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Abstract

This thesis concerns the study of Strichartz estimates for different dispersive systems and their applications. The structure is as follows. In the introduction we present the problems treated in this manuscript, describing how they insert in the existing literature. Then, we describe in simple terms the main contributions of the works on which the thesis is based on, leaving the details to the successive chapters.

In Chapter 1, we prove generalized Strichartz estimates for the massless 2D and 3D Dirac equation with the Coulomb potential. This chapter is based on the published work [53] using tools developed in [33, 34]. Moreover, as an application, we prove local well posedness for an Hartree-type nonlinear system in dimension 3. In the last section on this chapter we introduce an ongoing project, in collaboration with Federico Cacciafesta and Junyong Zhang concerning the analysis of the Dirac-Coulomb equation with a positive mass.

Chapter 2, based on the published work [28] in collaboration with Federico Cacciafesta and Long Meng, is dedicated to the study of the dispersive behavior, via local in time Strichartz estimates, of the half wave and half Klein-Gordon equations on compact smooth Riemannian manifolds without boundary. As an application, we derive similar estimates for the Dirac equation on the same setting, whose definition is introduced in Section 0.2.4. The strategy of the proof follows the ones introduced in [23, 56], combined with a refined version of the WKB approximation.

In Chapter 3 we present joint work with Charles Collot, Anne-Sophie de Suzzoni and Cyril Malézé, [46], which will be published soon. We study the Hartree-Fock equation, which admits homogeneous states that model infinitely many particles at equilibrium. We prove their asymptotic stability in large dimensions, under assumptions on the linearized operator. Perturbations are moreover showed to scatter to linear waves. We obtain this result for the equivalent formulation of the Hartree-Fock equation in the framework of random fields. The main novelty is to consider the full Hartree-Fock equation, including for the first time the exchange term in the study of these equilibria. The proof relies on dispersive estimates for the study of the linearized operator around the equilibrium and perturbative techniques.

Ringraziamenti

This thesis is the outcome of three years full of unsolved mathematical questions, a spare of solved ones but mostly of new connections and relations, which I am very grateful for. I will try here to explain why. I apologize in advance if I do not acknowledge all the people that deserve to be; hopefully I will be able to recover this lack in the future. Moreover, I prefer to use, only in this chapter, different languages, choosing each time the one that would sound more familiar to me.

Without any doubt, this thesis would not be as it is if it were not for my supervisors, listed alphabetically, Federico Cacciafesta and Anne-Sophie de Suzzoni. I sincerely thank you for the interesting problems you suggested for me to look at and your complete availability to discuss the questions that necessarily arized. I am particularly appreciative for the constant support and motivation to go beyond the research per se and to dive into the mathematical community, by giving seminars and participating in workshops and conferences which improved my ability to discuss math and to play board games; especially after some beer and wine, something I now recognize to be an important skill for a good mathematician.

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[1] e Jack

Introduction

Dispersive equations are PDEs characterized by the fact that different frequencies propagates in different directions. More precisely, one can associate to each equation the so called *dispersive relation* ω, which is the velocity at which each elementary components of a wave packet travels. If the gradient of $\omega(\xi)$ is still a function that depends on the frequency ξ the equation is said to be dispersive. The most classical examples, with corresponding dispersion relations are given by

- *Schrödinger equation*

$$i\hbar\partial_t u - \frac{\hbar^2}{2m}\Delta_x u = 0 \quad \to \quad \omega(\xi) = \frac{\hbar}{2m}|\xi|^2,$$

  with $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, $m > 0$ and $\hbar$ the Planck constant;

- *wave/Klein-Gordon equation*

$$\partial_{tt}^2 u - c^2\Delta_x u + \frac{m^2c^4}{\hbar^2}u = 0, \quad m \geq 0 \quad \to \quad \omega(\xi) = \pm\sqrt{c^2|\xi|^2 + \frac{m^2c^4}{\hbar^2}},$$

  with $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, $m \geq 0$, $\hbar$ as before and c the speed of light;

- *Korteweg-de Vries equation*

$$\partial_t u + \partial_{xxx}^3 u = 0 \quad \to \quad \omega(\xi) = \xi^3,$$

  with $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Notice that we can make a further distinction: if $\nabla_\xi\omega(\xi)$ is not bounded, this is the case for example of the Schrödinger equation, the relative equation is said to be dispersive with infinite speed of propagation. Otherwise, this is the case of the wave or KG equation, it is called dispersive with finite speed of propagation. Roughly speaking, this encodes information of the velocity at which the mass of the solution "escapes" bounded region. Another classical example is given by the *Dirac equation*. Due to its rich algebraic structure and its interesting physical derivation, we prefer to postpone its definition to Section 0.2. Moreover, even if they will not be studied in this manuscript, we recall that more examples of dispersive equations can be found in the field of water waves. We mention, as an example, the following dispersion relations

- *Deep gravity waves* $\rightarrow$ $\omega(\xi) = |\xi|^{\frac{1}{2}}$;
- *Capillary waves* $\rightarrow$ $\omega(\xi) = |\xi|^{\frac{3}{2}}$;
- *Shallow gravity waves* $\rightarrow$ $\omega(\xi) = \sqrt{|\xi| \tanh|\xi|}$;
- *Shallow capillary waves* $\rightarrow$ $\omega(\xi) = \sqrt{|\xi|^3 \tanh|\xi|}$.

Thanks to the research on nonlinear models, it has been understood that dispersion plays a fundamental role in the dynamics of a system. Consequently, a great effort has been devoted to studying the tools which permit to quantify dispersive phenomena in terms of estimates for the free flows. In the following section we describe two types of dispersive estimates, the *time-decay* and *Strichartz estimates*, which as we will see are not unrelated.

0.1 Dispersive estimates

In this section we focus on the Schrödinger and wave equations in order to present some classical tools that are widely used in this field. Since the results presented in this section are now fairy classical, we will not enter deep in the details. We refer for them to [6] (chapter 8).

0.1.1 Time-decay estimates

i) Schrödinger equation: Let us consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$. The goal is now to find a good representation of the solution. In order to avoid technical issues, we restrict our attention to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the vector space of smooth rapidly decreasing functions. Moreover, we adopt the following notation for the *Fourier transform* with respect to the space variable:

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int u(x) e^{-ix\cdot\xi} dx, \quad \forall \xi \in \mathbb{R}^n.$$

We recall the following useful property of the Fourier transform:

$$\mathcal{F}(\partial^\alpha u)(\xi) = i^{|\alpha|}\xi^\alpha \mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^n,$$

for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ with length $|\alpha|$. Therefore, by taking the Fourier transform of (1) we have

$$\begin{cases} i\partial_t \hat{u}(t,\xi) - i^2|\xi|^2 \hat{u}(t,\xi) = 0, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \end{cases}$$

which has solution

$$\hat{u}(t,\xi) = e^{it|\xi|^2}\hat{u}_0(\xi).$$

Taking the inverse Fourier transform we finally get

$$u(t,x) = \mathcal{F}^{-1}\Big(e^{it|\xi|^2}\hat{u}_0(\xi)\Big)(x) =: e^{-it\Delta}u_0(x).$$

Moreover, by explicit computations, one gets that

$$u(t,x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}}u_0(y)dy.$$

Therefore, it is straightforward to obtain the following time-decay estimate for the solution of the Schrödinger equation

$$\|e^{-it\Delta}u_0\|_{L^\infty_x} \leq C|t|^{-\frac{n}{2}}\|u_0\|_{L^1_x}, \tag{2}$$

for some constant $C>0$ which does not depend on t nor on the initial datum.

ii) Wave equation: Let us now focus on the Cauchy problem associated with the wave equation

$$\begin{cases} \partial^2_{tt}u - \Delta u = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^n,\\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x), \end{cases} \tag{3}$$

where as before $u\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$. We would like to investigate the validity of a polynomial time-decay estimate as in the case of Schrödinger. To do so, we play the same game as before. We pass to the Fourier transform in (3). We obtain

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}\big(\cos(t|\xi|)\hat{u}_0\big)(x) + \mathcal{F}^{-1}\bigg(\frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1(\xi)\bigg)(x) \\ &=: \cos(t\sqrt{-\Delta})u_0(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1(x). \end{aligned} \tag{4}$$

This representation suggests to focus on the study of

$$g := e^{it\sqrt{-\Delta}}f = \mathcal{F}^{-1}\big(e^{it|\xi|}\hat{f}\big)(x).$$

The analysis of the decay of this propagator is not as simple as the one of Schrödinger; in fact, it involves the study of oscillatory integrals via the method of stationary/non-stationary phase. An extensive treatment of this topic can be found in [78]. Nevertheless, it has been proved that the following time-decay holds

$$\|e^{it\sqrt{-\Delta}}u_0\|_{L^\infty_x} \leq C(1+|t|)^{-\frac{n-1}{2}}\|f\|_{L^1_x} \tag{5}$$

where $C>0$ if f is *frequency localized*, that is, if

$$\operatorname{supp}\hat{f} \subseteq \{r\leq|\xi|\leq R\} \quad \text{for some} \quad 0<r<R<+\infty. \tag{6}$$

0.1.2 Strichartz estimates

We observe that, by Plancherel's theorem, we have "for free" an identity for both Schrödinger and wave propagator. Indeed

$$\|e^{-it\Delta}u_0\|_{L^2_x} = \|e^{it|\xi|^2}\hat{u}_0\|_{L^2_\xi} = \|\hat{u}_0\|_{L^2_\xi} = \|u_0\|_{L^2_x}$$

and similarly

$$\|e^{it\sqrt{-\Delta}}f\|_{L^2_x} = \|f\|_{L^2_x}.$$

These estimates can be combined with the decay estimates found in the previous section to obtain a number of inequalities involving space-time Lebesgue norms, called Strichartz estimates. The classical method is based on complex interpolation and duality arguments. It is summarized in this result of Keel and Tao (see [85], Theorem 1.2). More general versions of this result will be presented in Chapters 2 and 3.
Let us assume that for each $t \in \mathbb{R}$ we have an operator $U(t)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

i) for some $c_1 > 0$

$$\|U(t)f\|_{L^2} \leq c_1\|f\|_{L^2},$$

ii) for some $\sigma > 0$ and $c_2 > 0$, one of the following decay estimates holds:

- for all $t \neq s$ and $f \in L^1(\mathbb{R}^n)$

$$\|U(s)U(t)^*f\|_{L^\infty} \leq c_2|t-s|^{-\sigma}\|f\|_{L^1} \quad \text{(untruncated decay)}$$

- for all t, s and $f \in L^1(\mathbb{R}^n)$

$$\|U(s)U(t)^*f\|_{L^\infty} \leq c_2(1+|t-s|)^{-\sigma}\|f\|_{L^1} \quad \text{(truncated decay)}$$

Then the estimate

$$\|U(t)f\|_{L^p_t L^q_x} \leq c\|f\|_{L^2_x}$$

holds for any $(p,q) \in [2,+\infty]^2$ $\sigma-$admissible, i. e. such that

$$\frac{1}{p} + \frac{\sigma}{q} \leq \frac{\sigma}{2}, \qquad (p,q,\sigma) \neq (2,\infty,1), \tag{7}$$

in the case of the truncated decay. If only the untrucated decay holds then the estimates are satisfied for any (p,q) for which equality in (7) holds. Notice that all the constants that appear do not depend on t and on f.
The mixed Lebesgue norms are defined as

$$\|f\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^n))} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|f|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \quad \forall p,q \in [1,+\infty),$$

with natural modifications if $p, q = +\infty$. Therefore, from (2) and Keel-Tao's result, we have the following family of Strichartz estimates for the Schrödinger propagator

$$\|e^{-it\Delta}u_0\|_{L^p_t L^q_x} \leq c\|u_0\|_{L^2}$$

for any $p, q \geq 2$ such that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad q < +\infty. \tag{8}$$

A couple of indexes satisfying (8) is said to be *Schrödinger admissible*. Here and in the following we use c to denote a positive constant which depends only on p, q, n. On the other side, for the propagator associated to the wave equation we have that if f is frequency localized (as in (6)) then

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t L^q_x} \leq c\|f\|_{L^2}$$

for any $p, q \geq 2$ such that

$$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, \quad q < \frac{2(n-1)}{n-3}. \tag{9}$$

A couple of indexes satisfying (9) is said to be *wave admissible*. The endpoint $q = \frac{2(n-1)}{n-3}$ is admissible if $n > 3$. It seems however too restrictive to consider only frequency localized initial data. To recover the general case, one relies on the *Paley-Littlewood decomposition*. We refer to [5] for more details and applications. Roughly speaking, the main idea of this procedure consists in sampling the frequencies by means of a decomposition in the frequency space in annuli $\{\xi \in \mathbb{R}^n : |\xi| \sim 2^j, j \in \mathbb{Z}\}$. In this way, one obtains a decomposition of the function into a sum of a countable number of functions whose Fourier transform is supported in an annulus. In this way it is possible to prove the following family of Strichartz estimates for a function f without any assumption of localization:

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t L^q_x} \leq c\|f\|_{\dot{H}^s}$$

where p, q are as in (9), $s = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p}$ and the norm on the RHS is the homogeneous Sobolev norm which can be defined for any $\gamma \in \mathbb{R}$ as

$$\|f\|_{\dot{H}^\gamma} = \||\xi|^\gamma \hat{f}(\xi)\|_{L^2_\xi} = \|(\sqrt{-\Delta})^\gamma f\|_{L^2}.$$

Finally, we can combine these estimates with the decomposition (4) to get the Strichartz estimates for the wave equation; let u be a solution of (3), then

$$\|u\|_{L^p_t L^q_x} \leq c(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}})$$

where p, q satisfy conditions (9) and $s = n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p}$. In a similar way it is possible to prove (we refer to [51], Appendix A) that this kind of estimates hold also for the Klein-Gordon equation. We recall them in Section 0.2.2.

As remarked by Tao [120], we can interfere two kind of information from them; locally in time, they describe a type of smoothing effect, but reflected in a gain of integrability rather than regularity (if the datum is in L^2_x, the solution $u(t)$ is in L^q_x with $q > 2$ for most of the time), and only if one averages in time. For fixed time, no gain in integrability is possible (see Exercise 2.35 in [120]). Globally in time, they describe a decay effect: the L^q_x norm of a solution $u(t)$ must decay to zero as $t \to \infty$, at least in some L^p_t-averaged sense. Both effects of the Strichartz estimates reflect the dispersive nature of the equation (i.e. that different frequencies propagate in different directions); it is easy to verify that no such estimates are available for the dispersionless equations (e.g. transport equation), except for the trivial pair of exponents $(p, q) = (\infty, 2)$. Moreover, we will see in Section 0.3 that this kind of estimates are an effective tool in the study of dynamics of nonlinear systems.

To conclude, we mention that the dispersion can be also measured via *(Kato) smoothing estimates*. In the literature, different versions of these estimates can be found. For the Schrödinger equation, Kato and Yajima in [83] proved the following inequality

$$\|\langle x\rangle^{-\frac{1}{2}-}|D|^{\frac{1}{2}}e^{it\Delta}f\|_{L^2_t(\mathbb{R},L^2_x(\mathbb{R}^n))} \leq c\|f\|_{L^2(\mathbb{R}^n)}$$

which encodes smoothing effects, frequently observed for dispersive equations with infinite speed of propagation. For the Klein-Gordon and wave equations similar estimates hold, without gain of derivatives. We refer, respectively to [98, 106]. We recall moreover that they turn to be particularly useful to handle potential-type perturbations and can be used to derive Strichartz estimates, see e.g. [49] and the references therein for a general review.

0.2 The Dirac equation

0.2.1 The Dirac equation on $\mathbb{R}^n$

The Dirac equation was introduced by Paul Dirac in 1928 to describe the motion of fermions, such as electrons, which move freely in $\mathbb{R}^3$. In order to explain how it was derived, we start from the relativistic energy-momentum relation:

$$E^2 = p^2c^2 + m^2c^4 \tag{10}$$

where $p = (p_1, p_2, p_3)$ and m are respectively the momentum and the mass of the particle and c is the speed of light. Then, formally, the transition from classical to quantum mechanics can be accomplished by substituting appropriate operators for the classical quantities. In particular,

$$E =: p_0 \to i\hbar\partial_t, \qquad p_j \to -i\hbar\partial_{x_j}, \quad j = 1, 2, 3, \tag{11}$$

where $\hbar$ is the Planck's constant. Therefore, one obtains the Klein-Gordon equation

$$\partial^2_{tt}\psi - c^2\Delta\psi + \frac{m^2c^4}{\hbar^2}\psi = 0$$

with $\psi(t,x)$ a scalar function. The resulting equation is Lorentz covariant, but it does not allow to describe the internal structure of the electrons, namely the spin. Moreover, if one tries to construct a conserved current as for the Schrödinger equation, one obtains

$$\psi^* \partial_t \psi - \psi \partial_t \psi^* = 0.$$

However the quantity defined on the LHS is not positive definite, so it is impossible to interpret it as a probability density. The goal is then to find a first order in time equation which admits a straightforward interpretation as in the Schrödinger equation. The first idea would be to take the square-root of (10)

$$E = \sqrt{p^2c^2 + m^2c^4}$$

and quantized as before. In this way one gets the following equation

$$i\hbar \partial_t \psi = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\,\psi.$$

This forces to face the problem of interpreting the square-root operator on the RHS. In order to solve this problem, Dirac's idea was to look for a linearized equation of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + m\alpha_0)\psi = 0,$$

where $\{\alpha_i\}_{i=0}^3$ are some dynamical variables or operators that are independent of t, x_1, x_2, x_3. That is to say, by "squaring" the equation we should obtain the energy-momentum relation. Recalling (11) we formally impose

$$(i\partial_t - i\sum_{j=1}^{3} \alpha_j \partial_{x_j} + m\alpha_0)(-i\partial_t - i\sum_{j=1}^{3} \alpha_j \partial_{x_j} + m\alpha_0) = \left(\partial^2_{tt} - \Delta + m^2\right) \otimes \mathbb{1}. \qquad (12)$$

Here and in the following, to lighten the notation, we set $c = \hbar = 1$. We compute the LHS and get

$$\partial^2_{tt} - \sum_{j=1}^{3} \alpha_j^2 \partial^2_{x_j x_j} + m^2\alpha_0^2 - \sum_{i,j=1, i<j}^{3} (\alpha_i\alpha_j + \alpha_j\alpha_i)\partial_{x_i}\partial_{x_j} - im\sum_{j=1}^{3}(\alpha_j\alpha_0 + \alpha_0\alpha_j)\partial_{x_j}.$$

Therefore, in order to get the Klein-Gordon equation (or, better, a system of decoupled equations) we have to look for $\{\alpha_j\}_{j=0}^3$ such that

$$\{\alpha_i, \alpha_j\} := \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \qquad \forall i,j = 0,\dots 3, \qquad (13)$$

where δ_{ij} is the Kronecker delta. From this relation it is clear that α_j's cannot be scalars, but matrices. The smallest dimension in which these four matrices can be realized is $N = 4$. In a particular and widely used representation the α's matrices, called Dirac matrices, are given by

$$\alpha_j = \begin{pmatrix} 0_{2\times 2} & \sigma_j \\ \sigma_j & 0_{2\times 2} \end{pmatrix}, \quad j = 1,2,3, \qquad \alpha_0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & -\mathbb{1}_{2\times 2} \end{pmatrix},$$

Here $\sigma_j \in \mathcal{M}_{2\times 2}(\mathbb{C})$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

At last, the Dirac equation reads as

$$i\partial_t \psi + \mathcal{D}\psi + m\alpha_0 \psi = 0, \tag{15}$$

where $\psi := \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4$, $m \geq 0$ and $\mathcal{D}$ is the Dirac operator defined as

$$\mathcal{D} := -i\alpha \cdot \nabla_{\mathbb{R}^3} = -i \sum_{j=1}^{3} \alpha_j \partial_{x_j}.$$

The vector-valued wavefunction ψ on which the Dirac operator acts is called *spinor*. They are characterized by how they transforms under Lorentz transformations. Details can be found in the Appendix A, where a proof of the Lorentz covariance of the derived equation is also presented. We remark also that Dirac's modification provides a way to incorporate external electro-magnetic fields in a manner compatible with the relativistic theory. This will be exploit in Section 0.2.3 where we study the equation with linear perturbations.

This construction can be generalized for $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. In this case, the $n+1$ α's matrices are taken in $M_{N\times N}(\mathbb{C})$ with $N = 2^{\lceil \frac{n}{2} \rceil}$. This fact is related to an underlying algebraic structure. Indeed, the problem of finding the α matrices satisfying the anti-commutation relation (13) is strictly connected to the one of finding a representation for the Clifford algebra $Cl_{1,n}(\mathbb{R})$.
In particular, in dimension $n = 2$ the Dirac operator can be written in terms of the Pauli matrices (14) as

$$\mathcal{D}_2 = -i\sigma \cdot \nabla_{\mathbb{R}^2} = -i(\sigma_1 \partial_1 + \sigma_2 \partial_2).$$

According to quantum mechanics, we should work with self-adjoint operators in $L^2(\mathbb{R}^n; \mathbb{C}^N)$. We recall that the free Dirac operator is defined on $C^\infty_c(\mathbb{R}^n; \mathbb{C}^N)$ and it admits only one self-adjoint extension, its closure, with domain $H^1(\mathbb{R}^n; \mathbb{C}^N)$. Moreover, the spectrum is purely essential spectrum, given by $\sigma_{ess}(\mathcal{D}) = (-\infty, -m] \cup [m, +\infty)$ in the massive case and it extends to the whole real line if $m = 0$ (see [123], Section 1.4).

0.2.2 The Dirac equation as a dispersive PDE

As we saw before, the dynamics of the Dirac equation is strictly connected to the one of the wave or Klein-Gordon equation, respectively if $m = 0$ or $m > 0$. Indeed, by the identity (12) we obtain that $u(t, x) := e^{it(\mathcal{D}+m\alpha_0)} u_0(x)$ satisfies the Cauchy problem

$$\begin{cases} (\partial^2_{tt} - \Delta + m^2)\mathbb{1}_N u = 0, \\ u(0, x) = u_0(x), \\ \partial_t u(0, x) = i(\mathcal{D} + m\alpha_0) u_0(x). \end{cases}$$

Hence, each component of u satisfies the same Strichartz estimates as for the $n-$dimensional wave or Klein-Gordon equation. Let us briefly recall them, for the sake of completeness. Let u be the solution of the Cauchy problem associated with the massless Dirac equation

$$\begin{cases} i\partial_t u + \mathcal{D}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n \\ u(0,x) = u_0(x). \end{cases}$$

Then, for any (p,q) wave admissible

$$\|u\|_{L^p_t L^q_x} \leq c\|u_0\|_{\dot{H}^s}$$

where $s = n\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}$. Instead, if v is a solution of the Cauchy problem associated with the massive Dirac equation

$$\begin{cases} i\partial_t v + \mathcal{D}v + m\alpha_0 v = 0, \\ v(0,x) = v_0(x). \end{cases}$$

the following estimates hold for any (p,q) Schrödinger admissible

$$\|v\|_{L^p_t L^q_x} \leq c\|v_0\|_{H^s}$$

with $s = (n+1)\big(\frac{1}{2} - \frac{1}{q}\big) - \frac{1}{p}$. We observe that in the latter case the norm of the RHS is the non-homogeneous Sobolev norm. It can be defined for any $\gamma \in \mathbb{R}$ as

$$\|f\|_{H^\gamma} = \|\langle\xi\rangle^\gamma \hat{f}(\xi)\|_{L^2_\xi} = \|\langle\sqrt{-\Delta}\rangle^\gamma f\|_{L^2},$$

where $\langle\cdot\rangle$ denotes the Japanese bracket, $\langle x\rangle := \sqrt{1+x^2}$.

Remark 0.2.1. As we have seen, there exists a strong link between the massless/massive Dirac equation and the wave/Klein-Gordon equation. We should however remark that the Dirac equation is also connected with the Schrödinger equation. Indeed, in the *non-relativistic limit* $c \to +\infty$ it is possible to find solutions of the Dirac equation that resemble suitably rescaled and modulated solutions of the Schrödinger equation and viceversa. In order to understand the link, we present an heuristic argument and we suggest the interested reader to look at [120] (chapter 2, Ex. 2.8) for more details. To simplify the notation we restrict to 3-dimensional case. Let ψ be a four-component spinor solution of equation (15), that is, after reintroducing the constants $c, \hbar$, a solution of

$$i\hbar\partial_t\psi - ic\hbar\alpha\cdot\nabla\psi + mc^2\alpha_0\psi = 0. \tag{16}$$

We consider the rescaled function

$$\psi' = \psi e^{-i\frac{mc^2}{\hbar}t}.$$

The rescaled spinor ψ' is of particular use when evaluating the non-relativistic limit, since it is defined by "subtracting" from the time evolution of ψ the part due to its rest energy

mc^2, so that its time evolution is generated by the kinetic energy operator only. Moreover, we introduce φ, χ to denote its two two-components spinors, i.e. $\psi' = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$. The spinors φ, χ are called respectively large and small components of the Dirac four-component spinor, since, as we now show, in the non-relativistic limit χ becomes negligible with respect to φ. With the new functions, the equation (16) becomes

$$i\hbar\partial_t\varphi - ic\hbar\sigma \cdot \nabla\chi = 0, \tag{17a}$$

$$i\hbar\partial_t\chi - 2mc^2\chi = ic\hbar\sigma \cdot \nabla\varphi, \tag{17b}$$

where $\sigma = (\sigma_1, \sigma_2)$, σ_j $j = 1, 2$ are the Pauli matrices defined above. We now observe that the first term in the LHS of (17b) is negligible compared to the others. Therefore, we can substitute

$$\chi = -i\frac{\hbar}{2mc}\sigma \cdot \nabla\varphi \tag{18}$$

in (17a), obtaining that the two-component spinor satisfies

$$i\hbar\partial_t\varphi - \frac{\hbar^2}{2m}\Delta\varphi = 0.$$

To sum up, heuristically, in the non-relativistic limit the Dirac equation reduces to the Schrödinger equation for the two component spinor wave function φ. Moreover, from (18), we realize that the lower components χ of the Dirac spinor are of sub leading order $O(\frac{1}{c})$ with respect to the upper ones φ and therefore vanish in the non-relativistic limit $c \to +\infty$.

0.2.3 Linear perturbations: electromagnetic potentials

Since the full range of Strichartz estimates are available for the free Dirac equation one may investigate what happens when the equation is perturbed with a potential. That is, one consider the following equation

$$i\partial_t\psi - i\alpha \cdot (\nabla_x - iA(x))\psi + m\alpha_0\psi + V(x)\psi = 0, \quad x \in \mathbb{R}^n \tag{19}$$

where $V(x) \in M_{N\times N}(\mathbb{C})$ Hermitial represents an electric potential and $A(x) = (A^1(x), \dots, A^n(x))$ is the magnetic vector potential. In the physical three dimensional case, the magnetic vector potential A produces a magnetic field B, given by $B = \nabla \times A(x)$. In arbitrary dimension $n \geq 2$, the natural analogue of B is the matrix-valued field $B\colon \mathbb{R}^n \to M_{n\times n}(\mathbb{R})$ defined by

$$B_{ij} = \frac{\partial A^i}{\partial x_j} - \frac{\partial A^j}{\partial x_i}.$$

Lot of works are devoted to the study of the Strichartz estimates, and more in general time decay or local smoothing estimates for these systems, under different choices of the perturbation. See for example [50, 19, 26] for the cases of small potentials and [52, 60, 58, 59, 61] for large potentials. Particularly interesting for the point of view of dispersive analysis appears to be the case of scaling critical potentials. We notice, indeed, that the massless Dirac equation is left invariant under the action of the scaling given by $u_\lambda(t,x) = u(\lambda^{-1}t, \lambda^{-1}x)$, $\lambda > 0$. That is, if u is a solution of the massless (15), then u_λ solves the same equation. It is possible to choose potentials that preserves this invariance. This prevents, typically, the use of perturbative methods in the study of the effect of these potentials, requiring the development of new tools. Among these potentials physically relevant examples are given by the so called Aharonov-Bohm and Coulomb potentials. The Aharonov-Bohm 2D magnetic potential is defined as

$$A\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad A(x) = \alpha\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big), \quad x = (x_1, x_2),$$

where $\alpha \in \mathbb{R}$ is called the magnetic flux, while the Coulomb one is given by

$$V(x) = -\frac{\nu}{|x|}\mathbb{1}, \quad x \in \mathbb{R}^n, \quad n = 2,3, \quad \nu \in \mathbb{R},$$

where $\mathbb{1}$ is the identity matrix. They seem to appear as a natural threshold, in their decays at infinity, for the validity of global in time Strichartz estimates. Indeed, in [51], the authors consider the 3D Dirac equation $i\partial_t u + \mathcal{D}u + V(x)u = 0$ where $V(x) = V(x)^*$ is a 4×4 complex valued matrix, decaying (slightly) faster at infinity with respect to the Coulomb potential, precisely such that

$$|V(x)| \leq \frac{\delta}{|x|(1 + |\log|x||)^\sigma}, \quad \sigma > 1$$

where $\delta \ll 1$, in order to have a self-adjoint operator on $L^2(\mathbb{R}^3)$. They showed that the dispersion of the system is preserved, i.e. the solution u enjoys the same Strichartz estimates, as the free one. The same result can be also extended in the 2-dimensional setting. Instead, if one consider potential decaying slower at infinity, it is possible to construct potentials such that the associated system is no more dispersive (in the sense described above). More precisely, in [3] the authors consider the 3D massless magnetic Dirac equation, i. e. (19) with $m, V(x) = 0$ and they defined the vector field A as

$$A(x) = |x|^{-\delta} M x, \quad 1 < \delta < 2, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the complete set of Strichartz estimates fails. In particular, the mass of the solution is localized around a non-dispersive function, namely a standing wave generated by an eigenfunction of a suitable harmonic oscillator. To complete the picture, we remark that in the range $\delta \leq 1$ the spectrum of $\mathcal{D}_A = -i\alpha \cdot (\nabla_{\mathbb{R}^2} - iA(x))$ is purely discrete so any standing wave $u(t,x) = e^{it\lambda}Q(x)$, where λ is an eigenvalue of $\mathcal{D}_A$ and Q a corresponding eigenfunction is a solution which cannot verify any global Strichartz estimate.

Main contribution of Chapter 1:
Chapter 1 of the present thesis is devoted to the study of generalized Strichartz estimates

for solutions of the massless Dirac-Coulomb equation in dimension $n = 2, 3$. We proved that in both cases the solutions satisfy the following estimates

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L^p_t(\mathbb{R};L^q_{r^{n-1}dr}((0,+\infty);L^2_\theta(\mathbb{S}^{n-1})))} \leq C\|u_0\|_{\dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^n)}$$

where the norm on the RHS is defined with the action of the Dirac-Coulomb operator $\mathcal{D}_\nu := \mathcal{D} - \frac{\nu}{|x|}$:

$$\|f\|_{\dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^n)} = \||\mathcal{D}_\nu|^s f\|_{L^2(\mathbb{R}^n)}$$

and s is as in the free case. The couples (p, q) must satisfy some admissibility conditions which are stated precisely in Theorems 1.1.1 and 1.1.2. We remark that in both cases we require an upper bound for q, $q < q_c$ and in dimension 2 also a lower bound on p, $p_c < p$. Both p_c, q_c are explicit and depend only on the strength of the potential ν. Moreover, this condition can be removed if the initial datum is orthogonal to the singular part of the flow.

Regarding the Aharonov-Bohm field, we mention the work [32] where the authors proved local smoothing estimates and weighted generalized Strichartz estimates for the system. Moreover, very recently in [27] pointwise decay estimates and a full range family of Strichartz estimates are presented. In particular, after a careful analysis for the choice of the distinguished self-adjoint extension to work with $\mathcal{D}_{A,\gamma}$, they proved the following (Corollary 1.1); assume $\alpha \in (0, 1)$, (p, q) wave admissible and in addition that $q < q(\alpha)$ then

$$\|e^{it\mathcal{D}_{A,\gamma}} f\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^2))} \leq c\|f\|_{\dot{H}^s_A(\mathbb{R}^2)}, \quad s = 1 - \frac{1}{p} - \frac{2}{q} \tag{20}$$

where

$$q(\alpha) = \begin{cases} \frac{2}{\alpha} & \text{if } \alpha \in (0, \frac{1}{2}], \\ \frac{2}{1-\alpha} & \text{if } \alpha \in (\frac{1}{2}, 1) \end{cases}$$

and the norm on the RHS is a Sobolev norm adapted to the operator $\mathcal{D}_A$. Moreover, the restriction on q is necessary in the sense that (20) fails if $q \geq q(\alpha)$. However, if one projects onto the regular component of the flow, which in fact does not contain singularities, then the condition on q is no more necessary (Thm 1.2). This result and the one on the Coulomb potential suggest the fact that, as observed in [27] Rmk 1.4, singularities of the generalized eigenfunctions are an obstruction for the validity of Strichartz estimates, namely the stronger is the singularity of the generalized eigenfunctions, the smaller is the range (from the above) for the admissible Strichartz exponents. We would like to investigate it further in the future.

It would be also interesting to study the dispersion of solutions of the Dirac-Coulomb operator when the mass is non zero. In the free case, the presence of the mass modifies the spectrum of the operator, opening a "gap" given by the set $(-m, m)$. Moreover, as we saw before, it connects the Dirac operator to the Klein-Gordon operator rather then wave one, for which different, in terms of loss of admissibility conditions on the indexes, Strichartz estimates hold. We dedicate the last Section of Chapter 1 to this topic, for which we have some preliminary results on the validity of Strichartz estimates. As we will see, in the presence of the Coulomb potential the point spectrum of the operator is no more empty and it is contained in $(-m, m)$. Therefore, there exist eigenfunctions associated with eigenvalues in that range. These are not dispersive functions, by definition. That is why we consider spinors projected on the essential spectrum of the operator. Moreover, we have started by restricting to "radial" initial data, with the hope to be able to extend the results by refining the techniques.

0.2.4 The Dirac equation in curved backgrounds

In this section we describe how the definition of the Dirac equation can be extended in order to be adapted to curved spacetimes. We remark that this construction is now classical, but it required the introduction of new objects, because of the heavy algebraic structure of the Dirac operator. To lighten the notation, we perform the construction in dimension $n = 3$, however it holds in every dimension.

Let us first rewrite the equation (15) in a more compact way; we define the following matrices

$$\gamma^0 := \alpha_0, \qquad \gamma^j := \alpha_0\alpha_j, \quad j = 1, 2, 3.$$

Therefore, by multiplying (15) by α_0 and recalling that $\alpha_0^2 = \mathbb{1}$ we get the equivalent formulation

$$i\gamma^j\partial_j\psi - m\psi = 0.$$

Notice that we use here the Einstein notation for the sum on same indexes, i. e. $\gamma^j\partial_j = \sum_{j=0}^{3}\gamma^j\partial_j$, with $\partial_0 := -\partial_t$. We observe that this new matrices satisfy the following anticommutation relation

$$\{\gamma^i, \gamma^j\} = -2m^{ij}\mathbb{1},$$

where m^{ij} is the inverse of the Minkowski metric defined as $m = \mathrm{diag}(-1, 1, 1, 1)$. Let now $(\mathcal{M}, g)$ be a Lorentzian manifold endowed with a spin structure[2]. The introduction of the Dirac equation in curved space is due to Weyl [125] and Fock [66] in 1929. We remark however that this formalism was already introduced by Cartan in 1913 in his studies of the matrix representations of orthogonal groups and others. We refer the reader with historical interests to [84] and the references therein.
The starting point was to look for some matrices γ^μ such that

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = -2g^{\mu\nu}(x)\mathbb{1} \tag{21}$$

where $g^{\mu\nu}$ is the local inverse of the metric g. Because the RHS depends on x, the objects γ^μ on the LHS also depend on x. One can then expand $\gamma^\mu(x)$ in terms of the constant Dirac matrices γ^j of the flat spaces as follows

$$\gamma^\mu(x) = \gamma^j e_j^\mu(x). \tag{22}$$

[2] details in the Appendix A.

The matrices $e_j^\mu(x)$ are called the (inverse) *vierbein* (or in general *n-bein*) fields. Substitution of (22) into (21) shows that the metric is the product of two vierbeins, or that the vierbein fields e_j^μ are the square root of the metric

$$e_j^\mu m^{ij} e_i^\nu = g^{\mu\nu}.$$

Defining e_μ^j as the matrix inverse of e_j^μ (so that $e_\mu^j e_j^\nu = \delta_\mu^\nu$ and $e_j^\nu e_\nu^i = \delta_j^i$) we also have

$$g_{\mu\nu} = e_\nu^j e_\nu^i m_{ij}.$$

We observe that the choice of the vierbein is not unique. Indeed, if e changes into e' by a local Lorentz transform L, that is $(e')_\mu^a = L_b^a e_\mu^b$, the fact that L belongs locally to $O(1,3)$ ensures that $(e')_a^\mu m^{ab} (e')_b^\nu = g^{\mu\nu}$. Notice that we use the latin indexes when referring to the flat case, i. e. to the Minkowski background and the greek indexes for the curved setting. In particular, we raise (lower) latin and greek indexes multiplying respectively by m^{ij} (m_{ij}) and $g^{\mu\nu}$ ($g_{\mu\nu}$). Then, the Dirac operator is defined as

$$\mathcal{D}_\mathcal{M} = -i\gamma^\mu D_\mu \tag{23}$$

where D_μ is the covariant derivative acting on spinor fields. It is defined as

$$D_\mu = \partial_\mu + i\omega_\mu^{ab}\Sigma_{ab}$$

where ω_μ^{ab} is a purely geometric factor called the *spin connection* and Σ_{ab} is a purely algebraic factor depending only on the algebraic structure of Dirac spinors. This algebraic factor is defined as

$$\Sigma_{ab} = -\frac{i}{8}[\gamma_a, \gamma_b]$$

where the γ matrices are the ones defined in (22). The spin connection takes the form

$$\omega_\mu^{ab} = e_\nu^a(\partial_\mu e^{\nu b} + \Gamma_{\mu\sigma}^\nu e^{\sigma b})$$

where $\Gamma_{\mu\sigma}^\nu$ denotes the usual Christoffel symbol for the metric connection. It is possible to show (see [107], section 5.6) that the Dirac equation, defined as

$$i\gamma^\mu(x) D_\mu\psi + m\psi = 0, \quad m \geq 0 \tag{24}$$

is covariant. Lastly, we recall the Schrödinger-Lichnerowicz formula (see [114]), which allow to compute the “square” of the Dirac operator. This tool is, as we saw in the flat case, very useful in the study of the dynamics of the system. Therefore the squared equation becomes

$$D^\mu D_\mu \psi + m^2\psi + \frac{1}{4}\mathcal{R}_g\psi = 0 \tag{25}$$

where $\mathcal{R}_g$ is the scalar curvature associated with the metric g. More details on the first term are given in the following subsection.

Time and space decoupled

We restrict now to the case where time and space are decoupled. That is, we take the metrics $g_{\mu\nu}$ with the following structure

$$g_{\mu\nu} = \begin{cases} -1 & \text{if } \mu = \nu = 0, \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu, \\ h_{\mu\nu}(x) & \text{otherwise.} \end{cases}$$

It is proved (see [30], Section 2) that if time and space are decorrelated in the metrics, then they also are in the Dirac equation. The idea is that if f_a^i is a dreibein, i.e.n-bein with $n = 3$, hence satisfying

$$h^{ij} = f_a^i \delta^{ab} f_b^j \tag{26}$$

then the matrix e_a^μ defined as

$$e_a^\mu = \begin{cases} 1 & \text{if } \mu = a = 0 \\ 0 & \text{if } \mu a = 0 \text{ and } \mu \neq a \\ f_a^\mu & \text{otherwise.} \end{cases}$$

is a vierbein for g. Notice that in (26) a and b are taken only between 1 and 3. Moreover, both the Christoffel symbols and the spin connections for g can be defined in terms of the corresponding terms for h. The Dirac operator becomes

$$\mathcal{D} = i\gamma^0 \partial_0 + i\gamma^a f_a^j D_j.$$

Therefore the Dirac equation reads as

$$i\partial_t \psi + i\gamma^0 \gamma^a f_a^j D_j \psi + m\gamma^0 \psi = 0.$$

We notice that the squared equation (25) becomes

$$\partial_{tt}^2 \psi - \Delta^S \psi + \frac{1}{4} \mathcal{R}_h \psi + m^2 \psi = 0,$$

where $\mathcal{R}_h$ is the scalar curvature associated with the metric h. It is important to stress the fact that Δ^S is not the Laplace-Beltrami operator, but the *spinorial laplacian*. It can be expanded in terms of the Laplace-Beltrami operator, denoted Δ_h:

$$\Delta^S = \Delta_h - \Omega_1 - \Omega_2$$

where Ω_j, $j = 1, 2$ are terms of order, respectively one and zero:

$$\Omega_1 = 2B^\mu \partial_\mu, \quad \Omega_2 = -\partial^\mu B_\mu + B_\mu B^\mu - \Gamma_\nu^{\mu\nu} B_\mu, \tag{27}$$

where B_μ is such that $D_\mu = \partial_\mu + B_\mu$. This means that it is not possible to apply effortless, as in the flat case, the available results for the wave/Klein-Gordon equations to the

Dirac setting.

The study of dispersive equations in a non flat setting is a topic that has attracted significant interest in the last years. The literature concerning the Schrödinger, wave or Klein-Gordon equations is extensive and to provide a complete picture of it is not the aim of this thesis. Much less is know about the dynamics of the Dirac spinors on curved backgrounds. We mention however that this is nowadays a very active field of research, motivated for example by the study of interactions on spin-$\frac{1}{2}$ particles with gravitational fields, described by the coupling of the Dirac equation with the Einstein equation.
Let us now present some recent results concerning the validity of Strichartz estimates under different choices of the Riemanian metrics. We remark that it is not possible to rely on the classical Duhamel argument in order to obtain Strichartz estimates for the flow, due to the fact that, even in the asymptotically flat case, the perturbative term can not be regarded as a zero-order perturbation of the flat dynamics.
In the following, we will consider settings where time and space are decoupled. Therefore, we take $(t, x) \in \mathbb{R} \times \Sigma$ where (Σ, h) is a *complete* manifold of dimension n. The completeness of the manifold ensures that the Dirac operator is self-adjoint, see [44]. Moreover, we adopt the following notations; we denote as $\mathcal{D}$ the Dirac operator on Σ, which is defined in terms of the $n-$beins as described in the previous section. We use $L^p(\Sigma)$, $H^s(\Sigma)$, $\dot{H}^s(\Sigma)$, $W^{p,q}(\Sigma)$ to denote the Lebesgue and homogeneous/non-homogeneous Sobolev spaces on the spatial manifold (Σ, h). In particular, the norm $L^p(\Sigma)$ is given by

$$\|f\|_{L^p(\Sigma)}^p := \int |f(x)|^p \sqrt{\det(h(x))}dx, \quad \forall p \in [1, +\infty)$$

and similarly for $p = +\infty$. The space $\dot{H}^1(\Sigma)$ is induced by the norm

$$\|f\|_{\dot{H}^1(\Sigma)}^2 := \|\sqrt{h^{ij}\langle D_i f, D_j f\rangle_{\mathbb{C}^N}}\|_{L^2(\Sigma)}$$

where the D_j are the covariant derivatives for spinors. More in general, the space $W^{1,p}(\Sigma)$, $p \in [1, +\infty]$ is induced by

$$\|f\|_{W^{1,p}(\Sigma)} = \|\sqrt{h^{ij}\langle D_i f, D_j f\rangle_{\mathbb{C}^N}}\|_{L^p(\Sigma)} + \|f\|_{L^p(\Sigma)}.$$

The spaces $\dot{H}^s(\Sigma)$ and $W^{s,p}(\Sigma)$ with $s \in [-1, 1]$ are then defined by interpolation and duality. We remark that, in the case of asymptotically flat or compact manifolds (see respectively [31], Appendix A and [28], Section 3.2) these norms are equivalent to the ones defined with the action of the Laplace-Beltrami operator.

Spherically symmetric manifolds. In [10] the authors consider manifolds $\Sigma = \mathbb{R}_r^+ \times \mathbb{S}_{\theta,\phi}^2$ equipped with the Riemannian metrics

$$d\sigma = dr^2 + \varphi(r)^2 d\omega_{\mathbb{S}^{n-1}}^2$$

where $d\omega^2_{\mathbb{S}^2} = (d\theta^2 + \sin^2\theta d\phi^2)$ is the Euclidean metric on the 2D sphere $\mathbb{S}^2$. Let $\varphi \in C^\infty(\mathbb{R}^+)$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$, and for all $k \in \mathbb{N}$, $\varphi^{(2k)}(0) = 0$. We assume that there exists $\varphi_1 \in C^\infty(\mathbb{R}^+)$ non-negative such that

$$\varphi\colon r \mapsto r(1 + \varphi_1(r))$$

and

$$\sup_{r\geq 0}(|\varphi_1(r)| + |r\varphi_1(r)'| + |r^2\varphi_1''(r)|) \ll 1.$$

Then, they proved the validity of the following family of Strichartz estimates; let (p, q) wave or Schrödinger admissible if respectively $m = 0$ or $m > 0$ and $a, b \geq 0$. Assume either when $m = 0$, $n = 3$, that $\frac{5}{pb} + \frac{1}{2a} < 1$ or when $m \neq 0$ or $n > 3$ that $\frac{5}{pb} + \frac{1}{2a} \leq 1$. Then the solutions of the Dirac equation with initial data $u_0 \in H^{a,b}(\Sigma)$ satisfy

$$\left\| \left(\frac{\varphi(r)}{r} \right)^{\frac{n-1}{2}(1-\frac{2}{q})} u \right\|_{L^p_t(\mathbb{R}, W^{\frac{1}{q}-\frac{1}{p},q}(\Sigma))} \leq c\|u_0\|_{H^{a,b}(\Sigma)}.$$

The norm $H^{a,b}(\Sigma)$ is defined as

$$\|f\|_{H^{a,b}(\Sigma)} = \left(\|f\|^2_{H^a(\Sigma)} + \|(-\Delta_{\mathbb{S}^{n-1}})^{\frac{b}{2}} f\|^2_{L^2(\Sigma)} \right)^{\frac{1}{2}}.$$

which can be seen as a standard Sobolev norm with additional angular derivatives.
In the proof, strongly inspired by [7], the authors exploit the symmetry and decompose of the Dirac operator in a in a sum of radial operators. Note that this can be done if one replaces $\mathbb{S}^{n-1}$ with a smooth compact manifold (Thm 5.27 in [81]). Focusing on one level, they squared the radial operator in order to reduce to a system of Klein-Gordon equations and then, via Kato smoothing arguments, rely on the existing theory for this dynamics. Paley-Littlewood theory on the sphere is used to sum back the estimates obtained on each radial level. We remark that within this setting, in [29] local-in-time, weighted Strichartz estimates for the Dirac dynamics were proved, under some more general assumptions on the function φ. Also in that case the main strategy consisted in exploiting the spherical symmetry of the space in order to separate variables and to reduce the problem to a "sum" of much easier radial equations that could be regarded, after introducing weighted bispinors, as Dirac equations on the flat space perturbed with potentials, for which several results are available. Nevertheless, global in time Strichartz estimates turned out to be out of reach, the main problem being the lack of existence of dispersive estimates for the Dirac equation with scaling critical potentials in the Euclidean setting.
We notice that, the standard loss of derivatives would be recovered taking $a = \frac{1}{2}$ and $b = 0$. However this choice is excluded and this is due to technicalities, based on Paley-Littlewood arguments, of the proof. The estimates are indeed sharp at the 'radial' level.

Asymptotically flat manifolds. We now consider the special case of asymptotically flat manifolds, referring to [30, 31]. Let $h \in C^\infty(\mathbb{R}^3)$. Assume that there exists a constant

C_h small enough and $\sigma \in (0,1)$ such that for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq 3$ and all x,

$$|\partial^\alpha (h_{ij}(x) - \delta_{ij})| \leq C_h \langle x \rangle^{-|\alpha|-1-\sigma}$$

where $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$. We refer to [31] (Rmk 1.1) for the optimality of the required decay. Then, the massless Dirac flow satisfies the Strichartz estimates

$$\|e^{it\mathcal{D}} u_0\|_{L^p(\mathbb{R}, \dot{H}_q^{1-s}(\Sigma))} \lesssim \|u_0\|_{\dot{H}^1(\Sigma)}$$

for all wave admissible triple (s,p,q), while in the massive case we have

$$\|e^{it\mathcal{D}_m} u_0\|_{L^p(\mathbb{R}, H_r^{\frac{1}{2}-s}(\Sigma))} \lesssim \|u_0\|_{H^1(\Sigma)}$$

for all Schrödinger admissible triple (s,p,q) such that $p > 2$.
In this case, no symmetries of the manifolds can be exploited. Therefore, the strategy consisted in squaring the equation in order to represent the solution as

$$u = e^{it\mathcal{D}_m} = \dot{W}_m(t) u_0 + i W_m(t) \mathcal{D}_m u_0 + \int_0^t W_m(t-s)(\Omega_1(u)(s) + \Omega_2 u(s)) ds \qquad (28)$$

where

$$W_m(t) = \frac{\sin(t\sqrt{m^2 - \Delta_h})}{\sqrt{m^2 - \Delta_h}}, \quad \dot{W}_m = \partial_t W_m,$$

where Δ_h is the Laplace-Beltrami operator on (Σ, h) and Ω_j $j = 1,2$ are defined in (27). Then combine standard local smoothing estimates for the wave/KG equations with local smoothing estimates for Dirac (proved by the same authors in [30]). Let us observe that in the massless case no additional loss of derivatives, compared to the flat case, is required.

Remark 0.2.2. Let us mention that physically interesting models among the spherically symmetric manifolds can be found studying the dynamics of spinors in the outer regions of black holes. We recall that a spherically symmetric black hole can be described as a 4−dimensional manifold in the form $M = \mathbb{R}_t \times \mathbb{R}_r \times \mathbb{S}^2$ equipped with a Lorentzian metric

$$dg = F(r)^2 dt^2 - (F(r))^{-1} dr^2 - r^2 d\omega_{\mathbb{S}^2}^2.$$

The nature of the black hole will dictate the choice of the function $F(r)$. We refer, e.g., to [103] for the Schwarzchild, [95, 54] for the Reissener-Nördstrom and [76, 65] for the Kerr black holes. It is seen that both in the Schwarzschild and in the Reissner-Nordrström cases in the Regge-Wheeler variable (see the mentioned articles) the problem is reduced to the analysis on manifolds with an asymptotically flat and an asymptotically hyperbolic end. It would be interesting to focus on the study of the dispersive estimates in the asymptotically hyperbolic case, for which only partial results are available and combine them with the above presented results to treat this kind of models.

Main contribution of Chapter 2:
Compact manifolds without boundary. In [28], on which Chapter 2 of this work is based, we considered Σ to be a smooth compact Riemannian manifold without boundary of dimension $n \geq 2$. We proved that the following Strichartz estimates hold; let $I \subset \mathbb{R}$ be a bounded interval. Then, for any $m \geq 0$ and for any wave admissible pair (p, q) , we have

$$\|e^{it\mathcal{D}_m} u_0\|_{L^p(I, L^q(\Sigma))} \leq C(I) \|u_0\|_{H^s(\Sigma)},$$

where s is as for the wave equation on $\mathbb{R}^n$, while for any Schrödinger admissible pair (p, q), we have

$$\|e^{it\mathcal{D}_m} u_0\|_{L^p(I, L^q(\Sigma))} \leq C(I) \|u_0\|_{H^{s+\frac{1}{2p}}(\Sigma)},$$

where s is as for the Klein-Gordon equation on $\mathbb{R}^n$. Notice that in these case the estimates are local in time. We recall that the spectrum of the Dirac operator on compact manifolds coincides with the point spectrum and it is a discrete set, see [68] Section 4.2. Therefore one cannot hope to have global-in-time estimates satisfied by every solutions. The proof, as the previous case, relies on the representation (28) of the solutions of the Dirac equation. We remark that thanks to the smoothness and compactness of the manifold, the coefficients $\Omega_{1,2} \in C_c^\infty(\mathbb{R}^n)$, locally. Therefore the perturbative terms can be easily handle once one has Strichartz estimates for the half wave/Klein-Gordon propagators. These were not presented in the literature (at least for the massive case). The study of these estimates, to which Section 2.2 of this manuscript is devoted, represent indeed the most difficult part of the proof and an interesting result per se in the context of Strichartz estimates for dispersive equations on compact spaces.
We observe that we have a loss of $\frac{1}{2p}$ derivatives if we take the Schrödinger admissible exponents, while no loss is required in the wave case. We recall that also in [23], for the Schrödinger equation a loss of $\frac{1}{p}$ derivatives has been observed. They proved that the estimate is sharp for $p = 2$ in the case of the spheres. We obtain the same result for any sphere $\mathbb{S}^n$, $n \geq 4$. However, as observed in [7], the Strichartz estimates may also be related to the sign of the curvature of the manifold. Indeed, on compact manifolds with other types of geometries, the estimates for the Schrödinger flow can be improved, see e.g., [18, 55] for the torus and [15, 77] for general compact manifolds with non-positive sectional curvatures. Improvements were also shown for the wave equation on (non-compact) hyperbolic spaces. We refer to [2] and the references therein. It would be interesting to investigate this behavior for the Dirac equation in future works.

