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Coherent Quantum Dynamics of Bosons in Measure

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Abstract

This thesis is devoted to the study of discrete models for Quantum Dynamics of Bosons.

In the first part we provide rigorous mathematical study of the dynamics of trapped spinless atoms, when the initial state is chosen to be a coherent state distributed according to specific invariant measures. The model considered in the Thesis is the standard one in the literature, namely the so-called Bose-Hubbard (BH) model, whose general Hamiltonian is a gauge-invariant polynomial in the Dirac ladder operators.

In the second part we consider some aspects of finite dimensional reduction of relevant objects introduced in the theory of Bose gases. In particular we consider the approximation by means of finite dimensional subspaces with $\dim = L$ of the boson one-particle density operator and the solution of Hartree equation by providing explicit estimates as $L \to \infty$.

Riassunto

L'argomento principale di questa tesi è lo studio di modelli discreti per la dinamica di bosoni
Nella prima parte della tesi viene fornito uno studio rigoroso della dinamica di bosoni di spin zero intrappolati in un reticolo, quando gli stati iniziali sono presi come stati coerenti distribuiti secondo una specifica misura invariante per la dinamica. Il modello standard per questo tipo di problemi è il modello di Bose-Hubbard, la cui Hamiltoniana è in generale un polinomio gauge-invariante degli operatori di scala di Dirac.
Nella seconda parte approfondiamo alcuni aspetti del concetto di riduzione finito dimensionale, che viene applicata ad alcuni oggetti rilevanti nello studio dei gas di Bose. In particolare consideriamo l'approssimazione su uno spazio finito dimensionale di $\dim = L$ dell'operatore densità di singola particella e della soluzione dell'equazione di Hartree, fornendo stime esplicite per $L \to \infty$.

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Introduction

The dynamical study of physical systems composed by many *bosons* is a problem which dates back to the early days of quantum mechanics, mainly due to the theoretical prediction of the phenomenon today known as *Bose-Einstein condensation* (BEC) for a non-interacting gas in the late '20s of the last century. At the time the observation of superfluidity in liquid Helium at very low temperatures (less than 4 K at standard pressure) was interpreted by London as the occurrence of BEC ([42], 1938), although this vision was highly criticized since liquid Helium is very far from being a non-interacting system of particles.

Since then some phenomenological theories were introduced, including the well known Tisza two-fluids model ([56], 1938) which was later developed by Landau ([37], 1941), but it was the russian physicist and mathematician N. N. Bogoliubov who, in a pioneristic work of 1947 on superfluidity, began to systematically attack the problem from a microscopic point of view (see [19]), that is using basic principles of *Quantum Field Theory*. Despite giving some correct predictions, for example in determining the approximate low energy excitation spectrum of the gas, his theory was not flawless and lacked rigour from a mathematical point of view. During the 1950s and 1960s a lot of effort was put in understanding the behavior of interacting Bose gases from a theoretical physics perspective ([38, 48] and [49], in which the concept of *off-diagonal long range order* was introduced), but the mathematical structure did not significantly improve.

In 1995, the experimental observation of BEC - outside the realm of Helium - made for Rubidium-87 by Cornell and Wieman [7], and for Sodium-23 by Ketterle [25], led to an explosion of activity in the physics of Bose gases.[1] Their approach was based on laser cooling techniques and magneto-optical traps, firstly introduced in the 1980s, and represents a cornerstone in the field of interacting bosons, whose study has attracted increasing interest from experimental, numerical and theoretical communities (an excellent review on the subject is contained in the book [50]). Nowadays there exist a large literature on rigorous results in BEC theory, focused on the ground state properties of many-body Hamiltonians (see [41] and references

[1]The number of papers up to 2005 on BECs can be found at `https://core.ac.uk/reader/11880460`.

therein), on the quantitative derivation of effective evolution equations of many-body systems, such as the Gross-Pitaevskii and the Hartree equation, based on rigorous techniques of quantum dynamics (see [51, 13, 18]) and so on, making BEC theory a rich and active research field in modern mathematical physics.

Along with the study of Bose gases grounded on the microscopic description, in the last few decades *discrete* models have gained a lot of attention, mainly due to the experimental studies of BECs in optical lattices formed by counter-propagating laser beams, and to the subsequent investigations of tunneling processes, Josephson junctions, Mott insulators and so on (see for example [17, 6]).
The fundamental example of discrete model is the *Bose-Hubbard model* (BH), historically introduced in a study on granular superconductors by Gersh and Knollmann [32] as a boson version of the Hubbard model for electrons in metals [34]. Loosely speaking, the BH model describes the competition between tunneling of particles between neighboring potential wells (or *sites*) and their on-site interaction, and has proved to be promising in view of applications to quantum computing and quantum information [36]. On the other hand, the scalar (semi-classical, in a sense) counterpart of the BH model is the so called Discrete Nonlinear Schödinger equation (DNLS) - also known as Discrete Gross-Pitaevskii equation in physics - which basically replaces the purely quantum setting based on operators with a description in terms of complex scalar fields on each site. The DNLS equation was first heuristically obtained starting from the Gross-Pitaevskii equation in BEC theory within the *tight-binding approximation* [58], to be rigorously justified quite lately in [52] by means of semi-classical analysis techniques.
On the other hand, another approach is possible, where the DNLS is obtained starting from the BH model and taking *coherent expectations* of the basic operators of the theory (see below). Such an approach, however, turns to be a very rough approximation since it usually disregards the correlations between different sites (see [27]).
Actually, the understanding of how this procedure could be made rigorous was precisely the starting point of the research work contained in the present thesis.

The thesis is structured as follows.

Chapter 1: We give a brief review of the technical background needed in the subsequent chapters. In particular, in Section 1.1 we introduce the Bargmann spaces and the algebra of Wick operators on them, which can be seen as part of Pseudo-differential theory on (anti-)holomorphic spaces of functions. In Section 1.2 we recall some basic facts of boson Fock spaces which for our purposes can be seen as a useful computational tool when dealing with many-body particle systems, especially in applications to *one-body density operators*. We conclude by describing a *finite dimensional reduction* on Fock spaces, which will play a central role in interfacing discrete models with those constructed in Fock theory. In particular we construct a unitary operator mapping the the Bargmann space $\mathcal{H}L^2$ to the boson Fock space over a finite dimensional Hilbert space $\mathcal{F}_B(V)$

$$U\colon \mathcal{H}L^2(\mathbb{C}^L) \to \mathcal{F}_B(V).$$

We would like to point out that this procedure is somehow sketched and scattered in literature (see the references within the chapter) but, to our knowledge, never practically employed in the study of quantum models.

Chapter 2: We deduce the DNLS equation[2]

$$i\frac{d}{dt}w_k(t) = E_k w_k(t) + J(w_{k+1}(t) + w_{k-1}(t)) + U|w_k(t)|^2 w_k(t)$$

as a *Mean-Field limit* of coherent expectations of annihilation operators on the Bargmann space, allowed to evolve in time according to the Heisenberg equations of motion. The hamiltonian operator generating the quantum dynamics is given by the BH operator

$$H_N(A) = \sum_{j=1}^{L} \big(E_j A_j^* A_j + J(A_{j-1}^* A_j + A_{j+1}^* A_j)\big) + \frac{U}{2N}\sum_{j=1}^{L} A_j^* A_j^* A_j A_j$$

where the A_j^*'s, A_j's are the creation and annihilation operators on the Bargmann space. The novelty of our analysis relies mainly on two aspects. Firstly, we interpret the coherent expectation of operators involved as *symbols* of Wick operators, consistently with Berezin's approach depicted in Chapter 1 as it is usually done in Egorov-type theorems of semi-classical analysis, while in physics literature coherent expectations are usually introduced by physical motivation. Secondly, we do not actually fix a single coherent state, but we measure how the DNLS flow and the symbol of the evolved annihilation operator differ in time on average with respect to a suitable metric on the space of symbols, i.e. by distributing in measure the family of all coherent states. This allows us to obtain a linear in time estimate on the divergence of the two evolved quantities, which becomes more significant as the number of particles in the lattice model grows. Here the choice of the metric is not arbitrary:

[2]Here U is a coupling constant with the non-linear term and *not* the unitary operator of the previous equation.

we can show indeed how it is possible to interpret it as a manifestation of quantum averages taken with respect to a particular density operator.

Chapter 3: We apply the finite dimensional reduction procedure to the *one-body density operator*

$$\gamma_{\Psi}^{(1)} = \frac{1}{\langle \Psi, \mathbf{N}\Psi \rangle} \sum_{j,k \geq 0} \langle \Psi, b(u_j)^* b(u_k)\Psi \rangle \langle u_j, \cdot \rangle u_k,$$

where $b(u_j)^*$'s and $b(u_j)$'s are the creation and annihilation operators on the Fock space and $\mathbf{N}$ is the *boson number operator* (see Chapter 1). We remark that this operator is at the core of the modern definition of BECs according to [49], based on the concept of *off-diagonal long range order*. Moreover, during our analysis we are forced to study how the quantum dynamics of a many-body state function is affected by the finite dimensional reduction. The generator of dynamics here is a quantum hamiltonian consisting of an external potential which models an optical lattice confined by an anharmonic trap, in which particles are allowed to interact via a bounded potential.

Chapter 4: We extend the idea of finite dimensional reduction to the Hartree equation,

$$i\dot{\varphi}(t) = H\varphi(t) + (V * |\varphi(t)|)^2 \varphi(t),$$

which can be seen as a smoothed version of the Gross-Pitaevskii equation. Again the non-linear model here takes into account the ideal experimental setting in which particles move inside an optical lattice as previously described, where the non-linearity is given by the interaction potential. In particular, we study how the solution of the Hartree equation differs from its reduced version by quantitative estimates (an approach which is reminiscent of the work of Bourgain [21] on the approximation of solutions of the Nonlinear Schrödinger Equation on the flat torus). With the extra assumption of *large trap regime* we obtain a linear in time approximation of the solution of the Hartree equation.

Chapter 1

Setting and Preliminaries

Bargmann space (also known as Segal-Bargmann or Fock-Bargmann) was introduced in 1961 by V. Bargmann as a realization of a separable Hilbert space in terms of holomorphic functions [10], realizing that it is the correct setting to represent the Canonical Commutation Relations (CCR) in terms of complex variables as first discussed by Fock [29]. In early 1970s Berezin was able to lay the basis of a Pseudo-Differential theory on Bargmann space known today as *Wick calculus* [14], which has been proven useful also outside the realm of Quantum Mechanics (see e.g. [30] for applications in Harmonic Analysis). Very recently Berezin's ideas has been furthere developed in the context of Gelfand-Shilov generalized functions [57, 55].

1.1 Bargmann Space and Wick Quantization

The *Bargmann space* $\mathcal{H}L^2(\mathbb{C}^n)$ considered here is the space of antiholomorphic functions on $\mathbb{C}^n$ in n complex variables which satisfy the integrability condition

$$\|f\|^2_{\mathcal{H}L^2} := \frac{1}{\pi^n}\int_{\mathbb{C}^n} |f(\bar{z})|^2 e^{-|z|^2} dz d\bar{z} < \infty, \tag{1.1}$$

where $dzd\bar{z}$ is the usual Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. We review here some elementary facts of such space needed in the following. We remark that we follow here the same conventions as in [14], while in the Bargmann's original paper [10] the functions considered are *holomorphic*, but this difference does not play any role for our purposes.
For an entire function f with Taylor expansion $\sum_{\alpha\in\mathbb{N}^n} c_\alpha \bar{z}^\alpha$, an equivalent definition of the Bargmann norm is given by

$$\|f\|^2_{\mathcal{H}L^2} = \sum_{\alpha\in\mathbb{N}^n} \alpha! |c_\alpha|^2, \tag{1.2}$$

whenever any of the two sides is finite. Here we used the notation $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$

and $\alpha! = \alpha_1! \cdots \alpha_n!$. By using polarization identity, $\mathcal{H}L^2(\mathbb{C}^n)$ inherits the structure of a Hilbert space with inner product

$$\langle f, g\rangle_{\mathcal{H}L^2} = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \overline{f(\bar{z})} g(\bar{z}) e^{-|z|^2} \mathrm{d}z\mathrm{d}\bar{z} = \sum_{\alpha\in\mathbb{N}^n} \alpha! \bar{c}_\alpha d_\alpha, \tag{1.3}$$

where the d_k's are the coefficients of Taylor expansion of g. To simplify notation we make use of the symbol $\langle \cdot, \cdot\rangle$ for the Bargmann inner product when there is no ambiguity for its meaning.

Remark 1.1.1. (i) It is well known that $\mathcal{H}L^2(\mathbb{C}^n)$ is unitarily equivalent to $L^2(\mathbb{R}^n)$, and we shall denote as $U_{\mathcal{B}}$ the isomorphism $L^2(\mathbb{R}^n) \to \mathcal{H}L^2(\mathbb{C}^n)$ (*Bargmann transform*).

(ii) From (1.3) we see that the set of vectors

$$\left\{ u_\alpha(\bar{z}) = \frac{\bar{z}^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^n \right\} \tag{1.4}$$

forms a complete orthonormal system in $\mathcal{H}L^2(\mathbb{C}^n)$. This set is the image via $U_{\mathcal{B}}$ of the set of all Hermite functions $\{\psi_\alpha\}_{\alpha\in\mathbb{N}^n} \subset L^2(\mathbb{R}^n)$ and in particular for any multi-index α

$$U_{\mathcal{B}}\psi_\alpha = u_\alpha. \tag{1.5}$$

(iii) Consider the space of Schwartz functions $\mathscr{S} \subset L^2(\mathbb{R}^n)$. It's clear that the set

$$\mathscr{E} = U_{\mathcal{B}}(\mathscr{S}) \tag{1.6}$$

is a dense linear subspace of $\mathcal{H}L^2(\mathbb{C}^n)$ containing all vectors of the set in (1.4). The properties of $\mathscr{E}$ and some other related subspaces were studied in [11].

(iv) The fact that simple monomials form a basis of $\mathcal{H}L^2$ allows us to represent it as a direct sum of simpler subspaces. Indeed, define the subspace $\mathrm{Poly}_n^k \subset \mathcal{H}L^2(\mathbb{C}^n)$ as the set of anti-holomorphic *homogeneous* polynomials of degree k in n complex variables. For every pair of natural numbers $k \neq l$, $\mathrm{Poly}_n^k \perp \mathrm{Poly}_n^l$ with respect to Bargmann product, so that

$$\mathcal{H}L^2(\mathbb{C}^n) = \bigoplus_{k\geq 0} \mathrm{Poly}_n^k. \tag{1.7}$$

For any $k \in \mathbb{N}$, the set $\{u_\alpha,\ |\alpha| = k\}$ is an orthonormal basis of Poly_n^k as well and it is a classic result in combinatorics that

$$\dim \mathrm{Poly}_n^k = \binom{n+k-1}{k}. \tag{1.8}$$

In the following we shall deal with a certain class of integral operators on Bargmann space which play the role of *quantization* of classical observables (an excellent review is given in [30]).

Definition 1.1.2. *For an entire function $\sigma\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ we define its associated* Wick operator *as*

$$(Op^W(\sigma)f)(\bar z) := \frac{1}{\pi^n}\int_{\mathbb{C}^n} e^{\bar z\cdot w}\sigma(\bar z, w)f(\bar w)e^{-|w|^2}dwd\bar w. \tag{1.9}$$

The function $z \to \sigma(\bar z, z)$ (restriction of σ to the diagonal $\{z = w\}$) is called Wick symbol *of $Op^W(\sigma)$.*

Conversely, let $\tilde\sigma\colon \mathbb{C}^n \to \mathbb{C}$ be a smooth function. The anti-Wick *operator associated with $\tilde\sigma$ is defined as*

$$(Op^{AW}(\tilde\sigma)f)(\bar z) := \frac{1}{\pi^n}\int_{\mathbb{C}^n} e^{\bar z\cdot w}\tilde\sigma(\bar w, w)f(\bar w)e^{-|w|^2}dwd\bar w. \tag{1.10}$$

In this case, $\tilde\sigma$ is called anti-Wick symbol *of $Op^{AW}(\tilde\sigma)$.*

Remark 1.1.3. (i) Although the notation may be a bit unclear and misleading, it is a tradition in literature to emphasize the non-holomorphicity of $\tilde\sigma$ in (1.10) by writing $\tilde\sigma(\bar w, w)$ (see e.g. [30] and [14]).
(ii) Clearly σ and $\tilde\sigma$ must have some suitable growth restriction so that the integrals in (1.9)-(1.10) converge and the domains of such operators are clearly the set of all $f \in \mathcal{H}L^2(\mathbb{C}^n)$ such that both $\mathrm{Op}^W(\sigma)f$ and $\mathrm{Op}^{AW}(\tilde\sigma)f$ are in $\mathcal{H}L^2(\mathbb{C}^n)$.
(iii) As an elementary example of Wick operators we consider differential operators with polynomial coefficients. Indeed given any polynomial function

$$p(\bar z, w) = \sum_{\alpha,\beta} c_{\alpha,\beta}\bar z^\alpha w^\beta, \tag{1.11}$$

where $c_{\alpha,\beta}$ are complex numbers such that $c_{\alpha,\beta} = 0$ definitely if the multi-indices α and β are sufficiently large, it is easy to see that

$$(\mathrm{Op}^W(p)f)(\bar z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\bar z^\alpha \frac{\partial^\beta f}{\partial \bar z^\beta}. \tag{1.12}$$

On the other hand, if we consider the function $z \to p(\bar z, z)$ we see that

$$(\mathrm{Op}^{AW}(p)f)(\bar z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\frac{\partial^\beta}{\partial \bar z^\beta}(\bar z^\alpha f), \tag{1.13}$$

that is all the derivations are pushed to the left of multiplication with coordinates. In both cases $p(\bar z, z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\bar z^\alpha z^\beta$

The question about which functions may serve as Wick or anti-Wick symbols of some operator has been addressed in [14] and we recall here the minimal setting.

Lemma 1.1.4. *Let* $\tilde{\sigma}\colon \mathbb{C}^n \to \mathbb{C}$ *be a function such that for* $p > 2$

$$\int_{\mathbb{C}^n} |\tilde{\sigma}(\bar{z}, z)|^p e^{-|z|^2} dz d\bar{z} < \infty.$$

Then $\tilde{\sigma}$ *is the anti-Wick symbol of some operator* T *on* $\mathcal{H}L^2(\mathbb{C}^n)$ *whose domain is dense. Moreover,* T *has a Wick symbol given by*

$$\sigma(\bar{z}, z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \tilde{\sigma}(\bar{w}, w) e^{-|z-w|^2} dw d\bar{w}. \tag{1.14}$$

We often make use of the opposite viewpoint, that is asking whether an operator $T\colon D(T) \to \mathcal{H}L^2(\mathbb{C}^n)$ possesses a Wick symbol. A sufficient condition (see [14]) is that if the domains of T and T^* contain all vectors of the form[1]

$$\phi_w(\bar{z}) := e^{-\frac{1}{2}|w|^2 + \bar{z}\cdot w}, \qquad w \in \mathbb{C}^n, \tag{1.15}$$

then T has a Wick symbol given by

$$\sigma_T(\bar{z}, w) = \frac{\langle \phi_z, T\phi_w \rangle}{\langle \phi_z, \phi_w \rangle}, \tag{1.16}$$

that is $T \equiv \mathrm{Op}^W(\sigma_T)$. In particular every bounded operator on $\mathcal{H}L^2(\mathbb{C}^n)$ is a Wick operator. Vectors in the set $\{\phi_z\}_{z\in\mathbb{C}^n}$ are called *coherent states* and it is immediate to see that $\|\phi_z\|_{\mathcal{H}L^2} = 1$ for all $z \in \mathbb{C}^n$.
Similarly to usual standard or Weyl quantization on L^2 spaces, Wick operators are somehow a closed family in the algebraic sense, as established in the next result.

Lemma 1.1.5. *The composition* $Op^W(\sigma) Op^W(\tau)$ *of two operators with Wick symbols* σ *and* τ *is a Wick operator with symbol*

$$\sigma \star \tau(\bar{z}, z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \sigma(\bar{z}, w)\tau(\bar{w}, z) e^{-|z-w|^2} d\bar{w} dw. \tag{1.17}$$

As an easy corollary we get that the *commutator* of two Wick operators has a Wick symbol

$$[\mathrm{Op}^W(\sigma), \mathrm{Op}^W(\tau)] = \mathrm{Op}^W(\{\sigma, \tau\}_\star) \tag{1.18}$$

where

$$\{\sigma, \tau\}_\star := \sigma \star \tau - \tau \star \sigma. \tag{1.19}$$

[1] For example, if both $D(T)$ and $D(T^*)$ contain $\mathscr{E}$ (cfr. formula 1.6).

From (1.17) we see that the correspondences $(\sigma,\tau) \to \sigma \star \tau$ and $(\sigma,\tau) \to \{\sigma,\tau\}_\star$ define bilinear maps on symbols, which will be a central tool in applications in Chapter 3.

Definition 1.1.6. *Given two Wick symbols σ and τ, the bilinear, associative and non-commutative product at right-hand side of (1.17) is the* Wick $\star$-product *of σ and τ.*
The application in (1.19) is the Wick bracket *of σ and τ and has the following properties*

(i) bilinearity;

(ii) skew-symmetry;

(iii) satisfies Jacoby identity

$$\{\sigma_1, \{\sigma_2, \sigma_3\}_\star\}_\star + \{\sigma_2, \{\sigma_3, \sigma_1\}_\star\}_\star + \{\sigma_3, \{\sigma_1, \sigma_2\}_\star\}_\star = 0;$$

(iv) it is Leibniz with respect to the Wick $\star$-product

$$\{\sigma_1, \sigma_2 \star \sigma_3\}_\star = \sigma_2 \star \{\sigma_1, \sigma_3\}_\star + \{\sigma_1, \sigma_2\}_\star \star \sigma_3.$$

The expression in (1.17) is not very useful for our purposes. We make use instead of a formal asymptotic series expansion of the $\star$-product and brackets (see e.g. [15], formula 2.34 in Section 5.2), namely

$$\sigma \star \tau \simeq \sum_{|I|\geq 0} \frac{1}{I!} \frac{\partial^I \sigma}{\partial z^I} \frac{\partial^I \tau}{\partial \bar{z}^I} \tag{1.20}$$

$$\{\sigma,\tau\}_\star \simeq \sum_{|I|\geq 0} \frac{1}{I!} \left(\frac{\partial^I \sigma}{\partial z^I} \frac{\partial^I \tau}{\partial \bar{z}^I} - \frac{\partial^I \sigma}{\partial \bar{z}^I} \frac{\partial^I \tau}{\partial z^I} \right) \tag{1.21}$$

Here we do not address the problem of convergence of expressions in (1.20)-(1.21), as it is done in [12], nor we try get some rigorous estimates on the remainder of such asymptotic expressions at every order, a question which to our knowledge seems to be lacking in literature. The motivation is twofold: as we see in Chapter 3 , all we need in our applications is the Wick bracket of two symbols in which the second one is a polynomial, so that the series become finite sums. On the other hand, in the general context of Deformation Quantization the asymptotic series above are treated as formal algebraic objects for which convergence issues are not relevant (see [20]). To conclude this brief review we would like to fix the notation for the most basic Wick operators, that is the multiplication and derivation with respect to coordinates.

