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Some new regularity results for length minimizing curves in sub-Riemannian geometry

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Introduction

The regularity of geodesics (length-minimizing curves) in sub-Riemannian geometry is an open problem since many years. Its difficulty is due to the presence of the singular (or abnormal) extremals, i.e., curves where the differential of the end-point map is singular (it is not surjective). Normal length minimizing curves, corresponding to regular points of the end-point map, are instead C^∞ regular.

Roughly speaking, a sub-Riemannian manifold is a smooth manifold where the admissible directions in the tangent bundle are limited to a smooth distribution. We assign infinite length to other directions and then admissible curves are the ones having their velocity in the distribution.

In this thesis, we mainly consider classical sub-Riemannian manifolds, i.e triplets (M, Δ, g), where M is a smooth manifold, $\Delta \subset TM$ is a smooth distribution of rank $2 \leq d < \dim(M)$ (the distribution of admissible directions), and g is a smooth metric on Δ. In a neighborhood $U \subset M$ of any point $q \in M$, there exist vector-fields $f_1, \dots, f_d \in \mathrm{Vec}(U)$ such that $\Delta = \mathrm{span}\{f_1, \dots, f_d\}$ on U. Since our considerations are local, we can assume in the sequel that $U = M$. We also assume that Δ satisfies Hörmander's condition

$$\mathrm{Lie}\{f_1, \dots, f_d\}(p) = T_pM, \quad p \in M, \tag{0.1}$$

i.e., that Δ is completely non-integrable.

Let $I = [0, 1]$ be the unit interval. A curve $\gamma \in AC(I; M)$ is admissible or horizontal if $\dot\gamma \in \Delta_\gamma$ a.e. on I, that is

$$\dot\gamma(t) = \sum_{i=1}^{d} u_i(t) f_i(\gamma(t)), \quad \text{for a.e. } t \in I, \tag{0.2}$$

for some unique $u = (u_1, \dots, u_d) \in L^1(I; \mathbb{R}^d)$, called control of γ. Without loss of generality, we can assume that g makes $f_1, \dots, f_d$ orthonormal, in which case the length of γ is the L^1-norm of its control, i.e.

$$L(\gamma) = \int_I g(\dot\gamma(t), \dot\gamma(t))^{1/2}.$$

We can also replace the Banach space $L^1(I;\mathbb{R}^d)$ with the smaller Hilbert space $X = L^2(I;\mathbb{R}^d)$. Finally, condition (0.1) ensures that M is connected by admissible curves (Chow-Rashevskii Theorem).

The end-point map $E_q : X \to M$ with base-point $q \in M$ is defined letting $E_q(u) = \gamma_u(1)$, where γ_u is the unique solution to (0.2) with $\gamma_u(0) = q$. The point $\bar{q} = E_q(u)$ is the end-point of the curve γ_u. Since $q \in M$ is fixed, we shall simply write $E = E_q$. Controls $u \in X$ where the differential d_uE is not surjective are called singular and horizontal curves driven by singular controls are called singular or abnormal extremals. The first chapter of this thesis is devoted to a full and completely general description of this kind of arguments, summarizing the general theory known to date.

The existence of singular curves arose for the first time after the paper [24] by Strichartz was published, more than thirty years ago. In this paper it was claimed, due to an incomplete application of the Pontryagin Maximum Principle, that all minimizing curves are normal and, therefore, that they are always smooth, as in the case of Riemannian geometry where abnormal curves do not appear. Later, the author admitted that his paper contained an irreparable mistake: he forgot to treat abnormal singularities of the end-point map.

The first example of a singular curve that is as a matter of fact length-minimizing was discovered in [20] by Richard Montgomery. After that, many other examples were discovered, such as in [19]. All such examples are smooth curves, showing that the nature of abnormal length minimizers is very difficult: it is still an open problem if constant-speed geodesics, which a priori are only absolutely or Lipschitz continuous, are always smooth (i.e. C^∞-regular) in any sub-Riemannian manifold. If the assigned distribution has step at most 2, then all geodesics are smooth (and, in fact, the claim by Strichartz is true in this case). It is also an open question even if they are always C^1-regular. A positive answer to this question is given in [10], but only when the manifold is analytic of dimension 3.

The problem is open also in the model case of Carnot groups. This special class of sub-Riemannian manifolds consists of Lie groups whose Lie algebra is stratified, and it deserves a particular mention since Carnot groups provide an infinitesimal model for any sub-Riemannian manifold, near any given point (provided the point satisfies a technical condition, which holds generically). In the context of Carnot groups, the regularity problem has recently been solved also when the step is at most 3 (independently by Tan-Yang in [27] and by Le Donne-Leonardi-Monti-Vittone in [17]).

The aim of this thesis is to give some new partial results about this regularity problem. Necessary conditions for the minimality of singular extremals can be obtained from the differential study of the end-point map. The theory is well-known till the second order and was initiated by Goh [11] and developed by Agrachev and Sachkov in [3].

Instead, a recent approach to the regularity problem of length-minimizing curves is based on the analysis of specific singularities such as corners, spiral-like curves or curves with no straight tangent line. This approach does not use general open mapping theorems but it rather relies on the ad hoc construction of shorter competitors, see [4, 12, 18, 21, 22].

The differential analysis of the end-point to deduce necessary conditions for the minimality and the direct study of singularity points of abnormal curves are the main approaches in literature that have been pursued in an attempt to solve the problem. They are the guideline, respectively, of Chapters 2 and 3, which are described more in detail below.

Finally, in Chapter 4, we collect some, in our opinion, interesting open questions that could be investigated concerning the regularity problem of sub-Riemannian geodesics.

Chapter 2. Using second order open mapping theorems (index theory), for a strictly singular length minimizing curve γ and for some adjoint curve λ, in [1] the authors prove the validity of the following Goh conditions:

$$\langle \lambda, [f_i, f_j](\gamma) \rangle = 0, \qquad i, j = 1, \dots, d. \tag{0.3}$$

The first order conditions $\langle \lambda, f_i(\gamma) \rangle = 0$ are ensured by Pontryagin Maximum Principle. Partial necessary conditions of the third order are obtained in [7].

Our goal, in this chapter, is to extend the second order theory of [1] to any order $n \geq 3$ and to get necessary conditions as in (0.3) involving brackets of n vector fields. The results that we present in this chapter were proved in [8] thanks to the collaboration of Francesco Boarotto.

There is a clear connection between the geometry of Δ and the expansion of the end-point map F. In particular, the commutators of length n should appear in the n-th order term of the expansion of F. In Section 2.4, we provide a first positive answer to this idea. In the case of a singular curve γ_u of corank-one, i.e., such that $\mathrm{Im}(d_u F)$ has codimension 1 in $T_{\gamma_u(1)}M$, we get the following result. For the definition of adjoint curve, see Section 2.8.

Theorem 0.1. *Let (M, Δ, g) be a sub-Riemannian manifold, $\gamma = \gamma_u \in AC(I; M)$ be a strictly singular length minimizing curve of corank 1, and assume that*

$$\mathcal{D}_u^h F = 0, \quad h = 2, \dots, n-1. \tag{0.4}$$

Then any adjoint curve $\lambda \in AC(I;T^*M)$ satisfies

$$\langle \lambda(t), [f_{j_n}, [\dots [f_{j_2}, f_{j_1}] \dots]](\gamma(t))\rangle = 0, \tag{0.5}$$

for all $t \in I$ and for all $j_1, \dots, j_n = 1, \dots, d$.

We call the differentials $\mathcal{D}^h_u F$ appearing in (0.4) *intrinsic differentials* of F. For $v_1, \dots, v_h \in X$, at the point $u = 0$ we define

$$D^h_0 F(v_1, \dots, v_h) = \frac{d^h}{dt^h} F\Big(\sum_{h=1}^{n} \frac{t^h v_h}{h!}\Big)\Big|_{t=0}.$$

We first restrict $D^h_0 F$ to a suitable domain $\mathrm{dom}(\mathcal{D}^h_0 F) \subset X^{h-1}$ that, roughly speaking, consists of points where the lower order differentials $d_0 F, D^2_0 F, \dots, D^{h-1}_0 F$ vanish. Then we define $\mathcal{D}^h_0 F = \mathrm{proj}(D^h_0 F)$, with proj projection onto $\mathrm{coker}(d_0 F)$, see Definition 2.6. A motivation for this definition is the fact that $\mathcal{D}^h_0 F$ behaves covariantly, in the sense that, for a given diffeomorphism $P \in C^\infty(M;M)$, $\mathcal{D}^h_0(P \circ F)$ depends only on the first order derivatives of P.

The proof of Theorem 0.1 relies on an open mapping argument applied to the extended end-point map $F_J = (F, J) : X \to M \times \mathbb{R}$, where $J(u) = \frac{1}{2}\|u\|^2_{L^2(I;\mathbb{R}^d)}$ is the energy of $\gamma = \gamma_u$. Minimizing the energy is in fact equivalent to minimizing the length, because for horizontal curves parameterized by arc-length the L^2-norm of the control coincides with its L^1-norm.

Motivated by this application, in Section 2.2 we develop a theory about open mapping theorems of order n for functions $F : X \to \mathbb{R}^m$ between Banach spaces. In our opinion, this preliminary study is worth of interest on its own. It adapts in a geometrical perspective some ideas presented in [25].

Theorem 0.2. *Let X be a Banach space and let $F \in C^\infty(X;\mathbb{R}^m)$, $m \in \mathbb{N}$, be a smooth mapping such that $\mathcal{D}^h_0 F = 0$ for all $2 \le h < n$, for some $n \ge 2$, and with regular $\mathcal{D}^n_0 F$ at the critical point $0 \in X$. Then F is open at 0.*

We refer to Section 2.2 for precise definitions. The notion of "regularity" used in Theorem 0.2 is delicate because the intrinsic differential $\mathcal{D}^n_0 F : \mathrm{dom}(\mathcal{D}^n_0 F) \to \mathrm{coker}(d_0 F)$ is a non-linear mapping defined on a domain without linear structure. The notion depends on an extension theorem for $\mathrm{dom}(\mathcal{D}^n_0 F)$ and is fixed in Definition 2.13. When $0 \in X$ is a critical point of corank 1 it becomes more effective, see Corollary 2.16.

The rest of the chapter is devoted to the study of the open mapping property for the extended end-point map F_J around a singular control u. In fact, we will study the auxiliary map G, called *variation map*, defined by $G(v) = F(u + v)$, in order to move the base-point from u to 0.

A crucial ingredient in our analysis is the definition of non-linear sets $V_h \subset X$, $h \in \mathbb{N}$, consisting of controls with vanishing iterated integrals for any order $h \leq n-1$, see (2.23). Using such controls we are able to catch the geometric structure of the n-th differential $\mathcal{D}_0^n G$ in terms of Lie brackets. The algebraic properties of the sets V_h are studied in Section 2.3.

In Sections 2.4 and 2.5, we use the formalism of chronological calculus [3, Chapter 2] to compute the n-th differential $\mathcal{D}_0^n G$ of the variation map and the final outcome is formula (2.36). This formula contains a localization parameter $s > 0$ that can be used to shrink the support of the control in a neighborhood of some point $t_0 \in [0, 1)$. Passing to the limit as $s \to 0^+$, we obtain a new map $\mathcal{G}_{t_0}^n : X \to \mathbb{R}$:

$$\mathcal{G}_{t_0}^n(v) = \int_{\Sigma_n} \langle \lambda, [g_{v(t_n)}^{t_0}, [\dots, [g_{v(t_2)}^{t_0}, g_{v(t_1)}^{t_0}]] \dots](\bar{q}) \rangle dt_1 \dots dt_n,$$

where $\Sigma_n = \{0 \leq t_n \leq t_{n-1} \leq \dots \leq t_1 \leq 1\}$ is the standard simplex, $\lambda \in T_{\bar{q}}^* M$ is a fixed covector orthogonal to $\mathrm{coker}(d_u F)$, $\bar{q} = F(q) \in M$ is the end-point, and $g_{v(t_i)}^{t_0}$ is the pull-back of the time-dependent vector field $f_v = v^1 f_1 + \dots + v^d f_d$ along the flow of u. In the corank 1 case, we show that if there exists $v \in V_{n-1}$ with $\mathcal{G}_{t_0}^n(v) \neq 0$ then the extended map G_J is open at 0, see our crucial Theorems 2.28 and 2.29. So $\mathcal{G}_{t_0}^n = 0$ on V_{n-1} becomes a necessary condition for the length-minimality of singular extremals.

In Sections 2.6 and 2.7, we study the geometric implications of equation $\mathcal{G}_{t_0}^n = 0$. First, we explore the symmetries of $\mathcal{G}_{t_0}^n$, showing how the shuffle algebra of iterated integrals interacts with generalized Jacobi identities of order n, see Theorem 2.31. In spite of the non-linear structure of V_{n-1}, we are able to polarize the equation $\mathcal{G}_{t_0}^n = 0$ on linear subspaces of V_{n-1} of arbitrarily large dimension, thus de facto bypassing the non-linearity of the problem.

At this point, we regard the quantities in (0.5) as unknowns of a nonsingular system of linear equations, thus proving their vanishing. To get this nonsingularity, we work with families of trigonometric functions having sparse and high frequences, see Theorems 2.36 and 2.37.

Our argument leading to the final proof of Theorem 0.1 is summarized in Section 2.8. In Section 2.9 we complete the study of a well-known example of singular curve. In $M = \mathbb{R}^3$ we consider the distribution spanned by the vector-fields

$$f_1 = \frac{\partial}{\partial x_1} \quad \text{and} \quad f_2 = (1 - x_1) \frac{\partial}{\partial x_2} + x_1^n \frac{\partial}{\partial x_3},$$

where $n \in \mathbb{N}$ is a parameter. Using the theory developed in Sections 2.1-2.5, we show that for odd n the curve $\gamma(t) = (0, t, 0)$, $t \in [0, 1]$, is not length minimizing. This is interesting because, for even n, this singular curve is on the contrary locally length minimizing. In fact, for even n the n-th differential of the end-point map is not regular according to Definition 2.13 part i) and our open mapping theorem does not apply. See our discussion in Remark 2.47.

Concerning the assumptions made in Theorem 0.1, hypothesis (0.4) is restrictive. Its geometric meaning is explained in Remark 2.42: in fact, it implies (but is not equivalent to) $\langle \lambda, [f_{j_h}, [\dots [f_{j_2}, f_{j_1}] \dots]](\gamma) \rangle = 0$ for $h \leq n-1$. Weakening (0.4) requires a substantial improvement of Theorem 0.2. The corank 1 assumption on the length minimizing curve is used when Theorem 0.2 is applied to the end-point map. We think that it should be possible to drop this assumption, but this certainly requires some new deep idea.

Chapter 3. The most elementary kind of singularity for a Lipschitz curve is of the corner-type: at a given point, the curve has a left and a right tangent that are linearly independent. In [18] and [12] it was proved that length minimizers cannot have singular points of this kind. These results have been improved in [22]: at any point, the tangent cone to a length-minimizing curve contains at least one line (a half line, for extreme points). The uniqueness of this tangent line for length minimizers is an open problem. Indeed, there exist other types of singularities related to the non-uniqueness of the tangent. In particular, there exist spiral-like curves whose tangent cone at the center contains many and in fact all tangent lines, see Example 3.14 below. These curves may appear as Goh extremals in Carnot groups, see [16] and [17, Section 5]. For these reasons, the results of [22] are not enough to prove the nonminimality of spiral-like extremals. Goal of this chapter is to show that curves with this kind of singularity are not length-minimizing. The results that we present in this chapter were proved in [23].

Our notion of horizontal spiral in a sub-Riemannian manifold of rank 2 is fixed in Definition 0.3. We will show that spirals are not length-minimizing when the horizontal distribution $\mathcal{D}$ satisfies the following commutativity condition. Fix two vector fields $f_1, f_2 \in \mathcal{D}$ that are linearly independent at some point $p \in M$. For $k \in \mathbb{N}$ and for a multi-index $J = (j_1, \dots, j_k)$, with $j_i \in \{1, 2\}$, we denote by $f_J = [f_{j_1}, [\dots, [f_{j_{k-1}}, f_{j_k}] \cdots]]$ the iterated commutator associated with J. We define its length as the length of the multi-index J, i.e., $\mathrm{len}(f_J) = \mathrm{len}(J) = k$. Then, our commutativity assumption is that, in a neighborhood of the point p,

$$[f_I, f_J] = 0 \quad \text{for all multi-indices with } \mathrm{len}(I), \mathrm{len}(J) \geq 2. \tag{0.6}$$

This condition is not intrinsic and depends on the basis f_1, f_2 of the distribution $\mathcal{D}$.

After introducing exponential coordinates of the second type, the vector fields f_1, f_2 can be assumed to be of the form (3.5) below, and the point p will be the center of the spiral. In coordinates we have $\gamma = (\gamma_1, \ldots, \gamma_n)$ and, by (3.5), the γ_j's satisfy for $j = 3, \ldots, n$ the following integral identities

$$\gamma_j(t) = \gamma_j(0) + \int_0^t a_j(\gamma(s))\dot{\gamma}_2(s)ds, \quad t \in [0,1].$$

When $\gamma(0)$ and γ_1, γ_2 are given, these formulas determine in a unique way the whole horizontal curve γ. We call $\kappa \in AC([0,1];\mathbb{R}^2)$, $\kappa = (\gamma_1, \gamma_2)$, the horizontal coordinates of γ.

Definition 0.3 (Spiral)**.** We say that a horizontal curve $\gamma \in AC([0,1];M)$ is a *spiral* if, in exponential coordinates of the second type centered at $\gamma(0)$, the horizontal coordinates $\kappa \in AC([0,1];\mathbb{R}^2)$ are of the form

$$\kappa(t) = t\mathrm{e}^{i\varphi(t)}, \quad t \in]0,1],$$

where $\varphi \in C^1(]0,1];\mathbb{R}^+)$ is a function, called *phase* of the spiral, such that $|\varphi(t)| \to \infty$ and $|\dot{\varphi}(t)| \to \infty$ as $t \to 0^+$. The point $\gamma(0)$ is called center of the spiral.

A priori, Definition 0.3 depends on the basis f_1, f_2 of $\mathcal{D}$, see however our comments about its intrinsic nature in Remark 3.13. Without loss of generality, we shall focus our attention on spirals that are oriented clock-wise, i.e., with a phase satisfying $\varphi(t) \to \infty$ and $\dot{\varphi}(t) \to -\infty$ as $t \to 0^+$. Such a phase is decreasing near 0. Notice that if $\varphi(t) \to \infty$ and $\dot{\varphi}(t)$ has a limit as $t \to 0^+$ then this limit must be $-\infty$.

The main result in Chapter 3 is the following

Theorem 0.4. *Let $(M, \mathcal{D}, g)$ be an analytic sub-Riemannian manifold of rank 2 satisfying* (0.6)*. Any horizontal spiral $\gamma \in AC([0,1];M)$ is not length-minimizing near its center.*

Differently from [18, 12, 22] and similarly to [21], the proof of this theorem cannot be reduced to the case of Carnot groups, the infinitesimal models of equiregular sub-Riemannian manifolds. This is because the blow-up of the spiral could be a horizontal line, that is indeed length-minimizing.

The nonminimality of spirals combined with the necessary conditions given by Pontryagin Maximum Principle is likely to give new regularity results on classes of sub-Riemannian manifolds, in the spirit of [4]. We think, however, that the main interest of Theorem 0.4 is in the deeper understanding that it provides on the loss of minimality caused by singularities.

The proof of Theorem 0.4 consists in constructing a competing curve shorter than the spiral. The construction uses exponential coordinates of the second type and our first step is a review of Hermes' theorem on the structure of vector-fields in such coordinates. In this situation, the commutativity condition (0.6) has a clear meaning explained in Theorem 3.8, that may be of independent interest.

In Section 3.2, we start the construction of the competing curve. Here we use the specific structure of a spiral. The curve obtained by cutting one spire near the center is shorter. The error appearing at the end-point will be corrected modifying the spiral in a certain number of locations with "devices" depending on a set of parameters. The horizontal coordinates of the spiral are a planar curve intersecting the positive x_1-axis infinitely many times. The possibility of adding devices at such locations arbitrarily close to the origin will be a crucial fact.

In Section 3.3, we develop an integral calculus on monomials that is used to estimate the effect of cut and devices on the end-point of the modified spiral. Then, in Section 3.4, we fix the parameters of the devices in such a way that the end-point of the modified curve coincides with the end-point of the spiral. This is done in Theorem 3.23 by a linearization argument. Sections 3.2-3.4 contain the technical core and computations of Chapter 3.

We use the specific structure of the length-functional in Section 3.5, where we prove that the modified curve is shorter than the spiral, provided that the cut is sufficiently close to the origin. This will be the conclusion of the proof of Theorem 0.4.