0.3 Nonlinear applications

The Strichartz estimates are widely used in the study of local and global well posedness and scattering results for nonlinear systems. Classical references of this topic for the Schrödinger equation are given by [39, 90]. Roughly speaking, nonlinearities naturally appear when describing interaction among particles or an external force acting on the systems. The first natural problem one has to face concerns the (local) well posedness of the system in some functional space. Then, one standard way to proceed is to construct a solution map and find complete metric spaces where to perform a fixed point argument. Very often, this metric spaces are the ones for which some Strichartz estimates for the linear flow hold. With this idea in mind, in Section 1.4, we prove local well posedness in $\dot{H}^{\frac{1}{p}}(\mathbb{R}^3)$ of the system with an Hartree-type nonlinearity

$$\begin{cases} i\partial_t u + \mathcal{D}_\nu u = \big(\omega * \langle \beta u, u\rangle\big)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3 \\ u(x,0) = u_0(x), \end{cases}$$

for a special class on initial data. Here, $\mathcal{D}_\nu$ is the Dirac-Coulomb operator, $\beta = \mathrm{diag}(1,1,-1-1)$ and $\omega \in L^p(\mathbb{R}^3)$ is a radially symmetric function. We refer the reader to Section 1.4 for a discussion on the choice of the initial data and the potential ω. We recall however that the case $\omega = \delta_0$, excluded here, would recover the cubic Dirac equation. We refer to the seminal works [62] and [9] of for the study of global well posedness for the cubic equation, without potentials, in $H^s(\mathbb{R}^3)$ respectively in the subcritical and critical regimes.

Main contribution of Chapter 3:
Moreover, in Chapter 3 Strichartz estimates are a key tool to prove stability of non-localized equilibria, via scattering, for the Hartree-Fock equation, in its formulation for random fields:

$$\begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}[|X|^2])X - \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{X(y)}X(x)]X(y)\,dy, \\ X(t=0) = X_0. \end{cases} \tag{29}$$

where $X : [0,T]\times\mathbb{R}^d\times\Omega \mapsto \mathbb{C}$ is a random field defined over a probability space $(\Omega,\mathcal{A},\mathbb{P})$, w is an even pairwise interaction potential, $\mathbb{E}$ denotes the expectation on Ω. This equation, whose formal derivation is presented in Section 3.1.3, models the dynamics of a system of many, possibly infinite, interacting fermions in a mean field limit. The equation admits non-localized equilibria Y_f (see Section 3.1.2 for the definition). We consider small perturbations of these equilibria with initial data

$$X_0 = Y_f + Z_0 \tag{30}$$

and we look at asymptotic stability of the equilibrium via scattering to linearized waves. These are given by the free evolution

$$S(t)Z(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(\eta\cdot x-\theta(\eta))}\hat{Z}(\eta)d\eta.$$

where θ satisfies some regularity and ellipticity assumption. We prove that the linear flow disperses, in terms of time decay and Strichartz estimates, as the Schrödinger linear flow. Then, using perturbative techniques and choosing carefully the functional set up, we show that (this is the content of Theorem 3.1.1) for $d \geq 4$ under smallness hypothesis on the equilibrium and interaction potential w, the Cauchy problem (29) with initial data (30) has a solution $X \in Y_f + \mathcal{C}(\mathbb{R}, L^2(\Omega, H^{s_c}(\mathbb{R}^d)))$. Moreover, there exists $Z_\pm \in L^2(\Omega, H^{s_c}(\mathbb{R}^d))$ such that

$$X(t) = Y_f(t) + S(t)Z_\pm + o_{L^2(\Omega, H^{s_c}(\mathbb{R}^d))}(1), \qquad \text{as } t \to \pm\infty.$$

Observe that the exponent of Sobolev space $H^{s_c}(\mathbb{R}^d)$ is the critical Sobolev regularity for the cubic Schrödinger equation in dimension d, that is $s_c = \frac{d}{2} - 1$.

Chapter 1

Strichartz estimates for the Dirac-Coulomb equation

1.1 Introduction

As explained in the Introduction, this chapter is concerned with the Cauchy problem associated with the massless Dirac equation with an electric Coulomb potential in 2 and 3 spatial dimensions. For the sake of completeness, we recall it. It reads as

$$\begin{cases} i\partial_t u + \mathcal{D}_n u - \frac{\nu}{|x|} u = 0, \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

where $u(t,x)\colon \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{C}^N$, $n \in \{2,3\}$, $N = 2^{\lceil \frac{n}{2} \rceil}$, $\nu \in \big[-\frac{n-1}{2}, \frac{n-1}{2} \big]$ and $\mathcal{D}_n$ is the Dirac operator on $\mathbb{R}^n$. As described in Section 0.2.1, it is defined on $\mathbb{R}^2$ as

$$\mathcal{D}_2 = -i\sigma \cdot \nabla_{\mathbb{R}^2} = -i(\sigma_1 \partial_x + \sigma_2 \partial_y)$$

and on $\mathbb{R}^3$ as

$$\mathcal{D}_3 = -i\alpha \cdot \nabla_{\mathbb{R}^3} = -i \sum_{j=1}^{3} \alpha_j \partial_j,$$

where $\{\sigma_j\}_{j=1,2}$ are the Pauli matrices and $\{\alpha_j\}_{j=1,2,3}$ are the Dirac matrices. Let us emphasize that we take the Coulomb potential in dimension 2 to be $\frac{1}{|x|}$ and not $-\log|x|$

The massless Dirac equation is widely used to describe physical systems from Quantum Mechanics; the 3D equation is a model for the dynamics of massless fermions, such as the neutrinos. The 2D equation appears in the study of propagation of waves spectrally concentrated near some singular points on 2-dimensional honeycomb structures. We remark that among the materials which enjoy this structure one finds the *graphene*, a single-layer sheet of hexagonally-arranged carbon atoms, that is attracting a lot of interest in the recent years due to its countless technological applications (see [64] and references therein for a survey). Notice that also with the Coulomb potential, describing interactions between particles, the system remains physically interesting (e.g., non-perfect graphene, see [38]).

The restriction on the parameter ν comes from the fact that, according to Quantum Mechanics, we should work with self-adjoint operators on $L^2(\mathbb{R}^n;\mathbb{C}^N)$. We recall that $\mathcal{D}_n - \frac{\nu}{|x|}$, defined on $C_0^\infty(\mathbb{R}^n\backslash\{0\};\mathbb{C}^N)$, is essentially self-adjoint[1] with domain $H^1(\mathbb{R}^n;\mathbb{C}^N)$ if and only if $n=2$ and $\nu=0$ or $n=3$ and $|\nu| < \frac{\sqrt{3}}{2}$. If $n=3$ and $|\nu| = \frac{\sqrt{3}}{2}$ the Dirac-Coulomb operator is still essentially self-adjoint but with domain contained in $H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^4)$. For any other values of ν there exist infinitely many different self-adjoint extensions. However, it has been shown that for ν in the range we consider it is possible to define a distinguished (i. e. "physically relevant") self-adjoint extension. In particular, for $|\nu| < \frac{n-1}{2}$ one can choose the self-adjoint extension with domain contained in $H^{\frac{1}{2}}(\mathbb{R}^n;\mathbb{C}^N)$ (see the recent works [63], [100] and references therein).

In this Chapter, we are interested in the study of the above mentioned systems from the point of view of the dispersive analysis. We mentioned in Section 0.1.2 the Strichartz estimates as a tool to quantify the dispersion of a system and we recalled in Section 0.2.2 the ones that hold for the massless (and massive) Dirac equation. However, it has been observed (see [80] for a survey and the references therein) that one can enlarge the set of admissible couples, requiring some additional integrability in the angular variable. For the free Dirac equation one has the following *generalized Strichartz estimates*

$$\|e^{it\mathcal{D}_n}u_0\|_{L^p_t L^q_{|x|} L^2_\theta(\mathbb{R}\times\mathbb{R}^n)} \leq C\|u_0\|_{\dot{H}^{\frac{n}{2}-\frac{n}{q}-\frac{1}{p}}(\mathbb{R}^n)} \tag{1.2}$$

for (p,q) satisfying

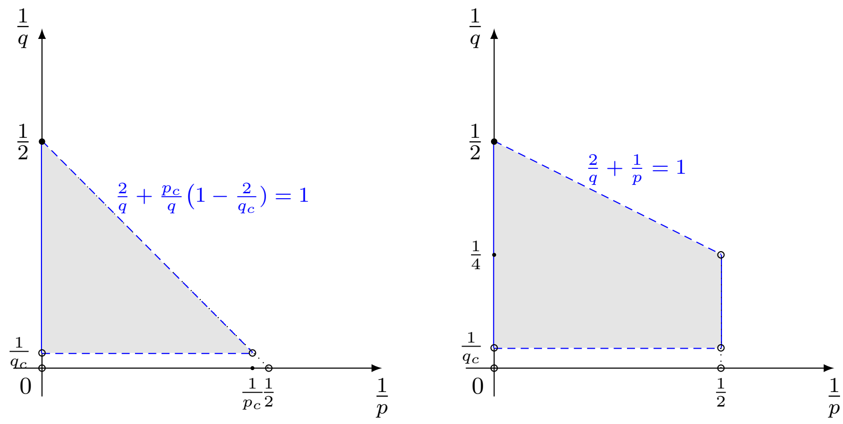
$$p,q\in[2,+\infty], \quad \frac{n-1}{q}+\frac{1}{p}<\frac{n-1}{2} \text{ or } (p,q)=(\infty,2), \quad (p,q)\neq(2,\infty),(\infty,\infty).$$

The main results we present here are concerned with this kind of estimates for the system perturbed with the Coulomb potential. We remark that the analysis of the dynamics of this systems has been started in [33]. The authors showed the validity of the following local smoothing type estimates for solutions u of (1.1)

$$\left\| |x|^{-\alpha} \left|\mathcal{D}_n - \tfrac{\nu}{|x|}\right|^{\frac{1}{2}-\alpha} u \right\|_{L^2_t L^2_x} \leq C\|u_0\|_{L^2} \tag{1.3}$$

where $\frac{1}{2} < \alpha < \sqrt{k_n^2-\nu^2}+\frac{1}{2}$ and $k_n = \frac{n-1}{2}$. However, the case $\alpha = \frac{1}{2}$ is excluded, therefore it does not allow to deduce Strichartz estimates using the standard Duhamel formulation and the combination of it with the Strichartz estimates for the free flow. Then, in [34], the authors proved asymptotic estimates for the generalized eigenfunctions of the Dirac-Coulomb operator on $\mathbb{R}^3$ and, as an application, some generalized Strichartz estimates for the solutions of (1.1) on $\mathbb{R}^3$.

[1] That is, it admits only one self-adjoint extension, its closure.



Case $n = 2$, $\nu = \frac{1}{4}$. Case $n = 3$, $\nu = \frac{1}{2}$.

Figure 1.1: Generalized Strichartz estimates

The aim of this presentation is two-folded; firstly, we continue the analysis of the dispersion of (1.1). We extend, compared to the result in [34], the set of admissible couples for the validity of generalized Strichartz estimates in 3D, we prove similar estimate for the 2D system and we also provide new local smoothing estimates. Then, we apply the obtained results to the study of local well posedness of nonlinear systems. Before we state the results we introduce the following notations that will be used throughout the paper.

Notations. We denote with $\mathcal{D}_{n,\nu}$ the Dirac-Coulomb operator acting on $\mathbb{R}^n$, that is $\mathcal{D}_{n,\nu} := \mathcal{D}_n - \frac{\nu}{|x|}$ for $n = 2, 3$ and, with an abuse of notation, we will omit the index n when will be clear from the context.

We denote with $\dot{H}^s$, $s \in \mathbb{R}$, the standard homogeneous Sobolev spaces with the norm $\|u\|_{\dot{H}^s} = \big\||D|^s u\big\|_{L^2}$ where $|D| = \sqrt{-\Delta}$. Instead we use $\dot{H}^s_{\mathcal{D}_\nu}$ to denote the homogeneous Sobolev spaces induced by the action of the Dirac-Coulomb operator, i.e., with norm $\|u\|_{\dot{H}^s_{\mathcal{D}_\nu}} = \big\||\mathcal{D}_\nu|^s u\big\|_{L^2}$.

We say that $u \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ is *Dirac-radial* if it coincides with his projection on the first partial wave subspaces. Then, we call $u \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ *Dirac-non radial* if it is orthogonal to Dirac-radial functions. We postpone to Subsection 1.2.1 (Remark 1.2.1) the precise definitions.
With an abuse of notation, we use the term function to refer to both scalars and vector-valued functions. Their nature will be clear from the context.

Our first main result concerns Strichartz estimates with an additional angular regularity for solutions of (1.1), as explained above. As we will see we will have some technical conditions that will force us to slightly restrict the set of admissible ν. Moreover, with respect to the free case, we obtain a smaller set of admissibility for the indexes p, q (as shown in Figure 1.1). However we can recover the classical range (9) in two cases: if $\nu = 0$[2]or, for all $\nu \in \left[-\frac{n-1}{2}, \frac{n-1}{2}\right]$, if the initial datum is Dirac-non radial.The reason behind this additional restriction on the indexes will appear clearly in the proofs of theorems 1.1.1 and 1.1.2; it is due to the behavior near the origin of the "first" generalized eigenfunctions. We separate the cases $n = 2$ and $n = 3$ in order to lighten the notations. We have the following

Theorem 1.1.1 (Strichartz estimates 2D)**.** *Let $|\nu| < \frac{\sqrt{24}}{10}$ and (p, q) such that*

$$p_c < p \leq +\infty, \quad 2 \leq q < q_c, \quad \frac{2}{q} + \frac{p_c}{p}\left(1 - \frac{2}{q_c}\right) < 1 \text{ or } (p, q) = (\infty, 2), \tag{1.4}$$

where $q_c = \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}$, $p_c = \frac{\gamma_{\frac{1}{2}} + \frac{1}{2}}{\gamma_{\frac{1}{2}}}$ and $\gamma_{\frac{1}{2}} = \sqrt{\frac{1}{4} - \nu^2}$. Then, there exists a constant $C > 0$ such that for any $u_0 \in \dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^2; \mathbb{C}^2)$ the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L^p_t L^q_{rdr} L^2_\theta} \leq C \|u_0\|_{\dot{H}^s_{\mathcal{D}_\nu}} \tag{1.5}$$

provided $s = 1 - \frac{1}{p} - \frac{2}{q}$.
Moreover, if u_0 is Dirac-non radial, then the Strichartz estimates hold for all $|\nu| \leq \frac{1}{2}$ and (p, q) satisfying the admissibility condition (1.4) with $(p_c, q_c) = (2, +\infty)$, $q = q_c$ included.

Theorem 1.1.2 (Strichartz estimates 3D)**.** *Let $|\nu| < \frac{\sqrt{15}}{4}$ and (p, q) such that*

$$2 \leq p \leq +\infty, \quad 2 \leq q < q_c, \quad \frac{2}{q} + \frac{1}{p} < 1 \text{ or } (p, q) = (\infty, 2), \tag{1.6}$$

where $q_c = \frac{3}{1 - \sqrt{1 - \nu^2}}$. Then, there exists a constant $C > 0$ such that for any $u_0 \in \dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^3; \mathbb{C}^4)$, the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L^p_t L^q_{r^2dr} L^2_\theta} \leq C \|u_0\|_{\dot{H}^s_{\mathcal{D}_\nu}} \tag{1.7}$$

provided $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$.
Moreover, if u_0 is Dirac-non radial, then the Strichartz estimates hold for all $|\nu| \leq 1$ and (p, q) satisfying the admissibility condition (1.6) with $q = q_c = +\infty$, included.

Observe that the admissibility condition for ν is slightly smaller than the one for which the Dirac-Coulomb operator has a distinguished self-adjoint extension. We believe that this is due to technicalities and could be enlarged using refined estimates.
We describe in Figure 1.1 the admissible range (region in grey) in the cases $n = 2$, $\nu = \frac{1}{4}$ and $n = 3$, $\nu = \frac{1}{2}$. Notice that if $\nu = 0$ we can extend the region up to the segments $\overline{0P}$ where $P = \left(\frac{1}{2}, 0\right)$, reaching the classical range.

[2]except for the segment corresponding to $q = +\infty$.

Remark 1.1.3. Observe that in the 2D case the norm in the R. H. S. is the one induced by the Dirac-Coulomb operator and it is not, in general, equivalent to the standard Sobolev norm. A more detailed discussion is to be found in Subsection 1.2.4.

Remark 1.1.4. We notice that from the Strichartz estimates (1.5), (1.7) it is possible to deduce standard Strichartz estimates with an additional loss of angular derivatives on u_0. The idea is to combine the estimates above with Sobolev embeddings on the sphere of dimension $n-1$. We denote with Λ_θ^s the angular derivative operator, which is defined in terms of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$

$$\Lambda_\theta^s = (1 - \Delta_{\mathbb{S}^{n-1}})^{\frac{s}{2}}.$$

This operator does not commute with the Dirac operator $\mathcal{D}_n$. However, it has been observed in [26] (see formula (2.45))[3] that it is possible to define a modified operator $\tilde{\Lambda}_\theta^s$ commuting with $\mathcal{D}_n$ and such that

$$\|\tilde{\Lambda}_\theta^s f\|_{L^2_\theta} \simeq \|\Lambda_\theta^s f\|_{L^2_\theta}.$$

Then, by Sobolev embeddings $H^\sigma_\theta(\mathbb{S}^{n-1}) \hookrightarrow L^q_\theta(\mathbb{S}^{n-1})$, we get

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L^p_t L^q_x} \lesssim \|\Lambda_\theta^\sigma e^{it\mathcal{D}} u_0\|_{L^p_t L^q_{r^{n-1}dr}} \simeq \|e^{it\mathcal{D}} \tilde{\Lambda}_\theta^\sigma u_0\|_{L^p_t L^q_{r^{n-1}dr}} \lesssim \|\tilde{\Lambda}_\theta^\sigma u_0\|_{\dot{H}_s}$$

where $\sigma = \frac{n-1}{2} - \frac{n-1}{q}$, $0 < s < \frac{n-1}{2}$ and ν, (p, q) as in Theorems 1.1.1, 1.1.2.

The strategy developed in order to prove the above results is a refinement of the one in [34]; let us briefly describe it. Firstly, we exploit the radial structure of the Dirac-Coulomb operator, using the partial wave decomposition to reduce the Dirac-Coulomb operator to a differential operator acting only on the radial component; then the "relativistic" Hankel transform (see Section 1.2.2 for the definition) allows us to reduce the radial operator to a multiplicative and diagonal one. This gives us an explicit representation for the solutions of (1.1) (see (1.3)). Then, we use the pointwise estimates of the generalized eigenfunctions to estimate the $L^p_t(\mathbb{R})L^q_r([R, 2R])$-norm of the solution, where $R > 0$, on a fixed angular level for frequency localized initial data. Finally we conclude using the orthogonality of the partial waves, scaling argument and dyadic decompositions of the frequencies.

Remark 1.1.5. We mention that the same strategy has been later used in [27] to study another kind of physically relevant scale-invariant perturbation of the Dirac equation, the Aharonov-Bohm potential, as described in Section 0.2.3. The authors are able to prove the validity of the Strichartz estimates for that system without the angular regularity assumption. Let us observe, however, that, contrary to the Coulomb case, the structure of A-B potential allows the use of the "squaring trick" for the reduced radial equation.

[3] In [26] the authors work in the 3D setting. However, one can use the same definition to build a modified operator, with the same properties, also in the 2D setting.

Moreover, while here the "relativistic" Hankel transform is build using confluent hypergeometric functions, in the other case the generalized eigenfunctions are made by Bessel functions, whose behavior is more studied in the literature.

With the same tools, we can also show the validity of a new smoothing estimate for solutions of (1.1).

Proposition 1.1.6. *Let $u_0 \in L^2(\mathbb{R}^n;\mathbb{C}^N)$. Then the following estimate holds*

$$\sup_{R>0} R^{-\frac{1}{2}} \|e^{it\mathcal{D}_\nu} u_0\|_{L^2_t L^2_{|x|\le R}} \lesssim \|u_0\|_{L^2_x}. \tag{1.8}$$

Remark 1.1.7. We observe that the estimate in the Morrey spaces $L^{1,2}(\mathbb{R}^n)$ can be viewed as a limiting case of the local smoothing estimates (1.3) as $\alpha \to \frac{1}{2}$. We mention also that the same estimate has been proved in [19] for solutions of small magnetic Dirac equations with completely different techniques and in [32] for the Dirac equation in the Aharonov-Bohm magnetic field.

As mentioned in the introduction, Strichartz estimates can be used in the study of the dynamics of dispersive nonlinear systems. Therefore, as an application of Theorem 1.1.2 for Dirac-radial functions, we discuss a local well posedness result for the following 3D nonlinear system

$$\begin{cases} i\partial_t u + \mathcal{D}_\nu u = N(u), \\ u(x,0) = u_0(x), \end{cases} \tag{1.9}$$

where $|\nu| < \frac{\sqrt{3}}{2}$, u_0 is Dirac-radial and the nonlinearity is of the form

$$N(u) = \big(\omega * \langle \beta u, u\rangle\big)u, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix} \tag{1.10}$$

with ω radially symmetric, i.e., $\omega(x) = \omega(|x|)$, $\omega \in L^p(\mathbb{R}^3)$ for some $p > 1$.

We have the following

Theorem 1.1.8. *Let ω be a radial function, $\omega \in L^p(\mathbb{R}^3)$, $p \in \big[\frac{3}{2}, +\infty\big]$. Let also $u_0 \in \dot{H}^s(\mathbb{R}^3)$, $s = \frac{3}{2p}$, be Dirac-radial. Then there exists a positive time $T = T(\|u_0\|_{\dot{H}^s}, \|\omega\|_{L^p})$ such that (1.9) has a unique solution $u \in C([0,T]; \dot{H}^s(\mathbb{R}^3))$.*

Theorem 1.1.9. *Let ω be a radial function, $\omega \in L^p(\mathbb{R}^3)$, $p \in \big(\frac{3}{1+2\sqrt{1-\nu^2}}, +\infty\big)$. Let also $u_0 \in \dot{H}^s(\mathbb{R}^3)$, $s = \frac{1}{p}$, be Dirac-radial. Then there exists a positive time $T = T(\|u_0\|_{\dot{H}^s}, \|\omega\|_{L^p})$ such that (1.9) has a unique solution $u \in C([0,T]; \dot{H}^s(\mathbb{R}^3)) \cap L^{2p}([0,T]; L^{2p'}(\mathbb{R}^3))$, where p' is the conjugate exponent of p.*

The choice of such initial datum is twofoldedly motivated. First, we notice that (see Proposition 1.4.1) if u_0 is Dirac-radial then the Strichartz estimates hold in the classical way. Second, the Dirac-radiality of the solution is preserved by the nonlinearity (as observed in Remark 1.4.4). Then, the proofs of Theorems 1.1.8 and 1.1.9 will be based on standard fixed-point arguments on suitable complete metric spaces (see [45], [93] and references therein for related results). Observe that in the first one we do not use any kind of Strichartz estimates; they come into play if we want to require less integrability for ω. However, we shall remark that, in both cases, we strongly exploit the equivalence between $\dot{H}^s$ and $\dot{H}^s_{\mathcal{D}_\nu}$ norms, which holds for every $s \in [0,1]$ if $|\nu| < \frac{\sqrt{3}}{2}$. The failing of this equivalence in the general 2D setting (see Subsection 1.2.4), prevents us from extending the result from $\mathbb{R}^3$ to $\mathbb{R}^2$. To conclude, notice that the conditions on ω are satisfied by the Yukawa potential $V_b(x) = c\frac{e^{-b|x|}}{|x|} \in L^p(\mathbb{R}^3)$ for all $p < 3$, $b > 0$ and by $\omega(x) = \langle x\rangle^{-\alpha} \in L^p(\mathbb{R}^3)$ for all $p > \frac{3}{\alpha}$, $\alpha > 0$. Let us underline that with this choice of potentials there is no null structure to exploit in order to obtain a better decay. It would be interesting to consider the case $\omega = \delta_0$, in order to recover the standard cubic nonlinearity $N(u) = \langle \beta u, u\rangle u$. However, one expects to obtain local well posedness results in $H^s(\mathbb{R}^3)$ in the subcritical case $s > s_c$. The critical exponent s_c is given by the homogeneity of the Cauchy problem and it can be obtained by scaling arguments. In this case $s_c = 1$. However, for $s > 1$ we do not have anymore the equivalence between $\dot{H}^s$ and $\dot{H}^s_{\mathcal{D}_\nu}$ norms. This would prevent the use of standard tools developed in Sobolev spaces, e.g. Sobolev embeddings, and would require additional work to adapt them in the spaces obtained by the action of the Dirac-Coulomb operator. This would be the object of future works.

1.2 Preliminaries

1.2.1 Partial wave decomposition

A crucial aspect of the Dirac-Coulomb operator is that it can be seen as a radial operator with respect to some suitable decomposition, both in dimension 2 and 3. We recall these decompositions, refering to [123] for all the details.
We use spherical coordinates to write

$$L^2(\mathbb{R}^n;\mathbb{C}^N) \cong L^2((0,\infty), r^{n-1}dr) \otimes L^2(\mathbb{S}^{n-1};\mathbb{C}^N).$$

Then, we use the partial wave decompositions to define the isomorphisms

$$L^2(\mathbb{S}^1;\mathbb{C}^2) \cong \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathfrak{h}^2_k,$$

$$L^2(\mathbb{S}^2;\mathbb{C}^4) \cong \bigoplus_{k\in\mathbb{Z}^*} \bigoplus_{m=-|k|+\frac{1}{2}}^{|k|-\frac{1}{2}} \mathfrak{h}^3_{k,m_k}$$

where each subspace $\mathfrak{h}^n$ is two-dimensional and it is left invariant under the action of the Dirac-Coulomb operator. The orthonormal basis of each $\mathfrak{h}^2_k$, $\{\Xi^+_k(\theta), \Xi^-_k(\theta)\}$ is expressed in terms of the "classic Fourier basis", that is

$$\Xi^+_k(\theta) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} e^{i(k-1/2)\theta} \\ 0 \end{pmatrix}, \quad \Xi^-_k(\theta) = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} 0 \\ e^{i(k+1/2)\theta} \end{pmatrix}, \tag{1.11}$$

analogously, the orthonormal basis of each $\mathfrak{h}^3_{k,m}$, $\{\Xi^+_{k,m}(\theta_1,\theta_2),\Xi^-_{k,m}(\theta_1,\theta_2)\}$ is expressed in terms of the standard spherical harmonics $Y^m_l(\theta_1,\theta_2)$, that is

$$\Xi^+_{k,m}(\theta_1,\theta_2)=\begin{pmatrix} i\Omega^m_k \\ 0_2\end{pmatrix}, \quad \Xi^-_{k,m}=\begin{pmatrix} 0_2 \\ \Omega^m_{-k}\end{pmatrix} \tag{1.12}$$

where

$$\Omega_{k,m}=\frac{1}{\sqrt{|2k+1|}}\begin{pmatrix} \sqrt{|k-m+\frac12|}Y^{m-\frac12}_{|k+\frac12|-\frac12} \\ \operatorname{sgn}(-k)\sqrt{|k+m+\frac12|}Y^{m+\frac12}_{|k+\frac12|-\frac12}\end{pmatrix}.$$

We thus have the unitary isomorphisms

$$L^2(\mathbb{R}^2;\mathbb{C}^2)\cong\bigoplus_k L^2((0,\infty),rdr)\otimes\mathfrak{h}^2_k$$

$$L^2(\mathbb{R}^3;\mathbb{C}^4)\cong\bigoplus_{k,m} L^2((0,\infty),r^2dr)\otimes\mathfrak{h}^3_{k,m}$$

given, respectively, by the decompositions

$$\begin{aligned}\Phi^2(x)&=\sum_{k\in\mathbb{Z}+\frac12}\phi^+_k(r)\,\Xi^+_k(\theta)+\phi^-_k(r)\,\Xi^-_k(\theta)\\ \Phi^3(x)&=\sum_{k\in\mathbb{Z}^*}\sum_{m=-|k|+\frac12}^{|k|-\frac12}\phi^+_{k,m}(r)\Xi^+_{k,m}(\theta_1,\theta_2)+\phi^-_{k,m}(r)\Xi^-_{k,m}(\theta_1,\theta_2).\end{aligned} \tag{1.13}$$

The action of $\mathcal{D}_{n,\nu}$ on each partial wave subspace $L^2((0,\infty),r^{n-1}dr)^2\otimes\mathfrak{h}^n$ can be represented by the radial matrices

$$d^n_{\nu,k}=\begin{pmatrix} -\frac{\nu}{r} & -(\frac{d}{dr}+\frac{n-1}{2r})+\frac{k}{r} \\ \frac{d}{dr}+\frac{n-1}{2r}+\frac{k}{r} & -\frac{\nu}{r}\end{pmatrix}, \tag{1.14}$$

which are well defined on $C^\infty_0((0,\infty),r^{n-1}dr)^2\subset L^2((0,\infty),r^{r-1}dr)^2$.
Notice that formulas (1.14) only depend on k and not on m_k. Then, in order to give a unified treatment of the two cases $n=2,3$, in what follows we will maintain only the dependence on the parameter k for $n=3$. Moreover, we will omit the dependence on n of the angular part. Thus, if $\Phi\in L^2(\mathbb{R}^n;\mathbb{C}^4)$ we will decompose it as

$$\Phi(x)=\sum_{k\in\mathcal{A}_n}\varphi^+_k(r)\Xi^+_k(\theta)+\varphi^-_k(r)\Xi^-_k(\theta)=\sum_{k\in\mathcal{A}_n}\varphi_k(r)\cdot\Xi_k(\theta) \tag{1.15}$$

where if $n=2$, $\mathcal{A}_2=\mathbb{Z}+\frac12$, $\theta\in\mathbb{S}^1$ and the functions Ξ_k are the ones defined in (1.11), instead if $n=3$, $\mathcal{A}_3=\mathbb{Z}^*$, $\theta\in\mathbb{S}^2$ and the functions Ξ_k are as in (1.12) (omitting the dependence of m). With this decomposition, by Stone's theorem, the propagator is given by

$$e^{it\mathcal{D}_\nu}\Phi(x)=\sum_{k\in\mathcal{A}_n}\varphi^+_k(t,r)\,\Xi^+_k(\theta)+\varphi^-_k(t,r)\,\Xi^-_k(\theta)$$

where

$$\begin{pmatrix} \varphi_k^+(t,r) \\ \varphi_k^-(t,r) \end{pmatrix} = e^{itd_{\nu,k}} \begin{pmatrix} \varphi_k^+(r) \\ \varphi_k^-(r) \end{pmatrix}.$$

Remark 1.2.1. Observe that the radial functions (meaning a vector of four radial functions) are contained in the firsts eigenspaces (corresponding to $k = \pm\frac{1}{2}$ if $n = 2$ and to $k = \pm 1$ if $n = 3$) but they are not left invariant, in general, by the Dirac operator. In order to consider invariant sets of functions, we call $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ *Dirac-radial* if in the decomposition given by (1.15), for $n = 2$ $u_k^\pm(r) = 0$ for all $|k| > \frac{1}{2}$ and for $n = 3$ $u_{k,m}^\pm(r) = 0$ for all $|k| > 1$. On the contrary, $u \in L^2(\mathbb{R}^n; \mathbb{C}^N)$ is *Dirac-non radial* if it is orthogonal to the first partial wave subspaces; more precisely if, in the decomposition given by (1.13), for $n = 2$, $u_k^\pm(r) = 0$ if $|k| = \frac{1}{2}$ and, for $n = 3$, $u_{k,m}^\pm(r) = 0$ if $|k| = 1$.

1.2.2 Relativistic Hankel transform

Once one has decomposed the Dirac-Coulomb operator in a sum of radial operators, the key idea is to look for an isometry that transforms each radial differential operator into a multiplication operator. This is the role of the “relativistic” Hankel transform, which is built with the generalized eigenfunctions $\psi_{k,E}$ of $\mathcal{D}_{n,\nu}$. The idea of this construction was borrowed in [24] in which the author considered the Hankel transform. This is built with the Bessel functions that are the generalized eigenfunctions for the radial Schrödinger operator (see e.g. [24], Section 2.1). In this sense the transform we consider can be viewed as a relativistic counterpart of the standard one. For the sake of completeness we recall in this Subsection the definition and its properties, without proofs. We refer to [33] (Section 2.2) for a complete presentation. Please notice that the definition we present here is a slight modification of the one introduced in [33], which we believe to be more readable.

Definition 1.2.2. Let $\Phi \in L^2((0,\infty), r^{n-1}dr) \otimes \mathfrak{h}_k^n$ for some fixed k and let $\varphi(r) = (\varphi_1(r), \varphi_2(r))$ be the vector of its radial coordinates in decomposition (1.15).
We define the *relativistic Hankel transform* as a sequence of operators

$$\mathcal{P}_k\colon [L^2((0,+\infty), r^{n-1}dr)]^2 \to L^2(\mathbb{R}, E^{n-1}dE)$$

acting on spinors $\varphi(r)$ as

$$\mathcal{P}_k\varphi(E) = \int_0^{+\infty} \psi_{k,E}(r)^T \cdot \varphi(r) r^{n-1} dr, \quad E \in \mathbb{R}.$$

Here

$$\psi_{k,E}^n(r) = \begin{pmatrix} F_{k,E}^n(r) \\ G_{k,E}^n(r) \end{pmatrix}, \tag{1.16}$$

represents the vector of radial coordinates of the generalized eigenfunctions. That is, in the notation of (1.15), $\Psi^n(x) := \sum_{k\in\mathcal{A}_n} \psi_{k,E}^n(r) \cdot \Xi_k(\theta)$ “solves”

$$\left(\mathcal{D}_n - \frac{\nu}{|x|}\right)\Psi^n = E\Psi^n, \quad E > 0$$

and, in particular, for each k

$$d^n_{\nu,k}\psi^n_{k,E} = E\psi^n_{k,E}.$$

For the sake of completeness we recall the formulas for $F^n_{k,E}, G^n_{k,E}$, that are

$$\begin{aligned} F^n_{k,E}(r) &= \frac{\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\frac{\pi\nu}{2}}(2Er)^{\gamma-\frac{n-1}{2}} Re\{e^{i(Er+\xi)}{}_1F_1(\gamma-i\nu, 2\gamma+1, -2iEr)\}, \\ G^n_{k,E}(r) &= \frac{i\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\frac{\pi\nu}{2}}(2Er)^{\gamma-\frac{n-1}{2}} Im\{e^{i(Er+\xi)}{}_1F_1(\gamma-i\nu, 2\gamma+1, -2iEr)\} \end{aligned} \tag{1.17}$$

where ${}_1F_1(a,b,z)$ are confluent hypergeometric functions, $\gamma = \sqrt{k^2-\nu^2}$ and $e^{-2i\xi} = \frac{\gamma-i\nu}{k}$ is a phase shift.

Remark 1.2.3. We recall that the spectrum of $\mathcal{D}_n$ is purely absolutely continuous and it is the whole real line $\mathbb{R}$. The formulas in (1.17) give the generalized eigenstates of the continuous spectrum corresponding to the positive energies $E > 0$. The ones corresponding to negative energies can be obtained using a charge conjugation argument. Then, one gets that

$$\psi^n_{k,-E}(r) = \begin{pmatrix} F^n_{k,-E}(r) \\ G^n_{k,-E}(r) \end{pmatrix} = \begin{pmatrix} \tilde{F}^n_{-k,E}(r) \\ \tilde{G}^n_{-k,E}(r) \end{pmatrix}$$

where $\tilde{F}^n$, $\tilde{G}^n$ are the functions obtained by (1.17) by changing the sign of ν.
We also want to underlain that the homogeneity of the generalized eigenfunctions $\Psi^n_{k,E}(r)$ with respect to E and r is the same, in particular we have that $\psi^n_{k,E}(r) = \psi^n_{k,1}(Er)$. The lack of this homogeneity in the massive case prevents us from extending the results to that case.

To conclude, we list some properties of the transform that will be exploited in the following. We refer to [33] for the proof.

Proposition 1.2.4. *The following properties hold:*

i) $\mathcal{P}_k$ *is an* L^2*-isometry,*

ii) $(\mathcal{P}_k d_{\nu,k}\varphi)(E) = \sigma_3 E(\mathcal{P}_k\varphi)(E)$, $E > 0$,

iii) The inverse transform of $\mathcal{P}_k$ *is, for fixed* k, *an operator*

$$\mathcal{P}_k^{-1} \colon L^2(\mathbb{R}, E^{n-1}dE) \to [L^2((0,+\infty), r^{n-1}dr)]^2$$

acting on scalar functions $f(E)$ *as*

$$\mathcal{P}_k^{-1}f(r) = \int_{-\infty}^{+\infty} \psi_{k,E}(r)f(E)E^{n-1}dE, \quad r \in \mathbb{R}^+. \tag{1.18}$$

1.2.3 Asymptotic estimates of the generalized eigenfunctions

Other fundamental tools that we will use are some asymptotic estimates of the generalized eigenfunctions $\psi^n_{k,E}$: these will allow us to obtain an estimate on the L^q norm of such functions on fixed interval, uniformly with respect to the parameter k. Notice that this is another substantial difference respect to the cases of the wave equation with an inverse square potential ([96]) and the Dirac equation with Aharonov-Bohm magnetic potential (see [32]). In these cases, indeed, the generalized eigenfunctions are basically Bessel functions for which asymptotic estimates are well known (see Remark 1.2.6 below).

In the following we will use the notation ψ^n_k to indicate the generalized eigenfunction of eigenvalue 1. That is $\psi^n_k := \psi^n_{k,1}$. Recall that generalized eigenfunction of eigenvalue E are derived by $\psi^n_k(r)$. that is, $\psi^n_{k,E}(r) = \psi^n_k(Er)$ for any $E > 0$.

Proposition 1.2.5. *Given $\nu \in \left[-\frac{n-1}{2}, \frac{n-1}{2}\right]$ and $k \in \mathcal{A}_n$, let $\gamma = \sqrt{k^2-\nu^2}$ and consider the generalized eigenfunctions ψ^n_k of $\mathcal{D}_{\nu,n}$ with eigenvalue $E = 1$ given by formulas (1.16) and (1.17). Then there exist positive constants C, D independent of k, ν such that the following pointwise estimate holds for all $\rho \in \mathbb{R}\backslash\{0\}$:*

$$\left|\psi^n_k(\rho)\right| \leq C \begin{cases} (\min\{\frac{|\rho|}{2},1\})^{\gamma-\frac{n-1}{2}} e^{-D|k|}, & 0<|\rho|\leq \max\{\frac{|k|}{2},2\}, \\ |k|^{-\frac{2n-3}{4}}\left(\left||k|-|\rho|\right|+|k|^{\frac{1}{3}}\right)^{-\frac{1}{4}}, & \frac{|k|}{2}\leq|\rho|\leq 2|k|, \\ |\rho|^{-\frac{n-1}{2}}, & |\rho|\geq 2|k|, \end{cases} \tag{1.19}$$

Moreover, the following estimate holds

$$\left|(\psi^n_k)'(\rho)\right| \leq C \begin{cases} (\min\{\frac{|\rho|}{2},1\})^{\gamma-\frac{n+1}{2}} e^{-D|k|}, & 0<|\rho|\leq \max\{\frac{|k|}{2},2\}, \\ |k|^{-\frac{2n-3}{4}}\left(\left||k|-|\rho|\right|+|k|^{\frac{1}{3}}\right)^{-\frac{1}{4}}, & \frac{|k|}{2}\leq|\rho|\leq 2|k|, \\ |\rho|^{-\frac{n-1}{2}}, & |\rho|\geq 2|k|, \end{cases} \tag{1.20}$$

with C, $D > 0$ different from the ones in (1.19) but still independent of k, ν,

Proof. The proof for the case $n = 3$ is given in [34], Thm. 1.1. For the case $n = 2$ we observe that, from formulas (1.17), we have a relation between the generalized eigenfunctions in dimensions 2 and 3; they differ from a factor $(2Er)^{\frac{1}{2}}$ and from the range where the parameter k lives. Then, it is possible to adapt the proof in [34], with minor modifications, to deduce the estimates in dimension 2. □

Remark 1.2.6. Notice that the obtained estimates are the same, for large value of k, that holds for the generalized eigenfunctions of the wave equation with inverse square potentials; in fact, in that case the functions are given by

$$\tilde{\psi}^n_\rho(r) = (\rho r)^{-\frac{n-2}{2}} J_{\nu(k)}(\rho r)$$

where J_ν is the Bessel function of order ν, $\nu(k) = \sqrt{\mu(k)^2 + a}$, $\mu(k) = \frac{n-2}{2} + k$, $k \in \mathbb{N}$ and $a > -\frac{(n-2)^2}{4}$. The Bessel functions J_ν, for $\nu \geq 2$, enjoy the pointwise bounds

$$|J_\nu(\rho)| \leq C \begin{cases} e^{-d\nu}, & 0 < \rho < \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(|\rho - \nu| + \nu^{\frac{1}{3}})^{-\frac{1}{4}}, & \frac{\nu}{2} < \rho < 2\nu, \\ \rho^{-\frac{1}{2}}, & 2\nu < \rho < \infty, \end{cases}$$

for some C and d non depending on ν (see, e.g., [118]).

1.2.4 Equivalence of norms

As stated in the Introduction, in order to prove the local well posedness of the non-linear system we exploit the equivalence between the Sobolev norms H^s and the ones induced by the action of the Dirac-Coulomb operator $\mathcal{D}_{n,\nu}$. In this Subsection we recall it and we also add some results in this direction about the 2D setting.
For what concerns the 3D case we have the following

Proposition 1.2.7. *Let $|\nu| < \frac{\sqrt{3}}{2}$ and $u \in \mathrm{dom}(\mathcal{D}_{3,\nu})$. Then for every $s \in [0,1]$ there exist two positive constants C_1, C_2 such that*

$$C_1 \|u\|_{\dot{H}^s(\mathbb{R}^3)} \leq \|u\|_{\dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^3)} \leq C_2 \|u\|_{\dot{H}^s(\mathbb{R}^3)}.$$

Proof. We start observing that if $|\nu| < \frac{\sqrt{3}}{2}$ then the domain of the self-adjoint extension of $\mathcal{D}_{3,\nu}$ is $H^1(\mathbb{R}^3;\mathbb{C}^4)$ (see [123] Section 4.3.3). Moreover, for ν in this range, $\min\{1, \frac{1}{2} + \sqrt{1-\nu^2}\} = 1$. Then, the statement comes directly from the following Lemma (it is Corollary 1.8 in [67]).

Lemma 1.2.8. *Let $|\nu| \leq 1$. For any $f \in C_c^\infty(\mathbb{R}^3;\mathbb{C}^4)$ we have*

i) if $s \in [0, \min\{1, \frac{1}{2} + \sqrt{1-\nu^2}\}]$, then

$$\left\| |-\Delta|^{\frac{s}{2}} f \right\|_{L^2} \lesssim_{\nu,s} \left\| |\mathcal{D}_\nu|^s f \right\|_{L^2};$$

ii) if $s \in [0,1]$, then

$$\left\| |\mathcal{D}_\nu|^s f \right\|_{L^2} \lesssim_{\nu,s} \left\| |-\Delta|^{\frac{s}{2}} f \right\|_{L^2}$$

□

In the 2D case we cannot hope to obtain the same results; firstly, we recall that the domain of the distinguished self-adjoint extension of $\mathcal{D}_{2,\nu}$ is not contained in $H^1(\mathbb{R}^2;\mathbb{C}^2)$ (see [99], Corollary 16), for any $\nu \neq 0$. Secondly, the property *ii*) is proven using the Hardy's inequality, which we know to fail on $\mathbb{R}^2$.
However, the distinguished self-adjoint extension is chosen such that its domain is contained in $H^{\frac{1}{2}}(\mathbb{R}^2;\mathbb{C}^4)$, if $|\nu| < \frac{1}{2}$. This suggests that at least inequality *i*) in Lemma 1.2.8 could hold if we take $s \in [0, \frac{1}{2}]$. In fact, we have the following

Proposition 1.2.9. *Let* $|\nu| < \frac{1}{2}$. *For any* $u \in \mathrm{dom}(\mathcal{D}_{2,\nu})$ *and* $s \in \left[0, \frac{1}{2}\right]$, *we have*

$$\left\|u\right\|_{\dot{H}^s} \lesssim_{\nu,s} \left\||\mathcal{D}_\nu|^s u\right\|_{L^2}.$$

Proof. From [99] [Thm 1] we have that for every $\nu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ there exists $C_\nu > 0$ such that

$$|\mathcal{D}_\nu| \geq C_\nu \sqrt{-\Delta} \otimes \mathbb{1}_2.$$

Then, by operator monotonicity of the map $f(t) = t^p$, $p \in [0,1]$ (see [113] [Thm 12.12]), we conclude that

$$|\mathcal{D}_\nu|^p \geq C_\nu^p \sqrt{-\Delta}^p \otimes \mathbb{1}_2,$$

that is the estimate with $2s = p$. □

For what concerns the reverse inequality, we observe that even if the Hardy's inequality fails in 2D, it still holds for non-radial functions (see [24], Prop. 1 pag 8). Then we have the following

Proposition 1.2.10. *Let* $|\nu| < \frac{1}{2}$. *For any* $u \in \mathrm{dom}(\mathcal{D}_{2,\nu})$, *u Dirac-non radial and* $s \in [0, \frac{1}{2}]$ *we have*

$$\left\||\mathcal{D}_\nu|^s u\right\|_{L^2} \lesssim_{\nu,s} \left\||-\Delta|^{\frac{s}{2}} u\right\|_{L^2}$$

Proof. We first prove that, if $f \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2)$ and Dirac-non radial, then the Hardy's inequality holds, i.e.

$$\left\||x|^{-1} f\right\|_{L^2} \leq \left\|\nabla f\right\|_{L^2}. \tag{1.21}$$

We can decompose f as:

$$f(x) = \sum_{k \in \mathbb{Z}+\frac{1}{2}, k \neq \frac{1}{2}} f_k^+(r)\Xi_k^+(\theta) + f_k^-(r)\Xi_k^-(\theta) =: f^+(r,\theta) + f^-(r,\theta).$$

and we recall that $\partial_\theta \Xi_k^\pm(\theta) = \left(k \mp \frac{1}{2}\right)\Xi_k^\pm(\theta)$. Then,

$$\int_0^{2\pi} |f^\pm|^2 d\theta = \sum_{k \in \mathbb{Z}+\frac{1}{2}, k \neq \frac{1}{2}} |f_k^\pm|^2 \leq \sum_{k \in \mathbb{Z}+\frac{1}{2}, k \neq \frac{1}{2}} (k \mp \tfrac{1}{2})^2 |f_k^\pm|^2 = \int_0^{2\pi} |\partial_\theta f^\pm|^2 d\theta$$

and thus

$$\left\||x|^{-1} f\right\|_{L^2}^2 \leq \int_0^\infty \int_0^{2\pi} \frac{1}{r^2}\left[|\partial_\theta f^+|^2 + |\partial_\theta f^-|^2\right] d\theta r dr \lesssim \left\|\nabla f\right\|_{L^2}^2.$$

Hence, by the Cauchy-Schwarz inequality and by (1.21), we have

$$(\mathcal{D}_\nu)^2 \lesssim (-\Delta) \otimes \mathbb{1}_2.$$

Then, by monotonicity[4], we get that for any $f \in C_c^\infty(\mathbb{R}^2 : \mathbb{C}^2)$, f Dirac-non radial and for any $s \in [0,1]$

$$\|f\|_{\dot{H}^s_{\mathcal{D}_\nu}(\mathbb{R}^2)} \lesssim_{\nu,s} \|f\|_{\dot{H}^s(\mathbb{R}^2)}.$$

We conclude recalling that if $|\nu| < \frac{1}{2}$ then $\mathrm{dom}(\mathcal{D}_{2,\nu}) \subset H^{\frac{1}{2}}(\mathbb{R}^2; \mathbb{C}^2)$, but not in $H^1(\mathbb{R}^2; \mathbb{C}^2)$ (see [100]). □

[4] as in the proof of Proposition 1.2.9.

1.3 Proofs of the results

Before going into the proofs of the above mentioned results, we exploit the tools described in Subsections 1.2.1 and 1.2.2 in order to work with a useful representation of the solution. Let $u_0 \in L^2(\mathbb{R}^n;\mathbb{C}^N)$, by (1.15) and property $ii)$ in Proposition 1.2.4, we have

$$u_0(x) = \sum_{k\in\mathcal{A}_n} u_{0,k}(r)\cdot \Xi_k(\theta),$$
$$e^{it\mathcal{D}_\nu}u_0(x) = \sum_{k\in\mathcal{A}_n} \mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\cdot \Xi_k(\theta) \tag{1.22}$$

where $x=(r,\theta)\in\mathbb{R}^n$. Moreover, thanks to the L^2-orthogonal decomposition,

$$\|u_0\|_{L^2_x} = \Big(\sum_{k\in\mathcal{A}_n}\|u_{0,k}(r)\|^2_{L^2_{r^{n-1}dr}}\Big)^{\frac12} \tag{1.23}$$

$$\begin{aligned}\big\|e^{it\mathcal{D}_\nu}u_0\big\|^2_{L^\infty_t L^2_x} &= \Big\|\sum_{k\in\mathcal{A}_n}\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\cdot\Xi_k(\theta)\Big\|^2_{L^\infty_t L^2_{r^{n-1}dr}L^2_\theta}\\ &= \sum_{k\in\mathcal{A}_n}\Big\|\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\Big\|^2_{L^\infty_t L^2_{r^{n-1}dr}}.\end{aligned} \tag{1.24}$$

More generally, for $p,q\geq 2$, by Minkowski's inequality $\|\cdot\|_{L^q l^2}\leq\|\cdot\|_{l^2L^q}$ $\forall q\geq 2$, we get

$$\begin{aligned}\big\|e^{it\mathcal{D}_\nu}u_0\big\|_{L^p_t L^q_{r^{n-1}dr}L^2_\theta} &= \Big\|\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\Big\|_{L^p_t L^q_{r^{n-1}dr}l^2_k(\mathcal{A}_n)}\\ &\leq\Big(\sum_{k\in\mathcal{A}_n}\|\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\|^2_{L^p_t L^q_{r^{n-1}dr}}\Big)^{\frac12}.\end{aligned} \tag{1.25}$$

1.3.1 Proof of Proposition 1.1.6

Thanks to (1.23) and (1.24), it suffices to show that, for any fixed $k\in\mathcal{A}_n$,

$$R^{-1}\Big\|\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\big](r)\Big\|^2_{L^2_tL^2_{r^{n-1}dr}([0,R])}\leq C\|u_{0,k}(r)\|^2_{L^2_{r^{n-1}dr}}$$

where C is a positive constant independent of k and R.

Let $g_k(\rho)=\mathcal{P}_k u_{0,k}(\rho)$. Then, from (1.18) and Plancherel's theorem in t (on each component), we get

$$\begin{aligned}\Big\|\mathcal{P}_k^{-1}\big[e^{it\rho\sigma_3}(g_k(\rho)\big](r)\Big\|^2_{L^2_tL^2_{r^{n-1}dr}([0,R])} &= \Big\|\mathscr{F}^{-1}_{\rho\sigma_3\to t}\big\{H_k^*(r\rho)\cdot g_k(\rho)\rho^{n-1}\chi_{\mathbb{R}^+}(\rho)\big\}\Big\|^2_{L^2_{rdr}([0,R])L^2_t}\\ &\leq\int_0^{+\infty}\int_0^R\big|H_k^*(r\rho)g_k(\rho)\rho^{n-1}\big|^2r^{n-1}drd\rho\\ &\lesssim\int_0^{+\infty}\int_0^R\big(|\psi_k^n(r\rho)|^2+|\psi_k^n(-r\rho)|^2\big)|g_k(\rho)|^2\rho^{2(n-1)}r^{n-1}drd\rho.\end{aligned}$$

We thus need to estimate

$$\int_0^{+\infty} \left(\int_0^R |\psi_k^n(r\rho)|^2 r^{n-1} dr \right) |g_k(\rho)|^2 \rho^{2(n-1)} d\rho. \tag{1.26}$$

Lemma 1.3.1. *Let ψ_k^n be the generalized eigenfunction of $\mathcal{D}_{n,\nu}$ with eigenvalue 1. Then, there exists a positive constant C depending on n but not on k and ν, such that*

$$\frac{1}{R}\int_0^R |\psi_k^n(r)|^2 r^{n-1} dr \leq C. \tag{1.27}$$

Remark 1.3.2. We observe that the same estimate holds for $\psi_k^n(-r)$, that is the generalized eigenfunction with eigenvalue -1. In fact, as stated in Remark 1.2.3, such eigenfunctions are obtained by ψ_k^n by changing the sign of k and ν. However, all the estimates in the proof will be independent by the sign of ν and k.

Proof. Let $R > 0$. We use (1.19) to estimate the integral in (1.27); we need to consider three different cases :

i) if $0 < R < \frac{|k|}{2}$, then we have

$$\begin{aligned} \frac{1}{R}\int_0^R |\psi_k^n(r)|^2 r^{n-1} dr &\lesssim \frac{1}{R}\int_0^R (\min\{\tfrac{r}{2},1\})^{2\gamma-(n-1)} e^{-2D|k|} r^{n-1} dr \\ &\leq \frac{1}{R}\int_0^R (\min\{\tfrac{r}{2},1\})^{-(n-1)} e^{-2D|k|} r^{n-1} dr, \end{aligned} \tag{1.28}$$

if $\min\{\frac{r}{2},1\} = 1$ then

$$(1.28) \lesssim \frac{1}{R} R^n e^{-2DR} \leq C;$$

otherwise, if $\frac{r}{2} \leq 1$,

$$(1.28) \leq \frac{1}{R} e^{-2D|k|} 2^{n-1} R \leq C;$$

ii) if $\frac{|k|}{2} < R < 2|k|$, then we split the interval $[0,R] = [0,|k|/2] + [|k|/2, R] = I_1 + I_2$:

$$\frac{1}{R}\int_{I_1} |\psi_k^n(r)|^2 r^{n-1} dr \lesssim \frac{1}{R} e^{-2D|k|} \frac{|k|}{2}^n \leq C,$$

$$\begin{aligned} \frac{1}{R}\int_{I_2} |\psi_k^n(r)|^2 r^{n-1} dr &\lesssim R^{-1} \int_{I_2} |k|^{-\frac{2n-3}{2}} \left(\big||k| - r\big| + |k|^{\frac{1}{3}} \right)^{-\frac{1}{2}} r^{n-1} dr \\ &\lesssim |k|^{-\frac{1}{2}} \int_{\frac{|k|}{2}}^{2|k|} \big||k| - r\big|^{-\frac{1}{2}} dr \leq C; \end{aligned}$$

iii) if $R > 2|k|$, then $[0,R] = [0,|k|/2] + [|k|/2, 2|k|] + [2|k|, R] = I_1 + I_2 + I_3$: estimates over I_1, I_2 as before,

$$R^{-1} \int_{I_3} |\psi_k^n(r)|^2 r^{n-1} dr \lesssim R^{-1} \int_{2|k|}^R dr \leq C.$$

□

Then, coming back to (1.26), we have, after a change of variable $\xi = r\rho$,

$$
\begin{aligned}
(1.26) &= \int_0^{+\infty} \left(\int_0^{R\rho} |\psi_k(\xi)|^2 \xi d\xi \right) |g_k(\rho)|^2 \rho^{n-2} d\rho \\
&\lesssim R \int_0^{+\infty} |g_k(\rho)|^2 \rho^{n-1} d\rho = CR \big\| \mathcal{P}_k u_{0,k} \big\|^2_{L^2_{\rho d\rho}},
\end{aligned}
$$

then the claim since $\mathcal{P}_k$ is an L^2-isometry.