Definition 1.1.7. *For any $1 \leq j \leq n$ consider the dense subspaces of $\mathcal{H}L^2(\mathbb{C}^n)$*

$$D(A_j^\dagger) = \left\{ f \in \mathcal{H}L^2(\mathbb{C}^n) \;\middle|\; \int_{\mathbb{C}^n} |\bar{z}_j f(\bar{z})|^2 e^{-|z|^2} dz d\bar{z} < \infty \right\}, \tag{1.22}$$

$$D(A_j) = \left\{ f \in \mathcal{H}L^2(\mathbb{C}^n) \;\middle|\; \int_{\mathbb{C}^n} \left| \frac{\partial f}{\partial \bar{z}_j}(\bar{z}) \right|^2 e^{-|z|^2} dz d\bar{z} < \infty \right\}. \tag{1.23}$$

The creation *and* annihilation *operators on $\mathcal{H}L^2$ are defined respectively as*

$$(A_j^\dagger f)(\bar{z}) = \bar{z}_j f(\bar{z}), \qquad \forall f \in D(A_j^\dagger) \tag{1.24}$$

$$(A_j f)(z) = \frac{\partial f}{\partial \bar{z}_j}(\bar{z}), \qquad \forall f \in D(A_j). \tag{1.25}$$

We recall some known facts about these operators (see [10], Theorem 3.1).

Theorem 1.1.8. *For all $1 \leq j \leq n$*

(i) $A_j, A_j^\dagger$ *are closed.*

(ii) $D(A_j) = D(A_j^\dagger)$.

(iii) $A_j^* = A_j^\dagger$ *and* $(A_j^\dagger)^* = A_j$.

(iv) $[A_j, A_k^*] = \delta_{jk}$ *on* $\mathscr{E}$.

Thanks to property (iii) we shall denote the creation operator using the adjoint notation A_j^*. As an easy application of Remark 1.1.3 (iii) we see that the Wick symbols of A_j^* and A_j are given by the functions $z \to \bar{z}_j$ and $z \to z_j$ respectively, for all $1 \leq j \leq n$.

1.2 Boson Fock spaces

The aim of this section is to collect some known definitions and results of *Fock spaces*. Main references are the books [8] and [24]. Given a Hilbert space $\mathfrak{h}$, we define the *full Fock space* by

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}^{\otimes n} \tag{1.26}$$

where $\mathfrak{h}^{\otimes 0} = \mathbb{C}$ and $\mathfrak{h}^{\otimes n}$ is the n-fold tensor product of $\mathfrak{h}$ with itself. Any element $\Psi \in \mathcal{F}(\mathfrak{h})$ can be regarded as a sequence

$$\Psi = (\Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(n)}, \ldots), \qquad \Psi^{(n)} \in \mathfrak{h}^{\otimes n} \quad \forall n \in \mathbb{N}$$

such that the quantity

$$\|\Psi\|_{\mathcal{F}} = \sqrt{\sum_{n\geq 0} \|\Psi^{(n)}\|^2} \tag{1.27}$$

is finite. The full Fock space is a Hilbert space itself with inner product

$$\langle \Phi, \Psi \rangle = \sum_{n\geq 0} \langle \Phi^{(n)}, \Psi^{(n)} \rangle$$

inducing precisely the norm in Eq. (1.27). For any $N \geq 0$ and $\psi \in \mathfrak{h}^{\otimes N}$ there is a vector $\widetilde{\psi}$ defined as follows: $\widetilde{\psi}^{(n)} = 0$ if $n \neq N$ and $\widetilde{\psi}^{(N)} = \psi$. The correspondence

$$\psi \to \widetilde{\psi} = (0, \dots, 0, \psi, 0, \dots) \tag{1.28}$$

is an isometric embedding of $\mathfrak{h}^{\otimes N}$ into $\mathcal{F}(\mathfrak{h})$. In other words, $\mathfrak{h}^{\otimes N}$ is unitarily equivalent to the N-particle sector

$$\mathcal{F}^{(N)} = \big\{ (0, \dots, 0, \psi, 0, \dots) \;\; | \;\; \psi \in \mathfrak{h}^{\otimes N} \big\}$$

and we shall use freely this identification in the following. Lastly, one can construct a Hilbert basis of $\mathfrak{h}^{\otimes N}$ by taking all possible simple tensor products of vectors in a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset \mathfrak{h}$

$$\{e_{i_1} \otimes \cdots \otimes e_{i_N}, \; i_l \in \mathbb{N}\} \subset \mathfrak{h}^{\otimes N}. \tag{1.29}$$

Then, it can be proved that the set

$$\{e_{i_1} \widetilde{\otimes \cdots \otimes} e_{i_n} \;\; | \;\; n \in \mathbb{N}, \quad i_l \in \mathbb{N}, \; l = 1, \dots, n \; \} \subset \mathcal{F}(\mathfrak{h})$$

is a basis of the full Fock space (the *tilde character* refers to the embedding in 1.28). The *Boson Fock space* is the subspace $\mathcal{F}_B(\mathfrak{h}) \subset \mathcal{F}(\mathfrak{h})$ obtained as the action of orthogonal projection

$$\Pi_B := \bigoplus_{n\geq 0} S_n \tag{1.30}$$

on the full Fock space, that is $\mathcal{F}_B(\mathfrak{h}) = \Pi_B(\mathcal{F}(\mathfrak{h}))$. Here S_n is the *symmetrizer* on $\mathfrak{h}^{\otimes n}$, whose action on simple tensor products is given by

$$S_n(\psi_1 \otimes \cdots \otimes \psi_n) = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}, \tag{1.31}$$

where $\mathfrak{S}_n$ is the set of permutations of n objects and all the ψ_n's are in $\mathfrak{h}$. We shall denote the orthonormal basis of $\mathfrak{h}_B^{\otimes N} := S_n(\mathfrak{h}^{\otimes N})$ and $\mathcal{F}_B(\mathfrak{h})$ respectively as

$$\{e_{i_1} \vee \cdots \vee e_{i_N} \mid i_l \in \mathbb{N}, \ 1 \le l \le N\}, \tag{1.32}$$

$$\{e_{i_1} \widetilde{\vee \cdots \vee} e_{i_n} \mid n \in \mathbb{N}, \ i_l \in \mathbb{N}, \ l = 1, \ldots, n \}, \tag{1.33}$$

where the symmetrized product $\vee$ is considered here as normalized

$$e_{i_1} \vee \cdots \vee e_{i_N} = C_{i_1,\ldots,i_N} S_N(e_1 \otimes \cdots \otimes e_{i_N}) \tag{1.34}$$

with

$$C_{i_1,\ldots,i_N} = \sqrt{\frac{N!}{i_1! \cdots i_N!}} \, .$$

A notion which is useful in computations is the *finite particle subspace* of $\mathcal{F}_B$. It is defined as

$$\mathcal{F}_{B,0}(\mathfrak{h}) = \left\{ \Psi \in \mathcal{F}_B(\mathfrak{h}) \mid \exists r \in \mathbb{N} \text{ such that } \Psi^{(n)} = 0 \text{ if } n > r \right\}, \tag{1.35}$$

and it is easy to verify that it is dense in $\mathcal{F}_B(\mathfrak{h})$.

On $\mathcal{F}_B(\mathfrak{h})$ some operators interlacing different sectors can be defined.

Definition 1.2.1. *For any vector $f \in \mathfrak{h}$, the* boson creation operator *associated with f is the operator $b(f)^\dagger \colon D(b(f)^\dagger) \to \mathcal{F}(\mathfrak{h})$ given by*

$$D(b(f)^\dagger) = \left\{ \Psi \in \mathcal{F}_B(\mathfrak{h}) \mid \sum_{n \ge 1} n \|\Psi^{(n-1)}\|^2 < \infty \right\} \tag{1.36}$$

$$(b(f)^\dagger \Psi)^{(n)} = \begin{cases} 0, & \text{if} \quad n = 0, \\ \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), & \text{if} \quad n > 0. \end{cases} \tag{1.37}$$

The boson annihilation operator *is $b(f)$ the adjoint of $b(f)^\dagger$.*

Similarly to creation and annihilation operators on Bargmann space, we list some useful properties of $b(f)$ and $b(f)^\dagger$.

Lemma 1.2.2. *For any $f \in \mathfrak{h}$*

(i) $\mathcal{F}_{B,0}(\mathfrak{h}) \subset D(b(f)) \cap D(b(f)^\dagger)$, so the boson creation and annihilation operators are densely defined.

(ii) $b(f)$ and $b(f)^\dagger$ are closed, so $b(f)^* = b(f)^\dagger$.

(iii) $D(b(f)) = D(b(f)^\dagger)$.

(iv) $[b(f), b(g)^\dagger] = \langle f, g \rangle$ on $\mathcal{F}_{B,0}(\mathfrak{h})$.

Due to part (ii) of previous Lemma we shall denote the creation operator simply as $b(f)^*$. Lastly, we briefly review some elementary concepts of *second quantization operators*. These are defined as a natural extension of operators acting on $\mathfrak{h}^{\otimes n}$. For simplicity, consider a self-adjoint operator H on $\mathfrak{h}$ and construct its extension on $\mathfrak{h}^{\otimes n}$ (a *one-body operator*) by

$$H^{(n)} := \sum_{j=1}^{n} \mathbb{I} \otimes \cdots \otimes \underset{\substack{\downarrow \\ j-\text{th}}}{H} \otimes \cdots \otimes \mathbb{I}. \tag{1.38}$$

An example of a one-body operator is given in the case in which $\mathfrak{h} = L^2(\mathbb{R}^d)$ and H is the Schrödinger hamiltonian ($\hbar = m = 1$)

$$H = -\frac{1}{2}\Delta + V(x). \tag{1.39}$$

In this setting, $H^{(n)}$ is easily found to be the operator on $\otimes^n L^2(\mathbb{R}^d) \simeq L^2(\mathbb{R}^{nd})$

$$H^{(n)} = -\frac{1}{2}\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} V(x_j), \tag{1.40}$$

that is a usual n-particle Schrödinger operator with external potential V. In general, the *second quantization* of H is then defined as

$$d\Gamma(H) := \bigoplus_{n\geq 0} H^{(n)}, \tag{1.41}$$

that is, the action of $d\Gamma(H)$ on a Fock space vector is given component-wise by

$$(d\Gamma(H)\Psi)^{(n)} = H^{(n)}\Psi^{(n)}. \tag{1.42}$$

An important example is given by the second quantization of the identity on $\mathfrak{h}$ which is usually called *number operator* $\mathbf{N}$ and it is easily computed as

$$\begin{cases} \mathbf{N} := d\Gamma(\mathbb{I}) \\ (\mathbf{N}\Psi)^{(n)} = n\Psi^{(n)}. \end{cases} \tag{1.43}$$

A similar class of operator can be constructed on Fock space from a single-particles ones. For any operator S on $\mathfrak{h}$, define its n-fold tensor product by

$$\otimes^n S := \underbrace{S \otimes \cdots \otimes S}_{n-\text{times}}. \tag{1.44}$$

Then the Γ-operator associated with S is defined as

$$\Gamma(S) := \bigoplus_{n\geq 0} \otimes^n S. \tag{1.45}$$

It can be shown that both $d\Gamma$ and Γ operators both commute with Π_B, so that their bosonic version is simply given by a reduction procedure (see next Section) on $\mathcal{F}_B(\mathfrak{h})$, that is

$$\begin{aligned} d\Gamma_B(H)\Psi &:= d\Gamma(H)\Psi,\ \forall\Psi\in\mathcal{F}_B(\mathfrak{h})\cap D(d\Gamma(H)) \\ \Gamma_B(S)\Psi &:= \Gamma(S)\Psi,\ \forall\Psi\in\mathcal{F}_B(\mathfrak{h})\cap D(\Gamma(S)). \end{aligned} \tag{1.46}$$

Second quantizations of one-particle operators usually model Hamiltonians of *free* (non-interacting) particles. However, in the concrete case in which $\mathfrak{h} = L^2(\mathbb{R}^d)$ we can construct a *two*-body operator for any sector of boson Fock space if the interaction is given in terms of a multiplication operator. More precisely, let $\mathcal{U}$ a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ which is almost everywhere finite and symmetric, that is $\mathcal{U}(x,y) = \mathcal{U}(y,x)$ for almost every pair $(x,y) \in \mathbb{R}^{2d}$. The two-body operator $\mathcal{U}^n$ associated with $\mathcal{U}$ is defined on the space $L^2_s(\mathbb{R}^{nd})$ of symmetric L^2 functions as the multiplication operator

$$\mathcal{U}^{(n)}(x_1,\ldots,x_n) := \frac{1}{2}\sum_{1\leq j<k\leq n}\mathcal{U}(x_j,x_k). \tag{1.47}$$

The natural extension of $\mathcal{U}^{(n)}$ to the whole Fock space is given by

$$\mathbf{U} := \bigoplus_{n\geq 0}\mathcal{U}^{(n)}, \tag{1.48}$$

which physically describes an interaction of (possibly) infinitely many bosons. A typical Hamiltonian on $\mathcal{F}_B(L^2(\mathbb{R}^d))$ is then defined by

$$\mathbf{H} = d\Gamma(H) + \mathbf{U} = \bigoplus_{n\geq 0}\left(\sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial x_j^2} + V(x_j)\right) + \mathcal{U}^{(n)}(x_1,\ldots,x_n)\right). \tag{1.49}$$

Remark 1.2.3. (i) If an operator $\mathbf{A}$ on $\mathcal{F}_B(\mathfrak{h})$ is defined as $\mathbf{A} = \oplus_{n\geq 0}A_n$ where each A_n is a self-adjoint operator on $\mathfrak{h}_B^{\otimes n}$, then $\mathbf{A}$ is self-adjoint and (see [8])

$$e^{-it\mathbf{A}} = \bigoplus_{n\geq 0} e^{-itA_n}. \tag{1.50}$$

In particular this means that the quantum evolution generated by $\mathbf{H}$ preserves sectors of $\mathcal{F}_B(\mathfrak{h})$, that is if $\Psi \in \mathcal{F}_B^{(N)}$, then $e^{-it\mathbf{H}}\Psi \in \mathcal{F}_B^{(N)}$.

(ii) There exist an important relation connecting operators of the form in (1.49) creation and annihilation operators. Using the explicit form of b and b^* on $\mathcal{F}(L^2_s(\mathbb{R}^d))$

$$(b(f)\Psi)^{(n)}(x_1,\dots,x_n) = \sqrt{n+1}\int_{\mathbb{R}^d} \overline{f(x)}\Psi^{(n+1)}(x,x_1,\dots,x_n)dx \tag{1.51}$$

$$(b(f)^*\Psi)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(x_j)\Psi^{(n-1)}(x_1,\dots,\hat{x}_j,\dots,x_n), \tag{1.52}$$

it can be shown that (see [24], Section 10.2.2)[2]

$$d\Gamma(H) = \sum_{j,k}\langle e_j, He_k\rangle b(e_j)^* b(e_k) \tag{1.53}$$

$$\mathbf{U} = \frac{1}{2}\sum_{j,k,l,m}\langle e_j\vee e_k, \mathcal{U}(e_l\vee e_m)\rangle b(e_j)^* b(e_k)^* b(e_l) b(e_m), \tag{1.54}$$

where $\{e_j\}$ is a orthonormal basis in $L^2(\mathbb{R}^d)$ and the series converge in the strong sense on $\mathcal{F}_{B,0}(L^2(\mathbb{R}^d))$.

1.3 Finite dimensional reduction of creation and annihilation operators

Restricting a linear operator on $\mathfrak{h}$ to a target closed subspace M is not always possible, since its domain may have zero intersection with M. However in some important cases this can be done. Consider the projector P onto M, then a linear operator $T\colon D(T)\to\mathfrak{h}$ is said to be *reduced* by M if $PT\subset TP$,[3] that is for any $f\in D(T)$, $Pf\in D(T)$ and $PTf = TPf$. If this happens, the *reduction* of T to M is the operator defined by

$$\begin{cases} D(T_M) = D(T)\cap M \\ T_M f = Tf, \ \ \forall f\in D(T_M). \end{cases} \tag{1.55}$$

As we saw in previous section, examples of reduced operator are provided by $d\Gamma$ and Γ operators, whose reduction to $\mathcal{F}_B(\mathfrak{h})$ is given by $d\Gamma_B$ and Γ_B operators respectively.

Let us now consider, for any fixed $L\in\mathbb{N}$, a finite dimensional subspace $V\subset\mathfrak{h}$ with $\dim V = L$. Then the following result holds.

[2] Equation (1.53) actually holds also in the case in which $\mathfrak{h}$ does not have a concrete L^2 realization.

[3] Here the symbol $A\subset B$ clearly means that A is extended by B.

Lemma 1.3.1. *Let $Q_L\colon \mathfrak{h} \to V$ be the orthogonal projection associated with V. Then*[4]

(i) $\otimes^n Q_L$ *is an orthogonal projection with range* $\otimes^n V$.

(ii) $\otimes^n Q_L$ *is reduced by* $\otimes_s^n \mathfrak{h}$.

(iii) $\Gamma_B(Q_L)$ *is an orthogonal projection with range* $\mathcal{F}_B(V)$.

(iv) $b(f), b(f)^*$ *are reduced by* $\mathcal{F}_B(V)$ *for all* $f \in V$.

Proof. (i) See Theorem 3.2 (vii) in [8].
(ii) We prove the equivalent condition $(\otimes^n Q_L)S_n = S_n \otimes^n Q_L$. Let $\psi \in \otimes^n \mathfrak{h}$ and define the basis $\{\varphi_j\}_{j\in\mathbb{N}}$ of $\mathfrak{h}$ in the following way. The first L vectors of the φ_j's are an orthonormal basis of V, while the others are obtained by taking the completition to a basis of $\mathfrak{h}$. Since the choice of ψ does not depend on the basis, we can write

$$\psi = \sum_{\alpha_1,\dots,\alpha_n} c_{\alpha_1,\dots,\alpha_n} \varphi_{\alpha_1} \otimes \cdots \otimes \varphi_{\alpha_n}. \tag{1.56}$$

The action of $\otimes^n Q_L$ to ψ is simply given by truncating the series in (1.56), so we have

$$\begin{aligned}
S_n(\otimes^n Q_L)\psi &= S_n \sum_{\alpha_1,\dots,\alpha_n \le L} c_{\alpha_1,\dots,\alpha_n} \varphi_{\alpha_1} \otimes \cdots \otimes \varphi_{\alpha_n} \\
&= \sum_{\alpha_1,\dots,\alpha_n \le L} c_{\alpha_1,\dots,\alpha_n} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \varphi_{\sigma(\alpha_1)} \otimes \cdots \otimes \varphi_{\sigma(\alpha_n)} \\
&= (\otimes^n Q_L) \sum_{\alpha_1,\dots,\alpha_n} c_{\alpha_1,\dots,\alpha_n} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \varphi_{\sigma(\alpha_1)} \otimes \cdots \otimes \varphi_{\sigma(\alpha_n)} \\
&= (\otimes^n Q_L) S_n \psi,
\end{aligned}$$

where in the third line we used the fact that, for any permutation $\sigma \in \mathfrak{S}_n$

$$(\otimes^n Q_L)(\varphi_{\sigma(\alpha_1)} \otimes \cdots \otimes \varphi_{\sigma(\alpha_n)}) \equiv 0, \quad \text{if} \quad \sigma(\alpha_j) > L$$

for some $j = 1, \dots, n$. It is easy to see that this happens if and only if $\alpha_j > L$ for some $j = 1, \dots, n$.
(iii) An easy application of Theorem 5.6 (v) in [8] if the closed subspace considered is V.
(iv) See again [8], Theorem 5.15, in the case in which the closed subspace is V. □

[4] recall (1.44) for the definition of the product $\otimes^n S$ for some operator S on $\mathfrak{h}$.

Statement (iv) is particularly useful when considering concrete realizations on some Hilbert space of a finite set of creation and annihilation operators. For all $f \in V$ we can define the reduced operators by

$$\begin{aligned} D(b_{red}(f)^*) &= D(b(f)^*) \cap \mathcal{F}_B(V), \\ b_{red}(f)^*\Psi &= b(f)^*\Psi \quad \forall \Psi \in D(b_{red}(f)^*) \end{aligned} \tag{1.57}$$

and

$$\begin{aligned} D(b_{red}(f)) &= D(b(f)) \cap \mathcal{F}_B(V), \\ b_{red}(f)\Psi &= b(f)\Psi \quad \forall \Psi \in D(b_{red}(f)) \end{aligned}. \tag{1.58}$$

Given a basis $\{e_j\}_{j=1}^L$ of V we will use the following notation: for all $j = 1, \dots, L$

$$\begin{aligned} b_j &:= b_{red}(e_j) \\ b_j^* &:= b_{red}(e_j)^* \end{aligned} \tag{1.59}$$

Remark 1.3.2. In general a second quantization operator cannot be reduced by means of Lemma 1.3.1. However, we can consider hamiltonian operators on $\mathcal{F}_B(V)$ built with reduced creation and annihilation operators as in (1.53)-(1.54)

$$H_L := Q_L \mathbf{H} Q_L = \sum_{j,k=1}^{L} \tau_{jk} b_j^* b_k + \frac{1}{2} \sum_{j,k,l,m=1}^{L} \Lambda_{jklm} b_j^* b_k^* b_m b_l. \tag{1.60}$$

In Chapter 3 we will study the dynamics of such a hamiltonian in the simple case in which the only non-zero coefficients of the quartic term are the diagonal ones $\Lambda_{jjjj} \neq 0$ by using a representation on Bargmann space.

1.4 Isomorphism of $\mathcal{F}_B(V)$ and Bargmann space

In this section we construct a unitary operator mapping the boson Fock space over a finite dimensional Hilbert space to $\mathcal{H}L^2$. We remark that this construction is briefly mentioned in [24] and sketched in [30], although the computations are not explicitly done.

Let us then consider $\mathcal{F}_B(V)$, $\dim V = L$, which is naturally isomorphic to

$$\mathcal{F}_B(\mathbb{C}^L) = \bigoplus_{k \geq 0} \otimes_s^k \mathbb{C}^L.$$

Given a basis $\{e_j\}_{1 \leq j \leq L} \subset \mathbb{C}^L$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_L)$, we denote as $e_j^{\alpha_j}$

the α_j-times repetition of e_j, that is

$$e_j^{\alpha_j} = \underbrace{e_j \otimes \cdots \otimes e_j}_{\alpha_j \text{ times}},$$

so that an orthonormal basis of $\otimes_s^k \mathbb{C}^L$ can be simply taken as the set

$$\mathcal{B}_k = \left\{ \sqrt{\frac{k!}{\alpha!}} S_k(e_1^{\alpha_1} \otimes \cdots \otimes e_L^{\alpha_L}),\ |\alpha| = k \right\}. \tag{1.61}$$

Counting all the distinct elements in $\mathcal{B}_k$ we see that

$$\dim \otimes_s^k \mathbb{C}^L = \binom{L+k-1}{k}$$

and recalling the definition of Poly_L^k (see Remark 1.1.1 (iv)), there exist a linear space isomorphism $\mathrm{Poly}_L^k \simeq \otimes_s^k \mathbb{C}^L$ (cfr. (1.8) for $n = L$). Such a map can be actually taken to be *unitary* with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}L^2}$ and the natural inner product on $\otimes_s^k \mathbb{C}^L$. Indeed, define the application U_k sending the basis defined in (1.4) into a basis of $\otimes_s^k \mathbb{C}^L$

$$U_k(u_\alpha) = \sqrt{\frac{k!}{\alpha!}} S_k(e_1^{\alpha_1} \otimes \cdots \otimes e_L^{\alpha_L}), \quad |\alpha| = k, \tag{1.62}$$

and extend it by linearity on the whole Poly_L^k. Obviously

$$\|U_k(u_\alpha)\|_{\mathcal{F}_B} = \|u_\alpha\|_{\mathcal{H}L^2}$$

and as a consequence, for every homogeneous polynomial $p \in \mathrm{Poly}_L^k$, one has

$$\|U_k(p)\|_{\mathcal{F}_B} = \|p\|_{\mathcal{H}L^2},$$

thus U_k is an isometry. It is easily verified that U_k is surjective, so it is a unitary map $\mathrm{Poly}_L^k \to \otimes_s^k \mathbb{C}^L$. By a standard theorem (see [8], Theorem 4.1) the linear operator

$$U := \bigoplus_{k \geq 0} U_k \colon \mathcal{H}L^2(\mathbb{C}^L) \to \mathcal{F}_B(\mathbb{C}^L)$$

turns to be unitary on the whole spaces.
We see now how the creation and annihilation operators of $\mathcal{H}L^2(\mathbb{C}^L)$ (see Definition 1.1.7) are unitarily equivalent to the ones of $\mathcal{F}_B(\mathbb{C}^L)$.

