist two permutations α and β of $\{1, 2, \dots, r\}$ such that

$M_i \sim_a N_{\alpha(i)}$ and $M_i \sim_b N_{\beta(i)}$ for every $i = 1, 2, \ldots, r$. So, we can say that the uniqueness of the decomposition is given not up to one permutation, but up to two permutations. It is worth noting that the invariants needed to describe direct-sum decompositions are closely related to the maximal ideals of the endomorphism rings of the modules.

General theories that include all these cases has been recently developed by Facchini, Perone and Příhoda (cf. [45], [42]). They investigate the Krull-Schmidt Theorem in suitable categorical frameworks, making use of the so-called ideals in a category and factor categories. These notions were first introduced by Mitchell [85]. Roughly speaking, these ideals allow to treat all the endomorphism rings of the objects at the same time and to consider "more simple" categories (i.e. factor categories) in which the endomorphism rings of the indecomposable objects become local rings.

In this chapter, we study some additive categories in which it is possible to find objects whose behaviour with respect to direct-sum decompositions is very similar to those we have just mentioned. The main difference is that we treat cases in which the number of invariants needed to describe finite direct-sum decompositions is bigger than two, and, in some constructions, this number can be arbitrarily large.

This chapter is organized as follows.

We list in **Section 1.1** all the notations and the terminology that will be used in this chapter.

Section 1.2 collects all preliminary results. We start by recalling the notions of Goldie dimension and dual Goldie dimension of a module, as well as some basics on semilocal rings and modules with a semilocal endomorphism ring. In the second part of this section, we focus our attention on modules whose endomorphism rings have two maximal right ideals. If a ring has exactly two maximal right ideals, they are necessarily two-sided. These rings are called rings of type 2 and a module M whose endomorphism ring is of type 2 (that is, $\mathrm{End}_R(M)$ has exactly two maximal right ideals) is said to be a module of type 2. More generally, if a ring R has exactly n maximal right ideals (for a positive integer n) and they are all two-sided, then we say that R is of type n. A module is said to be of type n if $\mathrm{End}_R(M)$ is a ring of type n. This notion was introduced by Facchini and Příhoda in [45]. We consider several classes of modules whose endomorphism rings are either local or of type 2, namely the classes of uniserial modules, cyclically presented modules over a local ring, couniformly presented modules, and kernels of morphisms between indecomposable injective modules. As we have said before, in all these cases direct-sum decompositions are described by two invariants, closely related to the maximal ideals of the endomorphism rings of the modules. Finally, we recall two results of Facchini, Perone and Příhoda (Theorems 1.35 and 1.36), that provide generalizations of these cases from a categorical point of view.

Section 1.3 is based on a joint work with A. Facchini and S. F. El-Deken [20]. In this section, we focus our attention on the category Morph(Mod-R), whose objects are morphisms $\mu_M : M_0 \to M_1$ between right R-modules. It is well known [50, 65] that this category is equivalent to the category of right modules over the triangular matrix ring $T := \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$. We provide several results about the endomorphism rings of objects of Morph(Mod-R). For instance, in Proposition 1.42 we show that if $\mu_M : M_0 \to M_1$ is an object of Morph(Mod-R) and M_0 and M_1 are right R-modules of type m and n respectively, then the endomorphism ring $\mathrm{End}_{\mathrm{Morph(Mod}\text{-}R)}(\mu_M)$ of μ_M in Morph(Mod-R) is a ring of type $\leq n + m$. Moreover, it is possible to give a precise description of the maximal ideals of $\mathrm{End}_{\mathrm{Morph(Mod}\text{-}R)}(\mu_M)$ in terms of the maximal ideals of $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$. Then, we continue by investigating morphisms between modules of type 1 (i.e., between modules with a local endomorphism ring) and we show that a weak version of the Krull-Schmidt Theorem also holds in this setting (Theorem 1.49). The last part of this section is devoted to the study of morphisms between uniserial modules. We describe the endomorphism rings of such objects and we state a weak version of the Krull-Schmidt Theorem (that is proved in Section 1.5), in which the behaviour of direct-sum decompositions is described by 4 invariants (Theorem 1.53). These invariants generalize those of monogeny class and epigeny class defined in [34] for uniserial modules.

Section 1.4 deals with other particular additive categories in which it is possible to detect objects of finite type and study the behaviour of their direct-sums. These categories are defined as follows. Let R be a ring and let n be a fixed positive integer. We consider the category $\mathcal{E}_n$ whose objects are right R-modules M with a fixed chain of submodules

$$0 = M^{(0)} \leq M^{(1)} \leq M^{(2)} \leq \cdots \leq M^{(n)} = M.$$

With abuse of notation, we simply denote by M such an object. A morphism in $\mathcal{E}_n$ between two chains M and N is a right R-module morphism $f : M \to N$ such that $f(M^{(i)}) \subseteq N^{(i)}$ for every $i = 1, \dots, n$. This category was introduced in [19] with the purpose to generalize some results given in [21] for short exact sequences of right R-modules. Particular attention is given to chains M whose consecutive factor modules $M^{(i)}/M^{(i-1)}$ $(i = 1, \dots, n)$ are non-zero uniserial modules. Following the same pattern of Section 1.3, we describe the endomorphism rings of chains with non-zero uniserial factors and state a weak version of the Krull-Schmidt Theorem (that will be proved in Section 1.5) in which direct-sum decompositions are described via $2n$ invariants (Theorem 1.60). Then, we provide some examples and conclude by studying what occurs for chains with some zero factors.

Section 1.5 is entirely devoted to the proof of Theorems 1.53 and 1.60. We merge together different techniques, notions and ideas taken from several works and papers (cf. [19, 21, 31, 41, 44, 45]). We start this section by underlining the common patterns presented in Sections 1.3 and 1.4. Then we describe the interplay between the maximal ideals of the endomorphism rings of the objects and the invariants needed to describe their direct-sums. In our results, we largely make use of a particular class of ideals in a category. These ideals were introduced in [43] for any preadditive category and are defined as follows. Let A be an object of a preadditive category $\mathcal{A}$ and I be a two-sided ideal of the ring $\mathrm{End}_{\mathcal{A}}(A)$. Let $\mathcal{I}$ be the ideal of the category $\mathcal{A}$ defined as follows. A morphism $f\colon X \to Y$ in $\mathcal{A}$ belongs to $\mathcal{I}(X, Y)$ if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{A}$. The ideal $\mathcal{I}$ is called the ideal of $\mathcal{C}$ associated to I.

To conclude, in **Section 1.6** we discuss other cases. In Sections 1.3 and 1.4 we almost always deal with objects related to uniserial modules (for instance, Theorem 1.53 is stated for morphisms between non-zero uniserial modules) and all our results are, in some sense, a generalization of those proved in [34] for uniserial modules. So, it is natural to try to go back over the results presented in this chapter and replace the class of uniserial modules with one of the other classes mentioned in Section 1.2, for instance, the class of couniformly presented modules or the class of modules with a local endomorphism ring. For this reason, in the last few pages of this chapter we sketch how to adapt the proof given in Section 1.5 to all these other cases.

Chapter 2

The second chapter is based on joint papers with C.A. Finocchiaro [22, 23] and it is devoted to the study of Prüfer rings in several different constructions. In 1932, H. Prüfer introduced a new class of integral domains, namely the domains in which every finitely generated ideal is invertible [88]. These rings were named in his honour by W. Krull in 1936 [78], two years after Prüfer's death. As noted by R. Gilmer in the introduction of his book, *Multiplicative Ideal Theory* [58], the notion of Prüfer domain has a predominant role in the classical ideal theory. In the literature, it is possible to find plenty of equivalent characterizations of Prüfer domains, confirming the fact that this notion has played (and is still playing) a central role in non-Noetherian commutative ring theory. For instance, we can think of a Prüfer domain as the non-Noetherian version of a Dedekind domain, or as the global version of a valuation domain. Among natural examples of non-Noetherian (and non-local) Prüfer domains, we mention the ring

$$\mathrm{Int}(\mathbb{Z}) := \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

of integer-valued polynomials over $\mathbb{Z}$ and the ring of entire functions on the complex plane $\mathbb{C}$. Though Prüfer domains are not necessarily Noetherian, they must be coherent, because finitely generated invertible ideals are finitely related.

Another evidence of the ubiquity of this notion relies in the fact that its many characterizations fall into very different areas of commutative algebra. Indeed, Prüfer domains can be characterized in terms of arithmetical ideal properties, localizations, flatness and other homological properties, integral closures, etc.

Because of the great number of equivalent definitions of Prüfer domains, a lot of extensions of this notion can be found in the literature both in the case of domains and rings with zero divisors. In our exposition, we discuss some "Prüfer-like conditions", considering the following extensions of the notion of Prüfer domain:

1. R is a semihereditary ring;
2. $\mathrm{w.gl.dim}(R) \leq 1$;
3. R is an arithmetical ring;
4. R is a Gaussian ring;
5. R is a Prüfer ring.

The study of these notions goes from the first papers of Jersen [71, 72], Butts and Smith [17] and Griffin [68] up to the recent works of Bazzoni and Glaz [8, 9, 64], passing through the efforts of several other authors, too many to cite them all. We will mention some of them in this thesis, and we refer to the excellent survey of Bazzoni and Glaz [9] for a more comprehensive list.

There is another way to generalize the concept of Prüfer rings, recently developed by M. Knebush and D. Zhang [76]. Notions of regularity and invertibility of ideals are relativized to arbitrary ring extensions $R \subseteq T$. Such extensions are called *Prüfer extensions* if the inclusion $R \hookrightarrow T$ is a flat epimorphism (in the category of rings) and every finitely generated T-regular ideal of R is T-invertible. This notion recover the classical one of Prüfer ring if $T = \mathrm{Tot}(R)$, that is, $R \subseteq \mathrm{Tot}(R)$ is a Prüfer extension if and only if R is a Prüfer ring.

It is clear that Prüfer extensions have an intrinsic ring-theoretic meaning. Nevertheless, it is worth mentioning that they find many applications in real and p-adic geometry, as largely explained in the introduction of Knebush and Zhang's book [76]. Among all, particularly interesting is the following example [76, Chapter 1, Example 14]: let X be a topological space and let $C(X, \mathbb{R})$ [resp. $C^b(X, \mathbb{R})$] denote the ring of continuous [resp. bounded and continuous] $\mathbb{R}$-valued functions on X. Then $C^b(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$ is a Prüfer extension.

In this chapter, we investigate Prüfer-like conditions from several different points of view. A substantial part of this Chapter is devoted to the study of these properties in constructions related to fiber products. It is worth mentioning that Greenberg and Vasconcelos (1974 [66], 1976 [67]) were probably the first authors who studied pullbacks of rings with zero-divisors. They considered very particular conductor squares with the aim of classifying rings of weak global dimension 2. An important input towards the investigation of the ideal theory in pullbacks was given by Fontana (1980 [48]) and by Gabelli, Houston and Taylor(1997 [56], 2000 [57], 2007 [69]). Prüfer conditions in fiber products of rings with zero-divisors were recently considered by Bakkari and Mahdou in some $D + \mathfrak{m}$ constructions (2009 [6]) and by Boynton in regular conductor squares (2007 [14], 2008 [15], 2011 [16]).

It is worth recalling that the great interest in pullbacks is due to the fact that this kind of constructions are important tools in the arsenal of commutative algebraists, because of their ability of producing rings (with zero-divisors) with certain predetermined properties. For this reason, they provide a rich source of examples and counterexamples in Commutative Algebra. Lucas' paper [80] is probably the best reference for those who want to have a deep insight in this circle of ideas.

This chapter is organized as follows.

In **Section 2.1** we collect basic notations and terminology we will use in the sequel.

Section 2.2 contains almost all preliminary results we need in this chapter, except for some basics on bi-amalgamated algebras that we decided to place in Section 2.5, because of their technical features. In the first part of this section, we outline the main properties of the "Prüfer-like conditions" mentioned before, displaying the relations between them. The second part contains some background on Prüfer extensions. In the last part of Section 2.2, we recall some basic properties of fiber products and we briefly list several classical constructions in Commutative Algebra, like the Nagata idealization, the $D + \mathfrak{m}$, $A + XB[X]$ and $A + XB[[X]]$ constructions, the CPI-extensions and the amalgamated algebras, with a mention to bi-amalgamated algebras, that will be largely studied in Section 2.5.

In **Section 2.3** we prove that if $A \subseteq B$ is a Prüfer extension and A is a local ring, then the set of all elements of A that are not invertible in B is a prime ideal of A. Moreover, if $B = A_S$ for some multiplicatively closed subset of A, then any subring R of B containing A must be the form $R = A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of A. These two results generalized those of Boynton [15] for local Prüfer rings.

In **Section 2.4** we study the transfer of Prüfer-like conditions in pullbacks of the form

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ B & \xrightarrow{\pi} & T \end{array}$$

where $\pi : B \to T$ is a surjective homomorphism whose kernel is a regular ideal, and T is an overring of R. We prove that A has a certain Prüfer-like condition (i.e. A is Prüfer, Gaussian, arithmetical, of weak global dimension ≤ 1, semihereditary) whenever B and T have the same condition, and the converse holds for Prüfer, Gaussian and arithmetical rings (Theorem 2.27). We observe that the assumption that T is an overring of R is actually a necessary condition in many cases, for instance if B is a localization of A or if $\mathrm{Tot}(R)$ is an absolutely flat ring (so, in particular, when R is a domain). We provide new examples of Prüfer rings, with an application to Prüfer Manis rings. It is worth noting that Theorem 2.27 covers several different results studied up to now by Bakkari and Mahdou [6, Theorem 2.1], Boynton [15, Theorem 4.1], Houston and Taylor [69, Theorem 1.3] and Fontana [48, Theorem 2.4 (3)]). Using Theorem 2.27, it is also possible to give a shorter proof of the following result of Boisen and Sheldon [11, Theorem 2]: a Prüfer ring is the homomorphic image of a Prüfer domain if and only if its total quotient ring is the homomorphic image of a Prüfer domain.

We conclude this section by proving that, in a generic pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \alpha \\ B & \xrightarrow{\beta} & C \end{array}$$

if both $\ker(\alpha)$ and $\ker(\beta)$ are regular ideals, then R turns out to be a Prüfer ring only in the trivial case, namely (if and) only if A and B are both Prüfer rings and $R = A \times B$ (Theorem 2.39). The same holds also for the other Prüfer-like conditions.

Section 2.5 is devoted to the study of bi-amalgamated algebras, a construction introduced by Kabbaj, Louartiti and Tamekkante [73] as a generalization of that of amalgamated algebras [27]. These kind of rings, built starting by two ring homomorphisms $f : A \to B$, $g : A \to C$, and ideals $\mathfrak{b}$ and $\mathfrak{c}$ of B and C respectively, will be denoted by $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. We first outline some basic properties of bi-amalgamated algebras, using its fiber product presentation in order to give further remarks on the topological nature of its prime spectrum. Then we investigate the problem of when $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring. A precise answer can be given if both the ideals $\mathfrak{b}, \mathfrak{c}$ are regular. The general case is much more difficult. Necessary and sufficient conditions for $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ to be Prüfer are provided under a good behaviour of the regular elements of the factor ring $A/f^{-1}(\mathfrak{b})$. Finally, we give conditions for a bi-amalgamation to be a Gaussian ring and we exhibit new examples.

In **Section 2.6** we consider the interplay between the condition of a ring to be a Prüfer ring and homomorphic images. According to [46, Proposition 4.4], if A is a Prüfer ring and $\mathfrak{a}$ is an ideal of A, then $A/\mathfrak{a}$ is a Prüfer ring whenever $\mathfrak{a}$ is regular. The first goal of this section is to present a notion that allow to consider homomorphic images of Prüfer rings that are still Prüfer, without taking regular ideals. We consider surjective ring homomorphism $f : A \to B$ with the following property: pre-images of regular elements of B are regular elements of A. We provide several examples of this kind of morphisms and we show that if such a homomorphism from A onto B exists, and A is a Prüfer ring, then so is B (Theorem 2.66). As a consequence, if A is a local Prüfer ring, then so is $A/Z(A)$. The second part of this section is devoted to the study of the ideal theory of pre-Prüfer rings, that is, those rings whose nontrivial homomorphic images are Prüfer ring [11]. We show that any two given prime ideals of a localization of a pre-Prüfer ring are comparable, provided at least one of them is regular (Theorem 2.70). This is a generalization of [11, Theorem 1.2]. Furthermore, we show that any Noetherian pre-Prüfer ring has Krull dimension ≤ 1 and we observe that the converse does not hold, unlike to the integral case.

CHAPTER 1

Weak forms of the Krull-Schmidt Theorem

1.1 Notations and terminology

Throughout this chapter all rings are assumed to be associative, with an identity element and not necessarily commutative. A two-sided ideal I of a ring R is said to be *completely prime* if, for every $x, y \in R$, $xy \in I$ implies $x \in I$ or $y \in I$. We denote by $J(R)$ the *Jacobson radical* of R, that is, the intersection of all maximal right (or left) ideals of R. A ring R is *semisimple artinian* if it is right (or left) artinian and $J(R) = 0$. This is equivalent to say that R is a direct sum of finitely many simple right (or left) R-modules. A ring R is *semilocal* if $R/J(R)$ is a semisimple artinian ring. We denote by Mod-R the category of right R-modules over a ring R. If M_R is a right R-module, we often omit the subscript R, especially if other subscripts are needed. $E(M_R)$ denotes the injective envelope of M_R. A module M_R is said to be *local* if it is non-zero and has a greatest proper submodule. A submodule N_R of right R-module M_R is *essential in* M_R (and we write $N_R \leq_e M_R$) if we have $N_R \cap K_R \neq 0$ for every non-zero submodule K_R of M_R. A right R-module M_R is said to be *uniform* if it is non-zero and every proper submodule of M is essential, that is, if the intersection of any two non-zero submodules is non-zero. Dually, a submodule N_R of M_R is *superfluous in* M_R (and we write $N_R \leq_s M_R$) if whenever K_R is any submodule of M_R such that $N_R + K_R = M_R$, then $K_R = M_R$. A right R-module M_R is said to be *couniform* (or *hollow*) if it is non-zero and every proper submodule of M is superfluous. It is clear that uniform and couniform modules are necessarily indecomposable and that the two notions are not preserved under direct sums or quotients.

1.2 Preliminaries

Goldie dimension and dual Goldie dimension

It is well known that it is possible to use the notion of uniform module to "measure" the dimension of a module, in a way that generalizes, in some sense, the notion of dimension of a vector space. In this subsection, we briefly recall this definition and some basic properties, and we refer to [35, Sections 2.6÷2.8] for more details.

Let M_R be a right R-module. A set $\{N_\lambda \mid \lambda \in \Lambda\}$ of non-zero submodules of M_R is said to be *independent* if

$$\sum_{\lambda\in\Lambda} N_\lambda = \bigoplus_{\lambda\in\Lambda} N_\lambda.$$

It can be proved that M does not have an infinite independent set of non-zero submodules if and only if there exists an integer n such that M_R has an independent set $\{U_1, \dots, U_n\}$ of uniform submodules of M_R with $\bigoplus_{i=1}^n U_i \leq_e M_R$. In this case, the integer n is uniquely determined and it is said to be the *Goldie dimension of* M_R. It will be denoted by $\dim(M_R)$. If $\dim(M_R) = n$, then n is the maximum of the integers m such that M_R has an independent set of m non-zero proper submodules. If M_R admits an infinite independent set of non-zero submodules, then M_R is said to have *infinite Goldie dimension*.

If N_R is an essential submodule of M_R, then $\dim(M_R) = \dim(N_R)$, so that, in particular, the injective envelope of M_R has the same Goldie dimension of M_R. The following proposition collects some basic properties of the Goldie dimension.

Proposition 1.1. Let M_R be a right R-module.

1. $\dim(M) = 0$ if and only if $M = 0$.
2. $\dim(M) = 1$ if and only if M is a uniform module.
3. If $N \leq M$ and $\dim(M)$ is finite, then $\dim(N) \leq \dim(M)$ and the equality holds if and only if N is essential in M.

4. If M_1 and M_2 are right R-modules of finite Goldie dimension, then $\dim(M_1 \oplus M_2) = \dim(M_1) + \dim(M_2)$.

As one might expect, the notion of Goldie dimension can be dualized. We say that a finite set $\{N_1, \dots, N_k\}$ of proper submodules of M_R is *coindependent* if the canonical injective map

$$M / \bigcap_{i=1}^{k} N_i \hookrightarrow \bigoplus_{i=1}^{k} M/N_i$$

is an isomorphism. An arbitrary set of proper submodules of M_R is said to be *coindependent* if all its finite subsets are coindependent. It can be proved that M does not have an infinite coindependent set of proper submodules if and only if there exists an integer n such that M_R has a coindependent set $\{N_1, \dots, N_n\}$ of proper submodules of M_R with all M/N_i couniform modules for every $i = 1, \dots, n$ and $\bigcap_{i=1}^{n} N_i \leq_s M_R$. In this case, n is the maximum of the set of all the cardinalities of coindependent sets of proper submodules of M_R and n is said to be the *dual Goldie dimension of* M_R (it will be denoted by $\operatorname{codim}(M_R)$). If M_R admits an infinite coindependent set of proper submodules, then M_R is said to have *infinite dual Goldie dimension.*

Proposition 1.2. Let M_R be a right R-module.

1. $\operatorname{codim}(M) = 0$ if and only if $M = 0$.
2. $\operatorname{codim}(M) = 1$ if and only if M is a couniform module.
3. If $N \leq M$ and $\operatorname{codim}(M)$ is finite, then $\operatorname{codim}(M/N) \leq \operatorname{codim}(M)$ and the equality holds if and only if N is superfluous in M.
4. If M_1 and M_2 are right R-modules of finite dual Goldie dimension, then $\operatorname{codim}(M_1 \oplus M_2) = \operatorname{codim}(M_1) + \operatorname{codim}(M_2)$.

It is worth noting that the notions of Goldie dimension and dual Goldie dimension are based on lattice-theoretical properties of the set of submodules of M_R and therefore they can also be treated in terms of modular lattices (see [35, Section 2.6]). We conclude this subsection with a result on uniform and couniform modules that will be largely used in the sequel.

Lemma 1.3. [35, Lemma 6.26] Let A, B and C be non-zero right R-modules and let $f : A \to B$ and $g : B \to C$ be right R-module homomorphisms. Then:

1. If B is uniform, then $g \circ f$ is injective if and only if both f and g are injective.

2. If B is couniform, then $g \circ f$ is surjective if and only if both f and g are surjective.

Local morphisms, semilocal rings and rings of finite type

Definition 1.4. [24] Let $\varphi : R \to S$ be a ring morphism. Then φ is a *local* morphism if for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R.

It is clear that if $\varphi : R \to S$ and $\psi : S \to T$ are two ring morphisms and both φ and ψ are local morphisms, then so is $\psi \circ \varphi$. Conversely, if the composite morphism $\psi \circ \varphi$ is local, then φ is a local morphism. Moreover, if φ is a surjective morphism, then φ is local if and only if $\ker(\varphi) \subseteq J(R)$ (cf. [39, Lemma 3.1]). Local morphisms preserve the dual Goldie dimensions of the rings, in the following sense.

Proposition 1.5. [24, Corollary 2] If $\varphi : R \to S$ is a local morphism, then $\operatorname{codim}(R) \leq \operatorname{codim}(S)$.

The next proposition shows that semilocal rings can be characterized in terms of local morphisms and that they are precisely those of finite dual Goldie dimension.

Proposition 1.6. The following conditions are equivalent for a ring R.

1. R is semilocal;

2. the ring $R/J(R)$ is right [resp. left] artinian;

3. the ideal $J(R)$ is the intersection of finitely many maximal right [resp. left] ideals of R;

4. [24, Theorem 1] there exists a local morphism from R into a semisimple artinian ring or, equivalently, into a semilocal ring;

5. [91, Corollary 1.14] R has finite dual Goldie dimension $\operatorname{codim}(R_R)$.

Example 1.7. (for more details, see [35, pages 6 and 7])

1. A commutative ring is semilocal if and only if it has finitely many maximal ideals.
2. Every right (or left) artinian ring is semilocal.
3. If R is semilocal, then the ring $M_n(R)$ of all $n \times n$ matrices over R is semilocal. In particular, $M_n(\mathbb{Q})$ is a semilocal ring with infinitely many maximal right ideals.
4. The direct product $R_1 \times \cdots \times R_n$ of finitely many semilocal rings $R_1, \dots, R_n$ is semilocal.
5. Every homomorphic image of a semilocal ring is semilocal.
6. The endomorphism ring of an artinian right R-module is a semilocal ring.

For a semilocal ring R, the dual Goldie dimension $\operatorname{codim}(R_R)$ coincides with the composition length of the semisimple artinian ring $R/J(R)$. Moreover, $\operatorname{codim}(R_R)$ is the smallest possible number of maximal right [resp. left] ideals such that $J(R)$ can be represented as their intersections. As far as the maximal two-sided ideals of R are concerned, we have the following

Proposition 1.8. A semilocal ring R has at most $\operatorname{codim}(R_R)$ maximal two-sided ideals, and the intersection of the maximal two-sided ideals is $J(R)$.

Remark 1.9. If R is a ring with at most two maximal right [resp. left] ideals, then the maximal right [resp. left] ideals are two-sided, but this is not true in general. As a matter of fact, consider the ring $R := M_2(\mathbb{F}_2)$. Then R is a semilocal ring with exactly three maximal right ideals, namely $I_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) R$, $I_2 := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) R$ and $I_3 := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right) R$, that are not two-sided. Moreover, $0 = J(R) = I_1 \cap I_2 = I_2 \cap I_3 = I_1 \cap I_3$ and $\operatorname{codim}(R_R) = 2$.

By the Artin-Wedderburn's Theorem, if R is a semilocal ring, then $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$ for suitable non-negative integers $t, n_1, \dots, n_t$

and division rings $D_1, \dots, D_t$. The rings $M_{n_i}(D_i)$ are simple rings but, of course, not necessarily division rings. This behaviour seems to be not properly fulfilling compared to that of commutative rings, in which $R/J(R)$ is isomorphic to the direct product of finitely many fields. This fact motivates the following definition.

Definition 1.10. [44] Let $n \geq 1$ be an integer. A ring R is of *type* n if $R/J(R)$ is isomorphic to the direct product of n division rings. We say that R is of *finite type* if it is of type n for some $n \geq 1$.

It is clear that if R is a ring of finite type, then it is semilocal, and that for commutative rings, R is semilocal if and only if R is of finite type. Moreover, if R is a ring of type n, then $n = \operatorname{codim}(R_R)$.

Proposition 1.11. [44, Proposition 2.1] Let R be a ring and let $n \geq 1$ be an integer. The following conditions are equivalent.

1. R is a ring of type n.
2. n is the smallest of the positive integers m for which there is a local morphism from R into a direct product of m division rings.
3. The ring R has exactly n distinct maximal right [resp. left] ideals, and they are all two-sided ideals of R.

In view of this proposition, for a ring of finite type we can simply talk about "maximal ideals", because we have

$$\{ \text{ max. right ideals } \} = \{ \text{ max. left ideals } \} = \{ \text{ max. two-sided ideals } \}.$$

Modules with a semilocal endomorphism ring

Let M_R be a right R-module. In order to consider the dual Goldie dimension of the endomorphism ring of M_R, we define

$$\delta(M_R) := \begin{cases} \operatorname{codim}(\operatorname{End}(M_R)) & \text{if } M_R \neq 0 \\ 0 & \text{if } M_R = 0 \end{cases}$$

It is immediate that for a non-zero module M_R, the endomorphism ring of M_R is semilocal if and only if $\delta(M_R)$ is finite. In particular, a module has a local endomorphism ring if and only if $\delta(M_R) = 1$. The following proposition shows that the class of modules with semilocal endomorphism rings is closed under direct summands and finite direct sums.

Proposition 1.12. (see [38, Introduction, p. 575]) Let M_R and N_R be right R-modules. Then $\delta(M_R \oplus N_R) = \delta(M_R) + \delta(N_R)$. In particular, $M_R \oplus N_R$ has a semilocal endomorphism ring if and only if both M_R and N_R have a semilocal endomorphism ring.

Modules with a semilocal endomorphism ring cancel from direct sums. The following proposition can be found in [35, page 105] as a corollary of two results of Bass [7] and Evans [33].

Proposition 1.13. (Cancellation property) Let A_R, B_R and C_R be right R-modules and assume that $\delta(A_R)$ is finite. Then $A_R \oplus B_R \cong A_R \oplus C_R$ implies $B_R \cong C_R$.

Remark 1.14. It is worth noting that this result is stated only for modules, but its proof shows that it holds in any additive category (cf. [35, pages 104 and 105]). Hence, if $\mathcal{C}$ is an additive category, A, B, C are three objects of $\mathcal{C}$ and $\mathrm{End}_{\mathcal{C}}(A)$ is semilocal, then $A \oplus B \cong A \oplus C$ implies $B \cong C$.

Among all modules with a semilocal endomorphism ring, those whose endomorphism ring is of finite type will be of particular interest.

Definition 1.15. [45] Let $n \geq 1$ be an integer. A non-zero right R-module M_R is a module of type n [resp. of finite type] if its endomorphism ring $\mathrm{End}(M_R)$ is a ring of type n [resp. finite type]. The zero module is defined to be of type 0. More generally, if $\mathcal{A}$ is an additive category and $M \in \mathrm{Ob}(\mathcal{A})$ is a non-zero object, then we say that M is an object of type n [resp. of finite type] if $\mathrm{End}_{\mathcal{A}}(M)$ is a ring of type n [resp. finite type]. Zero objects in $\mathcal{A}$ are defined to be of type 0.

Several classes of modules of finite type will be presented in the following subsections, while in Sections 1.3 and 1.4 we will consider additive categories in which is it possible to detect some objects of finite type.

Krull-Schmidt Theorems for modules

In 1909, J.H. Maclagan-Weddenburn [82] stated a first version of the so called “Classical Krull-Schmidt Theorem for finite groups“. In his work, he published the following result.

Theorem 1.16. If a finite group G has two direct-product decompositions $G = G_1 \times \cdots \times G_r = H_1 \times \cdots \times H_t$ into indecomposable groups, then $r = t$ and there exist an automorphism φ of G and a permutation σ of $\{1, 2, \ldots, r\}$ such that $\varphi(G_i) = H_{\sigma(i)}$ for every $i = 1, 2 \ldots, r$.

Nevertheless, his proof had some gaps. Two years later, in 1911, R. Remak proved Weddenburn's result in his Ph.D. dissertation [89], also showing that the automorphism φ in the statement of Theorem 1.16 can be chosen to be central (recall that an automorphism of a group G is said to be central if it induces the identity $G/Z(G) \to G/Z(G)$, where $Z(G)$ denotes the center of G). Krull and Schmidt transferred this result to modules of finite composition length [77, 92].

Theorem 1.17. (Classical Krull-Schmidt Theorem for modules) Let R be a ring and M be a module of finite length. Then there exists a decomposition

$$M = M_1 \oplus \cdots \oplus M_r$$

into indecomposable submodules. Moreover, if $M = N_1 \oplus \cdots \oplus N_t$ is another decomposition of M into indecomposable submodules, then $r = t$ and there exists a permutation σ of $\{1, 2, \ldots, r\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \ldots, r$.

Notice that any module with a local endomorphism ring is necessarily indecomposable and by Fitting's Lemma, the converse holds as well for modules of finite composition length. Theorem 1.17 was extended by G. Azumaya [5] to the case of possibly infinite direct sums of modules with a local endomorphism ring.

Theorem 1.18. (Krull-Schmidt-Remak-Azumaya Theorem) Let R be a ring and let M be a module that is a direct sum of modules with local endomorphism rings. Then M is a direct sum of indecomposable modules in an essentially unique way in the following sense. If

$$M = \bigoplus_{i \in I} M_i \cong \bigoplus_{j \in J} N_j$$

where all the submodules M_i, $i \in I$ and N_j, $j \in J$ are indecomposable, then there exists a bijection $\sigma : I \to J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.

In the following subsections, we will consider finite direct-sums of modules and we will see what happens if we drop the assumption that the direct summands have a local endomorphism ring. We consider several classes of modules whose endomorphism rings have two maximal right ideals. As we have seen, if a ring has exactly two maximal right ideals, they are necessarily two-sided, and therefore the ring is of type 2.

Uniserial modules

Recall that a right R-module is *uniserial* if the lattice of its submodules is linearly ordered under inclusion, that is, for every $V, W \leq U$, either $V \subseteq W$ or $W \subseteq V$. The next result shows that the endomorphism ring of a uniserial module has at most two maximal right (left) ideals, hence uniserial modules are modules of type at most 2.

Theorem 1.19. [34, Theorem 1.2] The endomorphism ring $\mathrm{End}(U_R)$ of a non-zero uniserial module U_R has at most two maximal right ideals: the two-sided completely prime ideals $I_U := \{\, f \in \mathrm{End}(U_R) \mid f \text{ is not injective} \,\}$ and $K_U := \{\, f \in \mathrm{End}(U_R) \mid f \text{ is not surjective} \,\}$, or only one of them. In particular, uniserial modules have type at most 2.

In order to discuss direct-sum decompositions of uniserial modules, we need to introduce the notions of monogeny class and epigeny class. These notions will turn out to be the "invariants" needed to classify direct-sum decompositions of uniserial modules (see Theorem 1.21 below).

Definition 1.20. [34] Two right R-modules M_R and N_R are said to have the same *monogeny class*, denoted by $[M_R]_m = [N_R]_m$, if there exist two right R-module monomorphisms $f : M_R \to N_R$ and $g : N_R \to M_R$. Similarly, M_R and

N_R are said to have the same *epigeny class*, denoted by $[M_R]_e = [N_R]_e$, if there exist two right R-module epimorphisms $f : M_R \to N_R$ and $g : N_R \to M_R$.

For uniserial modules, the monogeny class and the epigeny class can be expressed in terms of the maximal right ideals of the endomorphism rings of the modules. To be more precise, two uniserial right R-modules U and V have the same monogeny class [resp. epigeny class] if and only if there exist two morphisms $f : U_R \to V_R$ and $g : V_R \to U_R$ such that $gf \notin I_U$ [resp. $gf \notin K_U$].

For uniserial modules, we have the following weak form of the Krull-Schmidt Theorem.

Theorem 1.21. [34, Theorem 1.9] Let $U_1, \ldots, U_r, V_1, \ldots, V_t$ be uniserial modules over an arbitrary ring R. Then

$$U_1 \oplus \cdots \oplus U_r \cong V_1 \oplus \cdots \oplus V_t$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \ldots, r\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \ldots, r$.

Let $n \geq 2$ be an integer. In [34, Example 2.1], Alberto Facchini constructed an example of n^2 pairwise non-isomorphic finitely presented uniserial modules $U_{j,k}$ $(j, k = 1, 2, \ldots, n)$ over a suitable serial ring R in order to show that a module that is a direct sum of n uniserial modules can have $n!$ pairwise non-isomorphic direct-sum decompositions into indecomposables. Here we briefly recall the construction of the ring R and the uniserial right R-modules $U_{i,j}$, and we refer to [34, Example 2.1] and [35, Example 9.20] for more details.

Example 1.22. Fix an integer $n \geq 2$. Let $\mathbf{M}_n(\mathbb{Q})$ be the ring of all $n \times n$-matrices over the field $\mathbb{Q}$ of rational numbers. Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}_p, \mathbb{Z}_q$ be the localizations of $\mathbb{Z}$ at two distinct maximal ideals (p) and (q) of $\mathbb{Z}$ (here $p, q \in \mathbb{Z}$ are distinct prime numbers). Let Λ_p denote the subring of $\mathbf{M}_n(\mathbb{Q})$ whose elements are the $n \times n$-matrices with entries in $\mathbb{Z}_p$

on and above the diagonal and entries in $p\mathbb{Z}_p$ under the diagonal, that is,

$$\Lambda_p := \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \dots & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \dots & \mathbb{Z}_p \\ \vdots & & \ddots & \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \dots & \mathbb{Z}_p \end{pmatrix} \subseteq \mathbf{M}_n(\mathbb{Q}).$$

Similarly, set

$$\Lambda_q := \begin{pmatrix} \mathbb{Z}_q & \mathbb{Z}_q & \dots & \mathbb{Z}_q \\ q\mathbb{Z}_q & \mathbb{Z}_q & \dots & \mathbb{Z}_q \\ \vdots & & \ddots & \\ q\mathbb{Z}_q & q\mathbb{Z}_q & \dots & \mathbb{Z}_q \end{pmatrix} \subseteq \mathbf{M}_n(\mathbb{Q}).$$

If

$$R := \begin{pmatrix} \Lambda_p & 0 \\ \mathbf{M}_n(\mathbb{Q}) & \Lambda_q \end{pmatrix},$$

then R is a subring of the ring $\mathbf{M}_{2n}(\mathbb{Q})$ of $2n \times 2n$-matrices with rational entries.

