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Topics in large-volume behavior of interacting particle systems: emergent rhythms and propagation of chaos

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Abstract

This thesis concerns the large-volume behavior of systems of stochastic interacting particles. The first two chapters deal with emergent self-sustained rhythmic behaviors and investigate some possible mechanisms at their origin, whereas the third one is dedicated to the conditional propagation of chaos.

In Chapter 1 we investigate the emergence of collective periodic behaviors from the combination of specific interaction networks and Brownian noise. We consider a toy model of two populations of frustratedly interacting diffusions. We show that, in the thermodynamic limit, a periodic law might arise, in some parameter regimes, for an intermediate noise intensity. Despite the frustrated interaction network, no rhythmic behavior is present in the absence of noise. Thus, this phenomenon goes under the name of noise-induced periodicity. In Chapter 2 we investigate the emergence of collective oscillations in a mean-field contact process where the dynamics of the interaction terms is subject to dissipation. We show that, in the thermodynamic limit, the system is attracted towards a stable equilibrium point, but persistent, noise-induced, oscillations might arise for the correspondent fluctuation process in an appropriate regime of the parameters. Therefore, self-sustained rhythmic behaviors survive as finite-size effects when the number of individuals in the system is large but finite. In Chapter 3 we consider a system of SDEs driven by Poisson random measures and whose coefficients depend on the empirical measure of the system. In such finite system, particles are subject to simultaneous jumps whose distribution lies in the normal domain of attraction of a strictly α-stable law. We prove strong existence and uniqueness of the limit McKean-Vlasov system, which is an infinite-exchangeable system of SDEs driven by a common stable process and whose coefficients depend on the conditional law of any of its coordinates, given such common process. We then show the convergence of the finite to the limit system, also providing explicit error bounds for finite time marginals.

Sommario

L'argomento di questa tesi sono i comportamenti di larga scala di sistemi stocastici costituiti da un grande numero di componenti interagenti. Nei primi due capitoli ci occupiamo dei comportamenti emergenti periodici auto-sostenuti e approfondiamo alcuni meccanismi che potrebbero generarli, mentre il terzo capitolo è dedicato al fenomeno della propagazione del caos.

Nel Capitolo 1 studiamo l'emergenza di comportamenti collettivi periodici risultanti dalla combinazione di particolari tipi di interazioni tra particelle e rumore Browniano. Consideriamo un sistema costituito da due popolazioni di diffusioni con interazioni frustrate. Mostriamo che, nel limite termodinamico, in alcuni regimi dei parametri del modello, un'intensità intermedia del rumore potrebbe originare una legge periodica. Nonostante la rete di interazioni frustrate, in assenza di rumore non si verificano comportamenti periodici. Pertanto, si parla di periodicità indotta dal rumore. Nel Capitolo 2 studiamo l'emergenza di oscillazioni collettive in un processo di contatto a campo medio in cui la dinamica dei termini di interazione è soggetta a dissipazione. Mostriamo che, nel limite termodinamico, il sistema è attratto verso un punto di equilibrio stabile, ma che il processo di fluttuazioni del sistema intorno a tale equilibrio mostra oscillazioni persistenti in alcuni regimi dei parametri del modello. Pertanto, i comportamenti oscillatori auto-sostenuti in questo modello sono il risultato di effetti di taglia finita e si osservano solo quando il numero di particelle del sistema è grande ma finito. Nel Capitolo 3 consideriamo un sistema di equazioni differenziali stocastiche diretto da misure random di Poisson e i cui coefficienti sono funzioni della misura empirica del sistema. Le particelle di tale sistema finito sono soggette a salti simultanei la cui distribuzione appartiene al dominio normale di attrazione di una legge strettamente α-stabile. Dimostriamo l'esistenza e l'unicità forti del sistema di McKean-Vlasov limite. Questo risulta essere un sistema di equazioni differenziali stocastiche dirette da un medesimo processo α-stabile, invariante in legge per permutazioni finite delle componenti e con coefficienti dipendenti dalla legge condizionale di qualsiasi coordinata, dato il processo stabile comune. Mostriamo infine la convergenza del sistema finito a quello limite, ottenendo anche rate di convergenza espliciti per le marginali temporali dei processi.

Introduction

This thesis deals with the large-volume behavior of systems of stochastic interacting particles. In particular, it focuses on two main topics: the emergence of collective self-sustained rhythmic behaviors and the property of the propagation of chaos.

Collective self-sustained rhythms are among the most commonly observed dynamical patterns in real-world systems comprised of many interacting components (see e.g. [49, 27, 39, 64, 18] and references therein). Loosely speaking, they occur when some averaged quantity of a system exhibits periodic or almost-periodic oscillations that are persistent in time, despite the fact that the individual components of the system neither have themselves a periodic behavior nor are subject to any external periodic forcing. Indeed, such rhythmic behaviors are only perceivable on a macroscopic scale (i.e. not when looking at the single constituents of the system) and result from the spontaneous organization of the single components of the system, which is only possible in the presence of specific features in their dynamics (e.g., particular kinds of interactions among them, and/or noise affecting their dynamics). Two examples pertinent to this thesis are neural oscillations ([70] and references therein) and cycles in endemic infectious diseases ([5] and references therein).

Despite their ubiquity, the origin of collective self-sustained rhythms is still poorly understood from a theoretical standpoint. From a mathematical point of view, a natural approach to investigate their emergence is to consider large families of interacting units and to model their dynamics through (possibly noisy) dynamical systems coupled by interaction terms. Then, one usually considers the limit when the number of units diverges (*thermodynamic* limit) and studies its behavior over long times ([38, 52, 49]). From this theoretical perspective, collective periodic behaviors are generally characterized by the presence of a periodic limit law ([61, 62]).

This is the strategy that we adopt in Chapters 1 and 2 of this thesis, which are devoted to the analysis of some possible mechanisms through which a family of non-oscillating interacting units can generate a rhythm in the absence of external periodic forcings. Various stylized models have been proposed to this aim (we refer the reader to [38, 49] and references therein). In most of them rigorous results are hard to obtain, as their study ends up in looking for stable attractors of nonlinear infinite-dimensional systems ([38, 49]). Therefore, in this thesis we focus on more analytically tractable mean-field models. Within this context, we inspect in particular the roles of noise, of the interaction network, and of dissipation added to the interaction terms in the dynamics of a system.

In Chapter 1, we investigate the emergence of self-sustained rhythms in a two-population mean-field system of Itô diffusions with a frustrated interaction network. Several works have hinted at the fact that the interplay between interaction network and noise might lead to the appearance of persistent oscillatory behaviors. Indeed, on the one hand, it has been pointed out ([18, 27, 66, 4]) that a specific network structure may favor the emergence of collective rhythms. In particular, in [18] it has been proven that the presence of two communities with possibly different sizes and inter- and intra-population interactions may suffice for the emergence of collective oscillations in a mean-field Ising model. On the other hand, several works ([52, 53, 49, 38, 61, 62, 67]) have shown how noise added to a system may destabilize fixed points and create limit cycles. In particular, in [61, 62] it has been shown that noise can lead to oscillatory laws in systems of nonlinear diffusions whose deterministic counterparts do not display any periodic behavior. In the light of these results, in Chapter 1 we investigate the joint action of Brownian noise and a particular kind of interaction network (which we call *frustrated*) in a toy model with two populations of particles. This chapter is based on [54].

Chapter 2 is dedicated to the emergence of self-sustained oscillations in a dissipative version of the mean-field contact process. The existence of collective periodic behaviors has already been proven for some systems derived as perturbations of classical reversible models from statistical physics by adding dissipation in the interaction term ([20, 17, 12]). Within a continuous-time Markovian dynamics, dissipation dampens the strength of the interaction among particles during the time when no transition occurs and breaks time-reversibility, which is incompatible with limit cycles ([38]). In the spirit of these considerations, in Chapter 2 we modify the dynamics of the contact process on the complete graph by providing interaction terms with a dissipative evolution. This chapter is based on [23].

According to the approach outlined above, our first step to investigate the emergence of collective periodic behaviors for the systems studied in Chapters 1 and 2 was to consider the dynamics of the averaged quantities of those systems in the thermodynamic limit. To rigorously derive such limit, we proved that the systems in Chapters 1 and 2 have the propagation of chaos property. This has motivated a more in-depth study of this notion.

The topic of propagation of chaos has already been extensively addressed in the literature. To cite just a few works relevant to this thesis where propagation of chaos is proved in different settings see [13, 65, 42, 41, 6, 45, 10]. The idea behind it is the following: if one considers a finite system of particles which evolve coupled by interaction terms and with independent identically distributed initial conditions, then, for any fixed number of components, the interactions generally destroy the independence property the system has at time zero. On the contrary, if we let the number of components of the system diverge, under appropriate hypotheses, independence propagates in time, in the sense that, within the infinite-particle limit system, any particle evolves independently of any finite subset of the others. In addition, all particles have the same law in the limit system. When this happens, we say that propagation of chaos occurs. Different (more or less strong) versions of the propagation of chaos property exist that formalize this intuitive notion. We refer to [13] for rigorous definitions and a review of classical methods to prove propagation of chaos.

In Chapter 3, we extend our study to the *conditional* propagation of chaos property, proving that it holds for a class of mean-field systems of particles subject to simultaneous jumps in the domain of attraction of an α-stable law. The topic of conditional propagation of chaos is not new to literature either (see for instance [69, 9, 16, 63]). If conditional propagation of chaos holds for a system of interacting particles, then, in the infinite-particle limit, the components of the system are conditionally independent and identically distributed given some common noise that couples their dynamics. Such common noise consists in a common random process - in most cases a Brownian motion ([16, 26, 9, 69]), sometimes some finite-activity jump process ([63]) - which might be thought of as modeling the influence of a random environment, and it is typically already present in the SDEs of the finite-particle system. In Chapter 3 instead, we consider a system where no common noise is present in the finite-particle dynamics, but particles are subject to *simultaneous jumps* (in the spirit of [6, 25, 35]), and the common noise which entails conditional propagation of chaos arises only in the thermodynamic limit, as a result of an *appropriate scaling* for the simultaneous jumps. A similar model is considered in [29, 30], where simultaneous jumps are assumed to have finite second moment. In Chapter 3, we extend these works to the setting where the law of the simultaneous jumps is heavy-tailed and belongs to the normal domain of attraction of a strictly α-stable law. This chapter is based on [50].

Outline of the thesis

In the following, we present the main results of each chapter of this thesis in more detail.

Chapter 1: noise-induced periodicity in a frustrated network of interacting diffusions

In Chapter 1 we investigate how the joint effect of the topology of the interaction network and Brownian noise can lead to self-sustained periodic behaviors in systems of interacting particles. To this aim, and building on [18], we design a toy model of frustratedly interacting diffusions: we consider a system of N Itô diffusions divided into two communities and subject to mean-field interaction, where the interaction terms are such that the particles of one community want to conform to the average position of the particles of the other community, whereas the particles in the latter want to do the contrary.

Specifically, the equations describing the dynamics of the system are

$$\begin{aligned} dx_i^{(N)} &= \left(-\left(x_i^{(N)}\right)^3 + x_i^{(N)}\right)dt - \alpha\,\theta_{11}\left(x_i^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\,\theta_{12}\left(x_i^{(N)} - m_2^{(N)}\right)dt + \sigma dw_i, \qquad \text{for } i = 1,\dots,N_1, \\[2ex] dy_i^{(N)} &= \left(-\left(y_i^{(N)}\right)^3 + y_i^{(N)}\right)dt - \alpha\,\theta_{21}\left(y_i^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\,\theta_{22}\left(y_i^{(N)} - m_2^{(N)}\right)dt + \sigma dw_{N_1+i}, \qquad \text{for } i = 1,\dots,N_2. \end{aligned} \tag{1}$$

Here, $(x_i)_{i=1}^{N_1} \in \mathbb{R}^{N_1}$ (resp. $(y_i)_{i=1}^{N_2} \in \mathbb{R}^{N_2}$) denotes the positions of the particles of the first (resp. second) community, $\alpha := N_1/N$, $m_1^{(N)} := \frac{1}{N_1}\sum_{i=1}^{N_1} x_i^{(N)}$ and $m_2^{(N)} := \frac{1}{N_2}\sum_{i=1}^{N_2} y_i^{(N)}$ denote the centers of mass of the two communities, θ_{11} and θ_{22} tune intra-community interactions (that is, the strengths of the interactions between any pair of particles belonging to the first and second community respectively), θ_{12} and θ_{21} tune inter-community interactions (that is, θ_{ij} tunes how strongly any particle of population j influences any particle of population i, for $i \in \{1,2\}$, $j \neq i$), $((w_i(t))_{t\geq 0})_{i=1}^N$ are independent standard Brownian motions, and $\sigma \geq 0$ tunes the amount of noise in the system. We make the specific choices θ_{11}, $\theta_{22} > 0$, which implies that each particle wants to conform to the average position of the particles of its own community, and $\theta_{12}\theta_{21} < 0$, which means that the particles of one population are attracted towards the center of mass of the other population, whereas particles of this latter are repulsed by the center of mass of the former (*frustration* of the interaction network).

We argue that this system features the phenomenon of noise-induced periodicity: in the limit $N \to +\infty$ and in appropriate regimes of the interaction parameters, an intermediate noise intensity induces periodic oscillations in the average positions of the two populations.

Intuitively, the emergence of periodicity in this model can be explained as follows. Imagine first to initialize all the particles so that $x_i^{(N)} = m_1^{(N)}$ for all $i = 1, \dots, N_1$ and $y_i^{(N)} = m_2^{(N)}$ for all $i = 1, \dots, N_2$, and set $\sigma = 0$. Then the interaction terms will be zero, each population will behave as a single macro-particle and the system will end up in one of the equilibria of the two-dimensional vector field

$$\begin{aligned} \dot{x} &= -x^3 + x - (1-\alpha)\theta_{12}(x - y) \\ \dot{y} &= -y^3 + y - \alpha\theta_{21}(y - x). \end{aligned}$$

On the contrary, if $\sigma > 0$, even if we initialize all the particles of both populations at one of the stable fixed points of the cubic vector field in system (1), particles will start to diffuse away from the equilibrium in different, independent directions. Consequently, the interaction terms in (1) will become different from zero, and, as the inter-population interaction coefficients have opposite signs, the rest states of the two populations will become incompatible. The two populations will form a frustrated pair of systems and will keep oscillating: one population will always try to move away from the other towards its own equilibrium, but the latter will always try to close the gap.

This heuristic picture is confirmed by numerical simulations of the finite system (1) for large N, which show that, in appropriate parameter regimes, $m_1^{(N)}$ and $m_2^{(N)}$ behave almost periodically. These are discussed in Section 1.3.

More rigorously, the first step to prove noise-induced periodicity is the derivation of the thermodynamic limit of system (1), which is given in Theorem 1.4.2. Theorem 1.4.2 claims that system (1) has the multi-class version of the propagation of chaos property ([41]): in the limit $N \to \infty$, particles are independent of each other within any finite subsystem and identically distributed within each population, so that each particle of the first (resp. second) community obeys the first (resp. second) equation of the following system:

$$\begin{aligned} dx &= [-x^3 + x - \alpha\theta_{11}(x - \mathbb{E}[x]) - (1-\alpha)\theta_{12}(x - \mathbb{E}[y])]dt + \sigma dw_1 \\ dy &= [-y^3 + y - \alpha\theta_{21}(y - \mathbb{E}[x]) - (1-\alpha)\theta_{22}(y - \mathbb{E}[y])]dt + \sigma dw_2, \end{aligned} \tag{2}$$

where $\mathbb{E}$ stands for the expectation with respect to the probability measure $Q(t;\cdot) = \text{Law}(x(t), y(t))$, for every $t \in [0, T]$ and $T > 0$, and $(w_1(t))_{0\leq t\leq T}$ and $(w_2(t))_{0\leq t\leq T}$ are two independent standard Brownian motions.

As a next step to prove noise-induced periodicity, we look for the presence of stable attractors for system (2).

In Section 1.4.2 we show that, despite the frustrated interaction network, the limit dynamics (2) is attracted to a stable fixed point if $\sigma = 0$. This implies that periodic behaviors, if any, can only emerge when the noise crosses a certain threshold.

Moreover, in Theorem 1.4.3, we prove that, for sufficiently small σ, the limiting positions of a pair of representative particles, one for each population, evolve approximately as a pair of independent Gaussian processes. In Section 1.4.4 we derive an explicit system of ODEs describing the means and variances of these Gaussian processes and in Section 1.5 we show that such system undergoes a Hopf bifurcation at a critical noise value $\sigma_c > 0$, hence it has a limit cycle as a long-time attractor. This implies that the laws of the Gaussian processes approximating the limit system are indeed periodic. Therefore, the collective periodic phenomena exhibited by our model can be qualitatively explained in this approximation.

Chapter 2: noise-induced oscillations for the mean-field dissipative contact process

In Chapter 2, we investigate how a dissipation term added to the dynamics of a reversible Markovian system of interacting particles can lead to persistent oscillations. In particular, we study a dissipative version of the classical contact process with mean-field interaction.

The contact process can be seen as the microscopic counterpart of one of the most basic epidemiological models, the SIS (susceptible-infectious-susceptible) model. Its simplest non-trivial version is the mean-field contact process: we consider a population of N individuals, sitting at the vertices of a complete graph; any individual can be either healthy and susceptible (and in this case we associate to her a variable $x_i = 0$) or infected and non-susceptible ($x_i = 1$); the configuration of the system $(x_i)_{i=1}^N$ evolves as a continuous time Markov chain with the following rates:

- each infected individual recovers with rate 1, and becomes immediately susceptible again;
- each susceptible individual becomes infected with a rate equal to a given constant $\lambda > 0$ times the fraction of her neighbors in the graph that are infected, that is, the average number of infected individuals in the system.

It turns out ([48]) that the study of this model reduces to the analysis of the one-dimensional Markov chain $m_N := \frac{1}{N}\sum_{i=1}^N x_i$ representing the evolution of the average number of infected individuals in the system, and that the process $(x_i)_{i=1}^N$ is absorbed in the null state in finite time for any finite N. We say that the epidemics *dies out*. We refer to [48] for definitions and properties of the contact process in more general settings.

In Chapter 2 of this thesis we introduce a modification of the just described classical contact process on the complete graph, which we call *dissipative* contact process. Such a modification takes into account the temporal decay of the individuals' viral loads, and their consequent reduced infectiousness. The state x_i, $i = 1, \ldots, N$, of each individual takes values in the interval $[0,1]$, and is interpreted as her viral load; we say that an individual is healthy/susceptible if $x_i = 0$, and infected/non-susceptible if $x_i > 0$. The dynamics goes as follows:

- each infected individual recovers with rate $r > 0$, and becomes immediately susceptible again;
- each susceptible individual becomes infected with a rate equal to $\lambda > 0$ times the average viral load of the whole population. As soon as this happens, her viral load jumps to the value 1;
- between jumps, the viral load of any individual decays exponentially in time with rate $\alpha > 0$.

The SDEs describing the dynamics of the system read

$$\begin{aligned} x_i(t) = x_i(0) - \int_0^t \alpha x_i(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,\mathbb{1}_{\{x_i(s^-)=0\}}\lambda v_N(s^-)]}(u) N_i^{\uparrow}(ds,du) \\ - \int_{[0,t]\times[0,+\infty)} x_i(s^-)\mathbb{1}_{[0,\mathbb{1}_{\{x_i(s^-)>0\}}(s^-)r]}(u) N_i^{\downarrow}(ds,du). \end{aligned} \tag{3}$$

Here $v_N(t) := \frac{1}{N}\sum_{i=1}^N x_i(t)$ denotes the average viral load of the population and $(N_i^{\uparrow})_{i=1}^N$ and $(N_i^{\downarrow})_{i=1}^N$ are families of independent Poisson random measures on $[0,+\infty)\times[0,+\infty)$ with intensity measure the Lebesgue measure $dtdu$. Moreover, $N_i^{\uparrow}$ and $N_j^{\downarrow}$ are independent for all $i, j \in \{1, \ldots, N\}$.

Notice that in this model we have two time scales, with a useful interpretation: r^{-1} is the time scale at which infected individuals lose their immunity (it is the expected time needed for their viral loads to reset to zero), while α^{-1} is the time scale at which infected individuals remain contagious (i.e. their viral load remains significantly different from zero). In many real epidemics $\alpha^{-1} \ll r^{-1}$: for a relatively long time non-susceptible individuals are immune, and do not contribute to the propagation of the disease (since their viral load has dissipated and reached a negligible value).

We derive the thermodynamic limit of system (3), proving a propagation of chaos statement (see Theorem 2.2.1). As a corollary (Corollary 2.2.1.1), we obtain that the empirical average process $(m_N(t), v_N(t))_{t\geq 0}$, where $m_N(t) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{x_i(t)>0\}}$ denotes the average number of infected individuals in the system at time t, converges to the solution to the following ODE:

$$\begin{aligned} \dot{m}(t) &= \lambda(1-m(t))v(t) - rm(t) \\ \dot{v}(t) &= -\alpha v(t) + \lambda(1-m(t))v(t) - rv(t), \end{aligned} \tag{4}$$

where $m(t) := \mathbb{P}(\bar{x}(t) > 0)$ and $v(t) := \mathbb{E}[\bar{x}(t)]$ and $\bar{x}$ is the limit process of system (3), obtained from the propagation of chaos.

In Section 2.3.1 we show that, under suitable conditions on the parameters including the case $\alpha^{-1} \ll r^{-1}$, the limit system (4) has a unique globally stable fixed point, which is approached by damped oscillations. In particular, no persistent rhythmic behaviors are present in this limit.

In Section 2.4, we turn to the study of the stochastic corrections to the limit deterministic behavior of (m_N, v_N) given in (4). The convergence rate yielded by the propagation of chaos (Corollary 2.2.1.1) suggests that these are of order $1/\sqrt{N}$. Therefore, we define the fluctuation process as

$$\begin{bmatrix} \xi^N(t) \\ \eta^N(t) \end{bmatrix} := \begin{bmatrix} \sqrt{N}(m_N(t) - m(t)) \\ \sqrt{N}(v_N(t) - v(t)) \end{bmatrix}. \tag{5}$$

We show in Theorem 2.4.1 that this process converges in law to a Gaussian process $(\xi(t), \eta(t))_{t\geq 0}$, which is solution to a two-dimensional linear SDE driven by a standard Brownian motion.

In Section 2.4.1, we compute analytically the power spectral density of the limit fluctuation process (ξ, η) at equilibrium and we prove that it has a peak around a deterministic, non-zero frequency. This means that the fluctuation process (5) exhibits persistent oscillations when N is large. This result, together with numerical simulations of the finite particle system (3), suggests that, despite the dampening found in the thermodynamic limit, collective rhythmic behaviors might arise for (m_N, v_N) when the effect of stochastic fluctuations is not negligible, that is, when N is large but finite. This is why we call the oscillations in our model *noise-induced*.

In Section 2.5, we investigate further the nature of the oscillations in our model by letting the parameters λ and α diverge with N. This is motivated by the observation that the peak of the power spectral density of the limit fluctuation process becomes sharper for increasing values of λ. Furthermore, by letting α diverge appropriately with λ, we remain in a parameter regime where the non-zero fixed point of (4) exists and is a globally stable spiral.

We prove in Theorem 2.5.1 that, if properly rescaled, the dynamics of (m_N, v_N) converges to a harmonic oscillator affected by noise. The joint effect of dissipation and noise becomes apparent from this analysis: the period of the emergent oscillations is nearly-deterministic and comes essentially from the infinite-volume dynamics obtained with the propagation of chaos, whereas the amplitude of oscillations varies irregularly and randomly, as it is significantly affected by noise.

This qualitative description is in agreement with what observed in real epidemics. Furthermore, in their epidemiological interpretation, these results suggest that cycles in infectious diseases might be induced by *finite-size effects*: in large populations, the noise, which is intrinsic in the individuals' dynamics, excites frequencies close to a characteristic frequency of the system, producing collective sustained oscillations. This finite-size effect appears as a universal phenomenon, similar to what observed in models of population dynamics (see [5, 55]).

Chapter 3: strong propagation of chaos for systems of interacting particles with nearly stable jumps

In Chapter 3 we prove conditional propagation of chaos for a system of interacting particles subject to simultaneous jumps, and where the distribution of such simultaneous jumps lies in the domain of attraction of a strictly stable law of index $\alpha \in (0, 2) \setminus \{1\}$. For any $N \geq 2$ we consider a system $(X^{N,i})_{i=1}^N$ of N particles described by their positions $X_t^{N,i} \in \mathbb{R}$, $i = 1, \dots, N$. The system evolves as a piecewise deterministic Markov process and its dynamics can be described as follows:

- At any time t, any particle can jump with a rate depending on its position according to a rate function $f : \mathbb{R} \to \mathbb{R}_+$, which in particular will be assumed to be bounded and Lipschitz continuous. When such a jump occurs for some particle, we say that the particle performs a main jump and its position changes by an amount ψ, where $\psi : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ depends on the position of the particle under consideration and on the empirical measure of the system right before the jump, $\mathcal{P}_1(\mathbb{R})$ denoting the set of probability measures on $\mathbb{R}$ having finite first moment. ψ will be assumed in particular to be bounded and Lipschitz continuous with respect to some convenient Wasserstein distance on $\mathcal{P}_1(\mathbb{R})$, depending on α.

- Each time some particle i in the system has a main jump, all the other particles $j \neq i$ receive at the same time the same small random kick, called collateral jump. That is, their positions are changed simultaneously by the same amount $U/N^{1/\alpha}$, where U is a random variable whose distribution lies in the domain of attraction of a strictly stable law of index α (see Def. 3.2.1). The variable U determining the amplitude of the collateral jumps is re-sampled independently from the same distribution each time some particle in the system performs a main jump.

- In between consecutive jumps, each particle follows a deterministic flow with drift b, where $b : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is a deterministic function of the particle's position and of the empirical measure of the system, and will be assumed to be bounded and Lipschitz continuous, again with respect to some convenient Wasserstein distance on $\mathcal{P}_1(\mathbb{R})$.

Similar systems are employed in neuroscience to model families of interacting neurons (see [27, 29] and references therein). In that context, the variables $X^{N,i}$ represent the membrane potentials of the neurons, a main jump corresponds to a spike and collateral jumps represent the synaptic inputs received by post-synaptic neurons from pre-synaptic ones. The system is described by the following SDEs:

$$\begin{aligned} X_t^{N,i} &= X_0^{N,i} + \int_0^t b(X_s^{N,i}, \mu_s^N)ds + \int_{[0,t]\times\mathbb{R}_+} \psi(X_{s^-}^{N,i}, \mu_{s^-}^N) \mathbb{1}_{\{z \le f(X_{s^-}^{N,i})\}} \bar{\pi}^i(ds, dz) \\ &+ \frac{1}{N^{1/\alpha}} \sum_{j \neq i} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}^*} u \mathbb{1}_{\{z \le f(X_{s^-}^{N,j})\}} \pi^j(ds, dz, du), \qquad i = 1, \dots, N, \end{aligned} \tag{6}$$

where $(\pi^i)_{i=1}^N$ is a family of independent Poisson random measures on $\mathbb{R}_+^2 \times \mathbb{R}$ having intensity $dsdz\nu(du)$ - ν denoting a heavy-tailed law in the domain of attraction of a strictly stable law with index $\alpha \in (0,2) \setminus \{1\}$ (see Def. 3.2.1) -, and $\bar{\pi}^i(ds, dz) := \int_{\mathbb{R}} \pi^i(ds, dz, du)$, $i = 1, \dots, N$, are their projections. Therefore, π^i rules the main jumps of particle i and the amplitudes of the collateral jumps triggered by them, whereas the last integral encodes the fact that each particle i is induced to jump every time that any other particle $j \neq i$ performs a main jump.

The main result in Chapter 3 (Theorem 3.2.10) claims that, as $N \to +\infty$, the particle system (6) converges to the following system

$$\begin{aligned}\bar{X}^i_t = \bar{X}^i_0 + \int_0^t b(\bar{X}^i_s, \bar{\mu}_s)ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}^i_{s^-}, \bar{\mu}_{s^-})\mathbb{1}_{\{z\le f(\bar{X}^i_{s^-})\}}\bar{\pi}^i(ds,dz) \\ + \int_0^t \bar{\mu}^{1/\alpha}_{s^-}(f)\, dS^\alpha_s, \qquad i \ge 1,\end{aligned} \tag{7}$$

where $(\bar{\pi}^i)_{i=1}^N$ are independent Poisson random measures on $\mathbb{R}^2_+$ with Lebesgue intensity and independent of S^α, which is a strictly α-stable process, and $\bar{\mu}_t(f)$ denotes the expectation of f computed with respect to the conditional law $\bar{\mu}_t := \mathcal{L}(\bar{X}^1_t \,|\, \sigma(S^\alpha_s,\ s \le t))$. Notice that in system (7) S^α is the same for all $i \ge 1$. For this reason, we say that it constitutes a source of common noise for the particles' dynamics. Indeed, one can show that it is the only source of common noise in the limit system: particles in (7) are conditionally independent (and have the same law $\bar{\mu}$) given S^α. Hence, conditional propagation of chaos holds for system (6).

Before proving this result, in Section 3.3 we show strong well-posedness of the limit system (7).

Then, Theorem 3.2.10 states conditional propagation of chaos: for all $N \ge 1$, we can construct, on an extension of the probability space where (6) is defined, a version $S^{N,\alpha}$ of S^α, independent of $(\bar{\pi}^i)_i$, such that - denoting by $\bar{X}^N$ the unique strong solution to the limit system (7) driven by $S^{N,\alpha}$ and $(\bar{\pi}^i)_i$, and by $T^N_K := \inf\{t > 0 : |S^{N,\alpha}_t - S^{N,\alpha}_{t^-}| > K\}$ the time of the first jump of $S^{N,\alpha}$ exceeding K - we can provide an explicit upper bound to

$$\mathbb{E}[\mathbb{1}_{\{t<T^N_K\}}|X^{N,i}_t - \bar{X}^i_t| \wedge |X^{N,i}_t - \bar{X}^i_t|^{\alpha_- \wedge 1}]$$

in terms of α, N, t and K and for any $t > 0$, $i = 1, \ldots, N$ and $K > 0$. Here $\alpha_- \in (0, \alpha)$ ($\alpha_- > 1$ if $\alpha > 1$) is fixed and its meaning will become clear in Chapter 3.

The core of the proof consists in showing that the collateral jump terms converge to the stochastic integral in (7) and to find an appropriate norm to quantify the rate of convergence. Here, we provide a heuristic outline of the key steps in the proof. Consider (an approximation of) the collateral jump term of a fixed particle i in (6) (denoted by A^N_t as in Section 3.4):

$$A^N_t := \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u\mathbb{1}_{\{z\le f(X^{N,j}_{s^-})\}}\pi^j(ds,dz,du). \tag{8}$$

If the jump rates of all the particles were constant over $[0,t]$ - i.e., $f(x) \equiv \lambda > 0$ -, then the total number $P^N_{0,t}$ of main jumps in $[0,t]$ would be a Poisson random variable with expectation $N\lambda t$, where $N\lambda$ would be the total rate of main jumps of the system over $[0,t]$. The total number of summands in A_t^N would also be equal to $P_{0,t}^N$. Then, using the basic properties of Poisson random measures, we could rewrite A_t^N a.s. as a compound Poisson variable:

$$A_t^N = \frac{1}{N^{1/\alpha}} \sum_{k=1}^{P_{0,t}^N} U^k,$$

with $(U^k)_{k=1}^{P_{0,t}^N}$ i.i.d. $\sim \nu$ representing the collateral jumps of particle i during $[0,t]$. Employing the stable central limit theorem for the random sum $\sum_{k=1}^{P_{0,t}^N} U^k$, we would obtain that, as $N \to +\infty$, $(P_{0,t}^N)^{-1/\alpha} \sum_{k=1}^{P_{0,t}^N} U^k \xrightarrow{d} S_1^\alpha$, S_1^α being a strictly α-stable random variable, and in turn

$$A_t^N \sim \left(\frac{P_{0,t}^N}{N}\right)^{1/\alpha} S_1^\alpha.$$

If we also employed the law of large numbers to approximate $P_{0,t}^N$ by its average $N\lambda t$, we would get

$$A_t^N \sim \left(\frac{N\lambda t}{N}\right)^{1/\alpha} S_1^\alpha = \lambda^{1/\alpha} t^{1/\alpha} S_1^\alpha \overset{d}{=} \lambda^{1/\alpha} S_t^\alpha,$$

where we have used that the strictly stable variable S_1^α yielded by the stable CLT has the same distribution of a strictly α-stable process at time $t = 1$ and the self-similarity property of strictly α-stable processes (see for instance Proposition 13.5 in [60]) to write $t^{1/\alpha} S_1^\alpha \overset{d}{=} S_t^\alpha$, with S_t^α denoting the increment of a strictly stable process over $[0,t]$.

To deal with our case, where the jump rate f is generally non-constant, we divide $[0,t]$ into small slots of length δ, $([k\delta,(k+1)\delta[)_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1}$, we freeze the particles' jump rates within each interval, taking their values $f(X_{k\delta}^{N,i})$ at the beginning of the intervals, and we perform on each interval k the approximations just described, working on the terms

$$A_{k\delta,(k+1)\delta}^N := \frac{1}{N^{1/\alpha}} \sum_{j=1}^{N} \int_{]k\delta,(k+1)\delta]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z \le f(X_{k\delta}^{N,j})\}} \pi^j(ds,dz,du).$$

In this case, within each interval k, the total number of main jumps $P_{k\delta,(k+1)\delta}^N$ is (conditionally on the information up to time $k\delta$) Poisson distributed with mean $\sum_{i=1}^N f(X_{k\delta}^{N,i})\delta$, since the (constant) total main jump rate inside the k-th interval is now $\sum_{i=1}^N f(X_{k\delta}^{N,i})$ in place of $N\lambda$. Therefore, analogously as above, the total contribution of all collateral jumps inside any interval k will be approximately given by

$$A_{k\delta,(k+1)\delta}^N \sim \left(\frac{P_{k\delta,(k+1)\delta}^N}{N}\right)^{1/\alpha} S_{k\delta,(k+1)\delta}^{N,\alpha} \sim \left(\frac{\sum_{j=1}^N f(X_{k\delta}^{N,j})\delta}{N}\right)^{1/\alpha} S_{k\delta,(k+1)\delta}^{N,\alpha} \overset{d}{=} (\mu_{k\delta}^N(f))^{1/\alpha}[S_{(k+1)\delta}^{N,\alpha} - S_{k\delta}^{N,\alpha}],$$

where $S^{N,\alpha}_{k\delta,(k+1)\delta}$ is the strictly stable variable yielded by the stable central limit theorem applied in the k-th interval (see Proposition 3.4.2), $\mu^N_{k\delta}(f) := \frac{\sum_{j=1}^N f(X^{N,j}_k)\delta}{N}$ denotes the integral of a function f against the empirical measure of the finite system at time $k\delta$, and $S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta} \stackrel{d}{=} \delta^{1/\alpha} S^{N,\alpha}_{k\delta,(k+1)\delta}$ denotes the increment of a strictly stable process over $[k\delta, (k+1)\delta[$, obtained from the self-similarity property. In Section 3.4.3, we will paste together the (independent) increments $(S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta})_k$ to construct a version of the limit α-stable common noise S^α in (7).

Last, adding up the contributions of all the time intervals $k = 0, \dots, \lceil t/\delta \rceil - 1$ (see Section 3.4.5), we expect that, as $N \to +\infty$ and $\delta = \delta(N) \to 0$,

$$A^N_t \sim \sum_k A^N_{k\delta,(k+1)\delta} \sim \sum_k (\mu^N_{k\delta}(f))^{1/\alpha} [S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta}] \to \int_{[0,t]} \lim_{N\to+\infty} (\mu^N_{s^-}(f))^{1/\alpha} dS^\alpha_s,$$

where we have formally rewritten each increment $S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta}$ as an integral and, in view of our convergence result Theorem 3.2.10, $(\mu^N)_N$ is expected to converge to the random measure $\bar{\mu}$.

This heuristic construction will be made rigorous in Section 3.4 and clarifies how the common noise arises in the limit as a result of the stable central limit theorem, from the joint contribution of the collateral jumps and thanks to the chosen scaling.

Section 3.5 completes the convergence proof employing some auxiliary results and intermediate useful representations for the particle and the limit systems.

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Chapter 1

Noise-induced periodicity in a frustrated network of interacting diffusions

In this chapter, based on [54], we investigate the emergence of a collective periodic behavior in a frustrated network of interacting diffusions. Particles are divided into two communities depending on their mutual couplings. On the one hand, both intra-population interactions are positive: each particle wants to conform to the average position of the particles in its own community. On the other hand, inter-population interactions have different signs: the particles of one population want to conform to the average position of the particles of the other community, while the particles in the latter want to do the opposite. We show that this system features the phenomenon of noise-induced periodicity: in the infinite volume limit, in a certain range of interaction strengths, although the system has no periodic behavior in the zero-noise limit, a moderate amount of Brownian noise may generate an attractive periodic law.

1.1 Introduction

Robust periodic behaviors are frequently encountered in life sciences and are indeed one of the most commonly observed self-organized dynamics. For instance, spontaneous brain activity exhibits rhythmic oscillations called alpha and beta waves [28]. From a theoretical standpoint, the mechanism driving the emergence of periodic behaviors in such systems is poorly understood. For example, neurons neither have any tendency to behave periodically on their own, nor are subject to any periodic forcing; nevertheless, they organize to produce a regular motion perceived at the macroscopic scale [70]. Various models of large families of interacting particles showing self-sustained oscillations have been proposed; we refer the reader to [27, 4, 3, 7, 12, 38, 22, 21, 20, 52, 53], where possible mechanisms leading to a rhythmic behavior are discussed and many related references are given.

Here we mention two mechanisms - which are of interest to us - capable to induce or enhance periodic behaviors in stochastic systems with many degrees of freedom. The first one is noise. The role of the noise is twofold: on the one hand, it can lead to oscillatory laws in systems of nonlinear diffusions whose deterministic counterparts do not display any periodic behavior [61, 62]; on the other hand, it can facilitate the transition from an equilibrium solution to macroscopic self-organized oscillations [17, 49, 67].

The second mechanism is the topology of the interaction network. It has been recently pointed out in [27, 66, 18] that a specific network structure may favor the emergence of collective rhythms. In particular, in [66, 18], the large volume dynamics of a two-population generalization of the mean field Ising model is considered. The system is shown to undergo a transition from a disordered phase, where the magnetizations of both populations fluctuate around zero, to a phase in which they both display a macroscopic regular rhythm. Such a transition is driven by inter- and intra-population interactions of different strengths and signs leading to dynamical frustration.

In the present chapter we combine the two mechanisms described above and we design a toy model of frustratedly interacting diffusions that shows noise-induced periodicity, in the sense that periodic oscillations appear for an intermediate amount of noise. The peculiar feature of the model under consideration is that the structure of the interaction network depends on the noise in that it is the noise that switches on the interaction terms, thus leading to periodic dynamics.

1.2 Description of the model and outline of the results

We consider a system of N particles with states in $\mathbb{R}$, so that $\mathbf{z}^{(N)} \in \mathbb{R}^N$ denotes the N-particle configuration. We divide the N particles into two disjoint communities of sizes N_1 and N_2 respectively and we denote by I_1 (resp. I_2) the set of sites belonging to the first (resp. second) community. We have $|I_1| = N_1$ and $|I_2| = N_2$, with $N = N_1 + N_2$. In this setting, we indicate with $(x_j^{(N)}(t))_{j=1}^{N_1} \in \mathbb{R}^{N_1}$ the positions at time t of the particles of population I_1 and with $(y_j^{(N)}(t))_{j=1}^{N_2} \in \mathbb{R}^{N_2}$ the positions at time t of the particles of population I_2, so that

$$\mathbf{z}^{(N)}(t) = \Big(\overbrace{x_1^{(N)}(t), x_2^{(N)}(t), \dots, x_{N_1}^{(N)}(t),}^{\text{Community } I_1} \overbrace{y_1^{(N)}(t), y_2^{(N)}(t), \dots, y_{N_2}^{(N)}(t)}^{\text{Community } I_2} \Big) \in \mathbb{R}^N$$

represents the state of the whole system at time t.

Particles in our model are coupled according to the interaction network sketched in Fig. 1.1. In particular, the strength of the interaction between any pair of particles in the system depends on the community they belong to: the parameters θ_{11} and θ_{22} control the interaction strength between any pair of particles within the same community, whereas θ_{12} (resp. θ_{21}) tunes the strength of the influence of any particle of population I_2 (resp. I_1) over any particle of population I_1 (resp. I_2).

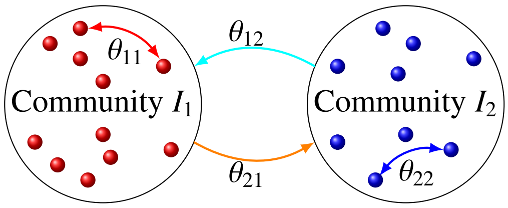


Figure 1.1: A schematic representation of the interaction network. Particles are divided into two communities, I_1 and I_2. Ignoring inter-population interactions, each community taken alone is a mean-field system with interaction strength θ_{ii} ($i = 1$ or 2). When we couple the two communities, population I_1 (resp. I_2) influences the dynamics of population I_2 (resp. I_1) through the average position of its particles with strength θ_{21} (resp. θ_{12}).

Now we introduce the microscopic dynamics we are interested in. Let

$$m_1^{(N)}(t) := \frac{1}{N_1}\sum_{j=1}^{N_1} x_j^{(N)}(t) \quad \text{and} \quad m_2^{(N)}(t) := \frac{1}{N_2}\sum_{j=1}^{N_2} y_j^{(N)}(t)$$

be the empirical means of the positions of the particles in populations I_1 and I_2, respectively, at time t. Moreover, denote by $\alpha := \frac{N_1}{N}$ the fraction of sites belonging to the first group. Then, omitting time dependence for notational convenience, the interacting particle system we are going to study reads

$$\begin{aligned} dx_j^{(N)} &= \left(-\left(x_j^{(N)}\right)^3 + x_j^{(N)}\right)dt - \alpha\,\theta_{11}\left(x_j^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\,\theta_{12}\left(x_j^{(N)} - m_2^{(N)}\right)dt + \sigma dw_j, \qquad \text{for } j = 1,\ldots,N_1, \\ dy_j^{(N)} &= \left(-\left(y_j^{(N)}\right)^3 + y_j^{(N)}\right)dt - \alpha\,\theta_{21}\left(y_j^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\,\theta_{22}\left(y_j^{(N)} - m_2^{(N)}\right)dt + \sigma dw_{N_1+j}, \qquad \text{for } j = 1,\ldots,N_2, \end{aligned} \tag{1.1}$$

where $((w_j(t))_{t\geq 0})_{j=1}^N$ are N independent copies of a standard Brownian motion. Here the diffusion coefficient $\sigma \geq 0$ is the parameter that tunes the amount of noise in the system, since it is the same for each coordinate.

Remark. Existence and uniqueness of a strong solution to (1.1) can be established via the Khasminskii criterion (see Theorem 3.5 in [47]). In particular, this criterion relies on the existence of a nonnegative, C^2 function $V : \mathbb{R}^N \to \mathbb{R}$ such that $\inf_{|\mathbf{z}|>R} V(\mathbf{z}) \to +\infty$ as $R \to +\infty$ and such that, for some constant $c > 0$, $\mathcal{L}V \leq cV$, where $\mathcal{L}$ denotes the infinitesimal generator of the diffusion (1.1).

It is easy to check that, by taking the function

$$\tilde{V}(\mathbf{z}) := \frac{1}{N_1} \sum_{i=1}^{N_1} \left[\frac{x_i^4}{4} + \frac{x_i^2}{2} \right] + \frac{1}{N_2} \sum_{i=1}^{N_2} \left[\frac{y_i^4}{4} + \frac{y_i^2}{2} \right],$$

with $\mathbf{z} = (x_1, \dots x_{N_1}, y_1, \dots, y_{N_2})$, one obtains an inequality of the form $\mathcal{L}\tilde{V}(\mathbf{z}) \leq c\tilde{V}(\mathbf{z}) + d$, for some $c, d \geq 0$. Then one concludes by noticing that this implies $\mathcal{L}V \leq cV$, if we define $V := \tilde{V} + \frac{d}{c}$.

Notice that in system (1.1) the two groups of particles interact only through their empirical means. This makes our model mean-field and, in particular, when $\theta_{11} = \theta_{22} = \theta_{12} = \theta_{21} = \theta > 0$, the system of equations (1.1) reduces to the mean-field interacting diffusions considered in [24]. In a general setting, all the interaction parameters can be either positive or negative allowing both cooperative/conformist and uncooperative/anti-conformist interactions. In our case, as we will argue below, a necessary feature for the system to show periodic behavior is *frustration of the network*, i.e. the inter-community interactions must have opposite signs. Therefore, here we focus on the case $\theta_{11} > 0$, $\theta_{22} > 0$ and $\theta_{12}\theta_{21} < 0$. Moreover, without loss of generality, we make the specific choice $\theta_{12} > 0$ and $\theta_{21} < 0$, which means that particles in I_1 tend to conform to the average particle position of community I_2, whereas particles in I_2 prefer to differ from the average particle position of community I_1 (see Eq. (1.1)).

Numerical simulations of system (1.1) with large N show that $m_1^{(N)}(t)$ and $m_2^{(N)}(t)$ display an oscillatory behavior in appropriate regions of the parameter space (see Section 1.3). This led us to investigate the thermodynamic limit of our system of interacting diffusions. It is known ([61, 62, 47, 56]) that only *nonlinear* SDEs - that is, SDEs whose coefficients depend on the law of the solution itself - can have solutions with a time-periodic law. Therefore, the law of the solution to (1.1) cannot be periodic. However, the mean-field interaction in (1.1) has a peculiar feature. When the interaction is of this type, at any time t, the empirical average of the particle positions in (1.1) is expected to converge, as the number of particles goes to infinity, to the first moment of the law at time t of the solution of a nonlinear SDE. Such nonlinear SDE might have solutions with periodic law, hence, with periodic first moment (see [62]). Therefore, the oscillations in the trajectories of $m_1^{(N)}(t)$ and $m_2^{(N)}(t)$ shown by simulations can be theoretically explained via the thermodynamic limit of the system.

We outline here the main results presented in the sequel. We follow an approach similar to the one adopted in [17].

1. In Section 1.4.1 we prove that, starting from i.i.d. initial conditions, independence propagates in time when taking the infinite-volume limit. In particular, as N grows large, up to any finite time horizon T, the time evolution of a pair of representative particles, one for each population, is described by the limiting dynamics

$$
\begin{aligned}
dx &= \Big[-x^3 + x - \alpha\theta_{11}\,(x - \mathbb{E}[x]) - (1-\alpha)\theta_{12}\,(x - \mathbb{E}[y])\Big]\,dt + \sigma dw_1 \\
dy &= \Big[-y^3 + y - \alpha\theta_{21}\,(y - \mathbb{E}[x]) - (1-\alpha)\theta_{22}\,(y - \mathbb{E}[y])\Big]\,dt + \sigma dw_2,
\end{aligned}
\tag{1.2}
$$

   where notation $\mathbb{E}$ stands for the expectation with respect to the probability measure $Q(t;\cdot) = \mathrm{Law}(x(t), y(t))$, for every $t \in [0,T]$, and $(w_1(t))_{t\in[0,T]}$ and $(w_2(t))_{t\in[0,T]}$ are two independent standard Brownian motions.
   In particular, we show that, for all $T > 0$ and for all $t \in [0,T]$, any random vector of the form $(x_{i_1}^{(N)}(t), \dots, x_{i_{k_1}}^{(N)}(t), y_{j_1}^{(N)}(t), \dots, y_{j_{k_2}}^{(N)}(t))$ converges in distribution, as N goes to infinity, to a vector $(x_1(t), \dots, x_{k_1}(t),\ y_1(t), \dots, y_{k_2}(t))$, whose entries are independent random variables such that $(x_i)_{i=1}^{k_1}$ are copies of the solution to the first equation in (1.2) and $(y_i)_{i=1}^{k_2}$ are copies of the solution to the second equation in (1.2). This is the multi-class version of the propagation of chaos property ([41, 27]): in the large-population limit, particles within the same community converge in law to independent identically distributed copies of the same limit process, and different communities are independent.