Lemma 1.4.1. *Let $b_j^* = b(e_j)^*$ and $b_j = b(e_j)$ be the creation and annihilation operators on $\mathcal{F}_B(\mathbb{C}^L)$, associated with a basis $\{e_j\}_{1 \leq j \leq n}$ of $\mathbb{C}^L$. Then $D(b_j^*) = UD(A_j^*)$,*

$D(b_j) = UD(A_j)$ and

$$b_j^* = UA_j^*U^{-1}, \tag{1.63}$$

$$b_j = UA_jU^{-1}, \tag{1.64}$$

on $D(b_j) = D(b_j^*)$.

Proof. Let us prove Eq. (1.63) on $\mathcal{F}_{b,0}(\mathbb{C}^L)$. It is sufficient to consider vectors in $\mathcal{B}_k$ for some $k \in \mathbb{N}$ (recall 1.61) and extend the argument by linearity. Given a multi-index α with $|\alpha| = k$, denote as E_k^α a fixed element in $\mathcal{B}_k$. By Eq. (1.62) we have

$$\begin{aligned}
(A_j^*U^{-1}E_k^\alpha)(z) &= (A_j^*u_\alpha)(z) \\
&= \frac{z_1^{\alpha_1}\cdots z_j^{\alpha_j+1}\cdots z_n^{\alpha_n}}{\sqrt{\alpha_1!\cdots\alpha_n!}} \\
&= \sqrt{\alpha_j+1}\,\frac{z_1^{\alpha_1}\cdots z_j^{\alpha_j+1}\cdots z_n^{\alpha_n}}{\sqrt{\alpha_1!\cdots(\alpha_j+1)!\cdots\alpha_n!}} \\
&= \sqrt{\alpha_j+1}u_{\alpha+\mathbf{1}_j}(z),
\end{aligned}$$

where $\alpha + \mathbf{1}_j$ is the multi-index $(\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_n)$. Since $|\alpha + \mathbf{1}_j| = k + 1$, U sends $u_{\alpha+\mathbf{1}_j}$ to the $(k+1)$-particle sector of $\mathcal{F}_B(\mathbb{C}^L)$, in particular $U(u_{\alpha+\mathbf{1}_j}) = E_{k+1}^{\alpha+\mathbf{1}_j}$. We then have

$$\begin{aligned}
UA_j^*U^{-1}E_k^\alpha &= \sqrt{\alpha_j+1}E_{k+1}^{\alpha+\mathbf{1}_j} \\
&= \sqrt{\alpha_j+1}\cdot\sqrt{\frac{(k+1)!}{(\alpha+\mathbf{1}_j)!}}\cdot S_{k+1}(e_1^{\alpha_1}\otimes\cdots\otimes e_j^{\alpha_j+1}\otimes\cdots\otimes e_n^{\alpha_L}) \\
&= \sqrt{k+1}\cdot\sqrt{\frac{k!}{\alpha!}}\cdot S_{k+1}(e_1^{\alpha_1}\otimes\cdots\otimes e_j^{\alpha_j+1}\otimes\cdots\otimes e_n^{\alpha_L}) \\
&= b_j^*\left(\sqrt{\frac{k!}{\alpha!}}S_k(e_1^{\alpha_1}\otimes\cdots\otimes e_j^{\alpha_j}\otimes\cdots\otimes e_n^{\alpha_L})\right) \\
&= b_j^*E_k^\alpha,
\end{aligned}$$

so Eq. (1.63) is true on $\mathcal{F}_{b,0}(\mathbb{C}^L)$. By simply taking the adjoint, the same conclusion holds for all the b_j's restricted to $\mathcal{F}_{b,0}(\mathbb{C}^L)$. Consider now $\Psi \in \mathcal{F}_B(\mathbb{C}^L)$ and its expansion on the Fock basis $\Psi = \sum_\alpha c_\alpha \widetilde{E_k^\alpha}$. If $\Psi \in D(b_j^*)$ then

$$\|b_j^*\Psi\|_{\mathcal{F}_B}^2 = \sum_\alpha |c_\alpha|^2(\alpha_j+1) < \infty,$$

so if we define the function $f \in \mathcal{H}L^2(\mathbb{C}^L)$

$$f = \sum_\alpha c_\alpha u_\alpha$$

it is clear that $Uf = \Psi$ and

$$\|A_j^* f\|^2_{\mathcal{H}L^2} = \sum_\alpha |c_\alpha|^2(\alpha_j + 1).$$

It then follows that $f \in D(A_j^*)$ and since $\Psi \in D(b_j^*)$ is arbitrary one has $D(b_j^*) = UD(A_j^*)$. To prove Eq. (1.63) on the whole domain of b_j^*, take any $\Psi \in D(b_j^*)$. Since $\mathcal{F}_{b,0}(\mathbb{C}^L)$ is a core for b_j^*, there exist a sequence $\{\Psi_N\}_{N\geq 0} \subset \mathcal{F}_{b,0}(\mathbb{C}^L)$ such that

$$\begin{cases} \Psi_N \to \Psi \\ b_j^*\Psi_N \to b_j^*\Psi \end{cases} \qquad \text{for } N \to +\infty$$

in $\mathcal{F}_b(\mathbb{C}^L)$. By the preceeding discussion, there exist a unique sequence $\{f_N\}_{N\geq 0}$, with $f_N = U^{-1}\Psi_N$ such that $\lim_{N\to+\infty} f_N = f = U^{-1}\Psi$ in $\mathcal{H}L^2(\mathbb{C}^L)$. Since $\Psi_N \in \mathcal{F}_{b,0}(\mathbb{C}^L)$, it follows that

$$A_j^* f_N = U^{-1}b_j^* U f_N = U^{-1}b_j^*\Psi_N \to U^{-1}b_j^*\Psi = U^{-1}b_j^* U f.$$

By the closedness of A_j^*, we have that $A_j^* f_N \to A_j^* f$, so

$$A_j^* f = U^{-1}b_j^* U f,$$

or equivalently

$$UA_j^*U^{-1}\Psi = b_j^*\Psi, \qquad \forall \Psi \in D(b_j^*),$$

that is Eq. (1.63). Using a similar argument for b_j, Eq. (1.64) follows on the whole $D(b_j)$. □

Chapter 2

Mean Field derivation of DNLS equation

The Bose-Hubbard model (BH) has gained a massive attention since the discovery of BECs in optical lattices, even though it has been studied prior to this as a quantum version of Discrete Non-linear Schrödinger equation (see e.g. [16]). In its simplest form, the BH hamiltonian for L lattice sites is given by

$$H = \sum_{j=1}^{L} \left(E_j A_j^* A_j + J(A_{j-1}^* A_j + A_{j+1}^* A_j) \right) + \frac{U}{2} \sum_{j=1}^{L} A_j^* A_j^* A_j A_j \tag{2.1}$$

where the A_j's and A_j^*'s are operators on some Hilbert space which satisfy the commutation rules $[A_j, A_k^*] = \delta_{jk}$. We show here how the flow of the discrete Nonlinear Schrödinger equation (DNLS) as a Mean Field limit of the quantum dynamics of a BH model for N interacting particles. By representing the operators involved as creation and annihilation operator on Bargmann space, we show that the Wick symbol of the annihilation operators evolved by the Heisenberg picture converges, as N becomes large, to the solution of the DNLS. A quantitative estimate is written in terms of the parameters of the model and a linear behavior in time is proved thanks to a Gaussian measure on initial data.

2.1 The mean field setting

The DNLS considered in this chapter is the ordinary differential equation on $\mathbb{C}^L$

$$i\frac{d}{dt} w_k(t) = E_k w_k(t) + J(w_{k+1}(t) + w_{k-1}(t)) + U|w_k(t)|^2 w_k(t) \tag{2.2}$$

with periodic boundary conditions (see [2])

$$\begin{cases} w_L(t) = w_0(t), \\ w_{L+1}(t) = w_1(t) \end{cases} \qquad \forall t \in \mathbb{R} \tag{2.3}$$

where $w_k(t) \in \mathbb{C}$, $1 \leq k \leq L$ and $\{E_1, \dots, E_L, J, U\}$ are positive parameters. Equation (2.2) is a particular case of Discrete Self-Trapping equation (DST) (see [26])

$$i\frac{d}{dt}w_k(t) = \sum_{j=1}^{L} \tau_{jk}\bar{w}_j(t)w_k(t) + U|w_k(t)|^2 w_k(t) \tag{2.4}$$

in the case in which the symmetric matrix $\tau_{jk} = \tau_{kj}$ is

$$\tau = \begin{pmatrix} E_1 & J & 0 & \cdots & 0 & J \\ J & E_2 & J & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & & J \\ J & 0 & \cdots & 0 & J & E_L \end{pmatrix} \tag{2.5}$$

It is convenient to rewrite (2.2) as a Hamiltonian system by considering its complex conjugate

$$-i\frac{d}{dt}\bar{w}_k(t) = E_k\bar{w}_k(t) + J(\bar{w}_{k+1}(t) + \bar{w}_{k-1}(t)) + U|w_k(t)|^2\bar{w}_k(t). \tag{2.6}$$

It is immediate to see that given the (real) function $h = h(\bar{w}, w)$

$$h(\bar{w}, w) := \sum_{j=1}^{L} \left(E_j|w_j|^2 + J(\bar{w}_{j-1}w_j + \bar{w}_{j+1}w_j)\right) + \frac{U}{2}\sum_{j=1}^{L}|w_j|^4 \tag{2.7}$$

then (2.2) and (2.6) can be written concisely as

$$\begin{cases} \dfrac{d}{dt}\bar{w}_k = i\dfrac{\partial h}{\partial w_k} \\ \dfrac{d}{dt}w_k = -i\dfrac{\partial h}{\partial \bar{w}_k} \end{cases} \tag{2.8}$$

The model we expect to have the DNLS as Mean Field limit is the *rescaled* Bose-Hubbard hamiltonian acting on $\mathcal{H}L^2(\mathbb{C}^L)$

$$H_N(A) = \sum_{j=1}^{L} \left(E_jA_j^*A_j + J(A_{j-1}^*A_j + A_{j+1}^*A_j)\right) + \frac{U}{2N}\sum_{j=1}^{L} A_j^*A_j^*A_jA_j \tag{2.9}$$

for positive parameters $\{E_1, \dots, E_L, J, U\}$ where the A_j's were defined in Section 1.1.

The literature on the Mean Field derivation of the Nonlinear Schrödinger Equation (NLS), Hartree equation, and more in general about the study of many body quantum mechanical systems, is quite rich (see for example [4, 13, 28, 31, 33, 35, 39, 40, 44, 53] and references therein). However, it seems that a direct Mean Field derivation of the DNLS together with quantitive, explicit estimates is missing. In some works (see [1, 47, 52] and references therein) the DNLS is obtained directly from the NLS equation, in the framework of the tight-binding approximation. However, combining these two kinds of results, a growth exponential in time of the Mean Field estimate for DNLS follows, essentially due to the usage of Grönwall lemma. Here instead a growth linear in time of the Mean Field estimate is provided. In [47], the validity of the discrete nonlinear Schrödinger equation is studied for the tight-binding approximation of the Gross–Pitaevskii equation with a periodic potential, where the approach is based on Wannier functions associated to it. In [52] the author consider, without the use of the Wannier functions, the nonlinear one-dimensional time-dependent Schrödinger equation with a periodic potential plus a bounded perturbation. In the limit of large potential the wave function can be approximated by the solution to the DNLS. In [1] the validity of tight-binding approximation is studied for the NLS with a two-dimensional optical lattice.

2.2 Setting and results

To avoid any possible confusion about the notation, let us denote as $u_k(t, \bar{w}, w)$ the k-th component of the solution of DNLS equation with initial conditions $u_k(0, \bar{w}, w) = w_k$ for all $1 \leq k \leq L$. Here we show that the coherent quantum expectation of $A_k/\sqrt{N}$ is close to $u_k(t, w)$ in a suitable L^p-measure sense when N is large, which is precisely what we mean by the Mean Field limit of (2.9). This is achieved thanks to an explicit estimate expressed in terms of the parameters of the model, globally in time.

Our first step to deal with the Mean Field asymptotics is to consider the rescaled operators

$$\hat{a}_k := \frac{A_k}{\sqrt{N}}, \quad \hat{a}_k^* := \frac{A_k^*}{\sqrt{N}}, \quad [\hat{a}_j, \hat{a}_k^*] = \frac{\delta_{jk}}{N}\,\mathrm{Id}, \tag{2.10}$$

and the Heisenberg equation $i\frac{d}{dt}A_k(t) = [A_k(t), H_N(A)]$ that reads

$$i\frac{d}{dt}\hat{a}_k(t) = E\hat{a}_k(t) + J(\hat{a}_{k+1}(t) + \hat{a}_{k-1}(t)) + U\hat{a}_k^*(t)\hat{a}_k(t)\hat{a}_k(t)\ , \tag{2.11}$$

where $\hat{a}_k(0) := \hat{a}_k$ and $1 \leq k \leq L$. Notice that equation (2.11) is clearly the operator counterpart of (2.2). We now rewrite equation (2.11) in terms of the Wick symbols (see Section 1.1) of the operators $\hat{a}_k(t)$ and $\widehat{H}$. Let $\phi_w(\bar{z}) := e^{w\cdot\bar{z}-\frac{1}{2}|w|^2}$ be the normalized coherent states in $\mathcal{H}L^2(\mathbb{C}^L)$ and recall that $\hat{a}_k\phi_{\sqrt{N}w} = w_k\phi_{\sqrt{N}w}$. Define the symbol

$$\rho_k(t,\bar{w},w) := \langle \phi_{\sqrt{N}w}, \hat{a}_k(t)\phi_{\sqrt{N}w}\rangle \tag{2.12}$$

and notice that

$$\begin{aligned}\langle \phi_{\sqrt{N}w}, H_N(A)\phi_{\sqrt{N}w}\rangle &= N\sum_{1\le j\le L}\left(E_j|w_j|^2 + J(\bar{w}_{j+1}w_j + \bar{w}_j w_{j+1}) + \frac{U}{2}|w_j|^4\right)\\ &= Nh(\bar{w},w).\end{aligned} \tag{2.13}$$

Then taking the expectation along a fixed coherent state φ_w, that is taking the Wick symbol associated with the Heisenberg equation (2.11), we get

$$\frac{i}{N}\frac{d}{dt}\rho_k = \{\rho_k, h\}_\star, \tag{2.14}$$

with initial data $\rho_k(0,w,\bar{w}) = w_k$, where the Wick bracket has been defined in Section 1.1. It is readily seen that, as an asymptotic series,

$$\{\rho_k, h\}_\star \;\simeq\; \sum_{|r|\ge 1}^{\infty}\frac{1}{r!}\left(\frac{1}{N}\right)^{|r|}\left(\frac{\partial^r\rho_k}{\partial w^r}\frac{\partial^r h}{\partial\bar{w}^r} - \frac{\partial^r h}{\partial w^r}\frac{\partial^r \rho_k}{\partial\bar{w}^r}\right), \tag{2.15}$$

In view of (2.14) - (2.15) we recognize the role of $1/N$ as a semiclassical parameter. Since h is a second order polynomial of $\bar{w}$ and w separately, it follows that the Wick bracket is a finite sum $\{\cdot,h\}_\star = N^{-1}\mathcal{L}_1 + N^{-2}\mathcal{L}_2$ where

$$\mathcal{L}_1 := \sum_{j=1}^{L}\left(\frac{\partial h}{\partial\bar{w}_j}\frac{\partial}{\partial w_j} - \frac{\partial h}{\partial w_j}\frac{\partial}{\partial\bar{w}_j}\right), \tag{2.16}$$

$$\mathcal{L}_2 := \frac{1}{2}\sum_{j,k=1}^{L}\left(\frac{\partial^2 h}{\partial\bar{w}_j\partial\bar{w}_k}\frac{\partial^2}{\partial w_j\partial w_k} - \frac{\partial^2 h}{\partial w_j\partial w_k}\frac{\partial^2}{\partial\bar{w}_j\partial\bar{w}_k}\right). \tag{2.17}$$

Thus, it follows that $\mathcal{L}_1$ is precisely the generator of DNLS flow (2.2)

$$i\frac{d}{dt}u = \mathcal{L}_1(u). \tag{2.18}$$

By denoting $\Delta := \{(\bar{w},w)\mid w\in\mathbb{C}^L\}\subset\mathbb{C}^{2L}$, $(\bar{\Phi}_t,\Phi_t):\Delta\subset\mathbb{C}^{2L}\to\mathbb{C}^{2L}$ the flow of

$\dot{\gamma} = i(\partial_w h(\gamma), -\partial_{\bar{w}} h(\gamma))$, it follows that

$$u(t, w) = \Phi_t(\bar{w}, w) \ . \tag{2.19}$$

This observation, together with equation (2.14), tells us that $\rho_k - u_k$ is a kind of semiclassical perturbation term, and thus $\rho_k - u_k \to 0$ as $N \to +\infty$. Indeed, we will prove such a result with respect to an $L^p(\mu_N)$-norm, where $p \geq 1$ and μ_N is a suitable Gaussian measure, invariant under the DNLS flow. With respect to this target, remind that the total number operator defined as $\widehat{N} := \sum_{k=1}^{L} A_k^* A_k = N \sum_{k=1}^{L} \hat{a}_k^* \hat{a}_k$ fulfills $[H_N(A), \widehat{N}] = 0$ and whence the Euclidean norm

$$|w|^2 := \sum_{k=1}^{L} |w_k|^2 \tag{2.20}$$

of any initial datum $w \in \mathbb{C}^L$ is conserved by the quantum flow, since

$$\langle \phi_{\sqrt{N}w}, \widehat{N} \phi_{\sqrt{N}w} \rangle = N|w|^2, \tag{2.21}$$

or in other terms

$$\{h, |w|^2\}_\star = 0. \tag{2.22}$$

Moreover, the well known ℓ^2 - conservation law for the DNLS can be rewritten as

$$\mathcal{L}_1(|w|^2) = 0. \tag{2.23}$$

Both these two important properties will be used in the proof of the Theorem 2.2.1, and for this reason we define the invariant Gaussian probability measure

$$d\mu_N(\bar{w}, w) := \mathcal{N}^L \ e^{-N|w|^2} dw d\bar{w}, \tag{2.24}$$

where $w = x + iy$ and $\mathcal{N}^L := \pi^{-L} N^L$ is the normalization constant. This measure is linked (see Prop. 2.3.1) to a weighted Trace formula involving Wick operators that will be an important tool to our approach. We are now ready to state the main result of this chapter.

Theorem 2.2.1. *Let $u(t, w)$ be the flow of the DNLS equation (2.2) and let $\rho_k(t, w)$ be the solution of (2.14) for $1 \leq k \leq L$. Then $\forall$ $p \geq 1$ we have $u_k, \rho_k \in L^p(\mu_N)$ and that there exist a positive constant A_p depending only on p such that*

$$\|\rho_k - u_k\|_{L^p(\mu_N)} \leq A_p \, \frac{L}{N} \frac{U \ t}{\sqrt{N}}, \qquad \forall t \geq 0, \tag{2.25}$$

We notice that (2.25) can be written with the condition $L/N \leq D < +\infty$, namely

the number of *sites* L in the Bose-Hubbard model can be an increasing function of N up to a linear behavior.
We also stress that the L^p - norm used above allows to discuss, in the measure sense, the pointwise estimate for $|\rho_k - u_k|(t, \bar{w}, w)$. Indeed, we have the following

Corollary 2.2.2. *Fix a parameter* $0 < \epsilon < \frac{1}{2}$ *and define the set*

$$w_k := \Big\{(\bar{w}, w) \mid \quad |\rho_k - u_k|(t, \bar{w}, w) > A_p \frac{L}{N} \frac{U\, t}{N^{\epsilon}}, \quad \forall t \geq 0\Big\}. \tag{2.26}$$

Then, for any $1 \leq k \leq L$

$$\mu_N(w_k) \leq N^{-p\cdot(\frac{1}{2}-\epsilon)}, \quad \forall p \geq 1 \quad \forall N \geq 1. \tag{2.27}$$

Moreover, when $L/N \leq D < +\infty$ *the set*

$$\mathcal{U}_k := \Big\{(\bar{w}, w) \mid \quad |\rho_k - u_k|(t, \bar{w}, w) > A_p\, D\, U\ t, \qquad \forall t \geq 0\Big\} \tag{2.28}$$

fulfills $\mathcal{U}_k \subset w_k$ *and hence* $\mu_N(\mathcal{U}_k) \leq \mu_N(w_k)$.

Notice that $p \mapsto A_p$ is an increasing function, whence inequality (2.27) provides, when N is fixed and p is large, a measure of the region where $|\rho_k - u_k|$ is large. On the other hand, in the case of fixed p and large N we have a vanishing measure of the region where $|\rho_k - u_k|$ is superlinear in time.
The BH model considered here is actually simpler than the ones usually employed in Quantum Field Theory. However, the explicit estimate in terms of the parameters of the model and its linear dependence on time in (2.25) seem to be a novel and promising result with respect to other kind of Mean Field estimates on the NLS.
Furthermore, we stress that Theorem 2.2.1 can be seen as an Egorov type result (see [59]), written to the first order and with respect to the L^p-norm, for Wick symbols. With respect to this observation, we recall Proposition 5.1 in [22] where it is proved the convergence, as $\hbar \to 0$, of the Wick symbol of an evolved quantum observable towards the Weyl symbol composed with the Hamiltonian flow. In this result, the well known bound of the Ehrenfest time $|t| < T_\hbar$ is shown. We also recall Proposition 5.10 and Theorem 5.6 in [5] where, in the framework of evolved Wick operators on the Fock space and with a quantum dynamics much more general than our, it is proved the convergence towards the solution of the Hartree equation as $1/N \to 0$, but the estimate on the remainder in Theorem 5.6 is again local in time. In our main result we avoid locality in time by making use of the $L^p(\mu_N)$-norm (the meaning and the properties of the measure μ_N are clarified in Prop. 2.3.1 and 2.3.3 below).
To conclude, we stress the absence in (2.25) of the parameters E and J involved

in the quadratic part of the operator $H_N(A)$ in (2.9), in agreement with a well known elementary result: any quantum expectation of the Heisenberg equations of a linear system (quadratic Hamiltonian) yields the classical equations of motion. Thus, the distance between the Wick symbol ρ_k solving the equation (2.14) and the k-th component u_k of the flow for the equation (2.18) is ruled only by nonlinearity, namely by the parameter U. As a consequence, Theorem 2.2.1 holds for Hamiltonian operators H with a completely general quadratic part, such as the DST equation discussed above. This is not the primary target of the present work, but we observe here that a more general setting of H ensures a larger set of invariant measures for the DNLS flow and whence an interesting open problem is to study the link between this kind of Mean Field estimates and the various invariant measures, which will be treated in a forthcoming paper.