1.3.2 Proofs of Strichartz estimates

In order to lighten the notation, in the following we will treat separately the cases $n = 2, 3$ and we will omit the dependence on n of the generalized eigenfunctions. We start with the 2-dimensional case.

Proof of Theorem 1.1.1. The proof is divided in four steps. The key step is the second one in which we prove the following Lemma. This provides us Strichartz estimates on a fixed radial interval for a frequency localized solution on a fixed angular level. The restriction of the estimates of intervals $[R, 2R]$ allow us to study the different behavior of the eigenfunctions "close" or "far" to the origin. Then the linear estimates in Theorem 1.1.1 will follow by scaling arguments and interpolation with the standard $L^\infty_t L^2_x$-estimate.

Lemma 1.3.3. *Let $(p, q) \in (2, \infty] \times [2, +\infty)$ and $k \in \mathbb{Z} + \frac{1}{2}$. Let $I = \left[\frac{1}{2}, 1\right]$ and $g_k \in L^2_{\rho d\rho}((0, +\infty))$ such that* $\mathrm{supp}(g_k) \subset I$. *Then*

$$
\left\| \mathcal{P}_k^{-1} \left[e^{it\rho\sigma_3} g_k(\rho) \right](r) \right\|_{L^p_t L^q_{rdr}([R,2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \times \begin{cases} R^{\gamma - \frac{1}{2} + \frac{2}{q}}, & R \leq 1 \\ R^{\frac{1}{q} + \beta(p)(1 - \frac{2}{q})}, & R \geq 2, \end{cases}
$$

where

$$
\beta(p) = \begin{cases} \frac{1}{p} - \frac{1}{2} & \text{if } p \in [2, 4), \\ \frac{1}{p} - \frac{1}{3} & \text{if } p \in [4, +\infty] \end{cases}
$$

and the constant C is independent of k, ν, but eventually depends on p and q. Moreover, if $q = +\infty$, for all $p > 2$ we have

$$
\left\| \mathcal{P}_k^{-1} \left[e^{it\rho\sigma_3} g_k(\rho) \right](r) \right\|_{L^p_t L^\infty_{dr}([R,2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \times \begin{cases} R^{\gamma - 1}, & R \leq 1 \\ R^{\beta(p)}, & R \geq 2, \end{cases}
$$

Step 1: By relying on the pointwise estimates of Proposition 1.2.5, we estimate the L^q norms of the generalized eigenfunctions on a fixed interval of length R;

Lemma 1.3.4. *Let $k \in \mathbb{Z} + \frac{1}{2}$, $\gamma = \sqrt{k^2 - \nu^2}$, $|\nu| \le \frac{1}{2}$ and $q \in [2, +\infty]$. Then, the following estimates hold*

$$\left\|\psi_k\right\|_{L^q([R,2R])} \le C \times \begin{cases} R^{\gamma + \frac{1}{q} - \frac{1}{2}}, & \text{if } R \le 1, \\ R^{\beta(q)}, & \text{if } R \ge 1, \end{cases}$$

and

$$\left\|\psi_k'\right\|_{L^q([R,2R])} \le C \times \begin{cases} R^{\gamma + \frac{1}{q} - \frac{3}{2}}, & \text{if } R \le 1, \\ R^{\beta(q)}, & \text{if } R \ge 1, \end{cases}$$

where

$$\beta(q) = \begin{cases} \frac{1}{q} - \frac{1}{2} & \text{if } q \in [2,4), \\ \frac{1}{q} - \frac{1}{3} & \text{if } q \in [4, +\infty] \end{cases}$$

and all the constants are independent of γ, k, but eventually dependent on q.

Proof. We split the proof in two cases:

i) $R \le 1$:
Let $q \in [2, +\infty)$; observing that $R \le r \le 2R$ implies $r \le 2$, we have

$$\int_R^{2R} |\psi_k(r)|^q dr \lesssim \int_R^{2R} (\min\{\tfrac{r}{2}, 1\})^{q\gamma - \frac{q}{2}} e^{-Dq|k|} dr \lesssim \int_R^{2R} r^{q\gamma - \frac{q}{2}} dr \lesssim R^{q\gamma - \frac{q}{2} + 1},$$

and

$$\int_R^{2R} |\psi_k'(r)|^q dr \lesssim \int_R^{2R} (\min\{\tfrac{r}{2}, 1\})^{q\gamma - \frac{3}{2}q} e^{-Dq|k|} dr \lesssim \int_R^{2R} r^{q\gamma - \frac{3}{2}q} dr \lesssim R^{q\gamma - \frac{3}{2}q + 1}.$$

Now let $q = +\infty$. Then, as before,

$$\sup_{r \in [R,2R]} |\psi_k(r)| \lesssim \sup_{r \in [R,2R]} r^{\gamma - \frac{1}{2}} 2^{-\gamma + \frac{1}{2}} \le 2^{\frac{1}{2}} R^{\gamma - \frac{1}{2}}.$$

In the same way we estimate $\|\psi'\|_{L^\infty}$.

ii) $R \ge 1$:
We write the interval of integration as $[R, 2R] = I_1 + I_2 + I_3$, where

$$I_1 = [R, 2R] \cap \left[0, \tfrac{|k|}{2}\right], \quad I_2 = [R, 2R] \cap \left[\tfrac{|k|}{2}, 2|k|\right], \quad I_3 = [R, 2R] \cap [2|k|, +\infty)$$

and we estimate each interval separately.
For I_1 we can assume $2R \le |k|$, otherwise $I_1 = \emptyset$. Let $q \in [2, +\infty)$, then

$$\int_{I_1} |\psi_k(r)|^q dr \lesssim \int_{I_1} (\min\{\tfrac{r}{2}, 1\})^{q\gamma - \frac{q}{2}} e^{-Dq|k|} dr \le C_\alpha R^{-\alpha}, \quad \forall\, \alpha > 0;$$

in fact, if $\min\{\frac{r}{2},1\}=1$, then

$$\int_{I_1}|\psi_k(r)|^q dr \lesssim e^{-Dq|k|}|k|^{\alpha}|k|^{-\alpha}\leq C_\alpha R^{-\alpha},$$

otherwise $r\leq 2$ and, again

$$\int_{I_1}|\psi_k(r)|^q dr \lesssim \int_R^{2R} r^{-\frac{q}{2}}e^{-Dqr}dr\leq C_\alpha R^{-\alpha};$$

Let now $q=+\infty$, then

$$|\psi_k(r)|\lesssim \begin{cases} e^{-DR} & \text{if } r\geq 2,\\ r^{-\frac{1}{2}}2^{\frac{1}{2}}e^{-DR} & \text{if } r\leq 2\end{cases}$$

and we get the claim.
For I_2 we can assume $\frac{|k|}{4}\leq R\leq 2|k|$, otherwise $I_2=\emptyset$. Let $q\in[2,4)$, we compute the integral and obtain

$$\begin{aligned}\int_{I_2}|\psi_k(r)|^q dr &\lesssim \int_{\frac{|k|}{2}}^{2|k|}|k|^{-\frac{q}{4}}\big(\big||k|-r\big|+|k|^{\frac{1}{3}}\big)^{-\frac{q}{4}}dr\\ &=|k|^{1-\frac{q}{4}}\int_{\frac{1}{2}}^{2}\big(|k|^{\frac{1}{3}}(|k|^{\frac{2}{3}}|1-\tau|+1)\big)^{-\frac{q}{4}}d\tau\\ &=|k|^{1-\frac{q}{4}-\frac{q}{12}}\int_{-\frac{1}{2}}^{1}\big(|k|^{\frac{2}{3}}|y|+1\big)^{-\frac{q}{4}}dy=|k|^{1-\frac{q}{4}-\frac{q}{12}-\frac{2}{3}}\int_{-\frac{|k|^{\frac{2}{3}}}{2}}^{|k|^{\frac{2}{3}}}\big(1+|x|\big)^{-\frac{q}{4}}dx\\ &=|k|^{\frac{1}{3}-\frac{q}{3}}\frac{4}{4-q}\Big[\big(1+\tfrac{|k|^{\frac{2}{3}}}{2}\big)^{1-\frac{q}{4}}+\big(1+|k|^{\frac{2}{3}}\big)^{1-\frac{q}{4}}-2\Big]\\ &\lesssim |k|^{\frac{1}{3}-\frac{q}{3}+\frac{2}{3}-\frac{q}{6}}=|k|^{1-\frac{q}{2}}\simeq R^{1-\frac{q}{2}}.\end{aligned} \tag{1.29}$$

For $q\in[4,+\infty)$ we estimate the norm as

$$\int_{I_2}|\psi_k(r)|^q dr\lesssim\int_{I_2}|k|^{-\frac{3}{4}q}\big(\big||k|-r\big|+|k|^{\frac{1}{3}}\big)^{-\frac{q}{4}}r^{\frac{q}{2}}dr\lesssim|k|^{-\frac{5}{6}q}\int_R^{2R}r^{\frac{q}{2}}\lesssim R^{1-\frac{1}{3}q}. \tag{1.30}$$

It remains to estimate the L^∞ norm, for which is enough to observe that, if $r\in I_2$

$$|\psi_k(r)|\lesssim|k|^{-\frac{1}{4}-\frac{1}{12}}\simeq R^{-\frac{1}{3}}.$$

Lastly, for I_3 we get

$$\int_{I_3}|\psi_k(r)|^q dr\lesssim\int_R^{2R}r^{-\frac{q}{2}}dr\lesssim R^{1-\frac{q}{2}}$$

and

$$\sup_{r\in[R,2R]}|\psi_k(r)|\lesssim R^{-\frac{1}{2}}.$$

Similar computations bring to the estimates of $\|\psi_k'\|_{L^q([R,2R])}$ for $R\geq 1$.

□

Step 2: We can now provide an estimate for the localized solution on a fixed angular level k and with the radius r varying on a fixed interval of length R.

In order to lighten the notation, in the following we will write $e^{it\rho}$ instead of $e^{it\sigma_3\rho}$ since all the estimates are to be understood on every component of the functions with values on $\mathbb{C}^N$. Moreover, we observe that all the estimates hold also for the generalized eigenfunctions corresponding to negative energies.

Proof of Lemma 1.3.3. i) $R \le 1$:

Let $p \in [2, +\infty]$ and Ω be an interval. Then by the embedding $\dot{H}^{\frac{1}{2}-\frac{1}{q}}(\Omega) \hookrightarrow L^q(\Omega)$, that holds for every $q \in [2, +\infty)$, and interpolation we have the following chain on inequalities.[5]

$$\begin{aligned}
&\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t L^q_{dr}([R,2R])} \\
&\lesssim \left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t \dot{H}^{\frac{1}{2}-\frac{1}{q}}([R,2R])} \\
&\lesssim \left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|^{\frac{1}{2}+\frac{1}{q}}_{L^p_t L^2_{dr}([R,2R])} \times \left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|^{\frac{1}{2}-\frac{1}{q}}_{L^p_t \dot{H}^1([R,2R])} \\
&\lesssim R^{\gamma-\frac{1}{2}+\frac{1}{q}} \|g_k(\rho)\|_{L^{p'}_{\rho d\rho}(I)}.
\end{aligned}$$

Where the last inequality comes from Minkowski and Hausdorff-Young inequalities and Lemma 1.3.4[6]:

$$\begin{aligned}
\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t L^2_{dr}([R,2R])} &\lesssim \|\psi_k(r\rho)g_k(\rho)\|_{L^{p'}_{d\rho}(I) L^2_{dr}([R,2R])} \\
&\lesssim \left\| |g_k(\rho)| \|\psi_k\|_{L^2_{dr}[R\rho,2R\rho]} \right\|_{L^{p'}_{d\rho}(I)} \\
&\lesssim R^\gamma \|g_k(\rho)\|_{L^{p'}_{\rho d\rho}(I)},
\end{aligned}$$

and

$$\begin{aligned}
\left\| \int_0^\infty e^{it\rho}\psi'_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t L^2_{dr}([R,2R])} &\lesssim \|\psi'_k(r\rho)g_k(\rho)\|_{L^{p'}_{\rho d\rho}(I) L^2_{dr}([R,2R])} \\
&\lesssim \left\| |g_k(\rho)| \|\psi'_k\|_{L^2_{dr}[R\rho,2R\rho]} \right\|_{L^{p'}_{\rho d\rho}(I)} \\
&\lesssim R^{\gamma-1} \|g_k(\rho)\|_{L^{p'}_{\rho d\rho}(I)}.
\end{aligned}$$

[5] Here with $\dot{H}^k(\Omega)$, $k \in \mathbb{Z}$ we denote the space of functions whose weak derivaties of order k are in $L^2(\Omega)$. We recall that, the same space can be also defined by extending the functions from Ω to $\mathbb{R}$ and then taking the $\dot{H}^k(\mathbb{R})$ norm. This construction works also if k is not an integer. We refer to [121], chapter 4.

[6] Notice that since $\rho \in [\frac{1}{2}, 1]$ then $\rho R \le 1$ and $\rho^\gamma \le 1$.

Let us observe that here and in the following we do not play particular attention to the power ok ρ that appears in the integrals, since $\rho \in [\frac{1}{2}, 1]$. Therefore we obtain

$$\left\| \int_0^\infty e^{it\rho}\psi_k'(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{rdr}^q([R,2R])} \lesssim R^{\gamma-\frac{1}{2}+\frac{2}{q}} \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)}.$$

ii) $R \geq 2$:
Let $p \in [2, +\infty]$. We prove the following estimates

$$\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{rdr}^\infty([R,2R])} \lesssim R^{\beta(p)} \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \tag{1.31}$$

and

$$\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{rdr}^2([R,2R])} \lesssim R^{\frac{1}{2}} \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)}, \tag{1.32}$$

then the result will follow by interpolation.
We first estimate (1.31). From Minkowski and Hausdorff-Young inequalities and Lemma 1.3.4, we have[7]

$$\left\| \int_0^{+\infty} e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{dr}^p([R,2R])} = \left\| \mathcal{F}_{t\mapsto\rho}\big(\psi_k(r\rho)g_k(\rho)\rho\chi_{\mathbb{R}^+}(\rho)\big)\right\|_{L^p_{dr}([R,2R])L_t^p}$$
$$\lesssim \left\| \|\psi_k(r\rho)g_k(\rho)\|_{L^{p'}_{d\rho}(I)} \right\|_{L^p_{dr}([R,2R])}$$
$$\lesssim \left(\int_I |g_k(\rho)|^{p'} \left(\int_R^{2R} |\psi_k(r\rho)|^p dr \right)^{\frac{1}{p}} d\rho \right)^{\frac{1}{p'}} \lesssim R^{\beta(p)} \|g_k(\rho)\|_{L^{p'}_{\rho d\rho}(I)}.$$

The same holds also for $\psi_k'(r\rho)$. Then, from the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{rdr}^\infty([R,2R])} = \left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L_t^p L_{dr}^\infty([R,2R])}$$
$$\lesssim \left(\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|^p_{L_t^p L_{dr}^p([R,2R])} + \left\| \int_0^\infty e^{it\rho}\psi_k'(r\rho)g_k(\rho)\rho d\rho \right\|^p_{L_t^p L_{dr}^p([R,2R])} \right)^{\frac{1}{p}}$$
$$\lesssim R^{\beta(p)} \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)}.$$

The estimate (1.32) comes easily observing that $R\rho \geq 1$ then from Lemma 1.3.4 we have

$$\left(\int_R^{2R} |\psi_k(r\rho)|^2 r dr \right)^{\frac{1}{2}} \leq C R^{\frac{1}{2}}.$$

[7] We observe that since $\rho \in [\frac{1}{2}, 1]$ then $R\rho \geq 1$.

Lastly, we can estimate the $L^\infty_{dr}([R,2R])$ norm as

$$\begin{aligned}
\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t L^\infty_{dr}([R,2R])} &\lesssim R^{\gamma-1}\|g_k(\rho)\|_{L^2_{\rho d\rho}(I)}, \quad \text{for } R \leq 1,\\
\left\| \int_0^\infty e^{it\rho}\psi_k(r\rho)g_k(\rho)\rho d\rho \right\|_{L^p_t L^\infty_{rdr}([R,2R])} &\lesssim R^{\beta(p)}\|g_k(\rho)\|_{L^2_{\rho d\rho}(I)}, \quad \text{for } R \geq 2,
\end{aligned}$$

where for the former estimate we use the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$. □

Step 3: We remove assumptions on the localizations on the solution in order to get the generalized Strichartz estimates with the indexes p, q in the range of admissibility given by the following

Lemma 1.3.5. *Let*

$$p \in (2,+\infty], \qquad 5 < q < \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}, \qquad \frac{1}{q} + \left(\frac{1}{p} - \frac{1}{2}\right)\left(1 - \frac{2}{q}\right) < 0.$$

Then

$$\|e^{it\mathcal{D}_\nu}u_0\|_{L^p_t L^q_{rdr} L^2_\theta} \leq C\|u_0\|_{\dot{H}^s_{\mathcal{D}_\nu}}$$

where $s = 1 - \frac{1}{p} - \frac{2}{q}$ *and* $\gamma_{\frac{1}{2}} = \sqrt{\frac{1}{4} - \nu^2}$. *Moreover, if* u_0 *is Dirac-non radial we can take* $q \in (5, +\infty]$.

We remove the assumption of the frequency localization of the solution, to get the Strichartz estimates for (p, q) admissible for Lemma 1.3.5; Thanks to (1.25), it suffices to show that

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} \left\| \mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \right\|^2_{L^p_t L^q_{rdr}} \leq C\|u_0\|^2_{\dot{H}^s_{\mathcal{D}_\nu}}.$$

Let N, R be dyadic numbers and $\phi \in C_c^\infty\left(\left[\frac{1}{2}, 1\right]\right)$ such that $\sum_{N \in 2^{\mathbb{Z}}} \phi(\rho N^{-1}) = 1$. Then, for $q < +\infty$, from the embedding $l^2 \hookrightarrow l^q$ and scaling we have the following

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}+\frac{1}{2}} \left\|\mathcal{P}_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r)\right\|^2_{L^p_t L^q_{rdr}} \\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \Big\| \sum_{N\in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|^2_{L^p_t L^q_{rdr}} \\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \Big\|\Big(\sum_{R\in 2^{\mathbb{Z}}} \Big\| \sum_{N\in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|^q_{L^q_{rdr}([R,2R])}\Big)^{\frac{1}{q}}\Big\|^2_{L^p_t} \\
&\lesssim \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{R\in 2^{\mathbb{Z}}} \Big\| \sum_{N\in 2^{\mathbb{Z}}} \mathcal{P}_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|^2_{L^p_t L^q_{rdr}([R,2R])} \\
&\lesssim \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{R\in 2^{\mathbb{Z}}} \Big(\sum_{N\in 2^{\mathbb{Z}}} \Big\|\mathcal{P}_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|_{L^p_t L^q_{rdr}([R,2R])}\Big)^2 \\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{R\in 2^{\mathbb{Z}}} \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2-\frac{1}{p}-\frac{2}{q}} \Big\|P_k^{-1}[e^{-it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(N\rho)\phi(\rho)](r)\Big\|_{L^p_t L^q_{rdr}([NR,2NR])}\Big)^2.
\end{aligned} \tag{1.33}$$

Moreover, from Lemma 1.3.3 we can continue the chain of inequalities and we get

$$\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{R\in 2^{\mathbb{Z}}} \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2-\frac{1}{p}-\frac{2}{q}} Q(NR) \big\|(\mathcal{P}_k u_{0,k})(N\rho)\phi(\rho)\big\|_{L^2_{\rho d\rho}}\Big)^2 \\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{R\in 2^{\mathbb{Z}}} \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{1-\frac{1}{p}-\frac{2}{q}} Q(NR) \big\|(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})\big\|_{L^2_{\rho d\rho}}\Big)^2
\end{aligned}$$

where

$$Q(NR) = \begin{cases} (NR)^{\gamma-\frac{1}{2}+\frac{2}{q}} & \text{if } NR \le 1, \\ (NR)^{\frac{1}{q}+\beta(p)(1-\frac{2}{q})} & \text{if } NR \ge 2. \end{cases}$$

We note that if we take p, q such that

$$\gamma - \frac{1}{2} + \frac{2}{q} > 0, \quad \frac{1}{q} + \beta(p)\big(1 - \frac{2}{q}\big) < 0, \tag{1.34}$$

then

$$\sup_{R\in 2^{\mathbb{Z}}} \sum_{N\in 2^{\mathbb{Z}}} Q(NR) < +\infty, \quad \sup_{N\in 2^{\mathbb{Z}}} \sum_{R\in 2^{\mathbb{Z}}} Q(NR) < +\infty.$$

Recall that $\gamma = \sqrt{k^2 - \nu^2} \ge \sqrt{2}$ if $|k| > \frac{1}{2}$. Then, the first condition in (1.34) becomes: $\gamma_{\frac{1}{2}} - \frac{1}{2} + \frac{2}{q} > 0$.

Let $A_{N,k} = N^{1-\frac{1}{p}-\frac{2}{q}} \big\|(\mathcal{P}_k u_{0,k})(\rho)\phi(\frac{\rho}{N})\big\|_{L^2_{\rho d\rho}}$, we use the Schur test Lemma:

$$\Big(\sum_{R\in 2^{\mathbb{Z}}} \big(\sum_{N\in 2^{\mathbb{Z}}} Q(NR) A_{N,k}\big)^2 \Big)^{\frac{1}{2}} = \sup_{\|B_R\|_{l^2}\le 1} \sum_{R\in 2^{\mathbb{Z}}} \sum_{N\in 2^{\mathbb{Z}}} Q(NR) A_{N,k} B_R$$

which is bounded by

$$\leq C\Big(\sum_{R\in 2^{\mathbb{Z}}}\sum_{N\in 2^{\mathbb{Z}}} Q(NR)|A_{N,k}|^2\Big)^{\frac12}\Big(\sum_{R\in 2^{\mathbb{Z}}}\sum_{N\in 2^{\mathbb{Z}}} Q(NR)|B_R|^2\Big)^{\frac12}$$
$$\leq C\big(\sup_{R\in 2^{\mathbb{Z}}}\sum_{N\in 2^{\mathbb{Z}}} Q(NR)\sup_{N\in 2^{\mathbb{Z}}}\sum_{R\in 2^{\mathbb{Z}}} Q(NR)\big)^{\frac12}\Big(\sum_{N\in 2^{\mathbb{Z}}}|A_{N,k}|^2\Big)^{\frac12}\Big(\sum_{R\in 2^{\mathbb{Z}}}|B_R|^2\Big)^{\frac12}$$
$$\leq C\Big(\sum_{N\in 2^{\mathbb{Z}}}|A_{N,k}|^2\Big)^{\frac12}.$$

Putting the estimates together, we have obtained

$$\begin{aligned}\|e^{it\mathcal{D}_\nu}u_0\|^2_{L^p_tL^q_{rdr}L^2_\theta} &\leq C\sum_{k\in\mathbb{Z}+\frac12}\sum_{R\in 2^{\mathbb{Z}}}\Big(\sum_{N\in 2^{\mathbb{Z}}} Q(NR)A_{N,k}\Big)^2\\ &\leq C\sum_{k\in\mathbb{Z}+\frac12}\sum_{N\in 2^{\mathbb{Z}}} N^{2\left(1-\frac1p-\frac2q\right)}\big\|(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})\big\|^2_{L^2_{\rho d\rho}}\\ &\leq C\|u_0\|^2_{\dot H^{1-\frac1p-\frac2q}_{\mathcal{D}_\nu}}.\end{aligned}$$

Let us now suppose that u_0 is Dirac-non radial, that is $\mathcal{P}_k u_{0,k}=0$ for $|k|=\frac12$. Then the first condition in (1.34) is satisfied for all $q\leq+\infty$. Then, the previous estimates prove the claim for any $q\in(5,+\infty)$. If $q=+\infty$, we argue as before; we take N,R dyadic numbers and $\phi\in C^\infty_c\big(\big[\frac12,1\big]\big)$ such that $\sum_{N\in 2^{\mathbb{Z}}}\phi(\rho N^{-1})=1$. By the immersion $l^2\hookrightarrow l^\infty$, the Minkowski's inequality and scaling we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}+\frac12}\Big\|\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r)\Big\|^2_{L^p_tL^\infty_{dr}}\\
&\lesssim\sum_{k\in\mathbb{Z}+\frac12}\Big\|\sum_{N\in 2^{\mathbb{Z}}}\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|^2_{L^p_tL^\infty_{dr}}\\
&\lesssim\sum_{k\in\mathbb{Z}+\frac12}\Big\|\sup_{R\in 2^{\mathbb{Z}}}\Big\|\sum_{N\in 2^{\mathbb{Z}}}\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|_{L^\infty_{dr}([R,2R])}\Big\|^2_{L^p_t}\\
&\lesssim\sum_{k\in\mathbb{Z}+\frac12}\Big\|\Big(\sum_{R\in 2^{\mathbb{Z}}}\Big\|\sum_{N\in 2^{\mathbb{Z}}}\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|^2_{L^\infty_{dr}[R,2R]}\Big)^{\frac12}\Big\|^2_{L^p_t}\\
&\lesssim\sum_{k\in\mathbb{Z}+\frac12}\sum_{R\in 2^{\mathbb{Z}}}\Big(\sum_{N\in 2^{\mathbb{Z}}}\Big\|\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)\phi(\tfrac{\rho}{N})](r)\Big\|_{L^p_tL^\infty_{dr}([R,2R])}\Big)^2\\
&=\sum_{k\in\mathbb{Z}+\frac12}\sum_{R\in 2^{\mathbb{Z}}}\Big(\sum_{N\in 2^{\mathbb{Z}}}N^{2-\frac1p}\Big\|P_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(N\rho)\phi(\rho)](r)\Big\|_{L^p_tL^\infty_{dr}([NR,2NR])}\Big)^2.\end{aligned}\tag{1.35}$$

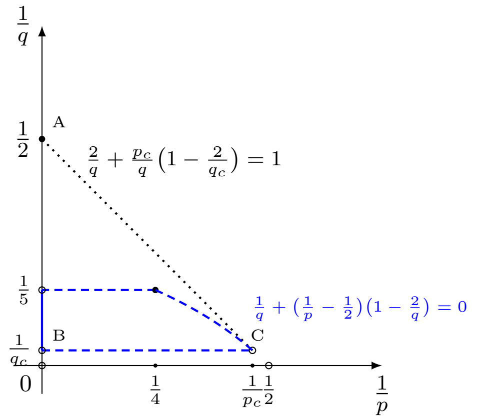


Figure 1.2: case $n = 2$, $\nu = \frac{1}{4}$.

for every $p > 2$. Then, we conclude using the Schur test Lemma as before.
Step 4: We observe that the set of admissible exponents is not empty if and only if

$$5 < \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}} \quad \Longleftrightarrow \quad |\nu| < \frac{\sqrt{24}}{10}.$$

In this case, we have the validity of the Strichartz estimates in the set with blue boundary described in Figure 1.2, where we define $q_c := \frac{2}{\frac{1}{2} - \gamma_{\frac{1}{2}}}$ and the corresponding endpoint index satisfying (1.34) as p_c, that is $p_c := \frac{\gamma_{\frac{1}{2}} + \frac{1}{2}}{\gamma_{\frac{1}{2}}}$. Then, we can interpolate between the estimate in Lemma 1.3.5 with (p, q_c), $p \in (p_c, +\infty]$ and the standard estimate

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L_t^\infty L^2_{rdr} L^2_\theta} \leq C \|u_0\|_{L^2}$$

to reach the triangle $\widehat{ABC}$ (Figure 1.2). Lastly, the estimates for Dirac-non radial solutions follows from the same argument. □

Remark 1.3.6. Let us observe that the estimate in Lemma 1.3.4 is not optimal in the range $R \geq 1$ and $p > 4$. Indeed, if we use (1.29) to estimate the L^p norm for $p > 4$ we get the improved exponent

$$\tilde{\beta}(q) = \begin{cases} \frac{1}{q} - \frac{1}{2}, & \text{if } q \in [2, 4), \\ -\frac{1}{12}, & \text{if } q = 4, \\ \frac{1}{3q} - \frac{1}{3}, & \text{if } q \in (4, +\infty). \end{cases}$$

However, this does not improve the Strichartz estimates.

Strichartz estimates 3D case

We now discuss the 3-dimensional case. We retrace the proof of the 2D case. For the sake of brevity we omit the proofs of the first three steps since the computations are similar.
Step 1:

Lemma 1.3.7. *Let $k \in \mathbb{Z}^*$, $\gamma = \sqrt{k^2 - \nu^2}$, $|\nu| \leq 1$ and $q \in [2, +\infty]$. Then, the following estimates hold*

$$\|\psi_k\|_{L^q([R,2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - 1}, & \text{if } R \leq 1, \\ R^{\beta(q)}, & \text{if } R \geq 1, \end{cases}$$

and

$$\|\psi_k'\|_{L^q([R,2R])} \leq C \times \begin{cases} R^{\gamma + \frac{1}{q} - 2}, & R \leq 1, \\ R^{\beta(q)}, & R \geq 1, \end{cases}$$

where

$$\beta(q) = \begin{cases} \frac{1}{q} - 1 & \text{if } q \in [2,4), \\ \frac{1}{q} - \frac{5}{6} & \text{if } q \in [4, +\infty] \end{cases}$$

and all the constants are independent of γ, k, but eventually dependent on q.

Step 2:

Lemma 1.3.8. *Let $(p,q) \in [2,+\infty] \times [2,+\infty)$ and $k \in \mathbb{Z}^*$. Let $I = [\frac{1}{2}, 1]$ and $g_k \in L^2_{\rho^2 d\rho}((0,+\infty))$ such that $\mathrm{supp}(g_k) \subset I$. Then*

$$\left\| \mathcal{P}_k^{-1}\left[e^{it\rho\sigma_3} g_k(\rho)\right](r) \right\|_{L^p_t L^q_{r^2 dr}([R,2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho^2 d\rho}(I)} \times \begin{cases} R^{\gamma - 1 + \frac{3}{q}}, & R \leq 1 \\ R^{\frac{1}{q} + \beta(p)(1 - \frac{2}{q})}, & R \geq 2, \end{cases}$$

where

$$\beta(p) = \begin{cases} \frac{1}{p} - 1 & \text{if } p \in [2,4), \\ \frac{1}{p} - \frac{5}{6} & \text{if } p \in [4, +\infty] \end{cases}$$

and the constant C is independent of k, ν, but eventually depends on p and q. Moreover, if $q = +\infty$, for all $p \geq 2$ we have

$$\left\| \mathcal{P}_k^{-1}\left[e^{it\rho\sigma_3} g_k(\rho)\right](r) \right\|_{L^p_t L^\infty_{dr}([R,2R])} \leq C \|g_k(\rho)\|_{L^2_{\rho d\rho}(I)} \times \begin{cases} R^{\gamma - \frac{3}{2}}, & R \leq 1 \\ R^{\beta(p)}, & R \geq 2, \end{cases}$$

Notice that in this case we can take $p = 2$ since $\beta(2) < 0$.
Step 3:

Lemma 1.3.9. *Let*

$$p \in [2, +\infty], \qquad \frac{16}{5} < q < \frac{3}{1 - \gamma_1}, \qquad \frac{1}{q} + \left(\frac{1}{p} - 1\right)\left(1 - \frac{2}{q}\right) < 0.$$

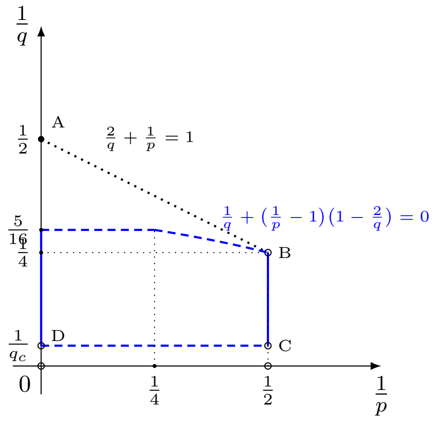


Figure 1.3: case $n = 3$, $\nu = \frac{1}{2}$.

Then

$$\|e^{it\mathcal{D}_\nu}u_0\|_{L^p_t L^q_{r^2dr} L^2_\theta} \leq C\|u_0\|_{\dot{H}^s_{\mathcal{D}_\nu}}$$

where $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$ *and* $\gamma_1 = \sqrt{1-\nu^2}$. *Moreover, if* u_0 *is Dirac-non radial we can take* $q \in \left(\frac{16}{5}, +\infty\right]$.

Proof. Notice that it suffices to show that

$$\sum_{k\in\mathbb{Z}^*} \left\| \mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \right\|^2_{L^p_t L^q_{r^2dr}} \leq C\|u_0\|^2_{\dot{H}^s_{\mathcal{D}_\nu}}.$$

We proceed as in Lemma 1.3.5, with suitable modifications. □

Step 4: In the last step we want to combine the previous estimate with the conservation of mass in order to get the Strichartz estimates in Theorem 1.1.2. We notice that $q = 4$ is admissible if and only if

$$4 < \frac{3}{1-\gamma_1} = q_c \iff |\nu| < \frac{\sqrt{15}}{4}.$$

In this case, we have the validity of the Strichartz estimates in the set with blue boundary described in Figure 1.3. Then we interpolate between the estimates in Lemma 1.3.9 with exponent $(p, 4)$, $p > 2$ and the $L^\infty_t L^2_{r^2dr} L^2_\theta$-estimate, widening the admissible set up to cover the region $ABCD$ in Figure 1.3.

1.4 Nonlinear system

This Section is dedicated to the proof of the well posedness results for the system (1.9) with Dirac-radial initial datum. We start by proving the classical Strichartz estimates for Dirac-radial functions (Proposition 1.4.1 below) as well as two Lemmas that we will use throughout the proofs of the nonlinear results.

Proposition 1.4.1. *Let $|\nu| < \frac{\sqrt{15}}{4}$ and (p,q) such that*

$$2 \le p \le +\infty, \quad 2 \le q < q_c, \quad \frac{2}{q} + \frac{1}{p} < 1 \text{ or } (p,q) = (\infty, 2), \tag{1.36}$$

where $q_c = \frac{3}{1-\sqrt{1-\nu^2}}$. Then, there exists a constant $C > 0$ such that for any u_0 Dirac-radial, $u_0 \in \dot{H}^s(\mathbb{R}^3;\mathbb{C}^4)$, the following Strichartz estimates hold

$$\|e^{it\mathcal{D}_\nu} u_0\|_{L^p_t L^q_x} \le C\|u_0\|_{\dot{H}^s} \tag{1.37}$$

provided $s = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}$.

Corollary 1.4.2. *The Strichartz estimate (1.37) implies the estimate*

$$\|u\|_{L^p([0,T],L^q(\mathbb{R}^3))} \le C\|f\|_{L^1([0,T],\dot{H}^s(\mathbb{R}^3))}, \tag{1.38}$$

for any u solution of

$$i\partial_t u + \mathcal{D}_\nu u = f, \quad u(0) = 0$$

in the time interval $[0,T]$, $T > 0$ and p, q, s as in Proposition 1.4.1.

Proof of Proposition 1.4.1. Let u_0 be Dirac-radial. From (1.3) and observing that the functions Ξ_k are bounded, we get

$$\begin{aligned}
\|u\|_{L^p_t L^q_x} &= \Big\| \sum_{|k|=1} \mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r) \cdot \Xi_k(\theta) \Big\|_{L^p_t L^q_{r^2dr} L^q_\theta} \\
&\le \sum_{|k|=1} \|\Xi_k\|_{L^q_\theta} \big\|\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r)\big\|_{L^p_t L^q_{r^2dr}} \\
&\lesssim \Big(\sum_{|k|=1} \big\|\mathcal{P}_k^{-1}[e^{it\rho\sigma_3}(\mathcal{P}_k u_{0,k})(\rho)](r)\big\|^2_{L^p_t L^q_{r^2dr}} \Big)^{\frac{1}{2}}.
\end{aligned}$$

Then we can proceed as in the proof of Lemma 1.3.9 and we conclude by interpolation with the $L^\infty_t L^2_x$ estimate. □

Proof of Corollary 1.4.2. We observe that u can be written, by Duhamel formula, as

$$u(t,x) = -i \int_0^t e^{i(t-s)\mathcal{D}_\nu} f(s) ds.$$

Then, the estimates (1.38) follows using the Minkowski inequality and (1.37). □

Remark 1.4.3. We observe that Proposition 1.4.1 holds the same if the initial datum u_0 is such that, in the decomposition given by (1.3), $u_{0,k} \neq 0$ only for a finite number of $k \in \mathbb{Z}^*$. In fact, inequality (1.4) remains true if we sum over a finite number of indexes. Moreover, a similar result can be proved for the 2D case.

Remark 1.4.4. We observe that $N(u) = \big(\omega * \langle \beta u, u\rangle\big)u$ preserves the Dirac-radiality of u. More precisely, it has been shown in [25] (Lemma 5.5) that if u is a Dirac-radial function, then $\langle \beta u, u\rangle$ is a radial scalar function (in the classical sense). Moreover, we recall that ω is radial, then $\omega * \langle \beta u, u\rangle$ is a radial function which implies that $N(u)$ is still Dirac-radial.

Lemma 1.4.5. *Let $u \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$ and $\omega \in L^\gamma(\mathbb{R}^3) \cap L^\alpha(\mathbb{R}^3)$. Then, the following inequality holds*

$$\big\|\big(\omega * \langle \beta u, u\rangle\big)u\big\|_{\dot H^s} \lesssim \|\omega\|_{L^\gamma}\|u\|_{\dot H^s}\|u\|_{L^{\mu_2}}\|u\|_{L^{p_2}} + \|\omega\|_{L^\alpha}\|u\|_{\dot H^s}\|u\|^2_{L^{2\beta}}, \tag{1.39}$$

where $s \geq 0$, $\alpha, \beta, \gamma \in [1, +\infty]$, $p_2, \mu_2 \in (1, +\infty]$ such that

$$\frac{1}{\gamma} + \frac{1}{p_2} + \frac{1}{\mu_2} = 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Proof. Let $D^s := (-\Delta)^{\frac{s}{2}}$. In the following all the estimates have to be thought component by component and then summed back together. From generalized fractional Leibniz rule (see e.g. [73] Thm. 1),

$$\begin{aligned}\Big\|\big(\omega * \langle \beta u, u\rangle\big)u\Big\|_{\dot H^s} &= \Big\|D^s\big[\big(\omega * \langle \beta u, u\rangle\big)u\big]\Big\|_{L^2} \\ &\lesssim \Big\|D^s\big(\omega * \langle \beta u, u\rangle\big)\Big\|_{L^{p_1}}\|u\|_{L^{p_2}} + \big\|\omega * \langle \beta u, u\rangle\big\|_{L^\infty}\big\|D^s u\big\|_{L^2},\end{aligned} \tag{1.40}$$

where

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 \in (1, +\infty),\ p_2 \in (1, +\infty].$$

We estimate separately:

- Using Young's inequality

  $$\big\|\omega * \langle \beta u, u\rangle\big\|_{L^\infty} \lesssim \|\omega\|_{L^\alpha}\big\|\langle \beta u, u\rangle\big\|_{L^\beta} \lesssim \|\omega\|_{L^\alpha}\|u\|^2_{L^{2\beta}},$$

  where

  $$1 = \frac{1}{\alpha} + \frac{1}{\beta}, \quad \alpha, \beta \in [1, \infty];$$

- Using Young's inequality and generalized Leibniz rule

  $$\begin{aligned}\Big\|D^s\big(\omega * \langle \beta u, u\rangle\big)\Big\|_{L^{p_1}} &= \Big\|\omega * D^s\langle \beta u, u\rangle\Big\|_{L^{p_1}} \lesssim \|\omega\|_{L^\gamma}\big\|D^s\langle \beta u, u\rangle\big\|_{L^\mu} \\ &\lesssim \|\omega\|_{L^\gamma}\big\|D^s u\big\|_{L^2}\|u\|_{L^{\mu_2}},\end{aligned}$$

  where

  $$1 + \frac{1}{p_1} = \frac{1}{\gamma} + \frac{1}{\mu}, \qquad \frac{1}{\mu} = \frac{1}{2} + \frac{1}{\mu_2}, \qquad \mu, \gamma \in [1, +\infty],\ \mu_2 \in (1, \infty].$$

  Summing up, we obtain the result.

□

Lemma 1.4.6. *Let $u, v \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$, then*

$$\big\|D^s\big(\langle\beta u, u\rangle - \langle\beta v, v\rangle\big)\big\|_{L^\mu} \lesssim \big\|D^s(u-v)\big\|_{L^{\mu_1}}(\|u\|_{L^{\mu_2}} + \|v\|_{L^{\mu_2}})+ \\ + \big(\big\|D^s u\big\|_{L^{\gamma_1}} + \big\|D^s v\big\|_{L^{\gamma_1}}\big)\|u-v\|_{L^{\gamma_2}}$$

where $s \geq 0$ and

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Proof. We notice that, if u, v are scalar functions, we have

$$D^s(|u|^2 - |v|^2) = D^s(u\bar{u} - v\bar{v} \pm v\bar{u}) = D^s((u-v)\bar{u}) + D^s(v(\bar{u} - \bar{v}))$$

Then we conclude arguing component by component and by the fractional Leibniz rule.

□

We can now prove Theorem 1.1.8.

Proof. Let $T, M > 0$, we define the space

$$X_{T,M} := \{u \in L^\infty_T \dot{H}^s \colon \|u\|_{L^\infty_T \dot{H}^s} \leq M\},$$

that, endowed with the metric

$$d(u,v) = \|u - v\|_{L^\infty_T \dot{H}^s},$$

is a complete metric space, and the map

$$\Phi(u)(t,x) := e^{it\mathcal{D}_\nu} u_0(x) - i \int_0^t e^{i(t-s)\mathcal{D}_\nu} N(u)(s,x) ds. \tag{1.41}$$

Then, the proof relies on standard contraction argument; we want to find T, M such that $\Phi : X_{T,M} \to X_{T,M}$ is a contraction map on $(X_{T,M}, d)$.

Step 1: We observe that, since $s < \frac{3}{2}$, $\dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}}$ and that Lemma 1.4.5 holds with the choice

$$\begin{cases} \mu_2 = p_2 = 2\beta = \frac{6}{3-2s} = 2p', \\ \alpha = \gamma = \frac{3}{2s} = p. \end{cases}$$

Moreover, since $s \leq 1$ and $|\nu| < \frac{\sqrt{3}}{2}$, $\|u\|_{\dot{H}^s} \simeq \|u\|_{\dot{H}^s_{\mathcal{D}_\nu}}$ (Proposition 1.2.7). Then, from Minkowski's inequality and (1.39), we have

$$\begin{aligned} \big\|\Phi(u)\big\|_{L^\infty_T \dot{H}^s} &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \big\|e^{i(t-\tau)\mathcal{D}_\nu} N(u)(\tau,x)\big\|_{L^\infty_T \dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \int_0^T \big\|N(u)\big\|_{\dot{H}^s} d\tau \\ &\lesssim \|u_0\|_{\dot{H}^s} + \|\omega\|_{L^p} \int_0^T \|u\|_{\dot{H}^s} \|u\|^2_{L^{\frac{6}{3-2s}}} d\tau \end{aligned}$$

and, from the Sobolev's embedding, we get

$$\big\|\Phi(u)\big\|_{L^\infty_T\dot{H}^s} \lesssim \|u_0\|_{\dot{H}^s} + \|\omega\|_{L^p}T\|u\|^3_{L^\infty_T\dot{H}^s} \leq C\|u_0\|_{\dot{H}^s} + C\|\omega\|_{L^p}TM^3,$$

for some $C > 0$.
Step 2: Proceeding as before, we get

$$\begin{aligned}\big\|\Phi(u)-\Phi(v)\big\|_{L^\infty_T\dot{H}^s} &\lesssim \int_0^T \big\|N(u)-N(v)\big\|_{\dot{H}^s}d\tau\\ &\lesssim \int_0^T \Big[\big\|\big(\omega*\big(\langle\beta u,u\rangle-\langle\beta v,v\rangle\big)\big)u\big\|_{\dot{H}^s} + \big\|\big(\omega*\langle\beta v,v\rangle\big)(u-v)\big\|_{\dot{H}^s}\Big]d\tau.\\ &= \int_0^T (I+II)(\tau)\,d\tau.\end{aligned}$$

We estimate separately I and II; from Lemma 1.4.5 and Young's inequality, we have

$$\begin{aligned}I &\lesssim \big\|\omega*|D|^s\big(\langle\beta u,u\rangle-\langle\beta v,v\rangle\big)\big\|_{L^{\frac{3}{s}}}\|u\|_{L^{\frac{6}{3-2s}}} + \big\|\omega*\big(\langle\beta u,u\rangle-\langle\beta v,v\rangle\big)\big\|_{L^\infty}\|u\|_{\dot{H}^s}\\ &\lesssim \|\omega\|_{L^p}\big\||D|^s\big(\langle\beta u,u\rangle-\langle\beta v,v\rangle\big)\big\|_{L^{\frac{3}{3-s}}}\|u\|_{L^{\frac{6}{3-2s}}} + \|\omega\|_{L^p}\|u-v\|_{L^{\frac{6}{3-2s}}}\big\||u|+|v|\big\|_{L^{\frac{6}{3-2s}}}\|u\|_{\dot{H}^s}\end{aligned}$$

where for the second term we have used that, for two scalars functions f, g, $\big||f|^2-|g|^2\big| \leq |f-g|(|f|+|g|)$[8]. To estimate the first term we observe that Lemma 1.4.6 holds taking

$$\begin{cases}\mu_1 = \gamma_1 = 2,\\ \mu_2 = \gamma_2 = \frac{6}{3-2s},\end{cases}$$

then by the Sobolev embedding, we have

$$I \lesssim \|\omega\|_{L^p}\|u-v\|_{\dot{H}^s}\|u\|_{\dot{H}^s}\big(\|u\|_{\dot{H}^s}+\|v\|_{\dot{H}^s}\big)$$

In addition, we have

$$\begin{aligned}II &\lesssim \big\|\omega*|D|^s\langle\beta v,v\rangle\big\|_{L^{\frac{3}{s}}}\|u-v\|_{L^{\frac{6}{3-2s}}} + \big\|\omega*\langle\beta v,v\rangle\big\|_{L^\infty}\|u-v\|_{\dot{H}^s}\\ &\lesssim \|\omega\|_{L^p}\|v\|_{L^{\frac{6}{3-2s}}}\|v\|_{\dot{H}^s}\|u-v\|_{L^{\frac{6}{3-2s}}} + \|\omega\|_{L^p}\|v\|^2_{L^{\frac{6}{3-2s}}}\|u-v\|_{\dot{H}^s}.\end{aligned}$$

Putting all the estimates together, we have obtained that there exists $\tilde{C} > 0$ such that

$$\big\|\Phi(u)-\Phi(v)\big\|_{L^\infty_T\dot{H}^s} \leq \tilde{C}\|\omega\|_{L^p}\|u-v\|_{L^\infty_T\dot{H}^s}\int_0^T \big(\|u\|_{\dot{H}^s}+\|v\|_{\dot{H}^s}\big)^2 d\tau.$$

Then, if we choose T, M such that

$$\begin{cases}C\|u_0\|_{\dot{H}^s} \leq \frac{M}{2},\\ T < \min\big\{\frac{1}{2C\|\omega\|_{L^p}M^2}, \frac{1}{8\tilde{C}\|\omega\|_{L^p}M^2}\big\},\end{cases}$$

Φ maps $X_{T,M}$ into itself and it is a contraction on $(X_{T,M}, d)$ and applying the Banach fixed-point theorem we get the claim. □

[8]Then, for u, v is true that $\big|\langle\beta u,u\rangle-\langle\beta v,v\rangle\big| \leq 4|u-v|(|u|+|v|)$.

Now we turn to the proof of Theorem 1.1.9; notice that in the proof above we didn't use any kind of Strichartz estimates. We will exploit them in the following.

Proof. The idea is to apply the Banach contraction theorem on a ball in the space $Y_T = L^\infty\big([0,T];\dot H^s(\mathbb{R}^3)\big) \cap L^r([0,T];L^q(\mathbb{R}^3))$, $T>0$, endowed with the norm

$$\|u\|_{Y_T} := \sup_{t\in[0,T]} \|u\|_{\dot H^s(\mathbb{R}^3)} + \|u\|_{L^r([0,T],L^q(\mathbb{R}^3))},$$

for some (r,q) admissible for Proposition 1.4.1 so that

$$\|e^{it\mathcal{D}_\nu}u_0\|_{L^r_tL^q_x} \leq C\|u_0\|_{\dot H^s},$$

where $s=\frac{3}{2}-\frac{1}{r}-\frac{3}{q}$ is such that $s\leq 1$. We choose $(r,q)=(2p,2p')$, where p' is the conjugate exponent of p. We claim that this is an admissible couple and that we can find $T,M>0$ such that the map Φ is as in (1.41), is a contraction on the complete metric space $(Y_{T,M},d)$, where

$$Y_{T,M} := \big\{u\in L^\infty_T\dot H^s \cap L^{2p}_TL^{2p'} \colon \|u\|_{Y_T}\leq M\big\}, \quad d(u,v):=\|u-v\|_{Y_T}.$$

Let us now prove the claim. We notice that if $p\in\big(\frac{3}{1+2\sqrt{1-\nu^2}},+\infty\big)$ then $(2p,2p')$ satisfies conditions (1.36) and $s=\frac{3}{2}-\frac{1}{2p}-\frac{3}{2p'}=\frac{1}{p}\leq 1$. Moreover, estimate (1.39) holds with the following choice of indexes

$$\begin{cases} \mu_2 = p_2 = 2\beta = 2p', \\ \alpha=\gamma=p, \end{cases}$$

Then, proceeding as before and by (1.38), we get

$$\begin{aligned}
\big\|\Phi(u)\big\|_{\dot H^s} &\lesssim \|u_0\|_{\dot H^s} + \int_0^T \big\|e^{i(t-\tau)\mathcal{D}_\nu}N(u)(\tau,x)\big\|_{L^\infty_T\dot H^s}d\tau \\
&\lesssim \|u_0\|_{\dot H^s} + \int_0^T \|N(u)\|_{\dot H^s}d\tau \\
&\lesssim \|u_0\|_{\dot H^s} + \|\omega\|_{L^p}\|u\|_{L^\infty_T\dot H^s}\int_0^T\|u(\tau)\|^2_{L^{2p'}}d\tau \\
&\lesssim \|u_0\|_{\dot H^s} + T^{1-\frac{1}{p}}\|\omega\|_{L^p}\|u\|_{L^\infty_T\dot H^s}\|u\|^2_{L^{2p}_TL^{2p'}}
\end{aligned}$$

and

$$\big\|\Phi(u)\big\|_{L^{2p}_TL^{2p'}} \lesssim \|u_0\|_{\dot H^s} + T^{1-\frac{1}{p}}\|\omega\|_{L^p}\|u\|_{L^\infty_T\dot H^s}\|u\|^2_{L^{2p}_TL^{2p'}}.$$

Moreover

$$\begin{aligned}
\big\|\Phi(u)-\Phi(v)\big\|_{Y_T} &\lesssim \int_0^T \big\|N(u)-N(v)\big\|_{\dot H^s}d\tau \\
&\lesssim \int_0^T \Big[\big\|\big(\omega*(\langle\beta u,u\rangle-\langle\beta v,v\rangle)\big)u\big\|_{\dot H^s} + \big\|\big(\omega*\langle\beta v,v\rangle\big)(u-v)\big\|_{\dot H^s}\Big]d\tau \\
&= \int_0^T (I+II)(\tau)\,d\tau.
\end{aligned}$$

We estimate I, II separately; from Lemma 1.4.5 and Young's inequality, we have

$$\begin{aligned} I &\lesssim \big\|\omega * D^s\big(\langle\beta u, u\rangle - \langle\beta v, v\rangle\big)\big\|_{L^{2p}}\|u\|_{L^{2p'}} + \big\|\omega * \big(\langle\beta u, u\rangle - \langle\beta v, v\rangle\big)\big\|_{L^\infty}\|u\|_{\dot H^s} \\ &\lesssim \|\omega\|_{L^p}\big\|D^s\big(\langle\beta u, u\rangle - \langle\beta v, v\rangle\big)\big\|_{L^\mu}\|u\|_{L^{2p'}} + \|\omega\|_{L^p}\big\|\langle\beta u, u\rangle - \langle\beta v, v\rangle\big\|_{L^{p'}}\|u\|_{\dot H^s}, \end{aligned}$$

where $\mu = (2p)'$. From Lemma 1.4.6, we can estimate the L^μ norm as

$$\big\|D^s\big(\langle\beta u, u\rangle - \langle\beta v, v\rangle\big)\big\|_{L^\mu} \lesssim \|u-v\|_{\dot H^s}(\|u\|_{L^{2p'}} + \|v\|_{L^{2p'}}) + \|u-v\|_{L^{2p'}}(\|u\|_{\dot H^s} + \|v\|_{\dot H^s}).$$

Then, we can continue the chain of inequalities

$$\begin{aligned} I &\lesssim \|\omega\|_{L^p}\|u-v\|_{L^\infty_T \dot H^s}(\|u\|_{L^{2p'}} + \|v\|_{L^{2p'}})\|u\|_{L^{2p'}} + \\ &+ \|\omega\|_{L^p}\|u-v\|_{L^{2p'}}\big[\big(\|u\|_{L^\infty_T \dot H^s} + \|v\|_{L^\infty_T \dot H^s}\big)\|u\|_{L^{2p'}} + \|u\|_{L^\infty_T \dot H^s}\||u| + |v|\|_{L^{2p'}}\big] \end{aligned}$$

We estimate II using again Young's inequality and fractional Leibniz rule, getting

$$II \leq \|\omega\|_{L^p}\|u-v\|_{L^\infty_T \dot H^s}\|v\|^2_{L^{2p'}} + \|\omega\|_{L^p}\|u-v\|_{L^{2p'}}\|u\|_{L^\infty_T \dot H^s}\|v\|_{L^{2p'}}.$$

Summing up, there exist two positive constants $C, \tilde C$ such that

$$\begin{aligned} \big\|\Phi(u)\big\|_{Y_T} &\leq C\|u_0\|_{Y_T} + CT^{1-\frac1p}\|\omega\|_{L^p}\|u\|^3_{L^{2p}_T L^{2p'}}, \\ \big\|\Phi(u) - \Phi(v)\big\|_{Y_T} &\leq \tilde C T^{1-\frac1p}\|\omega\|_{L^p}(\|u\|_{Y_T} + \|v\|_{Y_T})^2\|u-v\|_{Y_T}. \end{aligned}$$

To conclude, if we choose $T, M > 0$ such that

$$\begin{cases} C\|u_0\|_{\dot H^s} \leq \frac{M}{2}, \\ T^{1-\frac1p} < \min\big\{\frac{1}{2C\|\omega\|_{L^p}M^2}, \frac{1}{8\tilde C\|\omega\|_{L^p}M^2}\big\}, \end{cases}$$

we get the claim. □

1.5 Hints on the massive case

The aim of this section is to describe an ongoing project which is concerned with the investigation of the dispersive behavior of the solutions of the massive Dirac-Coulomb system, and more precisely the validity of Strichartz estimates. Seeing that this project is not yet ready for the publication, we will describe the results and the strategy of their proof without entering in technical details.