2. Being nonlinear, system (1.2) is a good candidate for having a solution with periodic law. It is however very hard to gain insight into its long-time behavior or to find periodic solutions as the problem is infinite-dimensional, due to the presence of nonlinearity and noise. As a first step, in Section 1.4.2 we study the limiting system (1.2) in the absence of noise and, in particular, we argue that oscillatory behaviors are not observed when $\sigma = 0$. This remains true for small values of $\sigma > 0$ in some parameter regimes. See Section 1.3 for details.

3. In Sections 1.4.3 and 1.4.4 we tackle system (1.2) with noise. We show that, in the presence of a small amount of noise, the limiting positions of representa-

tive particles of the two populations evolve approximately as a pair of independent Gaussian processes (*small-noise Gaussian approximation*). This reduces the problem to a finite-dimensional one, since we provide the explicit (deterministic) closed equations for the mean and variance of those processes. In Section 1.5 we argue that the dynamical system describing the time evolution of the means and the variances has a Hopf bifurcation and, as a consequence, in a certain range of the noise intensity, it has a limit cycle as a long-time attractor, implying that the laws of the previously mentioned Gaussian processes are periodic. Thus, the small-noise Gaussian approximation gives a good qualitative description of the emergence of the self-sustained oscillations observed for system (1.1) (see Section 1.3).

Intuitively, the mechanism behind the emergence of periodicity in our system is similar to the one in [18] and can be described as follows. Imagine to start with two independent communities, that is, particles evolve according to system (1.1) with $\theta_{12} = \theta_{21} = 0$. When the intra-population interaction strengths θ_{11} and θ_{22} are large enough, each population tends to its own rest state, that one may guess to be (close to) one of the minima of the double well potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ (see [24]). The key aspect, which makes the model under consideration interesting, is that linking the two populations together within an interaction network with $\theta_{12}\theta_{21} < 0$ is not enough for periodic behaviors to appear. Dynamical frustration and, in turn, oscillations arise only when the noise intensity is large enough, as the interaction terms in system (1.1) are switched on by the noise. Indeed, when $\sigma = 0$ and all the particles in a same population share the same initial condition, the system is attracted to an equilibrium point where $x_j^{(N)} = m_1^{(N)}$ for all $j \in \{1, \ldots, N_1\}$ and $y_k^{(N)} = m_2^{(N)}$ for all $k \in \{1, \ldots, N_2\}$ (see Fig. 1.2). It follows that the zero-noise dynamics does not display any periodic behavior. On the contrary, if σ is positive and sufficiently large, particles do not get stuck at equilibrium points, as diffusion is enhanced, and the interaction terms start playing a role, generating dynamical frustration. The two populations form now a frustrated pair of systems where the rest state of the first is not compatible with the rest position of the second. As a consequence, the dynamics does not settle down to a fixed equilibrium and keeps oscillating. Therefore, the noise is responsible for the emergence of a stable rhythm (see Section 1.4). This feature is the hallmark of the phenomenon of noise-induced periodicity.

1.3 Noise-induced periodicity: numerical study

In this section, we present numerical simulations of the finite-size system (1.1), aimed at giving evidences of the phenomenon of noise-induced periodicity.
In the setting introduced in Section 1.2, we ran several simulations of (1.1) for different choices of σ and several values of the interaction strengths. In all cases, we performed simulations with 10^6 iterations with time-step $dt = 0.005$ for a system of 1000 particles equally divided between the two populations ($\alpha = 0.5$). All particles in the same population were given the same initial condition. We fixed $\theta_{11} = \theta_{22} = 8$ and let $A := (1 - \alpha)\,\theta_{12} > 0$ and $B := -\alpha\theta_{21} > 0$ vary. The results are displayed in Fig. 1.2, Fig. 1.3 and Table 1.1, where also the specific values we employed for A, B and σ are reported. The choices of the parameters are discussed in more detail in Section 1.4, as they correspond to different regimes of the limiting noiseless dynamics (i.e., system (1.2) with $\sigma = 0$), namely, $A - 1 < B < A + 2$, $B = A + 2$ and $B > A + 2$.
We observe the following:

1. If $\sigma = 0$ the system is attracted to a fixed point (see the first column of Fig. 1.2). Numerical evidences support the idea that, in the regimes $A - 1 < B < A + 2$ and $B > A + 2$, this behavior persists for small $\sigma > 0$.

2. When the intensity of the noise is tuned to an intermediate range of values, an oscillatory behavior is observed in the $(m_1^{(N)}, m_2^{(N)})$ plane throughout the duration of the simulation, suggesting the presence of a periodic limit law (see the second column of Fig. 1.2). Thus, our model seems to exhibit noise-induced periodicity. This phenomenon, which at the best of our knowledge lacks a full theoretical comprehension, can be loosely described in the following terms: an intermediate amount of noise may create/stabilize some attractors and destabilize others. In our case it seems that the noise destabilizes (some of the) fixed points and generates a rhythmic behavior of the empirical averages of the particle positions of the two communities.
   We would like to mention that, in the regime $A = B + 2$, an arbitrarily small value of $\sigma > 0$ seems to be sufficient to induce periodicity.

3. Letting $\sigma \gg 1$ completely alters the dynamics: the system approaches the fixed point $(0, 0)$ and then essentially behaves as a Brownian motion (see the third column of Fig. 1.2).

Noise	Coupling strengths	Period of Poincaré map	Period fft
$\sigma = 0.5$	$A = 2, B = 2.5$	19.35 ± 0.18	19.31 ± 0.77
$\sigma = 0.1$	$A = 2, B = 4$	29.34 ± 0.29	28.90 ± 0.92
$\sigma = 0.6$	$A = 2, B = 7$	6.45 ± 0.01	6.45 ± 1.28

Table 1.1: Period of the rhythmic oscillations of system (1.1) in the $(m_1^{(N)}, m_2^{(N)})$ plane in the various regimes and in the presence of an intermediate amount of noise. First and second column: values of the parameters. Third column: period of $t \mapsto (m_1^{(N)}(t), m_2^{(N)}(t))$ obtained by computing the average passage time from positive zero to negative zero of $m_2^{(N)}$; i.e., the average return time to the Poincaré section $\{m_2^{(N)} = 0, m_1^{(N)} > 0\}$. Fourth column: period of $t \mapsto m_2^{(N)}(t)$ estimated from the Fourier spectra considering a sampling period equal to $dt = 0.005$.

In Fig. 1.3 and Table 1.1 the oscillatory behavior emerging in system (1.1) is analyzed further. We computed the average return time of the system to the Poincaré section $\{m_2^{(N)} = 0, m_1^{(N)} > 0\}$ and its standard deviation, in the various regimes. These are reported in the third column of Table 1.1. The Poincaré section is plotted as a red line in Fig. 1.3. In addition, we computed the discrete Fourier transform, averaged over $M = 50$ simulations, for the average particle position of the second population, $m_2^{(N)}$. From the peak of the Fourier transform we recovered the period of the trajectory of $m_2^{(N)}(t)$. The average period and its standard deviation are reported in the fourth column of Table 1.1 for different values of the parameters.

1.4 Propagation of chaos and small-noise approximation

In this section we give our main results. We begin with a propagation of chaos statement, allowing to get the macroscopic description (1.2) of our system. Then, we analyze the noiseless version of the macroscopic dynamics and we show the absence of limit cycles as attractors. Finally, in a small-noise regime, we derive a Gaussian approximation of the infinite-volume evolution (1.2) that displays an oscillatory behavior.

1.4.1 Propagation of chaos

Propagation of chaos claims that, as $N \to \infty$, the evolution of each particle remains independent of the evolution of any finite subset of the others. This is coherent with the fact that individual units interact only through the empirical means of the two populations, over which the influence of a finite number of particles becomes negligible when taking the infinite-volume limit. In our case the limiting evolution of a pair of representative particles, one for each population, is the process $((x(t), y(t))_{t\in[0,T]}$ described by the stochastic differential equation (1.2).

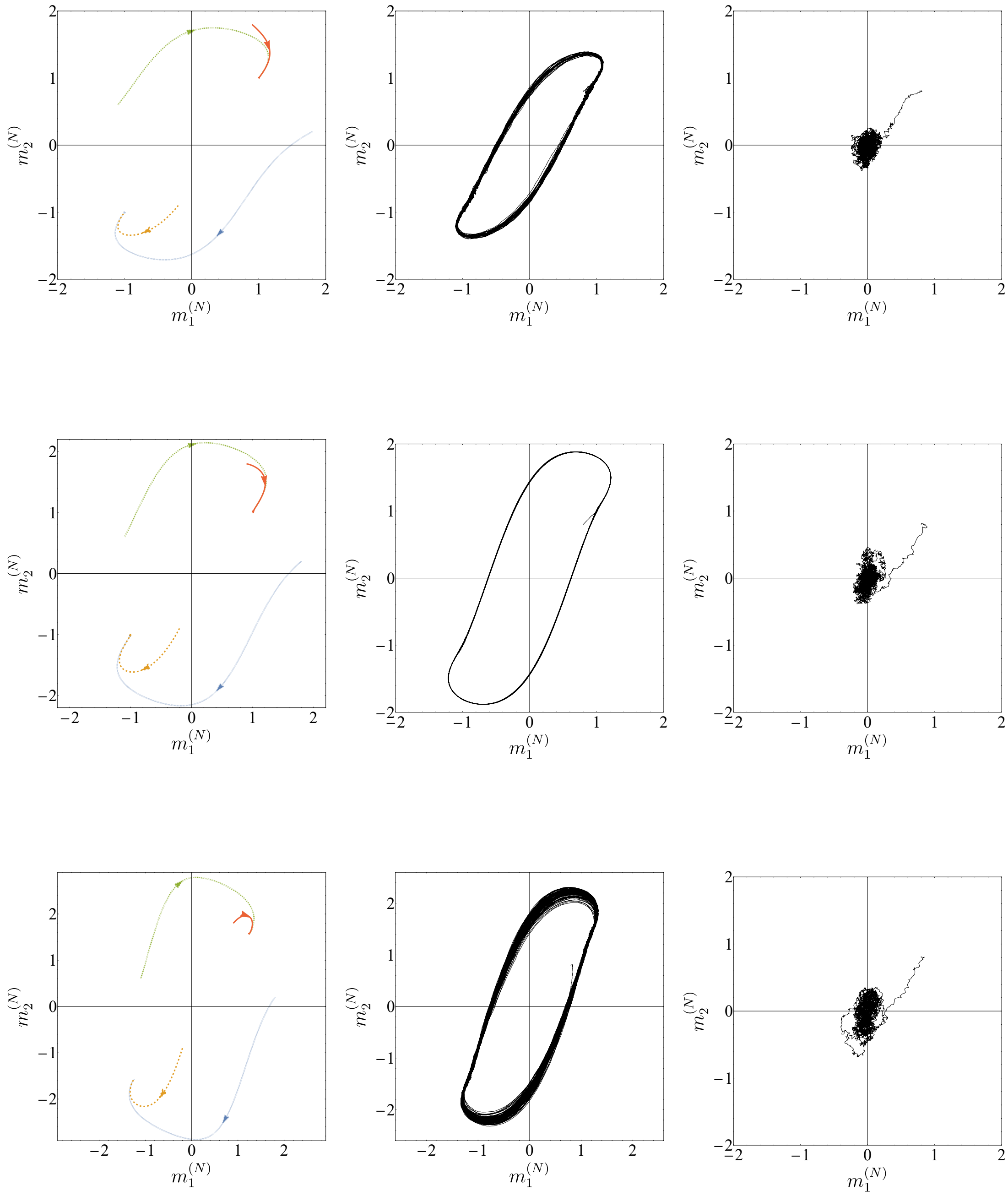


Figure 1.2: Trajectories of $(m_1^{(N)}(t), m_2^{(N)}(t))$ obtained with numerical simulations of system (1.1), in the absence of noise (first column), in the presence of an intermediate amount of noise (second column) and of a high-intensity noise (third column). In all cases, we considered 10^6 iterations with a time-step $dt = 0.005$, 1000 particles, $\alpha = 0.5$, $\theta_{11} = \theta_{22} = 8$. From top to bottom: $A - 1 < B < A + 2$, in particular, $A = 2$ and $B = 2.5$; $B = A + 2$, in particular, $A = 2$ and $B = 4$; $B > A + 2$, in particular, $A = 2$ and $B = 7$. We see that, during a time interval of the same length (namely, 10^6 iterations), when the intensity of the noise is below a certain threshold (first column, $\sigma = 0$ in all the three panels) no periodic behavior arises in any of the three considered cases and the system ends up in one of the stable equilibria. On the contrary, when the intensity of the noise is large (third column, $\sigma = 5$ in all the three panels), the zero-mean Brownian disturbance dominates and the trajectories resemble random excursions around the origin. Whenever the amount of noise is intermediate (second column, from top to bottom: $\sigma = 0.5$, $\sigma = 0.1$ and $\sigma = 0.6$), self-sustained oscillations appear; for further details about this scenario see Fig. 1.3.

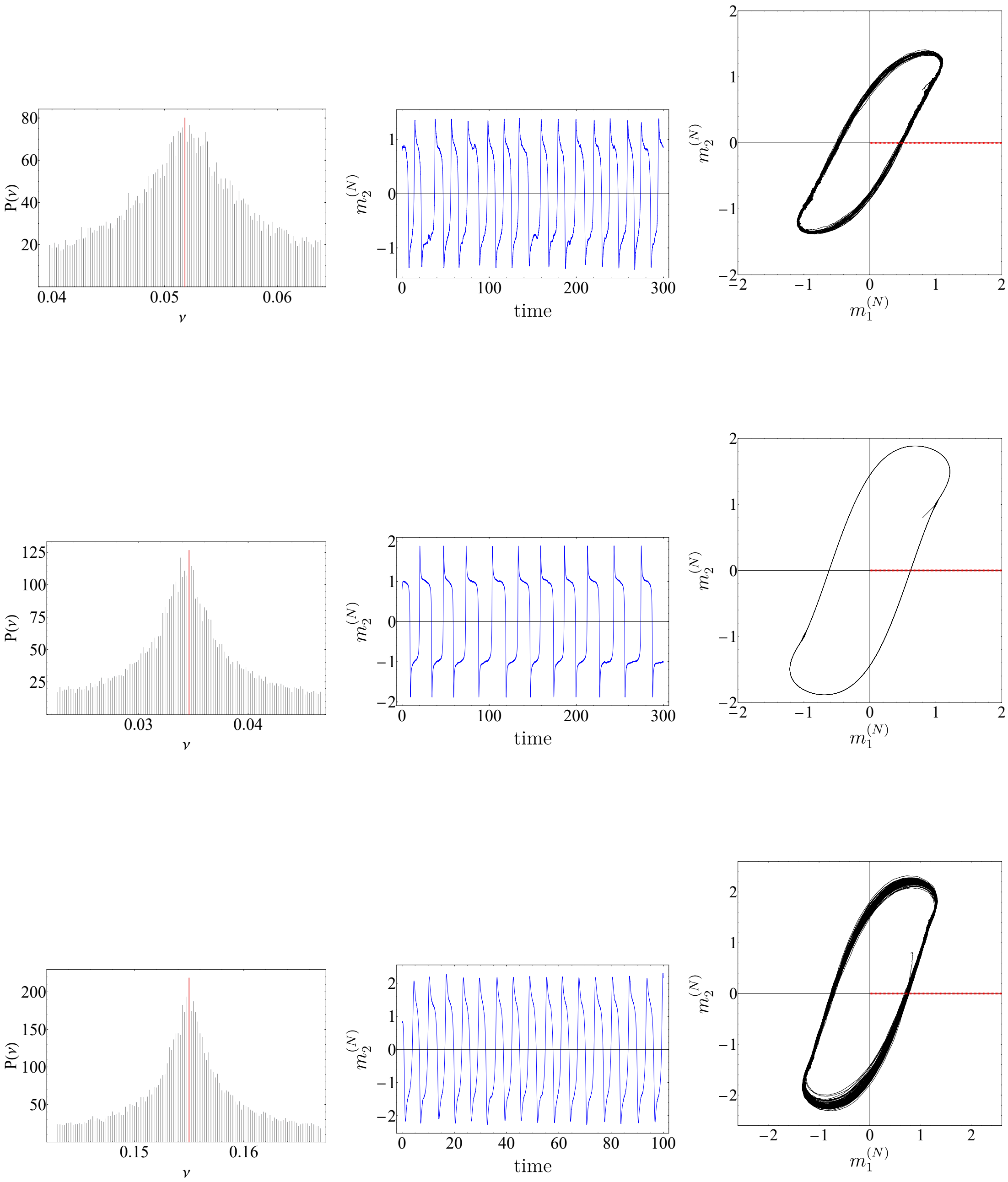


Figure 1.3: Analysis of the period of the trajectories of $(m_1^{(N)}(t), m_2^{(N)}(t))$ obtained via numerical simulations of system (1.1), in the presence of an intermediate amount of noise. In all cases, we considered 10^6 iterations with a time-step $dt = 0.005$, 1000 particles, $\alpha = 0.5$, $\theta_{11} = \theta_{22} = 8$. From top to bottom: $A - 1 < B < A + 2$, in particular, $A = 2$ and $B = 2.5$, with $\sigma = 0.5$; $B = A + 2$, in particular, $A = 2$ and $B = 4$, with $\sigma = 0.1$; $B > A + 2$, in particular, $A = 2$ and $B = 7$, with $\sigma = 0.6$.
In the first column, we plotted the relevant spectral region of the averaged modulus of the discrete Fourier transform $P(\nu)$ of $m_2^{(N)}$ against the frequencies ν. For these figures we employed the Fourier function of Mathematica applied to a trajectory of $m_2^{(N)}$ over 10^6 steps and averaged the obtained spectrum over $M = 50$ simulations. The average periods in the three cases were obtained as the reciprocals of the frequencies highlighted by the red peaks. In the second column, we plotted the time evolution of $m_2^{(N)}$. The third column shows a trajectory of $(m_1^{(N)}(t), m_2^{(N)}(t))$. There, red dashed horizontal lines mark the Poincaré sections we employed for the computation of the average period.

Before proving our main result (Theorem 1.4.2), we begin by proving strong well-posedness of this limit system.

Theorem 1.4.1 (Well-posedness of system (1.2))**.** *Fix $T > 0$. For any initial condition $(x(0), y(0)) = (\xi_x, \xi_y)$, with ξ_x, ξ_y real random variables having finite first moment and being independent of the Brownian motions $(w_i(t))_{t\in[0,T]}$, $i = 1, 2$, system* (1.2) *has a unique strong solution.*

Proof. We follow the argument in [24], based on a Picard iteration. We define recursively two sequences of stochastic processes $(x_n(t))_{t\in[0,T]}$ and $(y_n(t))_{t\in[0,T]}$, indexed by $n \geq 1$, via their Itô's differentials

$$\begin{aligned} dx_n(t) = \Big\{ &-x_n^3(t) + x_n(t) - \alpha\theta_{11}\Big(x_n(t) - \mathbb{E}\left[x_{n-1}(t)\right]\Big) \\ &- (1-\alpha)\,\theta_{12}\Big(x_n(t) - \mathbb{E}\left[y_{n-1}(t)\right]\Big)\Big\}\, dt + \sigma dw_1(t) \end{aligned}$$

$$\begin{aligned} dy_n(t) = \Big\{ &-y_n^3(t) + y_n(t) - \alpha\theta_{21}\Big(y_n(t) - \mathbb{E}\left[x_{n-1}(t)\right]\Big) \\ &- (1-\alpha)\,\theta_{22}\Big(y_n(t) - \mathbb{E}\left[y_{n-1}(t)\right]\Big)\Big\}\, dt + \sigma dw_2(t), \end{aligned}$$

all with the same initial condition $(x_n(0), y_n(0)) = (\xi_x, \xi_y)$. By subtracting two subsequent sets of equations, written in integral form, for every $t \in [0, T]$, we obtain

$$\begin{aligned} x_{n+1}(t) - x_n(t) = &\int_0^t \left(x_{n+1}(s) - x_n(s)\right)\left[1 - f_n(s) - \alpha\theta_{11} - (1-\alpha)\,\theta_{12}\right] ds \\ &+ \int_0^t \left(\alpha\theta_{11}\mathbb{E}\left[x_n(s) - x_{n-1}(s)\right] + (1-\alpha)\,\theta_{12}\mathbb{E}\left[y_n(s) - y_{n-1}(s)\right]\right) ds \end{aligned} \tag{1.3}$$

$$\begin{aligned} y_{n+1}(t) - y_n(t) = &\int_0^t \left(y_{n+1}(s) - y_n(s)\right)\left[1 - \alpha\theta_{21} - g_n(s) - (1-\alpha)\,\theta_{22}\right] ds \\ &+ \int_0^t \left(\alpha\theta_{21}\mathbb{E}\left[x_n(s) - x_{n-1}(s)\right] + (1-\alpha)\,\theta_{22}\mathbb{E}\left[y_n(s) - y_{n-1}(s)\right]\right) ds, \end{aligned} \tag{1.4}$$

where we have employed the identity $a^3 - b^3 = (a-b)\,(a^2 + b^2 + ab)$ and we have set $f_n(s) := x_{n+1}^2(s) + x_n^2(s) + x_{n+1}(s)x_n(s)$ and $g_n(s) := y_{n+1}^2(s) + y_n^2(s) + y_{n+1}(s)y_n(s)$. Observe that $f_n(t), g_n(t) \geq 0$ for all $t \in [0, T]$.

Eq. (1.3) and Eq. (1.4) are of the form $\varphi(t) = \int_0^t \varphi(s)H(s)ds + \int_0^t Q(s)ds$, where $\varphi(t)$ is given by $x_{n+1}(t) - x_n(t)$ and $y_{n+1}(t) - y_n(t)$, respectively. The solution to an equation of this form can be explicitly written as $\varphi(t) = \varphi(0) + \int_0^t Q(s)e^{\int_s^t H(r)dr}ds$. In our case, $\varphi(0) = 0$ since $x_n(0) = \xi_x$ and $y_n(0) = \xi_y$ for all $n \geq 1$ by assumption. Therefore, for

$t \in [0, T]$, the solutions of Eq. (1.3) and Eq. (1.4) are

$$x_{n+1}(t) - x_n(t) = \int_0^t \{\alpha\theta_{11}\mathbb{E}[x_n(s) - x_{n-1}(s)] + (1-\alpha)\,\theta_{12}\mathbb{E}[y_n(s) - y_{n-1}(s)]\} \times e^{\int_s^t (1-f_n(r)-\alpha\theta_{11}-(1-\alpha)\theta_{12})dr} ds \tag{1.5}$$

$$y_{n+1}(t) - y_n(t) = \int_0^t \{\alpha\theta_{21}\mathbb{E}[x_n(s) - x_{n-1}(s)] + (1-\alpha)\,\theta_{22}\mathbb{E}[y_n(s) - y_{n-1}(s)]\} \times e^{\int_s^t (1-\alpha\theta_{21}-g_n(r)-(1-\alpha)\theta_{22})dr} ds. \tag{1.6}$$

We now get into the core of the proof.

Step 1*: auxiliary property.* We will show that, for every $T > 0$, the sequences $\left((\mathbb{E}[x_n(t)])_{t\in[0,T]}\right)_{n\geq 1}$ and $\left((\mathbb{E}[y_n(t)])_{t\in[0,T]}\right)_{n\geq 1}$ are Cauchy sequences in the space $C([0,T])$, equipped with the supremum norm

$$d(f, g) := \sup_{t\in[0,T]} |f(t) - g(t)| \quad \forall f,\ g \in C([0,T]).$$

As a consequence, since $(C([0,T]), d)$ is a complete metric space, we will obtain convergence to elements $(m_x(t))_{t\in[0,T]}, (m_y(t))_{t\in[0,T]} \in C([0,T])$.

We take the absolute value and the expectation in both Eq. (1.5) and Eq. (1.6). If we denote by $\phi_n(t) := \sup_{s\in[0,t]} \mathbb{E}[|x_{n+1}(s) - x_n(s)|]$ and $\psi_n(t) := \sup_{s\in[0,t]} \mathbb{E}[|y_{n+1}(s) - y_n(s)|]$, from Eq. (1.5) we obtain

$$\phi_n(t) \leq \tilde{C}_t \int_0^t \phi_{n-1}(r)dr + \tilde{D}_t \int_0^t \psi_{n-1}(r)dr, \tag{1.7}$$

for some positive constants $\tilde{C}_t$ and $\tilde{D}_t$. From Eq. (1.6) we get an analogous inequality for $\psi_n(t)$. The inequalities are valid for all $t \in [0, T]$. Iteratively employing inequality (1.7) gives

$$\phi_n(T) \leq C_T\phi_1(T)\frac{T^{n-1}}{(n-1)!} + D_T\psi_1(T)\frac{T^{n-1}}{(n-1)!},$$

for suitable positive constants C_T and D_T. Similarly we bound $\psi_n(T)$. Hence, $\phi_n(T)$ and $\psi_n(T)$ go to zero as $n \to +\infty$. Thus, it follows that $\left((\mathbb{E}[x_n(t)])_{t\in[0,T]}\right)_{n\geq 1}$ and $\left((\mathbb{E}[y_n(t)])_{t\in[0,T]}\right)_{n\geq 1}$ are Cauchy sequences in $C([0,T])$ and converge to the continuous limits $(m_x(t))_{t\in[0,T]}$ and $(m_y(t))_{t\in[0,T]}$, respectively.

Step 2*: existence of the solution to* (1.2). Consider the following system of stochastic differential equations

$$
\begin{aligned}
dx(t) &= \Big[-x^3(t) + x(t) - \alpha\theta_{11}\big(x(t) - m_x(t)\big) - (1-\alpha)\,\theta_{12}\big(x(t) - m_y(t)\big)\Big]\,dt + \sigma dw_1(t)\\
dy(t) &= \Big[-y^3(t) + y(t) - \alpha\theta_{21}\big(y(t) - m_x(t)\big) - (1-\alpha)\,\theta_{22}\big(y(t) - m_y(t)\big)\Big]\,dt + \sigma dw_2(t),
\end{aligned}
\tag{1.8}
$$

with initial condition $(x(0), y(0)) = (\xi_x, \xi_y)$. Since the functions m_x and m_y are bounded for every $t \in [0, T]$, existence and uniqueness of a strong solution for (1.8) follows from a Khasminskii's test with norm-like function $V(x, y) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{y^4}{4} + \frac{y^2}{2}$. See [47, 56].

Let $((x(t), y(t))_{t\in[0,T]}$ be the unique strong solution for (1.8). We construct the differences

$$
\begin{aligned}
x_{n+1}(t) - x(t) = &\int_0^t (x_{n+1}(s) - x(s))\,[1 - f_n(s) - \alpha\theta_{11} - (1-\alpha)\,\theta_{12}]\,ds\\
&+ \int_0^t \Big(\alpha\theta_{11}\mathbb{E}\,[x_n(s) - m_x(s)] + (1-\alpha)\,\theta_{12}\mathbb{E}[y_n(s) - m_y(s)]\Big)\,ds
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1}(t) - y(t) = &\int_0^t (y_{n+1}(s) - y(s))\,[1 - \alpha\theta_{21} - g_n(s) - (1-\alpha)\,\theta_{22}]\,ds\\
&+ \int_0^t \Big(-\alpha\theta_{21}\mathbb{E}\,[x_n(s) - m_x(s)] - (1-\alpha)\,\theta_{22}\mathbb{E}[y_n(s) - m_y(s)]\Big)\,ds,
\end{aligned}
$$

with $f_n(s) := x_{n+1}^2(s) + x^2(s) + x_{n+1}(s)x(s)$ and $g_n(s) := y_{n+1}^2(s) + y^2(s) + y_{n+1}(s)y(s)$, and we repeat the same argument as in Step 1. As a consequence, we get that

$$
(\mathbb{E}\,[x_n(t)])_{t\in[0,T]} \xrightarrow{n\to\infty} (\mathbb{E}\,[x(t)])_{t\in[0,T]}
$$

and

$$
(\mathbb{E}\,[y_n(t)])_{t\in[0,T]} \xrightarrow{n\to\infty} (\mathbb{E}\,[y(t)])_{t\in[0,T]}
$$

in $C\,([0, T])$. Therefore, since we have already showed that

$$
(\mathbb{E}\,[x_n(t)])_{t\in[0,T]} \xrightarrow{n\to\infty} (m_x(t))_{t\in[0,T]}
$$

and

$$
(\mathbb{E}\,[y_n(t)])_{t\in[0,T]} \xrightarrow{n\to\infty} (m_y(t))_{t\in[0,T]},
$$

we have that $\mathbb{E}\,[x(t)] = m_x(t)$ and $\mathbb{E}\,[y(t)] = m_y(t)$ for all $t \in [0, T]$. Hence, system (1.8) coincides with system (1.2) and its solution $(x(t), y(t))_{t\in[0,T]}$ provides a solution for (1.2).

Step 3*: uniqueness of the solution to* (1.2). Let $(u(t), v(t))_{t\in[0,T]}$ be another solution to (1.2). We write the integral equations for $x(t) - u(t)$ and $y(t) - v(t)$ and we use them to get estimates for $\phi(t) = |\mathbb{E}[x(t) - u(t)]|$ and $\psi(t) = |\mathbb{E}[y(t) - v(t)]|$. By mimicking the computations above, we obtain

$$\phi(t) \leq \hat{C}_T \int_0^t (\phi(s) + \psi(s))\, ds \quad \text{and} \quad \psi(t) \leq \hat{D}_T \int_0^t (\phi(s) + \psi(s))\, ds,$$

for suitable positive constants $\hat{C}_T$ and $\hat{D}_T$. Summing up the two previous inequalities and using Gronwall's lemma yields $\phi(t) + \psi(t) \leq 0$ for all $t \in [0, T]$. This shows that $\mathbb{E}[x(t)] = \mathbb{E}[u(t)]$ and $\mathbb{E}[y(t)] = \mathbb{E}[v(t)]$ for all $t \in [0, T]$. Thus, $(x(t), y(t))_{t\in[0,T]}$ and $(u(t), v(t))_{t\in[0,T]}$ are both solutions to (1.8) with the same pair (m_x, m_y) and the same initial condition. It follows that $(x(t), y(t)) = (u(t), v(t))$ for all $t \in [0, T]$. □

Now we can prove the following theorem, which is the central result of the current section.

Theorem 1.4.2. *Fix* $T > 0$*. Let* $(x_1^{(N)}(t), \dots, x_{N_1}^{(N)}(t), y_1^{(N)}(t), \dots, y_{N_2}^{(N)}(t))_{t\in[0,T]}$ *be the solution to Eq.* (1.1) *with an initial condition satisfying the following requirements:*

- *the collection* $(x_1^{(N)}(0), \dots, x_{N_1}^{(N)}(0), y_1^{(N)}(0), \dots, y_{N_2}^{(N)}(0))$ *is a family of independent random variables.*
- *the random variables* $(x_1^{(N)}(0), \dots, x_{N_1}^{(N)}(0))$ *(resp.* $(y_1^{(N)}(0), \dots, y_{N_2}^{(N)}(0))$*) are identically distributed with law* λ_x *(resp.* λ_y*). We assume that* λ_x *and* λ_y *have finite second moment.*
- *the random variables* $x_j^{(N)}(0)$ *and* $y_k^{(N)}(0)$ *are independent of the Brownian motions* $((w_i(t))_{t\in[0,T]})_{i=1}^N$ *for all* $j = 1, \dots, N_1$ *and* $k = 1, \dots, N_2$*.*

Moreover, let $(x_1(t), \dots, x_{N_1}(t), y_1(t), \dots, y_{N_2}(t))_{t\in[0,T]}$ *be the process whose entries are independent and such that* $((x_j(t))_{t\in[0,T]})_{j=1}^{N_1}$ *(resp.* $((y_k(t)_{t\in[0,T]})_{k=1}^{N_2}$*) are copies of the solution to the first (resp. second) equation in* (1.2)*, with the same initial conditions and the same Brownian motions used to define system* (1.1)*. Here, "the same" means component-wise equality.*
Define the index sets $\mathcal{I} = \{i_1, \dots, i_{k_1}\} \subseteq \{1, \dots, N_1\}$*, with* $|\mathcal{I}| = k_1$*, and* $\mathcal{J} = \{j_1, \dots, j_{k_2}\} \subseteq \{1, \dots, N_2\}$*, with* $|\mathcal{J}| = k_2$*. Then, we have*

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathbf{z}_{k_1,k_2}^{(N)}(t) - \mathbf{z}_{k_1,k_2}(t)\right|\right] = 0, \tag{1.9}$$

with $|\mathbf{z}|$ *the* ℓ^1*-norm of a vector* $\mathbf{z}$*,* $\mathbf{z}^{(N)}_{k_1,k_2}(t) = (x^{(N)}_{i_1}(t), \ldots, x^{(N)}_{i_{k_1}}(t), y^{(N)}_{j_1}(t), \ldots, y^{(N)}_{j_{k_2}}(t))$ *and* $\mathbf{z}_{k_1,k_2}(t) = (x_1(t), \ldots, x_{k_1}(t), y_1(t), \ldots, y_{k_2}(t))$*.*

Remark. Recall that the convergence in Theorem 1.4.2 implies, for $t \in [0, T]$, convergence in distribution of any finite-dimensional vector $\mathbf{z}^{(N)}_{k_1,k_2}(t)$ to $\mathbf{z}_{k_1,k_2}(t)$.

Proof. The proof of Theorem 1.4.2 is standard and it relies on a coupling method [17, 6]. We want to prove (1.9). Without loss of generality, we take $\mathcal{I} = \{1, \ldots, k_1\}$ and $\mathcal{J} = \{1, \ldots, k_2\}$. Since

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathbf{z}^{(N)}_{k_1,k_2}(t) - \mathbf{z}_{k_1,k_2}(t)\right|\right] \leq \sum_{j=1}^{k_1} \mathbb{E}\left[\sup_{t\in[0,T]} \left|x^{(N)}_j(t) - x_j(t)\right|\right] + \sum_{j=1}^{k_2} \mathbb{E}\left[\sup_{t\in[0,T]} \left|y^{(N)}_j(t) - y_j(t)\right|\right],$$

to conclude it suffices to show that each of the $k_1 + k_2$ terms goes to zero in the limit $N \to \infty$.

In the sequel we will consider only the term $\mathbb{E}\left[\sup_{t\in[0,T]} \left|x^{(N)}_1(t) - x_1(t)\right|\right]$; the other terms can be dealt with similarly. We only sketch our computations as they use the very same tricks as in the proof of Theorem 1.4.1. Since the processes $(x^{(N)}_1(t))_{t\in[0,T]}$ and $(x_1(t))_{t\in[0,T]}$ are initiated at the same position, we obtain

$$x^{(N)}_1(t) - x_1(t) = \int_0^t \left[\left(x^{(N)}_1(s) - x_1(s)\right)(1 - f(s) - \alpha\theta_{11} - (1-\alpha)\theta_{12}) + \mu(s)\right] ds, \quad (1.10)$$

where we have employed the identity $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ and we have set $f(s) := \left(x^{(N)}_1(s)\right)^2 + x^2_1(s) + x^{(N)}_1(s)x_1(s)$, and where

$$\mu(s) := \alpha\theta_{11}\left(m^{(N)}_1(s) - \mathbb{E}[x_1(s)]\right) + (1-\alpha)\theta_{12}\left(m^{(N)}_2(s) - \mathbb{E}[y_1(s)]\right).$$

Observe that Eq. (1.10) is of the form $\varphi(t) = \int_0^t \varphi(s)H(s)ds + \int_0^t Q(s)ds$, with $\varphi(t) = x^{(N)}_1(t) - x_1(t)$. Therefore, as the solution of the latter equation is $\varphi(t) = \varphi(0) + \int_0^t Q(s)e^{\int_s^t H(r)dr}ds$ and, in our case, $x^{(N)}_1(0) = x_1(0)$ by assumption, for every $t \in [0, T]$, we can estimate

$$\left|x^{(N)}_1(t) - x_1(t)\right| \leq \int_0^t |\mu(s)|\, e^{\int_s^t (1 - f(r) - \alpha\theta_{11} - (1-\alpha)\theta_{12})dr} ds \leq C_T \int_0^t |\mu(s)|\, ds,$$

for some positive constant C_T. At this point, taking the supremum and the expectation of both sides of the previous inequality, for $\tilde{t} \in [0, T]$, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]} \left|x_1^{(N)}(t) - x_1(t)\right|\right] \le C_T \int_0^{\tilde{t}} \mathbb{E}\left[|\mu(s)|\right] ds. \tag{1.11}$$

We need an upper bound for $\mathbb{E}\left[|\mu(s)|\right]$. We have

$$\mathbb{E}\left[|\mu(s)|\right] \le \mathbb{E}\left[\alpha\theta_{11}\left|m_1^{(N)}(s) - \mathbb{E}[x_1(s)]\right| + (1-\alpha)\theta_{12}\left|m_2^{(N)}(s) - \mathbb{E}[y_1(s)]\right|\right]$$

and, by adding and subtracting $\frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s)$ (resp. $\frac{1}{N_2}\sum_{i=1}^{N_2} y_i(s)$) inside the first (resp. second) absolute value, we get

$$\begin{aligned}\mathbb{E}\left[|\mu(s)|\right] \le \alpha\theta_{11}&\left\{\frac{1}{N_1}\sum_{i=1}^{N_1}\mathbb{E}\left[\left|x_i^{(N)}(s) - x_i(s)\right|\right] + \mathbb{E}\left[\left|\frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s) - \mathbb{E}[x_1(s)]\right|\right]\right\}\\ &+ (1-\alpha)\theta_{12}\left\{\frac{1}{N_2}\sum_{i=1}^{N_2}\mathbb{E}\left[\left|y_i^{(N)}(s) - y_i(s)\right|\right] + \mathbb{E}\left[\left|\frac{1}{N_2}\sum_{i=1}^{N_2} y_i(s) - \mathbb{E}[y_1(s)]\right|\right]\right\}.\end{aligned}$$

Since the limiting variables $(x_i(t))_{i=1}^{N_1}$ and $(y_i(t))_{i=1}^{N_2}$ are i.i.d. families and have uniformly bounded second moments for all $t \in [0, T]$ (due to the well-posedness of system (1.2)), the standard CLT assures that there exists a positive constant K_T such that, uniformly for all $s \in [0, T]$, it holds

$$\mathbb{E}\left[\left|\frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s) - \mathbb{E}[x_1(s)]\right|\right] \le \frac{K_T}{\sqrt{N_1}}$$

and

$$\mathbb{E}\left[\left|\frac{1}{N_2}\sum_{i=1}^{N_2} y_i(s) - \mathbb{E}[y_1(s)]\right|\right] \le \frac{K_T}{\sqrt{N_2}}.$$

Moreover, we have

$$\mathbb{E}\left[\left|x_i^{(N)}(s) - x_i(s)\right|\right] \le \mathbb{E}\left[\sup_{r\in[0,s]}\left|x_i^{(N)}(s) - x_i(s)\right|\right]$$

and

$$\mathbb{E}\left[\left|y_i^{(N)}(s) - y_i(s)\right|\right] \le \mathbb{E}\left[\sup_{r\in[0,s]}\left|y_i^{(N)}(s) - y_i(s)\right|\right].$$

These last terms are in fact independent of the index i due to the symmetry of the system which, in turn, is due to the choice of the initial conditions and the mean-field assumption. Thus, recalling that $\alpha = \frac{N_1}{N}$, we obtain

$$\mathbb{E}\left[|\mu(s)|\right] \leq \alpha\theta_{11}\mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r) - x_1(r)\right|\right] + \frac{\sqrt{\alpha}\,\theta_{11}K_T}{\sqrt{N}}$$
$$+ (1-\alpha)\theta_{12}\mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r) - y_1(r)\right|\right] + \frac{\sqrt{1-\alpha}\,\theta_{12}K_T}{\sqrt{N}}. \quad (1.12)$$

By employing Eq. (1.12) in Eq. (1.11), it is easily seen that there exists a constant D, depending on T and on the parameters $\alpha, \theta_{11}, \theta_{22}, \theta_{12}$, and θ_{21}, such that

$$\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|x_1^{(N)}(t) - x_1(t)\right|\right] \leq D\int_0^{\tilde{t}} \mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r) - x_1(r)\right|\right] ds$$
$$+ D\int_0^{\tilde{t}} \mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r) - y_1(r)\right|\right] ds$$
$$+ \frac{D}{\sqrt{N}}, \quad (1.13)$$

Similarly, we obtain also

$$\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|y_1^{(N)}(t) - y_1(t)\right|\right] \leq D\int_0^{\tilde{t}} \mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r) - x_1(r)\right|\right] ds$$
$$+ D\int_0^{\tilde{t}} \mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r) - y_1(r)\right|\right] ds$$
$$+ \frac{D}{\sqrt{N}}. \quad (1.14)$$

If we define

$$g(\tilde{t}) := \mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|x_1^{(N)}(t) - x_1(t)\right|\right] + \mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|y_1^{(N)}(t) - y_1(t)\right|\right],$$

by summing up the inequalities (1.13) and (1.14), we obtain

$$g(\tilde{t}) \leq 2\,D\int_0^{\tilde{t}} g(s)ds + 2\,\frac{D}{\sqrt{N}}.$$

An application of Gronwall's lemma leads to the conclusion. Indeed, we get the inequality $g(T) \leq \frac{2De^{2DT}}{\sqrt{N}}$, whose right-hand side goes to zero as $N \to +\infty$. □

1.4.2 Analysis of the zero-noise dynamics

In this section we consider system (1.2) with $\sigma = 0$. Notice that, in the zero-noise version of (1.2), the terms $\alpha\theta_{11}(x - \mathbb{E}[x])$ and $(1-\alpha)\theta_{22}(y - \mathbb{E}[y])$ are both zero. Thus, setting

$$A := (1-\alpha)\,\theta_{12} > 0 \qquad \text{and} \qquad B := -\alpha\theta_{21} > 0,$$

system (1.2) reduces to

$$\begin{aligned} \dot{x} &= -x^3 + x - A\,(x-y) \\ \dot{y} &= -y^3 + y - B\,(x-y)\,. \end{aligned} \tag{1.15}$$

We make the following assumption at this point. We will focus on the case

(H) $A > 1$ and $B > A - 1$.

The reason for this choice is that in this parameter regime one can obtain an analytic characterization of the phase portrait of system (1.15), still displaying a rich variety of cases. The central concern in the subsequent sections will be the investigation of the conditions under which noise-induced periodicity occurs.

To this end, we studied the location and the nature of the fixed points of system (1.15) by varying A and B under the regime given by hypothesis **(H)** and checked that no local bifurcation generating limit cycles occurs. Unfortunately, the global analysis of the system turns out to be very involved and we are able to exclude the existence of limit cycles only by numerical evidences (see Fig. 1.4).

System (1.15) admits the following equilibria:

- The fixed points $(0,0)$ and $\pm(1,1)$ are present for any value of A and B. However, their nature changes depending on the parameters. More specifically,
  - when $A - 1 < B < A + 2$, $(0,0)$ is an unstable node and $\pm(1,1)$ are stable nodes.
  - when $B = A + 2$, $(0,0)$ is an unstable node and $\pm(1,1)$ have a neutral and a stable direction.
  - for $B > A + 2$, $(0,0)$ is an unstable node and $\pm(1,1)$ are saddle points.
- Depending on the values of A and B, there may be two additional equilibria. In particular, three situations may arise:

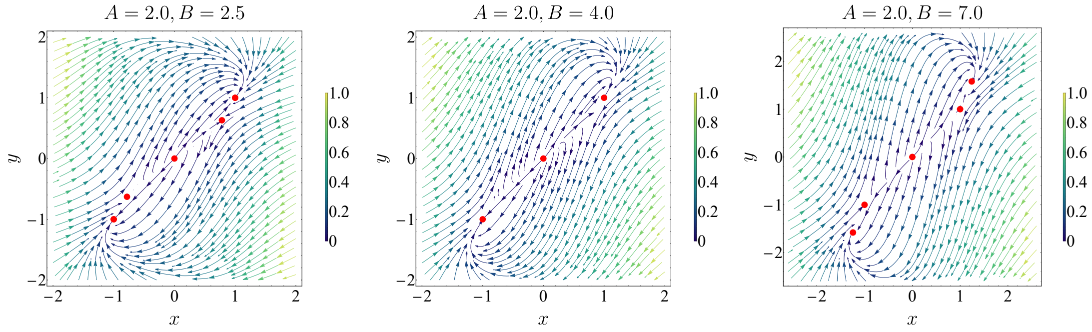
Parameter regime $A > 1$, $B > A - 1$			
Parameters \ Equilibria	$(0,0)$	$\pm(1,1)$	$\pm(x, \beta x)$
$A - 1 < B < A + 2$	unstable node	stable nodes	$0 < x < 1, 0 < \beta < 1$, saddle points
$B = A + 2$	unstable node	one negative and one null eigenvalue	−
$B > A + 2$	unstable node	saddle points	$x > 1, \beta > 1$, stable nodes

Table 1.2: Regime $A > 1$, $B > A - 1$. Nature of the fixed points of system (1.15) for different values of the parameters.

- when $A - 1 < B < A + 2$, there exists $\beta > 0$ such that the points $\pm(x, \beta x)$ are fixed points for (1.15), with $0 < x < 1$ and $\beta < 1$. That is, the equilibria are $(0, 0)$, $\pm(1, 1)$ and $\pm(x, \beta x)$, symmetrically located in the first and the third quadrants. The fixed points $\pm(x, \beta x)$ are saddle points.
- when $B = A + 2$, no other fixed points are present apart from $(0, 0)$ and $\pm(1, 1)$.
- when $B > A + 2$, there exists $\beta > 0$ such that $\pm(x, \beta x)$ are fixed points for (1.15), with $x > 1$ and $\beta > 1$. That is, system (1.15) has five equilibria: $(0, 0)$, $\pm(1, 1)$ and $\pm(x, \beta x)$, symmetrically located in the first and the third quadrants. The fixed points $\pm(x, \beta x)$ are stable nodes.

The depicted scenarios are summarized in Table 1.2. We refer the reader to Appendix A for a detailed proof. In Fig. 1.4, we display numerically obtained phase portraits for specific values of the parameters in the three cases $A - 1 < B < A + 2$, $B = A + 2$ and $B > A + 2$. In all these cases, numerical investigations strongly corroborate the absence of limit cycles for the limit system in the absence of noise, system (1.15). However, on the other hand, Fig. 1.4 gives insights into the phenomenon of noise-induced periodicity: the heteroclinic orbits that seem to be present in the phase portraits of the limit system suggest that, for $\sigma > 0$, the limit system might have limit cycles similar to the ones observed in Section 1.3 when simulating the dynamics of the finite system (1.1) for N large.