2.3 Proof of main theorems

In this Section we provide the proof of the main Theorem we have stated in previous one. To such a purpose, we will need some preliminary Lemmas and Propositions. Among them, Lemma 1 and 2 are just quoted and used, their statements and proofs being reported at the end of the Chapter.

The following result provides a weighted Trace Formula for Wick operators, involving the positive definite operator $e^{-\lambda\widehat{N}}$ with $\lambda > 0$. In order to make a link with the gaussian measure μ_N given in (2.24), we have to write a bijective relation between the parameters λ and N. This result will be useful for the subsequent result on expectation values of Wick operators under quantum dynamics.

Proposition 2.3.1. *Let μ_N be as in (2.24). Let $Op^W(g)$ be a Wick operator on $\mathcal{F}_B(\mathbb{C}^L)$ such that $g \in L^1(\mu_N)$. Let $\widehat{N} := \sum_{k=1}^{L} A_k^* A_k$. Then*

$$\mathrm{Tr}\left(\frac{e^{-\lambda\widehat{N}}}{\gamma_\lambda} Op^W(g)\right) = \int g \; d\mu_N \tag{2.29}$$

where $\gamma_\lambda := \mathrm{Tr}(e^{-\lambda\widehat{N}})$ and $e^\lambda = N + 1$.

Proof. We begin by the equality

$$\mathrm{Tr}\left(\frac{e^{-\lambda\widehat{N}}}{\gamma_\lambda} \mathrm{Op}^W(g)\right) = \int_{\mathbb{C}^L} \sigma\left(\frac{e^{-\lambda\widehat{N}}}{\gamma_\lambda} \mathrm{Op}^W(g)\right)(\bar{w}, w)\; dw d\bar{w} \tag{2.30}$$

and notice that the Wick symbol of $e^{-\lambda\widehat{N}}$ reads

$$\sigma\left(e^{-\lambda\widehat{N}}\right)(\bar{w}, w) = e^{-\mu|w|^2}, \quad \mu := 1 - e^{-\lambda}. \tag{2.31}$$

Equality (2.31) allows to write the constant

$$\gamma_\lambda = \mathrm{Tr}(e^{-\lambda\widehat{N}}) = \int \sigma\left(e^{-\lambda\widehat{N}}\right)(\bar{w}, w)\ dwd\bar{w} = \left(\frac{1}{\mu}\right)^L. \tag{2.32}$$

Thanks to the Wick-$\star$ product, (2.30) can be rewritten as

$$\frac{1}{\gamma_\lambda}\int e^{-\mu|w|^2} \star g(\bar{w}, w)\ dwd\bar{w}. \tag{2.33}$$

We also remind formula (2.38) in [13] that provides a link between Wick and anti-Wick symbols

$$e^{-\mu|w|^2} = e^{\Delta_{\bar{w}w}}\sigma_{\mathrm{AW}}(e^{-\lambda\widehat{N}}) = \int e^{-(z-w)(\bar{z}-\bar{w})}\ \sigma_{\mathrm{AW}}(z, \bar{z})\ dzd\bar{z}, \tag{2.34}$$

where $\Delta_{\bar{w}w} := \sum_{k=1}^L \frac{\partial^2}{\partial\bar{w}_k\partial w_k}$, and remind that Wick and anti-Wick symbols of $e^{-\lambda\widehat{N}}$ are unique. Integration by parts in (2.33) gives (formally)

$$\frac{1}{\gamma_\lambda}\int \left(e^{-\Delta_{\bar{w}w}}e^{-\mu|w|^2}\right) g(\bar{w}, w)\ dwd\bar{w}. \tag{2.35}$$

Recall that

$$d\mu_N(\bar{w}, w) = N^L\ e^{-N|w|^2}dwd\bar{w} \tag{2.36}$$

where N^L is a normalization constant. Our target is thus to prove the well posed equation

$$\gamma_\lambda^{-1}e^{-\mu|w|^2} = N^L\ e^{\Delta_{\bar{w}w}}e^{-N|w|^2}, \tag{2.37}$$

namely

$$\begin{aligned} e^{-\mu|w|^2} &= \gamma_\lambda\ N^L \int e^{-(z-w)(\bar{z}-\bar{w})}\ e^{-N|z|^2}\ dzd\bar{z} && (2.38)\\ &= \mu^{-L}N^L e^{-\frac{N}{N+1}|w|^2}\int e^{-(N+1)|z|^2}\ dzd\bar{z} && (2.39)\\ &= \mu^{-L}\left(\frac{N}{N+1}\right)^L e^{-\frac{N}{N+1}|w|^2} && (2.40)\end{aligned}$$

which is solved by $\mu = N/(N+1)$, and since $\mu = 1 - e^{-\lambda}$ we recover $e^\lambda = N+1$. □

Remark 2.3.2. We now recall that $A_j^*A_k = \mathrm{Op}^W(g)$ when $g = \bar{w}_j w_k$, (see Section 1.1). For these Wick operators, Proposition 2.3.1 reads

$$\mathrm{Tr}\left(\frac{e^{-\lambda\widehat{N}}}{\gamma_\lambda}A_j^*A_k\right) = \int \bar{w}_j w_k d\mu_N. \tag{2.41}$$

Since the monomials $\{w_k\}_{k=1}^L$ are orthonormal with respect to Gaussian measure, an easy computation shows that

$$\mathrm{Tr}\left(\frac{e^{-\lambda\widehat{N}}}{\gamma_\lambda}A_j^*A_k\right) = \frac{\delta_{jk}}{N} = \frac{\delta_{jk}}{e^\lambda - 1}. \tag{2.42}$$

Thus, equality (2.42) can be considered as the version, in the Fock-Bargmann space, of the Quantum Wick Theorem showed in [31] that works in the Fock space and with the related bosonic creation and annihilation operators of Quantum Field Theory.

In the next we provide a kind of quantum mean value formula for the time evolved $G(s) := U_s^* G U_s$ where $U_s = e^{-isH}$ where to simplify notations we write H instead of H_N (recall (2.9)) and $G = \mathrm{Op}^W(g)$ are Wick operators. This result will be applied within the proof of Theorem 2.2.1 for operators of type $G = (\hat{a}_k^*\hat{a}_k + \frac{1}{N})^p$ with rescaled creation and annihilation operators as in (2.10). This tool allows to avoid, in our setting and for our estimates, the well known problem of Ehrenfest time, as well as to avoid the application of Grönwall Lemma (and thus exponential in time upper bounds).

Proposition 2.3.3. *Let $G = Op^W(g)$ be a Wick operator on $\mathcal{H}L^2(\mathbb{C}^L)$ such that $g \in L^1(\mu_N)$. Let $G(s) := U_s^* G U_s$. Define $g(s, \bar{w}, w) := \langle \phi_w, G(s)\phi_w\rangle$. Then, $\forall s \geq 0$*

$$\int g(s, \bar{w}, w) d\mu_N(\bar{w}, w) = \int g(0, \bar{w}, w) d\mu_N(\bar{w}, w). \tag{2.43}$$

Proof. We apply Proposition 2.3.1

$$\int g(0, \bar{w}, w)\, d\mu_N(\bar{w}, w) = \mathrm{Tr}\left(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op}^W(g)\right), \tag{2.44}$$

and recall that the trace is invariant by unitary conjugations of operators, so that

$$\mathrm{Tr}\left(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op}^W(g)\right) = \mathrm{Tr}\left(U_s^*\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op}^W(g)U_s\right). \tag{2.45}$$

Now we recall that $[\widehat{N}, H] = 0$ and whence $[\widehat{N}, U_s] = 0$, which gives

$$\mathrm{Tr}\left(U_s^*\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}\mathrm{Op}^W(g)U_s\right) = \mathrm{Tr}\left(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}U_s\mathrm{Op}^W(g)U_s^*\right) \tag{2.46}$$

and applying again Proposition 2.3.1 to $G(s)$ and, in view of Remark 2.3.4, we conclude

$$\mathrm{Tr}\left(\frac{e^{-\beta\widehat{N}}}{\gamma_\beta}G(s)\right) = \int g(s, \bar{w}, w) d\mu_N(\bar{w}, w) \tag{2.47}$$

since by definition $g(s,\cdot)$ must be the Wick symbol of $G(s)$ (recall also the definitions in Chapter 1). □

Remark 2.3.4. Since U_s is bounded on $\mathcal{H}L^2(\mathbb{C}^L)$, it follows that it is a Wick operator (see Section 1.1). This implies that $G(s)$ is Wick operator itself, since it is a composition of Wick operators.

Remark 2.3.5. The Prop. 2.3.3 works also with $g(s,a\bar{w},aw)$ and $g(0,a\bar{w},aw)$ for any fixed $a>0$. Indeed, for $\widetilde{N}:=N/a^2$ we have

$$\begin{aligned} \int g(s,a\bar{w},aw)\,d\mu_N(\bar{w},w) &= a^{-2L}\int g(s,\bar{v},v)\,d\mu_{\widetilde{N}}(\bar{v},v) = a^{-2L}\int g(\bar{v},v)\,d\mu_{\widetilde{N}}(\bar{v},v) \\ &= \int g(a\bar{w},aw)\,d\mu_N(\bar{w},w). \end{aligned} \tag{2.48}$$

This observation will be useful in the application of this equality with $a=\sqrt{N}$.

In what follows we get an estimate for $|\rho_k(t,\bar{w},w)-u_k(t,w)|$ for any fixed $w\in\mathbb{C}^L$. This will be used, in the proof of Theorem 2.2.1, to have the $L^p(\mu_N)$ estimate.

Proposition 2.3.6. *Let* $\Delta := \{(\bar{w},w)\mid w\in\mathbb{C}^L\}\subset\mathbb{C}^{2L}$, $(\bar{\Phi}_t,\Phi_t):\Delta\subset\mathbb{C}^{2L}\to\mathbb{C}^{2L}$ *the flow of* $\dot{\gamma}=i(\partial_w h(\gamma),-\partial_{\bar{w}}h(\gamma))$ *with* h *as in (2.13). Let* $u(t,w):=(u_1,\dots u_L)(t,w)$ *be the solution of (2.2), and*

$$\rho_k(t,\bar{w},w) := \langle\phi_{\sqrt{N}w},\hat{a}_k(t)\phi_{\sqrt{N}w}\rangle \tag{2.49}$$

$$n_k(t,\bar{w},w) := \langle\phi_{\sqrt{N}w},\hat{a}_k^*(t)\hat{a}_k(t)\phi_{\sqrt{N}w}\rangle. \tag{2.50}$$

$$\mathcal{P}(\bar{v},v) := \sum_{1\le j\le L}\Big[3N\,|v_j|^4+4\sqrt{N}|v_j|^3+\sqrt{2}|v_j|^2\Big]. \tag{2.51}$$

Then,

$$|\rho_k(t,\bar{w},w)-u_k(t,w)|\le U\int_0^t \mathcal{P}(\bar{v},v)\left(n_k(s,\bar{v},v)+\frac{1}{N}\right)^{\frac{1}{2}}\Big|_{v=\Phi_{t-s}(\bar{w},w)}\,ds. \tag{2.52}$$

Proof. Since $\exp\{-it\mathcal{L}_1\}(w_k)$ is the k-th component of the solution of DNLS, a standard semigroup argument allows us to write

$$\rho_k(t,\bar{w},w)-u_k(t,w) = -\frac{i}{N}\int_0^t \mathcal{L}_2\rho_k(s,\bar{v},v)\Big|_{v=\Phi_{t-s}(\bar{w},w)}\,ds, \tag{2.53}$$

where $\mathcal{L}_2$ has been defined in (2.17). Then using explicitly the definition of h in (2.7)

$$-\frac{i}{N}\mathcal{L}_2\rho_k = -i\frac{U}{2N}\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho_k}{\partial v_j^2}-\bar{v}_j^2\frac{\partial^2\rho_k}{\partial\bar{v}_j^2}\right). \tag{2.54}$$

We now recall the setting

$$\rho_k(s,\bar{v},v) := \langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle \tag{2.55}$$

where $\phi_{\sqrt{N}v}(\bar{z}) = e^{\sqrt{N}v\bar{z}-\frac{1}{2}N|v|^2}$ and notice that

$$\frac{\partial}{\partial v_j}\phi_{\sqrt{N}v}(\bar{z}) = \left(\sqrt{N}\bar{z}_j - \frac{N}{2}\bar{v}_j\right)\phi_{\sqrt{N}v}(\bar{z}), \tag{2.56}$$

$$\frac{\partial}{\partial \bar{v}_j}\phi_{\sqrt{N}v}(\bar{z}) = -\frac{N}{2}v_j\phi_{\sqrt{N}v}(\bar{z}). \tag{2.57}$$

Thanks to Lemma 2.3.8, we have

$$\begin{aligned}\frac{\partial \rho_k}{\partial v_j} &= \langle -\frac{N}{2}v_j\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\left(\sqrt{N}\bar{z}_j - \frac{N}{2}\bar{v}_j\right)\phi_{\sqrt{N}v}\rangle & (2.58)\\ &= -N\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \sqrt{N}\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\bar{z}_j\phi_{\sqrt{N}v}\rangle. & (2.59)\end{aligned}$$

Notice that $\bar{z}_j\phi_{\sqrt{N}v}(\bar{z}) = \sqrt{N}\hat{a}_j^*\phi_{\sqrt{N}v}(\bar{z})$ and thus

$$\frac{\partial \rho_k}{\partial v_j} = -N\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + N\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle. \tag{2.60}$$

Applying twice this formula we get

$$\begin{aligned}\frac{\partial^2 \rho_k}{\partial v_j^2} &= N^2\bar{v}_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + & (2.61)\\ &\quad -N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + N^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\rangle \\ &= N^2\bar{v}_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2\bar{v}_j\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle \\ &\quad +N^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\rangle.\end{aligned}$$

Applying the same computations for the derivatives on $\bar{v}_j$ we get

$$\begin{aligned}\frac{\partial^2 \rho_k}{\partial \bar{v}_j^2} &= N^2 v_j^2\langle \phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2 v_j\langle \hat{a}_j^*\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle & (2.62)\\ &\quad +N^2\langle \hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}, \hat{a}_k(s)\phi_{\sqrt{N}v}\rangle.\end{aligned}$$

The sum in (2.54) can now be rewritten as

$$\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho_k}{\partial v_j^2}-\bar{v}_j^2\frac{\partial^2\rho_k}{\partial \bar{v}_j^2}\right)= \tag{2.63}$$

$$\begin{aligned}
= \ & \sum_{j=1}^{L} N^2|v_j|^4\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2 v_j|v_j|^2\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + \\
& + N^2 v_j^2\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + \qquad (2.64)\\
& - \sum_{j=1}^{L} N^2|v_j|^4\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - 2N^2 \bar{v}_j|v_j|^2\langle\hat{a}_j^*\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle + \\
& + N^2\bar{v}_j^2\langle\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle, \qquad (2.65)
\end{aligned}$$

which simplifies to

$$\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho_k}{\partial v_j^2}-\bar{v}_j^2\frac{\partial^2\rho_k}{\partial \bar{v}_j^2}\right)= \tag{2.66}$$

$$\begin{aligned}
= \ & \sum_{j=1}^{L} -2N^2 v_j|v_j|^2\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + N^2 v_j^2\langle\phi_{\sqrt{N}v},\hat{a}_k(s)\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\rangle \\
& + \sum_{j=1}^{L} 2N^2\bar{v}_j|v_j|^2\langle\hat{a}_j^*\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle - N^2\bar{v}_j^2\langle\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v},\hat{a}_k(s)\phi_{\sqrt{N}v}\rangle.
\end{aligned}$$

The sum exhibits the following upper bound

$$\left|\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho}{\partial v_j^2}-\bar{v}_j^2\frac{\partial^2\rho}{\partial \bar{v}_j^2}\right)\right|\leq \tag{2.67}$$

$$\begin{aligned}
\leq \ & \sum_{j=1}^{L} 2N^2|v_j|^3\|\hat{a}_k^*(s)\phi_{\sqrt{N}v}\| \ \|\hat{a}_j^*\phi_{\sqrt{N}v}\| + N^2|v_j|^2\|\hat{a}_k^*(s)\phi_{\sqrt{N}v}\| \ \|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\| \\
& + \sum_{j=1}^{L} 2N^2|v_j|^3\|\hat{a}_j^*\phi_{\sqrt{N}v}\| \ \|\hat{a}_k(s)\phi_{\sqrt{N}v}\| + N^2|v_j|^2\|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\| \ \|\hat{a}_k(s)\phi_{\sqrt{N}v}\|,
\end{aligned}$$

namely

$$\begin{aligned}
\leq \ & \sum_{j=1}^{L}\left(2N^2|v_j|^3 \ \|\hat{a}_j^*\phi_{\sqrt{N}v}\| + N^2|v_j|^2 \ \|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\|\right)\|\hat{a}_k^*(s)\phi_{\sqrt{N}v}\| \\
+ \ & \sum_{j=1}^{L}\left(2N^2|v_j|^3\|\hat{a}_j^*\phi_{\sqrt{N}v}\| + N^2|v_j|^2\|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\|\right)\|\hat{a}_k(s)\phi_{\sqrt{N}v}\|.
\end{aligned}$$

We need to get an estimate for $\|\hat{a}_j^*\phi_{\sqrt{N}v}\|$ and $\|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\|$.

$$\begin{aligned}\|\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 &= \langle\phi_{\sqrt{N}v}, \hat{a}_k\hat{a}_j^*\phi_{\sqrt{N}v}\rangle = \langle\phi_{\sqrt{N}v}, \left(\hat{a}_j^*\hat{a}_j + \frac{1}{N}\right)\phi_{\sqrt{N}v}\rangle \\ &= \langle\phi_{\sqrt{N}v}, \hat{a}_j^*\hat{a}_j\phi_{\sqrt{N}v}\rangle + \frac{1}{N}. \end{aligned} \tag{2.68}$$

Since $\hat{a}_j\phi_{\sqrt{N}v} = v_j\phi_{\sqrt{N}v}$ and recalling that $\phi_{\sqrt{N}v}$ are normalized, it follows

$$\begin{aligned}\|\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 &= \langle\phi_{\sqrt{N}v}, \hat{a}_j\hat{a}_j^*\phi_{\sqrt{N}v}\rangle = \langle\phi_{\sqrt{N}v}, \left(\hat{a}_j^*\hat{a}_j + \frac{1}{N}\right)\phi_{\sqrt{N}v}\rangle \\ &= \langle\phi_{\sqrt{N}v}, \hat{a}_j^*\hat{a}_j\phi_{\sqrt{N}v}\rangle + \frac{1}{N} = |v_j|^2 + \frac{1}{N}, \end{aligned} \tag{2.69}$$

and thus

$$\|\hat{a}_j^*\phi_{\sqrt{N}v}\| = \left(|v_j|^2 + \frac{1}{N}\right)^{\frac{1}{2}} \leq |v_j| + \frac{1}{\sqrt{N}}. \tag{2.70}$$

We now look at

$$\begin{aligned} & \|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 = \langle\phi_{\sqrt{N}v}, \hat{a}_j\hat{a}_j\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\rangle \qquad (2.71) \\ = \; & \langle\phi_{\sqrt{N}v}, \hat{a}_j\left(\hat{a}_j^*\hat{a}_j + \frac{1}{N}\right)\hat{a}_j^*\phi_{\sqrt{N}v}\rangle \\ = \; & \langle\phi_{\sqrt{N}v}, \hat{a}_j\hat{a}_j^*\hat{a}_j\hat{a}_j^*\phi_{\sqrt{N}v}\rangle + \frac{1}{N}\|\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 \\ = \; & \langle\phi_{\sqrt{N}v}, \left(\hat{a}_j^*\hat{a}_j + \frac{1}{N}\right)\left(\hat{a}_j^*\hat{a}_j + \frac{1}{N}\right)\phi_{\sqrt{N}v}\rangle + \frac{1}{N}\|\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 \\ = \; & \|\hat{a}_j^*\hat{a}_j\phi_{\sqrt{N}v}\|^2 + \frac{2}{N}\langle\phi_{\sqrt{N}v}, \hat{a}_j^*\hat{a}_j\phi_{\sqrt{N}v}\rangle + \frac{1}{N^2} + \frac{1}{N}\|\hat{a}_j^*\phi_{\sqrt{N}v}\|^2. \end{aligned}$$

By using again $\hat{a}_j\phi_{\sqrt{N}v} = v_j\phi_{\sqrt{N}v}$ and (2.69), we have

$$\begin{aligned} & \|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\|^2 \qquad (2.72) \\ = \; & |v_j|^2\left(|v_j|^2 + \frac{1}{N}\right) + \frac{2}{N}|v_j|^2 + \frac{1}{N^2} + \frac{1}{N}\left(|v_j|^2 + \frac{1}{N}\right) \\ = \; & |v_j|^4 + \frac{4}{N}|v_j|^2 + \frac{2}{N^2}, \end{aligned}$$

and hence

$$\|\hat{a}_j^*\hat{a}_j^*\phi_{\sqrt{N}v}\| \leq |v_j|^2 + \frac{2}{\sqrt{N}}|v_j| + \frac{\sqrt{2}}{N}. \tag{2.73}$$

Inserting (2.70) - (2.73) into (2.68) we get

$$\Big|\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho}{\partial v_j^2}-\bar v_j^2\frac{\partial^2\rho}{\partial \bar v_j^2}\right)\Big| \tag{2.74}$$
$$\leq \sum_{j=1}^{L}\left(2N^2|v_j|^3\left(|v_j|+\frac{1}{\sqrt N}\right)+N^2|v_j|^2\left(|v_j|^2+\frac{2}{\sqrt N}|v_j|+\frac{\sqrt2}{N}\right)\right)\|\hat a_k^*(s)\phi_{\sqrt N v}\|$$
$$+ \sum_{j=1}^{L}\left(2N^2|v_j|^3\left(|v_j|+\frac{1}{\sqrt N}\right)+N^2|v_j|^2\left(|v_j|^2+\frac{2}{\sqrt N}|v_j|+\frac{\sqrt2}{N}\right)\right)\|\hat a_k(s)\phi_{\sqrt N v}\|.$$

Thus

$$\Big|\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho}{\partial v_j^2}-\bar v_j^2\frac{\partial^2\rho}{\partial \bar v_j^2}\right)\Big| \tag{2.75}$$
$$\leq N^2\sum_{j=1}^{L}\left(3|v_j|^4+\frac{4}{\sqrt N}|v_j|^3+\frac{\sqrt2}{N}|v_j|^2\right)(\|\hat a_k^*(s)\phi_{\sqrt N v}\|+\|\hat a_k(s)\phi_{\sqrt N v}\|).$$

We observe that

$$\begin{aligned}\|\hat a_k(s)\phi_{\sqrt N v}\| &= (\langle\phi_{\sqrt N v},\hat a_k^*(s)\hat a_k(s)\phi_{\sqrt N v}\rangle)^{\frac12}\\ &\leq \left(\langle\phi_{\sqrt N v},\hat a_k^*(s)\hat a_k(s)\phi_{\sqrt N v}\rangle+\frac1N\right)^{\frac12},\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}\|\hat a_k^*(s)\phi_{\sqrt N v}\| &= (\langle\phi_{\sqrt N v},\hat a_k(s)\hat a_k^*(s)\phi_{\sqrt N v}\rangle)^{\frac12}\\ &= \left(\langle\phi_{\sqrt N v},\hat a_k^*(s)\hat a_k(s)\phi_{\sqrt N v}\rangle+\frac1N\right)^{\frac12}.\end{aligned} \tag{2.77}$$

As a consequence

$$\Big|\sum_{j=1}^{L}\left(v_j^2\frac{\partial^2\rho}{\partial v_j^2}-\bar v_j^2\frac{\partial^2\rho}{\partial \bar v_j^2}\right)\Big| \tag{2.78}$$
$$\leq 2N^2\sum_{j=1}^{L}\left(3|v_j|^4+\frac{4}{\sqrt N}|v_j|^3+\frac{\sqrt2}{N}|v_j|^2\right)\left(n_k(s,\bar v,v)+\frac1N\right)^{\frac12}.$$

Now define $\mathcal{P}(\bar v,v):=\sum_{1\leq j\leq L}(3N\,|v_j|^4+4\sqrt N|v_j|^3+\sqrt2|v_j|^2)$ and recall equalities (2.53) - (2.54) which imply the statement (2.52). □

In view of previous propositions, we can now provide the proof of the main result.

