Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”
Corso di Dottorato di Ricerca in Scienze Matematiche
Curricolo Matematica
Ciclo XXXVI

Specializations of quaternionic big Heegner classes

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Abstract

In this thesis, we construct generalized Heegner classes associated to quaternionic modular forms and the big Heegner classes that p-adically interpolate them along a Coleman (finite slope) family of quaternionic modular forms, following the motivic approach introduced by Jetchev–Loeffler–Zerbes in [JLZ21]. We also describe the specialization of the big Heegner classes in classical weights.

Keywords: generalized Heegner classes, big Heegner classes, quaternionic modular forms, quaternionic multiplication abelian surfaces.

Sommario

In questa tesi, costruiamo classi di Heegner generalizzate associate a forme modulari quaternioniche e le classi grosse di Heegner che le interpolano p-adicamente lungo una famiglia di Coleman (quelle a pendenza finita) di forme modulari quaternioniche, seguendo l'approccio motivico introdotto da Jetchev–Loeffler–Zerbes in [JLZ21]. Descriviamo inoltre la specializzazione delle classi grosse di Heegner in pesi classici.

Parole chiave: classi di Heegner generalizzate, classi grosse di Heegner, forme modulari quaternioniche, superfici abeliane a moltiplicazione quaternionica.

Contents

Introduction

Ever since its introduction in Kolyvagin's work on the Birch–Swinnerton-Dyer (BSD) conjecture, the concept of Heegner points has seen many generalizations to different contexts, always bearing in its core the idea of using algebraic objects (cohomological classes) to bound the size of Selmer groups of elliptic curves and obtain important results toward the BSD conjecture. In their original appearance, the crucial inputs to obtain these results are, on the one hand, the theory of Euler systems of Heegner points, as developed by Kolyvagin [Kol90], and on the other hand, the relation proved by Gross–Zagier [GZ86] between the height of Heegner points and derivatives of complex L-functions of elliptic curves, base changed to suitable imaginary quadratic fields.

Generalizations of Heegner points to the setting of higher weight modular forms were obtained by Nekovář [Nek95], introducing Heegner cycles; the analogue of the Gross–Zagier relation has been obtained in this setting by Zhang [Zha97]. Under a p-adic Abel–Jacobi map, these cycles map into étale cohomology classes, which can be associated to elliptic modular forms via a projection into their respective p-adic Galois representations. These classes also form Euler systems and, as in the case of elliptic curves, in combination with Zhang's result can be used to prove important results toward the Bloch–Kato conjecture for modular forms, a wide generalization of the BSD conjecture.

More recently, Bertolini–Darmon–Prasanna [BDP13] introduced yet another class of algebraic cycles, the generalized Heegner cycles, defined over a generalized Kuga–Sato variety consisting of the canonical desingularization of $\mathcal{E}^k \times_Y E^k$, where $\mathcal{E}$ is the universal elliptic curve over some modular curve Y, E is a fixed elliptic curve and k is the (even) weight of some level N cuspidal eigenform. These cycles enjoy interesting p-adic properties, and can be directly related to certain values of p-adic analogues of complex L-functions, called BDP p-adic L-functions; notably, these values are obtained by p-adic limit processes at points outside the range of p-adic interpolation of complex L-values. The relation between Abel–Jacobi images of generalized Heegner cycles and values of the BDP p-adic L-function outside the range of interpolation is known as explicit reciprocity law, and plays a fundamental role in all applications of generalized Heegner cycles to the Bloch–Kato conjecture.

A natural next step is to wonder how those generalized Heegner classes p-adically interpolate along a p-adic family passing through a fixed modular form, a parallel line of investigation that has received increasing attention in the last twenty years. One of the applications of p-adic interpolation is toward the analogue of the Bloch–Kato conjecture for Galois representations attached to p-adic families of modular forms; other beautiful applications arise from the possibility of using geometric constructions available for infinitely many classical forms in a p-adic family to generate, via p-adic approximation, algebraic objects at points where at first no such constructions are available.

Regarding the p-adic interpolation of generalized Heegner cycles attached to elliptic modular forms, the ordinary (zero slope, Hida family) case is due to Howard [How07]: after establishing a relation between the generalized Heegner classes and Heegner points in a certain tower of modular curves, a big Heegner point, playing the role of p-adic family of Heegner points, allows for the p-adic interpolation along a Hida family of classes associated to elliptic modular forms. The finite slope (Coleman family) case was done independently by Büyükboduk–Lei [BL21] and Jetchev–Loeffler–Zerbes [JLZ21], the first using a similar approach to Howard's, the latter introducing a new "motivic" approach where Heegner points are out of the picture: instead, base vectors of representations associated to certain Chow motives over the base modular curve from which the generalized Heegner classes themselves are defined take their place, being interpolated into p-adic families, and transfering naturally that interpolation via Gysin maps to the generalized Heegner classes. The p-adic family of generalized Heegner classes is called big Heegner class, as they play in this scenario the same role big Heegner points did in Howard's approach.

Generalized Heegner classes have also been considered in the quaternionic case, being introduced by Brooks [HB15] and Magrone [Mag22] in this setting. While in the elliptic case a so called Heegner hypothesis stating that all primes dividing the tame level N of the modular forms involved are split in the base quadratic imaginary field K, here we allow some to be inert, and the product of those inert prime factors of N becomes the discriminant of a rational quaternion algebra B. This setting allows new applications to Bloch–Kato conjectures for families of Galois representations, as a wider range of imaginary quadratic fields is now available with the relaxed hypothesis. The base modular curve classifying moduli of elliptic curves is then replaced by a Shimura curve associated to B, classifying moduli of abelian surfaces with quaternionic multiplication by B. This means that varieties here are double the size in comparison to the elliptic case, but since modular forms remain one-dimensional global sections of the bundle of relative differentials, and so of the same size as in the elliptic case, one needs to halve dimensions in order to achieve objects that are the same size, and thus behave similarly, to their elliptic counterpart. This is done by means of an idempotent applied to the Chow motives coming from abelian varieties, so they essentialy behave like motives coming from elliptic curves.

This thesis regards the p-adic interpolation of quaternionic generalized Heegner classes along Coleman families of quaternionic modular forms, adapting the motivic approach of Jetchev–Loeffler–Zerbes, which extends more naturally to the quaternionic Coleman case than the "classical" one. For Hida families of quaternionic modular forms, a "classical" approach with big Heegner points, introduced in the quaternionic case by Longo–Vigni [LV11], is also possible, and is the subject of joint work in progress with Matteo Longo and Paola Magrone [LMW23], where we also show that, for quaternionic Hida families, both approaches yield the same big Heegner classes; to make this comparison effective, we need to compare higher weight specializations of both objects. Higher weights are naturally encompassed in the motivic side, and on the Big Heegner points side, this is made possible by an extension to the quaternionic setting of some results by Castella [Cas20] and Ota [Ota20] exploiting a careful analysis of their weight 2 specializations.

Explicit reciprocity laws associating big Heegner classes to p-adically interpolated L-functions were done in the elliptic case by Jetchev–Loeffler–Zerbes. Using Salazar–Gao's quaternionic version of Andreatta–Iovita–Stevens's Eichler–Shimura isomorphism ([BSG17]) and Magrone's construction of the quaternionic families of p-adic L-functions from [LMW23], we expect, in a complementary article to this thesis, to derive a quaternionic counterpart of the explicit reciprocity law satisfied by the quaternionic big Heegner classes constructed here.

Overview of the thesis

This work is organized in four chapters, as follows:

1. The first chapter follows well established literature in recalling generalities on quaternion algebras; quaternionic multiplication abelian surfaces; level structures, arithmetic trivializations and moduli problems of QM abelian surfaces equipped with such structures; relative Chow motives of universal abelian surfaces;

2. The second chapter presents the construction of generalized Heegner classes in the quaternionic case;

3. The third chapter includes preliminaries on modular forms, Galois representations and Coleman families in the quaternionic case, and how they relate to their elliptic counterparts through various instances of the Jacquet–Langlands correspondence;

4. The fourth and final chapter presents generalized Heegner classes associated to modular forms and their p-adic interpolation along Coleman families into big Heegner classes; closes with a theorem on the specialization of big Heegner points at classical weights.

We proceed to briefly describe the contents of this thesis. Let B be a rational quaternion algebra of square-free discriminant N^-, p be a prime not dividing N^+N^- and K a quadratic imaginary field of discriminant $-D_K$ satisfying the generalized Heegner hypothesis, that is, all primes dividing pN^+ (resp. N^-) are split (resp. inert) in K. For each $m \geq 0$, there is a Shimura curve X_m associated to B playing the role of quaternionic counterpart of the tame level N^+ and wild level p^m modular curve. Let $\pi_{\mathcal{A}}\colon \mathcal{A}_m \to X_m$ be the universal abelian variety over X_m and let $\mathcal{F}$ be a quaternionic eigenform over X_m of tame level N^+ and weight $k = 2r+2$ with $r \in \mathbb{Z}_{\geq 0}$.

Brooks [HB15] has introduced the quaternionic analogue of Bertolini–Darmon–Prasanna's generalized Heegner cycles: let $A := (\mathbb{C}/\mathcal{O}_K) \times (\mathbb{C}/\mathcal{O}_K)$ be a fixed abelian surface with an action by a maximal order of B that comes from the fact that B splits in K (such action is called quaternionic multiplication), $\mathcal{O}_{cp^n}$ be the order in K of conductor cp^n for some $c \geq 0$ prime to $pD_K N^- N^+$ and consider the natural isogeny $\phi\colon A \to A_{cp^n} := (\mathbb{C}/\mathcal{O}_{cp^n}) \times (\mathbb{C}/\mathcal{O}_{cp^n})$. Then

$$\operatorname{graph}(\phi)^r \in \mathrm{CH}^{k-1}(W_{k,m} \otimes_H F)_{\mathbb{Q}}$$

defines a cycle over the generalized Kuga–Sato variety $W_{k,m} := \mathcal{A}_m^r \times_{X_m} A^r$ over a suitably large extension F of K, and over which a projector ϵ_W, introduced in the quaternionic context by Besser [Bes95], acts as an "average over all permutations of the factors of the fiber product", defining the generalized Heegner cycle

$$\Delta_{cp^n,m}^{[k]} := \epsilon_W \operatorname{graph}(\phi)^r \in \epsilon_W \mathrm{CH}^{k-1}(W_{k,m} \otimes_H F)_{\mathbb{Q}},$$

which can be mapped into a class in $\epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m} \otimes_H F, \mathbb{Q}_p(k-1))$ via the étale realization map. A technique known as Lieberman's trick allows us to replace the complicated Kuga–Sato variety $W_{k,m}$ with the more simple Shimura curve X_m for the price of a slightly more complicated coefficient sheaf:

$$\begin{aligned}\epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m}, \mathbb{Q}_p(k-1)) \longrightarrow \\ H^2_{\text{ét}}\left(X_m, \mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p) \otimes \mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)(2r+1)\right),\end{aligned}$$

where e is a "dimension halver" idempotent which splits $R^1\pi_{\mathcal{A},*}\mathbb{Q}_p$ into two isomorphic factors each of the size of the correspondent object arising from an elliptic curve.

The image of $\Delta_{cp^n,m}^{[k]}$ under the above map is the generalized Heegner class $z_{cp^n,m}^{[k]}$. We wish to define one such class associated to the quaternionic modular form $\mathcal{F}$, but the coefficient sheaf is still "too big": the first factor is correct, and the p-adic Galois representation associated to $\mathcal{F}$ (or rather, its dual) is naturally a projection from it, but the second factor, from which the twist by Hecke characters arise, is $2r$-dimensional as opposed to the 1-dimensional factor we need for a character. The factor $\mathrm{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)$ is indeed a direct sum of $2r$ characters: for each $0 \leq j \leq 2r$, choose one of those summands, call it $h^{(2r-j,j)}_{\text{ét}}$ and consider the map in cohomology naturally induced by the projection of $\mathrm{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)$ onto $h^{(2r-j,j)}_{\text{ét}}$. The image of $z_{cp^n,m}^{[k]}$ under this projection is its j-component $z_{cp^n,m}^{[k,j]}$, now of suitable size.

As we just pointed out, the dual p-adic Galois representation $V^*_{\mathcal{F}}$ is naturally a quotient of

$$H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)(2r+1)).$$

The p-adic Abel–Jacobi map derived from the Hochschild–Lyndon–Serre spectral sequence maps $z^{[k,j]}_{cp^n,m}$ into a class in

$$H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)(2r+1) \otimes \sigma^{2r-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}})\right),$$

where $\sigma^{2r-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}$ is a character arising from the embedding of K into a p-adic field L over which our p-adic sheaves are being considered. Projecting this class under the map induced by the projection onto $V^*_{\mathcal{F}}$ gives a class $z^{[\mathcal{F},j]}_{cp^n,m}$, which happens to be defined over H_{cp^n}, the ring class field of K of conductor cp^n. If ξ is a Hecke character that restricts to $\sigma^{2r-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}$ in $\mathrm{Gal}(F/H_{cp^n})$, we can then see $z^{[\mathcal{F},j]}_{cp^n,m}$ as a class in $H^1(H_{cp^n}, V^*_{\mathcal{F}} \otimes \xi)$.

To p-adically interpolate those classes along a Coleman family $\mathscr{F}$ passing by a p-stabilized quaternionic modular form $\mathcal{F}$ over the Shimura curve X_1 and defined over a suitably small affinoid $\mathscr{U}$ of the weight space $\mathscr{W}$, we transfer this problem to the p-adic interpolation of basis vectors of the representation associated to the motive (whose étale realization is $\mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)(2r+1) \otimes \sigma^{2r-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}$) under Ancona's functor [Anc15]. The basis vector $e^{[k,j]}_{cp^n}$ maps under a Gysin map to the generalized Heegner class $z^{[k,j]}_{cp^n}$, and their p-adic interpolation over $\mathscr{U}$, $\mathbf{e}^{[j]}_{\mathscr{U},cp^n}$ maps under an interpolated Gysin map to the j-component of the big Heegner class $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}$, which satisfies, for varying n and m, Euler system properties. The association of this big Heegner class to the Coleman family is done in a similar vein to the single form case: the distributions module that interpolate the coefficient sheaves above projects onto (the dual of) a big Galois representation $\mathbf{V}_{\mathscr{F}}$ which specializes at weight k to $V_{\mathcal{F}_k}$, where $\mathcal{F}_k$ denotes the specialization of $\mathscr{F}$ at weight $k \in \mathbb{Z} \cap \mathscr{U}$. This projection defines a class $\mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^n,1}$, which specializes at weight $k \geq j$ to $\binom{2r}{j}^{-1} z^{[\mathcal{F}_k,j]}_{cp^n,1}$, the binomial factor arising from the slope of the specialization at weight k of $\mathscr{F}$.

Finally, the big Heegner class associated to $\mathscr{F}$, denoted $\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp}$ and defined over F_{cp}, is defined by interpolating j-components defined above in j. As our main result, we describe the specialization of the big Heegner class $\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp}$:

Theorem (Theorem 4.4.8)**.** The specialization of $\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp}$ at $0 \leq j \leq 2r$, weight $k = 2r+2 \in \mathbb{Z} \cap \mathscr{U}$ and Hecke character of ξ of infinity type $(2r-j, j)$ is

$$a_p(\mathcal{F}_k)^{-n} \binom{2r}{j}^{-1} z^{[\mathcal{F}_k,j,\xi]}_{cp^n,1} \in H^1(H_{cp}, V^*_{\mathscr{F}} \otimes \xi),$$

where $a_p(\mathcal{F}_k)$ denotes the eigenvalue of $\mathcal{F}_k$ with respect to the Hecke operator U at p.

Chapter 1

Moduli of QM abelian varieties

Let us first set some global notation. Fix algebraic closures $\overline{\mathbb{Q}}$ for $\mathbb{Q}$ with an embedding $\iota_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and, for any prime ℓ, $\overline{\mathbb{Q}}_\ell$ for $\mathbb{Q}_\ell$ with an embedding $\iota_\ell\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. For any prime ℓ, v_ℓ denotes the ℓ-adic valuation normalized so that $v_\ell(\ell) = 1$. Next, we consider a quadratic imaginary field K of discriminant $-D_K$ and a positive integer N such that the following conditions hold:

(Heg1) A prime $\ell \mid N$ either splits or remains inert in K;

(Heg2) Being N^+ the product of all primes $\ell \mid N$ that split in K, $N^+ \geq 4$;

(Heg3) Being N^- the product of all primes $\ell \mid N$ inert in K, N^- is square-free and has an even and non-zero number of prime divisors.

Since no prime factor of N ramifies in K, we have the decomposition $N = N^+N^-$. We also fix a rational prime p, coprime to N and splitting as $p = \mathfrak{p}\bar{\mathfrak{p}}$ in K.

Remark 1.0.1. Condition (Heg1) is known as the *generalized Heegner hypothesis* on K respect to N and extends the classical Heegner hypothesis in allowing a prime $\ell \mid N$ to remain inert in K. Whether there are inert primes or not is what distinguishes between the elliptic and the quaternionic case: condition (Heg3) implies the existence of an indefinite rational quaternion algebra of discriminant N^- which, in the former case ($N^- = 1$), is $\mathrm{M}_2(\mathbb{Q})$, and the arising Shimura varieties are modular curves classifying moduli of elliptic curves. In the latter case ($N^- > 1$), we get division algebras, and the arising Shimura varieties are Shimura curves classifying moduli of QM abelian varieties, the dimension 2 analogues of elliptic curves. Condition (Heg2) is needed for the representability of the moduli problems we are going to consider (see Remark 1.4.4).

As usual, we write $\mathcal{O}_K$ for the ring of integers of K. For $c \geq 1$ an integer coprime with p and $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{O}_{cp^n} := \mathbb{Z} + cp^n\mathcal{O}_K$ the order of $\mathcal{O}_K$ of conductor cp^n. We write H_{cp^n} for the ring class field of K of conductor cp^n; put $H := H_1$ and

$$H_{cp^\infty} := \bigcup_{n\geq 1} H_{cp^n}.$$

Notice that, since $\gcd(c,p)=1$, we have that $H_c \cap H_{p^\infty} = H$, and so

$$\mathrm{Gal}(H_{cp^\infty}/K) \cong \mathrm{Gal}(H_c/K) \times \mathrm{Gal}(H_{p^\infty}/K).$$

It follows from the Artin reciprocity law that

$$\mathrm{Gal}(H_{cp^n}/K) \cong \mathrm{Pic}(\mathcal{O}_{cp^n}).$$

Since p splits in K, $\mathrm{Gal}(H_{p^\infty}/H) \cong \mathbb{Z}_p^\times$, for which we have the standard decomposition $\mathbb{Z}_p^\times \cong \Gamma \times \Delta$, with $\Gamma = 1+p\mathbb{Z}_p$ and $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$. The field fixed by Δ in the extension H_{p^∞}/K, called the anticyclotomic $\mathbb{Z}_p$-extension of K, is denoted by K_∞, and for its Galois group we write

$$\Gamma_\infty^{\mathrm{acyc}} := \mathrm{Gal}(K_\infty/K) \cong \Gamma \cong \mathbb{Z}_p.$$

Denote by K_n the subfield of K_∞ such that $\mathrm{Gal}(K_n/K) \cong \mathbb{Z}/p^n\mathbb{Z}$.

As a last set of notation, $\widehat{\mathbb{Z}} := \varprojlim_{n\in\mathbb{Z}_{\geq 1}} \mathbb{Z}/n\mathbb{Z}$ denotes the profinite integers, and for any $\mathbb{Z}$-module M, we write $\widehat{M} := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. If F is any number field, we denote its ring of adeles by

$$\mathbb{A}_F := \prod_{v \text{ finite place of } F} (F_v, \mathcal{O}_v) \times \prod_{w \text{ place at infinity of } F} F_w,$$

where the restricted product in the first term means that each vector $(a_v) \in \mathbb{A}_F$, $a_v \in \mathcal{O}_v$ for all but a finite number of places. The first component of the product above is the ring of finite adeles of F, denoted by $\mathbb{A}_{F,\mathrm{fin}}$, and the second component is the ring of infinite adeles of F, denoted F_∞.

Remark 1.0.2. The Artin reciprocity map (see, for example, [Mil20, Theorem 5.3])

$$\mathrm{rec}_K \colon \mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K),$$

where K^{ab} as usual denotes the maximal abelian extension of K, will be always taken to be geometrically normalized, that is, uniformizers map to *geometric* Frobenii instead of the arithmetic Frobenii, which is done by just composing the original map with the inversion map $\sigma \mapsto \sigma^{-1}$.

1.1 Quaternion algebras

Let F be a field of characteristic 0. A *quaternion algebra* over F is a 4-dimensional central simple algebra over F. If the field F is not specified, we always mean a rational quaternion algebra, that is, $F = \mathbb{Q}$. An alternative, more concrete characterization can be derived from the following statement (see [Voi21, Theorem 7.1.1, Corollary 7.1.2]):

Proposition 1.1.1 (Wedderburn–Artin)**.** Let B a finite-dimensional algebra over F. Then B is simple if and only if $B = \mathrm{M}_n(D)$, where n is a positive integer and D a division algebra over F.

Corollary 1.1.2. Let B be an F-algebra. Then B is a quaternion algebra if, and only if, B is spanned over F by 1, i, j and ij for which there are $a, b \in F^\times$ such that

$$i^2 = a, \quad j^2 = b, \text{ and } ji = -ij.$$

Another consequence of Proposition 1.1.1 is that a quaternion algebra B is either a division algebra or a matrix ring: since $\dim_F(B) = 4$, then either $n = 1$ and we have $B = D$, or $n = 2$ and D has dimension 1, so $B = \mathrm{M}_2(F)$. Therefore the difference is in whether B has zero divisors or not, which can be introduced when extending scalars:

Definition 1.1.3. Let F' be an extension of F and B be a quaternion algebra over F. The algebra B is said to *split* over F if $B \otimes_F F' \cong \mathrm{M}_2(F')$.

For a place v over F, finite or infinite, we write $B_v := B \otimes_F F_v$. As we just pointed out, B_v is either the matrix ring $M_2(F_v)$ (the *split* case, according to the above definition) or a division algebra (which we name the *ramified* case). The Hilbert symbol

$$(a, b)_v := \begin{cases} 1, & \text{if } ax^2 + by^2 = z^2 \text{ has non-zero solutions in } F_v \\ -1, & \text{otherwise} \end{cases}$$

detects whether B_v is split or ramified, respectively. By the Hilbert reciprocity law, the symbol $(a, b)_v$ is -1 only for finitely many places v, and they balance each other out, always appearing in even quantity:

$$\prod_v (a, b)_v = 1.$$

The product of the finite places over which B is ramified is the *disciminant* of B.

For $F = \mathbb{Q}$ (over which Hilbert reciprocity law is equivalent to the quadratic reciprocity law), the behavior at the only place at infinity can be used to classify quaternion $\mathbb{Q}$-algebras.

Definition 1.1.4. A quaternion algebra B over $\mathbb{Q}$ is said to be *indefinite* if it splits at ∞, being *definite* otherwise.

In particular, a quaternion algebra is indefinite (resp. definite) if and only if its discriminant has an even (resp. odd) number of prime factors.

Remark 1.1.5. For each square-free positive integer D there is a quaternion algebra with discriminant D, and all such quaternion algebras are isomorphic (see [Voi21, Theorem 14.6.1], see also [Lem11, §2]).

Condition (Heg3) in the beginning of this chapter says that N^- is square-free, so there is a quaternion algebra B of discriminant N^- (which we fix and always refer to it as "the" quaternion algebra of discriminant N^-). Furthermore, since N^- has an even number of prime divisors, B is indefinite.

For more details on quaternion algebras, refer to [Voi21].

1.1.1 Involutions on quaternion algebras

Definition 1.1.6. An *involution* $\dagger$ on a quaternion algebra B over a field F is a map $\bullet^\dagger \colon B \to B$ that is a

- self-inverse: $(b^\dagger)^\dagger = b$ for all $b \in B$;
- anti-automorphism: a bijective F-linear map such that $(b_1 b_2)^\dagger = b_2^\dagger b_1^\dagger$.

A *standard involution* is an involution such that $bb^\dagger \in F$ for all $b \in B$.

The *main involution* of a quaternion algebra B is a standard involution that is the quaternionic counterpart of the complex conjugation, given by

$$\overline{\alpha + \beta i + \gamma j + \delta ij} := \alpha - \beta i - \gamma j - \delta ij.$$

Remark 1.1.7. All the involutions on a quaternion algebra B are conjugated to each other, that is, if $\dagger$ and $\ddagger$ are two involutions, there exists $b_0 \in B$ such that, for all $b \in B$, $b^\ddagger = b_0^{-1} b^\dagger b_0$. This is an immediate consequence of the Skolem–Nother theorem (see [Voi21, Theorem 7.1.3]).