Last, we remark that the main results of this chapter, given in Sections 1.4.1 and 1.4.4, hold for all A, $B > 0$, as one can see from the proofs. Furthermore, qualitatively analogous behaviors to the ones described in the current section were numerically observed in the case $0 < A \leq 1$, $B > 0$, when extra fixed points for system (1.15) may



(a) (b) (c)

Figure 1.4: Phase portraits of system (1.15) for diverse values of A and B. (a) Case $A - 1 < B < A + 2$ with $A = 2$ and $B = 2.5$. Fixed points: $(0, 0)$ is an unstable node, $\pm(1, 1)$ are stable nodes and $\pm(0.78, 0.63)$ (numerically obtained coordinates) are saddle points. (b) Case $B = A + 2$ with $A = 2$ and $B = 4$. Fixed points: $(0, 0)$ is an unstable node and $\pm(1, 1)$ have a negative and a zero eigenvalue. (c) Case $B > A + 2$ with $A = 2$ and $B = 7$. Fixed points: $(0, 0)$ is an unstable node, $\pm(1, 1)$ are saddle points and $\pm(1.24, 1.58)$ (numerically obtained coordinates) are stable spirals. Red dots mark the equilibria. Streamline colors correspond to the magnitude of the vector field scaled to $[0, 1]$ (relative magnitude). A detailed analysis of the nature of the fixed points in the three regimes can be found in Appendix A.

exist.

1.4.3 The Fokker-Planck equation

The long-time behavior of the law of the solution to system (1.2) may be investigated by considering the corresponding Fokker-Planck equation, that reads as

$$\begin{aligned}\frac{\partial q_1}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2 q_1}{\partial z^2} - \frac{\partial}{\partial z}\left\{\left[(1-\alpha\theta_{11} - (1-\alpha)\,\theta_{12})z - z^3\right]q_1\right\}\\ &\quad - \alpha\theta_{11}\langle z, q_1\rangle\frac{\partial q_1}{\partial z} - (1-\alpha)\,\theta_{12}\langle z, q_2\rangle\frac{\partial q_1}{\partial z}\\ \frac{\partial q_2}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2 q_2}{\partial z^2} - \frac{\partial}{\partial z}\left\{\left[(1-\alpha\theta_{21} - (1-\alpha)\theta_{22})z - z^3\right]q_2\right\}\\ &\quad - \alpha\theta_{21}\langle z, q_1\rangle\frac{\partial q_2}{\partial z} - (1-\alpha)\theta_{22}\langle z, q_2\rangle\frac{\partial q_2}{\partial z},\end{aligned} \tag{1.16}$$

where time and space dependencies have been left implicit for simplicity of notation. Here $\langle z, q_i\rangle := \int z q_i(z;t)dz$, with $i = 1, 2$. The regularizing effect of the second-order partial derivatives guarantees that, for $t \in [0, T]$, the laws of $x(t)$ and $y(t)$ have respective densities $q_1(\cdot;t)$ and $q_2(\cdot;t)$ solving (1.16). By using the finite element method ([59]), we performed numerical simulations of system (1.16) starting from the initial distributions $q_1(z;0) = q_2(z;0) = \delta_{0.8}(z)$. These initial conditions correspond to what we did in Section 1.3, where we initialized the particles of both groups at $z = 0.8$ in the

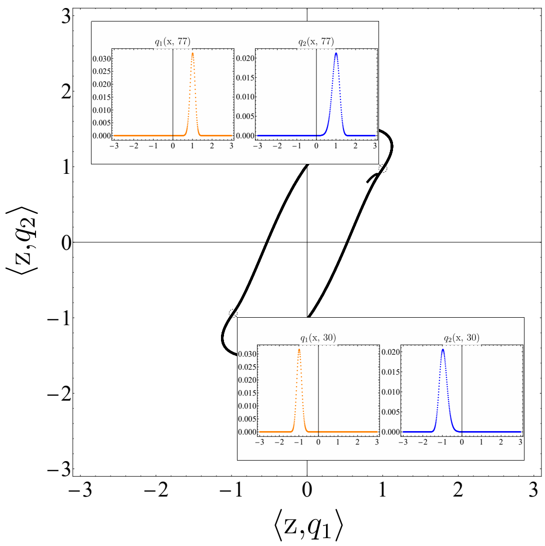


Figure 1.5: Temporal evolution of the average positions $\langle z, q_1 \rangle$ and $\langle z, q_2 \rangle$ of the two populations in the thermodynamic limit. Parameter values: $A = 2$ and $B = 2.5$; the other regimes are analogous. The insets show the densities q_1 (orange) and q_2 (blue) at some times during the simulation.

simulations of the microscopic system. We observed that q_1 and q_2 both assume a bell shape during the simulation, while the average positions of the two populations, $\langle z, q_i \rangle$ ($i = 1, 2$), computed numerically, display an oscillatory behavior. We show the results of these simulations in Fig. 1.5. The above considerations justify the idea of the Gaussian approximation for system (1.2) that will be analyzed in the following section.

1.4.4 Small-noise approximation

In this section we derive a small-noise approximation of system (1.2). In particular, motivated by what we observed in Section 1.4.3 and Fig. 1.5, we build a pair of independent Gaussian processes $(\tilde{x}(t), \tilde{y}(t))_{t\in[0,T]}$ that closely follows $(x(t), y(t))_{t\in[0,T]}$, unique solution to (1.2), when the noise is small. Although such an approximation holds rigorously true in the limit of vanishing noise, numerical simulations suggest it remains valid also beyond the assumption $\sigma \ll 1$.

Notice that, in general, $x(t)$ and $y(t)$ are not Gaussian nor independent random variables. Hence, we cannot derive closed equations for their means, as the equations for these latter involve moments of all orders, yielding an infinite-dimensional system of ODEs (see (1.19) in the proof below, where it is clear that the p-th moments depend on the $(p + 2)$-th moments). The result in Theorem 1.4.3 allows us to reduce the study of such an infinite-dimensional system to the analysis of the four-dimensional system describing the time evolution of the mean and the variance of each approximating Gaussian process (Eq. (1.17)). As we will show in Section 1.5, this system has a Hopf bifurcation at a critical value σ_c of the noise intensity. Therefore, the Gaussian approximation derived in this section explains the qualitative behavior of system (1.1) shown in Section 1.3.

The precise statement of our result is the following.

Theorem 1.4.3. *Fix $T > 0$. Let $((x(t), y(t))_{t\in[0,T]}$ solve Eq. (1.2) with deterministic initial conditions $x(0) = x_0$ and $y(0) = y_0$. There exists a Gaussian Markov process $(\tilde{x}(t), \tilde{y}(t))_{t\in[0,T]}$, with independent components, with $\tilde{x}(0) = x_0$ and $\tilde{y}(0) = y_0$ satisfying the properties:*

1. *The first two moments of $\tilde{x}(t)$ and $\tilde{y}(t)$ satisfy the equations satisfied by the respective moments of x and y, but with Gaussian-like higher-order moments. In particular, denoting by $m_1(t)$ (resp. $m_2(t)$) the expectation of $\tilde{x}(t)$ (resp. $\tilde{y}(t)$) and by $v_1(t)$ (resp. $v_2(t)$) the variance of $\tilde{x}(t)$ (resp. $\tilde{y}(t)$), it holds*

$$\begin{aligned}
\frac{dm_1}{dt} &= -m_1^3 + m_1(1-3v_1) - (1-\alpha)\theta_{12}(m_1 - m_2)\\
\frac{dm_2}{dt} &= -m_2^3 + m_2(1-3v_2) - \alpha\theta_{21}\,(m_2 - m_1)\\
\frac{dv_1}{dt} &= -6v_1^2 - 6m_1^2 v_1 + 2v_1 - 2\alpha\theta_{11}v_1 - 2(1-\alpha)\theta_{12}v_1 + \sigma^2\\
\frac{dv_2}{dt} &= -6v_2^2 - 6m_2^2 v_2 + 2v_2 - 2\alpha\theta_{21}v_2 - 2\,(1-\alpha)\,\theta_{22}v_2 + \sigma^2.
\end{aligned} \tag{1.17}$$

2. *For all $T > 0$, there exists a constant $C_T > 0$ such that, for every $\sigma > 0$, it holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \{|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)|\}\right] \le C_T\sigma^2.$$

   This means that the processes $(\tilde{x}(t))_{t\in[0,T]}$ and $(\tilde{y}(t))_{t\in[0,T]}$ are simultaneously σ-close to the solutions of (1.2).

Proof. The proof of Theorem 1.4.3 follows the strategy used in [17], where an analogous result for a one population system of mean-field interacting particles with dissipation is given.

Throughout the proof, we will denote as above $A := (1-\alpha)\theta_{12}$ and $B := -\alpha\theta_{21}$.

We start by deriving the equations of the moments of $x(t)$ and $y(t)$ in system (1.2). By applying Itô's rule to system (1.2), we can obtain the SDEs solved by $x^p(t)$ and $y^p(t)$

for any $p \geq 1$. This yields

$$\begin{aligned}
dx^p &= \sigma p x^{p-1} dw_1 + \Big[-p x^{p+2} + p x^p - \alpha\theta_{11} p \left(x - \mathbb{E}[x]\right) x^{p-1} \\
&\quad - Ap\left(x - \mathbb{E}[y]\right) x^{p-1} + \tfrac{\sigma^2}{2} p(p-1) x^{p-2}\Big] dt \\
dy^p &= \sigma p y^{p-1} dw_2 + \Big[-p y^{p+2} + p y^p + Bp \left(y - \mathbb{E}[x]\right) y^{p-1} \\
&\quad - (1-\alpha)\,\theta_{22} p \left(y - \mathbb{E}[y]\right) y^{p-1} + \tfrac{\sigma^2}{2} p(p-1) y^{p-2}\Big] dt.
\end{aligned} \tag{1.18}$$

Let $m_p^x(t) = \mathbb{E}[x^p(t)]$ and $m_p^y(t) = \mathbb{E}[y^p(t)]$ be the p-th moments of the variables $x(t)$ and $y(t)$ solving system (1.2), respectively. Taking the expectation in (1.18), we obtain

$$\begin{aligned}
\frac{dm_p^x}{dt} &= -p m_{p+2}^x + p m_p^x - \alpha\theta_{11} p \left(m_p^x - m_1^x m_{p-1}^x\right) \\
&\quad - Ap\left(m_p^x - m_1^y m_{p-1}^x\right) + \tfrac{\sigma^2}{2} p(p-1) m_{p-2}^x \\
\frac{dm_p^y}{dt} &= -p m_{p+2}^y + p m_p^y + Bp\left(m_p^y - m_1^x m_{p-1}^y\right) \\
&\quad - (1-\alpha)\theta_{22} p \left(m_p^y - m_1^y m_{p-1}^y\right) + \tfrac{\sigma^2}{2} p(p-1) m_{p-2}^y.
\end{aligned} \tag{1.19}$$

If $Z \sim N(\mu, v)$ is a Gaussian random variable with mean μ and variance v, we have that $\mathbb{E}(Z^3) = \mu^3 + 3\mu v$ and $\mathbb{E}(Z^4) = \mu^4 + 6\mu^2 v + 3v^2$. Therefore, plugging these identities into Eq. (1.19), for $p = 1, 2$ we get closed differential equations for the mean and the variance of two processes which have Gaussian law at each time $t > 0$ and whose first and second moments evolve in time as the first and second moments of x and y respectively. This is how we obtained system (1.17). Therefore, the first point of Theorem 1.4.3 will be true by construction. We remark that, because of the specific form of the equations in (1.2) that do not have terms of the type $x^n y^m$, we can close the equations (1.17) without computing mixed moments. This is why it is possible to obtain an approximation for x and y via a Gaussian process with independent components.

We turn to the second part of the statement and we actually construct the Gaussian processes. Notice that system (1.17) has a unique global solution, since the four-dimensional vector field is continuous in each variable and has continuous partial derivatives at each point. Let $(m_1(t), m_2(t), v_1(t), v_2(t))_{t\in[0,T]}$ be the unique solution to (1.17), with initial conditions $m_1(0) = x(0)$, $m_2(0) = y(0)$, $v_1(0) = v_2(0) = 0$, and set $V_i(t) := \sigma^{-2} v_i(t)$ $(i = 1, 2)$. The first step of the proof is to define two centered Gaussian processes, $(\xi_1(t))_{t\in[0,T]}$ and $(\xi_2(t))_{t\in[0,T]}$, so that $\mathbb{E}[\xi_i^2(t)] = V_i(t)$ for all $t \in [0, T]$ $(i = 1, 2)$. We consider the process $(\xi_1(t), 0 \leq t \leq T)$ first. If we write its differential as that of a generic Itô's process, i.e. $d\xi_1(t) = \psi(t)dt + \phi(t)\, dw_1(t)$, with ϕ, ψ suitable functions and

$(w_1(t), 0 \leq t \leq T)$ a standard Brownian motion, by Itô's formula we get

$$d\xi_1^2(t) = \Big(2\xi_1(t)\psi(t) + \phi^2(t)\Big)\, dt + 2\xi_1(t)\phi(t)\, dw_1(t).$$

In turn, we obtain

$$\frac{d\mathbb{E}[\xi_1^2(t)]}{dt} = 2\mathbb{E}[\xi_1(t)\psi(t)] + \mathbb{E}[\phi^2(t)]$$

and we can impose $\phi(t) = 1$ and $\xi_1(t)\psi(t) = \xi_1^2(t)\tilde{\psi}(t)$, where $\tilde{\psi}(t)$ is a deterministic factor such that $\mathbb{E}[\xi_1^2(t)]$ satisfies the equation for $V_1(t)$, obtained from Eq. (1.17). Namely, we must require that $\tilde{\psi}(t) = -3\sigma^2 V_1(t) - 3\,(m_1(t))^2 + 1 - \alpha\theta_{11} - A$. With straightforward modifications, we also obtain a differential characterization for the process $(\xi_2(t))_{t\in[0,T]}$. Putting everything together, we get the following system of SDEs

$$\begin{aligned}
d\xi_1(t) &= \Big(-3\sigma^2 V_1(t) - 3m_1^2(t) + 1 - \alpha\theta_{11} - A\Big)\,\xi_1(t)\, dt + dw_1(t)\\
d\xi_2(t) &= \Big(-3\sigma^2 V_2(t) - 3m_2^2(t) + 1 + B - (1-\alpha)\,\theta_{22}\Big)\,\xi_2(t)\, dt + dw_2(t)\\
\xi_1(0) &= \xi_2(0) = 0.
\end{aligned} \tag{1.20}$$

The processes $((\xi_i(t))_{t\in[0,T]})_{i=1,2}$ are both Gaussian Markov processes, with zero mean and such that $\mathrm{Var}\,[\xi_i(t)] = V_i(t)$ for all $t \in [0,T]$ $(i = 1,2)$. Moreover, they are well-defined, since we have uniqueness of the solution for system (1.17).
Now we define two new processes:

$$\tilde{x}(t) := m_1(t) + \sigma\,\xi_1(t) \quad \text{and} \quad \tilde{y}(t) := m_2(t) + \sigma\,\xi_2(t), \tag{1.21}$$

which can be easily seen to be Markovian and Gaussian. Moreover, their respective means $m_1(t)$, $m_2(t)$ and their respective variances $v_1(t) = \sigma^2 V_1(t)$, $v_2(t) = \sigma^2 V_2(t)$ satisfy Eq. (1.17) by construction. Overall, the constructed processes $(\tilde{x}(t))_{t\in[0,T]}$ and $(\tilde{y}(t))_{t\in[0,T]}$ have first and second moments satisfying Eq. (1.19) for $p = 1,2$ and are Gaussian.
To conclude the proof we need to upper bound the right-hand side of the following inequality

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \{|x(t) - \tilde{x}(t)| + |y(t) - \tilde{y}(t)|\}\right]\\
&\qquad \leq \mathbb{E}\left[\sup_{t\in[0,T]} |x(t) - \tilde{x}(t)|\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |y(t) - \tilde{y}(t)|\right].
\end{aligned} \tag{1.22}$$

By using together Eq. (1.20), Eq. (1.21) and Eq. (1.17), we obtain

$$
\begin{aligned}
d\tilde{x}(t) &= dm_1(t) + \sigma\, d\xi_1(t) \\
&= \left[-m_1^3(t) - 3\sigma^2 m_1(t)V_1(t) + m_1(t) - A\left(m_1(t) - m_2(t)\right)\right] dt \\
&+ \sigma\xi_1(t)\left[-3\sigma^2 V_1(t) - 3m_1^2(t) + 1 - \alpha\theta_{11} - A\right] dt + \sigma\, dw_1(t) \\
&= \left[-\tilde{x}^3(t) + \sigma^3\xi_1^3(t) + 3\sigma^2 m_1(t)\xi_1^2(t) + \left(1 - 3\sigma^2 V_1(t) - A\right)\tilde{x}(t)\right] dt \\
&+ \left[Am_2(t) - \alpha\theta_{11}\sigma\xi_1(t)\right] dt + \sigma\, dw_1(t)
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
d\tilde{y}(t) &= \left[-\tilde{y}^3(t) + \sigma^3\xi_2^3(t) + 3\sigma^2 m_2(t)\xi_2^2(t) + \left(1 + B - 3\sigma^2 V_2(t)\right)\tilde{y}(t)\right] dt \\
&+ \left[-Bm_1(t) - (1-\alpha)\,\theta_{22}\sigma\xi_2(t)\right] dt + \sigma\, dw_2(t).
\end{aligned}
$$

At this point we follow the very same steps we used before in the proofs of Theorems 1.4.1 and 1.4.2. We have

$$
\begin{aligned}
x(t) - \tilde{x}(t) &= \int_0^t \left(x(s) - \tilde{x}(s)\right)\left[1 - f_1(s) - \alpha\theta_{11} - A\right] ds \\
&- \sigma^2 \int_0^t \left(\sigma\xi_1^3(s) + 3m_1(s)\xi_1^2(s) - 3m_1(s)V_1(s) - 3\sigma V_1(s)\xi_1(s)\right) ds, \qquad (1.23)
\end{aligned}
$$

with $f_1(s) = x^2(s) + \tilde{x}^2(s) + x(s)\tilde{x}(s)$. Eq. (1.23) is of the form $\varphi(t) = \int_0^t \varphi(s)H(s)ds + \int_0^t Q(s)ds$, with $\varphi(t) = x(t) - \tilde{x}(t)$. As $\varphi(0) = 0$, the solution to Eq. (1.23) is given by $\varphi(t) = \int_0^t Q(s)e^{\int_s^t H(r)dr}ds$, where

$$
H(s) = 1 - f_1(s) - \alpha\theta_{11} - A
$$

and

$$
Q(s) = -\sigma^2\left(\sigma\xi_1^3(s) + 3m_1(s)\xi_1^2(s) - 3m_1(s)V_1(s) - 3\sigma V_1(s)\xi_1(s)\right).
$$

Hence, we have

$$
|x(t) - \tilde{x}(t)| \le \int_0^t |Q(s)|\, e^{\int_s^t (1 - f_1(r) - \alpha\theta_{11} - A)dr} ds
$$

and, therefore,

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |x(t) - \tilde{x}(t)|\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \int_0^t |Q(s)|\, e^{\int_s^t (1 - f_1(r) - \alpha\theta_{11} - A)dr} ds\right]
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\int_0^T |Q(s)| \sup_{t\in[0,T]} e^{\int_s^t (1-f_1(r)-\alpha\theta_{11}-A)dr} ds\right] \\
&\leq C_T \int_0^T \mathbb{E}\left[|Q(s)|\right] ds \\
&\leq \tilde{C}_T \sigma^2, \qquad (1.24)
\end{aligned}
$$

for some $\tilde{C}_T > 0$. The last inequality follows from the fact that we have introduced $\tilde{Q}(s) = \sigma^{-2}Q(s)$, that, being a polynomial function of a Gauss-Markov process, has a time-locally bounded L^1-norm.

An analogous estimate holds for the second term in the right-hand side of (1.22), so that

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |y(t) - \tilde{y}(t)|\right] \leq \tilde{D}_T \sigma^2, \qquad (1.25)
$$

for a suitable positive constant $\tilde{D}_T$. Putting together Eq. (1.22), Eq. (1.24) and Eq. (1.25) yields the conclusion of the proof of part 2 in Theorem 1.4.3.

□

Remark. Notice that, in general, $x(t)$ and $y(t)$ are not Gaussian nor independent random variables. Hence, we cannot derive closed equations for their means, as the equations for these latter involve moments of all orders, yielding an infinite-dimensional system of ODEs (see (1.19) in the above proof, where it is clear that the p-th moments depend on the $(p + 2)$-th moments).

The result in Theorem 1.4.3 allows us to reduce the study of such an infinite-dimensional system to the analysis of the four-dimensional system describing the time evolution of the mean and the variance of each approximating Gaussian process (Eq. (1.17)).

1.5 Subcritical Hopf bifurcation

In this section, we will provide numerical evidence of the presence of a *subcritical Hopf bifurcation* for the dynamical system (1.17) for a critical value σ_c of the noise size, in all the three parameter regimes examined above (see Table 1.1 and Fig. 1.4, recalling that $A := (1 - \alpha)\theta_{12}$ and $B := -\alpha\theta_{21}$ as in Section 1.4.2).

Recall that, in a dynamical system, a Hopf bifurcation occurs when a stable periodic orbit arises from an equilibrium point as, at some critical value of some parameter of the system, it loses stability. Subcritical means that - as in the present case - such a transition happens when moving the parameter from larger to smaller values. A Hopf bifurcation can be detected by checking whether a pair of complex eigenvalues of the linearized system around the equilibrium crosses the imaginary axis as the parameter changes (see Theorem 2, Chapter 4.4 in [57]).

Therefore we consider the dynamical system (1.17) with $\theta_{11} = \theta_{22} = 8$ and A and B chosen in the regime given by assumption **(H)**. We study the nature of the equilibrium point $(m_1, m_2, v_1, v_2) = (0, 0, \tilde{v}_1, \tilde{v}_2)$, where

$$\tilde{v}_1 = \frac{1}{6}\left(-3 - A + \sqrt{(-3-A)^2 + 6\sigma^2}\right)$$
$$\tilde{v}_2 = \frac{1}{6}\left(-3 + B + \sqrt{(-3+B)^2 + 6\sigma^2}\right),$$

as the noise intensity $\sigma > 0$ varies. The Jacobian matrix relative to system (1.17) at $(0, 0, \tilde{v}_1, \tilde{v}_2)$ reads

$$J(\sigma) = \begin{bmatrix} 1 - A - 3\tilde{v}_1 & A & 0 & 0 \\ -B & 1 + B - 3\tilde{v}_2 & 0 & 0 \\ 0 & 0 & -6 - 2A - 12\tilde{v}_1 & 0 \\ 0 & 0 & 0 & -6 + 2B - 12\tilde{v}_2 \end{bmatrix}$$

and its eigenvalues are

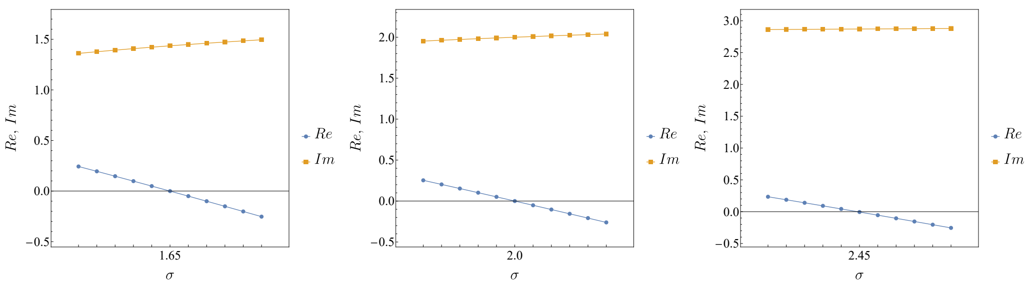
$$\begin{aligned}\lambda_1 = \frac{1}{4}\Bigg(&10 - A + B - \sqrt{(A+3)^2 + 6\sigma^2} - \sqrt{(B-3)^2 + 6\sigma^2} \\ &- \sqrt{2}\Bigg[A^2 + A\left(\sqrt{(A+3)^2 + 6\sigma^2} - \sqrt{(B-3)^2 + 6\sigma^2} - 7B + 3\right) \\ &- \left(\sqrt{(B-3)^2 + 6\sigma^2} - B\right)\left(\sqrt{(A+3)^2 + 6\sigma^2} + B\right) - 3B + 6\sigma^2 + 9\Bigg]^{\frac{1}{2}}\Bigg)\end{aligned}$$

$$\begin{aligned}\lambda_2 = \frac{1}{4}\Bigg(&10 - A + B - \sqrt{(A+3)^2 + 6\sigma^2} - \sqrt{(B-3)^2 + 6\sigma^2} \\ &+ \sqrt{2}\Bigg[A^2 + A\left(\sqrt{(A+3)^2 + 6\sigma^2} - \sqrt{(B-3)^2 + 6\sigma^2} - 7B + 3\right) \\ &- \left(\sqrt{(B-3)^2 + 6\sigma^2} - B\right)\left(\sqrt{(A+3)^2 + 6\sigma^2} + B\right) - 3B + 6\sigma^2 + 9\Bigg]^{\frac{1}{2}}\Bigg)\end{aligned}$$

$$\lambda_3 = -6 - 2A - 12\tilde{v}_1$$

$$\lambda_4 = -6 + 2B - 12\tilde{v}_2.$$

The eigenvalues λ_3 and λ_4 are negative for all $\sigma > 0$. The eigenvalues λ_1 and λ_2 are complex conjugate when (a) $A = 2$ and $B = 2.5$, (b) $A = 2$ and $B = 4$, (c) $A = 2$ and $B = 7$, and we checked numerically that they cross the imaginary axis with negative derivative at the respective critical values (a) $\sigma_c \simeq 1.65$, (b) $\sigma_c \simeq 2$, (c) $\sigma_c \simeq 2.45$ (see Fig. 1.6). Hence, when the noise intensity *decreases* to cross the threshold value σ_c, the fixed point $(0, 0, \tilde{v}_1, \tilde{v}_2)$ changes its nature from stable to unstable.



(a) (b) (c)

Figure 1.6: Real parts (blue) and absolute values of the imaginary parts (orange) of the eigenvalues λ_1, λ_2 of the Jacobian matrix $J(\sigma)$ as functions of σ with: (a) $A = 2$, $B = 2.5$, σ ranging from 1.4 to 1.9 with step 0.05; (b) $A = 2$, $B = 4$, σ ranging from 1.75 to 2.25 with step 0.05; (c) $A = 2$, $B = 7$, σ ranging from 2.2 to 2.7 with step 0.05.

Furthermore, simulations of system (1.17), with values of A and B as in Table 1.1 and Fig. 1.2, give the results shown in Fig. 1.7 and Fig. 1.8, where rhythmic oscillations for intermediate values of noise are detected. These evidences confirm the presence of a subcritical Hopf bifurcation at the equilibrium $(m_1, m_2, v_1, v_2) = (0, 0, \tilde{v}_1, \tilde{v}_2)$ for a critical value $\sigma_c = \sigma_c(\theta_{11}, \theta_{22}, A, B)$ for the regimes examined above.

Notice that the oscillations displayed by system (1.17) when decreasing σ below σ_c disappear for $\sigma = 0$ (see Fig. 1.8). Indeed, when $\sigma = 0$, $v_1 = v_2 = 0$ is a fixed point of the subsystem formed by the third and fourth equations in (1.17). As a consequence, the zero-noise limit of the first two equations in (1.17) reduces to the noiseless version of system (1.2), which does not display any oscillatory behavior.

Overall, our analysis shows that the behavior of system (1.1) for different noise sizes is well described, at least qualitatively, by the Gaussian approximation (1.17).

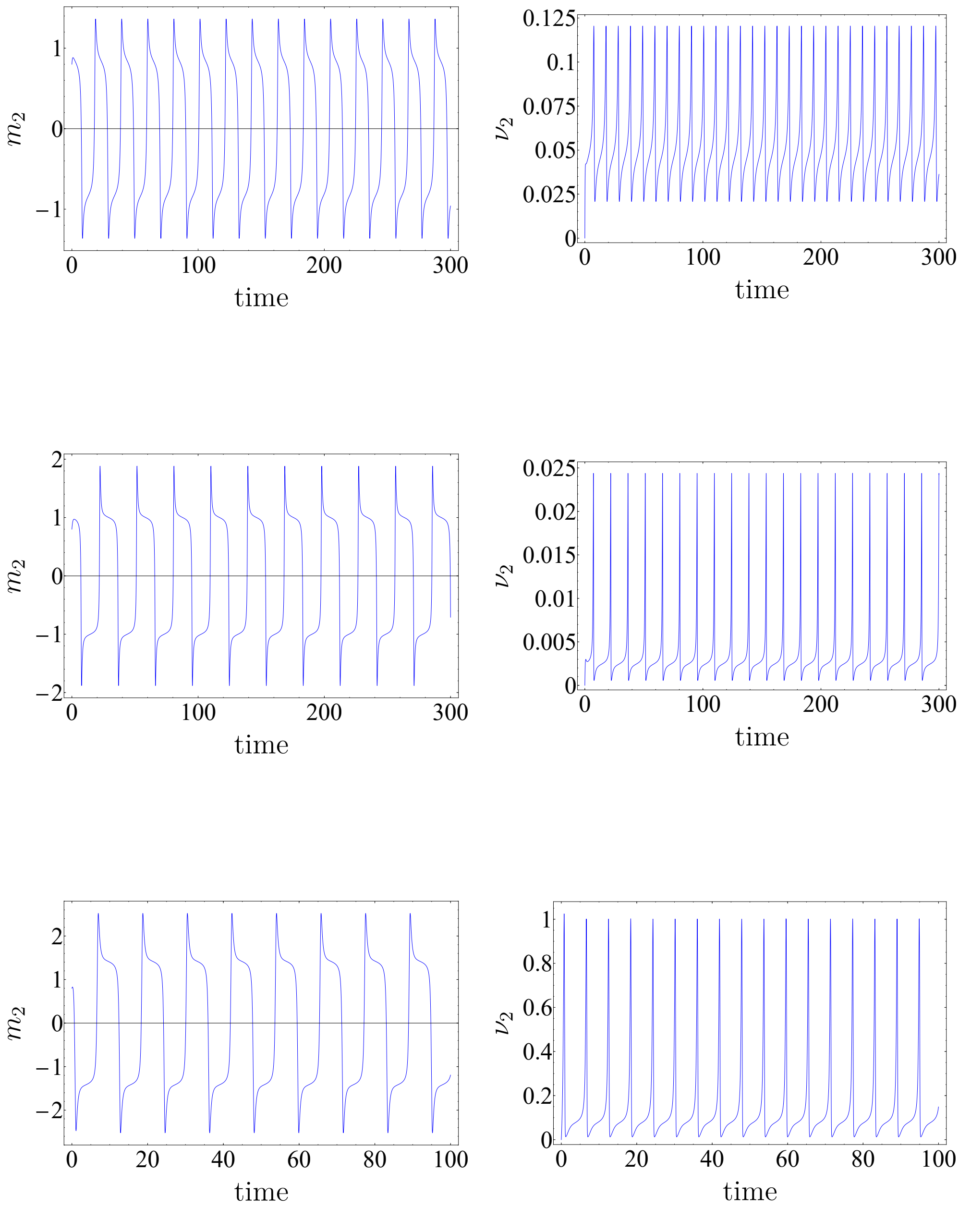


Figure 1.7: Time evolution of the mean m_2 and the variance v_2 according to the dynamical system (1.17). In all cases, we considered 10^6 iterations with a time-step $dt = 0.005$, $\alpha = 0.5$, $\theta_{11} = \theta_{22} = 8$. From top to bottom: $A - 1 < B < A + 2$, in particular, $A = 2$ and $B = 2.5$, with $\sigma = 0.5$; $B = A + 2$, in particular, $A = 2$ and $B = 4$, with $\sigma = 0.1$; $B > A + 2$, in particular, $A = 2$ and $B = 7$, with $\sigma = 0.6$.

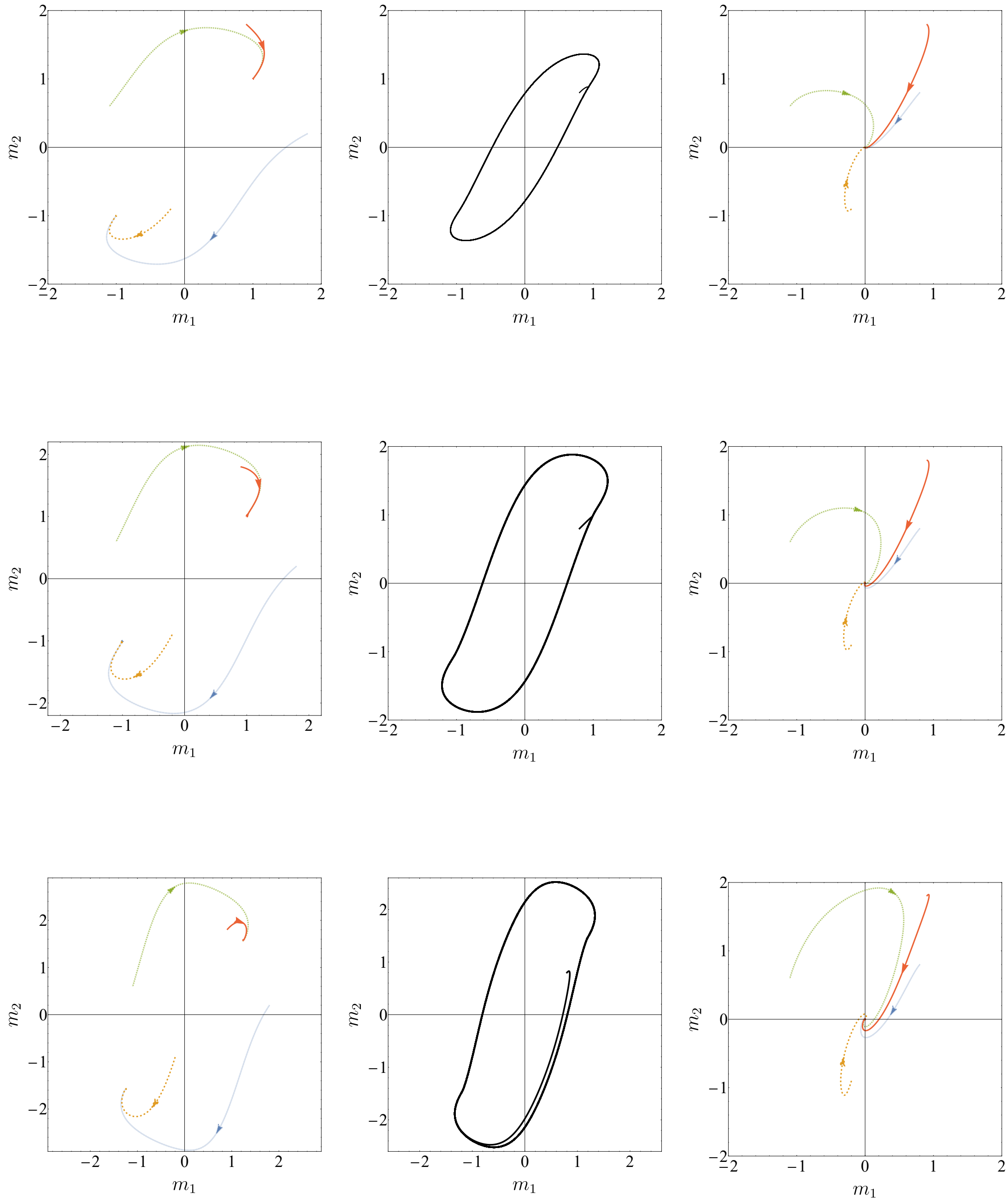


Figure 1.8: Projected dynamics of system (1.17) in the (m_1, m_2) plane. In all cases, we considered 10^6 iterations with a time-step $dt = 0.005$, $\alpha = 0.5$, $\theta_{11} = \theta_{22} = 8$. From top to bottom: $A - 1 < B < A + 2$, in particular, $A = 2$ and $B = 2.5$; $B = A + 2$, in particular, $A = 2$ and $B = 4$; $B > A + 2$, in particular, $A = 2$ and $B = 7$. In the first column we plot the trajectories of the system in the zero-noise case ($\sigma = 0$), in the second column we consider an intermediate intensity for the noise ($\sigma = 0.5$, 0.1 and 0.6 respectively) and in the third one we set $\sigma = 5$.

Chapter 2

Noise-induced oscillations for the mean-field dissipative contact process

In this chapter, based on [23], we consider a modification of the classical contact process on the complete graph obtained by adding dissipation to the interaction terms. We prove that such particle system has the propagation of chaos property and that the limit (deterministic) dynamics has a unique, globally stable fixed point. On the other hand, the stochastic fluctuations of the system around its deterministic limit converge to a Gaussian process whose power spectral density has a peak at a non-zero frequency. Therefore, the propagation of chaos and the corresponding normal fluctuations reveal that the noise, which is only present in the finite-size system, induces persistent oscillations in the model for large but finite population sizes.

2.1 Introduction

The contact process is the microscopic counterpart of one of the most basic epidemiological models, the SIS (susceptible-infectious-susceptible) model. Suppose the individuals of a population are placed at the vertices of a graph (V, E); the individual at the vertex $i \in V$ can be either *susceptible* ($x_i = 0$) or *non-susceptible* ($x_i = 1$). The configuration $x = (x_i)_{i \in V}$ evolves as a continuous time Markov chain with the following rates:

- each non-susceptible individual becomes susceptible with rate 1;
- each susceptible individual becomes non-susceptible with a rate equal to a given constant λ times the fraction of non-susceptible neighbors.

This definition suffices in finite graphs, while further care is needed in countable graphs (see [48]). In finite graphs, the chain is absorbed in the null state ($x_i \equiv 0$) in finite time (we say that the epidemics *dies out*). The *time to absorption* could however vary from a *fast absorption* (at most polynomial in $|V|$) to a slow absorption (exponential in $|V|$). For many (sequences of) graphs the transition from fast to slow absorption occurs as the *infection constant* λ crosses a critical value $\overline{\lambda}$. The simplest nontrivial example is the case in which (V, E) is the *complete graph* of N vertices: $V = \{1, 2, \dots, N\}$, $E = V \times V$. The study of this model reduces to the analysis of the one-dimensional Markov process

$$m_N := \frac{1}{N} \sum_{i=1}^{N} x_i,$$

and the critical infection constant is $\overline{\lambda} = 1$. Infinite countable graphs require a more detailed analysis, leading to results in several directions, including ergodicity, convergence, and local and global survival of the epidemics; such rich behavior could imply the existence of more than one critical value for the infection constant.

In this chapter, we introduce a modification of the contact process. The state x_i of each individual takes values in the interval $[0, 1]$, and is interpreted as her *viral load*; we say that an individual is susceptible if $x_i = 0$. The dynamics goes as follows:

- each non-susceptible individual becomes susceptible with rate $r > 0$;
- each susceptible individual becomes non-susceptible with a rate equal to $\lambda > 0$ times the arithmetic mean of the viral loads of her neighbors, and her viral load jumps to the value 1;
- between jumps, the viral load decays exponentially with rate $\alpha > 0$.

Note that the model is overparametrized, as we could rescale the time to have $r = 1$. We keep however all parameters as they have a useful interpretation: r^{-1} is the time scale at which infected (i.e. non-susceptible) individuals lose their immunity, while α^{-1} is the time scale at which infected individuals remain contagious. In many real epidemics $\alpha^{-1} \ll r^{-1}$: for a relatively long time non-susceptible individuals are *immune*, and do not contribute to the propagation of the disease. Note that many of the useful mathematical properties of the contact process are lost. For instance, monotonicity breaks due to the presence of immune individuals that block contagion.

In this chapter we study this modified contact process, which we call *dissipative*, on the complete graph. Unlike the corresponding classical contact process, the dissipative version does not admit a finite-dimensional reduction: any finite-dimensional functional of the empirical measure is non-Markovian. We particularly deal with the thermodynamic limit of the process ($N \to +\infty$), and we obtain a law of large numbers (propagation of chaos) and a central limit theorem. Despite the lack of finite-dimensional reduction, both the *limit process* corresponding to the law of large numbers and the limit *fluctuation process* corresponding to the central limit theorem are two-dimensional. Under suitable conditions on the parameters, including the case $\alpha^{-1} \ll r^{-1}$, the limit process has a unique stable fixed point, which is approached by damped oscillations. The limit fluctuation process reveals, via a Fourier analysis, that noise induces persistent oscillations, despite of the damping exhibited by the limit process. This appears as a universal phenomenon, similar to what is observed in models of population dynamics (see [55, 1]). The nature of these oscillations is then investigated by letting the parameters α and λ to diverge (slowly) with N: if properly rescaled, the dynamics of the fraction of non-susceptible individuals and of the mean viral load converges to a harmonic oscillator affected by noise. In their epidemiological interpretation, these results suggest that pandemic waves are induced by finite-size effects, allowing the random noise to excite frequencies close to a characteristic value. The resulting motion is however not strictly periodic: though the period is nearly constant, the amplitude of each oscillation is significantly affected by noise, in agreement with what is observed in real epidemics.

2.2 Microscopic dynamics and propagation of chaos

In this section, we introduce the N-particle system which we call dissipative contact process and we derive the dynamics of a representative individual in the limit as $N \to +\infty$.

We consider a population of N individuals whose states $(x_i)_{i=1}^N$ take values in $[0, 1]$ and represent the individuals' viral loads. The microscopic dynamics of the system is sketched in Figure 2.1.

We define $m_N(t)$ and $v_N(t)$ to be the average number of infected individuals and the average viral load of the N-particle system at time t respectively, namely,

$$m_N(t) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_i(t)>0\}} = \int_0^1 \mathbb{1}_{\{y>0\}} \rho_t^N(dy) \tag{2.1}$$

and

$$v_N(t) := \frac{1}{N} \sum_{i=1}^{N} x_i(t) = \int_0^1 y \rho_t^N(dy), \tag{2.2}$$

where ρ_t^N is the empirical measure of $(x_i)_{i=1}^N$ at time t, i.e., the measure-valued process

$$\rho_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}. \tag{2.3}$$

The state $\mathcal{X}^N := (x_i)_{i=1}^N$, taking values on $[0,1]^N$, is a Markov process whose infinitesimal generator is the closure of the operator $\mathcal{L}^N$ acting on continuously differentiable functions $\phi \in C^1\left([0,1]^N\right)$ as

$$\begin{aligned}\mathcal{L}^N\phi\left(\mathcal{X}^N\right) &= \sum_{k=1}^N -\alpha x_k \partial_{x_k}\phi\left(\mathcal{X}^N\right)\\ &+ \sum_{k=1}^N \mathbb{1}_{\{x_k(t)=0\}}\lambda\nu_N\left[\phi\left(\mathcal{X}^{N,\uparrow,k}\right)-\phi\left(\mathcal{X}^N\right)\right]\\ &+ \sum_{k=1}^N \mathbb{1}_{\{x_k(t)>0\}} r\left[\phi\left(\mathcal{X}^{N,\downarrow,k}\right)-\phi\left(\mathcal{X}^N\right)\right],\end{aligned} \tag{2.4}$$

where $\mathcal{X}^{N,\uparrow,k}$ denotes a state equal to $\mathcal{X}^N$, except for the k-th coordinate, which is 1, and $\mathcal{X}^{N,\downarrow,k}$ denotes a state equal to $\mathcal{X}^N$, except for the k-th coordinate, which is 0.

Equivalently, the state $x_i(t)$ of each individual evolves according to

$$\begin{aligned}x_i(t) = x_i(0) - \int_0^t \alpha x_i(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,\mathbb{1}_{\{x_i(s^-)=0\}}\lambda\nu_N(s^-)]}(u)N_i^{\uparrow}(ds,du)\\ - \int_{[0,t]\times[0,+\infty)} x_i(s^-)\mathbb{1}_{[0,\mathbb{1}_{\{x_i(s^-)>0\}}r]}(u)N_i^{\downarrow}(ds,du).\end{aligned} \tag{2.5}$$

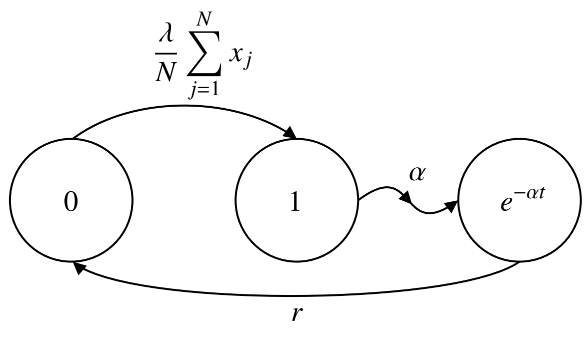


Figure 2.1: An individual i who is initially susceptible ($x_i = 0$), can become infectious with rate $\frac{\lambda}{N}\sum_{j=1}^N x_j$ (mean-field interaction). When this happens, her viral load x_i jumps to the value 1. When infectious, individual i can recover with rate r and, in this case, she becomes immediately susceptible again. In the meantime, the viral load of the individual decreases deterministically in time with rate α. Notice that this dynamics is invariant under permutations of the particles. This model is a piecewise deterministic Markov process, as the x_is have a deterministic continuous dynamics with discontinuities determined by random jump events.

Here $(N_i^{\uparrow})_{i=1}^N$ and $(N_i^{\downarrow})_{i=1}^N$ are families of independent Poisson random measures on $[0,+\infty)\times[0,+\infty)$ with intensity measure the Lebesgue measure $dt\times du$. Moreover, $N_i^{\uparrow}$ and $N_j^{\downarrow}$ are independent for all $i,j\in\{1,\dots,N\}$. Notice that integrating over $[0,t]$ makes the processes x_i càdlàg. Also, here $f(t^-):=\lim_{s\to t^-} f(s)$ is standard notation for the left limit of a function f at t. Strong existence and uniqueness for system (2.5) can be proven as in [43].

We will prove that the evolution of a representative component in the limit as $N\to+\infty$ is given by the stochastic differential equation

$$\begin{aligned}\bar{x}(t) = \bar{x}(0) - \int_0^t \alpha\bar{x}(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,\mathbb{1}_{\{\bar{x}(s^-)=0\}}\lambda v(s^-)]}(u)N^{\uparrow}(ds,du)\\ - \int_{[0,t]\times[0,+\infty)} \bar{x}(s^-)\mathbb{1}_{[0,\mathbb{1}_{\{\bar{x}(s^-)>0\}}r]}(u)N^{\downarrow}(ds,du),\end{aligned} \tag{2.6}$$

where $N^{\uparrow}$ and $N^{\downarrow}$ are independent Poisson random measures on $[0,+\infty)\times[0,+\infty)$, both having intensity measure $dt\times du$, and $v(t):=\int_{[0,+\infty)} u\rho_{\bar{x}(t)}(du)$ is the first moment of the law $\rho_{\bar{x}}$ of the variable $\bar{x}$ at time t.

Notice that Eq. (2.6) is a nonlinear, also called McKean-Vlasov, SDE, as the law of its solution appears as an argument of its coefficients through its first moment v. Existence and pathwise uniqueness of a strong solution to system (2.6) follow straightforwardly from [19] and [43].

Theorem 2.2.1 (Propagation of chaos). *Suppose $(x_i(0))_{i=1}^N$ are independent identically distributed random variables with values in $[0,1]$ and law μ_0. Denote by $((x_i(t))_{t\geq 0})_{i=1}^N$ the corresponding solution to system (2.5). Also, consider N independent copies of the solution to Eq. (2.6), $((\bar{x}_i(t))_{t\geq 0})_{i=1}^N$, with the same Poisson random measures $N_i^{\uparrow}$, $N_i^{\downarrow}$, $i=1,\dots,N$ and the same initial conditions of system (2.5).*

Then, for all $i=1,\dots,N$ and for all $T>0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\{|x_i(t)-\bar{x}_i(t)| + \left|\mathbb{1}_{\{x_i(t)>0\}} - \mathbb{1}_{\{\bar{x}_i(t)>0\}}\right|\right\}\right] \leq \frac{A_T}{\sqrt{N}} \tag{2.7}$$

where

$$A_T = \lambda T e^{(\alpha+3\lambda+2r)T}. \tag{2.8}$$

The proof of Theorem 2.2.1 is a standard application of coupling arguments and Gronwall's Lemma. We however include it here since the specific form of the constant A_T will be useful in Section 2.5.