Proof. [of Theorem 2.2.1] Recalling (2.52), we define the positive function

$$\psi(s) := U\mathcal{P}(\bar{v},v)\left(n_k(s,\bar{v},v)+\frac{1}{N}\right)^{\frac{1}{2}}\Big|_{(\bar{v},v)=\Phi_{t-s}(\bar{w},w)}, \tag{2.79}$$

and for the sake of simplicity we avoid to write the dependence from $(\bar{w},w)$. Thus, $|\rho_k - u_k| \leq \int_0^t \psi(s)\,ds$ and

$$\|\rho_k - u_k\|_{L^p(\mu_N)} \leq \Big\| \int_0^t \psi(s)\,ds \Big\|_{L^p(\mu_N)}. \tag{2.80}$$

More in details,

$$\Big\| \int_0^t \psi(s)\,ds \Big\|^p_{L^p(\mu_N)} = \int \left(\int_0^t \psi(s)\,ds\right)^p d\mu_N. \tag{2.81}$$

The Hölder inequality $\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$ with $1/q+1/p=1$, allows

$$\int_0^t \psi(s)\,ds \leq \left(\int_0^t \psi^p(s)\,ds\right)^{\frac{1}{p}} t^{1-\frac{1}{p}}, \tag{2.82}$$

and hence

$$\left(\int_0^t \psi(s)\,ds\right)^p \leq \int_0^t \psi^p(s)\,ds\ t^{p-1}. \tag{2.83}$$

This gives

$$\int \left(\int_0^t \psi(s)\,ds\right)^p d\mu_N \leq t^{p-1}\int_0^t \left(\int \psi^p(s)\,d\mu_N\right) ds. \tag{2.84}$$

We now focus our attention to

$$\int \psi^p(s)\,d\mu_N = \int \left(U\mathcal{P}(\bar{v},v)\left(n_k(s,\bar{v},v)+\frac{1}{N}\right)^{\frac{1}{2}}\Big|_{(\bar{v},v)=\Phi_{t-s}(\bar{w},w)}\right)^p d\mu_N. \tag{2.85}$$

The invariance of μ_N under the flow Φ_{t-s} implies

$$\begin{aligned}
\int \psi^p(s)\,d\mu_N &= \int \left(U\mathcal{P}(\bar{w},w)\left(n_k(s,\bar{w},w)+\frac{1}{N}\right)^{\frac{1}{2}}\right)^p d\mu_N && (2.86)\\
&= \int \left(U^2\mathcal{P}(\bar{w},w)^2\left(n_k(s,\bar{w},w)+\frac{1}{N}\right)\right)^{\frac{p}{2}} d\mu_N && (2.87)\\
&= \int \left(\langle \phi_{\sqrt{N}w}, B_k(s)\phi_{\sqrt{N}w}\rangle\right)^{\frac{p}{2}} d\mu_N && (2.88)
\end{aligned}$$

where we have just defined the positive definite operator

$$B_k(s) := U^2\mathcal{P}(\bar{w},w)^2\left(\hat{a}_k^*(s)\hat{a}_k(s)+\frac{1}{N}\right). \tag{2.89}$$

Now assume that $p = 2^m$ with $m \in \mathbb{N}$ so that

$$\left(\langle \phi_{\sqrt{N}w}, B_k(s)\phi_{\sqrt{N}w}\rangle\right)^{\frac{p}{2}} \leq \langle \phi_{\sqrt{N}w}, B_k^p(s)\phi_{\sqrt{N}w}\rangle^{\frac{1}{2}}. \tag{2.90}$$

We get, thanks to the normalization of μ_N,

$$\int \psi^p(s)\, d\mu_N \leq \int \langle \phi_{\sqrt{N}w}, B_k^p(s)\phi_{\sqrt{N}w}\rangle^{\frac{1}{2}}\, d\mu_N \leq \left(\int \langle \phi_{\sqrt{N}w}, B_k^p(s)\phi_{\sqrt{N}w}\rangle\, d\mu_N\right)^{\frac{1}{2}}, \tag{2.91}$$

and recalling (2.89),

$$\int \psi^p(s)\, d\mu_N \leq \left(\int U^{2p}\mathcal{P}(\bar{w}, w)^{2p}\langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*(s)\hat{a}_k(s) + \frac{1}{N}\right)^p \phi_{\sqrt{N}w}\rangle\, d\mu_N\right)^{\frac{1}{2}}. \tag{2.92}$$

The Cauchy-Schwartz inequality gives

$$\leq \left(\int U^{4p}\mathcal{P}(\bar{w}, w)^{4p}\, d\mu_N\right)^{\frac{1}{4}} \left(\int \langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*(s)\hat{a}_k(s) + \frac{1}{N}\right)^p \phi_{\sqrt{N}w}\rangle^2\, d\mu_N\right)^{\frac{1}{4}}. \tag{2.93}$$

Since $\hat{a}_k^*(s)\hat{a}_k(s) + \frac{1}{N}$ is positive definite, we have the upper bound

$$\leq \left(\int U^{4p}\mathcal{P}(\bar{w}, w)^{4p}\, d\mu_N\right)^{\frac{1}{4}} \left(\int \langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*(s)\hat{a}_k(s) + \frac{1}{N}\right)^{2p} \phi_{\sqrt{N}w}\rangle\, d\mu_N\right)^{\frac{1}{4}}. \tag{2.94}$$

Easily observe that, since $U^\star(s)U_s = \mathrm{Id}$, we have

$$\left(\hat{a}_k^*(s)\hat{a}_k(s) + \frac{1}{N}\right)^{2p} = U^\star(s)\left(\hat{a}_k^*\hat{a}_k + \frac{1}{N}\right)^{2p} U_s. \tag{2.95}$$

Now apply Proposition 2.3.3 and Remark 2.3.5 in order to rewrite (2.94) as

$$= \left(\int U^{4p}\mathcal{P}(\bar{w}, w)^{4p}\, d\mu_N\right)^{\frac{1}{4}} \left(\int \langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*\hat{a}_k + \frac{1}{N}\right)^{2p} \phi_{\sqrt{N}w}\rangle\, d\mu_N\right)^{\frac{1}{4}}. \tag{2.96}$$

Integrating this terms, see Lemma 2.3.7, we have constants

$$C_{1,p} := \left(\sum_{\alpha_1+\alpha_2+\alpha_3=4p} \binom{4p}{\alpha_1\ \alpha_2\ \alpha_3} 3^{\alpha_1} 2^{2\alpha_2+\alpha_3/2}\Gamma\left(2\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 + 1\right)\right)^{\frac{1}{4}}, \tag{2.97}$$

$$C_{2,p} := \left(\sum_{\alpha=1}^{2p}\binom{2p}{\alpha}\sum_{\beta=1}^{\alpha} S(\alpha,\beta)\beta!\right)^{\frac{1}{4}}, \tag{2.98}$$

where $S(\alpha,\beta)$ are Stirling numbers of second kind and Γ is the Euler Gamma function, such that

$$\left(\int \mathcal{P}(\bar{w},w)^{4p}\, d\mu_N\right)^{\frac{1}{4}} \leq C_{1,p}\left(\frac{L}{N}\right)^p, \tag{2.99}$$

and

$$\left(\int \langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*\hat{a}_k + \frac{1}{N}\right)^{2p} \phi_{\sqrt{N}w}\rangle\, d\mu_N\right)^{\frac{1}{4}} = C_{2,p}\left(\frac{1}{\sqrt{N}}\right)^p. \tag{2.100}$$

Thus,

$$\int \psi^p(s)\, d\mu_N \leq U^p\ C_{1,p}\ \left(\frac{L}{N}\right)^p C_{2,p}\ \left(\frac{1}{\sqrt{N}}\right)^p. \tag{2.101}$$

We are now in the position to conclude

$$\begin{aligned} \|\rho_k - u_k\|^p_{L^p(\mu_N)} &\leq t^{p-1}\int_0^t U^p\ C_{1,p}\ C_{2,p}\ \left(\frac{1}{\sqrt{N}}\right)^p ds & (2.102)\\ &= t^p\ U^p\ C_{1,p}\ \left(\frac{L}{N}\right)^p\ C_{2,p}\ \left(\frac{1}{\sqrt{N}}\right)^p, & (2.103)\end{aligned}$$

so that by defining, with $C_{1,p}$ and $C_{2,p}$ as in (2.97)-(2.98),

$$B_p := (C_{1,p}\ C_{2,p})^{\frac{1}{p}}, \tag{2.104}$$

we have, in the case $p = 2^m$ with $m \in \mathbb{N}$,

$$\|\rho_k - u_k\|_{L^p(\mu_N)} \leq tUB_p\frac{L}{N}\ \frac{1}{\sqrt{N}}. \tag{2.105}$$

Now observe that, thanks to normalization of μ_N and a simple application of Hölder inequality, we have $\|\rho_k - u_k\|_{L^p(\mu_N)} \leq \|\rho_k - u_k\|_{L^\alpha(\mu_N)}$ for any $\alpha \geq p$. Thus, fix $\alpha := 2^p$ so that

$$A_p := B_{2^p}, \tag{2.106}$$

ensures now $\forall p \geq 1$ the inequality

$$\|\rho_k - u_k\|_{L^p(\mu_N)} \leq t\ U\ A_p\ \frac{L}{N}\ \frac{1}{\sqrt{N}}. \tag{2.107}$$

It remains to prove that ρ_k and u_k are in $\in L^p(\mu_N)$. Recall that $u_k(t,w) = \Phi_t^{(k)}(\bar{w},w)$ and that μ_N is invariant under Φ_t. Hence,

$$\begin{aligned} \int |u_k(t,w)|^p d\mu_N(\bar{w},w) &= \int |\Phi_t^{(k)}(\bar{w},w)|^p d\mu_N(\bar{w},w) & (2.108)\\ = \int |w^k|^p\ (\Phi_t)_\star d\mu_N(\bar{w},w) &= \int |w^k|^p d\mu_N(\bar{w},w) < +\infty. & (2.109)\end{aligned}$$

where the last inequality is guaranteed since μ_N is a gaussian type measure and $|w^k|^p$ is a polynomial term. Inequality (2.107) gives $\|\rho_k - u_k\|_{L^p(\mu_N)} < +\infty$ and thus $\|\rho_k\|_{L^p(\mu_N)} < +\infty$. □

An immediate consequence of Theorem 2.2.1 is Corollary 2.2.2.

Proof. [of Corollary 2.2.2] Let $0 < \epsilon < \frac{1}{2}$ and define the set

$$w_k := \Big\{(w,\bar{w}) \mid \quad |\rho_k - u_k|(t,w,\bar{w}) > A_p \frac{L}{N}\frac{U\,t}{N^\epsilon}, \quad \forall t \geq 0\Big\}. \tag{2.110}$$

Then,

$$\mu_N(w_k)\left(A_p \frac{L}{N}\frac{U\,t}{N^\epsilon}\right)^p < \int_{w_k} |\rho_k - u_k|^p \; d\mu_N(\bar{w},w) \leq \int |\rho_k - u_k|^p \; d\mu_N(\bar{w},w). \tag{2.111}$$

Recalling inequality (2.107), we get

$$\mu_N(w_k)\left(A_p \frac{L}{N}\frac{U\,t}{N^\epsilon}\right)^p \leq \left(A_p \frac{L}{N}\frac{U\,t}{\sqrt{N}}\right)^p, \tag{2.112}$$

hence

$$\mu_N(w_k) \leq N^{-p\cdot(\frac{1}{2}-\epsilon)} \quad \forall p \geq 1 \quad \forall N \geq 1. \tag{2.113}$$

To conclude, define

$$\mathcal{U}_k := \Big\{(\bar{w},w) \mid \quad |\rho_k - u_k|(t,\bar{w},w) > A_p\, D\, U\; t, \qquad \forall t \geq 0\Big\} \tag{2.114}$$

and easily observe that, since $N/L \leq D$ and $1/N^\epsilon \leq 1$ we have $\mathcal{U}_k \subset w_k$. This gives $\mu_N(\mathcal{U}_k) \leq \mu_N(w_k)$. □

Before ending this chapter we provide the proof of the two technical Lemmas we used above.

Lemma 2.3.7. *Let $\mathcal{P}(\bar{w},w)$ be as in (2.51), then for any $1 \leq p < \infty$ there exists a positive constant $C_{1,p}$ such that*

$$\left(\int \mathcal{P}(\bar{w},w)^{4p} d\mu_N\right)^{\frac{1}{4}} \leq C_{1,p}\left(\frac{L}{N}\right)^p. \tag{2.115}$$

Moreover,

$$\left(\int \langle \phi_{\sqrt{N}w}, \left(\hat{a}_k^*\hat{a}_k + \frac{1}{N}\right)^{2p} \phi_{\sqrt{N}w}\rangle d\mu_N\right)^{\frac{1}{4}} = C_{2,p}\left(\frac{1}{\sqrt{N}}\right)^p \tag{2.116}$$

for a positive constant $C_{2,p}$.

Proof.

We first notice that for any fixed $w \in \mathbb{C}^L$, $\mathcal{P}(\bar{w}, w)$ is a sum of real non-negative numbers

$$\mathcal{P}(\bar{w}, w) = \sum_{j=1}^{L} f(\bar{w}_j, w_j) \tag{2.117}$$

where $f(\bar{w}_j, w_j) := 3N|w_j|^4 + 4\sqrt{N}|w_j|^3 + \sqrt{2}|w_j|^2$, so by using Hölder inequality we get

$$P(\bar{w}, w)^{4p} \leq L^{4p-1} \sum_j f(\bar{w}_j, w_j)^{4p}. \tag{2.118}$$

Since for any $v \in \mathbb{C}$, $f(\bar{v}/\sqrt{N}, v/\sqrt{N}) = N^{-1} g(\bar{v}, v)$ for $g(\bar{v}, v) = 3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2$, integrating with respect to gaussian measure and performing the change of variables $w'_j = \sqrt{N} w_j$ we have

$$\begin{aligned} \int_{\mathbb{C}^L} P(\bar{w}, w)^{4p} d\mu_N &\leq L^{4p-1} \sum_j N^L \int_{\mathbb{C}^L} f(\bar{w}_j, w_j)^{4p} e^{-N|w|^2} d\bar{w} dw = \\ &= \frac{L^{4p-1}}{N^{4p}} \sum_j \int_{\mathbb{C}^L} \left(3|w'_j|^4 + 4|w'_j|^3 + \sqrt{2}|w'_j|^2\right)^{4p} e^{-|w'|^2} d\bar{w}' dw'. \end{aligned} \tag{2.119}$$

For each $j = 1, \ldots, L$ we factorize the integrals not containing w_j, so introducing the variable $v \in \mathbb{C}$ and its corresponding measure $d\bar{v}dv$ we have

$$\begin{aligned} &= \frac{L^{4p-1}}{N^{4p}} \sum_j \left(\int_{\mathbb{C}} e^{-|v|^2} d\bar{v} dv\right)^{L-1} \times \\ &\qquad \times \left(\int_{\mathbb{C}} \left(3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2\right)^{4p} e^{-|v|^2} d\bar{v} dv\right) \\ &= \frac{L^{4p}}{N^{4p}} \int_{\mathbb{C}} \left(3|v|^4 + 4|v|^3 + \sqrt{2}|v|^2\right)^{4p} e^{-|v|^2} d\bar{v} dv \\ &= \frac{L^{4p}}{N^{4p}} \sum_{\alpha_1+\alpha_2+\alpha_3=4p} \binom{4p}{\alpha_1\ \alpha_2\ \alpha_3} 3^{\alpha_1} 2^{2\alpha_2+\alpha_3/2} \times \\ &\qquad \times \int_{\mathbb{C}} |v|^{4\alpha_1+3\alpha_2+2\alpha_3} e^{-|v|^2} d\bar{v} dv \\ &= \frac{L^{4p}}{N^{4p}} \sum_{\alpha_1+\alpha_2+\alpha_3=4p} \binom{4p}{\alpha_1\ \alpha_2\ \alpha_3} 3^{\alpha_1} 2^{2\alpha_2+\alpha_3/2} \Gamma\left(2\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 + 1\right) \\ &= \frac{L^{4p}}{N^{4p}} \sum_{\alpha_1+\alpha_2+\alpha_3=4p} \binom{4p}{\alpha_1\ \alpha_2\ \alpha_3} 3^{\alpha_1} 2^{2\alpha_2+\alpha_3/2} \Gamma\left(2\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 + 1\right) \end{aligned} \tag{2.120}$$

where the Euler Gamma function has been introduced. Notice that in the second line we used the fact that we have exactly L equal integrals in the expression. Taking the fourth of root of last expression we get inequality (2.115) with

$$C_{1,p} := \left(\sum_{\alpha_1+\alpha_2+\alpha_3=4p} \binom{4p}{\alpha_1\ \alpha_2\ \alpha_3} 3^{\alpha_1} 2^{2\alpha_2+\alpha_3/2} \Gamma\left(2\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 + 1\right)\right)^{\frac{1}{4}} . \quad (2.121)$$

To get (2.116) we need to compute the mean value $\langle \phi_{\sqrt{N}w}, \hat{n}_k^\alpha \phi_{\sqrt{N}w}\rangle =: \langle \hat{n}_k^\alpha \rangle$ for any positive integer α, where $\hat{n}_k = \hat{a}_k^*(0)\hat{a}_k(0)$. We find that from the definition of Wick-$*$ product there exists a recurrence relation between these quantities

$$\langle \hat{n}_k^\alpha \rangle = \left(|w_k|^2 + \frac{1}{N}\bar{w}_k \frac{\partial}{\partial \bar{w}_k}\right) \langle \hat{n}_k^{\alpha-1} \rangle = \left(|w_k|^2 + \frac{1}{N}\bar{w}_k \frac{\partial}{\partial \bar{w}_k}\right)^\alpha \mathbf{1} \quad (2.122)$$

where $\mathbf{1}$ is the constant function $\mathbf{1}(\bar{w}, w) \equiv 1$, so that in general

$$\langle \hat{n}_k^\alpha \rangle = \sum_{\beta=1}^{\alpha} S(\alpha,\beta) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta}} \quad (2.123)$$

where $S(\alpha, \beta)$ is the *Stirling number of the second kind* with integer parameters α and β (see computations below). Since $\hat{n}_k$ and N^{-1} commute as operators we can expand $(\hat{n}_k + N^{-1})^{2p}$ using the binomial theorem

$$\begin{aligned}
&\int_{\mathbb{C}^L} \langle \phi_{\sqrt{N}w}, \left(\hat{n}_k + \frac{1}{N}\right)^{2p} \phi_{\sqrt{N}w}\rangle d\mu_N = \\
&= \sum_{\alpha=1}^{2p} \binom{2p}{\alpha} N^{-2p+\alpha} N^L \int_{\mathbb{C}^L} \langle \phi_{\sqrt{N}w}, \hat{n}_k^\alpha \phi_{\sqrt{N}w}\rangle e^{-N|w|^2} d\bar{w}dw \\
&= \sum_{\alpha=1}^{2p} \binom{2p}{\alpha} \sum_{\beta=1}^{\alpha} S(\alpha,\beta) N^{-2p+\beta} N^L \int_{\mathbb{C}^L} |w_k|^{2\beta} e^{-N|w|^2} d\bar{w}dw \\
&= N^{-2p} \sum_{\alpha=1}^{2p} \binom{2p}{\alpha} \sum_{\beta=1}^{\alpha} S(\alpha,\beta)\beta!\,.
\end{aligned} \quad (2.124)$$

Taking again the fourth root, we get (2.116) with constant

$$C_{2,p} := \left(\sum_{\alpha=1}^{2p} \binom{2p}{\alpha} \sum_{\beta=1}^{\alpha} S(\alpha,\beta)\beta! \right)^{\frac{1}{4}} . \quad (2.125)$$

We now complete the proof of this Lemma, showing that the coefficients of the polynomial in (2.123) are the Stirling number of the second kind (see [3], Par. 24.1.4

for their definition and properties). By the recurrence relation (2.122) we have

$$\begin{aligned}
\langle \hat{n}_k^{\alpha+1} \rangle &= \left(|w_k|^2 + \frac{1}{N} \bar{w}_k \frac{\partial}{\partial \bar{w}_k} \right) \langle \hat{n}_k^{\alpha} \rangle \\
&= \left(|w_k|^2 + \frac{1}{N} \bar{w}_k \frac{\partial}{\partial \bar{w}_k} \right) \sum_{\beta=1}^{\alpha} S(\alpha,\beta) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta}} \\
&= \sum_{\beta=1}^{\alpha} S(\alpha,\beta) \frac{|w_k|^{2\beta+2}}{N^{\alpha-\beta}} + \sum_{\beta=1}^{\alpha} \beta S(\alpha,\beta) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta+1}} \qquad (2.126) \\
&= \sum_{\beta=2}^{\alpha+1} S(\alpha,\beta-1) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta+1}} + \sum_{\beta=1}^{\alpha} \beta S(\alpha,\beta) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta+1}} \\
&= \sum_{\beta=1}^{\alpha+1} \left(S(\alpha,\beta-1) + \beta S(\alpha,\beta) \right) \frac{|w_k|^{2\beta}}{N^{\alpha-\beta+1}},
\end{aligned}$$

where we used the fact that $S(\alpha,\alpha) = S(\alpha,1) = 1$, as it is verified using (2.122). Comparing last expression with the general expansion of $\langle \hat{n}_k^{\alpha+1} \rangle$ as in (2.123) with exponent $\alpha+1$, we see that

$$S(\alpha+1,\beta) = S(\alpha,\beta-1) + \beta S(\alpha,\beta)$$

which is precisely the recurrence relation defining Stirling numbers. □

Lemma 2.3.8. *Let T be a Wick operator $T = Op^W(g)$. Then*

$$\frac{\partial g}{\partial w_j} = \langle \left(\frac{\partial \phi_w}{\partial \bar{w}_j} \right), T\phi_w \rangle + \langle \phi_w, T \left(\frac{\partial \phi_w}{\partial w_j} \right) \rangle. \qquad (2.127)$$

Proof. Since $g(\bar{w}, w) = \langle \varphi_w, T\varphi_w \rangle$ we have

$$\frac{\partial g}{\partial w_j} = \langle \left(\frac{\partial \phi_w}{\partial \bar{w}_j} \right), T\phi_w \rangle + \langle \phi_w, \frac{\partial}{\partial w_j} (T\phi_w) \rangle. \qquad (2.128)$$

Now recalling the general form of Wick operators

$$\begin{aligned}
\frac{\partial}{\partial w_j} (T\phi_w)(\bar{z}) &= \frac{\partial}{\partial w_j} \int_{\mathbb{C}^L} e^{\bar{z}\cdot v} g(\bar{z}, v) \phi_w(\bar{v}) d\mu(v) \\
&= \int_{\mathbb{C}^L} e^{\bar{z}\cdot v} g(\bar{z}, v) \frac{\partial \phi_w}{\partial w_j}(\bar{v}) d\mu(v) \qquad (2.129) \\
&= T \left(\frac{\partial \phi_w}{\partial w_j} \right) (\bar{z}),
\end{aligned}$$

hence we have (2.127). □

Chapter 3

Applications of finite dimensional reduction

In this section we apply the finite dimensional reduction procedure to the *one-particle density operator* $\gamma_\Psi^{(1)}$ in the case in which Ψ lies the N-particle sector $\mathcal{F}^{(N)}$ (see Appendix A). Even though we are intersted in N-particles states it is convenient to consider them as embedded in Fock space, allowing the following expression for $\gamma_\Psi^{(1)}$. In general, it can be defined as the operator on $\mathfrak{h}$ which satisfies the sesquilinear form condition on $\mathfrak{h}$

$$\langle f, \gamma_\Psi^{(1)} g\rangle := \frac{\langle \Psi, b(g)^* b(f)\Psi\rangle}{\langle \Psi, \mathbf{N}\Psi\rangle} \tag{3.1}$$

where $\mathbf{N}$ is the boson number operator (see 1.43). Usually $\gamma_\Psi^{(1)}$ is interpreted as a *marginal* one-particle quantum state being a positive and trace-class operator with $\mathrm{Tr}(\gamma_\Psi^{(1)}) = 1$. Fix a basis of $\{u_j\}_{j\in\mathbb{N}} \subset \mathfrak{h}$ and consider the projector Q_L on the subspace generated by the first L elements of the basis

$$Q_L \colon \mathfrak{h} \to \mathrm{span}\,\{u_0, \dots, u_{L-1}\}\,. \tag{3.2}$$

Given a reduced Fock state $\Gamma(Q_L)\Psi$, our target is to prove by an explicit estimate that

$$\left|\left|\gamma_\Psi^{(1)} - \gamma_{\Gamma_B(Q_L)\Psi}^{(1)}\right|\right| \to 0, \quad L \to +\infty \tag{3.3}$$

in the operator norm.