1.1.2 Splitting and embeddings

Consider the indefinite rational quaternion algebra B of discriminant N^-.

Proposition 1.1.8. Let B be a quaternion algebra over a number field F and F'/F be a degree 2 extension. The following are equivalent:

(i) B splits over F';

(ii) For each place w of F', B splits over F'_w;

(iii) There is an embedding $F' \hookrightarrow B$ of F-algebras;

(iv) For each place v of F, there is an embedding of F_v-algebras $F'_v := F' \otimes_F F_v \hookrightarrow B_v$;

(v) All places v of F where B is ramified are not split on F'.

(see [Voi21, Proposition 14.6.7]).

The fact that all primes dividing N^- are inert, thus not split, in K implies that B splits over K: in particular, the above proposition gives an embedding of $\mathbb{Q}$-algebras $\iota_K \colon K \hookrightarrow B$, which we fix.

Define $\delta := \sqrt{-D_K}$ and

$$\vartheta := \frac{\delta + D'}{2}, \quad \text{where } D' = \begin{cases} D_K, & \text{if } 2 \nmid D_K \\ D_K/2, & \text{if } 2 \mid D_K. \end{cases}$$

As a $\mathbb{Q}$-algebra, K embeds into $\mathrm{M}_2(\mathbb{Q})$ by extending $\mathbb{Q}$-linearly the map defined by

$$i\colon K \hookrightarrow \mathrm{M}_2(\mathbb{Q})\colon \vartheta \mapsto \begin{pmatrix} \mathrm{trace}_{K/\mathbb{Q}}(\vartheta) & -\mathrm{norm}_{K/\mathbb{Q}}(\vartheta) \\ 1 & 0 \end{pmatrix},$$

and we denote the composition of i with the canonical inclusion $\mathrm{M}_2(\mathbb{Q}) \hookrightarrow \mathrm{M}_2(K)$ by the same symbol, as context will always make it clear which map we are referring to. We may split i as a composition

$$i\colon K \overset{\iota_K}{\hookrightarrow} B \overset{\iota_B}{\hookrightarrow} \mathrm{M}_2(K)$$

as follows: since B splits over K, there is an isomorphism $I_B\colon B \otimes_{\mathbb{Q}} K \xrightarrow{\sim} \mathrm{M}_2(K)$, and composing the natural inclusion $B \hookrightarrow B \otimes_{\mathbb{Q}} K$ with I_B gives a map $\iota_B\colon B \hookrightarrow \mathrm{M}_2(K)$. Then choose I_B such that $i = \iota_B \circ \iota_K$: all options for I_B are conjugated to each other, so one just have to match the image of $\iota_K(\vartheta)$ under ι_B with $i(\theta)$.

For any finite place $\ell \nmid N^-$, where B splits, we fix $i_\ell\colon B_v := B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Q}_\ell)$ such that $i_\ell(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) \subseteq \mathrm{M}_2(\mathbb{Z}_\ell)$. If $v \mid pN^+\infty$, we take i_v to be the only isomorphism satisfying

$$i_v(\vartheta) = \begin{pmatrix} \mathrm{trace}_{K/\mathbb{Q}}(\vartheta) & -\mathrm{norm}_{K/\mathbb{Q}}(\vartheta) \\ 1 & 0 \end{pmatrix}.$$

In particular, if ℓ is a finite place splitting B, i_ℓ is an isomorphism between $\mathcal{O}_{B,\ell} := \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $\mathrm{M}_2(\mathbb{Z}_\ell)$, which reduces modulo ℓ to an isomorphism

$$\bar{\imath}_\ell\colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Z}/\ell\mathbb{Z}).$$

Since each $\ell \mid pN^+$ splits at K, for each divisor M of $p^m N^+$ with $m \in \mathbb{Z}_{\geq 0}$, the Chinese Remainder Theorem allows us to combine the above reduced isomorphisms into

$$\bar{\imath}_M\colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Z}/M\mathbb{Z}).$$

1.1.3 Idempotents

With the notation of the previous subsection, define the following elements of $K \otimes_{\mathbb{Q}} K$:

$$e := \frac{\vartheta \otimes 1 - 1 \otimes \vartheta}{\delta \otimes 1} \quad \text{and} \quad \bar{e} := \frac{1 \otimes \vartheta - \bar{\vartheta} \otimes 1}{\delta \otimes 1} \tag{1.1}$$

(notice that $\delta = \vartheta - \bar{\vartheta}$). Via direct computation, one can show that those elements are orthogonal idempotents, thus satisfying

$$e^2 = e,\ \ \bar{e}^2 = \bar{e},\ \ e\bar{e} = 0 \text{ and } e + \bar{e} = 1.$$

The importance of those global elements is that their local realizations at splitting places are the idempotent matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$: if we again consider $\ell \mid pN^+$, it splits as

$\ell = \mathfrak{l}\bar{\mathfrak{l}}$ in K, where $\mathfrak{l}$ is the prime corresponding to the fixed embedding $\iota_\ell \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ from the beginning of the chapter. Thus $K_\ell := K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ splits as $K_{\mathfrak{l}} \oplus K_{\bar{\mathfrak{l}}} \cong \mathbb{Q}_\ell e_{\mathfrak{l}} \oplus \mathbb{Q}_\ell e_{\bar{\mathfrak{l}}}$ and so $K \otimes_{\mathbb{Q}} K$ embeds into $K \otimes_{\mathbb{Q}} K_{\mathfrak{l}} \cong K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Composing this embedding with the one coming from Proposition 1.1.8.(iv) and i_ℓ gives a map

$$j_\ell \colon K \otimes_{\mathbb{Q}} K \longleftrightarrow K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longleftrightarrow B_\ell \xrightarrow{i_\ell} \mathrm{M}_2(\mathbb{Q}_\ell).$$

Via the normalization choice of i_ℓ, one computes that

$$j_\ell(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad j_\ell(\bar{e}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Much like the other splittings of B, we can consider the map

$$j \colon K \otimes_{\mathbb{Q}} K \longleftrightarrow \mathrm{M}_2(K) \colon x \otimes y \mapsto i(x)y$$

and once again obtain that

$$j(e) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad j(\bar{e}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 1.1.9. Some references, such as [HB15], fix a *Hashimoto model* for B, that is, the choice of a certain $\mathbb{Q}$-basis $\{1, s_1, s_2, s_1 s_2\}$ and a totally real quadratic field M defined after it which splits B. This allows the definition of a global idempotent $e \in K \otimes_{\mathbb{Q}} B$ that is fixed by a certain involution of B obtained by conjugating the main involution of B by s_1.

1.1.4 Orders in quaternion algebras

Let B for now be any quaternion algebra over a field F of characteristic 0.

Definition 1.1.10. An *order* O is a subring of B which is also an $\mathcal{O}_F$-lattice, that is, $O \otimes_{\mathcal{O}_F} F = B$.

An order is said to be *maximal* if it is not properly contained in another order. One often fixes a maximal order in B and denote it by $\mathcal{O}_B$.

Definition 1.1.11. The intersection of two maximal orders of B is an *Eichler order*.

Now let B again denote the indefinite quaternion algebra of discriminant N^- and $\mathcal{O}_B$ a fixed maximal order. If M is an integer coprime to N^-, we can define

$$O_M := \{b \in B;\ i_\ell(b) \text{ is upper triangular mod } \ell^{v_\ell(M)},\ \forall \ell \mid M \text{ prime}\},$$

which is another maximal order. The order $\mathcal{O}_B \cap O_M$ is called the *standard Eichler order* of level M. For $m \in \mathbb{Z}_{\geq 0}$ and $M = p^m N^+$, we denote it by R_m (*cf.* [LV11, §2]).

Remark 1.1.12. The Eichler orders defined above depend on the choice of the maximal order $\mathcal{O}_B$ and the isomorphisms i_ℓ for all finite places $\ell \nmid N^-$.

1.2 QM abelian surfaces

For this section let B be a rational quaternion algebra of discriminant D_B.

Definition 1.2.1. Let S be a $\mathbb{Z}[1/D_B]$-scheme. An *abelian surface with quaternionic multiplication* by $\mathcal{O}_B$ is a pair (A, ι) consisting of an abelian scheme $A \to S$ of relative dimension 2 and an injective algebra morphism $\iota\colon \mathcal{O}_B \hookrightarrow \operatorname{End}_S(A)$.

Remark 1.2.2. Some sources, such as [HB15] and [Buz97], refer to QM abelian surfaces as *false ellptic curves*, stressing the similarities with the elliptic case.

The action of $\mathcal{O}_B$ induced by ι is called *quaternionic multiplication* by $\mathcal{O}_B$ (which we always shorten to QM, and often leave $\mathcal{O}_B$ implicit). It is costumary to drop ι when it is clear and to replace the spectrum of a ring $S = \operatorname{Spec}(R)$ by the ring R itself.

Definition 1.2.3. An *isogeny* (resp. an *isomorphism*) of QM abelian surfaces is an isogeny (resp. an isomorphism) of abelian schemes that commute with the QM action.

For a QM abelian surface (A, ι) and s a geometric point of S, let A_s denote the fiber of $A \to S$ over s. Let $t \in \mathcal{O}_B$ be such that $t^2 = -D_K < 0$, which exists because B splits over K, and define the involution $\dagger$ given by $b^\dagger := t^{-1}\bar{b}t$, where $\bar{\cdot}$ denotes the main involution on B. Then there is a unique principal polarization on $\lambda\colon A \to A^\vee$ such that, for geometric point s of S, the restriction of the Rosati involution of $\operatorname{End}(A_s)$ to $\mathcal{O}_B$ coincides with $\dagger$ (see [BC91, §III.1.5]).

Now take two QM abelian surfaces A and B, λ_A and λ_B the unique principal polarizations described above and $\phi\colon A \to B$ an isogeny, which induces a dual isogeny $\phi^\vee\colon B^\vee \to A^\vee$. The map $\phi^{\mathrm{T}} := \lambda_B^\vee \circ \phi^\vee \circ \lambda_A\colon B \to A$ is an isogeny in the reverse order. Then $\phi^{\mathrm{T}} \circ \phi$ is locally multiplication by an integer d, the *degree* of ϕ (see [Buz97, §1]).

Remark 1.2.4. Henceforth $\dagger$ will always denote the above involution on B.

1.3 Shimura curves from quaternion algebras

In this section, following [LV11, §2.2], we treat a specific example of Shimura curves that arise from indefinite quaternion algebras. For the sake of brevity, we refrain from presenting the basic concepts of the theory of Shimura varieties and instead point the reader to the relevant topics in [Mil05].

Loosely speaking, to define a Shimura variety one needs:

(i) A reductive algebraic group G over $\mathbb{Q}$;

(ii) A $G(\mathbb{R})$-conjugacy class D of morphisms $h\colon \mathbb{S} \to G(\mathbb{R})$, where $\mathbb{S}$ denotes the Deligne torus $\operatorname{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ - that is, a hermitian symmetric domain;

both satisfying the conditions for a *Shimura datum* as in *ibid.*, Definition 5.5, and

(iii) A compact open subgroup U of $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$,

which is used to determine the *level* of the Shimura variety. With those ingredients in hand, one defines, as a complex manifold, a *Shimura variety*

$$\mathrm{Sh}_U(G,D) := G(\mathbb{Q})\backslash D \times G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})/U,$$

where $G(\mathbb{Q})$ acts on D and $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ via multiplication from the left and U acts trivially on D and by multiplication from the right on $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$; see *ibid.*, §5 for the details.

We now define Shimura varieties associated to B, the indefinite rational quaternion algebra of discriminant N^-. Starting with (i), define the algebraic group $\mathbf{G}$ such that

$$\mathbf{G}(F) := (B \otimes_{\mathbb{Q}} F)^\times$$

for any extension $F/\mathbb{Q}$. Since B splits at ∞, this is a reductive algebraic group. Since K is the smallest field over which B splits (as B doesn't split over $\mathbb{Q}$ and $K/\mathbb{Q}$ is as small as it can be), we have an alternative description of $\mathbf{G}$ as $\mathrm{Res}_{K/\mathbb{Q}}(B^\times)$.

For item (ii), we first notice that $\mathbf{G}(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times = \mathrm{GL}_2(\mathbb{R})$, as $\mathbb{Q}_\infty = \mathbb{R}$ and B splits at ∞. Then we take

$$h\colon \mathbb{S} \to \mathrm{GL}_2(\mathbb{R})\colon a+bi \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right),$$

for the morphism, so its $G(\mathbb{R})$-conjugacy classes become

$$\begin{aligned} D = \{ghg^{-1}\}_{g\in\mathrm{GL}_2(\mathbb{R})} &\xrightarrow{\sim} \mathcal{H}^\pm \\ ghg^{-1} &\longleftrightarrow g\cdot i, \end{aligned}$$

writing $\mathcal{H}^\bullet := \{\tau \in \mathbb{C};\ \mathrm{sign}(\mathrm{Im}(\tau)) = \bullet\}$ for $\bullet \in \{-,+\}$ and $\mathcal{H}^\pm := \mathcal{H}^- \cup \mathcal{H}^+$. Verifying that $(\mathbf{G},\mathcal{H}^\pm)$ is a Shimura datum goes like Example 5.6 in *op. cit.*

Remark 1.3.1. The datum $(\mathbf{G},\mathcal{H}^\pm)$ is of PEL type (see [Mil05, §8] for the definition; see also [Tor20, §7]). In fact, a PEL datum for it is

$$\left(B, \bar{\cdot}, \mathbb{Q}^4, \begin{pmatrix} 0_{2\times2} & \mathrm{Id}_{2\times2} \\ -\mathrm{Id}_{2\times2} & 0_{2\times2} \end{pmatrix}, h\colon a+bi \mapsto \begin{pmatrix} a\mathrm{Id}_{2\times2} & b\mathrm{Id}_{2\times2} \\ -b\mathrm{Id}_{2\times2} & a\mathrm{Id}_{2\times2} \end{pmatrix}\right).$$

The third item in this quintuple is the *standard representation* of $\mathbf{G}$ over $\mathbb{Q}$, which over another field $F/\mathbb{Q}$ is given by

$$\mathbf{G}(F) = (B \otimes_{\mathbb{Q}} F)^\times \cong \mathrm{GL}(F^4) \implies V(F) = F^4,$$

where the isomorphism is between vector spaces.

Finally, the levels are related with the Eichler orders from §1.1.4: for an integer $m \geq 0$, we define $U_m := \widehat{R}_m^\times$ and let $\widetilde{U}_m$ be the subgroup of U_m consisting of the elements g whose p-component is congruent to a matrix of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$ modulo p^m.

With those ingredients, we get two families of Shimura varieties for the same Shimura datum $(\mathbf{G}, \mathcal{H}^{\pm})$ and varying level U_m and $\widetilde{U}_m$, respectively:

$$X_m(\mathbb{C}) := B^{\times}\backslash(\mathcal{H}^{\pm} \times \widehat{B}^{\times})/U_m \quad \text{and} \quad \widetilde{X}_m(\mathbb{C}) := B^{\times}\backslash(\mathcal{H}^{\pm} \times \widehat{B}^{\times})/\widetilde{U}_m. \tag{1.2}$$

The finite index inclusion $U_m \supseteq \widetilde{U}_m$ induces a finite covering $\varsigma_m \colon \widetilde{X}_m \twoheadrightarrow X_m$. The double coset represented by a point $(x, g) \in \mathcal{H}^{\pm} \times \widehat{B}^{\times}$ is written $[(x, g)]$.

For any $m \geq 0$ both double coset spaces $B^{\times}\backslash\widehat{B}^{\times}/U_m$ and $B^{\times}\backslash\widehat{B}^{\times}/\widetilde{U}_m$ consist of one element, so [Mil05, Lemma 5.13] implies that

$$X_m(\mathbb{C}) \cong \Gamma_m\backslash\mathcal{H}^{+} \quad \text{and} \quad \widetilde{X}_m(\mathbb{C}) \cong \widetilde{\Gamma}_m\backslash\mathcal{H}^{+} \tag{1.3}$$

where Γ_m (resp. $\widetilde{\Gamma}_m$) is the subgroup of norm 1 elements in $B^{\times} \cap U_m$ (resp. $B^{\times} \cap \widetilde{U}_m$). So both $X_m(\mathbb{C})$ and $\widetilde{X}_m(\mathbb{C})$ are Riemann surfaces, thus being algebraic curves defined over $\mathbb{C}$. Furthermore, there are algebraic curves X_m and $\widetilde{X}_m$ defined over $\mathbb{Q}$ whose $\mathbb{C}$-rational points match $X_m(\mathbb{C})$ and $\widetilde{X}_m(\mathbb{C})$ respectively, tying the notation together.

1.4 Moduli problems

In this section, we describe various structures that can be attached to QM abelian varieties and their respective moduli problems.

1.4.1 Naïve level structures

Let $M \mid pN^{+}$ be a positive integer, S be a $\mathbb{Z}[1/M]$-scheme and (A, ι) a QM abelian surface over S.

Definition 1.4.1. A *naïve full level M structure* on A is an isomorphism

$$\alpha \colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} A[M]$$

of S-group schemes locally for the étale topology of S which commutes with the left actions of $\mathcal{O}_B$ given by ι on $A[M]$, and the multiplication from the left of $\mathcal{O}_B$ on the constant S-group scheme $\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z})$.

Via the isomorphism $i_M \colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Z}/M\mathbb{Z})$ defined in §1.1.2, giving a full level M-structure on A is equivalent to giving an isomorphism $\mathrm{M}_2(\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} A[M]$ of finite flat group schemes over S commuting with the left action of $\mathcal{O}_B$ given by ι on $A[M]$ and by left matrix multiplication on $\mathrm{M}_2(\mathbb{Z}/M\mathbb{Z})$.

There is a left action of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^{\times}$ on the set of full level M structures on a QM abelian surface (A, ι) as follows. If $g \in (\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^{\times}$, then right multiplication $r_g(x) = xg$ by g defines an automorphism of the group $(\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^{\times}$ which commutes with the left action of $(\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^{\times}$ on itself by left multiplication;

for a naïve full level M structure $\alpha\colon (\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^\times \xrightarrow{\sim} A[M]$ on (A, ι), we see that $\alpha_g = \alpha \circ r_g$ is a naïve full level M structure on (A, ι), and the map $\alpha \mapsto \alpha_g$ gives a left action of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ on the set of naïve full level M structures on (A, ι). For any subgroup U of $\widehat{\mathcal{O}}_B^\times$, we obtain an action of U on full level M structures by composing the action of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ with the canonical projection

$$\widehat{\pi}_M\colon \widehat{\mathcal{O}}_B^\times \longrightarrow\!\!\!\!\rightarrow (\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times.$$

Definition 1.4.2. A *naïve level U structure* α on A is an equivalence class of full level M structures on A under the left action of U.

We write (A, ι, α) when equipping (A, ι) with a naïve level structure α. We say that two such triples (A, ι, α) and (A', ι', α') are *isomorphic* if there is an isomorphism of QM abelian surfaces $\varphi\colon A \to A'$ such that $\varphi \circ \alpha = \alpha'$.

Proposition 1.4.3. The functor which takes a $\mathbb{Z}[1/(MN^-)]$-scheme S to the set of isomorphism classes of such triples (A, ι, α) over S is representable by a $\mathbb{Z}[1/(MN^-)]$-scheme $\mathcal{X}_U$, which is projective, smooth, of relative dimension 1 and geometrically connected (see [Buz97, Theorem 2.1]).

Remark 1.4.4. The representability result is due to Morita [Mor81, Main Theorem 1] for naïve full level M structures. A complete proof of the general case can be found in [Buz97, §2] (see specially [Buz97, Corollary 2.3 and Propositions 2.4 and 2.5]) combining the representability result of [BBG+79, Theorem §14, Exposé III] and the proof in [Buz97, Lemma 2.2] that the moduli problem $\mathcal{F}_U$ is rigid. Both the original and the general case require $N^+ \geq 4$; see also [DT94, §4] and [HB15, Theorem 2.2].

Example 1.4.5. We now define two types of level structures. Consider

$$V_0(M) := \{g \in \widehat{\mathcal{O}}_B^\times;\ \exists a, b, d \text{ such that } i_M \circ \widehat{\pi}_M(g) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mod M\}$$

and its subgroup

$$V_1(M) := \{g \in \widehat{\mathcal{O}}_B^\times;\ \exists a, b \text{ such that } i_M \circ \widehat{\pi}_M(g) \equiv \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mod M\}.$$

Similarly, we consider the "transposed along the secondary diagonal" versions

$$U_0(M) := \{g \in \widehat{\mathcal{O}}_B^\times;\ \exists a, b, d \text{ such that } i_M \circ \widehat{\pi}_M(g) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mod M\} = V_0(M)$$

and its subgroup

$$U_1(M) := \{g \in \widehat{\mathcal{O}}_B^\times;\ \exists b, d \text{ such that } i_M \circ \widehat{\pi}_M(g) \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \mod M\}.$$

The map $g \mapsto g' = \mathrm{norm}(g)g^{-1}$ defines an anti-isomorphism of $V_0(M)$ to $U_0(M)$ that restricts to an anti-isomorphism from $V_1(M)$ to $U_1(M)$. We thus get an induced right action of $U_0(M)$ and $U_1(M)$ on full naïve level M structures, and two such structures are equivalent under the right action of $U_0(M)$ (resp. of $U_1(M)$) if and only if they are equivalent under the left action of $V_0(M)$ (resp. of $V_1(M)$). In this case, we say that two full naïve level structures are $U_0(M)$ or $U_1(M)$-equivalent, respectively.

Remark 1.4.6. The levels of the Shimura curves defined in §1.3 relate to the above levels as follows:

$$U_m := U_0(N^+p^m) \quad \text{and} \quad \widetilde{U}_m := U_0(N^+) \cap U_1(p^m).$$

The curves X_m and $\widetilde{X}_m$ are the generic fibers of $\mathcal{X}_m = \mathcal{X}_{U_m}$ and $\widetilde{\mathcal{X}}_m = \mathcal{X}_{\widetilde{U}_m}$, respectively.

1.4.2 Drinfeld level structures

Let $m \geq 1$ (our main reference [Buz97, §3] only deals with the case $m = 1$, but everything promptly generalizes to $m \geq 1$). Again, S is a $\mathbb{Z}[1/pN^+]$-scheme and (A, ι) is a QM abelian surface over S.

The fixed maximal order $\mathcal{O}_B$ acts on $A[p^m]$ via quaternionic multiplication. Then so does $\mathcal{O}_{B,p} := \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, as p splits, and $\mathrm{M}_2(\mathbb{Z}_p)$ via i_p, the latter by left multiplication. The idempontents $e, \bar{e} \in K \otimes_{\mathbb{Q}} K$ from §1.1.3 (which can be viewed as elements in $\mathcal{O}_{B,p}$ via $i_p^{-1} \circ j_p$) were defined to have matrix realizations as projections into the first and second component, respectively, that is, $j_p(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $j_p(\bar{e}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The kernel of the action of e on $A[p^m]$ is isomorphic as a group scheme to the image of the action of $\bar{e}$ on $A[p^m]$ and vice-versa. Furthermore, an element $w \in \mathcal{O}_{B,p}$ such that $j_p(w) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives an isomorphism of group schemes between $\ker(e)$ and $\ker(\bar{e})$. Therefore, we have a decomposition into isomorphic factors

$$A[p^m] = \ker(e) \oplus \ker(\bar{e}). \tag{1.4}$$

Definition 1.4.7. A $\Gamma_1(p^m)$-*level structure on* A is a pair (H, P) consisting of a cyclic finite flat S-subgroup scheme H of $eA[p^m]$ which is locally free of rank p^m and a choice of a generator P of H.

Again, we denote by $(A, \iota, \alpha, (H, P))$ the QM abelian variety (A, ι) equipped with a naïve level U structure α and a $\Gamma_1(p^m)$-level structure (H, P). We say that two such quadruples $(A, \iota, \alpha, (H, P))$ and $(A', \iota', \alpha', (H', P'))$ are *isomorphic* if there is an isomorphism of QM abelian surfaces $\varphi\colon A \to A'$ such that $\varphi \circ \alpha = \alpha'$, $\varphi(H) = H'$ and $\varphi(P) = P'$.