For every $i = 1, 2, \dots, n$, set

$$V_i := (\underbrace{\mathbb{Q}, \dots, \mathbb{Q}}_{n}, \underbrace{q\mathbb{Z}_q, \dots, q\mathbb{Z}_q}_{i-1}, \underbrace{\mathbb{Z}_q, \dots, \mathbb{Z}_q}_{n-i+1})$$

and

$$X_i := (\underbrace{p\mathbb{Z}_p, \dots, p\mathbb{Z}_p}_{i-1}, \underbrace{\mathbb{Z}_p, \dots, \mathbb{Z}_p}_{n-i+1}, \underbrace{0, \dots, 0}_{n}).$$

Then V_i and the X_j are right R-modules with right R-module structure given by matrix multiplication. Moreover, the unique infinite composition series of V_1 is

$$\begin{aligned} & V_1 \supset V_2 \supset \cdots \supset V_n \supset qV_1 \supset \cdots \supset qV_n \\ & \supset q^2V_1 \supset \cdots \supset q^2V_n \supset \cdots \supset p^{-2}X_1 \supset \cdots \supset p^{-2}X_n \\ & \supset p^{-1}X_1 \supset \cdots \supset p^{-1}X_n \supset X_1 \supset X_2 \supset \cdots \supset X_n \\ & \supset pX_1 \supset \dots pX_n \supset p^2X_1 \supset \dots p^2X_n \supset \cdots \supset 0 \end{aligned}$$

The right R-modules X_i/X_{i+1} and V_i/V_{i+1} are simple. Define the n^2 uniserial R-modules $U_{j,k} := V_k/X_j$. The modules $U_{j,k}$ satisfy the following properties:

(a) for every $j, k, h, l = 1, 2, \dots, r$, $[U_{j,k}]_m = [U_{h,l}]_m$ if and only if $j = h$;

(b) for every $j, k, h, l = 1, 2, \dots, r$, $[U_{j,k}]_e = [U_{h,l}]_e$ if and only if $k = l$.

Hence, for every pair of permutations σ, τ of $\{1, 2, \dots, n\}$ we have

$$U_{1,1} \oplus U_{2,2} \oplus \dots \oplus U_{n,n} \cong U_{\sigma(1),\tau(1)} \oplus U_{\sigma(2),\tau(2)} \oplus \dots, \oplus U_{\sigma(n),\tau(n)}$$

In particular, the two permutations σ, τ in Theorem 1.21 can be completely arbitrary.

It is worth mentioning that in [87], Pavel Příhoda proved that every indecomposable direct summand of a direct sum of finitely many uniserial modules is necessarily a uniserial module. Therefore, Theorem 1.21 describes the behaviour of <u>all</u> direct-sum decompositions of a direct sum of finitely many uniserial modules.

Cyclically presented modules

The behaviour of uniserial modules we have just described is enjoyed by other classes of modules. Let us start by presenting the case of cyclically presented modules, which was first studied in [3]. In this subsection R will be a local ring with maximal ideal $J(R)$. A right R-module is called *cyclically presented* if it is isomorphic to the cyclic module R_R/aR for some $a \in R$.

The endomorphism ring of a non-zero cyclically presented module R/aR is isomorphic to the ring E/aR, where $E := \{r \in R \mid ra \in aR\}$ is the *idealizer* of aR (notice that aR is a two-sided ideal of the ring E).

Theorem 1.23. [3, Theorem 2.1] Let $0 \neq a$ be a non-invertible element of a local ring R and let E denote the idealizer of aR. Set $I_a := \{r \in E \mid ra \in aJ(R)\}$ and $K_a := E \cap J(R)$. The endomorphism ring $\mathrm{End}(R/aR) \cong E/aR$ of the cyclically presented module R/aR has at most two maximal right ideals: the completely prime two-sided ideals I_a/aR and K_a/aR, or only one of them.

Definition 1.24. [3, Section 4] If R/aR and R/bR are two cyclically presented modules, we say that R/aR and R/bR *have the same lower part*, and we write $[R/aR]_\ell = [R/bR]_\ell$, if there exist elements $r, s, u, v \in R$ with u, v invertible, such that $ra = bu$ and $sb = av$.

For cyclically presented modules over a local ring, this notion dualizes the notion of having the same epigeny class. Indeed, it is straightforward to show that $[R/aR]_e = [R/bR]_e$ if and only if there exist elements $r, s, u, v \in R$ with u, v invertible such that $ua = br$ and $vb = as$. Moreover, R/aR and R/bR have the same lower part [resp. the same epigeny class] if and only if there exist two morphisms $f : R/aR \to R/bR$ and $g : R/bR \to R/aR$ such that $gf \notin I_a/aR$ [resp. $gf \notin K_a/aR$], or, equivalently, such that $fg \notin I_b/bR$ [resp. $fg \notin K_b/bR$].

We have the following weak form of the Krull-Schmidt Theorem.

Theorem 1.25. [3, Theorem 5.3] Let $a_1, \dots, a_r, b_1, \dots, b_t$ be non-invertible elements of a local ring R. Then

$$R/a_1R \oplus \cdots \oplus R/a_rR \cong R/b_1R \oplus \cdots \oplus R/b_tR$$

if and only if $r = t$ and there exist two permutations σ and τ of $\{1, 2, \dots, r\}$ such that $[R/a_iR]_\ell = [R/b_{\sigma(i)}R]_\ell$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, r$.

Couniformly presented modules

The next example we want to consider is that of couniformly presented modules, studied in [37]. We say that a right R-module M_R is *couniformly presented* [37] if it is non-zero and there exists a short exact sequence (called a *couniform presentation of* M_R)

$$0 \to C_R \to P_R \to M_R \to 0$$

where P_R is projective and both C_R and P_R are couniform right R-modules. Every couniformly presented module is local (that is, cyclic, non-zero and with a unique maximal proper submodule), hence indecomposable. Moreover, since every proper submodule of P_R is superfluous, $P_R \to M_R$ is necessarily a projective cover of M_R. It is clear that this notion generalizes that of cyclically presented modules introduced above. To be more precise, a cyclically presented module over a local ring R is either zero, or isomorphic to R or couniformly presented. If R is a right chained ring (that is, its lattice of right ideals is totally ordered under inclusion), then every couniformly presented module is uniserial.

The following lemma describes the projective modules that are couniform.

Lemma 1.26. [3, Lemma 8.7] Let P_R be a projective right R-module. Then, the following conditions are equivalent.

1. P_R is couniform.
2. P_R is a projective cover of a simple module.
3. $\delta(P_R) = 1$, that is, the endomorphism ring $\mathrm{End}(P_R)$ of P_R is local.
4. There exists an idempotent element e of R such that $P_R \cong eR$ and eRe is a local ring.
5. P_R is a finitely generated module with a unique maximal submodule.

Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module M_R. Then any endomorphism $f \in \mathrm{End}(M_R)$ lifts to an endomorphism $f_P : P_R \to P_R$. If we denote by $f_C : C_R \to C_R$ the restriction of f_P to C_R, we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_P} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0. \end{array}$$

The morphisms f_P and f_C are not uniquely determined by f. Nevertheless, if $f'_P : P_R \to P_R$ and $f'_C : C_R \to C_R$ are two other morphisms that make the above diagram commute, then f_P is surjective if and only if f'_P is surjective and f_C is surjective if and only if f'_C is surjective. It follows that the sets $K_M := \{f \in \mathrm{End}(M_R) \mid f \text{ is not surjective }\}$ and $I_M := \{f \in \mathrm{End}(M_R) \mid f_C \text{ is not surjective }\}$ are well-defined. Moreover, if $0 \to C'_R \to P'_R \to M_R \to 0$ is another couniform presentation of M_R, then it can be proved that the set $\{f \in \mathrm{End}(M_R) \mid f_C$ is not surjective $\}$ is equal to $\{f \in \mathrm{End}(M_R) \mid f_{C'}$ is not surjective $\}$, so that, I_M does not depend on the couniform presentation of M_R.

Theorem 1.27. [37, Theorem 2.5] The endomorphism ring $\mathrm{End}(M_R)$ of a couniformly presented module M_R (with a given couniform presentation $0 \to C_R \to P_R \to M_R \to 0$) has at most two maximal right ideals: the completely prime two-sided ideals

$$K_M := \{f \in \mathrm{End}(M_R) \mid f \text{ is not surjective }\}$$

and

$$I_M := \{f \in \mathrm{End}(M_R) \mid f_C \text{ is not surjective }\},$$

or only one of them. In particular, M_R is a module of type at most 2.

The notion of "having the same lower part" can be extended to arbitrary couniformly presented modules, as follows.

Definition 1.28. [37, Section 3] Let M_R and M'_R be two couniformly presented modules with two couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$ respectively. Then M_R and M'_R are said to *have the same lower part*, and we write $[M_R]_\ell = [M'_R]_\ell$, if there exist two right R-module morphisms $\varphi : P_M \to P'_M$ and $\psi : P'_M \to P_M$ such that $\varphi(C_R) = C'_R$ and $\psi(C'_R) = C_R$.

It is easy to see that $[M_R]_\ell = [M'_R]_\ell$ if and only if there exist two right R-module morphisms $f : M_R \to M'_R$ and $g : M'_R \to M_R$ such that $g \circ f \notin I_M$ (or equivalently, such that $f \circ g \notin I_{M'}$). Since the ideal I_M does not depend on the couniform presentation, the notion of having the same lower part is well-defined. Notice also that this definition generalize that given for cyclically presented modules over a local ring.

In a similar way, two couniformly presented modules M_R and M'_R have the same epigeny class if and only if there exist two right R-module morphisms $f : M_R \to M'_R$ and $g : M'_R \to M_R$ such that $g \circ f \notin K_M$ (or equivalently, such that $f \circ g \notin K_{M'}$).

We have the following weak form of the Krull-Schmidt Theorem for couniformly presented modules.

Theorem 1.29. [37, Theorem 4.3] Let $M_1, \dots, M_r, N_1, \dots, N_t$ be $r+t$ couniformly presented right R-modules. Then

$$M_1 \oplus \cdots \oplus M_r \cong N_1 \oplus \cdots \oplus N_t$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \dots, r\}$ such that $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for every $i = 1, 2, \dots, r$.

Kernels of morphisms between indecomposable injective modules

In this subsection we present another class of modules that dualizes, in some sense, that of couniformly presented modules. The precise meaning of this "dualization" is explained in [37], while all results we are going to present can be found in [36]. We will see that, also in this case, we can find the pattern that characterizes all our examples: at most two maximal ideals and a weak form of the Krull-Schmidt Theorem. Let $\varphi : E_1 \to E_2$ be a non-zero and non-injective right R-module morphism between indecomposable injective modules. Then $\ker(\varphi)$ is a uniform (hence indecomposable) module with injective envelope $E(\ker(\varphi)) \cong E_1$. If $f \in \mathrm{End}(\ker(\varphi))$, then f extends to a morphism $f_1 : E_1 \to E_1$. The induced morphism $\tilde{f}_1 : E_1/\ker(\varphi) \to E_1/\ker(\varphi)$ extends to a morphism $f_2 : E_2 \to E_2$. Hence, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\varphi) & \longrightarrow & E_1 & \xrightarrow{\varphi} & E_2 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\
0 & \longrightarrow & \ker(\varphi) & \longrightarrow & E_1 & \xrightarrow{\varphi} & E_2.
\end{array}$$

The morphisms f_1 and f_2 are not uniquely determined. Nevertheless, it can be proved that if f_1' and f_2' are two other morphisms that make the above diagram commute, then both $f_1 - f_1'$ and $f_2 - f_2'$ have non-zero kernel. Since the endomorphism ring $\mathrm{End}(E_R)$ of an indecomposable injective module E_R is a local ring with maximal ideal $\{f \in \mathrm{End}(E_R) \mid f \text{ is not injective}\}$, it follows that f_1 is injective if and only if f_1' is injective and similarly, f_2 is injective if and only if f_2' is injective. Therefore, are well-defined the sets

$$\begin{aligned} I_\varphi : &= \{f \in \mathrm{End}(\ker(\varphi)) \mid f \text{ is not injective}\} \\ &= \{f \in \mathrm{End}(\ker(\varphi)) \mid f_1 : E_1 \to E_1 \text{ is not injective}\} \end{aligned}$$

and

$$\begin{aligned} K_\varphi : &= \{f \in \mathrm{End}(\ker(\varphi)) \mid f_1^{-1}(\ker(\varphi)) \supsetneqq \ker(\varphi)\} \\ &= \{f \in \mathrm{End}(\ker(\varphi)) \mid f_2 : E_2 \to E_2 \text{ is not injective}\}. \end{aligned}$$

Theorem 1.30. [36, Theorem 2.1] Let $\varphi : E_1 \to E_2$ be a non-injective right R-module morphism between indecomposable injective modules. Then $\mathrm{End}(\ker(\varphi))$ has at most two maximal right ideals: the completely prime two-sided ideals K_φ and I_φ, or only one of them. In particular, $\ker(\varphi)$ is a module of type at most 2.

Definition 1.31. [36, Section 2] Two right R-modules M_R and N_R are said to have *the same upper part* (and we write $[M_R]_u = [N_R]_u$) if there exist two right R-modules morphisms $\varphi : E(M_R) \to E(N_R)$ and $\psi : E(N_R) \to E(M_R)$ such that $\varphi^{-1}(N) = M$ and $\psi^{-1}(M) = N$.

If $\varphi : E_1 \to E_2$ and $\varphi' : E_1' \to E_2'$ are two non-injective morphisms between indecomposable injective modules, then $[\ker(\varphi)]_u = [\ker(\varphi')]_u$ if and only if there exist two right R-module morphisms $\alpha : \ker(\varphi) \to \ker(\varphi')$ and $\beta : \ker(\varphi') \to \ker(\varphi)$ such that $\beta \circ \alpha \notin K_\varphi$ (or equivalently, such that $\alpha \circ \beta \notin K_{\varphi'}$). In a similar way, $[\ker(\varphi)]_m = [\ker(\varphi')]_m$ if and only if there exists a morphism $\lambda : \mathrm{End}(\ker(\varphi)) \setminus I_\varphi$ that factors through $\ker(\varphi')$ (or symmetrically, if and only if there exists a morphism $\lambda' : \mathrm{End}(\ker(\varphi')) \setminus I_{\varphi'}$ that factors through $\ker(\varphi)$).

Theorem 1.32. [36, Theorem 2.7] Let $\varphi_i : E_{i,1} \to E_{i,2}$ ($i = 1, 2, \dots, r$) and $\varphi_j' : E_{j,1}' \to E_{j,2}'$ ($j = 1, 2, \dots, t$) be $r + t$ non-injective morphisms between indecomposable injective modules over an arbitrary ring. Then

$$\ker(\varphi_1) \oplus \cdots \oplus \ker(\varphi_r) \cong \ker(\varphi_1') \oplus \cdots \oplus \ker(\varphi_t')$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \dots, r\}$ such that $[\ker(\varphi_i)]_u = [\ker(\varphi_{\sigma(i)})]_u$ and $[\ker(\varphi_i)]_m = [\ker(\varphi_{\tau(i)})]_m$ for every

$i = 1, 2, \dots, r$.

Ideals in a category and Krull-Schmidt Theorems

We now present two categorical frameworks in which it is possible to treat all the examples we have seen at the same time. First of all, we recall the definition of ideal in a category.

Definition 1.33. [85, p. 18] Let $\mathcal{C}$ be any preadditive category. An *ideal* of $\mathcal{C}$ assigns to every pair A, B of objects of $\mathcal{C}$ a subgroup $\mathcal{I}(A, B)$ of the abelian group $\mathrm{Hom}_{\mathcal{C}}(A, B)$ with the property that, for all morphisms $\varphi\colon C \to A$, $\psi\colon A \to B$ and $\omega\colon B \to D$ with $\psi \in \mathcal{I}(A, B)$, one has that $\omega\psi\varphi \in \mathcal{I}(C, D)$.

For any ideal $\mathcal{I}$ of $\mathcal{C}$, we can consider the *factor category* $\mathcal{C}/\mathcal{I}$, which is the category having the same objects as $\mathcal{C}$ and, for $A, B \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}/\mathcal{I})$, the group of morphisms $A \to B$ in $\mathcal{C}/\mathcal{I}$ is defined to be the factor group

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B)/\mathcal{I}(A, B).$$

The composition is that induced by the composition of $\mathcal{C}$. For any ideal $\mathcal{I}$ of $\mathcal{C}$, we have a canonical additive functor $F_{\mathcal{I}}\colon \mathcal{C} \to \mathcal{C}/\mathcal{I}$.

The following definition is due to Facchini and Příhoda [45].

Definition 1.34. [45, p. 565] A *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ consists of a subgroup $\mathcal{P}(A, B)$ of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for every pair of objects of $\mathcal{C}$, such that:
(1) for every objects A, B, C of $\mathcal{C}$, for every $f : A \to B$ and for every $g : B \to C$, one has that $gf \in \mathcal{P}(A, C)$ if and only if either $f \in \mathcal{P}(A, B)$ or $g \in \mathcal{P}(B, C)$, and
(2) $\mathcal{P}(A, A)$ is a proper subgroup of $\mathrm{End}_{\mathcal{C}}(A, A)$ for every object A of $\mathcal{C}$.

By definition, if $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$, then $\mathcal{P}(A, A)$ is a completely prime two-sided ideal of $\mathrm{End}_{\mathcal{C}}(A, A)$ for every $A \in \mathrm{Ob}(\mathcal{C})$.

If A, B are objects of $\mathcal{C}$, we say that A and B *have the same* $\mathcal{P}$*-class*, and we write $[A]_{\mathcal{P}} = [B]_{\mathcal{P}}$, if there exist two morphisms $f : A \to B$ and $g : B \to A$ such that $gf \notin \mathcal{P}(A, A)$ and $fg \notin \mathcal{P}(B, B)$. It is clear that "having the same $\mathcal{P}$-class" turns out to be an equivalence relation on $\mathrm{Ob}(\mathcal{C})$.

Completely prime ideals can be used to describe a general framework that includes all the examples of the weak forms of the Krull-Schmidt Theorem we have seen in this section (Subsections 1.2.5, 1.2.6, 1.2.7 and 1.2.8).

Theorem 1.35. [45, Theorem 6.2] Let $\mathcal{C}$ be a full subcategory of Mod-R in which all objects are indecomposable right R-modules and let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{C}$ such that, for every $A \in \mathcal{C}$, $\mathcal{P}(A, A) \cup \mathcal{Q}(A, A) = \{f \in \mathrm{End}(A) \mid f \text{ is not an automorphism}\}$. Let $A_1, \ldots, A_r, B_1, \ldots, B_t$ be $r + t$ objects of $\mathcal{C}$. Then

$$A_1 \oplus \cdots \oplus A_r \cong B_1 \oplus \cdots \oplus B_t$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \ldots, r\}$ such that $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[M_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i = 1, 2, \ldots, r$.

Notice that in Theorem 1.35, the assumption that $\mathrm{End}(A) \setminus (\mathcal{P}(A, A) \cup \mathcal{Q}(A, A))$ is the set of all the automorphisms of $\mathrm{End}(A)$ implies that all modules in $\mathcal{C}$ have type at most two, and that the maximal right (left) ideals of $\mathrm{End}_{\mathcal{C}}(A)$ are both $\mathcal{P}(A, A)$ and $\mathcal{Q}(A, A)$ or only one of them.

Let us see how to apply Theorem 1.35 to the case of uniserial modules. Let $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all nonzero uniserial right R-modules. Then, for any $U, V \in \mathrm{Ob}(\mathcal{C})$,

$$\mathcal{P}(U, V) := \{f \in \mathrm{Hom}(U, V) \mid f \text{ is not injective}\}$$

and

$$\mathcal{Q}(U, V) := \{f \in \mathrm{Hom}(U, V) \mid f \text{ is not surjective}\}$$

turn out to be two completely prime ideals of $\mathcal{C}$ that satisfy the condition of the theorem. Moreover, $[U]_m = [V]_m$ if and only if $[U]_{\mathcal{P}} = [V]_{\mathcal{P}}$ and, similarly, $[U]_e = [V]_e$ if and only if $[U]_{\mathcal{Q}} = [V]_{\mathcal{Q}}$.

There is another result, due to Facchini and Perone [41], strictly related to the weak versions of the Krull-Schmidt Theorem we have seen before. In order to discuss about it, we first recall the definitions of local functor and Jacobson radical of an additive category.

The *Jacobson radical of* $\mathcal{C}$ is the ideal $\mathcal{J}$ of $\mathcal{C}$ defined as follows. For any pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{J}(A, B)$ is the set of all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$

that satisfy the following equivalent conditions [85, p. 21]:
(a) $1_A - gf$ has a left inverse for every morphism $g : B \to A$;
(b) $1_B - fg$ has a left inverse for every morphism $g : B \to A$;
(c) $1_A - gf$ has a two-sided inverse for every morphism $g : B \to A$.
Notice that $\mathcal{J}(A, A) = J(\mathrm{End}_{\mathcal{C}}(A))$ for every $A \in \mathrm{Ob}(\mathcal{C})$. The quotient category $\mathcal{C}/\mathcal{J}$ has zero Jacobson radical.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two additive categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be a *local functor* if, for every $A, B \in \mathrm{Ob}(\mathcal{C})$ and for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, if $F(f)$ is an isomorphism in $\mathcal{C}'$, then f is an isomorphism in $\mathcal{C}$.

If $\mathcal{I}$ is an ideal of $\mathcal{C}$, the canonical functor $F_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ is local if and only if $\mathcal{I}$ is contained in the Jacobson radical $\mathcal{J}$ of $\mathcal{C}$. More generally, a functor $F : \mathcal{C} \to \mathcal{C}'$ between additive categories is local if and only if the kernel of F is contained in the Jacobson radical $\mathcal{J}$ of $\mathcal{C}$ (cf. [42]).

Starting from a pair of ideals, we can consider the following weak version of the Krull-Schmidt Theorem in additive categories.

Theorem 1.36. [42, Theorem 3.4] Let $\mathcal{C}$ be an additive category and let $(\mathcal{I}_1, \mathcal{I}_2)$ be a pair of ideals such that the canonical functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{I}_1 \times \mathcal{C}/\mathcal{I}_2$ is a local functor. Let $A_1, \dots, A_r, B_1, \dots, B_t$ be $r + t$ objects of $\mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}/\mathcal{I}_k}(A_i)$ and $\mathrm{End}_{\mathcal{C}/\mathcal{I}_k}(B_j)$ are local rings for $i = 1, \dots, r$, $j = 1, \dots, t$ and $k = 1, 2$. Then

$$A_1 \oplus \cdots \oplus A_r \cong B_1 \oplus \cdots \oplus B_t$$

if and only if $r = t$ and there are two permutations σ, τ of $\{1, 2, \dots, r\}$ such that $A_i \cong B_{\sigma(i)}$ in $\mathcal{C}/\mathcal{I}_1$ and $A_i \cong B_{\tau(i)}$ in $\mathcal{C}/\mathcal{I}_2$ for every $= 1, \dots, r$.

We can apply this theorem to all the situations we have seen before. Let us see what happens in the case of uniserial modules. For any pair of right R-modules A_R and B_R define:

$$\Delta(A, B) := \{f \in \mathrm{Hom}(M, N) \mid \ker(f) \leq_e A\}$$

and

$$\Sigma(A, B) := \{f \in \mathrm{Hom}(M, N) \mid \mathrm{Im}(f) \leq_s B\}.$$

Then, Δ and Σ are two ideals of Mod-R and it can be seen that the functor

$$F: \text{Mod-}R \to \text{Mod-}R/\Delta \times \text{Mod-}R/\Sigma$$

is a local functor (cf. [42, Section 4]). If U_R and V_R are two uniserial right R-modules, then $U \cong V$ in Mod-R/Δ if and only if $[U]_m = [V]_m$. Similarly, $U \cong V$ in Mod-R/Σ if and only if $[U]_e = [V]_e$. Moreover, in the notation of Theorem 1.19, $\Delta(U,U) = I_U$ and $\Sigma(U,U) = K_U$. Therefore, both $\text{End}_{\text{Mod-}R/\Delta}(U)$ and $\text{End}_{\text{Mod-}R/\Sigma}(U)$ are local rings and we can apply Theorem 1.36 to deduce the weak form of the Krull-Schmidt Theorem for uniserial modules (Theorem 1.21). We refer to [42] for a description of the ideals needed to discuss the cases of couniformly presented modules and kernels between indecomposable injective modules.

In the next two sections we will consider some additive categories in which it is possible to state a weak version of the Krull-Schmidt Theorem for suitable objects of finite type (see Theorems 1.53 and 1.60). It is worth noting that, in the situations we are going to study, the invariants needed to describe direct-sum decompositions are more than two and come from ideals of the endomorphism rings that are two-sided, completely prime, but not necessarily all maximal. It means that we must be careful when we move to quotient categories, because the endomorphism rings of the objects could be not local, so generalizations of Theorem 1.35 and Theorem 1.36 cannot be directly applied. Nevertheless, we will use some techniques that involve ideals and factor categories and that are largely based on the ideas used in the cases of objects of type at most 2.

1.3 The category of morphisms between modules

The content of this section is based on a joint work with Susan F. El-Deken and Alberto Facchini [20].

Let R be an associative ring with identity and Mod-R the category of right R-modules. Let Morph(Mod-R) denote the *morphism category*. The objects of this category are the R-module morphisms between right R-modules. We will denote by M and N generic objects $\mu_M \colon M_0 \to M_1$ and $\mu_N \colon N_0 \to N_1$ of

Morph(Mod-R) (this abuse of notation is justified by Theorem 1.37 below). A morphism $u\colon M \to N$ in the category Morph(Mod-R) is a pair of R-module morphisms (u_0, u_1) that makes the following diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{\mu_M} & M_1 \\ {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u_1} \\ N_0 & \xrightarrow{\mu_N} & N_1 \end{array}$$

commute, that is, such that $u_1\mu_M = \mu_N u_0$. The composition law is the obvious one: for every $u = (u_0, u_1) : M \to N$ and $v = (v_0, v_1) : N \to L$ we have $v \circ u := (v_0 \circ u_0, v_1 \circ u_1)$. Moreover, for every pair M, N of objects of Morph(Mod-R), the group $\mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M, N)$ is a subgroup of the cartesian product $\mathrm{Hom}_{\mathrm{Mod}\text{-}R}(M_0, N_0) \times \mathrm{Hom}_{\mathrm{Mod}\text{-}R}(M_1, N_1)$. Thus, for every $u = (u_0, u_1), u' = (u'_0, u'_1) \in \mathrm{Hom}_{\mathrm{Morph(Mod}\text{-}R)}(M, N)$, we have $u + u' = (u_0 + u'_0, u_1 + u'_1)$.

We will denote by E_M the endomorphism ring of the object $\mu_M\colon M_0 \to M_1$ in the category Morph(Mod-R).

The next result is well known (cf. [50, 65]).

Theorem 1.37. The category Morph(Mod-R) is equivalent to the category of right modules over the triangular matrix ring $T := \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$.

Remark 1.38. Let $\{\, M_\lambda \mid \lambda \in \Lambda \,\}$ be a family of objects of Morph(Mod-R), where λ ranges in an index set Λ. Thus M_λ is an object $\mu_{M_\lambda}\colon M_{0,\lambda} \to M_{1,\lambda}$ for every $\lambda \in \Lambda$. The coproduct of the family $\{\, M_\lambda \mid \lambda \in \Lambda \,\}$ is the object $\bigoplus_{\lambda\in\Lambda} M_\lambda$, where

$$\mu_{\bigoplus_{\lambda\in\Lambda} M_\lambda}\colon \bigoplus_{\lambda\in\Lambda} M_{0,\lambda} \to \bigoplus_{\lambda\in\Lambda} M_{1,\lambda}$$

is defined component-wise, with the canonical embeddings $e_{\lambda_0}\colon M_{\lambda_0} \to \bigoplus_{\lambda\in\Lambda} M_\lambda$ for every $\lambda_0 \in \Lambda$.

There is a canonical faithful local functor

$$Q\colon \ \mathrm{Morph(Mod}\text{-}R) \to \mathrm{Mod}\text{-}R \times \mathrm{Mod}\text{-}R,$$

which assigns to every object M of Morph(Mod-R) the object (M_0, M_1) of

Mod-$R \times$ Mod-R and to every morphism $u = (u_0, u_1)$ in Morph(Mod-R) the morphism (u_0, u_1) in Mod-$R \times$ Mod-R.

Lemma 1.39. For every object M of Morph(Mod-R), the canonical ring morphism $\varepsilon\colon E_M \to \mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1)$, defined by $\varepsilon\colon (u_0, u_1) \mapsto (u_0, u_1)$, is a local morphism.

Proof. A morphism (u_0, u_1) in Morph(Mod-R) is an isomorphism if and only if both u_0 and u_1 are right R-module isomorphisms. □

Proposition 1.40. Let M be an object of Morph(Mod-R) with $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ semilocal rings. Then the endomorphism ring E_M of M is semilocal.

Proof. By Lemma 1.39, the ring morphism

$$\varepsilon\colon E_M \to \mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1), \qquad \varepsilon\colon (u_0, u_1) \mapsto (u_0, u_1),$$

is a local morphism. Since $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ are semilocal rings, their direct product $\mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1)$ is semilocal [35, (4) on page 7], so that E_M is semilocal by [24, Corollary 2]. □

Proposition 1.41. If $u = (u_0, u_1)\colon M \to N$ is a morphism in the category Morph(Mod-R), $u_0 \in \mathcal{J}_{\text{Mod-}R}(M_0, N_0)$ and $u_1 \in \mathcal{J}_{\text{Mod-}R}(M_1, N_1)$, then

$$u = (u_0, u_1) \in \mathcal{J}_{\text{Morph(Mod-}R)}(M, N).$$

(Here, $\mathcal{J}_{\text{Mod-}R}$ and $\mathcal{J}_{\text{Morph(Mod-}R)}$ denote the Jacobson radicals of the two categories Mod-R and Morph(Mod-R) respectively.)

Proof. Both the functors

$$Q\colon \text{Morph(Mod-}R) \to \text{Mod-}R \times \text{Mod-}R$$

and

$$P\colon \text{Mod-}R \times \text{Mod-}R \to \text{Mod-}R/\mathcal{J}_{\text{Mod-}R} \times \text{Mod-}R/\mathcal{J}_{\text{Mod-}R}$$

are local functors, so that the composite functor

$$PQ\colon \text{Morph(Mod-}R) \to \text{Mod-}R/\mathcal{J}_{\text{Mod-}R} \times \text{Mod-}R/\mathcal{J}_{\text{Mod-}R}$$

is a local functor. Kernels of local functors are contained in the Jacobson radical, and the kernel of the composite functor PQ consists exactly of the morphisms $u = (u_0, u_1)\colon M \to N$ in the category Morph(Mod-R) with $u_0 \in \mathcal{J}_{\text{Mod-}R}(M_0, N_0)$ and $u_1 \in \mathcal{J}_{\text{Mod-}R}(M_1, N_1)$. $\square$

Morphisms between modules of finite type

Proposition 1.42. Let M be an object of Morph(Mod-R). If $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ are rings of type m and n respectively, then E_M has type $\leq m+n$. Moreover, if $I_1, \dots, I_n$ are the n maximal ideals of $\mathrm{End}_R(M_0)$ and $K_1, \dots, K_m$ are the m maximal ideals of $\mathrm{End}_R(M_1)$, then the at most $n+m$ maximal ideals of E_M are among the completely prime ideals $(I_t \times \mathrm{End}_R(M_1)) \cap E_M$ $(t = 1, \dots, n)$ and $(\mathrm{End}_R(M_0) \times K_q) \cap E_M$ $(q = 1, \dots, m)$.

Proof. Both the canonical projections

$$\mathrm{End}_R(M_0) \to \mathrm{End}_R(M_0)/J(\mathrm{End}_R(M_0)) \cong \prod_{t=1}^{n} \mathrm{End}_R(M_0)/I_t$$

and

$$\mathrm{End}_R(M_1) \to \mathrm{End}_R(M_1)/J(\mathrm{End}_R(M_1)) \cong \prod_{q=1}^{m} \mathrm{End}_R(M_0)/K_q.$$

are local morphisms. Therefore there is a canonical local morphism

$$E_M \to \prod_{t=1}^{n} \mathrm{End}_R(M_0)/I_t \times \prod_{q=1}^{m} \mathrm{End}_R(M_0)/K_q$$

into the direct product of $n+m$ division rings. By Proposition 1.11, E_M is a ring of type $\leq n+m$.

Let us prove that the maximal ideals of E_M are among the kernels of the $n+m$ canonical projections. First of all, notice that, for $t = 1, \dots, n$ and $q = 1, \dots, m$, the ideals $(I_t \times \mathrm{End}_R(M_1)) \cap E_M$ and $(\mathrm{End}_R(M_0) \times K_q) \cap E_M$ are the kernels of the canonical projections $E_M \to \mathrm{End}_R(M_0)/I_t$ and $E_M \to \mathrm{End}_R(M_1)/K_q$ respectively. Therefore, it is immediate that they are $n+m$ completely prime two-sided ideals of E_M. Moreover, the non-invertible elements of $\mathrm{End}_R(M_0)$ are exactly the elements of $I_1 \cup \cdots \cup I_t$ and similarly, the non-invertible elements of $\mathrm{End}_R(M_1)$ are exactly the elements of $K_1 \cup \cdots \cup K_q$. Since we have a local morphism $E_M \to \mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1)$, it follows that the non-invertible elements of E_M are precisely those in the union of the $n+m$ ideals $(I_t \times \mathrm{End}_R(M_1)) \cap E_M$ $(t = 1, \ldots, n)$ and $(\mathrm{End}_R(M_0) \times K_q) \cap E_M$ $(q = 1, \ldots, m)$. To conclude, it suffices to notice that these $n + m$ ideals are completely prime two-sided ideals of E_M, hence every proper right (left) ideal of E_M must be contained in one of them, by the Prime Avoidance Lemma. □

Example 1.43. We have just seen that the inclusion

$$\varepsilon\colon E_M \to \mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1)$$

is a local morphism. If we identify E_M with its image in $\mathrm{End}_R(M_0) \times \mathrm{End}_R(M_1)$, then we have that

$$(J(\mathrm{End}_R(M_0)) \times J(\mathrm{End}_R(M_1))) \cap E_M \subseteq J(E_M).$$

Moreover, if both $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ are rings of finite type, then so is E_M. The following example shows that (1) the previous inclusion involving the Jacobson radicals can be proper and (2) it can occur that E_M is a ring of finite type but neither $\mathrm{End}_R(M_0)$ nor $\mathrm{End}_R(M_1)$ are. Let k be any field. Consider the object $\mu_M\colon k^2 \to k^2$ of Morph(Mod-k) given by $(x, y) \mapsto (x, 0)$. Then μ_M is represented by the 2×2 matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The endomorphism ring E_M of M is given by the set of all pairs of matrices $(A_0, A_1) \in M_2(k) \times M_2(k)$ such that $MA_0 = A_1 M$. An easy computation shows that E_M consists exactly of all the pairs $(A_0, A_1) \in M_2(k) \times M_2(k)$ of the form

$$(A_0, A_1) = \left(\begin{pmatrix} u & 0 \\ v & w \end{pmatrix}, \begin{pmatrix} u & x \\ 0 & y \end{pmatrix} \right) \quad \text{for some } u, v, w, x, y \in k.$$

In particular, E_M is a subring of $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right) \times \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$. The nilpotent ideal $\left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right)$ of E_M is contained in the Jacobson radical of E_M. It follows that

$0 = E_M \cap (J(M_2(k)) \times J(M_2(k))) \subset J(E_M)$. Moreover, it is easy to see that the ring E_M is a ring of type 3. Its maximal right ideals are the completely prime two-sided ideals

$$I_1 := \{\left(\begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}\right) \in E_M \mid v, w, x, y \in k\},$$

$$I_2 := \{\left(\begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix}, \begin{pmatrix} u & x \\ 0 & y \end{pmatrix}\right) \in E_M \mid u, v, x, y \in k\},$$

$$I_3 := \{\left(\begin{pmatrix} u & 0 \\ v & w \end{pmatrix}, \begin{pmatrix} u & x \\ 0 & 0 \end{pmatrix}\right) \in E_M \mid u, v, w, x \in k\}.$$

We conclude this subsection characterizing morphisms with local endomorphism rings.

Theorem 1.44. Let M be any object of Morph(Mod-R), E_M its endomorphism ring in Morph(Mod-R), $\varepsilon\colon E_M \to \operatorname{End}_R(M_0) \times \operatorname{End}_R(M_1)$ denote the inclusion, $\pi_i\colon \operatorname{End}_R(M_0) \times \operatorname{End}_R(M_1) \to \operatorname{End}_R(M_i)$, $i = 0, 1$, be the canonical projections and $E_i := \pi_i\varepsilon(E_M)$. Then the endomorphism ring E_M of the object M is local if and only if one of the following three conditions holds:

1. $M_0 = 0$ and $\operatorname{End}_R(M_1)$ is a local ring.
2. $M_1 = 0$ and $\operatorname{End}_R(M_0)$ is a local ring.
3. $M_0 \neq 0$, $M_1 \neq 0$ and, for every endomorphism $u = (u_0, u_1) \in E_M$:
   (a) either u_0 or $1 - u_0$ is invertible in E_0, and
   (b) u_0 is invertible in E_0 if and only if u_1 is invertible in E_1.