3.1 One particle density - stationary case

Firstly, we notice that from (3.1) one can write the one particle density operator as

$$\gamma_{\Psi}^{(1)} = \frac{1}{\langle \Psi, \mathbf{N}\Psi\rangle} \sum_{j,k\geq 0} \langle \Psi, b(u_j)^* b(u_k)\Psi\rangle \langle u_j, \cdot\rangle u_k, \tag{3.4}$$

which is a convenient expression to compute estimates.

For simplicity we denote the reduction of the Fock state as and its orthogonal component as $\Psi_L := \Gamma_B(Q_L)\Psi$ and $\Psi_L^\perp := (\mathbb{I} - \Gamma_B(Q_L))\Psi$ so that $\Psi = \Psi_L + \Psi_L^\perp$. Since $\Psi \in \mathcal{F}^{(N)}$ we have that

$$\langle \Psi, \mathbf{N}\Psi\rangle = \langle \Psi_L, \mathbf{N}\Psi_L\rangle = N \tag{3.5}$$

and by definition of Q_L that

$$b(u_k)\Psi_L = 0, \qquad \text{if } k \geq L, \tag{3.6}$$

$$b(u_k)\Psi_L^\perp = 0, \qquad \text{if } k < L. \tag{3.7}$$

We then see that the reduction of Ψ simply truncates the series in (3.4)

$$\begin{aligned} \gamma_{\Psi_L}^{(1)} &= \frac{1}{N} \sum_{j,k\geq 0} \langle \Psi_L, b(u_j)^* b(u_k)\Psi_L\rangle \langle u_j, \cdot\rangle u_k \\ &= \frac{1}{N} \sum_{j,k< L} \langle \Psi_L, b(u_j)^* b(u_k)\Psi_L\rangle \langle u_j, \cdot\rangle u_k \\ &= \frac{1}{N} \sum_{j,k< L} \langle \Psi, b(u_j)^* b(u_k)\Psi\rangle \langle u_j, \cdot\rangle u_k, \end{aligned} \tag{3.8}$$

where the last line follows from (3.7). Apply the difference of the two operators to a function $f \in \mathfrak{h}$ to get

$$\begin{aligned} \gamma_{\Psi}^{(1)}(f) - \gamma_{\Psi_L}^{(1)}(f) = &\frac{1}{N} \sum_{j,k\geq L} \langle \Psi, b(u_j)^* b(u_k)\Psi\rangle \langle u_j, f\rangle u_k \\ &+ \frac{1}{N} \sum_{j\geq 0} \sum_{k\geq L} \langle \Psi, b(u_j)^* b(u_k)\Psi\rangle \langle u_j, f\rangle u_k \\ &+ \frac{1}{N} \sum_{j\geq L} \sum_{k\geq 0} \langle \Psi, b(u_j)^* b(u_k)\Psi\rangle \langle u_j, f\rangle u_k \end{aligned} \tag{3.9}$$

so that

$$\begin{aligned}
\|\gamma^{(1)}_{\Psi}(f) - \gamma^{(1)}_{\Psi_L}(f)\| &\leq \frac{1}{N}\sum_{j\geq 0}\|b(u_j)\Psi\|\sum_{k\geq L}\|b(u_k)\Psi\|\,|\langle u_k, f\rangle| \\
&\quad + \frac{1}{N}\sum_{j\geq L}\|b(u_j)\Psi\|\sum_{k\geq 0}\|b(u_k)\Psi\|\,|\langle u_k, f\rangle| \\
&\leq \frac{1}{N}\sum_{j\geq 0}\|b(u_j)\Psi\|\left(\sum_{k\geq L}\|b(u_k)\Psi\|^2\right)^{\frac{1}{2}}\|f\| \\
&\quad + \frac{1}{N}\sum_{j\geq L}\|b(u_j)\Psi\|\left(\sum_{k\geq 0}\|b(u_k)\Psi\|^2\right)^{\frac{1}{2}}\|f\|,
\end{aligned} \tag{3.10}$$

where in third line we used standard Cauchy-Schwartz and Bessel inequalities. By (1.43) we have

$$\sum_{k\geq 0}\|b(u_k)\Psi\|^2 = \sum_{k\geq 0}\langle\Psi, b(u_k)^* b(u_k)\Psi\rangle = N, \tag{3.11}$$

while

$$\left(\sum_{k\geq L}\|b(u_k)\Psi\|^2\right)^{\frac{1}{2}} \leq \sum_{k\geq L}\|b(u_k)\Psi\|. \tag{3.12}$$

To simplify notations, for any $M \in \mathbb{N}$, we define

$$G_{\Psi,M} := \sum_{j\geq M}\|b(u_j)\Psi\| \tag{3.13}$$

so that (3.10) gives the operator norm estimate

$$\|\gamma^{(1)}_{\Psi} - \gamma^{(1)}_{\Psi_L}\| = \sup_{\|f\|=1}\|\gamma^{(1)}_{\Psi}(f) - \gamma^{(1)}_{\Psi_L}(f)\| \leq \frac{G_{\Psi,L}}{N}\left(G_{\Psi,0} + \sqrt{N}\right). \tag{3.14}$$

Remark 3.1.1. In the next paragraph we consider also the *non-diagonal* one-particle density operator

$$\langle f, \gamma^{(1)}_{\Psi,\Phi} g\rangle := \frac{\langle\Psi, b(g)^* b(f)\Phi\rangle}{N} \tag{3.15}$$

for which the above computations can be repeated, giving the estimate

$$\|\gamma^{(1)}_{\Psi,\Phi} - \gamma^{(1)}_{\Psi_L,\Phi_L}\| \leq \frac{1}{N}\left(G_{\Psi,0}G_{\Phi,L} + G_{\Psi,L}\sqrt{N}.\right) \tag{3.16}$$

It is clear that (3.14) is meaningful only if Ψ is taken so that $G_{\Psi,0} < \infty$.[1] Moreover we would like to see how different choices of Ψ can affect the rate of convergence of $G_{\Psi,L}$ to 0 in terms of L. We see now how an explicit analysis can be given if Ψ is a

[1] This obviously implies that $G_{\Psi,L} \to 0$ as $L \to +\infty$.

symmetrized simple state whose Fourier coefficient have some decaying properties.

Proposition 3.1.2. *Let* $\Psi \in \mathcal{F}_B^{(N)}$ *be given by*

$$\Psi = S_N(\psi_1 \otimes \cdots \otimes \psi_N) \tag{3.17}$$

where all the $\{\psi_1, \dots, \psi_N\}$ *are normalized vectors in* $\mathfrak{h}$ *that satisfy*

$$|\langle u_k, \psi_j \rangle| \leq \frac{C}{(k+1)^r} \tag{3.18}$$

for some $r > 1$*. Then* $G_{\Psi,0} < \infty$ *and*

$$\|\gamma_\Psi^{(1)} - \gamma_{\Psi_L}^{(1)}\| \leq \frac{rC}{(r-1)}(C\zeta(r)+1)L^{-r+1} \tag{3.19}$$

where $\zeta(r)$ *is the Riemann zeta function.*

Proof. Using well-known formulas regarding annihilation operators (see [8], Lemma 2.7 and Lemma 5.5), we notice that

$$\begin{aligned} \langle b(u_k)\Psi, b(u_k)\Psi \rangle &= \frac{1}{N}\sum_{j,l=1}^{N} \langle u_k, \psi_j \rangle \langle \psi_l, u_k \rangle \left\langle S_{N-1}\left(\otimes_{i_1 \neq l}\psi_{i_1}\right), S_{N-1}\left(\otimes_{i_2 \neq j}\psi_{i_2}\right)\right\rangle \\ &= \frac{1}{N}\sum_{j,l=1}^{N} \langle u_k, \psi_j \rangle \langle \psi_l, u_k \rangle \frac{1}{(N-1)!} \sum_{\sigma \in \mathfrak{S}_l'} \prod_{i \neq j} \langle \psi_i, \psi_{\sigma(i)} \rangle, \end{aligned} \tag{3.20}$$

where $\mathfrak{S}_l'$ is permutation group of $\{1, \dots, l-1, l+1, \dots, N\}$. Since all the ψ_i's are normalized and using the hypotesis in (3.18), we get

$$\begin{aligned} \|b(u_k)\Psi\| &\leq \left(\frac{1}{N} \sum_{j,k=1}^{N} |\langle u_k, \psi_j \rangle|\, |\langle \psi_l, u_k \rangle| \frac{1}{(N-1)!} \cdot (N-1)! \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |\langle u_k, \psi_j \rangle| \leq \sqrt{N} \frac{C}{(k+1)^r} \end{aligned} \tag{3.21}$$

and so

$$G_{\Psi,0} \leq C\sqrt{N} \sum_{k \geq 0} \frac{1}{(k+1)^r} = C\zeta(r)\sqrt{N}, \tag{3.22}$$

where the Riemann zeta function is introduced. A similar result obviously holds for $G_{\Psi,L}$ with $\zeta(r)$ replaced by $\sum_{k \geq L}(k+1)^{-r}$. Using a standard integral test for series we can get the desired estimate on $G_{\Psi,L}$

$$\sum_{k\geq L}\frac{1}{(k+1)^r}\leq\frac{1}{(L+1)^r}+\int_{L+1}^{+\infty}\frac{1}{x^r}dx\leq\frac{r}{r-1}L^{-r} \tag{3.23}$$

and putting all together we get (3.19). □

Remark 3.1.3. (i) Notice that (3.19) is independent of N, that is Ψ can be taken in an arbitrary sector of Fock space, provided we can find a orthonormal basis for which (3.18) holds.
(ii) A slightly relaxed version of hypothesis (3.18) is satisfied if $\mathfrak{h}=\mathcal{H}L^2(\mathbb{C}^N)$ and Ψ is a symmetrized product of coherent states

$$\Psi\equiv\Psi_{\mathbf{z}}=S_N\left(\varphi_{z_1}\otimes\cdots\otimes\varphi_{z_N}\right),$$

where the $\mathbf{z}=(z_1,\ldots,z_N)\in\mathbb{C}^N$. Indeed, for any $r>1$, we can take $\{u_k\}_{k\geq0}$ to be the set of Hermite functions and it is immediate to verify that

$$\exists K_r>0\quad\text{such that}\quad|\langle u_k,\varphi_{z_j}\rangle|\leq\frac{C_{\mathbf{z}}}{(k+1)^r},\quad\forall k>K_r,$$

where for come constant $C_{\mathbf{z}}$ depending on $\mathbf{z}$.
(iii) From a different viewpoint, the basis $\{u_k\}_{k\geq0}$ can be taken to be the completion of $\{\psi_1,\ldots,\psi_N\}$. In this case $G_{\Psi,0}$ is obviously finite since $b(u_k)\Psi=0$ and $G_{\Psi,L}=0$ for k and L sufficiently large respectively.

3.2 Reduction of the quantum dynamics to a finite size model

We now deal with the case in which the N-particle state Ψ evolves in time. Let us consider the Hamiltonian on $\mathfrak{h}_B^{\otimes N}=L^2_s(\mathbb{R}^N)$

$$H=\sum_{j=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_j^2}+V_{ext}(x_j)\right)+\sum_{1\leq j<k\leq N}V_{int}(x_j-x_k)=:H_0+H_{int}, \tag{3.24}$$

where the external potential $V_{ext}\in C^\infty(\mathbb{R},\mathbb{R}_+)$ is a super-quadratic confining potential

$$V_{ext}(x)=x^{2\sigma},\quad\mathbb{N}\ni\sigma>1, \tag{3.25}$$

while V_{int} is a bounded function for every $i,j=1,\ldots,N$, so that

$$\|H_{int}\| = \sup_{\|\Phi\|=1} \|H_{int}\Phi\| < \infty. \tag{3.26}$$

We focus the attention on the quantum dynamics generated by H

$$\Psi(t) := e^{-iHt}\Psi \tag{3.27}$$

and we how it is affected by the finite dimensional reduction outlined at the beginning of the chapter (see (3.2)). In this case the basis $\{u_k\}_{k\in N} \subset L^2(\mathbb{R})$ chosen to construct Q_L is the set of eigenfunctions of the single particle operator $h_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_{ext}$, that is

$$h_0 u_k = E_k u_k, \quad \forall k \in \mathbb{N}, \tag{3.28}$$

where $E_k > 0$, so that

$$H_0(u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}) = \left(\sum_{j=1}^{N} E_{\alpha_j}\right) u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}. \tag{3.29}$$

Moreover, for simplicity we shall write

$$\otimes^N Q_L =: \mathcal{Q}_L \tag{3.30}$$

and employ the same notation of previous section $\Psi(t)_L = \mathcal{Q}_L\Psi(t)$.

Proposition 3.2.1. *Let H be as in (3.24) with V_{ext} and H_{int} as in (3.25) and (3.26) respectively. Then the projected dynamics is governed by a projected hamiltonian $\mathcal{Q}_L H \mathcal{Q}_L$ plus a remainder*

$$\Psi(t)_L = e^{-i\mathcal{Q}_L H \mathcal{Q}_L t}\mathcal{Q}_L\Psi + \Omega_L(t) \tag{3.31}$$

where $\Omega_L(t)$ satisfies, for sufficiently large L,

$$\|\Omega_L(t)\| \leq \frac{Ct}{L^{\frac{2\sigma}{\sigma+1}}} \tag{3.32}$$

for some constant $C > 0$ depending on N, Ψ and σ (see formula (3.44)).

Proof. We start proving (3.31)-(3.32) by writing

$$\begin{aligned} i\frac{d}{dt}\mathcal{Q}_L\, e^{-iHt}\,\Psi &= \mathcal{Q}_L\mathcal{H}\, e^{-iHt}\,\Psi = \mathcal{Q}_L\, H(\mathcal{Q}_L + \mathbb{I} - \mathcal{Q}_L)\, e^{-iHt}\,\Psi \\ &= (\mathcal{Q}_L H \mathcal{Q}_L)\, \mathcal{Q}_L e^{-iHt}\,\Psi + \mathcal{Q}_L H(\mathbb{I} - \mathcal{Q}_L)\, e^{-iHt}\,\Psi \\ &=: (\mathcal{Q}_L H \mathcal{Q}_L)\, \mathcal{Q}_L e^{-iHt}\,\Psi + W_L(t). \end{aligned} \tag{3.33}$$

A simple semigroup argument (see [43]) shows that

$$\Psi(t)_L = e^{-it\mathcal{Q}_L H \mathcal{Q}_L}\mathcal{Q}_L\Psi + \int_0^t e^{-i(t-\tau)\mathcal{Q}_L H \mathcal{Q}_L} W_L(\tau) d\tau =: e^{-it\mathcal{Q}_L H \mathcal{Q}_L}\mathcal{Q}_L\Psi + \Omega_L(t), \tag{3.34}$$

so that $\Omega_L(t)$ has the bound

$$\|\Omega_L(t)\| \leq \int_0^t \|W_L(\tau)\| d\tau. \tag{3.35}$$

To obtain a bound on $W_L(t)$ we observe that since $\mathcal{Q}_L$ is constructed starting from the eigenfunctions of H_0 we have easily

$$[\mathcal{Q}_L, H_0] = 0 \qquad \Longrightarrow \qquad [\mathcal{Q}_L, H] = [\mathcal{Q}_L, H_{int}], \tag{3.36}$$

so that

$$\begin{aligned} \mathcal{Q}_L H(\mathbb{I} - \mathcal{Q}_L) &= [\mathcal{Q}_L, H](\mathbb{I} - \mathcal{Q}_L) = [\mathcal{Q}_L, H_{int}](\mathbb{I} - \mathcal{Q}_L) \\ &= \mathcal{Q}_L H_{int}(\mathbb{I} - \mathcal{Q}_L), \end{aligned} \tag{3.37}$$

which implies

$$\|W_L(t)\| \leq \|\mathcal{Q}_L\| \; \|H_{int}\| \; \|(\mathbb{I} - \mathcal{Q}_L)\Psi(t)\| \leq \|H_{int}\| \; \|(\mathbb{I} - \mathcal{Q}_L)\Psi(t)\|. \tag{3.38}$$

Expanding $\Psi(t)$ over the basis $\{u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}\}$ and using (3.29) we get

$$\begin{aligned} \|(\mathbb{I} - \mathcal{Q}_L)\Psi(t)\|^2 &= \sum_{\alpha}{}^{(L)} \; |\langle u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}, \Psi(t)\rangle|^2 \\ &= \sum_{\alpha}{}^{(L)} \left(\sum_{j=1}^N E_{\alpha_j}\right)^{-2} |\langle H_0(u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}), \Psi(t)\rangle|^2 \\ &= \sum_{\alpha}{}^{(L)} \left(\sum_{j=1}^N E_{\alpha_j}\right)^{-2} |\langle u_{\alpha_1} \vee \cdots \vee u_{\alpha_N}, H_0\Psi(t)\rangle|^2 \end{aligned} \tag{3.39}$$

where the symbol $\sum^{(L)}$ means that *at least one* of the indices $\alpha_1, \ldots, \alpha_N$ is equal or greater than L. By this fact and since all the E_{α_j} are positive, we have that

$$\left(\sum_{j=1}^N E_{\alpha_j}\right)^{-1} \leq c_\sigma L^{-\frac{2\sigma}{\sigma+1}} \tag{3.40}$$

for some constant $c_\sigma > 0$ (see Appendix B). Thus

$$\begin{aligned}\|(\mathbb{I}-\mathcal{Q}_L)\Psi(t)\| &\leq c_\sigma L^{-\frac{2\sigma}{\sigma+1}} \left(\sum_\alpha{}^{(L)} |\langle u_{\alpha_1}\vee\cdots\vee u_{\alpha_N}, H_0\Psi(t)\rangle|^2\right)^{\frac{1}{2}} \\ &\leq c_\sigma L^{-\frac{2\sigma}{\sigma+1}} \|H_0\Psi(t)\|.\end{aligned} \tag{3.41}$$

Writing $H_0 = H - H_{int}$ we have easily that $\|H_0\Psi(t)\| \leq \|H\Psi\| + \|H_{int}\|$, so putting all together

$$\|W_L(t)\| \leq c_\sigma L^{-\frac{2\sigma}{\sigma+1}} \ \|H_{int}\| \ \left(\|H\Psi\| + \|H_{int}\|\right). \tag{3.42}$$

Since last bound is uniform in t, we have from (3.35)

$$\|\Omega_L(t)\| \leq \frac{c_\sigma t}{L^{\frac{2\sigma}{\sigma+1}}} \ \|H_{int}\| \ \left(\|H\Psi\| + \|H_{int}\|\right) \tag{3.43}$$

and (3.32) is proven with

$$C = c_\sigma \, \|H_{int}\| \ \left(\|H\Psi\| + \|H_{int}\|\right). \tag{3.44}$$

□

Remark 3.2.2. (i) During the proof we made use of the spectral estimates derived in Appendix B. For these to hold we can relax the hypotheses on the external potential and take V to *behave* like a super-quadratic function at infinity (see Chapter 4 and Appendix B).
(ii) For the same estimates to hold we need a smooth potential, a choice which leaves us only with positive integer values of σ.

3.3 One particle density - evolutive case

We go back now to the analysis of the one-particle density operator in the case in which the reference vector Ψ evolves in time according to (3.27). The generator of quantum dynamics is the hamiltonian in (3.24). The result we want to prove is the following: given an initial state $\Psi \in D(H)$ and its quantum evolution $\Psi(t)$ how much does the one-particle density associated to $\Psi(t)$ deviates from the one of $\Psi(t)_L$? That is, can we still prove a convergence result similar to Proposition (3.1.2) also in the non-stationary case and perform some quantitative estimates? The answer is positive and the proof of this fact turns out to be a simple application of the results of previous section.