Proposition 1.4.8. The functor which takes a $\mathbb{Z}_{(p)}$-scheme S to the set of isomorphism classes of such quadruplets $(A, \iota, \alpha, (H, P))$ over S is representable by a $\mathbb{Z}_{(p)}$-scheme $\mathcal{X}_{U,\Gamma_1(p^m)}$, which is proper and finite over $\mathcal{X}_U$, here viewed as a $\mathbb{Z}_{(p)}$-scheme. Moreover, there is a canonical isomorphism of $\mathbb{Q}$-schemes between the generic fiber of $\mathcal{X}_{U,\Gamma_1(p^m)}$ and the generic fiber of $\mathcal{X}_{U \cap U_1(p^m)}$ (see [Buz97, Proposition 4.1]).

Remark 1.4.9. If S is a $\mathbb{Q}_p$-scheme, there is a canonical bijection between $\Gamma_1(p^m)$-level structures on A and $V_1(p^m)$-level structures on A (see [Buz97, Lemma 4.4]).

1.4.3 Arithmetic trivializations

Let (A, ι) be a QM abelian surface over a $\mathbb{Z}_p$-scheme S. For this subsection only $\mathbf{m}$ denotes *either* a positive integer m or ∞. We denote by

$$A[p^\infty] = \varinjlim_{m \geq 1} A[p^m]$$

the p-primary part of A and by $A[p^{\mathbf{m}}]^0$ of the connected component of the identity in $A[p^{\mathbf{m}}]$. We also denote by $\boldsymbol{\mu}_{p^m}$ the S-group scheme p^m-th roots of unity and

$$\boldsymbol{\mu}_{p^\infty} = \varinjlim_{m \geq 1} A[p^m].$$

We recall that $\mathcal{O}_{B,p} := \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ acts on $A[p^{\mathbf{m}}]$ by quaternionic multiplication, and on $\boldsymbol{\mu}_{p^{\mathbf{m}}}$ by left matrix multiplication via ι_p.

Definition 1.4.10. An *arithmetic trivialization* on $A[p^{\mathbf{m}}]$ is an isomorphism of finite flat group schemes over S

$$\beta \colon \boldsymbol{\mu}_{p^{\mathbf{m}}} \times \boldsymbol{\mu}_{p^{\mathbf{m}}} \xrightarrow{\sim} A[p^{\mathbf{m}}]^0$$

which is equivariant with respect to the action of $\mathcal{O}_{B,p}$.

Remark 1.4.11. The existence of an arithmetic trivialization on $A[p^\infty]$ implies that A is an ordinary abelian scheme over S.

Via the decomposition into isomorphic factors (1.4) induced by e and $\bar{e}$ (which commutes with inverse limits and thus extends to $\mathbf{m} = \infty$), giving an arithmetic trivialization of $A[p^{\mathbf{m}}]$ is equivalent to giving an isomorphism $\beta \colon \boldsymbol{\mu}_{p^{\mathbf{m}}} \xrightarrow{\sim} eA[p^{\mathbf{m}}]^0$ of finite flat connected group schemes over S, equivariant for the action of $e\mathcal{O}_{B,p}$.

An arithmetic trivialization β of $A[p^\infty]$ induces for each integer $m \geq 1$ an arithmetic trivialization $\bar{\beta}^{(m)} \colon \boldsymbol{\mu}_{p^m} \xrightarrow{\sim} eA[p^m]^0$. An arithmetic trivialization β_m of $A[p^m]$ is said to be *compatible* with a given arithmetic trivialization β of $A[p^\infty]$ if the composition

$$\boldsymbol{\mu}_{p^m} \xrightarrow{\beta_m} eA[p^m]^0 \xrightarrow{(\bar{\beta}^{(m)})^{-1}} \boldsymbol{\mu}_{p^m}$$

is the identity.

1.4.4 Igusa towers

Let us consider $\mathcal{X}_0 = \mathcal{X}_{U_0(N^+)}$ as a $\mathbb{Z}_{(p)}$-scheme and let $\mathbb{X}_0$ denote the special fiber of $\mathcal{X}_0$. Denote by $\mathbf{Ha}$ the Hasse invariant of $\mathbb{X}_0$ (*cf.* [Kas04, §6]) and by $\widetilde{\mathbf{Ha}}$ a lift of $\mathbf{Ha}$ to $\mathcal{X}_0$ (*cf. ibid.* §7). Let $\mathcal{X}_0^{\mathrm{ord}} := \mathcal{X}_0[1/\widetilde{\mathbf{Ha}}]$, an affine open $\mathbb{Z}_{(p)}$-subscheme of $\mathcal{X}_0$.

Proposition 1.4.12. The moduli problem which associates to any $\mathbb{Z}_{(p)}$-scheme S the isomorphism classes of triplets (A, ι, α) where (A, ι) is an ordinary QM abelian surface over S and α a naïve $U_0(N^+)$-level structure is represented by $\mathcal{X}_0^{\mathrm{ord}}$.

Let $\mathcal{A}^{\mathrm{ord}}$ be the universal ordinary abelian variety over $\mathcal{X}_0^{\mathrm{ord}}$. For any $\mathbb{Z}_{(p)}$-algebra R, denote $\mathcal{A}_R^{\mathrm{ord}} := \mathcal{A}^{\mathrm{ord}} \otimes_{\mathbb{Z}_{(p)}} R$, and $\mathcal{A}_n^{\mathrm{ord}} := \mathcal{A}^{\mathrm{ord}}_{\mathbb{Z}/p^n\mathbb{Z}}$. Let $\mathcal{A}_n^{\mathrm{ord}}[p^m]^0$ be the connected component of identity in $\mathcal{A}_n^{\mathrm{ord}}[p^m]$, the p^m-torsion subgroup scheme of $\mathcal{A}_n^{\mathrm{ord}}$.

For two group schemes G and H equipped with a left $\mathcal{O}_{B,p}$-action, $\mathrm{Isom}_{\mathcal{O}_{B,p}}(G,H)$ denotes the set of isomorphisms of groups schemes $G \to H$ which are equivariant for the action of $\mathcal{O}_{B,p}$. Let $m \geq 1$ and $n \geq 1$ be integers. Consider

$$\mathcal{P}_{m,n}(S) = \mathrm{Isom}_{\mathcal{O}_{B,p}}\left(\boldsymbol{\mu}_{p^m} \times \boldsymbol{\mu}_{p^m}, \mathcal{A}_n^{\mathrm{ord}}[p^m]^0\right),$$

the set of arithmetic trivializations on $\mathcal{A}_n^{\mathrm{ord}}[p^m]$.

Proposition 1.4.13. The moduli problem $\mathcal{P}_{m,n}$ is represented by a $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\mathrm{Ig}_{m,n}$, the *p^m-layer of the Igusa tower over $\mathbb{Z}/p^n\mathbb{Z}$*, which is finite étale over $\mathrm{Ig}_{0,n}$.

By the universality of $\mathcal{A}_n^{\mathrm{ord}}$, the $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\mathrm{Ig}_{m,n}$ represents the moduli problem which associates to any $\mathbb{Z}/p^n\mathbb{Z}$-scheme S the set of isomorphism classes of quadruplets (A,ι,α,β) consisting of a QM abelian surface (A,ι) over S equipped with a $U_1(N^+)$-level structure α and an arithmetic trivialization β of $A[p^m]$.

For integers $m \geq 0$ and $n \geq 1$ and a $\mathbb{Z}/p^n\mathbb{Z}$-scheme S, the canonical monomorphism $\boldsymbol{\mu}_{p^m} \hookrightarrow \boldsymbol{\mu}_{p^{m+1}}$ of S-group schemes induces a canonical map $\mathrm{Ig}_{m+1,n} \to \mathrm{Ig}_{m,n}$. We can therefore consider the $\mathbb{Z}/p^n\mathbb{Z}$-scheme

$$\widehat{\mathrm{Ig}}_n = \varinjlim_m \mathrm{Ig}_{m,n},$$

called the *Igusa tower over* $\mathbb{Z}/p^n\mathbb{Z}$.

Proposition 1.4.14. The $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\widehat{\mathrm{Ig}}_n$ represents the moduli problem over $\mathbb{Z}/p^n\mathbb{Z}$

$$\mathcal{P}_n(S) = \mathrm{Isom}_{\mathcal{O}_{B,p}}\left(\boldsymbol{\mu}_{p^\infty} \times \boldsymbol{\mu}_{p^\infty}, \mathcal{A}_n^{\mathrm{ord}}[p^\infty]^0\right)$$

classifying the set of arithmetic trivializations of $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$, or, equivalently, the moduli problem which associates to a $\mathbb{Z}/p^n\mathbb{Z}$-scheme S the set of isomorphism classes of quadruplets (A,ι,α,β) for each integer $m \geq 1$ consisting of a QM abelian surface (A,ι) over S equipped with a $U_1(N^+)$-level structure α and a family of arithmetic trivializations β_m of $\mathcal{A}_n^{\mathrm{ord}}[p^m]$, one for each integer $m \geq 1$, such that there is a trivialization β of $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$ for which β_m is compatible with β, for all $m \geq 1$.

Define finally the *Igusa tower over* $\mathbb{Z}_p$ to be the $\mathbb{Z}_p$-formal scheme

$$\widehat{\mathrm{Ig}} = \varprojlim_n \widehat{\mathrm{Ig}}_n = \varprojlim_n \varinjlim_m \mathrm{Ig}_{m,n}$$

where the direct limit is computed with respect to the canonical maps induced by the canonical projection maps $\mathbb{Z}/p^{n+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ for each $n \geq 1$. Refer to [Hid04, Chapter 8], [Hid02, §2.1] and [Bur17, §2.5] for details on Igusa towers.

1.5 Heegner points

We recall Heegner points as introduced in [LV11, §3.1]. An embedding $f\colon K \hookrightarrow B$ induces a multiplicative map $f^\times\colon K^\times \hookrightarrow B^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R})$, which, after extending scalars to $\mathbb{R}$, can be seen as a point in $\mathcal{H}^+$ following the interpretation given in §1.3. Let $m \geq 0$ and $\widetilde{c} \geq 1$ be integers, with $\widetilde{c}$ coprime with pND_K.

Definition 1.5.1. Let $O \subseteq \mathcal{O}_K$ be an order and $R \subseteq B$ be an Eichler order. An embedding $f\colon K \hookrightarrow B$ is an *optimal embedding of* O *into* R if $f^{-1}(R) = O$.

Definition 1.5.2. A *Heegner point* of conductor $\widetilde{c}$ in X_m is a point $P = [(f, g)] \in X_m(\mathbb{C})$ where $g \in \widehat{B}^\times$ and f is an optimal embedding of $\mathcal{O}_{\widetilde{c}}$ into the Eichler order $g^{-1}U_m g \cap B$.

Heegner points in X_m are defined over $\overline{\mathbb{Q}}$. More precisely, since B is indefinite:

Proposition 1.5.3. If P is a Heegner point of conductor $\widetilde{c}$ in X_m, then $P \in X_m(H_{\widetilde{c}})$.

Since K is dense in $\widehat{K}$, the embedding $f\colon K \hookrightarrow B$ induces a map $\widehat{f}\colon \widehat{K} \hookrightarrow \widehat{B}$ called the *adelization* of f. For any finite place ℓ we denote by f_ℓ the restriction to the ℓ-component of $\widehat{K}$.

Definition 1.5.4. A *Heegner point* of conductor $\widetilde{c}$ in $\widetilde{X}_m$ is a point $\widetilde{P} = [(f, g)] \in \widetilde{X}_m(\mathbb{C})$ such that $\varsigma_m(P)$ is a Heegner point of conductor $\widetilde{c}$ in X_m and

$$f_p^{-1}\left(f_p\left((\mathcal{O}_{\widetilde{c}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times\right) \cap g_p^{-1}U_{m,p}g_p\right) = (\mathcal{O}_{\widetilde{c}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cap (1 + p^m\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times.$$

Simply put, a Heegner point in $\widetilde{X}_m$ is a lift of a Heegner point in X_m subject to some local condition on p. Heegner points in $\widetilde{X}$ are also defined over $\overline{\mathbb{Q}}$. More precisely, each point $[(f, g)]$ is defined over an extension of H_c that depends on f:

Proposition 1.5.5. If $P = [(f, g)]$ is a Heegner point of conductor $\widetilde{c}$ in $\widetilde{X}_m$, then $P \in \widetilde{X}_m\left(H_{cp^n}(\boldsymbol{\mu}_{cp^n})\right)$ (see *ibid.*, Proposition 3.2).

Proposition 1.5.6 (Shimura reciprocity law)**.** Let $x = [(f, g)]$ be a point in X_m or $\widetilde{X}_m$. For any $\sigma \in \mathrm{Gal}(\widehat{K}/K)$, let $a_\sigma \in \widehat{K}$ correspond to it under the Artin reciprocity map (see Remark 1.0.2). We have that

$$x^\sigma = [(f, \widehat{f}(a^{-1})g)]$$

(see [Shi71, Theorem 9.6]).

1.6 CM points

Let $m \geq 0$ and $c \geq 1$ be integers, with c coprime with pND_K. We now construct Heegner points of conductor $\widetilde{c} = cp^n$. Decompose $c = c^+c^-$ into the product c^+ of primes dividing c that are split in K and into the product c^- of primes dividing c which are inert in K. Since all primes dividing c^+ and N^+ are split, we can decompose $c^+ = \mathfrak{c}^+\bar{\mathfrak{c}}^+$ and $N^+ = \mathfrak{N}^+\bar{\mathfrak{N}}^+$ as a product of ideals. For each prime number ℓ and each integer $n \geq 0$, define

- $\xi_\ell := 1$ if $\ell \nmid N^+cp$;
- $\xi_p^{(n)} := \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{p}) = \mathrm{GL}_2(\mathbb{Q}_p)$;
- $\xi_\ell := \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell^s & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{l}) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid c^+$ and $s = v_\ell(c^+)$; where $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ splits in K and $\mathfrak{l} \mid \mathfrak{c}^+$;
- $\xi_\ell = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell^s & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid c^-$ and $s = v_\ell(c^-)$;
- $\xi_\ell := \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{l}) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid N^+$, where $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ is a factorization into prime ideals in $\mathcal{O}_K$ and $\mathfrak{l} \mid \mathfrak{N}^+$.

All those elements can be seen inside $\widehat{B}^\times$ under the isomorphisms i_ℓ defined in §1.1.2. Denote by $\xi^{(n)}$ the point whose p-component is $\xi_p^{(n)}$ and ℓ-component is ξ_ℓ for $\ell \neq p$. Next we define maps that generate Heegner points.

Define $x_{cp^n,m}\colon \mathrm{Pic}(\mathcal{O}_{cp^n}) \to X_m(\mathbb{C})$ by $[\mathfrak{a}] \mapsto [(\iota_K, a\xi^{(n)})]$, where for each class $[\mathfrak{a}]$ we take an element $\mathfrak{a}$ and write it as $\mathfrak{a} = a\widehat{\mathcal{O}}_{cp^n} \cap K$ for some $a \in \widehat{K}^\times$, seen as an element of $\widehat{B}^\times$ via the adelization of ι_K. We often replace $[\mathfrak{a}]$ in $x_{cp^n,m}([\mathfrak{a}])$ by just $\mathfrak{a}$ or a.

Proposition 1.6.1. For all $a \in \mathrm{Pic}(\mathcal{O}_{cp^n})$, the point $x_{cp^n,m}(a)$ is a Heegner point of conductor cp^n in $X_m(H_{cp^n})$.

Now lift the previous map to $\tilde{x}_{cp^n,m}\colon K^\times\backslash\widehat{K}^\times \to \widetilde{X}_m(\mathbb{C})$ by $\tilde{x}_{cp^n,m}(a) = [(\iota_K, a\xi^{(n)})]$.

Proposition 1.6.2. For all $a \in K^\times\backslash\widehat{K}^\times$, the point $\tilde{x}_{cp^n,m}(a)$ is a Heegner point of conductor cp^n in $\widetilde{X}_m(H_{cp^n})$.

A further generalization of the above maps can be obtained by considering the pro-$\mathbb{Z}_p$-scheme

$$\widetilde{X}_\infty = \varinjlim_m \widetilde{X}_m,$$

where $\widetilde{X}_m$ is viewed as a $\mathbb{Z}_p$-scheme by scalar extension. Then we have a uniformization map $\mathcal{H}^\pm \times \widehat{B}^\times \to \widetilde{X}_\infty(\mathbb{C})$ taking a point (x, g) to its classes in each layer $\widetilde{X}_m$. For $\mathfrak{a}$ an integral ideal of $\mathcal{O}_c$ such that $(\mathfrak{a}, \mathfrak{N}^+\mathfrak{p}) = 1$, write it as $\mathfrak{a} = a\widehat{\mathcal{O}}_K \cap \mathcal{O}_c$. We then define $x(\mathfrak{a}) := [(\iota_K, a^{-1}\xi)]$, which we often write $x(a)$. None of the above definitions depend on the choice of a. If follows from the previous proposition that $x(\mathfrak{a})$ is rational over H_{cp^∞} in the sense that the canonical projection $x_m(\mathfrak{a})$ of $x(\mathfrak{a})$ to $\widetilde{X}_m(\mathbb{C})$ belongs to $\widetilde{X}_m(H_{cp^\infty})$ for each $m \geq 0$. Let $x_m(\mathfrak{a})$ correspond to a quadruplet $(A_\mathfrak{a}, \iota_\mathfrak{a}, \alpha_\mathfrak{a}, \beta_{\mathfrak{a},m})$ consisting of a naïve level triplet classified by $\widetilde{X}_m$ with an additional arithmetic trivialization. The abelian variety $A_\mathfrak{a}$ can be defined over $\mathcal{V} = K^{\mathrm{ab}} \cap \mathbb{Z}_p^{\mathrm{unr}}$, where $\mathbb{Z}_p^{\mathrm{unr}}$ denotes the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. Since p is split in K, $A_\mathfrak{a}$ is p-ordinary and so there exists a unique arithmetic trivialization $\beta_\mathfrak{a}$ compatible with the arithmetic trivializations defined by $\beta_{\mathfrak{a},m}$. Therefore $x(\mathfrak{a}) \in \widetilde{X}_\infty$ gives a well-defined point $x(\mathfrak{a}) = (A_\mathfrak{a}, \iota_\mathfrak{a}, \alpha_\mathfrak{a}, \beta_\mathfrak{a})$ in the Igusa tower $\widehat{\mathrm{Ig}}$.

1.7 Symmetric tensors

This intermission follows [Kin15, §12.2.2] in remarking a seemingly not so well spread piece of notation. Let $n \geq 1$ be an integer, $\mathfrak{S}_n$ be the group of permutations on n letters and H be a free abelian module of finite rank over a ring R of characteristic 0.

The group $\mathfrak{S}_n$ acts on the n-th tensor power $H^{\otimes n}$ by permutation of the factors of each elementary tensor. The quotient of $H^{\otimes n}$ by the action of $\mathfrak{S}_n$ (that is, the coinvariant elements under this action) is the symmetric n-th power

$$\mathrm{Sym}^n(H) := H^{\otimes n}/\{h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)} - h_1 \otimes \cdots \otimes h_n;\ \forall \sigma \in \mathfrak{S}_n,\ h_i \in H\},$$

while the invariant elements under the action of $\mathfrak{S}_n$ are the symmetric tensors, forming a submodule of $H^{\otimes n}$ denoted by

$$\mathrm{TSym}^n(H) := \{\eta \in H^{\otimes n};\ \sigma(\eta) = \eta\}.$$

An element of $\mathrm{TSym}^n(H)$ is then a R-linear combination of the sums of all distinct permutations of an elementary tensor. To be more precise, for $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$, we split the multiset $S := \{h_1, \cdots, h_n\}$ into a disjoint union of multisets $S_1, \ldots, S_N$ such that for $a \in S_i$ and $b \in S_j$, $a = b \iff i = j$. Define

$$\alpha(S) := \binom{n}{\#S_1, \ldots, \#S_N} \cdot (n!)^{-1}$$

to be the redundancy of the permutations of S. Then all elements of $\mathrm{TSym}^n(H)$ are of the form

$$\eta = \sum_{j=0}^{N} t_j \cdot \alpha(h_1^{(j)}, \ldots, h_n^{(j)}) \sum_{\sigma \in \mathfrak{S}_n} h_{\sigma(1)}^{(j)} \otimes \cdots \otimes h_{\sigma(n)}^{(j)}, \quad t_j \in \mathbb{Z}.$$

Similarly to the symmetric algebra,

$$\mathrm{TSym}^\bullet(H) := \bigoplus_{n\geq 0} \mathrm{TSym}^n(H)$$

is a graded algebra with the usual sum and with the symmetric product $\odot$, the symmetrization of the tensor product, which extends linearly after defining

$$\left(\alpha(a_1,\dots,a_m)\sum_{\sigma\in\mathfrak{S}_m} a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(m)}\right)\odot\left(\alpha(b_1,\dots,b_n)\sum_{\sigma\in\mathfrak{S}_n} b_{\sigma(1)}\otimes\cdots\otimes b_{\sigma(n)}\right) :=$$
$$\frac{\alpha(a_1,\dots,a_m)\cdot\alpha(b_1,\dots,b_n)}{\alpha(c_1,\dots,c_{m+n})}\sum_{\sigma\in\mathfrak{S}_{m+n}} c_{\sigma(1)}\otimes\cdots\otimes c_{\sigma(m+n)},\ \text{where } c_i = \begin{cases} a_i & i\leq m \\ b_{i-m} & i>m. \end{cases}$$

In particular,

$$h^{\otimes m}\odot h^{\otimes n} = \frac{(m+n)!}{m!n!}h^{\otimes(m+n)}$$

for all $h\in H$.

Identifying H with $\mathrm{TSym}^1(H)$, H can be embedded in $\mathrm{TSym}^\bullet(H)$, and therefore the universal property of the symmetric algebra gives a graded algebra morphism $\mathrm{Sym}^\bullet(H)\to\mathrm{TSym}^\bullet(H)$, which specializes in degree n to

$$\begin{aligned} \mathrm{Sym}^n(H) &\longrightarrow \mathrm{TSym}^n(H) \\ h_1^{\cdot e_i}\cdots h_d^{\cdot e_d} &\longmapsto e_1!\cdots e_d!\,h_1^{\odot e_i}\odot\cdots\odot h_d^{\odot e_d} \end{aligned}$$

If $n!$ is invertible in R, the above map can be inverted and so becomes an isomorphism.

There is another natural link between Sym and TSym: if $\bullet^\vee := \mathrm{Hom}(\bullet, R)$ denotes the linear dual of an R-module, we have a canonical isomorphism

$$\mathrm{TSym}^n(H^\vee)\cong\left(\mathrm{Sym}^n(H)^\vee\right).$$

If $\mathscr{F}$ is a locally free sheaf on a variety X over a field of characteristic 0, $\mathrm{Sym}^\bullet(\mathscr{F})$ makes sense as a locally free sheaf on X (see [Sta, §01CK]), but that is not always the case with TSym. However, since H is a free module, $\mathrm{TSym}^\bullet(H)$ coincides with $\Gamma^\bullet(H)$, the divided power algebra of H (see [Lun08] for a definition and more details on the relation between these two objects) which does sheafify well: as pointed out in [KLZ17, §2.2], if $\mathscr{F}$ is a locally free sheaf on a variety X over a field of characteristic 0, $\Gamma^n(\mathscr{F})$ and therefore $\mathrm{TSym}^n(\mathscr{F})$ defines a locally free sheaf on X and, in particular, one can talk about symmetric tensors of coefficient sheaves of étale cohomology.

1.8 Motives of universal abelian varieties

1.8.1 Relative Chow motives

We begin by recalling general definitions on relative Chow motives as in [CH00].

Definition 1.8.1. Let F be a field and S be a smooth projective scheme over F. The category $\mathrm{CHM}_F(S)$ of *relative Chow motives over S with coefficients in F* consists of

- Objects: triples (V, p, m) where $V \to S$ is a smooth projective scheme, p is an element of $\mathrm{Corr}^0_S(V)_F = \mathrm{CH}^{\dim(V)}(V \times_S V) \otimes_{\mathbb{Z}} F$ satisfying $p^2 = p$ and m is an integer (the *Tate twist*);

- Morphisms: if $V = \coprod_i V_i$ is the decomposition of V in connected components,

$$\mathrm{Hom}((V, p, m), (W, q, n)) := q \circ \mathrm{Corr}^{s-r}_S(V_i, W) \circ p.$$

The category $\mathrm{CHM}_F(S)$ is F-linear and has a symmetric tensor product defined as

$$(V, p, m) \otimes (W, q, n) := (V \times_S W, pq, m + n).$$

Furthermore, this category is also equipped with a contravariant "varieties-to-motives" functor $M\colon \mathrm{Var}_S \to \mathrm{CHM}_F(S)$ mapping a smooth proejctive variety V over S to $(V, \mathrm{id}, 0)$ and a map $f\colon V \to W$ defined over F to $\mathrm{graph}(f) \in \mathrm{CH}^{\dim(V)}(V \times_S W)_F$.