The starting point to study the massive case, in analogy to the massless one, is the spectral analysis of the Dirac operator

$$\mathcal{D}_{m,\nu} := \mathcal{D} + m\alpha_0 - \frac{\nu}{|x|}$$

where $\mathcal{D}$ is the 3-dimentional Dirac operator defined in Section 0.2.1, $\alpha_0 = \mathrm{diag}(1, 1, -1, -1)$, $m \geq 0$ is the mass of the particle and $\nu \in \mathbb{R}$ is the strength of the interaction potential. In this section we focus on the 3d case since more literature can be found and it seems to be the most relevant from the physics point of view. Indeed it is used to describe the motion of an electron in the field of an atomic nucleus. Moreover, we will restrict to *attractive* Coulomb potentials, that is $\nu \geq 0$.
The operator $\mathcal{D}_{\mu,\nu}$, as seen for the case $m = 0$, is defined on $C_c^\infty(\mathbb{R}^3; \mathbb{C}^4)$ and it is essentially self-adjoint with domain $H^1(\mathbb{R}^3; \mathbb{C}^4)$ if $\nu < \frac{\sqrt{3}}{2}$. Moreover, if $\frac{\sqrt{3}}{2} \leq \nu \leq 1$ it is possible to choose, among the infinitely many self-adjoint extensions, a distinguished one such that the domain is included in $H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{C}^4)$. We denote the self-adjoint extension with the same symbol. The first difference with respect to the massless case is found in the spectrum. It is shown that the essential spectrum is given by $(-\infty, -m] \cup [m, +\infty)$ for all self-adjoint extensions. The discrete spectrum is contained in $(-m, m)$. The energy levels can be computed explicitly, they are infinitely many and accumulates in m (see [123], Section 7.4). Moreover, it is proved in [20] (Proposition 4.1) that the operator has no eigenvalue in $\pm m$.

In order to study the dynamics of the model, first we look for a spectral representation of the operator. We remark that, since we are interested in the dispersive behavior of solutions of the Dirac-Coulomb equation, we will take functions spectrally concentrated into the essential spectrum of the operator.
As in the massless case, we separate the physical variable $x \in \mathbb{R}^3$ into its angular and radial part. So that, exploiting the partial wave decomposition, one has the following decomposition

$$\Phi(x) = \sum_{k \in \mathbb{Z}^*} \sum_{m=-|k|+\frac{1}{2}}^{|k|-\frac{1}{2}} \phi^+_{k,m_k}(r) \Xi^+_{k,m_k}(\theta_1, \theta_2) + \phi^-_{k,m_k}(r) \Xi^-_{k,m_k}(\theta_1, \theta_2)$$

where $\Phi \in L^2(\mathbb{R}^3; \mathbb{C}^4) \cong \bigoplus_{k,m_k} L^2((0, \infty), r^2 dr) \otimes \mathfrak{h}^3_{k,m_k}$. We recall that the definitions of $\Xi^\pm_{k,m_k}(\theta)$ and $\mathfrak{h}^3_{k,m_k}$ can be found in Section 1.2.1. Moreover, for any fixed $k \in \mathbb{Z}^*$ the action of the Dirac-Coulomb operator is given by the radial matrix

$$d_{\nu,k} = \begin{pmatrix} -\frac{\nu}{r} + m & -(\frac{d}{dr} + \frac{1}{r}) + \frac{k}{r} \\ \frac{d}{dr} + \frac{1}{r} + \frac{k}{r} & -\frac{\nu}{r} - m \end{pmatrix}.$$

In order to diagonalize the radial Dirac-Coulomb operator we define, as before, the *massive relativistic Hankel transform* as a sequence of operators, $k \in \mathbb{Z}^*$

$$\mathcal{P}_k : [L^2(\mathbb{R}^+, r^2 dr)]^2 \to L^2(\mathbb{R}, dp) \tag{1.42}$$

acting on spinors $f(r) = \binom{f_1(r)}{f_2(r)}$, $r > 0$, as

$$\mathcal{P}_k f(E) = \int_0^{+\infty} \Psi_k^T(r, E) \cdot f(r) r^2 dr, \qquad E \in \mathbb{R}. \tag{1.43}$$

where $\Psi_k(r,E) = \begin{pmatrix} F_k(r,E) \\ G_k(r,E) \end{pmatrix}$ represents the generalized eigenfunction of the operator. Note that $\mathcal{P}_k$ takes spinors into scalar functions. In order to provide an explicit representation for the functions F_k and G_k we follow [86] (Section 36). It is seen that for $E > m$ these functions take the form

$$F_k^+(E;r) := 2\sqrt{\frac{m+E}{\pi E}} e^{\frac{\pi \alpha_E}{2}} \frac{|\Gamma(\gamma+1+i\alpha_E)|}{\Gamma(2\gamma+1)} \frac{(2pr)^\gamma}{r} Im\{e^{ipr+i\xi(E)}{}_1F_1(\gamma - i\alpha_E, 2\gamma+1, -2ipr)\} \tag{1.44}$$

$$G_k^+(E;r) := 2\sqrt{\frac{m-E}{\pi E}} e^{\frac{\pi \alpha_E}{2}} \frac{|\Gamma(\gamma+1+i\alpha_E)|}{\Gamma(2\gamma+1)} \frac{(2pr)^\gamma}{r} Re\{e^{ipr+i\xi(E)}{}_1F_1(\gamma - i\alpha_E, 2\gamma+1, -2ipr)\} \tag{1.45}$$

where

$$\alpha_E := \frac{\nu E}{p}, \quad \gamma = \sqrt{k^2-\nu^2}, \quad e^{2i\xi(E)} = \frac{k - \frac{i\alpha_E}{E} m}{\gamma - i\alpha_E} = \frac{k - \frac{i\nu m}{p}}{\gamma - i\alpha_E}$$

and

$$p = \sqrt{E^2 - m^2} \quad \text{if } E > 0 \quad \text{and} \quad p = -\sqrt{E^2-m^2} \quad \text{if } E < 0.$$

Notice in particular that F^+ is real and G^+ is purely imaginary. An explicit representation for negative energies $E < -m$ can be deduced by the positive one and by making use of a *charge conjugation argument*: in fact the negative eigenfunctions can be written in terms of the positive ones as follows

$$\Psi_k^-(-E,r,\nu) = i\alpha_2\alpha_0 \overline{\Psi}_{-k}(E,r,-\nu).$$

In other words, one has to rely on the following substitutions

$$E \to -E, \qquad \nu \to -\nu, \qquad k \to -k. \tag{1.46}$$

and "exchange the roles of F and G". Therefore, we have that if $E > 0$

$$\Psi_k^-(-E,r,\nu) = \begin{pmatrix} F_k^-(r,-E,\nu,k) \\ G_k^-(r,-E,\nu,k) \end{pmatrix} = \begin{pmatrix} G_{-k}^+(r,E,-\nu) \\ F_{-k}^+(r,E,-\nu) \end{pmatrix} \tag{1.47}$$

where for $E < 0$ and $p < 0$

$$F_k^-(E,r) := 2\sqrt{\frac{m-E}{\pi E}} e^{\frac{-\pi \alpha_E}{2}} \frac{|\Gamma(\gamma+1-i\alpha_E)|}{\Gamma(2\gamma+1)} \frac{(-2pr)^\gamma}{r} Re\{e^{-ipr+i\tilde{\xi}(E)}{}_1F_1(\gamma + i\alpha_E, 2\gamma+1, 2ipr)\} \tag{1.48}$$

$$G_k^-(E,r) := 2\sqrt{\frac{m+E}{\pi E}} e^{\frac{-\pi \alpha_E}{2}} \frac{|\Gamma(\gamma+1-i\alpha_E)|}{\Gamma(2\gamma+1)} \frac{(-2pr)^\gamma}{r} Im\{e^{-ipr+i\tilde{\xi}(E)}{}_1F_1(\gamma + i\alpha_E, 2\gamma+1, 2ipr)\} \tag{1.49}$$

Notice that with the substitutions the phase shift gets modified as follows

$$e^{2i\xi(E)} \to e^{2i\tilde{\xi}(E)} = \frac{-k - \frac{i\alpha_E}{E} m}{\gamma + i\alpha_E} = e^{\pi - 2i\xi(E)}. \tag{1.50}$$

Notice also in particular that F^- is purely imaginary and G^- is real. With some computations it is possible to prove that the inverse of $\mathcal{P}_k$, $k \in \mathbb{Z}^*$, can be defined for each k:

$$\mathcal{P}_k^{-1} : L^2(\mathbb{R}, dp) \to [L^2(\mathbb{R}^+, r^2 dr)]^2$$

acting on scalars as

$$\mathcal{P}_k^{-1}\phi(s) = \left(\int_{-\infty}^{0} + \int_{0}^{+\infty}\right) \overline{\Psi_k(s,E)}\phi(E)dp. \tag{1.51}$$

Following the strategy of the massless case, the next step would be pointwise estimates uniformly in k of the functions, as in Proposition 1.2.5. Let us remark that the presence of the mass breaks the homogeneity of the generalized eigenfunctions. Therefore it is not straightforward to extend the pointwise estimates for the generalized eigenfunctions associated with the massless Dirac-Coulomb operator to the massive ones. Having in mind that our last goal would be to obtain strong estimates, that is uniform in k, on Ψ_k, $k \in \mathbb{Z}^*$, we restrict to the case $|k| = 1$. Observe that this correspond to consider Dirac-radial functions.

Therefore, if $f \in D(\mathcal{D}_{\nu,m})$ is a Dirac-radial function, spectrally localize on $[m, +\infty)$ we have the following decomposition:

$$e^{it\mathcal{D}_{\nu,m}} f(r) = \int_m^\infty \Psi(E; p(E)r) e^{itE} (\mathcal{P}_k f)(E) \frac{E}{p} dE$$

where $p = \sqrt{E^2 - m^2}$. We analyze separately the case where the spectral parameter E is close to m and where it is far. This is because, as for the free Klein-Gordon equation, one expects two different behaviors, closer, respectively, to the Schrödinger and the wave propagators. Therefore, we let $\phi(\rho) = \chi_{[1,2)}(\rho)$, so that $\sum_{N\in 2^{\mathbb{Z}}} \phi(\frac{E-m}{N}) = 1$ for any $E \in (m, +\infty)$ and we write

$$e^{it\mathcal{D}_{\nu,m}} f(r) =$$
$$\sum_{N\in 2^{\mathbb{Z}}} \int_0^\infty \Psi(N\tilde{E} + m; p(N\tilde{E} + m)r) e^{it(N\tilde{E}+m)} (Pf)(N\tilde{E} + m))\phi(\tilde{E}) \frac{N\tilde{E} + m}{p(N\tilde{E} + m)} N d\tilde{E}. \tag{1.52}$$

Then, the we differentiate among two cases: $i)$ $N = 2^{\mathbb{N}}$, $ii)$ $N = 2^{-\mathbb{N}^*}$.
In the first case, which correspond to high frequencies, we write

$$p(N\tilde{E} + m) = \sqrt{N^2\tilde{E}^2 + 2mN\tilde{E}} = N\sqrt{\tilde{E}^2 + \frac{2m}{N}\tilde{E}} =: N\tilde{p}. \tag{1.53}$$

We recall that $\tilde{E} \in [1, 2]$ and we observe that there exists $c > 0$, $c \neq c(N)$ such that

$$\tilde{E} \leq \tilde{p} \leq c\tilde{E}.$$

Moreover, $\alpha_{\tilde{E}} = \frac{\nu(N\tilde{E}+m)}{\sqrt{N^2\tilde{E}^2+2mN\tilde{E}}}$

$$c_1 \le |\alpha_{\tilde{E}}| \le c_2 \tag{1.54}$$

for some c_1, c_2 independent of N. Exploiting the integral representation of the Whittaker functions (see [34], Section 2.2) we get the following asymptotic estimates.

Lemma 1.5.1. *Let $\Psi_k(E;r)$ be the generalized eigenfunction of $\mathcal{D}_{\nu,m}$, with $E \gg m$ and $-1 < \nu < 0$. We define $\gamma_1 := \sqrt{1-\nu^2}$. Then, the following estimates hold:*

i) There exists a constant C independent of E (it depends on γ_1, m) such that for all $r \in (0, +\infty)$

$$\left|\Psi_k(E;\tilde{p}r)\right| \le C(\tilde{p}r)^{\gamma_1 - 1}.$$

$$\left|\Psi_k'(E;\tilde{p}r)\right| \le C(\tilde{p}r)^{\gamma_1 - 1}(r^{-1} + \tilde{p}).$$

ii) There exists a constant C independent of E (it depends on γ_1, m) such that for all $r \gg 1$

$$\left|\Psi_k(E;\tilde{p}r)\right|, \left|\Psi_k'(E;\tilde{p}r)\right| \le C(\tilde{p}r)^{-1},$$

with $\tilde{p}$ uniformly bounded with respect to E.

We look now at the low frequencies case, that is if $N = 2^{-n}, n \in \mathbb{N}^*$. We write

$$p(N\tilde{E} + m) = \sqrt{N^2\tilde{E}^2 + 2mN\tilde{E}} = \sqrt{N}\sqrt{2m\tilde{E} + N\tilde{E}^2} =: \sqrt{N}\tilde{p}.$$

We observe that, for any $c > \sqrt{2m+1}$ the following holds

$$\sqrt{\tilde{E}} \le \tilde{p} \le c\sqrt{\tilde{E}}.$$

We notice that in this case $\alpha_{\tilde{E}}$ is unbounded above. This is different from the massless case or the high frequencies case. In particular,

$$|\alpha_{\tilde{E}}| \in O(N^{-\frac{1}{2}}) \quad \text{as N goes to 0.} \tag{1.55}$$

Arguing as before, it can be proved

Lemma 1.5.2. *Let $\Psi_k(E;r)$ be the generalized eigenfunction of $\mathcal{D}_{\nu,m}$, with $E \sim m$ and $-1 < \nu < 0$. We define $\gamma_1 := \sqrt{1-\nu^2}$. Then, the following estimates hold:*

i) There exists a constant C independent of E (it depends on γ_1, m) such that for all $r \in (0, +\infty)$

$$\left|\Psi_k(E;\tilde{p}r)\right| \le C(\tilde{p}r)^{\gamma_1 - 1}.$$

$$\left|\Psi_k'(E;\tilde{p}r)\right| \le C(\tilde{p}r)^{\gamma_1 - 1}(r^{-1} + \tilde{p}).$$

ii) There exists a constant C independent of E (it depends on γ_1, m) such that for all $r \gg 1$

$$\left|\Psi_k(E;\tilde{p}r)\right|, \left|\Psi_k'(E;\tilde{p}r)\right| \le C|\alpha_E|(\tilde{p}r)^{-1},$$

with $\tilde{p}$ uniformly bounded with respect to E.

By combining these estimates with the decomposition (1.52) as in the massless case, we claim that the following Strichartz estimates hold:
let $\phi = \chi_{(m,m+1)}$ and $f \in D(\mathcal{D}_{\nu,m})$ Dirac-radial and spectrally localize on (m, ∞) then

i) low frequencies estimate

$$\|e^{it\mathcal{D}_{\nu,m}}\phi(\mathcal{D}_{\nu,m})f\|_{L^p_t L^q_{r^2dr}} \leq \||\mathcal{D}_{\nu,m}|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}f\|_{L^2};$$

ii) high frequencies estimate

$$\|e^{it\mathcal{D}_{\nu,m}}(1-\phi)(\mathcal{D}_{\nu,m})f\|_{L^p_t L^q_{r^2dr}} \leq \||\mathcal{D}_{\nu,m}|^{\frac{3}{2}-\frac{1}{p}-\frac{3}{q}}f\|_{L^2},$$

for any p, q such that

$$q < \frac{3}{1-\gamma_1}, \qquad \frac{1}{p} + \frac{2}{q} < 1 \cup (\infty, 2). \tag{1.56}$$

To conclude, we observe that these estimates are the same, for the number of derivatives and admissibility condition on the indexes of the radial Strichartz estimates which are derived in [105] for the Klein-Gordon equation with radially symmetric initial data.

Chapter 2

Strichartz estimates for the Dirac equation in compact manifolds without boundary

2.1 Introduction

This chapter is devoted to the study of the half wave/Klein-Gordon and Dirac equation on compact manifolds. We have already discussed in the introduction some recent results concerning the study of the Dirac equation on non-flat settings. Let us recall here, in a non exhaustive way, the literature concerning the study of Strichartz estimates for other dispersive equations on compact manifolds without boundary. We should mention the seminal works [82] for the wave and [23] for the Schrödinger equation respectively. In the former, the author shows that due to the finite speed of propagation, Strichartz estimates are the same as the estimates on flat Euclidean manifolds, while in the latter the authors prove Strichartz estimates with some additional loss of derivatives for the Schrödinger equation. In both cases, the estimates are only local in time, as indeed the compactness of the manifold prevents from having global dispersion. More recently, in [56] the author extended these results to deal with the fractional Schrödinger propagator $e^{it(-\Delta_g)^{\sigma/2}}$ for $\sigma \in [0, +\infty)\backslash\{1\}$. All of these results are essentially based on the so-called *WKB approximation*, that will be the key tool in our strategy as well.

The aim of the present Chapter is two-folded: as a first result, we investigate the dispersive properties of the "half" wave/Klein-Gordon equation on a compact Riemannian manifold without boundary $(\mathcal{M}, g)$ of dimension $d \geq 2$, that is for system

$$\begin{cases} i\partial_t u(t,x) + P_m^{1/2} u(t,x) = 0 \qquad u(t,x) : \mathbb{R}_t \times \mathcal{M} \to \mathbb{C}, \\ u(0,x) = u_0(x) \end{cases} \tag{2.1}$$

where $P_m = -\Delta_g + m^2$, $m \geq 0$ and Δ_g denotes the Laplace-Beltrami operator on $(\mathcal{M}, g)$. Notice that the solution $u = e^{itP_m^{1/2}} u_0$ to system (2.1) is classically connected to the standard wave/Klein-Gordon equations: the function

$$u(t,x) = \cos(tP_m^{1/2})u_0(x) + \frac{\sin(tP_m^{1/2})}{P_m^{1/2}}u_1(x) = \mathcal{R}(e^{itP_m^{1/2}})u_0(x) + \frac{\mathcal{I}(e^{itP_m^{1/2}})}{P_m^{1/2}}u_1(x)$$

indeed solves the system

$$\begin{cases} \partial_t^2 u(t,x) + P_m u(t,x) = 0, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x). \end{cases} \tag{2.2}$$

In particular, we shall prove that solutions to (2.1) satisfy local in time Strichartz estimates both for wave and Schrödinger admissible pairs: these estimates, whose proof as we shall see requires a refined version of the WKB approximation, improve on the existing results provided by [82]. As a second result, we will prove Strichartz estimates for the Dirac equation on compact manifolds, that is for system

$$\begin{cases} i\partial_t u - \mathcal{D}_m u = 0, \quad u : \mathbb{R}_t \times \mathcal{M} \to \mathbb{C}^N, \\ u(0,x) = u_0(x) \end{cases} \tag{2.3}$$

where again $(\mathcal{M}, g)$ is a compact Riemannian manifold without boundary of dimension $d \geq 2$ equipped with a spin structure, $\mathcal{D}_m$ represents the Dirac operator and the dimension of the target space $N = N(d) = 2^{\lfloor \frac{d}{2} \rfloor}$ depends on the parity of d. The estimates, in this case, can be somehow deduced, as we shall see, from the ones for (2.1), after "squaring" system (2.3). We recall that the construction of the Dirac operator on non-flat backgrounds is presented in Section 0.2.4.

Before stating our main Theorems, let us recall the definitions of *admissible pairs*:

Definition 2.1.1 (Wave admissible pair)**.** We say a pair (p,q) is wave admissible if

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q,d) \neq (2,\infty,3), \quad \frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}.$$

Definition 2.1.2 (Schrödinger admissible pair)**.** We say a pair (p,q) is Schrödinger admissible if

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

We also denote

$$\gamma_{p,q}^{\mathrm{KG}} := (1+d)\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p}, \quad \gamma_{p,q}^{\mathrm{W}} := d\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{p}.$$

In what follows, we shall use standard notation for the Sobolev spaces, that is

$$\|u\|_{H^s(\mathcal{M})} := \|(1-\Delta_g)^{s/2}u\|_{L^2(\mathcal{M})}.$$

Also, we shall use the classical Strichartz spaces $X(I, Y(\mathcal{M}))$ where the X norm is taken in the time variable and the Y norm in the space variable

We are now in a position to state our main results.

2.1.1 Main results

Theorem 2.1.3 (Strichartz estimates for wave and Klein-Gordon). *Let $\mathcal{M}$ be a Riemannian compact manifold without boundary of dimension $d \geq 2$. Let $I \subset \mathbb{R}$ be a bounded interval. Then, for any $m \geq 0$ the following estimates hold:*

1. *for any wave admissible pair (p,q), we have*

$$\|e^{itP_m^{1/2}}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C(I)\|u_0\|_{H^{\gamma^{\mathrm{W}}_{p,q}}(\mathcal{M})}; \tag{2.4}$$

2. *for any Schrödinger admissible pair (p,q), we have*

$$\|e^{itP_m^{1/2}}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C(I)\|u_0\|_{H^{\gamma^{\mathrm{KG}}_{p,q}+\frac{1}{2p}}(\mathcal{M})}. \tag{2.5}$$

Remark 2.1.4. Let us compare this result with the one in [82]. In fact, from Theorem 2 of [82] it is possible to deduce Strichartz estimates for a solution u to the half wave/Klein-Gordon equation (2.1) with $m \geq 0$, $d \geq 2$. We observe that the principal symbol of $hP_m^{1/2}$ is $q_0(x,\xi) = \sqrt{g^{i,j}(x)\xi_i\xi_j}$ and $\operatorname{rank}\partial_\xi^2 q_0(x,\xi) = d-1$; then, Theorem 2 of [82] states that

$$\|u\|_{L^p(I;B^r_{q,q_1}(\mathcal{M}))} \leq C(I)\|u_0\|_{H^s}$$

for some s, r, p, q and q_1. Here $B^s_{p,q}$ denote the standard Besov spaces, we refer to [6] (Section 2.7) for the definition. In particular, from the embedding $B^0_{q,2}(\mathcal{M}) \hookrightarrow L^q(\mathcal{M})$ that holds for every $q \in [2,+\infty]$, we get

$$\|e^{itP_m^{1/2}}u_0\|_{L^p(I;L^q(\mathcal{M}))} \leq C(I)\|u_0\|_{H^{\gamma^{\mathrm{W}}_{p,q}}(\mathcal{M})}$$

provided that $p \in [2,+\infty], q \in [2,+\infty]$ and

$$\begin{cases} p > 2 & \text{if} \quad (d-1)\left(\frac{1}{2}-\frac{1}{q}\right) \geq 1, \\ \frac{2}{p}+\frac{d-1}{q} \leq \frac{d-1}{2} & \text{otherwise.} \end{cases}$$

This recovers estimate (2.4).

On the other hand, in order to prove estimate (2.5), we have to consider an "h-dependent principal symbol" of $hP_m^{1/2}$ which is $q_{\tilde{m},h}(x,\xi) = \sqrt{g^{i,j}(x)\xi_i\xi_j + h^2\tilde{m}^2}$ with $\tilde{m} = m$ if $m > 0$ and $\tilde{m} = 1$ if $m = 0$ (as in definition (2.10)). Then, $\operatorname{rank}\partial_\xi^2 q_{\tilde{m},h}(x,\xi) = d$ for any $h > 0$ rather than $\operatorname{rank}\partial_\xi^2 q_0(x,\xi) = d-1$ as mentioned above. This will give us the Schrödinger admissible pairs as on the flat manifolds. The $\frac{1}{2p}$ loss of regularity is a consequence of the delicate analysis of the term $q_{\tilde{m},h}$ with respect to $h \in (0,1]$. For the details, see the end of this section and Remark 2.1.11.

Theorem 2.1.5 (Strichartz estimates for Dirac). *Let $\mathcal{M}$ be a Riemannian compact manifold without boundary of dimension $d \geq 2$ equipped with a spin structure. Let $I \subset \mathbb{R}$ be a bounded interval. Then, for any $m \geq 0$ the following estimates hold:*

1. *for any wave admissible pair* (p, q) *, we have*

$$\|e^{it\mathcal{D}_m}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C(I)\|u_0\|_{H^{\gamma^{\mathrm{W}}_{p,q}}(\mathcal{M})}; \tag{2.6}$$

2. *for any Schrödinger admissible pair* (p, q)*, we have*

$$\|e^{it\mathcal{D}_m}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C(I)\|u_0\|_{H^{\gamma^{\mathrm{KG}}_{p,q}+\frac{1}{2p}}(\mathcal{M})}. \tag{2.7}$$

Remark 2.1.6. Notice that our argument could be adapted with minor modifications to prove the same Strichartz estimates for equations posed on $\mathbb{R}^d$ with metrics g satisfying the following assumptions:

1. There exists $C > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k \leq C|\xi|^2; \tag{2.8}$$

2. For all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha g^{jk}(x)| \leq C_\alpha, \quad j, k \in \{1, \cdots, d\}. \tag{2.9}$$

We should stress the fact that assumptions (2.8)-(2.9) are much weaker than the classical "asymptotically flatness" assumptions, for which global in time Strichartz estimates have been proved for several dispersive flows, (see in particular [31] for the Dirac equation). On the other hand, in our weaker assumptions above we are only able to prove local-in-time Strichartz estimates.

Remark 2.1.7. We stress the fact that, to the very best of our knowledge, Theorem 2.1.5 is the first result concerning the dispersive dynamics of the Dirac equation on compact manifolds. We should point out the fact that it is not a trivial consequence of Theorem 2.1.3, as it would be in the Euclidean setting. Indeed, as recalled in Section 0.2.4, while in the flat case the relation $\mathcal{D}_m^2 = -\Delta + m^2$ directly connects the solutions to the Dirac equation to a system of decoupled Klein-Gordon equations, in a non-flat setting, as the definition of the Dirac operator requires to rely on a different connection, the *spin connection*, this identity becomes $\mathcal{D}_m^2 = -\Delta_S + \frac{1}{4}\mathcal{R} + m^2$ where Δ_S is the spinorial (not the scalar) Laplace operator and $\mathcal{R}$ is the scalar curvature of the manifold.

As a final result, we will show that the estimates (2.6) are sharp in the case of the spheres in dimension $d \geq 4$: this requires writing explicitly the eigenfunctions of the Dirac operator on the sphere and to prove some asymptotic estimates for them; as we will see, these will be a consequence of some well known asymptotic estimates for Jacobi polynomials.

2.1.2 Overview of the strategy

The strategy for proving Theorem 2.1.3 follows a well-established path based on WKB approximation: in fact, our proof is strongly inspired by the one of Theorem 1 in [23] and the one of Theorem 1.2 in [56]. As a consequence, we shall omit some of the proofs that can be found in those papers. On the other hand, in order to obtain our Strichartz estimates we will need some "refined" version of the WKB approximation: let us briefly try to review the main ideas.

Recall that $P_m = -\Delta_g + m^2$ and $P_0 = -\Delta_g$. The first ingredient that we need is the following standard Littlewood-Paley decomposition:

Proposition 2.1.8. *Let* $\widetilde{\phi} \in C_0^\infty(\mathbb{R})$ *and* $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ *such that*

$$\widetilde{\phi}(\lambda) + \sum_{k=1}^{\infty} \phi(2^{-2k}\lambda) = 1, \quad \lambda \in \mathbb{R}.$$

Then for all $q \in [2, \infty)$, *we have*

$$\|f\|_{L^q(\mathcal{M})} \leq C_q \left(\|\widetilde{\phi}(P_0)f\|_{L^q(\mathcal{M})} + \left(\sum_{k=1}^{\infty} \|\phi(2^{-2k}P_0)f\|^2_{L^q(\mathcal{M})} \right)^{1/2} \right).$$

Proof. See, e.g., Corollary 2.3 in [23]. □

The second ingredient is the following TT^* criterion:

Proposition 2.1.9. *Let* $(X, \mathcal{S}, \mu)$ *be a* σ*-finite measured space, and* $U : \mathbb{R} \to \mathcal{B}(L^2(X, \mathcal{S}, \mu))$ *be a weakly measurable map satisfying, for some constants* $C, \gamma, \delta > 0$,

$$\|U(t)\|_{L^2(X) \to L^2(X)} \leq C, \quad t \in \mathbb{R},$$
$$\|U(t)U(s)^*\|_{L^1(X) \to L^\infty(X)} \leq Ch^{-\delta}(1 + |t-s|h^{-1})^{-\tau}, \quad t, s \in \mathbb{R}.$$

Then for all pair (p, q) *satisfying*

$$p \in [2, \infty], \quad q \in [1, \infty], \quad (p, q, \tau) \neq (2, \infty, 1), \quad \frac{1}{p} \leq \tau \left(\frac{1}{2} - \frac{1}{q} \right),$$

we have

$$\|U(t)u\|_{L^p(\mathbb{R}, L^q(X))} \leq Ch^{-\kappa}\|u\|_{L^2(X)}$$

where $\kappa = \delta(\frac{1}{2} - \frac{1}{q}) - \frac{1}{p}$.

Proof. See [85] or Proposition 4.1 in [127] for a semiclassical version. □

Then, the third main ingredient we need is given by the following proposition. Here and in what follows, we shall denote with

$$\tilde{m} = \begin{cases} m & \text{if } m > 0 \\ 1 & \text{if } m = 0. \end{cases} \tag{2.10}$$

Proposition 2.1.10 (Dispersive estimates). *Let $m \geq 0$, and $\phi \in C_0^\infty(\mathbb{R}\backslash[-\tilde{m},\tilde{m}])$ with $\tilde{m}$ given by (2.10). Then, for any $t \in [-t_0, t_0]$,*

$$\|e^{itP_m^{1/2}}\phi(-h^2\Delta_g)u_0\|_{L^\infty(\mathcal{M})} \leq Ch^{-d}(1+|t|h^{-1})^{-(d-1)/2}\|u_0\|_{L^1(\mathcal{M})}; \tag{2.11}$$

for any $t \in h^{\frac{1}{2}}[-t_0, t_0]$,

$$\|e^{itP_m^{1/2}}\phi(-h^2\Delta_g)u_0\|_{L^\infty(\mathcal{M})} \leq Ch^{-d-1}(1+|t|h^{-1})^{-d/2}\|u_0\|_{L^1(\mathcal{M})}. \tag{2.12}$$

Let us quickly show how Theorem 2.1.3 can be derived from these three Propositions.

Proof of Theorem 2.1.3. We first consider the Strichartz estimates for wave admissible pair by using (2.11). From Proposition 2.1.9 and (2.11), we infer that

$$\|e^{itP_m^{1/2}}\phi(-h^2\Delta_g)u_0\|_{L^p([-t_0,t_0],L^q(\mathcal{M}))} \leq Ch^{-\gamma_{p,q}^{\mathrm{W}}}\|u_0\|_{L^2(\mathcal{M})}.$$

By writing I as a union of N intervals $I_c = [c-t_0, c+t_0]$ of length $2t_0$ with $N \leq C$, we have

$$\|e^{itP_m^{1/2}}\phi(-h^2\Delta_g)u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq Ch^{-\gamma_{p,q}^{\mathrm{W}}}\|u_0\|_{L^2(\mathcal{M})}.$$

Taking $h = 2^{-k}$, Proposition 2.1.8 and the Minkowski inequality give

$$\begin{aligned}\|e^{itP_m^{1/2}}u_0\|_{L^p(I,L^q(\mathcal{M}))} &\leq C\|e^{itP_m^{1/2}}\widetilde{\phi}(P_0)u_0\|_{L^p(I,L^q(\mathcal{M}))}+ \\ &\quad + C\left(\sum_{k=1}^{\infty}\|e^{itP_m^{1/2}}\phi(2^{-2k}P_0)u_0\|_{L^p(I,L^q(\mathcal{M}))}\right)^{1/2} \\ &\leq C\|u_0\|_{L^2(\mathcal{M})} + C\left(\sum_{k=1}^{\infty}2^{-2k\gamma_{p,q}^{\mathrm{W}}}\|\phi(2^{-2k}P_0)u_0\|_{L^2(\mathcal{M})}\right)^{1/2} \\ &\leq C\|u_0\|_{H^{\gamma_{p,q}^{\mathrm{W}}}(\mathcal{M})}\end{aligned}$$

since $[P_m, P_0] = 0$, and where we have used that

$$\begin{aligned}\|e^{itP_m^{1/2}}\widetilde{\phi}(P_0)u_0\|_{L^q(\mathcal{M})} &\lesssim \|e^{itP_m^{1/2}}\widetilde{\phi}(P_0)u_0\|_{H^s(\mathcal{M})} = \|(1-\Delta_g)^{s/2}\widetilde{\phi}(P_0)u_0\|_{L^2(\mathcal{M})} \\ &\lesssim C\|\widetilde{\phi}(P_0)u_0\|_{L^2(\mathcal{M})}\end{aligned}$$

as $\widetilde{\phi}(\lambda) \in C_0^\infty(\mathbb{R})$.

We now turn to the proof for Schrödinger admissible pairs; here we make use of (2.12). We write I as a union of $N = N_h$ intervals $I_{c_n} = [c_n - h^{1/2}t_0, c_n + h^{1/2}t_0]$, $c_n \in \mathbb{R}$, of length $2h^{\frac{1}{2}}t_0$ with $N \leq Ch^{-\frac{1}{2}}$. Using Proposition 2.1.9, we infer that

$$\begin{aligned}\|e^{itP_m^{1/2}}\phi(-h^2\Delta_g)u_0\|_{L^p(I,L^q(\mathcal{M}))} &\leq \left(\sum_{n=1}^{N}\int_{I_{c_n}}\|e^{itP_m^{1/2}}\phi(-h^2P_0)u_0\|^p_{L^q(\mathcal{M})}dt\right)^{1/p} \\ &\leq CN^{1/p}h^{-\gamma_{p,q}^{\mathrm{KG}}}\|u_0\|_{L^2(\mathcal{M})} \leq Ch^{-\gamma_{p,q}^{\mathrm{KG}}-\frac{1}{2p}}\|u_0\|_{L^2(\mathcal{M})}.\end{aligned}$$

Arguing as for the wave admissible pairs case, we conclude that

$$\|e^{itP_m^{1/2}}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C\|u_0\|_{H^{\gamma_{p,q}^{\mathrm{KG}}+\frac{1}{2p}}(\mathcal{M})}.$$

□

Therefore, the only thing we need to prove is Proposition 2.1.10: Section 2.2 will be devoted to this. As the proof is quite technical and involved, before entering the details let us try to explain the main ideas and the main improvements with respect to the existing results.

We are going to prove the dispersive estimates (2.11) and (2.12) by making use of the WKB approximation and stationary phase theorem (see [111] for generalities). For (2.11), one can obtain the estimate by using the "standard" WKB approximation, as done in [23, 56] after a slight refinement of the stationary phase method. However, for (2.12), a more structural modification is needed: roughly speaking, the standard WKB approximation uses that any h-dependent symbol A_h can be written asymptotically as follows

$$A_h \sim \sum_{j=0}^{N-1} h^j a_j + \mathcal{O}(h^N),$$

for some $N \in \mathbb{N}$, where here the terms $(a_j)_j$ are independent of h. In order to obtain the dispersive estimate (2.12), we consider instead an h-dependent WKB approximation, that is,

$$A_h \sim \sum_{j=0}^{N-1} h^j a_{j,h} + \mathcal{O}(h^N).$$

The difference is that after the asymptotic expansion $a_{j,h}$ will still be h-dependent, but their values and all the derivatives will be uniformly bounded w.r.t. $h \in (0,1]$.

To explain it better, let us consider the following semiclassical half Klein-Gordon equation (i.e., $m > 0$) on the flat manifold $(\mathbb{R}^d, \delta_{jk})$:

$$ih\partial_t\widetilde{u} + h\sqrt{m^2-\Delta}\widetilde{u} = 0, \quad \widetilde{u}(0,x) = \phi(-h^2\Delta)u_0(x). \tag{2.13}$$

We seek $\widetilde{u}$ as the following oscillatory integral

$$\widetilde{u}(s,x) = \int_{\mathbb{R}^d} e^{\frac{i}{h}S_h(s,x,\xi)} a(s,x,\xi,h)\widehat{u}_0\left(\frac{\xi}{h}\right)\frac{d\xi}{(2\pi h)^d} \tag{2.14}$$

where

$$a(s,x,\xi,h) = \sum_{j=0}^{N} h^j a_{j,h}(s,x,\xi), \quad a_{0,h}(0,x,\xi) = \phi(\xi), \quad a_{j,h}(0,x,\xi) = 0 \text{ for } j \geq 1$$

and

$$S_h(0, x, \xi) = x \cdot \xi.$$

We first consider the standard h-independent WKB approximation. Proceeding as in [56] using the fact that the principal symbol of $h^2 P_m$ is $p_{0,0}(x, \xi) = |\xi|^2$, we know that S_h satisfies $S_h(t, x, \xi) = x \cdot \xi + t|\xi|$ which solves the following Hamilton-Jacobi equation

$$\partial_t S_h - \sqrt{|\nabla_x S|^2} = 0, \quad S_h(0, x, \xi) = x \cdot \xi$$

and $(a_{j,h}(t, x, \xi))_j$ independent of h exist for t small enough. Then the problem that $\widetilde{u}$ solves is indeed a wave equation which is essentially equivalent to the following one:

$$\partial_t^2 \widetilde{u} - \Delta \widetilde{u} = f(\widetilde{u})$$

where $f(\widetilde{u}) := -m^2 \widetilde{u}$ plays the role of an inhomogeneous term. Obviously, the Strichartz estimates obtained by this h-independent WKB approximation are far from optimal for the massive case.
Now we turn to the h-dependent WKB approximation that we shall use in Subsection 2.2.2. Taking $p_{m,h} = |\xi|^2 + h^2 m^2$ as the principal symbol, as we will see in (2.30), the phase S_h now takes the form $S_h = x \cdot \xi + t\sqrt{h^2 m^2 + |\xi|^2}$ for $m > 0$. Then we will get that

$$\partial_t a_{0,h} - \nabla_\xi \sqrt{h^2 m^2 + h^2 |\xi|^2} \cdot \nabla_x a_{0,h} = 0,$$

which yields that $a_{0,h}(t, x, \xi) = \phi(\xi)$ for any $t \in \mathbb{R}$. Analogously, we have $a_{k,h}(t, x, \xi) = 0$ for $k = 1, \cdots, N$ and $t \in \mathbb{R}$. As a result, we deduce the following oscillatory integral representation for $\widetilde{u}$:

$$\widetilde{u}(t, x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ih^{-1}[(x-y)\cdot\xi + t\sqrt{h^2 m^2 + |\xi|^2}]} \phi(|\xi|^2) u_0(y) d\xi dy.$$

This formula holds for any $t \in \mathbb{R}$. Thus the Strichartz estimates that this WKB approximation produces are really the "standard" ones for Klein-Gordon equation in the flat Euclidean case.

We can conclude: compared to the standard WKB approximation, this h-dependent version gives the exact integral formula for the half Klein-Gordon equation on $(\mathbb{R}^d, \delta_{jk})$. Then the Strichartz estimates that we deduce directly is exactly the one for the Klein-Gordon equation rather than the one for the wave equation. Furthermore, on the flat Euclidean manifold, we can get the global in time Strichartz estimates by using this h-dependent WKB approximation (see, e.g., [101, Chp. 2.5]) while only local-in-time Strichartz estimates will be obtained by using the h-independent WKB approximation.

We conclude the introduction with the following remark, that is technical:

Remark 2.1.11. Compared with the Klein-Gordon Strichartz estimates on flat manifold $(\mathbb{R}^d, \delta_{jk})$, we will lose some regularity on the initial datum (see (2.7)) on compact manifolds $(\mathcal{M}, g)$. As we will see later (formula (2.31)), on the compact manifold, the phase term S_h satisfies

$$\partial_t S_h(t,x,\xi) - \nabla_\xi \sqrt{h^2 m^2 + h^2 g^{jk}(x)\partial_j S_h \partial_k S_h} = 0$$

and takes the form

$$S_h(t,x,\xi) = x\cdot\xi + t\sqrt{g^{ij}\xi_i\xi_j + h^2\tilde{m}^2} + \mathcal{O}(t^2).$$

Compared with the phase term on the flat manifold, we have an error term $\mathcal{O}(t^2)$ which will complicate our argument when considering the stationary phase theory, and this will eventually produce the additional loss of regularity.

We divide this chapter in the following way: Section 2.2 is devoted to the proof of Proposition 2.1.10, while Section 2.3 contains the proof of Theorem 2.1.5 as well as a discussion on the sharpness of these latter estimates on the sphere.

2.2 Proof of the dispersive estimates

This section is devoted to the proof of Proposition 2.1.10. Let us start by recalling some basic results about coordinate charts and semiclassical calculus.

2.2.1 Preliminaries: coordinate charts, Laplace-Beltrami operator and semiclassical functional calculus.

A coordinate chart $(U_\kappa, V_\kappa, \kappa)$ on $\mathcal{M}$ comprises an homeomorphism κ between an open subset U_κ of $\mathcal{M}$ and an open subset V_κ of $\mathbb{R}^d$. Given $\chi \in C_0^\infty(U_\kappa)$ (resp. $\zeta \in C_0^\infty(V_\kappa)$), we define the pushforward of χ (resp. pullback of ζ) by $\kappa_*\chi = \chi\circ\kappa^{-1}$ (resp. $\kappa^*\zeta = \zeta\circ\kappa$). For a given finite cover of $\mathcal{M}$, namely $M = \cup_{\kappa\in\mathcal{F}} U_\kappa$ with $\#\mathcal{F} < \infty$, there exist $\chi_\kappa \in C_0^\infty(U_\kappa)$, $\kappa \in \mathcal{F}$ such that $1 = \sum_\kappa \chi_\kappa(x)$ for all $x \in \mathcal{M}$.

For all coordinate chart $(U_\kappa, V_\kappa, \kappa)$, there exists a symmetric positive definite matrix $g_\kappa(x) := (g^\kappa_{j\ell})_{1\le j,\ell\le d}$ with smooth and real valued coefficients on V_κ such that the Laplace-Beltrami operator $P_0 = -\Delta_g$ reads in $(U_\kappa, V_\kappa, \kappa)$ as

$$P_0^\kappa := -\kappa_*\Delta_g\kappa^* = -\sum_{j,\ell=1}^d |g_\kappa(x)|^{-1}\partial_j(|g_\kappa(x)|g_\kappa^{j\ell}(x)\partial_\ell),$$

where $|g_\kappa(x)| = \sqrt{\det(g_\kappa(x))}$ and $(g_\kappa^{j\ell}(x))_{1\le j,\ell\le d} := (g_\kappa(x))^{-1}$. Thus in the chart $(U_\kappa, V_\kappa, \kappa)$, the Klein-Gordon operator reads as $P_m^\kappa = \kappa_* P_m \kappa^*$.

We now recall some results from the semiclassical functional calculus that will be used throughout the paper. For any $\nu \in \mathbb{R}$, we consider the symbol class $\mathcal{S}(\nu)$ the space of smooth functions a_h on $\mathbb{R}^{2d}$ (may depend on h) satisfying

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta a_h(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{\nu-|\beta|},$$

for any $x, \xi \in \mathbb{R}^d$ and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. We also need $\mathcal{S}(-\infty) := \cap_{\nu\in\mathbb{R}} \mathcal{S}(\nu)$. We define the semiclassical pseudodifferential operator on $\mathbb{R}^d$ with a symbol $a_h \in S(\nu)$ by

$$\mathrm{Op}_h(a_h)u(x) := \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(x-y)\cdot\xi} a_h(x,\xi)u(y)dyd\xi \tag{2.15}$$

where $u \in \mathscr{S}(\mathbb{R}^d)$ the Schwartz space.

On a manifold $\mathcal{M}$, for a given $a_h \in \mathcal{S}(\nu)$ the semiclassical pseudo-differential operator is defined as follows

$$\mathrm{Op}_h^\kappa(a_h) := \kappa^* Op_h(a_h)\kappa_*.$$

If nothing is specified, the operator $\mathrm{Op}_h^\kappa(a_h)$ maps $C_0^\infty(U_\kappa)$ to $C^\infty(U_\kappa)$. In the case $\mathrm{supp}(a_h) \subset V_\kappa \times \mathbb{R}^d$, we have $\mathrm{Op}_h^\kappa(a_h)$ maps $C_0^\infty(U_\kappa)$ to $C_0^\infty(U_\kappa)$ hence to $C^\infty(\mathcal{M})$.

We are going to construct an h-dependent WKB approximation in order to obtain an h-dependent phase term S_h. To do so, we first introduce the following h-dependent symbol $p_{\tilde{m},h}^\kappa$:

$$p_{\tilde{m},h}^\kappa(x,\xi) := g_\kappa^{j\ell}(x)\xi_j\xi_\ell + h^2\tilde{m}^2 \tag{2.16}$$

with the choice of $\tilde{m}$ given by (2.10). In order to obtain the dispersive estimate (2.12), we will have to slightly modify the principal symbol $p_{0,0}^\kappa$ of the operator $h^2 P_m^\kappa$ into an "h-dependent principal symbol" $p_{\tilde{m},h}^\kappa$.

Let us now describe the relationship between the general operator $f(h^2 P_m)$ and the h-dependent symbol $f(p_{\tilde{m},h}^\kappa)$. In what follows, several cut-off functions will appear; we will denote them by $\chi^{(j)}$ for $j = 1, 2, 3, \ldots$ with the spirit that, as we shall see, $\chi_\kappa^{(n)} = 1$ near $\mathrm{supp}(\chi_\kappa^{(n-1)})$.

Lemma 2.2.1. *Let $\chi_\kappa^{(1)} \in C_0^\infty(U_\kappa)$ be an element of a partition of unity on $\mathcal{M}$ and $\tilde{\chi}_\kappa^{(2)} \in C_0^\infty(U_\kappa)$ be such that $\chi_\kappa^{(2)} = 1$ near $\mathrm{supp}(\chi_\kappa^{(1)})$. Then for $f \in C_0^\infty(\mathbb{R})$, $m, m' \geq 0$ and any $N \geq 1$,*

$$f(h^2 P_m)\chi_\kappa^{(1)} = \sum_{j=0}^{N-1} h^j \chi_\kappa^{(2)} \mathrm{Op}_h^\kappa(q_{j,h}^\kappa)\chi_\kappa^{(1)} + h^N R_{\kappa,N}(h), \tag{2.17}$$

where $q_{j,h}^\kappa \in \mathcal{S}(-\infty)$ with $\mathrm{supp}(q_{j,h}^\kappa) \subset \mathrm{supp}(f \circ p_{m',h}^\kappa)$ for $j = 0, \cdots, N-1$. Moreover, $q_{0,h}^\kappa = f \circ p_{m',h}^\kappa$ and, for any integer $0 \leq n \leq \frac{N}{2}$, there exists $C > 0$ such that for all $h \in (0,1]$,

$$\|R_{\kappa,N}(h)\|_{H^{-n}(\mathcal{M})\to H^n(\mathcal{M})} \leq Ch^{-2n}. \tag{2.18}$$

Proof. The proof closely follows the one of [23, Proposition 2.1] or [56, Proposition 3.2], and we only need to change the principal symbol of h^2P_m in [23, Proposition 2.1] with our symbol $p^\kappa_{m',h}$ as defined in (2.16). We omit the details. □

Before going further, let us introduce the following auxiliary functions: for a given $\phi \in C_0^\infty(\mathbb{R}\backslash[-2\tilde m^2, 2\tilde m^2])$ we take

$$\widetilde\psi \in C_0^\infty(\mathbb{R}\setminus\{0\}) : \forall\, h \in (0,1] \text{ and } \lambda \in \mathrm{supp}(\phi),\ \widetilde\psi(\lambda + h^2\tilde m^2) = 1 \tag{2.19}$$

and

$$\psi(\lambda) = \widetilde\psi(\lambda)\lambda^{1/2}. \tag{2.20}$$

Obviously, $\psi \in C_0^\infty(\mathbb{R})$. The idea is that the function ψ helps regularize the square root of the operator P_m, in view of applying Lemma 2.2.1. We have that

$$e^{ih^{-1}t\psi(h^2P_m)}\phi(-h^2\Delta_g) = e^{itP_m^{1/2}}\phi(-h^2\Delta_g). \tag{2.21}$$

According to the partition of unity and (2.21), it suffices to consider the operator $e^{itP_m^{1/2}}\phi(-h^2\Delta_g)$ on a chart, i.e.,

$$e^{itP_m^{1/2}}\phi(-h^2\Delta_g)\chi_\kappa^{(1)} = e^{ih^{-1}t\psi(h^2P_m)}\phi(-h^2\Delta_g)\chi_\kappa^{(1)}, \quad \kappa \in \mathcal{F}$$

where $\chi_\kappa^{(1)} \in C_0^\infty(U_\kappa)$ is an element of a partition of unity on $\mathcal{M}$. Using Lemma 2.2.1, we infer that there is a symbol $a_\kappa \in \mathcal{S}(-\infty)$ satisfying $\mathrm{supp}(a_\kappa) \subset \mathrm{supp}(\phi \circ p^\kappa_{0,0})$ and an operator $R_{1,\kappa,N}$ satisfying (2.18) such that

$$e^{ih^{-1}t\psi(h^2P_m)}\phi(-h^2\Delta_g)\chi_\kappa^{(1)} = e^{ih^{-1}t\psi(h^2P_m)}\chi_\kappa^{(2)}\mathrm{Op}_h^\kappa(a_\kappa)\chi_\kappa^{(1)} + h^N e^{ih^{-1}t\psi(h^2P_m)}R_{1,\kappa,N}(h) \tag{2.22}$$

with $\chi_\kappa^{(2)}$ given in Lemma 2.2.1. Let

$$u(t) = e^{ih^{-1}t\psi(h^2P_m)}\chi_\kappa^{(2)}\mathrm{Op}_h^\kappa(a_\kappa)\chi_\kappa^{(1)};$$

then u solves the following semi-classical evolution equation

$$\begin{cases} (ih\partial_t + \psi(h^2P_m))u(t) = 0, \\ \qquad\qquad u|_{t=0} = \chi_\kappa^{(2)}\mathrm{Op}_h^\kappa(a_\kappa)\chi_\kappa^{(1)}u_0. \end{cases} \tag{2.23}$$

We can now decompose the operator $\psi(h^2P_m)$ on manifold $\mathcal{M}$: letting $\chi_\kappa^{(3)}, \chi_\kappa^{(4)} \in C_0^\infty(U_\kappa)$ such that $\chi_\kappa^{(3)} = 1$ near $\mathrm{supp}(\chi_\kappa^{(2)})$ and $\chi_\kappa^{(4)} = 1$ near $\mathrm{supp}(\chi_\kappa^{(3)})$, and letting $\tilde m$ be given by (2.10), Lemma 2.2.1 yields

$$\psi(h^2P_m)\chi_\kappa^{(3)} = \chi_\kappa^{(4)}\mathrm{Op}_h^\kappa(q^\kappa(h))\chi_\kappa^{(3)} + h^N R_{2,\kappa,N}(h), \tag{2.24}$$

where

$$q^\kappa(h) = \psi(p^\kappa_{\tilde m,h}) + \sum_{j=1}^{N-1} h^j q^\kappa_{j,h} \tag{2.25}$$

with $q^\kappa_{j,h} \in \mathcal{S}(-\infty)$ and $R_{2,\kappa,N}(h)$ satisfies (2.18).