We briefly comment on the assumptions made in Theorem 0.4. The analyticity of M and $\mathcal{D}$ is needed only in Section 3.1. In the analytic case, it is known that length-minimizers are smooth in an open and dense set, see [26]. See also [10] for a C^1-regularity result when M is an analytic manifold of dimension 3.

The assumption that the distribution $\mathcal{D}$ has rank 2 is natural when considering horizontal spirals. When the rank is higher there is room for more complicated singularities in the horizontal coordinates, raising challenging questions about the regularity problem.

Dropping the commutativity assumption (0.6) is a major technical problem: getting sharp estimates from below for the effect produced by cut and devices on the end-point seems extremely difficult when the coefficients of the horizontal vector fields depend also on nonhorizontal coordinates, see Remark 3.18.

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Chapter 1

The end-point map in sub-Riemannian geometry

In this chapter, we discuss the general notions concerning sub-Riemannian manifolds. Despite our definitions given here are quite general, permitting to include all the classical notions appearing in the literature, we introduce the reader to sub-Riemannian geometry with an example taken from the real life.

After having introduced the sub-Riemannian (or Carnot-Carathéodory) distance, we discuss its finiteness and continuity, also known as Chow-Rashevskii theorem. Then we move to metric properties of sub-Riemannian manifold as metric spaces, proving in particular existence of length-minimizers.

In the final part of the chapter we introduce the crentral object of this thesis, namely the end-point map. Its first order analysis leads to Pontryagin extremals, which are, in fact, curves in the cotangent space corresponding to the critical points of the end-point map.

The notions and the results presented in this chapter are classical facts of sub-Riemannian geometry. For this reason we will omit most of the proofs of the statements contained in this chapter. The reader can find all the details of these arguments in Chapters 2, 3, 6 and 8 of [2].

1.1 Prelimirary notions

In this section we recall some general facts of differential geometry we need in the sequel. Since a sub-Riemannian manifold is smooth (see definition 1.10), our definitions in this section are given in the smooth setting.

We point out that in this section we are not developing the standard theory of differential geometry, rather just fixing our notations. We assume the reader to be very confident with the topics covered in this section.

From now on, if it is not specified, M denotes a smooth and connected manifold of dimension $n \in \mathbb{N}$, and $C^\infty(M)$ denotes the set of smooth functions from M to $\mathbb{R}$..

1. Smooth maps, differentials and vector bundles

We assume the reader to be confident with all the notions and properties concerning vector bundles. The rank of a vector bundle (E, π) over M, is simply denoted with $\mathrm{rank}E$.

The tangent bundle of a smooth manifold M is denoted with TM and it is a vector bundle over M. Its fiber at a point $q \in M$, denoted with T_qM, is the tangent space of M at q. Any smooth section $X : M \to TM$ is a smooth vector field over M. We denote with $\mathrm{Vec}(M)$ the set of smooth vector fields over M. We write $X(q)$ or X_q for denoting the evaluation of X at q.

The cotangent bundle of M is denoted with T^*M and it is a vector bundle over M. Its fiber at a point $q \in M$, denoted with T^*_qM, is the cotangent space of M at q. Any smooth section $\omega : M \to \Lambda^k(T^*M)$ is a smooth k-form over M, where here $\Lambda^k(T^*M)$ is the exterior algebra over T^*M. We denote with $\Omega^k(M)$ the set of smooth k-forms over M. We write $\omega(q)$ or ω_q for denoting the evaluation of X at q.

The cotangent bundle is the "dual" of the tangent bundle in the sense that any $\lambda \in T^*_qM$ is a linear form $\lambda : T_qM \to \mathbb{R}$, and we use the notation $\lambda(v) = \langle \lambda, v\rangle$, $v \in T_qM$.

If $\varphi : M \to N$ is a smooth map between two smooth manifolds, its differential is denoted with $d\varphi : TM \to TN$. We denote with $d_q\varphi : T_qM \to T_{\varphi(q)}N$ the differential of φ at a point $q \in M$, i.e. the restriction of $d\varphi$ on T_qM. We also use the notations φ_* and $\varphi_{*,q}$ for the differential.

We denote with $\varphi^* : T^*N \to T^*M$ the pull-back of φ, i.e. the unique map satisfying

$$\langle \lambda, \varphi_* v\rangle = \langle \varphi^*\lambda, v\rangle, \quad v \in T_qM,\ \lambda \in T^*_{\varphi(q)}N.$$

Notice that, in this case, $\varphi^*_q : T^*_{\varphi(q)}N \to T^*qM$ since it is the dual map of $\varphi_{*,q}$.

If $f \in C^\infty(N)$, $\varphi^* f = f \circ \varphi$ denotes the pull-back of f through φ.

2. Flow of smooth vector fields

We assume that the reader is familiar with the notions of vector field, complete vector field and flows of vector fields. In particular, we recall that the assumption for a vector field to be complete is not restrictive thanks to standard compactness arguments.

Given a complete vector field $X \in \mathrm{Vec}(M)$, the flow generated by X is a smooth map denoted with $P_t = e^{tX} : M \to M$. The exponential notation is due to the following properties

$$e^{0X} = id, \qquad e^{tX} \circ e^{sX} = e^{(t+s)X}, \qquad (e^{tX})^{-1} = e^{-tX}, \tag{1.1}$$
$$\frac{d}{dt} e^{tX}(q) = X(e^{tX}(q)). \tag{1.2}$$

We resume properties (1.1) and (1.2) saying that the family $\{e^{tX}, t \in \mathbb{R}\}$ is a one parameter subgroup of $\mathrm{Diff}(M)$.

A nonautonomous vector field is a family of vector fields $\{X_t\}_{t\in\mathbb{R}}$ such that the map $(t, q) \to X_t(q)$ is measurable in t and smooth in q. We point out our attention in nonautonomous vector fields of the form

$$X_t(q) = \sum_{i=1}^{m} u_i(t) f_i(q), \tag{1.3}$$

where u_i are functions in L^p, $p \in [1, +\infty]$, and f_i are smooth vector fields.

When X_t is complete, its flow is denoted with $P_{s,t}$. We assume all the properties of nonautonomous flows to be well-known. In particular, we highlight the following ones

$$P_{t,t} = id, \tag{1.4}$$
$$P_{t_1,t_2} \circ P_{t_2,t_3} = P_{t_1,t_3}, \quad \forall t_1, t_2, t_3 \in \mathbb{R}. \tag{1.5}$$

Conversely, every two-parameters family of smooth maps $P_{s,t} : M \to M$ satisfying (1.4) and (1.5) is called a flow in M, and its infinitesimal generator is the nonautonomous vector field

$$X_t(q) = \left.\frac{d}{ds}\right|_{s=0} P_{t,t+s}(q), \quad q \in M.$$

The relationship between autonomous and nonautonomous vector fields is contained in the following statement

Lemma 1.1. *Let* $\{P_{s,t}\}_{s,t\in\mathbb{R}}$ *be a family of smooth diffeomorphisms satisfying* (1.4)-(1.5)*. Its infinitesimal generator is an autonomous vector field* $X \in \mathrm{Vec}(M)$ *if and only if*

$$P_{0,t} \circ P_{0,s} = P_{0,t+s}, \quad \forall s, t \in \mathbb{R}. \tag{1.6}$$

Remark 1.2 (On the notation). When the infinitesimal generator of the flow is an autonomous vector field $X \in \mathrm{Vec}(M)$, we have $P_{s,t} = P_{t-s} = e^{(t-s)X}$. With this notation, (1.6) is exactly the second identity in (1.1). Moreover, property (1.5) for autonomous flows is a direct consequence of the second identity in (1.1).

3. Lie brackets and Hörmander condition

For $X, Y \in \mathrm{Vec}(M)$, we denote their Lie bracket with $[X, Y]$. We also denote $\mathrm{ad}X : \mathrm{Vec}(M) \to \mathrm{Vec}(M)$ the adjoint map $(\mathrm{ad}X)Y = [X, Y]$. The reader is assumed to be familiar with Lie brackets and all their differential and algebraic properties.

For a family $\mathcal{F} \subset \mathrm{Vec}(M)$ of smooth vector fields we denote with $\mathcal{F}_q$ the set of its vector fields evaluated at q, namely $\mathcal{F}_q = \{X(q) \mid X \in \mathcal{F}\} \subset T_qM$, and

$$(\mathrm{ad}\mathcal{F})\mathcal{F} = [\mathcal{F}, \mathcal{F}] = \mathrm{span}\{[X, Y] \mid X, Y \in \mathcal{F}\}.$$

The Lie algebra generated by $\mathcal{F}$, denoted by $\mathrm{Lie}\mathcal{F}$, is the smallest sub-algebra of $\mathrm{Vec}(M)$ containing $\mathcal{F}$, namely

$$\mathrm{Lie}\mathcal{F} = \bigoplus_{i=1}^{\infty} (\mathrm{ad}\mathcal{F})^i \mathcal{F}.$$

Similarly, we define

$$\mathrm{Lie}^s\mathcal{F} := \bigoplus_{i=1}^{s} (\mathrm{ad}\mathcal{F})^i \mathcal{F}.$$

Definition 1.3. Let $\mathcal{F} \subset \mathrm{Vec}(M)$ be a family of smooth vector fields. $\mathcal{F}$ is said to be bracket generating (or that satisfies the Hörmander condition) if

$$\mathrm{Lie}_q\mathcal{F} = T_qM, \quad \forall q \in M.$$

For a bracket generating family $\mathcal{F}$, its step $s \in \mathbb{N}$ at a point $q \in M$ is the minimal integer satisfying

$$\mathrm{Lie}_q^s\mathcal{F} = T_qM.$$

Notice that, in general, the step may depends on the point $q \in M$, and $s = s(q)$ can be unbounded on M, even for bracket-generating families.

A class of families of vector fields which we are interested in are the distributions.

Definition 1.4. A smooth distribution $\mathcal{D}$ of rank m on M is a smooth vector sub-bundle of rank m of the tangent bundle TM.

A metric g on $\mathcal{D}$ is a smooth map $q \mapsto g_q$ that assigns to each point $q \in M$ a positive definite scalar product g_q on the vector space $\mathcal{D}_q$.

4. Poisson bracket and Hamiltonian vector fields

We identify any smooth function $a \in C^\infty(M)$ with the constant on fibers function $\pi^*a \in C^\infty(T^*M)$, and any vector field $X \in \mathrm{Vec}(M)$ with the linear on fibers function

$a_X \in C^\infty(T^*M)$, where $a_X(\lambda) = \langle \lambda, X(q) \rangle$, $q = \pi(\lambda)$. Here $\pi : T^*M \to M$ is as usual the standard projection.

For $a, b \in C^\infty(T^*M)$ their Poisson bracket is denoted with $\{a, b\}$. The reader is assumed to be familiar with Poisson brackets and all their differential and algebraic properties. The Hamiltonian vector filed $\vec{a} \in \mathrm{Vec}(T^*M)$ associated to a is defined by

$$\vec{a} : C^\infty(T^*M) \to C^\infty(T^*M), \quad \vec{a}(b) = \{a, b\}.$$

Hamiltonian vector fields completely characterize the pull-back of flows of vector fields on M.

Proposition 1.5. *Let $X \in \mathrm{Vec}(M)$ be a complete vector field with flow $P_{0,t} = e^{tX}$. The flow on T^*M defined by $(P_{0,t}^{-1})^* = (e^{-tX})^*$ is generated by the Hamiltonian vector field $\overrightarrow{a_X}$, where $a_X(\lambda) = \langle \lambda, X(q) \rangle$ and $q = \pi(\lambda)$.*

This construction can be extended to the case of nonautonomous vector fields of the form (1.3).

Proposition 1.6. *Let X_t be a nonautonomous vector field as in* (1.3)*. Denote by $P_{0,t}$ the flow of X_t on M. Then the nonautonomous vector field on T^*M*

$$V_t = \overrightarrow{a}_{X_t}, \quad a_{X_t}(\lambda) = \langle \lambda, X_t(q) \rangle,$$

is the generator of the flow $(P_{0,t}^{-1})^*$.

Finally, we denote with $s \in \Lambda^1(T^*M)$ the tautological (or Liouville) 1-form

$$s : \lambda \mapsto s_\lambda \in T_\lambda(T^*M), \quad \langle s_\lambda, w \rangle = \langle \lambda, \pi_* w \rangle, \quad \lambda \in T^*M, \ w \in T_\lambda(T^*M),$$

and with $\sigma = ds \in \Lambda^2(T^*M)$ the canonical symplectic form of M.

The following statement establish the connection among the canonical symplectic form and Poisson brackets.

Proposition 1.7. *For every $\lambda \in T^*M$ and every $a, b \in C^\infty(T^*M)$ we have the identity*

$$\sigma_\lambda(\vec{a}(\lambda), \vec{b}(\lambda)) = d_\lambda b(\vec{a}(\lambda)) = \{a, b\}(\lambda).$$

1.2 The model of the rototranslations in the plane

In this section we present an easy sub-Riemannian model coming from real life: we study the trajectories of a car moving on the road. Such a model can be viewed as $\mathbb{R}^3 = (x, y, \theta)$, where the first two coordinates (x, y) determine the position of the car in the plane, and the third coordinate θ define the angle of the car's wheels, namely it defines the moving direction of the the car.

When θ is fixed, our car moves in the plane following the direction $(\cos\theta, \sin\theta)$, i.e. it solves the following PDE

$$\begin{cases} \dot{x}(t) = \cos\theta, \\ \dot{y}(t) = \sin\theta, \\ \dot{\theta}(t) = 0. \end{cases}$$

In other words we are moving along the v.f. $X := \cos\theta\partial_x + \sin\theta\partial_y$.

When the car is stopped, we can steer the wheels, namely we are changing the value of θ, solving then the following PDE

$$\begin{cases} \dot{x}(t) = 0, \\ \dot{y}(t) = 0, \\ \dot{\theta}(t) = 1. \end{cases}$$

In this case we are following the v.f. $Y := \partial_\theta$.

Finally, what it is not allowed for our car, is to follow the orthogonal vector to $(\cos\theta, \sin\theta)$. In terms of vector fields, the non-allowed direction is $Z := -\sin\theta\partial_x + \cos\theta\partial_y$.

Remark 1.8. The vector fields X and Y do not commute and, in particular, their Lie bracket is given by $[X, Y] = Z$. Moreover, the set $\{X_p, Y_p, Z_p\}$ is a basis of $\mathbb{R}^3$ for any point $p = (x, y, \theta)$. In other words, the distribution defined by $D_q = \{X_q, Y_q\}$ is bracket generating.

The distribution D_q defined above equipped with any smooth metric $\langle\cdot,\cdot\rangle_q$ is an example of a sub-Riemannian structure on $\mathbb{R}^3$, and the triplet $(\mathbb{R}^3, \mathcal{D}, \langle\cdot,\cdot\rangle)$ is an example of sub-Riemannian manifold. In Section 1.3 below we give the general notions of a sub-Riemannian structure over a smooth manifold M and of sub-Riemannian norm, studying their general properties.

Until now, we were supposing that both the velocity of the car and the velocity of the angle θ is constant and unit. To make our model complete, we have to make these parameters dynamic, namely all the admissible trajectories for the car are such curves $\gamma : I \to \mathbb{R}^3$ that satisfy the following PDE

$$\dot{\gamma}(t) = u_1(t)X(\gamma(t)) + u_2(t)Y(\gamma(t)), \tag{1.7}$$

where u_1 and u_2 are arbitrary essentially bounded functions from $[0, T]$ to $\mathbb{R}$, called controls. Any curve satisfying condition (1.7) is called horizontal or admissible.

Remark 1.9. In spite of we restrict the set of admissible curves, it is easy to see that we can join anyway any couple of point of $\mathbb{R}^3$. On the other hand, the presence of a metric $\langle\cdot,\cdot\rangle$ on the distribution D and condition (1.7) allows us to measure length of admissible curves. Then, one may define the distance between any couple of points as the infimum among the length of admissible curves joining such points.

The previous Remark is known as the Chow-Rashevskii theorem. In Section 1.4 below we will generalize the definition of admissible curves and we will study their main properties.

1.3 Sub-Riemannian structures

In this section we are going to generalize the example described in the previous section. We recall that M denotes a smooth connected manifold of dimension n.

Definition 1.10. A sub-Riemannian structure on M is a pair $(\mathbf{U}, f)$ where:

i) $\mathbf{U}$ is an Euclidean vector bundle with base M and Euclidean fiber $\mathbf{U}_q$, i.e., for every $q \in M$, $\mathbf{U}_q$ is a vector space equipped with a scalar product $\langle\cdot,\cdot\rangle_q$ which is smooth with respect to q. For any $u \in \mathbf{U}_q$ we denote with $|u| := \langle u, u\rangle^{1/2}$ the norm of u.

ii) The map $f : \mathbf{U} \to TM$ is smooth, it is a morphism of vector bundles, and it is fiber-wise linear. In particular, the following diagram is commutative

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[dr, "\pi_{\mathbf{U}}"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

where $\pi_{\mathbf{U}} : \mathbf{U} \to M$ and $\pi : TM \to M$ are the standard projections.

iii) The family of vector fields $\mathcal{D} := \{f(\sigma) | \sigma : M \to \mathbf{U} \text{ smooth section}\} \subset TM$ is bracket generating. Any vector field belonging to $\mathcal{D}$ is called horizontal. We also call step of the sub-Riemannian structure at q the step of $\mathcal{D}$.

When $\mathbf{U}$ admits a global trivialization, i.e. $\mathbf{U} = M \times \mathbb{R}^n$, we say that $(\mathbf{U}, f)$ is a free sub-Riemannian structure.

A smooth manifold M endowed with a sub-Riemannian structure $(\mathbf{U}, f)$ is called sub-Riemannian manifold, and it is denoted as the triple $(M, \mathbf{U}, f)$. When the map f is fiberwise surjective, $(M, \mathbf{U}, f)$ is called a Riemannian manifold.

Definition 1.11. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold. The smooth distribution on M defined by

$$q \mapsto \mathcal{D}_q = f(\mathbf{U}_q),$$

is called the distribution.

Remark 1.12. The set of horizontal vector fields has the structure of a finitely generate module over $C^\infty(M)$.

Remark 1.13. The distribution $\{\mathcal{D}_q\}_{q\in M}$ can be written in terms of the horizontal vector fields as

$$\mathcal{D}_q = \{X(q)|X \in \mathcal{D}\}.$$

In particular, if $(\mathbf{U}', f')$ is another sub-Riemannian structure on M with $\mathcal{D} = \mathcal{D}'$, then $\mathcal{D}_q = \mathcal{D}'_q$, for any $q \in M$. Conversely, condition $\mathcal{D}_q = \mathcal{D}'_q$, for any $q \in M$, does not imply $\mathcal{D} = \mathcal{D}'$ (see Example 1.35 below).

Definition 1.14. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold and let $q \in M$. We define the bundle rank of M as $m := \operatorname{rank}(\mathbf{U})$, and the rank of M at the point q as $r(q) := \dim \mathcal{D}_q$. When $r(q) = r$ is constant, we say that the sub-Riemannian structure $(\mathbf{U}, f)$ is regular of constant rank r, otherwise we say that $(\mathbf{U}, f)$ is rank-varying.

The rank of a sub-Riemannian manifold $(M, \mathbf{U}, f)$ satisfies

$$r(q) \leq \min\{m, n\}, \quad \text{where } m = \operatorname{rank}(\mathbf{U}),\ n = \dim(M). \tag{1.8}$$

From now on, if it is not specified, a sub-Riemannian manifold $(M, \mathbf{U}, f)$ is supposed to have dimension n and bundle rank m.

Remark 1.15. If the distribution has constant rank and $(\mathbf{U}', f')$ is another sub-Riemannian structure on M, then we have $\mathcal{D}_q = \mathcal{D}'_q$, for any $q \in M$, if and only if $\mathcal{D} = \mathcal{D}'$. In this case, $\mathcal{D} \subset TM$ is a smooth distribution of M, called horizontal distribution. Conversely, if $\mathcal{D}$ is a smooth distribution of M equipped with a smooth metric as in Definition 1.4, and $i : \mathcal{D} \to TM$ is the standard inclusion map, then $(M, \mathcal{D}, i)$ is a constant-rank sub-Riemannian manifold.

Definition 1.16. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold and let $q \in M$. The flag of M at the point q is the sequence of subspaces $\{\mathcal{D}^i_q\}_{i\in\mathbb{N}}$ of T_qM defined by

$$\mathcal{D}^i_q = \operatorname{span}\{[X_{i_1}, [\dots [X_{i_{\ell-1}}, X_{i_\ell}]] \dots]|X_{i_1}, \dots, X_{i_\ell} \in \mathcal{D}, \ell \leq i\}.$$

By the bracket generating assumption on the family $\mathcal{D}$ of horizontal vector fields, there exists a minimal integer $s = s(q)$ such that $\mathcal{D}^s_q = T_qM$, which is called the step of M at q.

Notice that, by construction, $\mathcal{D}^i_q \subset \mathcal{D}^{i+1}_q$.