Definition 1.8.2. Let $M = (V, p, m)$ be a motive. We define its *dual motive* to be

$$M^\vee := (V, p^{\mathrm{T}}, \dim(V) - m),$$

where p^{T} denotes the transpose of p as a correspondence.

Let $n \geq 1$ be an integer. As before, the group of permutations on n letters $\mathfrak{S}_n$ acts on $M^{\otimes n}$. The image of M under the projector $\pi_{\mathfrak{S}_n} = \sum_{\sigma \in \mathfrak{S}_n} \sigma$ is the n-th symmetric power of M, denoted $\mathrm{Sym}^n(M) = (V, \pi_{\mathfrak{S}_n} \circ p, m)$. In view of section 1.7, we define

$$\mathrm{TSym}^n(M) := \left(\mathrm{Sym}^n(M^\vee)\right)^\vee.$$

Remark 1.8.3. The motive (V, id, m) is more commonly written as $h(V)(m)$. Other two notable motives have custom notation: the *trivial motive* $\mathbb{1} := (\mathbb{P}^1_S, \mathrm{id}, 0)$ and the *Lefschetz motive* $\mathbb{L} := (\mathbb{P}^1_S, \mathrm{id}, -1) = \mathbb{1}(-1)$. Notice that, for $M \in \mathrm{CHM}_F(S)$,

$$M \otimes \mathbb{1} = M \quad M \otimes \mathbb{L} = M(-1) \text{ and } M \otimes \mathbb{L}^\vee = M(1).$$

1.8.2 Decomposition of motives of universal abelian varieties

Until the end of this section, let

- B be an indefinite quaternion algebra;
- G be an algebraic group, D be a hermitian symmetric domain so that (G, D) is a Shimura datum of PEL type arising from the quaternion algebra B;
- U be a compact open subgroup of $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$;
- X be the canonical model of the Shimura variety of level U associated to (G, D) over reflex field F (see [Mil05, §12]);
- $\pi\colon \mathcal{A} \to X$ be the universal PEL abelian variety over X, of genus g;
- V_G be the standard representation of G arising from the PEL datum.

Consider the motive $h(\mathcal{A})$. As shown in [DM91, Theorem 3.1], such motive can be decomposed into eigenspaces of the multiplication by n map. More precisely, if $n \in \mathbb{Z}$, the isogeny $[n]\colon A \to A$ induces a correspondence $(\mathrm{id} \times [n])^* \in \mathrm{CH}^{2g}(\mathcal{A} \times_X \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and there are unique projectors π_i for $0 \leq i \leq 2g$ such that $i \neq j \implies \pi_i \neq \pi_j$ and

$$(\mathrm{id} \times [n])^* \pi_i = n^i \pi_i.$$

Denoting the eigenspaces by $h^i(\mathcal{A}) := (\mathcal{A}, \pi_i, 0)$, we get the decomposition

$$h(\mathcal{A}) = \bigoplus_{i=0}^{2g} h^i(\mathcal{A}).$$

The next proposition summarizes basic properties related to this decomposition (see [Anc15, Théorème 2.4, Corollaire 2.6]):

Proposition 1.8.4. The following properties hold:

(i) For all $0 \leq i \leq 2g$, $\mathrm{Sym}^i(h^1(\mathcal{A}))$ is canonically isomorphic to $h^i(\mathcal{A})$. Furthermore, $\mathrm{Sym}^i(h^1(\mathcal{A})) = 0$ for $i > 2g$;

(ii) $h^0(\mathcal{A})$ is the trivial motive $\mathbb{1}$ and $h^{2g}(\mathcal{A})$ is $\mathbb{L}^{\otimes g}$, where $\mathbb{L}$ is the Lefschetz motive;

(iii) Poincaré duality: there are canonical isomorphisms $h^{2g-i}(\mathcal{A})^\vee \cong h^i(\mathcal{A})(g)$;

(iv) Lefschetz isomorphisms: there are non-canonical isomorphisms

$$h^i(\mathcal{A}) \cong h^{2g-i}(\mathcal{A})(g-i);$$

(v) There is a Poincaré–Lefschetz non-canonical isomorphism $h^1(\mathcal{A})(1) \cong h^1(\mathcal{A})^\vee$.

1.8.3 Realization of motives into Weil cohomologies

Recall that a *Weil cohomology theory* relative to X and with coefficients in F is a functor $H^\bullet\colon \mathrm{Var}_X \to \mathrm{Vec}_F^{\pm}$ from the category of varieties over X to the category of finite dimensional $\mathbb{Z}$-graded F-vector spaces satisfying, for each $V, W \in \mathrm{Var}_X$,

(i) Poincaré duality, that is, a perfect pairing $H^i(X) \times H^{2g-i}(X) \to H^{2g}(X) \cong F(g)$;

(ii) Künneth decomposition, that is, an isomorphism $H^\bullet(V) \otimes H^\bullet(W) \xrightarrow{\sim} H^\bullet(V \times_S W)$;

(iii) Cycle class map, that is, a map $\mathrm{cl}_X\colon \mathrm{CH}^i(X) \to H^{2i}(X)(i)$ compatible with the Künneth decomposition.

For example, the étale cohomology is a Weil cohomology: see [Mil80, Chapter VI] for the proofs of it satisfying the above requirements.

Fulfilling the motivation behind motives as lietmotifs of Weil cohomologies, for each such $H^\bullet$ there is a realization functor

$$R\colon \mathrm{CHM}_F(X) \to \mathrm{Vec}_F^{\pm}$$

compatible with the varieties-to-motives functor, that is, giving a commutative diagram

$$\begin{tikzcd} \mathrm{CHM}_F(X) \arrow[r, "R"] & \mathrm{Vec}_F^{\pm} \\ \mathrm{Var}_X \arrow[u, "M"] \arrow[ur, "H^\bullet"'] & \end{tikzcd}$$

and being such that $R(h^i(\mathcal{A})) = H^i(\mathcal{A})$ for all $0 \le i \le 2g$ ([Anc15, Proposition 3.5]). Two realizations will be most relevant to us (see *ibid.*, Exemples 3.3):

Definition 1.8.5. Let F be a number field with a fixed embedding into $\mathbb{C}$ and fix a base point $x \in X(F)$. The *Hodge realization* of a motive is the realization into the Weil cohomology

$$H^\bullet\colon (f\colon V \to X) \mapsto \bigoplus_i (R^i f_* F_V)_x.$$

Definition 1.8.6. Let ℓ be a rational prime, F be a finite extension of $\mathbb{Q}_\ell$ embedded into a fixed algebraic closure $\overline{\mathbb{Q}}_\ell$ and fiz a base geometric point $\bar{x} \in X(\overline{\mathbb{Q}}_\ell)$. The *étale realization* of a motive is the realization into the Weil cohomology

$$H^\bullet\colon (f\colon V \to X) \mapsto \bigoplus_i \left(R^i f_* F_V\right)_{\bar{x}}.$$

The following result works for any realization, although we are going to use it mostly for the two cases above:

Proposition 1.8.7 ([Anc15, Théorème 6.1])**.** Let $n \in \mathbb{Z}_{>0}$ and $H^\bullet$ be a Weil cohomology. Then every decomposition into direct summands of $H^1(\mathcal{A})^{\otimes n}$ lifts canonically to a decomposition into direct summands of $h^1(\mathcal{A})^{\otimes n}$.

1.8.4 Ancona functor

Let $\mathrm{Rep}_F(G)$ denote the category of F-representations of the algebraic group G, that is, the continuous group morphisms $G(F) \to \mathrm{GL}(V)$, where V is a finite dimensional F-vector space, and $\mathrm{VHS}_F(X(\mathbb{C}))$ the category of variations of F-Hodge structures over $X(\mathbb{C})$. In [Anc15], Ancona has shown that the canonical construction functor

$$\mu\colon\ \mathrm{Rep}_F(G) \longrightarrow \mathrm{VHS}_F(X(\mathbb{C}))$$

(see [Pin90, §1.18] and [Tor20, §4] for details) lifts through the Hodge realization functor (Definition 1.8.5) to a functor

$$\mathrm{Anc}_G\colon\ \mathrm{Rep}_F(G) \longrightarrow \mathrm{CHM}_F(X),$$

which we call the *Ancona functor* of G (with, implicitly, coefficients in F).

Proposition 1.8.8 ([Anc15, Théorème 8.6])**.** The functor Anc_G is F-linear and preserves duals and tensor products. Furthermore, the standard representation $V_G(F)$ of G maps into $h^1(\mathcal{A})$.

Remark 1.8.9. The original functor maps $V_G(F)$ to $h^1(\mathcal{A})^\vee$. Here, we are following the normalization in [LSZ22, Remark 6.2.3].

Now consider two Shimura varieties X and X' and a morphism between their Shimura data $f\colon (G', D') \to (G, D)$, that is, an algebraic group morphism $G' \to G$ taking D' to D ([Mil05, Definition 5.15]). We also assume that $f(U') \subseteq U$ so a map $f\colon X' \to X$ makes sense. Base change by f induces pullbacks at the levels of representations and of motives. Torzewski has shown that, for PEL Shimura data, the Ancona functor is independent of choice of data and commutes with morphisms that are *admissible* in the sense of [Tor20, Definition 9.1]: that $f^*V_{G'}$ is a direct summand of $V_G^{\oplus n}$ for some n.

Proposition 1.8.10 ([Tor20, Theorem 9.7])**.** Let $f\colon (G', D') \to (G, D)$ be an admissible morphism of PEL Shimura data. Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Rep}_F(G) \arrow[r, "f^*"] \arrow[d, "\mathrm{Anc}_G"'] & \mathrm{Rep}_F(G') \arrow[d, "\mathrm{Anc}_{G'}"] \\
\mathrm{CHM}_F(G) \arrow[r, "f^*"'] & \mathrm{CHM}_F(G').
\end{tikzcd}$$

Chapter 2

Generalized Heegner classes

For the whole chapter fix integers $m \geq 0$, $k \geq 2$ and $r := k/2 - 1$ (that is, $k = 2r + 2$). Recall the Shimura curve X_m of level U_m introduced in §1.3 and let $\mathcal{A}_m$ be the the correspondent universal abelian variety (see Proposition 1.4.3).

Via the main theorem of complex multiplication (see [Sil94, §II.8]), the elliptic curve $E := \mathbb{C}/\mathcal{O}_K$ has a model defined over H. We shall denote by (A, ι, α) a fixed QM abelian variety, consisting of $A := E \times E$, the embedding $\iota\colon \mathcal{O}_B \hookrightarrow \mathrm{End}(A) = \mathcal{O}_K \oplus \mathcal{O}_K$ induced by ι_K, and equipped with a level U_m structure α.

Remark 2.0.1. The fixed QM abelian variety A can also be taken to be the product of two different elliptic curves E_1 and E_2 with CM by $\mathcal{O}_K$. The relevant property is that the idempotents e and $\bar{e}$ must act as projections into each factor, that is $eA = E_1$ and $\bar{e}A = E_2$. Since $j(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $j(\bar{e}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it suffices to notice that the decomposition should be equivariant respect to the QM action of $\mathcal{O}_B$. This is clear when $E_1 = E_2$. For $E_1 \neq E_2$ the equivariance respect to the QM action of B follows from [Kan11, Theorem 2]; see also [Ufe12, Theorem 4.4].

2.1 Generalized Kuga–Sato variety

We recall the generalized Kuga–Sato variety following [HB15, §2.6].

Definition 2.1.1. The *Kuga–Sato variety* of weight k over X_m is $\mathcal{A}_m^r$, the r-fold fiber product of $\mathcal{A}_m$ over X_m. The *generalized Kuga–Sato variety* of weight k over X_m is

$$W_{k,m} := \mathcal{A}_m^r \times_{X_m} A^r.$$

Since A is defined over H and both $\mathcal{A}_m$ and X_m have models over $\mathbb{Q}$, $W_{k,m}$ has a model over H, which we fix henceforth. The generalized Kuga–Sato variety $W_{k,m}$ has relative dimension $4r + 1 = 2k - 1$ over X_m, the first factor contributing with $2r + 1$ and the second with $2r$.

We now recall a projector on the ring of correspondences of $\mathcal{A}_m^r$ studied in [Bes95]. Denote by $\pi_{\mathcal{A}}\colon \mathcal{A}_m \to X_m$ the canonical projection. Since the complex structure commutes with the action of B, the idempotents e and $\bar{e}$ from 1.1.3 induce a decomposition of variations of Hodge structures

$$R^1\pi_{\mathcal{A},*}\mathbb{Q} = eR^1\pi_{\mathcal{A},*}\mathbb{Q} \oplus \bar{e}R^1\pi_{\mathcal{A},*}\mathbb{Q}$$

into isomorphic factors. In [Bes95, Theorem 5.8.iii], a projector $\epsilon_{\mathcal{A}} \in \mathrm{Corr}^0_X(\mathcal{A}_m^r, \mathcal{A}_m^r)$ is defined which induces a projection from $R^{2r}\pi_{\mathcal{A},*}\mathbb{Q}$ to $\mathrm{Sym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$: after finding $\mathrm{Sym}^2(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$ inside $R^2\pi_{\mathcal{A},*}\mathbb{Q}$ (*ibid.*, Theorem 5.8.ii), we have that

$$\begin{aligned}
\mathrm{Sym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}) &\hookrightarrow \mathrm{Sym}^r(\mathrm{Sym}^2(eR^1\pi_{\mathcal{A},*}\mathbb{Q})) && (\textit{ibid.}, \text{5.6.vii})\\
&\hookrightarrow (\mathrm{Sym}^2(eR^1\pi_{\mathcal{A},*}\mathbb{Q}))^{\otimes r}\\
&\hookrightarrow (R^2\pi_{\mathcal{A},*}\mathbb{Q})^{\otimes r} && (\textit{ibid.}, \text{5.8.ii})\\
&\hookrightarrow R^{2r}\pi_{\mathcal{A},*}\mathbb{Q}
\end{aligned}$$

the last step following from Künneth formula. Essentially, $\epsilon_{\mathcal{A}}$ kills all factors in Künneth formula apart from $(R^2\pi_{\mathcal{A},*}\mathbb{Q})^{\otimes r}$, which gets projected into $\mathrm{Sym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$. We let $\epsilon_A \in \mathrm{Corr}^{2r}(A^r, A^r)$ to be the projector obtained specializing each factor or $\mathcal{A}_m^r$ to A. We also define $\epsilon_W := \epsilon_{\mathcal{A}}\epsilon_A \in \mathrm{Corr}^r_{X_m}(W_{k,m}, W_{k,m})$.

Remark 2.1.2. Although the projector originally has its image in $\mathrm{Sym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$, it will be more convenient for us to identify it with $\mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$, defined in §1.7. The reason is that in what follows, the image of said projector, after a twist, will be the symmetric power of $(eR^1\pi_{\mathcal{A},*}\mathbb{Q})^\vee$ so, under the above identification, the twisted image $\mathrm{TSym}^{2r}((eR^1\pi_{\mathcal{A},*}\mathbb{Q})^\vee)$ more directly identifies to the dual of $\mathrm{Sym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q})$, from which representations of modular forms will arise.

2.2 Generalized Heegner cycles

Fix also integers $c \geq 1$ prime to ND_Kp and $n \geq m$. Let $F_{\mathfrak{N}^+}$ be the ray class field of K modulo $\mathfrak{N}^+$ and $F_{\mathfrak{N}^+,cp^n}$ the smallest abelian extension of K containing both $F_{\mathfrak{N}^+}$ and H_{cp^n}. Under the reciprocity map, $F_{\mathfrak{N}^+,cp^n}$ corresponds to the group

$$U_{\mathfrak{N}^+,cp^n} = \{x \in \widehat{\mathcal{O}}^\times_{cp^n}; x \equiv 1 \mod \mathfrak{N}^+\}.$$

We often ignore the dependence on $\mathfrak{N}^+$ and just write $F_{cp^n} := F_{\mathfrak{N}^+,cp^n}$.

Following [HB15, §6.1], let F be a finite extension of F_{cp^n}, (A', ι') be a QM abelian surface and $\phi\colon A \to A'$ be an isogeny defined over F whose kernel intersects trivially the image of α, the level U_m structure of the fixed QM abelian surface A fixed before. Under these conditions, $\alpha' := \varphi \circ \alpha$ gives a level U_m structure on A', and therefore a point (A', ι', α') in $\mathcal{A}_m$ defined over F (*cf.* [Shi67, Theorem 3.2]), which allows the graph of φ to be embedded into $A \times_X \mathcal{A}_m$ via the "universality" inclusion $A' \hookrightarrow \mathcal{A}_m$, giving an embedding

$$\mathrm{graph}(\phi)^r \hookrightarrow W_{k,m}.$$

The graph of ϕ has dimension $2r$ so, as a rational cocycle in $W_{k,m}$ defined over F, it has codimension $2r+1 = k-1$.

Definition 2.2.1. We define the *generalized Heegner cycle* associated to the isogeny $\phi\colon A \to A'$ to be

$$\Delta_{\phi,m}^{[k]} := \epsilon_W \,\mathrm{graph}(\phi)^r \in \epsilon_W \,\mathrm{CH}^{k-1}(W_{k,m} \otimes_H F)_{\mathbb{Q}}.$$

Remark 2.2.2. Take A' to be the QM abelian surface $A_{cp^n} := \mathbb{C}/\mathcal{O}_{cp^n} \times \mathbb{C}/\mathcal{O}_{cp^n}$. The multiplication by cp^n isogeny $[cp^n]\colon E \to \mathbb{C}/\mathcal{O}_{cp^n}$ gives an isogeny $\phi_{cp^n}\colon A \to A_{cp^n}$ which is defined over F_{cp^n}. We simply write $\Delta_{cp^n,m}^{[k]}$ for the associated generalized Heegner cycle.

Chow groups can be identified with motivic cohomology groups (*cf.* [MVW06, Corollary 19.2]), so there is an isomorphism

$$r_{\mathrm{mot}}\colon \epsilon_W \,\mathrm{CH}^{k-1}(W_{k,m} \otimes_H F)_{\mathbb{Q}} \xrightarrow{\sim} \epsilon_W H^{2k-2}_{\mathrm{mot}}(W_{k,m} \otimes_H F, \mathbb{Q}(k-1)). \tag{2.1}$$

Composing it with the étale realization map (that is, the cycle map from étale cohomology),

$$r_{\text{ét}}\colon \epsilon_W H^{2k-2}_{\mathrm{mot}}(W_{k,m} \otimes_H F, \mathbb{Q}(k-1)) \longrightarrow \epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m} \otimes_H F, \mathbb{Q}_p(k-1)) \tag{2.2}$$

gives an étale class $\Delta_{\phi,m,\text{ét}}^{[k]} := r_{\text{ét}} \circ r_{\mathrm{mot}}(\Delta_{\phi,m}^{[k]})$ associated to $\Delta_{\phi,m}^{[k]}$.

2.3 Lieberman trick

By a technique known as Lieberman trick, one can replace the base scheme $W_{k,m}$ with the simpler Shimura variety X_m, to the cost of having a slightly more complicated coefficient system. The trick goes as follows: by Künneth decomposition theorem in étale cohomology (see [Mil80, Theorem 8.21]), we have

$$H^{2k-2}_{\text{ét}}(W_{k,m}, \mathbb{Q}_p) = \bigoplus_{i+j=2k-2} H^i_{\text{ét}}(\mathcal{A}_m^r, \mathbb{Q}_p) \otimes H^j_{\text{ét}}(A^r, \mathbb{Q}_p).$$

Let $\pi_{\mathcal{A}}\colon \mathcal{A}_m \to X_m$ and $\pi_A\colon A \hookrightarrow \mathcal{A}_m \xrightarrow{\pi_{\mathcal{A}}} X_m$ be the canonical projections. Since the Leray spectral sequence degenerates at page 2 (*cf.* [Del68, §2.4]), each of the groups in the right-hand side decomposes as

$$H^i_{\text{ét}}(\mathcal{A}_m^r, \mathbb{Q}_p) = \bigoplus_{a+b=i} H^a_{\text{ét}}(X_m, R^b\pi_{\mathcal{A},*}\mathbb{Q}_p) \text{ and } H^j_{\text{ét}}(A^r, \mathbb{Q}_p) = \bigoplus_{a+b=j} H^a_{\text{ét}}(X_m, R^b\pi_{A,*}\mathbb{Q}_p)$$

and the image of the projectors $\epsilon_{\mathcal{A}}$ and ϵ_A are motives whose Betti realizations are of type $((k-1,0),(0,k-1))$, as in [Bes95] (see the proof of Theorem 5.8 and the paragraph after the proof of Proposition 5.9 in *op. cit.*). Therefore, the only summand remaining after applying the projectors corresponds to the indexes $i=j=k-1$, so

$$\begin{array}{rl} \epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m},\mathbb{Q}_p) = & H^1_{\text{ét}}\left(X_m, \operatorname{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)\right) \otimes H^1_{\text{ét}}\left(X_m, \operatorname{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)\right) \\ & \downarrow{\scriptstyle \mathrm{PD}} \\ & H^2_{\text{ét}}\left(X_m, \operatorname{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p) \otimes \operatorname{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)\right), \end{array}$$

where PD is the Poincaré duality map (see for example [Mil80, Corollary 11.2]).

2.4 Generalized Heegner classes

Twisting the above cohomology groups by $k-1=2r+1$ gives a map

$$\begin{split} \epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m},\mathbb{Q}_p(k-1)) \longrightarrow \\ H^2_{\text{ét}}\left(X_m, \operatorname{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p) \otimes \operatorname{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)(2r+1)\right). \end{split} \tag{2.3}$$

It will be convenient to have the twists distributed in the following way: denoting

$$\mathscr{M}_{\text{ét}} := \operatorname{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p(1)) \otimes \operatorname{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p),$$

we have that

$$\operatorname{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p) \otimes \operatorname{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)(2r+1) \cong \mathscr{M}_{\text{ét}}(1). \tag{2.4}$$

The composition of r_{mot}, $r_{\text{ét}}$, (2.3) and the isomorphism induced by (2.4) gives a map

$$\epsilon_W H^{2k-2}_{\text{mot}}(W_{k,m} \otimes_H F, \mathbb{Q}(k-1)) \longrightarrow H^2_{\text{ét}}\left(X \otimes_H F, \mathscr{M}_{\text{ét}}(1)\right). \tag{2.5}$$

Definition 2.4.1. The image of $\Delta^{[k]}_{\phi,m}$ by (2.5) is the *generalized Heegner class* $z^{[k]}_{\phi,m}$.

2.5 j-components of generalized Heegner classes

We wish to define a motive $\mathscr{M}$ whose étale realization is $\mathscr{M}_{\text{ét}}$ (see Definition 1.8.6), and associate to it a representation under the Ancona functor. The reason is to make way for representations associated to modular forms to appear and, as mentioned in Remark 2.1.2, they will arise from the first factor of $\mathscr{M}_{\text{ét}}$. The second factor should work as a twist by a character, with an eye towards obtaining an anticyclotomic Euler system. However, the latter factor will be found to be associated not to one but a sum of characters, which can be projected into each of its 1-dimensional factors to find what in the next section will be called j-comopnents of a generalized Heegner class. But at first glance, it is not clear what the underlying Shimura variety of $\mathscr{M}$ should be. To make sense of it, let us consider each factor of $\mathscr{M}_{\text{ét}}$ separately and then combine them (which is allowed, since the Ancona functor is compatible with tensor products). This motivic approach is based on [JLZ21, §3.4].