2.2.2 The WKB approximation and semiclassical dispersive estimates

Inserting (2.24) into (2.23), the main operator we are going to study is

$$ih\partial_t + \mathrm{Op}_h^\kappa(q^\kappa(h))$$

on $\mathcal{M}$ which is equivalent to

$$ih\partial_t + \mathrm{Op}_h(q^\kappa(h))$$

on $\mathbb{R}^d$. Then the following result represents the key ingredient in the proof of Proposition 2.1.10.

Lemma 2.2.2. *Let $\phi \in C_0^\infty(\mathbb{R}\setminus\{0\})$, K be a small neighborhood of* $\mathrm{supp}(\phi)$ *not containing the origin, $a \in \mathcal{S}(-\infty)$ with* $\mathrm{supp}(a) \subset (p_{0,0}^\kappa)^{-1}(\mathrm{supp}(\phi))$ *and let $v_0 \in C_0^\infty(\mathbb{R}^d)$. Then there exist $t_0 > 0$ small enough, $S_h \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ and a sequence of functions $a_{j,h}(t, \cdot, \cdot)$ satisfying* $\mathrm{supp}(a_{j,h}(t, \cdot, \cdot)) \subset (p_{0,0}^\kappa)^{-1}(K)$ *uniformly w.r.t. $t \in [-t_0, t_0]$ and w.r.t. $h \in (0, 1]$ such that for all $N \geq 1$,*

$$(ih\partial_t + \mathrm{Op}_h(q^\kappa(h)))J_N(t) = R_N(t)$$

where q^κ is given by (2.25),

$$\begin{aligned} J_N(t)v_0(x) &= \sum_{j=0}^{N} h^j J_h(S_h(t), a_{j,h}(t))v_0(x) \\ &= \sum_{j=0}^{N} h^j \left[(2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S_h(t,x,\xi) - y\cdot\xi)} a_{j,h}(t, x, \xi)v_0(y)\, dyd\xi \right], \end{aligned} \tag{2.26}$$

$J_N(0) = \mathrm{Op}_h(a)$ and the remainder $R_N(t)$ satisfies that for any $t \in [-t_0, t_0]$, $h \in (0, 1]$ and $n \leq \frac{N}{2}$

$$\|R_N(t)\|_{H^{-n}(\mathbb{R}^d)\to H^n(\mathbb{R}^d)} \leq Ch^{N-2n}. \tag{2.27}$$

Moreover, there exists a constant $C > 0$ such that

1. *for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$,*

$$\|J_N(t)\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ch^{-d}(1 + |t|h^{-1})^{-\frac{d-1}{2}}; \tag{2.28}$$

2. *for all $t \in h^{1/2}[-t_0, t_0]$ and all $h \in (0, 1]$,*

$$\|J_N(t)\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ch^{-d-1}(1 + |t|h^{-1})^{-\frac{d}{2}}. \tag{2.29}$$

Remark 2.2.3. Compared with the existing results on the dispersive estimates for J_N-type oscillatory integrals (see, e.g., [82, 23, 56]), (2.29) is much more complicated even if eventually all the results are based on the stationary phase theorem. In fact, estimate (2.29) involves a much deeper insight into the behavior of the eigenvalues of the Hessian matrix $\nabla^2_\eta \widetilde{\Phi}_h$ where, as we shall see,

$$\widetilde{\Phi}_h(t,x,y,\eta) = t^{-1}\sqrt{g(x)}(x-y)\cdot\eta + \sqrt{|\eta|^2 + h^2\tilde{m}^2}.$$

More precisely, $\nabla^2_\eta \widetilde{\Phi}_h$ has $d-1$ eigenvalues away from 0 uniformly w.r.t. h and it has a unique eigenvalue of the size $\mathcal{O}(h^2)$. In order to apply the stationary phase theorem for (2.29), we will first need to use the stationary phase theorem to deal with a submatrix of $\nabla^2_\eta \widetilde{\Phi}_h$ associated with the $d-1$ eigenvalues which are away from 0 uniformly w.r.t h, and then use the Van der Corput lemma in order to deal with the remaining terms associated with the eigenvalue of size $\mathcal{O}(h^2)$. This strategy has been used to deal with the Klein-Gordon equations [101, 128].

Proof. We split the proof into three steps: the construction of the WKB approximation, the estimates for the remainder R_N for (2.27) and the semiclassical dispersive estimates (2.28) and (2.29). For the reader's convenience, we will omit the index κ since the chart has been fixed and we will borrow the notations and the results from [56, Step 1 and Step 2, Proof of Theorem 2.7] directly. The arguments of Step 1. and Step 2. below are essentially the same as in [56, Step 1 and Step 2, Proof of Theorem 2.7] (except taking the supremum over $h \in (0,1]$), thus we only give the sketch of the proof of these two steps.

Step 1: the WKB approximation.

We are going to seek for $J_N(t)$ satisfying (2.26). Before going further, we look for S_h satisfying the following Hamilton-Jacobi equation

$$\partial_t S_h(t) - \psi(p_{\tilde{m},h})(x, \nabla_x S_h(t)) = 0, \tag{2.30}$$

with $S_h(0) = x\cdot\xi$.

Proposition 2.2.4. *Let ψ be given by (2.19)-(2.20). There exists $t_0 > 0$ small enough and a unique solution $S_h \in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ to the Hamilton-Jacobi equation*

$$\begin{cases} \partial_t S_h(t,x,\xi) - \psi(p_{\tilde{m},h})(x,\nabla_x S_h) = 0, \\ \qquad\qquad\qquad\qquad S_h(0,x,\xi) = x\cdot\xi. \end{cases} \tag{2.31}$$

Moreover, for all $\alpha,\beta\in\mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ independent of h (with $h\in(0,1]$) such that for all $t\in[-t_0,t_0]$ and all $x,\xi\in\mathbb{R}^d$,

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta (S_h(t,x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}, \quad |\alpha+\beta|\ge 1, \tag{2.32}$$

$$\sup_{h\in(0,1]} \left|\partial_x^\alpha\partial_\xi^\beta\left(S_h(t,x,\xi) - x\cdot\xi - t\psi(p_{\tilde{m},h})(x,\xi)\right)\right| \le C_{\alpha,\beta}|t|^2. \tag{2.33}$$

Proof. This proposition holds since $\psi(p_{\tilde{m},h})$ satisfies the following condition: for all $\alpha, \beta \in \mathbb{N}^d$ there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta q_{0,h}| \leq C_{\alpha,\beta}.$$

Indeed, it satisfies the condition (A.2) in [56, Appendix A] uniformly w.r.t. $h \in (0,1]$. Then following the argument in [56, Appendix A], we get a unique solution S_h to the Hamilton-Jacobi equation (2.31) and S_h satisfies [56, Eqns. (2.19) and (2.20)] uniformly w.r.t. $h \in (0,1]$. Hence (2.32) and (2.33). □

In the next Proposition, we describe the action of a pseudodifferential operator on a Fourier integral operator.

Proposition 2.2.5. *Let $b_h \in \mathcal{S}(-\infty)$ and $c_h \in \mathcal{S}(-\infty)$ and $S_h \in C^\infty(\mathbb{R}^{2d})$ such that for all $\alpha, \beta \in \mathbb{N}^d$, $|\alpha + \beta| \geq 1$, there exists $C_{\alpha\beta} > 0$,*

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta (S_h(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}, \quad \textit{for all } x, \xi \in \mathbb{R}^d. \tag{2.34}$$

Then

$$Op_h(b_h) \circ J_h(S_h, c_h) = \sum_{j=0}^{N-1} h^j J_h(S_h, (b_h \triangleleft c_h)_j) + h^N J_h(S_h, r_N(h)),$$

where $(b_h \triangleleft c_h)_j$ is an universal linear combination of

$$\partial_\eta^\beta b_h(z, \nabla_x S_h(x,\xi)) \partial_x^{\beta-\alpha} c_h(x,\xi) \partial_x^{\alpha_1} S_h(x,\xi) \cdots \partial_x^{\alpha_k} S_h(x,\xi),$$

with $\alpha \leq \beta$, $\alpha_1 + \cdots + \alpha_k = \alpha$ and $|\alpha_l| \geq 2$ for all $l = 1, \cdots, k$ and $|\beta| = j$. The map $(b_h, c_h) \mapsto (b_h \triangleleft c_h)$ and $(b_h, c_h) \mapsto r_N(h)$ are continuous from $\mathcal{S}(-\infty) \times \mathcal{S}(-\infty)$ to $\mathcal{S}(-\infty)$ and $\mathcal{S}(-\infty)$ respectively. In particular, we have

$$\begin{aligned}
(b_h \triangleleft c_h)_0(x,\xi) &= b_h(x, \nabla_x S_h(x,\xi)) c_h(x,\xi), \\
i(b_h \triangleleft c_h)_1(x,\xi) &= \nabla_\eta b_h(x, \nabla_x S_h(x,\xi)) \cdot \nabla_x c_h(x,\xi) + \\
&\quad + \frac{1}{2} \mathrm{Tr}\left(\nabla_\eta^2 b_h(x, \partial_s S_h(x,\xi)) \cdot \nabla_x^2 S_h(x,\xi)\right) \cdot c(x,\xi).
\end{aligned}$$

Proof. This is a variant of [56, Proposition 2.9] (see also in [111, Théorème IV.19], [112, Lemma 2.5]) and [17, Appendix] . From [56, Proposition 2.9], we know that this proposition holds if b_h, c_h and S_h are h-independent. Then for any $\widetilde{h} \in (0,1]$, this proposition holds for $b_{\widetilde{h}}$, $c_{\widetilde{h}}$ and $S_{\widetilde{h}}$. Finally, this proposition holds for h-dependent symbols by taking $\widetilde{h} = h$. □

We are now in a position to explicitly write down the WKB approximation. From (2.30), Proposition 2.2.2 and Proposition 2.2.5, we infer that

$$(ih\partial_t + \mathrm{Op}_h(q(h)))J_N = -\sum_{r=0}^{N} h^r J_h(S_h(t), c_{r,h}(t)) + h^{N+1} J_h(S_h(t), r_{N+1}(h,t)),$$

(we recall that the symbol $q(h)$ is defined by (2.25)), where $r_{N+1} \in \mathcal{S}(-\infty)$ and

$$\begin{aligned} c_{0,h}(t) &= \partial_t S_h(t) a_{0,h}(t) - \psi(p_{\tilde{m},h})(x, \nabla_x S_h(t)) a_{0,h}(t) = 0, \\ -c_{r,h}(t) &= i\partial_t a_{r-1,h}(t) + (\psi(p_{\tilde{m},h}) \lhd a_{r-1,h})_1 + (q_{1,h} \lhd a_{r-1,h})_0 \\ &\quad + \sum_{k+j+l=r, j\le r-2} (q_{k,h} \lhd a_{j,h}(t))_l, \quad r = 1, \cdots, N-1, \\ -c_{N,h}(t) &= i\partial_t a_{N-1,h} + (\psi(p_{\tilde{m},h}) \lhd a_{N-1,h})_1 + (q_{1,h} \lhd a_{N-1,h})_0 + \sum_{\substack{k+j+l=N,\\ j\le N-2}} (q_{k,h} \lhd a_{j,h})_l. \end{aligned}$$

This leads to the following transport equations

$$i\partial_t a_{0,h}(t) + (\psi(p_{\tilde{m},h}) \lhd a_{0,h})_1 + (q_{1,h} \lhd a_{0,h})_0 = 0, \tag{2.35}$$

$$i\partial_t a_{r,h}(t) + (\psi(p_{\tilde{m},h}) \lhd a_{r,h})_1 + (q_{1,h} \lhd a_{r,h})_0 = -\sum_{k+j+l=r+1, j\le r-1} (q_{k,h} \lhd a_{j,h})_l \tag{2.36}$$

and

$$R_N(t) := h^{N+1} J_h(S_h(t), r_{N+1}(h,t)) \tag{2.37}$$

with

$$a_{0,h}(0,x,\xi) = a(x,\xi), \quad a_{r,h}(0,x,\xi) = 0 \quad \text{for} \quad r = 1, \cdots, N. \tag{2.38}$$

We rewrite the equations on $a_{r,h}$ as follows

$$\begin{aligned} &\partial_t a_{0,h} - V_h(t,x,\xi,h) \cdot \nabla_x a_{0,h} - f_h a_{0,h} = 0, \\ &\partial_t a_{r,h} - V_h(t,x,\xi,h) \cdot \nabla_x a_{r,h} - f_h a_{r,h} = g_{r,h}(h) \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} V_h(t,x,\xi) &= (\partial_\xi \psi(p_{\tilde{m},h}))(x, \nabla_x S_h(t,x)), \\ f_h(t,x,\xi) &= \frac{1}{2} tr[\nabla_\xi^2 \psi(p_{\tilde{m},h})(x, \nabla_x S_h) \cdot \nabla_x^2 S_h] + i q_{1,h}(x, \nabla_x S_h), \\ g_{r,h}(t,x,\xi) &= i \sum_{k+j+l=r+1, j\le r-1} (q_{k,h} \lhd a_{j,h})_l. \end{aligned}$$

We now construct $a_{r,h}$ by the method of characteristics and by induction as follows. Let $Z_h(t,s,x,\xi)$ be the flow associated with V_h, i.e.,

$$\partial_t Z_h = -V_h(t, Z_h), \quad Z_h(s,s,x,\xi) = x.$$

As $\psi(p_{\tilde{m},h}) \in \mathcal{S}(-\infty)$ and using the same trick as in [56, Lemma A.1], from (2.32) we infer

$$\sup_{h\in(0,1]} |\partial_x^\alpha \partial_\xi^\beta (Z_h(t,s,x,\xi) - x)| \le C_{\alpha\beta} |t-s| \tag{2.40}$$

for all $|t|, |s| \leq t_0$. Then by iteration, the solutions to (2.35) and (2.36) with initial data (2.38) are

$$a_{0,h}(t,x,\xi) = a(Z_h(0,t,x,\xi),\xi)\exp\left(\int_0^t f_h(s,Z_h(s,t,x,\xi),\xi)ds\right),$$

$$a_{r,h}(t,x,\xi) = \int_0^t g_{r,h}(s,Z_h(s,t,x,\xi),\xi)\exp\left(\int_s^t f_h(\tau,Z_h(\tau,t,x,\xi),\xi)d\tau\right)ds,$$

for $r = 1, \cdots, N-1$.

Using the fact that $a, q_{k,h}, f_h \in \mathcal{S}(-\infty)$, it is easy to see that $a_{0,h} \in \mathcal{S}(-\infty)$. Then $g_{1,h} \in \mathcal{S}(-\infty)$ and $a_{1,h} \in \mathcal{S}(-\infty)$. By iteration, we infer that $a_{r,h} \in \mathcal{S}(-\infty)$ for any $r = 1, \cdots, N-1$. On the other hand, $\mathrm{supp}(a) \subset p_{0,0}^{-1}(\mathrm{supp}(\phi))$. According to (2.40), this implies that, for $t_0 > 0$ small enough and for any $(x,\xi) \in p_{0,0}^{-1}(\mathrm{supp}(\phi))$, we have $(Z(t,s,x,\xi),\xi) \in p_{0,0}^{-1}(K)$ for all $|t|, |s| \leq t_0$. Thus, $a_{0,h}(t,x,\xi) = 0$ for $(x,\xi) \notin p_{0,0}^{-1}(\mathrm{supp}(\phi))$ since $\mathrm{supp}(g_{r,h}(t,\cdot,\cdot)) \subset \cup_{0\leq j\leq r-1}\mathrm{supp}(a_{j,h})$. This shows that $\mathrm{supp}(a_{r,h}(t,\cdot,\cdot)) \subset p_{0,0}^{-1}(K)$ uniformly w.r.t. $t \in [-t_0, t_0]$.

Step 2: L^2-boundedness of the remainder. The proof of the boundedness of the remainder is the same as in [56, Step 2, Page 8819-8820]. We use the notations therein and only need to point out that there exists $t_0 > 0$ small enough such that for all $t \in [t_0, t_0]$,

$$\sup_{h\in(0,h_0]} \|\nabla_x\nabla_\xi S_h(t,x,\xi) - \mathbb{1}_{\mathbb{R}^d}\| \ll 1, \quad \text{for all } x,\xi \in \mathbb{R}^d.$$

As a result, for any $\alpha, \alpha', \beta \in \mathbb{N}^d$, there exists $C_{\alpha\alpha'\beta} > 0$ such that

$$\sup_{h\in(0,h_0]} |\partial_x^\alpha \partial_y^{\alpha'} \partial_\xi^\beta (\Lambda^{-1}(t,x,y,\xi) - \xi)| \leq C_{\alpha\alpha'\beta}|t|$$

for any $t \in [-t_0, t_0]$. Here Λ is given by

$$\Lambda(t,x,y,\xi) = \int_0^1 \nabla_x S(t, y + s(x-y), \xi)ds.$$

Then by changing variable $\xi \mapsto \Lambda^{-1}(t,x,y,\xi)$, the action $J_h(S(t), r_{N+1}) \circ J_h(S(t), r_{N+1})^*$ becomes a semiclassical pseudodifferential operator. Then the proof in [56] gives the boundedness from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Concerning the boundedness from $H^{-n}(\mathbb{R}^d) \to H^n(\mathbb{R}^d)$, according to (2.37), we only need to point out that, for any $\alpha, \beta \in \mathbb{N}^d$ and $|\alpha|, |\beta| \leq n$, there exists a symbol $r_{N+1,\alpha,\beta} \in \mathcal{S}(-\infty)$ such that

$$\begin{aligned}\partial_x^\alpha R_N(t) \circ (\partial_x^\beta v_0) &= ih^{N+1}(2\pi h)^{-d}\partial_x\left(\iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S_h(t,x,\xi)-y\cdot\xi)} r_{N+1,\alpha,\beta}(t,x,\xi)\partial_y^\beta v_0(y)dyd\xi\right) \\ &= ih^{N+1-|\alpha|-|\beta|}(2\pi h)^{-d}\iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S_h(t,x,\xi)-y\cdot\xi)} r_{N+1,\alpha,\beta}(t,x,\xi)v_0(y)dyd\xi\end{aligned}$$

thanks to the fact that $r_{N+1} \in \mathcal{S}(-\infty)$ and Proposition 2.2.4. Then, repeating the proof above by replacing r_{N+1} by $r_{N+1,\alpha,\beta}$, we get (2.27).

Step 3: semiclassical dispersive estimates.

The kernel of $J_h(S_h(t), a_h(t))$ reads

$$L_h(t,x,y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ih^{-1}(S_h(t,x,\xi)-y\cdot\xi)} a_h(t,x,\xi)d\xi, \tag{2.41}$$

where $a_h(t) = \sum_{r=0}^{N-1} h^r a_{r,h}(t)$ and $(a_h(t))_{t\in[-t_0,t_0]}$ is bounded in $\mathcal{S}(-\infty)$ satisfying $\mathrm{supp}(a_h(t,\cdot,\cdot)) \in p_{0,0}^{-1}(K)$ for some small neighborhood K of $\mathrm{supp}(\phi)$ not containing the origin uniformly w.r.t. $t \in [-t_0, t_0]$.

It suffices to consider the case $t \geq 0$, as the case $t \leq 0$ can be dealt with in a similar way. If $0 \leq t \leq h$ or $1+th^{-1} \leq 2$, as S_h is compactly supported in ξ and a_h are uniformly bounded in t, x, y, we get

$$|L_h(t,x,y)| \leq Ch^{-d}(1+th^{-1})^{-(d-1)/2}. \tag{2.42}$$

Now let us consider the case $h \leq t \leq t_0$. Set $\lambda := th^{-1} \geq 1$. Then

$$S_h(t,x,\xi) = x\cdot\xi + t\sqrt{g^{ij}\xi_i\xi_j + h^2\tilde{m}^2} + t^2\int_0^1 (1-\theta)\partial_t^2 S_h(\theta t, x, \xi)d\theta$$

since $\psi(p_{\tilde{m},h})(x,\xi) = \sqrt{g^{j\ell}\xi_j\xi_\ell + h^2\tilde{m}^2}$ on $p_{0,0}^{-1}(K)$.

Setting $p(x,\xi) = \xi^t G(x)\xi = |\eta|^2$ with $\eta = \sqrt{G(x)}\xi$ or $\xi = \sqrt{g(x)}\eta$, where $g(x) = \big(g_{j\ell}(x)\big)_{j\ell}$ and $G(x) = \big(g(x)\big)^{-1} = \big(g^{j\ell}(x)\big)_{j\ell}$, the kernel L_h can be written as

$$L_h(t,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ith^{-1}\Phi_h(t,x,y,\eta)} a_h(t,x,\sqrt{g(x)}\eta)\sqrt{g(x)}d\eta, \tag{2.43}$$

where $\sqrt{g(x)} = \sqrt{\det g(x)}$ and

$$\Phi_h(t,x,y,\eta) = \frac{\sqrt{g(x)}(x-y)\cdot\eta}{t} + \sqrt{|\eta|^2 + h^2\tilde{m}^2} + t\int_0^1 (1-\theta)\partial_t^2 S_h(\theta t, x, \sqrt{g(x)}\eta)d\theta.$$

Now, let us deal with the wave and Klein-Gordon-type dispersive estimates separately.

Wave type dispersive estimates: proof of (2.28). Let us start with the case $\tilde{m} > 0$. The gradient of the phase Φ_h is

$$\nabla_\eta\Phi_h(t,x,y,\eta) = \frac{\sqrt{g(x)}(x-y)}{t} + \frac{\eta}{\sqrt{|\eta|^2+h^2\tilde{m}^2}} + t\sqrt{g(x)}\int_0^1 (1-\theta)(\nabla_\eta\partial_t^2 S_h)(\theta t, x, \sqrt{g(x)}\eta)d\theta.$$

If $|\sqrt{g(x)}(x-y)/t| \geq C$ for some constant C large enough, we use the non-stationary phase method which gives for any $N > \frac{d-1}{2}$,

$$|L_h(t,x,y)| \leq Ch^{-d}\lambda^{-N} \leq Ch^{-(d+1)/2}t^{-(d-1)/2}. \tag{2.44}$$

Here we recall that $\lambda = th^{-1}$.

We now deal with the case $|\sqrt{g(x)}(x-y)/t| < C$ by using the stationary phase method. For any $|\eta_j| \geq \varepsilon$ with some ε small but independent of t, we have

$$\nabla^2_{\eta^{(j)}} \Phi_h = \frac{1}{\sqrt{|\eta|^2 + h^2\tilde{m}^2}} \left[\mathbb{1}_{(d-1)\times(d-1)} - \frac{\eta^{(j)} \otimes \eta^{(j)}}{|\eta|^2 + h^2\tilde{m}^2} \right] + \mathcal{O}(t)$$

where $\eta^{(j)} = (\eta_1, \cdots, \eta_{j-1}, \eta_{j+1}, \cdots, \eta_{d-1})$. Then for any $j = 1, \cdots, N$ and t_0 small enough, we have

$$|\det \nabla^2_{\eta^{(j)}} \Phi_h| = (|\eta_j|^2 + h^2\tilde{m}^2)(|\eta|^2 + h^2\tilde{m}^2)^{(-d+1)/2} + \mathcal{O}(t) \geq C \tag{2.45}$$

independently of h. Let us now take a cover $\chi_j(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\sum_{j=1}^{d} \chi_j(x,\eta) = 1 \quad \text{on } p_{0,0}^{-1}(K). \tag{2.46}$$

Notice that for any $\eta \in \operatorname{supp}(\chi_j)$ we have $|\eta_j| \geq \varepsilon$. Let

$$L_{j,h}(t,x,y,\eta_j) = (2\pi h)^{-d} \int_{\mathbb{R}^{d-1}} e^{ith^{-1}\Phi_h(t,x,y,\eta)} \chi_j(x,\eta) a(t,x,\sqrt{g(x)}\eta) \sqrt{g(x)} d\eta^{(j)}$$

We need the following parameter-dependent stationary phase theorem as in [82].

Theorem 2.2.6. *Let $\Phi(x,y)$ be a real valued C^∞ function in a neighborhood of $(x_0,y_0) \in \mathbb{R}^{n+m}$. Assume that $\nabla_x \Phi(x_0,y_0) = 0$ and that $\nabla_x^2 \Phi(x_0,y_0)$ is non-singular, with signature σ. Denote by $x(y)$ the solution to the equation $\nabla_x \Phi(x,y) = 0$ with $x(y_0) = x_0$ given by the implicit function theorem. Then when $a \in C_0^\infty(K)$, K close to (x_0,y_0),*

$$\left| \int e^{i\lambda\Phi(x,y)} a(x,y)dx - \lambda^{-n/2} e^{i\lambda\Phi(x(y),y)} |\det(\nabla_x^2 \Phi(x(y),y))|^{-1/2} \times e^{\pi i \sigma/4} a(x(y),y) \right| \\ \leq C\lambda^{-1-\frac{n}{2}} \sum_{|\alpha| \leq 3+n} \sup_x |\partial_x^\alpha a(x,y)|.$$

Proof. See Theorem 7.7.6 in [78]. □

Applying this stationary phase theorem and choosing $x = \eta^{(j)}$, $y = \eta_j$, we have

$$\begin{aligned} |L_h(t,x,y)| &\leq \sum_{j=1}^{d} \left| \int_{\mathbb{R}} L_{j,h}(t,x,y,\eta_j) d\eta_j \right| \leq C \sum_{j=1}^{d} \|L_{j,h}(t,x,y,\cdot)\|_{L^\infty(\mathbb{R})} \\ &\leq Ch^{-d}\lambda^{-(d-1)/2} = Ch^{-(d+1)/2} t^{-(d-1)/2}. \end{aligned} \tag{2.47}$$

Recall that $\lambda = th^{-1}$ for $h \leq t \leq t_0$. Combining (2.42), (2.44) and (2.47), we conclude that

$$|L_h(t,x,y)| \leq h^{-d}(1+th^{-1})^{-(d-1)/2}.$$

If we take $\tilde{m} = 0$, as estimate (2.45) still holds, the proof works in the same way.

Klein-Gordon type dispersive estimates: proof of (2.29) Arguing as above for the wave one, we only need to consider the case $|t^{-1}\sqrt{g(x)}(x-y)| \leq C$. Unfortunately, in this case

$$\nabla_\eta^2 \Phi_h = \frac{1}{\sqrt{|\eta|^2 + h^2\tilde{m}^2}} \left[\mathbb{1}_{d\times d} - \frac{\eta \otimes \eta}{|\eta|^2 + h^2\tilde{m}^2} \right] + \mathcal{O}(t)$$

from which we infer that

$$|\det \nabla_\eta^2 \Phi_h| = h^2\tilde{m}^2 \big(|\eta|^2 + h^2\tilde{m}^2 \big)^{-\frac{d}{2}} + \mathcal{O}(t) \geq Ch^2\tilde{m}^2 + \mathcal{O}(t).$$

Notice now that, differently from (2.45), we may not be able to control the above term from below for $t \in [h, t_0]$ when h is small enough. To overcome this problem, we split the phase term Φ_h into two parts:

$$\Phi_h = \widetilde{\Phi}_h(t,x,y,\eta) + t\int_0^1 (1-\theta)\partial_t^2 S_h(\theta t, x, \sqrt{g(x)}\eta)d\theta$$

where

$$\widetilde{\Phi}_h(t,x,y,\eta) = t^{-1}\sqrt{g(x)}(x-y)\cdot\eta + \sqrt{|\eta|^2 + h^2\tilde{m}^2}. \tag{2.48}$$

Let

$$\widetilde{a}_h(t,x,\sqrt{g(x)}\eta) = e^{ih^{-1}t^2\int_0^1(1-\theta)\partial_t^2 S_h(\theta t,x,\sqrt{g(x)}\eta)d\theta} a_h(t,x,\sqrt{g(x)}\eta),$$

then we can write

$$L_h(t,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{i\lambda\widetilde{\Phi}_h(t,x,y,\eta)}\widetilde{a}_h(t,x,\sqrt{g(x)}\eta)\sqrt{g(x)}d\eta. \tag{2.49}$$

Then we turn to study this new oscillatory integral problem for any $t \in [0, h^{1/2}t_0]$. The advantage is that for $t \in [0, h^{1/2}t_0]$,

$$|\partial_\eta^\alpha \widetilde{a}_h| \leq C_\alpha (h^{-1}t^2)^{|\alpha|} \leq C'_\alpha$$

independently of h. So we only consider the interval $t \in [h, h^{1/2}t_0]$.

We can also write $L_{j,h}$ as

$$L_{j,h}(t,x,y,\eta_j) = (2\pi h)^{-d}\int_{\mathbb{R}^{d-1}} e^{i\lambda\widetilde{\Phi}_h(t,x,y,\eta)}\chi_j(x,\eta)\widetilde{a}_h(t,x,\sqrt{g(x)}\eta)\sqrt{g(x)}d\eta^{(j)}.$$

As explained in Remark 2.2.3, applying Theorem 2.2.6 as for the wave dispersive case we infer that

$$\int_{\mathbb{R}} L_{j,h}(t,x,y,\eta_j)d\eta_j = h^{-d}\lambda^{-\frac{d-1}{2}}\int_{\mathbb{R}} e^{i\lambda F_h(\eta_j)}A_h(t,x,\eta_j)d\eta_j + \mathcal{O}(h^{-d}\lambda^{-\frac{d+1}{2}}). \tag{2.50}$$

where

$$F_h(\eta_j) = \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j), \quad A_h(t, x, \eta_j) := \frac{e^{\pi i \sigma/4} \chi_j(\zeta(\eta_j), \eta_j) \widetilde{a}_h(\zeta(\eta_j), \eta_j)}{|\det(\nabla_x^2 \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j))|^{1/2}}$$

and, given by implicit function theorem, $\zeta(\eta_j)$ is the solution to the equation

$$\nabla_{\eta^{(j)}} \widetilde{\Phi}_h(\zeta, \eta_j) = 0 \quad \text{with} \quad \zeta(\eta_{j,0}) = \eta_0^{(j)} \tag{2.51}$$

with the point $(\eta_0^{(j)}, \eta_{j,0}) \in \mathbb{R}^{d-1} \times \mathbb{R}$ satisfying $\nabla_{\eta^{(j)}} \widetilde{\Phi}_h(\eta_0^{(j)}, \eta_{j,0}) = 0$. Furthermore, by implicit function theorem, we know that ζ is smooth and satisfies

$$\zeta'(\eta_j) = -[\nabla^2_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j)]^{-1} [\partial_{\eta_j} \nabla_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j)]. \tag{2.52}$$

Now we are going to study (2.50) by using the following Van der Corput lemma, see [117].

Lemma 2.2.7 (Van der Corput)**.** *Let ϕ be a real-valued smooth function in (a, b) such that $|\phi^{(k)}(x)| \geq c_k$ for some integer $k \geq 1$ and all $x \in (a, b)$. Then*

$$\left| \int_a^b e^{i\lambda\phi} \psi(x) dx \right| \leq C(c_k \lambda)^{-1/k} \left(|\psi(b)| + \int_a^b |\psi'| dx \right)$$

holds when (i) $k \geq 2$ or (ii) $k = 1$ and $\phi'(x)$ is monotone.

To apply this lemma to (2.50), we are going to verify that $|F''(\eta_j)| \geq C$ on $(\zeta(\eta_j), \eta_j) \in \operatorname{supp}(\chi_j)$. Using (2.51) and (2.52), we know that

$$\begin{aligned} F''(\eta_j) &= \nabla^2_{\eta^{(j)}} \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j) \zeta'(\eta_j) \cdot \zeta'(\eta_j) + 2\nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j) \zeta'(\eta_j) + \partial^2_{\eta_j} \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j) \\ &= -\nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j) [\nabla^2_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j)]^{-1} \partial_{\eta_j} \nabla_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j) + \partial^2_{\eta_j} \widetilde{\Phi}_h(\zeta(\eta_j), \eta_j). \end{aligned} \tag{2.53}$$

Notice that

$$\nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h = -\frac{\eta_j}{(|\eta|^2 + h^2 \tilde{m}^2)^{3/2}} \eta^{(j)},$$

and $\nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h$ is an eigenvector of $\nabla^2_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j)$. More precisely,

$$\nabla^2_{\eta^{(j)}} \widetilde{\Phi}_h(\eta^{(j)}, \eta_j) \nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h = \frac{h^2 \tilde{m}^2 + |\eta_j|^2}{(|\eta|^2 + h^2 \tilde{m}^2)^{3/2}} \nabla_{\eta^{(j)}} \partial_{\eta_j} \widetilde{\Phi}_h.$$

Thus,

$$F''(\eta_j) = \frac{1}{(|\zeta|^2 + |\eta_j|^2 + h^2 \tilde{m}^2)^{3/2}} (|\zeta|^2 + h^2 \tilde{m}^2 - \frac{|\eta_j|^2 |\zeta|^2}{|\eta_j|^2 + h^2 \tilde{m}^2}) \geq C h^2 \tilde{m}^2$$

for $(\zeta(\eta_j), \eta_j) \in \operatorname{supp}(\chi_j)$. Using now Lemma 2.2.7 with $k = 2$ into (2.50) yields

$$|L_h(t, x, y)| \leq \sum_{j=1}^{d} | \int_{\mathbb{R}} L_{j,h}(t, x, y, \eta_l) d\eta_j| \\ \leq C h^{-d} \lambda^{-(d-1)/2} (h^2 \lambda)^{-\frac{1}{2}} + C h^{-d} \lambda^{-\frac{d+1}{2}} \leq C h^{-d/2-1} t^{-d/2} \tag{2.54}$$

for any $t \in [h, h^{1/2} t_0]$ and $(x, \eta) \in p_{0,0}^{-1}(K)$. Gathering together (2.42) and (2.54), we conclude that for any $t \in h^{\frac{1}{2}}[-t_0, t_0]$ with t_0 small enough,

$$|L_h| \leq C h^{-d-1} (1 + t h^{-1})^{-d/2}. \tag{2.55}$$

This concludes the proof. □

2.2.3 Conclusion of the proof of Proposition 2.1.10

We are finally in position to prove Proposition 2.1.10. We need this additional result:

Lemma 2.2.8. *Let $\chi^{(1)}, \chi^{(2)} \in C_0^\infty(\mathbb{R}^d)$ such that $\chi^{(2)} = 1$ near $\operatorname{supp}(\chi^{(1)})$. Let K, $a_{j,h}(t, \cdot, \cdot) \in \mathcal{S}(-\infty)$, $S_h \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ and J_h be given as in Lemma 2.2.2. Then for $t_0 > 0$ small enough,*

$$J_h(S_h(t), a_h(t, x)) \chi^{(1)} = \chi^{(2)} J_h(S_h(t), a_h(t)) \chi^{(1)} + \tilde{R}(t)$$

where $\tilde{R}(t) = \mathcal{O}_{H^{-n}(\mathbb{R}^d) \to H^n \mathbb{R}^d)}(h^\infty)$.

Proof. The proof follows the one of [56, Lemma 3.6]; we omit the details. □

We now turn to the

Proof of Proposition 2.1.10. Let $J_N^\kappa(t) = \kappa^* J_N(t) \kappa_*$, $R_{3,\kappa,N} = \kappa^* R_N \kappa_*$ with J_N and R_N being given by Lemma 2.2.2.

Notice that

$$\frac{d}{ds} \left(e^{-ish^{-1}\psi(h^2 P_m)} \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)} \right) = -ih^{-1} e^{-ish^{-1}\psi(h^2 P_m)} (ih\partial_s + \psi(h^2 P_m)) \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)},$$

and $J_N^\kappa(0) = \operatorname{Op}_h^\kappa(a_\kappa)$. Integrating the above equation over $[0, t]$, we infer

$$e^{ith^{-1}\psi(h^2 P_m)} \chi_\kappa^{(2)} \operatorname{Op}_h^\kappa(a_\kappa) \chi_\kappa^{(1)} u_0 = \chi_\kappa^{(2)} J_N^\kappa(t) \chi_\kappa^{(1)} u_0 + \\ + ih^{-1} \int_0^t e^{i(t-s)h^{-1}\psi(h^2 P_m)} (ih\partial_s + \psi(h^2 P_m)) \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)} u_0 \, ds. \tag{2.56}$$

We now consider the terms inside the integral for the above formula. From (2.24), we infer

$$(ih\partial_s + \psi(h^2 P_m)) \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)} \\ = ih \chi_\kappa^{(2)} \partial_s J_N^\kappa(s) \chi_\kappa^{(1)} + \chi_\kappa^{(3)} \operatorname{Op}_h^\kappa(q^\kappa(h)) \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)} + h^N R_{2,\kappa,N}(t) \chi_\kappa^{(2)} J_N^\kappa(s) \chi_\kappa^{(1)}.$$

Then using Lemma 2.2.2 and Lemma 2.2.8,

$$\begin{aligned}(ih\partial_s + \psi(h^2 P_m))\chi_\kappa^{(2)} J_N^\kappa(s)\chi_\kappa^{(1)} \\ &= \chi_\kappa^{(3)}\kappa^*(ih\partial_s + \mathrm{Op}_h(q^\kappa(h)))J_N(s)\kappa_*\chi_\kappa^{(1)} + R_{4,\kappa,N}(s) + h^N R_{2,\kappa,N}(t)\chi_\kappa^{(2)} J_N^\kappa(s)\chi_\kappa^{(1)} \\ &= -\chi_\kappa^{(3)} R_{3,\kappa,N}(s)\chi_\kappa^{(1)} + R_{4,\kappa,N}(s) + h^N R_{2,\kappa,N}(t)\chi_\kappa^{(2)} J_N^\kappa(s)\chi_\kappa^{(1)}\end{aligned}$$

where $R_{4,\kappa,N}(s) = \mathcal{O}_{H^{-n}(\mathcal{M})\to H^n(\mathcal{M})}(h^\infty)$. Thus (2.22), (2.56) and this give

$$e^{ih^{-1}t\psi(h^2 P_m)}\phi(-h^2\Delta_g)\chi_\kappa u_0 = \tilde{\chi}_\kappa J_N^\kappa(t)\chi_\kappa u_0 + R_{\kappa,N} u_0 \tag{2.57}$$

with

$$\begin{aligned}R_{\kappa,N} &:= h^N e^{ih^{-1}t\psi(h^2 P_m)} R_{1,\kappa,N}(h) \\ &- ih^{-1}\int_0^t e^{i(t-s)h^{-1}\psi(h^2 P_m)}\left(\chi_\kappa^{(3)} R_{3,\kappa,N}(s)\chi_\kappa^{(1)} - R_{4,\kappa,N}(s) - h^N R_{2,\kappa,N}(t)\chi_\kappa^{(2)} J_N^\kappa(s)\chi_\kappa^{(1)}\right) ds.\end{aligned}$$

It follows from the Sobolev inequality and the fact $R_{j,\kappa,N} = \mathcal{O}_{H^{-n}(\mathcal{M})\to H^n(\mathcal{M})}(h^{N-2n})$ for any $n \le \frac{N}{2}$ that

$$\|R_{\kappa,N}u_0\|_{L^\infty(\mathcal{M})} \le C\|R_{\kappa,N}u_0\|_{H^d(\mathcal{M})} \le Ch^{N-2d-1}\|u_0\|_{H^{-d}(\mathcal{M})} \le Ch^{N-2d-1}\|u_0\|_{L^1(\mathcal{M})}.$$

Taking N large enough, we infer that for any $t \in [-t_0, t_0]$,

$$\|R_{\kappa,N}\|_{L^\infty(\mathcal{M})} \le Ch^{-d}(1 + |t|h^{-1})^{-\frac{d}{2}}.$$

From (2.57), Lemma 2.2.2 and this, we obtain (2.11) and (2.12). This completes the proof. □

2.3 Strichartz estimates for the Dirac equation

In this section, we show how to deduce Strichartz estimates for the Dirac flow from estimates of Theorem 2.1.3.

For the sake of completeness and to fix the notations, we begin with a brief overview of the construction of the Dirac equation in a non-flat (or non-Lorentzian) setting, as done in Section 0.2.4. For any $d \ge 2$ let us consider a $(d+1)$-dimensional manifold in the form $\mathbb{R}_t \times \mathcal{M}$ with $(\mathcal{M}, g)$ a compact Riemannian manifold of dimension d endowed with a spin structure; then, the Dirac operator on $\mathcal{M}$ can be written as

$$\mathcal{D}_m = -i\gamma^a e^i{}_a D_i - m\gamma^0 \tag{2.58}$$

with $m \ge 0$ is the mass and γ^j, $j = 1, \ldots, d$ is a set of matrices that satisfy the condition

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}, \qquad i,j = 1,\ldots d, \quad \gamma^0 = \begin{pmatrix} I_{\frac{N}{2}} & 0 \\ 0 & -I_{\frac{N}{2}} \end{pmatrix}$$

There are few different possible choices for the γ matrices; notice anyway that the explicit choice of the basis will play no role in our argument. Following [35], let us define these matrices recursively as follows (in computations below, the index d will be added to the γ matrices in order to keep track of the dimensions):

- *Case* $d = 2$. We set

$$\gamma_2^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_2^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- *Case* $d = 3$. We set

$$\gamma_3^1 = \gamma_2^1, \quad \gamma_3^2 = \gamma_2^2, \quad \gamma_3^3 = (-i)\gamma_2^1\gamma_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- *Case* $d > 3$ *even.* We set

$$\gamma_d^j = \begin{pmatrix} 0 & i\gamma_{d-1}^j \\ -i\gamma_{d-1}^j & 0 \end{pmatrix}, \quad j = 1, \dots, d-1, \qquad \gamma_d^d = \begin{pmatrix} 0 & I_{2^{\frac{d-2}{2}}} \\ I_{2^{\frac{d-2}{2}}} & 0 \end{pmatrix}.$$

- *Case* $d > 3$ *odd.* We set

$$\gamma_d^j = \gamma_{d-1}^j, \quad j = 1, \dots, d-1, \qquad \gamma_d^d = i^{\frac{d-1}{2}}\gamma_{d-1}^1 \cdot \dots \cdot \gamma_{d-1}^{d-1} = i^{\frac{d-1}{2}} \begin{pmatrix} I_{2^{\frac{d-3}{2}}} & 0 \\ 0 & -I_{2^{\frac{d-3}{2}}} \end{pmatrix}.$$

The matrix bundle ${e^i}_a$ is called *n-bein* and it is defined as follows

$$g^{ij} = {e^i}_a \delta^{ab} {e^j}_b \tag{2.59}$$

where δ is the Kronecker symbol, and in fact it connects the "spatial" metrics to the Euclidean one. Finally, the covariant derivative for spinors D_i is defined by

$$D_0 = \partial_0, \quad D_j = \partial_j + B_j, \quad j = 1, 2, \dots, d \tag{2.60}$$

where B_j writes

$$B_j = \frac{1}{8}[\gamma^a, \gamma^b]{\omega_j}^{ab}$$

and ${\omega_j}^{ab}$, called the *spin connection*, is given by

$${\omega_j}^{ab} = {e^i}_a \partial_j e^{ib} + {e_i}^a {\Gamma^i}_{jk} e^{kb} \tag{2.61}$$

with the Christoffel symbol (or affine connection) ${\Gamma^i}_{jk}$

$${\Gamma^i}_{jk} := \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{2.62}$$

We stress the fact that in the rest of this section we shall abuse notation by calling functions what should be more precisely called spinors.

We are now in a position to prove Strichartz estimates for the solutions to the Dirac equation (2.3), deducing them from the ones for the Klein-Gordon that we have proved in Section 1. The starting point is the following explicit formula, that has been proved in [30]:

$$\mathcal{D}^2 := m^2 + \frac{1}{4}\mathcal{R}_g - \Delta^{\mathcal{S}} = -\Delta_g + B^i\partial_i + \widetilde{D}^i B_i + B^i B_i + \frac{1}{4}\mathcal{R}_g + m^2 \tag{2.63}$$

where the spinorial laplacian $\Delta^{\mathcal{S}} = D^j D_j$, $\widetilde{D}^i\Psi_k = \partial^i\Psi_k - \Gamma^{l\,i}_{\,k}\Psi_l$, $B^i = h^{ij}B_j$ and $\mathcal{R}_g$ denotes the scalar curvature on $(\mathcal{M}, g)$. As a consequence, the solution u to the Dirac equation can be written as follows:

$$u(t,x) := e^{it\mathcal{D}_m}u_0 = \dot{W}_m(t)u_0 + iW_m(t)\mathcal{D}_m u_0 + \int_0^t W_m(t-s)(\Omega_1(u)(s) + \Omega_2 u(s))ds \tag{2.64}$$

where

$$W_m(t) = \frac{\sin(t\sqrt{m^2-\Delta_g})}{\sqrt{m^2-\Delta_g}}, \quad \dot{W}_m = \partial_t W_m$$

and

$$\Omega_1(u) := 2B^i\partial_i u, \quad \Omega_2 := -\partial^i B_i + B^i B_i - \Gamma^{j\,i}_{\,i} B_j - \frac{1}{4}\mathcal{R}_g. \tag{2.65}$$

Notice that as the manifold $\mathcal{M}$ is assumed to be smooth, the terms B_i, Γ^{ji}_i and $\mathcal{R}_g$ are smooth.

We first consider the case $m > 0$. We set $\widetilde{\gamma}_{pq} := \gamma^{\mathrm{W}}_{pq}$ for wave admissible pair (p,q), and $\widetilde{\gamma}_{pq} := \gamma^{\mathrm{KG}}_{pq}$ for Schrödinger admissible pair (p,q). Using Theorem 2.1.3 for wave admissible pair or Schrödinger admissible pair (p,q), we infer

$$\|e^{it\sqrt{m^2-\Delta_g}}v_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C\|v_0\|_{H^{\widetilde{\gamma}_{pq}}(\mathcal{M})}.$$

Thus for $m > 0$,

$$\|e^{it\mathcal{D}_m}u_0\|_{L^p(I,L^q(\mathcal{M}))} \leq C\|u_0\|_{H^{\widetilde{\gamma}_{pq}}(\mathcal{M})} + t_0 C\sup_s \|B^i\partial_i u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})} + t_0 C\sup_s \|\Omega_2 u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})}.$$

It remains to study the terms $\|B^i\partial_i u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})}$ and $\|\Omega_2 u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})}$. We first show that

$$\|B^i\partial_i u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})} \lesssim \|u(s)\|_{H^{\widetilde{\gamma}_{pq}}(\mathcal{M})}, \quad \|\Omega_2 u(s)\|_{H^{\widetilde{\gamma}_{pq}-1}(\mathcal{M})} \lesssim \|u(s)\|_{H^{\widetilde{\gamma}_{pq}}(\mathcal{M})}.$$

Using standard interpolation theory (see, e.g., [122, Proposition 2.1 and Proposition 2.2, Chp.4]), it suffices to show that

$$\|B^i\partial_i f\|_{H^{-1}(\mathcal{M})} \lesssim \|f\|_{L^2(\mathcal{M})}, \quad \|B^i\partial_i f\|_{H^{n-1}(\mathcal{M})} \lesssim \|f\|_{H^n(\mathcal{M})} \tag{2.66}$$

where $n > \widetilde{\gamma}_{pq}$ is an integer. As $B_1 \in C^\infty(\mathcal{M})$, we infer that

$$\|B^i\partial_i f\|_{H^{n-1}(\mathcal{M})} \lesssim \|f\|_{H^n(\mathcal{M})}.$$

On the other hand, as $B^i\partial_i u = \partial_i(B^i u) - (\partial_i B^i)u$, we have

$$\begin{aligned}\|B^i\partial_i f\|_{H^{-1}(\mathcal{M})} &\le \|(1-\Delta_g)^{-1/2}B^i\partial_i\|_{L^2(\mathcal{M})\to L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})}\\ &\le \|(1-\Delta_g)^{-1/2}[\partial_i B^i - (\partial_i B^i)]\|_{L^2(\mathcal{M})\to L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})} \lesssim \|f\|_{L^2(\mathcal{M})}.\end{aligned}$$

The above two estimates and the interpolation theory show that

$$\|B^i\partial_i f\|_{H^{\tilde{\gamma}_{pq}-1}(\mathcal{M})} \lesssim \|f\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})}.$$

Analogously, as $\Omega_2 \in C^\infty(\mathcal{M})$, we also have that

$$\|\Omega_2 f\|_{H^{\tilde{\gamma}_{pq}-1}(\mathcal{M})} \le \|\Omega_2 f\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} \le C\|f\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})}.$$

Thus,

$$\begin{aligned}\|e^{it\mathcal{D}_m}u_0\|_{L^p(I,L^q(\mathcal{M}))} &\le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} + Ct_0\sup_s \|u(s)\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})}\\ &\le \|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} + Ct_0\sup_s \||\mathcal{D}_m|^{\tilde{\gamma}_{pq}}u(s)\|_{L^2(\mathcal{M})}.\end{aligned}$$

The operator $|\mathcal{D}_m|$ is defined as follows

$$|\mathcal{D}_m| = \mathcal{D}_m[\mathbb{1}_{[0,+\infty)}(\mathcal{D}_m) - \mathbb{1}_{(-\infty,0)}(\mathcal{D}_m)] \tag{2.67}$$

and here we use the fact that for any $s \ge 0$,

$$C_1\|(1-\Delta_g)^s f\|_{L^2(\mathcal{M})} \le \||\mathcal{D}_m|^{2s}f\|_{L^2(\mathcal{M})} \le C_2\|(1-\Delta_g)^s f\|_{L^2(\mathcal{M})},$$

which is obtained by using the interpolation theory again and the fact that there are constants $C_1', C_2' > 0$ such that for any $n \in \mathbb{N}$,

$$C_1'\|(1-\Delta_g)^n f\|_{L^2(\mathcal{M})} \le \||\mathcal{D}_m|^{2n}f\|_{L^2(\mathcal{M})} \le C_2'\|(1-\Delta_g)^n f\|_{L^2(\mathcal{M})}.$$

According to (2.67), $[|\mathcal{D}_m|, \mathcal{D}_m] = 0$. As a result,

$$\begin{aligned}\|e^{it\mathcal{D}_m}u_0\|_{L^p(I,L^q(\mathcal{M}))} &\le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} + Ct_0\sup_s \||\mathcal{D}_m|^{\tilde{\gamma}_{pq}}e^{it\mathcal{D}_m}u_0\|_{L^2(\mathcal{M})}\\ &\le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} + C\||\mathcal{D}_m|^{\tilde{\gamma}_{pq}}u_0\|_{L^2(\mathcal{M})} \le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})}.\end{aligned}$$

This gives (2.6) and (2.7) for $m > 0$.

It remains to show the case $m = 0$. By the Duhamel formula, for $\tilde{m}$ given by (2.10), we have

$$e^{it\mathcal{D}_0}u_0 = e^{it\mathcal{D}_{\tilde{m}}}u_0 - \tilde{m}\int_0^t e^{i(t-s)\mathcal{D}_{\tilde{m}}}u(s)ds.$$

Repeating the above proof for the case $m > 0$, we infer

$$\|e^{it\mathcal{D}_0}u_0\|_{L^p(I,L^q(\mathcal{M}))} \le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})} + Ct_0\sup_s \||\mathcal{D}_m|^{\tilde{\gamma}_{pq}}e^{it\mathcal{D}_0}u_0\|_{L^2(\mathcal{M})} \le C\|u_0\|_{H^{\tilde{\gamma}_{pq}}(\mathcal{M})}.$$

This concludes the proof.

Remark 2.3.1. As it is seen, by making use of formulas (2.63)-(2.64), we have been able to deduce the Strichartz estimates for the Dirac flow from the ones for the half Klein-Gordon equation with a rather simple argument. In fact, it would have been much more complicated to tackle directly the study of the Dirac flow: if we studied the half-spinor-Klein-Gordon equation, then the proof of the existence of solutions for a_r in equation (2.39) would have been significantly more involved. Indeed, in the spinorial case, the equations on a_r turn out to be first-order ODE systems in the form $\partial_t a_r - \mathcal{A} a_r = \mathcal{F}_r$, with a_r, $\mathcal{A}$ and $\mathcal{F}_r$ matrices, rather than simple transport equations. On the one hand, the matrix $\mathcal{A}$ may not be self-adjoint, so the solution $a_r(t)$ may not have a bounded compact support; on the other hand, for a general t-dependent matrix $\mathcal{F}(t)$, we do not even know the formula for $\frac{d}{dt}e^{\mathcal{F}(t)}$, so we do not know the formula for $a_{0,h}$ and $a_{r,h}$. Finally, let us mention that it might be possible to rely on an explicit WKB approximation directly on the Dirac equation (see, e.g.,[16] for its construction in the flat case), but this seems to require a significant amount of technical work, and therefore we preferred to rely on the strategy above.

2.3.1 The case of the sphere.

As a final result, as done in [23] for the Schrödinger equation, we would like to test the sharpness of the Strichartz estimates proved in Theorem 2.1.5 in the case of the Riemannian sphere. In this case, the spectrum and the eigenfunctions of the Dirac operator are indeed explicit and well known (see, e.g.,[35],[124]); we include here a short review of the topic, as indeed an explicit representation of these eigenfunctions will be needed for our scope. Notice that in this section we will be considering the massless Dirac operator, that is the case $m = 0$, and the subscript on the Dirac operator will be used to keep track of the dimension.

As seen before, the definition of the Dirac matrices (and thus of the Dirac operator) is slightly different depending on whether the dimension d of the sphere is even or odd: it is thus convenient to discuss the two cases separately.