Proof. It is convenient to set $\sigma_i(t) := \mathbb{1}_{\{x_i(t)>0\}}$, so that (2.5) can be split into the system

$$\begin{aligned} x_i(t) &= x_i(0) + \int_0^t -\alpha x_i(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u)N_i^{\uparrow}(ds,du) \\ &\quad - \int_{[0,t]\times[0,+\infty)} x_i(s^-)\mathbb{1}_{[0,\sigma_i(s^-)r]}(u)N_i^{\downarrow}(ds,du) \\ \sigma_i(t) &= \sigma_i(0) + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u)N_i^{\uparrow}(ds,du) \\ &\quad - \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,\sigma_i(s^-)r]}(u)N_i^{\downarrow}(ds,du). \end{aligned} \tag{2.9}$$

A similar splitting applies to the limiting equation (2.6) if we define $\bar{\sigma}_i(t) := \mathbb{1}_{\{\bar{x}_i(t)>0\}}$:

$$\begin{aligned} \bar{x}_i(t) &= x_i(0) + \int_0^t -\alpha \bar{x}_i(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)N_i^{\uparrow}(ds,du) \\ &\quad - \int_{[0,t]\times[0,+\infty)} \bar{x}_i(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)N_i^{\downarrow}(ds,du) \\ \bar{\sigma}_i(t) &= \sigma_i(0) + \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)N_i^{\uparrow}(ds,du) \\ &\quad - \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)N_i^{\downarrow}(ds,du). \end{aligned} \tag{2.10}$$

It follows that

$$\begin{aligned} |x_i(t) - \bar{x}_i(t)| \le \alpha \int_0^t |x_i(s) - \bar{x}_i(s)|ds &+ \int_{[0,t]\times[0,+\infty)} \Big|\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u) \\ &- \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)\Big|\, N_i^{\uparrow}(ds,du) \\ &+ \int_{[0,t]\times[0,+\infty)} \Big|x_i(s^-)\mathbb{1}_{[0,\sigma_i(s^-)r]}(u) - \bar{x}_i(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\Big|\, N_i^{\downarrow}(ds,du) \end{aligned} \tag{2.11}$$

and similarly

$$\begin{aligned} |\sigma_i(t) - \bar{\sigma}_i(t)| \le \int_{[0,t]\times[0,+\infty)} &\Big|\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u) - \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)\Big|\, N_i^{\uparrow}(ds,du) \\ &+ \int_{[0,t]\times[0,+\infty)} \Big|\mathbb{1}_{[0,\sigma_i(s^-)r]}(u) - \mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\Big|\, N_i^{\downarrow}(ds,du). \end{aligned} \tag{2.12}$$

Note that both in (2.11) and (2.12), by increasingness of the integrals in the r.h.s, the l.h.s. can be replaced by $\sup_{s\le t}|x_i(s)-\bar{x}_i(s)|$ and $\sup_{s\le t}|\sigma_i(s)-\bar{\sigma}_i(s)|$ respectively. Taking expectation and letting

$$\varphi(t) = \mathbb{E}\left[\sup_{0\le s\le t}\left(|x_i(s) - \bar{x}_i(s)| + |\sigma_i(s) - \bar{\sigma}_i(s)|\right)\right],$$

we obtain

$$\begin{aligned}
\varphi(t) \le \alpha & \int_0^t \varphi(s)ds \\
&+ \mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u) - \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)\right| N_i^{\uparrow}(ds,du)\right] \\
&+ \mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|x_i(s^-)\mathbb{1}_{[0,\sigma_i(s^-)r]}(u) - \bar{x}_i(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\right| N_i^{\downarrow}(ds,du)\right] \\
&+ \mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u) - \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)\right| N_i^{\uparrow}(ds,du)\right] \\
&+ \mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|\mathbb{1}_{[0,\sigma_i(s^-)r]}(u) - \mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\right| N_i^{\downarrow}(ds,du)\right].
\end{aligned} \tag{2.13}$$

We now use the following facts: if $f(s,u)$ is bounded, positive and predictable then

$$\mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} f(s,u)N_i^{\uparrow}(ds,du)\right] = \mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} f(s,u)duds\right] \tag{2.14}$$

and, for $a, a', b, b' \ge 0$

$$\int_0^{+\infty} \left|a\mathbb{1}_{[0,b]}(u) - a'\mathbb{1}_{[0,b']}(u)\right| du \le a|b-b'| + b'|a-a'|.$$

This yields

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda\nu_N(s^-)]}(u) - \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda\nu(s^-)]}(u)\right| N_i^{\uparrow}(ds,du)\right] \\
&\quad = \lambda\mathbb{E}\left[\int_0^t |(1-\sigma_i(s))\nu_N(s) - (1-\bar{\sigma}_i(s))\nu(s)|\, ds\right] \\
&\quad \le \lambda\mathbb{E}\left[\int_0^t |\sigma_i(s) - \bar{\sigma}_i(s)|ds\right] + \lambda\mathbb{E}\left[\int_0^t |\nu_N(s) - \nu(s)|ds\right] \\
&\quad \le \lambda\int_0^t \varphi(s)ds + \lambda\mathbb{E}\left[\int_0^t |\nu_N(s) - \nu(s)|ds\right].
\end{aligned} \tag{2.15}$$

Similarly

$$\mathbb{E}\left[\int_{[0,t]\times[0,+\infty)} \left|x_i(s^-)\mathbb{1}_{[0,\sigma_i(s^-)r]}(u) - \bar{x}_i(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\right| N_i^{\downarrow}(ds,du)\right]$$

$$\leq r\mathbb{E}\left[\int_0^t |x_i(s) - \bar{x}_i(s)|ds\right] + r\mathbb{E}\left[\int_0^t |\sigma_i(s) - \bar{\sigma}_i(s)|ds\right] \tag{2.16}$$
$$\leq r\int_0^t \varphi(s)ds.$$

Estimating in the same way the other two terms in the r.h.s. of (2.13), from (2.13), (2.15) and (2.16) we obtain

$$\varphi(t) \leq (\alpha + 2\lambda + 2r)\int_0^t \varphi(s)ds + \lambda\mathbb{E}\left[\int_0^t |v_N(s) - v(s)|ds\right]. \tag{2.17}$$

Noticing that, letting $\bar{v}_N(s) := \frac{1}{N}\sum_{i=1}^N \bar{x}_i(s)$,

$$\mathbb{E}\left[|v_N(s) - v(s)|\right] \leq \mathbb{E}\left[|v_N(s) - \bar{v}_N(s)|\right] + \mathbb{E}\left[|\bar{v}_N(s) - v(s)|\right] \leq \phi(s) + (\mathrm{Var}(\bar{v}_N(s)))^{1/2}$$
$$\leq \phi(s) + \frac{1}{\sqrt{N}},$$

we finally obtain from (2.17)

$$\varphi(t) \leq (\alpha + 3\lambda + 2r)\int_0^t \varphi(s)ds + \frac{\lambda T}{\sqrt{N}}.$$

A direct application of Gronwall's Lemma provides

$$\varphi(T) \leq \frac{\lambda T}{\sqrt{N}}e^{(\alpha+3\lambda+2r)T},$$

and the proof is complete.

□

Theorem 2.2.1, together with the fact that the i.i.d. processes $\bar{x}_i$ satisfy a standard law of large numbers, implies the following result.

Corollary 2.2.1.1. *For every $T > 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |m_N(t) - \mathbb{P}\left[\bar{x}(t) > 0\right]|\right] \leq \frac{B_T}{\sqrt{N}} \qquad \mathbb{E}\left[\sup_{t\in[0,T]} |v_N(t) - \mathbb{E}\left[\bar{x}(t)\right]|\right] \leq \frac{B_T}{\sqrt{N}} \tag{2.18}$$

where

$$B_T = \left[\lambda T + 1 + K_1\sqrt{T(\lambda + r)}\right]e^{(\alpha+3\lambda+2r)T}, \tag{2.19}$$

and K_1 is the best constant in the Burkholder-Davis-Gundy inequality in L^1. Moreover, *defining*

$$v_{N,2}(t) := \frac{1}{N}\sum_{i=1}^{N} x_i^2(t), \tag{2.20}$$

we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|v_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right] \le \frac{C_T}{\sqrt{N}}, \tag{2.21}$$

where

$$C_T = 2A_T + (1 + \lambda T + K_1(3\sqrt{\lambda} + \sqrt{r})\sqrt{T})e^{(2\alpha+r)T}. \tag{2.22}$$

Proof. By adding and subtracting $\bar{m}_N(t) := \frac{1}{N}\sum_{i=1}^{N}\bar{\sigma}_i(t)$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}|m_N(t) - \mathbb{P}[\bar{x}(t) > 0]|\right] \\
&\qquad\le \mathbb{E}\left[\sup_{t\in[0,T]}|m_N(t) - \bar{m}_N(t)|\right] + \mathbb{E}\left[\sup_{t\in[0,T]}|\bar{m}_N(t) - m(t)|\right] \\
&\qquad\le \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\sup_{t\in[0,T]}|\sigma_i(t) - \bar{\sigma}_i(t)|\right] + \mathbb{E}\left[\sup_{t\in[0,T]}|\bar{m}_N(t) - m(t)|\right].
\end{aligned} \tag{2.23}$$

The first term in the r.h.s. of (2.23), by Theorem 2.2.1, is bounded by $\frac{A_T}{\sqrt{N}}$. So we show that a similar bound holds for the second term in the r.h.s. of (2.23). Averaging over i in (2.10) we get

$$\bar{m}_N(t) = \bar{m}_N(0) + \int_0^t (\lambda v(s)(1 - \bar{m}_N(s)) - r\bar{m}_N(s))\,ds + \frac{1}{N}M_N(t) \tag{2.24}$$

with

$$\begin{aligned}
M_N(t) = \sum_{i=1}^{N}\Bigg[\int_{[0,t]\times[0,+\infty)} &\mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda v(s^-)]}(u)\tilde{N}_i^{\uparrow}(ds,du) \\
&- \int_{[0,t]\times[0,+\infty)}\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\tilde{N}_i^{\downarrow}(ds,du)\Bigg],
\end{aligned} \tag{2.25}$$

where we have introduced the compensated Poisson random measures defined by

$$\begin{aligned}
\tilde{N}_i^{\uparrow}(ds,du) &:= N_i^{\uparrow}(ds,du) - dsdu \\
\tilde{N}_i^{\downarrow}(ds,du) &:= N_i^{\downarrow}(ds,du) - dsdu.
\end{aligned} \tag{2.26}$$

We recall that ([44]) for a predictable, positive and bounded $f(s,u)$ such that

$$\mathbb{E}\left[\int_{[0,T]\times[0,+\infty)} f^2(s,u)dsdu\right] < +\infty,$$

the integrals

$$I_i(t) := \int_{[0,t]\times[0,+\infty)} f(s,u)\tilde{N}_i^{\uparrow}(ds,du)$$

define orthogonal martingales with quadratic variation $\int_{[0,t]\times[0,+\infty)} f^2(s,u)dsdu$ ([44]). It follows that $M_N(t)$ is a martingale having quadratic variation equal to $\sum_{i=1}^N \int_0^t [\lambda v(s)(1 - \bar{\sigma}_i(s)) + r\bar{\sigma}_i(s)]ds$. From (2.37) we also obtain

$$m(t) = m(0) + \int_0^t (\lambda v(s)(1 - m(s)) - rm(s))\, ds. \tag{2.27}$$

From (2.24) and (2.27) we get

$$\begin{aligned} |\bar{m}_N(t) - m(t)| &\leq |\bar{m}_N(0) - m(0)| + \int_0^t (\lambda v(s) + r)|\bar{m}_N(s) - m(s)|ds + \frac{1}{N}|M_N(t)| \\ &\leq |\bar{m}_N(0) - m(0)| + \int_0^t (\lambda + r)|\bar{m}_N(s) - m(s)|ds + \frac{1}{N}|M_N(t)|. \end{aligned} \tag{2.28}$$

Observe that, by independence of the components,

$$\mathbb{E}\left[|\bar{m}_N(0) - m(0)|\right] \leq (\mathrm{Var}(\bar{m}_N(0)))^{1/2} \leq \frac{1}{\sqrt{N}}.$$

Letting

$$\varphi(t) := \mathbb{E}\left[\sup_{s\in[0,t]} |\bar{m}_N(s) - m(s)|\right],$$

using the Burkholder-Davis-Gundy inequality in L^1 and Jensen's inequality we obtain from (2.28)

$$\begin{aligned} \varphi(t) &\leq \frac{1}{\sqrt{N}} + (\lambda + r)\int_0^t \varphi(s)ds + \frac{1}{N}\mathbb{E}\left[\sup_{t\in[0,T]} |M_N(t)|\right] \\ &\leq \frac{1}{\sqrt{N}} + (\lambda + r)\int_0^t \varphi(s)ds + \frac{K_1}{N}\mathbb{E}\left[\left(\sum_{i=1}^N \int_0^t [\lambda v(s)(1 - \bar{\sigma}_i(s)) + r\bar{\sigma}_i(s)]ds\right)^{1/2}\right] \\ &\leq \frac{1}{\sqrt{N}} + (\lambda + r)\int_0^t \varphi(s)ds + \frac{K_1}{\sqrt{N}}\left(\int_0^T [\lambda v(s)(1 - \bar{m}_N(s)) + r\bar{m}_N(s)]ds\right)^{1/2} \\ &\leq \frac{1}{\sqrt{N}} + (\lambda + r)\int_0^t \varphi(s)ds + \frac{K_1\sqrt{T(\lambda + r)}}{\sqrt{N}}. \end{aligned}$$

giving

$$\varphi(T) \leq \frac{1 + K_1\sqrt{T(\lambda + r)}}{\sqrt{N}} e^{(\lambda+r)T}. \tag{2.29}$$

By this estimate, (2.23) and Theorem 2.2.1, we obtain the desired estimate for

$$\mathbb{E}\left[\sup_{t\in[0,T]} |m_N(t) - \mathbb{P}[\bar{x}(t) > 0]|\right].$$

The estimate for $\mathbb{E}\left[\sup_{t\in[0,T]} |v_N(t) - \mathbb{E}[\bar{x}(t)]|\right]$ follows exactly the same steps, and it is omitted. So we are left to show (2.21). We begin by observing that, letting $\bar{v}_{N,2}(t) := \frac{1}{N}\sum_{i=1}^{N} \bar{x}_i^2(t)$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left|v_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right] &\leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[\sup_{t\in[0,T]} |x_i^2(t) - \bar{x}_i^2(t)|\right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\bar{v}_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right] \\
&\leq \frac{2}{N}\sum_{i=1}^{N} \mathbb{E}\left[\sup_{t\in[0,T]} |x_i(t) - \bar{x}_i(t)|\right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\bar{v}_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right] \\
&\leq 2\frac{A_T}{\sqrt{N}} + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\bar{v}_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right],
\end{aligned} \tag{2.30}$$

where we have used estimate (2.7). To estimate $\mathbb{E}\left[\sup_{t\in[0,T]} \left|\bar{v}_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right]$ we first apply Itô's formula to (2.6) and obtain

$$\begin{aligned}
\bar{x}_i^2(t) &= \bar{x}_i^2(0) - \alpha\int_0^t 2\bar{x}_i^2(s)ds \\
&\quad + \int_{[0,t]\times[0,+\infty)} \left[(\bar{x}_i(s^-) + 1)^2 - \bar{x}_i^2(s^-)\right] \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda v(s^-)]}(u) N_i^{\uparrow}(ds, du) \\
&\quad + \int_{[0,t]\times[0,+\infty)} -\bar{x}_i^2(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u) N_i^{\downarrow}(ds, du) \\
&= \bar{x}_i^2(0) + \int_0^t \left[-2\alpha\bar{x}_i^2(s) + \lambda(1 - \bar{\sigma}_i(s))v(s) - r\bar{x}_i^2(s)\right] ds \\
&\quad + \int_{[0,t]\times[0,+\infty)} (2\bar{x}_i(s^-) + 1)\, \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda v(s^-)]}(u) \tilde{N}_i^{\uparrow}(ds, du) \\
&\quad - \int_{[0,t]\times[0,+\infty)} \bar{x}_i^2(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u) \tilde{N}_i^{\downarrow}(ds, du).
\end{aligned} \tag{2.31}$$

On the one hand, averaging Eq. (2.31) over i, we obtain $\bar{v}_{N,2}(t)$. On the other hand, taking the expectation in Eq. (2.31), we obtain the equation for $\mathbb{E}[\bar{x}^2(t)]$:

$$\mathbb{E}[\bar{x}^2(t)] = \mathbb{E}[\bar{x}^2(0)] - (2\alpha + r)\int_0^t \mathbb{E}[\bar{x}^2(s)]ds + \int_0^t \lambda(1 - m(s))v(s)ds. \tag{2.32}$$

Hence, defining

$$\varphi(t) := \mathbb{E}\left[\sup_{s\in[0,t]} |\bar{v}_{N,2}(s) - \mathbb{E}[\bar{x}^2(s)]|\right],$$

we obtain

$$\begin{aligned}\varphi(t) \le \mathbb{E}[|\bar{v}_{N,2}(0) - \mathbb{E}[\bar{x}^2(0)]|] &+ (2\alpha + r)\int_0^t \varphi(s)ds + \lambda \int_0^t \mathbb{E}\left[|m(s) - \bar{m}_N(s)|\right] ds \\ &+ \mathbb{E}\left[\sup_{s'\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_{[0,s']\times[0,+\infty)} (2\bar{x}_i(s^-) + 1)\, \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda v(s^-)]}(u)\tilde{N}_i^{\uparrow}(ds,du)\right|\right] \\ &+ \mathbb{E}\left[\sup_{s'\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_{[0,s']\times[0,+\infty)} \bar{x}_i^2(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\tilde{N}_i^{\downarrow}(ds,du)\right|\right].\end{aligned} \tag{2.33}$$

First we note that

$$\int_0^t \mathbb{E}\left[|m(s) - \bar{m}_N(s)|\right] ds \le \int_0^t (\mathrm{Var}(\bar{m}_N(s)))^{1/2}\, ds \le \frac{T}{\sqrt{N}}.$$

The two martingale terms in (2.33) can be bounded by the Burkholder-Davis-Gundy inequality:

$$\begin{aligned}E&\left[\sup_{s'\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_{[0,s']\times[0,+\infty)} (2\bar{x}_i(s^-) + 1)\, \mathbb{1}_{[0,(1-\bar{\sigma}_i(s^-))\lambda v(s^-)]}(u)\tilde{N}_i^{\uparrow}(ds,du)\right|\right] \\ &\le K_1\mathbb{E}\left[\left(\int_0^t \frac{1}{N^2}\sum_{i=1}^N (2\bar{x}_i(s) + 1)^2\,(1 - \bar{\sigma}_i(s))\lambda v(s)ds\right)^{1/2}\right] \le \frac{3K_1\sqrt{\lambda T}}{\sqrt{N}},\end{aligned}$$

$$\begin{aligned}\mathbb{E}&\left[\sup_{s'\in[0,t]}\left|\frac{1}{N}\sum_{i=1}^N \int_{[0,s']\times[0,+\infty)} \bar{x}_i^2(s^-)\mathbb{1}_{[0,\bar{\sigma}_i(s^-)r]}(u)\tilde{N}_i^{\downarrow}(ds,du)\right|\right] \\ &\le K_1\mathbb{E}\left[\left(\int_0^T \frac{1}{N^2}\sum_{i=1}^N \bar{x}_i^4(s)rds\right)^{1/2}\right] \le \frac{K_1\sqrt{rT}}{\sqrt{N}}.\end{aligned}$$

Thus we obtain

$$\varphi(t) \leq (2\alpha + r) \int_0^t \varphi(s) ds + \frac{D_T}{\sqrt{N}},$$

where

$$D_T = 1 + \lambda T + K_1(3\sqrt{\lambda} + \sqrt{r})\sqrt{T},$$

from which

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\bar{v}_{N,2}(t) - \mathbb{E}[\bar{x}^2(t)]|\right] \leq \frac{D_T}{\sqrt{N}} e^{(2\alpha+r)T} \tag{2.34}$$

follows by Gronwall's Lemma. Going back to (2.30) we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|v_{N,2}(t) - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right] \leq \frac{2A_T + D_T e^{(2\alpha+r)T}}{\sqrt{N}},$$

which completes the proof.

□

2.3 Macroscopic dynamics of aggregated variables

In this section, we study the limit dynamics and its long-time behavior in detail.

2.3.1 Evolution of the mean values

By taking the expectation in Eq. (2.6), it is easy to obtain the evolution equations for

$$m(t) := \mathbb{P}\left[\bar{x}(t) > 0\right] \tag{2.35}$$

and

$$v(t) := \mathbb{E}\left[\bar{x}(t)\right], \tag{2.36}$$

which are given by

$$\begin{aligned} \dot{m}(t) &= \lambda(1 - m(t))v(t) - rm(t) \\ \dot{v}(t) &= -\alpha v(t) + \lambda(1 - m(t))v(t) - rv(t). \end{aligned} \tag{2.37}$$

It is easily seen that $(0, 0)$ is always a fixed point for (2.37). Under the condition $\frac{r+\alpha}{\lambda} < 1$ a nonzero fixed point exists:

$$(m^*, v^*) := \left(1 - \frac{r+\alpha}{\lambda}, \frac{r}{r+\alpha}\left(1 - \frac{r+\alpha}{\lambda}\right)\right). \tag{2.38}$$

Linear stability for these fixed points is readily checked with the Jacobian matrix of the vector field in (2.37):

$$J(m, v) = \begin{bmatrix} -\lambda v - r & \lambda(1 - m) \\ -\lambda v & \lambda(1 - m) - (r + \alpha) \end{bmatrix}. \tag{2.39}$$

As

$$J(0, 0) = \begin{bmatrix} -r & \lambda \\ 0 & \lambda - (r + \alpha) \end{bmatrix}, \tag{2.40}$$

we see that the origin is linearly stable for $\frac{r+\alpha}{\lambda} > 1$. Its asymptotic global stability for $\frac{r+\alpha}{\lambda} \geq 1$ follows from the absence of limit cycles, that can be proved by employing Dulac's criterion: defining $g(m, v) := v^{-1}$, it holds

$$div\left(g(m, v)\begin{bmatrix} \dot{m} \\ \dot{v} \end{bmatrix}\right) = -\lambda - \frac{r}{v} < 0 \quad \forall\, 0 \leq m \leq 1,\, 0 < v \leq 1. \tag{2.41}$$

Under the condition $\frac{r+\alpha}{\lambda} < 1$, the Jacobian matrix

$$J(m^*, v^*) = \begin{bmatrix} -\frac{r\lambda}{r+\alpha} & r + \alpha \\ -\frac{r\lambda}{r+\alpha}\left(1 - \frac{r+\alpha}{\lambda}\right) & 0 \end{bmatrix} \tag{2.42}$$

has eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(-\frac{r\lambda}{r + \alpha} \pm \frac{r\lambda}{r + \alpha}\sqrt{1 - \frac{4(r + \alpha)^2}{r\lambda}\left(1 - \frac{r + \alpha}{\lambda}\right)}\right), \tag{2.43}$$

which have negative real part. Thus the fixed point (m^*, v^*) is linearly stable whenever it exists. Its global stability follows from the above Dulac's criterion. Note also that the eigenvalues $\lambda_{\pm}$ are real and negative when

$$\frac{4(r + \alpha)^2}{r\lambda}\left(1 - \frac{r + \alpha}{\lambda}\right) \leq 1,$$

which happens when

$$\lambda \leq \lambda_- := \frac{2(r + \alpha)^2}{r}\left(1 - \sqrt{\frac{\alpha}{r + \alpha}}\right) \text{ or } \lambda \geq \lambda_+ := \frac{2(r + \alpha)^2}{r}\left(1 + \sqrt{\frac{\alpha}{r + \alpha}}\right), \tag{2.44}$$

and are complex and conjugate with negative real part when

$$\lambda_- < \lambda < \lambda_+.$$

Notice that $r + \alpha < \lambda_-$. Thus, (m^*, v^*) is a stable node when $r + \alpha < \lambda \leq \lambda_-$ or $\lambda \geq \lambda_+$ and a stable spiral when $\lambda_- < \lambda < \lambda_+$.

Remark. In the contact process without dissipation there is correspondence between the Markovianity of the one-dimensional process $m_N(t)$ and the fact that its limit is *causal*, i.e. solves a well-posed one-dimensional Cauchy problem. In the dissipative contact process, the limit process $(m(t), v(t))$ is causal, but its pre-limit counterpart is not Markovian. Indeed, the presence of the term

$$\int_{[0,t]\times[0,+\infty)} x_i(s^-)\mathbb{1}_{[0,\mathbb{1}_{\{x_i(s^-)>0\}}r]}(u)N_i^{\downarrow}(ds, du)$$

in the evolution of x_i, produces for v_N a martingale with a quadratic variation containing a term proportional to

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^t x_i^2(s)ds,$$

which is not a function of (m_N, v_N). The contribution of this term, however, vanishes in the limit as $N \to +\infty$. It can be shown that the dissipative contact process does not admit any finite-dimensional reduction, in the sense that no finite-dimensional statistics of the microscopic variables has the Markov property for all N.

2.3.2 The distribution of $\bar{x}$

Our aim in this subsection is to determine the asymptotic distribution of the viral load. We denote by $(\rho^{\mu_0}_{\bar{x}(t)})_{t\geq 0}$ the flow of the distributions of $\bar{x}$, with the initial condition $\bar{x}(0) \sim \mu_0$. Moreover, let $(m(t), v(t))$ be the solution of (2.37) with initial conditions $m(0) = \mathbb{P}(\bar{x}(0) > 0)$, $v(0) = \mathbb{E}[\bar{x}(0)]$.

Theorem 2.3.1. *For any initial distribution μ_0 on $[0, 1]$, the distribution at time t of the limit process $\bar{x}$ when the initial distribution is $\bar{x}(0) \sim \mu_0$ is given by*

$$\rho^{\mu_0}_{\bar{x}(t)}(d\bar{x}) = \int_0^1 \rho^{\bar{x}_0}_{\bar{x}(t)}(d\bar{x})\mu_0(d\bar{x}_0), \tag{2.45}$$

with

$$\rho_{\bar{x}(t)}^{\bar{x}_0}(d\bar{x}) := e^{-rt}\mathbb{1}_{\bar{x}_0>0}\delta_{\bar{x}_0 e^{-\alpha t}}(d\bar{x}) + g_t(\bar{x})\mathbb{1}_{(e^{-\alpha t},1]}(\bar{x})d\bar{x} + k(t)\delta_0(d\bar{x}) \tag{2.46}$$

the marginal law at time t of the limit $\bar{x}$ process started at the deterministic initial condition $\bar{x}(0) = \bar{x}_0 \in [0, 1]$, with

$$g_t(\bar{x}) := \frac{\lambda\, k\left(t + \frac{1}{\alpha}\ln(\bar{x})\right) v\left(t + \frac{1}{\alpha}\ln(\bar{x})\right)}{\alpha}\bar{x}^{\frac{r-\alpha}{\alpha}} \tag{2.47}$$

and $k(t)$ being the solution to

$$\begin{cases} \dot{k}(t) + \lambda v(t)k(t) = r\,(1 - k(t)) \\ k(0) = \mathbb{1}_{\{\bar{x}_0=0\}}. \end{cases} \tag{2.48}$$

Besides δ_0, the unique stationary distribution is given by

$$\begin{aligned} \rho^*(d\bar{x}) &= \frac{\lambda\, r}{\alpha}\frac{v^*}{\lambda\, v^* + r}\bar{x}^{\frac{r-\alpha}{\alpha}}d\bar{x} + k^*\delta_0(d\bar{x}) \\ &= \frac{r}{\alpha}\left(1 - \frac{r+\alpha}{\lambda}\right)\bar{x}^{\frac{r-\alpha}{\alpha}}d\bar{x} + \frac{r+\alpha}{\lambda}\delta_0(d\bar{x}) \end{aligned} \tag{2.49}$$

where $k^* = \lim_{t\to+\infty} k(t) = \frac{r}{\lambda\, v^*+r}$ is the stationary solution to Eq. (2.48). *Moreover, ρ^* is the limit distribution (as $t \to +\infty$) for all initial distributions μ_0 different from δ_0.*

Remark. We give a heuristic interpretation of (2.46). The first summand arises as, starting from $\bar{x}_0 > 0$, the limit process $\bar{x}$ evolves deterministically and decreases exponentially fast to zero until a jump to $\bar{x} = 0$ occurs. This happens with rate r, so the probability that, at any time $t > 0$ and starting from $\bar{x}_0 > 0$, the process $\bar{x}$ has not jumped to zero yet is e^{-rt}. As soon as $\bar{x}$ undergoes a jump to zero, the initial condition $\bar{x}_0$ is forgotten.

This is when the second and the third summands come into play. The second summand constitutes the absolutely continuous part of $\rho_{\bar{x}(t)}^{\bar{x}_0}$. It is turned on as $\bar{x}$, after having reached zero for the first time, jumps for the first time to one. Indeed, the only way $\mathbb{1}_{(e^{-\alpha t},1]}(\bar{x}(t))$ can be different from zero is when $\bar{x}$ decays deterministically from $\bar{x}_0$ for a time $s \leq t$, then jumps to zero and then to one before time t, and from one resumes a deterministic decay (possibly with other extra jumps to zero and one) so that, by time t, $\bar{x}(t) \in (e^{-\alpha t}, 1]$. In fact, if $\bar{x}$ underwent only a deterministic decay from $\bar{x}_0$ at time zero to time t, we could only find $\bar{x}(t) \leq e^{-\alpha t}$.

The last summand accounts for the fact that the process can jump to zero and stay in zero for some time.

Proof. Recall that the limit process $\bar{x} = (\bar{x}(t))_{t\geq 0}$ is a piecewise deterministic Markov process. We first show that, for every $\bar{x}_0 \in [0,1]$ and $t \geq 0$, $\rho^{\bar{x}_0}_{\bar{x}(t)}$ is indeed a probability. Non negativity of $k(t)$ follows from (2.48) observing that $\dot{k}(t)$ would be strictly positive if $k(t) < 0$. Thus we only have to check that

$$\int_{e^{-\alpha t}}^{1} g_t(x)dx = 1 - k(t) - e^{-rt}\mathbb{1}_{\bar{x}_0>0}. \tag{2.50}$$

Using the fact that, from (2.48),

$$\lambda e^{rt} v(t)k(t) = \frac{d}{dt}(1 - k(t))e^{rt},$$

we have, by the change of variable $y = t + \frac{1}{\alpha}\ln x$,

$$\int_{e^{-\alpha t}}^{1} g_t(x)dx = e^{-rt}\int_0^t \lambda e^{ry}k(y)v(y)dy = 1 - k(t) - e^{-rt}(1 - k(0)),$$

from which (2.50) follows.

Now, it suffices to show that, for every $\bar{x}_0 \in [0,1]$, the distribution $\rho^{\bar{x}_0}_{\bar{x}(t)}$ given in (2.46) is such that for every differentiable $f : [0,1] \to \mathbb{R}$

$$\int_0^1 \mathcal{L}_t f(\bar{x})\rho^{\bar{x}_0}_{\bar{x}(t)}(d\bar{x}) = \frac{d}{dt}\int_0^1 f(\bar{x})\rho^{\bar{x}_0}_{\bar{x}(t)}(d\bar{x}), \tag{2.51}$$

where

$$\mathcal{L}_t f(\bar{x}) = -\alpha\bar{x}f'(\bar{x}) + \lambda v(t)\left[f(1) - f(0)\right]\mathbb{1}_{\{\bar{x}=0\}} + r\left[f(0) - f(\bar{x})\right]\mathbb{1}_{\{\bar{x}>0\}} \tag{2.52}$$

is the (time-dependent) infinitesimal generator of the process $\bar{x}$. We show the details for $\bar{x}_0 > 0$; the case $\bar{x}_0 = 0$ is similar and it is omitted.

Employing Eq.s (2.52) and (2.46) we have that

$$\begin{aligned}
\int_0^1 \mathcal{L}_t f(\bar{x})\rho^{\bar{x}_0}_{\bar{x}(t)}(d\bar{x}) &= \int_0^1 \left(-\alpha\bar{x}f'(\bar{x}) + \lambda v(t)\left[f(1) - f(0)\right]\mathbb{1}_{\{\bar{x}=0\}} + r\left[f(0) - f(\bar{x})\right]\mathbb{1}_{\{\bar{x}>0\}}\right)\rho^{\bar{x}_0}_{\bar{x}(t)}(d\bar{x}) \\
&= -\alpha e^{-rt}\bar{x}_0 e^{-\alpha t} f'(\bar{x}_0 e^{-\alpha t}) - \alpha\int_{e^{-\alpha t}}^1 \bar{x}f'(\bar{x})g_t(\bar{x})d\bar{x} \\
&+ \lambda v(t)\left[f(1) - f(0)\right]e^{-rt}\int_0^1 \mathbb{1}_{\{\bar{x}=0\}}\delta_{\bar{x}_0 e^{-\alpha t}}(d\bar{x}) + \lambda v(t)\left[f(1) - f(0)\right]k(t) \\
&+ re^{-rt}\int_0^1 \left[f(0) - f(\bar{x})\right]\mathbb{1}_{\{\bar{x}>0\}}\delta_{\bar{x}_0 e^{-\alpha t}}(d\bar{x}) + r\int_{e^{-\alpha t}}^1 \left[f(0) - f(\bar{x})\right]\mathbb{1}_{\{\bar{x}>0\}}g_t(\bar{x})d\bar{x}
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha \bar{x}_0 e^{-(\alpha+r)t} f'(\bar{x}_0 e^{-\alpha t}) - \alpha \left(\left[\bar{x} f(\bar{x}) g_t(\bar{x}) \right]_{e^{-\alpha t}}^1 - \int_{e^{-\alpha t}}^1 f(\bar{x}) \left(g_t(\bar{x}) + \bar{x} \partial_{\bar{x}} g_t(\bar{x}) \right) d\bar{x} \right) \\
&+ \lambda v(t) \left[f(1) - f(0) \right] k(t) \\
&+ r e^{-rt} \left[f(0) - f(\bar{x}_0 e^{-\alpha t}) \right] + r \int_{e^{-\alpha t}}^1 \left[f(0) - f(\bar{x}) \right] g_t(\bar{x}) d\bar{x}.
\end{aligned}
$$

On the other hand, we have that

$$
\begin{aligned}
\frac{d}{dt} \int_0^1 f(\bar{x}) \rho_{\bar{x}(t)}^{\bar{x}_0}(d\bar{x}) &= \frac{d}{dt} \int_0^1 f(\bar{x}) e^{-rt} \delta_{\bar{x}_0 e^{-\alpha t}}(d\bar{x}) + \frac{d}{dt} \int_{e^{-\alpha t}}^1 f(\bar{x}) g_t(\bar{x}) d\bar{x} + \dot{k}(t) f(0) \\
&= \frac{d}{dt} \left[e^{-rt} f(\bar{x}_0 e^{-\alpha t}) \right] + \int_{e^{-\alpha t}}^1 f(\bar{x}) \partial_t g_t(\bar{x}) d\bar{x} + \alpha e^{-\alpha t} f(e^{-\alpha t}) g_t(e^{-\alpha t}) \\
&+ \dot{k}(t) f(0) \\
&= -r e^{-rt} f(\bar{x}_0 e^{-\alpha t}) - \alpha \bar{x}_0 e^{-(\alpha+r)t} f'(\bar{x}_0 e^{-\alpha t}) + \int_{e^{-\alpha t}}^1 f(\bar{x}) \partial_t g_t(\bar{x}) d\bar{x} \\
&+ \alpha e^{-\alpha t} f(e^{-\alpha t}) g_t(e^{-\alpha t}) + \dot{k}(t) f(0).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{d}{dt} \int_0^1 f(\bar{x}) \rho_{\bar{x}(t)}^{\bar{x}_0}(d\bar{x}) - \int_0^1 \mathcal{L}_t f(\bar{x}) \rho_{\bar{x}(t)}^{\bar{x}_0}(d\bar{x}) \\
&= \int_{e^{-\alpha t}}^1 f(\bar{x}) \left[\partial_t g_t(\bar{x}) - (\alpha - r) g_t(\bar{x}) - \alpha \bar{x} \partial_{\bar{x}} g_t(\bar{x}) \right] d\bar{x} \\
&+ f(0) \left[\dot{k}(t) - r e^{-rt} - r \int_{e^{-\alpha t}}^1 g_t(\bar{x}) d\bar{x} + \lambda v(t) k(t) \right] \\
&+ f(1) \left[\alpha g_t(1) - \lambda v(t) k(t) \right] \\
&= \int_{e^{-\alpha t}}^1 f(\bar{x}) \left[\partial_t g_t(\bar{x}) - (\alpha - r) g_t(\bar{x}) - \alpha \bar{x} \partial_{\bar{x}} g_t(\bar{x}) \right] d\bar{x} \\
&+ f(0) \left[\dot{k}(t) - r e^{-rt} - r(1 - k(t) - e^{-rt}) + \lambda v(t) k(t) \right] \\
&+ f(1) \left[\alpha g_t(1) - \lambda v(t) k(t) \right] \\
&= \int_{e^{-\alpha t}}^1 f(\bar{x}) \left[\partial_t g_t(\bar{x}) - (\alpha - r) g_t(\bar{x}) - \alpha \bar{x} \partial_{\bar{x}} g_t(\bar{x}) \right] d\bar{x} \\
&+ f(0) \left[\dot{k}(t) - r(1 - k(t)) + \lambda v(t) k(t) \right] \\
&+ f(1) \left[\alpha g_t(1) - \lambda v(t) k(t) \right].
\end{aligned}
$$

By assumption $\dot{k}(t) - r(1 - k(t)) + \lambda v(t)k(t) = 0$. Moreover the equation

$$\partial_t g_t(\bar{x}) = (\alpha - r)g_t(\bar{x}) + \alpha\bar{x}\partial_{\bar{x}} g_t(\bar{x})$$
$$g_t(1) = \frac{\lambda}{\alpha} v(t)k(t)$$

can be solved by the method of characteristics, and it is easy to checked that it is solved by (2.47). This completes the proof of (2.51). The remaining statements concerning the limiting distribution are simple consequences of the stability of the fixed point k^* of (2.48).

□

2.4 The fluctuation process and its Gaussian limit

In this section, we study the stochastic fluctuations of the process (m_N, v_N) around its deterministic limit given by the solution to (2.37).

We define the fluctuation process to be

$$X^N(t) := \begin{bmatrix} \xi^N(t) \\ \eta^N(t) \end{bmatrix} := \begin{bmatrix} \sqrt{N}\,(m_N(t) - m(t)) \\ \sqrt{N}\,(v_N(t) - v(t)) \end{bmatrix}. \tag{2.53}$$

As observed in Section 2.3.1, this process is not Markovian. However, we prove a central limit theorem that implies its convergence to a Gauss-Markov process.

Theorem 2.4.1 (**Diffusion approximation of the fluctuation process**). *As in Theorem 2.2.1 we assume that $(x_i(0))_{i=1}^N$ are independent identically distributed random variables with values in $[0,1]$ and law μ_0. In particular, $X^N(0)$ converges in distribution to a Gaussian random variable X_0. Then X^N converges in distribution, in any interval $[0,T]$, to the solution $X(t) = [\xi(t), \eta(t)]^T$ to the following linear stochastic differential equation*

$$\begin{aligned} dX(t) &= F(t)X(t)dt + \Sigma(t)dW(t) \\ X(0) &= X_0 \end{aligned} \tag{2.54}$$

where

$$F(t) := \begin{bmatrix} -(\lambda v(t) + r) & \lambda(1 - m(t)) \\ -\lambda v(t) & \lambda(1 - m(t)) - r - \alpha \end{bmatrix} \tag{2.55}$$

is the drift matrix, $W(t)$ is a two-dimensional standard Brownian motion and the diffusion matrix $\Sigma(t)$ is the (unique, symmetric) square root of the matrix $A(t)$ given by

$$A(t) = \Sigma(t)\Sigma^T(t) = \begin{bmatrix} \lambda(1-m(t))\,v(t) + rm(t) & \lambda(1-m(t))\,v(t) + r\lambda v(t) \\ \lambda(1-m(t))\,v(t) + rv(t) & \lambda(1-m(t))\,v(t) + r\mathbb{E}\left[\bar{x}^2(t)\right] \end{bmatrix} \tag{2.56}$$

where $\bar{x}(t)$ is the limit process defined in (2.6).

Proof. We will employ the following theorem:

Theorem 2.4.2 (Diffusion approximation (Theorem VII, 4.1 [31])). *Let $\left(X^N\right)_{N\in\mathbb{N}}$ and $\left(B^N\right)_{N\in\mathbb{N}}$ $\mathbb{R}^d$-valued processes with càdlàg sample paths and let $A^N = \left(A^N_{ij}\right)$ be a symmetric $d\times d$ matrix-valued process such that A^N_{ij} has càdlàg sample paths in $\mathbb{R}$ and $A^N(t) - A^N(s)$ is non-negative definite for all $t > s \geq 0$. Let $\mathcal{F}^N_t := \sigma\left(X^N(s), B^N(s), A^N(s) \;:\; s \leq t\right)$. Let $\tau^N_h := \inf\left\{t > 0 \;:\; |X^N(t)| \geq h \text{ or } |X^N(t^-)| \geq h\right\}$. Assume that:*

- *$M^N := X^N - B^N$ and $M^N_i M^N_j - A^N_{ij}$ $i, j = 1, \ldots, d$ are $\left(\mathcal{F}^N_t\right)$-local martingales;*
- *for each $T > 0$, $h > 0$*

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{t\leq T\wedge\tau^N_h} |B^N(t) - B^N(t^-)|^2\right] = 0; \tag{2.57}$$

- *for each $T > 0$, $h > 0$, $i, j = 1, \ldots, d$*

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{t\leq T\wedge\tau^N_h} |A^N_{ij}(t) - A^N_{ij}(t^-)|\right] = 0; \tag{2.58}$$

- *for each $T > 0$, $h > 0$*

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{t\leq T\wedge\tau^N_h} |X^N(t) - X^N(t^-)|^2\right] = 0; \tag{2.59}$$

- *there exist a continuous, symmetric, non-negative definite $d \times d$ matrix-valued function on $\mathbb{R}^d$, $a = \left(a_{ij}\right)$, and a continuous function $b : \mathbb{R}^d \to \mathbb{R}^d$ such that, for each $h > 0$, $T > 0$ and $i, j = 1, \ldots, d$, and for all $\epsilon > 0$,*

$$\lim_{N\to+\infty} \mathbb{P}\left(\sup_{t\leq T\wedge\tau^N_h} \left|A^N_{ij}(t) - \int_0^t a_{ij}(X^N(s))ds\right| > \epsilon\right) = 0 \tag{2.60}$$

and

$$\lim_{N\to+\infty}\mathbb{P}\left(\sup_{t\le T\wedge\tau_h^N}\left|B_i^N(t)-\int_0^t b_i(X^N(s))ds\right|>\epsilon\right)=0; \tag{2.61}$$

- *the* $C_{\mathbb{R}^d}([0,+\infty))$ *martingale problem for*

$$\tilde{A}:=\left\{\left(f, Gf:=\frac{1}{2}\sum_{i,j}a_{ij}\partial_i\partial_j f+\sum_i b_i\partial_i f\right)\;:\; f\in C_c^\infty\left(\mathbb{R}^d\right)\right\} \tag{2.62}$$

  is well-posed;

- *the sequence of the initial laws of the* X^N*s converges in distribution to some probability distribution on* $\mathbb{R}^d$*,* ν*.*

Then $\left(X^N\right)_N$ *converges in distribution to the solution of the martingale problem for* $(\tilde{A},\nu)$*. That is, the laws of the processes* $\left(X^N\right)_N$ *converge weakly to the law of a process* X *which is a weak solution of the SDE*

$$dX(t)=b(X(t))dt+\Sigma(X(t))dW(t) \tag{2.63}$$

where $b=(b_i)_i$ *and* $\left(\Sigma\Sigma^T\right)_{ij}=a_{ij}$ *are the drift vector and the diffusion coefficient in* (2.62).

We are actually going to apply this theorem to a case in which the functions b_i and a_{ij} have an explicit, continuous dependence on the time t. This generalization is trivial as it amounts to add one dimension to the state space $\mathbb{R}^d$, introducing the deterministic extra variable $Y(t)=t$. Moreover, for next application of this theorem, the localization given by the stopping times τ_h^N will not be necessary, and it will be omitted.