2.5.1 The first factor

We consider $\mathscr{V}_{\text{ét}} := \mathrm{TSym}^{2r}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p(1))$ first. It follows by definition that $R^1\pi_{\mathcal{A},*}\mathbb{Q}_p$ is the étale realization of $h^1(\mathcal{A}_m) \in \mathrm{CHM}_{\mathbb{Q}}(X_m)$. By Proposition 1.8.7, decompositions of the étale realization of a motive lift to decompositions of the motive itself, so e induces a decomposition

$$h^1(\mathcal{A}_m) = eh^1(\mathcal{A}_m) \oplus \bar{e}h^1(\mathcal{A}_m) \tag{2.6}$$

as long as we take our motives with coefficients in K so e is defined, which is no problem as the canonical model we chose for X_m is defined over $\mathbb{Q}$. Furthermore, one can use Proposition 1.8.4 to write

$$h^1(\mathcal{A}_m)(1) \cong h^1(\mathcal{A}_m)^\vee \implies eh^1(\mathcal{A}_m)(1) \cong \left(e^\dagger h^1(\mathcal{A}_m)\right)^\vee,$$

where $\dagger$ is the involution of B defined in §1.2. Thus, we find $\mathscr{V}_{\text{ét}}$ to be the étale realization of

$$\mathscr{V} := \mathrm{TSym}^{2r}\left((e^\dagger h^1(\mathcal{A}_m))^\vee\right) \in \mathrm{CHM}_K(X_m).$$

So $\mathscr{V}$ comes from a K-representation of $\mathbf{G}$ defined in §1.3. The standard K-representation of such group is $K^4 \cong B \otimes_{\mathbb{Q}} K$ (as K-vector spaces), as

$$\mathbf{G}(K) = (B \otimes_{\mathbb{Q}} K)^\times = \mathrm{GL}(B \otimes_{\mathbb{Q}} K).$$

The idempontents $e^\dagger$ and $\bar{e}^\dagger$ induce a decomposition of K^4 into two isomorphic factors

$$K^4 = e^\dagger K^4 \oplus \bar{e}^\dagger K^4 \text{ with } e^\dagger K^4 \cong \bar{e}^\dagger K^4 \cong K^2.$$

By Proposition 1.8.8, $\mathrm{Anc}_{\mathbf{G}}(K^4) = h^1(\mathcal{A}_m)$ and so

$$\begin{aligned}
\mathscr{V}^{2r} &= \mathrm{TSym}^{2r}\left((e^\dagger h^1(\mathcal{A}_m))^\vee\right) \\
&= \mathrm{TSym}^{2r}\left((e^\dagger \mathrm{Anc}_{\mathbf{G}}(K^4))^\vee\right) \\
&= \mathrm{TSym}^{2r}\left((\mathrm{Anc}_{\mathbf{G}}(e^\dagger K^4))^\vee\right) \\
&= \mathrm{Sym}^{2r}\left(\mathrm{Anc}_{\mathbf{G}}(K^2)\right)^\vee \\
&= \mathrm{Anc}_{\mathbf{G}}\left((\mathrm{Sym}^{2r}(K^2))^\vee\right).
\end{aligned}$$

Therefore, we find that under Ancona functor, $h^1(\mathcal{A}_m)$ corresponds to the K-representation of $\mathbf{G}$ given by $(\mathrm{Sym}^{2r}(K^2))^\vee$.

2.5.2 The second factor

The procedure of the first case leads us to conjecture that the motive whose étale realization is $\mathrm{TSym}^{2r}(eR^1\pi_{A,*}\mathbb{Q}_p)$ is $\mathrm{TSym}^{2r}(eh^1(A))$, but A is not a universal abelian surface over a Shimura curve. Still, since $A \cong E \times E$ and $eA = E$ (see Remark 2.0.1), we have that $eh^1(A) = h^1(E)$, so it suffices to interpret the latter as a relative Chow motive.

The key point is that E is an elliptic curve defined over F_{cp^n} with CM by $\mathcal{O}_K$. The Shimura variety classifying the $\mathrm{Gal}(F_{cp^n}/K)$-orbits of elliptic curves with CM by $\mathcal{O}_K$ is the zero-dimensional PEL Shimura variety associated to the torus $\mathbf{H} := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ with level $U_{\mathfrak{N}^+,cp^n}$, the open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ corresponding to F_{cp^n} under Artin reciprocity, which has a canonical model S_{cp^n} defined over K (see the paragraph "CM-tori" at [Mil05, p. 347]). Denote by $\mathcal{E}$ the universal abelian variety over S_{cp^n}, which has dimension $[K : \mathbb{Q}]/2 = 1$ thus being an elliptic curve. By universality, the embedding $E \hookrightarrow \mathcal{E}$ allows us to see E as a scheme over S_{cp^n}, so $h^1(E) \in \mathrm{CHM}_K(S_{cp^n})$. Furthermore, since there are only finitely many moduli classes, $h^1(E)$ is a direct summand of $h^1(\mathcal{E})$. Then there is some direct summand W_1 of the standard K-representation of $\mathbf{H}$ (which is K, as $\mathbf{H}(K) = \mathrm{GL}(K)$) such that $\mathrm{Anc}_{\mathbf{H}}(W_1) = h^1(E) = eh^1(A)$, and therefore

$$\mathrm{TSym}^{2r}(eh^1(A)) = \mathrm{Anc}_{\mathbf{H}}\left(\mathrm{TSym}^{2r}(W_1)\right) \in \mathrm{CHM}_K(S_{cp^n}).$$

We are not interested in the precise description of W_1. What we care about is that the $2r+1$-dimensional representation $W_{2r} := \mathrm{TSym}^{2r}(W_1)$ splits into a direct sum of 1-dimensional representations (characters) $W_{2r}^{[j]}$ for $0 \le j \le 2r$ in which complex multiplication by $x \in \mathcal{O}_K$ acts as multiplication by $x^{2r-j}\bar{x}^j$, by which we mean the correspondences induced by both maps act the same way over the motive $eh^1(A)$. This is because a K-representation of $\mathbf{H}$ is the direct sum of representations of the form $\sigma^{i_1} \otimes \bar{\sigma}^{i_2}$, where i_1, i_2 are two integers and $\sigma\colon K \hookrightarrow K$ is a fixed embedding seen as the standard representation of $\mathbf{H}$ (see [Anc12, Remarque 4.8]). More generally, an embedding $\sigma_F\colon K \hookrightarrow F$ can be seen as an F-representation of $\mathbf{H}$.

Finally, define

$$h^{(2r-j,j)} := \mathrm{Anc}_{\mathbf{H}}(W_{2r}^{[j]}).$$

2.5.3 Combining both factors

Consider the algebraic representation $V^{(2r-j,j)}$ of $\mathbf{G} \times \mathbf{H}$ which over F is given by

$$V^{(2r-j,j)}(F) = \left(\mathrm{Sym}^{2r}(F^2)\right)^\vee \boxtimes \left(\sigma_F^{2r-j} \otimes \bar{\sigma}_F^j\right),$$

where $\boxtimes$ denotes the external tensor product. Define also the motive

$$\mathscr{M}^{(2r-j,j)} := \mathscr{V}^{2r} \otimes h^{(2r-j,j)}(A) \in \mathrm{CHM}_K(X_m \times S_{cp^n}).$$

Combining all of the above, it follows from the tensoriality of the Ancona functor that

$$\mathscr{M}^{(2r-j,j)} = \mathrm{Anc}_{\mathbf{G}\times\mathbf{H}}\left(V^{(2r-j,j)}(K)\right).$$

2.5.4 Definition of the j-components

The projection $\mathrm{TSym}^{2r}(eh^1(A)) \twoheadrightarrow h^{(2r-j,j)}(A)$ gives a correspondence $\mathscr{M} \to \mathscr{M}^{(2r-j,j)}$ in $\mathrm{Corr}^0_{X_m}(W_{k,m})$ which induces a map $\mathscr{M}_{\text{ét}} \to \mathscr{M}^{(2r-j,j)}_{\text{ét}}$ under the étale realization of motives and therefore a pushforward map in the étale cohomology

$$H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}_{\text{ét}}(1)\right) \longrightarrow H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}^{(2r-j,j)}_{\text{ét}}(1)\right), \tag{2.7}$$

Definition 2.5.1. The image of $z^{[k]}_{\phi,m}$ under (2.7) is a class $z^{[k,j]}_{\phi,m}$, the *j-component* of the generalized Heegner class $z^{[k]}_{\phi,m}$.

Remark 2.5.2. In view of Remark 2.2.2, for the generalized Heegner class associated to the isogeny $\phi_{cp^n}\colon A \to A_{cp^n}$ from before we simply write $z^{[k]}_{cp^n,m}$, and $z^{[k,j]}_{cp^n,m}$ for its j-component.

2.6 The Abel–Jacobi map

Let L be a p-adic field. Since p splits in K, there exists an embedding $\sigma_L\colon K \hookrightarrow L$. The degeneration at page 2 of the Hochschild–Lyndon–Serre spectral sequence [Nek00, §1.2] yields an isomorphism

$$H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}^{(2r-j,j)}_{\text{ét}}(1)\right) \xrightarrow{\sim} H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{M}^{(2r-j,j)}_{\text{ét}}(1))\right). \tag{2.8}$$

Recall from §2.5 the motive $\mathrm{TSym}^{2r}(eh^1(\mathcal{A}_m)(1)) = \mathscr{V}^{2r}$ and its étale realization $\mathscr{V}^{2r}_{\text{ét}}$, an étale lisse sheaf over L. As before, the embedding σ_L and its complex conjugate $\bar\sigma_L$ give an $\mathbf{H}$-representation $\sigma_L^{2r-j} \otimes \bar\sigma_L^j$. Each of those embeddings, seen as Galois characters, give characters $\sigma_{\text{ét}}, \bar\sigma_{\text{ét}}\colon \mathrm{Gal}(K^{\mathrm{ab}}/F_{\mathfrak{N}^+}) \to \mathbb{Q}_p^\times$ defined by $x \mapsto \sigma^{-1}(x_p)$ and $x \mapsto \bar\sigma^{-1}(x_p)$, where x_p is the p-component of x as an element of $(1 + \mathfrak{N}^+\widehat{\mathcal{O}}_K)^\times$, which is isomorphic to $\mathrm{Gal}(K^{\mathrm{ab}}/F_{\mathfrak{N}^+})$ via Artin reciprocity.

The right-hand side of (2.8) can then be further rewritten as

$$\begin{aligned} &H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{M}^{(2r-j,j)}_{\text{ét}}(1))\right) \cong \\ &\qquad H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^{2r}_{\text{ét}}(1) \otimes \sigma_{\text{ét}}^{2r-j}\bar\sigma_{\text{ét}}^j)\right). \end{aligned} \tag{2.9}$$

Definition 2.6.1. The composition of the maps (2.5), (2.7), (2.8) and (2.9) is the *p-adic Abel–Jacobi map*

$$\Phi^{[k,j]}_m\colon \epsilon_W\, \mathrm{CH}^{k-1}(W_{k,m} \otimes_H F_{cp^n})_{\mathbb{Q}} \longrightarrow H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^{2r}_{\text{ét}}(1) \otimes \sigma_{\text{ét}}^{2r-j}\bar\sigma_{\text{ét}}^j)\right). \tag{2.10}$$

In particular, $z^{[k,j]}_{\phi,m} = \Phi^{[k,j]}_m(\Delta^{[k]}_{\phi,m})$.

2.7 Basis vectors from CM points

We now use CM points from §1.6 to construct basis elements of the representation $V^{(2r-j,j)}$ base-changed to suitable p-adic fields.

As observed before, the image of the embedding $\sigma_{\overline{\mathbb{Q}}_p}\colon K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is contained in $\mathbb{Q}_p$ because p is split in K. The choice of a subfield $L \subseteq \overline{\mathbb{Q}}_p$ gives two embeddings $\sigma_L\colon K \hookrightarrow L$ and $\bar\sigma_L\colon K \hookrightarrow L$ satisfying $\bar\sigma_L(x) = \sigma_L(\bar x)$. The map $\iota_K\colon K \hookrightarrow B$ induces an embedding of algebraic groups $i\colon \mathbf{H} \hookrightarrow \mathbf{G}$ and, as before, for each pair of integers (ℓ_1, ℓ_2), we have a one-dimensional L-representation $\sigma_L^{\ell_1} \otimes \bar\sigma_L^{\ell_2}$ of $\mathbf{H}$.

In §1.1.2, the embedding ι_K was normalized so that $\vartheta \in K$ for all $x \in K$ we have that $\iota_K(x)\binom{\vartheta}{1} = \sigma_L(x)\binom{\vartheta}{1}$. More generally, we can similarly define vectors $e_{cp^n} \in L^2$ attached to the CM points $x_{cp^n,m}(1) = [(\iota_K, \xi^{(n)})]$ as follows. First, write $\xi^{(n)} = b^{(n)}u^{(u)}$ for some $b^{(n)} \in B^\times$ and $u^{(n)} \in U_m$; then $\vartheta_{cp^n} := (b^{(n)})^{-1}(\vartheta)$ is an eigenvector for $i_{cp^n} = (b^{(n)})^{-1}\iota_K b^{(n)}$, and $i_{cp^n}(\mathbf{H})$ acts on $v_{cp^n} := \binom{\sigma_L(\vartheta_{cp^n})}{1} \in L^2$ as the representation σ_L; note that, under the isomorphism (of compact Riemann surfaces) $X_m(\mathbb{C}) \cong \Gamma_m\backslash\mathcal{H}$ in §1.3 the point $[(i_{cp^n}, \xi^{(n)})]$ is sent to the class of the point ϑ_{cp^n}.

Dually, setting $\vartheta^*_{cp^n} = -1/\bar\vartheta_{cp^n}$, the correspondent vector $e_{cp^n} := \binom{\sigma_L(\bar\vartheta^*_{cp^n})}{1}$ is an eigenvector for the dual $(L^2)^\vee$ of the standard representation L^2 of $\mathbf{H}$, *i.e* for each $x \in K$ we have $(i_{cp^n}(x)^{-1})^{\mathrm{T}}(e_{cp^n}) = \sigma_L^{-1}(x)e_{cp^n}$. Denote $\bar e_{cp^n} := \binom{\bar\sigma_L(\vartheta^*_{cp^n})}{1}$.

Finally, define

$$e_{cp^n}^{[k,j]} := (e_{cp^n})^{\odot(2r-j)} \odot (\bar e_{cp^n})^{\odot j} \in \mathrm{TSym}^{2r}((L^2)^\vee).$$

Then $e_{cp^n}^{[k,j]}$ defines an element in the L-representation

$$V_L^{(2r-j,j)} := \mathrm{TSym}((L^2)^\vee) \boxtimes (\sigma_L^{(2r-j)} \otimes \bar\sigma_L^j)$$

of $\mathbf{G} \times \mathbf{H}$, which is invariant under the diagonal action of K by $i_{cp^n} \otimes \mathrm{id}$.

Each of the embeddings i_{cp^n} induces an embedding of $\delta_{cp^n}\colon \mathbf{H} \hookrightarrow \mathbf{G} \times \mathbf{H}$, defined by $\delta_{cp^n} = (i_{cp^n}, \mathrm{id})$, where id is the identity map. By Proposition 1.8.10, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_K(\mathbf{G} \times \mathbf{H}) \arrow[r, "\delta^*_{cp^n}"] \arrow[d, "\mathrm{Anc}_{\mathbf{G}\times\mathbf{H}}"] & \mathrm{Rep}_K(\mathbf{H}) \arrow[d, "\mathrm{Anc}_{\mathbf{H}}"] \\
\mathrm{CHM}_K(X_m \times S_{cp^n}) \arrow[r, "\delta^*_{cp^n}"] & \mathrm{CHM}_K(S_{cp^n})
\end{tikzcd}$$

where the horizontal maps are pullbacks induced by δ_{cp^n}. Via the functoriality of the étale realization, the maps described above descend to maps of lisse étale shaves over L. To simplify the notation, denote $X_m \times S_{cp^n}$ the K-scheme $(X_m \otimes_{\mathbb{Q}} K) \times S_{cp^n}$ (*i.e.* we simply view X_m as a K-scheme and take the product as K-schemes).

Let $\mathscr{M}_{\text{ét}}^{(2r-j,j)}$ be the étale realization of the motive $\mathscr{M}^{(2r-j,j)} \in \mathrm{CHM}_K(X_m \times S_{cp^n})$ introduced in §2.5 and consider the motive $\delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(2r-j,j)})$ in $\mathrm{CHM}_K(S_{cp^n})$. Composing the Gysin map, that is, the pushforward

$$H^0_{\text{ét}}\left(S_{cp^n}, \delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(2r-j,j)})\right) \longrightarrow H^2_{\text{ét}}\left(X_m \times S_{cp^n}, \mathscr{M}_{\text{ét}}^{(2r-j,j)}(1)\right)$$

induced by δ_{cp^n} (see [KLZ20, Definition 3.1.2, §5.2]) with the isomorphism

$$H^2_{\text{ét}}\left(X_m \times S_{cp^n}, \mathscr{M}_{\text{ét}}^{(2r-j,j)}(1)\right) \xrightarrow{\sim} H^2_{\text{ét}}\left(X_m \otimes_K F_{cp^n}, \mathscr{M}_{\text{ét}}^{(2r-j,j)}(1)\right)$$

that comes from the identification of S_{cp^n} as a K-variety with the $\mathrm{Gal}(F_{cp^n}/K)$-orbit of ϑ_{cp^n}, we obtain a map

$$\delta_{cp^n,*}\colon H^0_{\text{ét}}\left(S_{cp^n}, \delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(2r-j,j)})\right) \longrightarrow H^2_{\text{ét}}\left(X_m \otimes_K F_{cp^n}, \mathscr{M}_{\text{ét}}^{(2r-j,j)}(1)\right), \tag{2.11}$$

which we also call the *Gysin map* induced by δ_{cp^n}. Unraveling the definition of this map (see [Mil80, §VI.5–6]) the importance of those base vectors reveals itself:

Proposition 2.7.1. The image under $\delta_{cp^n,*}$ of the basis vector $e_{cp^n}^{[k,j]}$ is $z_{cp^n,m}^{[k,j]}$.

As we are going to see in chapter 4, being able to transfer the interpolation problem to vectors and then get for free the interpolation of generalized Heegner classes is at the core of the strategy of [JLZ21], which we adapt here to the quaternionic case.

Remark 2.7.2. All of the above constructions can be performed exactly the same way over $\widetilde{X}_m$ instead of X_m: the CM points appearing in that case would be $\widetilde{x}_{cp^n,m}(1)$ instead of $x_{cp^n,m}(1)$ which we have here, which correspond to the same optimal embeddings.

Chapter 3

Quaternionic modular forms

In this chapter, we recall modular forms for the quaternion algebra B of discriminant N^-, their p-adic Galois representations and Coleman families, and how they all relate to their elliptic counterparts through various instances of the Jacquet–Langlands correspondence.

3.1 p-adic modular forms

Quaternionic p-adic modular forms were defined for general weight in [Bra14], our reference for this section. See also [Kas99].

Let R be a p-adic ring, that is, a $\mathbb{Z}_p$-algebra which is complete and Hausdorff with respect to the p-adic topology, so that $R \cong \varprojlim_n R/p^n R$. In this section, if S is a $\mathbb{Z}_p$-scheme, $S_R := S \otimes_{\mathbb{Z}_p} R$ indicates the coefficient extension to R.

Definition 3.1.1. A *p-adic modular form with respect to B of tame level N^+* is a global section of $\widehat{\mathrm{Ig}}_R$.

If the quaternion algebra B doesn't need to be specified (which is often the case, as from now on B is, unless otherwise said, always taken to be the fixed quaternion algebra of discriminant N^-), we just refer to the correspondent p-adic modular forms as *quaternionic p-adic modular forms*.

Let $V_p(N^+, R)$ denote the space of quaternionic modular forms of tame level N^+. In terms of the layers of the Igusa tower $\widehat{\mathrm{Ig}}$,

$$V_p(N^+, R) = H^0(\widehat{\mathrm{Ig}}, \mathcal{O}_{\widehat{\mathrm{Ig}}}) \cong \varprojlim_n \varinjlim_m H^0(\mathrm{Ig}_{m,n}, \mathcal{O}_{\mathrm{Ig}_{m,n}}).$$

where, as usual, $\mathcal{O}_S$ denotes the structure sheaf of a scheme S. In view of Proposition 1.4.13, a quaternionic p-adic modular form as above is a rule $\mathcal{F}$ that, for each $n, m \geq 1$, takes a quadruple $(A, \iota, \alpha, \beta)$ consisting of a QM abelian surface (A, ι) over a R-algebra $\mathcal{R}$ with $U_0(N^+)$-level structure α and an arithmetic trivialization β_m of $A[p^m]$ over

$\mathcal{R}/p^n\mathcal{R}$, and assigns a value $\mathcal{F}(A,\iota,\alpha,\beta) \in \mathcal{R}/p^n\mathcal{R}$ which is compatible with respect to the canonical maps used to compute the direct and inverse limit, depends only on the isomorphism class of the quadruplet and is compatible under base change given by continuous morphisms between R-algebras.

An element $u \in \Gamma = 1 + p\mathbb{Z}_p$ acts on $\boldsymbol{\mu}_{p^\infty}$ by left multiplication. Precomposing an arithmetic trivialization $\beta \in \mathcal{P}_{m,n}$ of $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$ with that action u gives a new arithmetic trivialization $u \cdot \beta = (u \times u) \circ \beta \in \mathcal{P}_{m,n}$ which specializes to an arithmetic trivialization of $A[p^\infty]$ for any QM abelian surface A over R. This gives a $R[\![\Gamma]\!]$-module structure over $V_p(N^+, R)$ denoted by $\mathcal{F} \mapsto \mathcal{F}|\langle\lambda\rangle$ for $\mathcal{F} \in V_p(N^+, R)$ and $\lambda \in \Lambda_R$, defined for $u \in \Gamma$ by

$$(\mathcal{F}|\langle u\rangle)(A,\iota,\alpha,\beta) = \mathcal{F}(A,\iota,\alpha,u\cdot\beta)$$

and for any other $\lambda \in R[\![\Gamma]\!]$ by extending R-linearly.

Definition 3.1.2. Let $\psi\colon \Gamma \to \boldsymbol{\mu}_{p^\infty}(\overline{\mathbb{Q}}_p)$ be a finite order character and $k \in \mathbb{Z}_p$ a p-adic integer. We say that a p-adic modular form $\mathcal{F}$ is of *signature* (k,ψ) if for every $u \in \Gamma$, we have $\mathcal{F}|\langle u\rangle = u^k\psi(u)\mathcal{F}$.

3.1.1 Classic p-adic modular forms

As explained in [Kas99], the classic quaternionic p-adic modular forms are those coming from geometry. Let $\pi\colon A \to S$ be a QM abelian surface over a R and $\Omega_{A/R}$ denote the bundle of relative differentials. Then quaternionic multiplication by $\mathcal{O}_B$, after extension of scalars to $\mathbb{Z}_p$, gives an action on $\pi_*\Omega_{A/R}$. In particular, the idempotent e from §1.1.3 acts over $\pi_*\Omega_{A/R}$ allowing us to define the invertible sheaf $\underline{\omega}_{A/R} := e\pi_*\Omega_{A/R}$.

Remark 3.1.3. This solves the problem of the varieties being twice as big in the quaternionic case respect to the elliptic case: sections of $\pi_*\Omega_{A/R}$ are too big to be modular forms. Applying the projector halves their dimension, giving the one-dimensional sections we need to properly define one variable modular forms.

Definition 3.1.4. Let R be a $\mathbb{Z}_p$-algebra. A *test object* over a R-algebra $\mathcal{R}$ is a quintuplet $T = (A,\iota,\alpha,(H,P),\omega)$ consisting of a QM abelian surface (A,ι) over $\mathcal{R}$, a $U_0(N^+)$-level structure α on A, a $\Gamma_1(p^m)$-level structure (H,P) on A and a non-vanishing global section of the line bundle $\underline{\omega}_{A/\mathcal{R}}$ of relative differentials.

Two test objects are *isomorphic* if there is an isomorphism of QM abelian surfaces which induces isomorphisms of $V_1(N^+)$ and $\Gamma_1(p^m)$-level structure and pulls back the generator of the differentials of A' to that of A.

Definition 3.1.5. A *R-valued geometric modular form on* $\widetilde{\mathcal{X}}_m$ is a rule $\mathcal{F}$ that assigns to every isomorphism class of test objects $T = (A,\iota,\alpha,(H,P),\omega)$ over an R-algebra $\mathcal{R}$ a value $\mathcal{F}(T) \in \mathcal{R}$ which is

- compatible with base changes: if $\varphi\colon \mathcal{R} \to \mathcal{R}'$ is a morphism of R-algebras,

$$\mathcal{F}(A', \iota', \alpha', (H', P'), \omega') = \varphi\big(\mathcal{F}(A, \iota, \alpha, (H, P), \varphi^*(\omega')\big),$$

  where $A' = A \otimes_{R,\varphi} \mathcal{R}'$, and ι', α' and (H', P') are obtained by base change from ι, α and (H, P), respectively;

- satisfies a weight condition: for any $\lambda \in \mathcal{R}^\times$

$$\mathcal{F}(A, \iota, \alpha, (H, P), \lambda\omega) = \lambda^{-k}\mathcal{F}(A, \iota, \alpha, (H, P), \omega).$$

We denote $M_k(N^+, p^m, R)$ the R-module of R-valued weight k modular forms on $\widetilde{\mathcal{X}}_m$.