- *Case d even.* In this case, the Dirac operator can be recursively defined as

$$\mathcal{D}_{\mathbb{S}^d} = \left(\partial_\theta + \frac{d-1}{2}\cot\theta\right)\gamma_d^d + \frac{1}{\sin\theta}\begin{pmatrix} 0 & \mathcal{D}_{\mathbb{S}^{d-1}} \\ -\mathcal{D}_{\mathbb{S}^{d-1}} & 0 \end{pmatrix}$$

where the matrix γ_d^d, as we have seen, is given in this case by $\gamma_d^d = \begin{pmatrix} 0 & I_{2^{\frac{d-2}{2}}} \\ I_{2^{\frac{d-2}{2}}} & 0 \end{pmatrix}$.

Now, let $\chi_{\ell m}^{\pm}$ be such that

$$\mathcal{D}_{\mathbb{S}^{d-1}}\chi_{\ell m}^{\pm} = \pm(\ell + \tfrac{1}{2}(d-1))\chi_{\ell m}^{\pm}, \tag{2.68}$$

where $\ell = 0, 1, 2\dots$ and m run from 1 to the degeneracy d_ℓ of the eigenfunction (notice that this parameter will play no role in our forthcoming argument). Then, we set

$$\Psi = \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix}.$$

One can separate variables as follows :

$$\Psi^{+,-}_{n\ell m}(\theta,\Omega)=\phi_{n\ell}(\theta)\chi^{-}_{\ell m}(\Omega),\quad \Psi^{+,+}_{n\ell m}(\theta,\Omega)=\psi_{n\ell}(\theta)\chi^{+}_{\ell m}(\Omega) \tag{2.69}$$

(notice that the first apex + in the above labels the first and second component of Ψ, while the second one distinguishes on the choice of the sign $\pm$ performed in (2.68)). Clearly, an analogous decomposition holds for the component Ψ^-. Then, plugging (2.69) into the squared equation $\mathcal{D}^2_{\mathbb{S}^d}\Psi=-\lambda^2_{n,d}\Psi$ yields the following

$$\left[\left(\frac{\partial}{\partial\theta}+\frac{d-1}{2}\cot\theta\right)^2-\frac{(\ell+\frac{d-1}{2})^2}{\sin^2\theta}+(\ell+\tfrac{d-1}{2})\frac{\cos\theta}{\sin^2\theta}\right]\phi_{n\ell}=-\lambda^2_{n,\ell}\phi_{n\ell}$$

which has a unique (up to a constant) regular solution

$$\phi_{n\ell}(\theta)=(\cos\tfrac{\theta}{2})^{\ell+1}(\sin\tfrac{\theta}{2})^{\ell}P^{\frac{d}{2}+\ell-1,\frac{d}{2}+\ell}_{n-\ell}(\cos\theta) \tag{2.70}$$

where $P^{\alpha,\beta}_n$ is a Jacobi polynomial with $n-\ell\geq 0$ (this condition is required in order to have regular eigenfunctions) and with eigenvalue $\lambda^2_{n,d}=(n+\frac{d}{2})^2$. Similarly, one gets

$$\psi_{n\ell}(\theta)=(\cos\tfrac{\theta}{2})^{\ell}(\sin\tfrac{\theta}{2})^{\ell+1}P^{\frac{d}{2}+\ell,\frac{d}{2}+\ell-1}_{n-\ell}(\cos\theta). \tag{2.71}$$

Then, the functions

$$\Psi^1_{\pm n\ell m}(\theta,\Omega):=\frac{C_d(n\ell)}{\sqrt{2}}\begin{pmatrix}\phi_{n\ell}(\theta)\chi^{-}_{\ell m}(\Omega)\\ \pm i\psi_{n\ell}(\theta)\chi^{-}_{\ell m}(\Omega)\end{pmatrix} \tag{2.72}$$

and

$$\Psi^2_{\pm n\ell m}(\theta,\Omega):=\frac{C_d(n\ell)}{\sqrt{2}}\begin{pmatrix}i\psi_{n\ell}(\theta)\chi^{+}_{\ell m}(\Omega)\\ \pm\phi_{n\ell}(\theta)\chi^{+}_{\ell m}(\Omega)\end{pmatrix} \tag{2.73}$$

both satisfy equation

$$\mathcal{D}_{\mathbb{S}^d}\Psi^j_{\pm n\ell m}(\theta,\Omega)=\pm(n+\tfrac{d}{2})\Psi^j_{\pm n\ell m}(\theta,\Omega),\qquad j=1,2. \tag{2.74}$$

The standard normalization condition

$$\langle\Psi^j_{\pm n\ell m},\Psi^{j'}_{\pm n'\ell' m'}\rangle_{L^2}=\delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}\delta_{jj'}$$

fixes the value of the constant $C_d(n\ell)$ to be

$$C_d(n\ell)=\frac{\sqrt{(n-\ell)!(n+\ell+1)!}}{2^{\frac{d}{2}-1}\Gamma(n+1)}. \tag{2.75}$$

- *Case d odd.* In this case, we can write the Dirac equation as

$$\mathcal{D}_{\mathbb{S}^d}=\left(\partial_\theta+\frac{d-1}{2}\cot\theta\right)\gamma^d_d+\frac{1}{\sin\theta}\mathcal{D}_{\mathbb{S}^{d-1}}$$

with $\gamma_d^d = \begin{pmatrix} I_{2^{(d-3)/2}} & 0 \\ 0 & -I_{2^{(d-3)/2}} \end{pmatrix}$. As done for the even case, taking $\chi^{\pm}_{\ell m}$ to be the eigenfunctions of the Dirac operator on the $(d-1)$-dimensional sphere, i.e. satisfying (2.68), the normalized eigenfunctions to the Dirac operator are given by

$$\Psi_{\pm n\ell m}(\theta,\Omega) = \frac{C_d(n\ell)}{\sqrt{2}}\left(\phi_{n\ell}(\theta)\tilde{\chi}^{-}_{\ell m}(\Omega) \pm i\psi_{n\ell}(\theta)\tilde{\chi}^{+}_{\ell m}(\Omega)\right) \tag{2.76}$$

with $\phi_{n\ell}$, $\psi_{n\ell}$ given by (2.70)-(2.71), with $\tilde{\chi}^{\pm}$ defined as

$$\tilde{\chi}_{\ell m}{}^{-} = \frac{1}{\sqrt{2}}(1+\Gamma^d)\chi^{-}_{\ell m}, \qquad \tilde{\chi}_{\ell m}{}^{+} = \Gamma^d\chi^{-}_{\ell m},$$

and where the normalization constant is given by (2.75). The functions given in (2.76) satisfy equation (2.74).

We are now in a position to show that our Strichartz estimates (2.6) are sharp in dimension $d \geq 4$. Let us consider system (2.3) on $\mathcal{M} = \mathbb{S}^d$ with $m = 0$, and let us take as initial condition u_0 an eigenfunction of the Dirac operator for a fixed eigenvalue $\lambda = \pm(n+\frac{d}{2})$, with $n \in \mathbb{N}$. Then, the solution u can be written as $u = e^{it\mathcal{D}_0}u_0 = e^{it\lambda}u_0$. By taking any admissible Strichartz pair we can write, given that the time interval is bounded,

$$\|e^{it\lambda}u_0\|_{L^p_I L^q(\mathbb{S}^d)} \sim \|u_0\|_{L^q(\mathbb{S}^d)} \tag{2.77}$$

Now, we need the following spinorial adaptation of a classical result due to Sogge (see [115]).

Lemma 2.3.2. *Let $d \geq 2$. For any $\lambda = \pm(n+\frac{d}{2})$ with $n \in \mathbb{N}$ such that $|\lambda|$ is sufficiently large, there exists an eigenfunction Ψ_λ of the Dirac equation on $\mathbb{S}^d$ such that the following estimate holds:*

$$\|\Psi_\lambda\|_{L^q(\mathbb{S}^d)} \sim |\lambda|^{s(q)}\|\Psi_\lambda\|_{L^2(\mathbb{S}^d)} \tag{2.78}$$

with $s(q) = \frac{d-1}{2} - \frac{d}{q}$, provided $\frac{2(d+1)}{d-1} \leq q \leq \infty$.

Proof. Let us deal with the case d even; the case d odd can be dealt with similarly. Let us take for any eigenvalue $\lambda = \pm(n+\frac{d}{2})$ an eigenfunction Ψ in the form (2.72)-(2.73) corresponding to the choice $\ell = 0$, which is always admissible. Notice that the functions χ do not depend on n. Then, taking advantage of the classical asymptotic estimates on Jacobi polynomials

$$\int_0^1 (1-x)^r |P_n^{\alpha,\beta}|^p dx \sim n^{\alpha p - 2r - 2}$$

provided $2r < \alpha p - 2 + p/2$ (see, e.g., [119] page 391), we get that

$$\|\Psi_\lambda\|_{L^q(\mathbb{S}^d)} \sim |\lambda|^{\frac{d-1}{2}-\frac{d}{q}}$$

for $|\lambda| \gg 1$ and $q \geq \frac{2(d+1)}{d-1}$.

□

By making use of this Lemma we can thus estimate further (2.77) as follows

$$\|u_0\|_{L^q(\mathbb{S}^d)} \sim |\lambda|^{s(q)} \|u_0\|_{L^2(\mathbb{S}^d)}.$$

Then, taking $d \geq 4$ and $p = 2$ in Strichartz estimates (2.6) yields $q = \frac{2(d-1)}{d-3}$, so that $s(q) = \frac{d+1}{2(d-1)}$ which is exactly $\gamma^{\mathrm{W}}_{2,\frac{2(d-1)}{d-3}}$ and thus estimates (2.6) are sharp provided $d \geq 4$.

Remark 2.3.3. Lemma 2.3.2 is the analog of Theorem 4.2 in [115], where the author proves the same bound for homogeneous harmonic polynomials. Anyway, as the eigenfunctions of the Dirac operator are not "pure" spherical harmonics, we cannot simply evoke this result.

Remark 2.3.4. Notice that the argument above relies on the "endpoint" $p = 2$, and this is the reason why we are only able to prove the sharpness in the case $d \geq 4$. Indeed, the same computations provide

- for $d = 2$, by taking p smallest possible, that is $p = 4$ and thus $q = \infty$:

$$\gamma^{\mathrm{W}}_{4,\infty} = \frac{3}{4} \qquad \text{and} \qquad s(\infty) = \frac{1}{2};$$

- for $d = 3$, as the endpoint $(p, q) = (2, \infty)$ has to be excluded, by taking $p = 2 + \varepsilon$ with $\varepsilon > 0$ small:

$$\gamma^{\mathrm{W}}_{2+\varepsilon,\frac{2(2+\varepsilon)}{\varepsilon}} = \frac{2}{2+\varepsilon} \qquad \text{and} \qquad s\left(\frac{2(2+\varepsilon)}{\varepsilon}\right) = \frac{2}{2+\varepsilon} - \frac{\varepsilon}{2(2+\varepsilon)}$$

which shows that the estimates are sharp in the limit $\varepsilon \to 0$.

Chapter 3

Stability of homogeneous equilibria of the Hartree-Fock equation

3.1 Introduction

3.1.1 The Hartree-Fock equation in the framework of random fields

We consider the following time-dependent Hartree-Fock equation for fermions

$$\begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}[|X|^2])X - \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{X(y)}X(x)]X(y)\,dy, \\ X(t=0) = X_0. \end{cases} \tag{3.1}$$

where $X : [0,T]\times\mathbb{R}^d\times\Omega \mapsto \mathbb{C}$ is a random field defined over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, w is an even pairwise interaction potential, $\mathbb{E}$ denotes the expectation on Ω. Equation (3.1) is an equivalent reformulation of the standard Hartree-Fock equation for density matrices

$$i\partial_t\gamma = [-\Delta + w * \rho + \mathcal{X}, \gamma] \tag{3.2}$$

for a nonnegative self-adjoint operator γ on $L^2(\mathbb{R}^d)$ with kernel k, density $\rho(x) = k(x,x)$ and where $\mathcal{X}$ denotes the exchange term operator with kernel $-w(x-y)k(x,y)$. Indeed, for a given solution X to (3.1), if γ denotes the operator whose kernel k is the correlation function $k(x,y) = \mathbb{E}(X(x)\overline{X(y)})$, then γ is a solution to (3.2). And conversely, for a nonnegative finite rank self-adjoint operator $\gamma = \sum_{i=1}^N |\varphi_i\rangle\langle\varphi_i|$ that is a solution of (3.2), the associated random field $X = \sum_{i=1}^N g_i\varphi_i$, where $g_1, \dots, g_N$ are centered normalised and independent gaussian variables, solves (3.1) (this generalises to infinite rank self-adjoint operators using the spectral theorem).

Our motivation to study the formulation (3.1) of the Hartree-Fock equation instead of (3.2) is to draw analogies with the nonlinear Schrödinger equation. The randomness in (3.1) lies in the use of the extra variable $\omega \in \Omega$ that we think of as a probability variable, which is a convenient way to represent the coupling for this system of equations. It is also convenient for obtaining representation formulas for the equilibria, see Subsection 3.1.3. This randomness in the present problem should however not be mistaken with the randomness in the initial data of an evolution problem: here the evolution equation (3.1) makes sense for a function of both variables x and ω and as such is deterministic.

The time-dependent Hartree-Fock equation is a mean field equation for the dynamics of large Fermi systems. We recall in Subsection 3.1.3 the formal derivation of (3.1) from the many body Schrödinger equation. The derivation of (3.2) in the mean field limit was first done in [8], and was extended to the case of unbounded interaction potentials, as the Coulomb one, in [70]. Estimates for the convergence in the semi-classical limit that arises for confined Fermi systems were proved in [57, 13], and were extended recently to mixed states, and conditionnaly to more singular interaction potentials, in [11, 109]. Another derivation by different techniques, and including another large volume regime for long-range potentials, was given in [108, 4].

Equations (3.1) and (3.2) have received less attention than the static Hartree-Fock equation, or the reduced Hartree-Fock equation (3.3). We mention that the exchange term in the Hartree equation does not appear in the case of systems of bosons, but always appears for fermions. It is due to the form of the canonical wave functions considered for each system. The exchange term for fermionic system is often negligible compared to the direct term (see [108] for example), but in some cases it is relevant to keep it and study it. Still the time-dependent Hartree-Fock equation is always a better approximation of the dynamics of the fermions than the reduced Hartree equation.

The Cauchy problem for localised solutions to (3.2) was studied in [21, 22, 40, 41, 126].

Stability of non-localised equilibria for the reduced Hartree-Fock equation

Most of the results describing the dynamics have so far been obtained for the equations without exchange term

$$i\partial_t X = -\Delta X + (w * \mathbb{E}[|X|^2])X, \tag{3.3}$$

and

$$i\partial_t \gamma = [-\Delta + w * \rho, \gamma], \tag{3.4}$$

which are sometimes referred to as the reduced Hartree-Fock equation as in [36]. Indeed, the contribution of the exchange term is negligible in certain regimes. This is the case in the semi-classical limit to the Vlasov equation, which was studied in [71] for a system with a finite number of particles, and in [12], [13] in the limit of number of particles going to infinity.

The exchange term can also be approximated as a function of the density $\mathbb{E}(|X|^2) = \rho$, as in Density Functional Theory (see e.g. [37] for a review and [79, 116] for the Cauchy problem of time-dependent Kohn-Sham equations).

The reduced Hartree equations (3.3) and (3.4) admit nonlocalised equilibria that models a space-homogeneous electron gas. The stability of such equilibria has been studied in [42, 43, 87, 88] for Equation (3.4) and [47, 48, 74] for Equation (3.3). The stability of the zero solution for the time-dependent Kohn-Sham equation was showed in [110]. In [75], Hadama proved the stability of steady states for the reduced Hartree equation in a wide class, which includes Fermi gas at zero temperature in dimension greater than 3, with smallness assumption on the potential function.

In a recent work [102], Nguyen and You proved that the symbol of the linearised problem could not be inverted in the case of the Coulomb interaction potential. However, they were still able to describe and to prove some time decay for the linearised dynamics.

As observed in [89], there is a natural way to associate nonlocalised equilibria of (3.4) to space-homogeneous equilibria of the Vlasov equation. In the same article the authors proved that, in the high density limit, the Wigner transforms (see [91] for a survey on the Wigner transform) of solutions of (3.4) close to equilibria converge towards solutions of the Vlasov equation. Moreover, these solutions remain close, in a suitable sense, to the corresponding classical equilibria of the Vlasov equation (see [89], Thm 2.22). As observed before, the exchange term can be neglected in the semi-classical limit; then we expect that a similar result could be proven considering the Hartree equation with the exchange term. This will be the object of future works.

3.1.2 Main result

In this article we study, to our knowledge for the first time, nonlinear asymptotic dynamics of the Hartree-Fock equation, including the exchange term. Our first result is that Equation (3.1) admits nonlocalised equilibria of the form

$$Y_f = \int_{\mathbb{R}^d} f(\xi) e^{i(\xi x - \theta(\xi) t)} dW(\xi). \tag{3.5}$$

Above, the momenta distribution function f is any nonnegative L^2 function, the phase is

$$\theta(\xi) = |\xi|^2 + \int_{\mathbb{R}^d} w dx \int_{\mathbb{R}^d} f^2 dx - (2\pi)^{d/2} (\hat{w} * f^2)(\xi), \tag{3.6}$$

where $\hat{w}(\xi) = (2\pi)^{-d/2} \int e^{-i\xi x} w(x) dx$ is the Fourier transform of w, and dW is the Wiener integral, i.e. $dW(\xi)$ are infinitesimal centred Gaussian variables characterized by

$$\mathbb{E}[\overline{dW(\xi)} dW(\xi')] = \delta(\xi - \xi') d\xi d\xi'.$$

The law of the Gaussian field Y_f is invariant under time and space translations. It is an equilibrium of (3.11) in the sense that it is a solution whose law does not depend on time. Complete details on these equilibria are given in Subsection 3.1.3. We take perturbations of these equilibria with initial data

$$X_{|t=0} = Y_f + Z_0. \tag{3.7}$$

We show asymptotic stability of the equilibrium via scattering to linearized waves given by the free evolution

$$S(t)\, Z(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(\eta x - \theta(\eta))} \hat{Z}(\eta)\, d\eta.$$

The linear flow $S(t)$ disperses as does the Schrödinger linear flow, both pointwise and regarding Strichartz estimates. Details are given in Section 3.3. We set $\tilde\theta = -(2\pi)^{d/2}\hat w * f^2$ and assume

$$\tilde\theta \in C^{d+2} \cap W^{d+2,\infty} \cap W^{4,1}, \tag{3.8}$$

and the uniform ellipticity assumption

$$\eta^\top \nabla^{\otimes 2}\theta(\xi)\,\eta > \lambda_* |\eta|^2 \tag{3.9}$$

for all $\xi, \eta \in \mathbb{R}^d$, $\eta \neq 0$, for a constant $\lambda_* > 0$. We denote by $\|\mu\|_M$ the total variation of a measure μ.

Theorem 3.1.1. *Let $d \geq 4$. Let w be a Borel measure with $\langle y\rangle w$ a finite measure, and $g = f^2$ that satisfies $g \in W^{3,1} \cap W^{3,\infty}$ with $\langle \xi\rangle^{2\lceil s_c\rceil} g \in W^{2,1}$, be such that θ satisfies (3.8) and (3.9).*

Then there exists $C(\|\nabla^{\otimes 3}\tilde\theta\|_{L^\infty}, \lambda_*) > 0$, that is increasing in $\|\tilde\theta\|_{W^{3,\infty}}$ and decreasing with λ_*, such that assuming

$$\|\langle y\rangle w\|_M \|\nabla g\|_{W^{2,1}} \leq C(\|\nabla^{\otimes 3}\tilde\theta\|_{L^\infty}, \lambda_*),$$

the following holds true. There exists $\delta > 0$ such that if

$$\|Z_0\|_{L^2(\Omega, H^{s_c}(\mathbb{R}^d)) \cap L^{2d/(d+2)}(\mathbb{R}^d, L^2(\Omega))} \leq \delta,$$

the Cauchy problem (3.1) with initial data (3.7) has a solution in

$$Y_f + \mathcal{C}(\mathbb{R}, L^2(\Omega, H^{s_c}(\mathbb{R}^d)))$$

and there exists $Z_\pm \in L^2(\Omega, H^{s_c}(\mathbb{R}^d))$ such that

$$X(t) = Y_f(t) + S(t)Z_\pm + o_{L^2(\Omega, H^{s_c}(\mathbb{R}^d))}(1)$$

when $t \to \pm\infty$.

Remark 3.1.2. The solution of Theorem 3.1.1 is unique in a certain class of global in time perturbations: that which are small in the space defined by (3.21) and whose density is small in the space given by (3.22). This is as a by product of our proof by fixed point. We believe uniqueness holds in $\mathcal{C}(\mathbb{R}, H^s) \cap L^{2\frac{d+2}{d}}(\mathbb{R}, W^{s_c, 2\frac{d+2}{d}}(\mathbb{R}^d))$ without smallness assumption by a local well posedness argument similar to that for the cubic NLS, see for instance [72].

Comments on the result

On the optimality of the regularity assumptions. We remark that s_c is the critical Sobolev regularity for the cubic Schrödinger equation in dimension d. Thus, our regularity assumption on Z_0 seems optimal.

On the perturbative nature of the result, compared with the reduced Hartree-Fock equations. This result is perturbative since the equilibrium has to satisfy the smallness assumption (3.1.1), which contrasts with the previous results of the first and third author, [47, 48]. The reason for this is twofold: because of the exchange term, first, the dispersion relation θ can loose its ellipticity, and second, the linearised equation for the density ρ around the equilibrium is no longer a Fourier multiplier, see Section 3.3. We rely on perturbative arguments to invert it. However, Theorem 3.1.1 covers "large" equilibria, in the sense that $\|g\|_{L^1}$ can be large so that the density is large.

Some examples covered by the result. We now describe a physically relevant example of such g for which Theorem 3.1.1 applies. For Fermi gases at density ρ and positive temperature, the function g depends on the temperature T and the chemical potential μ as

$$g[\rho, T, \mu](\xi) = \rho C_{\frac{\mu}{T}} T^{-\frac{d}{2}} \frac{1}{e^{(|\xi|^2-\mu)/T}+1}$$

where $C_{\tilde{\mu}} = (\int \frac{d\xi}{e^{|\xi|^2-\tilde{\mu}}+1})^{-1}$. We can figure several ways of satisfying the assumptions (3.9) and (3.1.1). Indeed one estimates that for $s = 1, 2, 3$

$$\|\nabla^s g\|_{L^1} = \rho T^{-\frac{s}{2}} \frac{\|\nabla^s \frac{1}{e^{|\xi|^2-\mu/T}+1}\|_{L^1}}{\|\frac{1}{e^{|\xi|^2-\mu/T}+1}\|_{L^1}} \approx \rho T^{-\frac{s}{2}} \left(\frac{\mu}{T}\right)_+^{\frac{s}{2}-1}$$

and for $s = 2, 3$

$$\|\nabla^s \tilde{\theta}\|_{L^\infty} \leq \|w\|_M \|\nabla^s g\|_{L^1} \lesssim \rho T^{-\frac{s}{2}} \left(\frac{\mu}{T}\right)_+^{\frac{s}{2}-1}.$$

The hypotheses of the Theorem are then met for a small enough density at fixed temperature and chemical potential, or at fixed density letting T go to infinity while maintaining $\frac{\mu}{T}$ negative or constant positive.

We remark moreover that our result holds in large dimensions, i.e. for $d \geq 4$. The cases $d = 2, 3$ have to be treated separately. Indeed, as observed in [48], in dimensions 2 and 3, a contraction argument using solely Strichartz estimates is not sufficient to prove scattering for quadratic Schrödinger-type equations. This problem was solved in the cited work, in the case of the reduced Hartree-Fock equation, exploiting the structure of the nonlinearity. We believe that similar arguments could be exploited to treat also this case. This will be the object of future works. To conclude, we mention the very recent work [94] where the author studied the 1D case, showing that there is no long range scattering due to a nonlinear cancellation between the direct term and the exchange term for plane waves.

Comments on the proof

As already mentioned, the result is perturbative and hence so is the proof. The articulation is the following, we write the equation as a system with one equation describing the full perturbation as a linear equation depending on its correlations and the other describing the evolution of the correlations. Contrary to [47, 48], one needs to describe the full set of correlations, that is $\mathbb{E}(\bar{X}(x)X(y))$ and not only the diagonal $\mathbb{E}(|X(x)|^2)$, see Section 3.2.

The treatment of the nonlinear terms is similar to [47] keeping in mind that we have two space variables instead of one.

What mostly differs is the treatment of the linearised around the equilibrium part. It requires proving Strichartz estimates for a propagator that is adapted to the exchange term, see Subsection 3.3.1. The rest of the treatment of the linearised equation is based on explicit computations and functional analysis see Subsection 3.3.2 and Proposition 3.3.13.

In the rest of the introductive section, we formally derive the Hartree equation with exchange term and we explain what are its equilibria. In Section 3.1.3, we give useful notations and conventions used throughout the paper. In Section 3.2, we rewrite the problem as a contraction argument and we specify the functional framework. In Section 3.3, we treat the linearised equation around its equilibria. In Section 3.4, we prove bilinear estimates that are sufficient to close the contraction argument. Finally, in Section 3.5, we perform the contraction argument and give the final arguments to prove Theorem 3.1.1.

3.1.3 Formal derivation of the equation from the N body problem

In this subsection, we present a quick and formal derivation of Equation (3.1) from the many body Schrödinger equation. The references for rigorous results are given in Subsection 3.1. We consider N particles, represented by a wave function $\psi : \mathbb{R}^{dN} \to \mathbb{C}$, with binary interactions through a pair potential w. The Hamiltonian of the system is

$$\begin{aligned}\mathcal{E}_N(\psi) &= -\int_{\mathbb{R}^{dN}} \bar{\psi} \sum_{i=1}^{N} \Delta_i \psi \, d\underline{x} + \int_{\mathbb{R}^{dN}} \sum_{i<j} w(x_i - x_j)|\psi|^2 \, d\underline{x} \\ &= \mathcal{E}_{kin} + \mathcal{E}_{pot}\end{aligned}$$

where $\underline{x} = (x_1, ..., x_N) \in \mathbb{R}^{dN}$, each x_i belongs to $\mathbb{R}^d$ and Δ_i is the Laplacian in the variable x_i. We consider for Fermions wave functions of the form of a Slater determinant

$$\psi(\underline{x}) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma)^{\iota} \prod_{j=1}^{N} u_{\sigma(j)}(x_j),$$

where $(u_i)_{1\leq i\leq N}$ is an orthonormal family in $L^2(\mathbb{R}^d)$, and the sum is over permutations σ of $\{1, N\}$ whose signature is denoted by $\varepsilon(\sigma)$. We recall that these standard Ansätze are driven by the ideas that bosons are indiscernible particles and that fermions satisfy Pauli's exclusion principle, which translates to the symmetry and skew symmetry of ψ respectively. Under these Ansätze, the expression of the Hamiltonian simplifies. Indeed, the potential energy becomes

$$\begin{aligned}\mathcal{E}_{pot}(\psi) &= \int_{\mathbb{R}^{dN}} \frac{1}{N!}\sum_{i<j} w(x_i - x_j) \sum_{\sigma_1,\sigma_2\in\mathfrak{S}_N} \varepsilon(\sigma_1\sigma_2) \prod_{k=1}^{N} \overline{u_{\sigma_1(k)}(x_k)} u_{\sigma_2(k)}(x_k)\, d\underline{x} \\ &= \frac{1}{N!} \sum_{\sigma_1,\sigma_2\in\mathfrak{S}_N} \varepsilon(\sigma_1\sigma_2) \sum_{i<j} \langle u_{\sigma_1(i)}\bar{u}_{\sigma_2(i)}, w * (\bar{u}_{\sigma_1(j)}u_{\sigma_2(j)})\rangle \prod_{k\neq i,j} \langle u_{\sigma_1(k)}, u_{\sigma_2(k)}\rangle\end{aligned}$$

where the scalar product is taken into $L^2(\mathbb{R}^d)$. Because the family $(u_k)_{1\leq k\leq N}$ is orthonormal we get that and

$$\prod_{k\neq i,j} \langle u_{\sigma_1(k)}, u_{\sigma_2(k)}\rangle = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 \text{ or } \sigma_2 = \sigma_1 \circ (ij), \\ 0 & \text{otherwise.}\end{cases}$$

This further yields

$$\mathcal{E}_{pot}(\psi) = \sum_{i<j} \Big(\langle |u_{\sigma_1(i)}|^2, w * |u_{\sigma_1(j)}|^2\rangle - \langle u_{\sigma_1(i)}\bar{u}_{\sigma_1(j)}, w * (\bar{u}_{\sigma_1(j)}u_{\sigma_1(i)})\rangle\Big).$$

We perform the change of variables $i' = \sigma_1(i)$ and $j' = \sigma_1(j')$ and finally get

$$\mathcal{E}_{pot}(\psi) = \frac{1}{2}\sum_{i'\neq j'} \Big(\langle |u_{i'}|^2, w * |u_{j'}|^2\rangle - \langle u_{i'}\bar{u}_{j'}, w * (\bar{u}_{j'}u_{i'})\rangle\Big).$$

A similar and simpler computation shows that the kinetic energy is

$$\mathcal{E}_{kin}(\psi) = -\sum_{j'=1}^{N} \langle u_{j'}, \triangle u_{j'}\rangle.$$

The Hamiltonian is therefore

$$\mathcal{E}_N(\psi) = -\sum_{j'=1}^{N} \langle u_{j'}, \triangle u_{j'}\rangle + \frac{1}{2}\sum_{i'\neq j'} \Big(\langle |u_{i'}|^2, w * |u_{j'}|^2\rangle - \langle u_{i'}\bar{u}_{j'}, w * (\bar{u}_{j'}u_{i'})\rangle\Big).$$

In the limit of large number of particles N, we formally replace the sum $\sum_{i'\neq j'}$ by $\sum_{i',j'}$ and we arrive at the following system of evolution equations for $(u_1, ..., u_N)$:

$$i\partial_t u_j(x) = -\Delta u_j(x) + \sum_k \int_{\mathbb{R}^d} w(x-y)\Big(|u_k(y)|^2 u_j(x) - \overline{u_k(y)} u_j(y) u_k(x)\Big)\, dy, \quad (3.10)$$

for $j = 1, ..., N$.

One can recast the above system as an equation on random fields. Indeed, considering $X : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{C}$ a time dependent random field over $\mathbb{R}^d$, where $(\Omega, \mathcal{A}, d\omega)$ is the underlying probability space, of the form $X(x, \omega) := \sum_k u_k(x) g_k(\omega)$ where $(g_k)_{1\leq k\leq N}$ is an orthonormal family of $L^2(\Omega)$. Then X is a solution of

$$i\partial_t X = -\Delta X + \Big(w * \mathbb{E}[|X|^2]\Big)X - \int_{\mathbb{R}^d} w(\cdot - y)\mathbb{E}[\overline{X(y)}X(\cdot)]X(y)dy. \quad (3.11)$$

Equilibria

We briefly study in this subsection the equilibria

$$Y_f = \int_{\mathbb{R}^d} f(\xi) e^{i(\xi x - \theta(\xi) t)} dW(\xi),$$

and show they are solutions to (3.1). Note that $f \in L^2(\mathbb{R}^d)$ can be chosen real and nonnegative without loss of generality. We set

$$g = f^2$$

and decompose the phase

$$\theta(\xi) = |\xi|^2 + \tilde{\theta}(\xi) + \theta_0 \tag{3.12}$$

where

$$\tilde{\theta} = -(2\pi)^{\frac{d}{2}} \hat{w} * g \qquad \text{and} \qquad \theta_0 = (2\pi)^d \hat{w}(0) \hat{g}(0). \tag{3.13}$$

The function Y_f is a Gaussian field, whose law is invariant under spatial translations, and its correlation function is

$$\mathbb{E}[\overline{Y_f(x)} Y_f(y)] = \int_{\mathbb{R}^d} f^2(\xi) e^{i\xi(y-x)} d\xi = (2\pi)^{d/2} \hat{g}(x-y). \tag{3.14}$$

This formula can be used to show that Y_f is a solution to (3.11). Indeed, it gives

$$\Big(w * \mathbb{E}[|Y_f|^2] \Big)(x) = (2\pi)^d \hat{w}(0) \hat{g}(0)$$

as well as

$$\begin{aligned}
\int_{\mathbb{R}^d} w(x-y) \mathbb{E}[\overline{Y_f(y)} Y_f(x)] Y(y) dy &= \int_{\mathbb{R}^{3d}} w(x-y) g(\xi') e^{i\xi'(x-y)} f(\xi) e^{i(\xi y - \theta(\xi) t)} d\xi' dW(\xi) dy \\
&= (2\pi)^{\frac{d}{2}} \int \hat{w}(\xi - \xi') g(\xi') f(\xi) e^{i(\xi x - \theta(\xi) t)} d\xi' dW(\xi) \\
&= (2\pi)^{\frac{d}{2}} (\hat{w} * g)(D) Y_f
\end{aligned}$$

and the result follows by injecting these two identities in (3.11). In addition, the law of Y_f is invariant under time translations, making it an equilibrium of the equation. Relevant equilibria to the present article are discussed in the comments after Theorem 3.1.1.

Notation

The scalar product on $\mathbb{R}^d$ is denoted by

$$\xi x = \sum_1^d \xi_i x_i.$$

Our notation for the Fourier transform is

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx.$$

Fourier multipliers by a symbol s are denoted by $s(D)$ and defined by

$$\mathcal{F}(s(D)f)(\xi) = s(\xi)\hat{f}(\xi).$$

By $\nabla^{\otimes 2}$, we denote the Hessian matrix.
In order to lighten the notations, we denote by $2^{\mathbb{N}}$ the set $\{2^n, n \in \mathbb{N}\}$.
For $z \in \mathbb{R}^d$, we write T_z the translation such that for any function h and any $x \in \mathbb{R}^d$, $T_z h(x) = h(x+z)$.
For $p, q \in [1, \infty]$ and for $s \in \mathbb{R}$ we denote $L_t^p W_x^{s,q} L_\omega^2$ the space

$$(1-\Delta_x)^{-\frac{s}{2}} L^p(\mathbb{R}, L^q(\mathbb{R}^d, L^2(\Omega))),$$

with the norm:

$$||u||_{L_t^p W_x^{s,q} L_\omega^2} = ||\langle \nabla \rangle^s u||_{L_t^p L_x^q L_\omega^2}.$$

In the case $s = 2$ we also write $L_t^p H_x^s, L_\omega^2 = L_t^p W_x^{s,2}, L_\omega^2$.
For $p, q \in [1, \infty]$, $s, t \in \mathbb{R}$ we denote by $L_t^p B_q^{s,t} L_\omega^2 = L^p B_q^{s,t} L_\omega^2$ the space induced by the norm:

$$||u||_{L^p B_q^{s,t} L_\omega^2} = \left\| \left(\sum_{j<0} 2^{2js} ||u_j||^2_{L_x^q L_\omega^2} + \sum_{j \geq 0} 2^{2jt} ||u_j||^2_{L_x^q L_\omega^2} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})}.$$

We write $\|\mu\|_M$ the total variation of the *signed* measure $\mu \in M$.

3.2 Setting the contraction argument

In this section, we write the problem at hand, namely solving a Cauchy problem and a scattering problem, as a fixed point problem. This fixed point problem will be solved using a contraction argument in Section 3.5, and the proof of Theorem 3.1.1 will follow. Because Y_f is not localised and thus not in any Sobolev space, neither is X, and we choose as a variable for the fixed point not the full solution X but its perturbation around Y_f, namely $Z = X - Y_f$.

We fix an equilibrium Y, and drop the f subscript to lighten the notation. We consider a perturbed solution $X = Y + Z$ to (3.11). We expand using (3.14) and (3.13)

$$\begin{aligned} \mathbb{E}[w * |X|^2]X - \mathbb{E}[w * |Y|^2]Y &= \mathbb{E}[|Y|^2]Z + \mathbb{E}[w * (|X|^2 - |Y|^2|)]X \\ &= \theta_0 Z + \mathbb{E}[w * (|X|^2 - |Y|^2|)]X \end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{X(y)}X(x)]X(y)\,dy - \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{Y(y)}Y(x)]Y(y)\,dy\\ &\quad= \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{Y(y)}Y(x)]Z(y)\,dy + \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{X(y)}X(x) - \overline{Y(y)}Y(x)]X(y)\,dy\\ &\quad= \tilde{\theta}(D)Z + \int_{\mathbb{R}^d} w(x-y)\mathbb{E}[\overline{X(y)}X(x) - \overline{Y(y)}Y(x)]X(y)\,dy.\end{aligned}$$

Hence Z satisfies

$$i\partial_t Z \;=\; \theta(D)Z + \mathbb{E}[w * (|X|^2 - |Y|^2|)]X - \int_{\mathbb{R}^d} w(\cdot - y)\mathbb{E}[\overline{X(y)}X(\cdot) - \overline{Y(y)}Y(\cdot)]X(y)\,dy.$$

We introduce the perturbation of the two-point correlation function

$$\begin{aligned}V(x,z) &= \mathbb{E}[\overline{X}(x+z)X(x) - \overline{Y}(x+z)Y(x)]\\ &= \mathbb{E}[\overline{Y(x+y)}Z(x) + \overline{Z(x+y)}Y(x) + \overline{Z(x+y)}Z(x)].\end{aligned} \tag{3.15}$$

The evolution equation for Z becomes

$$i\partial_t Z = \theta(D)Z + \Big(w * V(\cdot,0)\Big)(Y+Z) - \int_{\mathbb{R}^d} w(z)V(x,z)(Y+Z)(x+z)\,dz.$$

Introducing the group $S(t) = e^{-it\theta(D)}$, we obtain

$$\begin{aligned}Z = S(t)Z_0 &- i\int_0^t S(t-\tau)\Big[\big(w * V(\cdot,0)\big)Y - \int w(z)V(x,z)Y(x+z)\,dz\Big]\\ &- i\int_0^t S(t-\tau)\Big[\big(w * V(\cdot,0)\big)Z - \int w(z)V(x,z)Z(x+z)\,dz\Big].\end{aligned}$$

This can be written under the form

$$Z = S(t)Z_0 + L_1(V) + L_2(V) + Q_1(Z,V) + Q_2(Z,V) \tag{3.16}$$

where the linearised operators are

$$\begin{aligned}L_1(V) &= -i\int_0^t S(t-\tau)\big[w * V(\cdot,0)Y\big]d\tau,\\ L_2(V) &= i\int_0^t S(t-\tau)\big[\int dz w(z)V(x,z)Y(x+z)\big]d\tau,\end{aligned}$$

and the quadratic terms are

$$\begin{aligned}Q_1(Z,V) &= -i\int_0^t S(t-\tau)\big[w * V(\cdot,0)Z\big]d\tau,\\ Q_2(Z,V) &= i\int_0^t S(t-\tau)\big[\int dz w(z)V(x,z)Z(x+z)\big]d\tau.\end{aligned}$$

The perturbed correlation function V is given by

$$\begin{aligned} V &= \mathbb{E}(\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)) \\ &\quad + L_3(V) + L_4(V) \\ &\quad + Q_3(Z,V) + Q_4(Z,V) + \mathbb{E}(\bar{Z}(x+y)Z(x)) \end{aligned} \tag{3.17}$$

where the corresponding linearized operators and quadratic terms are for $k = 3, 4$,

$$\begin{aligned} L_k(V) &= \mathbb{E}[\overline{Y(x+y)}L_{k-2}(V)(x)] + \mathbb{E}[\overline{L_{k-2}(V)(x+y)}Y(x)], \\ Q_k(Z,V) &= \mathbb{E}[\overline{Y(x+y)}Q_{k-2}(Z,V)(x)] + \mathbb{E}[\overline{Q_{k-2}(Z,V)(x+y)}Y(x)]. \end{aligned}$$

Combining (3.16) and (3.17) we arrive at the following fixed point equation for (Z, V)

$$\begin{pmatrix} Z \\ V \end{pmatrix} = \mathcal{A}_{Z_0} \begin{pmatrix} Z \\ V \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{(1)}_{Z_0}(Z,V)) \\ \mathcal{A}^{(2)}_{Z_0}(Z,V) \end{pmatrix} \tag{3.18}$$

where we have set

$$\begin{aligned} \mathcal{A}^{(1)}_{Z_0}(Z,V) &= S(t)Z_0 + L_1(V) + L_2(V) + Q_1(Z,V) + Q_2(Z,V), \\ \mathcal{A}^{(2)}_{Z_0}(Z,V) &= \mathbb{E}[\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)] \\ &\qquad + L_3(V) + L_4(V) + Q_3(Z,V) + Q_4(Z,V) + \mathbb{E}[\overline{Z(x+y)}Z(x)]. \end{aligned}$$

We will solve the fixed point equation (3.18) via a contraction argument for the application $\mathcal{A}_{Z_0}$ in the following Banach spaces for (Z, V):

$$\begin{aligned} E_Z &= \mathcal{C}(\mathbb{R}, L^2(\Omega, H^{s_c}(\mathbb{R}^d))) \cap L^p(\mathbb{R}, W^{s_c,p}(\mathbb{R}^d, L^2(\Omega))) \\ &\quad \cap L^{d+2}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega)) \cap L^4(\mathbb{R}, L^q(\mathbb{R}^d, L^2(\Omega))), \end{aligned} \tag{3.19}$$

$$E_V = \mathcal{C}(\mathbb{R}^d, L^{\frac{d+2}{2}}(\mathbb{R}\times\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}^d, L^2(\mathbb{R}, B_2^{-\frac{1}{2},s_c}(\mathbb{R}^d))) \tag{3.20}$$

where $p = 2\frac{d+2}{d}$, $s_c = \frac{d}{2} - 1$, $q = \frac{4d}{d+1}$. We endow E_V and E_Z with the norms

$$\begin{aligned} \|\cdot\|_Z &= \|\cdot\|_{L^\infty(\mathbb{R},L^2(\Omega,H^s(\mathbb{R}^d)))} + \|\cdot\|_{L^p(\mathbb{R},W^{s_c,p}(\mathbb{R}^d,L^2(\Omega)))} \\ &\quad + \|\cdot\|_{L^{d+2}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))} + \|\cdot\|_{L^4(\mathbb{R},B_q^{0,\frac{1}{4}}(\mathbb{R}^d,L^2(\Omega)))}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \|\cdot\|_V &= \|\cdot\|_{\mathcal{C}(\mathbb{R}^d,L^{\frac{d+2}{2}}(\mathbb{R}\times\mathbb{R}^d))} + \|\cdot\|_{\mathcal{C}(\mathbb{R}^d,L^2(\mathbb{R},B_2^{-\frac{1}{2},s_c}(\mathbb{R}^d)))} \\ &\quad + \|\cdot\|_{C(\mathbb{R}^d,L^2(\mathbb{R},H^{s_c}(\mathbb{R}^d)))}. \end{aligned} \tag{3.22}$$

Remark 3.2.1. The spaces and norms we chose are driven by the following considerations. The regularity s_c is the critical regularity of the cubic Schrödinger equation in dimension d. The choice of the Lebesgue exponents p and $d+2$ for Z and $\frac{d+2}{2}$ and 2 for V are the ones required to put Q_1 and Q_2 in the target space for the solution $\mathcal{C}(\mathbb{R}, L^2(\Omega, H^{s_c}(\mathbb{R}^d)))$. The regularity in the low frequencies for V, namely the $-\frac{1}{2}$ in $B_2^{-\frac{1}{2},s_c}$ is due to a low frequencies singularity that we see appearing in Proposition 3.3.9.

The choice of the L^∞ norm in the variable y is due to the fact that Lebesgue and Sobolev norms are invariant under the action of translations and thus the norms of V in x should be uniformly bounded in the variable y.

3.3 Linear estimates

3.3.1 Strichartz estimates

Proposition 3.3.1 (Strichartz estimates)**.** *Let θ, $\tilde{\theta}$ defined as in (3.6), (3.13) such that*

i) θ satisfies the ellipticity assumption (3.9),

ii) $\tilde{\theta} \in C^{d+2}(\mathbb{R}^d) \cap W^{d+2,\infty}(\mathbb{R}^d)$.

Let $(p,q) \in [2,\infty]^2$ such that

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad (p,q) \neq (2,\infty) \text{ if } d = 2.$$

Then there exists a constant $C = C(p,q,d)$ such that for any $u_0 \in L^2(\mathbb{R}^d)$,

$$\left\| e^{-it\theta(D)} u_0 \right\|_{L_t^p L_x^q} \leq C \|u_0\|_{L_x^2}.$$

Remark 3.3.2. The regularity assumption ii) is satisfied, for example, in the case of a very short range interaction potential $\langle y \rangle^{d+2} w \in M$, or in the case of an integrable potential $w \in M$ and of a smooth distribution function $g \in W^{d+2,1}$. The uniform ellipticity is then satisfied, for example, if the interaction potential and the density are small enough $\|\langle y \rangle^2 w\|_M \|g\|_{L^1} < C(d)$ or the distribution function is spread enough $\|w\|_M \|\nabla^2 g\|_{L^1} < C(d)$, respectively.

The ellipticity (3.9) can be false at high densities, no matter the interaction potential. Indeed, consider a fixed potential w and for $\rho > 0$ an equilibrium $g = \rho g^*$, with g^* and w both Schwartz and nonzero. We have $\theta = \theta_0 + |\xi|^2 - (2\pi)^{d/2} \rho \hat{w} * g^*$. Since there exists a point at which the Hessian of $\hat{w} * g$ is not nonnegative, for large ρ the Hessian of θ is not positive definite at that point.

The failure of the ellipticity (3.9) would lead to different linearized dynamics, and would strongly differ from the reduced Hartree equation (3.3)-(3.4) where this issue is absent.

Proof. Let us divide the proof in steps.
Step 1: localization in frequencies.
Let $\chi_0, \chi \in C_c^\infty(\mathbb{R}^d)$ be such that

i) $0 \leq \chi_0, \chi \leq 1$ and $\operatorname{supp} \chi_0 \subset \{|\xi| \leq 1\}$, $\operatorname{supp} \chi \subset \{1 \leq |\xi| \leq 2\}$;

ii) for any $\xi \in \mathbb{R}^d$
$$\chi_0(\xi) + \sum_{\lambda \in 2^{\mathbb{N}}} \chi(\lambda^{-1}\xi) = 1;$$

iii) $\exists C \in (0,1)$ such that for any $\xi \in \mathbb{R}^d$
$$c \leq \chi_0^2(\xi) + \sum_{\lambda \in 2^{\mathbb{N}}} \chi^2(\lambda^{-1}\xi) \leq 1.$$

Let us call $\chi_\lambda(\xi) \coloneqq \chi(\lambda^{-1}\xi)$, $\lambda \in 2^{\mathbb{N}}$. We define the following frequency localised function

$$u_\lambda(t,x) \coloneqq e^{-it\theta(D)}\chi_\lambda(D)u_0(x), \quad \lambda \in 2^{\mathbb{N}} \cup \{0\}$$

which is given by

$$\begin{aligned} u_\lambda(t,x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-it\theta(\xi)+i(x-y)\cdot\xi}\chi_\lambda(\xi)d\xi u_0(y)dy \\ &= \int_{\mathbb{R}^d} I_\lambda(t,x-y)u_0(y)dy = [I_\lambda(t,\cdot) * u_0(\cdot)](x), \end{aligned}$$

where

$$I_\lambda(t,x) \coloneqq (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-it\theta(\xi)+ix\cdot\xi}\chi_\lambda(\xi)d\xi.$$

Step 2: frequency localised dispersive estimate.
We recall the following result, which is Theorem 3 in [1].

Theorem 3.3.3. *Consider*

$$I(\mu) = \int_{\mathbb{R}^d} e^{i\mu\phi(\xi)}\psi(\xi)d\xi,$$

with $\phi \in C^{d+2}(\mathbb{R}^d)$ real-valued, $\psi \in C_0^{d+1}(\mathbb{R}^d)$. Let us define $K \coloneqq \operatorname{supp}\psi$. Assume

i) $\mathcal{M}_{d+2}(\phi) = \sum_{2\le|\alpha|\le d+2}\sup_{\xi\in K_{\varepsilon_0}}|\nabla^{\otimes\alpha}\phi(\xi)| < +\infty$ *where K_{ε_0} is a neighbourhood of K,*

ii) $\mathcal{N}_{d+1}(\psi) = \sum_{|\alpha|\le d+1}\sup_{\xi\in K}|\nabla^{\otimes\alpha}\psi(\xi)| < +\infty$,

iii) $a_0 = \inf_{\xi\in K_{\varepsilon_0}}|\det\nabla^{\otimes 2}\phi(\xi)| > 0$ *and the map $\xi \mapsto \nabla\phi(\xi)$ is injective.*

Then there exists $C > 0$ depending only on the dimension d such that

$$|I(\mu)| \le \mu^{-\frac{d}{2}}Ca_0^{-1}(1+\mathcal{M}_{d+2}^{\frac{d}{2}})\mathcal{N}_{d+1}. \tag{3.23}$$

Let

$$\phi(t,x,\xi) \coloneqq -\theta(\xi) + \frac{x}{t}\cdot\xi,$$

so that

$$I_\lambda(t,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{it\phi(t,x,\xi)}\chi_\lambda(\xi)d\xi.$$

We use (3.23) to estimate the L_x^∞ norm of I_λ for every $\lambda \in 2^{\mathbb{N}} \cup \{0\}$ uniformly wrt λ and x. Let us show that ϕ_λ, χ satisfy hypothesis of Theorem 3.3.3 and that we can bound the RHS of (3.23) uniformly wrt λ;

i) from assumption *ii*) we have $\mathcal{M}_{d+2} < C$ where $C \neq C(\lambda)$;

ii) the boundness of $\mathcal{N}_{d+1}$ follows from definitions of χ_λ, χ_0 in the previous step;

iii) since $d \geq 3$, from assumption $ii)$ it follows that ϕ is at least C^3. Then $a_0 > 0$ iff

$$\inf_{\xi\in\mathbb{R}^d} \det \nabla^{\otimes 2}(-\phi(\xi)) \geq c > 0.$$

Then, by assumption $i)$, there exists $c_1 > 0$ such that

$$\inf_{\xi\in\mathbb{R}^d} |\det \nabla^{\otimes 2}\phi(\xi)| \geq c_1$$

uniformly wrt λ. Moreover, we show that the gradient of $\tilde{\phi} = -\phi$ is injective:

$$\nabla\tilde{\phi}(\xi_1) - \tilde{\nabla}\tilde{\phi}(\xi_2) = \int_0^1 \frac{d}{ds}[\nabla\tilde{\phi}_(s\xi_1 + (1-s)\xi_2)]ds \tag{3.24}$$

$$= \int_0^1 \nabla^{\otimes 2}\tilde{\phi}(s\xi_1 + (1-s)\xi_2)\cdot(\xi_1 - \xi_2)ds. \tag{3.25}$$

Then, by assumption $i)$ we get

$$|\nabla\tilde{\phi}(\xi_1) - \nabla\tilde{\phi}(\xi_2)||\xi_1 - \xi_2| \geq \langle\nabla\tilde{\phi}(\xi_1) - \nabla\tilde{\phi}(\xi_2), \xi_1 - \xi_2\rangle \geq c|\xi_1 - \xi_2|^2;$$

Moreover, the same bound holds if $\lambda = 0$.
Finally, combining the previous estimate with the Young's inequality, we obtain

$$\begin{aligned}\|e^{it\theta(D)}\chi_\lambda(D)u_0\|_{L^\infty_x} &\leq \|I_\lambda(t,\cdot)\|_{L^\infty_x}\|u_0\|_{L^1_x} \\ &\leq Ct^{-\frac{d}{2}}\|u_0\|_{L^1_x}\end{aligned} \tag{3.26}$$

where C depends only on d.
Step 3: Strichartz estimates.
We recall the following result, it is Theorem 1.2 in [85].

Theorem 3.3.4. *Let (X, dx) be a measure space and H an Hilbert space. Suppose that for all $t \in \mathbb{R}$ an operator $U(t)\colon H \to L^2(x)$ obeys the following estimates*

- *for all t and $f \in H$ we have*

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_H;$$

- *for all $t \neq s$ and $g \in L^1(X)$*

$$\|U(s)U^*(t)g\|_{L^\infty} \lesssim |t-s|^{-\sigma}\|g\|_{L^1},$$

  for some $\sigma > 0$.

Then

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_H,$$

for all $(q,r) \in [2,+\infty]^2$ such that

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}, \quad (q,r,\sigma) \neq (2,\infty,1)$$

where the endpoint $P = \left(2, \frac{2\sigma}{\sigma-1}\right)$ is admissible if $\sigma > 1$.