Proof. Suppose the endomorphism ring E_M in Morph(Mod-R) local. If $M_0 = 0$, then $\mu_M = 0$, and so $\operatorname{End}_R(M_1) \cong E_M$ is local. Similarly if $M_1 = 0$. Suppose $M_0 \neq 0$ and $M_1 \neq 0$. Notice that $M_0 \neq 0$ and $M_1 \neq 0$ implies that $1 \neq 0$ in both the rings $\operatorname{End}_R(M_0)$ and $\operatorname{End}_R(M_1)$, hence in both their subrings E_0 and E_1. Hence E_0 and E_1 are non-trival homomorphic images of the local ring E_M. If $u = (u_0, u_1) \in E_M$, and u_0 is not invertible in E_0, then $1 - u_0$ is invertible in E_0, because E_0 is local. This proves that condition (a) in (3) holds. Moreover, the rings E_i are homomorphic images of the local ring E_M, so that the kernel of the surjective morphism $E_M \to E_i$ is contained in the Jacobson radical (i.e. the maximal ideal) of E_M. Hence the image of the maximal ideal of E_M (i.e. the set of non-invertible elements of E_M) is mapped exactly onto the maximal ideal of E_i. It follows that $u = (u_0, u_1)$ is an automorphism of M if and only if u_i is invertible E_i. Thus u_0 is invertible in E_0 if and only if u is an automorphism of M, if and only if u_1 is invertible in E_1.

For the converse, it is clear that (1) and (2) imply E_M local. If (3) holds, for every endomorphism $u = (u_0, u_1) \in E_M$ that is not an automorphism, we have that either u_0 is not an automorphism of M_0 or u_1 is not an automorphism of M_1. Hence u_0 is not invertible in E_0 or u_1 is not invertible in E_1. By (b), both u_0 is not invertible in E_0 and u_1 is not invertible in E_1. Now E_0 is a local ring by (a). Similarly, E_1 is a local ring by (a) and (b). It follows that both $1 - u_0$ is invertible in E_0 and $1 - u_1$ is invertible in E_1. Thus $1 - u$ is invertible in E_M, i.e., the ring E_M is local. □

The ring E_M is a subdirect product of the two rings $E_i := \pi_i\varepsilon(E_M)$ (for $i = 0, 1$) and the embedding $E_M \hookrightarrow E_0 \times E_1$ is a local morphism. If $M_0 \neq 0$ and $M_1 \neq 0$, then the ring E_M is local if and only if both the rings E_0 and E_1 are local and $(\pi_0\varepsilon)^{-1}(J(E_0)) = (\pi_1\varepsilon)^{-1}(J(E_1))$. Moreover, E_M is semilocal if and only if the two rings E_0 and E_1 are semilocal (Proof: ($\Rightarrow$) Because both the rings E_i are homomorphic images of E_M. ($\Leftarrow$) Because the morphism $E_M \to E_0 \times E_1$ is local.). Notice that E_M always has the two two-sided ideals $\ker(\pi_0\varepsilon)$ and $\ker(\pi_1\varepsilon)$, whose intersection is the zero ideal.

Proposition 1.45. Let M be an object of Morph(Mod-R) and assume that $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ are rings of finite type. Then M has a local endomorphism ring if and only if there exists $i = 0, 1$ such that for every endomorphism $u = (u_0, u_1) \in E_M$ both the following conditions hold:

(a) either u_i or $1 - u_i$ is an automorphism of M_i, and

(b) if u_i is an automorphism of M_i, then u is an automorphism of M.

Proof. Assume that E_M is local. For every $u = (u_0, u_1) \in E_M$, either u or $1 - u$ is invertible, so either u_i or $1 - u_i$ is an automorphism of M_i for every

$i = 0, 1$.

Now, let n and m be the types of $\mathrm{End}_R(M_0)$ and $\mathrm{End}_R(M_1)$ respectively. As a trivial case, we have that if $n = 0$ (that is, if $M_0 = 0$), then $E_M \cong \mathrm{End}_R(M_1)$ is a local ring and (b) follows. Similarly for $m = 0$. Thus we can assume $n, m \geq 1$. In the notation of Proposition 1.42, the maximal ideal of E_M is either

$$J(E_M) = (I_t \times \mathrm{End}_R(M_1)) \cap E_M \text{ for some } t = 1, \dots, n \tag{0}$$

or

$$J(E_M) = (\mathrm{End}_R(M_0) \times K_q) \cap E_M \text{ for some } q = 1, \dots, m. \tag{1}$$

Assume that (0) holds and let $u = (u_0, u_1)$ be an element of E_M such that u_0 is an automorphism of M_0. Then $u \notin J(E_M)$, because $u_0 \notin I_t$ for every $t = 1, \dots, n$ (notice that $\bigcup_{t=1}^n I_t$ is the set of all non-invertible elements of $\mathrm{End}_R(M_0)$). In particular, u_1 is not in $\bigcup_{q=1}^m K_q$, that is, u_1 is an automorphism of M_1. This implies that u is invertible in E_M. In a similar way, we can prove that if (1) holds, then, for every $u = (u_0, u_1) \in E_M$, $u_1 \in \mathrm{Aut}(M_1)$ implies u invertible in E_M.

Conversely, we want to prove that for every $u = (u_0, u_1)$, either u or $1 - u$ is invertible in E_M. Assume that there exists $i = 0, 1$ such that both conditions (a) and (b) hold. By (a), either u_i or $1 - u_i$ is invertible, so by (b) either u or $1 - u$ is invertible in E_M. □

Morphisms between two modules of type 1

Let R be an arbitrary ring. We now consider the full subcategory $\mathcal{L}$ of Mod-R whose objects are all right R-modules with a local endomorphism ring. Let Morph($\mathcal{L}$) be the full subcategory of Morph(Mod-R) whose objects are all morphisms between two objects of $\mathcal{L}$. The canonical functor $Q\colon$ Morph(Mod-R) $\to$ Mod-$R \times$ Mod-R restricts to a functor

$$Q_{\mathcal{L}}\colon \mathrm{Morph}(\mathcal{L}) \to \mathcal{L} \times \mathcal{L}.$$

Hence, for every object M of Morph($\mathcal{L}$), the endomorphism ring of M in the category Morph($\mathcal{L}$) is of type ≤ 2, and has at most two maximal ideals: the completely prime two-sided ideals

$$I_{M,d} := \{(u_0, u_1) \in E_M \mid u_0 \text{ is not an automorphism of } M_0\},$$

and

$$I_{M,c} := \{(u_0, u_1) \in E_M \mid u_1 \text{ is not an automorphism of } M_1\}$$

(see Theorem 1.42). As a consequence, an object M of Morph($\mathcal{L}$) has a local endomorphism ring if and only if either $I_{M,d} \subseteq I_{M,c}$ or $I_{M,d} \supseteq I_{M,c}$. Therefore, we get that:

Lemma 1.46. An object M of Morph($\mathcal{L}$) has a local endomorphism ring if and only if one of the following two conditions holds:

1. For every morphism $(u_0, u_1) \in E_M$, if u_0 is an automorphism of M_0, then u_1 is an automorphism of M_1, or
2. For every morphism $(u_0, u_1) \in E_M$, if u_1 is an automorphism of M_1, then u_0 is an automorphism of M_0.

The following two examples show that conditions (1) and (2) in the previous lemma are independent, or, equivalently, that both proper inclusions $I_{M,d} \subset I_{M,c}$ and $I_{M,c} \subset I_{M,d}$ can occur.

Example 1.47. Let $\mathbb{Z}_p$ be the localization of $\mathbb{Z}$ at its maximal ideal (p), so that $\mathbb{Z}_p$ is a DVR, whose field of fractions is $\mathbb{Q}$. Consider the inclusion $\mu_M \colon \mathbb{Z}_p \hookrightarrow \mathbb{Q}$, viewed as a $\mathbb{Z}_p$-module morphism. Of course, $\mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p) = \mathbb{Z}_p$ and $\mathrm{End}_{\mathbb{Z}_p}(\mathbb{Q}) = \mathbb{Q}$, which are both local rings. It is immediate to see that the endomorphism ring of M in Morph(Mod-$\mathbb{Z}_p$) is $E_M \cong \mathbb{Z}_p$, and that $0 = I_{M,c} \subset I_{M,d} = p\mathbb{Z}_p$.

Example 1.48. Let $\mathbb{Z}(p^\infty)$ be the Prüfer group and $\mu_M \colon \mathbb{Q} \to \mathbb{Z}(p^\infty)$ be any group epimorphism, so that μ_M is an object M in Morph(Mod-$\mathbb{Z}$). It is easily seen that the endomorphism ring E_M of M is canonically isomorphic to the localization $\mathbb{Z}_p$ of $\mathbb{Z}$ at its maximal ideal (p). In this case, we have that $0 = I_{M,d} \subset I_{M,c} = p\mathbb{Z}_p$.

We will say that two objects M and N of Morph(Mod-R) belong to

(1) *the same domain class*, and write $[M]_d = [N]_d$, if there exist morphisms $u\colon M \to N$ and $u'\colon N \to M$ with $u_0\colon M_0 \to N_0$ and $u'_0\colon N_0 \to M_0$ isomorphisms;
(2) *the same codomain class*, and write $[M]_c = [N]_c$, if there exist morphisms $u\colon M \to N$ and $u'\colon N \to M$ with $u_1\colon M_1 \to N_1$ and $u'_1\colon N_1 \to M_1$ isomorphisms.

In Morph(Mod-$\mathcal{L}$), we have two completely prime ideals, defined, for every pair of objects $\mu_M\colon M_0 \to M_1$ and $\mu_N : N_0 \to N_1$, by

$$\mathcal{P}_0(M,N) := \{u = (u_0, u_1) : M \to N \mid u_0 \text{ is not an isomorphism}\}$$

and

$$\mathcal{P}_1(M,N) := \{u = (u_0, u_1) : M \to N \mid u_1 \text{ is not an isomorphism}\}.$$

It is immediate to see that M and N have the same domain [resp. codomain] class if and only if they have the same $\mathcal{P}_0$ [resp. $\mathcal{P}_1$] class (see page 18). Moreover, for every object $\mu_M : M_0 \to M_1$ of Morph(Mod-$\mathcal{L}$), $u \in E_M$ is an automorphism if and only if $u \notin \mathcal{P}_0(M,M) \cup \mathcal{P}_1(M,M)$. Then Theorem 1.35 implies:

Theorem 1.49. Let $\mu_{M_k}\colon M_{0,k} \to M_{1,k}$, $k = 1,\dots,r$, and $\mu_{N_\ell}\colon N_{0,\ell} \to N_{1,\ell}$, $\ell = 1,\dots,s$, be $r+s$ objects in the category Morph(Mod-$\mathcal{L}$). Then $\bigoplus_{k=1}^r M_k \cong \bigoplus_{\ell=1}^s N_\ell$ in the category Morph(Mod-R) if and only if $r = s$ and there exist two permutations φ_d, φ_c of $\{1,2,\dots,r\}$ such that $[M_k]_d = [N_{\varphi_d(k)}]_d$ and $[M_k]_c = [N_{\varphi_c(k)}]_c$ for every $k = 1,\dots,r$.

Let $n \geq 2$ be an integer. We now give an example of a semilocal ring R (of type $2n$) with $2n$ pairwise non-isomorphic right R-modules A_i, B_i ($i = 1,2,\dots,n$), all $2n$ of them uniserial modules with local endomorphism rings, and n^2 right R-module morphisms $\mu_{i,j}\colon A_i \to B_j$ ($i,j = 1,2,\dots,n$), that is, objects $M_{i,j}$ of Morph(Mod-R) ($i,j = 1,2,\dots,n$), such that $\oplus_{i=1}^n M_{i,i}$ has $n!$ pairwise non-isomorphic decompositions as a direct sum of n indecomposable objects of Morph(Mod-R). More precisely, we will see that the objects $M_{i,j}$ ($i,j = 1,2,\dots,n$) are such that:

(1) for every $i,j,k,\ell = 1,2,\dots,n$, $[M_{i,j}]_d = [M_{k,\ell}]_d$ if and only if $i = k$;

(2) for every $i, j, k, \ell = 1, 2, \ldots, n$, $[M_{i,j}]_c = [M_{k,\ell}]_c$ if and only if $j = \ell$.

Therefore

$$M_{1,1} \oplus M_{2,2} \oplus \cdots \oplus M_{n,n} \cong M_{\sigma(1),\tau(1)} \oplus M_{\sigma(2),\tau(2)} \oplus \cdots \oplus M_{\sigma(n),\tau(n)}$$

for every pair of permutations σ, τ of $\{1, 2, \ldots, n\}$. This example is similar to [34, Example 2.1].

Example 1.50. Let $p, q \in \mathbb{Z}$ be two distinct primes, $\mathbb{Z}_p, \mathbb{Z}_q$ be the localizations of $\mathbb{Z}$ at its maximal ideals (p) and (q), so that $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are DVRs contained in $\mathbb{Q}$, and let $\mathbb{Z}_{pq} := \mathbb{Z}_p \cap \mathbb{Z}_q$ be the subring of $\mathbb{Q}$ consisting of all rational numbers a/b, with $a, b \in \mathbb{Z}$ such that $p \nmid b$ and $q \nmid b$. Thus $\mathbb{Z}_{pq}$ is a subring of $\mathbb{Q}$ that contains $\mathbb{Z}$, is a PID, is the localization of $\mathbb{Z}$ at the multiplicatively closed subset $\mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$, is a semilocal ring of type 2, and all its non-zero ideals are of the form $p^i q^j \mathbb{Z}_{pq}$, with $i, j \geq 0$.

Let R denote the subring of $\mathbf{M}_n(\mathbf{Q})$ whose elements are $n \times n$-matrices with entries in $\mathbf{Z}_{pq}$ on and above the diagonal and entries in $pq\mathbf{Z}_{pq}$ under the diagonal, that is,

$$R = \begin{pmatrix} \mathbf{Z}_{pq} & \mathbf{Z}_{pq} & \ldots & \mathbf{Z}_{pq} \\ pq\mathbf{Z}_{pq} & \mathbf{Z}_{pq} & \ldots & \mathbf{Z}_{pq} \\ \vdots & & \ddots & \\ pq\mathbf{Z}_{pq} & pq\mathbf{Z}_{pq} & \ldots & \mathbf{Z}_{pq} \end{pmatrix} \subseteq \mathbf{M}_n(\mathbf{Q}).$$

The set $W := M_{1\times n}(\mathbb{Q})$ of all $1 \times n$ matrices with entries in $\mathbb{Q}$ is a right R-module under matrix multiplication. Set

$$V_i := (\underbrace{q\mathbf{Z}_q, \ldots, q\mathbf{Z}_q}_{i-1}, \underbrace{\mathbf{Z}_q, \ldots, \mathbf{Z}_q}_{n-i+1}), \quad i = 1, 2, \ldots, n,$$

and

$$X_j = (\underbrace{p\mathbf{Z}_p, \ldots, p\mathbf{Z}_p}_{j-1}, \underbrace{\mathbf{Z}_p, \ldots, \mathbf{Z}_p}_{n-j+1}), \quad j = 1, 2, \ldots, m,$$

so that V_i and X_j are R-submodules of W with $V_1 \supset V_2 \supset \cdots \supset V_n \supset qV_1$ and $X_1 \supset X_2 \supset \cdots \supset X_n \supset pX_1$. For every $i, j = 1, 2, \ldots, n$, let $\mu_{i,j} \colon V_i \to W/X_j$ be the composite mapping of the inclusion $V_i \to W$ and the canonical projection $W \to W/X_j$, so that $\mu_{i,j}$ can be viewed as an object $M_{i,j}$ of Morph(Mod-R).

The endomorphism ring of the right R-module V_i is isomorphic to the local ring $\mathbb{Z}_q$, because $V_i \cong e_{ii}R_{(q)}$ as an R-module, where $R_{(q)}$ denotes the localization of the $\mathbb{Z}_{pq}$-algebra R at the maximal ideal (q) of $\mathbb{Z}_{pq}$, so that

$$\mathrm{End}_R(V_i) = \mathrm{End}_{R_{(q)}}(V_i) = \mathrm{End}_{R_{(q)}}(e_{ii}R_{(q)}) \cong e_{ii}R_{(q)}e_{ii},$$

which is isomorphic to the localization $\mathbb{Z}_q$ of $\mathbb{Z}$ at its maximal ideal $q\mathbb{Z}$.

Let us prove that the endomorphism ring of the right R-module W/X_j is also local. The module W/X_j is isomorphic to $\mathbb{Z}(p^\infty)^n$ (direct sum of n copies of the Prüfer group $\mathbb{Z}(p^\infty)$) as an abelian group, hence is artinian as an abelian group, hence is artinian as a right R-module. For an artinian right R-module L_R, the restriction to the socle $\mathrm{soc}(L_R)$ is a local homomorphism $\mathrm{End}_R(L_R) \to \mathrm{End}_R(\mathrm{soc}(L_R))$, because every endomorphism of an artinian module L_R which restricted to the socle is an automorphism of the socle, is necessarily an automorphism of L_R. As pq is in the Jacobson radical of R, pq annihilates all simple right R-modules, so that $\mathrm{soc}(W/X_j)$ is contained in $(\mathbb{Z}/p\mathbb{Z})^n$. Now $(\mathbb{Z}/p\mathbb{Z})^n$ is a uniserial right R-module of finite composition length n, whose socle is $(0, \dots, 0, \mathbb{Z}/p\mathbb{Z})$. Thus $\mathrm{soc}(W/X_j) = (0, \dots, 0, \mathbb{Z}/p\mathbb{Z})$, and the endomorphism ring of the socle of W/X_j is isomorphic to the field $\mathbb{Z}/p\mathbb{Z}$ with p elements. Thus there is a surjective local morphism $\mathrm{End}_R(W/X_j) \to \mathbb{Z}/p\mathbb{Z}$, hence $\mathrm{End}_R(W/X_j))$ is a local ring.

Let us show that, for every $i, j, k, \ell = 1, 2, \dots, n$, $[M_{i,j}]_d = [M_{k,\ell}]_d$ if and only if $i = k$. The ring R has type $2n$, so that it has $2n$ pairwise non-isomorphic simple right R-modules, up to isomorphism, $S_1, S_2, \dots, S_n$ (with p elements each) and $T_1, T_2, \dots, T_n$ (with q elements each).

The modules V_i/qV_i are uniserial right R-modules of finite composition length n and q^n elements, their composition factors are the n simple right R-modules $T_1, T_2, \dots, T_n$ (each with multiplicity one), and with top $V_i/\mathrm{rad}(V_i)$ isomorphic to T_i. Similarly, the modules X_j/pX_j are uniserial right R-modules of finite composition length n and p^n elements, their composition factors are the n simple right R-module $S_1, S_2, \dots, S_n$ (each with multiplicity one), and with top $X_j/\operatorname{rad}(X_j)$ isomorphic to S_j.

It follows that the $2n$ right R-modules $V_1, \dots, V_n, W/X_1, \dots, W/X_n$ are pairwise non-isomorphic, that multiplication by q is an isomorphism of V_i onto qV_i, and that multiplication by p is an isomorphism of W/X_j onto W/pX_j.

From the fact that the $2n$ right R-modules $V_1, \dots, V_n, W/X_1, \dots, W/X_n$ are pairwise non-isomorphic, it follows that, for every $i, j, k, \ell = 1, 2, \dots, n$, $[M_{i,j}]_d = [M_{k,\ell}]_d$ implies $i = k$, and $[M_{i,j}]_c = [M_{k,\ell}]_c$ implies $j = \ell$.

Since multiplication by q is an isomorphism of V_i onto qV_i, we get, for every $j \leq \ell$, commutative squares

$$\begin{array}{ccc} V_i & \xrightarrow{\mu_{ij}} & W/X_j \\ & & \Big\downarrow{\scriptstyle p\,\cong} \\ {\scriptstyle p\,\cong}\Big\downarrow & & W/pX_j \\ & & \Big\downarrow{\scriptstyle \mathrm{can}} \\ V_i & \xrightarrow[\mu_{i\ell}]{} & W/X_\ell \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V_i & \xrightarrow{\mu_{i\ell}} & W/X_\ell \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{can}} \\ V_i & \xrightarrow[\mu_{ij}]{} & W/X_j \end{array} .$$

This shows that $[M_{i,j}]_d = [M_{i,\ell}]_d$ for every i, j, ℓ.

The fact that multiplication by p is an isomorphism of W/X_j onto W/pX_j implies that, for every $i \leq k$, there are commutative diagrams

$$\begin{array}{ccc} V_i & \xrightarrow{\mu_{ij}} & W/X_j \\ {\scriptstyle q}\Big\downarrow & & \Big\downarrow{\scriptstyle q\,\cong} \\ V_k & \xrightarrow[\mu_{kj}]{} & W/X_j \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V_k & \xrightarrow{\mu_{kj}} & W/X_j \\ \Big\downarrow & & \Big\| \\ V_i & \xrightarrow[\mu_{ij}]{} & W/X_j \end{array} .$$

These diagrams show that $[M_{i,j}]_c = [M_{k,j}]_c$ for every i, j, k.

Morphisms between uniserial modules

In this section, we want to focus our attention on morphisms between uniserial modules. Let $\mathcal{U}$ denote the full subcategory of Mod-R whose objects are all uniserial right R-modules and consider the full subcategory Morph($\mathcal{U}$) of

Morph(Mod-R) whose objects are all morphisms between two objects of $\mathcal{U}$. The canonical functor $Q\colon$ Morph(Mod-R) $\to$ Mod-$R \times$ Mod-R restricts to a functor

$$Q_{\mathcal{U}}\colon \operatorname{Morph}(\mathcal{U}) \to \mathcal{U} \times \mathcal{U}.$$

Hence, for every object M of Morph($\mathcal{U}$), the endomorphism ring of M in the category Morph($\mathcal{U}$) is of type ≤ 4, and has at most four maximal ideals.

Proposition 1.51. Let $\mu_M\colon M_0 \to M_1$ be an object of Morph(Mod-R) with M_0 and M_1 non-zero uniserial right R-modules. Then E_M has at most four maximal right (left) ideals, which are among the completely prime two-sided ideals

$I_{M,0,m} := \{(u_0, u_1) \in E_M \mid u_0$ is not a right R-module monomorphism$\}$,

$I_{M,1,m} := \{(u_0, u_1) \in E_M \mid u_1$ is not a right R-module monomorphism$\}$,

$I_{M,0,e} := \{(u_0, u_1) \in E_M \mid u_0$ is not a right R-module epimorphism$\}$,

and

$I_{M,1,e} := \{(u_0, u_1) \in E_M \mid u_1$ is not a right R-module epimorphism$\}$.

Proof. It immediately follows from Theorem 1.19 and Proposition 1.42. □

We can define four equivalence relations on Ob(Morph(Mod-R)) related to those of monogeny class and epigeny class we have seen for uniserial modules. For every pair of morphisms $\mu_M : M_0 \to M_1$ and $\mu_N : N_0 \to N_1$, we will write:

(1) $[M]_{0,m} = [N]_{0,m}$ if there exist two morphisms $(u_0, u_1) \in \operatorname{Hom}(M, N)$ and $(v_0, v_1) \in \operatorname{Hom}(N, M)$ such that both u_0 and v_0 are injective right R-modules morphisms;

(2) $[M]_{1,m} = [N]_{1,m}$ if there exist two morphisms $(u_0, u_1) \in \operatorname{Hom}(M, N)$ and $(v_0, v_1) \in \operatorname{Hom}(N, M)$ such that both u_1 and v_1 are injective right R-modules morphisms;

(3) $[M]_{0,e} = [N]_{0,e}$ if there exist two morphisms $(u_0, u_1) \in \operatorname{Hom}(M, N)$ and $(v_0, v_1) \in \operatorname{Hom}(N, M)$ such that both u_0 and v_0 are surjective right R-modules morphisms;

(4) $[M]_{1,e} = [N]_{1,e}$ if there exist two morphisms $(u_0, u_1) \in \operatorname{Hom}(M, N)$ and

$(v_0, v_1) \in \mathrm{Hom}(N, M)$ such that both u_1 and v_1 are surjective right R-modules morphisms.

Remark 1.52. By Lemma 1.3, it is immediate that if $\mu_M : M_0 \to M_1$ and $\mu_N : N_0 \to N_1$ are two morphisms between non-zero uniserial modules, then for $i = 0, 1$ and $a = m, e$, $[M]_{i,a} = [N]_{i,a}$ if and only if there exist two morphisms $u \in \mathrm{Hom}(M, N)$ and $v \in \mathrm{Hom}(N, M)$ such that $v \circ u \notin I_{M,i,a}$ or, equivalently, such that $u \circ v \notin I_{N,i,a}$.

For morphisms between uniserial modules, we have the following weak form of the Krull-Schmidt Theorem. We will give a proof of this result in Section 1.5.

Theorem 1.53. Let $\mu_{M_j} : M_{0,j} \to M_{1,j}$, $j = 1, \dots, r$, and $\mu_{N_k} : N_{0,k} \to N_{1,k}$, $k = 1, \dots, t$, be $r + t$ morphisms between non-zero uniserial right R-modules. Then $\bigoplus_{j=1}^{r} M_j \cong \bigoplus_{k=1}^{t} N_k$ in Morph(Mod-R) if and only if $r = t$ and there exist four permutations $\varphi_{0,m}, \varphi_{1,m}, \varphi_{0,e}, \varphi_{1,e}$ of $\{1, 2, \dots, r\}$ such that $[M_j]_{i,a} = [N_{\varphi_{i,a}(j)}]_{i,a}$ for every $j = 1, \dots, r$, $i = 0, 1$ and $a = m, e$.

1.4 A category of chain of modules

All results contained in this section are based on [19] and on a joint paper with Alberto Facchini [21].

Let R be a ring and let n be a fixed positive integer. We consider the category $\mathcal{E}_n$, defined as follows. The objects of $\mathcal{E}_n$ are right R-modules M with a fixed chain of submodules

$$0 = M^{(0)} \leq M^{(1)} \leq M^{(2)} \leq \cdots \leq M^{(n)} = M.$$

With abuse of notation, we simply denote by M such an object. A morphism in $\mathcal{E}_n$ between two chains M and N is a right R-module morphism $f : M \to N$ such that $f(M^{(i)}) \subseteq N^{(i)}$ for every $i = 1, \dots, n$. We will denote by E_M the endomorphism ring $\mathrm{End}_{\mathcal{E}_n}(M)$ of an object M in the category $\mathcal{E}_n$.

Notice that $\mathcal{E}_n$ is an additive category, whose zero object is given by the zero module with its trivial chain $0^{(i)} = 0$ for every $i = 1, \dots, n$. It will be denoted by 0.

A morphism $f : M \to N$ in $\mathcal{E}_n$ induces right R-module morphisms on the factors

$$f_i : \frac{M^{(i)}}{M^{(i-1)}} \longrightarrow \frac{N^{(i)}}{N^{(i-1)}} \quad \text{for every } i = 1, \ldots, n.$$

We call $M^{(i)}/M^{(i-1)}$ the i^{th}*-factor module* of M and f_i the i^{th}*-induced morphism.*

We have a canonical local functor

$$Q : \mathcal{E}_n \longrightarrow \underbrace{\text{Mod-}R \times \cdots \times \text{Mod-}R}_{n\text{-times}}$$

that sends any object $M \in \text{Ob}(\mathcal{E}_n)$ into the n-uple of factor modules $(M^{(1)}, M^{(2)}/M^{(1)}, \ldots, M^{(n)}/M^{(n-1)})$ and any morphism $f : M \to N$ between two objects $M, N \in \text{Ob}(\mathcal{E}_n)$ into the n-tuple of the induced morphisms $(f_1, \ldots, f_n)$. Notice that the functor Q is not faithful. Let us see an elementary example of this fact for $n = 3$. Consider three right R-modules A_R, B_R and C_R, and define two chains

$$M :\ 0 < A < A \oplus B < A \oplus B \oplus C \quad \text{and} \quad N :\ 0 < A < A \oplus C < A \oplus B \oplus C.$$

Then, the assignments $(a, b, c) \mapsto (a, 0, c)$ and $(a, b, c) \mapsto (a, 0, 0)$ define two morphisms $f, g \in \text{Hom}_{\mathcal{E}_3}(M, N)$ such that $f_i = g_i$ for every $i = 1, 2, 3$, but clearly $f \neq g$.

Proposition 1.54. Let M be an object of $\mathcal{E}_n$, with factor modules $U^{(i)} = M^{(i)}/M^{(i-1)}$, $i = 1, \ldots, n$. Then:

1. $\text{codim}(E_M) \leq \sum_{i=1}^{n} \text{codim}(\text{End}_R(U^{(i)}))$.
2. If $\text{End}_R(U^{(i)})$ is semilocal for every $i = 1 \ldots, n$, then the endomorphism ring E_M is also a semilocal ring.
3. Assume that for every $i = 1, \ldots, n$, $U^{(i)}$ is of type m_i. Then, the endomorphism ring E_M of M is of type $\leq m_1 + \cdots + m_n$.

Proof. There is a canonical ring morphism

$$\varphi \colon E_M \to \text{End}_R(U^{(1)}) \times \cdots \times \text{End}_R(U^{(n)})$$

defined by $f \mapsto (f_1, \dots, f_n)$, which is a local morphism. Now (1) follows from Proposition 1.5 (2), and (2) follows from the fact that a ring is semilocal if and only if its dual Goldie dimension is finite (Proposition 1.6).

For (3), suppose that for every $i = 1, \dots, n$, $U^{(i)}$ is of type m_i. By Proposition 1.11, there are local morphisms $\mathrm{End}_R(U^{(i)}) \to D_1^{(i)} \times \cdots \times D_{m_i}^{(i)}$ for suitable division rings $D_{j_i}^{(i)}$, $i = 1, \dots, n$, $j_i = 1, \dots, m_i$. Composing with φ, we get a local morphism from E_M into the product of $m_1 + \cdots + m_n$ division rings. Thus E_M has type $\leq m_1 + \cdots + m_n$ again by Proposition 1.11. $\square$

Remark 1.55. Let M be an object of $\mathcal{E}_n$, with factor modules $U^{(i)} = M^{(i)}/M^{(i-1)}$ of type m_i for $i = 1, \dots, n$ and let $I_1^{(i)}, \dots, I_{m_i}^{(i)}$ be the maximal ideals of $\mathrm{End}_R(U^{(i)})$. We can repeat step by step the proof of Proposition 1.42 to deduce that the maximal ideals of E_M are among the $m_1 + \cdots + m_n$ completely prime two-sided ideals:

$$I_{M,i,k} := \{f \in E_M \mid f_i \text{ belongs to } I_k^{(i)}\} \quad \text{for } i = 1, \dots, n \text{ and } k = 1, \dots, m_i.$$

Chains with uniserial factors

We now consider the full subcategory $\mathcal{U}_n$ of the category $\mathcal{E}_n$ consisting of the objects whose factor modules are all non-zero uniserial right R-modules. Among the examples of modules having a chain of submodules with all factor modules uniserial we mention modules of finite composition length, serial modules and, more generally, polyserial modules studied in [53], [54] and [55].

Since uniserial modules have type ≤ 2, the endomorphism ring E_M of an object in $\mathcal{U}_n$ has type at most $2n$. To be more precise, applying Proposition 1.54 and Remark 1.55 we get the following result.

Theorem 1.56. Let M be an object of $\mathcal{U}_n$, with factor modules $U^{(i)}$. For every $i = 1, \dots, n$, set

$$I_{M,i,m} := \{\, f \in E_M \mid f_i \text{ is not an injective right } R\text{-module morphism}\}$$

and

$$I_{M,i,e} := \{ f \in E_M \mid f_i \text{ is not a surjective right } R\text{-module morphism}\}.$$

Then $I_{M,1,m}, \dots, I_{M,n,m}, I_{M,1,e}, \dots, I_{M,n,e}$ are $2n$ two-sided completely prime ideals of E_M, and every proper right ideal of E_M and every proper left ideal of E_M is contained in one of these $2n$ ideals of E_M. Moreover, $E_M/J(E_M)$ is isomorphic to the direct product of k division rings $D_1, \dots, D_k$ for some k with $1 \le k \le 2n$, where $\{D_1, \dots, D_k\} \subseteq \{E_M/I_{M,1,m}, \dots, E_M/I_{M,n,m}, E_M/I_{M,1,e}, \dots, E_M/I_{M,n,e}\}$.

Remark 1.57. Notice that in Theorem 1.56, without the assumption $U^{(i)} \neq 0$, the sets $I_{M,i,m}$ and $I_{M,i,e}$ defined in the statement could be empty. To be more precise, for a fixed $i = 1, \dots, n$, the sets $I_{M,i,m}$ and $I_{M,i,e}$ are empty exactly when $U^{(i)} = 0$. However, under the hypothesis of Theorem 1.56, the zero morphism belongs to $I_{M,1,m}, \dots, I_{M,n,m}, I_{M,1,e}, \dots, I_{M,n,e}$ and these $2n$ sets are always proper ideals of E_M.

Similarly to the category of morphisms, we can extend the notions of monogeny and epigeny class to this context. Let M and N be two objects of $\mathcal{E}_n$ with factor modules $U^{(1)}, \dots, U^{(n)}$ and $V^{(1)}, \dots, V^{(n)}$ respectively. For $i = 1, \dots, n$, we will say that M and N belong to
(1) *the same i^{th}-monogeny class*, and we will write $[M]_{i,m} = [N]_{i,m}$, if there exist two morphisms $f \colon M \to N$ and $g \colon N \to M$ in the category $\mathcal{E}_n$ such that $f_i : U^{(i)} \to V^{(i)}$ and $g_i : V^{(i)} \to U^{(i)}$ are injective right R-module morphisms;
(2) *the same i^{th}-epigeny class*, and write $[M]_{i,e} = [N]_{i,e}$ if there exist two morphisms $f \colon M \to N$ and $g \colon N \to M$ in the category $\mathcal{E}_n$ such that $f_i : U^{(i)} \to V^{(i)}$ and $g_i : V^{(i)} \to U^{(i)}$ are surjective right R-module morphisms.

Notice that, in this notation, $[M]_{i,m} = [0]_{i,m}$ if and only if $[M]_{i,e} = [0]_{i,e}$, if and only if $U^{(i)}$ is the zero module.

Example 1.58. Let M and N be two objects of $\mathcal{E}_n$ with factor modules $U^{(1)}, \dots, U^{(n)}$ and $V^{(1)}, \dots, V^{(n)}$ respectively. If $[M]_{i,m} = [N]_{i,m}$ for some $i = 1, \dots, n$, then $[U^{(i)}]_m = [V^{(i)}]_m$, that is, $U^{(i)}$ and $V^{(i)}$ belong to the same monogeny class in the sense of [34]. Similarly, if $[M]_{i,e} = [N]_{i,e}$ for some $i = 1, \dots, n$, then $[U^{(i)}]_e = [V^{(i)}]_e$. Now we will see an elementary example in which the converse holds as well, that is, if $[U^{(i)}]_m = [V^{(i)}]_m$ for some $i = 1, \dots, n$, then $[M]_{i,m} = [N]_{i,m}$, and if $[U^{(i)}]_e = [V^{(i)}]_e$ for some $i = 1, \dots, n$, then $[M]_{i,e} = [N]_{i,e}$. Let $M_1, \dots, M_n$ and $N_1, \dots, N_n$ be right R-modules. Consider the following two objects of $\mathcal{E}_n$:

$$0 < M_1 < M_1 \oplus M_2 < \cdots < M_1 \oplus \cdots \oplus M_n = M$$

and

$$0 < N_1 < N_1 \oplus N_2 < \cdots < \cdots < N_1 \oplus \cdots \oplus N_n = N.$$

The i^{th}-factor module $U^{(i)}$ of M is isomorphic to M_i and the i^{th}-factor module $V^{(i)}$ of N is isomorphic to N_i. So, if $[U^{(i)}]_m = [V^{(i)}]_m$ for some $i = 1, \dots, n$, then there exist two injective right R-module morphisms $\varphi : M_i \to N_i$ and $\psi : N_i \to M_i$ that can be extended trivially on all the other direct summands of M and N, in order to get two morphisms $f : M \to N$ and $g : N \to M$ in $\mathcal{E}_n$ such that both the induced morphisms f_i and g_i are right R-module monomorphisms. Hence $[M]_{i,m} = [N]_{i,m}$. Similarly, if $[U^{(i)}]_e = [V^{(i)}]_e$, then $[M]_{i,e} = [N]_{i,e}$.

Remark 1.59. Let M and N be two objects of $\mathcal{U}_n$ with factor modules $U^{(1)}, \dots, U^{(n)}$ and $V^{(1)}, \dots, V^{(n)}$ respectively. For $a = m, e$, by Lemma 1.3, $[M]_{i,a} = [N]_{i,a}$ if and only if there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{U}_n$ such that $gf \notin I_{M,i,a}$ (or, equivalently, such that $fg \notin I_{N,i,a}$).

We have the following weak form of the Krull-Schmidt Theorem for objects in $\mathcal{U}_n$.