Proposition 3.3.1. *Let H and Ψ as in Proposition 3.2.1. Then there exist a constant $C > 0$ (see (3.51)) depending on N, Ψ and σ such that*

$$\|\gamma^{(1)}_{\Psi(t)} - \gamma^{(1)}_{\Psi(t)_L}\| \leq CL^{-\frac{2\sigma}{\sigma+1}} \left(CL^{-\frac{2\sigma}{\sigma+1}} + 2\right). \tag{3.45}$$

Proof. Firstly, notice that for any pair of N-particle vectors Ψ, Φ we have that

$$\begin{aligned} \|\gamma^{(1)}_{\Psi,\Phi}\| &= \sup_{\|f\|=\|g\|=1} |\langle f, \gamma^{(1)}_{\Psi,\Phi} g\rangle| \\ &= \sup_{\|f\|=\|g\|=1} \frac{1}{N} |\langle \Psi, b(g)^* b(f)\Phi\rangle| \\ &\leq \sup_{\|f\|=\|g\|=1} \frac{1}{N} \|b(g)\Psi\| \, \|b(f)\Phi\| \\ &\leq \frac{1}{N} \sqrt{N}\|\Psi\| \sqrt{N} \, \|\Phi\| \\ &= \|\Psi\| \, \|\Phi\|. \end{aligned} \tag{3.46}$$

where the second to last line follows from a well-know bound in Fock space theory (see [8], Lemma 5.7). Clearly this also implies that

$$\|\gamma^{(1)}_{\Psi}\| = \|\gamma^{(1)}_{\Psi,\Psi}\| \leq \|\Psi\|^2. \tag{3.47}$$

Now for any t we can split $\Psi(t)$ as $\Psi(t) = \Psi(t)_L + (\mathbb{I} - \mathcal{Q}_L)\Psi(t) =: \Psi(t)_L + \Psi(t)^\perp$, implying

$$\gamma^{(1)}_{\Psi(t)} = \gamma^{(1)}_{\Psi(t)_L} + \gamma^{(1)}_{\Psi(t)^\perp} + \gamma^{(1)}_{\Psi(t)_L,\Psi(t)^\perp} + \gamma^{(1)}_{\Psi(t)^\perp,\Psi(t)_L} \tag{3.48}$$

so that

$$\|\gamma^{(1)}_{\Psi(t)} - \gamma^{(1)}_{\Psi(t)_L}\| \leq \|\Psi(t)^\perp\|^2 + 2\|\Psi(t)_L\| \, \|\Psi(t)^\perp\| \tag{3.49}$$

The norm of $\Psi(t)^\perp$ has been estimated in previous section (see (3.41)) while $\|\Psi(t)_L\| \leq \|\Psi\| = 1$, thus

$$\|\gamma^{(1)}_{\Psi(t)} - \gamma^{(1)}_{\Psi(t)_L}\| \leq CL^{-\frac{2\sigma}{\sigma+1}} \left(CL^{-\frac{2\sigma}{\sigma+1}} + 2\right) \tag{3.50}$$

where

$$C = c_\sigma(\|H\Psi\| + \|H_{int}\|). \tag{3.51}$$

□

Chapter 4

Finite dimensional reduction of Hartree equation

The non-linear Hartree equation describes the effective dynamics of a large system of bosons (see [51]). Here we consider the case in which each particle is subjected by an external potential V_{ext} which is the sum of a *trapping* potential and a bounded one. Moreover the bosons are allowed to interact via a L^1 potential. Following an idea of Bourgain [21] in the case of Non-linear Schrödinger Equation, we study here how the solution of Hartree equation can be approximated by means of finite dimensional reduction, providing an explicit estimate in time.

4.1 Setting of the problem

Let H be the operator on $L^2(\mathbb{R})$ defined as

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V_{trap}(x) + W(x) \tag{4.1}$$

where V is given by

$$V_{trap}(x) := x^{2\sigma}, \tag{4.2}$$

for some integer $\sigma > 1$ and $W \in C^\infty(\mathbb{R})$ is a bounded and positive function which models some sort of multiple-well potential. A prototypical example may be given by the periodic function

$$W(x) = W_1 \sin^2\left(\frac{\pi}{2}(K+1)(x+1)\right) \tag{4.3}$$

for $W_1 > 0$ and $K \in \mathbb{N}$, which describes an even-spaced lattice of wells of depth W_1. The reason behind the $K+1$ factor is that W has exactly K minima in the interval $(-1, 1)$ (see Figure 4.1).

The Hartree equation we study is the following

$$\begin{cases} i\dot{\varphi}(t) = H\varphi(t) + (V * |\varphi(t)|)^2\varphi(t), \\ \varphi(0) =: \varphi_0 \in H^1(\mathbb{R};\mathbb{C}), \quad \|\varphi_0\|_{L^2} = 1. \end{cases} \tag{4.4}$$

where V is a $L^1(\mathbb{R})$ potential satisfying

$$V(x-y) = V(y-x), \quad \forall x, y \in \mathbb{R}, \tag{4.5}$$

and $*$ means the convolution between functions, that is $f * g(x) = \int f(x-y)g(y)dy$ for all f, g in some suitable functional space.
About the existence and uniqueness of the solution we refer to [23]. Our aim is to study a way to construct an approximation of the solution of (4.4) by means of a finite dimensional reduction. To do this we define the orthogonal projection Q_L as in Chapter 1

$$Q_L(\psi) := \sum_{k<L} \langle u_k, \psi \rangle u_k, \quad \forall \psi \in L^2(\mathbb{R}), \tag{4.6}$$

where $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R};\mathbb{C})$ is a complete orthonormal set in $L^2(\mathbb{R};\mathbb{C})$ of eigenfunctions of H, and $Hu_k = E_k u_k$ (the existence of such a basis and the spectral properties of H are discussed in Appendix B). For $t \in \mathbb{R}$ we take a family of maps $\xi \colon t \to \xi(t) \in L^2(\mathbb{R})$ satisfying the *reduced* Hartree equation

$$\begin{cases} i\dot{\xi}(t) = H\xi(t) + Q_L((V * |\xi(t)|^2)\xi(t)) \\ \xi(0) = \xi_0 := Q_L\varphi_0. \end{cases} \tag{4.7}$$

Since by construction $[Q_L, H] = 0$, from the usual semigroup argument it follows that the solution of (4.7) is contained in $Q_L(L^2(\mathbb{R}))$ for all times (see Lemma 4.2.2 (i) below).

4.2 The estimates

The approximation we would like to achieve is contained in the next result.

Proposition 4.2.1. *Let $\varphi_0 \in H^1(\mathbb{R}, \mathbb{C})$ and $\xi_0 = Q_L\varphi_0$. Then, if H is as in (4.1) and $V \in L^1(\mathbb{R})$, for any sufficiently large $L \in \mathbb{N}$ the solutions of equations (4.4) and (4.7) fulfill the estimate*

$$\|\varphi(t) - \xi(t)\|_2 \le C_\sigma \left(\sqrt{\mathcal{E}} + \mathcal{A}_\sigma\, te^{\mathcal{B}t}\right) L^{-\frac{\sigma}{\sigma+1}}. \tag{4.8}$$

for some positive constants $\mathcal{E}$, C_σ, $\mathcal{A}_\sigma$ and $\mathcal{B}$ (see (4.14), (4.18) and (4.31)).

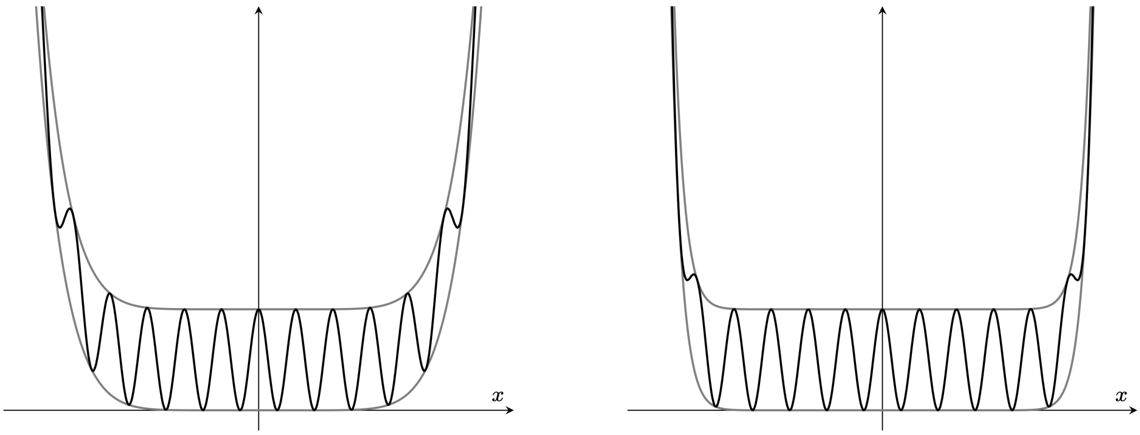

Figure 4.1: The potential W_1 in 4.3 for $\sigma = 4$ (left) and $\sigma = 10$ (right). The polynomial bounds are depicted in gray.

For the sake of notation we sometimes drop the t-dependence of the quantities involved during the proof.

Proof. The strategy of the proof goes as follows: along with $\varphi(t)$ and $\xi(t)$ we consider also the projected solution of (4.4) $Q_L\varphi(t)$ which satisfies the equation[1]

$$i\frac{d}{dt}Q_L\varphi = HQ_L\varphi + Q_L((V * |\varphi|^2)\varphi). \tag{4.9}$$

We introduce then the difference

$$z(t) := Q_L\varphi(t) - \xi(t) \in Q_L(L^2(\mathbb{R})), \tag{4.10}$$

which in turn is the solution of

$$\begin{cases} i\dot{z} = Hz + R(t) \\ z(0) = 0 \end{cases} \tag{4.11}$$

where the remainder is defined as

$$R(t) := Q_L((V * |\varphi|^2)\varphi) - Q_L((V * |\xi|^2)\xi). \tag{4.12}$$

Since

$$\|\varphi(t) - \xi(t)\|_2 \leq \|\varphi(t) - Q_L\varphi(t)\|_2 + \|z(t)\|_2 \tag{4.13}$$

our target is to estimate the two quantities at right-hand side.

(1) *Estimate for* $\|\varphi(t) - \pi\varphi(t)\|_2$. Observe that from the well-known conservation of energy of NLS equation

[1]Notice that this equation is *not* equivalent to (4.7).

$$\mathcal{E} = \mathcal{E}[\varphi_0] = \mathcal{E}[\varphi(t)], \quad \forall t \in \mathbb{R}, \tag{4.14}$$

where

$$\mathcal{E}[\varphi] := \langle \varphi, H\varphi \rangle + \frac{1}{2} \int dxdyV(x-y)|\varphi(x)|^2|\varphi(y)|^2 \geq \langle \varphi(t), H\varphi(t) \rangle. \tag{4.15}$$

Hence

$$\langle \varphi(t), H\varphi(t) \rangle = \sum_k E_k |\langle u_k, \varphi(t) \rangle|^2 =: \sum_k f_k(t) \leq \mathcal{E} \tag{4.16}$$

and we easily have

$$\begin{aligned} \|\varphi(t) - Q_L\varphi(t)\|_2^2 &= \sum_{k \geq L} |\langle u_k, \varphi(t) \rangle|^2 = \sum_{k \geq L} \frac{f_k(t)}{E_k} \\ &\leq \sup_{k \geq L} \left\{ E_k^{-1} \right\} \cdot \sum_{k \geq L} f_k(t) \leq \frac{\mathcal{E}}{E_L} \end{aligned} \tag{4.17}$$

and using the spectral bounds of Appendix B we infer that there exists a constant $C_\sigma > 0$ such that

$$\|\varphi(t) - Q_L\varphi(t)\|_2 \leq C_\sigma \frac{\sqrt{\mathcal{E}}}{L^{\frac{\sigma}{\sigma+1}}}. \tag{4.18}$$

(2) *Estimate for* $\|z(t)\|_2$. We start by deriving a evolutive equation for $\|z(t)\|_2$:

$$\begin{aligned} i\frac{d}{dt}\|z(t)\|_2^2 &= \langle z, i\dot{z} \rangle - \langle i\dot{z}, z \rangle \\ &= \langle z, R(t) \rangle - \langle R(t), z \rangle \\ &= 2i\Im(\langle z, R(t) \rangle). \end{aligned} \tag{4.19}$$

On the other hand

$$\frac{d}{dt}\|z(t)\|_2^2 = 2\|z(t)\|_2 \frac{d}{dt}\|z(t)\|_2,$$

so

$$\|z(t)\|_2 \frac{d}{dt}\|z(t)\|_2 \leq 2|\Im(\langle z, R(t) \rangle)| \leq 2|\langle z, R(t) \rangle| \leq 2\|z(t)\|_2 \; \|R(t)\|_2 \tag{4.20}$$

and

$$\|z(t)\|_2 = \int_0^t \frac{d}{d\tau}\|z(\tau)\|_2 d\tau \leq \int_0^t \|R(\tau)\|_2 d\tau. \tag{4.21}$$

By the definition of $R(t)$ (see (4.12)) we have

$$\begin{aligned}\|R(t)\|_2 &\leq \|(V*|\varphi|^2)\varphi) - (V*|\xi|^2)\xi\|_2 \\ &= \|(V*|\varphi|^2)\varphi) - (V*|Q_L\varphi|^2)Q_L\varphi) + (V*|Q_L\varphi|^2)Q_L\varphi) - (V*|\xi|^2)\xi\|_2 \\ &\leq \|(V*|\varphi|^2)\varphi) - (V*|Q_L\varphi|^2)Q_L\varphi)\|_2 + \|(V*|Q_L\varphi|^2)Q_L\varphi) - (V*|\xi|^2)\xi\|_2 \\ &=: \|R_1(t)\|_2 + \|R_2(t)\|_2.\end{aligned} \tag{4.22}$$

To provide a bound for R_1 we write the identity

$$\begin{aligned}(V*|\varphi|^2)\varphi) - (V*|Q_L\varphi|^2)Q_L\varphi) = (V*|\varphi|^2)(\varphi - Q_L\varphi) + \\ + (V*((\varphi - Q_L\varphi)\bar{\varphi})))Q_L\varphi \\ + \left(\left(V*\left((\bar{\varphi} - \overline{Q_L\varphi})Q_L\varphi\right)\right)\right)Q_L\varphi.\end{aligned} \tag{4.23}$$

The L^2-norm of $R_1(\tau)$ is easily estimated using the well-known inequality $\|f*g\|_p \leq \|f\|_1\|g\|_p$ for any functions $f \in L^1$, $g \in L^p$ and any $1 \leq p \leq \infty$. Indeed for $p = \infty$

$$\|(V*|\varphi|^2)(\varphi - Q_L\varphi)\|_2 \leq \|V*|\varphi|^2\|_\infty\|\varphi - Q_L\varphi\|_2 \leq \|V\|_1\|\varphi\|_\infty^2\|\varphi - Q_L\varphi\|_2, \tag{4.24}$$

while for $p = 2$

$$\|(V*((\varphi - Q_L\varphi)\bar{\varphi})))Q_L\varphi\|_2 \leq \|V\|_1\|\varphi\|_\infty\|Q_L\varphi\|_\infty\|\varphi - Q_L\varphi\|_2, \tag{4.25}$$

$$\|\left(\left(V*\left((\bar{\varphi} - \overline{Q_L\varphi})Q_L\varphi\right)\right)\right)Q_L\varphi\|_2 \leq \|V\|_1\|Q_L\varphi\|_\infty^2\|\varphi - Q_L\varphi\|_2. \tag{4.26}$$

Using Lemma 4.2.3 which guarantees the existence of both $\|\varphi\|_\infty$ and $\|Q_L\varphi\|_\infty$ and the estimates of point (1) we see that

$$\begin{aligned}\|R_1(t)\|_2 &\leq \|V\|_1\left(\|\varphi\|_\infty + \|Q_L\varphi\|_\infty\right)^2\|\varphi - Q_L\varphi\|_2 \\ &\leq 8\sqrt{2}\|V\|_1\frac{\mathcal{E}}{\sqrt{E_L}} \\ &\leq 8\sqrt{2}\|V\|_1 C_\sigma\frac{\mathcal{E}}{L^{\frac{\sigma}{\sigma+1}}}\end{aligned} \tag{4.27}$$

The same argument applies to $R_2(t)$. From the identity

$$(V * |Q_L\varphi|^2)Q_L\varphi - (V * |\xi|^2)\xi = (V * |Q_L\varphi|^2)^2 Q_L\varphi - (V * |\xi|^2)Q_L\varphi \\ + (V * |\xi|^2)Q_L\varphi - (V * |\xi|^2)\xi \tag{4.28}$$

and calling $\tilde{\mathcal{E}} = \mathcal{E}[\xi_0]$ the conserved energy for the reduced NLS equation (see Lemma 4.2.2), we get

$$\begin{aligned} \|R_2(t)\|_2 &\leq \|V\|_1(\|Q_L\varphi\|_\infty + \|\xi\|_\infty)^2 \|Q_L\varphi - \xi\|_2 \\ &= \|V\|_1(\|Q_L\varphi\|_\infty + \|\xi\|_\infty)^2 \|z(t)\|_2 \\ &\leq 2\sqrt{2}\|V\|_1 \left(\mathcal{E}^{1/4} + \tilde{\mathcal{E}}^{1/4}\right)^2 \|z(t)\|_2 \\ &\leq 2\sqrt{2}\|V\|_1 \left((\mathcal{E} + \tilde{\mathcal{E}})^{1/4} + (\mathcal{E} + \tilde{\mathcal{E}})^{1/4}\right)^2 \|z(t)\|_2 \\ &\leq 8\sqrt{2}\|V\|_1 \left(\mathcal{E} + \tilde{\mathcal{E}}\right)^{1/2} \|z(t)\|_2 \end{aligned} \tag{4.29}$$

Putting all together we have from (4.21)

$$\|z(t)\|_2 \leq \frac{\mathcal{A}_\sigma}{L^{\frac{\sigma}{\sigma+1}}} t + \mathcal{B}\int_0^t \|z(\tau)\|_2 d\tau. \tag{4.30}$$

where

$$\begin{cases} \mathcal{A}_\sigma := 8\sqrt{2}\|V\|_1 C_\sigma \mathcal{E} \\ \mathcal{B} := 8\sqrt{2}\|V\|_1 \left(\mathcal{E} + \tilde{\mathcal{E}}\right)^{1/2}, \end{cases} \tag{4.31}$$

so from the integral Gronwall inequality (see [45])

$$\|z(t)\|_{L^2} \leq \frac{\mathcal{A}_\sigma}{L^{\frac{\sigma}{\sigma+1}}}\ t\ e^{\mathcal{B}t} \tag{4.32}$$

which together with (4.18) gives (4.8).

□

We conclude by proving the two lemmas previously mentioned.

Lemma 4.2.2. *Let $\xi(t)$ be the solution of (4.7). Then*

(i) $\xi(t)$ is contained in $Q_L(L^2(\mathbb{R}))$, that is $\xi(t) = Q_L\xi(t)$.

(ii) The L^2-norm of $\xi(t)$ and the energy associated with (4.7)

$$\mathcal{E}[\xi] = \langle \xi, H\xi\rangle + \frac{1}{2}\int dxdyV(x-y)|\xi(x)|^2|\xi(y)|^2 \tag{4.33}$$

are constants of motion, that is

$$\|\xi(t)\|_2 = \|\xi_0\|_2, \tag{4.34}$$

$$\mathcal{E}[\xi(t)] = \mathcal{E}[\xi_0]. \tag{4.35}$$

Proof. (i) By writing $\xi(t)$ in integral form

$$\xi(t) = e^{-itH}\xi_0 + \int_0^t e^{-i(t-\tau)H}(Q_L((V * |\xi(\tau)|^2)\xi(\tau)))d\tau, \tag{4.36}$$

we see that the Fourier coefficients of $\xi(t)$ with respect to $\{u_k\}_{k\in\mathbb{N}}$ are

$$\langle u_k, \xi(t)\rangle = e^{-itE_k}\langle u_k, \xi_0\rangle + \int_0^t e^{-i(t-\tau)H}\langle Q_L(u_k), (V * |\xi(\tau)|^2)\xi(\tau)\rangle d\tau \tag{4.37}$$

which is 0 if $k \geq L$, so that $\xi(t)$ is a linear combination of the first L eigenfunctions u_k only.

(ii) To simplify notations we drop the t-dependence of ξ here. Just compute

$$\begin{aligned}
i\frac{d}{dt}\mathcal{E}[\xi] &= \langle \xi, H(i\dot{\xi})\rangle - \langle i\dot{\xi}, H\xi\rangle + \langle (V * |\xi|^2)\xi, i\dot{\xi}\rangle - \langle i\dot{\xi}, (V * |\xi|^2)\xi\rangle \\
&= \langle H\xi, H\xi + Q_L((V * |\xi|^2)\xi)\rangle - \langle H\xi + Q_L((V * |\xi|^2)\xi), H\xi\rangle + \\
&\qquad + \langle (V * |\xi|^2), H\xi + Q_L((V * |\xi|^2))\rangle - \langle H\xi + Q_L((V * |\xi|^2)\xi), (V * |\xi|^2)\xi\rangle \\
&= \|H\xi\|_2^2 + \langle H\xi, Q_L((V * |\xi|^2)\xi)\rangle - \|H\xi\|_2^2 - \langle Q_L((V * |\xi|^2)\xi), H\xi\rangle + \\
&\qquad + \langle (V * |\xi|^2)\xi, H\xi\rangle + \langle (V * |\xi|^2)\xi, Q_L((V * |\xi|^2)\xi)\rangle - \langle H\xi, (V * |\xi|^2)\xi\rangle + \\
&\qquad - \langle Q_L((V * |\xi|^2)\xi), (V * |\xi|^2)\xi\rangle \\
&= \langle H\xi, (V * |\xi|^2)\xi\rangle - \langle (V * |\xi|^2)\xi, H\xi\rangle + \langle (V * |\xi|^2)\xi, H\xi\rangle + \\
&\qquad + \langle (V * |\xi|^2)\xi, Q_L((V * |\xi|^2)\xi)\rangle - \langle H\xi, (V * |\xi|^2)\xi\rangle + \\
&\qquad - \langle (V * |\xi|^2)\xi, Q_L((V * |\xi|^2)\xi)\rangle \\
&= 0,
\end{aligned} \tag{4.38}$$

where the second to last line follows from $Q_L^* = Q_L$, $[Q_L, H] = 0$ and from point (i). The conservation of $\|\xi(t)\|_2$ is proven in a completely analogous manner. □

Lemma 4.2.3. *Let $\varphi(t)$ and $\xi(t)$ the solutions of (4.4) and (4.7) respectively. Then*

$$\|\varphi\|_\infty \le \sqrt{2}(2\mathcal{E})^{1/4}, \tag{4.39}$$

$$\|\xi\|_\infty \le \sqrt{2}(2\tilde{\mathcal{E}})^{1/4}, \tag{4.40}$$

$$\|Q_L\varphi(t)\|_\infty \le \sqrt{2}(2\mathcal{E})^{1/4}, \tag{4.41}$$

where $\mathcal{E} = \mathcal{E}[\varphi_0]$ and $\tilde{\mathcal{E}} = \mathcal{E}[\xi_0]$.