We observe that Theorem 3.3.4 holds with the choice $U(t) = e^{-it\theta(D)}\chi_\lambda(D)\colon L^2_x \to L^2_x$, $\sigma = \frac{d}{2}$. Then, we have that

$$\left\| e^{-it\theta(D)}\chi_\lambda(D)u_0 \right\|_{L^p_t L^q_x} \leq C\|u_0\|_{L^2_x} \tag{3.27}$$

for all $(p,q) \in [2,+\infty]^2$ such that

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad (p,q) \neq (2,\infty) \text{ if } d = 2. \tag{3.28}$$

Moreover, estimate (3.27) holds the same if we replace the RHS by $\|\chi_\lambda(D)u_0\|_{L^2_x}$. Indeed, we can replace χ_λ in (3.27) by a function $\tilde{\chi}_\lambda \in C^\infty_c$ such that $\tilde{\chi}_\lambda\chi_\lambda = \chi_\lambda$, then

$$\left\| e^{-it\theta(D)}\chi_\lambda(D)u_0 \right\|_{L^p_t L^q_x} = \left\| e^{-it\theta(D)}\tilde{\chi}_\lambda\chi_\lambda(D)u_0 \right\|_{L^p_t L^q_x} \leq C\|\chi_\lambda(D)u_0\|_{L^2_x}.$$

Then, since $[e^{-it\theta(D)}, \chi_\lambda(D)] = 0$, by Littlewood-Paley Theorem, Minkowski's inequality and (3.27), for all (p,q) satisfying (3.28), $q < \infty$, we have the following

$$\begin{aligned}
\left\| e^{-it\theta(D)}u_0 \right\|_{L^p_t L^q_x} &\simeq \left\| \left\| e^{-it\theta(D)}\chi_0(D)u_0 \right\|_{L^q_x} + \left\| \Big(\sum_{\lambda \in 2^{\mathbb{N}}} |e^{-it\theta(D)}\chi_\lambda(D)u_0|^2 \Big)^{\frac{1}{2}} \right\|_{L^q_x} \right\|_{L^p_t} \\
&\leq \left\| e^{-it\theta(D)}\chi_0(D)u_0 \right\|_{L^p_t L^q_x} + \left(\sum_{\lambda \in 2^{\mathbb{N}}} \left\| e^{-it\theta(D)}\chi_\lambda(D)u_0 \right\|^2_{L^p_t L^q_x} \right)^{\frac{1}{2}} \\
&\lesssim \left\| \chi_0(D)u_0 \right\|_{L^2_x} + \left(\sum_{\lambda \in 2^{\mathbb{N}}} \left\| \chi_\lambda(D)u_0 \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \\
&\simeq \|u_0\|_{L^2_x}.
\end{aligned}$$

□

Remark 3.3.5. We observe that it would be possible to adapt the proof of Theorem 1.2 in [104] in order to prove the Strichartz estimates with slightly different assumptions on θ. That is, $\theta \in C^{n_0}(\mathbb{R}^d)$, $n_0 > \frac{d+2}{2}$ and $\langle \xi \rangle^{n-2}\nabla^{\otimes n}\tilde{\theta} \in L^\infty$ for any $n = 2, \ldots, n_0$.

Proposition 3.3.1 leads classically, see e.g. [120] section 2.3, to the following results:

Corollary 3.3.6. *Let (q_1, r_1) admissible and (q_2, r_2, s_2) such that $r \geq 2$ where $\frac{1}{r} = \frac{1}{r_2} + \frac{s_2}{d}$ and $\frac{2}{q_2} + \frac{d}{r_2} + s_2 = \frac{d}{2}$. There exists $C > 0$ such that for all $u_0 \in H^{s_2}$ and $F \in L^{q_1'}_t H^{s_2, r_1'}_x$:*

$$\left\| S(t)u_0 - i\int_0^t S(t-\tau)F(\tau)d\tau \right\|_{L^{q_2}_t L^{r_2}_x} \leq C\Big[||u_0||_{H^{s_2}_x} + ||F||_{L^{q_1'}_t H^{s_2, r_1'}_x} \Big]. \tag{3.29}$$

3.3.2 Representation formulae for the linear terms

We record here suitable expressions for the operators L_1, L_2, L_3 and L_4. We introduce the semi-group of operators

$$\mathcal{T}_\xi(t)U(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int e^{-i(\theta(\eta+\xi)-\theta(\eta)-\theta(\xi))t} e^{i\eta x}\hat{U}(\eta)d\eta$$

Proposition 3.3.7. *One has the following formulas for V a Schwartz function:*

$$L_1(V) = -i\int\int_0^t f(\xi)e^{i\xi x - i\theta(\xi)t}\mathcal{T}_\xi(t-\tau)S(t-\tau)\big[w * V(\cdot,0,\tau)\big]\,d\tau dW(\xi), \tag{3.30}$$

$$L_2(V) = i\int\int\int_0^t f(\xi)e^{i\xi x - i\theta(\xi)t}e^{i\xi z}w(z)\mathcal{T}_\xi(t-\tau)S(t-\tau)V(\cdot,z,\tau)\,d\tau dz dW(\xi), \tag{3.31}$$

$$L_3(V) = i\iint\int_0^t (g(\xi+\zeta)-g(\xi))e^{-i(\theta(\xi+\zeta)-\theta(\xi))(t-\tau)}e^{i\zeta x - i\xi y}\hat{w}(\zeta)\hat{V}(\zeta,0,\tau)\,d\tau d\xi d\zeta, \tag{3.32}$$

and

$$L_4(V) = -i(2\pi)^{-\frac{d}{2}}\iiint\int_0^t (g(\xi+\zeta)-g(\xi))e^{-i(t-\tau)(\theta(\xi+\zeta)-\theta(\xi))}e^{i\zeta x + i\xi(z-y)}w(z)\hat{V}(\zeta,z,\tau)\,d\tau\,dz\,d\xi d\zeta. \tag{3.33}$$

Remark 3.3.8 (Formal properties of the formulae). The formulae (3.30) and (3.31) display a Galilei transformation-type effect. Indeed, the symbol of the group $\mathcal{T}_\xi(t)$ satisfies (due to the ellipticity condition (3.9) for θ)

$$|\nabla_\eta(\theta(\xi+\eta)-\theta(\eta)-\theta(\xi))| \approx |\xi| \qquad \text{and} \qquad \frac{\xi}{|\xi|}.\nabla_\eta(\theta(\xi+\eta)-\theta(\eta)-\theta(\xi)) \approx |\xi|$$

so that $\mathcal{T}_\xi(t)$ formally corresponds to transport with velocity ξ. This is clear when $\theta = |\xi|^2$ in which case $T_\xi(t)$ is the space translation of vector $2\xi t$.

From the formulae (3.30) and (3.31), one can then expect $L_1(V)$ and $L_2(V)$ to enjoy the same dispersive estimates as a solution to $i\partial_t u = \theta(D)u + V$. This is obtained by noticing that $T_\xi(t)S(t)$ enjoys the same dispersive estimates as $S(t)$, and by formally discarding the effects of the extra variables ξ and z.

These formulae also hint to the fact that $L_1(V)$ and $L_2(V)$ could enjoy improved dispersive estimates than that of the group $S(t)$ alone. Indeed, the operator $T_\xi(t)$ amounts to translating in the direction ξt. When averaging over ξ such transport effects in all directions, this should produce an additional damping mechanism. This is made rigorous in Proposition 3.3.9.

Proof. We first remark that from the definition (3.15) one has

$$\overline{V(x,y)} = V(x+y,-y)$$

which in Fourier translates into the relation

$$\overline{\hat{V}(\eta,y)} = \hat{V}(-\eta,-y)e^{-i\eta y}. \tag{3.34}$$

Formula for L_1. By the definition (3.5) of the equilibrium we have

$$L_1(V) = -i\int_0^t S(t-\tau)\big[w * V(\cdot,0)Y\big]d\tau,$$
$$= -i\int\int_0^t f(\xi)e^{-i\theta(\xi)\tau}S(t-\tau)\big[w*V(\cdot,0)e^{i\xi x}\big]d\tau dW(\xi).$$

We readily check that

$$e^{i\theta(\xi)t}S(t)(e^{i\xi x}U) = e^{i\xi x}\mathcal{T}_\xi(t)S(t)U, \tag{3.35}$$

which gives the desired identity (3.30).

Formula for L_2. Using again the definition (3.5) of the equilibrium we have

$$L_2(V) = -i\int_0^t S(t-\tau)\big[\int dz w(z)V(x,z)Y(x+z)\big]d\tau,$$
$$= -i\int\int\int_0^t f(\xi)e^{-i\theta(\xi)\tau}e^{i\xi z}w(z)S(t-\tau)\big[V(x,z)e^{i\xi x}\big]d\tau dz dW(\xi).$$

One then obtains (3.31) by appealing to (3.35).

Formula for L_3. We decompose

$$L_3(V) = \mathbb{E}[\overline{Y(x+y)}L_1(V)(x)] + \mathbb{E}[\overline{L_1(V)(x+y)}Y(x)]$$
$$= L_3^{(1)}(V) + L_3^{(2)}(V).$$

We notice that

$$\overline{L_3^{(1)}(V)(x,y)} = L_3^{(2)}(V)(x+y,-y)$$

which in Fourier gives

$$\widehat{L_3^{(1)}(V)}(\eta,y) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int \overline{L_3^{(2)}(V)(x+y,-y)}e^{-i\eta x}dx$$
$$= \frac{1}{(2\pi)^{\frac{d}{2}}}\overline{\int L_3^{(2)}(V)(z,-y)e^{i\eta(z-y)}dx}$$
$$= e^{i\eta y}\overline{\widehat{L_3^{(2)}}(-\eta,-y)}. \tag{3.36}$$

Hence it suffices to compute $\widehat{L_3^{(1)}(V)}$ in order to retrieve $\widehat{L_3(V)}$ as then

$$\widehat{L_3(V)}(\zeta,y) = \widehat{L_3^{(1)}(V)}(\zeta,y) + e^{i\zeta y}\overline{\widehat{L_3^{(1)}(V)}(-\zeta,-y)}. \tag{3.37}$$

We infer from (3.30) that

$$L_1(V) = -i\int\int\int_0^t f(\xi)e^{i\xi x - i\theta(\xi)t}e^{i\zeta x}e^{-i(t-\tau)(\theta(\zeta+\xi)-\theta(\xi))}\hat{w}(\zeta)\hat{V}(\zeta,0,\tau)\,d\tau d\zeta dW(\xi).$$

Using (3.5) yields

$$L_3^{(1)}(x,y) \;=\; -i\int\int_0^t f^2(\xi)e^{-i\xi y}e^{i\zeta x}e^{-i(t-\tau)(\theta(\zeta+\xi)-\theta(\xi))}\hat{w}(\zeta)\hat{V}(\zeta,0,\tau)\,d\tau d\zeta d\xi.$$

So we get:

$$\widehat{L_3^{(1)}}(V)(\zeta,y) = -i(2\pi)^{\frac{d}{2}}\hat{w}(\zeta)\int\int_0^t f^2(\xi)e^{-i(\theta(\xi+\zeta)-\theta(\xi))(t-\tau)}e^{-i\xi y}\hat{V}(\zeta,0,\tau)\,d\tau d\xi.$$

Injecting the above identity in (3.37) and using (3.34) finally gives

$$\widehat{L_3(V)}(\zeta,y) = -i(2\pi)^{\frac{d}{2}}\hat{w}(\zeta)\int_0^t \hat{V}(\zeta,0)\int(f(\xi)^2-f(\xi+\zeta)^2)e^{-i(\theta(\xi+\zeta)-\theta(\xi))(t-\tau)}e^{-i\xi y}d\tau\,d\xi.$$

This is (3.32).

Formula for L_4. It is very similar to L_3. We first decompose

$$\begin{aligned} L_4(V) &= \mathbb{E}[\overline{Y(x+y)}L_2(V)(x)] + \mathbb{E}[\overline{L_2(V)(x+y)}Y(x)] \\ &= L_4^{(1)}(V) + L_4^{(2)}(V). \end{aligned}$$

As for the analogue decomposition for L_3, we have

$$\widehat{L_4^{(1)}}(V)(\eta,y) = e^{i\eta y}\overline{\widehat{L_4^{(2)}}(-\eta,-y)}. \tag{3.38}$$

Hence it suffices to compute $L_4^{(1)}$. We infer from (3.31) that

$$L_2(V)(x) = \frac{i}{(2\pi)^{\frac{d}{2}}}\int_0^t\iiint f(\xi)w(z)\hat{V}(\eta-\xi,z,\tau)e^{i\eta x+i\xi z}e^{-i(t-\tau)\theta(\eta)-i\tau\theta(\xi)}\,dz\,d\eta\,d\tau\,dW(\xi)$$

so that using (3.5) one computes that

$$L_4^{(1)}(x,y) = \frac{i}{(2\pi)^{\frac{d}{2}}}\int_0^t\iiint f^2(\xi)w(z)\hat{V}(\eta-\xi,z,\tau)e^{i(\eta-\xi)x+i\xi(z-y)}e^{i(t-\tau)(\theta(\xi)-\theta(\eta))}\,dz\,d\eta\,d\tau\,d\xi.$$

One thus obtains, using a change of variables, that

$$\widehat{L_4^{(1)}}(\zeta,y) = i\int_0^t\iint f^2(\xi)w(z)\hat{V}(\zeta,z,\tau)e^{i\xi(z-y)}e^{-i(t-\tau)(\theta(\xi+\zeta)-\theta(\xi))}\,dz\,d\tau\,d\xi.$$

Using successively (3.38) and (3.34), and then changing variables

$$\begin{aligned} \widehat{L_4^{(2)}}(\zeta,y) &= e^{i\zeta y}\overline{\widehat{L_4^{(1)}}(-\zeta,-y)} \\ &= -ie^{i\zeta y}\int_0^t\iint f^2(\xi)w(z)\overline{\hat{V}(-\zeta,z,\tau)}e^{-i\xi(z+y)}e^{-i(t-\tau)(\theta(\xi)-\theta(\xi-\zeta))}\,dz\,d\tau\,d\xi \\ &= -i\int_0^t\iint f^2(\xi)w(z)\hat{V}(\zeta,-z,\tau)e^{i(\zeta-\xi)(z+y)}e^{-i(t-\tau)(\theta(\xi)-\theta(\xi-\zeta))}\,dz\,d\tau\,d\xi \\ &= -i\int_0^t\iint f^2(\xi+\zeta)w(z)\hat{V}(\zeta,z,\tau)e^{i\xi(z-y)}e^{-i(t-\tau)(\theta(\zeta+\xi)-\theta(\xi))}\,dz\,d\tau\,d\xi. \end{aligned}$$

Combining the two identities above concludes the proof of (3.33). □

3.3.3 The issue of low frequency regularity in the linear response

Proposition 3.3.9. *Let $\sigma, \sigma_1 \geq 0$, $\sigma_1 < \frac{d}{2}$, $p_1 > 2$, $q_1 \geq 2$ such that*

$$\frac{2}{p_1} + \frac{d}{q_1} = \frac{d}{2} - \sigma_1.$$

Assuming that $\langle \xi \rangle^{2\lceil \sigma \rceil} g \in W^{2,1}$ and $\tilde{\theta} \in W^{4,1}$ along with the ellipticity assumption, there exists a constant C_θ (decreasing with λ_* and increasing with $\|\tilde{\theta}\|_{W^{4,1}}$) such that for all $U \in L^2_t, B_2^{-1/2+\sigma_1, \sigma+\sigma_1-1/2}$,

$$\Big\| \int_0^\infty S(t-\tau)[U(\tau)Y(\tau)]d\tau \Big\|_{L^{p_1}(\mathbb{R}, W^{\sigma,q_1}(\mathbb{R}^d, L^2(\Omega)))} \leq C_\theta \|\langle \xi \rangle^{2\lceil \sigma \rceil} g\|_{W^{2,1}} \|U\|_{L^2_t, B_2^{-1/2+\sigma_1, \sigma+\sigma_1-1/2}}. \tag{3.39}$$

Proof. We start by taking U in the Schwartz class to give a sense to the computations and we conclude by density.

Set $L_1^\infty(U) := \int_0^\infty S(t-\tau)Y(\tau)U(\tau)d\tau$.

We denote for $\eta \in \mathbb{R}^d$, $t \in \mathbb{R}$, $S_\eta(t)$, the Fourier multiplier by

$$\xi \mapsto e^{-it(|\xi|^2 + 2\eta\cdot\xi + (2\pi)^d (-1)^\iota \hat{w} * g(\xi+\eta) - \hat{w}*g(\eta))} = e^{it(\theta(\eta) - \theta(\xi+\eta))}.$$

Step 1: We compute $\mathbb{E}[|L_1^\infty(U)|^2]$.

We have the commutation relation

$$S(t)(e^{i\eta x}U) = \mathcal{F}^{-1}\Big(e^{-it\theta(\xi)}\hat{U}(\xi - \eta)\Big) = e^{i\eta x}\mathcal{F}^{-1}\Big(e^{-it\theta(\xi+\eta)}\hat{U}(\xi)\Big)$$

and because the Fourier transform is taken only on the space variable, we get

$$S(t)(e^{i\eta x}U) = e^{i\eta x - it\theta(\eta)}\mathcal{F}^{-1}\Big(e^{-it(\theta(\xi+\eta)-\theta(\eta))}\hat{U}(\xi)\Big).$$

We recognize

$$S(t)(e^{i\eta x}U) = e^{i\eta x - it\theta(\eta)} S_\eta(t)U.$$

We deduce

$$\begin{aligned} S(t-\tau)(U(\tau)Y(\tau)) &= \int f(\eta) e^{-i\tau\theta(\eta)} S(t-\tau)(e^{i\eta x}U(\tau))dW(\eta) \\ &= \int f(\eta) e^{i\eta x} e^{-it\theta(\eta)} S_\eta(t-\tau)U(s)dW(\eta). \end{aligned}$$

Then we get, using the definition of Wiener integral:

$$\mathbb{E}[|L_1^\infty(U)|^2] = \int_{\eta\in\mathbb{R}^d} g(\eta) \Big| \int_0^\infty S_\eta(t-\tau)U(\tau)d\tau \Big|^2 d\eta.$$

This concludes Step 1.

Step 2: We claim that:

$$||L_1^\infty(U)||_{L_t^{p_1},L_x^{q_1},L_\omega^2} \leq C(\theta)\|g\|_{W^{2,1}}||U||_{L_t^2,B_2^{\sigma_1-\frac{1}{2},\sigma_1-\frac{1}{2}}}$$

where $C(\theta)$ is a constant depending on $\hat{w} * g$.

Using step 1 and Minkowski inequality we have:

$$||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2} \leq \int_{\eta\in\mathbb{R}^d} g(\eta)\left|\left|\int_0^\infty S_\eta(t-\tau)U(\tau)d\tau\right|\right|^2_{L_t^{p_1},L_x^{q_1}} d\eta.$$

By Strichartz's inequality and Bernstein's lemma, we have:

$$||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2} \leq \int_{\eta\in\mathbb{R}^d} g(\eta)\left|\left|\int_0^\infty S_\eta(-\tau)U(\tau)d\tau\right|\right|^2_{B_2^{\sigma_1,\sigma_1}} d\eta.$$

We introduce the variable U_1 defined by $\hat{U_1}(\xi) = |\xi|^{\sigma_1}\hat{U}(\xi)$ and we have, by Parceval's identity:

$$\begin{aligned}&||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\\ &\quad\leq \int_{\eta\in\mathbb{R}^d} g(\eta)\int_{\xi\in\mathbb{R}^d}\int_0^\infty\int_0^\infty e^{i(t_1-t_2)(\xi^2-2\xi.\eta-\tilde{\theta}_\xi(\eta))}\hat{U_1}(t_1,\xi)\overline{\hat{U_1}(t_2,\xi)}dt_2dt_1d\xi d\eta\end{aligned}$$

where $\tilde{\theta}_\xi(\eta) = \tilde{\theta}(\xi+\eta) - \tilde{\theta}(\eta)$. We define θ_ξ and g_ξ) is analogously.

We perform the change of variable $\eta_\xi = \eta - \tilde{\theta}_\xi(\eta)\frac{\xi}{2|\xi|^2}$. It is a C^1-diffeomorphism. Indeed, we have that the Jacobian matrix of $\eta \mapsto \eta_\xi$ is the identity minus the Jacobian matrix of $\eta \mapsto (2\pi)^d\tilde{\theta}_\xi(\eta)\frac{\xi}{2|\xi|^2}$. This last matrix is of rank 1 and writes $\frac{\xi}{2|\xi|}\nabla_\eta\frac{\tilde{\theta}_\xi}{|\xi|}$. We deduce that the Jacobian is invertible if $1 - \langle\frac{\xi}{2|\xi|}, \nabla_\eta\frac{\tilde{\theta}_\xi}{|\xi|}\rangle$ does not vanish. By definition of $\tilde{\theta}_\xi$, we have

$$\nabla\frac{\tilde{\theta}_\xi}{|\xi|} = \int_{0^1}\nabla^{\otimes 2}\tilde{\theta}(\eta+t\xi)\frac{\xi}{|\xi|}dt.$$

We deduce that

$$1 - \langle\frac{\xi}{2|\xi|}, \nabla_\eta\frac{\tilde{\theta}_\xi}{|\xi|}\rangle = \frac{1}{2}\int_0^1\frac{\xi^T}{|\xi|}\nabla^{\otimes 2}\theta(\eta+t\xi)\frac{\xi}{|\xi|}dt > \frac{1}{2}\lambda_*.$$

We denote $\phi_\xi = \tilde{\eta_\xi}^{-1}$. This gives, by also doing the change of variable $t = t_2 - t_1$:

$$\begin{aligned}&||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\\ &\leq \int_{\eta\in\mathbb{R}^d}\int_{\xi\in\mathbb{R}^d} g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\int_{\mathbb{R}}\int_{D_t} e^{-it(\xi^2-2\xi\eta)}\hat{U_1}(t_1,\xi)\overline{\hat{U_1}(t+t_1,\xi)}dt_1dtd\xi d\eta,\end{aligned}$$

where $D_t = [-t,\infty]$.

We integrate over η to get:

$$\begin{aligned}&||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\\ &\leq (2\pi)^{d/2}\int_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}}\int_{D_t}\mathcal{F}_\eta\Big(g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\Big)(-2\xi t)e^{-it\xi^2}\hat{U}_1(t_1,\xi)\overline{\hat{U}_1(t+t_1,\xi)}dt_1dtd\xi.\end{aligned}$$

We use Cauchy-Schwarz inequality over t_1 to get:

$$||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\leq\int_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}}\Big|\mathcal{F}_\eta\Big(g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\Big)(-2\xi t)\Big|||\hat{U}_1(t_1,\xi)||^2_{L^2_{t_1}}dtd\xi.$$

Then, we get by doing the change of variable $\tau=t|\xi|$:

$$||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\lesssim\int_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}}\Big|\mathcal{F}_\eta\Big(g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\Big)(-2\tau\frac{\xi}{|\xi|})\Big||\xi|^{-1}||\hat{U}_1(t_1,\xi)||^2_{L^2_{t_1}}d\tau d\xi.$$

We claim that $(\tau,\xi)\mapsto\mathcal{F}_\eta\Big(g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\Big)(-2\tau\frac{\xi}{|\xi|})$ belongs to $L_\xi^\infty(\mathbb{R}^d,L^1_\tau(\mathbb{R}))$. It is sufficient for this to prove that

$$(\xi,\eta)\mapsto g(\phi_\xi(\eta))jac(\phi_\xi(\eta))$$

belongs to $L_\xi^\infty(\mathbb{R}^d,W_\eta^{2,1}(\mathbb{R}^d))$. This is implied by the fact that $g\in W^{2,1}(\mathbb{R}^d)$ and $\phi_\xi\in L_\xi^\infty W^{3,\infty}$, the latter coming from the fact that $\tilde{\theta}\in W^{4,1}$ which implies that $\tilde{\theta}_\xi\in L_\xi^\infty(\mathbb{R}^d,W^3_\eta(\mathbb{R}^d))$. We therefore get

$$||L_1^\infty(U)||^2_{L_t^{p_1},L_x^{q_1},L_\omega^2}\leq C_\theta\|g\|_{W^{2,1}}\int_{\xi\in\mathbb{R}^d}|||\xi|^{\sigma_1-\frac12}\hat{U}(t_1,\xi)||^2_{L^2_{t_1}}dtd\xi=C_\theta\|g\|_{W^{2,1}}||U||_{L_t^2,B_2^{\sigma_1-\frac12,\sigma_1-\frac12}}.$$

Step 3: We first suppose that $\sigma\in\mathbb{N}$.
For $\alpha\in\mathbb{N}^d$ we write: $|\alpha|=\sum_{j=1}^d\alpha_j$ and $\partial^\alpha=\prod_{j=1}^d\partial^{\alpha_j}$.
For $\eta\in\mathbb{R}^d$ we write $\eta^\alpha=\prod_{j=1}^d\eta_j^{alpha_j}$.
We have for any $\alpha\in\mathbb{N}^d$:

$$\partial^\alpha L_1^\infty(U)=\sum_{\gamma+\beta=\alpha}C(\alpha,\beta)\int_0^t S(t-s)\partial^\beta Y(s)\partial^\gamma U(s)ds.$$

Indeed, Y is almost everywhere differentiable and there holds, for $|\beta|\leq\lceil s\rceil$:

$$\partial^\beta Y(s)=\int_{\mathbb{R}^d}i^{|\beta|}\eta^\beta f(\eta)e^{-is\theta(\eta)+i\eta x}dW(\eta).$$

Replacing Y by $\partial^\beta Y$ consists in replacing $f(\eta)$ by $i^{|\beta|}\eta^\beta f(\eta)$.
Thus, using step 2 we get:

$$||L_1^\infty(U)||^2_{L_t^{p_1},W_x^{\sigma,q_1},L_\omega^2}\lesssim||U||_{L_t^2,B_2^{\sigma_1-\frac12,\sigma_1+\sigma-\frac12}},\tag{3.40}$$

where the constants depends on $\sup_{|\alpha|\leq\sigma}\left|\left|\mathcal{F}_\eta\Big(\phi_\xi(\eta)^{2\alpha}g(\phi_\xi(\eta))jac(\phi_\xi(\eta))\Big)\right|\right|_{L^1}$.
We conclude by interpolation that inequalities (3.40) holds for any $\sigma \geq 0$.

□

Corollary 3.3.10. *With the notations of Proposition 3.3.9, we have that*

$$L_1^{\infty,0} : U \mapsto \int_0^t S(t-\tau)[U(\tau)Y(\tau)]d\tau, \quad L_1^{\infty,\infty} : U \mapsto \int_t^\infty S(t-\tau)[U(\tau)Y(\tau)]d\tau$$

are continuous operators from $L_t^2 B_2^{-1/2+\sigma_1,-1/2+\sigma_2+\sigma}$ *to* $L_t^{p_1} W_x^{\sigma,q_1}$.

Proof. This is a standard application of the Christ-Kiselev lemma. □

Corollary 3.3.11. *The operator*

$$L_1 \ : V \ \mapsto L_1(V) = -i\int_0^t S(t-\tau)\big[w * V(\cdot,0)Y\big]d\tau$$

is continuous from $L_y^\infty L_t^2 B_2^{-1/2,s_c}$ *to* E_Z *and thus from* E_V *to* E_Z.

Proof. We apply Corollary 3.3.10 to $(p_1, q_1, \sigma_1, \sigma)$ equal to either

$$(p, p, 0, s_c),\ (d+2, d+2, s_c, 0),\ (4, q, \frac{d-3}{4}, 0)\text{or } (\infty, 2, 0, s_c)$$

to get the result. □

Corollary 3.3.12. *The operator*

$$L_2 \ : V \ \mapsto L_2(V) = i\int_0^t S(t-\tau)\big[\int dz w(z)V(x,z)Y(x+z)\big]d\tau$$

is continuous from $L_y^\infty L_t^2 B_2^{-1/2,s_c}$ *to* E_Z *and thus from* E_V *to* E_Z.

Proof. We begin by writing that, as w is a finite measure:

$$||L_2(V)||_{E_Z} \lesssim \int_{z\in\mathbb{R}^d}\left|\left|\int_0^t S(t-\tau)V(\cdot,z)T_zY d\tau\right|\right|_{E_Z} w(z)dz.$$

For $z \in \mathbb{R}^d$, we have:

$$\left|\left|\int_0^t S(t-\tau)V(\cdot,z)T_zY d\tau\right|\right|_{E_Z} = \left|\left|\int_0^t S(t-\tau)T_{-z}V(\cdot,z)Y d\tau\right|\right|_{E_Z}.$$

By applying Corollary 3.3.10 as in Corollary 3.3.11 we get:

$$\left|\left|\int_0^t S(t-\tau)T_{-z}V(\cdot,z)Y d\tau\right|\right|_{E_Z} \lesssim ||T_{-z}V(\cdot,z)||_{L_t^2,B_2^{-\frac{1}{2},s_c}} = ||V(\cdot,z)||_{L_t^2,B_2^{-\frac{1}{2},s_c}},$$

and finally:

$$\left|\left|\int_0^t S(t-\tau)T_{-z}V(\cdot,z)Y d\tau\right|\right|_{E_Z} \lesssim ||V||_{E_V}$$

Then, we can conclude by writing that:

$$||L_2(V)||_{E_Z} \lesssim ||w||_M ||V||_{E_V}.$$

□

3.3.4 Estimates and invertibility on the last linear terms

Proposition 3.3.13. *The operators L_3 and L_4 are continuous on $\mathcal{C}_y L^2_{t,x} \cap L^\infty_y L^2_{t,x}$ with for $k = 3, 4$,*

$$\|L_k(V)\|_{L^\infty_y L^2_{t,x}} \leq C_{\theta,w,g} \|V\|_{L^\infty_y L^2_{t,x}}. \tag{3.41}$$

Moreover, we have that $L_4(V)$ and $L_3(V)$ belong to $\mathcal{C}_y L^2_{t,x}$.

The constant is of the form $C_{\theta,w,g} = C_\theta \|\langle y\rangle w\|_{L^1} \|\nabla g\|_{W^{2,1}}$ for a constant C_θ that is decreasing with λ_* and increasing with $\|\nabla^{\otimes 3}\tilde{\theta}\|_{L^\infty}$.

In order to prove Proposition 3.3.13, we need the following lemma.

Lemma 3.3.14 (Estimate for $\frac{e^{i\eta\alpha}}{\alpha}$-like principal values)**.** *Consider a function $\tilde{\Omega} \in C^2(\mathbb{R})$ satisfying $\tilde{\Omega}' > \lambda$ for some $\lambda > 0$ and $\tilde{\Omega}'' \in L^\infty$, then for any $q \in (1,\infty)$, for any $u \in W^{1,q}(\mathbb{R})$ and $\eta \in \mathbb{R}$ there holds*

$$\left| p.v. \int_{\mathbb{R}} u(\alpha) \frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} d\alpha \right| \leq \frac{C}{\lambda}\Big(1 + \frac{\|\tilde{\Omega}''\|_{L^\infty}}{\lambda}\Big) \|u\|_{W^{1,q}(\mathbb{R})} \tag{3.42}$$

for some universal C that is independent of $\tilde{\Omega}$ and η.

Remark 3.3.15. Note that the estimate is false at the endpoint cases $q = 1$ and $q = \infty$. To see it, it suffices to consider the Hilbert transform $\frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} = \frac{1}{\alpha}$. The Hilbert transform is ill-defined on L^∞ which invalidates the estimate for $q = \infty$. Moreover, if $u \in W^{1,1}$, then $(\frac{1}{\alpha} * u)' \in L^{1,w}$ where $L^{1,w}$ is the weak L^1 space. However, there exists unbounded functions v such that $v' \in L^{1,w}$, for example the log function. The failure of the estimate in the case $q = 1$ is the reason why $g \in W^{3,1}$ is required in the previous proposition.

Proof. Up to dividing $\tilde{\Omega}$ by λ, we assume $\lambda = 1$ without loss of generality. Then as $\tilde{\Omega}' > 1$ there exists a unique zero α_0 of $\tilde{\Omega}$. We decompose, omitting the $p.v.$ notation for simplicity,

$$\int_{\mathbb{R}} \frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} u(\alpha) d\alpha = I + II + III \tag{3.43}$$

with

$$I = \int_{|\alpha-\alpha_0|>1} \left(\frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} - \frac{e^{i\eta\alpha}}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} \right) u(\alpha)d\alpha,$$

$$II = \int_{\mathbb{R}} \frac{e^{i\eta\alpha}}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} u(\alpha)d\alpha,$$

$$III = \int_{|\alpha-\alpha_0|<1} \left(\frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} - \frac{e^{i\eta\alpha}}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} \right) u(\alpha)d\alpha.$$

To estimate I, note that $\tilde{\Omega}'(\alpha_0) > 1$, and that if $|\alpha-\alpha_0| \geq 1$ then $|\tilde{\Omega}(\alpha)| > |\alpha-\alpha_0|$, so that by the Hölder inequality,

$$|I| \leq 2\int_{|\alpha-\alpha_0|\geq 1} \frac{|u(\alpha)|}{|\alpha-\alpha_0|} d\alpha \lesssim \|u\|_{L^q}. \tag{3.44}$$

Next we have $II = \frac{e^{i\eta\alpha_0}}{\tilde{\Omega}'(\alpha_0)} v(\alpha_0)$ with $v = (\alpha \mapsto \frac{e^{i\eta\alpha}}{\alpha}) * u$. Since

$$|v(\tilde{\alpha})| = \left| \int_{\mathbb{R}} \frac{e^{i\eta(\alpha-\tilde{\alpha})}}{\alpha-\tilde{\alpha}} u(\alpha)d\alpha \right| \leq \left| \int_{\mathbb{R}} \frac{1}{\alpha-\tilde{\alpha}} e^{i\eta\alpha} u(\alpha)d\alpha \right|$$

and

$$|v'(\tilde{\alpha})| = \left| \int_{\mathbb{R}} \frac{e^{i\eta(\alpha-\tilde{\alpha})}}{\alpha-\tilde{\alpha}} u'(\alpha)d\alpha \right| \leq \left| \int_{\mathbb{R}} \frac{1}{\alpha-\tilde{\alpha}} e^{i\eta\alpha} u'(\alpha)d\alpha \right|,$$

the boundedness of the Hilbert transform on L^q implies that $v \in W^{1,q}$ with $\|v\|_{W^{1,q}} \lesssim \|u\|_{W^{1,q}}$. By the Sobolev embedding one therefore has $v \in L^\infty$ so that

$$|II| \lesssim \|u\|_{W^{1,q}}. \tag{3.45}$$

Finally, to bound III we note that $2|\tilde{\Omega}(\alpha) - \tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)| \leq \|\tilde{\Omega}''\|_{L^\infty}|\alpha-\alpha_0|^2$ by Taylor's formula. Hence if $|\alpha-\alpha_0| < \frac{|\tilde{\Omega}'(\alpha_0)|}{\|\tilde{\Omega}''\|_{L^\infty}}$ (or for all $\alpha \in \mathbb{R}$ if $\|\tilde{\Omega}''\|_{L^\infty} = 0$) then we have

$$\frac{1}{\tilde{\Omega}(\alpha)} = \frac{1}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} \frac{1}{1 + \frac{\tilde{\Omega}(\alpha)-\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)}} = \frac{1}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} + O(\|\tilde{\Omega}''\|_{L^\infty})$$

where we used $\tilde{\Omega}'(\alpha_0) > 1$. We recall $|\tilde{\Omega}(\alpha)| > |\alpha-\alpha_0|$ as $\tilde{\Omega}' > 1$. If $|\alpha-\alpha_0| > \frac{|\tilde{\Omega}'(\alpha_0)|}{\|\tilde{\Omega}''\|_{L^\infty}}$ there then holds $\frac{1}{|\Omega(\alpha)|} < \|\tilde{\Omega}''\|_{L^\infty}$. Combining, we get that

$$\frac{e^{i\eta\alpha}}{\tilde{\Omega}(\alpha)} - \frac{e^{i\eta\alpha}}{\tilde{\Omega}'(\alpha_0)(\alpha-\alpha_0)} = O(\|\tilde{\Omega}''\|_{L^\infty})$$

for all $|\alpha-\alpha_0| < 1$. Thus, by the Hölder inequality one gets

$$|III| \lesssim \|\tilde{\Omega}''\|_{L^\infty} \int_{|\alpha-\alpha_0|\leq 1} |u(\alpha)|d\alpha \lesssim \|\tilde{\Omega}''\|_{L^\infty}\|u\|_{L^q}. \tag{3.46}$$

Injecting the bounds (3.44), (3.45) and (3.46) in (3.43) shows the desired result (3.42). □

We can now turn to the proof of Proposition 3.3.13.

Proof of Proposition 3.3.13. In what follows $\lesssim$ denotes inequalities where the implicit constant solely depend on λ_* and $\|\theta\|_{W^{3,\infty}}$. The letter ω will denote the dual Fourier variable of t, and not the probability variable as in the rest of the article. This should create no confusion as no probability variable is involved in the proof. We introduce $U(x,z,t) = w(z)V(x,z,t)$, so that

$$\widehat{L_4(V)}(\zeta,y,t) = -i\int_0^t \iint (g(\xi+\zeta)-g(\xi))e^{-i(t-\tau)(\theta(\xi+\zeta)-\theta(\xi))}\hat{U}(\zeta,z,\tau)e^{i\xi(z-y)}dzd\xi d\tau.$$

We write the above under the form

$$\widehat{L_4(V)}(\zeta,y,t) = -i(2\pi)^{\frac{d}{2}}\int (g(\xi+\zeta)-g(\xi))e^{-i\xi y}\big(\mathcal{F}_{x,y}U(\zeta,-\xi,\cdot)*(\mathbf{1}(\cdot\geq 0)e^{-i(\theta(\xi+\zeta)-\theta(\xi))\cdot})\big)(t)d\xi$$

where we used that $U(t)=0$ for $t<0$. In a similar way, we have

$$\begin{aligned}&\widehat{L_3(V)}(\zeta,y,t)\\ &= i(2\pi)^{\frac{d}{2}}\int (g(\xi+\zeta)-g(\xi))e^{-i\xi y}\big(\hat{w}(\zeta)\mathcal{F}_xV(\zeta,0,\cdot)*(\mathbf{1}(\cdot\geq 0)e^{-i(\theta(\xi+\zeta)-\theta(\xi))\cdot})\big)(t)d\xi.\end{aligned}$$

We introduce

$$U_4(\zeta,\xi,\omega) = \mathcal{F}_{x,y,t}U(\zeta,-\xi,\omega), \quad U_3(\zeta,\xi,\omega) = \hat{w}(\zeta)\mathcal{F}_{x,t}V(\zeta,0,\omega).$$

Note that U_3 does not depend on ξ but it unifies notations.

We recall that

$$\mathcal{F}_t(\mathbf{1}(\cdot\geq 0))(\omega) = \frac{1}{2}\sqrt{2\pi}\delta(\omega) + \frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}.$$

Therefore, applying the Fourier transform in time we have for $k=3,4$,

$$L_k(V) = (-1)^{k+1}\frac{i(2\pi)^{\frac{d+1}{2}}}{2}L_{k,\delta}(V) + (-1)^{k+1}(2\pi)^{\frac{d-1}{2}}L_{k,p.v}(V) \tag{3.47}$$

where the Dirac part is

$$\mathcal{F}_{x,t}L_{k,\delta}(V)(\zeta,y,\omega) = \int (g(\xi+\zeta)-g(\xi))e^{-i\xi y}U_k(\zeta,\xi,\omega)\delta(\omega+\theta(\xi+\zeta)-\theta(\xi))d\xi$$

and the principal value part is

$$\mathcal{F}_{x,t}L_{k,p.v.}(V)(\zeta,y,\omega) = \int (g(\xi+\zeta)-g(\xi))e^{-i\xi y}U_k(\zeta,\xi,\omega)\frac{1}{\omega+\theta(\xi+\zeta)-\theta(\xi)}d\xi.$$

Step 1. *Intermediate bound for the Dirac part $L_{k,\delta}$.* For a fixed nonzero $\zeta \in \mathbb{R}^d$, we decompose any $\xi \in \mathbb{R}^d$ in the form $\xi = \xi^\perp + \tilde{\xi}$ where $\tilde{\xi} \in \mathrm{Span}\{\zeta\}$ and $\xi^\perp \in \{\zeta\}^\perp$. We write $\tilde{\xi} = \sigma \frac{\zeta}{|\zeta|}$. This gives

$$\mathcal{F}_{x,t} L_{k,\delta}(V)(\zeta, y, \omega) = \iint (g(\xi+\zeta) - g(\xi)) e^{-i\xi y} U_k(\zeta, \xi, \omega) \delta \left(\Omega(\zeta, \xi^\perp + \sigma \frac{\zeta}{|\zeta|}, \omega) \right) d\xi^\perp d\sigma$$

where

$$\Omega(\zeta, \xi, \omega) = \omega + \theta(\xi + \zeta) - \theta(\xi).$$

As $\nabla_\xi \Omega = \nabla\theta(\xi+\zeta) - \nabla\theta(\xi)$ the uniform ellipticity assumption (3.9) implies that

$$\frac{\zeta}{|\zeta|} . \nabla_\xi \Omega(\zeta, \xi, \omega) > \lambda_* |\zeta| \tag{3.48}$$

for all ζ, ξ and ω. Therefore, for each fixed ζ, $\xi^\perp$ and ω, Ω admits a unique zero of the form $\xi_0 = \xi^\perp + \tilde{\xi}_0$ with $\tilde{\xi}_0 = \sigma_0(\zeta, \xi^\perp, \omega)\frac{\zeta}{|\zeta|}$. Integrating along the σ variables then gives

$$\mathcal{F}_{x,t} L_{k,\delta}(V)(\zeta, y, \omega) = \int \frac{1}{\frac{\zeta}{|\zeta|} . \nabla_\xi \Omega(\zeta, \xi_0, \omega)} (g(\xi_0+\zeta) - g(\xi_0)) e^{-i\xi_0 y} U_k(\zeta, \xi_0, \omega) d\xi^\perp.$$

We introduce

$$g_\zeta(\xi) = \frac{g(\xi+\zeta) - g(\xi)}{|\zeta|}$$

and

$$\tilde{U}_{1,k}(\zeta, \xi, \omega) = \frac{|\zeta|}{\frac{\zeta}{|\zeta|} . \nabla_\xi \Omega(\zeta, \xi, \omega)} g_\zeta(\xi) U_k(\zeta, \xi, \omega)$$

and the above becomes

$$\mathcal{F}_{x,t} L_{k,\delta}(V)(\zeta, y, \omega) = \int e^{-i\xi^\perp y - i\sigma_0 \frac{\zeta}{|\zeta|} y} \tilde{U}_{1,k}(\zeta, \xi^\perp + \sigma_0 \frac{\zeta}{|\zeta|}, \omega) d\xi^\perp.$$

The integrand can be bounded using the one dimensional Sobolev embedding $W^{1,1}(\mathbb{R}) \to L^\infty(\mathbb{R})$:

$$|\tilde{U}_{1,k}(\zeta, \xi^\perp + \sigma_0 \frac{\zeta}{|\zeta|}, \omega)| \lesssim \|\tilde{U}_{1,k}(\zeta, \xi^\perp + \tilde{\xi}, \omega)\|_{L^1_{\tilde{\xi}}} + \|\nabla_\xi \tilde{U}_{1,k}(\zeta, \xi^\perp + \tilde{\xi}, \omega)\|_{L^1_{\tilde{\xi}}}.$$

After integration along the remaining $\xi^\perp$ variable, this leads to the intermediate bound:

$$|\mathcal{F}_{x,t} L_{k,\delta}(V)(\zeta, y, \omega)| \lesssim \|\tilde{U}_{1,k}(\zeta, \cdot, \omega)\|_{L^1_\xi} + \|\nabla_\xi \tilde{U}_{1,k}(\zeta, \cdot, \omega)\|_{L^1_\xi} \tag{3.49}$$

for any ζ, y and ω. Note that $\tilde{U}_{1,3}$ depends on ξ through g.

Step 2. *Intermediate bound for the principal value part $L_{k,p.v.}$.* The reasoning is very similar to the previous one concerning $L_{k,\delta}$. For fixed nonzero ζ and ω we change again variables and write $\xi = \xi^\perp + \tilde{\xi}$ with $\tilde{\xi} = \sigma\frac{\zeta}{|\zeta|}$ and $\xi^\perp \in \{\zeta\}^\perp$, so that

$$\mathcal{F}_{x,t}L_{k,p.v.}(V)(\zeta,y,\omega) = \iint g_\zeta(\xi)e^{-i\xi y}U_k(\zeta,\xi,\omega)\frac{|\zeta|}{\Omega(\zeta,\xi^\perp+\sigma\frac{\zeta}{|\zeta|},\omega)}d\xi^\perp d\sigma.$$

We introduce

$$\tilde{U}_{2,k}(\zeta,\xi,\omega) = g_\zeta(\xi)U_k(\zeta,\xi,\omega)$$

and the above becomes

$$\mathcal{F}_{x,t}L_{k,p.v.}(V)(\zeta,y,\omega) = \iint \tilde{U}_{2,k}(\zeta,\xi^\perp+\sigma\frac{\zeta}{|\zeta|},\omega)e^{-i\xi^\perp y - i\sigma y\frac{\zeta}{|\zeta|}}\frac{|\zeta|}{\Omega(\zeta,\xi^\perp+\sigma\frac{\zeta}{|\zeta|},\omega)}d\xi^\perp d\sigma.$$

We recall that Ω satisfies (3.48). Furthermore, as $\nabla^{\otimes 3}\tilde{\theta} \in L^\infty$, Ω satisfies by the mean value theorem that

$$\begin{aligned}\left|\nabla_\xi(\frac{\zeta}{|\zeta|}\cdot\nabla_\xi\Omega(\zeta,\xi,\omega))\right| &\lesssim \left|\nabla_\xi(\frac{\zeta}{|\zeta|}\cdot\nabla_\xi\theta(\xi+\zeta)) - \nabla_\xi(\frac{\zeta}{|\zeta|}\cdot\nabla_\xi\theta(\xi))\right| \\ &\leq C(\|\nabla^{\otimes 3}\tilde{\theta}\|_{L^\infty},\lambda_*)|\zeta|. \end{aligned} \tag{3.50}$$

One can thus integrate along the σ variable and apply the estimate (3.42) with $\lambda = \lambda_*|\zeta|$ to bound

$$\begin{aligned}&\left|\int \tilde{U}_{2,k}(\zeta,\xi^\perp+\sigma\frac{\zeta}{|\zeta|},\omega)e^{-i\xi^\perp y - i\sigma y\frac{\zeta}{|\zeta|}}\frac{|\zeta|}{\Omega(\zeta,\xi^\perp+\sigma\frac{\zeta}{|\zeta|},\omega)}d\sigma\right| \\ &\qquad \leq C(\|\nabla^{\otimes 3}\tilde{\theta}\|_{L^\infty},\lambda_*)\|\tilde{U}_{2,k}(\zeta,\xi^\perp+\tilde{\xi},\omega)\|_{L^2_{\tilde{\xi}}} + \|\nabla_\xi\tilde{U}_{2,k}(\zeta,\xi^\perp+\tilde{\xi},\omega)\|_{L^2_{\tilde{\xi}}}\end{aligned}$$

Integrating along the remaining $\xi^\perp$ variable, this leads to the intermediate bound

$$|\mathcal{F}_{x,t}L_{k,p.v.}(V)(\zeta,y,\omega)| \lesssim \|\tilde{U}_{2,k}(\zeta,\cdot,\omega)\|_{L^1_{\xi^\perp}L^2_{\tilde{\xi}}} + \|\nabla_\xi\tilde{U}_{2,k}(\zeta,\cdot,\omega)\|_{L^1_{\xi^\perp}L^2_{\tilde{\xi}}} \tag{3.51}$$

for any ζ, y and ω.

Step 3. *Bounds for $\tilde{U}_{1,k}$ and $\tilde{U}_{2,k}$.* By (3.48) one has the equivalence

$$|\tilde{U}_{1,k}(\zeta,\xi,\omega)| \approx |\tilde{U}_{2,k}(\zeta,\xi,\omega)| = |g_\zeta(\xi)U_k(\zeta,\xi,\omega)|.$$

We recall that ω denotes the dual Fourier variable of t. We have therefore by the Hölder inequalities

$$\|\tilde{U}_{1,k}(\zeta,\cdot,\cdot)\|_{L^1_\xi L^2_\omega} \lesssim \|g_\zeta\|_{L^1_\xi}\|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} \tag{3.52}$$

and similarly, for $q = 1, 2$,

$$\begin{aligned} \|\tilde U_{2,k}(\zeta,\cdot,\cdot)\|_{L^1_{\xi^\perp}L^q_{\tilde\xi}L^2_\omega} &\lesssim \|g_\zeta\|_{L^1_{\xi^\perp}L^q_{\tilde\xi}} \|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} \\ &\lesssim \|g_\zeta\|_{W^{1,1}_\xi} \|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} \end{aligned} \tag{3.53}$$

where we used the one-dimensional Sobolev embedding $W^{1,1}_{\tilde\xi} \to L^q_{\tilde\xi}$. Next, we differentiate

$$\nabla_\xi \tilde U_{2,k}(\zeta,\xi,\omega) = \nabla g_\zeta(\xi) U_k(\zeta,\xi,\omega) + g_\zeta(\xi)\nabla_\xi U_k(\zeta,\xi,\omega)$$

and estimate similarly that for $q = 1, 2$,

$$\begin{aligned} \|\nabla_\xi \tilde U_{2,k}(\zeta,\xi,\omega)\|_{L^1_{\xi^\perp}L^q_{\tilde\xi}L^2_\omega} &\lesssim \|\nabla g_\zeta\|_{L^1_{\xi^\perp}L^q_{\tilde\xi}} \|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} + \|g_\zeta\|_{L^1_{\xi^\perp}L^q_{\tilde\xi}} \|\nabla_\xi U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} \\ &\lesssim \|g_\zeta\|_{W^{2,1}} (\|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} + \|\nabla_\xi U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}). \end{aligned} \tag{3.54}$$

Finally, we decompose

$$\begin{aligned} \left|\nabla_\xi \tilde U_{1,k}(\zeta,\xi,\omega)\right| &= \left| \frac{|\zeta|}{\frac{\zeta}{|\zeta|}.\nabla_\xi \Omega(\zeta,\xi,\omega)} \nabla_\xi \tilde U_{2,k} + \nabla_\xi \left(\frac{|\zeta|}{\frac{\zeta}{|\zeta|}.\nabla_\xi\Omega(\zeta,\xi,\omega)} \right) \tilde U_{2,k} \right| \\ &\lesssim C(\|\nabla^{\otimes 3}\tilde\theta\|_{L^\infty}, \lambda_*)(|\nabla_\xi \tilde U_{2,k}| + |\tilde U_{2,k}|) \end{aligned}$$

where we used (3.48) and (3.50) to obtain that $|\frac{|\zeta|}{\frac{\zeta}{|\zeta|}.\nabla_\xi\Omega(\zeta,\cdot,\cdot)}| + |\nabla_\xi(\frac{|\zeta|}{\frac{\zeta}{|\zeta|}.\nabla_\xi\Omega(\zeta,\cdot,\cdot)})| \lesssim 1$. Using (3.53) and (3.54) with $q = 1$ then shows

$$\|\nabla_\xi \tilde U_{1,k}(\zeta,\xi,\omega)\|_{L^1_\xi L^2_\omega} \lesssim C(\|\nabla^{\otimes 3}\tilde\theta\|_{L^\infty}, \lambda_*) \|g_\zeta\|_{W^{2,1}} (\|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega} + \|\nabla_\xi U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}). \tag{3.55}$$

Step 4. *Final bound.* Pick $y_0 \in \mathbb{R}^d$. For any $\zeta \in \mathbb{R}^d$ and $\omega \in \mathbb{R}$, injecting the previous intermediate estimates (3.49) and (3.51) in (3.47) yields

$$|\mathcal{F}_{x,t} L_k(V)(\zeta, y_0, \omega)| \lesssim \sum_{j=0}^1 \|\nabla^j_\xi \tilde U_{1,k}(\zeta,\cdot,\omega)\|_{L^1_\xi} + \|\nabla^j_\xi \tilde U_{2,k}(\zeta,\cdot,\omega)\|_{L^1_{\xi^\perp}L^2_{\tilde\xi}}.$$

Hence, for any fixed $\zeta \in \mathbb{R}^d$, applying the Parseval and Minkowski inequalities gives

$$\begin{aligned} \|\mathcal{F}_x L_k(V)(\zeta, y_0, \cdot)\|_{L^2_t} &= \|\mathcal{F}_{x,t} L_k(V)(\zeta, y_0, \cdot)\|_{L^2_\omega} \\ &\lesssim \sum_{j=0}^1 \|\nabla^j_\xi \tilde U_{1,k}(\zeta,\cdot,\omega)\|_{L^2_\omega L^1_\xi} + \|\nabla^j_\xi \tilde U_{2,k}(\zeta,\cdot,\omega)\|_{L^2_\omega L^1_{\xi^\perp} L^2_{\tilde\xi}} \\ &\lesssim \sum_{j=0}^1 \|\nabla^j_\xi \tilde U_{1,k}(\zeta,\cdot,\omega)\|_{L^1_\xi L^2_\omega} + \|\nabla^j_\xi \tilde U_{2,k}(\zeta,\cdot,\omega)\|_{L^1_{\xi^\perp} L^2_{\tilde\xi} L^2_\omega}. \end{aligned}$$

Using (3.52), (3.53), (3.55) and (3.54) to bound the right-hand side above we get

$$\|\mathcal{F}_x L_k(V)(\zeta, y_0, \cdot)\|_{L^2_t} \lesssim C(\|\nabla^{\otimes 3}\tilde{\theta}\|_{L^\infty}, \lambda_*)\|g_\zeta\|_{W^{2,1}_\xi}(\|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}+\|\nabla_\xi U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}). \tag{3.56}$$

We now turn to bounding g_ζ. We have for $j = 0, 1, 2$

$$\nabla^j g_\zeta(\xi) = \frac{\nabla^j g(\xi+\zeta) - \nabla^j g(\xi)}{|\zeta|} = \int_0^1 \nabla^{j+1} g(\xi + t\zeta) \cdot \frac{\zeta}{|\zeta|}\, dt$$

so that by Minkowski,

$$\|\nabla^j g_\zeta\|_{L^1_\xi} = \frac{1}{|\zeta|}\|\nabla^j g(\cdot+\zeta) - \nabla^j g(\cdot)\|_{L^1} \leq \|\nabla g\|_{\dot{W}^{j,1}} \tag{3.57}$$

for any $\zeta \in \mathbb{R}^d$. Injecting (3.57) in (3.56), one obtains

$$\|\mathcal{F}_x L_k(V)(\zeta, y_0, \cdot)\|_{L^2_t} \lesssim C(\|\nabla^{\otimes 3}\tilde{\theta}\|_{L^\infty}, \lambda_*)\|\nabla g\|_{W^{2,1}}(\|U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}+\|\nabla_\xi U_k(\zeta,\cdot,\cdot)\|_{L^\infty_\xi L^2_\omega}).$$

For $k = 4$, we note that

$$U_4(\zeta,\xi,\omega) = \mathcal{F}_{x,y,t}(w(y)V(x,y,t))(\zeta,-\xi,\omega),$$
$$\nabla_\xi U_4(\zeta,\xi,\omega) = \mathcal{F}_{x,y,t}(iyw(y)V(x,y,t))(\zeta,-\xi,\omega).$$

We deduce

$$\|U_4\|_{L^2_\zeta L^\infty_\xi L^2_\omega} \leq \|\mathcal{F}_{t,x} V(\zeta, y, \omega)\|_{L^2_\zeta L^1_{|w(y)|dy} L^2_\omega}$$

and thus by Minkowski and Parseval inequality

$$\|U_4\|_{L^2_\zeta L^\infty_\xi L^2_\omega} \leq \|V\|_{L^1_{|w(y)|dy} L^2_{t,x}}.$$

For similar reasons

$$\|\nabla_\xi U_4\|_{L^2_\zeta L^\infty_\xi L^2_\omega} \leq \|V\|_{L^1_{|yw(y)|dy} L^2_{t,x}}.$$

It remains to use that $\langle y\rangle w$ is a finite measure to get

$$\|\nabla_\xi U_4\|_{L^2_\zeta L^\infty_\xi L^2_\omega} + \|U_4\|_{L^2_\zeta L^\infty_\xi L^2_\omega} \lesssim_w \|V\|_{L^\infty_y L^2_{t,x}}.$$

For $k = 3$, we note that

$$U_3(\zeta,\xi,\omega) = \hat{w}(\zeta)\mathcal{F}_{x,t}(V(x,0,t))(\zeta,\omega), \quad \nabla_\xi U_3(\zeta,\xi,\omega) = 0.$$

We deduce

$$\|U_3\|_{L^2_\zeta L^\infty_\xi L^2_\omega} \leq \|\hat{w}\|_{L^\infty}\|V(y=0)\|_{L^2_{t,x}}$$

which suffices to conclude.