Theorem 1.60. Let $M_1, M_2 \dots, M_r, N_1, N_2, \dots, N_s$ be $r + s$ objects of $\mathcal{U}_n$. Then $\bigoplus_{k=1}^r M_k \cong \bigoplus_{\ell=1}^s N_\ell$ in the category $\mathcal{E}_n$ if and only if $r = s$ and there exist $2n$ permutations $\varphi_{i,a}$ of $\{1, 2, \dots, r\}$, where $i = 1, \dots, n$ and $a = m, e$, such that $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$ for every $k = 1, \dots, r$.

We will give a proof of Theorem 1.60 in Section 1.5.

Example 1.61. Let $r \geq 2$ be an integer. We have seen in Example 1.22 that there exist r^2 pairwise non-isomorphic finitely presented uniserial modules $U_{j,k}$ $(j, k = 1, 2, \dots, r)$ over a suitable serial ring R satisfying the following properties:

1. for every $j, k, h, \ell = 1, 2, \dots, r$, $[U_{j,k}]_m = [U_{h,\ell}]_m$ if and only if $j = h$;
2. for every $j, k, h, \ell = 1, 2, \dots, r$, $[U_{j,k}]_e = [U_{h,\ell}]_e$ if and only if $k = \ell$.

Using the modules $U_{j,k}$, we want to show that the permutations $\varphi_{i,a}$ in the statement of Theorem 1.60 can be completely arbitrary. For any choice of $2r$ elements $k_1, \dots, k_{2r}$ of $\{1, \dots, r\}$, define the following object $M_{k_1,\dots,k_{2r}}$ of $\mathcal{U}_n$:

$$0 < U_{k_1,k_2} < U_{k_1,k_2} \oplus U_{k_3,k_4} < \dots < U_{k_1,k_2} \oplus \dots \oplus U_{k_{2r-1},k_{2r}} = M_{k_1,\dots,k_{2r}}.$$

For any two objects $M_{k_1,\dots,k_{2r}}$ and $M_{h_1,\dots,h_{2r}}$ of this form, using Example 1.58 and the properties (a) and (b), we get $[M_{k_1,\dots,k_{2r}}]_{i,m} = [M_{h_1,\dots,h_{2r}}]_{i,m}$ if and only if $[U_{k_{2i-1},k_{2i}}]_m = [U_{h_{2i-1},h_{2i}}]_m$, if and only if $k_{2i-1} = h_{2i-1}$ and $[M_{k_1,\dots,k_{2r}}]_{i,e} = [M_{h_1,\dots,h_{2r}}]_{i,e}$ if and only if $[U_{k_{2i-1},k_{2i}}]_m = [U_{h_{2i-1},h_{2i}}]_m$, if and only if $k_{2i} = h_{2i}$. In particular, from Theorem 1.60, the $r!$ objects $M_{k_1,\dots,k_{2r}}$ of $\mathcal{U}_n$ are pairwise non-isomorphic. Moreover, given $2n$ permutations $\sigma_1, \dots, \sigma_n, \tau_1 \dots, \tau_n$ of $\{1, 2, \dots, r\}$ we have the following isomorphism

$$M_{\sigma_1(1),\tau_1(1),\dots,\sigma_r(1),\tau_r(1)} \oplus M_{\sigma_1(2),\tau_1(2),\dots\sigma_r(2),\tau_r(2)} \oplus \dots \oplus M_{\sigma_1(r),\tau_1(r),\dots,\sigma_r(r),\tau_r(r)}$$

$$\cong M_{1,1,\dots,1} \oplus M_{2,2,\dots 2} \oplus \dots \oplus M_{r,r,\dots,r}$$

and the bijections $\varphi_{i,m}$ and $\varphi_{j,e}$ in the statement of Theorem 1.60 are the permutations σ_i and τ_j respectively.

The category of short exact sequences

If $n = 2$, the category $\mathcal{E}_2$ is equivalent to the category $\mathcal{S}$ of short exact sequences defined as follows. The objects of $\mathcal{S}$ are short exact sequences of right R-modules $0 \to A_R \to B_R \to C_R \to 0$. A morphism in $\mathcal{S}$ between two such exact sequences

$$0 \longrightarrow A_R \xrightarrow{\alpha} B_R \xrightarrow{\beta} C_R \longrightarrow 0$$

and

$$0 \longrightarrow A'_R \xrightarrow{\alpha'} B'_R \xrightarrow{\beta'} C'_R \longrightarrow 0$$

is a right R-module morphism $f\colon B_R \to B'_R$ that induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_R & \xrightarrow{\alpha} & B_R & \xrightarrow{\beta} & C_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f|_{A'}^{A}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & A'_R & \xrightarrow[\alpha']{} & B'_R & \xrightarrow[\beta']{} & C'_R & \longrightarrow & 0. \end{array}$$

Equivalently, a morphism in $\mathcal{S}$ can be viewed as a right R-module morphism $f\colon B_R \to B'_R$ such that $f(\alpha(A_R)) \subseteq \alpha'(A'_R)$.

The behaviour of short exact sequences as far as direct-sum decompositions is concerned was studied in [21]. Since almost all results are generalized for the categories $\mathcal{E}_n$, we only mention some examples.

For ease of notation, we identify $\mathcal{S}$ with $\mathcal{E}_2$, so that the full subcategory $\mathcal{U}_2$ consists of all short exact sequences $0 \to A \to B \to C \to 0$ with A and C uniserial modules.

Example 1.62. Let R be a ring having two non-isomorphic simple right R-modules S and S'. Consider the following two objects of $\mathcal{U}_2$:

$$M: \quad 0 \longrightarrow S \xrightarrow{\varepsilon_1} S \oplus S' \xrightarrow{\pi_1} S' \longrightarrow 0$$

and

$$N: \quad 0 \longrightarrow S' \xrightarrow{\varepsilon_2} S \oplus S' \xrightarrow{\pi_2} S \longrightarrow 0,$$

where ε_i and π_j are the embeddings and the canonical projections, respectively. These two objects have the same endomorphism ring in $\mathcal{E}_2$:

$$E_M = \begin{pmatrix} \mathrm{End}_R(S) & 0 \\ 0 & \mathrm{End}_R(S') \end{pmatrix} = E_N.$$

Moreover, the maximal right ideals of E are:

$$I_{M,2,m} = I_{M,2,e} = \begin{pmatrix} \mathrm{End}_R(S) & 0 \\ 0 & 0 \end{pmatrix} = I_{N,1,m} = I_{N,1,e}$$

and

$$I_{M,1,m} = I_{M,1,e} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{End}_R(S') \end{pmatrix} = I_{N,2,m} = I_{N,2,e}.$$

So, for any fixed $(i, a) \in \{1, 2\} \times \{m, e\}$, $I_{M,i,a}$ is a maximal right ideal of E_M

and $I_{N,i,a}$ is a maximal ideal of E_N, but $[M]_{i,a} \neq [N]_{i,a}$, because otherwise S and S' would be isomorphic, contrary to the hypothesis. In particular, the two objects are not isomorphic in $\mathcal{E}_2$ (Theorem 1.60).

Example 1.63. Let R be a ring and let U, V be two non-zero uniserial right R-modules. Consider the following object of $\mathcal{U}_2$:

$$M: \quad 0 \longrightarrow U \xrightarrow{\varepsilon} U \oplus V \xrightarrow{\pi} V \longrightarrow 0,$$

where ε and π are the natural embedding and the canonical projection respectively. The endomorphism ring of M is

$$E_M = \begin{pmatrix} \mathrm{End}_R(U) & 0 \\ \mathrm{Hom}_R(V,U) & \mathrm{End}_R(V) \end{pmatrix}.$$

For any endomorphism $f = \begin{pmatrix} f_{1,1} & 0 \\ f_{2,1} & f_{2,2} \end{pmatrix} \in E_B$, we have $f|_U^U = f_{1,1}$ and $\overline{f} = f_{2,2}$. So, in the notation of Theorems 1.19 and 1.56, $f \in I_{M,1,m}$ [resp. $f \in I_{M,1,e}$] if and only if $f_{1,1} \in I_U$ [resp. $f_{1,1} \in K_U$]. Similarly, $f \in I_{M,2,m}$ [resp. $f \in I_{M,2,e}$] if and only if $f_{2,2} \in I_V$ [resp. $f_{2,2} \in K_V$]. In particular, the type of E_M is exactly the sum of the types of $\mathrm{End}_R(U)$ and $\mathrm{End}_R(V)$. Therefore, choosing suitable uniserial R-modules, it is possible to obtain objects of $\mathcal{U}_2$ whose endomorphism ring has exactly 2, 3 or 4 maximal ideals.

Example 1.64. We have seen in Example 1.22 the construction of uniserial right R-modules $U_{i,j}$ used to show that a module that is a direct sum of n uniserial modules can have $n!$ pair-wise non-isomorphic direct-sum decompositions into indecomposables. All those modules $U_{i,j}$ have exactly two maximal ideals and we will now show that they are extensions of two uniserial modules with local endomorphism ring. We preserve the notations of Example 1.22. Notice that

$$X := (\underbrace{\mathbb{Q}, \dots, \mathbb{Q}}_{n}, \underbrace{0, \dots, 0}_{n}) = \bigcap_{m \geq 0} q^m V_j = \sum_{m \geq 0} p^{-m} X_i$$

for every $i, j = 1, 2, \dots, n$. Thus we have a short exact sequence

$$M: \quad 0 \to X/X_i \to U_{i,j} \to V_j/X \to 0 \tag{1.4.1}$$

The right R-module X/X_i is an artinian uniserial module whose lattice of submodules is order isomorphic to the ordinal number $\omega+1$, where ω denotes the first infinite ordinal. Since X/X_i is an artinian uniserial R-module, its endomorphism ring $\mathrm{End}_R(X/X_i)$ is a local ring whose maximal ideal consist of all the endomorphisms of X/X_i that are zero on the socle of X/X_i.

The right R-module V_j/X is a noetherian uniserial module, whose lattice of submodules is order anti-isomorphic to the ordinal number $\omega+1$. The cyclic uniserial module V_j/X is annihilated by the two-sided ideal $I := \begin{pmatrix} \Lambda_p & 0 \\ \mathbf{M}_n(\mathbb{Q}) & 0 \end{pmatrix}$ of R, so that it is a module over $R/I \cong \Lambda_q$. In particular, $\mathrm{End}_R(V_j/X) = \mathrm{End}_{\Lambda_q}(V_j/X)$. As a module over Λ_q, V_j/X turns out to be a cyclic projective uniserial module, hence $V_j/X \cong e\Lambda_q$ for some idempotent $e \in \Lambda_q$. Thus $\mathrm{End}_R(V_j/X) = \mathrm{End}_{\Lambda_q}(V_j/X) \cong \mathrm{End}_{\Lambda_q}(e\Lambda_q) \cong e\Lambda_q e \cong \mathbb{Z}_q$ is a local ring.

Finally, every endomorphism of the R-module $U_{i,j} := V_j/X_i$ maps X/X_i into X/X_i, i.e., X/X_i is a fully invariant submodule of V_j/X_i, because X/X_i is the Loewy submodule of V_j/X_i. Thus the endomorphism ring of the short exact sequence M in the category $\mathcal{E}_2$ is canonically isomorphic to the endomorphism ring of the right R-module $U_{i,j}$, which is a ring of type 2.

Example 1.65. Let $R = \mathbb{Z}$ be the ring of integers and let $p \in \mathbb{Z}$ be a prime. Consider the following object of $\mathcal{U}_2$:

$$M: \quad 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\iota} \mathbb{Z}(p^\infty) \xrightarrow{\beta} \mathbb{Z}(p^\infty) \longrightarrow 0,$$

where $\mathbb{Z}(p^\infty)$ denotes the Prüfer group and β is any surjective endomorphism of $\mathbb{Z}(p^\infty)$ having its kernel equal to $\mathrm{soc}(\mathbb{Z}(p^\infty)) \cong \mathbb{Z}/p\mathbb{Z}$. It is easy to check that $I_{M,2,e} \subseteq I_{M,2,m} \subseteq I_{M,1,m} = I_{M,1,e}$, and therefore E_M has exactly one maximal right ideal.

Objects with some zero factors

In this subsection we fix a simple right R-module S and we denote by S^n the following object of $\mathcal{E}_n$:

$$0 < S < S \oplus S < S \oplus S \oplus S < \cdots < \underbrace{S \oplus \cdots \oplus S}_{n\text{-times}} = S^n.$$

For any object $M \in \mathcal{E}_n$ such that all factor modules $U^{(i)}$ are uniserial (possibly zero), consider the subset $A \subseteq \{1, \dots, n\}$ such that $U^{(i)} = 0$ if and only if $i \in A$. We want to define the following object $S(M)$ of $\mathcal{E}_n$: for every $i = 1, \dots, n$, $S(M)^{(i)}$ is the direct sum of finitely many copies of S in such a way that $S(M)^{(i)}/S(M)^{(i-1)} = 0$ if $i \notin A$ and $S(M)^{(i)}/S(M)^{(i-1)} \cong S$ when $i \in A$. So, $M \oplus S(M)$ is an object of $\mathcal{U}_n$. Notice that $S^n = S(0)$ and $S(M) = 0$ if M is an object of $\mathcal{U}_n$. Using the canonical embeddings and the canonical projections, it is immediate to check that for every $a = m, e$ we have:

- $[M]_{i,a} = [0]_{i,a}$ for all $i \in A$;
- the i-th factor of $M \oplus S(M)$ is isomorphic to $U^{(i)}$ if $i \notin A$ and it is isomorphic to S if $i \in A$;
- $[M \oplus S(M)]_{i,a} = [M]_{i,a}$ for all $i \notin A$;
- $[M \oplus S(M)]_{i,a} = [S(M)]_{i,a} = [S^n]_{i,a}$ for all $i \in A$.

Remark 1.66. If M is an object of $\mathcal{E}_n$ with all the factor modules $U^{(i)}$ uniserial, then M can be viewed as an object of $\mathcal{U}_{n-d}$ for a suitable integer d (namely, the number of indexes such that $U^{(i)} = 0$). It follows that M is of finite type, because $E_M = \mathrm{End}_{\mathcal{E}_n}(M) \cong \mathrm{End}_{\mathcal{E}_{n-d}}(M)$ as rings. Moreover, the direct sum of finitely many object of $\mathcal{E}_n$ with all factor modules uniserial has a semilocal endomorphism ring (Remark 1.79).

Theorem 1.67. Let $M_1, \dots, M_r, N_1, \dots, N_s$ be non-zero objects of $\mathcal{E}_n$ with uniserial factor modules $U_k^{(i)} = M_k^{(i)}/M_k^{(i-1)}$ for $k = 1, \dots, r$ and $V_l^{(i)} = N_l^{(i)}/N_l^{(i-1)}$ for $l = 1, \dots, s$. Define

$$X_i := \{k \mid k = 1, \dots r,\ U_k^{(i)} \neq 0\} \quad \text{and} \quad Y_i := \{l \mid l = 1, \dots s,\ V_l^{(i)} \neq 0\}.$$

Then $\bigoplus_{k=1}^r M_k \cong \bigoplus_{l=1}^s N_l$ in $\mathcal{E}_n$ if and only if there exist $2n$ bijections $\varphi_{i,a} : X_i \to Y_i$ such that $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$ for every $i = 1, \dots, n$, $a = m, e$.

Proof. Assume that $\bigoplus_{k=1}^r M_k \cong \bigoplus_{l=1}^s N_l$ in $\mathcal{E}_n$. For any $i = 1, \dots, n$, we have the following isomorphisms in Mod-R:

$$\bigoplus_{k\in X_i} U_k^{(i)} \cong \bigoplus_{k=1}^{r} U_k^{(i)} \cong \frac{(\bigoplus_{k=1}^{r} M_k)^{(i)}}{(\bigoplus_{k=1}^{r} M_k)^{(i-1)}} \cong$$

$$\cong \frac{(\bigoplus_{l=1}^{s} N_l)^{(i)}}{(\bigoplus_{l=1}^{s} N_l)^{(i-1)}} \cong \bigoplus_{l=1}^{s} V_l^{(i)} \cong \bigoplus_{l\in Y_i} V_l^{(i)}.$$

So, looking at the Goldie dimension, we get that $|X_i| = |Y_i|$ for every $i = 1, \dots, n$. Assume that $r \geq s$. Then, we have the following isomorphism in $\mathcal{E}_n$

$$\bigoplus_{k=1}^{r} S(M_k) \cong \left(\bigoplus_{l=1}^{s} S(N_l)\right) \oplus \left(\underbrace{S^n \oplus S^n \oplus \cdots \oplus S^n}_{(r-s)\text{-times}}\right) \tag{1.4.2}$$

(notice that the i^{th} submodule of both sides is isomorphic to the direct sum of $ir - (|X_1| + \cdots + |X_i|)$ copies of S) and so, by hypothesis, we have an isomorphism in $\mathcal{E}_n$

$$\bigoplus_{k=1}^{r} (M_k \oplus S(M_k)) \cong \left(\bigoplus_{l=1}^{s} (N_l \oplus S(N_l))\right) \oplus \left(\underbrace{S^n \oplus S^n \oplus \cdots \oplus S^n}_{(r-s)\text{-times}}\right)$$

where all the direct summands are in $\mathcal{U}_n$. Write $M'_k = M_k \oplus S(M_k)$ for $k = 1, \dots, r$, $N'_l = N_l \oplus S(N_l)$ for $l = 1, \dots, s$ and $N'_l = S^n$ for $l = s+1, \dots, r$. From Theorem 1.60, there exist $2n$ permutations $\varphi_{i,a}$ of $\{1, 2, \dots, r\}$, where $i = 1, \dots, n$ and $a = m, e$, such that $[M'_k]_{i,a} = [N'_{\varphi_{i,a}(k)}]_{i,a}$. Let $k \in X_i$ and assume that $\varphi_{i,a}(k) \notin Y_i$. Since $|X_i| = |Y_i|$, there exists $j \notin X_i$ such that $\varphi_{i,a}(j) \in Y_i$. By construction, we have

$$[M_k]_{i,a} = [M'_k]_{i,a} = [N'_{\varphi_{i,a}(k)}]_{i,a} = [S(N_{\varphi_{i,a}(k)})]_{i,a} = [S(M_j)]_{i,a} = [M'_j]_{i,a} =$$

$$[N'_{\varphi_{i,a}(j)}]_{i,a} = [N_{\varphi_{i,a}(j)}]_{i,a}$$

and therefore, we can rearrange the permutation $\varphi_{i,a}$ in order to map k into $\varphi_{i,a}(j)$ and j into $\varphi_{i,a}(k)$, without changing its property of preserving the classes. In particular, we can always assume that $\varphi_{i,a}$ maps X_i into Y_i, so that $\varphi_{i,a}\,|_{X_i}\colon X_i \to Y_i$ are the sought bijections. Similarly, for $r < s$.

Conversely, assume that there exist $2n$ bijections $\varphi_{i,a} : X_i \to Y_i$ such that $[M_k]_{i,a} = [N_{\varphi_{i,a})k=}]_{i,a}$ for $i = 1, \dots, n$, $a = m, e$. Assume that $r \geq s$. Notice that the isomorphism in (1.4.2) depends only on the fact that $|X_i| = |Y_i|$ for every $i = 1, \dots, n$, so it holds also in this case. Consider the two direct sums

$$\bigoplus_{k=1}^{r} (M_k \oplus S(M_k))$$

and

$$\left(\bigoplus_{l=1}^{s} (N_l \oplus S(N_l))\right) \oplus \left(\underbrace{S^n \oplus S^n \oplus \cdots \oplus S^n}_{(r-s)\text{-times}}\right)$$

and define, as before, $M'_k = M_k \oplus S(M_k)$ for $k = 1, \dots, r$, $N'_l = N_l \oplus S(N_l)$ for $l = 1, \dots, s$ and $N'_l = S^n$ for $l = s+1, \dots, r$. Fix $i = 1, \dots, n$ and $a = m, e$. Then, for any $k \notin X_i$ and any $l \notin Y_i$, we have $[M'_k]_{i,a} = [S(M_k)]_{i,a} = [S(N_l)]_{i,a} = [N'_l]_{i,a}$, and therefore we can extend the bijection $\varphi_{i,a}$ to a bijection $\{1, 2, \dots, r\} \to \{1, 2, \dots, s, s+1, \dots, r\}$. Using Theorem 1.60, we get

$$\bigoplus_{k=1}^{r} (M_k \oplus S(M_k)) \cong \left(\bigoplus_{l=1}^{s} (N_l \oplus S(N_l))\right) \oplus \left(\underbrace{S^n \oplus S^n \oplus \cdots \oplus S^n}_{(r-s)\text{-times}}\right)$$

in $\mathcal{E}_n$. Now, applying [35, Theorem 4.5], we can cancel the terms $\bigoplus_{k=1}^{r} S(M_k)$ and $(\bigoplus_{l=1}^{s} S(N_l)) \oplus (S^n \oplus S^n \oplus \cdots \oplus S^n)$ from direct sums, because they are isomorphic objects with semilocal endomorphism ring (see Remarks 1.66 and 1.14). □

Remark 1.68. Notice that, in Theorem 1.67, dropping the assumption that all the objects are in $\mathcal{U}_n$, it could occur that $r \neq s$. A very simple example is the following. Let M be any object of $\mathcal{E}_n$ not in $\mathcal{U}_n$ with all factor modules uniserial modules (some of whom are necessarily zero). Set $M_1 = M \oplus S(M)$, $N_1 = M$ and $N_2 = S(M)$. Then, M_1, N_1 and N_2 are non-zero objects of $\mathcal{E}_n$ and $M_1 \cong N_1 \oplus N_2$, but $r = 1$ and $s = 2$. In particular, this example shows that the objects of $\mathcal{U}_n$ are indecomposable objects in the category $\mathcal{U}_n$, but they are not necessarily indecomposable in the category $\mathcal{E}_n$.

1.5 Factor categories and finite direct sums

In this section, we want to prove Theorems 1.53 and 1.60. First of all, notice that it is possible to outline a common pattern for both the situations, whose main properties are listed in the next remark.

Remark 1.69. We have:

1. A fixed positive integer n and an additive category $\mathcal{C}$ (for example, if $\mathcal{C} = \text{Morph}(\text{Mod-}R)$ we have $n = 2$, while for $\mathcal{C} = \mathcal{E}_n$, the integer n coincides with the "length" of the chains).

2. A full subcategory $\mathcal{U}$ of $\mathcal{C}$ such that for every $M \in \text{Ob}(\mathcal{U})$, the endomorphism ring $E_M := \text{End}_{\mathcal{C}}(M)$ has type at most $2n$, and the maximal ideals are among $2n$ completely prime two-sided ideals $I_{M,i,a}$ for $i = 1, \ldots, n$, $a = m, e$.

3. $2n$ equivalence relations $\sim_{i,a}$ ($i = 1, \ldots, n$, $a = m, e$) on $\text{Ob}(\mathcal{C})$ such that for every $M, N \in \text{Ob}(\mathcal{U})$, $[M]_{i,a} = [N]_{i,a}$ if and only if there exist two morphisms $f : M \to N$ and $g : N \to M$ in $\mathcal{C}$ such that $g \circ f \notin I_{M,i,a}$, or equivalently, such that $f \circ g \notin I_{N,i,a}$.

4. n additive functors $Q_i : \mathcal{C} \to \text{Mod-}R$, for $i = 1, \ldots, n$. For every object $M \in \text{Ob}(\mathcal{U})$ and $i = 1, \ldots, n$, $Q_i(M)$ is a non-zero uniserial right R-module.

5. For every $M, N \in \text{Ob}(\mathcal{U})$ and for every $i = 1, \ldots, n$, we have that $[M]_{i,a} = [N]_{i,a}$ if and only if there exist two morphisms $f : M \to N$ and $g : N \to M$ in $\mathcal{C}$ such that both $Q_i(f)$ and $Q_i(g)$ are right R-module monomorphisms if $a = m$, or right R-module epimorphisms if $a = e$.

6. A weak form of the Krull-Schmidt Theorem, that can be stated as follows: let $M_1, M_2 \ldots, M_r, N_1, N_2, \ldots, N_s$ be $r + s$ objects of $\mathcal{U}$. Then $\bigoplus_{k=1}^r M_k \cong \bigoplus_{\ell=1}^s N_\ell$ in the category $\mathcal{C}$ if and only if $r = s$ and there exist $2n$ permutations $\varphi_{i,a}$ of $\{1, 2, \ldots, r\}$, where $i = 1, \ldots, n$ and $a = m, e$, such that $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$ for every $k = 1, \ldots, r$.

Notice that for $n = 1$ and $\mathcal{C} = \text{Mod-}R$, we recover the situation described for uniserial modules. The functor $Q_1 : \text{Mod-}R \to \text{Mod-}R$ is simply the identity functor.

Let us start to outline the connection between the equivalence relations $\sim_{i,a}$ and the completely prime ideals $I_{M,i,a}$ of the endomorphism rings of the objects of $\mathcal{U}$.

Lemma 1.70. Let M and N be two objects of $\mathcal{U}$. Fix $i = 1, \dots, n$ and $a = m, e$ and assume that $[M]_{i,a} = [N]_{i,a}$. Then the following properties hold:

1. For every $j = 1, \dots, n$ and $b = m, e$, one has that $I_{M,j,b} \subseteq I_{M,i,a}$ if and only if $I_{N,j,b} \subseteq I_{N,i,a}$.
2. For every $j = 1, \dots, n$ and $b = m, e$, $I_{M,j,b} \subseteq I_{M,i,a}$ implies $[M]_{j,b} = [N]_{j,b}$.

Proof. (1) It suffices to show that if $j = 1, \dots, n$, $b = m, e$ and $I_{M,j,b} \subseteq I_{M,i,a}$, then $I_{N,j,b} \subseteq I_{N,i,a}$. Fix $j = 1, \dots, n$ and $b = m, e$ and suppose that $I_{N,j,b} \nsubseteq I_{N,i,a}$. Let $\varphi\colon N \to N$ be a morphism in $I_{N,j,b}$ not in $I_{N,i,a}$. Since $[M]_{i,a} = [N]_{i,a}$, there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{U}$ such that $gf \notin I_{M,i,a}$. In particular, by Lemma 1.3, $g\varphi f \in I_{M,j,b}$ and $g\varphi f \notin I_{M,i,a}$, which implies that $I_{M,j,b} \nsubseteq I_{M,i,a}$.

(2) Suppose that $I_{M,j,b} \subseteq I_{M,i,a}$ for some $j = 1, \dots, n$ and $b = m, e$. Since $[M]_{i,a} = [N]_{i,a}$, there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{U}$ such that $gf \notin I_{M,i,a}$. In particular $gf \notin I_{M,j,b}$, and therefore $[M]_{j,b} = [N]_{j,b}$ by Remark 1.69 (3).

□

Corollary 1.71. Let M and N be two objects of $\mathcal{U}$ and suppose that $[M]_{i,a} = [N]_{i,a}$ for every $i = 1, \dots, n$ and every $a = m, e$. Consider the sets $\mathcal{S}_M := \{I_{M,i,a} \mid i = 1, \dots, n \text{ and } a = m, e\}$ and $\mathcal{S}_N := \{I_{N,i,a} \mid i = 1, \dots, n \text{ and } a = m, e\}$ partially ordered by set inclusion. Then the canonical mapping

$$\Phi\colon \mathcal{S}_M \longrightarrow \mathcal{S}_N, \qquad \text{defined by} \qquad I_{M,i,a} \mapsto I_{N,i,a}$$

is a partially ordered set isomorphism.

Proof. It immediately follows from Lemma 1.70 (1). □

It is worth noting that at this point we are able to prove that the equivalence relations $\sim_{i,a}$ classify the isomorphism classes of objects of $\mathcal{U}$.

Proposition 1.72. Let M and N be two objects of $\mathcal{U}$. Then $M \cong N$ in $\mathcal{U}$ if and only if $[M]_{i,a} = [N]_{i,a}$ for every $i = 1, \dots, n$ and $a = m, e$.

Proof. Assume that $[M]_{i,a} = [N]_{i,a}$ for every $i = 1, \dots, n$ and $a = m, e$. By Remark 1.69 (2), the maximal right ideals of E_M are the ideals $I_{M,i,a}$, where (i,a) ranges in a non-empty subset S of $\{(1,m), \dots, (n,m), (1,e), \dots, (n,e)\}$. Consequently, the maximal right ideals of E_N are the ideals $I_{N,i,a}$ for the same pairs (i,a) in S (Corollary 1.71). By hypothesis, there are morphisms $f_{(i,a)} \colon M \to N$ and $f'_{(i,a)} \colon N \to M$ such that $f'_{(i,a)} f_{(i,a)} \notin I_{M,i,a}$ for every $(i,a) \in S$. Moreover, since $E_M/J(E_M)$ and $\prod_{(i,a)\in S} E_M/I_{M,i,a}$ are canonically isomorphic, for every $(i,a) \in S$ there exists $\varepsilon_{(i,a)} \in E_M$ such that $\varepsilon_{(i,a)} \equiv \delta_{(i,a),(j,b)} \pmod{I_{M,j,b}}$ for every $(j,b) \in S$. Similarly, for every $(i,a) \in S$ there exists $\varepsilon'_{(i,a)} \in E_N$ such that $\varepsilon'_{(i,a)} \equiv \delta_{(i,a),(j,b)} \pmod{I_{N,j,b}}$ for every $(j,b) \in S$. The two morphisms $\varphi\colon = \sum_{(i,a)\in S} \varepsilon'_{(i,a)} f_{(i,a)} \varepsilon_{(i,a)}$ and $\psi\colon = \sum_{(i,a)\in S} \varepsilon_{(i,a)} f'_{(i,a)} \varepsilon'_{(i,a)}$ are such that $\psi\varphi \in E_M$ is invertible modulo $J(E_M)$ and $\varphi\psi \in E_N$ is invertible modulo $J(E_N)$. Hence $\psi\varphi$ and $\varphi\psi$ are invertible in E_M and E_N respectively, and therefore $\varphi\colon M \to N$ is an isomorphism.

The other implication is clear. □

We now introduce another class of ideals, defined starting from two-sided ideals of the endomorphism ring of an object.

Definition 1.73. [43, 44] Let A be an object of a preadditive category $\mathcal{A}$ and I be a two-sided ideal of the ring $\mathrm{End}_{\mathcal{A}}(A)$. Let $\mathcal{I}$ be the ideal of the category $\mathcal{A}$ defined as follows. A morphism $f\colon X \to Y$ in $\mathcal{A}$ belongs to $\mathcal{I}(X,Y)$ if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{A}$. The ideal $\mathcal{I}$ is called the *ideal of* $\mathcal{C}$ *associated to* I.

It is easy to see that $\mathcal{I}$ turns out to be the greatest of the ideals $\mathcal{I}'$ of $\mathcal{A}$ with $\mathcal{I}'(A,A) \subseteq I$. Moreover $\mathcal{I}(A,A) = I$ and therefore, if A is an object of $\mathcal{A}$, the ideals associated to two distinct ideals of $\mathrm{End}_{\mathcal{A}}(A)$ are distinct ideals of the category $\mathcal{A}$.

We now go back to our category. For any object M of $\mathcal{C}$, if $I_{M,i,a}$ is a maximal right ideal of E_M, we denote by $\mathcal{I}_{M,i,a}$ the ideal of $\mathcal{C}$ associated to $I_{M,i,a}$. Set

$$V(M) := \{ \mathcal{I}_{M,i,a} \mid I_{M,i,a} \text{ is a maximal ideal of } E_M \}.$$

By Remark 1.69 (2), $V(M)$ has at most $2n$ elements.

Lemma 1.74. Fix $i = 1, \dots, n$ and $a = m, e$. Let M be an object of $\mathcal{U}$, $I_{M,i,a}$ be a maximal ideal of E_M, $\mathcal{I}_{M,i,a}$ its associated ideal in $\mathcal{C}$ and $F \colon \mathcal{C} \to \mathcal{C}/\mathcal{I}_{M,i,a}$ be the canonical functor.

1. For any non-zero object N of $\mathcal{C}$ such that E_N is a semilocal ring, either $\mathcal{I}_{M,i,a}(N,N) = E_N$ or $\mathcal{I}_{M,i,a}(N,N)$ is a maximal ideal of E_N. In this second case, the ideal of $\mathcal{C}$ associated to $\mathcal{I}_{M,i,a}(N,N)$ is equal to $\mathcal{I}_{M,i,a}$.
2. For any object N of $\mathcal{C}$ such that E_N is of finite type, either $F(N) = 0$ or $F(N) \cong F(M)$. Moreover, if $[M]_{i,a} = [N]_{i,a}$, then $F(N) \cong F(M)$.
3. If $F(M)^t \cong F(M)^s$ in the factor category $\mathcal{C}/\mathcal{I}_{M,i,a}$ for integers $t, s \geq 0$, then $t = s$.

Proof. (1) Immediately follows from [41, Lemma 2.1(ii)].

(2) We know that either $F(N) = 0$ or the endomorphism ring of $F(N)$ is a division ring. Let us consider the latter case. As $1_N \notin \mathcal{I}_{M,i,a}(N,N)$, there are $\alpha \colon M \to N$ and $\beta \colon N \to M$ such that $\beta\alpha \notin I_{M,i,a}$. Since $I_{M,i,a}$ is a completely prime ideal of E_M, also $\beta(\alpha\beta)\alpha \notin I_{M,i,a}$, hence $\alpha\beta \notin \mathcal{I}_{M,i,a}(N,N)$. By the previous part, both $\mathrm{End}_{\mathcal{C}}(M)/\mathcal{I}_{M,i,a}(M,M)$ and $\mathrm{End}_{\mathcal{C}}(N)/\mathcal{I}_{M,i,a}(N,N)$ are division rings and therefore $\alpha\beta$ and $\beta\alpha$ become automorphisms in $\mathcal{C}/\mathcal{I}_{M,i,a}$. It means that $F(M) \cong F(N)$ in $\mathcal{C}/\mathcal{I}_{M,i,a}$. For the last assertion, note that, also in the case $[M]_{i,a} = [N]_{i,a}$, there are morphisms $\alpha \colon M \to N$ and $\beta \colon N \to M$ such that $\beta\alpha \notin I_{M,i,a}$. So, as before, we can conclude that $F(M) \cong F(N)$.

(3) First notice that $F(M) \neq 0$ in $\mathcal{C}/\mathcal{I}_{M,i,a}$ because $\mathcal{I}_{M,i,a}(M,M) = I_{M,i,a}$. By the previous parts, $D := \mathrm{End}_{\mathcal{C}/\mathcal{I}_{M,i,a}}(M) = \mathrm{End}_{\mathcal{C}}(M)/I_{M,i,a}$ is a division ring. To conclude, it suffices to apply the functor

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}_{M,i,a}}(F(M), -) : \mathcal{C}/\mathcal{I}_{M,i,a} \to \text{Mod-}D.$$

□

Corollary 1.75. Fix $i = 1, \dots, n$ and $a = m, e$. Let M be an object of $\mathcal{U}$, $I_{M,i,a}$ be a maximal ideal of E_M, $\mathcal{I}_{M,i,a}$ its associated ideal in $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}/\mathcal{I}_{M,i,a}$ be the canonical functor.

1. For any object N of $\mathcal{U}$, the following conditions are equivalent:

(a) $F(N) = 0$ in $\mathcal{C}/\mathcal{I}_{M,i,a}$.

(b) $\mathcal{I}_{M,i,a}(N, N) = E_N$.

(c) $I_{N,i,a} \subsetneq \mathcal{I}_{M,i,a}(N, N)$.

(d) $[M]_{i,a} \neq [N]_{i,a}$.

2. For any object N of $\mathcal{U}$, the following conditions are equivalent:

(a) $F(M) \cong F(N)$ in $\mathcal{C}/\mathcal{I}_{M,i,a}$.

(b) $\mathcal{I}_{M,i,a}(N, N)$ is a proper ideal of E_N.

(c) $I_{N,i,a} = \mathcal{I}_{M,i,a}(N, N)$.

(d) $[M]_{i,a} = [N]_{i,a}$.

Moreover, in this second case, $I_{N,i,a}$ is a maximal right ideal of E_N and $\mathcal{I}_{M,i,a} = \mathcal{I}_{N,i,a}$.

Proof. (1). (a) $\Leftrightarrow$ (b) $F(N) = 0$ if and only if the endomorphism ring $E_N/\mathcal{I}_{M,i,a}(N, N)$ of $F(N)$ in the factor category $\mathcal{C}/\mathcal{I}_{M,i,a}$ is the zero ring.

(b) $\Rightarrow$ (c) It suffices to notice that $I_{N,i,a}$ is always a proper ideal of E_N.

(c) $\Rightarrow$ (d) Let φ be an element in $\mathcal{I}_{M,i,a}(N, N)$ not in $I_{N,i,a}$. For any two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{C}$, one has $g\varphi f \in I_{M,i,a}$, by definition of associated ideal. Since $\varphi \notin I_{N,i,a}$, it follows from Lemma 1.3 that $gf \in I_{M,i,a}$. This means that $[M]_{i,a} \neq [N]_{i,a}$.