An element $u \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ acts on a $\Gamma_1(p^m)$-level structure (H, P) of a QM abelian surface A over $\mathcal{R}$ by left multiplication on the generator section slot, that is, $u \cdot (H, P) := (H, u \cdot P)$. This action extends to a test object $T = (A, \iota, \alpha, u \cdot (H, P))$ by acting on the last slot and leaving the others unchanged. This gives an action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ on $M_k(N^+, p^m, R)$ by setting $(\mathcal{F}|\langle u\rangle)(T) = \mathcal{F}(u \cdot T)$.

Definition 3.1.6. Let $\psi\colon (\mathbb{Z}/p^m\mathbb{Z})^\times \to R$ be a finite order character. We say that a modular form $\mathcal{F} \in M_k(N^+, p^m, R)$ has *character* ψ if $\mathcal{F}|\langle u\rangle = \psi(u)\mathcal{F}$ for all $u \in (\mathbb{Z}/p^m\mathbb{Z})^\times$.

If a modular form $\mathcal{F}$ has trivial character, then it is invariant respect to choice of generator section. Denote by $M_k(N^+p^m, R)$ the R-submodule of $M_k(N^+, p^m, R)$ consisting of those modular forms with trivial character, that is, the R-module of R-valued weight k modular forms on $\mathcal{X}_m$.

In order to define a p-adic modular form from a geometric modular form, one needs to establish a relation between the test objects above with the quadruplets from before, which boils down to defining a $\Gamma_1(p^m)$-level structure (H, P) after an arithmetic trivialization β for each (A, ι, α). In view of §1.2, there is a unique principal polarization $\lambda\colon A \xrightarrow{\sim} A^\vee$ whose correspondent Rosati involution of $\mathrm{End}(A)$ coincides with the restriction to $\mathcal{O}_B$ of the involution $\dagger$ of B defined in §1.2.

As explained in [Mag22, §3.1], an arithmetic trivialization β determines by Cartier duality a point x_β in $e^\dagger \mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$, where Ta_p denotes the p-adic Tate module. By [Kat81, p. 150], we have an isomorphism of formal groups over $\overline{\mathbb{F}}_p$

$$\begin{aligned} \widehat{A} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p), \widehat{\mathbb{G}}_m) &\implies e\widehat{A} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Ta}_p((eA)^\vee)(\overline{\mathbb{F}}_p), \widehat{\mathbb{G}}_m) \\ &\implies e^\dagger \mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{A}, \widehat{\mathbb{G}}_m) \end{aligned}$$

Under the above isomorphism x_β corresponds to some $\varphi_\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{A}, \widehat{\mathbb{G}}_m)$. Conversely, one such morphism corresponds to a point in $e^\dagger \mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$ and again by Cartier duality to an arithmetic trivialization on A. The pullback of the standard differential dT/T of $\widehat{\mathbb{G}}_m$ under φ_β, denoted $\omega_\beta = \varphi_\beta^*(dT/T)$, defines a differential in $\underline{\omega}_{A/R}$.

Fix a generator of $\boldsymbol{\mu}_{p^\infty}$, which induces a generator ζ_{p^m} of $\boldsymbol{\mu}_{p^m}$ for all m. Then $\beta(\zeta_{p^m})$ gives a point P in $A[p^m]$ of exact order p^m and, taking H to be the subgroup scheme generated by P, (H,P) gives a $\Gamma_1(p^m)$-level structure on (A,ι,α). Define

$$\widehat{\mathcal{F}}(A,\iota,\alpha,\beta) := \mathcal{F}(A,\iota,\alpha,(H,P),\omega_\beta).$$

The map $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ is compatible with base change, only depends on the isomorphism class of (A,ι,α,β) and it is compatible with the maps occurring in the direct and inverse limit in the definition of p-adic modular forms; thus $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ establishes a map

$$M_k(N^+,p^m,R) \longrightarrow V_p(N^+,R) \tag{3.1}$$

sending a geometric modular form $\mathcal{F}$ is of weight k and character ψ to a p-adic modular form $\widehat{\mathcal{F}}$ is of signature (k,ψ).

Definition 3.1.7. A p-adic modular form is said to be *classic* if it belongs to the image of the map (3.1).

3.1.2 Hecke operators

In this section, we follow [Gou88, §II.2, II.3] and [HB15, §3.6] in defining the relevant Hecke operators for this work.

If (A,ι,α,β) is a quadruplet like in the previous section, since A has ordinary reduction at p (see Remark 1.4.11), the p-torsion subgroup H defined after β in the previous subsection is the *canonical subgroup*, that is, the one (and only one) which reduces modulo p to the kernel of the Frobenius morphism (see also [Kas99, Theorem 1.11]). Denoting by $\phi_0 \colon A \twoheadrightarrow A_0 := A/H$ the canonical projection, we can define another quadruplet $(A_0,\iota_0,\alpha_0,\beta_0)$ as follows:

- ι_0 is the pullback of ι via ϕ_0;
- Since ϕ_0 has degree p which is prime to N^+, ϕ_0 reduces to an isomorphism in the N^+-torsion, so $\alpha_0 := \phi_0 \circ \alpha$ is a well defined $U_0(N^+)$-level structure on A_0;
- As before, β corresponds to a morphism $\varphi_\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{A},\widehat{\mathbb{G}}_m)$. Since ϕ_0 is étale, it induces an isomorphism on formal completions $\widehat{\phi}_0 \colon \widehat{A}_0 \to \widehat{A}$. The morphism $\varphi := \varphi_\beta \circ (e\widehat{\phi}_0) \in \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{A}_0,\widehat{\mathbb{G}}_m)$ induces an arithmetic trivialization β_0 on A_0.

Definition 3.1.8. The operator $V \colon V_p(N^+,R) \to V_p(N^+,R)$ is defined for any p-adic modular form $\mathcal{F}$ over R by the equation $V \cdot \mathcal{F}(A,\iota,\alpha,\beta) := \mathcal{F}(A_0,\iota_0,\alpha_0,\beta_0)$.

Let H_i, for $1 \le i \le p$, denote all the other p-torsion subgroups of A. An equivalent construction as the one performed for the canonical subgroup gives quadruplets $(A_i,\iota_i,\alpha_i,\beta_i)$ where $\phi \colon A \twoheadrightarrow A_i := A/H_i$ is the quotient of A by H_i.

Definition 3.1.9. The operator $U\colon V_p(N^+,R)\to V_p(N^+,R)$ is defined for any p-adic modular form $\mathcal{F}$ over R by the equation $U\cdot\mathcal{F} := \dfrac{1}{p}\displaystyle\sum_{i=1}^{p}\mathcal{F}(A_i,\iota_i,\alpha_i,\beta_i)$.

It follows $V\circ U$ is just the identity map Id, which is not the case for $U\circ V$. In the elliptic modular forms case, the second composition kills all terms of index prime to p in the Fourier expansion. Motivated by this classical case, we define:

Definition 3.1.10. The *p-depletion* of a p-adic modular form $\mathcal{F}$ in $V_p(N^+,R)$ to be $\mathcal{F}^{[p]} := (\mathrm{Id}-UV)\cdot\mathcal{F}$.

If ℓ is any prime, the same construction as above gives for each of the ℓ-torsion groups $H_0,\dots,H_\ell$ and their quotients $\phi_i\colon A\twoheadrightarrow A_i := A/H_i$ a quadruplet $(A_i,\iota_i,\alpha_i,\beta_i)$.

Definition 3.1.11. The operator $T_\ell\colon V_p(N^+,R)\to V_p(N^+,R)$ is defined for any p-adic modular form $\mathcal{F}$ over R by the equation $T_\ell\cdot\mathcal{F} := \dfrac{1}{\ell}\displaystyle\sum_{i=1}^{\ell}\mathcal{F}(A_i,\iota_i,\alpha_i,\beta_i)$.

3.1.3 Jacquet–Langlands correspondence

Fix an isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$, the p-adic completion of $\overline{\mathbb{Q}}_p$, compatible with the embedding $\iota_p\colon\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. The reason why classical quaternionic modular forms are called so is that they come from actual classical modular forms via the Jacquet–Langlands correspondence (see [HB15, §2.7]):

Proposition 3.1.12 (Jacquet–Langlands)**.** There is an isomorphism

$$\mathrm{JL}\colon S_k(\Gamma_1(Np^m),\mathbb{C})^{N^-\text{-new}}\xrightarrow{\sim} M_k(N^+,p^m,\mathbb{C}_p),$$

where $S_k(\Gamma_1(Np^m),\mathbb{C})^{N^-\text{-new}}$ denotes the $\mathbb{C}$-vector space of classical cuspidal modular forms of level $\Gamma_1(Np^m)$ and that are new at all primes dividing N^-. Furthermore, if $f\in S_k(\Gamma_1(Np^m),\mathbb{C})^{N^-\text{-new}}$ has character ψ then so does $\mathcal{F}=\mathrm{JL}(f)$, and the eigenvalues of f and $\mathcal{F}$ respect to U and all Hecke operators T_ℓ away from $N=N^+N^-$ coincide.

3.1.4 Galois representations

Let L be a finite extension of $\mathbb{Q}_p$. Let $m\geq 0$, write $\mathcal{C}_m$ for X_m or $\widetilde{X}_m$ and, in both cases, $\pi_{\mathcal{A}}\colon\mathcal{A}_m\to\mathcal{C}_m$ for the universal abelian variety. For this subsection only, let

$$M_k := \begin{cases} M_k(N^+,p^m,L), & \text{if } \mathcal{C}_m=\widetilde{X}_m \\ M_k(N^+p^m,L), & \text{if } \mathcal{C}_m=X_m \end{cases}.$$

Consider $\mathcal{F} \in M_k$ an eigenform for all Hecke operators and let $F_{\mathcal{F}} \subseteq \overline{\mathbb{Q}}_p$ be its Hecke field. Here we take the *p-adic Galois representation* of $\mathcal{F}$ to be the Deligne p-adic representation

$$V_{\mathcal{F}} \subseteq H^1_{\text{ét}}\left(\mathcal{C} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, (\mathscr{V}^k_{\text{ét}})^\vee\right) \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$$

characterized by requiring that the trace of the geometric Frobenius element at a prime $\ell \nmid Np$ is equal to $a_\ell(\mathcal{F})$, the Hecke eigenvalue of T_ℓ. The dual, or contragredient, Galois representation $V^*_{\mathcal{F}}$ of $V_{\mathcal{F}}$ is then the maximal quotient

$$H^1_{\text{ét}}\left(\mathcal{C} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^k_{\text{ét}}(1)\right) \otimes_{\mathbb{Q}_p} F_{\mathcal{F}} \longrightarrow\!\!\!\!\rightarrow V^*_{\mathcal{F}},$$

where the dual Hecke operators T'_ℓ act by $a_\ell(\mathcal{F})$, for all $\ell \nmid Np$, and U acts by $a_p(\mathcal{F})$; see [KLZ17, §2.8].

Let f be a classical eigenform such that $\mathcal{F} = \mathrm{JL}(f)$ and denote by V_f its classical p-adic Galois representation with coefficients in $F_{\mathcal{F}} = F_f$. Since the eigenvalues respect to T_ℓ of f and $\mathcal{F}$ coincide in all but finitely many primes ℓ, their Hecke polynomial (that is, the characteristic polynomial of the image of the Frobenius element) at all but finitely many primes ℓ coincide. Since V_f and $V_{\mathcal{F}}$ are irreducible, they are then isomorphic.

3.1.5 p-stabilization of modular forms

Let $\mathcal{F} = \mathrm{JL}(f) \in M_k(N^+, 1, L)$ be a quaternionic modular form ($m = 0$). Let α and β be the roots of the Hecke polynomial of f (and of $\mathcal{F}$) at p. The p-stabilization of f respect to the root α (resp. β) is the modular form $f_\alpha(z) := f(z) - \beta f(pz)$ (resp. $f_\beta(z) := f(z) - \alpha f(pz)$).

Definition 3.1.13. The *p-stabilization* of $\mathcal{F} \in M_k(N^+, 1, L)$ respect to α (resp. β) is $\mathcal{F}_\alpha := \mathrm{JL}(f_\alpha)$ (resp. $\mathcal{F}_\beta := \mathrm{JL}(f_\beta)$), an element of $M_k(N^+, p, L)$.

Consider the canonical degeneration maps

- $\mathrm{pr}_1 \colon X_1 \to X_0$, which takes a quadruple $(A, \iota, \alpha, \beta)$ and maps into $(A, \iota, \alpha_p, \beta_p)$, where α_p is (the equivalence class of) $\psi_p \circ \alpha$ with ψ_p the map $(x, y) \mapsto (px, py)$; and $\beta_p = \beta \circ \varphi_p$ with $\varphi_p \colon \boldsymbol{\mu}_p \to \boldsymbol{\mu}_1$ being the map $x \mapsto x^p$, thus trivial;
- $\mathrm{pr}_2 \colon X_1 \to X_0$, which takes a quadruple $(A, \iota, \alpha, \beta)$ and outputs the quadruplet $(A_0, \iota_0, \alpha_0, \beta_0)$ associated to the canonical subgroup of A as explained in §3.1.2.

Proposition 3.1.14. The map

$$(\mathrm{pr}_\alpha)_* := (\mathrm{pr}_1)_* - \frac{(\mathrm{pr}_2)_*}{\alpha} \colon V_{\mathcal{F}_\alpha} \longrightarrow V_{\mathcal{F}} \tag{3.2}$$

is an isomorphism, where $(\mathrm{pr}_1)_*, (\mathrm{pr}_2)_* \colon V_{\mathcal{F}_\alpha} \to V_{\mathcal{F}}$ are induced by the canonical degeneration maps $\mathrm{pr}_1, \mathrm{pr}_2 \colon X_1 \to X_0$.

Proof. One can use the Jacquet–Langlands correspondence to reduce it to the elliptic version of this statement, which is [KLZ17, Proposition 7.3.1]. □

3.2 Coleman families

In this section we recall the concept of finite slope p-adic families of modular forms, first in the classical GL_2 case and then in the quaternionic case. For this section, let L be a finite extension of $\mathbb{Q}_p$.

3.2.1 Slope of modular forms

Definition 3.2.1. Let $f \in S_k(\Gamma_1(Np))$ be an eigenvector of U, with eigenvalue α_f. The *slope* of f is $v_p(\alpha_f)$.

If f is a normalized cuspidal eigenform, then $T_\ell \cdot f = a_\ell(f)f$, where $a_\ell(f)$ is the ℓ-th coefficient of the Fourier expansion of f. In particular, the slope of f is $v_p(a_p(f))$.

Let $\mathcal{F} = \mathrm{JL}(f)$ be a quaternionic modular form. We abuse the notation and write $a_\ell(\mathcal{F})$ for the eigenvalue of $\mathcal{F}$ respect to T_ℓ; the Jacquet–Langlands lift (Proposition 3.1.12) then gives $a_\ell(\mathcal{F}) = a_\ell(f)$.

Definition 3.2.2. The *slope* of a modular form $\mathcal{F} \in M_k(N^+, p, L)$ is $v_p(a_p(\mathcal{F}))$.

3.2.2 Eigencurves

Recall that the weight space $\mathscr{W} := \mathrm{Hom}_{\mathrm{cont},\mathbb{Z}_p}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, the rigid analytic variety associated to the Iwasawa algebra $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$. For $N \in \mathbb{Z}_{\geq 2}$, we also define the level N Iwasawa algebra to be $\Lambda_N := \mathbb{Z}_p[\![(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times]\!]$, whose associated level N weight space is $\mathscr{W}_N := \mathrm{Hom}_{\mathrm{cont},\mathbb{Z}_p}((\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, and there is a natural projection $\mathscr{W}_N \to \mathscr{W}$. We embed $\mathbb{Z}$ into those weight spaces by sending k into the character $x \mapsto x^{k-2}$. This normalization is so $k = 2r+2$ "maps into $2r$".

Let $\mathcal{E}(N)$ denote the Coleman–Mazur eigencurve of tame level N defined in [CM98], a rigid analytic variety parametrizing finite slope overconvergent p-adic eigenforms of tame level N, and in particular, the p-stabilized eigenforms $f \in S_k(\Gamma_1(Np), \mathbb{C})^{N^-\text{-new}}$ from Proposition 3.1.12. There exists a projection of the eigencurve $\mathcal{E}(N)$ into the level N weight space $\mathscr{W}_N$.

Using Buzzard's eigenvariety machine from [Buz07], Brasca (in [Bra16]) has constructed the quaternionic analogue of the eigencurve, parametrizing cuspidal quaternionic modular forms over certain PEL Shimura varieties. Since the Shimura curves we consider here are PEL and compact, no cusps exist and therefore any modular form defined over them is cuspidal, so Brasca's construction apply. We denote by $\mathscr{E}(N^+)$ the Brasca–Buzzard eigencurve, whose L-rational points parametrize modular forms of tame level N^+.

A general result by Chenevier, when adapted to our context, implies a p-adic Jacquet–Langlands correspondence between the elliptic and quaternionic eigencurves:

Proposition 3.2.3. The Coleman–Mazur eigencurve $\mathcal{E}(N)$ and the Brasca–Buzzard eigencurve $\mathscr{E}(N^+)$ are isomorphic as rigid analytic varieties; see [Che05, Théorème 3].

3.2.3 Elliptic and quaternionic Coleman families

Coleman families of any positive slope, generalizing the ordinary (zero slope) families defined by Hida, were first defined in [Col97] for elliptic modular forms. We follow the exposition in [NO19, §2.2] for the elliptic case.

Let f be a classical level N new normalized eigenform of even weight k_0, such that the roots of its Hecke polynomial, α and β are distinct and $v_p(\alpha) < k_0 - 1$. These conditions are enough for us to define after f_α a point in $\mathcal{E}(N)$ in which the projection to the weight space is smooth and étale (see [Han15, §1.1] and [Bel12, §2.3]).

For an affinoid $\mathscr{U} \subseteq \mathscr{W}_N$, we write $\mathscr{A}^\circ_{\mathscr{U}}$ for the ring of power bounded rigid analytic functions defined over $\mathbb{Q}_p$ (see [NO19, §2.1]).

Proposition 3.2.4. Let f be a p-stabilized modular form of slope λ in the conditions above. Then there exists an affinoid $\mathscr{U} \subseteq \mathscr{W}_N$ and, for each $n \in \mathbb{Z}_{>0}$, $\mathbf{a}_n \in \mathscr{A}^\circ_{\mathscr{U}} \otimes_{\mathbb{Q}_p} L$ rigid analytic functions with coefficients in a finite extension L of $\mathbb{Q}_p$ such that, for each $k \in \mathscr{U} \cap \mathbb{Z}$ with $\lambda < k - 1$, the specialization at weight k

$$f_k := \sum_{n=1}^{\infty} \mathbf{a}_n(k) q^n \in L[\![q]\!]$$

is a classical normalized cuspidal eigenform of level Np, weight k and slope λ, and for $k = k_0$, $f_{k_0} = f$; see also [LZ16, Theorem 4.6.4].

The family of functions $\{\mathbf{a}_n\}_n$ is the *(elliptic) Coleman family* defined over $\mathscr{U}$ passing by f. In other words, a Coleman family is an affinoid $\{f_k\}_{k\in\mathscr{U}}$ in the Coleman–Mazur eigencurve $\mathcal{E}(N)$ defined over $\mathscr{U}$. Let $\Lambda_{\mathscr{U}}$ be the Iwasawa algebra associated to $\mathscr{U}$, over which $\mathscr{A}^\circ_{\mathscr{U}}$ is finite and flat module. The (at first) formal series

$$\mathscr{f} := \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathscr{A}^\circ_{\mathscr{U}}[\![q]\!]$$

is a well defined element in $V_p(N^+, \mathscr{A}^\circ_{\mathscr{U}}) := V_p(N^+, L)\widehat{\otimes}_{\Lambda_{\mathscr{U}}} \mathscr{A}^\circ_{\mathscr{U}}$.

Let f be a classic p-stabilized eigenform as above and let $\mathcal{F} := \mathrm{JL}(f)$ be the correspondent p-stabilized quaternionic modular form, and let $\mathscr{f}$ be the Coleman family passing by f defined over some affinoid $\mathscr{U} \subseteq \mathscr{W}$ (after projection from $\mathscr{W}_N$). Via the p-adic Jacquet–Langlands correspondence (Proposition 3.2.3), $\mathscr{f}$, as an open neighborhood of f in $\mathcal{E}(N)$ maps into an open neighborhood $\mathscr{F}$ of $\mathcal{F}$ in $\mathscr{E}(N^+)$, which we call the *(quaternionic) Coleman family* passing by $\mathcal{F}$ defined over $\mathscr{U}$. As an abuse of notation, we write $\mathscr{F} = \mathrm{JL}(\mathscr{f})$ to emphasize the relation between the elliptic and quaternionic Coleman families. As a consequence of Propositions 3.2.3 and 3.2.4:

Corollary 3.2.5. Let $\mathscr{F} = \mathrm{JL}(\mathscr{f})$ be the Coleman family passing by $\mathcal{F}$ over $\mathscr{U} \subseteq \mathscr{W}$. For each $k \in \mathscr{U} \cap \mathbb{Z}$ with $\lambda < k - 1$, the specialization at weight k denoted $\mathcal{F}_k$ is the quaternionic modular form $\mathrm{JL}(f_k)$, where f_k is the specialization at weight k of $\mathscr{f}$. At weight $k = k_0$, the specialization is $\mathcal{F}$ itself.

Chapter 4

Big Heegner classes

In this chapter we define big Heegner classes associated to Coleman families of quaternionic modular forms through the p-adic interpolation of generalized Heegner classes associated to quaternionic modular forms, following the approach from [JLZ21, §4]. As seen in §2.7, the basis vectors we constructed after CM points correspond under the Gysin map to the generalized Heegner classes (Proposition 2.7.1), so the strategy is to interpolate p-adically the basis vectors and the Gysin map; the p-adic interpolation of generalized Heegner classes follows automatically.

Fix a p-adic ring R, let L be its fraction field and fix an embedding $\sigma_L\colon K \hookrightarrow L$. Let $k_0 = 2r_0 + 2 \in \mathbb{Z}$ and as before, denote by $\mathscr{W}$ the weight space, inside of which we find k_0 as the character $z \mapsto z^{2r_0}$. For an integer $s \geq 0$, denote by $\mathscr{W}^{(s)}$ the locus of the s-analytic characters, that is, the $\kappa \in \mathscr{W}$ such that

$$v_p(\kappa(1+p) - 1) > (p^{s-1}(p-1))^{-1}.$$

Let $\mathscr{U}$ be an open neighborhood of z_0 in $\mathscr{W}^{(s)}$ defined over L, $\Lambda_{\mathscr{U}} := R[\![u]\!]$ be its Iwasawa algebra and let $\kappa_{\mathscr{U}}\colon \Gamma \hookrightarrow \Lambda_{\mathscr{U}}^{\times}$ be its universal character. For any character $\sigma\colon R^{\times} \to \Gamma$ we write $\sigma^{\kappa_{\mathscr{U}}} := \kappa_{\mathscr{U}} \circ \sigma$, $\sigma^k := k \circ \sigma$ for any $k \in \mathbb{Z}$ and $\sigma^{\pm\kappa_{\mathscr{U}}+\ell_1}\bar{\sigma}^{\ell_2} := (\sigma^{\kappa_{\mathscr{U}}})^{\pm 1}\sigma^{\ell_1}\bar{\sigma}^{\ell_2}$ for any two integers ℓ_1 and ℓ_2.

Remark 4.0.1. For an extension F of K, let Σ_F denote the set of primes of F dividing pN. Since all Kuga–Sato and Shimura varieties we are considering have smooth models over $\mathcal{O}_F[1/\Sigma_F]$, all classes we defined in chapter 2 and all classes to be defined in the current chapter whose field of definition is F are actually defined over $\mathcal{O}_F[1/\Sigma_F]$, that is, they are elements of the finite dimensional subspace corresponding to the unramified cohomology outside of Σ_F. In particular, if the class is invariant respect to the action of $\mathrm{Gal}(F/F')$, then it is also defined over F' (and more precisely, in the unramified cohomology outside of $\Sigma_{F'}$). We follow the abuse of notation in [JLZ21] to write $H^1(F,-)$ when we actually mean $H^1(\mathcal{O}_F[1/\Sigma_F],-)$.