Definition 1.17. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold and let $\{\mathcal{D}^i_q\}_{i\in\mathbb{N}}$ the flag of M at q. Let $r_i(q) = \dim \mathcal{D}^i_q$. We say that M is equiregular if, for every $i \in \mathbb{N}$, the integer $r_i(q) = r_i$ is constant and does not depend on q.

We emphasize that $\mathcal{D}^1_q = \mathcal{D}_q$ and $r_1(q) = r(q)$ is the rank of M at q. So an equiregular sub-Riemannian manifold is regular of constant rank.

In what follows we denote points in $\mathbf{U}$ as pairs (q, u), where $q \in M$ is an element of the base and $u \in \mathbf{U}_q$ is an element of the fiber. Following this notation we can write the value of f at this point as

$$f(q, u) \quad \text{or} \quad f_u(q).$$

We prefer the second notation to stress that, for each $q \in M$, $f_u(q)$ is a vector in T_qM.

Remark 1.18. Every sub-Riemannian manifold $(M, \mathbf{U}, f)$ is locally free in the following sense. Once we have chosen a local trivialization $O_q \times \mathbb{R}^m$ for the vector bundle $\mathbf{U}$, where O_q is a neighborhood of $q \in M$, the following diagram is commutative

$$\begin{tikzcd}
O_q \times \mathbb{R}^m \arrow[r, "\psi"] \arrow[dr, "\pi_1"'] & \pi_{\mathbf{U}}^{-1}(O_q) \arrow[r, "f"] \arrow[d, "\pi_{\mathbf{U}}"] & TO_q \arrow[dl, "\pi"] \\
& O_q &
\end{tikzcd} \tag{1.9}$$

Here π_q is the standard projection, and ψ is the diffeomorphism of the local trivialization. Then, defining

$$\tilde{f} = f \circ \psi,$$

$(O_q, O_q \times \mathbb{R}^m, \tilde{f})$ is a free sub-Riemannian manifold.

Remark 1.19. The local characterization of the previous remark leads to the following local canonical construction. Once we have chosen a local trivialization $O_{q_0} \times \mathbb{R}^m$ for the vector bundle $\mathbf{U}$, we can choose a basis $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$ for $\mathbb{R}^m$. Using the same notation as in (1.9), the vectors $e_i = \psi(\tilde{e}_i)$ form a basis for the fibers $\mathbf{U}_q$, $q \in O_{q_0}$. Then, we can define the vector fields

$$f_i(q) := f(q, e_i) = f(q, \psi(\tilde{e}_i)), \quad i = 1, \ldots, m \tag{1.10}$$

By the linearity on the fibers of the maps f and ψ we have, for every $q \in O_q$ and every $u \in U_q$

$$f_u(q) = \sum_{i=1}^{m} u_i f_i(q), \quad q \in O_{q_0},$$

where $(u_1, \dots, u_m) \in \mathbb{R}^m$ are the coordinates both of $u \in \mathbf{U}_q$ and of $\tilde{u} = \psi^{-1}(u) \in \mathbb{R}^m$ in this trivialization, namely $u = \sum_{i=1}^m u_i e_i$ and $\tilde{u} = \sum_{i=1}^m u_i \tilde{e}_i$.

The local arguments of Remarks 1.18 and 1.19 are the basis for nilpotentization of sub-Riemannian manifolds. In Chapter 3, Theorem 3.5, we describe the nilpotentization of a specific kind of sub-Riemannian manifolds.

From now on, we forgot about the "tilde" notation (and then also the composition with ψ) when working with local trivializations.

Definition 1.20. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold. A family of (horizontal) vector field $\{f_1, \dots, f_m\}$, $m \geq \operatorname{rank}(\mathbf{U})$, is called a generating family for M if for any $q \in M$ and $u \in \mathbf{U}_q$ there exist $u_1, \dots, u_m \in \mathbb{R}$ such that

$$f(q, u) = \sum_{i=1}^m u_i f_i(q).$$

We will discuss in a moment when a sub-Riemannian manifolds admit a generating family. Before, we introduce the "natural" metric in sub-Riemannian manifolds and the notion of equivalence of sub-Riemannian structures.

Definition 1.21. Let $v \in \mathcal{D}_q$. We define the sub-Riemannian norm of v as follows

$$\|v\| = \min\{|u| : u \in \mathbf{U}_q,\ v = f_u(q)\}. \tag{1.11}$$

Notice that, since f is linear, $f(q, \cdot)^{-1}(v) \subset \mathbf{U}_q$ is an affine subspace of $\mathbf{U}_q$ (which is nonempty because $v \in \mathcal{D}_q$). Thus, the minimum in (1.11) is uniquely attained at the orthogonal projection of the origin onto this subspace.

The function defined in (1.11) is in fact a norm, which moreover satisfies the parallelogram identity. I.e., it is induced by the scalar product on $\mathcal{D}_q$

$$\langle v, w \rangle_{\mathcal{D}_q} = \frac{1}{4}\big(\|v + w\|^2 - \|v - w\|^2\big), \quad v, w \in \mathcal{D}_q.$$

Remark 1.22. If the map $f(q, \cdot) : \mathbf{U}_q \to \mathcal{D}_q$ is injective (and then necessarily surjective by (1.8)), then it is automatically an isometry, that is

$$\langle v_1, v_2 \rangle_{\mathcal{D}_q} = \langle u_1, u_2 \rangle_{\mathbf{U}_q}, \quad v_i = f(q, u_i),\ i = 1, 2.$$

When f is fiberwise injective, we have an isometry of Euclidean bundle between $\mathbf{U}$ and the distribution $\{\mathcal{D}_q\}_{q \in M}$. In this case, the sub-Riemannian norm of any vector $v \in \mathcal{D}_q$ can be equivalently defined as

$$\|v\| = |u|, \quad v = f(q, u).$$

Definition 1.23. Let $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ be two sub-Riemannian structures on the same smooth manifold M. They are said to be equivalent as distribution if there exist an Euclidean bundle $\mathcal{V}$ and two surjective morphisms of vector bundles $p : \mathbf{V} \to \mathbf{U}$, $p' : \mathbf{V} \to \mathbf{U}'$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathbf{V} \arrow[r, "p"] \arrow[d, "p'"] & \mathbf{U} \arrow[d, "f"] \\
\mathbf{U}' \arrow[r, "f'"] & TM
\end{tikzcd}$$

Remark 1.24. Two sub-Riemannian manifolds $(M, \mathbf{U}, f)$ and $(M, \mathbf{U}', f')$ are equivalent as distribution if and only if the two moduli of horizontal vector fields $\mathcal{D}$ and $\mathcal{D}'$ coincide. In fact, if $(M, \mathbf{U}, f)$ and $(M, \mathbf{U}', f')$ are equivalent as distribution, then the following diagram is commutative

$$\begin{tikzcd}
\mathbf{V} \arrow[rr, "p"] \arrow[dd, "p'"] & & \mathbf{U} \arrow[dl, "\pi_{\mathbf{U}}"] \arrow[dd, "f"] \\
& M & \\
\mathbf{U}' \arrow[ur, "\pi_{\mathbf{U}'}"] \arrow[rr, "f'"] & & TM \arrow[ul, "\pi"]
\end{tikzcd}$$

and the two moduli of horizontal vector fields $\mathcal{D}$ and $\mathcal{D}'$ coincide. In particular, we deduce that also the distributions $\mathcal{D}_q$ and $\mathcal{D}'_q$ coincide.

Conversely, if $\mathcal{D} = \mathcal{D}'$, then $(M, \mathbf{U}, f)$ and $(M, \mathbf{U}', f')$ are equivalent as distribution taking $\mathbf{V} = \mathbf{U} \times \mathbf{U}'$ and p, p' the standard projections. Notice that, according to Remark 2.7, the equivalence of distributions $\mathcal{D}_q$ and $\mathcal{D}'_q$ does not implies the equivalence of the sub-Riemannian structures as distribution. This is true only for constant-rank sub-Riemannian manifolds.

Definition 1.25. Let $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ be two sub-Riemannian structures on the same smooth manifold M equivalent as distributions. If the projections p, p' are compatible with the scalar product, i.e.

$$|u| = \min\{|v| : p(v) = u\}, \quad \forall u \in \mathbf{U},$$
$$|u'| = \min\{|v| : p'(v) = u'\}, \quad \forall u' \in \mathbf{U}',$$

then $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ are said to be equivalent.

If $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ are two equivalent sub-Riemannian structures on M, then the norms induced on the distribution coincides, i.e. for each $w \in \mathcal{D}_q$ we have $\|w\| = \|w\|'$, where $\|w\|$ is the norm induced on $\mathcal{D}_q$ by $(\mathbf{U}, f)$, and $\|w\|'$ is the norm induced on $\mathcal{D}_q$ by $(\mathbf{U}, f')$.

Definition 1.26. Let $\mathcal{R}$ be a family of sub-Riemannian structures on a smooth manifold M. We call $\mathcal{R}$ a class of equivalence for M if all the sub-Riemannian manifolds $(M, \mathbf{U}, f)$ are equivalent for any $(\mathbf{U}, f) \in \mathcal{R}$. A class of equivalence $\mathcal{R}$ is said to be maximal if there is no other class $\mathcal{R}'$ such that $\mathcal{R} \subset \mathcal{R}'$.

Any maximal class of equivalence of sub-Riemannian structures contains a representative $(\bar{\mathbf{U}}, \bar{f})$ where $\bar{\mathbf{U}}$ is a trivial bundle, as stated by the following theorem.

Theorem 1.27. *Every sub-Riemannian structure* $(\mathbf{U}, f)$ *on a smooth manifold* M *is equivalent to a sub-Riemannian structure* $(\bar{\mathbf{U}}, \bar{f})$ *on* M *where* $\bar{\mathbf{U}}$ *is a trivial bundle.*

The importance the of free structures is explained by the next lemma.

Lemma 1.28. *Every free sub-Riemannian manifold admits a generating family.*

Proof. Let $(M, \mathbf{U}, f)$ be a free sub-Riemannian manifold of bundle rank m, i.e. $\mathbf{U} = M \times \mathbb{R}^m$. A point in $\mathbf{U}$ can be written as (q, u), where $q \in \mathbf{U}$ and $u = (u_1, \dots, u_m) \in \mathbb{R}^m$. If we denote by $\{e_1, \dots, e_m\}$ an orthonormal basis of $\mathbb{R}^m$, then we can define globally the m smooth (horizontal) vector fields on M

$$f_i(q) := f(q, e_i). \tag{1.12}$$

This construction is similar to Remark 1.19 and the triviality of $\mathbf{U}$ make it global. Then we have, by the linearity on fibers of f

$$f(q, u) = \sum_{i=1}^{m} u_i f_i(q), \quad q \in M. \tag{1.13}$$

According to Definition 1.20, the vector fields f_i are a generating family. In particular, the modulus of horizontal vector fields $\mathcal{D}$ is globally generated by the f_i. $\square$

The vector fields defined in (1.12) are the canonical generating family associated to an orthonormal basis $\{e_1, \dots e_m\}$ of $\mathbb{R}^m$. Notice that, if f is not injective, then the vectors $f_i(q)$ are not necessarily orthonormal in $\mathcal{D}_q$.

Combining Theorem 1.27 and Lemma 1.28 one obtains the following result.

Corollary 1.29. *Every sub-Riemannian manifold admits a generating family.*

Remark 1.30. Thanks to this result, algebraic and geometric properties of horizontal vector fields are totally described by the algebraic and geometric properties of a generating family. We will use this fundamental fact in Section 1.5 below and in Chapter 2 to study the end-point map, and in Chapter 3 to describe the behavior of spirals.

Any maximal class of equivalence $\mathcal{R}$ on M contains an infinite number of free structures. Indeed, every free sub-Riemannian structure $(M \times \mathbb{R}^{m'}, f)$ for M is equivalent to $(M \times \mathbb{R}^m, f)$ for any $m' \geq m$ if $f'|_{M\times\mathbb{R}^m} = f$. In this case, in (1.13) we have $u_i = 0$, for $i = m+1, \dots, m'$. This fact suggest the following definition.

Definition 1.31. Let $(M, \bar{\mathbf{U}}, \bar{f})$ be a sub-Riemannian manifold and let $\mathcal{R}$ be the maximal class of equivalence containing $(\bar{\mathbf{U}}, \bar{f})$. We define the minimal bundle rank of M as

$$\operatorname{minrank}(M) = \min\{\operatorname{rank}(\mathbf{U}) \,|\, (\mathbf{U}, f) \in \mathcal{R}\}.$$

The minimal bundle rank of a sub-Riemannian manifold M is actually the bundle rank of the free sub-Riemannian structure of minimal bundle rank inducing an equivalent structure on M. From now on we suppose, without loss of generality, for any sub-Riemannian manifold $(M, \mathbf{U}, f)$ that the vector bundle $\mathbf{U}$ has minimal bundle rank.

1.3.1 Examples

Our approach to sub-Riemannian geometry is quite general. We end this section providing some classical examples which are included in our setting. In particular, Example 1.33 below is the classical setting appearing in the literature for sub-Riemannian manifolds. The new results concerning the regularity problem of sub-Riemannian geodesics we will provide in Chapters 2 and 3, are given indeed in this setting.

Example 1.32 (Riemannian Manifolds)**.** Classically, a Riemannian manifold is defined as a pair $(M, \langle\cdot,\cdot\rangle)$, where M is a smooth manifold and $\langle\cdot,\cdot\rangle_q$ is a scalar product on T_qM, smoothly depending on $q \in M$. This definition, as pointed out in Remark 2.7, is included in Definition 1.10 by taking $\mathbf{U} = D$ endowed with the structure induced by $\langle\cdot,\cdot\rangle$ and $f : D \to TM$ the canonical inclusion map.

Example 1.33 (Constant rank sub-Riemannian manifolds)**.** We emphasized in Remark 2.7 that a constant-rank sub-Riemannian manifold is given by $(M, \mathcal{D}, i)$, where $\mathcal{D} \subset TM$ is a smooth Euclidean distribution and i is the inclusion map. Classically, a constant-rank sub-Riemannian manifold is presented as a triple $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle_q$ is a scalar product on $\mathcal{D}_q$, smoothly depending on $q \in M$.

In particular, let us analyze the model presented in paragraph 1.2. There, on $M = \mathbb{R}^3$ we had the smooth distribution D defined by $D_q = \operatorname{span}\{X(q), Y(q)\}$. Then, any scalar product $\langle\cdot,\cdot\rangle_q$ defined on D_q defines a sub-Riemannian metric. The triplet $(\mathbb{R}^3, D, \langle\cdot,\cdot\rangle)$ is a free sub-Riemannian manifold of constant rank 2.

Example 1.34 (Carnot groups)**.** A Carnot group $(\mathbb{G}, \cdot)$ is a connected, simply connected and nilpotent Lie group $\mathbb{G}$, with a smooth group operation $\cdot$, whose Lie algebra $\mathfrak{g}$ is stratified, i.e., there exists a (fixed) decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$, such that $\mathfrak{g}_i = [\mathfrak{g}_1, \mathfrak{g}_{i-1}]$ for any $i = 2, \dots, s$ and $[\mathfrak{g}, \mathfrak{g}_s] = 0$.

The left translation based at $a \in \mathbb{G}$, defined by $L_a(b) = a \cdot b$, for $b \in \mathbb{G}$, is a diffeomorphism of $\mathbb{G}$. A vector field X on $\mathbb{G}$ is left-invariant if $(L_a)_* X = X$ for every $a \in \mathbb{G}$. Through the differetial at the identity $e \in \mathbb{G}$ of left translations, we have $T_a\mathbb{G} = (L_a)_{*,e}\mathfrak{g}$, i.e., the tangent bundle is generated by left-invariant vector fields. In particular, $D_a = (L_a)_{*,e}\mathfrak{g}_1$ is a left-invariant and bracket-generating smooth distribution of $T\mathbb{G}$. Similarly, any scalar product $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g}_1$ defines a left-invariant sub-Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{G}$ and $(\mathbb{G}, D, \langle \cdot, \cdot \rangle)$ is an equiregular (and then also a constant-rank) sub-Riemannian manifold.

In Chapter 3 we will see that Carnot groups are the nilpotent (i.e. first order) approximations of equiregular sub-Riemannian manifolds.

Example 1.35 (Same distributions with different horizontal vector fields)**.** Let us consider on $M = \mathbb{R}^2$ the two free sub-Riemannian structures of bundle rank 2 $(\mathbb{R}^2 \times \mathbb{R}^2, f)$ and $(\mathbb{R}^2 \times \mathbb{R}^2, f')$ giving by

$$f(x, y, u_1, u_2) = (x, y, u_1, u_2 x), \qquad f'(x, y, u_1, u_2) = (x, y, u_1, u_2 x^2).$$

The last coordinate of f and f' vanish if and only if $x = 0$, making $\mathcal{D}_{(x,y)} = \mathcal{D}'_{(x,y)}$ for every $(x, y) \in \mathbb{R}^2$.

On the other hand, $\mathcal{D}' \subsetneq \mathcal{D}$. In fact, if $X \in \mathcal{D}'$, then $X = f' \circ \sigma$, where $\sigma : M \to \mathbb{R}^2 \times \mathbb{R}^2$ is an arbitrary smooth section $\sigma(x, y) = (x, y, u_1(x, y), u_2(x, y))$, for some smooth u_1, u_2. Then, if $X \in \mathcal{D}'$, we have $X(x, y) = (u_\sigma, v_\sigma x^2)$, for some smooth functions u_1, u_2. So $X \in \mathcal{D}$, because $X = f \circ \bar{\sigma}$ with $\bar{\sigma}(x, y) = (x, y, u_1, u_2 x)$.

Finally, the vector field $X(x, y) = (1, x)$ belongs to $\mathcal{D}$, since it is the composition of f with $\sigma(x, y) = (x, y, 1, x)$, but it does not belong to $\mathcal{D}'$.

1.4 Admissible curves and sub-Riemannian distance

In this section we are going to generalize the second part of the example given in Section 1.2. Namely, we illustrate the general theory concerning admissible curves in sub-Riemannian geometry.

Definition 1.36. A Lipschitz curve $\gamma : [0, T] \to M$ is said to be admissible or horizontal for a sub-Riemannian structure $(\mathbf{U}, f)$ on M if there exists a measurable and essentially bounded function

$$u : [0,T] \to \mathbf{U}, \quad u(t) \in \mathbf{U}_{\gamma(t)} \tag{1.14}$$

called control, such that

$$\dot{\gamma}(t) = f(\gamma(t), u(t)), \text{ for a.e. } t \in [0,T]. \tag{1.15}$$

In this case, we say that u is a control corresponding or associated to γ.

In other words, an admissible curve γ is driven by a measurable and essentially bounded function u lying on the moving fibers of $\mathbf{U}$ along γ. We point out the attention on several facts regarding the definition of horizontal curves, collecting them in the following remarks

Remark 1.37. A curve $\gamma : [0,T] \to M$ such that $\gamma(t) = \gamma(0)$ for every $t \in [0,T]$ is always admissible

Remark 1.38. When $f : \mathbf{U} \to TM$ is fiberwise injective, the control u associated to a horizontal curve is unique.

Lemma 1.39. *Let $O_q \subset M$ be a neighborhood of $q \in M$, and let $O_q \times \mathbb{R}^m$ be a local trivialization of $\mathbf{U}$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$ and let f_i, $i = 1, \ldots, m$, be the horizontal vector fields defined as in* (1.10)*. A Lipschitz curve $\gamma : [0,T] \to O_q$ is horizontal if and only if there exists $u \in L^\infty([0,T], \mathbb{R}^m)$ such that*

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t) f_i(\gamma(t)), \quad \textit{for a.e. } t \in [0,T].$$

Proof. The proof is a direct consequence of Remark 1.19. $\square$

Thanks to this local characterization and the Carathéodory Theorem for nonautonomous vector fields (see [2, Theorem 2.15]), it follows that, for each initial condition $q \in M$ and for each control $u \in L^\infty([0,T], \mathbb{R}^m)$, the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = f_{u(t)}(\gamma(t)), \\ \gamma(0) = q, \end{cases}$$

always admits a solution defined on a sufficiently small interval.

Corollary 1.40. *For every $q \in M$ and for each control $u \in L^\infty([0,T], \mathbb{R}^m)$, there exists an admissible curve γ with control u and $\gamma(0) = q$.*

Remark 1.41. An admissible curve $\gamma : [0,T] \to M$ is Lipschitz, hence differentiable at almost every point. Therefore, combining (1.14) and (1.15), we have that its derivative satisfies

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}, \quad \text{for a.e. } t \in [0,T]. \tag{1.16}$$

This is, in general, not also a necessary condition for a curve to be horizontal. According to Remark 1.15, a curve is horizontal if and only if it satisfies condition (1.16) just for constant-rank sub-Riemannian manifolds.

We resume Lemma 1.39 and Remark 1.41 giving equivalent conditions for a Lipschitz curve $\gamma : [0,T] \to M$ to be horizontal in Remarks 1.42 and 1.43 below.