(d) $\Rightarrow$ (b) By Remark 1.69 (3), if $[M]_{i,a} \neq [N]_{i,a}$, then, for any pair of morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{U}$, we have $g 1_N f = gf \in I_{M,i,a}$. Therefore $1_N \in \mathcal{I}_{M,i,a}(N, N)$, so that $\mathcal{I}_{M,i,a}(N, N) = E_N$.

(2). First notice that, by Lemma 1.74 (2), either $F(M) = 0$ or $F(M) \cong F(N)$ in $\mathcal{C}/\mathcal{I}_{M,i,a}$. Moreover, $I_{N,i,a}$ is always contained in $\mathcal{I}_{M,i,a}(N,N)$. As a matter of fact, suppose $\varphi \in I_{N,i,a}$. By Lemma 1.3, for any $f\colon M \to N$ and any $g\colon N \to M$, we have that $g\varphi f \in I_{M,i,a}$, so $\varphi \in I_{M,i,a}(N,N)$. Now all the implications follow from part (1).

For the last assertion, apply Lemma 1.74 (1). □

Corollary 1.76. Let M and N be two objects in the category $\mathcal{U}$. Fix $i = 1, \dots, n$ and $a = m, e$. Suppose that $[M]_{i,a} = [N]_{i,a}$. Then the following properties hold:

1. $I_{M,i,a}$ is a maximal right ideal of E_M if and only if $I_{N,i,a}$ is a maximal right ideal of E_N.
2. Suppose that $I_{M,i,a}$ is a maximal right ideal of E_M. Then, for every $j = 1, \dots, n$ and $b = m, e$, $I_{M,j,b} = I_{M,i,a}$ if and only if $I_{N,j,b} = I_{N,i,a}$.

Proof. (1) It suffices to show that $I_{M,i,a}$ maximal implies $I_{N,i,a}$ maximal. From Corollary 1.75 (2), $I_{N,i,a} = \mathcal{I}_{M,i,a}(N,N)$ is a proper ideal of E_N, and it is a maximal right ideal of E_N by Lemma 1.74 (1).

(2) By (1), the hypotheses on M and N are symmetrical. Therefore it suffices to show that $I_{M,j,b} = I_{M,i,a}$ implies $I_{N,j,b} = I_{N,i,a}$. The inclusion $I_{N,j,b} \subseteq I_{N,i,a}$ follows from Lemma 1.70 (1). Moreover, by Lemma 1.70 (2), we can interchange the role of (i,a) and (j,b) and deduce the opposite inclusion applying Lemma 1.70 (1) again. □

Corollary 1.77. Let M and N be two objects of $\mathcal{U}$. Then $V(M) \cap V(N) \neq \emptyset$ if and only if there exists a pair $(j,a) \in \{1, 2 \dots, n\} \times \{m, e\}$ such that $I_{M,j,a}$ is a maximal right ideal of E_M, $I_{N,j,a}$ is a maximal ideal of E_N and $\mathcal{I}_{M,j,a} = \mathcal{I}_{N,j,a}$. Moreover, for such a pair, $[M]_{j,a} = [N]_{j,a}$.

Proof. Let $\mathcal{P} \in V(M) \cap V(N)$. Then there exist two pairs (j,a) and (k,b) in $\{1, 2, \dots, n\} \times \{m, e\}$ such that $I_{M,j,a}$ is a maximal right ideal of E_M, $I_{N,k,b}$ is a maximal ideal of E_N and $\mathcal{P} = \mathcal{I}_{M,j,a} = \mathcal{I}_{N,k,b}$ is the ideal of $\mathcal{C}$ associated both to $I_{M,j,a}$ and $I_{N,k,b}$. We have $I_{N,j,a} \subseteq \mathcal{I}_{M,j,a}(N,N) = \mathcal{I}_{N,k,b}(N,N) = I_{N,k,b}$. If the inclusion is proper, then $I_{N,k,b} = E_N$ by Corollary 1.75 (1), a contradiction.

It follows that $I_{N,j,a} = I_{N,k,b}$, so we can replace the pair (k, b) with (j, a). Clearly, the converse holds as well. Finally, the last assertion follows from Corollary 1.75 (2). $\square$

Remark 1.78. According to [41], we say that an ideal $\mathcal{M}$ of a preadditive category $\mathcal{A}$ is *maximal* if the improper ideal of $\mathcal{A}$ is the unique ideal of the category $\mathcal{A}$ properly containing $\mathcal{M}$. Moreover, we say that a preadditive category $\mathcal{A}$ is *simple* [41] if it has exactly two ideals, necessarily the trivial ones. Hence, a simple category has non-zero objects. By [41, Lemma 2.4], $\bigcup_{M \in \mathrm{Ob}(\mathcal{U})} V(M)$ turns out to be a class of maximal ideals of $\mathcal{U}$. Moreover, applying [41, Theorem 3.2], we can immediately deduce that if M is an object of $\mathcal{U}$ and $\mathcal{I} \in V(M)$, then the factor category $\mathcal{U}/\mathcal{I}$ is a simple category and the endomorphism ring of all its non-zero objects are division rings.

Remark 1.79. Let M and N be two objects of $\mathcal{U}$. For every $i = 1, \ldots, n$, $Q_i(M \oplus N) \cong Q_i(M) \oplus Q_i(N)$ and hence, applying Proposition 1.12, we have $\mathrm{codim}(\mathrm{End}(Q_i(M) \oplus Q_i(N))) = \mathrm{codim}(\mathrm{End}(Q_i(M))) + \mathrm{codim}(\mathrm{End}(Q_i(N)))$. In particular, $\mathrm{codim}(E_{M \oplus N})$ is finite, and therefore $E_{M \oplus N}$ is semilocal by Proposition 1.6.

Lemma 1.80. Let $M_1, \ldots, M_r$ be objects of $\mathcal{U}$. Then every maximal two-sided ideal of $E_{\bigoplus_{k=1}^r M_k}$ is of the form $\mathcal{I}_{M_h,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$ for some $h = 1, \ldots, r$, $i = 1, \ldots, n$ and $a = m, e$ such that $I_{M_h,i,a}$ is a maximal right ideal of E_{M_h}. Conversely, if (h, i, a) is a triple such that $h = 1, \ldots, r$, $i = 1, \ldots, n$, $a = m, e$ and $I_{M_h,i,a}$ is a maximal right ideal of E_{M_h}, then $\mathcal{I}_{M_h,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$ is a maximal two-sided ideal of $E_{\bigoplus_{k=1}^r M_k}$.

Proof. First, let I be a maximal two-sided ideal of $E_{\oplus_{k=1}^r M_k}$ and let $\mathcal{I}$ be its associated ideal on $\mathcal{C}$. Using Lemma 1.74 (1), we get that for any $M = M_1, \ldots, M_r$, either $\mathcal{I}(M, M) = E_M$ or $\mathcal{I}(M, M)$ is a maximal two-sided ideal of E_M. If $\mathcal{I}(M_h, M_h) = E_{M_h}$ for all $h = 1, \ldots, r$, by definition of associated ideal, it follows that $\varepsilon_h \pi_h \in I$ for every $h = 1, \ldots, r$, where $\varepsilon_h : M_h \to \oplus_{k=1}^r M_k$ and $\pi_h : \oplus_{k=1}^r M_k \to M_h$ are the canonical embedding and the canonical projection respectively. In particular $1_{\oplus_{k=1}^r M_k} = \sum_{h=1}^r \varepsilon_h \pi_h \in I$, which is absurd. It follows that there exists a triple (h, i, a) such that $\mathcal{I}(M_h, M_h) = I_{M_h,i,a}$ is a maximal right ideal of E_{M_h}. By Lemma 1.74 (1), $\mathcal{I}_{M_h,i,a} = \mathcal{I}$, so that, in particular, $I = \mathcal{I}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k) = \mathcal{I}_{M_h,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$.

Conversely, let (h, i, a) be a triple such that $h = 1, \dots, r$, $i = 1, \dots, n$, $a = m, e$ and $I_{M_h,i,a}$ is a maximal right ideal of E_{M_h}. Denote by $\mathcal{Q}$ the quotient category $\mathcal{C}/\mathcal{I}_{M_h,i,a}$ and consider the canonical functor $F : \mathcal{C} \to \mathcal{Q}$. Then $F(\bigoplus_{k=1}^r M_k) \cong F(M_h)^m$, where $m := m_{h,i,a} = |\{k \mid k = 1, \dots, r, \ [M_k]_{i,a} = [M_h]_{i,a}\}|$ (Corollary 1.75). It follows that $\mathrm{End}_{\mathcal{Q}}(\bigoplus_{k=1}^r M_k) \cong M_m(D)$, where $D = \mathrm{End}_{\mathcal{Q}}(M_h)$ is a division ring. In particular, $\mathrm{End}_{\mathcal{Q}}(\bigoplus_{k=1}^r M_k)$ is a simple artinian ring and so the kernel of $E_{\bigoplus_{k=1}^r M_k} \to \mathrm{End}_{\mathcal{Q}}(\bigoplus_{k=1}^r M_k)$, which is $\mathcal{I}_{M_h,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$, is a maximal two-sided ideal of $E_{\bigoplus_{k=1}^r M_k}$.

□

Corollary 1.81. Let $M_1, \dots, M_r$ be objects of $\mathcal{U}$. Then there is a one-to-one correspondence between $\bigcup_{k=1}^r V(M_k)$ and the maximal two-sided ideals of $\bigoplus_{k=1}^r M_k$ given by

$$\Psi : \mathcal{I}_{M_h,i,a} \mapsto \mathcal{I}_{M_h,i,a}(\oplus_{k=1}^r M_k, \oplus_{k=1}^r M_k).$$

Proof. By Lemma 1.80, Ψ is well defined. We can construct the inverse map of Ψ as follows. Let I be a maximal two-sided ideal of $E_{\oplus_{k=1}^r M_k}$ and let $\mathcal{I}$ be its associated ideal on $\mathcal{C}$. As in the proof of Lemma 1.80, there exists a triple (h, i, a) such that $\mathcal{I}(M_h, M_h) = I_{M_h,i,a}$ is a maximal right ideal of E_{M_h} and $\mathcal{I}_{M_h,i,a} = \mathcal{I}$. So, $\mathcal{I} \in \bigcup_{k=1}^r V(M_k)$ and $I \mapsto \mathcal{I}$ is the inverse of Ψ (apply Lemma 1.74 (1)). □

We now make use of some techniques, notations and ideas taken from [31]. If X and Y are finite disjoint sets, we will denote by $D(X, Y; E)$ the bipartite digraph (= directed graph) having X and Y as disjoint sets of non-adjacent vertices and E as set of edges. Equivalently, $V = X \cup Y$ is the vertex set of $D(X, Y; E)$, $E \subseteq X \times Y \cup Y \times X$ is the set of its edges, and $X \cap Y = \emptyset$. For every subset $T \subseteq V$, let $N^+(T) = \{\, w \in V \mid (v, w) \in E$ for some $v \in T \,\}$ be the *out-neighbourhood* of T ([30, introduction, p.184]). Define an equivalence relation $\sim_s$ on V by $v \sim_s w$ if there are both an oriented path from v to w and an oriented path from w to v ($v, w \in V$).

Proposition 1.82. ([30, Lemma 2.1], *Krull-Schmidt Theorem for bipartite digraphs*). Let X and Y be disjoint sets of cardinality n and m, respectively. Set $V := X \cup Y$. Let $D = D(X, Y; E)$ be a bipartite digraph having X and Y as disjoint sets of non-adjacent vertices. If $|T| \leq |N^+(T)|$ for every subset

T of V, then $n = m$ and, after a suitable relabelling of the indices of the elements $x_1, \dots, x_n$ of X and $y_1, \dots, y_n$ of Y, $x_i \sim_s y_i$ for every $i = 1, \dots, n$.

Remark 1.83. If T is any semilocal ring and I is a maximal two-sided ideal of T, then there exists an element $\delta_I \in T$ such that $\delta_I \equiv 1_T \pmod{I}$ and $\delta_I \equiv 0 \pmod{J}$ for every other maximal two-sided ideal J of T different from I. This follows from the fact that $T/J(T)$ is a direct product of finitely many simple rings.

Theorem 1.84. *Let $M_1, M_2 \dots, M_r, N_1, N_2, \dots, N_s$ be $r + s$ objects of $\mathcal{U}$. Then $\bigoplus_{k=1}^r M_k \cong \bigoplus_{\ell=1}^s N_\ell$ in the category $\mathcal{C}$ if and only if $r = s$ and there exist $2n$ permutations $\varphi_{i,a}$ of $\{1, 2, \dots, r\}$, where $i = 1, \dots, n$ and $a = m, e$, such that $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$ for every $k = 1, \dots, r$.*

Proof. First assume that $\bigoplus_{k=1}^r M_k \cong \bigoplus_{\ell=1}^s N_\ell$ in the category $\mathcal{C}$. For ease of notation, for every morphism f in $\mathcal{C}$, we will write $f^{(i)}$ instead of $Q_i(f)$. Since $\bigoplus_{k=1}^r Q_1(M_k) \cong \bigoplus_{\ell=1}^s Q_1(N_\ell)$ in the category Mod-R, looking at the Goldie dimension, we get that $r = s$. Let $\alpha : \bigoplus_{k=1}^r M_k \to \bigoplus_{\ell=1}^s N_\ell$ be an isomorphism in $\mathcal{C}$ with inverse $\beta : \bigoplus_{\ell=1}^s N_\ell \to \bigoplus_{k=1}^r M_k$. Denote by $\varepsilon_h : M_h \to \bigoplus_{k=1}^r M_k$, $\pi_h : \bigoplus_{k=1}^r M_k \to M_h$, $\varepsilon'_j : N_j \to \bigoplus_{\ell=1}^s N_\ell$ and $\pi'_j : \bigoplus_{\ell=1}^s N_\ell \to N_j$ the embeddings and the canonical projections and consider the composite morphisms $\chi_{h,j} := \pi'_j \alpha \varepsilon_h : M_h \to N_j$ and $\chi'_{j,h} := \pi_h \beta \varepsilon'_j : N_j \to M_h$. Fix $i = 1, \dots, n$. We want to prove the existence of the permutation $\varphi_{i,e}$ (dualizing the proof we get the existence of the permutation $\varphi_{i,m}$). Define a bipartite digraph $D = D(X, Y; E)$ (according on the choice of the pair (i, e)) having $X = \{M_1, \dots, M_r\}$ and $Y = \{N_1, \dots, N_r\}$ as disjoint sets of non-adjacent vertices, and the set E of edges defined as follows: one edge from M_h to N_j for each h and j such that $\chi_{h,j}^{(i)} : Q_i(M_h) \to Q_i(N_j)$ is a surjective right R-module morphism, and one edge from N_j to M_h for each h and j such that $(\chi'_{j,h})^{(i)} : Q_i(N_j) \to Q_i(M_h)$ is a surjective right R-module morphism.

We want to show that for every subset $T \subseteq X \cup Y$ of vertices, $|T| \leq |N^+(T)|$. Since the digraph is bipartite, we can suppose that $T \subseteq X$. If $p = |T|$ and $q = |N^+(T)|$, relabelling the indices we may assume that $T = \{M_1, \dots, M_p\}$ and $N^+(T) = \{N_1, \dots, N_q\}$. It means that $\chi_{h,j}^{(i)}$ are not surjective for every $h = 1, \dots, p$ and every $j = q + 1, \dots, r$. Since the modules $Q_i(N_j)$ are all uniserial, we have that $L_j := \bigcup_{h=1}^p \chi_{h,j}^{(i)}(Q_i(M_h)) \subsetneq Q_i(N_j)$ for every

$j = q+1, \dots, n$, and therefore all the quotient modules $Q_i(N_j)/L_j$ are nonzero for every $j = q+1, \dots, n$. Let $\pi : \bigoplus_{\ell=1}^r Q_i(N_\ell) \to \bigoplus_{j=q+1}^r Q_i(N_j)/L_j$ be the canonical projection. For every $h = 1, \dots, p$ and for every $j = q+1, \dots, r$, the composite morphism

$$Q_i(M_h) \xrightarrow{\varepsilon_h^{(i)}} \bigoplus_{k=1}^r Q_i(M_k) \xrightarrow{\alpha^{(i)}} \bigoplus_{\ell=1}^r Q_i(N_\ell) \xrightarrow{(\pi'_j)^{(i)}} Q_i(N_j) \to Q_i(N_j)/L_j$$

is zero because $(\pi'_j \alpha \varepsilon_h)^{(i)}(Q_i(M_h)) = \chi_{h,j}^{(i)}(Q_i(M_h)) \subseteq L_j$. It follows that for every $h = 1, \dots, p$, $\alpha_h^{(i)}(Q_i(M_h))$ is contained in the kernel of

$$\bigoplus_{\ell=1}^r Q_i(N_\ell) \xrightarrow{(\pi'_j)^{(i)}} Q_i(N_j) \to Q_i(N_j)/L_j$$

for every $j = q+1, \dots, r$. Since $\sum_{h=1}^p \varepsilon_h^{(i)}(Q_i(M_h)) = \bigoplus_{h=1}^p Q_i(M_h)$, it follows that there exists a morphism

$$\bigoplus_{k=1}^r Q_i(M_k) / \bigoplus_{h=1}^p Q_i(M_h) \cong \bigoplus_{m=p+1}^r Q_i(M_m) \longrightarrow Q_i(N_j)/L_j$$

making the following diagram

$$\begin{array}{ccc} \bigoplus_{k=1}^r Q_i(M_k) & \longrightarrow & \bigoplus_{m=p+1}^r Q_i(M_m) \\ {\scriptstyle \alpha^{(i)}} \downarrow & & \downarrow \\ \bigoplus_{\ell=1}^r Q_i(N_\ell) & \longrightarrow & Q_i(N_j)/L_j \end{array}$$

commute. Hence, there exist a morphism

$$\gamma : \bigoplus_{m=p+1}^r Q_i(M_m) \to \bigoplus_{j=q+1}^r Q_i(N_j)/L_j$$

and a commutative diagram

$$\begin{array}{ccc} \bigoplus_{k=1}^r Q_i(M_k) & \longrightarrow & \bigoplus_{m=p+1}^r Q_i(M_k) \\ {\scriptstyle \alpha^{(i)}} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \bigoplus_{\ell=1}^r Q_i(N_\ell) & \longrightarrow & \bigoplus_{j=q+1}^r Q_i(N_j)/L_j \end{array}$$

Since the horizontal arrows are the canonical projections, the morphism γ

must be surjective. Taking the dual Goldie dimension of the domain and the codomain of γ, we get $r - |T| \geq r - |N^+(T)|$, that is $|T| \leq |N^+(T)|$. To conclude, it suffices to apply Proposition 1.82 to the digraph D.

Conversely, assume that there exist $2n$ permutations $\varphi_{i,a}$ of $\{1, 2, \ldots, r\}$, where $i = 1, \ldots, n$ and $a = m, e$, such that $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$. Fix $M \in \{M_1, \ldots, M_r, N_1, \ldots, N_r\}$, $i = 1, \ldots, n$ and $a = m, e$ such that $I_{M,i,a}$ is a maximal right ideal of E_M. Consider the canonical functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{I}_{M,i,a}$. Then $F(\bigoplus_{k=1}^r M_k) \cong F(M)^m \cong F(\bigoplus_{\ell=1}^r N_\ell)$, where $m := m_{M,i,a} = |\{k \mid k = 1, \ldots, r,\ [M_k]_{i,a} = [M]_{i,a}\}| = |\{\ell \mid \ell = 1, \ldots, r,\ [N_\ell]_{i,a} = [M]_{i,a}\}|$ (Corollary 1.75). It means that there exists $f_{M,i,a} : \bigoplus_{k=1}^r M_k \to \bigoplus_{\ell=1}^r N_\ell$ in $\mathcal{C}$ which becomes an isomorphism in the factor category $\mathcal{C}/\mathcal{I}_{M,i,a}$. Notice that the triple (M, i, a) identifies a pair of objects $(M_k, N_{\varphi_{i,a}(k)})$ with the following properties: $[M_k]_{i,a} = [N_{\varphi_{i,a}(k)}]_{i,a}$, $I_{M_k,i,a}$ is a maximal right ideal of E_{M_k}, $I_{N_{\varphi_{i,a}(k)},i,a}$ is a maximal right ideal of $E_{N_{\varphi_{i,a}(k)}}$ and $\mathcal{I}_{M_k,i,a} = \mathcal{I}_{N_{\varphi_{i,a}(k)},i,a}$ (Corollary 1.75 and Corollary 1.76). So, according to Lemma 1.80, the triple (M, i, a) defines a maximal two-sided ideal $J_{M,i,a}$ of $E_{\bigoplus_{k=1}^r M_k}$, namely $J_{M,i,a} = \mathcal{I}_{M,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$. Since $E_{\bigoplus_{k=1}^r M_k}$ is a semilocal ring, there exists $\delta_{M,i,a}$ such that $\delta_{M,i,a} \equiv 1$ modulo $J_{M,i,a}$ and $\delta_{M,i,a} \equiv 0$ modulo all the other maximal two-sided ideals of $E_{\bigoplus_{k=1}^r M_k}$. Similarly, we can define an endomorphism $\delta'_{M,i,a}$ of $\bigoplus_{\ell=1}^r N_\ell$. Consider any subset Ω of $\{M_1, \ldots, M_r, N_1, \ldots, N_r\} \times \{1, \ldots, n\} \times \{m, e\}$ consisting of all the triples (M, i, a) such that $I_{M,i,a}$ is a maximal right ideal of E_M with the additional property that if $(M, i, a) \neq (M', j, b)$, then $\mathcal{I}_{M,i,a} \neq \mathcal{I}_{M',j,b}$. So, in particular $\{\mathcal{I}_{M,i,a} \mid (M, i, a) \in \Omega\} = (\bigcup_{k=1}^r V(M_k)) \cup (\bigcup_{\ell=1}^s V(N_\ell))$. Define the morphism $f := \sum_{(M,i,a)\in\Omega} \delta'_{M,i,a} f_{M,i,a} \delta_{M,i,a}$. By construction, f becomes an isomorphism in the factor category $\mathcal{C}/\mathcal{I}_{M,i,a}$ for every $(M, i, a) \in \Omega$. By [1, Proposition 5.2], f is an isomorphism in the category $\mathcal{C}/\mathcal{I}$, where $\mathcal{I}$ is the intersection of the finitely many ideals $\mathcal{I}_{M,i,a}$, with $(M, i, a) \in \Omega$. Since $\bigoplus_{k=1}^r M_k$ and $\bigoplus_{\ell=1}^r N_\ell$ are isomorphic in the category $\mathcal{C}/\mathcal{I}$, there exist two morphisms $\alpha : \bigoplus_{k=1}^r M_k \to \bigoplus_{\ell=1}^r N_\ell$ and $\beta : \bigoplus_{\ell=1}^r N_\ell \to \bigoplus_{k=1}^r M_k$ such that $\beta\alpha \equiv 1_{\bigoplus_{k=1}^r M_k}$ modulo $\mathcal{I}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$ and $\alpha\beta \equiv 1_{\bigoplus_{\ell=1}^r N_\ell}$ modulo $\mathcal{I}(\bigoplus_{\ell=1}^r N_\ell, \bigoplus_{\ell=1}^r N_\ell)$. But $E_{\bigoplus_{k=1}^r M_k}$ is semilocal (Remark 1.79), so its Jacobson radical $J(E_{\bigoplus_{k=1}^r M_k})$ is equal to the intersection of all its maximal two-sided ideals, which are the ideals $\mathcal{I}_{M,i,a}(\bigoplus_{k=1}^r M_k, \bigoplus_{k=1}^r M_k)$, where $(M,i,a) \in \Omega$. It means that

$$\mathcal{I}(\bigoplus_{k=1}^{r} M_k, \bigoplus_{k=1}^{r} M_k) = \bigcap_{(M,i,a)\in\Omega} \mathcal{I}_{M,i,a}(\bigoplus_{k=1}^{r} M_k, \bigoplus_{k=1}^{r} M_k) = J(E_{\bigoplus_{k=1}^{r} M_k}).$$

Similarly,

$$\mathcal{I}(\bigoplus_{\ell=1}^{r} N_\ell, \bigoplus_{\ell=1}^{r} N_\ell) = \bigcap_{(M,i,a)\in\Omega} \mathcal{I}_{M,i,a}(\bigoplus_{\ell=1}^{r} N_\ell, \bigoplus_{\ell=1}^{r} N_\ell) = J(E_{\bigoplus_{\ell=1}^{r} N_\ell}).$$

Thus $\alpha\beta$ and $\beta\alpha$ are invertible in the rings $E_{\bigoplus_{k=1}^{r} M_k}$ and $E_{\bigoplus_{\ell=1}^{r} N_\ell}$, respectively. In particular, α is both right invertible and left invertible in the category $\mathcal{C}$. It follows that α is an isomorphism in $\mathcal{C}$.

□

Corollary 1.85. Let $M_1, M_2 \dots, M_r, N_1, N_2, \dots, N_s$ be $r+s$ objects of $\mathcal{U}$. For every $i = 1, \dots, n$ and every $a = m, e$, define

$$X_{i,a} := \{k \mid k = 1, \dots, r,\ I_{M_k,i,a} \text{ is a maximal ideal of } E_{M_k}\}$$

and

$$Y_{i,a} := \{\ell \mid \ell = 1, \dots, s,\ I_{N_\ell,i,a} \text{ is a maximal ideal of } E_{N_\ell}\}.$$

Then $\bigoplus_{k=1}^{r} M_k \cong \bigoplus_{\ell=1}^{s} N_\ell$ in the category $\mathcal{C}$ if and only if $r = s$ and there exist $2n$ bijections $\psi_{i,a} : X_{i,a} \to Y_{i,a}$, where $i = 1, \dots, n$ and $a = m, e$, such that $[M_k]_{i,a} = [N_{\psi_{i,a}(k)}]_{i,a}$ for every $k \in X_{i,a}$.

Proof. If $\bigoplus_{k=1}^{r} M_k \cong \bigoplus_{\ell=1}^{s} N_\ell$ in the category $\mathcal{C}$, then we can restrict the permutations $\varphi_{i,a}$ of Theorem 1.60 to $X_{i,a}$, noting that $\varphi_{i,a}(X_{i,a}) = Y_{i,a}$ (Corollary 1.76).

For the other implication, observe that the proof of the "if part" of Theorem 1.60 works also in this case. □

1.6 Other constructions

We have introduced the categories Morph(Mod-R) and $\mathcal{E}_n$ (together with their full subcategories Morph($\mathcal{U}$) and $\mathcal{U}_n$) in order to obtain a framework in which it was possible to deal with object of finite type with the purpose to study the behaviour of their direct-sums. Up to now, we have almost always dealt with object related to uniserial modules. Nevertheless, it is possible to consider other different situations. Here we discuss the most natural ones.

As far as the category Morph(Mod-R) is concerned, we have seen in Subsection 1.3.2 the full subcategory Morph($\mathcal{L}$) of Morph(Mod-R) whose objects are all morphisms between modules of type 1 (i.e. modules with a local endomorphism ring). In a similar way, if $\mathcal{C}$ and $\mathcal{K}$ denote the full subcategories of Mod-R consisting of non-zero couniformly presented right R-modules and kernels of non-injective morphisms between indecomposable injective modules respectively, we can consider the following two full subcategories of Morph(Mod-R):

1. Morph($\mathcal{C}$), consisting of all morphisms between non-zero couniformly presented right R-modules;
2. Morph($\mathcal{K}$), whose objects are all morphisms between modules in $\mathcal{K}$.

Similarly, with respect to the category $\mathcal{E}_n$, we can define:

3. the full subcategory $\mathcal{L}_n$ whose objects have all factor modules with a local endomorphism ring;
4. the full subcategory $\mathcal{C}_n$ consisting of the objects whose factor modules are all non-zero couniformly presented right R-modules;
5. the full category $\mathcal{K}_n$ of $\mathcal{E}_n$ whose objects have all factor modules that are kernels of non-injective morphisms between indecomposable injective modules.

So, it is natural to try to adapt all we have seen also for these other constructions.

Mutatis mutandis, a version of Theorems 1.51 and 1.56 holds also for objects in these new subcategories. For instance, if M is an object of $\mathcal{C}_n$, we have $2n$ two-sided completely prime ideals of E_M, which are the ideals $I_{M,i,\ell} := \{f \in E_M \mid f_i \in I_{U^{(i)}}\}$ and $I_{M,i,e} := \{f \in E_M \mid f_i \in K_{U^{(i)}}\}$ for $i = 1, \dots, n$, where $I_{U^{(i)}}$ and $K_{U^{(i)}}$ are defined as in Theorem 1.27.

By continuing with this example, we have seen in Section 1.4 that the notions of i^{th}-monogeny class and i^{th}-epigeny class generalize those of monogeny and epigeny class introduced in [34]. The interplay between these notions and the objects of $\mathcal{U}_n$ looks like the "n-analogue" of that of uniserial modules. For objects in $\mathcal{C}_n$, we can generalize the notion of "having the same lower part" introduced in [37] as follows. Let M and N be two objects of $\mathcal{C}_n$ with factor modules $U^{(1)}, \dots, U^{(n)}$ and $V^{(1)}, \dots, V^{(n)}$ respectively. Fix a couniform presentation $0 \to C_M^{(i)} \to P_M^{(i)} \to U^{(i)} \to 0$ and $0 \to C_N^{(i)} \to P_N^{(i)} \to V^{(i)} \to 0$ for all the factor modules $U^{(1)}, \dots, U^{(n)}$ and $V^{(1)}, \dots, V^{(n)}$. For $i = 1, \dots, n$, we say that M and N *have the same* i^{th}*-lower part*, and we write $[M]_{i,\ell} = [N]_{i,\ell}$, if there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{E}_n$ such that both $f_i : U^{(i)} \to V^{(i)}$ and $g_i : V^{(i)} \to U^{(i)}$ lift to right R-module morphisms $\hat{f}_i : P_M^{(i)} \to P_N^{(i)}$ and $\hat{g}_i : P_N^{(i)} \to P_M^{(i)}$ with $\hat{f}_i(C_M^{(i)}) = C_N^{(i)}$ and $\hat{g}_i(C_N^{(i)}) = C_M^{(i)}$. As for the case of couniformly presented modules, this definition is independent from the choice of the couniform presentations. If M and N are two objects in $\mathcal{C}_n$, then $[M]_{i,\ell} = [N]_{i,\ell}$ if and only if there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{E}_n$ such that $gf \notin I_{M,i,\ell} = \{f \in E_M \mid f_i \in I_{U^{(i)}}\}$ (or, equivalently, such that $fg \notin I_{N,i,\ell} = \{f \in E_N \mid f_i \in I_{V^{(i)}}\}$). Similarly, $[M]_{i,e} = [N]_{i,e}$ if and only if there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{E}_n$ such that $gf \notin I_{M,i,e} = \{f \in E_M \mid f_i \in K_{U^{(i)}}\}$ (or, equivalently, such that $fg \notin I_{N,i,e} = \{f \in E_N \mid f_i \in K_{V^{(i)}}\}$). The corresponding arguments can be given for all the constructions listed at the beginning of this section.

Now, let us see if (and eventually how) it is possible to adapt what we have seen in Section 1.5, in order to prove the analogue of Theorems 1.53 and 1.60. The content of Remark 1.69 can be rephrased according to our previous considerations. Notice that we largely used Lemma 1.3, so we first need to replace that result with the suitable analogue for modules in $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{K}$. The easiest case is that of modules with a local endomorphism ring: if $f : A \to B$ and $g : B \to C$ are right R-module morphisms between modules in $\mathcal{L}$, then $g \circ f$ is an isomorphism if and only if both f and g are isomorphisms. Also the other cases are not difficult to treat. For instance, let M_1, M_2 and M_3 be couniformly presented modules, with couniform presentations $0 \to C_i \to P_i \to M_i \to 0$, for $i = 1, 2, 3$. Consider two right R-module morphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$, lift them to morphisms $\hat{f} : P_1 \to P_2$ and $\hat{g} : P_2 \to P_3$ and let $f' : C_1 \to C_2$ and $g' : C_2 \to C_3$ denote the restrictions of $\hat{f}$ and $\hat{g}$ respectively. Since couniformly presented modules are couniform, Lemma 1.3 ensures that $g \circ f$ is surjective if and only if both f and g are surjective. For the same reason, we get that $g' \circ f'$ is surjective if and only if both f' and g' are surjective. The case of kernels of non-injective morphisms between indecomposable injective modules can be treated in a similar way. At this point we are able to repeat step by step all the arguments given in Section 1.5 up to Corollary 1.81.

Now, looking at the proof of Theorem 1.84, we can notice that the "if-part" works in all these new cases, while the other implication require further comments.

Let us continue to describe the cases related to couniformly presented modules, that is, the categories $\mathrm{Morph}(\mathcal{C})$ and $\mathcal{C}_n$. Here we have $2n$ equivalence relations $\sim_{i,a}$ for $i = 1, \dots, n$ and $a = e, \ell$. In order to prove the existence of the permutations $\varphi_{i,e}$, $i = 1, \dots, n$, we can repeat exactly the proof given for Theorem 1.84. So, we only need to adapt the proof for the "lower-part case", that is, for the existence of the permutations $\varphi_{i,\ell}$, $i = 1, \dots, n$. In order to do that, we preserve all notations used in the proof. Fix an index $i = 1, \dots, n$ and, for every $k = 1, \dots, r$, couniform presentations $0 \to C_k \to P_k \to \mathcal{Q}_i(M_k) \to 0$ for every module $\mathcal{Q}_i(M_k)$ and couniform presentations $0 \to C'_k \to P'_k \to \mathcal{Q}_i(N_k) \to 0$ for every module $\mathcal{Q}_i(N_k)$. Every morphism $\chi^{(i)}_{h,j} : Q_i(M_h) \to Q_i(N_j)$ induces a right R-module morphism $\gamma_{h,j} : C_h \to C'_j$ and similarly, every morphism $(\chi'_{j,h})^{(i)} : Q_i(N_j) \to Q_i(M_h)$ induces a right R-module morphism $\gamma'_{j,h} : C'_j \to C_h$. Define a bipartite digraph $D = D(X, Y; E)$ having $X = \{M_1, \dots, M_r\}$ and $Y = \{N_1, \dots, N_r\}$ as disjoint sets of non-adjacent vertices, and the set E of edges defined as follows: one edge from M_h to N_j for each h and j such that $\gamma_{h,j}$ is a surjective right R-module morphism, and one edge from N_j to M_h for each h and j such that $\gamma'_{j,h}$ is a surjective right R-module morphism. Now, we can continue by arguing as for the permutations $\varphi_{i,e}$. Everything can be easily dualized for the categories Morph($\mathcal{K}$) and $\mathcal{K}_n$.

As far as the categories Morph($\mathcal{L}$) and $\mathcal{L}_n$ is concerned, we have a slight different situation. First off all, recall that we provided a weak form of the Krull-Schmidt Theorem for objects in Morph($\mathcal{L}$) in Theorem 1.49, as a consequence of Theorem 1.35. Therefore, we actually only need to prove a version of Theorem 1.84 for objects in $\mathcal{L}_n$ (but the same arguments give an alternative proof of Theorem 1.49). For objects M and N in $\mathcal{E}_n$, we can say that M and N have the same i^{th}-isomorphism class, and we will write $[M]_i = [N]_i$, if there exist two morphisms $f\colon M \to N$ and $g\colon N \to M$ in the category $\mathcal{E}_n$ such that both the i^{th}-induced morphisms $f_i : M^{(i)}/M^{(i-1)} \to N^{(i)}/N^{(i-1)}$ and $g_i : N^{(i)}/N^{(i-1)} \to M^{(i)}/M^{(i-1)}$ are isomorphisms. As before, we preserve the notations of Section 1.5. In order to prove the existence of the permutation φ_i for a fixed index $i = 1, \ldots, n$, define a bipartite digraph $D = D(X, Y; E)$ having $X = \{M_1, \ldots, M_r\}$ and $Y = \{N_1, \ldots, N_r\}$ as disjoint sets of non-adjacent vertices (notice that Theorem 1.17 ensures that $r = t$), and the set E of edges defined as follows: one edge from M_h to N_j for each h and j such that $\chi^{(i)}_{h,j} : Q_i(M_h) \to Q_i(N_j)$ is a bijective right R-module morphism, and one edge from N_j to M_h for each h and j such that $(\chi'_{j,h})^{(i)} : Q_i(N_j) \to Q_i(M_h)$ is a bijective right R-module morphism. In this case the modules $Q_i(M_h)$ and $Q_i(N_j)$ $(j, h = 1, \ldots, r)$ are not necessarily uniform or couniform, and therefore we can not argue as for the other cases. Nevertheless, we can apply [62, Theorem 2.2] to immediately deduce that for a fixed subset $T \subseteq X$, $|T| \leq |N^+(T)|$.