Proof. Firstly, recall the one-dimensional Gagliardo-Nirenberg inequality (see [46], Appendix B.5)

$$\|f\|_\infty \le \sqrt{2}\ \|\nabla f\|_2^{1/2}\ \|f\|_2^{1/2}, \quad \forall f \in H^1(\mathbb{R}). \tag{4.42}$$

In our case we have, from the conservation of energy $\mathcal{E}$ associated with (4.4)

$$\|\nabla\varphi(t)\|_2^2 = \langle\varphi(t), (-\Delta\varphi(t))\rangle \le 2\langle\varphi(t), H\varphi(t)\rangle \le 2\mathcal{E} \tag{4.43}$$

which implies, together with the L^2-norm conservation of NLS equation, that

$$\|\varphi\|_\infty \le \sqrt{2}(2\mathcal{E})^{1/4}. \tag{4.44}$$

The same argument obviously applies to $\xi(t)$, giving

$$\|\xi\|_\infty \le \sqrt{2}(2\tilde{\mathcal{E}})^{1/4}. \tag{4.45}$$

Regarding $Q_L\varphi(t)$ we write

$$\begin{aligned}\langle Q_L\varphi(t), HQ_L\varphi(t)\rangle &\le \langle Q_L\varphi(t), HQ_L\varphi(t)\rangle + \langle(\mathbb{I}-Q_L)\varphi(t), (\mathbb{I}-Q_L)H\varphi(t)\\ &= \langle\varphi(t), H\varphi(t)\rangle \le \mathcal{E},\end{aligned} \tag{4.46}$$

so the usual energy estimate $\|\nabla(Q_L\varphi(t))\|_2 \le \sqrt{2\mathcal{E}}$ and $\|Q_L\varphi(t)\| \le \|\varphi(t)\|$ give again

$$\|Q_L\varphi(t)\|_\infty \le \sqrt{2}(2\mathcal{E})^{1/4}. \tag{4.47}$$

□

Remark 4.2.4. (i)The coefficients $\{c_j(t)\}_{j=1}^L$ of $\xi(t) = \sum_{j=1}^L c_j(t)u_j$ obey the equation

$$i\dot{c}_j = E_jc_j + \sum_{k,l,m=1}^{L} \Lambda_{jklm}\bar{c}_kc_lc_m \tag{4.48}$$

for all $1 \leq j \leq L$, where

$$\Lambda_{jklm} = \int \overline{u_j(x)u_k(y)}\; V(x-y)u_l(y)u_m(x)dxdy, \tag{4.49}$$

that is a DNLS equation with diagonal quadratic term and *full* quartic term.
(ii) The same computations can be carried also in the case of Gross-Pitaevskii equation, *formally* the case in which $V(x-y) \propto \delta(x-y)$. In this case we get the same constants $\mathcal{A}_\sigma, \mathcal{B}$ without the $\|V\|_1$ factor.

4.3 Adding a control on trap size

Although inequality (4.8) is nice with respect to the L-dependency, it has a quite bad behavior in time since it becomes exponentially less significative as t grows. It is however an interesting fact that if we introduce a control on both the trap size and the depth of potential W something interesting can be said. More precisely, for $\lambda > 1$ let us define the Hamiltonian

$$H_\lambda := -\frac{1}{2}\frac{d^2}{dx^2} + V_\lambda(x) + \frac{1}{\lambda}W(x) \tag{4.50}$$

where $V_\lambda = (x/\lambda)^{2\sigma}$. Then if we allow the initial condition of (4.4) to be *well-behaved*, we have the following result.

Proposition 4.3.1. *Let H_λ be the operator on $L^2(\mathbb{R})$ defined as in (4.50). Let $\varphi(t)$ and $\xi(t)$ the solutions of the NLS equation and the reduced NLS with initial datum*

$$\varphi_0 = \sum_{0\leq k<q} \langle u_k, \varphi_0\rangle u_k, \quad \|\varphi_0\| = 1, \tag{4.51}$$

for some integer $q < L$ (only the first q levels of H are populated at $t=0$). Then there exist positive constants $a_{\sigma,q,\|W\|_\infty}, b_{\sigma,q,\|W\|_\infty}, f_{\sigma,q,\|W\|_\infty}$ depending only on σ, q and $\|W\|_\infty$ such that

$$\|\varphi(t)-\xi(t)\|_2 \leq b_{\sigma,q,\|W\|_\infty}\frac{1}{\lambda^{1/2}}\left(\frac{\lambda}{L}\right)^{\frac{\sigma}{\sigma+1}} + c_{\sigma,q,\|W\|_\infty}\frac{1}{\lambda}\left(\frac{\lambda}{L}\right)^{\frac{\sigma}{\sigma+1}} t + f_{\sigma,q,\|W\|_\infty}\frac{1}{\lambda^{1/2}}t. \tag{4.52}$$

Proof. First of all we notice that the hypothesis on φ_0 implies that $\xi_0 = Q_L\varphi_0 = \varphi_0$, so that

$$\tilde{\mathcal{E}} = \mathcal{E} = \mathcal{E}[\varphi_0]. \tag{4.53}$$

The conserved energy can be estimated with the new parameter λ using (B.18)

$$
\begin{aligned}
\mathcal{E} &= \sum_{0\le k<q} |\langle u_k, \varphi_0\rangle|^2 E_k + \frac{1}{2}\int V(x-y)|\varphi_0(x)|^2|\varphi_0(y)|^2 dxdy \\
&\le \max_{0\le k<q}\{E_k\}\cdot \sum_{0\le k<q} |\langle u_k, \varphi_0\rangle|^2 + \frac{1}{2}\int V(x-y)|\varphi_0(x)|^2|\varphi_0(y)|^2 dxdy \qquad (4.54)\\
&\le E_q + \int V(x-y)|\varphi_0(x)|^2|\varphi_0(y)|^2 dxdy.
\end{aligned}
$$

To find a bound on the integral at right-hand side we use again the Gagliardo-Nirenberg inequality in $\mathbb{R}$. In particular, calling $c_j := \langle u_j, \varphi_0\rangle$ we have

$$
\begin{aligned}
\int V(x-y)|\varphi_0(x)|^2|\varphi_0(y)|^2 dxdy \le \sum_{1\le j,k,l,m<q} |c_j c_k c_l c_m| \int V(x-y)|u_j(x)|\ |u_k(y)|\ \times \\
\times |u_l(x)|\ |u_m(y)| dxdy
\end{aligned}
\tag{4.55}
$$

and also

$$
\|u_j\|_\infty \le \sqrt{2}(2E_j)^{1/4} \le \sqrt{2}(2E_q)^{1/4} \tag{4.56}
$$

for all $1\le j<q$, hence

$$
\begin{aligned}
\int V(x-y)|u_j(x)|\ |u_k(y)||u_l(x)|\ |u_m(y)| dxdy &\le 8E_q\|V\|_1 \sum_{1\le j,k,l,m<q} |c_j c_k c_l c_m| \\
&\le 8E_q\|V\|_1 q^4
\end{aligned}
\tag{4.57}
$$

and finally

$$
\mathcal{E} \le (1+4q^4\|V\|_1)E_q. \tag{4.58}
$$

Now recalling the estimates in (B.18)

$$
\mathcal{E} \le (1+4q^4\|V\|_1)\left((\lambda^{-1}\|W\|_\infty + D_\sigma\left(\frac{q+1}{\lambda}\right)^{\frac{2\sigma}{\sigma+1}}\right) \le a_{\sigma,q,\|V\|_1,\|W\|_\infty}\lambda^{-1} \tag{4.59}
$$

for some positive constant $a_{\sigma,q,\|V\|_1,\|W\|_\infty}$.

We can now improve the estimate in time of previous section. Notice that from energy conservation we have for all $\tau \in \mathbb{R}$ that $\|z(\tau)\|_2 \leq \|\varphi(\tau)\|_2 + \|\xi(\tau)\|_2 \leq 2$, so using (4.53), (4.30) becomes

$$\|z(t)\|_2 \leq 8\sqrt{2}\|V\|_1 \frac{\mathcal{E}}{\sqrt{E_L}}\, t + 2\mathcal{B}\, t = 8\sqrt{2}\frac{\mathcal{E}}{\sqrt{E_L}} + 16\|V\|_1\sqrt{\mathcal{E}}t. \tag{4.60}$$

To obtain a bound on $\|\varphi - \xi\|_2$ we need to compute (recall (4.17) and (4.27))

$$\sqrt{\frac{\mathcal{E}}{E_L}} \leq b_{\sigma,q,\|V\|_1,\|W\|_\infty} \frac{1}{\lambda^{1/2}} \left(\frac{\lambda}{L}\right)^{\frac{\sigma}{\sigma+1}}, \tag{4.61}$$

$$\frac{\mathcal{E}}{\sqrt{E_L}} \leq c_{\sigma,q,\|V\|_1,\|W\|_\infty} \frac{1}{\lambda} \left(\frac{\lambda}{L}\right)^{\frac{\sigma}{\sigma+1}}, \tag{4.62}$$

$$2\mathcal{B} \leq f_{\sigma,q,\|V\|_1,\|W\|_\infty} \frac{1}{\lambda^{1/2}}, \tag{4.63}$$

for some positive constants $a_{\sigma,q,\|V\|_1,\|W\|_\infty}$, $b_{\sigma,q,\|V\|_1,\|W\|_\infty}$, $f_{\sigma,q,\|V\|_1,\|W\|_\infty}$. Putting last bounds together with (4.60), we get (4.3.1).

□

Remark 4.3.2. The hypotheses of Proposition 4.3.1 are quite peculiar, especially the one on the initial datum of NLS equation. The physical meaning however is clear: if our system is put at time zero in a low energy state, then the solution $\xi(t)$ of reduced NLS equation approximates $\varphi(t)$ up to linearly long times, provided that the ratio λ/L is bounded as both λ and L go to infinity.

Conclusions

Even though the main result of this work, namely Theorem 2.2.1 contained in Chapter 2, is appealing and represents a first step towards the rigorous understanding of the connection between the classical DNLS flow and the quantum dynamics generated by the Bose-Hubbard hamiltonian, it is still a bit far from being definitive and opens some interesting questions.

(i) Usually in literature quantitative bounds are derived for the difference of density matrices since they allow to compute the average of physical quantities, see e.g. [51], while here we consider the distance between symbols of certain evolving operators. The link between these two notions should be provided by putting together the results of 2, 3 and 4. More precisely if $\Psi(t)$ is some evolving N-body state and $\varphi(t)$ is the solution of the Hartree equation at time t, the idea is to compute the difference of the two quantities by means of the elementary inequality

$$\|\gamma^{(1)}_{\Psi(t)}-\langle\varphi(t),\cdot\rangle\varphi(t)\| \leq \|\gamma^{(1)}_{\Psi(t)}-\gamma^{(1)}_{\Psi(t)_L}\|+\|\gamma^{(1)}_{\Psi(t)_L}-\langle\xi(t),\cdot\rangle\xi(t)\|+\|\langle\xi(t),\cdot\rangle\xi(t)-\langle\varphi(t),\cdot\rangle\varphi(t)\|, \tag{4.64}$$

where L is a truncation index and $\xi(t)$ is the solution of the reduced Hartee equation defined in Chapter 4. While the first and the last term are easy to estimate using Propositions 3.3.1 and 4.2.1 respectively, the second one should involve the matching between the coefficients of ξ evolving according a modified DNLS equation (see 4.48) and the estimates of Chapter 2, a result which is missing at the moment.

(ii) The use of coherent states in Chapter 2 is of pure mathematical flavor, in the sense that it allows us to consider the closeness of classical and quantum flows in terms of symbols of some operators. It would be interesting however to understand how this can be linked with the pre-existing literature in which coherent states are an approximation of ground states of trapped bosonic systems (see again [51]).

(iii) While we are referring to the DNLS and BH model as describing bosons on a one-dimensional lattice with nearest-neighbor hopping term, our analysis can be carried also for the DST equation (see Section 2.1) allowing us to consider not only longer range interactions but also more complex topologies (see e.g. [26]).

Appendix A

Partial trace and one-particle density operator

In this brief appendix we recall a simple characterization of one-particle density operator. The main notion we need is the one of *partial trace*, which is quite hard to find in literature and in many instances it is treated only in finite dimensional case. Consider two separable Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ and their tensor product $\mathfrak{h}_1 \otimes \mathfrak{h}_2$. Given a trace class operator $\Gamma \in \mathfrak{B}_1(\mathfrak{h}_1 \otimes \mathfrak{h}_2)$ its partial trace with respect to $\mathfrak{h}_2$ is the unique trace class operator $\gamma^{(1)} \in \mathfrak{B}_1(\mathfrak{h}_1)$ such that (see e.g. [43], Section 13.4.6 or the lecture notes [9] for a deeper insight)

$$\langle f, \gamma^{(1)} g\rangle = \sum_{u \in \mathcal{B}} \langle f \otimes u, \Gamma(g \otimes u)\rangle, \qquad \forall f, g \in \mathfrak{h}_1, \tag{A.1}$$

for any arbitrary Hilbert basis $\mathcal{B}$ in $\mathfrak{h}_2$. Let us now consider the case in which $\mathfrak{h}_2 = \mathfrak{h}^{\otimes(n-1)}$ for a prescribed Hilbert space $\mathfrak{h} = \mathfrak{h}_1$ and take as the trace class operator a n-particle pure state $\Gamma_\Psi = \langle \Psi, \cdot\rangle \Psi$ for a certain $\Psi \in \mathfrak{h}^{\otimes n}$. It is convenient to rewrite (A.1) in terms of creation and destruction operators a^*, a on full Fock space. They are defined similarly to the boson case as (see [8])

$$D(a(f)^*) = \left\{ \Psi \in \mathcal{F}(\mathfrak{h}) \;\middle|\; \sum_{n \geq 0} n \|\Psi^{(n-1)}\|^2 < \infty \right\} \tag{A.2}$$

$$(a(f)^*\Psi)^{(n)} = \begin{cases} 0, & \text{if} \quad n = 0, \\ \sqrt{n}(f \otimes \Psi^{(n-1)}), & \text{if} \quad n > 0. \end{cases} \tag{A.3}$$

for any $f \in \mathfrak{h}$, while $a(f)$ is the adjoint of $a(f)^*$. As we saw previously, any vector in $\mathfrak{h}^{\otimes n}$ can be identified with the n-th element of a vector in the full Fock space. We shall exploit this fact by writing $\Psi = \widetilde{\Psi}^{(n)}$, so that $\Gamma_\Psi = \langle \widetilde{\Psi}^{(n)}, \cdot\rangle \widetilde{\Psi}^{(n)}$. Then by taking a Hilbert basis $\mathcal{B} \subset \mathfrak{h}^{\otimes(n-1)}$, the partial trace $\gamma_\Psi^{(1)}$ of Γ_Ψ with respect to $\mathfrak{h}$ satisfies

$$\begin{aligned}
\langle f, \gamma_\Psi^{(1)} g\rangle &= \sum_{u\in\mathcal{B}} \langle f\otimes u, \Gamma_\Psi(g\otimes u)\rangle \\
&= \sum_{u\in\mathcal{B}} \langle f\otimes u, \langle \Psi, g\otimes u\rangle \Psi\rangle \\
&= \sum_{u\in\mathcal{B}} \langle \Psi, g\otimes u\rangle\langle f\otimes u, \Psi\rangle \\
&= \frac{1}{n}\sum_{u\in\mathcal{B}} \langle \widetilde{\Psi}^{(n)}, (a^*(g)\widetilde{u})^{(n)}\rangle\langle (a^*(f)\widetilde{u})^{(n)}, \widetilde{\Psi}^{(n)}\rangle \\
&= \frac{1}{n}\sum_{u\in\mathcal{B}} \langle (a(g)\widetilde{\Psi})^{(n-1)}, u\rangle\langle u, (a(f)\widetilde{\Psi})^{(n-1)}\rangle.
\end{aligned} \tag{A.4}$$

Now, using the fact that $\mathcal{B}$ is a Hilbert basis of $\mathfrak{h}^{\otimes(n-1)}$, last equation reads

$$\begin{aligned}
&= \frac{1}{n}\left\langle (a(g)\widetilde{\Psi})^{(n)}, \sum_{u\in\mathcal{B}} \langle u, (a(f)\widetilde{\Psi})^{(n)}\rangle u \right\rangle \\
&= \frac{1}{n}\langle (a(g)\widetilde{\Psi})^{(n)}, (a(f)\widetilde{\Psi})^{(n)}\rangle \\
&= \frac{1}{n}\langle a(g)\widetilde{\Psi}, a(f)\widetilde{\Psi}\rangle \\
&= \frac{1}{n}\langle \widetilde{\Psi}, a^*(g)a(f)\widetilde{\Psi}\rangle.
\end{aligned} \tag{A.5}$$

We observe that since $\widetilde{\Psi}$ is in the n-particle sector $n = \langle \widetilde{\Psi}, \boldsymbol{N}\widetilde{\Psi}\rangle$, so that

$$\langle f, \gamma_\Psi^{(1)} g\rangle = \frac{\langle \widetilde{\Psi}, a^*(g)a(f)\widetilde{\Psi}\rangle}{\langle \widetilde{\Psi}, \boldsymbol{N}\widetilde{\Psi}\rangle}. \tag{A.6}$$

Notice that the right-hand side of last equation makes sense if $\tilde{\Psi}$ is any element in $D(\mathbf{N})^{\frac{1}{2}}$, in particular it can be taken to be a vector in $D(\mathbf{N})^{\frac{1}{2}} \cap \mathcal{F}_B(\mathfrak{h})$.

Appendix B

Asymptotic spectral bounds on trapping potentials

We derive here the asymptotic bound on the eigenvalues of the operator H on $L^2(\mathbb{R})$ defined as

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V_{trap}(x) + W(x) \tag{B.1}$$

needed in the estimates in Chapters (2) and (4). We recall that V is given by

$$V_{trap}(x) := x^{2\sigma}, \tag{B.2}$$

for some integer $\sigma > 1$ and $W \in C^\infty(\mathbb{R})$ is a bounded and positive function.
It is clear that within this setting $V_{trap} + W \to +\infty$ as $|x| \to +\infty$, so that the spectrum of H is bounded from below and consists only of discrete eigenvalues $\{E_k\}_{k\in\mathbb{N}}$ accumulating at infinity (see [54], Theorem 26.3). We want to prove that there exist an index $\bar{k}$ such that for all $k > \bar{k}$ the eigenvalues of H have the following bounds

$$\begin{cases} E_k \geq B_\sigma (k+1)^{\frac{2\sigma}{\sigma+1}} \\ E_k \leq \|W\|_\infty + D_{\sigma,W}(k+1)^{\frac{2\sigma}{\sigma+1}} \end{cases} \tag{B.3}$$

where B_σ and $D_{\sigma,W}$ are two positive constants.
We start noticing that the positivity of W implies that $V_{trap} + W$ is polynomially bounded by

$$x^{2\sigma} \leq (V_{trap}(x) + W(x)) \leq x^{2\sigma} + \|W\|_\infty, \quad \forall x \in \mathbb{R}. \tag{B.4}$$

Let $F(S)$ be the number of eigenvalues of H not exceeding S

$$F(S) := \sharp\{E_k \mid E_k \leq S\}, \tag{B.5}$$

which is a non-decreasing function $S \to F(S)$ and satisfies

$$
\begin{aligned}
F(E_0) &= 1, \\
F(E_1) &= 2, \\
&\dots \\
F(E_k) &= k+1
\end{aligned}
\tag{B.6}
$$

and so on.

Moreover let $\mathcal{H}(x,p) := -\frac{1}{2}p^2 + V(x) + W(x)$ the classical hamiltonian (Weyl symbol) associated with H and define the energy sub-level set Ω_S as

$$
\Omega_S := \left\{ (x,p) \in \mathbb{R}^2 \;\middle|\; \mathcal{H}(x,p) \leq S \right\}. \tag{B.7}
$$

then, thanks to (B.4), F and Ω turn to be of the same order as S goes to infinity (see [54], Theorem 30.1)

$$
\lim_{S\to+\infty} \frac{F(S)}{\frac{1}{2\pi}\int_{\Omega_S} dx dp} = \lim_{S\to+\infty} \frac{F(S)}{\frac{1}{2\pi}\operatorname{Area}(\Omega_S)} = 1. \tag{B.8}
$$

From the last condition we can recover an asymptotic bound on the eigenvalues E_k. Indeed (B.8) implies that there exist $\bar{S} > 0$ such that

$$
\frac{1}{2} \cdot \frac{1}{2\pi} \operatorname{Area}(\Omega_S) \leq F(S) \leq \frac{3}{2} \cdot \frac{1}{2\pi} \operatorname{Area}(\Omega_S), \quad \forall S > \bar{S}. \tag{B.9}
$$

On the other hand, thanks to (B.4) the following inclusions of sets in $\mathbb{R}^2$ holds for all $S > 0$

$$
\left\{ (x,p) \;\middle|\; \mathcal{H}(x,p) \leq S \right\} \subset \left\{ (x,p) \;\middle|\; \frac{1}{2}p^2 + x^{2\sigma} \leq S \right\} = \Omega^1_S, \tag{B.10}
$$

so that

$$
\begin{aligned}
\operatorname{Area}(\Omega_S) &\leq \operatorname{Area}(\Omega^1_S) \\
&= 4 \int_0^{S^{1/2\sigma}} \sqrt{2(S - x^{2\sigma})} dx \\
&= 4\sqrt{2}\, S^{\frac{\sigma+1}{2\sigma}} \int_0^1 \sqrt{1 - x^{2\sigma}} dx \\
&= 4\sqrt{2}\, \frac{\sqrt{\pi}\, \Gamma\left(1 + \frac{1}{2\sigma}\right)}{2\Gamma\left(\frac{1}{2}\left(3 + \frac{1}{\sigma}\right)\right)}\, S^{\frac{\sigma+1}{2\sigma}}.
\end{aligned}
\tag{B.11}
$$

Using (B.6) and the second inequality in (B.9) we find that for sufficiently large E_k, that is for $k > \bar{k}$ for some $\bar{k} \in \mathbb{N}$, there exist a constant $B_\sigma > 0$ such that

$$E_k \geq B_\sigma (k+1)^{\frac{2\sigma}{\sigma+1}}. \tag{B.12}$$

Notice that this estimate is a middle way between the two extreme cases $\sigma \to \infty$ (*particle-in-a-box* potential), for which $E_k \sim k^2$, and $\sigma = 1$ (quantum harmonic oscillator), for which $E_k \sim k$, since $1 < 2\sigma/(\sigma+1) < 2$ for $\sigma > 1$.
To prove the second estimate in (B.3) we use the same argument as above. From (B.4) we have the inclusion

$$\big\{(x,p) \mid \mathcal{H}(x,p) \leq S\big\} \supset \left\{(x,p) \mid \frac{1}{2}p^2 + x^{2\sigma} + \|W\|_\infty \leq S\right\} = \Omega_S^2, \tag{B.13}$$

so repeating the calculations in (B.11) we get

$$\mathrm{Area}(\Omega_S) \geq \mathrm{Area}(\Omega_S^2) = 4\sqrt{2}\,\frac{\sqrt{\pi}\;\Gamma\left(1+\frac{1}{2\sigma}\right)}{2\Gamma\left(\frac{1}{2}\left(3+\frac{1}{\sigma}\right)\right)}\,(S - \|W\|_\infty)^{\frac{\sigma+1}{2\sigma}} \tag{B.14}$$

and finally from the first inequality in (B.8) we establish the existence of a constant $D_{\sigma,W}$ such that

$$E_k \leq \|W\|_\infty + D_\sigma (k+1)^{\frac{2\sigma}{\sigma+1}} \tag{B.15}$$

for $k > \bar{k}$.

Remark B.0.1. We remark that the potential $V(x)$ models, roughly speaking, a system trapped in $[-1,1]$. If we want to control the size of the trap we may substitute V with a potential

$$V_\lambda(x) := \left(\frac{x}{\lambda}\right)^{2\sigma}, \quad \lambda > 0, \tag{B.16}$$

so that we may think the system to be trapped in $[-\lambda, \lambda]$. In this case, if we also allow the bounded potential to scale in magnitude according to

$$W_\lambda(x) := \frac{1}{\lambda} W(x), \tag{B.17}$$

inequalities (B.3) still hold with the following modification

$$\begin{cases} E_k \geq B_\sigma \left(\dfrac{k+1}{\lambda}\right)^{\frac{2\sigma}{\sigma+1}} \\[2ex] E_k \leq \dfrac{1}{\lambda}\,\|W\|_\infty + D_{\sigma,W}\left(\dfrac{k+1}{\lambda}\right)^{\frac{2\sigma}{\sigma+1}} . \end{cases} \tag{B.18}$$

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