Step 5. *Preservation of continuity.* Take $y_0, y \in \mathbb{R}^d$. We have

$$\mathcal{F}_{t,x} L_{k,\delta}(V)(\zeta, y_0 + y, \omega) - \mathcal{F}_{t,x} L_{k,\delta}(V)(\zeta, y_0, \omega) = \int \Big(e^{-i\xi^\perp y - i\sigma_0 \frac{\zeta}{|\zeta|} y} - 1\Big) e^{-i\xi^\perp y_0 - i\sigma_0 \frac{\zeta}{|\zeta|} y_0} \tilde{U}_{1,k}(\zeta, \xi^\perp + \sigma_0 \frac{\zeta}{|\zeta|}, \omega) d\xi^\perp.$$

We deduce that

$$\|L_{k,\delta}(V)(\cdot, y_0 + y, \cdot) - L_{k,\delta}(V)(\cdot, y_0, \cdot)\|_{L^2_{t,x}} \lesssim \|(e^{i\xi y} - 1)\tilde{U}_{1,k}\|_{L^2_{\zeta,\omega}, L^1_\xi} + \|(e^{i\xi y} - 1)\nabla_\xi \tilde{U}_{1,k}\|_{L^2_{\zeta,\omega}, L^1_\xi} + |y| \|\tilde{U}_{1,k}\|_{L^2_{\zeta,\omega}, L^1_\xi}.$$

Since $U_{1,k}$ and $\nabla_\xi U_{1,k}$ belong to $L^2_{\zeta,\omega}, L^1_\xi$ we get by the dominated convergence Theorem that

$$\|L_{k,\delta}(V)(\cdot, y_0 + y, \cdot) - L_{k,\delta}(V)(\cdot, y_0, \cdot)\|_{L^2_{t,x}} \to 0$$

as y goes to 0.

A similar reasoning yields the continuity of $L_{k,p.v.}$ in y. □

Proposition 3.3.16. *The linear operators L_3 and L_4 are continuous from E_V to E_V. Moreover, under the smallness assumption*

$$\|\langle y \rangle w\|_{L^1} \|\nabla g\|_{W^{2,1}} \leq C(\|\nabla^{\otimes 3} \tilde{\theta}\|_{L^\infty}, \lambda_*),$$

the linear operator $1 - L_3 - L_4$ is invertible on E_V.

Proof. Because L_3 and L_4 are bounded Fourier multipliers in x they commute with the laplacian and thus L_3 and L_4 are continuous from $\mathcal{C}_y L^2_t \dot{H}^{-1/2}_x \cap L^\infty_y L^2_t \dot{H}^{-1/2}_x$ to itself and from $\mathcal{C}_y L^2_t \dot{H}^{s_c}_x \cap L^\infty_y L^2_t \dot{H}^{s_c}_x$ to itself. We deduce that L_3 and L_4 are continuous from $\mathcal{C}_y L^2_t B_2^{-1/2,s_x} \cap L^\infty_y L^2_t B_2^{-1/2,s_c}$ to itself. Besides their operator norms go to 0 as the $W^{2,1}$ norm of ∇g goes to 0 which ensures the invertibility of $1 - L_3 - L_4$ on $\mathcal{C}_y L^2_t B_2^{-1/2,s_x} \cap L^\infty_y L^2_t B_2^{-1/2,s_c}$ as long as g is small enough. From this, we also deduce that L_3 and L_4 are continuous from E_V to $L^\infty_y L^2_{t,x}$.

From Corollaries 3.3.11 3.3.12, we know that L_1 and L_2 are continuous from $\mathcal{C}_y L^2_t B_2^{-1/2,s_x} \cap L^\infty_y L^2_t B_2^{-1/2,s_c}$ to E_Z, we get in particular that they are continuous from $\mathcal{C}_y L^2_t B_2^{-1/2,s_x} \cap L^\infty_y L^2_t B_2^{-1/2,s_c}$ to $L^{d+2}_{t,x} L^2_\omega$. Because for $k = 1, 2$,

$$|L_{k+2}(V)(x,y,t)| = |\mathbb{E}(\bar{Y}(x+y,t) L_1(V)(x,t)) + \mathbb{E}(\overline{L_1(V)}(x+y,t) Y(x,t))| \leq \Big(\int g\Big)(\|L_k(V)(x,t)\|_{L^2_\omega} + \|L_k(V)(x+y,t)\|_{L^2_\omega})$$

we deduce that

$$\|L_{k+2}(V)\|_{L^{d+2}_{t,x}} \lesssim_g \|L_k(V)\|_{E_Z} \lesssim_g \|V\|_{L^\infty_y L^2_t B_2^{-1/2,s_c}}.$$

Because $\frac{d+2}{2} \in [2, d+2]$, we deduce that for $k = 1, 2$,

$$\|L_{k+2}(V)\|_{L^{(d+2)/2}_{t,x}} \lesssim_g \|V\|_{L^\infty_y L^2_t B_2^{-1/2, s_c}}.$$

This ensures that L_3 and L_4 are continuous from $\mathcal{C}_y L^2_t B_2^{-1/2, s_x} \cap L^\infty_y L^2_t B_2^{-1/2, s_c}$ to E_V. Finally, noticing that

$$(1 - L_3 - L_4)^{-1} = 1 + (L_3 + L_4)(1 - L_3 - L_4)^{-1}$$

We get that the restriction of $(1 - L_3 - L_4)^{-1}$ to E_V is continuous from E_V to E_V which makes $1 - L_3 - L_4$ invertible on E_V. □

3.4 Bilinear estimates

3.4.1 Quadratic terms in the perturbation

Proposition 3.4.1. *There exists C such that for all $Z, V \in E_Z \times E_V$, and $k = 1, 2$, we have*

$$\|Q_k(Z, V)\|_Z \le C \|\langle y \rangle w\|_M \|Z\|_Z \|V\|_V.$$

Proof. Set

$$\tilde V_1 = w * V(y = 0) Z, \quad \tilde V_2(x) = -\int dz w(z) V(x, z) Z(x + z).$$

We have

$$Q_k(Z, V) = -i \int_0^t S(t - \tau) \tilde V_k(\tau) d\tau.$$

By Strichartz estimates (3.29), we have

$$\|Q_k(Z, V)\|_Z \lesssim \|\tilde V_k\|_{L^{p'}(\mathbb{R}, W^{s_c, p'}(\mathbb{R}^d, L^2(\Omega)))}.$$

We use the generalized Leibniz rule with

$$\frac{1}{p'} = \frac{1}{p} + \frac{2}{d+2}, \quad \frac{1}{p'} = \frac{1}{d+2} + \frac{1}{2}$$

and Hölder inequality in time to get

$$\begin{aligned}&\|\tilde V_1\|_{L^{p'}(\mathbb{R}, W^{s_c, p'}(\mathbb{R}^d, L^2(\Omega)))} \\ &\lesssim \|w * V(y = 0)\|_{L^{(d+2)/2}(\mathbb{R} \times \mathbb{R}^d) \cap L^2(\mathbb{R}, H^{s_c}(\mathbb{R}^d))} \|Z\|_{L^p(\mathbb{R}, W^{s_c, p}(\mathbb{R}^d, L^2(\Omega))) \cap L^{d+2}(\mathbb{R} \times \mathbb{R}^d, L^2(\Omega))}.\end{aligned}$$

Using that w is a finite measure, we get

$$\|\tilde V_1\|_{L^{p'}(\mathbb{R}, W^{s_c, p'}(\mathbb{R}^d, L^2(\Omega)))} \lesssim \|V\|_V \|Z\|_Z.$$

For $\tilde{V}_2$, we use the Minkowski's inequality to get

$$\|\tilde{V}_2\|_{L^{p'}(\mathbb{R},W^{s_c,p'}(\mathbb{R}^d,L^2(\Omega)))} \leq \int |w(z)| \|V(z)T_z Z\|_{L^{p'}(\mathbb{R},W^{s_c,p'}(\mathbb{R}^d,L^2(\Omega)))}$$

We use again the generalised Leibniz rule to get

$$\|\tilde{V}_2\|_{L^{p'}(\mathbb{R},W^{s_c,p'}(\mathbb{R}^d,L^2(\Omega)))} \lesssim \int |w(z)| \|V(z)\|_{L^{(d+2)/2}(\mathbb{R}\times\mathbb{R}^d)\cap L^2(\mathbb{R},H^{s_c}(\mathbb{R}^d))} \|T_z Z\|_Z$$

The Z norm being invariant under the action of translations, and because of the definition of V, we get

$$\|\tilde{V}_2\|_{L^{p'}(\mathbb{R},W^{s_c,p'}(\mathbb{R}^d,L^2(\Omega)))} \lesssim \int |w(z)| \langle z\rangle \|V\|_V \|Z\|_Z.$$

We use that $\langle z\rangle w$ is a finite measure to conclude. □

3.4.2 Quadratic terms in the correlation function

Proposition 3.4.2. *The bilinear map*

$$\begin{array}{lll} E_Z \times E_Z & \mapsto & E_V \\ (Z, Z') & \mapsto & ((x,y) \mapsto \mathbb{E}(Z(x+y)Z'(x)) \end{array}$$

is well-defined and continuous.

Proof. Estimate in $C_y L_{t,x}^{(d+2)/2}$. We write T_y the translation such that $T_y u(x) = u(x+y)$. For a given $y \in \mathbb{R}^d$, we have

$$\|\mathbb{E}(T_y Z Z')\|_{L^{(d+2)/2}(\mathbb{R}\times\mathbb{R}^d)} \leq \|T_y Z\|_{L^{d+2}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))} \|Z'\|_{L^{d+2}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))}.$$

We recall that Lebesgue norms are invariant under the action of translation, and that for any $u \in L_{t,x}^{d+2} L_\omega^2$ the curve $y \mapsto T_y u$ is continuous in $L_{t,x}^{d+2} L_\omega^2$. Thus, the above estimate implies

$$\|\mathbb{E}(T_y Z Z')\|_{L^{(d+2)/2}(\mathbb{R}\times\mathbb{R}^d)} \leq \|Z\|_Z \|Z'\|_Z$$

as well as $\mathbb{E}(T_y Z Z') \in C_y L_{t,x}^{(d+2)/2}$. We deduce that

$$\|\mathbb{E}(T_y Z Z')\|_{L^\infty(\mathbb{R}^d,L^{(d+2)/2}(\mathbb{R}^d\times\mathbb{R}))} \leq \|Z\|_Z \|Z'\|_Z.$$

Estimate in $C_y L_t^2 H_x^{s_c}$. We use the generalised Leibniz rule, with

$$\frac{1}{2} = \frac{d}{2(d+2)} + \frac{1}{d+2} = \frac{1}{p} + \frac{1}{d+2}$$

to get

$$\begin{aligned} \|\mathbb{E}(T_y Z Z')\|_{L^2(\mathbb{R},H^{s_c}(\mathbb{R}^d))} \leq \big(&\|T_y Z\|_{L^p(\mathbb{R},W^{s_c,p}(\mathbb{R}^d,L^2(\Omega)))} + \|T_y Z\|_{L^{d+2}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))}\big) \\ &\times\big(\|Z'\|_{L^p(\mathbb{R},W^{s_c,p}(\mathbb{R}^d,L^2(\Omega)))} + \|Z'\|_{L^{d+2}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))}\big). \end{aligned}$$

Again, we recall the invariance of Sobolev norms under translations, and that for any $u \in L_t^p W_x^{s_c,p} L_\omega^2$ the curve $y \mapsto T_y u$ is continuous in $L_t^p W_x^{s_c,p} L_\omega^2$. This shows

$$\sup_{y \in \mathbb{R}^d} \|\mathbb{E}(T_y Z Z')\|_{L^2(\mathbb{R}, H^{s_c}(\mathbb{R}^d))} \leq \|Z\|_Z \|Z'\|_Z$$

as well as $\mathbb{E}(T_y Z Z') \in C_y L_t^2 H_x^{s_c} L_\omega^2$.
Estimate in $C_y L_t^2 B_x^{-1/2,s_c}$. For this part of the V norm, we use that

$$\| \cdot \|_{L^2(\mathbb{R}, B_2^{-1/2,s_c}(\mathbb{R}^d))} \leq \| \cdot \|_{L^2(\mathbb{R}, \dot{H}^{-1/2}(\mathbb{R}^d))} + \| \cdot \|_{L^2(\mathbb{R}, H^{s_c}(\mathbb{R}^d))}.$$

The second term in the right-hand side has already been estimated. For the

$$L^2(\mathbb{R}, \dot{H}^{-1/2}(\mathbb{R}^d))$$

norm, we use homogeneous Sobolev estimates to get

$$\begin{aligned}\|\mathbb{E}(T_y Z Z')\|_{L^2(\mathbb{R}, \dot{H}^{-1/2}(\mathbb{R}^d))} &\lesssim \|\mathbb{E}(T_y Z Z')\|_{L^2(\mathbb{R}, L^{q/2}(\mathbb{R}^d))} \\ &\lesssim \|T_y Z\|_{L^4(\mathbb{R}, L^q(\mathbb{R}^d, L^2(\Omega)))} \|Z'\|_{L^4(\mathbb{R}, L^q(\mathbb{R}^d, L^2(\Omega)))}.\end{aligned}$$

Again, we recall the invariance of Lebesgue norms under the action of translations, and that for any $u \in L_t^4 L_x^q L_\omega^2$ the curve $y \mapsto T_y u$ is continuous in $L_t^4 L_x^q L_\omega^2$. This implies

$$\sup_{y \in \mathbb{R}^d} \|\mathbb{E}(T_y Z Z')\|_{L^2(\mathbb{R}, B_2^{-1/2,0}(\mathbb{R}^d))} \lesssim \|Z\|_Z \|Z'\|_Z.$$

as well as $\mathbb{E}(T_y Z Z') \in C_y L_t^2 B_2^{-1/2,0} L_\omega^2$.

□

Proposition 3.4.3. *There exists C such that for all $Z, V \in E_Z \times E_V$ and $k = 3, 4$, we have*

$$\|Q_k(Z, V)\|_V \leq C \|\langle y \rangle w\|_M \|Z\|_Z \|V\|_V.$$

Proof. As in the proof of Proposition 3.4.1, we set

$$\tilde{V}_1 = w * V(y = 0) Z, \quad \tilde{V}_2(x) = -\int dz w(z) V(x, z) Z(x + z).$$

We recall that Q_3 and Q_4 are defined as for $k = 3, 4$

$$Q_k(Z, V)(y) = \mathbb{E}(T_y \bar{Y} Q_{k-2}) + \mathbb{E}(T_y \bar{Q}_{k-2} Y).$$

Estimate in $\mathcal{C}_y L_{t,x}^{\frac{d+2}{2}}$. Using that $Q_{k-2}(Z, V)$ is continuous from $E_Z \times E_V$ to $L^{d+2}(\mathbb{R} \times \mathbb{R}^d)$, by Proposition 3.4.1, and since $Y \in L^\infty(\mathbb{R} \times \mathbb{R}^d, L^2(\Omega))$ and Lebesgue norms are invariant under the action of translations, we get that

$$\|Q_k(Z, V)(y)\|_{L^{d+2}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|Z\|_Z \|V\|_V$$

Moreover, again by Proposition 3.4.1, the same inequality holds with the $L^{\frac{2(d+2)}{d}}$ norm. Then, by interpolation we get $Q_k(Z,V) \in L^\infty_y, L^{(d+2)/2}_{t,x}$.
To get the continuity with respect to y we use that for any $u \in L^{(d+2)/2}_{t,x}$ the curve $y \mapsto T_y u$ is continuous in $L^{(d+2)/2}_{t,x}$.
Estimate in $C_y L^2_t B^{-1/2,s_c}_x$. We prove that

$$\|Q_k(Z,V)(y)\|_{L^2(\mathbb{R},B_2^{-1/2,s_c}(\mathbb{R}^d))} \lesssim \|Z\|_Z \|V\|_V.$$

We remark that because of the invariance of Besov norms under conjugation and translations, we have

$$\|\mathbb{E}(T_y\bar{Q}_{k-2}Y)\|_{L^2(\mathbb{R},B_2^{-1/2,s_c}(\mathbb{R}^d))} = \|\mathbb{E}(Q_{k-2}T_{-y}\bar{Y})\|_{L^2(\mathbb{R},B_2^{-1/2,s_c}(\mathbb{R}^d))}.$$

Therefore, we bound only uniformly in y,

$$\|\mathbb{E}(T_y\bar{Q}_{k-2}Y)\|_{L^2(\mathbb{R},B_2^{-1/2,s_c}(\mathbb{R}^d))}.$$

We proceed by duality and treat separately low and high frequencies. Take

$$U \in L^2(\mathbb{R}, B_2^{1/2}(\mathbb{R}^d))$$

such that U is localised in low frequencies. We have

$$\langle U, \mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle = \mathbb{E}(\langle UY, T_y Q_{k-2}\rangle).$$

Using the definition of Q_{k-2} and because the linear flow S commutes with translations we get

$$\langle U, \mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle = \mathbb{E}(\langle UY, -i\int_0^t S(t-\tau)T_y\tilde{V}_{k-2}(\tau)\rangle) = \mathbb{E}(\langle \int_\tau^\infty U(t)Y(t)dt, T_y\tilde{V}_{k-2}\rangle).$$

Set $p_1 = 2\frac{d+2}{d-2}$, we have

$$|\langle U, \mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle| \leq \|\int_\tau^\infty U(t)Y(t)dt\|_{L^{p_1}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))}\|T_y\tilde{V}_{k-2}(\tau)\|_{L^{p_1'}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))}.$$

We have that $\frac{2}{p_1} + \frac{d}{p_1} = \frac{d}{2} - 1$ and thus, by Proposition 3.3.9,

$$\|\int_\tau^\infty U(t)Y(t)dt\|_{L^{p_1}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))} \lesssim \|U\|_{L^2(\mathbb{R},B_2^{1/2,1}(\mathbb{R}^d))}$$

and given that U is localised in low frequencies, we have

$$\|\int_\tau^\infty U(t)Y(t)dt\|_{L^{p_1}(\mathbb{R}\times\mathbb{R}^d,L^2(\Omega))} \lesssim \|U\|_{L^2(\mathbb{R},B_2^{1/2,0}(\mathbb{R}^d))}.$$

We remark that $\frac{1}{p_1'} = \frac{1}{2} + \frac{2}{d+2}$, and thus by Hölder inequality, we get

$$\|\tilde{V}_1\|_{L^{p_1'}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))} \leq \|w * V(y=0)\|_{L^2(\mathbb{R}\times\mathbb{R}^d)}\|Z\|_{L^{(d+2)/2}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))}.$$

By definition of the V norm, we have

$$\|w * V(y=0)\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|V\|_V$$

and because in dimension higher than 4, $\frac{d+2}{2} \in [2\frac{d+2}{d}, d+2]$, we have

$$\|Z\|_{L^{(d+2)/2}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))} \leq \|Z\|_Z.$$

For the same reasons we have

$$\|\tilde{V}_2\|_{L^{p_1'}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))} \leq \int dz |w(z)| \|V(z)\|_{L^2(\mathbb{R}\times\mathbb{R}^d)}\|Z\|_Z$$

and we use the definition of the V norm and the fact that $\langle z\rangle w$ is a finite measure to conclude the analysis in low frequencies.
We turn to high frequencies, we take $U \in L^2(\mathbb{R}\times\mathbb{R}^d)$ localised in high frequencies.
We take $n \in \mathbb{N}$ and consider

$$\langle U, \nabla^{\otimes n}\mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle = \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}(\langle U, \nabla^{\otimes(n-j)}\bar{Y} \otimes \nabla^{\otimes j} T_y Q_{k-2}\rangle.$$

With the same computation as previously, we have

$$\langle U, \nabla^{\otimes n}\mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle = \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}(\langle \int_\tau^\infty S(\tau - t)U(t)\nabla^{\otimes(n-j)}Y(t)dt, \nabla^{\otimes j} T_y \tilde{V}_{k-2}\rangle.$$

By Hölder's inequality,

$$\begin{aligned}&|\langle U, \nabla^{\otimes n}\mathbb{E}(T_y Q_{k-2}\bar{Y})\rangle|\\ \leq& \sum_{j=0}^{n}\binom{n}{j} \|\int_\tau^\infty S(\tau-t)U(t)\nabla^{\otimes(n-j)}Y(t)dt\|_{L^p(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega)}\|\nabla^{\otimes j}T_y\tilde{V}_{k-2}\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))}.\end{aligned}$$

By Proposition 3.3.9, applied to $\nabla^{\otimes(n-j)}Y(t)$ we have

$$\|\int_\tau^\infty S(\tau - t)U(t)\nabla^{\otimes(n-j)}Y(t)dt\|_{L^p(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega)} \lesssim \|U\|_{L^2(\mathbb{R}, B_2^{-1/2,0}(\mathbb{R}^d))}.$$

We deduce that

$$\tilde{V}_{k-2} \mapsto \Pi\mathbb{E}(Y\int_0^t S(t-\tau)\tilde{V}_{k-2}(\tau)d\tau$$

where Π projects into high frequencies is continuous from $L^{p'}(\mathbb{R}, W^{n,p'}(\mathbb{R}^d, L^2(\Omega)))$ to $L_y^\infty, L^2(\mathbb{R}, H^n)$. By interpolation, we get that it is continuous from

$L^{p'}(\mathbb{R}, W^{s_c,p'}(\mathbb{R}^d, L^2(\Omega)))$ to $L^\infty_y, L^2(\mathbb{R}, H^{s_c})$.
We refer to the proof of Proposition 3.4.1 to get that

$$\|\tilde{V}_{k-2}\|_{L^{p'}(\mathbb{R}, W^{s_c,p'}(\mathbb{R}^d, L^2(\Omega)))} \lesssim \|Z\|_Z \|V\|_V.$$

Moreover, for any $u \in L^{p_1'}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))$ and for any $v \in L^{p'}(\mathbb{R}, W^{s_c,p'}(\mathbb{R}^d, L^2(\Omega)))$, the curves $y \mapsto T_y u$ and $y \mapsto T_y v$ are continuous respectively in $L^{p_1'}(\mathbb{R}\times\mathbb{R}^d, L^2(\Omega))$ and in $L^{p'}(\mathbb{R}, W^{s_c,p'}(\mathbb{R}^d, L^2(\Omega)))$. Thus we get the continuity with respect to y. □

3.5 Proof of the theorem

3.5.1 Free evolution of the initial data

Proposition 3.5.1. *Under the assumptions of Proposition 3.3.1, for any $Z_0 \in L^2_\omega H^{s_c}_x$ one has $S(t)Z_0 \in E_Z$ with*

$$\|S(t)Z_0\|_Z \lesssim \|Z_0\|_{L^2_\omega H^{s_c}_x}.$$

Proof. The proof is exactly the same as that of Proposition 4.2 in [47], since the group $S(t)$ enjoys the same Strichartz estimates as the usual Schrödinger group $e^{it\Delta}$ by Proposition 3.3.1.

□

Proposition 3.5.2. *Under the assumptions of Proposition 3.3.1, if $\langle\xi\rangle^{2\lfloor s_c\rfloor} g \in W^{2,1}$ and $\tilde{\theta} \in W^{4,1}$, then for any $Z_0 \in L^2_\omega H^{s_c}_x$ one has*

$$\|\mathbb{E}[\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)]\|_{L^\infty_y L^{\frac{d+2}{2}}_{t,x}} \lesssim \|Z_0\|_{L^2_\omega H^{s_c}_x}$$

and moreover if $Z_0 \in L_x^{\frac{2d}{d+2}} L^2_\omega$ then $\mathbb{E}[\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)] \in C_y L^2_t B_2^{-\frac{1}{2},s_c}$, with:

$$\|\mathbb{E}[\overline{Y(x)}S(t)T_y Z_0(x)]\|_{L^\infty_y L^2_t B_2^{-1/2,s_c}} \lesssim \|Z_0\|_{L_x^{2d/(d+2)} L^2_\omega} + \|Z_0\|_{H^{s_c}_x L^2_\omega}.$$

Proof. We recall that T_y is the translation $T_y u(x) = u(x+y)$. Noticing that the second term equals the first one up to complex conjugation and changing variables $(x,y) \mapsto (x+y, -y)$, it suffices to bound

$$\mathbb{E}[\overline{Y(x)}S(t)T_y Z_0(x)].$$

<u>The $C_y L^{(d+2)/2}_{t,x}$ estimate.</u> By Proposition 3.5.1, we have

$$\|S(t)Z_0\|_{L^p_{t,x}L^2_\omega} + \|S(t)Z_0\|_{L^{d+2}_{t,x}L^2_\omega} \lesssim \|Z_0\|_{L^2_\omega H^{s_c}_x}.$$

As $p \le \frac{d+2}{2} \le d+2$ this implies

$$\|S(t)Z_0\|_{L^{\frac{d+2}{2}}_{t,x} L^2_\omega} \lesssim \|Z_0\|_{L^2_\omega H^{s_c}_x}$$

by interpolation. From the formula (3.14) giving the correlation function of Y, we deduce $Y \in L^\infty_{t,x} L^2_\omega$. These estimates imply, via the Cauchy-Schwarz and Hölder inequalities,

$$\|\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)]\|_{L_{t,x}^{\frac{d+2}{2}}} \lesssim \|T_y Z_0\|_{L^2_\omega H^{s_c}}.$$

We recall that Lebesgue norms are invariant under translations, and that for any $u \in L^2_\omega H^{s_c}$, the curve $y \mapsto T_y u$ is continuous in $L^2_\omega H^{s_c}_x$. Hence $\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)] \in C_y L_{t,x}^{(d+2)/2}$ with

$$\|\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)]\|_{L^\infty_y L_{t,x}^{\frac{d+2}{2}}} \lesssim \|T_y Z_0\|_{L^2_\omega H^{s_c}}.$$

The $C_y L^2_t \dot H_x^{-1/2}$ estimate. We reason by duality. For $U \in L^2_t \dot H^{1/2}_x$ with $\|U\|_{L^2_t \dot H^{1/2}_x} = 1$ we will prove

$$\int_0^\infty \mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)] \bar U dx dt \lesssim \|T_y Z_0\|_{L_x^{2d/(d+2)} L^2_\omega} \tag{3.58}$$

what will imply

$$\|\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)]\|_{L^2_t \dot H_x^{-1/2}} \lesssim \|T_y Z_0\|_{L_x^{2d/(d+2)} L^2_\omega}.$$

The above estimate implies $\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)] \in C_y L^2_t \dot H_x^{-1/2}$ with

$$\|\mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)]\|_{L^\infty_y L^2_t \dot H_x^{-1/2}} \lesssim \|Z_0\|_{L_x^{2d/(d+2)} L^2_\omega}. \tag{3.59}$$

We are thus left to proving (3.58). Using Fubini and $S(t)^* = S(-t)$ we have

$$\int_0^\infty \mathbb{E}[\overline{Y(x)} S(t) T_y Z_0(x)] \bar U dx dt = \int \mathbb{E}\left[T_y Z_0 \int_0^\infty \overline{S(-t) Y U} dt\right] dx.$$

By Proposition 3.3.9 with $\sigma_1 = 1$, $\sigma = 0$, $p_1 = \infty$ and $q_1 = \frac{2d}{d-2}$ we have $\int_0^\infty S(-t) Y U dt \in L_x^{2d/(d-2)} L^2_\omega$ with $\| \int_0^\infty S(-t) Y U\|_{L_x^{2d/(d-2)} L^2_\omega} \lesssim 1$. This estimate implies (3.58) by Hölder's inequality.

The $\mathcal{C}_y L^2_t H^{s_c}_x$ estimate. We notice that by the Leibniz rule, in order to estimate

$$\nabla_x^{\lfloor s_c \rfloor} (Y S(t) T_y Z_0),$$

it is sufficient to estimate $\partial^\alpha Y S(t) T_y \partial^\beta Z_0$ for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ with $\sum_i \alpha_i + \sum_i \beta_i = \lfloor s_c \rfloor$. We pick such α and β, and, for again a duality argument, take $U \in H_x^{\lfloor s_c \rfloor - s_c} L^2_\omega$ with $\|U\|_{H_x^{\lfloor s_c \rfloor - s_c} L^2_\omega} = 1$. We assume U is supported away from the origin in frequencies, i.e. $\hat U(\xi) = 0$ for $|\xi| \leq 1$. We will show

$$\int_0^\infty \mathbb{E}[\overline{\partial^\alpha Y(x)} S(t) T_y \partial^\beta Z_0(x)] \bar U dx dt \lesssim \|T_y Z_0\|_{H_x^{s_c} L^2_\omega}. \tag{3.60}$$

This estimate, via duality, and combined with (3.59) for the low frequencies, will show

$$\|\mathbb{E}[\overline{Y(x)}S(t)T_yZ_0(x)]\|_{L_t^2H_x^{s_c}} \lesssim \|T_yZ_0\|_{L_x^{2d/(d+2)}L_\omega^2} + \|T_yZ_0\|_{H_x^{s_c}L_\omega^2}.$$

The above estimate implies $\mathbb{E}[\overline{Y(x)}S(t)T_yZ_0(x)] \in C_yL_t^2H_x^{s_c}$ with

$$\|\mathbb{E}[\overline{Y(x)}S(t)T_yZ_0(x)]\|_{L_y^\infty L_t^2H_x^{s_c}} \lesssim \|Z_0\|_{L_x^{2d/(d+2)}L_\omega^2} + \|T_yZ_0\|_{H_x^{s_c}L_\omega^2}. \tag{3.61}$$

It remains to show (3.60). We have

$$\int_0^\infty \mathbb{E}[\overline{\partial^\alpha Y(x)}S(t)T_y\partial^\beta Z_0(x)]\bar{U}dxdt = \int \mathbb{E}\left[T_y\partial^\beta Z_0\int_0^\infty \overline{S(-t)\partial^\alpha YU}dt\right]dx. \tag{3.62}$$

Note that

$$\partial^\alpha Y(t,x) = \int f_\alpha(\xi)e^{i(\xi x-\theta(\xi)t)}dW(\xi)$$

where $f_\alpha(\xi) = (i\xi_1)^{\alpha_1}...(i\xi_d)^{\alpha_d}f(\xi)$. We apply Proposition 3.3.9 with momenta distribution function f_α, $\sigma_1 = 0$, $\sigma = 0$, $p_1 = \infty$ and $q_1 = 2$ and obtain

$$\|\int_0^\infty S(-t)\partial^\alpha YU\|_{L^2_{x,\omega}} \lesssim \|U\|_{L_t^2H_x^{-1/2}} \lesssim \|U\|_{L_t^2H_x^{\lfloor s_c\rfloor-s_c}} = 1 \tag{3.63}$$

where for the before last inequality we used $\lfloor s_c\rfloor - s_c \in \{-1/2, 0\}$ and that U is located away from the origin in frequencies. Using the Cauchy-Schwarz inequality and (3.63), the identity (3.62) implies (3.60) as desired.

The $C_yL_t^2B_2^{-1/2,s_c}$ estimate. Combining the previous $C_yL_t^2\dot{H}_x^{-1/2}$ and $C_yL_t^2H_x^{s_c}$ estimates (3.59) and (3.61), it follows that $\mathbb{E}[\overline{Y(x)}S(t)T_yZ_0(x)] \in C_yL_t^2B_2^{-1/2,s_c}$ with

$$\|\mathbb{E}[\overline{Y(x)}S(t)T_yZ_0(x)]\|_{L_y^\infty L_t^2B_2^{-1/2,s_c}} \lesssim \|Z_0\|_{L_x^{2d/(d+2)}L_\omega^2} + \|Z_0\|_{H_x^{s_c}L_\omega^2}.$$

3.5.2 Conclusion and proof of the main Theorem

In this subsection, we finish the proof of Theorem 3.1.1. The proof of the theorem relies on solving the fixed point problem (3.18). First we write the fixed point argument and then we prove the sattering result.

We set:

$$L = \begin{pmatrix} 0 & L_1 + L_2 \\ 0 & L_3 + L_4 \end{pmatrix}.$$

Using Corollary 3.3.11 and 3.3.12 we have that $L_1 + L_2$ is continuous from E_V to E_Z, and using Proposition 3.3.16, $1 - L_3 - L_4$ is continuous, invertible and of continuous inverse on E_V. Then $1-L$ is continuous, invertible and of continuous inverse on $E_Z \times E_V$. Thus the problem (3.18) can be rewritten as:

$$\begin{pmatrix} Z \\ V \end{pmatrix} = (1-L)^{-1}\mathcal{B}_{Z_0}\begin{pmatrix} Z \\ V \end{pmatrix} = (1-L)^{-1}\begin{pmatrix} \mathcal{B}^{(1)}_{Z_0}(Z,V)), \\ \mathcal{B}^{(2)}_{Z_0}(Z,V) \end{pmatrix} \tag{3.64}$$

where we have set

$$\begin{aligned} \mathcal{B}^{(1)}_{Z_0}(Z,V) &= S(t)Z_0 + Q_1(Z,V) + Q_2(Z,V), \\ \mathcal{B}^{(2)}_{Z_0}(Z,V) &= \mathbb{E}[\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)] \\ &\quad + Q_3(Z,V) + Q_4(Z,V) + \mathbb{E}[\overline{Z(x+y)}Z(x)]. \end{aligned}$$

We define the mapping:

$$\Phi[Z_0] : \begin{cases} E_Z \times E_V & \to \quad E_Z \times E_V \\ \begin{pmatrix} Z \\ V \end{pmatrix} & \mapsto \quad (1-L)^{-1}\mathcal{B}_{Z_0}\begin{pmatrix} Z \\ V \end{pmatrix} \end{cases} .$$

Let us denote $E_0 := L^2_\omega, H^{s_c}$ the space for the initial datum. We are going to show that for Z_0 small enough in E_0, the mapping $\Phi[Z_0]$ is a contraction on $B_{E_Z\times E_V}(0, R||Z_0||_{E_0}) =: B$ for some constant $R > 0$. For simplification we denote the norm $||\cdot||_{E_Z\times E_V}$ by $||\cdot||$.

By Corollary 3.3.11 and 3.3.12 and Proposition 3.3.16 we have:

$$\left|\left|\Phi[Z_0]\begin{pmatrix} Z \\ V \end{pmatrix}\right|\right| \lesssim \left|\left|B_{Z_0}\begin{pmatrix} Z \\ V \end{pmatrix}\right|\right|.$$

We decompose $\mathcal{B}_{Z_0}$ as:

$$\mathcal{B}_{Z_0} = C_{Z_0} + Q,$$

where:

$$C_{Z_0} = \begin{pmatrix} S(t)Z_0 \\ \mathbb{E}[\overline{Y(x+y)}S(t)Z_0(x) + \overline{S(t)Z_0(x+y)}Y(x)] \end{pmatrix}$$

is the constant part and:

$$Q(Z,V) = \begin{pmatrix} Q_1(Z,V) + Q_2(Z,V) \\ Q_3(Z,V) + Q_4(Z,V) \end{pmatrix},$$

the quadratic part.

By Propositions 3.5.1 and 3.5.2 we have:

$$||C_{Z_0}|| \lesssim ||Z_0||_{Z_0}.$$

By Propositions 3.4.3 and 3.4.1 we have that for any $(Z,V) \in B$:

$$||Q(Z,V)|| \lesssim ||Z||_{E_Z}||V||_{E_V} \leq R^2||Z_0||^2_{E_0}.$$

Moreover, by bilinearity of $(Z,V) \mapsto Q_k(Z,V)$ for $k = 1,2,3,4$ we get that for any $(Z,V),(Z',V') \in B$:

$$||Q(Z,V) - Q(Z',V')|| \lesssim ||(Z,V) - (Z',V')||(||(Z,V)|| + ||(Z',V')||)$$

thus:

$$||Q(Z,V) - Q(Z',V')|| \lesssim R||Z_0||_{Z_0}||(Z,V) - (Z',V')||.$$

From the above estimates, one gets that $\Phi[Z_0]$ is a contraction on $B_{E_Z \times E_V}(0, R||Z_0||_{E_0})$, for some universal constant $R > 0$, for $||Z_0||_{Z_0}$ small enough. By the Banach's fixed point theorem, we get the existence and uniqueness of a solution to (3.18) in $B_{E_Z \times E_V}(0, R||Z_0||_{E_0})$.

We can now prove the scattering result of the theorem. We write:

$$\begin{aligned} Z(t) = S(t)\Big(Z_0 - i\int_0^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Y + \int w(z)V(x,z)Y(x+z)\,dz\Big] \\ - i\int_0^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Z + \int w(z)V(x,z)Z(x+z)\,dz\Big] \\ + i\int_t^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Y + \int w(z)V(x,z)Y(x+z)\,dz\Big] \\ + i\int_t^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Z + \int w(z)V(x,z)Z(x+z)\,dz\Big]\Big). \end{aligned}$$

By Corollary 3.3.11 and 3.3.12 we get that $\int_0^\infty S(-\tau)\Big[\big(w*V(\cdot,0)\big)Y + \int w(z)V(x,z)Y(x+z)\,dz\Big] \in L^2_\omega, H^{s_c}$ and that:

$$\Big|\Big|\int_t^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Y + \int w(z)V(x,z)Y(x+z)\,dz\Big]\Big|\Big|_{L^2_\omega, H^{s_c}} \to 0,$$

as $t \to +\infty$.

Moreover, by Proposition 3.4.1 we get that $\int_0^\infty S(-\tau)\Big[\big(w*V(\cdot,0)\big)Z + \int w(z)V(x,z)Z(x+z)\,dz\Big] \in L^2_\omega, H^{s_c}$ and that:

$$\Big|\Big|\int_t^\infty S(-\tau)\Big[\big(w * V(\cdot,0)\big)Z + \int w(z)V(x,z)Z(x+z)\,dz\Big]\Big|\Big|_{L^2_\omega, H^{s_c}} \to 0,$$

as $t \to +\infty$.

Therefore, there exists $Z_\infty \in L^2_\omega, H^{s_c}$ such that, as $t \to \infty$:

$$Z(t) = S(t)Z_\infty + o_{L^2_\omega, H^{s_c}}(1).$$

Thus, by definition of Z, we get:

$$X(t) = Y + S(t)Z_\infty + o_{L^2_\omega, H^{s_c}}(1),$$

concluding the proof of Theorem 3.1.1.

□

Appendix A

Appendix

A.1 Lorentz covariance of the Dirac equation

In order to be consistent with the principle of relativity, physical theories must have the same form in all Lorentz frames, i.e. they must be covariant. In this appendix we show how the covariance of the Dirac equation on $\mathbb{R} \times \mathbb{R}^3$ can be derived. In particular, we describe how a spinors transform under Lorentz transformations. We refer the reader to [14] (Chapter 2). Moreover, we suggest to look at [92] (Notes 47) for more details.

Let O and O' be two observers who are in different inertial reference frames which describe physical events with space-time coordinates x^μ and $(x')^\mu$, respectively. Here $\mu \in \{0, \ldots, 3\}$ where $t = x^0$. Moreover, we use Λ to denote a proper Lorentz transformation, that is $\Lambda \in SO^+(1,3)$, such that

$$(x')^\mu = \Lambda^\mu_\nu x^\nu.$$

We recall that $SO^+(1,3)$ denotes group of *proper* Lorentz transformations. That is, the connected component of $O(1,3)$ given by $SO^+(1,3) = \{\Lambda \in O(1,3) \colon \det \Lambda = 1, \Lambda^0_0 = +1\}$. Here and in the following, we use the Einstein notation, therefore the sum over same indexes is implied. Let us now recall the Dirac equation and rewrite it in a more convenient way. After reintroducing the physical constants $\hbar, c$, from (15) and using the Einstein notation, the Dirac equation on $\mathbb{R} \times \mathbb{R}^3$ can be written as

$$i\hbar\partial_t\psi - i\hbar\alpha_j\partial_j\psi + mc\alpha_0\psi = 0, \tag{A.1}$$

where the α matrices satisfy $\{\alpha^i, \alpha^j\} = 2\delta^{ij}\mathbb{1}$, $\forall i, j = 1, 2, 3$. We now define γ matrices as

$$\gamma^0 := -\alpha_0, \qquad \gamma^j := \alpha_0\alpha_j, \quad j = 1, 2, 3.$$

Notice that these matrices satisfy the anticommuting relation $\{\gamma^\mu, \gamma^\nu\} = -2m^{\mu\nu}\mathbb{1}$, $\forall \mu, \nu = 0, \ldots, 3$ where m is the Minkowski metric, $m = \mathrm{diag}(-1, 1, 1, 1)$. By multiplying equation (A.1) by α_0 we get the equivalent formulation

$$i\hbar\gamma^\mu\partial_\mu\psi + mc\psi = 0.$$

In order to establish Lorentz covariance of the Dirac equation, we must satisfy two requirements:

i) there must be an explicit transformation which allows O' observer, given the $\psi(x)$ of observer O, to compute the $\psi'(x')$ which describes to O' the same physical state;

ii) according to the relativity principle, $\psi'(x')$ will be a solution of the Dirac equation, written in the primed system, that is

$$\Big(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\Big)\psi'(x') = 0.$$

Therefore, we suppose that there exists a 4×4 matrix $S(\Lambda)$ such that

$$\psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x'). \tag{A.2}$$

The main problem is to find S. If we demand that condition ii) holds and observing that $\frac{\partial}{\partial x^\mu} = \Lambda^\nu_\mu \frac{\partial}{\partial x'^\nu}$, we get

$$\Big(i\hbar S(\Lambda)\gamma^\nu \Lambda^\mu_\nu S(\Lambda)^{-1} \frac{\partial}{\partial x'^\mu} - mc\Big)\psi'(x') = \Big(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\Big)\psi'(x').$$

That is, we find the following condition for $S(\Lambda)$

$$S(\Lambda)\gamma^\nu \Lambda^\mu_\nu S^{-1}(\Lambda) = \gamma^\mu. \tag{A.3}$$

Let us notice, moreover, that $S\colon SO^+(1,3) \to GL_4(\mathbb{C})$ should satisfies:

$$S(\Lambda_1)S(\Lambda_2) = \pm S(\Lambda_1\Lambda_2). \tag{A.4}$$

This means that is we apply a Lorentz transformation Λ_2 to a wave function then a second one Λ_1, the effect is the same as applying the single Lorentz transformation $\Lambda_1\Lambda_2$. We include a $\pm$ sign in (A.4) because we know this sign is necessary in the case of ordinary rotations of spin-$\frac{1}{2}$ particles, and because rotations are special cases of Lorentz transformations. Since a representation is caracterised by its derivative at the identity, to compute S one focus first on the case of infinitesimal Lorentz transformations. Then, to recover the general case it suffices to recall that any proper Lorentz transformation can be built up a product of infinitesimal ones. The infinitesimal transformations can be written as

$$\Lambda = \mathbb{1} + \frac{1}{2}\theta_{\mu\nu}J^{\mu\nu},$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor specifying the infinitesimal Lorentz transformation and for any μ, ν, $J^{\mu\nu}$ is a 4×4 matrix such that

$$(J^{\mu\nu})^\alpha_\beta = m^{\mu\alpha}\delta^\nu_\beta - m^{\nu\alpha}\delta^\mu_\beta,$$

where m is the Minkowski metric defined as $m = \mathrm{diag}(-1, 1, 1, 1)$. Because $\theta_\mu\nu = -\theta_{\nu\mu}$ there are only 6 independent components of $\theta_{\mu\nu}$, which are obtained if we restrict the indexes to $\mu < \nu$. Therefore we can think of Λ as functions of these independent $\theta_{\mu\nu}$. Then, $S(\Lambda)$ must also be a function of $\theta_\mu\nu$. Expanding S out to first order we get

$$S(\Lambda) = \mathbb{1} + \sum_{\mu<\nu} \theta_{\mu\nu} \frac{\partial S}{\partial \theta_{\mu\nu}}(0).$$

We define matrices $\sigma^{\mu\nu}$ for $\mu < \nu$ by

$$\frac{\partial S}{\partial \theta_{\mu\nu}}(0) = -\frac{i}{2}\sigma^{\mu\nu}$$

and then define $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ for $\mu \geq \nu$. Since the derivatives are evaluated at $\theta_{\mu\nu} = 0$, the matrices $\sigma^{\mu\nu}$ are independent of $\theta_{\mu\nu}$. The factor $-\frac{i}{2}$ is conventional. To determine $\sigma^{\mu\nu}$ we exploit the condition (A.3), which becomes

$$\Big(\mathbb{1} + \frac{i}{4}\theta_{\alpha\beta}\sigma^{\alpha\beta}\Big)\gamma^\mu\Big(\mathbb{1} - \frac{i}{4}\theta_{\alpha\beta}\sigma^{\alpha\beta}\Big) = \Big[\mathbb{1} + \frac{1}{2}\theta_{\alpha\beta}J^{\alpha\beta}\Big]^\mu_{\ \nu}\gamma^\nu.$$

Multiplying things out, using the explicit formula for $J^{\alpha\beta}$ and keeping terms of first order in $\theta^{\alpha\beta}$ one obtains that

$$\frac{i}{4}[\sigma^{\alpha\beta}, \gamma^\mu] = \frac{1}{2}(m^{\mu\alpha}\gamma^\beta - m^{\mu\beta}\gamma^\alpha)$$

must be solved for $\sigma^{\alpha\beta}$. By making the guess that $\sigma^{\alpha\beta} = k[\gamma^\alpha, \gamma^\beta]$ and after some algebra, we find

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

Summing up, we have obtained the following expression for S of an infinitesimal Lorentz transformation

$$S(\Lambda) = \mathbb{1} + \frac{1}{8}\theta_{\mu\nu}[\gamma^\mu, \gamma^\nu].$$

We suggest the reader to look at [92] (Sections 47.8, 47.9) for the explicit formula for pure rotations and boosts.

A.2 Spin manifolds

The aim of this appendix is to give to the reader an intuition about the definition of spin manifolds, and why this is necessary in order to define the Dirac operator on non flat backgrounds. We refer to [69, 97] for more details. In the following, as for the works presented in the introduction, we consider time and space to be decoupled.

We recall that on $\mathbb{R}^n$, in order to define the Dirac operator as a square-root of the Laplacian, we had to find n matrices $\gamma_1, \ldots, \gamma_n$ satisfying

$$\{\gamma_i, \gamma_j\} := \gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}, \quad \forall i, j = 1, \ldots, n.$$

The algebra multiplicatively generated by these elements is called *Clifford algebra* $\mathcal{C}_n$ of the negative definite quadratic form $(\mathbb{R}^n, -x_1^2 - \cdots - x_n^2)$. Therefore, from an algebraic point of view, the definition of the Dirac operator led to the study of complex representations $\kappa\colon \mathcal{C}_n \to End(V)$ of the Clifford algebra. It is shown that $\mathcal{C}_n$ has a smallest representation of dimension $\dim_{\mathbb{C}} V = 2^{\lceil \frac{n}{2} \rceil}$. Let us denote by Δ_n the corresponding vector space. Moreover, we can define a product, called Clifford multiplication, between a vector $x \in \mathbb{R}^n$ and an element $\psi \in \Delta_n$ as

$$x \cdot \psi = \sum_{i=1}^{n} x^i \kappa(\gamma_i)(\psi).$$

We would like to find a non trivial representation ε of the group $SO(n)$ in the space Δ_n that is compatible with this multiplication, i.e. which satisfies

$$A(x) \cdot \varepsilon(A)(\psi) = \varepsilon(A)(x \cdot \psi), \qquad \forall A \in SO(n), x \in \mathbb{R}^n, \psi \in \Delta_n.$$

Observe that this is equivalent to the relation (A.3). However, such representation does not exist. To overcome this problem, it has been observed that the universal cover of $SO(n)$, which is the group denoted by $Spin(n)$, is compact, it covers the group twice and there exists a representation $\tilde{\varepsilon}\colon Spin(n) \to GL(\Delta_n)$ which is compatible with the Clifford multiplication. Recalling the tangent bundle of M has the rotation group $SO(n)$ as structural group, the idea was then to consider those Riemannian manifolds (M, g) for which it is possible to replace $SO(n)$ by $Spin(n)$ as structural group.

Let us be now more precise. We begin by recalling that if $E \xrightarrow{\pi} B$ is a rank n vector bundle, then it is determined by its transitions functions g_{ij} mapping into the general linear group $GL(n, \mathbb{R})$. More precisely, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of B and let $\phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ be a local trivializations. Then the transition maps are given on the open set $U_{ij} = U_i \cap U_j$ by

$$\begin{aligned} \phi_j \phi_i^{-1}\colon U_{ij} \times \mathbb{R}^n &\to U_{ij} \times \mathbb{R}^n \\ (x, v) &\to (x, g_{ij}(x)v) \end{aligned}$$

with $g_{ij}\colon U_{ij} \to GL(n, \mathbb{R})$. If E is orientable, we can *reduce* the structure group to $SO(n)$, that is we can take these transition functions to map into $SO(n)$. We can associate to this vector bundle a principal $SO(n)-$bundle over B, the bundle of frames of E which we denote by $P_{SO}(E)$. Each fiber of this bundle is the set of orthonormal bases for a fiber of E.

Let (M, g) be an oriented Riemannian manifold of dimension n. We can consider the tangent vector bundle associated with M. That is, following the previous notation, we take B to be the manifold M and as topological space $E = TM := \coprod_{x \in M} T_x M$, where $T_x M$ is the tangent space in $x \in M$ of M. Let us denote by $P_{SO}TM \to M$ the $SO(n)-$principal bundle of positively oriented orthonormal frames on the tangent bundle of the manifold M.

Definition A.2.1 (Spin structure)**.** A *spin structure* on M is given by a $Spin(n)-$principal bundle $P_{Spin}TM \to M$ together with a $2-$fold covering map $P_{Spin}TM \overset{\eta}{\to} P_{SO}TM$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_{Spin}TM \times Spin(n) & \longrightarrow & P_{Spin}TM & & \\ \big\downarrow{\scriptstyle \eta\times\xi} & & \big\downarrow{\scriptstyle \eta} & \searrow & \\ & & & & M \\ & & & \nearrow & \\ P_{SO}TM \times SO(n) & \longrightarrow & P_{SO}TM & & \end{array}$$

where $\xi\colon Spin(n) \to SO(n)$ is a group homomorphism.

We will then call *spin manifold* an oriented Riemannian manifold which admits a spin structure. For this kind of manifolds we can define another bundle, called spinor bundle in the following way.

Definition A.2.2 (Spinor bundle)**.** Let (M, g) be a spin manifold. The spinor bundle of M is the vector bundle, denoted as ΣM, associated to the $Spin(n)$-principal bundle via the spinor representation

$$\Sigma M := P_{Spin}TM \times_{\tilde{\varepsilon}} \Delta_n = P_{Spin}TM \times \Delta_n/_{\sim}$$

where $(p, \sigma) \sim (p \cdot u, \tilde{\varepsilon}(u^{-1})(\sigma))$ for all $(p, \sigma) \in P_{Spin}TM \times \Delta_n$ and $u \in Spin(n)$.

Then we define the *spinors* as sections of ΣM.

Let us observe that if $x \in M$ and we look at a section ψ of ΣM this is locally given by a triple $\psi(x) = (x, A, \varphi)$, with $A \in Spin(n), \varphi \in \Delta_n$ which is equivalent to, that is in the same equivalent class of, $(x, \mathbb{1}, \tilde{\varepsilon}(A)\varphi)$, where we denote by $\mathbb{1}$ the identity matrix. In fact, in the definition above of the equivalence relation we can take $u = A^{-1}$. Therefore, locally we can identify a spinor with a function $\Psi\colon U_i \to \Delta_n$, where $\psi(x) = (x, \mathbb{1}, \Psi(x))$. To conclude, let us mentions that even if spinors cannot be introduced on every Riemannian space, they can be introduce for a large class. The existence of a spin structure can be translated into a topological condition on the manifold, that is the first two Stiefel-Whitney classes have to vanish. Among this class one finds for example the sphere $\mathbb{S}^n$, $n \geq 2$ which admits a unique spin structure. In general, any compact Riemann surface of genus g admits 2^{2g} non equivalent spin structures. However, non-spin manifolds do exist; the "simplest" example is the complex 2-dimensional projective space $\mathbb{C}P^2$.

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