CHAPTER 2

Prüfer rings in distinguished constructions

"It is possible to enumerate a few concepts which are central in our development of Multiplicative Ideal Theory. Quotient rings and rings of quotients fall into this category, they are basic to all subsequent considerations; invertible ideals also constitute a basic tool in the presentation of the theory. A third concept which plays a central role in the development of the classical ideal theory is that of Prüfer domain." [58, R. Gilmer, Multiplicative Ideal Theory]

2.1 Notations and terminology

In this chapter, all rings are commutative unitary rings. For a ring R, we denote by $Z(R)$ the set of all zero-divisors of R (so, in particular, $0 \in Z(R)$) and by $\mathrm{Reg}(R)$ the set of regular elements of R. An ideal $\mathfrak{i}$ of R is said to be *regular* if it contains a regular element of R. $\mathrm{Tot}(R)$ denotes the total quotient ring of R. In the case of integral domains, we write $Q(R)$ for the quotient field of R. An *overring* of R is a subring R' of $\mathrm{Tot}(R)$ containing R. More generally, if $R \subseteq T$ is a ring extension, a *T-overring of* R is a ring R' such that $R \subseteq R' \subseteq T$. Any ring homomorphism $f : R \to T$ maps the identity of R into the identity of T. The *conductor of f* is the ideal $\mathrm{Ann}_R(T/\mathrm{Im}(f))$. According to this definition, if $R \subseteq T$ is a ring extension, the *conductor* of T into R is the ideal of R $(R : T) := \{r \in R \mid rT \subseteq R\}$. It is the greatest common ideal of R and T. $\mathrm{Spec}(R)$ denotes the prime spectrum of R. An ideal $\mathfrak{i}$ of a ring R is *locally principal* if $\mathfrak{i}R_{\mathfrak{p}}$ is principal for every prime ideal $\mathfrak{p}$ of R.

2.2 Preliminaries

We start this section recalling some characterizations of Prüfer domains that lead to (different) generalizations of this concept to rings with zero-divisors. Regarding to these arguments, we provide several references in the sequel, but we want to emphatize some of them in advance, expecially to the benefit of those readers not familiar with these themes. Gilmer's book [60] contains several characterizations of Prüfer domains. It is certainly the most important reference for these topics, together with Fontana, Huckaba and Papick's book [49]. As far as this theory in the framework of rings of zero-divisors is concerned, we mention the books of Larsen and McCarthy [79] and Huckaba [70]. In particular, the last chapter of Huckaba's book contains some important constructions we will use later on. Finally, a good reference for the "homological notions" (e.g. weak global dimension, semihereditary rings, etc.) is Glaz's book [63].

Prüfer-like conditions

Recall that a (fractional) ideal $\mathfrak{a}$ of a ring R is *invertible* if there exists an R-submodule $\mathfrak{b}$ of $\mathrm{Tot}(R)$ such that $\mathfrak{a}\mathfrak{b} = R$. If such a submodule $\mathfrak{b}$ exists, then it necessarily coincides with $\mathfrak{a}^{-1} := \{x \in \mathrm{Tot}(R) \mid x\mathfrak{a} \subseteq R\}$, which is called the *inverse of* $\mathfrak{a}$ (cf. [60, §7]).

Definition 2.1. A domain D is called a *Prüfer domain* if every non-zero finitely generated ideal of D is invertible.

This is the original definition given by Prüfer in his article [88]. The first generalization we present can be found in a paper by Butts and Smith [17] under the name "α-rings". Such rings were later named Prüfer rings by Griffin in [68].

Definition 2.2. A ring R is a *Prüfer ring* if every finitely generated regular ideal is invertible.

Invertibility of ideals is strictly related to the notion of projectivity and the property of being locally principal. Indeed, for any ideal $\mathfrak{a}$ of R we have the following implications:

$$\mathfrak{a} \text{ invertible } \Rightarrow \mathfrak{a} \text{ projective } \Rightarrow \mathfrak{a} \text{ locally principal.}$$

For a finitely generated regular ideal, these three conditions coincide, that is, a finitely generated regular ideal is invertible if and only if it is projective if and only if it is locally principal (cf. [9, Theorem 2.5]).

This fact leads us to the second generalization of Prüfer domains to rings with zero-divisors. As suggested in [9], this notion seems to have made its first appearance in Cartan-Eilenberg's book in 1956 [25].

Definition 2.3. A ring R is called *semihereditary* if every finitely generated ideal of R is projective.

Among all properties of semihereditary rings, we mention a characterization due to Endo [32].

Theorem 2.4. [32, Theorem 5] A ring R is semihereditary if and only if $\mathrm{Tot}(R)$ is absolutely flat (i.e. every $\mathrm{Tot}(R)$-module is flat) and $R_{\mathfrak{p}}$ is a valuation domain for every prime (maximal) ideal $\mathfrak{p}$ of R.

This result is in line with the first characterization of Prüfer domains given by Krull in 1936. It is expressed in terms of localizations at prime ideals.

Theorem 2.5. [78] A domain D is a Prüfer domain if and only if $D_{\mathfrak{p}}$ is a valuation domain for every prime (resp. maximal) ideal of D.

It is well known that a domain D is a valuation domain if and only if the set of all ideals of D is totally ordered under inclusion. There are two natural ways to transfer this condition to rings with zero divisors.

Definition 2.6. A ring R is said to have *weak global dimension* ≤ 1 (and we write $\mathrm{w.gl.dim}(R) \leq 1$) if $\mathrm{Tor}_2^R(M,N) = 0$ for every pair of R-modules M and N.

This definition is equivalent to require that every finitely generated ideal of R is flat (see [63, Chapter 1]), so that, in particular, semihereditary rings have $\mathrm{w.gl.dim}(R) \leq 1$. This is a homological notion that can be expressed in terms of the Krull's characterization of Prüfer domains. Indeed, rings with $\mathrm{w.gl.dim}(R) \leq 1$ are precisely those rings for which $R_{\mathfrak{p}}$ is a valuation domain for every prime [resp. maximal] ideal $\mathfrak{p}$ of R [63, Corollary 4.2.6].

As a further generalization, it is possible to consider rings with the property that all localizations at prime ideals are *chained rings*, not necessarily domains (recall that a ring is chained if the set of its ideals is linearly ordered under inclusion).

Definition 2.7. A ring R is called *arithmetical* if $R_{\mathfrak{p}}$ is a chained ring for every prime [resp. maximal] ideal $\mathfrak{p}$ of R.

This definition was originally given by Fuchs [52] in a way that clearly explains the term "arithmetical": a ring R is arithmetical if the lattice of its ideals is distributive, that is, $\mathfrak{i} \cap (\mathfrak{j} + \mathfrak{k}) = (\mathfrak{i} \cap \mathfrak{j}) + (\mathfrak{i} \cap \mathfrak{k})$, for any three ideals $\mathfrak{i}, \mathfrak{j}$ and $\mathfrak{k}$ of R. The equivalent characterization we have presented in terms of localizations is due to Jersen [72]. He also proved that a ring R is arithmetical if and only if R is reduced and $\mathrm{w.gl.dim}(R) \leq 1$.

Although Prüfer rings are not necessarily arithmetical rings (see [9]), Griffin [68] showed that they are not so far from being locally chained, in the following sense. If $\mathfrak{p}$ is a prime ideal of a ring R, the pair $(R, \mathfrak{p})$ is said to have *the regular total order property* if, whenever $\mathfrak{a}, \mathfrak{b}$ are ideals of R, one at least of which is regular, then the ideals $\mathfrak{a}A_{\mathfrak{p}}, \mathfrak{b}A_{\mathfrak{p}}$ are comparable.

Theorem 2.8. [68, Theorem 13] A ring R is a Prüfer ring if and only if for any maximal ideal $\mathfrak{m}$ of R, the pair $(R, \mathfrak{m})$ has the regular total order property.

The next extension of the Prüfer domain notion to rings with zero divisors is due to Tsang [93]. Let R be a ring and X be an indeterminate over R. The *content of a polynomial* $f(X) \in R[X]$ is the ideal $c_R(f)$ of R generated by the coefficients of $f(X)$. We say that the polynomial f is *a Gaussian polynomial over* R if $c_R(fg) = c_R(f)c_R(g)$, for any polynomial $g \in R[X]$. The ring R is *a Gaussian (or Gauss) ring* if any polynomial $f(X) \in R[X]$ is a Gaussian polynomial [93]. In the following theorem, we summarize well-known results about the interplay between invertibility of ideals and Gaussian polynomials.

Theorem 2.9. Let R be a ring and let $f(X)$ be a polynomial in $R[X]$.

1. If $c_R(f)$ is locally principal, then $f(X)$ is a Gauss polynomial [93].

2. If $f(X)$ is a Gauss polynomial and $c_R(f)$ is a regular ideal of R, then $c_R(f)$ is an invertible ideal of R [9, Theorem 4.2(2)].

Any arithmetical ring is a Gaussian ring. Indeed, it is clear that every finitely generated ideal of an arithmetical ring R is locally principal (actually, this property characterizes arithmetical rings [72]) and by condition (1) of Theorem 2.9, R is a Gaussian ring. Moreover, from condition (2), it is immediate that any Gaussian ring is a Prüfer ring. In particular, we get another characterization of Prüfer domains, which was originally proved independently by Tsang [93] and Gilmer [58].

Theorem 2.10. [93, 58] An integral domain is a Prüfer domain if and only if it is a Gaussian domain.

To summarize, we have presented five conditions that extend the notion of Prüfer domain in the case of rings with zero-divisors. We refer to them as "Prüfer condition (n)", for $n = 1, \ldots, 5$, according to the following list:

1. R is a semihereditary ring;

2. $\operatorname{w.gl.dim}(R) \leq 1$;

3. R is an arithmetical ring;

4. R is a Gaussian ring;

5. R is a Prüfer ring.

Of course, these conditions are equivalent for Prüfer domains and we have seen that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold for rings with zero-divisors. Anyway, none of the implications can be reversed in general. Counterexamples are provided in [64]. Nevertheless, it is possible to reverse all the implications by adding extra conditions on the total ring of quotients of R. The result has the following very elegant formulation.

Theorem 2.11. [9, Theorem 5.7] Let R be a ring. Then, for $i = 1, \ldots, 4$:

1. R has the Prüfer condition (n) if and only if R has the Prüfer condition $(n+1)$ and $\operatorname{Tot}(R)$ has the Prüfer condition (n).

2. R has the Prüfer condition (n) if and only if R is a Prüfer ring and $\operatorname{Tot}(R)$ has the Prüfer condition (n).

3. If $\operatorname{Tot}(R)$ is absolutely flat, then all five Prüfer conditions on R are equivalent.

Every overring of a Prüfer ring is still Prüfer (see [79, Chapter X]). So, in view of the Theorem 2.11, we have the following corollary.

Corollary 2.12. Let R be a ring having Prüfer condition (n), for some $n = 1, \dots, 5$. Then, every overring of R has the same Prüfer condition (n).

Remark 2.13. For the purpose of our dissertation, it is worth noting that Prüfer conditions $(1) \div (4)$ are preserved under localizations, while condition (5) is not. Moreover, quotients of Gaussian rings [resp. arithmetical rings] are still Gaussian [resp. arithmetical]. The same holds for Prüfer rings if quotients are taken with respect to regular ideals [46, Proposition 4.4]. Nevertheless, Prüfer conditions (1) and (2) are, in general, not preserved under homomorphic images (e.g. quotients of valuation domains are not necessarily domains).

Prüfer extensions

There is another way to generalize the concept of Prüfer rings, recently developed by M. Knebush and D. Zhang [76]. In order to introduce this notion, we first underline that the invertibility of ideals is given in terms of the ring extension $R \subseteq \operatorname{Tot}(R)$, that is, in terms of R-submodules of $\operatorname{Tot}(R)$. This concept can be generalized with respect to any ring extension $A \subseteq B$ as follows. According to [76], an R-submodule $\mathfrak{a}$ of T is T*-regular* if $\mathfrak{a}T = T$. The ideal $\mathfrak{a}$ is called T*-invertible* if there exists an R-submodule $\mathfrak{b}$ of T such that $\mathfrak{a}\mathfrak{b} = R$. If $\mathfrak{a}$ is T-invertible, then it is T-regular. Indeed, if $\mathfrak{a}\mathfrak{b} = R$, by multiplying with the R-module T we get $\mathfrak{a}T = T$. Moreover, the R-module $\mathfrak{b}$ is uniquely determined, that is, $\mathfrak{b} = [R :_T \mathfrak{a}] := \{x \in T \mid x\mathfrak{a} \subseteq R\}$. It is called the T*-inverse of* $\mathfrak{a}$. With this notion in mind, it is possible to generalize the concept of Prüfer ring.

Definition 2.14. [76] Let $R \subseteq T$ be a ring extension. We say that R *is Prüfer in* T if the inclusion $R \hookrightarrow T$ is a flat epimorphism (in the category of rings) and every finitely generated T-regular ideal of R is T-invertible. We can also say that $R \subseteq T$ *is a Prüfer extension* or that R is T*-Prüfer*.

It is clear that this notion recover that of Prüfer ring if $T = \text{Tot}(R)$, that is, $R \subseteq \text{Tot}(R)$ is a Prüfer extension if and only if R is a Prüfer ring. Relations between T-regularity, T-invertibility, projectivity and the property of being locally principal are summarized in the following proposition. These relations generalize those for integral domains in the classical setting.

Proposition 2.15. Let $R \subseteq T$ be a ring extension and let $\mathfrak{a}$ be an ideal of R.

1. [76, §2 Remark 1.10] If $\mathfrak{a}$ is T-invertible, then it is T-regular and finitely generated.
2. [47, Proposition 2.1] $\mathfrak{a}$ is T-invertible if and only if $\mathfrak{a}$ is finitely generated, T-regular and a flat R-module.
3. [76, §2 Proposition 2.3] If $\mathfrak{a}$ is T-regular, then the following conditions are equivalent:
   (a) $\mathfrak{a}$ is T-invertible.
   (b) $\mathfrak{a}$ is finitely generated and locally principal.
   (c) $\mathfrak{a}$ is a projective R-module.

Here we present a bunch of criteria to characterize Prüfer extensions. We list only those that will be used in the sequel. Several other characterizations can be found in Knebush and Zhang's book [76].

Theorem 2.16. [76, Chapter I, Theorem 5.2 and Chapter II, Theorem 2.1] Let R be a subring of T. Then, the following conditions are equivalent.

1. R is Prüfer in T;
2. The inclusion $R \hookrightarrow B$ is a flat epimorphism for every T-overring B of R;

3. If B is any T-overring of R, then $(R:x)B = B$ for every $x \in T$.

4. Every T-overring of R is integrally closed in T;

5. R is integrally closed in T and $R[x] = R[x^n]$ for every $x \in T$ and for every $n \in \mathbb{N}$;

6. $(R:x) + x(R:x) = R$ for every $x \in T$;

7. R is integrally closed in T and for every overring B of T, the restriction map $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is injective.

In Section 2.4, permanence properties of Prüfer extensions will be largely used. In the next proposition, we collect some results that can be found in [76, pp. 50–52].

Theorem 2.17. The following properties hold.

1. If $R \subseteq T$ is a Prüfer extension, then for every T-overring S of R, R is Prüfer in S and S is Prüfer in T.

2. If $R \subseteq S$ and $S \subseteq T$ are Prüfer extensions, then $R \subseteq T$ is a Prüfer extension.

3. Let $R \subseteq T$ be a ring extension and $\mathfrak{i}$ an ideal of T contained in R. Then R is Prüfer in T if and only if $R/\mathfrak{i}$ is Prüfer in $T/\mathfrak{i}$.

We conclude this subsection by shortly presenting the notion of *Prüfer Manis rings*. We refer to [81], [10] and [70, Chapter 2] for an exhaustive description of this topic. A *Manis pair* $(A, \mathfrak{p})$ is a pair where A is a ring, $\mathfrak{p}$ is a prime ideal of A and for every $x \in \mathrm{Tot}(A) \setminus A$, there exists $y \in \mathfrak{p}$ such that $xy \in A \setminus \mathfrak{p}$. Given a ring A and a prime ideal $\mathfrak{m}$ of A, A is called a *Prüfer Manis ring* if the following equivalent conditions hold (see [10, Theorem 2.3]):

1. $(A, \mathfrak{m})$ is a Manis pair and A is a Prüfer ring.

2. A is a Prüfer ring and $\mathfrak{m}$ is the unique regular maximal ideal of A.

3. $(A, \mathfrak{m})$ is a Manis pair and $\mathfrak{m}$ is the unique regular maximal ideal of A.

Basics on pullbacks

Recall that if $\alpha : A \to C$ and $\beta : B \to C$ are ring homomorphisms, the *pullback* (or *fiber product*) of α and β is the subring

$$D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$$

of $A \times B$. There are two canonical morphisms $u : D \to A$ and $v : D \to B$ which are the restrictions to D of the canonical projections of $A \times B$ onto A and B respectively. The triple (D, u, v) is characterized by the property of being the solution of the universal problem of making the following diagram

$$\begin{array}{ccc} D & \xrightarrow{u} & A \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

commute. It is well known that if α is injective [resp. surjective, an isomorphism], then so if v. A typical application of pullbacks in commutative algebra is that of "attaching spectral spaces", a classical operation that allow to construct rings whose spectra have some predetermined properties. A good reference for these kind of topics is Fontana's work [48]. Here, we only present some results that will be used in this thesis. For a ring R, consider its prime spectrum $\mathrm{Spec}(R)$ endowed with the Zariski topology, that is, with the topology whose closed sets are the sets $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, where $\mathfrak{a}$ is an ideal of R. If $f : R \to T$ is a ring morphism, we denote by $f^* : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ the continuous map defined by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$. In reference to the pullback diagram displayed above, assume that β is surjective. We get the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(D) & \xhookleftarrow[u^*]{} & \mathrm{Spec}(A) \\ {\scriptstyle v^*}\uparrow & & \uparrow{\scriptstyle \alpha^*} \\ \mathrm{Spec}(B) & \xhookleftarrow[\beta^*]{} & \mathrm{Spec}(C) \end{array}$$

of topological spaces. In the following proposition, we collect some results that can be found in [48] and [27].

Proposition 2.18. Set $\mathfrak{b} := \ker(\beta)$, $\mathfrak{d} := \ker(u)$ and preserve the notations introduced above.

1. v induces an isomorphism of modules (subordinate to v) between $\mathfrak{d}$ and $\mathfrak{b}$. In particular, $\mathfrak{d}$ is contained in the conductor of v.
2. $u^* : \mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(D)$ is a closed topological embedding. Thus, $\mathrm{Spec}(A)$ is homeomorphic to its image $V(\mathfrak{d})$ under u^*.
3. For every $\mathfrak{p} \in \mathrm{Spec}(D)$, $\mathfrak{p} \not\supseteq \mathfrak{d}$, there exists a unique $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $v^{-1}(\mathfrak{q}) = \mathfrak{p}$. Moreover, $D_{\mathfrak{p}} \cong B_{\mathfrak{q}}$ under the canonical homomorphism induced by v.
4. The restriction of v^* to $\mathrm{Spec}(B) \setminus V(\mathfrak{b})$ induces an homeomorphism $\mathrm{Spec}(B) \setminus V(\mathfrak{b}) \cong \mathrm{Spec}(D) \setminus V(\mathfrak{d})$ (and, in particular, an isomorphism of partially ordered sets).
5. The homeomorphisms $\mathrm{Spec}(A) \cong V(\mathfrak{d})$ and $\mathrm{Spec}(B) \setminus V(\mathfrak{b}) \cong \mathrm{Spec}(D) \setminus V(\mathfrak{d})$ are, in particular, isomorphisms of partially ordered sets. Therefore, they preserve maximality.
6. D is a local ring if and only if A is a local ring and $\mathfrak{b} \subseteq \mathrm{Jac}(B)$. In particular, if A and B are local rings, then so is D. Moreover, if D is a local ring, its maximal ideal is the image of the maximal ideal of A via u^*.

Intuitively, $\mathrm{Spec}(D)$ is obtained by "attaching" $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ along $\mathrm{Spec}(C)$. To be more precise, $\mathrm{Spec}(D)$ is homeomorphic to the topological space defined by the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ modulo the equivalence relation generated by $\mathfrak{p} \sim \alpha^*(\mathfrak{p})$, for each $\mathfrak{p} \in \mathrm{Spec}(C)$ [48, Theorem 1.4].

Constructions related to pullbacks

Let $R \subseteq T$ be a ring extension with non-zero conductor $\mathfrak{c} = (R : T)$. Then $A := R/\mathfrak{c}$ is a subring of $B := T/\mathfrak{c}$ and the pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\pi} & B \end{array} \tag{2.2.1}$$

(where π is the canonical projection) is called a **conductor square**. If $\mathfrak{c}$ is a regular ideal of T, we say that the above diagram is a **regular** conductor square.

In [15], Boynton gives a precise description of when R has Prüfer condition (n) for $1 \leq n \leq 5$ in regular conductor squares. This characterization is expressed in terms of the local features of the pullback diagram, as follows:

Theorem 2.19. [15, Theorem 4.2] We preserve the notations of the regular conductor square presented above.

1. If R is a Prüfer ring, then A and T are Prüfer rings, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$ for each prime (maximal) ideal $\mathfrak{p}$ of R. Conversely, for each prime (maximal) ideal $\mathfrak{p}$ of R, if $A_{\mathfrak{p}}$ and $T_{\mathfrak{p}}$ are Prüfer rings, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$, then R is a Prüfer ring.

2. For $n = 1, 2, 3, 4$, R is a commutative ring with Prüfer condition (n) if and only if T has Prüfer condition (n), $A_{\mathfrak{p}}$ is a Prüfer ring, and $B_{\mathfrak{p}}$ is an overring of $A_{\mathfrak{p}}$ for each prime (maximal) ideal $\mathfrak{p}$ of R.

Conductor squares are particular pullbacks in which the morphisms α and β are injective and surjective respectively. Some authors refer to this latter type of fiber products as *Cartesian squares*, whereas some others use this terminology as a synonymous of (a generic) pullback. Because of this ambiguity, we prefer not to name these particular pullbacks. It is worth noting that such fiber products arise essentially in the following way. Let $\pi : T \to B$ be a surjective ring homomorphism, take a subring A of B and set $R := \pi^{-1}(A)$. Then:

1. $\ker(\pi)$ is contained in the conductor $\mathfrak{c} = (R : T)$. In particular, it is a common ideal of R and T.

2. R is canonically isomorphic to the fiber product $\iota \times_T \pi$, where ι is the inclusion $R \hookrightarrow T$, so that, we have the following pullback diagram

$$\begin{array}{ccc} R \cong \iota \times_T \pi & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \iota \\ T & \xrightarrow{\ \pi\ } & B \end{array}$$

Moreover,

3. If $\mathfrak{c}$ is a regular ideal of T, then T is an overring of R. Indeed, if $r \in \mathfrak{c}$ is a regular element of T, then $t \mapsto \frac{tr}{r}$ defines an injective ring homomorphism $T \hookrightarrow \mathrm{Tot}(R)$.

4. If $T = S^{-1}R$ for some multiplicatively closed subset $S \subseteq \mathrm{Reg}(R)$, then $B = (\bar{S})^{-1}A$. Moreover, B is an overring of A. Indeed, if there exists some non-zero element $a \in A$ such that $\bar{s}a = 0$ for some $s \in S$, then $\frac{a}{1} = 0$ in B, in contrast with the fact that A is a subring of B.

In Section 2.4, we study the transfer of Prüfer-like properties in such pullback diagrams, characterizing when the ring R has Prüfer condition (n) $(1 \leq n \leq 5)$ in terms of direct conditions on the given data A, B and T, under the reasonable (and quite natural) assumption that A is an overring of B.

Among all results in this direction, it is worth mention that of Houston and Taylor [69], that we will see can be deduced from Theorem 2.27. We preserve the notations of the original paper.

Proposition 2.20. [69, Theorem 1.3] Let T be a domain and let $\mathfrak{i}$ be an ideal of T. Let D be a domain contained in $E := T/\mathfrak{i}$ and let $\pi : T \to E$ denote the canonical projection. Then $R := \pi^{-1}(D)$ is a Prüfer ring if and only if both D and T are Prüfer rings, $\mathfrak{i}$ is a prime ideal of T and D, E have the same field of fractions.

We now briefly recall some classical constructions in commutative algebra, almost all related to fiber products. We will use them in some examples in Sections 2.4, 2.5 and 2.6 and we will discuss about Prüfer and Gaussian properties of bi-amalgamated algebras in Section 2.5.

1) Nagata idealization: Let A be a ring and let M be an A-module. The *Nagata idealization of M in A* is the ring $A(+)M$ defined as follows [86]: the product $A \times M$ is endowed with the ring structure where addition is induced componentwise from additions of A and M and multiplication is defined by setting $(a, x)(b, y) := (ab, ay + bx)$ for all $(a, x), (b, y) \in A \times M$. The identity of $A(+)M$ is $(1, 0)$. Moreover, A can be identified with a subring of $A(+)M$ via the map $a \mapsto (a, 0)$ and M is isomorphic, as an A-module, to the ideal

$0(+)M$ of $A(+)M$. This latter fact gives rise to the name "idealization". It is immediate that if $M \neq 0$, then the ideal $0(+)M$ of $A(+)M$ is nilpotent of index 2.

2) Amalgamated duplication along an ideal: In [30], D'Anna and Finocchiaro introduced a ring construction that arises, in some way, as a sort of generalization of the Nagata idealization $A(+)M$. Let A be a ring and let $\mathfrak{a}$ be an ideal of A. The *amalgamated duplication of A along $\mathfrak{a}$* is the subring

$$A \bowtie \mathfrak{a} := \{(a, a + x) \mid a \in A, x \in \mathfrak{a}\}$$

of the direct product $A \times A$. This construction is related to that of Nagata in the following sense: if $\mathfrak{a}$ is a nilpotent ideal of index 2, then $A \bowtie \mathfrak{a}$ is canonically isomorphic to $A(+)\mathfrak{a}$. The amalgamated duplication arises as the following pullback:

$$\begin{array}{ccc} A \bowtie \mathfrak{a} & \longrightarrow & A \\ \downarrow & & \downarrow \pi \\ A & \xrightarrow{\pi} & A/\mathfrak{a}, \end{array}$$

where $\pi : A \to A/\mathfrak{a}$ is the canonical projection.

3) Amalgamated algebras along an ideal: As it is immediately seen, whenever $\mathfrak{a}$ is a nonzero ideal of A, the ring $A \bowtie \mathfrak{a}$ is not an integral domain. A new ring construction, which is more general than the amalgamated duplication and can be an integral domain, was introduced in [27] as follows: starting from a ring homomorphism $f : A \longrightarrow B$ and from an ideal $\mathfrak{b}$ of B, D'Anna, Finocchiaro and Fontana defined *the amalgamation of A and B along $\mathfrak{b}$, with respect to f* to be the following subring

$$A \bowtie^f \mathfrak{b} := \{(a, f(a) + b) \mid a \in A, b \in \mathfrak{b}\}$$

of the direct product $A \times B$. Obviously, if $f = \mathrm{Id}_A : A \longrightarrow A$ and $\mathfrak{a}$ is an ideal of A, then $A \bowtie^f \mathfrak{a}$ is the classical amalgamated duplication. This construction can be studied in the frame of pullbacks as well. Indeed, let $\pi : B \to B/\mathfrak{b}$ be the canonical projection. We have the following pullback diagram:

$$\begin{array}{ccc} A \bowtie^f \mathfrak{b} & \longrightarrow & A \\ \downarrow & & \downarrow {\scriptstyle \pi\circ f} \\ B & \xrightarrow{\pi} & B/\mathfrak{b} \end{array}$$

The transfer of Prüfer-like conditions on amalgamated algebras were studied by Finocchiaro [46] and Azimi, Sahandi and Shirmohammadi [4].

4) Constructions of the type $A+XB[X]$ and $A+XB[[X]]$: Let $A \subseteq B$ be a ring extension and let $X := \{X_1, \ldots, X_n\}$ be a finite set of indeterminates over B. The subring $A + XB[X]$ of $B[X]$ arises from the following pullback diagram

$$\begin{array}{ccc} A + XB[X] & \longrightarrow & A \\ \big\downarrow & & \big\downarrow {\scriptstyle \iota} \\ B[X] & \xrightarrow{\pi} & B \end{array}$$

where π and ι are the canonical projection and the inclusion respectively. It is possible to recover this construction as a particular case of amalgamated algebra. Indeed, taking the ideal $\mathfrak{b} := XB[X]$ of $B[X]$ and the natural embedding $\varepsilon : A \hookrightarrow B[X]$, we have $A + XB[X] \cong A \bowtie^{\varepsilon} \mathfrak{b}$. In a similar way, it is possible to construct the subring $A + XB[[X]]$ of the ring of power series $B[[X]]$. Notice that, given any ring extension $A \subseteq B$ and an indeterminate X over B, we have an isomorphism $A(+)B \cong A + XB[X]/(X^2)$.

5) $D + \mathfrak{m}$ construction: Another construction covered by amalgamated algebras is the so called $D + \mathfrak{m}$ *construction* [58]. Let $\mathfrak{m}$ be a maximal ideal of a ring T and let D be a subring of T such that $D \cap \mathfrak{m} = (0)$. The ring $D + \mathfrak{m}$ defined by the pullback diagram

$$\begin{array}{ccc} D + \mathfrak{m} & \longrightarrow & D \\ \big\downarrow & & \big\downarrow {\scriptstyle \iota} \\ T & \xrightarrow{\pi} & T/\mathfrak{m} \end{array}$$

is canonically isomorphic to $D \bowtie^{\iota} \mathfrak{m}$, where $\iota : D \hookrightarrow T$ is the inclusion. Bakkari and Mahdou [6] studied the transfer of Prüfer-like conditions in a particular case of this construction.

Proposition 2.21. [6, Theorem 2.1] Let $(T, \mathfrak{m})$ be a local ring of the form

$T = k + \mathfrak{m}$, for some field k. Take a subring D of k such that $Q(D) = k$ and set $R := D + \mathfrak{m}$. Then R has Prüfer condition (n) if and only if T and D have Prüfer condition (n).

6) CPI-extensions: The following construction was introduced by Boisen and Sheldon [12]. Let $\mathfrak{p}$ be a prime ideal of a ring A. Set $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, and let $\pi : A_\mathfrak{p} \to k$ and $\lambda : A \to A_\mathfrak{p}$ be the canonical projection and the localization map respectively. It is well known that $Q(A/\mathfrak{p})$ is canonically isomorphic to k, so that $A/\mathfrak{p}$ can be identified as a subring of k. The subring $\lambda(A) + \mathfrak{p}A_\mathfrak{p} = \pi^{-1}(A/\mathfrak{p})$ is called the *CPI-extension of A with respect to* $\mathfrak{p}$ ("CPI" stands for "complete pre-image"). Its natural presentation as a pullback is the following:

$$\begin{array}{ccc} \lambda(A) + \mathfrak{p}A_\mathfrak{p} & \longrightarrow & A/\mathfrak{p} \\ \big\downarrow & & \big\downarrow \\ A_\mathfrak{p} & \xrightarrow{\ \pi\ } & k \end{array}$$

It can be seen that $\lambda(A) + \mathfrak{p}A_\mathfrak{p}$ is isomorphic to $(A \bowtie^\lambda \mathfrak{p}A_\mathfrak{p})/(\mathfrak{p} \times \{0\})$ [27, Example 2.7].

Even if presented in a very short way, it should be clear that amalgamated algebras have a significant role in commutative algebra, at least (but not only) for their ability of covering several different pullback constructions. We conclude this section by mentioning a further generalization of amalgamated algebras due to Kabbaj, Louartiti and Tamekkante [73]. This notion will be largely discussed in Section 2.5, so here we only give the definition for the sake of completeness.

7) Bi-amalgamated algebras: Let $f : A \longrightarrow B$, $g : A \longrightarrow C$ be ring homomorphisms and let $\mathfrak{b}$ [resp. $\mathfrak{c}$] be an ideal of B [resp. C] satisfying $f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$. The *bi-amalgamation of A with (B, C) along $(\mathfrak{b}, \mathfrak{c})$, with respect to* (f, g) [73] is the subring of $B \times C$ defined by

$$A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) := \{(f(a) + b, g(a) + c) \mid a \in A, b \in \mathfrak{b}, c \in \mathfrak{c}\}.$$

If $A = C$ and $g : A \longrightarrow A$ is the identity on A, we get the usual amalgamated algebra [73, Example 2.1]: $A \bowtie^f \mathfrak{b} = A \bowtie^{f,\mathrm{Id}_A} (\mathfrak{b}, \mathfrak{i}_0)$.

2.3 Some new results on Prüfer extensions

We start this section by giving the generalization to arbitrary ring extensions of a standard fact regarding the invertibility of ideals in local domains. This is probably a well-known fact, but, since we are not able to provide an appropriate reference, we include the proof for the convenience of the reader.

Proposition 2.22. Let $A \subseteq B$ be a ring extension where A is a local ring, and let $\mathfrak{f} := (f_1, \dots, f_n)A$ be a B-invertible A-submodule of B. Then $\mathfrak{f} = f_i A$, for some $1 \leq i \leq n$.

Proof. By assumption, there are elements $z_1, \dots, z_n \in [A :_B \mathfrak{f}]$ such that $\sum_{i=1}^{n} f_i z_i = 1$. Note that $f_i z_i \in A$ for each $1 \leq i \leq n$ and thus, since A is local, there is some index $\bar{i}$ such that $f_{\bar{i}} z_{\bar{i}}$ is a unit in A. It immediately follows that $F = f_{\bar{i}} A$ □

We now exhibit two results regarding Prüfer extensions, that generalize those for local Prüfer rings proved in [15].

Proposition 2.23. Let A be a local ring and let $A \subseteq B$ be a Prüfer extension. Then the set of all elements of A that are not invertible in B is a prime ideal of A.

Proof. Set $\mathfrak{p} := \{a \in A \mid a \notin \mathcal{U}(B)\}$. Then $A \backslash \mathfrak{p}$ is a saturated multiplicatively closed subset of A. By Zorn's Lemma, it is possible to find a prime ideal $\mathfrak{q} \in \operatorname{Spec}(A)$ maximal with respect to the property of being contained in $\mathfrak{p}$. If $\mathfrak{q} \subsetneq \mathfrak{p}$, take $p \in \mathfrak{p} \setminus \mathfrak{q}$ and consider the ideal $\mathfrak{a} := pA + \mathfrak{q}$ of A. Then there exist $a \in A$ and $q \in \mathfrak{q}$ such that $pa + q \in \mathcal{U}(B)$, which implies that the two-generated ideal $(p, q)A$ of A is B-regular and hence B-invertible. Since A is a local ring, we have either $(p, q)A = pA$ or $(p, q)A = qA$, by Proposition 2.22. In the first case we have that $B = (p, q)B = pB$, which is not possible since $p \in \mathfrak{p}$. In the second case we have $qA \subset \mathfrak{q}$, which implies that $p \in \mathfrak{q}$, a contradiction. It follows that $\mathfrak{p} = \mathfrak{q}$, hence $\mathfrak{p}$ is a prime ideal of A. □

Corollary 2.24. [15, Lemma 3.5] If A is a local Prüfer ring, then the set $Z(A)$ of zero-divisors is a prime ideal.

Proposition 2.25. Let $A \subseteq B$ be a Prüfer extension. Assume that A is a local ring and that $B = S^{-1}A$ for some multiplicatively closed subset $S \subseteq \operatorname{Reg}(A)$. If R is a subring of B containing A, then R is a local B-Prüfer ring and $R = A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of A.

Proof. We can assume, without loss of generality, that the multiplicatively closed subset S is saturated. Set $\mathfrak{p} := \{a \in A \mid a \notin \mathcal{U}(R)\}$. Since $A \subseteq R$ is a Prüfer extension, Proposition 2.23 ensures that $\mathfrak{p}$ is a prime ideal of A.

It is clear that $A_{\mathfrak{p}} \subseteq R$. Let $r \in R$ and write $r = a/s$ for some $a \in A$ and $s \in S$. Since A is a local B-Prüfer ring, we have $(a, s)A = aA$ or $(a, s)A = sA$. If $(a, s)A = aA$, then $a = sx$ for some $x \in A$ and $r = x/1 \in A_{\mathfrak{p}}$. If $(a, s)A = sA$, then we can write $s = ay$ for some $y \in A$ and since S is saturated, both a and y turn out to be in S. We have $r = a/s = a/ay = 1/y$, which implies $y \notin \mathfrak{p}$. Therefore, $r \in A_{\mathfrak{p}}$. □

Corollary 2.26. [15, Lemma 3.6] If A is a local ring with Prüfer condition (n) and if R is an overring of A, then R is a local ring with Prüfer condition (n). Moreover, $R = A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of A.

2.4 Prüfer conditions in distinguished pullbacks

Theorem 2.27. Let $\pi : B \to T$ be a surjective ring homomorphism, where T is an overring of some ring R. Assume that $\ker(\pi)$ is a regular ideal of B. Set $A := \pi^{-1}(R)$.

1. A is a Prüfer ring if and only if both B and R are Prüfer rings;
2. A is a Gaussian [resp. arithmetical] ring if and only if both B and R are Gaussian [resp. arithmetical] rings;
3. If both B and R are rings of weak global dimension ≤ 1 [resp. semi-hereditary rings], then so is A.

Proof. The ring A is defined by the following pullback diagram:

$$\begin{array}{ccc} A & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\pi} & T \end{array}$$

First assume that A is a Prüfer ring. By the fact that $\ker(\pi)$ is a regular ideal, contained in the conductor of the ring extension $A \subseteq B$, we have that B is an overring of A (see pag. 74) and hence B is a Prüfer ring by Corollary 2.12. Moreover, the regularity of $\ker(\pi)$ ensures that R is also a Prüfer ring (Remark 2.13). In the same way, we can prove the “only-if part” in (2).