4.1 Classes associated to quaternionic modular forms

Let $\mathcal{F} \in M_k(N^+p^m, L)$ be a quaternionic newform of weight k and trivial character over X_m. From §3.1.4, the dual Galois representation $V_{\mathcal{F}}^*$ of $\mathcal{F}$ is a quotient of $H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}_{\text{ét}}^{2r}(1))$, so we have a projection

$$\mathrm{pr}_{\mathcal{F}}^{[j]}\colon H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}_{\text{ét}}^{2r}(1)) \otimes \sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}\right) \longrightarrow H^1\left(F_{cp^n}, V_{\mathcal{F}}^* \otimes \sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}\right). \tag{4.1}$$

Definition 4.1.1. The class $z_{cp^n,m}^{[\mathcal{F},j]} := \mathrm{pr}_{\mathcal{F}}^{[j]}\left(z_{cp^n,m}^{[k,j]}\right)$ is the j-component of the *generalized Heegner class* associated to $\mathcal{F}$ and with $0 \le j \le 2r$.

Recall that an *(algebraic) Hecke character of infinity type* (ℓ_1, ℓ_2) is a character $\xi\colon K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that $\xi|_{K_\infty^\times}(x) = x^{\ell_1}\bar{x}^{\ell_2}$.

Lemma 4.1.2. Let ξ be a Hecke character of infinity type $(2r-j, j)$ of conductor c prime to Np. Then the generalized Heegner class $z_{cp^n,m}^{[\mathcal{F},j]}$ belongs to the $\mathrm{Gal}(F_{cp^n}/H_{cp^n})$-invariant subspace of $H^1(F_{cp^n}, V_{\mathcal{F}}^* \otimes \sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}) \cong H^1(F_{cp^n}, V_{\mathcal{F}}^* \otimes \xi)$.

Proof. The proof is similar to [JLZ21, Proposition 3.5.2]. The group $\widehat{\mathcal{O}}_{cp^n}^\times / U_{\mathfrak{N}^+,cp^n} \cong (\mathbb{Z}/N^+\mathbb{Z})^\times$ acts naturally on S_{cp^n} and by diamond operators on X_m, and both actions are compatible with the embedding $\delta_{cp^n}^*\colon S_{cp^n} \hookrightarrow X_m \times S_{cp^n}$ induced by the embedding from §2.7. Since ξ has conductor prime to N^+, ξ restricts to the character $\sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}$ on $\mathrm{Gal}(F_{cp^n}/H_{cp^n})$, thus extending $\sigma^{2r-j}\bar{\sigma}^j$ to $\mathrm{Gal}(K^{\text{ab}}/H_{cp^n})$. Since $z_{cp^n,m}^{[\mathcal{F},j]}$ lies in the finite dimensional subspace of classes which are unramified outside Np, the result then follows from Shimura reciprocity law ([Shi71, Theorem 9.6]). □

The inflation-restriction exact sequence and the irreducibility of $V_{\mathcal{F}}$ induce for all such ξ an isomorphism

$$\left(H^1(F_{cp^n}, V_{\mathcal{F}}^* \otimes \xi)\right)^{\mathrm{Gal}(F_{cp^n}/H_{cp^n})} \cong H^1(H_{cp^n}, V_{\mathcal{F}}^* \otimes \xi). \tag{4.2}$$

Definition 4.1.3. Let ξ be an algebraic Hecke character of infinity type $(2r-j, j)$ with $0 \le j \le 2r$. The *ξ-component* $z_{cp^n,m}^{[\mathcal{F},j,\xi]}$ of the generalized Heegner class is the image of $z_{cp^n,m}^{[\mathcal{F},j]}$ via the isomorphism (4.2).

Remark 4.1.4. The above construction can be adapted for the selfdual representation of $\mathcal{F}$ and central characters (*i.e.* Hecke characters of the form $(s, -s)$ with $s \in \mathbb{Z}$) as follows: since, as representations over F_{cp^n}, we have that the self-dual representation is given by

$$V_{\mathcal{F}}^\dagger \cong V_{\mathcal{F}}^*(-r) \cong V_{\mathcal{F}}(r+1),$$

and recalling that the twist of a representation is the power of the cyclotomic character χ_{cyc} with which said representation is tensored, we conclude that

$$V_{\mathcal{F}}^* \otimes \sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j} \cong V_{\mathcal{F}}^\dagger \otimes (\sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}\chi_{\text{cyc}}^{r}).$$

Thus $\mathrm{pr}_{\mathcal{F}}^{[j]}$ lands in $H^1\left(F_{cp^n}, V_{\mathcal{F}}^{\dagger} \otimes (\sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}\chi_{\text{cyc}}^{r})\right)$. Since χ_{cyc} has infinity type $(-1,-1)$, the character $\sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}\chi_{\text{cyc}}^{r}$ extends to a Hecke character of infinity type $(r-j, j-r)$ or, more precisely, to $\xi\chi_{\text{cyc}}^{r}$, where ξ is a Hecke character of infinity type $(2r-j, j)$. The two constructions are related by the bijection $\xi \mapsto \xi\chi_{\text{cyc}}^{s}$ between Hecke characters of infinity type (ℓ_1, ℓ_2) and those of infinity type $(\ell_1 - s, \ell_2 - s)$.

Let α and β be the roots of the Hecke polynomial of $\mathcal{F}$ at p and $\mathcal{F}_{\alpha}$ the p-stabilization of $\mathcal{F}$ respect to α. Proposition 3.1.14 gives an isomorphism $(\mathrm{pr}_{\alpha})_*\colon V_{\mathcal{F}_{\alpha}} \xrightarrow{\sim} V_{\mathcal{F}}$.

Proposition 4.1.5. The map $(\mathrm{pr}_{\alpha})_*$ induces an isomorphism

$$H^1\left(H_{cp^n}, V_{\mathcal{F}_{\alpha}}^{*} \otimes \xi\right) \xrightarrow{\sim} H^1\left(H_{cp^n}, V_{\mathcal{F}}^{*} \otimes \xi\right)$$

under which $z_{cp^n,m}^{[\mathcal{F}_{\alpha},j,\xi]}$ maps into

$$\left(1 - \frac{\xi(\mathfrak{p})}{\alpha}\right)\left(1 - \frac{\xi(\bar{\mathfrak{p}})}{\alpha}\right) z_{cp^n,m}^{[\mathcal{F},j,\xi]},$$

the coefficient being known as an *Euler factor*.

Proof. This is an explicit calculation as in [KLZ17, Theorem 5.7.6], see also [JLZ21, Proposition 3.5.5]. □

4.2 p-adic interpolation of basis vectors

Let $A_{\mathscr{U},s}$ be the $\Lambda_{\mathscr{U}}[1/p]$-module of continuous $\Lambda_{\mathscr{U}}[1/p]$-valued functions on $\mathbb{Z}_p$ of slope s, that is, of the form $\displaystyle\sum_{t\geq 0} a_t \left(\frac{Z}{p^s}\right)^t$ with $a_t \in \Lambda_{\mathscr{U}}[1/p]$.

Remark 4.2.1. For $s = 0$, $A_{\mathscr{U},0}$ is just the $\Lambda_{\mathscr{U}}[1/p]$-module of continuous $\Lambda_{\mathscr{U}}[1/p]$-valued functions on $\mathbb{Z}_p$, $\mathscr{C}(\mathbb{Z}_p, \Lambda[1/p])$.

The monoid $\Sigma = \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ acts on $A_{\mathscr{U},s}$ from the left by

$$\gamma \cdot \varphi(Z) = \kappa_{\mathscr{U}}(bZ+d)\varphi(Z \cdot \gamma),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $Z \cdot \gamma = \frac{aZ+c}{bZ+d}$. The space $\mathrm{Sym}^{2r}(L^2)$ of weight $2r$ homogeneous polynomials in two variables X and Y embeds into $A_{\mathscr{U},s}$ by $P(X,Y) \mapsto P(Z,1)$ with $Z = X/Y$. Such embedding is equivariant with respect to the action above and its restriction to $\mathrm{Sym}^{2r}(L^2)$ given by

$$\gamma \cdot \varphi(Z) = (bZ+d)^{2r}\varphi(Z \cdot \gamma).$$

The dual of this embedding gives the *moment map*

$$\mathrm{mom}^{2r}\colon D_{\mathscr{U},s} := \mathrm{Hom}_{\Lambda_{\mathscr{U}}[1/p]}(A_{\mathscr{U},s}, \Lambda_{\mathscr{U}}[1/p]) \longrightarrow \mathrm{TSym}^{2r}((L^2)^\vee)$$

defined by the integration formula

$$\left(\mathrm{mom}^{2r}(\mu)\right)(\varphi) = \int_{\mathbb{Z}_p} \varphi(x,1)d\mu(x), \tag{4.3}$$

for each $\varphi \in \mathrm{Sym}^{2r}(L^2)$.

Lemma 4.2.2. $\sigma_L(\vartheta^*_{cp^n}) = \alpha_{cp^n} p^n$ for some $\alpha_{cp^n} \in \mathbb{Z}_p^\times$.

Proof. The question is local. Write $b^{(n)} = i_p(b^{(n)})$ to simplify the notation. Recall that $\xi_p^{(n)} = b^{(n)}u^{(n)}$ and $\xi_p^{(n)} = \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar\vartheta \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right)$; since $\vartheta \in \mathbb{Z}_p$ and $\delta \in \mathbb{Z}_p^\times$, then, locally at p, we have $b^{(n)} = \left(\begin{smallmatrix} \vartheta & \bar\vartheta \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right)u_0$ for some element u_0 in $\mathrm{GL}_2(\mathbb{Z}_p)$. Therefore, using again that $\delta \in \mathbb{Z}_p^\times$, we see that $(b^{(n)})^{-1} = u_1\left(\begin{smallmatrix} p^{-n} & -p^{-n} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\bar\vartheta \\ -1 & \vartheta \end{smallmatrix}\right)$ for some $u_1 = u_0^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)$. It follows that $\vartheta_{cp^n} = p^{-n}u_1\left(\begin{smallmatrix} \vartheta-\bar\vartheta \\ 0 \end{smallmatrix}\right)$. Now recall that the pair $[(\iota_K, \xi^{(n)})]$ is an Heegner point on X_m for all $m \geq 0$, thus $u^{(n)}$ satisfies the congruence $u^{(n)} \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p^m$ for all $m \geq 0$, from which we conclude that $u_1 = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a \in \mathbb{Z}_p^\times$, $d \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$. Therefore we have $\vartheta_{cp^n} = p^{-n}a(\vartheta - \bar\vartheta)$ for a suitable $a \in \mathbb{Z}_p^\times$, and the result follows. □

In light of the above lemma, the following definition makes sense:

Definition 4.2.3. Define the distribution $\mathbf{e}_{\mathscr{U},cp^n} \in D_{\mathscr{U}}$ by the integration formula

$$\mathbf{e}_{\mathscr{U},cp^n}(\varphi) = \int_{\mathbb{Z}_p} \varphi(x)\mathbf{e}_{\mathscr{U},cp^n}(x) := \varphi(\sigma_L(\vartheta^*_{cp^n}))$$

for any $\varphi \in A_{\mathscr{U},s}$.

The distribution $\mathbf{e}_{\mathscr{U},cp^n}$ plays the role of a p-adic family over $\mathscr{U}$ of base vectors:

Proposition 4.2.4. The distribution $\mathbf{e}_{\mathscr{U},cp^n}$ has the following properties:

- The action of $i_{cp^n}\left((\mathcal{O}_{cp^n} \otimes \mathbb{Z}_p)^\times\right)$ on $\mathbf{e}_{\mathscr{U},cp^n}$ is via $\sigma_L^{-\kappa_{\mathscr{U}}}$.
- For all integers $k \geq 0$ we have $\mathrm{mom}^{2r}(\mathbf{e}_{\mathscr{U},cp^n}) = e^{[k,0]}_{cp^n}$.

Proof. For the first statement, recall that i_{cp^n} is an optimal embedding of $\mathcal{O}_{cp^n}$ into the Eichler order $R_m = B^\times \cap U_m$. For u in $(\mathcal{O}_{cp^n} \otimes \mathbb{Z}_p)^\times$, we have $(i_{cp^n}(u)^{-1})^{\mathrm{T}}\binom{\vartheta^*_{cp^n}}{1} = \sigma_L^{-1}(u)\binom{\vartheta^*_{cp^n}}{1}$ (see §2.7). Let u act by $\left(\begin{smallmatrix} a & b \\ cp^n & d \end{smallmatrix}\right)$. For each $\varphi \in A_{\mathscr{U},s}$ we then have

$$\begin{aligned} i_{cp^n}(u)^{-1}\mathbf{e}_{\mathscr{U},cp^n}(\varphi) &= \int_{\mathbb{Z}_p} \varphi(x) d(i_{cp^n}(u)^{-1}\mathbf{e}_{\mathscr{U},cp^n})(x) \\ &= \int_{\mathbb{Z}_p} \kappa_{\mathscr{U}}(bx+d)\varphi\left(\frac{ax+cp^n}{bx+d}\right)\mathbf{e}_{\mathscr{U},cp^n}(x) \\ &= \sigma_L^{\kappa_{\mathscr{U}}}(u)\varphi\left(\frac{a\sigma_L(\vartheta^*_{cp^n})+cp^n}{b\sigma_L(\vartheta^*_{cp^n})+d}\right) \\ &= \sigma_L^{\kappa_{\mathscr{U}}}(u)\varphi(\sigma_L(\vartheta^*_{cp^n})) \\ &= \sigma_L^{\kappa_{\mathscr{U}}}(u)\mathbf{e}_{\mathscr{U},cp^n}(\varphi). \end{aligned}$$

Therefore, we conclude that the action of $i_{cp^n}(u)$ on the measure $\mathbf{e}_{\mathscr{U},cp^n}$ is just the product $\sigma_L^{-\kappa_{\mathscr{U}}(u)}\mathbf{e}_{\mathscr{U},cp^n}$, which is the first statement. For the second, take $P \in \mathrm{Sym}^{2r}(L^2)$; then we have

$$\left(\mathrm{mom}^{2r}(\mathbf{e}_{\mathscr{U},cp^n})\right)(P) = \int_{\mathbb{Z}_p} P(x,1)d\mathbf{e}_{\mathscr{U},cp^n}(x) = P\left(\sigma_L(\vartheta^*_{cp^n}),1\right) = e^{[k,0]}_{cp^n}(P),$$

concluding the proof of the second equality. □

To interpolate the remaining vectors, we introduce some twist. Let

$$\Pi_j\colon D_{\mathscr{U}-j,m} \otimes \mathrm{TSym}^j\left((L^2)^\vee\right) \longrightarrow D_{\mathscr{U},m}$$

denote the *overconvergent projector* from [LZ16, Corollary 5.2.1], which basically acts as the usual symmetrized tensor product up to denominators: there is a map acting on evaluations at $v \in (L^2)^\vee$ by

$$\Pi_j^*\colon D_{\mathscr{U},m} \longrightarrow D_{\mathscr{U}-j,m} \otimes \mathrm{TSym}^j\left((L^2)^\vee\right) : \mathrm{ev}_v \mapsto \mathrm{ev}_v \circ v^{\odot j}$$

such that $\Pi_j \circ \Pi_j^*$ acts as multiplication by $\binom{\nabla}{j}$, where ∇ is an operator that interpolates multiplication by integers (see more details in §4.4.1). The denominators of Π_j are explicitly bounded in terms of j and m, see *ibid.* Remark 5.2.2.

Definition 4.2.5. Define the distribution

$$\mathbf{e}^{[j]}_{\mathscr{U},cp^n} := \Pi_j(\mathbf{e}_{\mathscr{U},cp^n} \odot (e^{-j}_{cp^n} \otimes \bar{e}^j_{cp^n})),$$

where $\odot$ is the symmetrized tensor product.

Proposition 4.2.6. The distribution $\mathbf{e}^{[j]}_{\mathscr{U},cp^n}$ has the following properties:

- The group $i_{cp^n}((\mathcal{O}_{cp^n}\otimes\mathbb{Z}_p)^\times)$ acts on $\mathbf{e}^{[j]}_{\mathscr{U},cp^n}$ via the representation $\sigma^{-(\kappa_{\mathscr{U}}-j)}\bar{\sigma}^j$
- For all integers $k\geq 0$ we have $\mathrm{mom}^{2r}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n})=e^{[k,j]}_{cp^n}$.

Proof. The first item follows immediately from Proposition 4.2.4. As for the second item, it suffices to notice that

$$\begin{aligned}\mathrm{mom}^{2r}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n}) &= \mathrm{mom}^{2r}(\mathbf{e}_{\mathscr{U},cp^n})\odot(e^{j}_{cp^n}\otimes\bar{e}^{j}_{cp^n})\\ &= e^{[k,0]}_{cp^n}\odot(e^{j}_{cp^n}\otimes\bar{e}^{j}_{cp^n})\\ &= e^{2r-j}_{cp^n}\otimes\bar{e}^{j}_{cp^n}=e^{[k,j]}_{cp^n},\end{aligned}$$

concluding the proof. □

4.3 p-adic interpolation of generalized Heegner classes

As in §2.6, let $\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\sigma_{\text{ét}}^{j}$ be the étale realization of $\sigma_L^{\kappa_{\mathscr{U}}-j}\sigma_L^{j}$. We then have a map

$$H^0_{\text{ét}}\left(S_{cp^n},\delta^*_{cp^n}(D_{\mathscr{U}}\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j})\right)\xrightarrow{\delta_{cp^n,*}}H^2_{\text{ét}}\left(X_m\otimes_K F_{cp^n},D_{\mathscr{U}}(1)\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}\right)\tag{4.4}$$

that interpolates the Gysin maps from §2.7, in the sense of being compatible with the one in (2.11) via the moment maps: for each $k\in\mathbb{Z}\cap\mathscr{U}$ with $k\geq j$ we have, writing $\overline{X}_m:=X_m\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ and $\mathscr{D}_{\text{ét}}^{(2r-j,j)}:=D_{\mathscr{U}}\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}$ to simplify the notation in the diagram,

$$\begin{tikzcd} H^0_{\text{ét}}\left(S_{cp^n},\delta^*_{cp^n}(\mathscr{D}_{\text{ét}}^{(2r-j,j)})\right) \arrow[r, "\mathrm{mom}^{2r}"] \arrow[d, "\delta_{cp^n,*}"] & H^0_{\text{ét}}\left(S_{cp^n},\delta^*_{cp^n}(\mathscr{M}_{\text{ét}}^{(2r-j,j)})\right) \arrow[d, "\delta_{cp^n,*}"] \\ H^2_{\text{ét}}\left(X_m\otimes F_{cp^n},\mathscr{D}_{\text{ét}}^{(2r-j,j)}\right) \arrow[r, "\mathrm{mom}^{2r}"] \arrow[d, "\sim"] & H^2_{\text{ét}}\left(X_m\otimes F_{cp^n},\mathscr{M}_{\text{ét}}^{(2r-j,j)}(1)\right) \arrow[d, "\sim"] \\ H^1\left(F_{cp^n},H^1_{\text{ét}}(\overline{X}_m,\mathscr{D}_{\text{ét}}^{(2r-j,j)})\right) \arrow[r, "\mathrm{mom}^{2r}"] & H^1\left(F_{cp^n},H^1_{\text{ét}}(\overline{X}_m,\mathscr{M}_{\text{ét}}^{(2r-j,j)}(1))\right) \end{tikzcd}\tag{4.5}$$

Definition 4.3.1. Let $j\geq 0$ and $n\geq m\geq 1$ be integers. Define the *j-component of the big Heegner class associated to $\mathscr{U}$* to be $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}:=\delta_{cp^n,*}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n})$.

By Diagram (4.5), $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}$ is an element of $H^1\big(F_{cp^n},H^1_{\text{ét}}(\overline{X}_m,\mathscr{D}_{\text{ét}}^{(2r-j,j)})\big)$.

Proposition 4.3.2. For each $k\in\mathbb{Z}\cap\mathscr{U}$ with $k\geq j$ we have $\mathrm{mom}^{2r}(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m})=z^{[k,j]}_{cp^n,m}$.

Proof. Again by Diagram (4.5) and Proposition 4.2.6,

$$\mathrm{mom}^{2r}(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}) = \mathrm{mom}^{2r}(\delta_{cp^n,*}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n})) = \delta_{cp^n,*}(\mathrm{mom}^{2r}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n})) = \delta_{cp^n,*}(e^{[k,j]}_{cp^n,m}),$$

and it follows then from Proposition 2.7.1. □

Next we derive the norm-compatibility relations satisfied by the set of big Heegner classes or varying n, m and j that turns it into an Euler system, arguing similarly to the elliptic counterpart in [JLZ21, Proposition 5.1.2]. We begin with an alternative description of the induced action of the Hecke operators in the cohomology level.

For $m \geq 1$, denote by U^1_m the subgroup of U_m whose elements' p-parts are *lower* triangular modulo p:

$$U^1_m := \{g \in U_m;\ g_p \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \mod p\}$$

(notice that those matrices are already upper triangular modulo p by being so modulo p^m, thus elements of U^1_m are diagonal modulo p). We define the correspondent Shimura curve to be

$$X^1_m := B^\times \backslash (\mathcal{H}^\pm \times \widehat{B}^\times)/U^1_m,$$

which, as a moduli space, classifies triplets (A, ι, α) consisting of a QM abelian surface (A, ι), a U-level structure α of full level N^+p^{m+1}, where, in the notation of §1.4.1

$$U = \{g \in \widehat{\mathcal{O}}^\times_B;\ \exists a, b, d \text{ such that } i_{N^+p^m} \circ \widehat{\pi}_{N^+p^m}(g) \equiv \left(\begin{smallmatrix} a & pb \\ 0 & d \end{smallmatrix}\right) \mod N^+p^m\},$$

and to which we also may attach an arithmetic trivialization $\beta\colon \boldsymbol{\mu}_{p^{m+1}} \to eA[p^{m+1}]^0$. This kind of Shimura curve relates to the other kind X_m by the following maps:

$$\begin{tikzcd} X^1_m \arrow[r, "\Phi_p"] \arrow[d, "\hat{\mathrm{pr}}"'] & X_{m+1} \arrow[d, "\mathrm{pr}"] \\ X_m & X_m, \end{tikzcd}$$

where

- pr takes a quadruple $(A, \iota, \alpha, \beta)$ and maps into $(A, \iota, \alpha_p, \beta_p)$, where $\beta_p = \beta \circ \varphi_p$ with $\varphi_p\colon \boldsymbol{\mu}_{p^{m+1}} \to \boldsymbol{\mu}_{p^m}$ being the map $x \mapsto x^p$, and α_p is (the equivalence class of) $\psi_p \circ \alpha$, where ψ_p is the map $(x, y) \mapsto (px, py)$;

- $\hat{\mathrm{pr}}$ acts similarly to pr with the correspondent changes in the level structure;

- Φ_p acts as $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on the level structure.

Those maps induce pullbacks and pushforwards (trace maps) in the cohomology of those Shimura curves. Following [KLZ17, §2.4] and [Kat04, §2.9], we define the Hecke operator

$$U := (\hat{\mathrm{pr}})_* \circ (\Phi_p)^* \circ (\mathrm{pr})^*,$$

which coincides with the Hecke operator U defined in the level of quaternionic modular forms in §3.1.2 in the following sense: in the notation of said section, $(\mathrm{pr})^*$ lifts a quadruplet $(A, \iota, \alpha, \beta)$ to $(A, \iota, \alpha_{p^{-1}}, \beta_{p^{-1}})$ by composing α with $\psi_{p^{-1}} \colon (x, y) \mapsto (p^{-1}x, p^{-1}y)$ and precomposing β with $\varphi_{p^{-1}} \colon x \mapsto x^{1/p}$; the isomorphism $(\Phi_p)^*$ introduces the averaging factor $\frac{1}{p}$; finally, the pushforward (trace) $(\hat{\mathrm{pr}})_*$ projects the quadruplets back to level $U_0(N^+p^m)$ by taking the trace of the quotients running over all p-torsion subgroups of A, recovering the original definition of U.

Under the Poincaré duality (refer again to [Mil80, Corollary 11.2]), pushforwards and pullbacks are dual to each other, and $(\Phi_p)_* = (\Phi_p^{-1})^*$ as the induced actions of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are dual to each other. Thus, the dual to the Hecke operator U is

$$U' := (\mathrm{pr})_* \circ (\Phi_p^{-1})^* \circ (\hat{\mathrm{pr}})^*.$$

Proposition 4.3.3 (Euler relations)**.** $\mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}}(\mathbf{z}^{[j]}_{\mathscr{U},cp^{n+1},m}) = U' \cdot \mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}$, for all $n \geq m \geq 1$ and all $j \geq 0$.