By using (2.9), (2.37), the identities $m_N(t)=\frac{\xi^N(t)}{\sqrt{N}}+m(t)$ and $v_N(t)=\frac{\eta^N(t)}{\sqrt{N}}+v(t)$, and recalling the compensated Poisson random measures defined in (2.26) we have that

$$\xi^N(t)=\int_0^t\left[-(\lambda v(s)+r)\xi^N(s)+\lambda(1-m(s))\eta^N(s)-\lambda\frac{\xi^N(s)\eta^N(s)}{\sqrt{N}}\right]ds+M_1^N(t)$$

$$\eta^N(t)=\int_0^t\left[-\lambda v(s)\xi^N(s)+(-(\alpha+r)+\lambda(1-m(s)))\,\eta^N(s)-\lambda\frac{\xi^N(s)\eta^N(s)}{\sqrt{N}}\right]ds+M_2^N(t), \tag{2.64}$$

where

$$
\begin{aligned}
M_1^N(t) &:= \xi^N(0) + \sum_{i=1}^N \Bigg[\int_{[0,t]\times[0,+\infty)} \frac{\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda v_N(s^-)]}(u)}{\sqrt{N}} \tilde{N}_i^{\uparrow}(ds,du) \\
&\qquad - \int_{[0,t]\times[0,+\infty)} \frac{\mathbb{1}_{[0,\sigma_i(s^-)r]}(u)}{\sqrt{N}} \tilde{N}_i^{\downarrow}(ds,du)\Bigg] \\
M_2^N(t) &:= \eta^N(0) + \sum_{i=1}^N \Bigg[\int_{[0,t]\times[0,+\infty)} \frac{\mathbb{1}_{[0,(1-\sigma_i(s^-))\lambda v_N(s^-)]}(u)}{\sqrt{N}} \tilde{N}_i^{\uparrow}(ds,du) \\
&\qquad - \int_{[0,t]\times[0,+\infty)} \frac{x_i(s^-)\mathbb{1}_{[0,\sigma_i(s^-)r]}(u)}{\sqrt{N}} \tilde{N}_i^{\downarrow}(ds,du)\Bigg]
\end{aligned}
\tag{2.65}
$$

are square-integrable martingales having predictable quadratic variation equal to

$$
\begin{aligned}
\langle M_1^N\rangle_t &= \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} \left(\frac{\mathbb{1}_{[0,(1-\sigma_i(s))\lambda v_N(s)]}(u)}{\sqrt{N}}\right)^2 duds \\
&\quad + \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} \left(\frac{\mathbb{1}_{[0,\sigma_i(s)r]}(u)}{\sqrt{N}}\right)^2 duds \\
&= \int_0^t (\lambda(1-m_N(s))v_N(s) + r m_N(s))\, ds \\
\langle M_2^N\rangle_t &= \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} \left(\frac{\mathbb{1}_{[0,(1-\sigma_i(s))\lambda v_N(s)]}(u)}{\sqrt{N}}\right)^2 duds \\
&\quad + \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} \left(\frac{x_i(s)\mathbb{1}_{[0,\sigma_i(s)r]}(u)}{\sqrt{N}}\right)^2 duds \\
&= \int_0^t \left(\lambda(1-m_N(s))v_N(s) + r\frac{\sum_{i=1}^N x_i^2(s)}{N}\right) ds.
\end{aligned}
$$

Also, notice that the quadratic covariation process between M_1^N and M_2^N is given by

$$
\begin{aligned}
\langle M_1^N, M_2^N\rangle_t &= \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} \left(\frac{\mathbb{1}_{[0,(1-\sigma_i(s))\lambda v_N(s)]}(u)}{\sqrt{N}}\right)^2 duds \\
&\quad + \sum_{i=1}^N \int_{[0,t]\times[0,+\infty)} x_i(s)\left(\frac{\mathbb{1}_{[0,\sigma_i(s)r]}(u)}{\sqrt{N}}\right)^2 duds \\
&= \int_0^t (\lambda(1-m_N(s))v_N(s) + r v_N(s))\, ds.
\end{aligned}
$$

We can therefore apply Theorem 2.4.2 with the following positions:

$$B^N(t) := \begin{bmatrix} \int_0^t \left[-(\lambda v(s) + r)\xi^N(s) + \lambda(1 - m(s))\eta^N(s) - \lambda\frac{\xi^N(s)\eta^N(s)}{\sqrt{N}}\right] ds \\ \int_0^t \left[-\lambda v(s)\xi^N(s) + (-(\alpha + r) + \lambda(1 - m(s)))\,\eta^N(s) - \lambda\frac{\xi^N(s)\eta^N(s)}{\sqrt{N}}\right] ds \end{bmatrix}$$

$$b(x,t) = b(\xi,\eta,t) := \begin{bmatrix} -(\lambda v(t) + r)\xi + \lambda(1 - m(t))\eta \\ -\lambda v(t)\xi + (-(\alpha + r) + \lambda(1 - m(t)))\,\eta \end{bmatrix}$$

$$A^N(t) := \begin{bmatrix} \int_0^t (\lambda(1 - m_N(s))v_N(s) + rm_N(s))\,ds & \int_0^t (\lambda(1 - m_N(s))v_N(s) + rv_N(s))\,ds \\ \int_0^t (\lambda(1 - m_N(s))v_N(s) + rv_N(s))\,ds & \int_0^t \left(\lambda(1 - m_N(s))v_N(s) + r\frac{\sum_{i=1}^N x_i^2(s)}{N}\right) ds \end{bmatrix}$$

$$a(x,t) = a(t) = \begin{bmatrix} \lambda\,(1 - m(t))\,v(t) + rm(t) & \lambda\,(1 - m(t))\,v(t) + r\lambda v(t) \\ \lambda\,(1 - m(t))\,v(t) + rv(t) & \lambda\,(1 - m(t))\,v(t) + r\mathbb{E}\left[\bar{x}^2(t)\right] \end{bmatrix}.$$

Conditions (2.57) and (2.58) are obvious, as $B^N(t)$ and $A^N(t)$ are continuous. Condition (2.59) is also simple: the jumps of ξ^N and η^N are bounded by $\frac{1}{\sqrt{N}}$. We now verify conditions (2.60) and (2.61). We begin by proving (2.61) for the first component of B^N.

$$\begin{aligned} \sup_{t\in[0,T]} \left| B_1^N(t) - \int_0^t b_1(X^N(s))ds \right| &= \sup_{t\in[0,T]} \left| \int_0^t \left[-(\lambda v(s) + r)\xi^N(s) + \lambda(1 - m(s))\eta^N(s) \right.\right. \\ &\left.\left. -\lambda\frac{\xi^N(s)\eta^N(s)}{\sqrt{N}} + (\lambda v(s) + r)\xi^N(s) - \lambda(1 - m(s))\eta^N(s) \right] ds \right| \\ &\le T\lambda \sup_{s\in[0,T]} \left| \frac{\xi^N(s)\eta^N(s)}{\sqrt{N}} \right|. \end{aligned} \tag{2.66}$$

Thus, using Corollary 2.2.1.1 and Markov inequality, for every $\epsilon > 0$,

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} \left| B_1^N(t) - \int_0^t b_1(X^N(s))ds \right| \ge \epsilon \right) \le \mathbb{P}\left(T\lambda \sup_{t\in[0,T]} \left| \frac{\xi^N(t)\eta^N(t)}{\sqrt{N}} \right| \ge \epsilon \right) \\ &= \mathbb{P}\left(\sup_{t\in[0,T]} |\xi^N(t)| \sup_{t\in[0,T]} |\eta^N(t)| \ge \frac{\epsilon\sqrt{N}}{T\lambda} \right) \\ &\le \mathbb{P}\left(\sup_{t\in[0,T]} |\xi^N(t)| \ge \sqrt{\frac{\epsilon}{T\lambda}} N^{1/4} \right) + \mathbb{P}\left(\sup_{t\in[0,T]} |\eta^N(t)| \ge \sqrt{\frac{\epsilon}{T\lambda}} N^{1/4} \right) \\ &= \mathbb{P}\left(\sup_{t\in[0,T]} |m_N(t) - m(t)| \ge \sqrt{\frac{\epsilon}{T\lambda}} N^{-1/4} \right) + \mathbb{P}\left(\sup_{t\in[0,T]} |v_N(t) - v(t)| \ge \sqrt{\frac{\epsilon}{T\lambda}} N^{-1/4} \right) \end{aligned}$$

$$\leq 2\frac{\frac{B_T}{\sqrt{N}}}{\sqrt{\frac{\epsilon}{T\lambda}}N^{-1/4}} \overset{N\to+\infty}{\longrightarrow} 0.$$

The proof for B_2^N is similar, and it is omitted. We now verify (2.60) for $i = j = 2$, all other cases being similar.

$$\begin{aligned}
\left|A_{22}^N(t) - \int_0^t a_{22}(s)ds\right| =& \left|\int_0^t \left(\lambda(1-m_N(s))v_N(s) + r\frac{\sum_{i=1}^N x_i^2(s)}{N}\right.\right.\\
&\left.\left.-\lambda(1-m(s))v(s) - r\mathbb{E}\left[\bar{x}^2(t)\right]\right)ds\right|\\
&\leq \int_0^t |\lambda(1-m_N(s))v_N(s) - \lambda(1-m(s))v_N(s)|\,ds\\
&+ \int_0^t |\lambda(1-m(s))v_N(s) - \lambda(1-m(s))v(s)|\,ds \qquad (2.67)\\
&+ r\int_0^t \left|\frac{\sum_{i=1}^N x_i^2(s)}{N} - \mathbb{E}\left[\bar{x}^2(t)\right]\right|ds\\
&\leq \lambda\int_0^t (|m_N(s) - m(s)| + |v_N(s) - v(s)|)\,ds\\
&+ r\int_0^t \left|\frac{\sum_{i=1}^N x_i^2(s)}{N} - \mathbb{E}\left[\bar{x}^2(t)\right]\right|ds.
\end{aligned}$$

By Markov inequality, we are left to show that

$$\lim_{N\to+\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_{22}^N(t) - \int_0^t a_{22}(s)ds\right|\right] = 0.$$

Using (2.67), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_{22}^N(t) - \int_0^t a_{22}(s)ds\right|\right]\\
&\quad\leq \mathbb{E}\left[\lambda\int_0^T (|m_N(s) - m(s)| + |v_N(s) - v(s)|)\,ds + r\int_0^T \left|\frac{\sum_{i=1}^N x_i^2(s)}{N} - \mathbb{E}\left[\bar{x}^2(t)\right]\right|\right]
\end{aligned}$$

which converges to zero as $N \to +\infty$ by Corollary 2.2.1.1. □

2.4.1 Power spectral density of the fluctuation process in the supercritical regime

The Gaussian process $X(t)$ defined in (2.54) describes, for N large, the fluctuations of the process (m_N, v_N) around its deterministic limit (m, v). If we assume $\lambda > r + \alpha$ we have that, except for the trivial case $m(0) = v(0) = 0$,

$$\lim_{t\to+\infty}(m(t), v(t)) = (m^*, v^*) := \left(1 - \frac{r+\alpha}{\lambda}, \frac{r}{r+\alpha}\left(1 - \frac{r+\alpha}{\lambda}\right)\right).$$

Moreover, the evolution of $X(t)$, as t converges to infinity, converges to the stationary solution of the linear stochastic equation

$$dX(t) = FX(t)dt + \Sigma dW(t),$$

where

$$F = \lim_{t\to+\infty} F(t) := \begin{bmatrix} -(\lambda v^* + r) & \lambda(1-m^*) \\ -\lambda v^* & 0 \end{bmatrix},$$

$$\Sigma\Sigma^T = \lim_{t\to+\infty} \Sigma(t)\Sigma^T(t) = \begin{bmatrix} \lambda(1-m^*)v^* + rm^* & \lambda(1-m^*)v^* + r\lambda v^* \\ \lambda(1-m^*)v^* + rv^* & \lambda(1-m^*)v^* + r\mathbb{E}_{\rho^*}[x^2] \end{bmatrix},$$

where the distribution ρ^* is given in (2.49) and $\mathbb{E}_{\rho^*}$ denotes the expectation taken with respect to it. By abuse of notation we still denote by $X(t)$ this stationary Gaussian process; it describes the long time fluctuations of the process (m_N, v_N) around the equilibrium (m^*, v^*). Several features of this process can be obtained by its *power spectral density matrix* $S(\omega)$ defined by

$$S(\omega) := \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}\left[\hat{X}_t(\omega)\hat{X}_t^T(\omega)\right], \tag{2.68}$$

where, for $\omega \in \mathbb{R}$,

$$\hat{X}_t(\omega) := \int_0^t X(s)e^{-i\omega s}ds.$$

In particular, the diagonal element $S_{11}(\omega)$ (resp. $S_{22}(\omega)$), provides the spectral profile of the frequencies of the two components of $X(t)$.

Proposition 2.4.3. *Denoting by $(a_{ij})_{i,j=1,2}$ the entries of the matrix $\Sigma\Sigma^T$ and by $(F_{ij})_{i,j=1,2}$ the entries of the matrix F, we have*

$$S_{11}(\omega) = \frac{a_{11}\omega^2 + a_{11}F_{22}^2 + a_{22}F_{12}^2 - 2F_{12}F_{22}a_{12}}{(\omega^2 - det(F))^2 + \omega^2 tr(F)^2} \tag{2.69}$$

$$S_{22}(\omega) = \frac{a_{22}\omega^2 + a_{22}F_{11}^2 + a_{11}F_{21}^2 - 2F_{11}F_{21}a_{21}}{(\omega^2 - det(F))^2 + \omega^2 tr(F)^2} \tag{2.70}$$

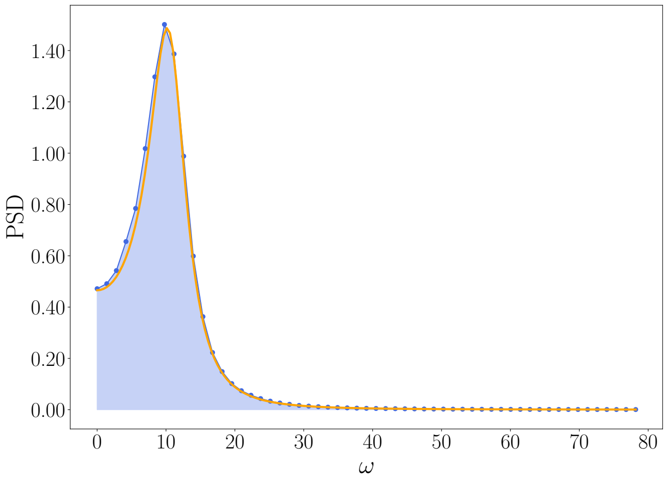


Figure 2.2: The spectral density of the process $\xi(t)$ in (2.54) (in orange) compared to the estimated spectrum of ξ_N in (2.53) (blue points) for the parameters $N = 10000$, $r = 5$, $\lambda = 100$, $\rho = 0.7$. To obtain the blue curve, 100 simulations of system (2.5) were performed employing an Euler scheme with time-step 0.001.

The proof of this proposition can be found for instance in [37], Section 4.4.6.

It is interesting to see the behavior of these spectral densities in the case the time scales r^{-1} (the expected time needed for an infected individual to lose immunity) and α^{-1} (the time an infected individual remains contagious) are well separated, i.e. $\alpha \gg r$. To describe the behavior of the spectral density in this regime we keep r and $\rho := \frac{\alpha}{\lambda} \in (0,1)$ fixed and let λ, and therefore also α, increase to infinity. Note that in this regime, using the notation in (2.44), $\lambda_- \sim \alpha = \rho\lambda < \lambda$ and $\lambda_+ \sim \frac{4\rho^2\lambda^2}{r} > \lambda$ for λ large. So the condition $\lambda_- < \lambda < \lambda_+$ is satisfied, and the equilibrium (m^*, v^*) is a stable spiral. Moreover

$$S_{11}(\omega) \sim \frac{2r(1-\rho)\,\omega^2 + r(1-\rho)\,\rho^2\lambda^2}{(\omega^2 - r(1-\rho)\,\lambda)^2 + \frac{r^2}{\rho^2}\omega^2}$$

and

$$S_{22}(\omega) \sim \frac{r(1-\rho)\,\omega^2 + r^3\frac{1-\rho}{\rho^2} + 2r^3\frac{(1-\rho)^3}{\rho^2} - 2r^3\frac{(1-\rho)^2}{\rho^2}}{(\omega^2 - r(1-\rho)\,\lambda)^2 + \frac{r^2}{\rho^2}\omega^2}.$$

It is easily seen that, for λ large, both spectral densities have a sharp peak around the frequency $\omega^* := \sqrt{r(1-\rho)\lambda}$, as shown in Figure 2.2: random fluctuations around equilibrium exhibit oscillations with a nearly deterministic period. This analysis motivates the scaling limit performed in next section.

2.5 Rescaling the parameters

The aim of this section is to obtain the limit behavior of the microscopic dynamics when we let the parameters α and λ diverge with N. Consistently with what we have seen in Section 2.4.1, we fix $r > 0, \rho \in (0,1)$, set $\alpha_N = \rho\lambda_N$ and let $\lambda_N \uparrow +\infty$ as $N \uparrow +\infty$. We have observed that this asymptotics produces a sharp peak in the power spectrum. On the one hand, since the power spectral density reflects the long-time behavior of a process, we do not expect the peak of S_{11} and S_{22} to be due to the damped oscillations which the limit system (2.37) exhibits when far from equilibrium. On the other hand, for $\lambda \gg r$, by looking at the linearization of system (2.37) around its stable equilibrium we might expect that (2.37) behaves as a deterministic harmonic oscillator, the dampening factor acting on very long time scales. In turn, one might think that the sharp peak of the power spectrum essentially captures deterministic oscillations with a negligible dampening. However, numerical simulations of system (2.53) show oscillations with a nearly deterministic period, but with a random amplitude, hence, different in nature from the ones of a deterministic oscillator. To explain this phenomenon, taking into account that we are interested in the long time behavior, we begin by assuming the initial state $(x_i(0))_{i=1}^N$ to be a system of i.i.d. variables, as in Theorem 2.4.1, but such that $\mathbb{P}(x_i(0) > 0) = m^*$, $\mathbb{E}[x_i(0)] = v^*$, i.e. the equilibrium values have been attained. For instance, we may assume $x_i(0)$ to be distributed according to the ρ^* defined in (2.49). The key point now is to find a suitable rescaling of the fluctuations $(m_N(t) - m^*)$, $(v_N(t) - v^*)$ to have a nontrivial limit. Note that even propagation of chaos is not obvious anymore: indeed, since some parameters of the model diverge with N, the possibility that propagation of chaos deteriorates in short time cannot be ignored. Moreover, if oscillations are detected in the limit, a specific time rescaling is forced by the the previous observation that the asymptotic frequency is of order $\sqrt{\lambda}$.

Theorem 2.5.1. *Define the rescaled fluctuation processes as follows:*

$$\begin{aligned} \hat{\xi}^N(t) &:= \sqrt{N}\frac{m_N\left(\frac{t}{\sqrt{\lambda_N}}\right) - m^*}{\lambda_N^{1/4}} \\ \hat{\eta}^N(t) &:= \lambda_N^{1/4}\sqrt{N}\left(v_N\left(\frac{t}{\sqrt{\lambda_N}}\right) - v^*\right) \end{aligned} \tag{2.71}$$

and assume the initial state $(x_i(0))_{i=1}^N$ to be a system of i.i.d. variables such that $\mathbb{P}(x_i(0) >$

0) $= m^*$, $\mathbb{E}[x_i(0)] = v^*$. *Moreover assume* $\alpha_N = \rho\lambda_N$, *and*

$$\lim_{N\to+\infty} \lambda_N = +\infty \qquad\qquad \lim_{N\to+\infty} \frac{\sqrt{\lambda_N}}{\log N} = 0.$$

Then the process $(\hat{\xi}^N(t), \hat{\eta}^N(t))^T$ *converges in distribution, in any interval* $[0, T]$, *to the random harmonic oscillator*

$$\begin{aligned} d\hat{\xi}(t) &= \rho\hat{\eta}(t)dt \\ d\hat{\eta}(t) &= -\frac{r}{\rho}(1-\rho)\hat{\xi}(t)dt + r(1-\rho)dW(t) \\ \hat{\xi}(0) &= \hat{\eta}(0) = 0 \end{aligned} \tag{2.72}$$

where $W(t)$ *is a standard Brownian motion.*

Proof. This proof is similar to that of Theorem 2.4.1. However, since some of the parameters are sent to infinity with N, we need to use the details of the upper bound for the propagation of chaos (see (2.7) and (2.8)).

We use here the standard scaling invariance of Poisson random measures in the following form. Let $N(dt, du)$ be a Poisson random measure of intensity $dt\,du$, and let $\Lambda : [0, +\infty) \to [0, +\infty)$ be right continuous. Then for $a > 0$

$$\int_{[0,at]\times\mathbb{R}^+} \mathbb{1}_{[0,\Lambda(s^-)]} N(ds, du) = \int_{[0,t]} \mathbb{1}_{[0,a\Lambda(as^-)]} \hat{N}(ds, du),$$

where $\hat{N}$ has the same distribution as N. This is checked by defining (interpreting point random measures as random sets)

$$\hat{N} := \{(t, u) : (t/a, au) \in N\}.$$

It is elementary to see that $\hat{N}$ is a Poisson random measure of intensity $dt\,du$.

We use below this property with $a = \frac{1}{\sqrt{\lambda_N}}$, and we will omit the superscript ˆ on the rescaled Poisson processes.

Using (2.9) and (2.26) we have

$$\begin{aligned} \hat{\xi}^N(t) &= \hat{\xi}^N(0) + \int_0^t d\hat{\xi}^N(s) = \hat{\xi}^N(0) + \frac{\sqrt{N}}{\lambda_N^{1/4}} \frac{1}{N} \sum_{i=1}^N \int_0^t d\sigma_i\left(\frac{s}{\sqrt{\lambda_N}}\right) \\ &= \hat{\xi}^N(0) + \frac{\sqrt{N}}{\lambda_N^{1/4}} \frac{1}{N} \sum_{i=1}^N \left\{ \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\left(1-\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right)\lambda_N \nu_N\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right]}(u) N_i^{\uparrow}(ds, du) \right. \end{aligned}$$

$$
\begin{aligned}
&- \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)r\right]}(u) N_i^{\downarrow}(ds, du) \Bigg\} \\
&= \hat{\xi}^N(0) + \frac{\sqrt{N}}{\lambda_N^{1/4}} \int_0^t \frac{1}{\sqrt{\lambda_N}} \left[\lambda_N \left(1 - m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right) v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) - r m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right] ds \\
&+ \frac{\sqrt{N}}{\lambda_N^{1/4}} \frac{1}{N} \sum_{i=1}^N \Bigg\{ \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\left(1-\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right)\lambda_N v_N\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right]}(u) \tilde{N}_i^{\uparrow}(ds, du) \\
&- \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)r\right]}(u) \tilde{N}_i^{\downarrow}(ds, du) \Bigg\} \\
&=: \hat{\xi}^N(0) + \hat{B}_1^N(t) + \hat{M}_1^N(t),
\end{aligned}
$$

where, recalling (2.71),

$$
\begin{aligned}
\hat{B}_1^N(t) &:= \frac{\sqrt{N}}{\lambda_N^{1/4}} \int_0^t \frac{1}{\sqrt{\lambda_N}} \left[\lambda_N \left(1 - m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right) v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) - r m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right] ds \\
&= \frac{\sqrt{N}}{\lambda_N^{1/4}} \int_0^t \frac{1}{\sqrt{\lambda_N}} \Bigg[\lambda_N \left(1 - m^* - \frac{\lambda_N^{1/4}}{\sqrt{N}} \hat{\xi}^N(s)\right)\left(v^* + \frac{1}{\sqrt{N}\lambda_N^{1/4}} \hat{\eta}^N(s)\right) \\
&\quad - r\left(m^* + \frac{\lambda_N^{1/4}}{\sqrt{N}} \hat{\xi}^N(s)\right)\Bigg] ds \qquad (2.73) \\
&\overset{(i)}{=} \int_0^t \tfrac{1}{\sqrt{\lambda_N}} \left[(-\lambda_N v^* - r)\hat{\xi}^N(s) + \left(\lambda_N(1-m^*)\tfrac{1}{\sqrt{\lambda_N}}\right)\hat{\eta}^N(s) - \tfrac{\lambda_N}{\sqrt{N}\lambda_N^{1/4}} \hat{\xi}^N(s)\hat{\eta}^N(s)\right] ds \\
&\overset{(ii)}{=} \int_0^t \tfrac{1}{\sqrt{\lambda_N}} \left[-\left(\tfrac{r}{\rho} + o_N(1)\right)\hat{\xi}^N(s) + \sqrt{\lambda_N}\,(\rho + o_N(1))\,\hat{\eta}^N(s) - \tfrac{\lambda_N^{3/4}}{\sqrt{N}} \hat{\xi}^N(s)\hat{\eta}^N(s)\right] ds,
\end{aligned}
$$

where in (i) we employed Eq.s (2.37) and in (ii) we employed (2.38), and

$$
\begin{aligned}
\hat{M}_1^N(t) &:= \hat{\xi}^N(0) \\
&\quad + \frac{\sqrt{N}}{\lambda_N^{1/4}} \frac{1}{N} \sum_{i=1}^N \Bigg\{ \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\left(1-\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right)\lambda_N v_N\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right]}(u) \tilde{N}_i^{\uparrow}(ds, du) \qquad (2.74) \\
&\quad - \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)r\right]}(u) \tilde{N}_i^{\downarrow}(ds, du) \Bigg\}
\end{aligned}
$$

is a martingale with quadratic variation equal to

$$
\begin{aligned}
\langle \hat{M}_1^N \rangle_t &= \frac{1}{N\sqrt{\lambda_N}} \sum_{i=1}^N \int_0^t \frac{1}{\sqrt{\lambda_N}} \left[\lambda_N \left(1 - \sigma_i\left(\frac{s}{\sqrt{\lambda_N}}\right)\right) v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) + r\sigma_i\left(\frac{s}{\sqrt{\lambda_N}}\right)\right] ds \\
&= \frac{1}{\lambda_N} \int_0^t \left[\lambda_N \left(1 - m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right) v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) + r m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right] ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \Bigg[\frac{1}{\lambda_N} \left(\lambda_N (1-m^*) v^* + r m^*\right) + \frac{1}{\sqrt{N}\lambda_N^{3/4}} (-\lambda_N v^* + r) \hat{\xi}^N(s) \\
&+ \lambda_N (1-m^*) \frac{1}{\sqrt{N}\lambda_N^{5/4}} \hat{\eta}^N(s) - \frac{\sqrt{\lambda_N}}{N} \hat{\xi}^N(s) \hat{\eta}^N(s) \Bigg] ds \\
&= \int_0^t \Bigg[2r\left(1-\rho+o_N(1)\right) \frac{1}{\lambda_N} + \left(-\frac{r}{\rho} + 2r + o_N(1)\right) \frac{1}{\sqrt{N}\lambda_N^{3/4}} \hat{\xi}^N(s) \\
&+ \left(\rho + o_N(1)\right) \frac{1}{\sqrt{N}\lambda_N^{1/4}} \hat{\eta}^N(s) - \frac{\sqrt{\lambda_N}}{N} \hat{\xi}^N(s) \hat{\eta}^N(s) \Bigg] ds.
\end{aligned} \tag{2.75}
$$

In (2.73) and (2.75) we denoted by $o_N(1)$ any sequence of real numbers that goes to zero as $N \to +\infty$. In a similar way, we can write

$$
\hat{\eta}^N(t) = \hat{\eta}^N(0) + \hat{B}_2^N(t) + \hat{M}_2^N(t),
$$

where

$$
\begin{aligned}
\hat{B}_2^N(t) &:= \int_0^t \left[-\lambda_N v^* \hat{\xi}^N(s) - \frac{\lambda_N^{3/4}}{\sqrt{N}} \hat{\xi}^N(s) \hat{\eta}^N(s) \right] ds \\
&= \int_0^t \left[-r\left(\frac{1}{\rho} - 1 + o_N(1)\right) \hat{\xi}^N(s) - \frac{\lambda_N^{3/4}}{\sqrt{N}} \hat{\xi}^N(s) \hat{\eta}^N(s) \right] ds,
\end{aligned} \tag{2.76}
$$

$$
\begin{aligned}
\hat{M}_2^N(t) &:= \hat{\eta}^N(0) \\
&+ \sqrt{N}\lambda_N^{1/4} \frac{1}{N} \sum_{i=1}^N \Bigg\{ \int_{[0,t]\times[0,+\infty)} \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\left(1-\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right)\lambda_N v_N\left(\frac{s^-}{\sqrt{\lambda_N}}\right)\right]}(u) \tilde{N}_i^{\uparrow}(ds, du) \\
&- \int_{[0,t]\times[0,+\infty)} x_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right) \mathbb{1}_{\left[0, \frac{1}{\sqrt{\lambda_N}}\sigma_i\left(\frac{s^-}{\sqrt{\lambda_N}}\right) r\right]}(u) \tilde{N}_i^{\downarrow}(ds, du) \Bigg\}
\end{aligned} \tag{2.77}
$$

is the remaining martingale after compensation of the PRM's and has quadratic variation equal to

$$
\begin{aligned}
\langle \hat{M}_2^N \rangle_t &= \int_0^t \left[\lambda_N \left(1 - m_N\left(\frac{s}{\sqrt{\lambda_N}}\right)\right) v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) + r\frac{1}{N}\sum_{i=1}^N x_i^2\left(\frac{s}{\sqrt{\lambda_N}}\right) \right] ds \\
&= \int_0^t \Bigg[\left(\lambda_N (1-m^*) v^* + r\frac{1}{N}\sum_{i=1}^N x_i^2\left(\frac{s}{\sqrt{\lambda_N}}\right) \right) - \lambda_N v^* \frac{\lambda_N^{1/4}}{\sqrt{N}} \hat{\xi}^N(s) \\
&+ (1-m^*) \frac{\lambda_N^{3/4}}{\sqrt{N}} \hat{\eta}^N(s) - \frac{\lambda_N}{N} \hat{\xi}^N(s) \hat{\eta}^N(s) \Bigg] ds
\end{aligned}
$$

$$= \int_0^t \left[\left(r(1-\rho) + r\frac{1}{N}\sum_{i=1}^N x_i^2\left(\frac{s}{\sqrt{\lambda_N}}\right) + o_N(1) \right) - r\left(\frac{1}{\rho} - 1 + o_N(1)\right)\frac{\lambda_N^{1/4}}{\sqrt{N}}\hat{\xi}^N(s) \right.$$

$$\left. +(\rho + o_N(1))\frac{\lambda_N^{3/4}}{\sqrt{N}}\hat{\eta}^N(s) - \frac{\lambda_N}{N}\hat{\xi}^N(s)\hat{\eta}^N(s) \right] ds. \tag{2.78}$$

We can also compute the covariation of M_1^N and M_2^N:

$$\begin{aligned}
\langle \hat{M}_1^N, \hat{M}_2^N \rangle_t &= \sum_{i=1}^N \int_0^t \frac{1}{N\sqrt{\lambda_N}} \left[\left(1 - \sigma_i\left(\frac{s}{\sqrt{\lambda_N}}\right)\right)\lambda_N v_N\left(\frac{s}{\sqrt{\lambda_N}}\right) + r x_i\left(\frac{s}{\sqrt{\lambda_N}}\right) \right] ds \\
&= \frac{1}{\sqrt{\lambda_N}} \int_0^t \left[\left(1 - m^* - \frac{\lambda_N^{1/4}}{\sqrt{N}}\hat{\xi}^N(s)\right)\lambda_N\left(v^* + \frac{1}{\sqrt{N}\lambda_N^{1/4}}\hat{\eta}^N(s)\right) \right. \\
&\quad \left. + r\left(v^* + \frac{1}{\sqrt{N}\lambda_N^{1/4}}\hat{\eta}^N(s)\right) \right] ds \\
&= \frac{1}{\sqrt{\lambda_N}} \int_0^t \left[r(1 - \rho + o_N(1)) + \frac{r^2}{\lambda_N}\left(\frac{1}{\rho} - 1 + o_N(1)\right) \right. \\
&\quad - r\left(\frac{1}{\rho} - 1 + o_N(1)\right)\frac{\lambda_N^{1/4}}{\sqrt{N}}\hat{\xi}^N(s) \\
&\quad \left. + (\lambda_N(\rho + o_N(1)) + r)\frac{1}{\sqrt{N}\lambda_N^{1/4}}\hat{\eta}^N(s) - \frac{1}{N}\hat{\xi}^N(s)\hat{\eta}^N(s) \right] ds.
\end{aligned} \tag{2.79}$$

We can therefore apply Theorem 2.4.2 with

$$b(x,t) = b(\xi,\eta) := \begin{bmatrix} \rho\eta \\ -\frac{r}{\rho}(1-\rho)\xi \end{bmatrix}$$

$$a(x,t) := \begin{bmatrix} 0 & 0 \\ 0 & r(1-\rho) \end{bmatrix}.$$

Using the localization by the stopping time τ_h^N defined in Theorem 2.4.2, all conditions required by Theorem 2.4.2 are readily checked except for the convergence of $\langle \hat{M}_2^N \rangle_t$, where the key point is to show that for $\varepsilon > 0$

$$\lim_{N\to+\infty} \mathbb{P}\left(\sup_{0 \le t \le \frac{T}{\sqrt{\lambda_N}}} \frac{1}{N}\sum_{i=1}^N x_i^2(t) > \varepsilon \right) = 0, \tag{2.80}$$

which in turn follows from

$$\lim_{N\to+\infty} \mathbb{E}\left[\sup_{0\le t\le \frac{T}{\sqrt{\lambda_N}}} \frac{1}{N}\sum_{i=1}^{N} x_i^2(t)\right] = 0. \tag{2.81}$$

By using (2.34) and observing that

$$\mathbb{E}[\bar{x}^2(t)] \le \mathbb{E}[\bar{x}(t)] = v^* = \frac{r}{r+\alpha_N}\left(1 - \frac{r+\alpha_N}{\lambda_N}\right) \to 0$$

as $N \to +\infty$, to prove (2.80) it is enough to show that if C_T is the time-dependent constant in (2.22), we have

$$\lim_{N\to+\infty} \frac{C_{T/\sqrt{\lambda_N}}}{\sqrt{N}} = 0. \tag{2.82}$$

Assuming

$$\lim_{N\to+\infty} \lambda_N = +\infty,$$

it is easily checked that the dominant term in $\frac{C_{T/\sqrt{\lambda_N}}}{\sqrt{N}}$ is of order

$$\frac{\sqrt{\lambda_N}}{\sqrt{N}} \exp\left[KT\sqrt{\lambda_N}\right] = \exp\left[KT\sqrt{\lambda_N} + \frac{1}{2}\log\lambda_N - \frac{1}{2}\log N\right],$$

for a suitable constant K, which goes to zero provided

$$\lim_{N\to+\infty} \frac{\sqrt{\lambda_N}}{\log N} = 0.$$

This completes the proof.

□

As the simulation in Figure 2.3 shows, the Brownian noise has little effect on the frequency of the oscillations, which is $\sqrt{r(1-\rho)}$ for the deterministic system, but substantially affects the amplitudes.

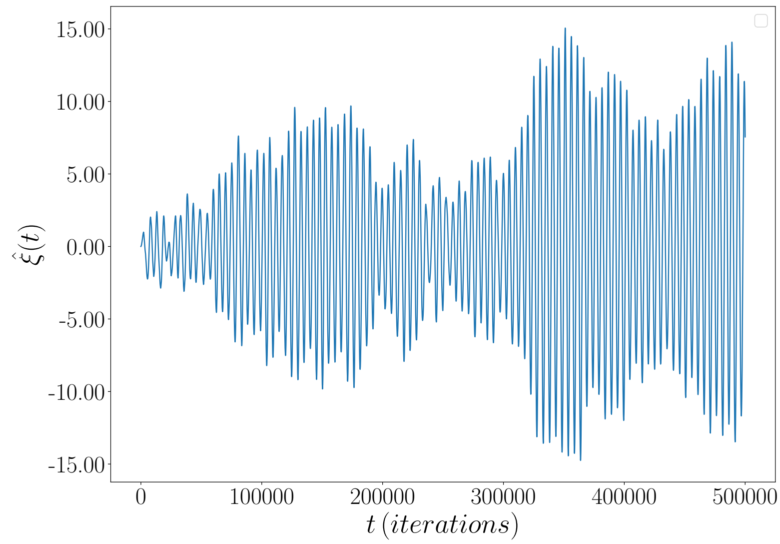


Figure 2.3: Sample trajectory of $\hat{\xi}(t)$ (system (2.72)) at stationarity.

Chapter 3

Strong propagation of chaos for a system of interacting particles with nearly stable jumps

In this chapter, based on [50], we consider a system of N interacting particles, described by SDEs driven by Poisson random measures, where the coefficients depend on the empirical measure of the system. Every particle jumps with a jump rate depending on its position. When this happens, all the other particles of the system simultaneously receive a small random kick which is distributed according to a heavy-tailed random variable belonging to the domain of attraction of an α-stable law and scaled by $N^{-1/\alpha}$, where $0 < \alpha < 2$. Moreover, in case $0 < \alpha < 1$, the jumping particle itself undergoes a macroscopic jump.

The particular scaling of the simultaneous jumps causes a common stochastic term to survive in the SDEs of the infinite-particle limit system. Conditionally to this common noise, particles in the limit system are independent and identically distributed. Thus the system considered in this chapter exhibits the conditional propagation of chaos property.

More precisely, the limit system turns out to be solution of a McKean-Vlasov SDE, driven by an α-stable process. We prove strong unique existence of the limit system and, by means of a suitable coupling, we show convergence of the finite to the limit system at the level of finite time marginals and with respect to a convenient distance on the state-space of the processes, also providing an explicit convergence rate.

3.1 Introduction

In this chapter we study the large population limit of the Markov process $X^N = (X_t^N)_{t\geq 0}$, $X_t^N = (X_t^{N,1}, \ldots, X_t^{N,N})$, which takes values in $\mathbb{R}^N$ and has generator A^N given by

$$A^N\varphi(x) = \sum_{i=1}^N \partial_{x^i}\varphi(x) b(x^i, \mu^{N,x}) + \sum_{i=1}^N f(x^i) \int_{\mathbb{R}} \nu(du) \left(\varphi\left(x + \psi(x^i, \mu^{N,x}) e_i + \sum_{j\neq i} \frac{u}{N^{1/\alpha}} e_j \right) - \varphi(x) \right), \quad (3.1)$$

for any smooth test function $\varphi : \mathbb{R}^N \to \mathbb{R}$. In the above formula, $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$, $\mu^{N,x} = \frac{1}{N}\sum_{j=1}^N \delta_{x^j}$ is the associated empirical measure, and e_j denotes the j−th unit vector in $\mathbb{R}^N$. Moreover, $b(x^i, \mu^{N,x})$ is a bounded drift function depending both of the position x^i of a fixed particle and of the empirical measure $\mu^{N,x}$ of the total system, $f : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz continuous bounded rate function, and ν is the law of a heavy-tailed random variable belonging to the domain of attraction of an α-stable law (see Definition 3.2.1 below); ν is supposed to be centered if $\alpha \in (1, 2)$. Since f is bounded, X^N is a piecewise deterministic Markov process. In between jumps, any particle $X^{N,i}$ follows a deterministic flow having drift b. Each particle jumps at rate $f(x^i)$ whenever its current position is x^i. When jumping, it receives an additional kick $\psi(x^i, \mu^{N,x})$ which is added to its position. Moreover, at the same time, all the other particles in the system receive the same *collateral jump* ([6]). This collateral jump is random, it has law ν, and it is renormalized by $N^{1/\alpha}$, where N is the system size. Therefore in our system there is coexistence of main jumps - the jumps of size $\psi(x^i, \mu^{N,x})$ - and of random and common small kicks that are received synchronously by all the other particles. Such systems have originally been introduced to model large biological neural nets such as the brain where the collateral jumps correspond to the synaptic weight of a neuron on its postsynaptic partner and the main jumps to the hyperpolarization of a neuron after a spike.

In [29] and [30], the authors have already studied the thermodynamic limit of such systems in a diffusive scaling, that is, when $\alpha = 2$ and ν is a centered probability measure on $\mathbb{R}$ having a second moment. In this case, the central limit theorem implies that the large population limit of the system having generator (3.1) is given by an infinite exchangeable system evolving according to

$$X_t^i = X_0^i + \int_0^t b(X_s^i, \mu_s) ds + \int_0^t \psi(X_{s-}^i, \mu_s) dZ_s^i + \sigma \int_0^t \sqrt{\mu_s(f)} dW_s, \; t \geq 0, i \in \mathbb{N}, \quad (3.2)$$

where $\sigma^2 = \int_{\mathbb{R}} u^2 \nu(du)$ and where W is a standard one-dimensional Brownian motion which is common to all particles. In (3.2) above, Z^i is the counting process associated to the jumps of particle i, having intensity $t \mapsto f(X^i_{t^-})$.

The presence of the Brownian motion W is a source of common noise in the limit system and implies that the *conditional propagation of chaos* property holds: in the limit system, particles are conditionally independent, if we condition on W. In particular, in [29] the authors have shown that the limit empirical measure μ_s is the directing measure of the infinite limit system (see Def. (2.6) in [2] for precise definition), which in turn is necessarily given by $\mu_s = \mathcal{L}(X^i_s | W_u, u \leq s)$, such that the stochastic integral term appearing in (3.2) is given by

$$\int_0^t \sqrt{\mathbb{E}(f(X^i_s)|W_u, u \leq s)} dW_s.$$

It is a natural question to ask what happens in the situation when ν does not belong to the domain of attraction of a normal law but of a stable law of index $\alpha < 2$. In this chapter we give an answer to this question. Not surprisingly, the limit Brownian motion will be replaced by a stable process S^α of index α such that the limit equation is now given by

$$\bar{X}^i_t = \bar{X}^i_0 + \int_0^t b(\bar{X}^i_s, \bar{\mu}_s) ds + \int_0^t \psi(\bar{X}^i_{s^-}, \bar{\mu}_{s^-}) d\bar{Z}^i_s + \int_0^t (\bar{\mu}_{s^-}(f))^{1/\alpha} dS^\alpha_s, \ t \geq 0, i \in \mathbb{N}, \quad (3.3)$$

with $\bar{\mu}_s = \mathcal{L}(\bar{X}^i_s | S^\alpha_u, u \leq s)$. In the case $1 < \alpha < 2$, we have to exclude main jumps; that is, we have to suppose that $\psi(\cdot) \equiv 0$. This is due to the fact that the stochastic integral with respect to $\bar{Z}^i$ has to be treated in the L^1- norm which is a norm not suited for the integral with respect to S^α, see Remark 3.3.1 below.

The present chapter establishes the proof of the strong convergence of the finite system to the limit system, with respect to a convenient distance. This is done by proposing a coupling of the finite system with the limit one. More precisely we will construct a particular version $S^{N,\alpha}$ of the stable process which is defined on an extension of the same probability space on which the finite system X^N is defined, and then we consider the limit system driven by $S^{N,\alpha}$. So we have to ensure first the existence of a unique strong solution of (3.3). This is relatively straightforward, when considering the system before the first big jump of the driving stable process, bigger than K for some fixed large K. This strategy is inspired by the study of classical SDEs driven by Lévy noise proposed by [33], it has been used in the framework of non-conditional McKean-Vlasov equations by [11], and it can be very easily extended to the present framework of conditional

McKean-Vlasov equations.

In a second step, we then prove the strong convergence of the finite system to the limit system. The main ingredient of this step is an explicit construction of the stable process $S^{N,\alpha}$ based on the random nearly stable heights of the collateral jumps present in the finite particle system. We discretize time and freeze the jump rate during small time intervals of length δ. Let us suppose for simplicity that ν is already the law of a strictly stable random variable (this assumption will not be needed in the sequel). Then the total contribution of collateral jumps during one such interval is a random sum, renormalized by $N^{1/\alpha}$, constituted of independent stable random variables, each representing one collateral jump. The total number of terms in the sum is Poisson distributed (conditionally). Thus we are able to use the self-similarity property of the $\alpha-$stable law: we know that, if Y_n are i.i.d. strictly $\alpha-$stable random variables, then

$$Y_1 + \ldots + Y_n \sim n^{1/\alpha} Y_1 [1].$$

The following trivial but very useful result says that this property survives for random sums, and it is the main argument of our coupling construction.

Proposition 3.1.1. *Let Y_n be i.i.d. strictly $\alpha-$stable random variables. Let P be an integer valued random variable, independent of $(Y_n)_n$. Then the following equality holds.*

$$\sum_{n=1}^{P} Y_n = P^{1/\alpha}\tilde{Y}_1, \quad \tilde{Y}_1 \sim Y_1,$$

where P and $\tilde{Y}_1$ are independent.

We apply this result on each time interval $[k\delta, (k+1)\delta[$, $k \geq 0$, such that the total number of jumps during this time interval, P, follows a Poisson distribution with parameter $\delta \sum_{i=1}^N f(x^i)$, conditionally to $\mathcal{F}_{k\delta}$. Here, $f(x^i)$ is the frozen jump rate of particle i at time $k\delta$. We then use the law of large numbers for the Poisson random variable P to replace P by its intensity $\delta \sum_{i=1}^N f(x^i) = N\delta \int f d\mu^{N,x}$ (where $\mu^{N,x}$ is the empirical measure $N^{-1}\sum \delta_{x^i}$), so that $N^{-1/\alpha}P^{1/\alpha} \sim (\delta \int f d\mu^{N,x})^{1/\alpha}$. So the contribution of collateral jumps, over one time interval, is approximately given by $(\int f d\mu^{N,x})^{1/\alpha}\delta^{1/\alpha}\tilde{Y}_1$, and since $\delta^{1/\alpha}\tilde{Y}_1 \sim S^\alpha_\delta$, it is then reasonable to expect that, as $\delta \to 0$, the joint contribution of all small intervals gives rise to the stochastic integral term $\int_0^t (\bar{\mu}_{s-}(f))^{1/\alpha} dS^\alpha_s$ that appears in

[1] Since we need this exact self-similarity property of the $\alpha-$stable law we are not able to truncate the stable random variables Y_n as it is often done in the literature.

(3.3) above, and this is precisely the strategy of our proof, up to technical details.

The main result of this chapter, Theorem 2.2.1, then shows that for all $t > 0$, for all $i = 1, \ldots, N$, for all truncation levels $K > 0$, we have a precise control on

$$\mathbb{E}[\mathbb{1}_{\{t<T_K^N\}}|X_t^{N,i} - \bar{X}_t^i| \wedge |X_t^{N,i} - \bar{X}_t^i|^{\alpha_- \wedge 1}]$$

in terms of α, N, t and of the truncation level K (see Theorem 2.2.1 below for the precise form of this control). Here, we fix some $0 < \alpha_- < \alpha$ ($\alpha_- > 1$ if $\alpha > 1$), and $T_K^N = \inf\{t > 0 : |\Delta S_t^{N,\alpha}| > K\}$, $\Delta S_t^{N,\alpha} := S_t^{N,\alpha} - S_{t^-}^{N,\alpha}$, is the time of the first jump of $S^{N,\alpha}$ exceeding K.

Comments on the norms we have used

Our error bounds contain three main contributions, and each of these contributions has to be treated using a different norm. The final strong error is then obtained by balancing all these error terms.

1. The error due to the stable central limit theorem is treated using the L^1-norm in case $\alpha > 1$, and the norm induced by $|\cdot| \wedge |\cdot|^{\alpha_-}$ (see (3.4)), with $0 < \alpha_- < \alpha$, in case $\alpha < 1$.

2. We need to study stochastic integral terms of the type $\int_0^t (\bar{\mu}_{s-}^N(f))^{1/\alpha} dS_s^{N,\alpha}$ ($\bar{\mu}^N$ being the empirical measure of the limit system (3.3)), $\int_0^t (\bar{\mu}_{s-}(f))^{1/\alpha} dS_s^{N,\alpha}$ and their convergence. Since small and big jumps of $S^{N,\alpha}$ are not integrable in the same norm, as usual, we cut big jumps and work on the event $\{t < T_K^N\}$, for some fixed K. To control the dependency on K in the case $\alpha > 1$ we then work in the $L^{\alpha_+}-$norm, $\alpha_+ > \alpha$. For technical reasons, this trick does not work in the case $\alpha < 1$, such that we work with the L^1-norm then.

3. Finally, most error terms that do only concern the finite particle system (such as time discretization errors) have to be controlled in the $L^{\alpha_-}-$norm, with $0 < \alpha_- < \alpha$ and $\alpha_- > 1$ for $\alpha > 1$, since $X_t^{N,i} \in L^{\alpha_-}$ does not belong to L^α.

Bibliographical comments

The property of *conditional propagation of chaos* is related to the existence of common noise in the limit system and has been a lot studied in the literature; see for instance [9], [16] and [26]. In these papers the common noise, which is most often a common, maybe infinite dimensional, Brownian motion, is already present at the level of the finite particle system, the mean field interactions act on the drift of each particle, and the scaling is the classical one in N^{-1}. On the contrary to this, in our model, the common noise is only present in the limit, and it is created by the $\alpha-$stable limit theorem as a consequence of the joint action of the collateral jumps of the finite size particle system and of the *scaling* in $N^{-1/\alpha}$.

In the classical setting of *unconditional propagation of chaos*, infinite-volume limits for mean-field particle systems driven by general Lévy noise have also been extensively studied. We refer the interested reader to the paper [42], where the author considers equations driven by (possibly compensated) Poisson random measures. He works under Lipschitz and integrability conditions with the L^1-norm - which is not possible when the driving noise is an $\alpha-$stable process with $\alpha < 1$. [45] have worked with general Lévy-noise, but mostly under L^2-conditions. They consider also the case when the driving process does only possess moments of order $\alpha < 1$, in which situation they are only able to obtain weak existence of the limit without proving uniqueness in law. In the recent paper [10], the author obtains a quantitative propagation of chaos result for systems driven by $\alpha-$stable subordinators, in the case $\alpha \in]1, 2[$. There is no measure dependent term within the stochastic integral term in [10], and the author is mostly interested in relaxing the regularity assumptions on the coefficients and works only under the assumption of Hölder continuity. Finally, several papers are devoted to the well-posedness of the limit equation. Let us mention [36] which treats the general case $0 < \alpha < 2$, under mild regularity assumptions, by means of an associated non-linear martingale problem. Finally, [11] treats the case $1 < \alpha < 2$, including a measure dependent term in the stochastic integral term, under general Lipschitz assumptions. All these papers are devoted to the unconditional framework.