Now assume that both R and B are Prüfer rings. Using the permanence properties of Prüfer extensions listed in Theorem 2.17 and keeping in mind that for a ring S, $S \subseteq \mathrm{Tot}(S)$ is a Prüfer extension if and only if S is a Prüfer ring, we get:

- since T is an overring of R, $R \subseteq T$ is a Prüfer extension;
- $A \subseteq B$ is a Prüfer extension;
- $B \subseteq \mathrm{Tot}(A) = \mathrm{Tot}(B)$ is a Prüfer extensions, so $A \subseteq \mathrm{Tot}(A)$ is a Prüfer extension. Hence, A is a Prüfer ring.

Finally, assume that both B and R are rings with Prüfer condition (n), for $n = 1, 2, 3, 4$. Then B and R are Prüfer rings and therefore so is A. Moreover, $\mathrm{Tot}(B) = \mathrm{Tot}(A)$ has Prüfer condition (n). By Theorem 2.11, A has Prüfer condition (n).

□

Corollary 2.28. Let R be a ring with total quotient ring $T = \mathrm{Tot}(R)$. Then $R + XT[X]$ is a semihereditary [resp. arithmetical] ring if and only if R is a semihereditary [resp. arithmetical] ring and T is absolutely flat.

Proof. Recall that a ring S is absolutely flat if and only if $S[X]$ is semihereditary if and only if $S[X]$ is arithmetical (see [84] and [18]). As we have seen on page 76, $R + XT[X]$ can be obtained by the following pullback:

$$\begin{array}{ccc} R + XT[X] & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \iota \\ T[X] & \xrightarrow{\ \pi\ } & T \end{array}$$

Assume that R is semihereditary and that T is absolutely flat. Then $T[X]$ is a semihereditary ring. Applying Theorem 2.27, we have that $R + XT[X]$ is semihereditary.

On the other hand, if $R + XT[X]$ is semihereditary, then so is $T[X]$. In particular, T is absolutely flat. Moreover R is a Prüfer ring (see Remark 2.13). Applying Theorem 2.11, we get that R is a semihereditary ring.

Ditto for the arithmetical case. □

The next application of Theorem 2.27 concerns the Prüfer Manis rings (for the definition, see the end of Subsection 2.2.2).

Corollary 2.29. Let B be a Prüfer Manis ring and let V be a valuation domain with quotient field $B/\mathfrak{m}$, where $\mathfrak{m}$ denotes the unique regular maximal ideal of B. Consider the canonical projection $\pi : B \to B/\mathfrak{m}$. Then $\pi^{-1}(V)$ is a Prüfer Manis ring.

Proof. Set $A := \pi^{-1}(V)$ and $k := B/\mathfrak{m}$. We have the following pullback diagram:

$$\begin{array}{ccc} A & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\ \pi\ } & B/\mathfrak{m} \end{array}$$

By Theorem 2.27, A is a Prüfer ring. Let $\mathfrak{p} := \pi^{-1}(\mathfrak{m}_V)$ be the contraction of the maximal ideal $\mathfrak{m}_V$ of V. We prove that $(A, \mathfrak{p})$ is a Manis pair. It suffices to show that for every $x \in \mathrm{Tot}(A) \setminus A$ there exists $y \in \mathfrak{p}$ such that $xy \in A \setminus \mathfrak{p}$. Let $x \in \mathrm{Tot}(A) \setminus A$. We distinguish two cases:

Case (1): $x \in B$. Then $\pi(x) \in k \setminus V$ and so $1/\pi(x) \in \mathfrak{m}_V$. It means that there exists $y \in \mathfrak{p}$ such that $\pi(y)\pi(x) = 1$. In particular, $xy \in A \setminus \mathfrak{p}$.

Case (2): $x \notin B$. Since $(B, \mathfrak{m})$ is a Manis pair, there exists $y \in \mathfrak{m}$ such that $xy \in B \setminus \mathfrak{m}$. Therefore $\pi(xy) \in k^*$ and so there exists $z \in B$ such that $\pi(xyz) = 1$. In particular, $yz \in \mathfrak{m} = \ker(\pi) \subseteq \mathfrak{p}$ and $xyz \in A \setminus \mathfrak{p}$. $\square$

Corollary 2.30. Let $A \subseteq B$ be a ring extension and assume that there is a regular ideal $\mathfrak{i}$ of B contained in the conductor $(A :_A B)$. Set $R := A/\mathfrak{i}$ and $T := B/\mathfrak{i}$. Then A is a local ring with Prüfer condition (n) if and only if B is a local ring with Prüfer condition (n), R is a local Prüfer ring and T is an overring of R (see [15, Theorem 4.1] for the case $\mathfrak{i} = (A :_A B)$).

Proof. Assume that A is a local ring with Prüfer condition (n) (for $1 \leq n \leq 5$). We can argue in the same way of [15, Theorem 4.1]. By Lemma 2.26, B is a local ring with Prüfer condition (n). Since $\mathfrak{i}$ is a regular ideal and A is a Prüfer ring, R is a Prüfer ring (Remark 2.13). Since B is a localization of A (Lemma 2.26), T is an overring of R (see pag. 74).

For the other implication, if both B and R are local rings, then so is A (Proposition 2.18 (6)). Since B and R are Prüfer ring, Theorem 2.27 implies that A is a Prüfer ring. Moreover, since B has Prüfer condition (n), $\mathrm{Tot}(A)$ has Prüfer condition (n). By Theorem 2.11, A has Prüfer condition (n). $\square$

At this point, the role of the assumption that T is an overring of R in Theorem 2.27 should be underlined. The following two examples (provided both for the case of integral domains and the case of rings with zero divisors) show that this assumption can not be dropped in the "if-parts" of Theorem 2.27.

Example 2.31. Let X, Y be two indeterminates over a field k. Consider the following pullback diagram:

$$\begin{array}{ccc} k + Yk(X)[Y]_{(Y)} & \longrightarrow & k \\ \big\downarrow & & \big\downarrow \\ k(X)[Y]_{(Y)} & \xrightarrow{\pi} & k(X) \end{array}$$

Both k and $k(X)[Y]_{(Y)}$ are (local) Prüfer rings, $\ker(\pi)$ is clearly a regular ideal of $k(X)[Y]_{(Y)}$, but $A := k + Yk(X)[Y]_{(Y)}$ is not a Prüfer ring. Indeed, A is a local domain by Proposition 2.18 (6), but it is not a valuation domain, since X, X^{-1} are in the quotient field of A but none of them belongs to A.

Example 2.32. Let X be an indeterminate over $\mathbb{Q}$. Consider the following pullback diagram:

$$\begin{array}{ccc} \mathbb{Z}+X^2\mathbb{Q}[X] & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}+X\mathbb{Q}[X] & \longrightarrow & \frac{\mathbb{Z}+X\mathbb{Q}[X]}{(X^2)} \end{array}$$

Both $\mathbb{Z}+X\mathbb{Q}[X]$ and $\mathbb{Z}$ are Prüfer rings, the kernel of the bottom morphism is regular, but $\mathbb{Z}+X^2\mathbb{Q}[X]$ is not a Prüfer ring by Proposition 2.20 (see also below). Notice that condition (1) of Proposition 2.33 gives an easy way to see that $\mathbb{Z} \subseteq \frac{\mathbb{Z}+X\mathbb{Q}[X]}{(X^2)}$ is not a Prüfer extension. Indeed, for every element $\bar{f} \in Z(\frac{\mathbb{Z}+X\mathbb{Q}[X]}{(X^2)}) = \frac{X\mathbb{Q}[X]}{(X^2)}$, we have $\mathrm{Ann}_{\mathbb{Z}}(\bar{f}) = 0$. Notice that $\frac{\mathbb{Z}+X\mathbb{Q}[X]}{(X^2)} \cong \mathbb{Z}(+)\mathbb{Q}$ (more generally, given any ring extension $A \subseteq B$ and an indeterminate X over B, we have an isomorphism $\frac{A+XB[X]}{(X^2)} \cong A(+)B$).

On the other hand, in the notation of Theorem 2.27, assume that A is a Prüfer ring and that $\ker(f)$ is a regular ideal of B. As we have seen in the proof of Theorem 2.27, using the permanence properties of Prüfer extensions, we get that both $A \subseteq B$ and $R \subseteq T$ are Prüfer extensions (cf. Theorem 2.17). As far as the total quotient rings of R and T is concerned, we have the following proposition.

Proposition 2.33. Let $A \subseteq B$ be a Prüfer extension. Then:

1. $\mathrm{Ann}_A(x) \neq 0$ for every $x \in Z(B)$;

2. $\mathrm{Tot}(A) \subseteq \mathrm{Tot}(B)$;

3. if B is a Prüfer ring, then $\mathrm{Tot}(A)$ is Prüfer in $\mathrm{Tot}(B)$;

4. If $\mathrm{Tot}(A)$ is absolutely flat and B is a Prüfer ring, then $\mathrm{Tot}(A) = \mathrm{Tot}(B)$.

5. If A is a domain and B is a Prüfer ring, then $\mathrm{Tot}(A) = \mathrm{Tot}(B)$. In particular, B must be a domain.

Proof. (1) Let x be an element in $Z(B)$. Pick an element $z \in Z(B)$ such that $zx = 0$. By Theorem 2.16 (6), we can write $1 = a_1 + a_2 z$ for suitable

$a_1, a_2 \in (R : z)$. It follows that $x(1 - a_1) = 0$. If $1 - a_1 = 0$, then $z \in R$, otherwise $1 - a_1 \in \mathrm{Ann}_A(x)$.

(2) It suffices to prove that $\mathrm{Reg}(A) \subseteq \mathrm{Reg}(B)$. Let $r \in \mathrm{Reg}(A)$ and assume that there exists $x \in Z(B)$ such that $rx = 0$. As before, by Theorem 2.16, there exist $a_1, a_2 \in (A \colon x)$ such that $1 = a_1 + xa_2$, so that $r = ra_1$. Since r is a regular element of A, it follows that $a_1 = 1$, that is $x \in A$, a contradiction.

(3) By assumption, both $A \subseteq B$ and $B \subseteq \mathrm{Tot}(B)$ are Prüfer extensions, so A is Prüfer in $\mathrm{Tot}(B)$. From (2) we have$A \subseteq \mathrm{Tot}(A) \subseteq \mathrm{Tot}(B)$, hence $\mathrm{Tot}(A) \subseteq \mathrm{Tot}(B)$ is a Prüfer extension.

(4) By (3), $\mathrm{Tot}(A) \subseteq \mathrm{Tot}(B)$ is a Prüfer extension, so that, in particular, $\mathrm{Tot}(A) \hookrightarrow \mathrm{Tot}(B)$ is a (flat) epimorphism. But a non-trivial epimorphism from an absolutely flat ring into an arbitrary ring is necessarily an isomorphism.

(5) Immediately follows from (4). □

In view of these results, a question arises naturally: in the notation of Theorem 2.27, assume that $\ker(\pi)$ is a regular ideal of B and that A is a Prüfer ring. In which cases is it possible to deduce that T is an overring of R?

For instance, if A is a local Prüfer ring, then B is a localization of A (cf. Corollary 2.26) and therefore T is an overring of R (see page 74). Moreover, as we have seen in Proposition 2.33, we can deduce that T is an overring of R also if $\mathrm{Tot}(R)$ is an absolutely flat ring (so, in particular, if R is a domain). In this latter case, we have the following result.

Proposition 2.34. Let $\pi : B \to T$ be a surjective ring homomorphism, and let R be a subring of T. Assume that $\mathrm{Tot}(R)$ is an absolutely flat ring and that $\ker(\pi)$ is a regular ideal of B. Set $A := \pi^{-1}(R)$. Then, for $n = 1, \dots, 5$, A has Prüfer condition (n) if and only if both B and R have the same Prüfer condition (n) and T is an overring of R.

Proof. If A has Prüfer condition (n) for some $n = 1, \dots, 5$ (and hence A is a Prüfer ring), Proposition 2.33 ensures that T is an overring of R. Moreover, all five Prüfer conditions are equivalent on R, because $\mathrm{Tot}(R)$ is absolutely flat (cf. Theorem 2.11). So, we can conclude that both B and R have the same Prüfer condition of A.

For the other implication, apply Theorem 2.27. □

It is worth noting that Proposition 2.20 (restated below) is a particular case of our previous result. Therefore, we can extend the list of corollaries of Theorem 2.27.

Corollary 2.35. [69, Theorem 1.3] Let T be a domain and let $\mathfrak{i}$ be an ideal of T. Let D be a domain contained in $E := T/\mathfrak{i}$ and let $\pi : T \to E$ denote the canonical projection. Then $R := \pi^{-1}(D)$ is a Prüfer ring if and only if both D and T are Prüfer rings, $\mathfrak{i}$ is a prime ideal of T and D, E have the same field of fractions.

Proof. It suffices to notice that, since D is a domain, E is an overring of D if and only if $\mathfrak{i}$ is a prime ideal of T and D and E have the same field of fractions. □

As a further specialization, we have:

Corollary 2.36. (see [48, Theorem 2.4 (3)] for the local case) Let D be a domain and let $\mathfrak{p}$ be a prime ideal of D. Let E be a domain having $D/\mathfrak{p}$ as field of quotients. Set $D_1 := \pi^{-1}(E)$, where $\pi : D \to D/\mathfrak{p}$ is the canonical projection. Then D_1 is a Prüfer domain if and only if both D and E are Prüfer domains.

Using Theorem 2.27, it is also possible to deduce the following result of Boisen and Larsen.

Corollary 2.37. [11, Theorem 2] A Prüfer ring is the homomorphic image of a Prüfer domain if and only if its total quotient ring is the homomorphic image of a Prüfer domain.

In their article, Boisen e Larsen provide a very elegant proof of this fact. Here, we deduce the "if-part" from Theorem 2.27 and we include the proof given in [11] of the other implication for the sake of completeness.

Proof. Let R be a Prüfer ring with total quotient ring T. Assume that T is the homomorphic image of a Prüfer domain B under the ring morphism f. Then $\ker(f)$ is a regular ideal of B and Theorem 2.27 implies that $f^{-1}(R)$ is a Prüfer domain.

For the other implication, suppose that R is the homomorphic image of the Prüfer domain D under the homomorphism θ. Consider the multiplicative subset $S := \theta^{-1}(\mathrm{Reg}(R))$ of D and let ϕ be the extension of θ to $S^{-1}D$ obtained by $\phi(d/s) := \theta(d)/\theta(s)$. Then ϕ becomes a homomorphism from the Prüfer domain $S^{-1}D$ onto T. □

We conclude this section with another result concerning Prüfer conditions for a distinguished class of fiber products. We will make use of an elementary result stated in the following proposition. Since we are not able to provide an appropriate reference, we include the proof for convenience.

Proposition 2.38. Let A, B be rings and $n = 1, \ldots, 5$. Then, $A \times B$ has Prüfer condition (n) if and only if both A and B have the same Prüfer condition (n).

Proof. By [61, Proposition 3], $A \times B$ is a Prüfer ring if and only if both A and B are Prüfer rings.

If $A \times B$ is a Gaussian ring , then both A and B are Gaussian rings, because quotients of Gaussian rings are still Gaussian. The other implication immediately follows from the fact that if $h = \sum_{i=0}^{n}(a_i, b_i)X^i$ is a polynomial in $(A\times B)[X]$ and we set $f := \sum_{i=1}^{n} a_i X^i \in A[X]$ and $g := \sum_{i=1}^{n} b_i X^i \in B[X]$, then $c_{A\times B}(h) = c_A(f) \times c_B(g)$.

Assume that $A \times B$ is an arithmetical ring. Then, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, $(A \times B)_{\mathfrak{p}\times B} \cong A_{\mathfrak{p}}$ is a chained ring. It follows that A is an arithmetical ring. The same holds for B. On the other hand, if both A and B are arithmetical rings, take $\mathfrak{m} \in \mathrm{Spec}(A \times B)$. Since $\mathfrak{m} = \mathfrak{p} \times A$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$ or $\mathfrak{m} = A \times \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Spec}(B)$, $(A \times B)_{\mathfrak{m}} \cong A_{\mathfrak{p}}$ or $B_{\mathfrak{q}}$. In both cases, $(A \times B)_{\mathfrak{m}}$ is a chained ring, hence $A \times B$ is arithmetical.

In the same way it can be proved that $\mathrm{w.gl.dim}(A \times B) \leq 1$ if and only if $\mathrm{w.gl.dim}(A) \leq 1$ and $\mathrm{w.gl.dim}(B) \leq 1$.

To conclude, recall that a ring R is semihereditary if and only if R is coherent and $\mathrm{w.gl.dim}(R) \leq 1$ [63, Corollary 4.2.19], and that $A \times B$ is a coherent ring if and only if both A and B are coherent rings [63, Theorems 2.4.1 and 2.4.3]. □

Proposition 2.39. Let $f : A \to C$ and $g : B \to C$ be ring morphisms, and assume that $\ker(f), \ker(g)$ are regular ideals of A, B, respectively. Then the following conditions are equivalent:

1. $f \times_C g$ is a Prüfer ring
2. A and B are Prüfer rings and $f \times_C g = A \times B$.

Proof. (2) ⇒ (1) The product of Prüfer rings is a Prüfer ring (cf. Proposition 2.38).

(1) ⇒ (2). First, it is easy to verify that the conductor of the ring extension $D := f \times_C g \subseteq A \times B$ is $\mathfrak{c} := \ker(f) \times \ker(g)$. Assume, by contradiction, that $D \subsetneq A \times B$, that is, $\mathfrak{c}$ is a proper ideal of D, and let $\mathfrak{m}$ be a maximal ideal of D containing $\mathfrak{c}$. If S_A (resp., S_B) is the image of $D \setminus \mathfrak{m}$ under the natural morphism $D \to A$ (resp., $D \to B$), $S_C := f(S_A) = g(S_B)$ and $\widetilde{f} : A_{S_A} \to C_{S_C}$, $\widetilde{g} : B_{S_B} \to C_{S_C}$ are the morphisms induced by f, g, respectively, on the localizations, then $D_\mathfrak{m}$ is canonically isomorphic to $R := \widetilde{f} \times_{C_{S_C}} \widetilde{g}$, by [48, Proposition 1.9]. By assumptions there are regular elements $a \in \ker(f), b \in \ker(g)$ and, in particular, their images $a/1 \in A_{S_A}$ and $b/1 \in B_{S_B}$ are regular. It follows that the element $(a/1, b/1) \in R$ is regular. According, to Theorem 2.8, the ideals $(a/1, b/1)R, (a/1, 0)R$ of R are comparable. If there is $(\rho, \sigma) \in R$ such that $(a/1, b/1) = (a/1, 0)(\rho, \sigma)$, we have $b/1 = 0$ in B_{S_B}, in particular, against the fact that b is regular in B. On the other hand, if there is an element $(\eta, \zeta) \in R$ such that $(a/1, 0) = (a/1, b/1)(\eta, \zeta)$. Keeping in mind that $a/1, b/1$ are regular in A_{S_A}, B_{S_B}, respectively, it follows that $\eta = 1, \zeta = 0$ and, since $(\eta, \zeta) \in R$,

$$1 = \widetilde{f}(\eta) = \widetilde{g}(\zeta) = 0.$$

This is a contradiction because since $\mathfrak{c} \subseteq \mathfrak{m}$ we easily infer that $0 \notin S_C$. This proves that $f \times_C g = A \times B$ and, by (1), it is Prüfer. The conclusion follows from Proposition 2.38. □

Remark 2.40. Let $R \subseteq T$ be a ring extension with conductor Γ. Then R is isomorphic to the fiber product $\pi \times_{T/\Gamma} \iota$, where $\pi : T \to T/\Gamma$ is the canonical projection and $\iota : R/\Gamma \hookrightarrow T/\Gamma$ is the canonical embedding. So, R can be viewed as a subring of $(R/\Gamma) \times T$ and the conductor of the ring extension $R \cong \pi \times_{T/\Gamma} \iota \subseteq (R/\Gamma) \times T$ is $0 \times \Gamma$, which is never regular in $(R/\Gamma) \times T$.

Keeping in mind that Prüfer conditions (n) (for $1 \leq n \leq 5$) are preserved under finite products (cf. Proposition 2.38) the following consequence of Proposition 2.39 is clear.

Corollary 2.41. Preserve the notation and the assumptions of Proposition 2.39. Then the following conditions are equivalent.

1. $f \times_C g$ has Prüfer condition (n).
2. A, B have Prüfer condition (n) and $f \times_C g = A \times B$.

2.5 Bi-amalgamated algebras

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be ring homomorphisms and let $\mathfrak{b}$ and $\mathfrak{c}$ be ideal of B and C respectively, satisfying $f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$. As we have seen in Section 2.2.3, Kabbaj, Louartiti and Tamekkante [73] defined and studied the following subring

$$A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) := \{(f(a) + b, g(a) + c) \mid a \in A, b \in \mathfrak{b}, c \in \mathfrak{c}\}$$

of $B \times C$, called the *bi-amalgamation of* A *with* (B, C) *along* $(\mathfrak{b}, \mathfrak{c})$*, with respect to* (f, g).

We start by recalling some basics on bi-amalgamated algebras. Other facts concerning Noetherian bi-amalgamated algebras can be found in [73] and in a joint work with Finocchiaro [22].

Proposition 2.42. [73] We preserve the notation at the beginning of this section. Set $\mathfrak{i}_0 := f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$.

1. Consider the ring homomorphisms $\alpha : f(A)+\mathfrak{b} \to A/\mathfrak{i}_0$, $f(a)+b \mapsto a+\mathfrak{i}_0$ and $\beta : g(A) + \mathfrak{c} \to A/\mathfrak{i}_0$, $g(a) + c \mapsto a + \mathfrak{i}_0$. Then the bi-amalgamation

is determined by the following pullback diagram;

$$\begin{array}{ccc} A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) & \longrightarrow & f(A) + \mathfrak{b} \\ \downarrow & & \downarrow \alpha \\ g(A) + \mathfrak{c} & \xrightarrow{\beta} & A/\mathfrak{i}_0, \end{array}$$

2. Consider the following ring homomorphisms

$$\iota_1 : A/\mathfrak{i}_0 \to \frac{f(A) + \mathfrak{b}}{\mathfrak{b}} \times \frac{f(A) + \mathfrak{c}}{\mathfrak{c}}, \qquad a + \mathfrak{i}_0 \mapsto (f(a) + \mathfrak{b}, g(a) + \mathfrak{c}),$$

and

$$\mu_2 : A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \to A/\mathfrak{i}_0, \qquad (f(a) + b, g(a) + c) \mapsto a + \mathfrak{i}_0.$$

Then, the diagram

$$\begin{array}{ccc} A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) & \xrightarrow{\mu_2} & A/\mathfrak{i}_0 \\ \downarrow \iota_2 & & \downarrow \iota_1 \\ (f(A) + \mathfrak{b}) \times (g(A) + \mathfrak{c}) & \xrightarrow{\mu_1} & \frac{f(A)+\mathfrak{b}}{\mathfrak{b}} \times \frac{f(A)+\mathfrak{c}}{\mathfrak{c}}, \end{array}$$

is a conductor square with conductor $\mathfrak{b} \times \mathfrak{c}$. Here, ι_2 and μ_1 are the natural embedding and the canonical surjection respectively.

3. $0 \times \mathfrak{c}$, $\mathfrak{b} \times 0$ and $\mathfrak{b} \times \mathfrak{c}$ are ideals of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. If $\mathfrak{a}$ is an ideal of A, the set

$$\mathfrak{a} \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) := \{(f(a) + b, g(a) + c \mid a \in \mathfrak{a}, b \in \mathfrak{b} \text{ and } c \in \mathfrak{c}\}$$

is an ideal of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. Moreover, we have the following canonical ring isomorphisms:

(a)

$$\frac{A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})}{\mathfrak{a} \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})} \cong \frac{A}{\mathfrak{a} + \mathfrak{i}_0}$$

(b)

$$\frac{A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})}{0 \times \mathfrak{c}} \cong f(A) + \mathfrak{b} \quad \text{and} \quad \frac{A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})}{\mathfrak{b} \times 0} \cong g(A) + \mathfrak{c}$$

(c)
$$\frac{A}{\mathfrak{i}_0} \cong \frac{A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})}{\mathfrak{b}\times\mathfrak{c}} \cong \frac{f(A)+\mathfrak{b}}{\mathfrak{b}} \cong \frac{g(A)+\mathfrak{c}}{\mathfrak{c}}$$

4. Let $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{i}_0$. Consider the multiplicative subsets $S_\mathfrak{p} := f(A \setminus \mathfrak{p}) + \mathfrak{b}$ of B and $T_\mathfrak{p} := g(A \setminus \mathfrak{p}) + \mathfrak{c}$ of C. Let $f_\mathfrak{p} : A_\mathfrak{p} \longrightarrow B_{S_\mathfrak{p}}$ and $g_\mathfrak{p} : A_\mathfrak{p} \longrightarrow C_{T_\mathfrak{p}}$ be the ring homomorphisms induced by f and g. Then $f_\mathfrak{p}^{-1}(\mathfrak{b}B_{S_\mathfrak{p}}) = g_\mathfrak{p}^{-1}(\mathfrak{c}C_{T_\mathfrak{p}}) = \mathfrak{i}_0 A_\mathfrak{p}$ and
$$A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})_{(\mathfrak{p}\bowtie^{f,g}(\mathfrak{b},\mathfrak{c}))} \cong A_\mathfrak{p} \bowtie^{f_\mathfrak{p},g_\mathfrak{p}} (\mathfrak{b}B_{S_\mathfrak{p}}, \mathfrak{c}C_{T_\mathfrak{p}}).$$

From now on, all results contained in this section (and in the next two subsections) are based on a joint work with Finocchiaro [22]. First of all, notice that there is another way to present the bi-amalgamated construction in terms of fiber products. Indeed, we have the following pullback diagram:

$$\begin{array}{ccc} A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c}) & \longrightarrow & A/\mathfrak{i}_0 \\ \downarrow & & \downarrow{\scriptstyle i_{f,g}} \\ B \times C & \xrightarrow{\pi} & \frac{B}{\mathfrak{b}} \times \frac{C}{\mathfrak{c}}, \end{array}$$

where $\pi : B \times C \longrightarrow B/\mathfrak{b} \times C/\mathfrak{c}$ is the projection and the vertical arrow $i := i_{f,g} : A/\mathfrak{i}_0 \longrightarrow B/\mathfrak{b} \times C/\mathfrak{c}$ is the canonical ring embedding defined by setting $i_{f,g}(a + \mathfrak{i}_0) := (f(a) + \mathfrak{b}, g(a) + \mathfrak{c})$, for any $a + \mathfrak{i}_0 \in A/\mathfrak{i}_0$.

In the following remark we summarize some basic properties of any bi-amalgamation, while Proposition 2.44 below provides a complete description of the prime spectrum of $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})$.

Remark 2.43. We preserve the notations at the beginning of this section.

1. If $\mathfrak{j}$ and $\mathfrak{j}'$ are ideals of B and C respectively, then the sets
$$\mathfrak{j}^{\sharp_B} := \{(f(a)+b,\ g(a)+c) \mid a \in A,\ (b,c) \in \mathfrak{b}\times\mathfrak{c}, f(a)+b \in \mathfrak{j}\}$$
and
$$\mathfrak{j}'^{\sharp_C} := \{(f(a)+b,\ g(a)+c) \mid a \in A,\ (b,c) \in \mathfrak{b}\times\mathfrak{c}, g(a)+c \in \mathfrak{j}'\}$$
are ideals of $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})$, since they are the contractions of the ideals

$\mathfrak{j} \times C$ and $B \times \mathfrak{j}'$, respectively, of $B \times C$ in $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$.

2. Set $\mathfrak{k} := \mathfrak{k}_{f,g} := \ker(f) \cap \ker(g)$. f and g induce a natural ring embedding $\iota := \iota_{f,g} : A/\mathfrak{k} \longrightarrow A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. It is defined by setting $\iota(a + \mathfrak{k}) := (f(a), g(a))$, for any $a + \mathfrak{k} \in A/\mathfrak{k}$.

3. If $\mathfrak{b}$ and $\mathfrak{c}$ are finitely generated A-modules (where the A-module structures are defined via f, g, respectively) then the ring embedding $\iota : A/\mathfrak{k} \longrightarrow A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is finite. As a matter of fact, if $\{b_1, \ldots, b_n\}$ (resp., $\{c_1, \ldots, c_m\}$) is a set of generators of $\mathfrak{b}$ (resp., $\mathfrak{c}$) as an A-module,

$$\{(1,1), (b_i, 0), (0, c_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

is a set of generators of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ as a $A/\mathfrak{k}$-module.

4. If f and g are finite ring homomorphisms, then the ring extension $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \subseteq B \times C$ is finite. Indeed, if $\{x_1, \ldots, x_n\}$ (resp., $\{y_1, \ldots, y_m\}$) is a set of generators of B (resp., C) as an A-module, then

$$\{(x_i, 0), (0, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$$

is a set of generators of $B \times C$ as a $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$-module.

5. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two ideals of A containing $\mathfrak{i}_0$ such that $\mathfrak{a}_1 \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \subseteq \mathfrak{a}_2 \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$, then $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$. Indeed, let x be an element of $\mathfrak{a}_1$. Since $(f(x), g(x)) \in \mathfrak{a}_1 \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \subseteq \mathfrak{a}_2 \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$, there exist elements $y \in \mathfrak{a}_2$, $b \in \mathfrak{b}$ and $c \in \mathfrak{c}$ such that $(f(x), g(x)) = (f(y)+b, g(y)+c)$. In particular, $x - y \in \mathfrak{i}_0 \subseteq \mathfrak{a}_2$, which implies that $x \in \mathfrak{a}_2$.

Proposition 2.44. The following statements hold.

1. The canonical surjective ring homomorphism

$$p : A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \longrightarrow A/\mathfrak{i}_0, \qquad (f(a) + b, g(a) + c) \mapsto a + \mathfrak{i}_0$$

induces the closed topological embedding

$$p^\star : V(\mathfrak{i}_0) \longrightarrow \mathrm{Spec}(A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})), \qquad \mathfrak{p} \mapsto \mathfrak{p} \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$$

establishing a homeomorphim between $V(\mathfrak{i}_0)$ and the image $p^\star(V(\mathfrak{i}_0)) = V(\mathfrak{b} \times \mathfrak{c}) \subseteq \mathrm{Spec}(A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}))$.

2. The inclusion $i : A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \longrightarrow B \times C$ induces a continuous map $i^\star : \mathrm{Spec}(B \times C) \longrightarrow \mathrm{Spec}(A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}))$, defined by contraction. The mapping $i^\star$ induces by restriction a homeomorphism

$$\mathrm{Spec}(B \times C) \setminus V(\mathfrak{b} \times \mathfrak{c}) \longrightarrow \mathrm{Spec}(A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})) \setminus V(\mathfrak{b} \times \mathfrak{c})$$

defined by

$$\mathfrak{q} \times C \mapsto \mathfrak{q}^{\sharp_B}, \qquad B \times \mathfrak{q}' \mapsto \mathfrak{q}'^{\sharp_C}$$

for any $\mathfrak{q} \in \mathrm{Spec}(B) \setminus V(\mathfrak{b})$ and $\mathfrak{q}' \in \mathrm{Spec}(C) \setminus V(\mathfrak{c})$ (here, $\mathfrak{q}^{\sharp_B}$ and $\mathfrak{q}'^{\sharp_C}$ are defined as in Remark 2.43 (1)).

3. The homeomorphisms defined in (1) and (2) preserve maximality.

4. $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a local ring if and only if $A/\mathfrak{i}_0$ is a local ring, $\mathfrak{b} \subseteq \mathrm{Jac}(B)$ and $\mathfrak{c} \subseteq \mathrm{Jac}(C)$.

Proof. It is an immediate consequence of Proposition 2.18. □

Arithmetical properties of bi-amalgamations were studied by Kabbaj, Mahdou and Moutui [74]. In what follows we investigate the problem of when a bi-amalgamation is a Prüfer or a Gaussian ring.

Prüfer bi-amalgamations

We start this subsection considering the case in which both $\mathfrak{b}$ and $\mathfrak{c}$ are regular ideals. In [22] a direct proof of the following fact is given for Prüfer rings. Here, we deduce this result from Proposition 2.39, including all five Prüfer-like conditions.

Proposition 2.45. Assume that $\mathfrak{b}$ and $\mathfrak{c}$ are regular ideals. Then, for $n = 1, \ldots, 5$ the following conditions are equivalent:

1. $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ has Prüfer condition (n);
2. B, C are rings having Prüfer condition (n) and $\mathfrak{b} = B$.

Proof. Fix $n = 1, \ldots, n$. First notice that $\mathfrak{b} = B$ if and only if $\mathfrak{c} = C$. In particular, if $\mathfrak{b} = B$ and B, C have Prüfer condition (n), then $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \cong$

$B \times C$ has the same Prüfer condition (n) and the implication $(2) \Rightarrow (1)$ follows from Proposition 2.38.

Now assume that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a ring having Prüfer condition (n). Observe that the conductor of the ring extension $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) \subseteq (f(A)+\mathfrak{b}) \times (g(A)+\mathfrak{c})$ is $\mathfrak{b} \times \mathfrak{c}$. Applying Proposition 2.39 to the pullback diagram

$$\begin{array}{ccc} A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) & \longrightarrow & f(A) + \mathfrak{b} \\ \downarrow & & \downarrow \\ g(A) + \mathfrak{c} & \longrightarrow & A/\mathfrak{i}_0, \end{array}$$

we get that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c}) = (f(A) + \mathfrak{b}) \times (g(A) + \mathfrak{c})$. In particular, we deduce that $\mathfrak{b} = B = f(A) + \mathfrak{b}$ and $\mathfrak{c} = C = g(A) + \mathfrak{c}$. Apply Proposition 2.38 to conclude. ☐

Corollary 2.46. [46, Theorem 3.1] Let $f : A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{b}$ be a regular ideal of B such that $f^{-1}(\mathfrak{b})$ is a regular ideal of A. Then $A \bowtie^{f} \mathfrak{b}$ is a Prüfer ring if and only if A, B are Prüfer rings and $\mathfrak{b} = B$.

Proof. Apply Proposition 2.45, keeping in mind the representation of $A \bowtie^{f} \mathfrak{b}$ as a bi-amalgamation (see [73, Example 2.1]). ☐

We have already seen that a precise answer of when $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring can be given if both $\mathfrak{b}$ and $\mathfrak{c}$ are regular ideals. The general case is much more difficult, in a way not so different from that of the arithmetical case (see [74]). Our first goal is to provide necessary conditions under some "regular properties" for f and g.

Let $\pi : A \longrightarrow A/\mathfrak{i}_0$ be the canonical projection. Consider the following properties:

$$f(\pi^{-1}(\mathrm{Reg}(A/\mathfrak{i}_0))) \subseteq \mathrm{Reg}(B) \text{ and } g(\pi^{-1}(\mathrm{Reg}(A/\mathfrak{i}_0))) \subseteq \mathrm{Reg}(C) \qquad (*)$$

and

$$f(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(B) \text{ and } g(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(C). \qquad (**)$$

Proposition 2.47. Assume that condition (*) holds. If $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring, then so is $A/\mathfrak{i}_0$.

Proof. During the proof, we denote $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ and $A/\mathfrak{i}_0$ simply by R and R_0 respectively. Let $\mathfrak{h} = (x_0 + \mathfrak{i}_0, x_1 + \mathfrak{i}_0, \dots, x_n + \mathfrak{i}_0)$ be a regular finitely generated ideal of R_0. Set $\eta_i := (f(x_i), g(x_i)) \in R$ for every $i = 0, \dots, n$. Then condition (*) implies that $\mathfrak{h}' := (\eta_0, \eta_1, \dots, \eta_n)$ is a regular finitely generated ideal of R. In particular the polynomial $P(T) := \sum_{i=0}^{n} \eta_i T^i \in R[T]$ is a Gauss polynomial. Our aim is to show that the polynomial $F(T) := \sum_{i=0}^{n}(x_i + \mathfrak{i}_0)T^i \in R_0[T]$ is also a Gauss polynomial. Let $G(T) = \sum_{j=0}^{m}(y_i + \mathfrak{i}_0)T^j$ be any other polynomial in $R_0[T]$. It suffices to prove that $c_{R_0}(F)c_{R_0}(G) \subseteq c_{R_0}(FG)$. If $a + \mathfrak{i}_0 \in c_{R_0}(F)c_{R_0}(G)$ and $Q(T) := \sum_{j=0}^{m}(f(y_j), g(y_j))T^j \in R[T]$, then $(f(a) + b, g(a) + c) \in c_R(P)c_R(Q) = c_R(PQ)$ for suitable elements $b \in \mathfrak{b}$ and $c \in \mathfrak{c}$. Looking only at the first coordinate of R, we get that there exist $a_0, \dots, a_{n+m} \in A$ and $b_0, \dots, b_{n+m} \in \mathfrak{b}$ such that

$$f(a) + b = \sum_{k=0}^{n+m} \left((f(a_k) + b_k) \sum_{i+j=k} f(x_i y_j) \right),$$

so, in particular,

$$f(a) - \sum_{k=0}^{n+m} \left(f(a_k) \sum_{i+j=k} f(x_i y_j) \right) \in \mathfrak{b}.$$

It means that $a - \sum_{k=0}^{n+m} \left(\sum_{i+j=k} x_i y_j \right) a_k \in \mathfrak{i}_0$, which implies that $a + \mathfrak{i}_0 \in c_{R_0}(FG)$. □

Proposition 2.48. Assume that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring, let $\mathfrak{a}$ be an ideal of A and let $p : A \longrightarrow A/\mathfrak{a}$ be the canonical projection. If $f(p^{-1}(\mathrm{Reg}(A/\mathfrak{a}))) \subseteq \mathrm{Reg}(B)$ and $g(p^{-1}(\mathrm{Reg}(A/\mathfrak{a}))) \subseteq \mathrm{Reg}(C)$, then $A/\mathfrak{a}$ is a Prüfer ring whenever one of the following conditions holds:

1. $\mathfrak{i}_0 \subseteq \mathfrak{a}$;
2. g is surjective and $\ker(g) \subseteq \mathfrak{a}$;
3. $\mathfrak{c} \subseteq g(A)$ and $\ker(g) \subseteq \mathfrak{a}$.