Remark 1.42. Let M be a free sub-Riemannian manifold with generating family $f_1, \dots, f_m$ (possibly associated in the canonical way to the standard basis of $\mathbb{R}^m$). Let $\gamma : [0,T] \to M$ be a Lipschitz curve. Then the following definitions are equivalent

H1) γ is horizontal.

H2) There exists a function $u \in L^\infty([0,T], \mathbb{R}^m)$ such that

$$\dot\gamma(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t)), \quad \text{for a.e. } t \in [0,T]$$

Remark 1.43. Let M be a constant rank sub-Riemannian manifold. Let $\gamma : [0,T] \to M$ be a Lipschitz curve. Then the following definitions are equivalent

H1) γ is horizontal;

H3) $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$, for a.e. $t \in [0,T]$.

Example 1.44. In the setting of Example 1.35, let us consider the curve $\gamma : t \in [-1,1] \mapsto (t, t^2) \in \mathbb{R}^2$. We have $\gamma(t) \in \mathcal{D}_{\gamma(t)}$ and $\gamma(t) \in \mathcal{D}'_{\gamma(t)}$ for all $t \in [-1,1]$.

On the other hand, γ is admissible for f, since its corresponding control is $(u_1, u_2) = (1,2)$, but it is not admissible for f', since its corresponding control is uniquely determined as $(u_1, u_2) = (1, 2/t)$ for a.e. $t \in [-1,1]$, which is not essentially bounded (and even not integrable). This situation is compatible with the inclusion $\mathcal{D}' \subset \mathcal{D}$, because a horizontal vector field for f is not necessarily horizontal for f'.

1.4.1 Length of admissible curves

When the derivative of an horizontal curve is defined, it belongs pointwisely to the fibers of $\mathbf{U}$ along γ. Then, at any differentiability point of γ, the velocity norm of any horizontal curve γ is defined by

$$\|\dot\gamma(t)\| = \min\{|u| : u \in \mathbf{U}_{\gamma(t)},\ \dot\gamma(t) = f_u(\gamma(t))\}. \tag{1.17}$$

Hence, it is well defined the function $t \mapsto u^*(t)$ which realizes the minimum in (1.17). In particular, the norm of $\dot{\gamma}$ is given, at any differentiability point of γ, by

$$\|\dot{\gamma}(t)\| = |u^*(t)|.$$

Notice that, if the function $u^*(t)$ is measurable and essentially bounded, then it is in fact a control associated to the curve γ.

Lemma 1.45. *Let $\gamma : [0,T] \to M$ be a horizontal curve and let $\mathrm{Diff}(\gamma) \subset [0,T]$ be set of differentiability points of γ. Let*

$$u^* : t \in \mathrm{Diff}(\gamma) \mapsto u^*(t) \in \mathbf{U}_{\gamma(t)}$$

be the function defined by

$$u^*(t) = \mathrm{argmin}\{|u| : u \in \mathbf{U}_{\gamma(t)},\ \dot{\gamma}(t) = f_u(\gamma(t))\}. \tag{1.18}$$

Then u^* is a control for γ, i.e. it is measurable and essentially bounded.

Actually, the set $[0,T]\backslash\mathrm{Diff}(\gamma)$ has null measure. Thus, from now on, with a slight abuse of notation we will consider the function u^* introduced in (1.18) as defined on the whole $[0,T]$.

Definition 1.46 (Minimal control)**.** Given a horizontal curve $\gamma : [0,T] \to M$, the control u^* defined in (1.18) is called the minimal control of γ.

Remark 1.47. If the admissible curve $\gamma : [0,T] \to M$ is differentiable, its minimal control is defined everywhere on $[0,T]$. Nevertheless, it could be not continuous, in general. Indeed consider, as in Example 1.35, the free sub-Riemannian structure on $M = \mathbb{R}^2$ given by

$$f(x, y, u_1, u_2) = (x, y, u_1, u_2 x),$$

and let $\gamma : [0,T] \to \mathbb{R}^2$ defined by $\gamma(t) = (t, t^2)$. Its minimal control $u^*(t)$ satisfies $(u_1^*(t), u_2^*(t)) = (1, 2)$, while $(u_1^*(t), u_2^*(t)) = (1, 0)$, hence is not continuous.

Definition 1.48. Let $\gamma : [0,T] \to M$ be an admissible curve and let $u^* : [0,T] \to \mathbf{U}$ be its minimal control. We define the sub-Riemannian length of γ as

$$L(\gamma) = \int_0^T \|\dot{\gamma}(t)\| dt = \int_0^T |u^*(t)| dt$$

We say that γ is parameterized by arc length (or arc length parameterized if $\|\dot{\gamma}(t)\| = 1$ for a.e. $t \in [0,T]$.

The length of an admissible curve is, in other words, the L^1-norm of its minimal control. Since $L^\infty([0,T],\mathbb{R}^m) \subset L^1([0,T],\mathbb{R}^m)$, any admissible curve has finite length and the definition is well posed. In particular, for an arc length parameterized curve we have that $L(\gamma) = T$.

Lemma 1.49. *The length of an admissible curve is invariant by Lipschitz reparameterization.*

Lemma 1.50. *Every admissible curve of positive length is a Lipschitz reparameterization of an arc length parameterized admissible one.*

Remark 1.51. Up to a reparameterization, we can always assume one of the following for a horizontal curve γ

i) γ is defined on I and has constant speed;

ii) γ is defined on $[0,T]$ and has unit speed, i.e. it is parameterized by arc-length.

1.4.2 Sub-Riemannian distance and Chow-Rashevskii theorem

Once defined horizontal curves with their length, one can define a distance in M in the following natural way

Definition 1.52. Let M be a sub-Riemannian manifold with $q_0, q_1 \in M$. We define the sub-Riemannian (or Carnot-Caratéodory) distance between q_0 and q_1 as

$$d(q_0,q_1) = \inf\{L(\gamma) \,|\, \gamma : [0,T] \to M \text{ admissible },\gamma(0) = q_0,\ \gamma(1) = q_1\}$$

Let us recall that, according to Remark 1.28, every sub-Riemannian manifold admits a generating family of horizontal vector fields $\{f_1,\dots,f_m\}$. In these optics, an admissible curve can only move along the directions of the generating vector fields, and namely the horizontal directions.

The Chow-Rashevskii Theorem ensures that the sub-Riemannian distance is well defined, in spite of we restricted the admissible moving directions for horizontal curves.

Theorem 1.53 (Chow-Rashevskii)**.** *Let M be a sub-Riemannian manifold. Then*

i) (M,d) is a metric space;

ii) The topology induced by (M,d) is equivalent to the manifold topology.

Remark 1.54. One of the main consequences contained in the proof of this result is that, thanks to the bracket-generating condition, every couple of points in M is always joined by an admissible curve. Hence, $d(q_0,q_1) < +\infty$, for every $q_0,q_1 \in M$.

1.4.3 Length-minimizers

The aim of this section is to discuss the existence of length minimizing curves, i.e. of those curves which, roughly speaking, realize the distance between couple of points (for example, segments in Euclidean spaces).

Definition 1.55. Let $\gamma : [0, T] \to M$ be an admissible curve. We say that γ is a length-minimizer (or a length-minimizing curve) if

$$L(\gamma) = d(\gamma(0), \gamma(T)).$$

In other words, a length minimizing curve minimizes the length among admissible curves with same endpoints. A key fact involving length minimizing curves is the lower semicontinuity of the sub-Riemannian distance.

Theorem 1.56 (Lower semicontinuity of the distance)**.** *Let $\gamma_n : [0, T] \to M$ be a sequence of admissible curves parameterized by arc-length. Suppose that $\gamma_n \to \gamma$ uniformly on $[0, T]$, with $\liminf_{n\to\infty} \gamma_n < +\infty$. Then γ is admissible and*

$$L(\gamma) \leq \liminf_{n\to\infty} \gamma_n$$

Corollary 1.57. *Let $\gamma_n : [0, T] \to M$ be a sequence of length-minimizers parameterized by arc-length. Suppose that $\gamma_n \to \gamma$ uniformly on $[0, T]$. Then γ is a length-minimizer.*

The (local) existence of length-minimizers is guaranteed by Theorem 1.58 below. The proof relies upon both the lower semicontinuity of the distance and a natural compactness assumption.

Theorem 1.58 (Existence of length-minimizers)**.** *Let M be a sub-Riemannian manifold and $q_0 \in M$. Assume that the closed ball $\overline{B}_{q_0}(r)$ is compact, for some $r > 0$. Then for all $q \in B_{q_0}(r)$ there exists a length minimizer joining q_0 and q.*

Corollary 1.59. *Let M be a sub-Riemannian manifold and $q_0 \in M$. There exists $\varepsilon > 0$ such that for every $q \in B_{q_0}(\varepsilon)$ there is a minimizing curve joining q_0 and q.*

Remark 1.60. The compactness assumption in Theorem 1.58 is completely natural and cannot be removed. In fact, the existence of length-minimizers between two points is not true in general, as it happens, for example, for two symmetric points with respect to the origin in $M = \mathbb{R}^n \setminus \{0\}$, endowed with the Euclidean metric.

On the other hand, when length-minimizers exist between two fixed, they may not be unique, as in the case of two antipodal points on the sphere $\mathbb{S}^2$.

The existence of length-minimizing curves leads to a characterization of the metric completeness of sub-Riemannian distance.

Proposition 1.61. *Let M be a sub-Riemannian manifold. Then the three following properties are equivalent*

i) (M,d) *is complete;*

ii) $\overline{B}_q(r)$ *is compact for every* $q \in M$ *and* $r > 0$*;*

iii) There exists $\varepsilon > 0$ *such that* $\overline{B}_q(r)$ *is compact for every* $q \in M$*.*

Remark 1.62. In the proof of proposition 1.61, the fact that the distance is sub-Riemannian is used only to prove that "(i) implies (ii)". Actually the same result holds true in the more general context of length metric space, see [9, Chapter 2].

Combining this result with Corollary 1.59 we obtain the following corollary.

Corollary 1.63. *Let* (M,d) *be a complete sub-Riemannian manifold. Then for every* $q_0, q_1 \in M$ *there exists a length minimizer joining* q_0 *and* q_1*.*

The results contained in this section provide the existence of length minimizers in sub-Riemannian geometry, but they do not provide any additional regularity with respect to admissible curves.

The regularity of length minimizing curves in sub-Riemannian geometry is an open problem since 40 years, and, actually, it is the problem we face-up in this thesis.

In Section 1.5 below, the first order analysis of the end-point map emphasizes that the root of the problem lies in the abnormal minimizers, i.e. in the singularities of the end-point map.

In Chapters 2 and 3, we provide new interesting results concerning the regularity of length minimizers.

1.4.4 Lipschitz vs absolutely continuous admissible curves

In Definition 1.36 we assumed admissible curves to be Lipschitz: we were in fact defining Lipschitz-admissible curves, requiring the control u to be an L^∞ function. Notice that the length of a Lipschitz admissible curve is well defined as the L^1-norm of its minimal control, since $L^\infty([0,T]) \subset L^1([0,T])$.

The same definitions can be given for absolutely continuous curves, as long as we require the control u to be an L^1 function.

Definition 1.64. An absolutely continuous curve $\gamma : [0, T] \to M$ is said to be *AC-admissible* if there exists an L^1 function $u : t \in [0, T] \mapsto u(t) \in U_{\gamma(t)}$ such that $\dot{\gamma}(t) = f(\gamma(t), u(t))$, for a.e. $t \in [0, T]$.

Once defined AC-admissible curve, the minimal control is defined in the same way as in (1.18) and Lemma 1.49 continues to hold. The length of an AC-admissible curve is defined again as the L^1-norm of its minimal control.

Being the set of absolutely continuous curve bigger than the set of Lipschitz ones, one could expect that the sub-Riemannian distance between two points is smaller when computed among all absolutely continuous admissible curves. However this is not the case thanks to the invariance by reparameterization. Indeed Lemmas 1.49 and 1.50 can be rewritten in the absolutely continuous framework in the following form.

Lemma 1.65. *The length of an AC-admissible curve is invariant by AC reparameterization.*

Lemma 1.66. *Any AC-admissible curve of positive length is a AC reparameterization of an arc length parameterized admissible one.*

As a consequence of these results, if we define

$$d_{AC}(q_0, q_1) = \inf\{L(\gamma) \mid \gamma : [0, T] \to M AC\text{-admissible }, \gamma(0) = q_0,\ \gamma(T) = q_1\},$$

we have the following statement

Proposition 1.67. *For every $q_0, q_1 \in M$, we have $d_{AC}(q_0, q_1) = d(q_0, q_1)$*

In a similar way, one can introduce $W^{1,2}$-admissible curve, i.e., those associated with L^2 controls. Since $L^2([0, T]) \subset L^1([0, T])$, their length remains defined as the L^1-norm of the minimal control. Again, with the same arguments as below, one obtains $d_{W^{1,2}} = d$.

Remark 1.68. As a consequence of these considerations, we can work indifferently with L^∞, L^2 or L^1 controls for horizontal curves. For example, in Section 1.4 below we define the end-point map in the setting of L^2 controls.

Remark 1.69. More in general, for any $p \in \mathbb{N}$, one can define $W^{1,p}$-admissible curves associated with L^p controls. The resulting distance $d_{W^{1,p}}$ coincides with the sub-Riemannian one.

1.5 End-Point map: first-order analysis and Pontryagin extremals

In this section we introduce the central object of our thesis: the end-point map. Its first order differential analysis leads to the Pontryagin maximum principle, showing the existence of two kinds of curves which are candidate to be length minimizing. In particular, the singular case is the deep reason behind the regularity problem of the geodesics.

Without loss of generality, we assume our sub-Riemannian manifold M to be free of bundle rank m, and $\{f_1, \dots, f_m\}$ will denote a generating family of smooth horizontal vector fields. We will also consider, in order to clean up the notation, horizontal curves defined on the interval $I = [0,1]$, instead of the interval $[0,T]$.

Fix $q_0 \in M$. For every control $u \in L^2(I;\mathbb{R}^m)$, the corresponding trajectory based at q_0 is the curve that solves the Cauchy problem

$$\dot\gamma(t) = f_{u(t)}(\gamma(t)) = \sum_{i=1}^{m} u_i(t) f_i(\gamma(t)), \quad \gamma(0) = q_0, \tag{1.19}$$

and it is denoted with $\gamma_u^{q_0}$, or, since $q_0 \in M$ is fixed, simply with γ_u.

Let $\mathcal{U}_{q_0} \subset L^2(I;\mathbb{R}^m)$ be the set of controls u such that the corresponding trajectory γ_u based at q_0 is defined on the whole interval I. It is a well known fact that $\mathcal{U}_{q_0}$ is an open subset of $L^2(I;\mathbb{R}^m)$.

Definition 1.70. Let $(M, \mathbf{U}, f)$ be a free sub-Riemannian manifold, and fix $q_0 \in M$. The end-point map based at q_0 is the map

$$E_{q_0} : \mathcal{U}_{q_0} \to M, \quad E_{q_0}(u) = \gamma_u(1),$$

where $\gamma_u : I \to M$ is the unique solution of the Cauchy problem (1.19).

A first property of the end-point map, which is a consequence of the Chow-Rashevskii theorem, is its openness.

Theorem 1.71. *Let $(M, \mathbf{U}, f)$ be a free sub-Riemannian manifold, and fix $q_0 \in M$. Then the end-point map E_{q_0} is open at every $u \in \mathcal{U}_{q_0}$.*

A second property of the end-point map is its smoothness in the Fréchet sense. The following theorem provides in fact an explicit formula for its first differential.

Theorem 1.72. *Let $q_0 \in M$. Then, the end-point map E_{q_0} is smooth on $\mathcal{U}_{q_0}$. In particular, its Fréchet differential $D_u E_{q_0} : L^2(I;\mathbb{R}^m) \to T_{\gamma_u(1)}M$ is given by*

$$D_u E_{q_0}(v) = \int_0^1 (P^u_{t,1})_* f_{v(t)}\big|_{\gamma_u(1)}\, dt. \tag{1.20}$$

Here $P^u_{t,1}$ denotes the flow generated by u, while $f_{v(t)} = \sum_{i=1}^m v_i(t) f_i$.

The proof of this theorem requires the introduction of a particular formalism of calculus, named the Chronological Calculus. It will be the object of the next section.

1.5.1 Chronological calculus

In this section we develop a language, called chronological calculus, that will allow us to work in an efficient way with flows of nonautonomous vector fields. The basic idea of chronological calculus is to replace a non-linear finite-dimensional object, the manifold M, with a linear infinite-dimensional one, the commutative algebra $C^\infty(M)$.

Definition 1.73. For every $q \in M$, we define the evaluation linear functional as

$$\widehat{q} : C^\infty(M) \to \mathbb{R}, \quad \widehat{q}(a) = a(q).$$

Notice that, since the set $C^\infty(M)$ of smooth functions on M is an $\mathbb{R}$-algebra with the usual operation of pointwise addition and multiplication, for every $q \in M$, the functional $\widehat{q}$ is a homomorphism of algebras, i.e., it satisfies $\widehat{q}(a \cdot b) = \widehat{q}(a) \cdot \widehat{q}(b)$.

We also emphasize that if $q(t) = q_t$ is a smooth curve in M, then it defines the curve of operator $\widehat{q}_t(a) = a(q_t)$.

Definition 1.74. For every diffeomorphism $P \in \mathrm{Diff}(M)$, we define the linear operator

$$\widehat{P} : C^\infty(M) \to C^\infty(M), \quad \widehat{P}(a) = a \circ P,$$

which is an automorphism of the algebra $C^\infty(M)$.

Remark 1.75. One can prove that for every nontrivial homomorphism of algebras $\varphi : C^\infty(M) \to \mathbb{R}$ there exists $q \in M$ such that $\varphi = \widehat{q}$. Analogously, for every automorphism of algebras $\varphi : C^\infty(M) \to C^\infty(M)$, there exists a diffeomorphism $P \in \mathrm{Diff}(M)$ such that $\widehat{P} = \varphi$. A proof of these facts is contained in [3, Appendix A].

The "hat" operator for tangent vectors and, then, for vector fields, is defined via the natural action of vectors of functions.

Definition 1.76. For every $q \in M$ and $v \in T_qM$ we define

$$\widehat{v} : C^\infty(M) \to \mathbb{R}, \quad \widehat{v}(a) = \langle d_q a, v \rangle.$$

Since the differential of a smooth function satisfies the Leibnitz rules, for every $v \in T_qM$ the linear operator $\widehat{v}$ is a derivation on $C^\infty(M)$,

$$\widehat{v}(a \cdot b) = \widehat{v}(a) b(q) + a(q) \widehat{v}(b), \quad \forall a, b \in C^\infty(M).$$

Remark 1.77. If $v \in T_qM$, then it is the tangent vector of a curve $q(t) = q_t$ such that $q(0) = q$. For every $a \in C^\infty(M)$, we have

$$\left.\frac{d}{dt}\right|_{t=0} a(q_t) = \langle d_{q(t)}a, \dot{q}_t\rangle|_{t=0} = \langle d_q a, v\rangle.$$

Thus, it holds the operator identity

$$\widehat{v} = \left.\frac{d}{dt}\right|_{t=0} \widehat{q}_t : C^\infty(M) \to \mathbb{R}.$$

Definition 1.78. Let $X \in \mathrm{Vec}(M)$ and let $P_t = e^{tX} : C^\infty(M) \to C^\infty(M)$ its flow. We define

$$\widehat{X} : C^\infty(M) \to C^\infty(M), \quad (\widehat{X}a)(q) = \langle d_q a, X(q)\rangle,$$

or, equivalently,

$$\widehat{X} = \left.\frac{d}{dt}\right|_{t=0} \widehat{P}_t : C^\infty(M) \to C^\infty(M).$$

Notice that the equivalence of the definitions given above is ensured by Remark 1.77. Consequently, we also have the Leibnitz rule

$$\widehat{X}(ab) = \widehat{X}(a)b + a\widehat{X}(b), \quad \forall a, b \in C^\infty(M).$$

Remark 1.79. In the following we will identify any object with its dual interpretation as operator on functions and stop to use a different notation for the same object when acting on the space of smooth functions.

If P is a diffeomorphism on M and q is a point on M the point $P(q)$ is simply represented by the usual composition $\widehat{q} \circ \widehat{P}$ of the corresponding linear operator. Thus, when using the operator notation, composition works in the opposite side. To simplify the notation in what follows we will remove the "hat" identifying an object with its dual, but use the symbol "$\odot$" to denote the composition of these object, so that $P(q)$ will be $q \odot P$.

Analogously, the composition $X \odot P$ of a vector field X and a diffeomorphism P will denote the linear operator $a \mapsto X(a \circ P)$.

Lemma 1.80. *Let $q \in M$ and let $v \in T_qM$. For every $P \in \mathrm{Diff}(M)$ we have*

$$d_qP(v) = v \odot P,$$

as operators on $C^\infty(M)$.