Let us come back to this chapter of the thesis, which deals with the conditional propagation of chaos property in a situation where the driving Lévy process appears only in the limit system. We have already mentioned that it continues and extends the diffusive setting studied in [29] and [30] to the framework of the $\alpha-$ stable limit theorem. As in [29] and [30], the basic strategy is to discretize time and construct a coupling with the driving noise of the limit equation explicitly within each time interval $[k\delta, (k + 1)\delta)$. This strategy is actually inspired by the approach proposed in [25].

We now quickly discuss the main differences with respect to the former diffusive setting. In the bounded variation regime $0 < \alpha < 1$, the error due to time discretization is now of order δ, as opposed to $\sqrt{\delta}$ which was the leading order in the diffusive scaling. Contrarily to the diffusive case, the main contribution to the error comes now from the error made when replacing the Poisson variable P by its expectation and from the quantified error in the stable limit theorem. The first error gives rise to a term of order $(N\delta)^{-\alpha/2}$, and this is a main contribution to the error. In the diffusive case this error was negligible. Concerning the quantified rate of convergence in the stable limit theorem, we suppose that the law of the collateral jumps is heavy-tailed and rely on recent results of [15]. Finally, there is the error which is due to the bound on the Wasserstein α_-−distance between the empirical measure $\bar{\mu}_s^N$ of the limit system and $\bar{\mu}_s$. We rely on results obtained by [34] to control this error.

In case $1 < \alpha < 2$, to control the error is much more complicated than in the original diffusive case. Indeed, the presence of the stochastic integral imposes that we have to deal with the small jumps of the driving stable process with respect to at least the L^{α_+}−norm. And this is what we do. However, all errors related to the finite particle system can only be treated with respect to the L^{α_-}−norm. So we use Hölder's inequality repeatedly - and each time we do this, we loose with respect to the original convergence rate. The error due to time discretization is now of order δ^{1/α_-}, which gives another important contribution. Concerning the quantified rate of convergence in the stable limit theorem, we rely on recent results of [14].

Last, we mention here the work [51], which studies the case $0 < \alpha < 1$, in a particular framework where all jumps are positive and where big jumps do not need to be cut.

General notation

Throughout this chapter we shall use the following notation. Given any measurable space $(S, \mathcal{S})$, $\mathcal{P}(S)$ denotes the set of all probability measures on $(S, \mathcal{S})$, endowed with the topology of weak convergence. For $p > 0$, $\mathcal{P}_p(\mathbb{R})$ denotes the set of probability measures on $\mathbb{R}$ that have a finite moment of order p. For two probability measures $\nu_1, \nu_2 \in \mathcal{P}_p(\mathbb{R})$, the Wasserstein distance of order p between ν_1 and ν_2 is defined as

$$W_p(\nu_1, \nu_2) = \inf_{\pi \in \Pi(\nu_1, \nu_2)} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^p \pi(dx, dy) \right)^{(1/p) \wedge 1},$$

where π varies over the set $\Pi(\nu_1, \nu_2)$ of all probability measures on the product space $\mathbb{R} \times \mathbb{R}$ with marginals ν_1 and ν_2. Notice that the Wasserstein distance of order p between ν_1 and ν_2 can be rewritten as the infimum of $(\mathbb{E}[|X - Y|^p])^{(1/p) \wedge 1}$ over all possible couplings (X, Y) of the random elements X and Y distributed according to ν_1 and ν_2 respectively, i.e.

$$W_p(\nu_1,\nu_2) = \inf\left\{(\mathbb{E}|X-Y|^p)^{(1/p)\wedge 1} : \mathcal{L}(X)=\nu_1 \text{ and } \mathcal{L}(Y)=\nu_2\right\}.$$

Moreover, the Kantorovitch-Rubinstein duality yields

$$W_1(\nu_1,\nu_2) = \sup\{\nu_1(\varphi)-\nu_2(\varphi) : \forall\, x,\ y\in\mathbb{R}\ |\varphi(x)-\varphi(y)|\le L|x-y| \text{ with } L\le 1\}.$$

We will also use the following notation. For any $q \le 1$,

$$\begin{aligned} &d_q(x,y) := |x-y|\wedge|x-y|^q, \\ &||x||_{d_q} := |x|\wedge|x|^q \end{aligned} \tag{3.4}$$

and, for any $\nu_1,\ \nu_2 \in \mathcal{P}_1(\mathbb{R})$,

$$W_{d_q}(\nu_1,\nu_2) = \inf_{\pi\in\Pi(\mu_1,\nu_2)}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} d_q(x,y)\pi(dx,dy)\right)$$

is the Wasserstein distance associated with the metric d_q (see [15]). As for the classical Wasserstein distance, the Kantorovitch-Rubinstein duality yields ([68], particular case 5.16)

$$W_{d_q}(\nu_1,\nu_2) = \sup\{\nu_1(\varphi)-\nu_2(\varphi) : \forall x,\ y\in\mathbb{R}\ |\varphi(x)-\varphi(y)|\le d_q(x,y)\}.$$

Notice that $W_{d_q}(\nu_1,\nu_2) \le W_1(\nu_1,\nu_2)$ for all $\nu_1,\ \nu_2 \in \mathcal{P}_1(\mathbb{R})$.

Moreover, $D(\mathbb{R}_+,\mathbb{R}_+)$ denotes the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, endowed with the Skorokhod metric, and C and K denote arbitrary positive constants whose values can change from line to line in an equation. We write C_θ and K_θ if the constants depend on some parameter θ.

Finally, $\alpha_- < \alpha$ and $\alpha_+ > \alpha$ will be two fixed constants belonging to $(0,2)$, one strictly smaller and the other strictly larger than α, the index of the driving stable process. We also suppose that $\alpha_- > 1$ in case $\alpha > 1$.

3.2 Model, assumptions, main results, organization of the chapter

3.2.1 The model

Throughout this chapter, $S^\alpha = (S^\alpha_t)_{t\geq 0}$ denotes an $\alpha-$stable Lévy process given by ([8], [60])

$$S^\alpha_t = \int_{[0,t]\times\mathbb{R}} z\tilde{M}(ds,dz), \text{ if } \alpha > 1 \tag{3.5}$$
$$S^\alpha_t = \int_{[0,t]\times\mathbb{R}} zM(ds,dz), \text{ if } \alpha < 1.$$

Its jump measure M is a Poisson random measure (PRM) on $\mathbb{R}_+ \times \mathbb{R}_*$ having intensity $ds\nu^\alpha(dz)$, with

$$\nu^\alpha(dz) = \frac{a_+}{z^{\alpha+1}}\mathbb{1}_{\{z>0\}}dz + \frac{a_-}{|z|^{\alpha+1}}\mathbb{1}_{\{z<0\}}dz,$$

where $a_+, a_- \geq 0$ are some fixed parameters, and $\tilde{M}(ds,dz) := M(ds,dz) - \nu^\alpha(dz)ds$ denotes the compensated PRM.

In what follows we will consider random variables which are distributed according to a heavy-tailed law which belongs to the domain of attraction of a stable law, according to the following definition.

Definition 3.2.1. Following Example 2 in [14] and [15], we say that a law is heavy-tailed with indices α, γ, β, A and $\tilde{A}$, with $0 \leq \alpha < 2, \alpha \neq 1, \gamma > 0$, if its distribution function G has the form

$$\begin{aligned} 1 - G(x) &= \frac{A}{|x|^\alpha}(1+\beta) + \frac{\tilde{A}}{|x|^{\alpha+\gamma}}(1+\beta), \qquad && x \geq K, \\ G(x) &= \frac{A}{|x|^\alpha}(1-\beta) + \frac{\tilde{A}}{|x|^{\alpha+\gamma}}(1-\beta), \qquad && x \leq -K, \end{aligned}$$

for some $K > 0$, where $\beta \in [-1,1]$ encodes the asymmetry in the distribution, $A, \tilde{A} > 0$ are such that $|K|^{-\alpha}(A + |K|^{-\gamma}\tilde{A}) \leq \frac{1}{2}$, and $\gamma > 0$. In particular, such law belongs to the domain of attraction of an α-stable law (see [32], IX.8, Theorem 1).

Remark. Since a law satisfying Definition 3.2.1 belongs to the domain of attraction of an α-stable law, Theorem 3 in [40], Part III, Chapter 7, Section 35 assures that its q-th absolute moments are finite for $q < \alpha$. This will be often employed in the sequel.

After these preliminary definitions, we now introduce our finite particle system. To do so, let $(\pi^i(ds,dz,du))_{i\geq 1}$ be a family of i.i.d. Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ having intensity measure $dsdz\nu(du)$, where, in the case $\alpha < 1$, ν satisfies Definition 3.2.1

with parameters $\alpha, \gamma, \beta, A, \tilde{A}$, and, in the case $\alpha > 1$, ν is the law that one obtains after centering a law whose distribution function satisfies Definition 3.2.1 with parameters $\alpha, \gamma, \beta, A, \tilde{A}$. Consider also an i.i.d. family $(X_0^i)_{i\geq 1}$ of $\mathbb{R}$-valued random variables independent of the Poisson measures, distributed according to some fixed probability measure ν_0 on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In what follows we write $(\Omega, \mathcal{A}, \mathbf{P})$ for the basic probability space on which are defined all π^i and all X_0^i, and we use the associated canonical filtration

$$\mathcal{F}_t = \sigma\{\pi^i([0,s] \times A \times B), s \leq t, A \in \mathcal{B}(\mathbb{R}_+), B \in \mathcal{B}(\mathbb{R}), i \geq 1\} \vee \sigma\{X_0^i, i \geq 1\},\ t \geq 0.$$

We will also use the projected Poisson random measures which are defined by

$$\bar{\pi}^i(ds, dz) = \pi^i(ds, dz, \mathbb{R}),$$

having intensity $dsdz$. For any $N \in \mathbb{N}$, we consider a system of interacting particles $(X_t^{N,i}), t \geq 0,\ 1 \leq i \leq N$, evolving according to

$$X_t^{N,i} = X_0^i + \int_0^t b(X_s^{N,i}, \mu_s^N) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(X_{s^-}^{N,i}, \mu_{s^-}^N) \mathbb{1}_{\{z \leq f(X_{s^-}^{N,i})\}} \bar{\pi}^i(ds, dz) + \frac{1}{N^{1/\alpha}} \sum_{j\neq i} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{s^-}^{N,j})\}} \pi^j(ds, dz, du), \quad (3.6)$$

where $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ is the empirical measure of the system at time t. In the above equation, $b : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$, $\psi : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ are measurable and bounded functions. In case $\alpha > 1$, we always suppose that $\psi(\cdot) \equiv 0$, that is, there are no main jumps.

Remark. Notice that, since for each i $\pi^i([0,t] \times [0, \|f\|_\infty] \times \mathbb{R})$ is Poisson distributed with parameter $\|f\|_\infty t \times \nu(\mathbb{R}) = \|f\|_\infty t$, the number of atoms of the measures π^i's in $[0,t] \times [0, \|f\|_\infty] \times \mathbb{R}$ is a.s. finite. Hence the integral in the r.h.s. of (3.6) is in fact a sum with a.s. a finite number of terms.

In what follows we will provide additional conditions on the functions b, f and ψ that, together with our preliminary considerations, in particular Proposition 3.1.1, allow to show that, as $N \to \infty$, the above particle system converges (in law) to an infinite exchangeable system $(\bar{X}^i)_{i\geq 1}$ solving

$$\bar{X}_t^i = X_0^i + \int_0^t b(\bar{X}_s^i, \bar{\mu}_s) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}_{s^-}^i, \bar{\mu}_{s^-}) \mathbb{1}_{\{z \leq f(\bar{X}_{s^-}^i)\}} \bar{\pi}^i(ds, dz) + \int_{[0,t]} (\bar{\mu}_{s^-}(f))^{1/\alpha} dS_s^\alpha, \quad (3.7)$$

where S^α, given by (3.5), is independent of the collection of Poisson random measures $(\bar{\pi}^i(ds, dz))_{i\geq 1}$ and of the initial values $(X_0^i)_{i\geq 1}$, and where $\bar{\mu}_s = \mathcal{L}(\bar{X}_s^1 | S_u^\alpha, u \leq s)$.

The main part of this chapter is devoted to the proof of (a quantified version of) the convergence of the finite system (3.6) to the limit system (3.7).

But, before doing so, we briefly discuss strong existence and uniqueness of the particle system and its associated limit system.

3.2.2 Assumptions

To prove well-posedness of the particle and limit systems, we will only need the following assumptions:

Assumption 3.2.1. a) b is bounded.

b) There exists a constant $C > 0$ such that for every $x, y \in \mathbb{R}$ and every $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, it holds $|b(x,\mu) - b(y,\nu)| \leq C\left(|x-y| + W_1(\mu,\nu)\right)$.

Assumption 3.2.2. a) f is lowerbounded by some strictly positive constant $\underline{f} > 0$.

b) f is bounded.

c) f is Lipschitz-continuous.

Recall that we assumed that there are no main jumps in case $\alpha > 1$. In case $\alpha < 1$, to deal with the main jumps, we also suppose that

Assumption 3.2.3. a) ψ is bounded.

b) There exists a constant $C > 0$ such that for every $x, y \in \mathbb{R}$ and every $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, it holds $|\psi(x,\mu) - \psi(y,\nu)| \leq C\left(|x-y| + W_1(\mu,\nu)\right)$.

Assumption 3.2.4. ν_0 admits a finite first moment in case $\alpha < 1$ and a finite second moment in case $1 < \alpha < 2$.

To prove the convergence to the limit system we also need to assume:

Assumption 3.2.5. $f \in C^1$.

In case $\alpha < 1$, we have to strengthen the Lipschitz assumptions 3.2.1b) and 3.2.3b) and suppose additionally that

Assumption 3.2.6. There exists a constant $C > 0$ such that for some fixed $\alpha_- \in (0, \alpha)$, for every $x, y \in \mathbb{R}$ and every $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, $|b(x,\mu) - b(y,\nu)| + |\psi(x,\mu) - \psi(y,\nu)| \leq C\left(d_{\alpha_-}(x,y) + W_{d_{\alpha_-}}(\mu,\nu)\right)$.

Assumption 3.2.7. a) ν_0 admits a finite moment of order $(2\alpha) \vee 1$ if $\alpha < 1$ and a finite moment of order p for some $p > 2$ if $1 < \alpha < 2$.

b) ν is heavy-tailed according to Definition 3.2.1, for some $\alpha \in]0, 2] \setminus \{1\}$ and for some γ such that $\alpha + \gamma \notin \{1, 2\}$. Furthermore, ν is centered if $\alpha > 1$, that is, for $\alpha > 1$ ν is the distribution that one obtains after centering a law with distribution function G as in Definition 3.2.1.

Remark. In what follows we will repeatedly use that Assumption 3.2.2 implies that for any two probability measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ and for any $p \geq 1$,

$$|(\mu(f))^{1/\alpha} - (\nu(f))^{1/\alpha}|^p \leq C|\mu(f) - \nu(f)|^p \leq C|\mu(f) - \nu(f)| \leq CW_1(\mu, \nu), \tag{3.8}$$

where C is a constant that may change from one occurence to another. Here we used that $z \mapsto z^{1/\alpha}$ is Lipschitz on $[0, \|f\|_\infty]$ in case $\alpha < 1$ (on $[\underline{f}, \infty]$ in case $\alpha > 1$) together with the boundedness of f. In the case $\alpha < 1$, we will also use that, for any $\alpha_- < 1$,

$$|(\mu(f))^{1/\alpha} - (\nu(f))^{1/\alpha}|^p \leq C|\mu(f) - \nu(f)|^p \leq C|\mu(f) - \nu(f)| \leq CW_{d_{\alpha_-}}(\mu, \nu). \tag{3.9}$$

This is derived similarly to (3.8) and observing that the boundedness and Lipschitz continuity of f imply that there exists $C > 0$ such that for every $x, y \in \mathbb{R}$ it holds $|f(x) - f(y)| \leq Cd_{\alpha_-}(x, y)$, so we can employ the Kantorovich-Rubinstein duality on f/C to conclude.

3.2.3 Main results

Theorem 3.2.8. *Grant Assumptions 3.2.1b), 3.2.2b), 3.2.3 and 3.2.4. Then system* (3.6) *admits a unique strong solution.*

The proof of Theorem 3.2.8 is given in the Appendix section B.

Theorem 3.2.9. *Grant Assumptions 3.2.1, 3.2.2, 3.2.3 and 3.2.4. Then* (3.7) *admits a unique strong solution.*

The proof of Theorem 3.2.9 is given in Section 3.3.

We may now state our main theorem. Remember that $(\Omega, \mathcal{A}, \mathbf{P})$ denotes the space on which all the (π_i) and $X_0^{N,i}, i \in \mathbb{N}^*, N \in \mathbb{N}^*$, are defined. Let X^N be the unique strong solution of (3.6) driven by (π_i).

Theorem 3.2.10. *Grant Assumptions 3.2.1–3.2.7. For any $N \in \mathbb{N}^*, \delta \in (0, 1)$ such that $2\delta\|f\|_\infty < 1$, we can construct, on an extension of $(\Omega, \mathcal{A}, \mathbf{P})$, a one-dimensional α-stable* *process $S^{N,\alpha,\delta}$, independent of the initial positions $(X_0^{N,i})_{i=1,\ldots,N}$ and of $(\bar{\pi}^i)_{i\geq 1}$, such that the following holds.*

If $(\bar{X}_t^N)_t$ denotes the unique strong solution to (3.7) *driven by $S^{N,\alpha,\delta}$ and $(\bar{\pi}^i)_{i\geq 1}$, and writing $T_K^N := \inf\{t \geq 0 \,:\, |\Delta S_t^{N,\alpha,\delta}| > K\}$, we have for all $t \geq 0$, $i = 1,\ldots,N$ and $K > 0$,*

i) for any $1 < \alpha_- < \alpha < \alpha_+ < 2$,

$$\mathbb{E}[\mathbb{1}_{\{t<T_K^N\}}|X_t^{N,i} - \bar{X}_t^{N,i}|] \leq e^{Ct\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}} \left(\left(N^{1-\frac{\alpha_-}{\alpha}}\delta\right)^{\frac{1}{\alpha_+}} + r(N,\delta) + N^{-1/2}\right)^{1/\alpha_+}, \qquad (3.10)$$

where

$$r(N,\delta) := N^{\frac{1}{\alpha_-}\left(1+\frac{1}{\alpha_-}\right)\left(1-\frac{\alpha_-}{\alpha}\right)}\delta^{\frac{1}{(\alpha_-)^2}} + \lceil\frac{t}{\delta}\rceil\delta^{\frac{1}{\alpha}}\left(g(N\delta) + (N\delta)^{-\frac{1}{2}}\right) + \delta^{\frac{1}{\alpha}},$$

and where the function g is given in (3.23) *below;*

ii) for any $0 < \alpha_- < \alpha < 1$,

$$\mathbb{E}[\mathbb{1}_{\{t<T_K^N\}}d_{\alpha_-}(X_t^{N,i}, \bar{X}_t^{N,i})] \leq e^{Ct\frac{K^{1-\alpha}}{1-\alpha}} \times$$
$$\times\left(r(N,\delta) + N^{-1/2}\mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}}\right), \qquad (3.11)$$

where

$$r(N,\delta) := N^{2\left(1-\frac{\alpha_-}{\alpha}\right)}\delta + \lceil\frac{t}{\delta}\rceil\delta^{\frac{\alpha_-}{\alpha}}\left(g(N\delta) + (N\delta)^{-\frac{\alpha_-}{2}}\right) + \delta^{\frac{\alpha_-}{\alpha}},$$

and where the function g is given in (3.25) *below.*

In the following we will drop the dependence of $S^{N,\alpha,\delta}$ on δ for convenience of notation, as we will always take $\delta = \delta(N)$, so that $\delta \to 0$ and $\delta N \to +\infty$ as $N \to +\infty$.

Remark. The rate of convergence stated in the above theorem depends on our choice of $\alpha_- < \alpha$ and $\alpha_+ > \alpha$. To get an idea of the leading term in the error, taking formally $\alpha_- \uparrow \alpha$, $\alpha_+ \downarrow \alpha$, we obtain the following rate of convergence (see Appendix section E).

1. For $\alpha > 1$ and $\gamma + \alpha < 2$,

$$N^{-\frac{\gamma}{\alpha^2(1-\alpha+\gamma\alpha+\alpha^2)}}\mathbb{1}_{\gamma<\alpha/2} + N^{-\frac{1}{2\alpha\left(1-\alpha+\frac{3}{2}\alpha^2\right)}}\mathbb{1}_{\gamma\in\left(\frac{\alpha}{2},2-\alpha\right)}.$$

2. For $\alpha > 1$ and $\gamma + \alpha > 2$,

$$N^{-\frac{1}{2\alpha\left(1-\alpha+\frac{3}{2}\alpha^2\right)}}\mathbb{1}_{\alpha\in(1,\frac{4}{3})} + N^{-\frac{2-\alpha}{\alpha^2(1+\alpha)}}\mathbb{1}_{\alpha>\frac{4}{3}}.$$

3. For $\alpha \in (0, \sqrt{3}-1)$ and $\gamma < \frac{\alpha^2}{2}$ or for $\alpha \in (\sqrt{3}-1, 1)$ and $\gamma < 1-\alpha$,

$$N^{-\frac{\gamma}{\alpha+\gamma}}.$$

4. For $\alpha \in (0, \sqrt{3}-1)$ and $\gamma > \frac{\alpha^2}{2}$,

$$N^{-\frac{\alpha}{2+\alpha}}.$$

5. For $\alpha \in (\sqrt{3}-1, 1)$ and $\gamma > 1-\alpha$,

$$N^{\alpha-1}.$$

3.2.4 Organization of the chapter

The rest of the chapter is organized as follows. Section 3.3 is devoted to the proof of pathwise uniqueness for the limit system. The proofs of well-posedness of the particle system and strong existence for the limit one are postponed to Appendix sections B and C, since they are based on standard techniques.

In Section 3.4, we provide a representation of the interaction term in (3.6) in terms of a stochastic integral with respect to an α-stable process. This representation entails a time discretization of the particle system (see in particular Subsection 3.4.5) and relies on (a generalization of) the stable CLT and on previously obtained bounds in [14, 15] (see Prop. 3.4.2, where the errors due to both the CLT and the rates of [14, 15] are introduced). The overall error that we make by all these approximations will be controlled using the L^1-norm in the case $\alpha \in (1,2)$ and the norm induced by the distance d_{α_-} in the case $\alpha \in (0,1)$ (see R_t^N in the statement of Theorem 3.4.1). In Section 3.4, we will also provide an explicit construction of the limit process driving system (3.7) (see Subsection 3.4.3).

Section 3.5 concludes the convergence proof employing some auxiliary results and intermediate useful representations for the particle and the limit systems. In particular, in that section additional error terms to the ones collected in Section 3.4 appear due to the need to approximate the conditional law of the limit system by its empirical measure. These errors are controlled in L^{α_+}-norm for $\alpha > 1$ and in L^1-norm for $\alpha < 1$ thanks to the properties of boundedness and Lipschitz continuity of the functions f and b, and using results in [34].

3.3 Strong existence and uniqueness for the limit system

In this section, we prove the well-posedness for the limit system (3.7). We consider one typical particle $\bar{X}_t$ representing the limit system (3.7). It evolves according to

$$\bar{X}_t = X_0 + \int_0^t b(\bar{X}_s, \bar{\mu}_s) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(\bar{X}_{s^-}, \bar{\mu}_{s^-}) \mathbb{1}_{\{z \le f(\bar{X}_{s^-})\}} \bar{\pi}(ds, dz) + \int_{[0,t]} (\bar{\mu}_{s^-}(f))^{1/\alpha} dS_s^\alpha, \quad (3.12)$$

where $X_0 \sim \nu_0$, $\bar{\mu}_s = \mathcal{L}(\bar{X}_s | S_u^\alpha, u \le s)$, $\bar{\pi}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ having intensity $dsdz$, and where $S^\alpha, \bar{\pi}$ and X_0 are independent. We use the above representation both for $\alpha < 1$ and $\alpha > 1$, keeping in mind that in the latter case, main jumps are excluded from our study, that is, $\psi(\cdot) \equiv 0$. See Remark 3.3.1 below. We also use the associated canonical filtration

$$\bar{\mathcal{F}}_t := \sigma\{\bar{\pi}([0,s] \times A), s \le t, A \in \mathcal{B}(\mathbb{R}_+)\} \vee \sigma\{X_0\} \vee \sigma\{S_s^\alpha, s \le t\}, \ t \ge 0.$$

Remark. Let $p > 0$. Notice that despite the presence of the integral against the stable process S^α in (3.12) above, $\bar{\mu}_t$ admits a finite moment of order p for any $t \ge 0$, whenever X_0 does so. This follows from the boundedness of b, f and ψ, since

$$|\bar{X}_t| \le |X_0| + ||b||_\infty t + ||\psi||_\infty \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \le ||f||_\infty\}} \bar{\pi}(ds, dz) + \sup_{s \le t} \left| \int_{[0,s]} \bar{\mu}_{v^-}(f)^{1/\alpha} dS_v^\alpha \right|.$$

Since $t \mapsto \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \le ||f||_\infty\}} \bar{\pi}(ds, dz)$ is a Poisson process of rate $||f||_\infty$, possessing all moments, we deduce that

$$\int_{\mathbb{R}} |x|^p \bar{\mu}_t(dx) = \mathbb{E}[|\bar{X}_t|^p | S^\alpha] \le C_p(t) \left(\mathbb{E}|X_0|^p + 1 + \sup_{s \le t} \left| \int_{[0,s]} \bar{\mu}_{v^-}(f)^{1/\alpha} dS_v^\alpha \right|^p \right) < \infty \quad (3.13)$$

almost surely.

3.3.1 Pathwise uniqueness for the limit system

The case $\alpha > 1$

We start discussing the case $\alpha > 1$. Fix some $K > 0$. By the Lévy-Itô decomposition (see [8], Theorem 2.4.16), S^α admits the pathwise representation

$$S_t^\alpha = \int_{[0,t]\times B_K} z\tilde{M}(ds,dz) + \int_{[0,t]\times B_K^c} zM(ds,dz) - \int_{[0,t]\times B_K^c} z\nu^\alpha(dz)ds, \tag{3.14}$$

$t \geq 0$, where $B_K := \{z \in \mathbb{R} : |z| \leq K\}$ and where $\tilde{M}$ denotes the compensated jump measure.

For any $K > 0$ define

$$T_K := \inf\{t \geq 0 : |\Delta S_t^\alpha| > K\},$$

that is, the first time the process S_t^α has a jump greater than K, $\Delta S_t^\alpha := S_t^\alpha - S_{t^-}^\alpha$. Notice that for any finite T, $\lim_{K\to\infty} \mathbb{P}(T_K > T) = 1$ as $K \to +\infty$.

Consider now two solutions to the limit system (3.6), $X = (X_t)_{t\geq 0}$ and $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$, with the same initial condition $X_0 = \tilde{X}_0$, and denote by μ_t and $\tilde{\mu}_t$ the conditional laws of X_t and $\tilde{X}_t$ respectively given S^α. Observing that on $\{t < T_K\}$, the stochastic integral term corresponding to big jumps in (3.14) equals zero, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{1}_{\{t<T_K\}} |X_t - \tilde{X}_t|^2 \leq \\
&\leq C\Bigg[T\int_0^T \mathbb{1}_{\{s<T_K\}} |b(X_s,\mu_s) - b(\tilde{X}_s,\tilde{\mu}_s)|^2 ds + TM_K^2 \int_0^T \mathbb{1}_{\{s<T_K\}} |\mu_s^{1/\alpha}(f) - \tilde{\mu}_s^{1/\alpha}(f)|^2 ds \\
&\qquad + \sup_{t\in[0,T]} \left| \int_{[0,t]\times B_K} \mathbb{1}_{\{s\leq T_K\}} [\mu_{s^-}^{1/\alpha}(f) - \tilde{\mu}_{s^-}^{1/\alpha}(f)] z\tilde{M}(ds,dz) \right|^2 \Bigg],
\end{aligned}$$

where

$$M_K := \int_{B_K^c} z\nu^\alpha(dz). \tag{3.15}$$

Using the Burkholder-Davis-Gundy inequality to deal with the stochastic integral term, we obtain

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{T<T_K\}} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|^2\right] \leq C_{K,T} \Bigg[&\int_0^T \mathbb{E}\left[\mathbb{1}_{\{s<T_K\}} |b(X_s,\mu_s) - b(\tilde{X}_s,\tilde{\mu}_s)|^2\right] ds \\
&+ \int_0^T \mathbb{E}\left[\mathbb{1}_{\{s<T_K\}} |\mu_s^{1/\alpha}(f) - \tilde{\mu}_s^{1/\alpha}(f)|^2\right] ds \Bigg].
\end{aligned}$$

Using the first inequality of (3.8) with $p = 2$, Jensen's inequality and the Lipschitz continuity of f, we obtain

$$|(\mu_s(f))^{1/\alpha} - (\tilde{\mu}_s(f))^{1/\alpha}|^2 \leq C \left|\mu_s(f) - \tilde{\mu}_s(f)\right|^2 = C \left(\mathbb{E}[f(X_s) \,|\, S^\alpha] - \mathbb{E}[f(\tilde{X}_s) \,|\, S^\alpha]\right)^2$$
$$\leq C\mathbb{E}[|f(X_s) - f(\tilde{X}_s)|^2 \,|\, S^\alpha] \leq C\mathbb{E}[|X_s - \tilde{X}_s|^2 \,|\, S^\alpha]. \quad (3.16)$$

We conclude that $\mathbb{1}_{\{s<T_K\}}|(\mu_s(f))^{1/\alpha} - (\tilde{\mu}_s(f))^{1/\alpha}|^2 \leq C\mathbb{E}[\mathbb{1}_{\{s<T_K\}}|X_s - \tilde{X}_s|^2 \,|\, S^\alpha]$, since $\{s < T_K\}$ is S^α–measurable.

Using this and Assumption 3.2.1b) together with the fact that $W_1(\mu_s, \tilde{\mu}_s) \leq W_2(\mu_s, \tilde{\mu}_s)$,

$$\mathbb{E}\left[\mathbb{1}_{\{T<T_K\}} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|^2\right] \leq C_{K,T}\left[\int_0^T \mathbb{E}\left[\mathbb{1}_{\{s<T_K\}}\left(|X_s - \tilde{X}_s|^2 + W_2^2(\mu_s, \tilde{\mu}_s)\right)\right] ds \right.$$
$$\left. + \int_0^T \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{s<T_K\}}|X_s - \tilde{X}_s|^2 \,|\, S^\alpha\right]\right] ds\right].$$

By definition of the Wasserstein-2 distance, $W_2^2(\mu_s, \tilde{\mu}_s) \leq \mathbb{E}\left[|X_s - \tilde{X}_s|^2 \,|\, S^\alpha\right]$, such that

$$\mathbb{E}\left[\mathbb{1}_{\{T<T_K\}} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|^2\right] \leq C_{K,T} \int_0^T \mathbb{E}\left[\mathbb{1}_{\{s<T_K\}} \sup_{r\in[0,s]} |X_r - \tilde{X}_r|^2\right] ds. \quad (3.17)$$

Notice that due to our a priori bounds, $\mathbb{E}\left[\mathbb{1}_{\{s<T_K\}} \sup_{r\in[0,s]} |X_r - \tilde{X}_r|^2\right]$ is finite, such that the above inequality implies, by Gronwall's inequality, that

$$\mathbb{E}\left[\mathbb{1}_{\{T<T_K\}} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|^2\right] = 0. \quad (3.18)$$

Since $\lim_{K\to\infty} \mathbb{1}_{\{T<T_K\}} = 1$ almost surely, the assertion follows by monotone convergence.

The case $\alpha < 1$

We now discuss the case $\alpha < 1$. In this case S^α is of bounded variation such that we may use the L^1-norm instead of the L^2-norm. We have

$$\sup_{t\in[0,T]} \mathbb{1}_{\{t<T_K\}}|X_t - \tilde{X}_t| \leq \int_0^T \mathbb{1}_{\{s<T_K\}}|b(X_s, \mu_s) - b(\tilde{X}_s, \tilde{\mu}_s)|ds$$
$$+ \int_{[0,T]\times\mathbb{R}_+} \mathbb{1}_{\{s\leq T_K\}}|\psi(X_{s^-}, \mu_{s^-})\mathbb{1}_{\{z\leq f(X_{s^-})\}} - \psi(\tilde{X}_{s^-}, \tilde{\mu}_{s^-})\mathbb{1}_{\{z\leq f(\tilde{X}_{s^-})\}}|\bar{\pi}(ds, dz)$$
$$+ \int_{[0,T]\times\mathbb{R}_*} \mathbb{1}_{\{s\leq T_K\}}|\mu_{s^-}^{1/\alpha}(f) - \tilde{\mu}_{s^-}^{1/\alpha}(f)||z|M(ds, dz).$$

Using Assumption 3.2.1b), (3.8) with $p = 1$ and similar arguments as in (3.16)-(3.17), now with the L^1-norm instead of the L^2-norm, we have that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{\{s<T_K\}}|b(X_s,\mu_s) - b(\tilde{X}_s,\tilde{\mu}_s)|ds \right. \\ \left. + \int_{[0,T]\times\mathbb{R}_*} \mathbb{1}_{\{s\le T_K\}}|\mu_{s^-}^{1/\alpha}(f) - \tilde{\mu}_{s^-}^{1/\alpha}(f)||z|M(ds,dz)\right] \\ \le C_{K,T}\mathbb{E}\left[\mathbb{1}_{\{T<T_K\}} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|\right].$$

Finally we use that both f and ψ are bounded and get

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_+} \mathbb{1}_{\{s\le T_K\}}|\psi(X_{s^-},\mu_{s^-})\mathbb{1}_{\{z\le f(X_{s^-})\}} - \psi(\tilde{X}_{s^-},\tilde{\mu}_{s^-})\mathbb{1}_{\{z\le f(\tilde{X}_{s^-})\}}|\bar{\pi}(ds,dz)\right] \\ \le \mathbb{E}\int_0^T \mathbb{1}_{\{s\le T_K\}}\left[\|\psi\|_\infty|f(X_{s^-}) - f(\tilde{X}_{s^-})| + \|f\|_\infty|\psi(X_{s^-},\mu_{s^-}) - \psi(\tilde{X}_{s^-},\tilde{\mu}_{s^-})|\right]ds.$$

We then use their Lipschitz continuity to conclude the proof as before using Gronwall's lemma.

The existence of a strong solution of (3.7) follows from a Picard iteration. The proof is postponed to the Appendix section C.

Remark. We stress that *main jumps* can only be treated using L^1-norm. This is why in the case $\alpha > 1$ we have to disregard main jumps, since we need to use L^2-norm to deal with the small jumps in this case.

3.4 Representing the interaction term of the finite particle system as a stochastic integral against a stable process

3.4.1 Main representation result

To prove Theorem 3.2.10, we cut time into time slots of length $\delta > 0$. We will choose $\delta = \delta(N)$ such that, as $N \to \infty$, $\delta(N) \to 0$ and (at least) $N\delta(N) \to \infty$. The precise choice will be given in Section E below. We write $\tau_s = k\delta$ for $k\delta < s \le (k+1)\delta$, $k \in \mathbb{N}$, $s > 0$. A first step of the proof is a representation of the interaction term

$$A_t^N := \frac{1}{N^{1/\alpha}}\sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u\mathbb{1}_{\{z\le f(X_{s^-}^{N,j})\}}\pi^j(ds,dz,du)$$

in terms of a stochastic integral against a stable process. Recall that $r(N,\delta)$ has been defined in Theorem 3.2.10.

Theorem 3.4.1. *Grant Assumptions 3.2.1–3.2.7. For any $N \in \mathbb{N}^*$ and $\delta \in (0,1)$, there exists an α-stable Lévy process $S^{N,\alpha}$, independent of the PRM's $(\bar{\pi}^i)_{i=1}^N$ and of the initial conditions $(X_0^{N,i})_{i=1}^N$, such that*

$$A_t^N = \int_{[0,t]} \left(\mu_{\tau_s}^N(f)\right)^{1/\alpha} dS_s^{N,\alpha} + R_t^N. \tag{3.19}$$

i) If $1 < \alpha < 2$, we have, for $\delta = \delta(N)$ sufficiently small,

$$\mathbb{E}[|R_t^N|] \leq C_t r(N,\delta).$$

ii) If $0 < \alpha < 1$, we have

$$\mathbb{E}[\|R_t^N\|_{d_{\alpha-}}] \leq C_t r(N,\delta).$$

In both cases, C_t is a constant which is non-decreasing in t.

The remainder of this section is devoted to the proof of Theorem 3.4.1 which is the main tool to prove Theorem 3.2.10. The proof is given in Section 3.4.5 and it uses the following steps.

Step 1. We replace A_t^N by its time discretized version

$$A_t^{N,\delta} := \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{\tau_s}^{N,j})\}} \pi^j(ds,dz,du).$$

This is done directly in the proof of Theorem 3.4.1, Section 3.4.5. The error made due to this time discretization is controlled thanks to a general error bound stated in Proposition 3.4.4 below.

Step 2. We show that any increment of $A^{N,\delta}$ can be represented as the product of a conditional Poisson random variable (the total number of jumps per time interval) and the increment of a stable process. This is the content of Proposition 3.4.2.

Step 3. We replace, in the proof of Theorem 3.4.1, Section 3.4.5, the suitably renormalized Poisson random variable by its expectation to conclude our proof.

3.4.2 Representation of the discretized increment of the interaction term

Let $0 \leq s < t$ and define

$$A^N_{s,t} := \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{]s,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z \leq f(X^{N,j}_s)\}} \pi^j(dr, dz, du) \tag{3.20}$$

and

$$P^N_{s,t} := \sum_{j=1}^N \int_{]s,t]\times\mathbb{R}_+\times\mathbb{R}} \mathbb{1}_{\{z \leq f(X^{N,j}_s)\}} \pi^j(dr, dz, du). \tag{3.21}$$

The following lemma combines a (conditional version of) Proposition 3.1.1 with a quantified version of the stable central limit theorem. Recall that we suppose that ν is heavy-tailed according to Definition 3.2.1 above. The parameter γ is the one of this definition.

Proposition 3.4.2. *For all $N \in \mathbb{N}^*$ and all $0 \leq s < t$, such that $2(t-s)\|f\|_\infty < 1$, there exists an α-stable random variable $S^{N,\alpha}_{s,t}$, defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$ of the original probability space $(\Omega, \mathcal{A}, \mathbf{P})$, independent of $P^N_{s,t}$, of $\mathcal{F}_s$ and of $(\bar{\pi}^i)_{i\geq 1}$, such that,*

i) if $1 < \alpha < 2$,

$$\mathbb{E}\left[\left| A^N_{s,t} - \left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} S^{N,\alpha}_{s,t} \right|\right] \leq C(t-s)^{\frac{1}{\alpha}} g(N(t-s)), \tag{3.22}$$

where

$$g(x) = \begin{cases} x^{-\frac{\gamma}{\alpha}} & 0 < \gamma < 2-\alpha \\ x^{-\frac{2-\alpha}{\alpha}} & \gamma > 2-\alpha, \end{cases} \tag{3.23}$$

ii) if $\alpha < 1$,

$$\mathbb{E}\left[d_{\alpha_-}\left(A^N_{s,t}, \left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} S^{N,\alpha}_{s,t}\right)\right] \leq C[(t-s)^{\frac{\alpha_-}{\alpha}} g(N(t-s)) + e^{-CN|t-s|}], \tag{3.24}$$

where

$$g(x) = x^{-1} + x^{-\frac{\gamma}{\alpha}} + x^{\frac{\alpha-1}{\alpha}}, \ \gamma \neq 1-\alpha. \tag{3.25}$$

Proof. Using basic properties of Poisson random measures, there exists an i.i.d. sequence of random variables $U^{(k)}_{s,t}, k \geq 1$, distributed as ν, independent of $P^N_{s,t}$, of $\mathcal{F}_s$ and of $\bar{\pi}^i, i \geq 1$, such that almost surely,

$$A^N_{s,t} = \frac{1}{N^{1/\alpha}} \tilde{S}_{P^N_{s,t}}, \text{ where we put } \tilde{S}_n := \sum_{k=1}^{n} U^{(k)}_{s,t},$$

which we rewrite as

$$A^N_{s,t} = \left(\frac{P^N_{s,t}}{N} \right)^{1/\alpha} \frac{1}{(P^N_{s,t})^{1/\alpha}} \tilde{S}_{P^N_{s,t}}. \tag{3.26}$$

Since ν belongs to the domain of attraction of an α-stable law, and since it is also centered for $\alpha > 1$, the weak limit, as $n \to +\infty$, of the sequence $\left(\frac{1}{n^{1/\alpha}} \tilde{S}_n \right)_{n \geq 1}$, is the law of an $\alpha-$stable random variable S^α.

Let μ_n be the optimal coupling minimizing $W_1 \left(\frac{1}{n^{1/\alpha}} \tilde{S}_n, S^\alpha \right)$ if $\alpha > 1$ and $W_{d_{\alpha_-}} \left(\frac{1}{n^{1/\alpha}} \tilde{S}_n, S^\alpha \right)$ for $\alpha < 1$, for any fixed $n \geq 1$. These optimal couplings exist by Theorem 4.1 in [68]. Denote by $\mu^{(1)}_n$ their first marginal in both cases. Then, by Lemma 3.12 in [58], there exists a measurable function $G_n : \mathbb{R} \times (0,1) \to \mathbb{R}$ such that $(X_n, G_n(X_n, V)) \sim \mu_n$ whenever $(X_n, V) \sim \mu^{(1)}_n \otimes U(0,1)$. In particular, this holds for $X_n = \frac{1}{n^{1/\alpha}} \tilde{S}_n$ and $V \sim U(0,1)$ independent of it. Since the second marginal of μ_n is $\mathcal{L}(S^\alpha)$, this means that

$$G_n \left(\frac{1}{n^{1/\alpha}} \tilde{S}_n, V \right) \sim S^\alpha. \tag{3.27}$$

We also set, for $n = 0$, $G_0(x, V) = G_0(V) \sim S^\alpha$. Let us finally introduce $G : \mathbb{N} \times \mathbb{R} \times (0,1) \to \mathbb{R}$ by putting

$$G(n, x, v) := \sum_{l=0}^{\infty} 1_{\{n=l\}} G_l(x, v).$$

Then, for any fixed $N \in \mathbb{N}$ and $0 \leq s < t$, and for $V \sim U(0,1)$ independent of the PRM's $(\pi^i)_i$, we define

$$S^{N,\alpha}_{s,t} := G \left(P^N_{s,t}, \frac{1}{(P^N_{s,t})^{1/\alpha}} \tilde{S}_{P^N_{s,t}}, V \right). \tag{3.28}$$

We now prove that $S^{N,\alpha}_{s,t}$ is an α-stable variable, which is independent of $P^N_{s,t}$, of $\mathcal{F}_s$ and of $\bar{\pi}^i, i \geq 1$. Let $A \in \sigma\{P^N_{s,t}\} \vee \sigma\{\bar{\pi}^i, i \geq 1\} \vee \mathcal{F}_s$. We have

$$\begin{aligned}
\mathbb{E}[\phi(S^{N,\alpha}_{s,t}) \mathbb{1}_A] &= \mathbb{E} \left[\phi \left(G \left(P^N_{s,t}, \frac{1}{(P^N_{s,t})^{1/\alpha}} \tilde{S}_{P^N_{s,t}}, V \right) \right) \mathbb{1}_A \right] \\
&= \sum_{n=0}^{\infty} \mathbb{E} \left[\phi \left(G_n \left(\frac{1}{n^{1/\alpha}} \tilde{S}_n, V \right) \right) \mathbb{1}_{\{P^N_{s,t}=n\}} \mathbb{1}_A \right] \\
&= \sum_{n=0}^{\infty} \mathbb{E} \left[\phi \left(G_n \left(\frac{1}{n^{1/\alpha}} \tilde{S}_n, V \right) \right) \right] \mathbb{P}(\{P^N_{s,t} = n\} \cap A)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[\phi\left(S^{\alpha}\right)\right] \mathbb{P}(\{P_{s,t}^{N} = n\} \cap A) = \mathbb{E}\left[\phi\left(S^{\alpha}\right)\right] \mathbb{P}(A),$$

for any measurable bounded test function ϕ. Here, we used that $((\tilde{S}_n)_n, V)$ is independent of $P_{s,t}^N$, of $\mathcal{F}_s$ and of $\bar{\pi}^i, i \geq 1$, to obtain the third equality. Thus, $S_{s,t}^{N,\alpha}$ is indeed an α-stable variable, which is independent of $P_{s,t}^N$, of $\mathcal{F}_s$ and of $\bar{\pi}^i, i \geq 1$.

Finally, the bounds in (3.22) and (3.24) follow using the representation (3.26) for $A_{s,t}^N$.

Case $\boldsymbol{\alpha > 1}$. Employing that, conditionally on $\{P_{s,t}^N = n\}$, $G\left(P_{s,t}^N, \frac{\tilde{S}_{P_{s,t}^N}}{(P_{s,t}^N)^{1/\alpha}}, V\right) = G_n\left(\frac{1}{n^{1/\alpha}}\tilde{S}_n, V\right)$, that $\mathcal{L}\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, G_n\left(\frac{1}{n^{1/\alpha}}\tilde{S}_n, V\right)\right)$ is the optimal coupling μ_n for the W_1 distance and (3.27),

$$\begin{aligned}
\mathbb{E}\left[\left|A_{s,t}^N - \left(\frac{P_{s,t}^N}{N}\right)^{1/\alpha} S_{s,t}^{N,\alpha}\right|\right] &= \sum_{n>0} \left(\frac{n}{N}\right)^{1/\alpha} \mathbb{E}\left[\left|\frac{\tilde{S}_n}{n^{1/\alpha}} - G_n\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, V\right)\right|\right] \mathbb{P}(P_{s,t}^N = n) \\
&= \frac{1}{N^{1/\alpha}} \sum_{n>0} n^{1/\alpha} W_1\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, G_n\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, V\right)\right) \mathbb{P}(P_{s,t}^N = n) \\
&= \frac{1}{N^{1/\alpha}} \sum_{n>0} n^{1/\alpha} W_1\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S^{\alpha}\right) \mathbb{P}(P_{s,t}^N = n). \quad (3.29)
\end{aligned}$$

Example 2 in [14] yields that, for any $n \in \mathbb{N}$, $W_1\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S^{\alpha}\right) \leq Cg(n)$, for a positive constant C, where the function g is given in (3.23) and satisfies that $x \mapsto x^{1/\alpha} g(x)$ is concave on $\mathbb{R}_+$ for any choice of $\gamma > 0$. As a consequence, we can apply Jensen's inequality in (3.29) to obtain

$$\begin{aligned}
&\frac{1}{N^{1/\alpha}} \sum_{n>0} n^{1/\alpha} W_1\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S^{\alpha}\right) \mathbb{P}(P_{s,t}^N = n) \leq \frac{C}{N^{1/\alpha}} \sum_{n>0} n^{1/\alpha} g(n) \mathbb{P}(P_{s,t}^N = n) \\
&= \frac{C}{N^{1/\alpha}} \mathbb{E}[(P_{s,t}^N)^{1/\alpha} g(P_{s,t}^N)] \leq \frac{C}{N^{1/\alpha}} \mathbb{E}[P_{s,t}^N]^{1/\alpha} g(\mathbb{E}[P_{s,t}^N]) \leq C\|f\|_{\infty}^{1/\alpha} |t-s|^{1/\alpha} g(\underline{f} N |t-s|),
\end{aligned}$$

where we have employed that, conditionnally on $\mathcal{F}_s$, $P_{s,t}^N \sim Pois((t-s)N\mu_s^N(f))$ and Assumption 3.2.2b) to upperbound $\mathbb{E}[P_{s,t}^N]$ and 3.2.2a) to upperbound $g(\mathbb{E}[P_{s,t}^N])$ (using that g is decreasing).