Proof. Since $\mathrm{Gal}(H_{cp^{n+1}}/H_{cp^n}) \cong U_m/U_m^1 \cong U_m/U_{m+1}$, the last isomorphism being a result of the action of Φ_p as $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the preimage of the (U_m/U_m^1)-orbit of $x_{cp^n,m}(1) = [(\iota_K, \xi^{(n)})] \in X_m$ under $\hat{\mathrm{pr}}$ is the $\mathrm{Gal}(H_{cp^{n+1}}/H_{cp^n})$ orbit of the CM point $[(\iota_K, \xi^{(n)})]$ in X_m^1. Repeating the construction of the j-component of the big Heegner class associated to $\mathscr{U}$ for the Shimura curve X_m^1 gives a class

$$\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m} = \delta_{cp^n,*}(\mathbf{e}^{1,[j]}_{\mathscr{U},cp^n}) \in H^1\left(F_{cp^{n+1}}, H^1_{\text{ét}}(X_m^1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, D_{\mathscr{U}} \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j})\right),$$

defined over $F_{cp^{n+1}}$, and thus satisfying

$$\mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}}(\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m}) = (\hat{\mathrm{pr}})^*(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}).$$

Therefore

$$\begin{aligned} U' \cdot \mathbf{z}^{[j]}_{\mathscr{U},cp^n,m} &= (\mathrm{pr})_* \circ (\Phi_p^{-1})^* \circ (\hat{\mathrm{pr}})^*(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}) \\ &= (\mathrm{pr})_* \circ (\Phi_p^{-1})^*(\mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}}(\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m})) \\ &= \mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}}\left((\mathrm{pr})_* \circ (\Phi_p^{-1})^*(\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m})\right) \end{aligned}$$

The morphism Φ_p maps the CM point $[(\iota_K, \xi^{(n)})] \in X_m^1$ into $[(\iota_K, \xi^{(n+1)})] \in X0_{m+1}$, so

$$(\Phi_p^{-1})^*(\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m}) = (\Phi_p)_*(\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m}) = \mathbf{z}^{[j]}_{\mathscr{U},cp^{n+1},m+1},$$

and finally $(\mathrm{pr})_*$ projects $[(\iota_K, \xi^{(n+1)})] \in X_{m+1}$ into the "same point" $[(\iota_K, \xi^{(n+1)})] \in X_m$ (this identity like behavior of the pushforward of the projection is akin to what happens in the elliptic case, as seen in [KLZ17, §2.4]). Piecing it all together gives

$$\begin{aligned} U' \cdot \mathbf{z}^{[j]}_{\mathscr{U},cp^n,m} &= \mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}} \left((\mathrm{pr})_* \circ (\Phi_p^{-1})^* (\mathbf{z}^{1,[j]}_{\mathscr{U},cp^n,m}) \right) \\ &= \mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}} \left((\mathrm{pr})_* (\mathbf{z}^{[j]}_{\mathscr{U},cp^{n+1},m+1}) \right) \\ &= \mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}} \left(\mathbf{z}^{[j]}_{\mathscr{U},cp^{n+1},m} \right), \end{aligned}$$

which is the relation we wanted. □

4.4 Big Heegner classes for Coleman families

Let $\mathcal{F}$ be a p-stabilized quaternionic modular form of weight $k_0 = 2r_0 + 2$, level N^+p and trivial character. Let as in §3.2.3 $\mathscr{F}$ be the fixed Coleman family passing through $\mathcal{F}$ and let $\mathscr{U}$ be an open neighborhood of k_0 small enough so that $\mathscr{F}$ specializes at every $k \in \mathscr{U} \cap \mathbb{Z}$ to a p-stabilized form. Finally, denote by f be the classical modular form which lifts to $\mathcal{F}$ via the Jacquet–Langlands correspondence and recall that $a_p(\mathcal{F}) = a_p(f)$ denotes the U-eigenvalue of $\mathcal{F}$.

4.4.1 Big Galois representations

Let $\mathscr{f}$ be the Coleman family defined over $\mathscr{U} \subseteq \mathscr{W}$ passing through f corresponding via the p-adic Jacquet–Langlands lift to $\mathscr{F}$. In the same way a Coleman family p-adically interpolates modular forms, it is possible to p-adically interpolate Galois representations:

Proposition 4.4.1. After shrinking $\mathscr{U}$, there is an affinoid $\widetilde{\mathscr{U}} \supseteq \mathscr{U}$ in $\mathscr{W}^{(1)}$ such that the $\mathcal{O}(\mathscr{U})$-module

$$H^1_{\text{ét}} \left(X_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, D_{\widetilde{\mathscr{U}},1}(1) \right) \widehat{\otimes}_{\Lambda_{\widetilde{\mathscr{U}}}[1/p]} \mathcal{O}(\mathscr{U})$$

has a rank 2 direct summand $\mathbf{V}^*_{\mathscr{F}}$ interpolating the dual p-adic Galois representations of the specializations at integral weights of $\mathscr{F}$ in the following sense: for each $k \in \mathscr{U} \cap \mathbb{Z}_{\geq 0}$, there is an isomorphism of L-representations

$$\mathbf{V}^*_{\mathscr{F}} / k \mathbf{V}^*_{\mathscr{F}} \cong V^*_{\mathcal{F}},$$

where k is seen as a character in $\mathscr{W}$.

Proof. Using the p-adic Jacquet–Langlands correspondence (Proposition 3.2.3), this reduces to the equivalent statement about $\mathscr{f}$, which is [LZ16, Theorem 4.6.6]. □

Definition 4.4.2. The $\mathcal{O}(\mathscr{U})$-module $\mathbf{V}^*_{\mathscr{F}}$ from Proposition 4.4.1 is called the *dual big Galois representation* associated to $\mathscr{F}$, and its linear dual $\mathbf{V}_{\mathscr{F}} \cong \mathbf{V}^*_{\mathscr{F}}(1)$ is the *big Galois representation* associated to $\mathscr{F}$.

We recall from [LZ16, §5.1] the operator ∇, which on locally analytic L-valued functions defined on $\mathscr{U}$ by

$$\nabla f(z) = \frac{\mathrm{d}}{\mathrm{d}t} f(tz)\Big|_{t=1}$$

(*ibid.* Proposition 5.1.2). For each character $\kappa \in \mathscr{W}$ defined over $\mathscr{U}$, $\nabla \circ \kappa|_{\mathscr{U}}$ acts as multiplication by $\kappa'(1)$ and, in particular, at weight $k \in \mathbb{Z} \cap \mathscr{U}$ corresponding to the character $x \mapsto x^{k-2}$, ∇ specializes to multiplication by $k-2$ (*ibid.* Proposition 5.2.5).

This allows for the definition of, for each $0 \leq j \leq 2r$, the operator

$$\binom{\nabla}{j} := \frac{1}{j!} \prod_{i=0}^{j-1} (\nabla - i),$$

which is acts invertibly on $\Lambda_{\mathscr{U}}[1/p]$ after shrinking $\mathscr{U}$ enough so to avoid all integers $0, \dots, j-1$ (*ibid.*, Remark 5.2.2) and specializes at weight k to $\binom{2r}{j}$.

Similarly to §4.1, the natural projection $D_{\widetilde{\mathscr{U}},1} \to \mathbf{V}^*_{\mathscr{F}} \cong \mathbf{V}_{\mathscr{F}}(-1)$ induces a map

$$\mathrm{pr}^{[j]}_{\mathscr{F}} \colon H^1\left(F_{cp^n}, D_{\widetilde{\mathscr{U}},1}(1) \otimes \sigma_{\text{ét}}^{\kappa_{\widetilde{\mathscr{U}}}-j} \bar{\sigma}_{\text{ét}}^{j}\right) \longrightarrow H^1\left(F_{cp^n}, \mathbf{V}_{\mathscr{F}} \otimes \sigma_{\text{ét}}^{\kappa_{\widetilde{\mathscr{U}}}-j} \bar{\sigma}_{\text{ét}}^{j}\right). \tag{4.6}$$

To control the growth of the classes we are going to define, in view of the proof of Proposition 4.4.6 to follow, it is convenient to introduce a factor of $\binom{\nabla}{j}^{-1}$ to the above map (*cf. ibid.* Proposition 5.2.5), leading to the following definition:

Definition 4.4.3. The *j-component* of the big Heegner class is defined by

$$\mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^n,1} := \binom{\nabla}{j}^{-1} \mathrm{pr}^{[j]}_{\mathscr{F}}\left(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,1}\right),$$

where $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,1}$ is the class from Definition 4.3.1.

These classes satisfy the following Euler relations:

Proposition 4.4.4. For all $0 \leq j \leq 2r$ and $n \geq 1$, we have that

$$\mathrm{norm}_{F_{cp^{n+1}}/F_{cp^n}}\left(\mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^{n+1},1}\right) = \mathbf{a}_p \cdot \mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^n,1}.$$

Proof. This follows directly from Proposition 4.3.3 and the fact that, by definition, U' acts over the dual Galois representation by the eigenvalue $\mathbf{a}_p$ (*cf.* §3.1.4). □

Proposition 4.4.5. Let $k \in \mathscr{U} \cap \mathbb{Z}$ with $k \geq j$. The specialization at weight k of $\mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^n,1}$ is $\binom{2r}{j}^{-1} z^{[\mathcal{F}_k,j]}_{cp^n,1}$.

Proof. By construction, the specialization at weight k of $\mathbf{z}^{[\mathscr{F},j]}_{\mathscr{U},cp^n,1}$ is the image of the specialization at weight k of $\binom{\nabla}{j}^{-1} \mathbf{z}^{[j]}_{\mathscr{U},cp^n,1}$ under $\mathrm{pr}^{[j]}_{\mathcal{F}_k}$, which is exactly $\binom{2r}{j}^{-1} z^{[\mathcal{F}_k,j]}_{cp^n,1}$. □

4.4.2 Definition of big Heegner classes

After knowing the j-components of the big Heegner classes, the remaining step towards the general definition is the interpolation of the characters $\sigma^{-j}\bar{\sigma}^j$ for $j \geq 0$.

Let $F_{cp^\infty} = \bigcup_{n=1}^{\infty} F_{cp^n}$. Since $H_{cp^n} \cap F_{\mathfrak{N}^+} = K$ for all $n \geq 0$, $\Gamma_\infty := \mathrm{Gal}(F_{cp^\infty}/F_c)$ is isomorphic to $\mathrm{Gal}(H_{cp^\infty}/H_c)$. For any integer $n \geq 1$, let Γ_n denote the subgroup $\mathrm{Gal}(F_{cp^\infty}/F_{cp^n}) \cong \mathrm{Gal}(H_{cp^\infty}/H_{cp^n})$ of Γ_∞. Via the Artin reciprocity map, there is an isomorphism $\Gamma_1 \cong (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times/\mathbb{Z}_p^\times$, so the character $\sigma/\bar{\sigma}\colon (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times/\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ induces a Galois character $\sigma_{\text{ét}}/\bar{\sigma}_{\text{ét}}\colon \Gamma_1 \to \mathbb{Z}_p^\times$.

The Iwasawa algebra $\Lambda_{\Gamma_1} := \mathbb{Z}_p[\![\Gamma_1]\!]$ can be seen as a $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{cp})$-module via the canonical projection into Γ_1, which embeds as group-like elements in Λ_{Γ_1}. Twisting Λ_{Γ_1} by the inverse of $\sigma_{\text{ét}}/\bar{\sigma}_{\text{ét}}$ gives a Galois module $\Lambda_{\Gamma_1}(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})$, whose elements can be seen as $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{cp})$-representations. Define the specialization at $j \in \mathbb{Z}_{\geq 0}$ to be the map

$$\Lambda_{\Gamma_1}(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}}) \longrightarrow \overline{\mathbb{Q}}\sigma_{\text{ét}}^{-j}\bar{\sigma}_{\text{ét}}^{j}\colon \mu \mapsto \left(\int_{\Gamma_{cp^n}} d\mu\right)(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})^j \tag{4.7}$$

Define $\sigma^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}^{\mathbf{j}} := \sigma^{\kappa_{\mathscr{U}}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda_{\Gamma_1}(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})$. Map (4.7) induces correspondent specialization at j maps for $\sigma^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}^{\mathbf{j}}$ and therefore a map

$$\mathrm{mom}_{j,n}\colon H^1\left(F_{cp}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right) \longrightarrow H^1(F_{cp^n}, \mathbf{V}_{\mathscr{F}} \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}). \tag{4.8}$$

Proposition 4.4.6. There exists an element $\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]} \in H^1\left(F_{cp}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right)$ such that $\mathrm{mom}_{j,n}(\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]}) = \mathbf{a}_p^{-n}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]}$, for all $n \geq 1$ and all $j \geq 0$.

Proof. The existence of the class will follow from a general construction of Loeffler–Zerbes, namely [LZ16, Proposition 2.3.3]. To check the conditions for it, endow $\mathcal{O}(\mathscr{U})$ with the supremum norm and take λ to be the slope of $\mathscr{F}$, that is $\log_p(\|\mathbf{a}_p^{-1}\|)$, a positive real number which by hypothesis is strictly smaller than $k-1$, thus $2r \geq \lfloor \lambda \rfloor$. Define the classes

$$c_{n,j} := \mathbf{a}_p^{-n}\binom{\nabla}{2r}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]} \in H^1(F_{cp^\infty}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}}),$$

considered over F_{cp^∞} where the characters $\sigma_{\text{ét}}$ and $\bar{\sigma}_{\text{ét}}$ are trivial. For the first condition in *loc. cit.*, it suffices to check that

$$\mathrm{norm}_{F_{cp^{n+1}}/F_{cp^n}}(c_{n+1,j}) = c_{n,j},$$

which is immediate from Proposition 4.4.4.

For the second condition, we wish to show that there exists a constant C such that

$$\left\| p^{-2rn}\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\mathrm{Res}_{F_{cp^\infty}/F_{cp^n}}(c_{m,j})\right\| \leq Cp^{\lfloor \lambda n \rfloor},$$

where $\|\cdot\|$ is the supremum seminorm on $H^1(F_{cp^\infty}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}\sigma^{\kappa_{\mathscr{U}}})$ coming from a choice of norm on $\mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}}$ as a Banach $\mathcal{O}(\mathscr{U})$-module. By Lemma 4.2.2, $e_{cp^n}-\bar{e}_{cp^n} \in p^n(\mathcal{O}_L^2)^\vee$, therefore

$$(e_{cp^n}-\bar{e}_{cp^n})^{\odot 2r} = \sum_{j=0}^{2r}(-1)^j(e_{cp^n}^{2r-j}\odot \bar{e}_{cp^n}^{j}) \in p^{2rn}\,\mathrm{TSym}^{2r}((\mathcal{O}_L^2)^\vee).$$

Taking the product of that sum with $\mathbf{e}_{\mathscr{U},cp^n}$ and applying the overconvergent projector Π_j gives

$$\sum_{j=0}^{2r}(-1)^j\Pi_j(\mathbf{e}_{\mathscr{U}-2r,cp^n}\odot(e_{cp^n}^{2r-j}\odot \bar{e}_{cp^n}^{j})) = \sum_{j=0}^{2r}(-1)^j\binom{\nabla-j}{2r-j}\mathbf{e}_{\mathscr{U},cp^n}^{[j]}, \tag{4.9}$$

where the equality (without the alternating sum) is [LZ16, Lemma 5.1.5]. Since Π_j has denominators bounded in terms of j and m, there is a constant $C_{m,j}$ such that

$$\Pi_j(\mathbf{e}_{\mathscr{U}-2r,cp^n}\odot(e_{cp^n}^{2r-j}\odot \bar{e}_{cp^n}^{j})) \in C_{m,j}p^{2rn}D_{\mathscr{U},m}^\circ,$$

where $D_{\mathscr{U},m}^\circ$ denotes the norm-1 distributions in $D_{\mathscr{U},m}$. Applying the big Gysin map (4.4) and $\mathrm{pr}_{\mathcal{F}}$ to the right-hand side of (4.9) and using the explicit calculation

$$\binom{\nabla}{j}\binom{\nabla-j}{2r-j} = \frac{\nabla\cdot\ldots\cdot(\nabla-j+1)}{j!}\cdot\frac{(\nabla-j)\cdot\ldots\cdot(\nabla-2r+1)}{(2r-j)!}\cdot\frac{(2r)!}{(2r)!} = \binom{2r}{j}\binom{\nabla}{2r},$$

we have that

$$\left\|\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\binom{\nabla}{2r}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]}\right\| \le C_{m,j}p^{2rn}.$$

To conclude the calculation,

$$\begin{aligned}\left\|p^{-2rn}\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\mathrm{Res}_{F_{cp^\infty}/F_{cp^n}}(c_{m,j})\right\| &\le p^{-2rn}\left\|\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\binom{\nabla}{2r}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]}\mathbf{a}_p^{-n}\right\| \\ &\le C_{m,j}\|\mathbf{a}_p^{-n}\| = C_{m,j}p^{\lambda n}\end{aligned}$$

which implies the inequality we want. Thus we may apply [LZ16, Proposition 2.3.3] to obtain a unique class $\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]} \in H^1(F_{cp^\infty}, D_{\mathscr{W}_1,\lambda}\widehat{\otimes}_{\Lambda_{\Gamma_1}}(\mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}}))$ interpolating the classes $\mathrm{Res}_{F_{cp^\infty}/F_{cp^n}}(c_{n,j})$, where $\mathscr{W}_1$ denotes the weight space associated to Λ_{Γ_1}. Since all quaternionic modular forms are cuspidal (as the unerlying Shimura varieties are compact) we have that $H^0(F_{cp^\infty}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}}) = 0$, which, via the inflation-restriction sequence, allows the above class to be defined over F_{cp}, which we can map to the larger submodule

$$\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]} \in H^1\left(F_{cp}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right).$$

It then follows by the interpolation property in the definition of the class that

$$\mathrm{mom}_{j,n}(\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]}) = \mathbf{a}_p^{-n}\binom{\mathbf{k}-2}{2r}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]},$$

where $\mathbf{k}$ is the weight of the Coleman family, which is constant and equal to $k = 2r+2$, thus making the binomial term vanish, concluding the proof. □

Definition 4.4.7. The class $\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]} \in H^1\left(F_{cp}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right)$ from Proposition 4.4.6 is called the *big Heegner class* associated to $\mathscr{F}$.

4.4.3 Specialization of big Heegner classes

For each $k \in \mathbb{Z}\cap\widetilde{\mathscr{U}}$, composing with the $\mathrm{mom}_{j,n}$ with the weight $k = 2r+2$ specialization, we also have a map

$$\mathrm{mom}_{j,n}^{2r}\colon H^1\left(F_{cp}, \mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right) \longrightarrow H^1(F_{cp^n}, V_{\mathcal{F}_k}^* \otimes \sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}).$$

Theorem 4.4.8. For each $k \in \mathbb{Z}\cap\mathscr{U}$ with $k \geq j$, we have that

$$\mathrm{mom}_{j,n}^{2r}\left(\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]}\right) = a_p(\mathcal{F}_k)^{-n}\binom{2r}{j}^{-1} z_{cp^n,1}^{[\mathcal{F}_k,j]} \in H^1(F_{cp}, V_{\mathscr{F}}^*\otimes\sigma_{\text{ét}}^{2r-j}\bar{\sigma}_{\text{ét}}^{j}).$$

Furthermore, if ξ is an algebraic Hecke character of infinity type $(2r-j, j)$, the specialization at ξ is

$$a_p(\mathcal{F}_k)^{-n}\binom{2r}{j}^{-1} z_{cp^n,1}^{[\mathcal{F}_k,j,\xi]} \in H^1(H_{cp}, V_{\mathscr{F}}^*\otimes\xi).$$

Proof. By Proposition 4.4.6, $\mathrm{mom}_{j,n}(\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]}) = \mathbf{a}_p^{-n}\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]}$. The specialization at weight k of $\mathbf{a}_p$ is by definition $a_p(\mathcal{F}_k)$ and the specialization at weight k of $\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathscr{F},j]}$ is $\binom{2r}{j}^{-1} z_{cp^n,1}^{[\mathcal{F}_k,j]}$ by Proposition 4.4.5. The second statement follows immediately. □

Corollary 4.4.9. Let $\mathcal{F}^\sharp \in M_k(N^+,1,L)$ be such that $\mathcal{F}$ as above is its p-stabilization respect to a root α of its Hecke polynomial. For each $k \in \mathbb{Z}\cap\mathscr{U}$ with $k \geq j$ and ξ an algebraic Hecke character of infinity type $(2r-j, j)$, the big Heegner class $\mathbf{z}_{\mathscr{U},cp}^{[\mathscr{F},\mathbf{j}]}$ specializes at weight k and character ξ to

$$\left(1-\frac{\xi(\mathfrak{p})}{\alpha}\right)\left(1-\frac{\xi(\bar{\mathfrak{p}})}{\alpha}\right)\alpha^{-n}\binom{2r}{j}^{-1} z_{cp^n,1}^{[\mathcal{F}^\sharp,j,\xi]} \in H^1(H_{cp}, V_{\mathscr{F}}^*\otimes\xi).$$

Proof. Follows directly from Theorem 4.4.8 and Proposition 4.1.5. □

Remark 4.4.10. Suppose we have an ordinary family of quaternionic modular forms (a Coleman family of slope 0, that is, a Hida family). As mentioned in the introduction, other than the motivic approach of Jetchev–Loefler–Zerbes ([JLZ21]) that we have just explained in the quaternionic context, another way to achieve big Heegner classes would be through big Heegner points, defined in the quaternionic context by Longo–Vigni ([LV11]). In joint work with Matteo Longo and Paola Magrone ([LMW23]) we show that both approaches lead to the same classes, as they coincide in a sufficiently small neighborhood of k_0.

4.4.4 Big Heegner classes defined over K

The Big Heegner class previously described can be further mapped into a class defined over K, however, this is not as straightforward as might appear at first: the character $\kappa_{\mathscr{U}}$ is not well-defined over H_{cp}, so trying to use Galois-invariance to lower from F_{cp} to H_{cp} will not work - not before a workaround that eliminates the dependency on $\kappa_{\mathscr{U}}$. The workaround is to reparametrize the underlying affinoid of the space of distributions $\mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}^{\mathbf{j}}$ by an appropriate affinoid over a suitable, bigger weight space.

The space $\mathbf{V}_{\mathscr{F}}\widehat{\otimes}_{\mathbb{Z}_p}\sigma^{\kappa_{\mathscr{U}}}$ is parametrized by the affinoid $\mathscr{U}\subseteq\mathscr{W}$ and the space $\sigma^{-\mathbf{j}}\bar{\sigma}^{\mathbf{j}}$ is a Λ_{Γ_1}-module, which corresponds to an affinoid $\mathscr{U}_1\subseteq\mathscr{W}_1$, which, for simplicity, we may take to be the whole weight space.

The following is a summary of [JLZ21, §5.4]:

Proposition 4.4.11. Let $\widetilde{\mathscr{W}}\to\mathscr{W}\times\mathscr{W}_1$ be the weight space parametrizing the continuous characters $K^\times\backslash\mathbb{A}^\times_{K,\mathrm{fin}}\to\mathbb{Q}_p^\times$ which restrict trivially to $(\widehat{\mathcal{O}_K})^\times_{(p)}$. Then the preimage of $\mathscr{U}\times\mathscr{W}_1$ in $\widetilde{\mathscr{W}}$ is an affinoid $\widetilde{\mathscr{U}}'\subseteq\widetilde{\mathscr{W}}$ isomorphic to $\mathscr{U}\times\mathscr{W}_1$ and whose universal character $\kappa_{\widetilde{\mathscr{U}}'}$ is well defined as a character $\mathrm{Gal}(K^{\mathrm{ab}}/K)\to\mathcal{O}(\widetilde{\mathscr{U}}')^\times$, and therefore is defined over K.

Let $\widetilde{\mathbf{V}}_{\mathscr{F}}:=\mathbf{V}_{\mathscr{F}}\otimes_{\mathcal{O}(\mathscr{U})}\mathcal{O}(\widetilde{\mathscr{U}}')(\kappa_{\widetilde{\mathscr{U}}'})$ and consider the image of the big Heegner class $\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp}$ in $H^1(F_{cp},\widetilde{\mathbf{V}}_{\mathscr{F}})$, denoted by the same symbol. By an argument similar to Lemma 4.1.2, $\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp}$ is found to be invariant respect to $\mathrm{Gal}(F_{cp}/H_{cp})$, so it can be seen as class in $H^1(H_{cp},\widetilde{\mathbf{V}}_{\mathscr{F}})$. Finally, define

$$\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U}}:=\mathrm{cores}_{H_{cp}/K}(\mathbf{z}^{[\mathscr{F},\mathbf{j}]}_{\mathscr{U},cp})\in H^1(K,\widetilde{\mathbf{V}}_{\mathscr{F}}),$$

which, by abuse of nomenclature, we also call the *big Heegner class* associated to $\mathscr{F}$, enjoying the same specializations as the "original" big Heegner class, with the difference that this one is defined over K.

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