Proof. We have $d_qP(v) \in T_{P(q)}M$. Thus, its action on a function $a \in C^\infty(M)$ is given by

$$d_qP(v)(a) = d_{P(q)}a(d_qP(v)) = d_q(a \circ P)(v) = v(a \circ P) = (v \odot P)(a).$$

$\square$

Remark 1.81. Recall that, if $X \in \mathrm{Vec}(M)$ is a smooth vector field, then its pushforward P_*X through a smooth diffeomorphism P is given by $(P_*X)(q) = d_{P^{-1}(q)}P(X(P^{-1}(q)))$. Therefore, as a consequence of Lemma 1.80, we have

$$P_*X = P^{-1} \odot X \odot P,$$

as operators on $C^\infty(M)$.

Definition 1.82. Let $P \in \mathrm{Diff}(M)$. We define the adjoint operator of P on smooth vector fields as

$$\mathrm{Ad}P : \mathrm{Vec}(M) \to \mathrm{Vec}(M), \quad \mathrm{Ad}P(X) = P \odot X \odot P^{-1}.$$

With this notation, we have

$$P_*X = (\mathrm{Ad}P^{-1})X.$$

Now consider a nonautonomous vector field X_t. Using the operatorial notation introduced above, the corresponding nonautonomous ODE on M reads

$$\frac{d}{dt}q(t) = q(t) \odot X_t. \tag{1.21}$$

As discussed in Section 1.1, the solution to (1.21) defines a flow, i.e. a family of diffeomorphisms $P_{s,t}$ on M. In terms of chronological calculus, it holds the following lemma.

Lemma 1.83. *Let $P_{s,t}$ be the flow defined by a nonautonomous vector field X_t. Then $P_{s,t}$ satisfies the operatorial Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}P_{s,t} = P_{s,t} \odot X_t, \\ P_{s,s} = id \end{cases} \tag{1.22}$$

Definition 1.84. We call the solution to (1.22) the right chronological exponential, and we use the notation

$$P_{s,t} = \overrightarrow{\exp} \int_s^t X_\tau d\tau. \tag{1.23}$$

Remark 1.85. The notation introduced in (1.23) is consistent with the exponential notation defined for autonomous vector fields. Indeed, if $X_t = X \in \mathrm{Vec}(M)$ is autonomous, we have

$$\overrightarrow{\exp} \int_s^t X d\tau = e^{(t-s)X},$$

which is in fact the flow of X based at the time s.

Remark 1.86. For a nonautonomous vector field X_t, we have in general

$$\overrightarrow{\exp} \int_s^t X_\tau d\tau \neq e^{\int_s^t X_\tau d\tau}. \tag{1.24}$$

In (1.24) it holds the equality if and only if the vector fields commute at every time, i.e. $[X_s, X_t] = 0$ for every $s, t \in \mathbb{R}$.

Remark 1.87 (Volterra series). Identity (1.23) can be rewritten in the integral form

$$P_{s,t} = id + \int_s^t P_{s,\tau} \odot X_\tau d\tau \tag{1.25}$$

Denoting $\Delta_k(s,t) = \{(t_1, \ldots, t_k) \in \mathbb{R}^k \,|\, s \leq t_k \leq \cdots \leq t_1 \leq t\}$ the k-dimensional simplex, the iteration of (1.25) N times gives

$$P_{s,t} = id + \sum_{k=1}^{N-1} \int_{\Delta_k(s,t)} X_{t_k} \odot \cdots \odot X_{t_1} d\mathcal{L}^k + R_N,$$

where the remainder R_N is defined by

$$R_N = \int_{\Delta_N(s,t)} P_{s,t_N} \odot X_{t_N} \odot \cdots \odot X_{t_1} d\mathcal{L}^N,$$

and $\mathcal{L}^k$ denotes the standard k-dimensional Lebesgue measure. Formally, letting $N \to \infty$ and assuming that $R_N \to 0$, we can write

$$P_{s,t} = id + \sum_{k=1}^{\infty} \int_{\Delta_k(s,t)} X_{t_k} \odot \cdots \odot X_{t_1} d\mathcal{L}^k.$$

The convergence of Volterra series and the estimates for the remainder term R_N plays a role in the proof of Theorem 1.72. Since these arguments are quite long, we do not develop them, and we refer the reader to [2, Chapter 6] for a very exhaustive and detailed discussion.

Proposition 1.88. *Assume that the flow $P_{s,t}$ satisfies the Cauchy problem* (1.22). *Then the inverse flow $Q_{s,t} := (P_{s,t})^{-1}$ satisfies the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt} Q_{s,t} = -X_t \odot Q_{s,t}, \\ Q_{s,s} = id. \end{cases} \tag{1.26}$$

Definition 1.89. We call a solution $Q_{s,t}$ to (1.26) the left chronological exponential, and we use the notation

$$Q_{s,t} = \overleftarrow{\exp} \int_s^t (-X_\tau) d\tau.$$

Lemma 1.90. *Let* $P_{s,t} = \overrightarrow{\exp} \int_s^t X_\tau d\tau$ *be a nonautonomous flow. Then* $\mathrm{Ad}P_{s,t}$ *is a solution to the Cauchy problem*

$$\dot{A}_t = A_t \odot \mathrm{ad}X_t, \quad A_s = id, \tag{1.27}$$

where, as usual, $\mathrm{ad}X : Y \mapsto [X, Y]$ *is the adjoint action on the Lie algebra of vector fields.*

Remark 1.91. As a chronological exponential, we have the identity

$$\mathrm{Ad}\left(\overrightarrow{\exp} \int_s^t X_\tau d\tau\right) = \overrightarrow{\exp} \int_s^t \mathrm{ad}X_\tau d\tau \tag{1.28}$$

In particular, combining (1.27) and (1.28), in the case of autonomous vector fields we have

$$e_*^{-tX} = e^{t\mathrm{ad}X}.$$

We conclude this section providing a Variation formula for the flow of the sum of two nonautonomous vector fields X_t, Y_t. Namely, let us consider the Cauchy problem

$$\dot{A}_t = A_t \odot (X_t + Y_t), \quad A_s = id. \tag{1.29}$$

Here, we see Y_t as a perturbation of X_t, and we want to describe the solution to the perturbed equation (1.29) as a perturbation of the solution to the original one.

Proposition 1.92 (Variation formula)**.** *Let* X_t, Y_t *be two nonautonomous vector fields. Then, denoting* $P_{s,t} = \overrightarrow{\exp} \int_s^t X_\tau d\tau$*, one has*

$$\begin{aligned}\overrightarrow{\exp} \int_s^t (X_\tau + Y_\tau) d\tau &= \overrightarrow{\exp} \int_s^t X_\tau d\tau \odot \overrightarrow{\exp} \int_s^t \left(\overrightarrow{\exp} \int_t^\tau \mathrm{ad}X_\sigma d\sigma\right) Y_\tau d\tau \\ &= P_{s,t} \odot \overrightarrow{\exp} \int_s^t (P_{\tau,t})_* Y_\tau d\tau.\end{aligned} \tag{1.30}$$

The whole formalism of chronological calculus, and in particular the Variation formula (1.30), will be used in the following and in Chapter 2 to compute the differentials of the end-point map.

1.5.2 Regularity of the end-point map: proof of Theorem 1.72

This section is devoted to the proof of Theorem 1.72. The proof is divided into two steps: in the first step, we compute the differential of the end-point map at $u = 0$, while in a second moment we prove in fact formula (1.20).

Step 1. Using the formalism of chronological calculus, the end-point map can be written as an operator on $C^\infty(M)$ in the following way

$$E_{q_0}(v) = q_0 \odot \overrightarrow{\exp} \int_0^1 f_{v(t)} dt, \quad v \in \mathcal{U}_{q_0}.$$

Then, for any $a \in C^\infty(M)$, we can write the first order Volterra expansion of $E_{q_0}(v)$ evaluated at a, namely

$$E_{q_0}(v)(a) = q_0 \odot \left(id + \int_0^1 f_{v(t)} dt + R_2(v) \right)(a).$$

The convergence of the Volterra series (discussed in the previous section) provides, for every $a \in C^\infty(M)$, the following estimate

$$|q_0 \odot R_2(v)(a)| = O(\|v\|_{L^2}^2),$$

whenever v is sufficiently small. Therefore, we have

$$\left| \left(E_{q_0}(v) - q_0 - q_0 \odot \int_0^1 f_{v(t)} dt \right)(a) \right| = O(\|v\|_{L^2}^2).$$

Since $q_0 = E_{q_0}(0)$ and the map

$$v \mapsto q_0 \odot \int_0^1 f_{v(t)} dt = \int_0^1 f_{v(t)}(q_0) dt$$

is linear, we have the differentiability of the end-point map at 0. In particular,

$$d_0 E_{q_0}(v) = \int_0^1 f_{v(t)}(q_0) dt = \int_0^1 (P^0_{t,1})_* f_{v(t)}(q_1) dt,$$

where the last equality follows from $P^0_{t,1} = id$.

Step 2. To compute the Taylor expansion of the end-point map at a general $u \in \mathcal{U}_{q_0}$, let us consider the map

$$v \mapsto E_{q_0}(u+v) = q_0 \odot \overrightarrow{\exp} \int_0^1 f_{u(t)+v(t)} dt,$$

for small $v \in L^2$. Using the variation formula, we get

$$E_{q_0}(u+v) = q_0 \odot P^u_{0,1} \odot \overrightarrow{\exp} \int_0^1 (P^u_{t,1})_* f_{v(t)} dt.$$

We set $q_1 = E_{q_0}(u) = q_0 \odot P^u_{0,1}$, and

$$G^u_{q_1}(v) = q_1 \odot \overrightarrow{\exp} \int_0^1 (P^u_{t,1})_* f_{v(t)} dt. \tag{1.31}$$

We proceed in a similar way as in Step 1, but for the map $G^u_{q_1}$ to compute its differential at 0. Namely, we replace q_0 with q_1, and $f_{v(t)}$ with $(P^u_{t,1})_* f_{v(t)}$, obtaining

$$d_0 G^u_{q_1}(v) = \int_0^1 (P^u_{t,1})_* f_{v(t)}(q_1) dt.$$

As operators on smooth functions, we have

$$\left| \left(G^u_{q_1}(v) - G^u_{q_1}(0) - d_0 G^u_{q_1}(v) \right)(a) \right| = O(\|v\|^2_{L^2}),$$

for every $a \in C^\infty(M)$. The last identity, read in terms of the end-point map E_{q_0}, gives

$$d_u E_{q_0}(v) = d_0 G^u_{q_1}(v) = \int_0^1 (P^u_{t,1})_* f_{v(t)}(q_1) dt.$$

This complete the proof of Theorem 1.72.

Definition 1.93. The map $G^u_{q_1}$ defined in (1.31) is called variation map.

Remark 1.94. The definition of variation map is naturally suggested by the Variation formula applied to the end-point map. It is used to move local differential properties of the end-point to the origin. Whit similar computations and arguments, one can deduce the the higher order degree of the Taylor expansion of the end-point map. Actually, the higher order differentials have to be defined in the correct domain. We develop this theory in the next chapter of this thesis.

1.5.3 Lagrange multiplier rule and Pontryagin extremals

The aim of this section is to apply the result of Theorem 1.95 below to the case when $F = E_{q_0}$ is the end-point map based at q_0 and $\varphi = J$ is the sub-Riemannian energy (see Definition 1.98 below), in order to obtain necessary condition for an admissible curve to be a length minimizer.

Theorem 1.95 (Lagrange multipliers rule)**.** *Let $\mathcal{H}$ be an Hilbert space and let M be a smooth n-dimensional manifold. Let $\mathcal{U} \subset \mathcal{H}$ be an open subset of $\mathcal{H}$. Consider two smooth maps $\varphi : \mathcal{U} \to \mathbb{R}$ and $F : \mathcal{U} \to M$. Fix a point $q \in M$ and assume that $u \in \mathcal{U}$ is a solution to the minimization problem*

$$\min\{\varphi(v) \mid F(v) = q\} = \min \varphi\big|_{F^{-1}(q)}.$$

Then there exists a non-null $(\lambda, \nu) \in T^*_q M \times \mathbb{R}$, i.e. $(\lambda, \nu) \neq (0,0)$, such that

$$\lambda d_u F + \nu d_u \varphi = 0. \tag{1.32}$$

Remark 1.96. We point out that formula (1.32) means that for every $v \in \mathcal{H} = T_u\mathcal{U}$ one has

$$\langle \lambda, d_uF(v)\rangle + \nu d_u\varphi(v) = 0.$$

The compact notation in (1.32) will be used also in the sequel, with the same meaning.

Notice that, in the setting of Theorem 1.95, the pair (λ, ν) has to satisfy, up to a scalar multiplication, at least one of the following condition:

(N) $\nu = -1$. Then from (1.32) we deduce

$$\lambda d_uF = d_u\varphi; \tag{1.33}$$

(A) $\nu = 0$. Then $\lambda \neq 0$, and from (1.32) we deduce

$$\lambda d_uF = 0.$$

The case (N) is called normal, while the case (A) is called abnormal or singular.

Remark 1.97. Classically, Theorem 1.95 appears with $\mathcal{H} = \mathbb{R}^n = (x_1, \dots, x_n)$, $M = \mathbb{R}^m = (x_1, \dots, x_m)$, $u = \bar{x} = \operatorname{argmin}\{\varphi(x)|F(x) = 0\} \in \mathbb{R}^n$, and one adds the assumption $J_{\bar{x}}F \neq 0$, removing the possibility for $\bar{x}$ to be abnormal. In this case, combining equation (1.33) and the bound $F(\bar{x}) = 0$, one gets

$$\nabla(\varphi - \lambda \cdot F)(\bar{x}) = \nabla L(\bar{x}, \lambda) = 0.$$

The vector λ is called Lagrange multiplier and the function $L(x, \lambda) = \varphi(x) - \lambda \cdot F(x)$ is called Lagrangian.

Definition 1.98. We define the energy functional in the following way

$$J : L^2(I, \mathbb{R}^m) \to \mathbb{R}, \quad J(u) = \frac{1}{2}\int_0^1 |u(t)|^2 dt = \|u\|^2_{L^2}.$$

When u is the control of an admissible curve γ, then $J(u)$ is called the sub-Riemannian energy of γ. Sometimes we also write $J(\gamma)$ instead of $J(u)$.

Remark 1.99. While the length is invariant by reparameterization (see Lemma 1.49), J is not. Indeed consider, for every $\alpha > 0$, the reparameterized curve $\gamma_\alpha : [0, 1/\alpha] \to M$, $\gamma_\alpha(t) = \gamma(\alpha t)$. Using that $\dot{\gamma}_\alpha(t) = \alpha\dot{\gamma}(\alpha t)$, we have

$$J(\gamma_a) = \alpha J(\gamma)$$

Thus, if the final time is not fixed, the infimum of J, among admissible curves joining two fixed points, is always zero.

The following lemma relates minimizers of the energy with fixed final time with minimizers of the length.

Lemma 1.100. *Fix two points $q_0, q_1 \in M$. Let L and J be, respectively, the sub-Riemannian length and energy functionals. An admissible curve $\gamma : I \to M$ is a solution to the minimum problem*

$$\min\{J(u)|E_{q_0}(u) = q_1\},$$

if and only if it solves the minimum problem

$$\min\{L(u)|E_{q_0}(u) = q_1\},$$

and has constant speed.

Notice that in the previous Lemma we have fixed the final time to be 1, avoiding the problem appearing in Remark 1.99. Then, by the Lagrange multiplier rule one obtains the following result as a necessary condition for an admissible curve to be a length minimizer.

Proposition 1.101. *Assume that an admissible curve $\gamma : I \to M$ u is a length minimizer. Let u be the minimal control associated to γ. Then there exists $(\lambda, \nu) \in T_q^*M \times \mathbb{R}$ such that $(\lambda, \nu) \neq (0, 0)$, and*

$$\lambda d_u E_{q_0} + \nu d_u J = 0. \tag{1.34}$$

Remark 1.102. Since $J(v) = \frac{1}{2}\|v\|^2_{L^2}$, then $d_u J(v) = (u, v)_{L^2}$ and, identifying L^2 with its dual, we have $d_u J = u$.

Theorem 1.103 (Pontryagin maximum principle)**.** *Let $q_0 \in M$. Let u be the minimal control corresponding to a curve $\gamma : I \to M$ based at q_0, namely*

$$\dot\gamma(t) = f_{u(t)}(\gamma(t)) \text{ a.e. on } I, \quad \gamma(0) = q_0 \in M.$$

Suppose that γ is length-minimizing and parameterized with constant speed (or, equivalently, energy-minimizing). Finally, let $q_1 = E_{q_0}(u) = \gamma(1)$. Then there exists $\lambda_1 \in T^*_{q_1}M$ such that, defining $\lambda(t) = (P^u_{t,1})^*\lambda_1$, we have at least one of the following

(N) For every $i = 1, \dots, m$ we have

$$u_i(t) = \langle \lambda(t), f_i(\gamma(t))\rangle, \text{ for a.e. } t \in I, \tag{1.35}$$

and this occurs if and only if u satisfies (1.34) with $(\lambda, \nu) = (\lambda_1, -1)$, namely

$$\lambda_1 d_u E_{q_0} = u \tag{1.36}$$

(A) For every $i = 1, \dots, m$ we have

$$0 = \langle \lambda(t), f_i(\gamma(t)) \rangle, \ \textit{for a.e. } t \in I,$$

and this occurs if and only if u satisfies (1.34) *with* $(\lambda, \nu) = (\lambda_1, 0)$*, namely*

$$\lambda_1 d_u E_{q_0} = 0$$

Remark 1.104. Since $q_1 = \gamma(1)$, $\lambda_1 \in T^*_{q_1}M$ and $P^u_{s,t}$ is the flow of γ in M, then $\lambda(t) = (P^u_{t,1})^* \lambda_1$ is a lift of $\gamma(t)$, i.e.

$$\pi(\lambda(t)) = \gamma(t), \quad \forall t \in I.$$

Proof. Let us prove (N). The proof of (A) is analogous.

Assume that u satisfies (1.35) for some λ_1, and let us prove that the curve defined by $\lambda(t) = (P^u_{t,1})^* \lambda_1$ satisfies (1.36), which means that for every $v \in L^2(I, \mathbb{R}^m)$ we have

$$\langle \lambda_1, d_u E_{q_0}(v) \rangle = (u, v)_{L^2}. \tag{1.37}$$

Using (1.20), the left hand side in (1.37) reads

$$\begin{aligned}
\langle \lambda_1, d_u E_{q_0}(v) \rangle &= \int_0^1 \langle \lambda_1, \big((P^u_{t,1})_* f_{v(t)}\big)(q_1) \rangle dt = \int_0^1 \langle \lambda_1, (P^u_{t,1})_* \big(f_{v(t)}((P^u_{t,1})^{-1} q_1)\big) \rangle dt \\
&= \int_0^1 \langle (P^u_{t,1})^* \lambda_1, f_{v(t)}(\gamma(t)) \rangle dt = \int_0^1 \langle \lambda(t), f_{v(t)}(\gamma(t)) \rangle dt \\
&= \int_0^1 \sum_{i=1}^m \langle \lambda(t), f_i(\gamma(t)) \rangle v_i(t) dt.
\end{aligned}$$

Then (1.37) becomes

$$\int_0^1 \sum_{i=1}^m \langle \lambda(t), f_i(\gamma(t)) \rangle v_i(t) dt = \int_0^1 \sum_{i=1}^m u_i(t) v_i(t) dt.$$

Since v is arbitrary, this implies (1.35).

Conversely, let us assume there exists $\lambda_1 \in T_{q_1}M$ such that the curve defined by $\lambda(t) = (P^u_{t,1})^* \lambda_1$ satisfies (1.35). Then, following the above computations in the opposite direction, one obtains exactly (1.36). □

The Pontryagin maximum principle provides a first-order necessary condition for a curve γ to be length-minimizing, leading to the following definition

Definition 1.105. Let $\gamma : I \to M$ be a horizontal curve based at q_0 with corresponding control u. Assume that γ and u satisfy at least one condition between (N) and (A) of Theorem 1.103 . Then we give the following definitions

i) The curve $\lambda : I \to T^*M$ is called adjoint curve;

ii) The couple $(u(t), \lambda(t))$ is called sub-Riemannian extremal. When the extremal satisfies (N) it is called normal extremal. Otherwise, it is said to be an abnormal or a singular extremal.

Remark 1.106. We emphasize that a length minimizer is necessarily an extremal (normal, abnormal or both), as a consequence of the Lagrange multipliers rule. Conversely, there are many examples of extremals which are not length minimizers.

1.5.4 The Hamiltonian viewpoint

In this section, we use the Hamiltonian viewpoint of Pontryagin extremals to prove regularity of normal ones, and to give some geometric characterization of abnormal extremals.