Case $\boldsymbol{\alpha < 1}$. In this case, we introduce $G := \left\{\frac{1}{2}|t-s| N \underline{f} \leq P_{s,t}^N \leq 2|t-s| N \|f\|_{\infty}\right\}$. It is easy to show, by the exponential Markov inequality, that $\mathbb{P}(G^c \,|\, \mathcal{F}_s) \leq ce^{-CN|t-s|}$, for some constants $c, C > 0$ not depending on N. Then we start from

$$\begin{aligned}
&\mathbb{E}\left[d_{\alpha_-}\left(A^N_{s,t}, \left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} S^{N,\alpha}_{s,t}\right)\right] \\
&\leq \mathbb{E}\left[\left(\left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} \vee \left(\frac{P^N_{s,t}}{N}\right)^{\alpha_-/\alpha}\right) d_{\alpha_-}\left(\frac{1}{(P^N_{s,t})^{1/\alpha}}\tilde{S}_{P^N_{s,t}}, S^{N,\alpha}_{s,t}\right)\mathbb{1}_G\right] \\
&\quad +\mathbb{E}\left[\left|A^N_{s,t} - \left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} S^{N,\alpha}_{s,t}\right|^{\alpha_-} \mathbb{1}_{G^c}\right] =: T_1 + T_2.
\end{aligned}$$

To control T_2, we use the sub-additivity of $x \mapsto x^{\alpha_-}$ and Hölder's inequality with q'/α_- where $q' \in]\alpha_-, \alpha[$ and associated conjugate exponent p' such that $1/p' + \alpha_-/q' = 1$, to upper bound

$$\begin{aligned}
T_2 &\leq \left(\mathbb{E}[|A^N_{s,t}|^{q'}]\right)^{\alpha_-/q'} \mathbb{P}(G^c)^{1/p'} + \left(\mathbb{E}[|S^{N,\alpha}_{s,t}|^{q'}]\right)^{\alpha_-/q'} \left(\mathbb{E}\left[\left(\frac{P^N_{s,t}}{N}\right)^{\alpha_- p'/\alpha} \mathbb{1}_{G^c}\right]\right)^{1/p'} \\
&\leq \left(CN^{1-q'/\alpha}|t-s|\right)^{\alpha_-/q'} \mathbb{P}(G^c)^{1/p'} + C\left(\mathbb{E}\left[\left(\frac{P^N_{s,t}}{N}\right)^{\alpha_- p'/\alpha} \mathbb{1}_{G^c}\right]\right)^{1/p'} \\
&\leq ce^{-\tilde{C}N|t-s|},
\end{aligned}$$

where C is an upper bound on $(\mathbb{E}[|S^{N,\alpha}_{s,t}|^{q'}])^{\alpha_-/q'}$ and where we have used Hölder's inequality once more to upper bound $\mathbb{E}(\left(P^N_{s,t}/N\right)^{\alpha_- p'/\alpha} \mathbb{1}_{G^c})$. To control T_1, notice that we have $\frac{P^N_{s,t}}{N} \leq 1$ on G (recall that we assumed that $2\|f\|_\infty(t-s) \leq 1$), such that

$$\left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} \vee \left(\frac{P^N_{s,t}}{N}\right)^{\alpha_-/\alpha} = \left(\frac{P^N_{s,t}}{N}\right)^{\alpha_-/\alpha}.$$

Therefore, using analogous arguments as in the case $\alpha > 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\left(\frac{P^N_{s,t}}{N}\right)^{1/\alpha} \vee \left(\frac{P^N_{s,t}}{N}\right)^{\alpha_-/\alpha}\right) d_{\alpha_-}\left(\frac{1}{(P^N_{s,t})^{1/\alpha}}\tilde{S}_{P^N_{s,t}}, S^{N,\alpha}_{s,t}\right)\mathbb{1}_G\right] \\
&\leq \mathbb{E}\left[\left(\frac{P^N_{s,t}}{N}\right)^{\alpha_-/\alpha} d_{\alpha_-}\left(\frac{1}{(P^N_{s,t})^{1/\alpha}}\tilde{S}_{P^N_{s,t}}, S^{N,\alpha}_{s,t}\right)\mathbb{1}_G\right] \\
&\leq \frac{1}{N^{\alpha_-/\alpha}} \sum_{n\geq 0} n^{q/\alpha}\mathbb{E}\left[d_{\alpha_-}\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, G_n\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, V\right)\right)\right]\mathbb{P}(P^N_{s,t} = n) \\
&= \frac{1}{N^{\alpha_-/\alpha}} \sum_{n\geq 0} n^{\alpha_-/\alpha} d_{W_{\alpha_-}}\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S^\alpha\right)\mathbb{P}(P^N_{s,t} = n).
\end{aligned}$$

Example 2 in [15] yields that, for any $n \in \mathbb{N}$, $d_{W_{\alpha-}}\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S^\alpha\right) \leq Cg(n)$, where g is given in (3.25) and satisfies that $x \mapsto x^{\alpha-/\alpha} g(x)$ is concave on $\mathbb{R}_+$ for any $\gamma > 0$. Then, as before, we apply Jensen's inequality to conclude the proof:

$$\frac{1}{N^{q/\alpha}} \sum_{n\geq 0} n^{\alpha-/\alpha} d_{W_{\alpha-}}\left(\frac{\tilde{S}_n}{n^{1/\alpha}}, S_1^\alpha\right) \mathbb{P}(P_{s,t}^N = n) \leq C|t-s|^{\alpha-/\alpha} g(\underline{f}N|t-s|).$$

□

3.4.3 Construction of an α-stable Lévy process

For any fixed $N \in \mathbb{N}^*$ and $\delta > 0$, Lemma 3.4.2 with $s = k\delta$ and $t = (k+1)\delta$, $k \geq 0$, yields the existence of a family of i.i.d. random variables $(S^{N,\alpha}_{k\delta,(k+1)\delta})_{k\geq 0}$, defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$ of the original probability space $(\Omega, \mathcal{A}, \mathbf{P})$, which all follow the law of S_1^α. Indeed, it suffices to consider an i.i.d. sequence $(V_k)_{k\geq 0}$ of uniform random variables which are independent of anything else and to put, recalling (3.28),

$$S^{N,\alpha}_{k\delta,(k+1)\delta} := G\left(P^N_{k\delta,(k+1)\delta}, \frac{1}{(P^N_{k\delta,(k+1)\delta})^{1/\alpha}} \tilde{S}_{P^N_{k\delta,(k+1)\delta}}, V_k\right).$$

By construction, for each $k \geq 0$, $S^{N,\alpha}_{k\delta,(k+1)\delta}$ is independent of $\mathcal{F}_{k\delta}$.

We have the following proposition.

Proposition 3.4.3. *For each $N \in \mathbb{N}^*$ there exists an α-stable process $S^{N,\alpha}$, constructed on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of $\mathcal{F}_0$ and of $\bar{\pi}^i, i \geq 1$, such that almost surely,*

$$S^{N,\alpha}_{k\delta} - S^{N,\alpha}_{(k-1)\delta} = \delta^{1/\alpha} S^{N,\alpha}_{(k-1)\delta,k\delta} \tag{3.30}$$

for all $k \geq 1$.

Proof. Fix any $\delta > 0$ and consider the stable process $(S_t^\alpha)_{t\geq 0}$, starting from $S_0^\alpha = 0$, defined on some probability space. Then, by Theorem 8.5 in [46], the joint law of $(S_\delta^\alpha, (S_t^\alpha)_{t\in[0,\delta]})$ can be disintegrated into the product of $\mathcal{L}(S_\delta^\alpha)$ and a probability kernel Q from $\mathbb{R}$ to $D([0,\delta],\mathbb{R})$ such that Q is $\mathcal{L}(S_\delta^\alpha)$-a.s. unique and satisfies

$$Q(S_\delta^\alpha, \cdot) = \mathcal{L}((S_t^\alpha)_{t\in[0,\delta]} | S_\delta^\alpha)(\cdot) \text{ a.s.}$$

Next, consider the product spaces $\mathbb{R}^\mathbb{N}$ and $\Omega' := \prod_{k\geq 1} D([0,\delta],\mathbb{R})$, endowed with the product σ-algebras, and define the product kernel

$$Q^\infty((x^{(k)})_{k\geq 1}, d\gamma) := \otimes_{k\geq 1} Q(x^{(k)}, d\gamma^{(k)}) \,.$$

Then $Q^\infty((x^{(k)})_{k\geq 1}, d\gamma)$ is a probability kernel, from $\mathbb{R}^{\mathbb{N}}$ to Ω', thanks to the extension theorem of Ionescu-Tulcea. Define now

$$\phi\ :\ \Omega' \to D(\mathbb{R}_+, \mathbb{R}),\ (\gamma^{(k)})_{k\geq 1} \mapsto (\phi_t)_{t\geq 0}$$

by $\phi_0 = 0$ and for any $t > 0$,

$$\phi_t := \sum_{k\geq 0} \mathbb{1}_{]k\delta,(k+1)\delta]}(t) \left(\sum_{l=1}^{k} \gamma^{(l)}(\delta) + \gamma^{(k+1)}(t - k\delta) \right),$$

with the convention $\sum_{l=1}^{0} = 0$.

Introduce finally

$$X^{(k)} = \delta^{1/\alpha} S^{N,\alpha}_{(k-1)\delta, k\delta}\ ,$$

defined thanks to Lemma 3.4.2 for any $k \geq 1$ on an extension $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$ of the original probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Then by construction, $(\phi_t)_{t\geq 0}$ is an α-stable process under $\tilde{\mathbf{P}} \otimes Q^\infty((X^{(k)})_{k\geq 1}, \cdot)$. Setting $(S^{N,\alpha}_t)_t := (\phi_t)_t$ concludes the proof.

□

3.4.4 Errors due to time discretization

We now state a generic result expressing the error that is due to time discretization.

Proposition 3.4.4. *Grant Assumptions 3.2.1a), 3.2.2b), 3.2.3a), 3.2.4 and 3.2.5. Then for any* $1 \leq i \leq N$, *for any* $s, t \geq 0$ *such that* $|s - t| \leq 1$, *and for* $\alpha_- < \alpha$, *(*$\alpha_- \geq 1$, *if* $\alpha > 1$*), we have*

(i)

$$\mathbb{E}[|f(X^{N,i}_t) - f(X^{N,i}_s)|] \leq C \begin{cases} N^{1/\alpha_- - 1/\alpha}|t-s|^{1/\alpha_-}, & \textit{if } \alpha \in (1,2) \\ N^{1-\alpha_-/\alpha}|t-s|, & \textit{if } \alpha \in (0,1). \end{cases}$$

(ii) Moreover, if $\alpha > \alpha_- > 1$, *then we also have*

$$\mathbb{E}[|X^{N,i}_t - X^{N,i}_s|^{\alpha_-}] \leq C N^{1-\alpha_-/\alpha}|t-s|.$$

Proof. By exchangeability, it suffices to consider $i = 1$. Suppose w.l.o.g. that $s \leq t$. By Ito's formula, recalling that $\psi \equiv 0$ if $\alpha > 1$,

$$f(X^{N,1}_t) - f(X^{N,1}_s) = \int_s^t b(X^{N,1}_v, \mu^N_v) f'(X^{N,1}_v) dv$$

$$+\int_{]s,t]\times\mathbb{R}_+}\mathbb{1}_{\{z\le f(X^{N,1}_{v^-})\}}[f(X^{N,1}_{v^-}+\psi(X^{N,1}_{v^-},\mu^N_{v^-}))-f(X^{N,1}_{v^-})]\bar{\pi}^i(dv,dz)$$

$$+\sum_{j\neq 1}\int_{]s,t]\times\mathbb{R}_+\times\mathbb{R}}[f(X^{N,1}_{v^-}+\frac{u}{N^{1/\alpha}})-f(X^{N,1}_{v^-})]\mathbb{1}_{\{z\le f(X^{N,j}_{v^-})\}}\pi^j(dv,dz,du)$$

$$=: B^N_{s,t}+\psi^N_{s,t}+I^N_{s,t}.$$

Using the Lipschitz continuity of f and the boundedness of f, f', b, ψ, we have that

$$\mathbb{E}(|B^N_{s,t}|+|\psi^N_{s,t}|)\le C|t-s|.$$

As for the term $I^N_{s,t}$, when $\alpha\in(0,1)$ so that $\alpha_-<1$ as well, we use that f is bounded and Lipschitz to obtain

$$\left|f\left(v+\frac{u}{N^{1/\alpha}}\right)-f(v)\right|\le C\left|f\left(v+\frac{u}{N^{1/\alpha}}\right)-f(v)\right|^{\alpha_-}\le C\frac{|u|^{\alpha_-}}{N^{\alpha_-/\alpha}},$$

such that

$$\mathbb{E}|I^N_{s,t}|\le CN^{1-\alpha_-/\alpha}|t-s|.$$

If $\alpha>1$, then ν is centered and the interaction term is a martingale. Thus we use the Burkholder-Davis-Gundy inequality in L^{α_-}, $1\le\alpha_-<\alpha$, and the Lipschitz continuity of f to obtain

$$\mathbb{E}[|I^N_{s,t}|]\le(\mathbb{E}[|I^N_{s,t}|^{\alpha_-}])^{1/\alpha_-}\le$$

$$C\left(\mathbb{E}\left[\left(\frac{1}{N^{2/\alpha}}\sum_{j=2}^N\int_{]s,t]\times\mathbb{R}_+\times\mathbb{R}^*}u^2\mathbb{1}_{\{z\le f(X^{N,j}_{v^-})\}}\pi^j(dv,dz,du)\right)^{\alpha_-/2}\right]\right)^{1/\alpha_-}.$$

Using that for $0<\alpha_-/2\le 1$ $\mathbb{R}_+\ni x\mapsto x^{\alpha_-/2}$ is sub-additive and the boundedness of f, we obtain

$$\mathbb{E}[|I^N_{s,t}|]\le C\left(\mathbb{E}\left[\frac{1}{N^{\alpha_-/\alpha}}\sum_{j=2}^N\int_{]s,t]\times\mathbb{R}_+\times\mathbb{R}^*}|u|^{\alpha_-}\mathbb{1}_{\{z\le f(X^{N,j}_{v^-})\}}\pi^j(dv,dz,du)\right]\right)^{1/\alpha_-}$$

$$\le C\left(N^{1-\alpha_-/\alpha}\int_{\mathbb{R}^*}|u|^{\alpha_-}\nu(du)\int_s^t\mathbb{E}\left[f(X^{N,1}_v)\right]dv\right)^{1/\alpha_-}\le CN^{1/\alpha_-(1-\alpha_-/\alpha)}|t-s|^{1/\alpha_-}. \quad (3.31)$$

Item (ii) follows similarly, by the BDG inequality in L^{α_-}, sub-additivity of $z\mapsto z^{\alpha_-/2}$ and boundedness of b. □

3.4.5 Proof of Theorem 3.4.1

In this subsection, we give the proof of Theorem 3.4.1. Let $S^{N,\alpha}$ be the α-stable process obtained as in Lemma 3.4.3 from the sequence of increments $(S^{N,\alpha}_{k\delta,(k+1)\delta})_{k\geq 0}$ given by Proposition 3.4.2. Remember that

$$A^N_t = \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z\leq f(X^{N,j}_{s-})\}} \pi^j(ds,dz,du)$$

and

$$A^{N,\delta}_t = \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z\leq f(X^{N,j}_{\tau_s})\}} \pi^j(ds,dz,du).$$

Putting

$$R^{N,1}_t := A^N_t - A^{N,\delta}_t,$$

$$R^{N,2}_t := A^{N,\delta}_t - \int_{[0,t]} \left(\mu^N_{\tau_s}(f)\right)^{1/\alpha} dS^{N,\alpha}_s,$$

and

$$R^N_t := R^{N,1}_t + R^{N,2}_t,$$

we obtain (3.19).

Study of $R^{N,1}_t$

Notice that, in case $\alpha > 1$, $R^{N,1}_t$ is a martingale, since $\int_{\mathbb{R}^*} u\nu(du) = 0$. Then, in this case, we can employ the Burkholder-Davis-Gundy inequality in L^{α_-} and similar arguments as those leading to (3.31). In case $\alpha < 1$, we use instead the sub-additivity of $x \mapsto x^{\alpha_-}$ on $\mathbb{R}_+$ for $\alpha_- < \alpha$. In both cases we obtain

$$\mathbb{E}[|R^{N,1}_t|^{\alpha_-}] \leq CN^{1-\alpha_-/\alpha} \int_{\mathbb{R}^*} |u|^{\alpha_-}\nu(du) \int_0^t \mathbb{E}\left[\left|f(X^{N,j}_s) - f(X^{N,j}_{\tau_s})\right|\right] ds.$$

Using Proposition 3.4.4, we obtain

$$\mathbb{E}[|R^{N,1}_t|^{\alpha_-}] \leq Ct \left\{ \begin{array}{ll} N^{(1+\frac{1}{\alpha_-})(1-\alpha_-/\alpha)}\delta^{1/\alpha_-}, & \text{if } \alpha > 1 \\ N^{2(1-\alpha_-/\alpha)}\delta & \text{if } \alpha < 1 \end{array} \right\}. \tag{3.32}$$

In particular, for $\alpha > 1$,

$$\mathbb{E}[|R_t^{N,1}|] \leq \left(\mathbb{E}[|R_t^{N,1}|^{\alpha_-}]\right)^{1/\alpha_-} \leq C_t N^{\frac{1}{\alpha_-}(1+\frac{1}{\alpha_-})(1-\alpha_-/\alpha)} \delta^{1/(\alpha_-)^2}, \tag{3.33}$$

and, for $\alpha < 1$, we use $\mathbb{E}[||R_t^{N,1}||_{d_{\alpha_-}}] \leq \mathbb{E}[|R_t^{N,1}|^{\alpha_-}]$.

Study of $R_t^{N,2}$

Recall (3.20) which we will apply for $s = k\delta$ and $t = (k+1)\delta$. Then

$$\begin{aligned} A_t^{N,\delta} &= \sum_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1} A_{k\delta,(k+1)\delta}^N - \frac{1}{N^{1/\alpha}} \sum_{j=1}^N \int_{]t, \lceil \frac{t}{\delta} \rceil \delta] \times \mathbb{R}_+ \times \mathbb{R}^*} u \mathbb{1}_{\{z \leq f(X^{N,j}_{(\lceil \frac{t}{\delta} \rceil - 1)\delta})\}} \pi^j(ds, dz, du) \\ &=: \sum_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1} A_{k\delta,(k+1)\delta}^N - E_t^1. \end{aligned}$$

Also, define

$$E_t^2 := \int_{]t, \lceil \frac{t}{\delta} \rceil \delta]} \left(\mu^N_{(\lceil \frac{t}{\delta} \rceil - 1)\delta}(f)\right)^{1/\alpha} dS_s^{N,\alpha}$$

and

$$E_{k\delta,(k+1)\delta}^N := \left| \left(\frac{P_{k\delta,(k+1)\delta}^N}{N\delta} \right)^{1/\alpha} - (\mu_{k\delta}^N(f))^{1/\alpha} \right| |S_{(k+1)\delta}^{N,\alpha} - S_{k\delta}^{N,\alpha}|,$$

where $P_{k\delta,(k+1)\delta}^N$ was defined in (3.21).

Then, for $1 < \alpha < 2$, we can write

$$\begin{aligned} \mathbb{E}[|R_t^{N,2}|] &\leq \mathbb{E}\left[\left| \sum_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1} \left\{ A_{k\delta,(k+1)\delta}^N - \int_{]k\delta,(k+1)\delta]} \left(\mu_{k\delta}^N(f)\right)^{1/\alpha} dS_s^{N,\alpha} \right\} - E_t^1 + E_t^2 \right| \right] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1} \left| A_{k\delta,(k+1)\delta}^N - \left(\frac{P_{k\delta,(k+1)\delta}^N}{N} \right)^{1/\alpha} S_{k\delta,(k+1)\delta}^{N,\alpha} \right.\right. \\ &\quad \left.\left. + \left(\frac{P_{k\delta,(k+1)\delta}^N}{N} \right)^{1/\alpha} S_{k\delta,(k+1)\delta}^{N,\alpha} - \int_{]k\delta,(k+1)\delta]} \left(\mu_{k\delta}^N(f)\right)^{1/\alpha} dS_s^{N,\alpha} \right| \right] \\ &\quad + \mathbb{E}[|E_t^1|] + \mathbb{E}[|E_t^2|] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{\lceil \frac{t}{\delta} \rceil - 1} \left| A_{k\delta,(k+1)\delta}^N - \left(\frac{P_{k\delta,(k+1)\delta}^N}{N} \right)^{1/\alpha} S_{k\delta,(k+1)\delta}^{N,\alpha} \right| \right] \end{aligned}$$

$$+\mathbb{E}\left[\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\left|\left(\frac{P^N_{k\delta,(k+1)\delta}}{N\delta}\right)^{1/\alpha}-(\mu^N_{k\delta}(f))^{1/\alpha}\right||S^{N,\alpha}_{(k+1)\delta}-S^{N,\alpha}_{k\delta}|\right]$$
$$+\mathbb{E}[|E^1_t|]+\mathbb{E}[|E^2_t|]$$
$$\leq\left(\lceil\frac{t}{\delta}\rceil-1\right)C\delta^{1/\alpha}g(N\delta)+\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\mathbb{E}[E^N_{k\delta,(k+1)\delta}]+\mathbb{E}[|E^1_t|]+\mathbb{E}[|E^2_t|], \tag{3.34}$$

where we have used that, thanks to (3.30), almost surely, $\delta^{1/\alpha}S^{N,\alpha}_{k\delta,(k+1)\delta}=S^{N,\alpha}_{(k+1)\delta}-S^{N,\alpha}_{k\delta}$, and the last inequality follows straight from Proposition 3.4.2 *i*).

For $\alpha<1$, using the sub-additivity of the function $x\mapsto x^{\alpha_-}$ on $\mathbb{R}_+$ for $\alpha_-<\alpha<1$, we can write

$$\mathbb{E}[\|R^{N,2}_t\|_{d_{\alpha_-}}]\leq\mathbb{E}\left[\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\left|A^N_{k\delta,(k+1)\delta}-\left(\frac{P^N_{k\delta,(k+1)\delta}}{N}\right)^{1/\alpha}S^{N,\alpha}_{k\delta,(k+1)\delta}\right|\right. \tag{3.35}$$
$$\left.\wedge\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\left|A^N_{k\delta,(k+1)\delta}-\left(\frac{P^N_{k\delta,(k+1)\delta}}{N}\right)^{1/\alpha}S^{N,\alpha}_{k\delta,(k+1)\delta}\right|^{\alpha_-}\right]$$
$$+\mathbb{E}\left[\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\left|\left(\frac{P^N_{k\delta,(k+1)\delta}}{N\delta}\right)^{1/\alpha}-(\mu^N_{k\delta}(f))^{1/\alpha}\right|^{\alpha_-}|S^{N,\alpha}_{(k+1)\delta}-S^{N,\alpha}_{k\delta}|^{\alpha_-}\right]$$
$$+\mathbb{E}[|E^1_t|^{\alpha_-}]+\mathbb{E}[|E^2_t|^{\alpha_-}]$$
$$\leq\left(\lceil\frac{t}{\delta}\rceil-1\right)C\delta^{\alpha_-/\alpha}g(N\delta)+\sum_{k=0}^{\lceil\frac{t}{\delta}\rceil-1}\mathbb{E}[(E^N_{k\delta,(k+1)\delta})^{\alpha_-}]+\mathbb{E}[|E^1_t|^{\alpha_-}]+\mathbb{E}[|E^2_t|^{\alpha_-}].$$

In the last inequality we employed Proposition 3.4.2 *ii*), and we used that the exponential factor is negligible compared to the other terms.

Now, using the same arguments as those used in the proof of Proposition 3.4.4, we have in both cases $\alpha<1$ and $1<\alpha<2$,

$$\mathbb{E}[|E^1_t|^{\alpha_-\wedge 1}]\leq C\left\{\begin{array}{ll}N^{1/\alpha_--1/\alpha}\delta^{1/\alpha_-} & \text{if } \alpha>1\\ N^{1-\alpha_-/\alpha}\delta & \text{if } \alpha<1\end{array}\right\}. \tag{3.36}$$

Moreover, since f is bounded and $S^{N,\alpha}_s-S^{N,\alpha}_r\overset{d}{=}|s-r|^{1/\alpha}S^\alpha$,

$$\mathbb{E}[|E^2_t|^{\alpha_-\wedge 1}]=\mathbb{E}\left[\left|\left(\mu^N_{(\lceil\frac{t}{\delta}\rceil-1)\delta}(f)\right)^{1/\alpha}\right|^{\alpha_-\wedge 1}\left|\int_{]t,\lceil\frac{t}{\delta}\rceil\delta]}dS^{N,\alpha}_s\right|^{\alpha_-\wedge 1}\right]$$
$$\leq C\mathbb{E}[|S^{N,\alpha}_{\lceil\frac{t}{\delta}\rceil\delta}-S^{N,\alpha}_t|^{\alpha_-\wedge 1}]\leq C\delta^{(\alpha_-\wedge 1)/\alpha}. \tag{3.37}$$

Last, we use once more deviation inequalities to deal with the (conditional) Poisson random variables: as before, since f is bounded and lowerbounded, for the event

$$G := \left\{ \frac{1}{2}\delta N \underline{f} \leq P^N_{k\delta,(k+1)\delta} \leq 2\delta N \|f\|_\infty \right\}, \text{ we have } \mathbb{P}(G^c) \leq ce^{-C\delta N}.$$

Notice that $0 < \underline{f} < \mu^N_{k\delta}(f) \leq \|f\|_\infty$. So we can use the Lipschitz property of $z \mapsto z^{1/\alpha}$ on $[\frac{1}{2}\underline{f}, \infty[$ in case $\alpha > 1$ and on $[0, 2\|f\|_\infty]$ in case $\alpha < 1$, Jensen's inequality and the fact that $P^N_{k\delta,(k+1)\delta}$ is conditionally Poisson distributed together with the boundedness of f to deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(\frac{P^N_{k\delta,(k+1)\delta}}{N\delta}\right)^{1/\alpha} - \left(\mu^N_{k\delta}(f)\right)^{1/\alpha}\right|^{\alpha_-\wedge 1} \mathbb{1}_G\right] \leq C\mathbb{E}\left[\left|\frac{P^N_{k\delta,(k+1)\delta}}{N\delta} - \mu^N_{k\delta}(f)\right|^{\alpha_-\wedge 1} \mathbb{1}_G\right] \\
&\leq C\mathbb{E}\left[\left|\frac{P^N_{k\delta,(k+1)\delta}}{N\delta} - \mu^N_{k\delta}(f)\right| \mathbb{1}_G\right]^{\alpha_-\wedge 1} \leq C\mathbb{E}\left[\left|\frac{P^N_{k\delta,(k+1)\delta}}{N\delta} - \mu^N_{k\delta}(f)\right|^2\right]^{\frac{1}{2}(\alpha_-\wedge 1)} \\
&\leq C\left(\frac{1}{N^2\delta^2} Var\left[Pois\left(N\delta\mu^N_{k\delta}(f)\right)\right]\right)^{\frac{1}{2}(\alpha_-\wedge 1)} \leq C(N\delta)^{-(\alpha_-\wedge 1)/2}.
\end{aligned}$$

Moreover, it is straightforward to show that

$$\mathbb{E}\left[\left|\left(\frac{P^N_{k\delta,(k+1)\delta}}{N\delta}\right)^{1/\alpha} - \left(\mu^N_{k\delta}(f)\right)^{1/\alpha}\right|^{\alpha_-\wedge 1} \mathbb{1}_{G^c}\right] \leq Ce^{-c\delta N}.$$

Therefore, using the independence of $S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta}$ of $\mathcal{F}_{k\delta}$ and of $P^N_{k\delta,(k+1)\delta}$, we conclude that

$$\begin{aligned}
\mathbb{E}[(E^N_{k\delta,(k+1)\delta})^{\alpha_-\wedge 1}] &= \mathbb{E}\left[\left|\left(\frac{P^N_{k\delta,(k+1)\delta}}{N\delta}\right)^{1/\alpha} - (\mu^N_{k\delta}(f))^{1/\alpha}\right|^{\alpha_-\wedge 1}\right] \mathbb{E}\left[|S^{N,\alpha}_{(k+1)\delta} - S^{N,\alpha}_{k\delta}|^{\alpha_-\wedge 1}\right] \\
&\leq C(N\delta)^{-(\alpha_-\wedge 1)/2}\delta^{(\alpha_-\wedge 1)/\alpha}.
\end{aligned} \tag{3.38}$$

Overall, putting together (3.34) ((3.35) if $\alpha < 1$), (3.36), (3.37) and (3.38), we have in the case $\alpha > 1$,

$$\mathbb{E}[|R^{N,2}_t|] \leq C_t\left[\lceil \frac{t}{\delta} \rceil \delta^{1/\alpha}\left(g(N\delta) + (N\delta)^{-1/2}\right) + N^{1/\alpha_- - 1/\alpha}\delta^{1/\alpha_-} + \delta^{1/\alpha}\right]$$

and

$$\mathbb{E}[\|R_t^{N,2}\|_{d_{\alpha_-}}] \le C_t \left[\lceil \frac{t}{\delta} \rceil \left(\delta^{\alpha_-/\alpha} g(N\delta) + (N\delta)^{-\alpha_-/2} \delta^{\alpha_-/\alpha} \right) + N^{1-\alpha_-/\alpha}\delta + \delta^{\alpha_-/\alpha} \right]$$

for $\alpha < 1$.

Recall the control on $R_t^{N,1}$ obtained in (3.33) above in case $\alpha > \alpha_- > 1$. We clearly have that $N^{1/\alpha_- - 1/\alpha}\delta^{1/\alpha_-} \le N^{\left(1+\frac{1}{\alpha_-}\right)\left(1-\frac{\alpha_-}{\alpha}\right)}\delta^{\frac{1}{\alpha_-}}$. Suppose now that $\delta = \delta(N)$ is small enough, such that the latter expression is smaller than 1. Since $\alpha_- > 1$, this implies that

$$N^{1/\alpha_- - 1/\alpha}\delta^{1/\alpha_-} \le N^{\left(1+\frac{1}{\alpha_-}\right)\left(1-\frac{\alpha_-}{\alpha}\right)}\delta^{\frac{1}{\alpha_-}} \le \left(N^{\left(1+\frac{1}{\alpha_-}\right)\left(1-\frac{\alpha_-}{\alpha}\right)}\delta^{\frac{1}{\alpha_-}} \right)^{1/\alpha_-} \le r(N,\delta).$$

Moreover, for $\alpha < 1$, recalling (3.32), clearly $N^{1-\alpha_-/\alpha}\delta \le N^{2(1-\frac{\alpha_-}{\alpha})}\delta \le r(N,\delta)$, which allows to conclude.

3.5 Convergence to the limit system

3.5.1 Some technical results

We provide some technical lemmas that we state in this subsection for generic processes X and $\tilde{X}$ with associated measures μ and $\tilde{\mu}$ which can be either the associated empirical distributions or the respective conditional laws given S^α. In the sequel, M denotes the jump measure of S^α and, as before, $T_K = \inf\{t \ge 0 : |\Delta S_t^\alpha| > K\}$. Moreover, we will always assume that Assumptions 3.2.1–3.2.7 are satisfied.

Lemma 3.5.1. *Let*

$$M_t(\mu, \tilde{\mu}) := \int_{[0,t]\times\mathbb{R}^*} \left(\mu_{s^-}^{1/\alpha}(f) - \tilde{\mu}_{s^-}^{1/\alpha}(f) \right) z\tilde{M}(ds, dz) \quad \text{if} \quad \alpha > 1,$$

and

$$M_t(\mu, \tilde{\mu}) := \int_{[0,t]\times\mathbb{R}^*} \left(\mu_{s^-}^{1/\alpha}(f) - \tilde{\mu}_{s^-}^{1/\alpha}(f) \right) zM(ds, dz) \quad \text{if} \quad \alpha < 1.$$

i) For all $1 < \alpha < \alpha_+ \le 2$, *we have for any* $K > 0$,

$$\begin{aligned} \mathbb{E}\left[|M_t(\mu,\tilde{\mu})|^{\alpha_+} \mathbb{1}_{\{t<T_K\}} \right] &\le C \frac{K^{\alpha_+ - \alpha}}{\alpha_+ - \alpha} \int_0^t \mathbb{E}\left[|\mu_s(f) - \tilde{\mu}_s(f)|^{\alpha_+} \mathbb{1}_{\{s<T_K\}} \right] ds \\ &\le C \frac{K^{\alpha_+ - \alpha}}{\alpha_+ - \alpha} \int_0^t \mathbb{E}\left[W_1(\mu_s, \tilde{\mu}_s) \mathbb{1}_{\{s<T_K\}} \right] ds. \end{aligned} \quad (3.39)$$

ii) For all $0 < \alpha_- < \alpha < 1$, *we have for any* $K > 0$,

$$\mathbb{E}\left[|M_t(\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}\right] \le C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}\left[|\mu_s(f)-\tilde{\mu}_s(f)|\mathbb{1}_{\{s<T_K\}}\right]ds$$
$$\le C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}\left[W_{d_{\alpha_-}}(\mu_s,\tilde{\mu}_s)\mathbb{1}_{\{s<T_K\}}\right]ds. \quad (3.40)$$

Proof. We write

$$M_t(\mu,\tilde{\mu}) =: M_t^1(\mu,\tilde{\mu}) + M_t^2(\mu,\tilde{\mu}),$$

where $M_t^1(\mu,\tilde{\mu})$ corresponds to the integral over $[0,t]\times B_K$ and $M_t^2(\mu,\tilde{\mu})$ to that over $[0,t]\times B_K^c$. Let us first consider the case $1<\alpha<\alpha_+\le 2$. Using first the Burkholder-Davis-Gundy inequality and then the sub-additivity of $x \to x^{\alpha_+/2}$, we obtain

$$\mathbb{E}[|M_t^1(\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}] \le \mathbb{E}\left[\left|\int_{[0,t\wedge T_K]\times B_K}\left(\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f)\right)z\tilde{M}(ds,dz)\right|^{\alpha_+}\right]$$
$$\le C\mathbb{E}\left[\left(\int_{[0,t\wedge T_K]\times B_K}\left(\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f)\right)^2 z^2 M(ds,dz)\right)^{\alpha_+/2}\right]$$
$$\le C\int_0^t \mathbb{E}\left[|\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f)|^{\alpha_+}\mathbb{1}_{\{s<T_K\}}\right]ds\int_{B_K}\frac{|z|^{\alpha_+}}{|z|^{1+\alpha}}dz$$
$$= C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f)|^{\alpha_+}\mathbb{1}_{\{s<T_K\}}]ds.$$

As for $M_t^2(\mu,\tilde{\mu})$, using Jensen's inequality,

$$\mathbb{E}[|M_t^2(\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}] \le \mathbb{E}\left[\left|\int_0^t(\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f))\mathbb{1}_{\{s<T_K\}}ds\int_{B_K^c}\frac{z}{|z|^{1+\alpha}}dz\right|^{\alpha_+}\right]$$
$$\le C_t\int_0^t \mathbb{E}\left[|\mu_s^{1/\alpha}(f)-\tilde{\mu}_s^{1/\alpha}(f)|^{\alpha_+}\mathbb{1}_{\{s<T_K\}}\right]ds.$$

Now, in both previous bounds we use the Lipschitz property of $z\mapsto z^{1/\alpha}$ on $[\underline{f},\infty)$ to obtain the first inequality in (3.39) and we use (3.8) to obtain the second bound in (3.39):

$$\mathbb{E}\left[|M_t(\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}\right] \le C\mathbb{E}\left[|M_t^1(\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}\right] + C\mathbb{E}\left[|M_t^2(\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}\right]$$
$$\le C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|\mu_s(f)-\tilde{\mu}_s(f)|^{\alpha_+}\mathbb{1}_{\{s<T_K\}}]ds \le C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}\left[W_1(\mu_s,\tilde{\mu}_s)\mathbb{1}_{\{s<T_K\}}\right]ds.$$

Let now $0<\alpha<1$ so that $M_t^2(\mu,\tilde{\mu})$, being a non-compensated integral, equals zero on the event $\{t < T_K\}$. Using that $z \to z^{1/\alpha}$ is Lipschitz on $[0, \|f\|_\infty]$, we obtain

$$\begin{aligned}
\mathbb{E}[|M_t(\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] &= \mathbb{E}[|M_t^1(\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] \\
&\leq \mathbb{E}\left[\int_{[0,t\wedge T_K]\times B_K} \left|\mu_s^{1/\alpha}(f) - \tilde{\mu}_s^{1/\alpha}(f)\right| |z| M(ds,dz)\right] \\
&\leq C\mathbb{E}\left[\int_{[0,t]} \left|\mu_s^{1/\alpha}(f) - \tilde{\mu}_s^{1/\alpha}(f)\right| \mathbb{1}_{\{s<T_K\}} ds \int_{B_K} \frac{|z|}{|z|^{\alpha+1}} dz\right] \\
&\leq C\frac{K^{1-\alpha}}{1-\alpha}\mathbb{E}\left[\int_{[0,t]} \left|\mu_s(f) - \tilde{\mu}_s(f)\right| \mathbb{1}_{\{s<T_K\}} ds\right].
\end{aligned}$$

Finally, using (3.9), this last expression is in turn upper bounded by

$$C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}\left[W_{d_{\alpha_-}}(\mu_s,\tilde{\mu}_s)\mathbb{1}_{\{s<T_K\}}\right] ds.$$

□

Lemma 3.5.2. *Let*

$$B_t(X,\tilde{X},\mu,\tilde{\mu}) := \int_0^t [b(X_s,\mu_s) - b(\tilde{X}_s,\tilde{\mu}_s)] ds.$$

i) If $1 < \alpha < 2$, then for any $1 < \alpha < \alpha_+ \leq 2$ and for any $K > 0$,

$$\mathbb{E}[|B_t(X,\tilde{X},\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}] \leq C_t \left\{ \begin{array}{l} \mathbb{E}\int_0^t \mathbb{1}_{\{s<T_K\}}\left(|X_s-\tilde{X}_s|^{\alpha_+} + W_1^{\alpha_+}(\mu_s,\tilde{\mu}_s)\right) ds \\ \mathbb{E}\int_0^t \mathbb{1}_{\{s<T_K\}}\left(|X_s-\tilde{X}_s| + W_1(\mu_s,\tilde{\mu}_s)\right) ds \end{array} \right\}.$$

ii) If $0 < \alpha < 1$, then for any $0 < \alpha_- < \alpha$ and for any $K > 0$,

$$\mathbb{E}[|B_t(X,\tilde{X},\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] \leq C\mathbb{E}\int_0^t \mathbb{1}_{\{s<T_K\}}(d_{\alpha_-}(X_s,\tilde{X}_s) + W_{d_{\alpha_-}}(\mu_s,\tilde{\mu}_s)) ds.$$

In particular, if $X = X^{N,i}$, $\tilde{X} = \tilde{X}^{N,i}$, where $(X^{N,i})_{i=1}^N$, $(\tilde{X}^{N,i})_{i=1}^N$ are two systems defined on the same probability space and $(X^{N,i},\tilde{X}^{N,i})_{i=1}^N$ is exchangeable, and if moreover $\mu = \mu^{N,X}$, $\tilde{\mu} = \tilde{\mu}^{N,X}$ are the empirical measures of X and $\tilde{X}$, then

$$\mathbb{E}[|B_t(X,\tilde{X},\mu,\tilde{\mu})|^{\alpha_+}\mathbb{1}_{\{t<T_K\}}] \leq C \left\{ \begin{array}{l} \int_0^t \mathbb{E}[|X_s-\tilde{X}_s|^{\alpha_+}\mathbb{1}_{\{s<T_K\}}] ds \\ \int_0^t \mathbb{E}[|X_s-\tilde{X}_s|\mathbb{1}_{\{s<T_K\}}] ds \end{array} \right\}, \tag{3.41}$$

respectively

$$\mathbb{E}[|B_t(X,\tilde{X},\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] \le C\int_0^t \mathbb{E}[d_{\alpha_-}(X_s,\tilde{X}_s)\mathbb{1}_{\{s<T_K\}}]ds. \tag{3.42}$$

Proof. The inequalities of item i) and item ii) follow immediately from the Lipschitz property of b, using moreover the fact that b is also bounded in case $\alpha > 1$. Equations (3.41) and (3.42) follow from the fact that $\frac{1}{N}\sum_{i=1}^N \delta_{(X_s^{N,i},\tilde{X}_s^{N,i})}$ is a coupling of $\frac{1}{N}\sum_{i=1}^N \delta_{X_s^{N,i}}$ and $\frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}_s^{N,i}}$ and the exchangeability of $(X^{N,i},\tilde{X}^{N,i})_{i=1}^N$. Of course both inequalities hold also without indicator, but in the sequel we need them in this form. □

Lemma 3.5.3. *For $\alpha < 1$, let*

$$\Psi_t(X,\tilde{X},\mu,\tilde{\mu}) := \int_{[0,t]\times\mathbb{R}_+}\Big[\psi(X_{s^-},\mu_{s^-})\mathbb{1}_{\{z\le f(X_{s^-})\}} - \psi(\tilde{X}_{s^-},\tilde{\mu}_{s^-})\mathbb{1}_{\{z\le f(\tilde{X}_{s^-})\}}\Big]\bar{\pi}(ds,dz).$$

Then, for any $0<\alpha_-<\alpha$ and any $K>0$,

$$\mathbb{E}[|\Psi_t(X,\tilde{X},\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] \le C\int_0^t \mathbb{E}\Big[(d_{\alpha_-}(X_s,\tilde{X}_s)+W_{d_{\alpha_-}}(\mu_s,\tilde{\mu}_s))\mathbb{1}_{\{t<T_K\}}\Big]ds. \tag{3.43}$$

In particular, if $X = X^{N,i}$, $\tilde{X} = \tilde{X}^{N,i}$, where $(X^{N,i})_{i=1}^N$, $(\tilde{X}^{N,i})_{i=1}^N$ are two systems defined on the same probability space and $(X^{N,i},\tilde{X}^{N,i})_{i=1}^N$ is exchangeable, and if moreover $\mu=\mu^{N,X}$, $\tilde{\mu}=\tilde{\mu}^{N,X}$ are the empirical measures of X and $\tilde{X}$,

$$\mathbb{E}[|\Psi_t(X,\tilde{X},\mu,\tilde{\mu})|\mathbb{1}_{\{t<T_K\}}] \le C\int_0^t \mathbb{E}\Big[d_{\alpha_-}(X_s,\tilde{X}_s)\mathbb{1}_{\{t<T_K\}}\Big]ds. \tag{3.44}$$

Proof. The proof is analogous to the one of Lemma 3.5.2, using Assumption 3.2.6 and using the boundedness and Lipschitz continuity of f and the Kantorovitch-Rubinstein duality.

□

3.5.2 Introducing an auxiliary process

In what follows, to clearly distinguish the empirical measures of the respective auxiliary processes that we shall consider, we write

$$\mu_s^{N,X} = \frac{1}{N}\sum_{i=1}^N \delta_{X_s^{N,i}}$$

and use a similar notation when replacing X by another process Y or $\bar{X}$.

Based on Theorem 3.4.1, we introduce for all $N \in \mathbb{N}^*$, $i = 1, \dots, N$ the auxiliary process

$$Y_t^{N,i} = X_0^{N,i} + \int_0^t b(X_s^{N,i}, \mu_s^{N,X}) ds + \int_{[0,t]\times\mathbb{R}_+} \psi(X_{s^-}^{N,i}, \mu_{s^-}^{N,X}) \mathbb{1}_{\{z \le f(X_{s^-}^{N,i})\}} \bar{\pi}^i(ds, dz) + \int_{[0,t]} \left(\mu_{s^-}^{N,X}(f)\right)^{1/\alpha} dS_s^{N,\alpha}, \quad (3.45)$$

where we recall that $\psi(\cdot) \equiv 0$ in case $\alpha > 1$. Let $T_K^N = \inf\{t \ge 0 : |\Delta S_t^{N,\alpha}| > K\}$. In what follows, C_t denotes a constant (that may change from line to line) which is non-decreasing as a function of t, and $r(N, \delta)$ is given in Theorem 3.2.10.

Proposition 3.5.4. *Grant Assumptions 3.2.1–3.2.7.*

i) *If* $\alpha > 1$, *then we have for any* α_-, α_+ *such that* $1 < \alpha_- < \alpha < \alpha_+ < 2$,

$$\mathbb{E}[|X_t^{N,i} - Y_t^{N,i}| \mathbb{1}_{\{t < T_K^N\}}] \le C_t r_t^{X,Y},$$

where

$$r_t^{X,Y} := \left(\frac{K^{\alpha_+ - \alpha}}{\alpha_+ - \alpha}\right)^{1/\alpha_+} \left(N^{1-\alpha_-/\alpha}\delta\right)^{1/\alpha_+} + r(N, \delta).$$

ii) *If* $0 < \alpha < 1$, *then we have for any* α_- *such that* $0 < \alpha_- < \alpha < 1$, *for any* $K > 0$,

$$\mathbb{E}[d_{\alpha_-}(X_t^{N,i}, Y_t^{N,i}) \mathbb{1}_{\{t < T_K^N\}}] \le C_t \frac{K^{1-\alpha}}{1-\alpha} r(N, \delta).$$

Proof of Proposition 3.5.4. Clearly,

$$X_t^{N,i} = Y_t^{N,i} + R_t^N - E_t^{N,i} - M_t^N,$$

where R_t^N is defined in Theorem 3.4.1, $M_t^N := M_t(\mu^{N,X}, \mu_\tau^{N,X})$ as in Lemma 3.5.1 and

$$E_t^{N,i} := \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}^*} \frac{u}{N^{1/\alpha}} \mathbb{1}_{\{z \le f(X_{s^-}^{N,i})\}} \pi^i(ds, dz, du).$$

$$\mathbb{E}[|E_t^{N,i}|] \le \frac{C}{N^{1/\alpha}} \mathbb{E}\left[\int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}^*} |u| \mathbb{1}_{\{z \le \|f\|_\infty\}} \pi^i(ds, dz, du)\right] \le CtN^{-1/\alpha}, \quad (3.46)$$

whereas if $0 < \alpha_- < \alpha < 1$, using sub-additivity,

$$\mathbb{E}[|E_t^{N,i}|^{\alpha_-}] \le \frac{C}{N^{\alpha_-/\alpha}} \mathbb{E}\left[\int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}^*} |u|^{\alpha_-} \mathbb{1}_{\{z \le \|f\|_\infty\}} \pi^i(ds, dz, du)\right] \le CtN^{-\alpha_-/\alpha}, \quad (3.47)$$

Finally, since $N\delta \to \infty$, it is easy to see that both upper bounds (3.46),(3.47) are bounded by $Cr(N, \delta)$.