Proof. It suffices to notice that the proof of Theorem 2.47 can be easily adapted in all three cases of the statement. □

Corollary 2.49. [46, Proposition 4.2] Let $f : A \longrightarrow B$ be a ring homomorphism such that $f(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(B)$ and let $\mathfrak{b}$ be an ideal of B such that $A \bowtie^f \mathfrak{b}$ is a Prüfer ring. Then A is a Prüfer ring.

Proof. Apply case (2) of Proposition 2.48 to $g := \mathrm{Id}_A : A \longrightarrow A$, $\mathfrak{a} := 0$ and $\mathfrak{c} := \mathfrak{i}_0$. Then such a bi-amalgamation is $A \bowtie^f \mathfrak{b}$. □

Example 2.50. Let $\mathbf{k}$ be a field and set $A = \mathbf{k}[X,Y]$, $\mathfrak{i}_0 = (Y)$, $B = C = A/\mathfrak{i}_0$, $\mathfrak{b} = \mathfrak{c} = 0$ and $f = g = \pi : A \longrightarrow A/\mathfrak{i}_0$. It is easy to see that condition (*) holds, $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c}) \cong A/\mathfrak{i}_0$ is a Prüfer ring, but A is not a Prüfer ring.

Proposition 2.51. Assume that condition (**) holds and preserve the notation of Remark 2.42 (4). If $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})$ is a Prüfer ring, then $\mathfrak{b}B_{S_\mathfrak{m}} = f_\mathfrak{m}(\frac{r}{1})\mathfrak{b}B_{S_\mathfrak{m}}$ and $\mathfrak{c}C_{T_\mathfrak{m}} = g_\mathfrak{m}(\frac{r}{1})\mathfrak{c}C_{T_\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(A) \cap V(\mathfrak{i}_0)$ and every regular element $r \in A$.

Proof. Fix a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{i}_0$ and a regular element r of A. Set $\sigma = r/1 \in A_\mathfrak{m}$. Condition (**) implies that $(f_\mathfrak{m}(\sigma), g_\mathfrak{m}(\sigma))$ is a regular element of the local ring $\hat{R} := A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})_{(\mathfrak{m}\bowtie^{f,g}(\mathfrak{b},\mathfrak{c}))}$. Fix an element $\tau \in \mathfrak{b}B_{S_\mathfrak{m}}$. By hypothesis, $(A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c}), \mathfrak{m} \bowtie^{f,g} (\mathfrak{b},\mathfrak{c}))$ has the regular total order property, so that, in particular, the principal ideals of $\hat{R}$ generated respectively by $(f_\mathfrak{m}(\sigma), g_\mathfrak{m}(\sigma))$ and $(\tau, 0)$ are comparable. Using the fact that $g_\mathfrak{m}(\sigma) \neq 0$, it is immediate to check that the inclusion $(\tau,0)\hat{R} \subseteq (f_\mathfrak{m}(\sigma), g_\mathfrak{m}(\sigma))\hat{R}$ holds. Thus there exist elements $\alpha \in A_\mathfrak{m}, \beta \in \mathfrak{b}B_{S_\mathfrak{m}}$ and $\gamma \in \mathfrak{c}C_{T_\mathfrak{m}}$ such that

$$(\tau, 0) = (f_\mathfrak{m}(\sigma), g_\mathfrak{m}(\sigma))(f_\mathfrak{m}(\alpha) + \beta, g_\mathfrak{m}(\alpha) + \gamma).$$

From $g_\mathfrak{m}(\sigma)(g_\mathfrak{m}(\alpha) + \gamma) = 0$ we get $g_\mathfrak{m}(\alpha) + \gamma = 0$, and so $\alpha \in \mathfrak{i}_0 A_\mathfrak{m}$. In particular, $\tau = f_\mathfrak{m}(\sigma)(f_\mathfrak{m}(\alpha) + \beta) \in f_\mathfrak{m}(\sigma)\mathfrak{b}B_{S_\mathfrak{m}}$. It follows that $\mathfrak{b}B_{S_\mathfrak{m}} = f_\mathfrak{m}(\frac{r}{1})\mathfrak{b}B_{S_\mathfrak{m}}$. The equality $\mathfrak{c}C_{T_\mathfrak{m}} = g_\mathfrak{m}(\frac{r}{1})\mathfrak{c}C_{T_\mathfrak{m}}$ can be proved similarly. □

Remark 2.52. There is an analogous of the previous proposition in the situation in which condition (*) holds. In this case, if $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})$ is a Prüfer ring, then $\mathfrak{b}B_{S_\mathfrak{m}} = f_\mathfrak{m}(\frac{r}{1})\mathfrak{b}B_{S_\mathfrak{m}}$ and $\mathfrak{c}C_{T_\mathfrak{m}} = g_\mathfrak{m}(\frac{r}{1})\mathfrak{c}C_{T_\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(A) \cap V(\mathfrak{i}_0)$ and every element $r \in A$ such that $r + \mathfrak{i}_0$ is a regular element in $A/\mathfrak{i}_0$. The proof works also in this case, since also condition (*) implies that the element $(f_\mathfrak{m}(\sigma), g_\mathfrak{m}(\sigma))$ is regular in $A \bowtie^{f,g} (\mathfrak{b},\mathfrak{c})_{(\mathfrak{m}\bowtie^{f,g}(\mathfrak{b},\mathfrak{c}))}$.

Proposition 2.53. Let $A/\mathfrak{k}$ be a total ring of fractions, where as usual $\mathfrak{k} := \ker(f) \cap \ker(g) \subseteq \mathfrak{i}_0$, and assume that $\mathfrak{b} \subseteq \mathrm{Jac}(B)$ and $\mathfrak{c} \subseteq \mathrm{Jac}(C)$. If both $\mathfrak{b}$ and $\mathfrak{c}$ are torsion $A/\mathfrak{k}$-modules (with the $A/\mathfrak{k}$-module structure inherited by f and g respectively), then $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a total ring of fractions.

Proof. Let $(f(a) + b, g(a) + c) \in A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ be a non-invertible element. We have to show that $(f(a) + b, g(a) + c)$ is a zero-divisor. Since $\mathfrak{b} \subseteq \mathrm{Jac}(B)$ and $\mathfrak{c} \subseteq \mathrm{Jac}(C)$, the maximal ideals of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ are exactly those of the form $\mathfrak{p} \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$, where $\mathfrak{p} \in \mathrm{Max}(A) \cap V(\mathfrak{i}_0)$. Hence, there exists a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{i}_0$ such that $(f(a) + b, g(a) + c) \in \mathfrak{m} \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. In particular, $a \in \mathfrak{m}$ is a zero-divisor modulo $\mathfrak{k}$, so there exists $a' \notin \mathfrak{k}$ such that $aa' \in \mathfrak{k}$. Since both $\mathfrak{b}$ and $\mathfrak{c}$ are torsion $A/\mathfrak{k}$-modules, there exist two regular elements $x_0 + \mathfrak{k}$ and $y_0 + \mathfrak{k}$ in $A/\mathfrak{k}$ such that $f(x_0)b = 0$ and $g(y_0)c = 0$. Of course, $a'x_0y_0 \notin \mathfrak{k}$, and so $(f(a'x_0y_0), g(a'x_0y_0))$ is a non-zero element such that $(f(a) + b, g(a) + c)(f(a'x_0y_0), g(a'x_0y_0)) = (0, 0)$. □

Lemma 2.54. Assume that $(f(a)+b, g(a)+c)$ is a zero-divisor of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. Then, at least one of the following conditions hold:

1. $a + \mathfrak{i}_0$ is a zero-divisor of $A/\mathfrak{i}_0$;
2. there exists $(b', c') \in \mathfrak{b} \times \mathfrak{c}$, with $(b', c') \neq (0, 0)$, such that $b'(f(a)+b) = 0$ and $c'(g(a) + c) = 0$.

Proof. Assume that $(f(a) + b, g(a) + c)(f(a') + x, g(a') + y) = 0$ for some non-zero element $(f(a')+x, g(a')+y)$ of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. Then $(f(a)+b)(f(a')+x) = 0$ implies that $aa' \in \mathfrak{i}_0$. If $a' \notin \mathfrak{i}_0$, then $a + \mathfrak{i}_0$ is a zero-divisor of $A/\mathfrak{i}_0$. Otherwise, $f(a') + x \in \mathfrak{b}$, $g(a') + y \in \mathfrak{c}$ and at least one of them is not zero. □

Notice that condition (2) of the previous lemma always implies that $(f(a) + b, g(a) + c)$ is a zero-divisor of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. If also condition (1) implies that $(f(a) + b, g(a) + c)$ is a zero-divisor of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$, we say that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ has the condition $(\star)$. So, $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ has condition $(\star)$ if, whenever $(f(a) + b, g(a) + c)$ is an element of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ and $a + \mathfrak{i}_0$ is a zero-divisor of $A/\mathfrak{i}_0$, then $(f(a) + b, g(a) + c)$ is a zero-divisor of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$.

Example 2.55. Assume that B and C are local total quotient rings with maximal ideals $\mathfrak{b}$ and $\mathfrak{c}$, respectively. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be ring homomorphisms such that $\mathfrak{i}_0 = f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$. Then condition $(\star)$ holds.

Theorem 2.56. (1) Assume that condition (*) holds. If $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring, then $A/\mathfrak{i}_0$ is a Prüfer ring and $\mathfrak{b}B_{S_\mathfrak{m}} = f_\mathfrak{m}(\frac{r}{1})\mathfrak{b}B_{S_\mathfrak{m}}$ and $\mathfrak{c}C_{T_\mathfrak{m}} = g_\mathfrak{m}(\frac{r}{1})\mathfrak{c}C_{T_\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(A) \cap V(\mathfrak{i}_0)$ and every element $r \in A$ such that $r + \mathfrak{i}_0$ is a regular element in $A/\mathfrak{i}_0$.

(2) Assume that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a local ring having condition $(\star)$. Let $\mathfrak{m}$ be the unique maximal ideal of A containing $\mathfrak{i}_0$. If $A/\mathfrak{i}_0$ is a Prüfer ring and $\mathfrak{b} = f(r)\mathfrak{b}$ and $\mathfrak{c} = g(r)\mathfrak{c}$ for every $r \in \pi^{-1}(\mathrm{Reg}(A/\mathfrak{i}_0))$, then $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Prüfer ring.

Proof. Part (1) follows from Proposition 2.47 and Remark 2.52.

(2) Let $(f(a_1) + b_1, g(a_1) + c_1), (f(a_2) + b_2, g(a_2) + c_2) \in A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ and assume that $(f(a_1)+b_1, g(a_1)+c_1)$ is a regular element. Condition $(\star)$ implies that $a_1 + \mathfrak{i}_0$ is a regular element of the Prüfer local ring $A/\mathfrak{i}_0$, and so the principal ideals of $A/\mathfrak{i}_0$ generated by $a_1 + \mathfrak{i}_0$ and $a_2 + \mathfrak{i}_0$ are comparable. There are two cases.

Case 1. There exist $x \in A$ and $u \in \mathfrak{i}_0$ such that $a_2 = a_1 x + u$. We can write $b_1 = f(a_1)b'_1$ for some $b'_1 \in \mathfrak{b}$. Notice that under our hypothesis $\mathfrak{b} \subseteq \mathrm{Jac}(B)$, so in particular, $1 + b'_1$ is an invertible element of B. It follows that we can also find $\beta \in \mathfrak{b}$ such that

$$f(a_1)\beta = \frac{b_2 + f(u) - b_1 f(x)}{1 + b'_1}.$$

Elements c'_1 and γ in $\mathfrak{c}$ can be defined in an analogous way, getting

$$g(a_1)\gamma = \frac{c_2 + g(u) - c_1 g(x)}{1 + c'_1}.$$

It is now straightforward to show that $(f(a_2) + b_2, g(a_2) + c_2) = (f(a_1) + b_1, g(a_1) + c_1)(f(x) + \beta, g(x) + \gamma)$, that is, the principal ideals of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ generated by $(f(a_1)+b_1, g(a_1)+c_1)$ and $(f(a_2)+b_2, g(a_2)+c_2)$ are comparable.

Case 2. There exist $x \in A$ and $u \in \mathfrak{i}_0$ such that $a_1 = a_2 x + u$. Notice that

$\mathfrak{b} = f(a_2x)\mathfrak{b}$ and $\mathfrak{c} = g(a_2x)\mathfrak{c}$, because also $a_2x+\mathfrak{i}_0$ is a regular element of $A/\mathfrak{i}_0$. As before, we can find elements $b_2' \in \mathfrak{b}$ and $c_2' \in \mathfrak{c}$ such that $b_2 = f(a_2x)b_2'$, $c_2 = g(a_2x)c_2'$. Moreover, there exist $\beta \in \mathfrak{b}$ and $\gamma \in \mathfrak{c}$ such that

$$f(a_2x)\beta = \frac{b_1 + f(u) - b_2f(x)}{1 + f(x)b_2'}$$

and

$$g(a_2x)\gamma = \frac{c_1 + g(u) - c_2g(x)}{1 + g(x)c_2'}.$$

Now, we can conclude noting that the following equality holds:

$$(f(a_1)+b_1, g(a_1)+c_1) = (f(a_2)+b_2, g(a_2)+c_2)(f(x)+\beta f(x), g(x)+\gamma g(x)).$$

□

Gaussian bi-amalgamations

We are going to study the transfer of Gaussian condition to bi-amalgamations in the local case. We will largely use the following characterization of local Gaussian rings.

Theorem 2.57. [93, Theorem 2.2] A local ring S is Gaussian if and only if for every two elements $x, y \in S$ the following two conditions hold:

(i) $(x, y)^2 = (x^2)$ or (y^2);

(ii) if $(x, y)^2 = (x^2)$ and $xy = 0$, then $y^2 = 0$.

It will be useful to observe that an ideal $\mathfrak{a}$ of a Gaussian local ring is nilpotent of index 2 if and only if so are all its elements. To be more precise, we have the following

Lemma 2.58. Let $\mathfrak{a}$ be an ideal of a Gaussian local ring S. Then $\mathfrak{a}^2 = 0$ if and only if $a^2 = 0$ for every $a \in \mathfrak{a}$.

Proof. For every $x, y \in \mathfrak{a}$, the ideal $(x, y)^2$ is equal to (x^2) or (y^2) (cf. Theorem 2.57). If $a^2 = 0$ for every $a \in \mathfrak{a}$, we have $(xy) = (x, y)^2 = 0$, which implies that $xy = 0$ and so $\mathfrak{a}^2 = 0$. □

Theorem 2.59. Assume that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Gaussian local ring. Then:

1. $A/\mathfrak{i}_0$, $f(A) + \mathfrak{b}$ and $g(A) + \mathfrak{c}$ are Gaussian local rings;
2. if $\mathfrak{b}^2 \neq 0$, then $\mathfrak{c}^2 = 0$;
3. if $\mathfrak{b}^2 = 0$, then $f(a)\mathfrak{b} \subseteq f(a^2)B$ for every $a \in A$.

Proof. (1) immediately follows from the fact that quotients of a Gaussian ring are Gaussian rings.

(2) Assume that $\mathfrak{b}^2 \neq 0$. Since $\mathfrak{b}$ is an ideal of $f(A)+\mathfrak{b}$, by Lemma 2.58 there exists an element $b \in \mathfrak{b}$ such that $b^2 \neq 0$. For any element $c \in \mathfrak{c}$ we have that the ideal $((b, 0), (0, c))^2 = ((b^2, 0), (0, c^2))$ of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ must be equal to $((b^2, 0))$ or $((0, c^2))$. Since $b^2 \neq 0$, $((b^2, 0), (0, c^2)) = ((b^2, 0))$ must be the case, which implies that $c^2 = 0$. We can conclude applying Lemma 2.58.

(3) Assume that $\mathfrak{b}^2 = 0$. For every $a \in A$ and $b \in \mathfrak{b}$ we have that the ideal $((f(a), g(a)), (b, 0))^2$ is equal to 0 or $(f(a^2), g(a^2))$. In the first case $f(a)b = 0$, while in the second one, there exist $\alpha \in A$, $\beta \in \mathfrak{b}$ and $\gamma \in \mathfrak{c}$ such that $f(a)b = f(a^2)(f(\alpha) + \beta)$ and $0 = g(a^2)(g(\alpha) + \gamma)$. In both cases $f(a)b \in f(a^2)B$. □

The following example shows that the converse of Theorem 2.59 does not hold.

Example 2.60. Let p be a prime number, set $A := \mathbb{Z}_{(p)}$ and $B := \mathbb{Z}_{(p)}/p^4\mathbb{Z}_{(p)}$, let $f : A \longrightarrow B$ be the canonical projection and let $\mathfrak{b}$ be the principal ideal of B generated by the class of p^2. By [73, Example 2.1], the bi-amalgamation $A \bowtie^{f,\mathrm{Id}_A} (\mathfrak{b}, \mathfrak{i}_0)$ is the standard amalgamation $A \bowtie^{f} \mathfrak{b}$ and, since $f(p)\mathfrak{b} \neq f(p^2)\mathfrak{b}$, it is not Gaussian, by [4, Theorem 4.1], but all the conditions of Theorem 2.59 are trivially satisfied.

Theorem 2.61. Assume that f and g are surjective. Then $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Gaussian local ring whenever the following conditions hold:

1. A is a Gaussian local ring;
2. $\mathfrak{b}^2 = \mathfrak{c}^2 = 0$;

3. $f(a)\mathfrak{b} = f(a^2)\mathfrak{b}$ and $g(a)\mathfrak{c} = g(a^2)\mathfrak{c}$ for every $a \in A$.

Proof. First notice that conditions (1) and (2) imply that $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a local ring (Proposition 2.44(4)). Moreover, since A is Gaussian and f and g are surjective, then B and C are Gaussian. Consider elements $(f(a_1) + b_1, g(a_1) + c_1)$ and $(f(a_2) + b_2, g(a_2) + c_2)$ of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$. Since A is Gaussian, we can assume that the ideal $(a_1, a_2)^2$ of A is equal to $(a_1)^2$, and so there exist elements $x, y \in A$ such that $a_2^2 = a_1^2 x$ and $a_1 a_2 = a_1^2 y$. We want to show that the ideal $((f(a_1)+b_1, g(a_1)+c_1), (f(a_2)+b_2, g(a_2)+c_2))^2$ of $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is equal to $((f(a_1) + b_1, g(a_1) + c_1)^2)$. In order to do that, we want to prove that there exist elements $\beta, \beta' \in \mathfrak{b}$ and $\gamma, \gamma' \in \mathfrak{c}$ such that

$$(f(a_2) + b_2, g(a_2) + c_2)^2 = (f(a_1) + b_1, g(a_1) + c_1)^2 (f(x) + \beta, g(x) + \gamma)$$

and

$$(f(a_1) + b_1, g(a_1) + c_1)(f(a_2) + b_2, g(a_2) + c_2) =$$
$$= (f(a_1) + b_1, g(a_1) + c_1)^2 (f(y) + \beta', g(y) + \gamma').$$

Using the fact that $\mathfrak{b}^2 = \mathfrak{c}^2 = 0$ and the relation $a_2^2 = a_1^2 x$, the first equality can be rewritten as

$$(2f(a_2)b_2, 2g(a_2)c_2) = (f(a_1^2)\beta + 2f(a_1 x)b_1, g(a_1^2)\gamma + 2g(a_1 x)c_1).$$

By (3), we have $2f(a_2)b_2 = f(a_2^2)b_2' = f(a_1^2 x)b_2'$ and $2g(a_2)c_2 = g(a_2^2)c_2' = g(a_1^2 x)c_2'$ for suitable elements $b_2' \in \mathfrak{b}$ and $c_2' \in \mathfrak{c}$. So, it suffices that β and γ satisfy the equality

$$(f(a_1^2)\beta, g(a_1^2)\gamma) = (f(a_1)(f(a_1 x)b_2' - 2f(x)b_1), g(a_1)(g(a_1 x)c_2' - 2g(x)c_1)).$$

The existence of the elements β and γ is now obvious, again by (3). For the second equality, we can argue in a similar way. Now, we want to prove that if $((f(a_1)+b_1, g(a_1)+c_1), (f(a_2)+b_2, g(a_2)+c_2))^2 = ((f(a_1)+b_1, g(a_1)+c_1)^2)$ and $(f(a_1) + b_1, g(a_1) + c_1)(f(a_2) + b_2, g(a_2) + c_2) = 0$, then $(f(a_2) + b_2, g(a_2) + c_2)^2 = 0$. Looking only at the first component, we have that the ideal $(f(a_1) + b_1, f(a_2) + b_2)^2$ of B is equal to $(f(a_1) + b_1)^2$ and that the product $(f(a_1)+b_1)(f(a_2)+b_2) = 0$. Since B is Gaussian, it implies that $(f(a_2)+b_2)^2 = 0$. Similarly, $(g(a_2) + c_2)^2 = 0$. To conclude, apply Theorem 2.57. $\square$

Remark 2.62. Notice that Example 2.50 shows that condition (1) in Theorem 2.61 is not necessary. To prove that also conditions (2) and (3) are not necessary for $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ to be a Gauss ring, let p be a prime number, $A := \mathbb{Z}_{(p)}$, $B := \mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)}$, let $f : A \longrightarrow B$ be the canonical projection and let $\mathfrak{b}$ (resp., $\mathfrak{m}$) be the maximal ideal of B (resp., of A). By [73, Example 2.1], the bi-amalgamation $A \bowtie^{f,\mathrm{Id}_A} (\mathfrak{b}, \mathfrak{m})$ is the amalgamated algebra $A \bowtie^f \mathfrak{b}$ and it is Gaussian, in view of [4, Theorem 4.1]. But $\mathfrak{m}^2 \neq 0$ and, for instance, $p\mathfrak{m} \neq p^2\mathfrak{m}$.

Example 2.63. Let $(V, \mathfrak{m})$ be a valuation domain such that $\mathfrak{m}^2 \neq \mathfrak{m}^3$. Set $A := V/\mathfrak{m}^3$, $B = C := V/\mathfrak{m}^2$ and consider the canonical morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$. If $\mathfrak{b}$ and $\mathfrak{c}$ are the maximal ideals of B and C respectively and $\mathfrak{i}_0 = f^{-1}(\mathfrak{b}) = g^{-1}(\mathfrak{c})$ is the maximal ideal of A, then it is immediate to check that the conditions of Theorem 2.61 are satisfied, and so $A \bowtie^{f,g} (\mathfrak{b}, \mathfrak{c})$ is a Gaussian local ring.

2.6 Prüfer rings and homomorphic images

Regular morphisms

According to [46, Proposition 4.4], if A is a Prüfer ring and $\mathfrak{a}$ is an ideal of A, then $A/\mathfrak{a}$ is a Prüfer ring whenever $\mathfrak{a}$ is regular. As we have seen, this fact is crucial in the proof of Theorem 2.27 and in several other results. The first goal of this section is to present a notion that allow to consider homomorphic images of Prüfer rings that are still Prüfer, without taking regular ideals. Our definition mimic that of local morphisms. Recall that a ring morphism $f : A \to B$ is said to be *local* if $f^{-1}(\mathcal{U}(B)) \subseteq \mathcal{U}(A)$, that is, if for every $a \in A$, $f(a)$ is invertible in B if and only if a is invertible in A [24]. A surjective ring homomorphism is local if and only if $\ker(f) \subseteq J(R)$.

Definition 2.64. Let $f : A \to B$ be a ring morphism. We say that f is a *regular morphism* if $f^{-1}(\mathrm{Reg}(B)) \subseteq \mathrm{Reg}(A)$. We say that a ring B is a *regular homomorphic image* of A if there exists a surjective regular morphism $f : A \to B$.

Let us start with the following elementary lemma.

Lemma 2.65. Let $f : A \to B$ be a ring morphism. Then the following properties hold.

1. B is a regular homomorphic image of A via f if and only if f is surjective and $Z(A) \subseteq f^{-1}(Z(B))$.

2. If f is a local morphism and B is a total ring of quotients, then f is a regular morphism.

3. If f is a regular morphism and A is a total ring of quotients, then f is a local morphism.

4. If f is surjective, then f is a regular morphism if and only if, for every $a \in A$, $(\ker(f) : a) \subseteq \ker(f)$ implies $a \in \mathrm{Reg}(A)$.

5. If $Z(A)$ is contained in a proper ideal $\mathfrak{i}$ of A, then $A \to A/\mathfrak{i}$ is a regular morphism.

6. If A is a ring in which every zero-divisor is nilpotent, then $A \to A/\mathfrak{p}$ is a regular morphism for every $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proof.

1. is clear.

2. If B is a total ring of quotients, then $\mathrm{Reg}(B) = \mathcal{U}(B)$. Since f is a local morphism, we have $f^{-1}(\mathrm{Reg}(B)) = f^{-1}(\mathcal{U}(B)) \subseteq \mathcal{U}(A) \subseteq \mathrm{Reg}(A)$.

3. In this case, we have $f^{-1}(\mathcal{U}(B)) \subseteq f^{-1}(\mathrm{Reg}(B)) \subseteq \mathrm{Reg}(A) = \mathcal{U}(A)$.

4. Immediately follows from the fact that for every $a \in A$, $f(a)$ is regular in B if and only if $(\ker(f) : a) \subseteq \ker(f)$.

5. By (4), it suffices to prove that for every $a \in A$, $(\mathfrak{i} : a) \subseteq \mathfrak{i}$ implies $a \in \mathrm{Reg}(A)$. If $(\mathfrak{i} : a) \subseteq \mathfrak{i}$, then $a \notin \mathfrak{i}$, and so $a \notin Z(A)$, that is a is a regular element of A.

6. If every zero-divisor is nilpotent, then $Z(A)$ is contained in all minimal primes of A. Apply (5) to conclude.

□

We have seen in Section 2.5 that necessary and sufficient conditions for the bi-amalgamated algebras to inherit the property of being Prüfer can be given by assuming some "regular properties" of the morphisms f and g involved in such constructions. Despite these assumptions, the volatility of the homomorphisms forces some necessary and sufficient conditions to be quite technical. These kind of difficulties disappear for regular morphisms.

Theorem 2.66. Every regular homomorphic image of a Prüfer ring is a Prüfer ring.

Proof. Let $f : A \to B$ be a regular surjective morphism and assume that A is a Prüfer ring. Let $\mathfrak{b} := (b_0, \dots, b_n)$ be a finitely generated regular ideal of B. For every $i = 0, \dots, n$, we can write $b_i = f(a_i)$ for some $a_i \in A$. Since f is a regular morphism, the finitely generated ideal $\mathfrak{a} := (a_0, \dots, a_n)$ of A is a regular ideal and, since A is a Prüfer ring, $\mathfrak{a}$ is invertible. In particular, by Proposition 2.9 (1), the polynomial $P(X) := \sum_{i=0}^{n} a_i X^i \in A[X]$ is a Gaussian polynomial. In view of Proposition 2.9 (2), it is enough to show that $F(X) := \sum_{i=0}^{n} b_i X^i \in B[X]$ is still a Gaussian polynomial. Let $G(X) := \sum_{j=0}^{m} \beta_j X^j$ be any other polynomial in $B[X]$. It suffices to verify that $c_B(F)c_B(G) \subseteq c_B(FG)$. Take any element $b = f(a) \in c_B(F)c_B(G)$ and consider the polynomial $Q(X) := \sum_{j=0}^{m} \alpha_j X^j \in A[X]$, where $f(\alpha_j) = \beta_j$ for $j = 0, \dots, m$. Then, we can pick some element $r \in \ker(f)$ such that $a + r \in c_A(P)c_A(Q) = c_A(PQ)$. In other words, we can write

$$a + r = \sum_{k=0}^{n+m} \eta_k \left(\sum_{i+j=k} a_i \alpha_j \right),$$

for suitable elements $\eta_0, \dots, \eta_{n+m} \in A$. Hence

$$b = f(a + r) = \sum_{k=0}^{n+m} f(\eta_k) \left(\sum_{i+j=k} b_i \beta_j \right) \in c_B(FG).$$

The conclusion follows. □

Corollary 2.67. Let A be a local Prüfer ring. Then $A/Z(A)$ is a Prüfer domain.

Proof. By [15, Lemma 3.5], $Z(A)$ is a prime ideal of A. The morphism

$A \to A/Z(A)$ is regular by Lemma 2.65 (5), and so $A/Z(A)$ is Prüfer by Theorem 2.66. $\square$

Example 2.68. Let A be a ring and let M be an A-module such that for every element $x \in M \setminus \{0\}$, $\mathrm{Ann}_A(x) \subseteq Z(A)$. Consider the idealization $A(+)M$ of M in A. Then it is easily seen that the canonical map $A(+)M \to A$ is a regular morphism. In particular, if $A(+)M$ is a Prüfer ring, then so is A.

Combining Theorems 2.27 and 2.66 it is possible to give a shortest proof of the result of Bakkari and Mahdou we have mentioned in the preliminary section (and restated below).

Corollary 2.69. [6, Theorem 2.1] Let $(T, \mathfrak{m})$ be a local ring of the form $T = k + \mathfrak{m}$, for some field k. Take a subring D of k such that $\mathrm{Q}(D) = k$ and set $R := D + \mathfrak{m}$. Then R has Prüfer condition (n) if and only if T and D have the same Prüfer condition (n).

Proof. We have the following pullback diagram:

$$\begin{array}{ccc} R & \xrightarrow{\pi_0} & D \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\pi} & k \end{array}$$

where π and π_0 are the canonical projections. If $\mathfrak{m}$ is a regular ideal of T, then it suffices to apply Theorem 2.27.

So, assume that $\mathfrak{m}$ consists only of zero divisors of T. It is immediate that T is a total ring of quotients, $\mathfrak{m} = Z(T)$ and $\mathrm{Tot}(R) = T$. Moreover, π_0 is a regular morphism, because $\pi_0^{-1}(D \setminus \{0\}) = R \setminus \mathfrak{m} \subseteq \mathrm{Reg}(R)$. So, if R is a Prüfer ring, then so are T and D.

For the other implication, assume that both D and T are Prüfer rings. Then, $D \subseteq k$ is a Prüfer extension, and by Theorem 2.17, so is $R \subseteq T$. Since $\mathrm{Tot}(R) = T$, R is a Prüfer ring.

To conclude, it suffices to notice that all five Prüfer conditions coincide on D and that if R is a Prüfer ring, then R has the Prüfer condition (n) if and only if $T = \mathrm{Tot}(R)$ has the same Prüfer condition (n) by Theorem 2.11. $\square$

Pre-Prüfer rings

In [13], the authors define a ring R to be a *pre-Prüfer ring* if every proper homomorphic image of R is a Prüfer ring (here "proper" means different from 0 and from R). They show that the prime spectrum of a pre-Prüfer domain forms a tree [13, Theorem 1.2] and that in the Noetherian case, pre-Prüfer domains are precisely those of dimension 1 [13, Corollary 1.3]. Here, we point out a generalization of these two results in the case of rings with zero divisors.

Let $\mathcal{I}$ be a family of ideals of a ring R. Inspired by the notion of the regular total order property for pairs given by Griffin [68], we say that $\mathcal{I}$ has the *regular total order property* if for every pair of ideals $I, J \in \mathcal{I}$, where at least one of them is regular, I and J are comparable. It is clear that if R is a domain, then $\mathcal{I}$ has the regular total order property if and only if $\mathcal{I}$ is a chain. The prime spectrum of a ring R forms a tree if and only if $\mathrm{Spec}(R_{\mathfrak{m}})$ is linearly ordered for each maximal ideal $\mathfrak{m}$ of R. For pre-Prüfer rings, we have the following result.

Theorem 2.70. *Let A be a pre-Prüfer ring. Then* $\mathrm{Spec}(A_{\mathfrak{m}})$ *has the regular total order property for every maximal ideal* $\mathfrak{m}$ *of A.*

Proof. In view of [13, Theorem 1.1], we can assume that A is a local pre-Prüfer ring. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{p}$ regular. We want to prove that $\mathfrak{p}$ and $\mathfrak{q}$ are comparable. If this is not the case, we can certainly assume that $\mathfrak{q}$ is nonzero. Since $\mathfrak{p}$ is regular, we must have $0 \neq \mathfrak{p}\mathfrak{q} \subseteq \mathfrak{i} := \mathfrak{p} \cap \mathfrak{q}$ and therefore $A/\mathfrak{i}$ is a local Prüfer ring, as A is local pre-Prüfer. It is straightforward to show that $\mathfrak{p}/\mathfrak{i} \cup \mathfrak{q}/\mathfrak{i} = Z(A/\mathfrak{i})$. By [15, Lemma 3.5] the set $Z(A/\mathfrak{i})$ of zero-divisors of $A/\mathfrak{i}$ is a prime ideal of $A/\mathfrak{i}$. Then $\mathfrak{p}/\mathfrak{i}$ and $\mathfrak{q}/\mathfrak{i}$ must be comparable, a contradiction. □

As we have already said, in the domain case the previous result has a simpler form.

Corollary 2.71. [13, Theorem 1.2] The prime spectrum of a pre-Prüfer domain is a tree.

Corollary 2.72. Let R be a pre-Prüfer ring. Then the following statements hold.

1. Two distinct minimal primes over a regular ideal of R are comaximal.

2. If R is local, then every regular ideal of R has a unique minimal prime.

Proof. We only need to prove (1), statement (2) being an immediate consequence. Let $\mathfrak{i}$ be a regular ideal of R and assume that $\mathfrak{p}, \mathfrak{q} \supseteq \mathfrak{i}$ are distinct (regular) minimal prime ideals over $\mathfrak{i}$ which are not comaximal. Let $\mathfrak{m}$ be a maximal ideal of R containing $\mathfrak{p} + \mathfrak{q}$. Then $\mathfrak{p}R_\mathfrak{m}, \mathfrak{q}R_\mathfrak{m}$ are both regular prime ideals of $R_\mathfrak{m}$ and they are not comparable, against Theorem 2.70 □

The next result is a slight generalization of one implication of [13, Corollary 1.3].

Proposition 2.73. Let R be a Noetherian pre-Prüfer ring. Then $\dim(R) \leq 1$.

Proof. Assume $\dim(R) > 1$ and let $\mathfrak{q} \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ be a chain of prime ideals. By [13, Theorem 1.1], there is no restriction in assuming that R is local with maximal ideal $\mathfrak{m}$. By [75, Theorem 144], the set $\mathcal{S}$ of all prime ideals $\mathfrak{h}$ of R such that $\mathfrak{q} \subsetneq \mathfrak{h} \subsetneq \mathfrak{m}$ is infinite. If $\mathfrak{q} = 0$, then Theorem 2.70 implies that $\mathcal{S}$ is an infinite chain and thus R is infinite-dimensional, against the fact that R is local and Noetherian. If $\mathfrak{q} \neq 0$, then $R/\mathfrak{q}$ is a local Dedekind domain (since R is pre-Prüfer) of dimension ≥ 2, another contradiction. □

We conclude this subsection by noting that in the integral case, Noetherian pre-Prüfer domains are precisely those of dimension at most 1, that is, a Noetherian domain D is a pre-Prüfer domain if and only if $\dim(D) \leq 1$ [13, Corollary 1.3]. Nevertheless, in rings with zero-divisors, there exist one-dimensional Noetherian rings that are not pre-Prüfer, as the following example shows.

Example 2.74. Let A be a Noetherian, one-dimensional local domain which is not Dedekind. Let $\mathbf{k}$ be the residue field of A, endowed with its natural structure of A-module. Consider the idealization $R := A(+)\mathbf{k}$ (see page 74). Then $\dim(A) = \dim(R) = 1$ (cf. [70, Chapter VI]) and R is a Noetherian ring by [2, Proposition 2.2]. But R is not a pre-Prüfer ring, since $A \cong R/(0(+)\mathbf{k})$ is not a Prüfer ring.

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