Definition 1.107. Let $(M, \mathbf{U}, f)$ be a free sub-Riemannian manifold with generating family $\{f_1, \dots, f_m\}$. We define the Hamiltonian linear on fibers functions of M as

$$h_i : T^*M \to \mathbb{R}, \quad h_i(\lambda) = \langle \lambda, f_i(q) \rangle, \ q = \pi(\lambda), \qquad i = 1, \dots, m.$$

The associated vector fields $\vec{h}_i \in \mathrm{Vec}(T^*M)$ are called the Hamiltonian vector fields of M.

Using Definition 1.107 and the language of Section 1.1, Theorem 1.103 immediately rewrites itself in the following way.

Theorem 1.108. *Let $\gamma : [0, T] \to M$ be an admissible curve which is a length-minimizer, parameterized by constant speed. Let $\bar{u}$ be the corresponding minimal control. Then there exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}M$ such that*

$$\dot{\lambda}(t) = \sum_{i=1}^{m} \bar{u}_i(t) \vec{h}_i(\lambda(t)), \quad \text{for a.e. } t \in [0, T],$$

and one of the following conditions is satisfied:

(N) $h_i(\lambda(t)) = \bar{u}_i(t)$, *for* $i = 1, \dots, m$;

(A) $h_i(\lambda(t)) = 0$, *for* $i = 1, \dots, m$.

Moreover in case (A) one has $\lambda(t) \neq 0$ for all $t \in [0, T]$.

Notice that Theorem 1.108 says that normal and abnormal extremals appear as solution of a Hamiltonian system. Nevertheless, this Hamiltonian system is a priori nonautonomous and depends on the trajectory itself by the presence of the control $\bar{u}$ associated with the extremal trajectory.

Moreover, the actual formulation of Theorem 1.108 for the necessary optimality condition still does not clarify if the extremals depend on the choice of a generating family $\{f_1, \ldots, f_m\}$ of the sub-Riemannian structure.

The rest of the section is devoted to the geometric intrinsic description of normal and abnormal extremals, which relies upon the study of the intrinsic sub-Riemannian Hamiltonian

Definition 1.109. Let M be a sub-Riemannian manifold. The (intrinsic) sub-Riemannian Hamiltonian is the function on T^*M defined as follows

$$H : T^*M \to T^*M, \quad H(\lambda) = \max_{u \in \mathbf{U}_q} \left(\langle \lambda, f_u(q) \rangle - \frac{1}{2}|u|^2 \right), \quad q = \pi(\lambda). \tag{1.38}$$

Proposition 1.110. *The sub-Riemannian Hamiltonian H is smooth and quadratic on fibers. Moreover, for every generating family $\{f_1, \ldots, f_m\}$ of the sub-Riemannian structure, the sub-Riemannian Hamiltonian H is written as follows*

$$H(\lambda) = \frac{1}{2}\sum_{i=1}^{m} \langle \lambda, f_i(q) \rangle^2 = \frac{1}{2}\sum_{i=1}^{m} h_i(\lambda)^2, \quad \lambda \in T_q^*M, \ q = \pi(\lambda). \tag{1.39}$$

Proof. In terms of a generating family $\{f_1, \ldots, f_m\}$, the sub-Riemannian Hamiltonian (1.38) is written as follows

$$H(\lambda) = \max_{u \in \mathbb{R}^m} \left(\sum_{i=1}^{m} \langle \lambda, f_i(q) \rangle - \frac{1}{2}\sum_{i=1}^{m} |u_i|^2 \right) \tag{1.40}$$

Differentiating (1.40) with respect to u_i, one gets that the maximum in the right hand side is attained for $u_i = h_i(\lambda)$, from which formula (1.39) follows. The fact that H is smooth and quadratic on fibers then easily follows from (1.39). □

Remark 1.111. The sub-Riemannian Hamiltonian H is intrinsic in the sense that equivalent sub-Riemannian structures induce the same Hamiltonian on M.

Remark 1.112 (Differential of the sub-Riemannian Hamiltonian)*.* Let us denote $H_q = H|_{T_q^*M}$ the restriction on fibers of the sub-Riemannian Hamiltonian. For $\lambda \in T_q^*M$, the differential $d_\lambda H_q : T_q^*M \to \mathbb{R}$ is a linear form, hence it can be canonically identified with an element of T_qM. By (1.39), it follows that, for every generating family $\{f_1, \ldots, f_m\}$, this canonical identification is given by

$$d_\lambda H_q = \sum_{i=1}^{m} h_i(\lambda) f_i(q)$$

and, namely

$$d_\lambda H_q : T_q^*M \to \mathbb{R}, \quad d_\lambda H_q = \sum_{i=1}^m h_i(\lambda) h_i.$$

Normal extremals

In this paragraph we summarize the fundamental properties of normal extremals, that are smoothness and minimality.

Theorem 1.113. *A Lipschitz curve $\lambda : [0,T] \to T^*M$ is a normal extremal if and only if it is a solution of the Hamiltonian system*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)).$$

Moreover, given a normal extremal, the corresponding normal extremal trajectory $\lambda(t) = \pi(\lambda(t))$ is smooth and has constant speed satisfying

$$\frac{1}{2}\|\dot{\gamma}(t)\|^2 = H(\lambda(t)), \quad \forall t \in [0,T]$$

Remark 1.114. In canonical coordinates $\lambda = (p, x)$ in T^*M, H is quadratic with respect to p and

$$H(p,x) = \frac{1}{2}\sum_{i=1}^m \langle p, f_i(x)\rangle^2.$$

The Hamiltonian system associated with H, in these coordinates, is written as follows

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \displaystyle\sum_{i=1}^m \langle p, f_i(x)\rangle f_i(x), \\ \dot{p} = -\dfrac{\partial H}{\partial x} = \displaystyle\sum_{i=1}^m \langle p, f_i(x)\rangle f_i(x) \end{cases} \tag{1.41}$$

From here it is easy to see that if $\lambda(t) = (p(t), x(t))$ is a solution of (1.41), then also the rescaled extremal $\alpha\lambda(\alpha t) = \alpha(p(\alpha t), x(\alpha t))$ is a solution of the same Hamiltonian system, for every $\alpha > 0$.

Corollary 1.115. *A normal extremal trajectory is parameterized by constant speed. In particular it is arc length parameterized if and only if its extremal lift is contained in the level set $H^{-1}(1/2)$.*

Finally, a fundamental result about local optimality of normal trajectories is that small pieces of a normal trajectory are length minimizers. The proof of the following statement is really nontrivial.

Theorem 1.116. *Let $\gamma : I \to M$ be a sub-Riemannian normal extremal. Then, for every $t \in [0,1)$ there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

i) $\gamma|_{[\tau,\tau+\varepsilon]}$ *is a length minimizer;*

ii) $\gamma|_{[\tau,\tau+\varepsilon]}$ *is the unique length minimizer joining $\gamma(\tau)$ and $\gamma(\tau+\varepsilon)$, up to reparameterization.*

Abnormal extremals

The regularity problem of sub-Riemannian geodesics lies, as emphasized many times in this thesis, on the presence of abnormal minimizers, since it is not possible to determine a priori their regularity. In the next chapters, we provide new regularity results for abnormal extremals when M is a sub-Riemannian manifold of constant rank.

In this section, we provide a simple geometric characterization of abnormal extremals, always if M has constant rank m. In spite of 0 is never a regular point for the sub-Riemannian Hamiltonian H, in this case, thanks to the constant rank theorem, the set $H^{-1}(0)$ defined by (1.40) is a smooth submanifold of T^*M of codimension m.

Let us recall that σ denotes the canonical symplectic form of M (see paragraph 4 of Section 1.1 above).

Proposition 1.117. *Let H be the sub-Riemannian Hamiltonian associated with a sub-Riemannian structure of constant rank. Then a Lipschitz curve on $H^{-1}(0)$ is a characteristic curve for $\sigma|_{H^{-1}(0)}$ if and only if it is the reparameterization of an abnormal extremal.*

Remark 1.118. From Proposition 1.117 it follows that abnormal extremals do not depend on the sub-Riemannian metric, but only on the distribution. Indeed the set $H^{-1}(0)$ is characterized as the annihilator $\Delta^\perp$ of the distribution

$$H^{-1}(0) = \{\lambda \in T^*M \mid \langle \lambda, v\rangle = 0, \forall\, v \in \Delta_{\pi(\lambda)}\} = \Delta^\perp \subset T^*M.$$

Here the orthogonal is meant in the duality sense. Then, the notion of being abnormal is an intrinsic notion of the sub-Riemannian structure.

Chapter 2

Higher order necessary conditions for the minimality of abnormal curves

Let (M, Δ, g) be a sub-Riemannian manifold of constant rank d with a fixed generating family $f_1, \ldots, f_d$. We also fix a point $q_0 \in M$ and we denote the end-point map $E = E_{q_0}$.

In the end of the previous chapter, we give some necessary conditions of the first order for the minimality of abnormal admissible curves, also known as the Pontryagin maximum principle (Theorem 1.103).

In this section, we develop a deeper study of he end-point map, looking for higher order necessary conditions for the minimality. Here, for "higher order" we mean conditions which comes from the study of the n-th differential of the end-point map.

Actually, the theory is well-known till the second order and was initiated by Goh [11] and developed by Agrachev and Sachkov in [3], and the complete proof of the following theorem is given in [1].

Theorem 2.1. *Let M be a sub-Riemannian manifold and $f_1, \ldots, f_m$ be a generating family. Let u be an abnormal minimizer, let $q_1 = E(u)$, and let $\lambda_1 \in T^*_{q_1}M$ satisfy $\lambda_1 d_u E = 0$. Assume that $\operatorname{ind}^- \lambda_1 \mathcal{D}^2_u E < +\infty$. Then the following conditions are satisfied:*

i) $\langle \lambda(t), [f_i, f_j](\gamma(t)) \rangle = 0$, for all $t \in I$, and for every $i, j = 1, \ldots, d$.

ii) $\langle \lambda(t), [f_{u(t)}, [f_v, f_v]](\gamma(t)) \rangle \geq 0$, for a.e. $t \in I$, and for every $v \in \mathbb{R}^d$.

Here we are using the notations of Theorem 1.103: $\lambda(t) = (P^u_{t,1})^* \lambda_1$, for $t \in I$, and $\gamma(t) = \pi(\lambda(t))$ are respectively the extremal and the trajectory associated with the *control u and with the final covector λ_1.*

We clarify the notations used in Theorem 2.1. The map $\mathcal{D}^2_u E : \ker(d_u E) \to \operatorname{coker}(d_u E)$ is the second intrinsic differential, also called the intrinsic Hessian, of the end-point map at the point u, and $\lambda_1 \mathcal{D}^2_u E : \ker d_u E \to \mathbb{R}$ is a scalar quadratic defined via the natural action of λ_1 on $\operatorname{Im}(d_u E)^\perp$. The general definition of the intrinsic differentials of a smooth map $F : X \to \mathbb{R}^m$ from a Banach space X onto $\mathbb{R}^m$ is given in Definition 2.6 below. Instead, $\operatorname{ind}^- \lambda_1 \mathcal{D}^2_u E$ denotes the (negative) index of the end-point map at the point u.

Definition 2.2. Let $Q : V \to R$ be a quadratic form defined on a vector space V . The index (or negative index) of Q is the maximal dimension of a negative subspace of Q:

$$\operatorname{ind}^- Q = \sup\{\dim W \mid Q|_{W \setminus 0} < 0\}.$$

Recall that in the finite-dimensional case, the negative index coincides with the number of negative eigenvalues in the diagonal form of Q.

Remark 2.3. When the index of the end-point map is finite, an oppen mapping argument can be applied to the extended end-point map, similarly to the case of the Pontryagin maximum principle, providing the results of Theorem 2.1.

The statements (i) and (ii) of Theorem 2.1 are called, respectively, Goh conditions and Generalized Legendre conditions for abnormal minimizers. In particular, Goh conditions are necessary conditions for the minimality of the second order, in the sense that a study of the second differential of E involves the Lie brackets of length 2 of the generating vector fields $f_1, \dots, f_d$.

The goal of this chapter is to extend the Goh conditions of order 2 to any order $n \geq 3$ and to get necessary conditions involving brackets of n vector fields through the study of higher order differentials of the end-point map. In particular, our theory shows a clear connection between the geometry of Δ and the expansion of the end-point map: the commutators of length n of the generating vector fields appear in the n-th order term of the Taylor expansion of the end-point map.

Our studies are limited to the case of abnormal curves of corank 1, i.e. such that $\operatorname{Im}(d_u F)$ has codimension 1 in $T_{\gamma_u(1)} M$. Our main result is the following.

Theorem 2.4. *Let (M, Δ, g) be a sub-Riemannian manifold, $\gamma = \gamma_u \in AC(I; M)$ be a strictly singular length minimizing curve of corank 1, and assume that*

$$\mathcal{D}^h_u E = 0, \quad h = 2, \dots, n-1. \tag{2.1}$$

Then any adjoint curve $\lambda \in AC(I; T^*M)$ satisfies

$$\langle \lambda(t), [f_{j_n}, [\dots [f_{j_2}, f_{j_1}] \dots]](\gamma(t)) \rangle = 0,$$

for all $t \in I$ and for all $j_1, \dots, j_n = 1, \dots, d$.

We call the differentials $\mathcal{D}_u^h F$ appearing in (2.1) intrinsic differentials of E. Their definition is given in Definition 2.6 below.

As in the cases of the Pontryagin maximum principle and of the Goh conditions, the proof of Theorem 2.4 relies on an open mapping argument applied to the extended end-point map $F_J = (F, J) : X \to M \times \mathbb{R}$, where $J(u) = \frac{1}{2}\|u\|^2_{L^2(I;\mathbb{R}^d)}$ is the energy of $\gamma = \gamma_u$. Motivated by this application, in Section 2.2 we develop a theory about open mapping theorems of order n for functions $F : X \to \mathbb{R}^m$ between Banach spaces. In our opinion, this preliminary study is worth of interest on its own. It adapts in a geometrical perspective some ideas presented in [25].

2.1 Intrinsic differentials

Let X be a Banach space and $F \in C^\infty(X; \mathbb{R}^m)$, $m \in \mathbb{N}$, be a smooth map. For any $n \in \mathbb{N}$ we define the n-th differential of F at $0 \in X$ as the map $d_0^n F : X \to \mathbb{R}^m$

$$d_0^n F(v) = \frac{d^n}{dt^n} F(tv)\Big|_{t=0}, \quad v \in X.$$

With abuse of notation, the associated n-multilinear differential is the map $d_0^n F : X^n \to \mathbb{R}^m$ defined in one of the following two equivalent ways, for $v_1, \dots, v_n \in X$,

$$\begin{aligned} d_0^n F(v_1, \dots, v_n) &= \frac{\partial^n}{\partial t_1 \dots \partial t_n} F\Big(\sum_{h=1}^n t_h v_h\Big)\Big|_{t_1 = \dots = t_n = 0} \\ &= \frac{1}{n!} \frac{\partial^n}{\partial t_1 \dots \partial t_n} d_0^n F\Big(\sum_{h=1}^n t_h v_h\Big)\Big|_{t_1 = \dots = t_n = 0}. \end{aligned} \tag{2.2}$$

We have the identity $d_0^n F(v) = d_0^n F(v, \dots, v)$. The differential $d_0^n F$ is symmetric, in the sense that $d_0^n F(v_1, \dots, v_n) = d_0^n F(v_{\sigma_1}, \dots, v_{\sigma_n})$ for any permutation $\sigma \in S_n$. Here and hereafter, we use the notation $\sigma_i = \sigma(i)$ for a permutation σ and for $i = 1, \dots, n$.

A different n-multilinear differential for F at 0 is the map $D_0^n F : X^n \to \mathbb{R}^m$ defined by the formula

$$D_0^n F(v_1, \dots, v_n) = \frac{d^n}{dt^n} F\Big(\sum_{h=1}^n \frac{t^h v_h}{h!}\Big)\Big|_{t=0}, \quad v_1, \dots, v_n \in X. \tag{2.3}$$

The multilinear differential $D_0^n F$ is not symmetric.

The n-multilinear differentials $d_0^n F$ and $D_0^n F$ are related via the Faà di Bruno formula [15]. We denote by $\mathscr{I}_n$ the set of n-multi-indices, i.e.,

$$\mathscr{I}_n = \{\alpha \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$

where $\mathbb{N} = \{1, 2, \dots\}$ starts from 1. When the naturals start from 0 we use the notation $\mathscr{I}_n^0$. Also, for $d \in \mathbb{N}$ we use the notation $\mathscr{I}_{n,d}$ for the sets of n-multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathscr{I}_n$ with values in $\{1, \dots, d\}^n$. For $\alpha \in \mathscr{I}_n$, we use the standard notation $|\alpha| = \alpha_1 + \dots + \alpha_n$ for the length (or weight) of α and $\alpha! = \alpha_1! \dots \alpha_n!$ for its factorial.

Proposition 2.5 (Faà di Bruno). *For any $n \in \mathbb{N}$ and $v_1, \dots, v_n \in X$ we have*

$$D_0^n F(v_1, \dots, v_n) = \sum_{h=1}^{n} \sum_{\alpha \in \mathscr{I}_h, |\alpha| = n} \frac{n!}{h! \alpha!} d_0^h F(v_\alpha), \tag{2.4}$$

where, for $\alpha \in \mathscr{I}_h$, we set $v_\alpha = (v_{\alpha_1}, \dots, v_{\alpha_h}) \in X^h$.

The n-differential $D_0^n F$, $n \geq 2$, does not transform covariantly, in the sense that, for a generic diffeomorphism $P \in C^\infty(\mathbb{R}^m; \mathbb{R}^m)$ the differential $D_0^n(P \circ F)$ depends also on the derivatives of P of order 2 and higher. In order to have an "intrinsic" n-differential, we need to restrict $D_0^n F$ to a suitable domain and project it onto $\mathrm{coker}(d_0 F)$. We denote by $\mathrm{proj} : \mathbb{R}^m \to \mathrm{coker}(d_0 F)$ the standard projection (quotient map). We can fix coordinates in X and $\mathbb{R}^m$ in order to have the splittings

$$X = \ker(d_0 F) \oplus \mathbb{R}^{m-\ell}, \qquad \mathbb{R}^m = \mathbb{R}^\ell \oplus \mathrm{Im}(d_0 F), \tag{2.5}$$

where $\ell = \dim(\mathrm{coker}(d_0 F))$ and $d_0 F : \mathbb{R}^{m-\ell} \to \mathrm{Im}(d_0 F)$ is a linear isomorphism.

Definition 2.6 (Intrinsic n-differential). Let $F \in C^\infty(X; \mathbb{R}^m)$. By induction on $n \geq 2$, we define a domain $\mathrm{dom}(\mathcal{D}_0^n F) \subset X^{n-1}$ and a map $\mathcal{D}_0^n F : \mathrm{dom}(\mathcal{D}_0^n F) \to \mathrm{coker}(d_0 F)$, called *intrinsic n-differential* of F at 0, in the following way.

When $n = 2$ we let $\mathrm{dom}(\mathcal{D}_0^2 F) = \{v \in X \mid D_0 F(v) = 0\} = \ker(d_0 F) \subset X$ and we define $\mathcal{D}_0^2 F : \mathrm{dom}(\mathcal{D}_0^2 F) \to \mathrm{coker}(d_0 F)$

$$\mathcal{D}_0^2 F(v) = \mathrm{proj}(D_0^2 F(v, *)), \quad v \in \mathrm{dom}(\mathcal{D}_0^2 F). \tag{2.6}$$

Inductively, for $n > 2$ we set

$$\mathrm{dom}(\mathcal{D}_0^n F) = \left\{v \in \mathrm{dom}(\mathcal{D}_0^{n-1} F) \times X \mid D_0^{n-1} F(v) = 0\right\} \subset X^{n-1},$$

and we define $\mathcal{D}_0^n F : \mathrm{dom}(\mathcal{D}_0^n F) \to \mathrm{coker}(d_0 F)$ as

$$\mathcal{D}_0^n F(v) = \mathrm{proj}(D_0^n F(v, *)), \quad v \in \mathrm{dom}(\mathcal{D}_0^n F). \tag{2.7}$$

Remark 2.7. The definition of $\mathcal{D}_0^n F$ in (2.6) and (2.7) does not depend on the last argument $* \in X$ of $D_0^n F$. Indeed, by formula (2.4) with $v = (v_1, \ldots, v_n)$, so with $* = v_n$ in the notation above, we have

$$D_0^n F(v) = d_0 F(v_n) + \sum_{h=2}^{n} \sum_{\alpha \in \mathscr{I}_h, |\alpha| = n} \frac{n!}{h!\alpha!} d_0^h F(v_\alpha), \tag{2.8}$$

where v_α does not contain v_n when $|\alpha| = n$ and $h \geq 2$, and $\operatorname{proj}(d_0 F(v_n)) = 0$.