Let $1 < \alpha_- < \alpha < \alpha_+ < 2$. Then using Lemma 3.5.1, Jensen's inequality, exchangeability, the boundedness of f and item ii) of Proposition 3.4.4,

$$\begin{aligned} \mathbb{E}\left[|M_t^N|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] &\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}\left[\left|\mu_s^{N,X}(f)-\mu_{\tau_s}^{N,X}(f)\right|^{\alpha_+}\right]ds \qquad (3.48)\\ &\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|f(X_s^{N,1})-f(X_{\tau_s}^{N,1})|^{\alpha_+}]ds \\ &\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|X_s^{N,1}-X_{\tau_s}^{N,1})|^{\alpha_-}]ds \\ &\leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}N^{1-\alpha_-/\alpha}\delta. \end{aligned}$$

Remember that Proposition 3.4.1 gives $\mathbb{E}[|R_t^N|] \leq C_{q,t}r(N, \delta)$ in case $\alpha > 1$. Using this bound and collecting (3.48), (3.46) we obtain the claim *i*). Whereas if $0 < \alpha < 1$, the claim *ii*) follows from upper bounding $d_{\alpha_-}(X_t^{N,i}, Y_t^{N,i}) \leq \|R_t^N\|_{d_{\alpha_-}} + |E_t^{N,i}|^{\alpha_-} + |M_t^N|$, using then Theorem 3.4.1, (3.47), Lemma 3.5.1, exchangeability and Proposition 3.4.4:

$$\begin{aligned} \mathbb{E}\left[|M_t^N|\mathbb{1}_{\{t<T_K^N\}}\right] &\leq C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}\left[\left|\mu_s^{N,X}(f)-\mu_{\tau_s}^{N,X}(f)\right|\right]ds \\ &\leq C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}[|f(X_s^{N,1})-f(X_{\tau_s}^{N,1})|]ds \leq C_t\frac{K^{1-\alpha}}{1-\alpha}N^{1-\alpha_-/\alpha}\delta, \end{aligned}$$

noticing that this last term is upper bounded by $\frac{K^{1-\alpha}}{1-\alpha}r(N, \delta)$. □

Proposition 3.5.5. *Grant Assumptions 3.2.1–3.2.7. Let Y given by* (3.45) *and* $r_t^{X,Y}$ *given by Proposition 3.5.4. Then Y can be represented as*

$$\begin{aligned} Y_t^{N,i} = X_0^{N,i} + \int_0^t b(Y_s^{N,i},\mu_s^{N,Y})ds + \int_{[0,t]\times\mathbb{R}_+}\psi(Y_{s^-}^{N,i},\mu_{s^-}^{N,Y})\mathbb{1}_{\{z\leq f(Y_{s^-}^{N,i})\}}\bar{\pi}^i(ds,dz) \\ + \int_{[0,t]}\left(\mu_{s^-}^{N,Y}(f)\right)^{1/\alpha}dS_s^{N,\alpha} + R_t^{N,Y}, \quad \textit{and} \end{aligned}$$

i) if $1 < \alpha < 2$, *then for any* $1 < \alpha < \alpha_+ \leq 2$,

$$\mathbb{E}[|R_t^{N,Y}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}] \leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}r_t^{X,Y};$$

ii) if $0 < \alpha < 1$, *then for any* $K > 0$,

$$\mathbb{E}[|R_t^{N,Y}|\mathbb{1}_{\{t<T_K^N\}}] \leq C_t\left(\frac{K^{1-\alpha}}{1-\alpha}\right)^2 r(N,\delta).$$

Proof. Let $1 < \alpha < 2$. Then $R_t^{N,Y} = B_t^N + M_t^N$, with $M_t^N := M_t(\mu^{N,X}, \mu^{N,Y})$ and $B_t^N = B_t(X^{N,i}, Y^{N,i}, \mu^{N,X}, \mu^{N,Y})$. Using first (3.41) of Lemma 3.5.2 and then Proposition 3.5.4, we obtain

$$\mathbb{E}[|B_t^N|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}] \leq C\int_0^t \mathbb{E}[|X_s^{N,i} - Y_s^{N,i}|\mathbb{1}_{\{s<T_K^N\}}]ds \leq Ctr_t^{X,Y}. \tag{3.49}$$

Moreover, using Lemma 3.5.1, Jensen's inequality, the exchangeability of $(X^{N,i}, Y^{N,i})_i$, together with the boundedness and the Lipschitz continuity of f,

$$\begin{aligned}
\mathbb{E}\left[|M_t^N|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] &\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}\left[\left|\mu_s^{N,X}(f) - \mu_s^{N,Y}(f)\right|^{\alpha_+}\mathbb{1}_{\{s<T_K^N\}}\right]ds \\
&\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N \left|f(X_s^{N,i}) - f(Y_s^{N,i})\right|^{\alpha_+}\mathbb{1}_{\{s<T_K^N\}}\right] \\
&\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|f(X_s^{N,i}) - f(Y_s^{N,i})|^{\alpha_+}\mathbb{1}_{\{s<T_K^N\}}]ds \\
&\leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}\int_0^t \mathbb{E}[|X_s^{N,i} - Y_s^{N,i}|\mathbb{1}_{\{s<T_K^N\}}]ds \leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}r_t^{X,Y}.
\end{aligned} \tag{3.50}$$

Finally, collecting (3.49) and (3.50) we obtain the claim *i*).

Let now $0 < \alpha_- < \alpha < 1$. Clearly $R_t^{N,Y} = B_t^N + \Psi_t^N + M_t^N$, with $M_t^N := M_t(\mu^{N,X}, \mu^{N,Y})$, $\Psi_t^N := \Psi_t(X^{N,i}, Y^{N,i}, \mu^{N,X}, \mu^{N,Y})$ and $B_t^N := B_t(X^{N,i}, Y^{N,i}, \mu^{N,X}, \mu^{N,Y})$. Now we have, by (3.42) of Lemma 3.5.2 and (3.44) of Lemma 3.5.3 respectively,

$$\mathbb{E}[|B_t^N|\mathbb{1}_{\{t<T_K^N\}}] \leq C\int_0^t \mathbb{E}[d_{\alpha_-}(X_s^{N,i}, Y_s^{N,i})\mathbb{1}_{\{s<T_K^N\}}]ds$$

and

$$\mathbb{E}[|\Psi_t^N|\mathbb{1}_{\{t<T_K^N\}}] \leq C\int_0^t \mathbb{E}[d_{\alpha_-}(X_s^{N,i}, Y_s^{N,i})\mathbb{1}_{\{s<T_K^N\}}]ds,$$

so that, using Proposition 3.5.4 in both the last inequalities, we obtain

$$\mathbb{E}[(|B_t^N| + |\Psi_t^N|)\mathbb{1}_{\{t<T_K^N\}}] \leq C_t\frac{K^{1-\alpha}}{1-\alpha}r(N,\delta).$$

Finally, (3.40) implies

$$\mathbb{E}[|M_t^N|\mathbb{1}_{\{t<T_K^N\}}] \quad \leq \quad C\frac{K^{1-\alpha}}{1-\alpha}\int_0^t \mathbb{E}\left[W_{d_{\alpha_-}}(\mu_s,\tilde{\mu}_s)\mathbb{1}_{\{s<T_K^N\}}\right]ds \quad \leq \quad C_t\left(\frac{K^{1-\alpha}}{1-\alpha}\right)^2 r(N,\delta),$$

concluding the proof. □

3.5.3 Representation for the limit system

In this subsection we prove the following representation result.

Proposition 3.5.6. *Grant Assumptions 3.2.1–3.2.7. Let $\bar{X}$ denote the unique solution to system* (3.7) *driven by $S^{N,\alpha}$, and $(\bar{X}^{N,i})_{i=1,\dots,N}$ be the first N coordinates of such solution. Then, for all $i = 1,\dots,N$,*

$$\bar{X}_t^{N,i} = \bar{X}_0^{N,i} + \int_0^t b(\bar{X}_s^{N,i},\mu_s^{N,\bar{X}})ds + \int_{[0,t]\times\mathbb{R}_+}\psi(\bar{X}_{s^-}^{N,i},\mu_{s^-}^{N,\bar{X}}(f))\mathbb{1}_{\{z\leq f(\bar{X}_{s^-}^{N,i})\}}\bar{\pi}^i(ds,dz) + \int_{[0,t]}\left(\mu_{s^-}^{N,\bar{X}}(f)\right)^{1/\alpha}dS_s^{N,\alpha} + R_t^{N,\bar{X}},$$

where

i) if $1<\alpha<\alpha_+<2$,

$$\mathbb{E}\left[|R_t^{N,\bar{X}}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] \leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}N^{-1/2};$$

ii) if $0<\alpha_-<\alpha<1$,

$$\mathbb{E}\left[|R_t^{N,\bar{X}}|\mathbb{1}_{\{t<T_K^N\}}\right] \leq C_t\frac{K^{1-\alpha}}{1-\alpha}(N^{-1/2}\mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}}).$$

To prove the above result we will need the following lemma.

Lemma 3.5.7. *Supposing that $E(|X_0|^{(2\alpha)\vee 1})<\infty$ if $\alpha<1$ and $\mathbb{E}(|X_0|^p)<\infty$ for some $p>2$ in case $\alpha>1$, we have for any $t\geq 0$ and $\alpha_-<\alpha$*

$$\mathbb{E}(W_{\alpha_-}(\mu_t^{N,\bar{X}},\bar{\mu}_t))\mathbb{1}_{\alpha<1} + \mathbb{E}(W_1(\mu_t^{N,\bar{X}},\bar{\mu}_t))\mathbb{1}_{\alpha>1} \leq C_t\left\{\begin{array}{ll} N^{-1/2} & \alpha>1 \\ N^{-1/2}, & 1>\alpha_->\frac{1}{2} \\ N^{-\alpha_-}, & \alpha_-<\frac{1}{2}\end{array}\right\},$$

where C_t is a positive constant which is non-decreasing in t.

The proof of this Lemma is given in the Appendix section D.

Proof of Prop. 3.5.6. Here, with $\Psi = 0$ in the case $1 < \alpha < 2$,

$$R_t^{N,\bar{X}} = B_t(\bar{X}^{N,i}, \bar{X}^{N,i}, \mu^{N,\bar{X}}, \bar{\mu}) + \Psi_t(\bar{X}^{N,i}, \bar{X}^{N,i}, \mu^{N,\bar{X}}, \bar{\mu}) + M_t(\mu^{N,\bar{X}}, \bar{\mu}).$$

We start with the proof of the case $1 < \alpha < 2$. Using Lemma 3.5.2,

$$\mathbb{E}[|B_t(\bar{X}^{N,i}, \bar{X}^{N,i}, \mu^{N,\bar{X}}, \bar{\mu})|^{\alpha_+} \mathbb{1}_{\{s<T_K^N\}}] \leq \int_0^t \mathbb{E}\left[W_1(\mu_s^{N,\bar{X}}, \bar{\mu}_s)\right] ds \leq CtN^{-1/2}.$$

Using Lemma 3.5.1 and Lemma 3.5.7, the result of *i*) follows from

$$\mathbb{E}\left[|M_t(\mu^{N,\bar{X}}, \bar{\mu})|^{\alpha_+} \mathbb{1}_{\{t<T_K^N\}}\right] \leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha} \int_0^t \mathbb{E}\left[W_1(\mu_s^{N,\bar{X}}, \bar{\mu}_s)\right] ds \leq C_t \frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha} N^{-1/2}.$$

We now deal with the case $0 < \alpha < 1$. From Lemma 3.5.2 and Lemma 3.5.7,

$$\begin{aligned}
&\mathbb{E}[|B_t(\bar{X}^{N,i}, \bar{X}^{N,i}, \mu^{N,\bar{X}}, \bar{\mu})| \mathbb{1}_{\{s<T_K^N\}}] \\
&\qquad \leq C \int_0^t \mathbb{E}\left[W_{d_{\alpha_-}}(\mu_s^{N,\bar{X}}, \bar{\mu}_s)\right] ds \leq C \int_0^t \mathbb{E}\left[W_{\alpha_-}(\mu_s^{N,\bar{X}}, \bar{\mu}_s)\right] ds \\
&\qquad\qquad \leq Ct(N^{-1/2} \mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-} \mathbb{1}_{\{\alpha_-<\frac{1}{2}\}}).
\end{aligned}$$

The same bound is true for $\Psi_t(\bar{X}^{N,i}, \bar{X}^{N,i}, \mu^{N,\bar{X}}, \bar{\mu})$, by Lemma 3.5.3. Moreover, from Lemma 3.5.1 and Lemma 3.5.7,

$$\begin{aligned}
\mathbb{E}\left[|M_t(\mu^{N,\bar{X}}, \bar{\mu}) \mathbb{1}_{\{t<T_K^N\}}\right] &\leq C\frac{K^{1-\alpha}}{1-\alpha} \int_0^t \mathbb{E}[W_{d_{\alpha_-}}(\mu_s^{N,\bar{X}}, \bar{\mu}_s)] ds \\
&\leq \frac{K^{1-\alpha}}{1-\alpha} Ct(N^{-1/2} \mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-} \mathbb{1}_{\{\alpha_-<\frac{1}{2}\}}),
\end{aligned}$$

concluding the proof.

□

3.5.4 Bounding the distance between $Y_t^{N,i}$ and $\bar{X}_t^{N,i}$

Proposition 3.5.8. *Grant Assumptions 3.2.1–3.2.3.*

Then, for all $N \in \mathbb{N}^*$ and $i = 1, \ldots, N$,

i) For $1 < \alpha < 2$, for all $\alpha < \alpha_+ < 2$ and for all $K > 0$,

$$\mathbb{E}\left[|Y_t^{N,i} - \bar{X}_t^{N,i}|^{\alpha_+} \mathbb{1}_{\{t<T_K^N\}}\right] \leq (r_t^{X,Y} + N^{-1/2}) e^{Ct\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}}.$$

ii) For $0 < \alpha < 1$*, for all* $0 < \alpha_- < \alpha$ *and all* $K > 0$*,*

$$\mathbb{E}\left[|Y_t^{N,i} - \bar{X}_t^{N,i}|\mathbb{1}_{\{t<T_K^N\}}\right] \leq (r(N,\delta) + N^{-1/2}\mathbb{1}_{\{\alpha>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}})e^{Ct\frac{K^{1-\alpha}}{1-\alpha}}.$$

Proof. Using the representations of $\bar{X}$ and of Y given by Propositions 3.5.6 and 3.5.5,

$$\begin{aligned} Y_t^{N,i} - \bar{X}_t^{N,i} = B(Y_t^{N,i}, \bar{X}_t^{N,i}, \mu^{N,Y}, \mu^{N,\bar{X}}) + \Psi(Y_t^{N,i}, \bar{X}_t^{N,i}, \mu^{N,Y}, \mu^{N,\bar{X}}) \\ + M_t(\mu^{N,Y}, \mu^{N,\bar{X}}) + R_t^{N,\bar{X}} + R_t^{N,Y}. \end{aligned}$$

We start with the proof of the case $1 < \alpha < 2$. In this case

$$\mathbb{E}\left[|R_t^{N,\bar{X}}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] \leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha}N^{-1/2}$$

and

$$\mathbb{E}[|R_t^{N,Y}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}] \leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha}r_t^{X,Y}.$$

Using Lemma 3.5.2,

$$\mathbb{E}\left[|B(Y_t^{N,i}, \bar{X}_t^{N,i}, \mu^{N,Y}, \mu^{N,\bar{X}})|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] \leq C\int_0^t \mathbb{E}[|Y_s^{N,i} - \bar{X}_s^{N,i}|^{\alpha_+}\mathbb{1}_{\{s<T_K^N\}}]ds.$$

Furthermore, using the first inequality in (3.39) in Lemma 3.5.1, Jensen's inequality and exchangeability,

$$\mathbb{E}\left[|M_t(\mu^{N,Y}, \mu^{N,\bar{X}})|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] \leq C\frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha}\int_0^t \mathbb{E}\left[|Y_s^{N,i} - \bar{X}_s^{N,i}|^{\alpha_+}\mathbb{1}_{\{s<T_K^N\}}\right]ds.$$

We conclude, using Gronwall's Lemma, that

$$\begin{aligned} \mathbb{E}\left[|Y_t^{N,i} - \bar{X}_t^{N,i}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}\right] &\leq C_t\frac{K^{\alpha_+-\alpha}}{\alpha_+ - \alpha}(r_t^{X,Y} + N^{-1/2})e^{\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}t} \\ &\leq C(r_t^{X,Y} + N^{-1/2})e^{Ct\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}}, \end{aligned}$$

where in the last inequality we have used that $xe^x \leq e^{Cx}$ for some $C > 0$, for all $x \geq 0$.

We now deal with the case $0 < \alpha < 1$. Then

$$\mathbb{E}\left[|R_t^{N,\bar{X}}|\mathbb{1}_{\{t<T_K^N\}}\right] \leq C_t\frac{K^{1-\alpha}}{1-\alpha}(N^{-1/2}\mathbb{1}_{\{\alpha>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}})$$

and

$$\mathbb{E}[|R_t^{N,Y}|\mathbb{1}_{\{t<T_K^N\}}] \leq C_t \left(\frac{K^{1-\alpha}}{1-\alpha}\right)^2 r(N,\delta).$$

By Lemma 3.5.2,

$$\mathbb{E}\left[|B(Y_t^{N,i}, \bar{X}_t^{N,i}, \mu^{N,Y}, \mu^{N,\bar{X}})|\mathbb{1}_{\{t<T_K^N\}}\right] \leq C \int_0^t \mathbb{E}[|Y_s^{N,i} - \bar{X}_s^{N,i}|\mathbb{1}_{\{s<T_K^N\}}]ds.$$

The same bound is true for Ψ, from Lemma 3.5.3. Using Lemma 3.5.1 and the boundedness of f,

$$\mathbb{E}\left[|M_t(\mu^{N,Y}, \mu^{N,\bar{X}})|\mathbb{1}_{\{t<T_K^N\}}\right] \leq C\frac{K^{1-\alpha}}{1-\alpha} \int_0^t \mathbb{E}\left[|Y_s^{N,i} - \bar{X}_s^{N,i}|\mathbb{1}_{\{s<T_K^N\}}\right].$$

We conclude similarly as above, using Gronwall's Lemma, that in the case $0 < \alpha < 1$,

$$\mathbb{E}\left[|Y_t^{N,i} - \bar{X}_t^{N,i}|\mathbb{1}_{\{t<T_K^N\}}\right] \quad \leq \quad C_t(r(N,\delta) \; + \; N^{-1/2}\mathbb{1}_{\{\alpha>\alpha_->\frac{1}{2}\}} \; + \; N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}})e^{Ct\frac{K^{1-\alpha}}{1-\alpha}}.$$

□

3.5.5 Proof of Theorem 3.2.10

Proof. We start with $\alpha > 1$. In this case, using Propositions 3.5.4 and 3.5.8 and supposing $r_t^{X,Y} \leq 1$ (which is true for N sufficiently large),

$$\begin{aligned}\mathbb{E}[|X_t^{N,i} - \bar{X}_t^{N,i}|\mathbb{1}_{\{t<T_K^N\}}] &\leq \mathbb{E}[|X_t^{N,i} - Y_t^{N,i}|] + \left(\mathbb{E}[|Y_t^{N,i} - \bar{X}_t^{N,i}|^{\alpha_+}\mathbb{1}_{\{t<T_K^N\}}]\right)^{1/\alpha_+} \\ &\leq C_t r_t^{X,Y} + \left((r_t^{X,Y} + N^{-1/2})e^{Ct\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}}\right)^{1/\alpha_+} \\ &\leq (r_t^{X,Y} + N^{-1/2})^{1/\alpha_+}e^{Ct\frac{K^{\alpha_+-\alpha}}{\alpha_+-\alpha}}.\end{aligned}$$

Let now $0 < \alpha < 1$. In this case, using Propositions 3.5.4 and 3.5.8,

$$\begin{aligned}\mathbb{E}[d_{\alpha_-}(X_t^{N,i}, \bar{X}_t^{N,i})\mathbb{1}_{\{t<T_K^N\}}] &\leq \mathbb{E}[d_{\alpha_-}(X_t^{N,i}, Y_t^{N,i})] + \mathbb{E}[|Y_t^{N,i} - \bar{X}_t^{N,i}|\mathbb{1}_{\{t<T_K^N\}}] \\ &\leq C_t\frac{K^{1-\alpha}}{1-\alpha}r(N,\delta) + (r(N,\delta) + N^{-1/2}\mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}})e^{Ct\frac{K^{1-\alpha}}{1-\alpha}} \\ &\leq C\left(r(N,\delta) + N^{-1/2}\mathbb{1}_{\{1>\alpha_->\frac{1}{2}\}} + N^{-\alpha_-}\mathbb{1}_{\{\alpha_-<\frac{1}{2}\}}\right)e^{Ct\frac{K^{1-\alpha}}{1-\alpha}},\end{aligned}$$

which finishes the proof. □

Appendices

A Equilibria of system (1.15)

In this section, we consider system (1.15) and study the nature of its fixed points, depending on the values of the parameters A and B. Throughout this analysis, we use the basic theory of dynamical systems as it can be found for instance in [57].

For the moment, we only assume $A > 1$ and $B > 0$. We will restrict later to the parameter regime given by hypothesis **(H)** in Section 1.4.2.

1. The fixed points $(0, 0)$ and $\pm(1, 1)$ are present for any values of A and B.

   (a) The linearized system around the origin has the eigenvalues

   $$\lambda_1 = 1 \text{ and } \lambda_2 = 1 - A + B = 1 - \gamma,$$

   where $\gamma := A - B$. Thus $(0, 0)$ is a saddle when $\gamma > 1$, it has an unstable and a neutral direction for $\gamma = 1$ and it is an unstable node otherwise.

   (b) Eigenvalues of the linearized system around $\pm(1, 1)$ are

   $$\lambda_1 = -2 \text{ and } \lambda_2 = -2 - A + B = -2 - \gamma.$$

   As a consequence, $\pm(1, 1)$ are stable nodes for $\gamma > -2$. They have a neutral and a stable direction when $\gamma = -2$ and they are saddle points otherwise.

   In conclusion: i) for $\gamma < -2$, $(0, 0)$ is unstable and $\pm(1, 1)$ are saddle points; ii) when $-2 < \gamma < 1$, $(0, 0)$ is unstable and $\pm(1, 1)$ are stable nodes; iii) for $\gamma > 1$, $(0, 0)$ is a saddle point and $\pm(1, 1)$ are stable nodes.

2. Depending on the values of the parameters A and B, there might be two additional equilibria. We search for equilibria of the form $(x, \beta x)$ with $\beta \neq 0$, $x \neq 0$. Notice

that all the possible equilibria except for $(0, 0)$ have such a form.
Substituting $y = \beta x$ in the first equation of (1.15), we get

$$\bar{x}_\beta = \pm\left(\sqrt{1 - A(1-\beta)}, \beta\sqrt{1 - A(1-\beta)}\right), \tag{51}$$

subject to the condition

$$\beta > \frac{A-1}{A}. \tag{52}$$

Notice that no $\beta < 0$ fulfills (52), since we have assumed $A > 1$. Therefore, system (1.15) can possibly have extra fixed points of the form $(x, \beta x)$ only if they lie in the first or the third quadrant.
The second equation in (1.15) leads to the fixed point equation

$$\beta = f(\beta) \quad \text{with} \quad f(\beta) := \sqrt{\frac{1 - B\frac{1-\beta}{\beta}}{1 - A(1-\beta)}}. \tag{53}$$

Observe that, for $\beta = 1$, we recover the equilibria $\pm(1, 1)$. Therefore, fixed points of the type $\bar{x}_\beta$ may exist if condition (52) is satisfied and Eq. (53) has real solutions, that is if

$$\beta > \max\left\{\frac{A-1}{A}, \frac{B}{1+B}\right\}, \tag{54}$$

which is equivalent to

$$\begin{cases} \beta > \frac{A-1}{A} = \frac{B}{1+B}, & \text{if } B = A-1 \\ \beta > \frac{A-1}{A}, & \text{if } B < A-1 \\ \beta > \frac{B}{1+B}, & \text{if } B > A-1. \end{cases}$$

Therefore, we have the following.

- If $B = A - 1$, Eq. (53) becomes $\beta = \frac{1}{\sqrt{\beta}}$, whose unique solution is $\beta = 1$. In this case, $\gamma = 1$, so $\pm(1, 1)$ are stable nodes and $(0, 0)$ has a zero eigenvalue, thus it is not a hyperbolic fixed point and the linearization cannot give information about the phase portrait close to it. The dynamical system (1.15) can be rewritten as

$$\begin{aligned} \dot{x} &= -x^3 - x(A-1) + Ay \\ \dot{y} &= -y^3 + x - A(x-y). \end{aligned} \tag{55}$$

Observe that the linear terms in both the components of the vector field in Eq. (55) are positive above the line $y = \frac{A-1}{A}x$ and negative below it. Thus, the third-order terms can be neglected close to the equilibrium and the linearization gives an accurate sketch of the phase portrait of the system locally. Along the line $y = \frac{A-1}{A}x$, that is the eigendirection of the zero eigenvalue, only the third-order terms count and we get $\dot{x} < 0, \dot{y} < 0$ in the first quadrant and $\dot{x} > 0, \dot{y} > 0$ in the third one. Overall, the eigendirection of the zero eigenvalue is locally stable.

- If $B < A - 1$, due to condition (54), we expect to find solutions to Eq. (53) only for $\beta > \frac{A-1}{A}$. Observe that $f(\beta)$ has a vertical asymptote to positive infinity as β approaches $\frac{A-1}{A}$ and it has a horizontal asymptote to zero as β grows to infinity. Moreover, $\frac{\partial f}{\partial \beta}(\beta) = 0$ when

$$\beta_{\pm} = \frac{AB \pm \sqrt{AB(B-(A-1))}}{A(1+B)}, \tag{56}$$

  which are complex for $B < A - 1$. Therefore, $f(\beta)$ is strictly decreasing. Thus, its graph cannot have more than one intersection with the line $y = \beta$ and this unique intersection must be at $\beta = 1$. Overall, when $B < A - 1$, no fixed points of the type $(x, \beta x)$ are present, except for $\pm(1, 1)$. In this setting, we already established that $(0, 0)$ is a saddle point and $\pm(1, 1)$ are stable nodes.

- If $B > A - 1$, we have already pointed out that $(0, 0)$ is unstable, while the nature of $\pm(1, 1)$ can change according to γ being less than or greater than -2. From condition (54), it follows that we have to look for solutions to Eq. (53) for $\beta > \frac{B}{B+1}$. Observe that $f\left(\frac{B}{1+B}\right) = 0$ and $\lim_{\beta \to +\infty} f(\beta) = 0$. The points $\beta_{\pm}$, given in (56), where $\frac{\partial f}{\partial \beta}(\beta) = 0$, are real and distinct in this case. Moreover, $\beta_{-} < \frac{B}{1+B}$. Hence, the function $f(\beta)$ has only a critical point (maximum) at $\beta = \beta_{+} > \frac{B}{1+B}$, so it may cross the line $y = \beta$ once, twice or never. In particular, since we know the solution $\beta = 1$ to be always present, the intersections might coincide ($\beta = 1$ itself) or be distinct ($\beta = 1$ and a second intersection, for β greater or less than 1).

  We distinguish three subcases.

  - The intersection at $\beta = 1$ is the unique solution to Eq. (53) if the graphs of $y = \beta$ and $y = f(\beta)$ are tangent at that point, i.e., if $\frac{\partial f}{\partial \beta}(1) = 1$. This holds only if $B = A + 2$. In this case, the analysis of the linearized

system tells us that $\pm(1,1)$ have a negative and a zero eigenvalue and to check stability one has to take into account higher-order terms.
We study only the point $(1,1)$, the analysis of $-(1,1)$ being similar. To make our computations easier, we translate the vector field so that the fixed point $(1,1)$ is shifted to $(0,0)$. We make the change of variables $u = x - 1$ and $v = y - 1$. In the new coordinates (u, v), system (1.15) becomes

$$\begin{aligned} \dot{u} &= -(A+2)u + Av - 3u^2 - u^3 \\ \dot{v} &= -(A+2)u + Av - 3v^2 - v^3. \end{aligned} \tag{57}$$

The line $v = \frac{A+2}{A}u$, along which the first-order terms in (57) vanish, that is the eigendirection of the zero eigenvalue, always lies above the line $v = u$, that is the eigendirection of the non-zero eigenvalue. The first-order terms in Eq. (57) are positive above the line $v = \frac{A+2}{A}u$ and negative below it. So, out of this line, higher-order terms can be neglected close to the origin, whereas the second-order terms become non-negligible as soon as we are along that line, where it is immediate to see that the vector field points downward-left.

- If $0 < \frac{\partial f}{\partial \beta}(1) < 1$, i.e., if $B < A + 2$, two intersections are present, one at $\beta = 1$ and one at $\beta = \beta_\times(A, B) < 1$. The smallest solution to Eq. (53) gives rise to two extra fixed points of the type (51), with both coordinates smaller than 1 in absolute value.
- If $\frac{\partial f}{\partial \beta}(1) > 1$, i.e., $B > A + 2$, in addition to the intersection at $\beta = 1$, we have a second intersection at $\beta = \beta_\times(A, B) > 1$ and we get two fixed points of the type (51), with both coordinates greater than 1 in absolute value.

From now on we restrict to the case $B > A - 1$ (hypothesis **(H)**) and we examine in more detail what happens in the three cases that we considered in Section 1.4.2 and that are shown in Fig. 1.4. The point $(0,0)$ is an unstable fixed point in all the scenarios. We will give information on the other equilibria.
Case 1. If $A = 2$, $B = 2.5$, Eq. (53) has the solutions $\beta = 1$ and $\beta = \beta_\times(A, B) < 1$, numerically obtained. From Eq. (51), we obtain respectively the fixed points $\pm\bar{x}_1 = \pm(1,1)$ and $\pm\bar{x}_{\beta_\times} = \pm(0.78, 0.63)$. The eigenvalues of the linearized system around $\pm\bar{x}_1$ are both real and negative, implying that the points are stable nodes. The fixed points $\pm\bar{x}_{\beta_\times}$ turn out to be saddle points. The phase portrait numerically obtained for this first

case is shown in Fig. 1.4(a).
Case 2. If $A = 2$, $B = 4$, Eq. (53) has the unique solution $\beta = 1$, so the only fixed points, apart from $(0, 0)$, are $\pm\bar{x}_1$. We refer the reader to the analysis above, which holds for any $A > 1$, $B = A + 2$, and to Fig. 1.4(b).
Case 3. If $A = 2$, $B = 7$, Eq. (53) has two solutions: $\beta = 1$ and $\beta = \beta_\times(A, B) > 1$. The fixed points $\pm\bar{x}_1$ can be easily seen to be saddle points, while the fixed points $\pm\bar{x}_{\beta_\times} = \pm(1.24, 1.58)$ have complex conjugate eigenvalues with negative real part, thus they are stable spirals. The phase portrait in this case is shown in Fig. 1.4(c).

B Proof of Theorem 3.2.8

Proof. Fix $N \in \mathbb{N}^*$ and let $\tau_0 := 0$ and $(\tau_n)_{n\geq 1}$ be the sequence of the jump times of the Poisson process $(\sum_{j=1}^N \pi^j([0, t] \times [0, \|f\|_\infty] \times \mathbb{R}))_{t\geq 0}$. Let moreover $(U^k)_{k\geq 1}$ be an i.i.d. family of real valued random variables, $\sim \nu$, such that U^k is the atom of $\sum_{j=1}^N \pi^j(\{\tau_k\} \times [0, \|f\|_\infty], \cdot)$. We construct the solution to (2.5) recursively on each interval $[\tau_n, \tau_{n+1}]$, $n \geq 0$. On $[\tau_0, \tau_1)$, the solution to (2.5) obeys

$$X_t^{N,i} = X_0^{N,i} + \int_0^t b(X_s^{N,i}, \mu_s^N)ds. \tag{58}$$

By Assumption 3.2.1b), (58) admits a unique solution on $[0, \tau_1[$ ([42]). Suppose that at τ_1, the i-th particle has a main jump. We put then $X_{\tau_1}^{N,i} = X_{\tau_1-}^{N,i} + \psi(X_{\tau_1-}^{N,i}, \mu_{\tau_1-}^N)$ in case $\alpha < 1$ and $X_{\tau_1}^{N,i} = X_{\tau_1-}^{N,i}$ if $\alpha > 1$. Moreover, we put $X_{\tau_1}^{N,j} = X_{\tau_1-}^{N,j} + \frac{U_1}{N^{1/\alpha}}$ for all $j \neq i$. We then solve the equation (58) on $[0, \tau_2 - \tau_1]$ with this new initial condition and so on. Since N is finite, $\tau_n \to \infty$ as $n \to \infty$, such that the above construction can be achieved on the whole positive time axis. □

C Proof of Theorem 3.2.9 (existence part)

In this section we prove strong existence for the limit system (3.7). We start discussing the case $\alpha > 1$ when there are no main jumps. Our argument is partially inspired by the one used in Proposition 2 in [33]. Fix a truncation level $K > 0$, recall that we assume $X_0 \in L^2$ and that we have introduced the constant M_K in (3.15). We define the following Picard iteration for all $n \geq 1$:

$$\begin{aligned} X_t^{[0],K} &\equiv X_0 \quad \forall\, t \geq 0, \\ X_0^{[n],K} &= X_0 \quad \forall\, n \geq 1, \end{aligned}$$

$$X_t^{[n],K} = X_0^{[n-1],K} + \int_0^t b(X_s^{[n-1],K}, \mu_s^{[n-1],K})ds + \int_{[0,t]\times B_K} \left(\mu_{s-}^{[n-1],K}(f)\right)^{1/\alpha} z\tilde{M}(ds,dz)$$
$$-M_K \int_0^t \left(\mu_s^{[n-1],K}(f)\right)^{1/\alpha} ds, \tag{59}$$

where $\mu_s^{[n],K} := \mathcal{L}(X_s^{[n],K} \,|\, S^\alpha)$.

Using similar arguments as in the proof of the uniqueness, we have the a priori bound

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{[n],K}\right|^2\right] \le C_{K,T}(1 + \mathbb{E}[|X_0|^2]) \tag{60}$$

for some constant $C_{K,T}$ which does only depend on the truncation level K and on T, but not on n.

We show now that, for any $T > 0$ and $t \in [0,T]$, the sequence $(X_t^{[n],K})_{n\ge 0}$ defined in (59) converges a.s. to a limit $\bar{X}_t^K$.

First notice that, analogously to what we obtained in the uniqueness proof (see (3.17)),

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{[n+1],K} - X_t^{[n],K}\right|^2\right] \le C_{K,T} \int_0^T \mathbb{E}\left[\sup_{t\in[0,s]} \left|X_t^{[n],K} - X_t^{[n-1],K}\right|^2\right] ds \tag{61}$$

for some constant $C_{K,T}$ non-decreasing with respect to T.

Introduce for any $n \ge 0$

$$u_T^{[n],K} := \mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{[n+1],K} - X_t^{[n],K}\right|^2\right] \tag{62}$$

and iterate (61) to obtain

$$u_T^{[n],K} \le C_{K,T}^2 \int_0^T ds_1 \int_0^{s_1} u_{s_2}^{[n-2],K} ds_2 \le C_{K,T}^n \int_0^T ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n u_{s_n}^{[0],K}$$
$$\le \frac{T^n}{n!} C_{K,T}^n u_T^{[0],K},$$

where $u_T^{[0],K} = \mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{[1],K} - X_t^{[0],K}\right|^2\right] \le C_T$ is bounded thanks to (60).

This implies that

$$\mathbb{E}\left[\sum_{n\ge 0} \sup_{t\in[0,T]} \left|X_t^{[n+1],K} - X_t^{[n],K}\right|\right] \le \sum_{n\ge 0} \sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{[n+1],K} - X_t^{[n],K}\right|^2\right]}$$

$$= \sum_{n\geq 0} \sqrt{u_T^{[n],K}} \leq \sqrt{C_T} \sum_{n\geq 0} C_{K,T}^{n/2} \frac{T^{n/2}}{\sqrt{n!}} < +\infty,$$

such that

$$\sup_{t\in[0,T]} \sum_{n\geq 0} \left| X_t^{[n+1],K} - X_t^{[n],K} \right| < +\infty \qquad \text{a.s..}$$

Hence the series $\sum_{n\geq 0} \left(X_t^{[n+1],K} - X_t^{[n],K} \right)$ converges a.s. and we can define a.s.

$$\bar{X}_t^K := X_0 + \sum_{n\geq 0} \left(X_t^{[n+1],K} - X_t^{[n],K} \right).$$

The next step is to prove that the a.s. limit of $(X_t^{[n],K})_{n\geq 0}$, $\bar{X}_t^K$, solves the same equation as the process $\bar{X}_t$ on $[0, T_K[$, that is, that almost surely,

$$\begin{aligned} \bar{X}_t^K &= X_0 + \int_0^t b(\bar{X}_s^K, \bar{\mu}_s^K) ds + \int_{[0,t]\times B_K} \left(\bar{\mu}_{s^-}^K(f) \right)^{1/\alpha} z\tilde{M}(ds, dz) \\ &\quad -M_K \int_0^t \left(\bar{\mu}_s^K(f) \right)^{1/\alpha} ds, \end{aligned} \tag{63}$$

where $\bar{\mu}_s^K := \mathcal{L}(\bar{X}_s^K \,|\, S^\alpha)$. This follows by taking the $n \to +\infty$ limit in (59), since

- by Assumption 3.2.1b), $\mathbb{E}[|b(X_t^{[n],K}, \mu_t^{[n],K}) - b(\bar{X}_t^K, \bar{\mu}_t^K)|] \leq C\mathbb{E}[|X_t^{[n],K} - \bar{X}_t^K|]$. On the other hand, $\mathbb{E}[|X_t^{[n],K} - \bar{X}_t^K|] \to 0$ by dominated convergence. Hence, from this L^1 convergence of $X_t^{[n],K}$ to $\bar{X}_t^K$, we obtain L^1 convergence of $b(X_t^{[n],K}, \mu_t^{[n],K})$ to $b(\bar{X}_t^K, \bar{\mu}_t^K)$. This latter yields in turn a.s. convergence, up to a subsequence.
- $\mu_t^{[n],K}(f) = \mathbb{E}\left[f(X_t^{[n],K}) \,|\, S^\alpha \right] \to \mathbb{E}\left[f(\bar{X}_t^K) \,|\, S^\alpha \right] =: \bar{\mu}_t^K(f)$ a.s. Moreover, we have that $|\mu_t^{[n],K}(f)| \leq \|f\|_\infty$ by Assumption 3.2.2b), hence we obtain the L^2-convergence of the stochastic integrals against $\tilde{M}$, whence the almost sure convergence, once again for a subsequence. The convergence of $\int_0^t \left(\mu_s^{[n],K}(f) \right)^{1/\alpha} ds$ follows similarly.

As a consequence of the above construction we dispose of a family of processes $(\bar{X}^K)_{K\in\mathbb{N}}$ such that

(i) for any K, $\bar{X}^K$ solves (63);

(ii) $\bar{X}_t^{K+1} = \bar{X}_t^K$ a.s. for all $t \in [0, T_K[$, since both are solution of the same equation on $[0, T_K[$.

So, letting $T_0 = 0$, the following process is well-defined

$$\bar{X}_t := \sum_{K\geq 1} \mathbb{1}_{[T_{K-1}, T_K[}(t) \bar{X}_t^K,$$

and it solves (3.7) on $[0, T]$ for any $T \geq 0$.

The same construction thanks to a Picard iteration works also in the case $\alpha < 1$, using L^1-norm instead of L^2-norm now. Details are omitted.

D Proof of Lemma 3.5.7

Proof. Under the law $\mathbb{P}(\cdot|S^{N,\alpha})$, that is, conditionally on $S^{N,\alpha}$, the N coordinates $\bar{X}_t^{N,1}, \ldots, \bar{X}_t^{N,N}$ are i.i.d. and distributed according to $\bar{\mu}_t$.

Let us first treat the case $\alpha_- < \frac{1}{2}$. We have already argued in Remark 3.3, see in particular (3.13), above that $\bar{\mu}_t$ admits a finite first moment. Theorem 1 of [34] (with their p replaced by α_- and their q replaced by 1, and using conditional expectation $\mathbb{E}(\cdot|S^{N,\alpha})$ instead of unconditional one) implies that, for a universal constant $C(\alpha_-)$,

$$\mathbb{E}\left(W_{\alpha_-}(\bar{\mu}_t^N, \bar{\mu}_t)|S^{N,\alpha}\right) \leq C(\alpha_-)\left(\int |x|\bar{\mu}_t(dx)\right)^{\alpha_-} \cdot N^{-\alpha_-}.$$

Relying on the upper bound obtained in Remark 3.3 above, we have that $\int |x|\bar{\mu}_t(dx) \leq \mathbb{E}(|X_0|) + Ct + \|f\|_\infty \sup_{s \leq t} |S_s^{N,\alpha}|$. Using the sub-additivity of the function $|\cdot|^{\alpha_-}$ and taking expectation then yields the result.

We now treat the case $1 > \alpha_- > \frac{1}{2}$, in which case $\alpha > \frac{1}{2}$ as well, since $\alpha > \alpha_-$. In this case, $\bar{\mu}_t$ admits a finite moment of order 2α, since X_0 does by assumption. We now apply Theorem 1 of [34] with their p replaced by α_- and their q replaced by 2α such that

$$\mathbb{E}\left(W_{\alpha_-}(\bar{\mu}_t^N, \bar{\mu}_t)|S^{N,\alpha}\right) \leq C(\alpha_-, \alpha)\left(\int |x|^{2\alpha}\bar{\mu}_t(dx)\right)^{\alpha_-/(2\alpha)} \cdot (N^{-1/2} + N^{-(1-\frac{\alpha_-}{2\alpha})}).$$

Since $\frac{1}{2} < 1 - \frac{\alpha_-}{2\alpha}$, the leading order of the above expression is given by $N^{-1/2}$. Moreover,

$$\int |x|^{2\alpha}\bar{\mu}_t(dx) \leq C_t(\mathbb{E}(|X_0|^{2\alpha}) + 1 + \sup_{s \leq t} |S_s^{N,\alpha}|^{2\alpha}.$$

The sub-additivity of the function $|\cdot|^{\alpha_-/(2\alpha)}$ (recall that $\alpha_- < \alpha$) implies then that, yet for another constant $\tilde{C}_t$,

$$\left(\int |x|^{2\alpha}\bar{\mu}_t(dx)\right)^{\alpha_-/(2\alpha)} \leq \tilde{C}_t(1 + \sup_{s \leq t} |S_s^{N,\alpha}|^{\alpha_-}),$$

and we take expectation to conclude.

Finally in case $\alpha > 1$, we use Theorem 1 of [34] with their p replaced by 1 and their

q replaced by our p such that

$$\mathbb{E}\left(W_1(\bar{\mu}_t^N,\bar{\mu}_t)|S^{N,\alpha}\right) \le C(p)\left(\int |x|^p \bar{\mu}_t(dx)\right)^{1/p} \cdot N^{-1/2},$$

and we conclude similarly as above, using the sub-additivity of the function $|\cdot|^{1/p}$. □

E Discussion of the convergence rate in Theorem 3.2.10

Case $1 < \alpha < 2$

According to Theorem 3.2.10, the error term is given, up to a constant, by (3.10), where we have to choose $\delta = \delta(N)$ such that $N\delta \to \infty$.

To understand formally what is the leading term in our error, we let $\alpha_- \uparrow \alpha$ and $\alpha_+ \downarrow \alpha$. In the limit $\alpha_- = \alpha_+ = \alpha$, we are left with an error term given by (up to a constant and to the common power $1/\alpha$)

$$\delta^{\frac{1}{\alpha^2}} + \delta^{\frac{1-\alpha}{\alpha}}\left(g(N\delta) + (N\delta)^{-1/2}\right).$$

Since $g(x) = x^{-B}$ for some $B > 0$, if we suppose $\delta = N^{-\eta}$ for some $\eta \in (0,1)$, we can write

$$\delta^{\frac{1-\alpha}{\alpha}}\left(g(N\delta) + (N\delta)^{-1/2}\right) = N^{\eta(1-1/\alpha+B)}g(N) + N^{\eta(3/2-1/\alpha)}N^{-1/2},$$

which is an increasing function of η. On the other hand, $\delta^{\frac{1}{\alpha^2}}$ is a decreasing function of η, and so we have to choose η such that the two terms which are left are equal, that is,

$$\delta^{\frac{1-\alpha+\alpha^2}{\alpha^2}} = g(N\delta) + (N\delta)^{-1/2}.$$

The leading term between $g(N\delta)$ and $(N\delta)^{-1/2}$ will asymptotically behave as $(N\delta)^{-C}$, with either $C = B$ or $C = 1/2$. Then we have to solve

$$\delta^{\frac{1-\alpha+\alpha^2}{\alpha^2}} = N^{-C}\delta^{-C}, \text{ which gives } \delta = N^{-\frac{C\alpha^2}{1-\alpha+C\alpha^2+\alpha^2}}.$$

By the equality we imposed, and re-introducing the $1/\alpha$ power, the rate will be

$$\delta^{1/\alpha^3} = N^{-\frac{C}{(1-\alpha+C\alpha^2+\alpha^2)\alpha}}.$$

Consider now the explicit form of the function g given in (3.23). We give the explicit rate (i.e. we choose C) in the two cases:

Case 1: $\gamma < 2 - \alpha$. Then, if $\gamma < \alpha/2$, $C = \gamma/\alpha$ and the rate is

$$N^{-\frac{\gamma}{\alpha^2(1-\alpha+\gamma\alpha+\alpha^2)}},$$

whereas, if $\gamma \in \left(\frac{\alpha}{2}, 2-\alpha\right)$, $C = 1/2$, such that the rate is

$$N^{-\frac{1}{2\alpha\left(1-\alpha+\frac{3}{2}\alpha^2\right)}}.$$

Case 2: $\gamma > 2 - \alpha$. Then, for $\alpha \in (1, \frac{4}{3})$, $C = 1/2$, such that the rate is

$$N^{-\frac{1}{2\alpha\left(1-\alpha+\frac{3}{2}\alpha^2\right)}}.$$

If $\alpha > \frac{4}{3}$, then $C = \frac{2-\alpha}{\alpha}$, such that the rate is

$$N^{-\frac{2-\alpha}{\alpha^2(1+\alpha)}}.$$

Case $\alpha < 1$

According to Theorem 3.2.10, the error term equals, up to a constant, to (3.11). Taking $\alpha_- = \alpha$ gives the sharpest possible bound, which includes the terms

$$\delta \qquad g(N\delta) \qquad (N\delta)^{-\frac{\alpha}{2}}$$

where $(N\delta)^{-\frac{\alpha}{2}}$ dominates the term $(N\delta)^{-1}$ which is present in the expression of g given in (3.25). Hence we are left with an error proportional to

$$\delta + \left[(N\delta)^{-\frac{\gamma}{\alpha}} + (N\delta)^{\frac{\alpha-1}{\alpha}}\right] + (N\delta)^{-\frac{\alpha}{2}}.$$

Take now $\delta = N^{-\eta}$ with $0 < \eta < 1$. Then $(N\delta)^{-\frac{\gamma}{\alpha}} + (N\delta)^{\frac{\alpha-1}{\alpha}} + (N\delta)^{-\frac{\alpha}{2}}$ is an increasing function of η whose leading order has the form $(N\delta)^{-C}$ with C one of those exponents, while δ is a decreasing function of η. Therefore, we impose

$$\delta = (N\delta)^{-C}, \text{ whence } \delta = N^{-\frac{C}{1+C}}.$$

We conclude by choosing C according to (3.25). If $\alpha \in (0, \sqrt{3} - 1)$ and $\gamma < \frac{\alpha^2}{2}$ or if $\alpha \in (\sqrt{3} - 1, 1)$ and $\gamma < 1 - \alpha$, the rate is

$$N^{-\frac{\gamma}{\alpha+\gamma}}.$$

If $\alpha \in (0, \sqrt{3} - 1)$ and $\gamma > \frac{\alpha^2}{2}$, the rate is

$$N^{-\frac{\alpha}{2+\alpha}}.$$

If $\alpha \in (\sqrt{3} - 1, 1)$ and $\gamma > 1 - \alpha$, the rate is

$$N^{\alpha-1}.$$

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Acknowledgments

To come.