While $\operatorname{dom}(\mathcal{D}_0^2 F) = \ker(d_0 F)$ has finite codimension in X, this might not be the case when $n > 2$. In order to develop the theory within a consistent setting we need some additional assumption on F.

Proposition 2.8. *Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map. If $\mathcal{D}_0^h F = 0$ for all $2 \leq h < n$, with $n \geq 3$, then $\operatorname{dom}(\mathcal{D}_0^n F) \subset X^{n-1}$ is a nonempty affine bundle over $\ker(d_0 F)$ that is diffeomorphic to $\ker(d_0 F)^{n-1}$.*

Proof. The proof is by induction on $n \geq 3$. When $n = 3$ the domain of $\mathcal{D}_0^3 F$ is

$$\operatorname{dom}(\mathcal{D}_0^3 F) = \big\{ (v_1, v_2) \in \ker(d_0 F) \times X \mid D_0^2 F(v_1, v_2) = 0 \big\},$$

where, as in (2.8), $D_0^2 F(v_1, v_2) = d_0 F(v_2) + d_0^2 F(v_1, v_1)$.

We use the splittings (2.5). Since the map $d_0 F : \mathbb{R}^{m-\ell} \to \operatorname{Im}(d_0 F)$ is invertible, we can define $\varphi \in C^\infty(\ker(d_0 F), \mathbb{R}^{m-\ell})$ letting

$$\varphi(v_1) = -(d_0 F)^{-1}(d_0^2 F(v_1, v_1)).$$

This is well-defined because, by assumption, we have $\mathcal{D}_0^2 F = 0$ and this implies $\operatorname{proj}(d_0^2 F(v_1, v_1)) = 0$. Now, letting

$$\Phi(v_1, v_2) = (v_1, v_2 + \varphi(v_1)),$$

we obtain a diffeomorphism $\Phi : \ker(d_0 F)^2 \to \operatorname{dom}(\mathcal{D}_0^3 F)$.

Suppose the theorem is true for n and let us prove it for $n+1$. Our inductive assumption is the existence of a diffeomorphism $\Phi \in C^\infty(\ker(d_0 F)^{n-1}, \operatorname{dom}(\mathcal{D}_0^n F))$. Now we have

$$\begin{aligned} \operatorname{dom}(\mathcal{D}_0^{n+1} F) &= \{ (v, w) \in \operatorname{dom}(\mathcal{D}_0^n F) \times X \mid D_0^n F(v, w) = 0 \} \\ &= \big\{ (\Phi(u), w) \in X^n \mid u \in \ker(d_0 F)^{n-1},\ D_0^n F(\Phi(u), w) = 0 \big\}, \end{aligned}$$

and, by (2.8), equation $D_0^n F(\Phi(u), w) = 0$ reads

$$w = \psi(u) = -(d_0 F)^{-1} \sum_{h=2}^{n} \sum_{\alpha \in \mathscr{I}_h, |\alpha| = n} \frac{n!}{h!\alpha!} d_0^h F(\Phi(u)_\alpha).$$

The function ψ is well-defined because $\mathcal{D}_0^n F = 0$. Now $\Psi(u, z) = (u, z + \psi(u))$ is a diffeomorphism from $\ker(d_0 F)^n$ to $\mathrm{dom}(\mathcal{D}_0^{n+1} F)$.

$\square$

The following theorem guaranties that for any finite set of elements in $\ker(d_0 F)$ there exists an extension with "polynomial coefficients" inside $\mathrm{dom}(\mathcal{D}_0^n F)$ that is linear in the first factor $\ker(d_0 F)$.

Proposition 2.9. *Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map such that $\mathcal{D}_0^h F = 0$ for all $2 \le h < n$, for some $n \ge 2$. For any $v_1^1, \ldots, v_1^\ell \in \ker(d_0 F) \subset X$, $\ell \in \mathbb{N}$, there exist vectors $v_j^\beta \in X$, $j = 1, \ldots, n-1$ and $\beta \in \mathcal{I}_\ell^0$ with $|\beta| = j$, such that the function $w \in C^\infty(\mathbb{R}^\ell; X^{n-1})$ with coordinates*

$$w_j(t) = \sum_{\beta \in \mathscr{I}_\ell^0, |\beta| = j} t^\beta v_j^\beta, \quad j = 1, \ldots, n-1, \tag{2.9}$$

satisfies $w(t) \in \mathrm{dom}(\mathcal{D}_0^n F)$ for all $t \in \mathbb{R}^\ell$. In particular, when e^i is the i-th vector of the standard basis of $\mathbb{R}^\ell$ we have $v_1^{\mathrm{e}^i} = v_1^i$.

Proof. The proof is by induction on $n \ge 2$.

For $n = 2$ the statement follows from the fact that $\mathrm{dom}(\mathcal{D}_0^2 F) = \ker(d_0 F)$ is a vector space. In this case, we have $j = 1$ and $\beta = \mathrm{e}^i$ for some i. Fixing $v_1^\beta = v_1^i$ with $v_1 = (v_1^1, \ldots, v_1^\ell)$, formula (2.9) gives a function w_1 with values in $\mathrm{dom}(\mathcal{D}_0^2 F)$.

We assume the claim for $n-1$ and we prove it for n. In particular, for $j \le n-2$, the vectors $v_j^\beta \in X$ are already fixed so that the functions defined in (2.9) with $j \le n-2$ satisfy $(w_1(t), \ldots, w_{n-2}(t)) \in \mathrm{dom}(\mathcal{D}_0^{n-1} F)$ for all $t \in \mathbb{R}^\ell$. Our goal is to find v_{n-1}^β, for $\beta \in \mathscr{I}_\ell^0$ with $|\beta| = n-1$, such that $w(t) = (w_1(t), \ldots, w_{n-1}(t)) \in \mathrm{dom}(\mathcal{D}_0^n F)$.

The condition $w(t) \in \mathrm{dom}(\mathcal{D}_0^n F)$ is equivalent to

1) $(w_1(t), \ldots, w_{n-2}(t)) \in \mathrm{dom}(\mathcal{D}_0^{n-1} F)$;

2) $D_0^{n-1} F(w(t)) = 0$.

The first condition is true by induction. By formula (2.8), the latter is equivalent to

$$d_0 F(w_{n-1}(t)) + \sum_{h=2}^{n-1} \sum_{\alpha \in \mathscr{I}_h, |\alpha| = n-1} \frac{(n-1)!}{h! \alpha!} d_0^h F(w_\alpha(t)) = 0. \tag{2.10}$$

We solve this equation to determine the vectors $v_{n-1}^\beta \in X$. By linearity, we have

$$d_0 F(w_{n-1}(t)) = \sum_{\beta \in \mathscr{I}_\ell^0, |\beta| = n-1} t^\beta d_0 F(v_{n-1}^\beta),$$

and

$$\begin{aligned}
d_0^h F(w_\alpha(t)) &= d_0^h F(w_{\alpha_1}(t), \ldots, w_{\alpha_1}(t)) \\
&= \sum_{\beta^1 \in \mathscr{I}_\ell^0, |\beta^1| = \alpha_1} \cdots \sum_{\beta^h \in \mathscr{I}_\ell^0, |\beta^h| = \alpha_h} t^{\beta^1 + \cdots + \beta^h} d_0^h F(v_{\alpha_1}^{\beta^1}, \ldots, v_{\alpha_h}^{\beta^h}).
\end{aligned}$$

By the identity principle of polynomials, solving equation (2.10) is equivalent to solving the set of equations

$$d_0 F(v_{n-1}^\beta) + \sum_{h=2}^{n-1} \sum_{\substack{\alpha \in \mathscr{I}_h, |\alpha| = n-1 \\ \beta^1 + \cdots + \beta^h = \beta}} \frac{(n-1)!}{h!\alpha!} d_0^h F(v_{\alpha_1}^{\beta^1}, \ldots, v_{\alpha_h}^{\beta^h}) = 0, \tag{2.11}$$

for $\beta \in \mathcal{I}_\ell^0$ with $|\beta| = n-1$. This is possible since, by assumption, we have $\mathcal{D}_0^h F = 0$ for $1 \le h \le n-1$. This implies that $\mathrm{proj}(d_0^h F(v_{\alpha_1}^{\beta^1}, \ldots, v_{\alpha_h}^{\beta^h})) = 0$, i.e., $d_0^h F(v_{\alpha_1}^{\beta^1}, \ldots, v_{\alpha_h}^{\beta^h}) \in \mathrm{Im}(d_0 F)$ and thus we can find $v_{n-1}^\beta \in X$ solving equation (2.11). $\square$

As soon as the splitting $X = \ker(d_0 F) \oplus X/\ker(d_0 F)$ is fixed, the differential $d_0 F : X/\ker(d_0 F) \to \mathrm{Im}(d_0 F)$ is a linear isomorphism. So the solutions v_{n-1}^β of equations (2.11) are unique in $X/\ker(d_0 F)$. In this sense, Proposition 2.9 gives a unique extension of the form (2.9) for vectors in $\ker(d_0 F)$.

For every $t \in \mathbb{R}^\ell$ with $t_i \neq 0$ we set $\mathrm{sgn}(t) = \big(\mathrm{sgn}(t_1), \ldots, \mathrm{sgn}(t_\ell)\big)$. We call orthant each subset of $\mathbb{R}^\ell$ where $\mathrm{sgn}(t)$ is constant. There are 2^ℓ orthants. Given 2ℓ elements $v_1^{1,\pm}, \ldots, v_1^{\ell,\pm} \in \ker(d_0 F)$, Proposition 2.9 gives an extension $w^{\mathrm{sgn}(t)} \in \mathrm{dom}(\mathcal{D}_0^n F)$ of $v_1^{1,\mathrm{sgn}(t_1)}, \ldots, v_1^{\ell,\mathrm{sgn}(t_\ell)}$ that in each orthant has coordinates

$$w_j^{\mathrm{sgn}(t)}(t) = \sum_{\beta \in \mathscr{I}_\ell^0, |\beta| = j} t^\beta v_j^{\beta,\mathrm{sgn}(t)}, \quad j = 1, \ldots, n-1. \tag{2.12}$$

The vectors $v_j^{\beta,\mathrm{sgn}(t)}$ are the solutions to (2.11) in the orthant of t.

Corollary 2.10. *Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map such that $\mathcal{D}_0^h F = 0$ for all $2 \le h < n$. For any 2ℓ elements $v_1^{1,\pm}, \ldots, v_1^{\ell,\pm} \in \ker(d_0 F)$, the function $w^{sgn(t)}$ defined for $t_i \neq 0$ in (2.12) extends to a continuous function $w \in C(\mathbb{R}^\ell; \mathrm{dom}(\mathcal{D}_0^n F))$.*

Proof. We split the set $\{1, \ldots, \ell\}$ into two subsets $1 \le i_1 < \cdots < i_k \le \ell$ and $1 \le j_1 < \ldots < j_{\ell-k} \le \ell$, for some $k \le \ell$. We consider the subspace $H = \{t \in \mathbb{R}^\ell \mid t_{i_1} = \cdots = t_{i_k} = 0\}$ and define the set of multi-indices

$$\mathcal{I}_{\ell,H}^0 = \{\beta \in \mathcal{I}_\ell^0 \mid \beta_{i_1} = \ldots = \beta_{i_k} = 0\}.$$

Let $\bar{t} \in H$ be such that $\bar{t}_{j_1} \cdots \bar{t}_{j_{\ell-k}} \neq 0$. We prove by induction on $j = 1, \ldots, n-1$ that for $\beta \in \mathscr{I}_{\ell,H}^0$ the vectors $v_j^{\beta,\mathrm{sgn}(t)}$ are constant when t is close to $\bar{t}$. When $j = 1$

we have $\beta = \mathrm{e}^{j_p}$ and we can use the expression for $v_1^{\beta,\mathrm{sgn}(t)}$ given by Proposition 2.9 to obtain $v_1^{\beta,\mathrm{sgn}(t)} = v_1^{\mathrm{e}^{j_p},\mathrm{sgn}(t_{j_p})} = v_1^{j_p,\mathrm{sgn}(\bar t_{j_p})}$, for t close to $\bar t$.

We check the inductive step. The vectors $v_{n-1}^{\beta,\mathrm{sgn}(t)}$ are the solutions to the equation

$$d_0F(v_{n-1}^{\beta,\mathrm{sgn}(t)}) + \sum_{h=2}^{n-1} \sum_{\substack{\alpha\in\mathscr{I}_h,|\alpha|=n-1\\ \beta^1+\cdots+\beta^h=\beta}} \frac{(n-1)!}{h!\alpha!} d_0^h F(v_{\alpha_1}^{\beta^1,\mathrm{sgn}(t)}, \dots, v_{\alpha_h}^{\beta^h,\mathrm{sgn}(t)}) = 0,$$

where the vectors $v_{\alpha_1}^{\beta^1,\mathrm{sgn}(t)}, \dots, v_{\alpha_h}^{\beta^h,\mathrm{sgn}(t)}$ are constant for t close to $\bar t$, by inductive assumption. In fact, $\beta^1, \dots, \beta^h \in \mathscr{I}^0_{\ell,H}$ if $\beta \in \mathscr{I}^0_{\ell,H}$.

Now the existence of the following limit concludes the proof:

$$\lim_{t\to\bar t} w_j^{\mathrm{sgn}(t)}(t) = \lim_{t\to\bar t} \sum_{\beta\in\mathscr{I}^0_{\ell,H},|\beta|=j} v_j^{\beta,\mathrm{sgn}(t)}, \quad j = 1,\dots,n-1.$$

□

Definition 2.11 (Regular extension)**.** We call the function $w \in C(\mathbb{R}^\ell; \mathrm{dom}(\mathcal{D}_0^n F))$ of Corollary 2.10 the *regular extension* of $v^{1,\pm}, \dots, v^{\ell,\pm} \in \ker(d_0F)$ to $\mathrm{dom}(\mathcal{D}_0^n F)$.

We will use this extension to define the notion of regular n-differential.

2.2 Open mapping theorem of order n

In this section, we prove an open mapping theorem with sufficient conditions on the differentials of the map up to order $n \in \mathbb{N}$.

Definition 2.12. Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map. We say that $0 \in X$ is a critical point of F with corank $\ell \in \{1, \dots, m\}$ if $\dim(\mathrm{coker}(d_0F)) = \ell$.

Definition 2.13 (Regular n-differential)**.** Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map such that $0 \in X$ is a critical point of corank $\ell \in \{1, \dots, m\}$ and $\mathcal{D}_0^h F = 0$ for all $2 \le h < n$, for some $n \ge 2$. We say that $\mathcal{D}_0^n F : \mathrm{dom}(\mathcal{D}_0^n F) \to \mathrm{coker}(d_0F)$ is regular if:

i) n is even and there exist 2ℓ elements $v^{1,\pm}, \dots, v^{\ell,\pm} \in \ker(d_0F)$ such that the function $f : \mathbb{R}^\ell \to \mathrm{coker}(d_0F)$

$$f(t) = \mathcal{D}_0^n F(w^{\mathrm{sgn}(t)}(\varrho(t))), \quad t \in \mathbb{R}^\ell, \tag{2.13}$$

is a bijection with bounded inverse at zero, i.e., there exists $0 < L < \infty$ such that

$$|f^{-1}(\tau)| \le L|\tau|, \quad \tau \in \mathrm{coker}(d_0F). \tag{2.14}$$

Above, $\varrho(t) = \big(\mathrm{sgn}(t_1)|t_1|^{1/n}, \dots, \mathrm{sgn}(t_\ell)|t_\ell|^{1/n}\big)$ and, for $t \in \mathbb{R}^\ell$, $w^{\mathrm{sgn}(t)}$ is the regular extension of $v^{1,\mathrm{sgn}(t_1)}, \dots, v^{\ell,\mathrm{sgn}(t_\ell)}$.

ii) n is odd and there exist $v^1, \dots, v^\ell \in \ker(d_0 F)$ such that the function f in (2.13) with $v^{i,\pm} = v^i$ is a bijection with bounded inverse at zero.

When the corank is $\ell = 1$ the notion of regular n-differential is effective.

Proposition 2.14. *Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map such that $0 \in X$ is a critical point of corank $\ell = 1$ and $\mathcal{D}_0^h F = 0$ for all $2 \leq h < n$, for some $n \geq 2$. Assume that:*

i) n is even and there exist 2 elements $v^\pm \in \ker(d_0 F)$ and $t \in \mathbb{R}$ such that $f(t)$ and $f(-t)$ have opposite sign; or,

ii) n is odd and there exists $v \in \ker(d_0 F)$ and $t \in \mathbb{R}$ such that $f(t) \neq 0$.

Then $\mathcal{D}_0^n F$ is regular. Above f is the function defined in (2.13).

Proof. When $\ell = 1$ and n is even, the function f in (2.13) is piece-wise linear. The assumption $f(t)f(-t) < 0$, for some $t \in \mathbb{R}$, makes it injective from $\mathbb{R}$ to $\mathbb{R}$, and hence with bounded inverse at zero.

When n is odd, the function f in (2.13) is linear and $f(t) \neq 0$ for some $t \in \mathbb{R}$ makes it invertible.

□

Theorem 2.15. *Let $F \in C^\infty(X; \mathbb{R}^m)$ be a smooth map such that $\mathcal{D}_0^h F = 0$ for all $2 \leq h < n$, for some $n \geq 2$, and with regular $\mathcal{D}_0^n F$ at the critical point $0 \in X$. Then F is open at* 0.

Proof. We prove the theorem when n is even. Let $0 \in X$ be a critical point for F of corank $\ell \in \{1, \dots, m\}$. We identify $\mathrm{coker}(d_0 F) = \mathbb{R}^\ell$ and we split $X = \mathbb{R}^{\ell - m} \oplus \ker(d_0 F)$.

The regularity of $\mathcal{D}_0^n F$ means that there exist $v^{1,\pm}, \dots, v^{\ell,\pm} \in \mathrm{dom}(\mathcal{D}_0^n F)$, with regular extension $w \in C(\mathbb{R}^\ell; \mathrm{dom}(\mathcal{D}_0^n F))$ as in Definition 2.11, such that the function $f : \mathbb{R}^\ell \to \mathbb{R}^\ell$ in (2.13) is bijective and satisfies (2.14).

By formula (2.9), the extension $w = (w_1, \dots, w_{n-1})$ is of the form

$$w_j(t) = \sum_{\beta \in \mathscr{I}_\ell^0, |\beta| = j} t^\beta v_j^{\beta, \mathrm{sgn}(t)}, \quad j = 1, \dots, n-1,$$

where the vectors $v_j^{\beta,\mathrm{sgn}(t)} \in X$ are fixed in the proof of Proposition 2.9.

We define the map $\Phi : \mathbb{R}^{m-\ell} \times \mathbb{R}^{\ell} \to X$ by

$$\Phi(r,t) = r + \sum_{j=1}^{n-1} \frac{w_j(t)}{j!}, \quad (r,t) \in \mathbb{R}^{m-\ell} \times \mathbb{R}^{\ell}.$$

Above and hereafter, we identify $r \in \mathbb{R}^{\ell-m}$ with $(r,0) \in X$, so that the sum $r+v$ with $v \in X$ is well defined. We claim that we have the expansion

$$F\left(\Phi(r,t)\right) = d_0F(r) + D_0^n F(w(t),0) + R(r,t), \tag{2.15}$$

where the remainder satisfies

$$\lim_{(r,t)\to 0} \frac{R(r,t)}{|r| + |t|^n} = 0. \tag{2.16}$$

For any positive scalar $s \geq 0$, using the homogeneity $w_j(st) = s^j w(t)$ we obtain the formula

$$\Phi(0,st) = \sum_{j=1}^{n-1} \frac{s^j}{j!} w_j(t),$$

and, for fixed t, the function $\varphi(s) = F(\Phi(0,st))$ has the Taylor expansion

$$\varphi(s) = \sum_{j=1}^{n} \frac{\varphi^{(j)}(0)}{j!} s^j + \frac{\varphi^{(n+1)}(\bar{s})}{(n+1)!} s^{n+1}, \quad s \in [.0,1], \tag{2.17}$$

for some $\bar{s} \in [0,s]$.

By definition (2.3), we have $\varphi^{(j)}(0) = D_0^j F(w_1(t),\dots,w_{j-1}(t))$ and since $w(t) \in \mathrm{dom}(\mathcal{D}_0^n F)$, we deduce that $\varphi^{(j)}(0) = 0$ for $j = 1,\dots,n-1$, while for $j = n$ we have

$$\varphi^{(n)}(0) = D_0^n F(w_1(t),\dots,w_{n-1}(t),0) = D_0^n F(w(t),0).$$

From the Taylor expansion (2.17) with $s = 1$, we obtain

$$F(\Phi(0,t)) = D_0^n F(w(t),0) + E(t), \quad t \in \mathbb{R}^{\ell}$$

where the error satisfies the estimate

$$|E(t)| = \left| \frac{\varphi^{(n+1)}(\bar{s})}{(n+1)!} \right| \leq C|t|^{n+1}, \tag{2.18}$$

for some constant $C > 0$.

Now, we obtain the expansion (2.15) adding a development in r of the first order. We are left with the proof of (2.16). Also by (2.18), the error $R(r,t)$ is estimated by a sum of the form

$$|R(r,t)| \leq \sum_{0\leq i\leq 2, 0\leq j\leq n+1} c_{ij} |r|^i |t|^j$$

with constants c_{ij} such that $c_{0j} = 0$ if $j \le n$ and $c_{10} = 0$. So we have $|R(r,t)| \le C(|r|^2 + |r||t| + |t|^{n+1})$ and the mixed term is estimated by Young inequality:

$$|r||t| \le \frac{n}{n+1}|r|^{(n+1)/n} + \frac{1}{n+1}|t|^{n+1}.$$

This finishes the proof of (2.16).

The map F is open at 0 if the map $F \circ \Phi$ is open at 0. And $F \circ \Phi$ is open at 0 if and only if the map

$$(r,t) \mapsto \Psi(r,t) = F(\Phi(r,\varrho(t))) = d_0 F(r) + D_0^n F(w(\varrho(t)),0) + R(r,\varrho(t))$$

is open at $(r,t) = 0$. We will show that Ψ is open at zero by a fixed point argument.

With respect to the factorization $(r,t) \in \mathbb{R}^{m-\ell} \times \mathbb{R}^\ell$, we introduce the norm $\|(r,t)\| = \max\{|r|, \lambda_0|t|\}$ and the balls

$$B_\delta = \{(r,t) \in \mathbb{R}^{m-\ell} \times \mathbb{R}^\ell : \|(r,t)\| \le \delta\}$$

for positive $\delta > 0$. The balls B_δ are compact and convex. The parameter $\lambda_0 > 0$ will be fixed later.

The map Ψ is open at 0 if for any (small) $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\delta \subset \Psi(B_\varepsilon)$. We pick $(\xi,\tau) \in B_\delta$ and we look for $(r,t) \in B_\varepsilon$ such that $\Psi(r,t) = (\xi,\tau)$. We factorize

$$D_0^n F(w(\varrho(t)),0) = (\mathcal{D}_0^n F(w(\varrho(t))), g(t)) = (f(t), g(t)),$$

and $R(r,\varrho(t)) = (R_1(r,t), R_2(r,t)) \in \mathbb{R}^{m-\ell} \times \mathbb{R}^\ell$. Here, with a slight abuse of notation, we are incorporating ϱ into R_1 and R_2.

Since g is continuous and homogeneous of degree 1, there exists a constant $C_1 > 0$ such that

$$|g(t)| \le C_1|t|. \tag{2.19}$$

By (2.16), for any $0 < \sigma \le 1$ there exists an $\varepsilon > 0$ such that for $|r| + |t| \le \varepsilon$ (in particular for $(r,t) \in B_\varepsilon$) we have

$$|R_1(r,t)| + |R_2(r,t)| \le \sigma(|r| + |t|). \tag{2.20}$$

We will fix σ in a while.

The equation $\Psi(r,t) = (\xi,\tau)$ is then equivalent to the system

$$\begin{cases} d_0 F(r) + g(t) + R_1(r,t) = \xi \\ f(t) + R_2(r,t) = \tau, \end{cases} \tag{2.21}$$

that reads as the following fixed-point system

$$\begin{cases} r = d_0F^{-1}\big(\xi - g(t) - R_1(r,t)\big) = h_1(r,t) \\ t = f^{-1}\big(\tau - R_2(r,t)\big) = h_2(r,t). \end{cases}$$

We claim that the map $h = (h_1, h_2)$ maps B_ε into itself, provided that $\delta > 0$, $\sigma > 0$, and $\lambda_0 > 0$ are small enough. Indeed, by (2.19) and (2.20) we have

$$\begin{aligned} |h_1(r,t)| &\le \|d_0F^{-1}\|(|\xi| + |g(t))| + |R_1(r,t)|) \\ &\le C_2(|\xi| + |t| + \sigma(|r| + |t|)) \\ &\le C_2(\delta + 2\lambda_0\varepsilon + \sigma\varepsilon). \end{aligned}$$

Choosing $\delta \le \varepsilon/3C_2$, $\lambda_0 = 1/6C_2$, and $\sigma \le 1/3C_2$ we obtain $|h_1(r,t)| \le \varepsilon$.

On the other hand, by (2.14) and (2.20)

$$\begin{aligned} |h_2(r,t)| &\le L(|\tau| + |R_2(r,t)|) \\ &\le L(|\tau| + \sigma(|r| + |t|)) \\ &\le L(\lambda_0\delta + 2\sigma\varepsilon). \end{aligned}$$

Choosing $\delta \le \varepsilon/2L$ and $\sigma \le \lambda_0/4L$ we obtain $|h_2(r,t)| \le \lambda_0\varepsilon$. This finishes the proof that for each $(\xi,\tau) \in B_\delta$ there exists $(r,t) \in B_\varepsilon$ solving the system (2.21).

□

When the corank is $\ell = 1$, by Proposition 2.13 we have the following:

Corollary 2.16. *Let $F \in C^\infty(X;\mathbb{R}^m)$ be a smooth map such that $0 \in X$ is a critical point of corank $\ell = 1$ and $\mathcal{D}_0^hF = 0$ for all $2 \le h < n$, for some $n \ge 2$. Assume that:*

i) n is even and there exist 2 elements $v^\pm \in \mathrm{dom}(\mathcal{D}_0^nF)$ such that $\mathcal{D}_0^nD(v^-)$ and $\mathcal{D}_0^n(v^+)$ have opposite sign; or,

ii) n is odd and there exists $v \in \mathrm{dom}(\mathcal{D}_0^nF)$ such that $\mathcal{D}_0^nF(v) \neq 0$.

Then F is open at 0.

2.3 Integrals on simplexes

In this section, we prove some elementary properties of integrals on simplexes that will be used in the analysis of the end-point map. Here and hereafter, $I = [0,1]$ denotes the unit interval. We fix $d \in \mathbb{N}$ (it will be the rank of the distribution of vector fields on the manifold) and in the rest of the Chapter we let

$$X = L^2(I;\mathbb{R}^d).$$

The tensor product $\otimes : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^{\ell m}$ is defined by

$$(v \otimes w)^k = v^i w^j, \quad k = m(i-1) + j,$$

where $1 \le i \le \ell$ and $1 \le j \le m$. Above, we are using the notation $v = (v^1, \dots, v^\ell) \in \mathbb{R}^\ell$, etc. The map $\otimes$ is associative but not commutative.

Definition 2.17. For $n \in \mathbb{N}$ and $t, s \in I$ such that $t + s \le 1$, we define the n-dimensional simplex

$$\Sigma_n(t,s) = \big\{(t_1, \dots, t_n) \in I^n \mid t < t_n < \cdots < t_1 < t + s\big\}.$$

When $t = 0$ and $s = 1$ we use the short notation $\Sigma_n = \Sigma_n(0,1)$. We also let

$$\Sigma_n^\flat(t,s) = \big\{(t_1, \dots, t_n) \in I^n \mid t < t_1 < \cdots < t_n < t + s\big\},$$

and $\Sigma_n^\flat = \Sigma_n^\flat(0,1)$.

For $n \in \mathbb{N}$ we define the subset of X

$$U_n = \left\{ v \in X \mid \int_{\Sigma_n} v(t_n) \otimes \cdots \otimes v(t_1) d\mathcal{L}^n = 0 \right\}, \tag{2.22}$$

Here and in the following, we denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$. We also set

$$V_n = \bigcap_{i=1}^n U_i. \tag{2.23}$$

For any multi-index $\alpha \in \mathcal{I}_{n,d} = \big\{\alpha \in \mathcal{I}_n \mid \alpha_i \in \{1, \dots, d\}\big\}$ and $v \in X$, we define the integral

$$I_n^\alpha(v) = \int_{\Sigma_n} v^{\alpha_n}(t_n) \dots v^{\alpha_1}(t_1) d\mathcal{L}^n.$$

Then $v \in U_n$ if and only if $I_n^\alpha(v) = 0$ for all $\alpha \in \mathcal{I}_{n,d}$.

For $v \in X$, $n \in \mathbb{N}$ and $t, s \in I$ such that $t + s \le 1$, we let

$$I_n(t,s;v) = \int_{\Sigma_n(t,s)} v(t_n) \otimes \cdots \otimes v(t_1) d\mathcal{L}^n,$$
$$I_n^\flat(t,s;v) = \int_{\Sigma_n^\flat(t,s)} v(t_n) \otimes \cdots \otimes v(t_1) d\mathcal{L}^n.$$

Lemma 2.18. *For any $v \in V_n$ and $t \in I$ we have*

$$I_n^\flat(0,t;v) = (-1)^n I_n(t, 1-t; v). \tag{2.24}$$

Proof. The proof is by induction on $n \in \mathbb{N}$. When $n = 1$ the claim reads

$$\int_0^t v(s)ds = -\int_t^1 v(s)ds, \quad t \in I,$$

that holds true because $v \in V_1$ means $\displaystyle\int_0^1 v(s)ds = 0$.

We assume that formula (2.24) holds for $n-1$ and we prove it for n. Indeed, using first $v \in U_n$ and then $v \in V_{n-1}$ we get

$$\begin{aligned}
I_n^\flat(0,t;v) &= \int_0^t v(t_n) \otimes I_{n-1}^\flat(0,t_n;v)dt_n \\
&= -\int_t^1 v(t_n) \otimes I_{n-1}^\flat(0,t_n;v)dt_n \\
&= (-1)^n \int_t^1 v(t_n) \otimes I_{n-1}(t_n, 1-t_n;v)dt_n \\
&= (-1)^n I_n(t, 1-t; v).
\end{aligned}$$

□

The reverse parameterization of a function $v \in X$ is the function $v^\flat \in X$ defined by the formula

$$v^\flat(t) = v(1-t), \quad t \in I.$$

Corollary 2.19. *Let $v \in X$. Then $v \in V_n$ if and only if $v^\flat \in V_n$.*

Proof. If $v \in V_n$, by Lemma 2.18 with $t = 0$ it follows that $v^\flat \in V_n$. The opposite implication follows from $v^{\flat\flat} = v$. □

The set V_n is stable also with respect to localization. Given $v \in X$, $s > 0$ and $t_0 \in I$ such that $t_0 + s \leq 1$, we define

$$v_{t_0,s}(t) = v\left(\frac{t-t_0}{s}\right)\chi_{[t_0,t_0+s]}(t), \quad t \in I. \tag{2.25}$$

Lemma 2.20. *If $v \in V_n$ then $v_{t_0,s} \in V_n$ for all $s > 0$ and $t \in I$ such that $t_0 + s \leq 1$.*

Proof. The claim $v_{t_0,s} \in X$ is clear. We prove that, for every $1 \leq i \leq n$,

$$I_i(t_0, s; v_{t_0,s}) = \int_{\Sigma_i(t_0,s)} v_{t_0,s}(t_i) \otimes \cdots \otimes v_{t_0,s}(t_1)d\mathcal{L}^n = 0.$$

Indeed, by the change of variable $(t_1, \ldots, t_i) = (s\tau_1 + t_0, \ldots, s\tau_i + t_0)$, we get

$$I_i(t_0, s; v_{t_0,s}) = s^i I_i(0,1;v) = 0.$$

□

The set $V_{n-1} \subset X$ is not a linear space and the map $v \mapsto I_n(v) = I_n(0,1;v)$ is not additive. However, we can construct linear subsets of V_{n-1} of any finite dimension starting from one function. Given $v \in X$, we define $v_1, v_2 \in X$ letting

$$v_1 = v_{0,1/2} \quad \text{and} \quad v_2 = v_{1/2,1/2}.$$

These are the localization of v with $t_0 = 0, 1/2$ and $s = 1/2$.

Proposition 2.21. *If $v \in V_{n-1}$ then $v_1, v_2, v_1 + v_2 \in V_{n-1}$ and*

$$I_n(v_1 + v_2) = I_n(v_1) + I_n(v_2).$$

Moreover, we have $I_n(v_1) = I_n(v_2) = \frac{1}{2^n} I_n(v)$.

Proof. The fact that $v_1, v_2 \in V_{n-1}$ is proved in Lemma 2.20. We show the remaining claims. For any multi-index $\alpha \in \mathcal{I}_{h,2}$, $1 \le h \le n-1$, consider the integral

$$I^\alpha(v_1, v_2) = \int_{\Sigma_h} v^1_{\alpha_1}(t_1) \dots v^h_{\alpha_h}(t_h) d\mathcal{L}^h. \tag{2.26}$$

Letting $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$, the support of the function $v^1_{\alpha_1}(t_1) \dots v^h_{\alpha_h}(t_h)$ is contained in the product $I_{\alpha_1} \times \dots \times I_{\alpha_h}$. If there exist $i < j$ such that $\alpha_i = 1 < \alpha_j = 2$, then $\Sigma_h \cap I_{\alpha_1} \times \dots \times I_{\alpha_h} = \emptyset$, and then $I^\alpha(v_1 + v_2) = 0$.

The complementary case is when $\alpha_1 = \dots = \alpha_k = 2$ and $\alpha_{k+1} = \dots = \alpha_h = 1$ for some $k = 0, 1, \dots, h$. In this case, the integral in (2.26) splits into the product of two integrals:

$$I^\alpha(v_1, v_2) = \Big(\int_{\Sigma_k} v^1_2(t_1) \dots v^k_2(t_k) d\mathcal{L}^k \Big) \Big(\int_{\Sigma_{h-k}} v^{k+1}_1(t_{k+1}) \dots v^h_1(t_h) d\mathcal{L}^{h-k} \Big).$$

If $v_1, v_2 \in V_{n-1}$ this shows that $I^\alpha(v_1, v_2) = 0$ for all $\alpha \in \mathcal{I}_{h,2}$ and for all $h \le n-1$. This proves that $v_1 + v_2 \in V_{n-1}$.

When $h = n$ the argument above also shows that for all $\alpha \in \mathcal{I}_{n,2}$ such that $\alpha \neq (1, \dots, 1)$ and $\alpha \neq (2, \dots, 2)$ we have $I^\alpha(v_1, v_2) = 0$. We conclude that

$$I_n(v_1 + v_2) = \sum_{\alpha \in \mathcal{I}_{n,2}} I^\alpha(v_1, v_2) = I_n(v_1) + I_n(v_2).$$

□

2.4 Expansion of the end-point map

In this section we expand the end-point map and we compute its n-th order term. The computations use the language of chronological calculus for non-autonomous vector fields developed in Section ... . Since our analysis is local, we shall without loss of generality identify our sub-Riemannian manifold M with $\mathbb{R}^m$. So M-valued maps will be in fact $\mathbb{R}^m$-valued, fitting the framework of Section 2.1.

We briefly recall the definitions of the end-point mapo E and of the variation map G, namely for $q \in M$ and $u, v \in X$ we have

$$E_q(u) = q \circ \overrightarrow{\exp} \int_0^1 f_{u(t)} dt, \quad u \in X,$$

and

$$G^u_{\bar{q}}(v) = \bar{q} \circ \overrightarrow{\exp} \int_0^1 g^{u,t}_{v(t)} dt, \quad v \in X,$$

where $g^{u,t}_{v(t)}$ is the time-dependent vector field

$$g^{u,t}_{v(t)} = \mathrm{Ad}\left(\overrightarrow{\exp} \int_1^t f_{u(\tau)} d\tau \right) f_{v(t)} = (P^u_{t,1})_* f_v. \tag{2.27}$$

Here P^u denotes as usual the flow of the vector field $f_u = \sum_{i=1}^d u_i f_i$.

Our goal is to compute the Taylor's expansion of the variation map. For $k \in \mathbb{N}$ and $v \in X$, we define the vector field $W_k(v)$ as

$$\begin{aligned} W_k(v) &= \int_{\Sigma_k} \mathrm{ad} g^{u,\tau_k}_{v(\tau_k)} \circ \cdots \circ \mathrm{ad} g^{u,\tau_2}_{v(\tau_2)} \big(g^{u,\tau_1}_{v(\tau_1)} \big) d\mathcal{L}^k \\ &= \int_{\Sigma_k} [g^{u,\tau_k}_{v(\tau_k)}, \ldots, g^{u,\tau_1}_{v(\tau_1)}] d\mathcal{L}^k. \end{aligned} \tag{2.28}$$

Here and hereafter, we use the following notation for the iterated bracket of vector fields $g_k, \ldots, g_1$:

$$[g_k, \ldots, g_1] = [g_k, [\cdots, [g_2, g_1] \cdots]] = \mathrm{ad} g_k \circ \cdots \circ \mathrm{ad} g_2 (g_1).$$

For a multi-index $\beta \in \mathcal{I}_k$ we define the operator (composition of vector fields)

$$W_\beta(v) = W_{\beta_1}(v) \circ \ldots \circ W_{\beta_k}(v). \tag{2.29}$$

The operator-valued map $v \mapsto W_\beta(v)$ introduced in (2.29) is polynomial in v with homogeneous degree $p = |\beta|$. Its p-polarization is defined via the formula

$$W_\beta(v_1, \ldots, v_p) = \frac{1}{p!} \frac{\partial^p}{\partial t_1 \ldots \partial t_p} W_\beta \Big(\sum_{i=1}^p t_i v_i \Big) \Big|_{t_1 = \cdots = t_p = 0}, \tag{2.30}$$

where $v_1, \ldots, v_p \in X$. This definition is consistent with (2.2).

By the argument of Lemma 3.3 and Remark 3.4 in [7], for any $p \in \mathbb{N}$ and for any $v \in X$ the p-differential of G has the representation

$$d_0^p G(v) = \sum_{k=1}^{p} \sum_{\beta \in \mathcal{I}_k, |\beta|=p} c_\beta W_\beta(v), \tag{2.31}$$

where, for any $\beta \in \mathcal{I}_k$, we set

$$c_\beta = |\beta|! \prod_{s=1}^{k} (\beta_1 + \cdots + \beta_s)^{-1} \in \mathbb{R}.$$

Using these formulas we obtain a representation for the differentials $D_0^h G$.

Lemma 2.22. *For any $h \in \mathbb{N}$ and for all $v = (v_1, \dots, v_h) \in X^h$ we have*

$$D_0^h G(v) = \sum_{p=1}^{h} \sum_{\alpha \in \mathcal{I}_p, |\alpha|=h} \frac{h!}{\alpha! p!} \sum_{k=1}^{p} \sum_{\beta \in \mathcal{I}_k, |\beta|=p} c_\beta W_\beta(v_\alpha),$$

where $v_\alpha = (v_{\alpha_1}, \dots, v_{\alpha_p})$ for $\alpha \in \mathcal{I}_p$.

Proof. Formula (2.4) reads

$$D_0^h G(v) = \sum_{p=1}^{h} \sum_{\alpha \in \mathcal{I}_p, |\alpha|=h} \frac{h!}{\alpha! p!} d_0^p G(v_\alpha), \tag{2.32}$$

and by (2.2), (2.31), and (2.30) we deduce that, for $w = (w_1, \dots, w_p) \in X^p$,

$$\begin{aligned} d_0^p G(w) &= \frac{1}{p!} \frac{\partial^p}{\partial t_1 \dots \partial t_p} d_0^p G\Big(\sum_{i=1}^{p} t_i w_i\Big)\Big|_{t_1=\cdots=t_p=0} \\ &= \frac{1}{p!} \sum_{k=1}^{p} \sum_{\beta \in \mathcal{I}_k, |\beta|=p} c_\beta \frac{\partial^p}{\partial t_1 \dots \partial t_p} W_\beta\Big(\sum_{h=1}^{p} t_h w_h\Big)\Big|_{t_1=\cdots=t_p=0} \\ &= \sum_{k=1}^{p} \sum_{\beta \in \mathcal{I}_k, |\beta|=p} c_\beta W_\beta(w). \end{aligned} \tag{2.33}$$

□

For a given $v \in X$ let us consider the localization $v_{t_0,s}$ for some $t_0 \in [0,1)$ and small $s > 0$, as in (2.25).

Proposition 2.23. *Let $h \in \mathbb{N}$, $v \in X$, and $t_0 \in (0,1)$. For any $s \in (0, 1-t_0)$ we have*

$$W_h(v_{t_0,s}) = s^h \int_{\Sigma_h} [g^{t_0}_{v(t_h)}, \dots, g^{t_0}_{v(t_1)}] d\mathcal{L}^h + O(s^{h+1}). \tag{2.34}$$

Moreover, there exists a constant $C > 0$ such that $|O(s^{h+1})| \le C s^{h+1}$ for all $v \in X$ with $\|v\|_X \le 1$.

Proof. With the notation introduced in (2.27) and omitting the superscript u, we have

$$\begin{aligned} g^{\tau}_{v_{t_0,s}(t)} &= \mathrm{Ad}\left(\overrightarrow{\exp}\int_1^{\tau} f_{u(\sigma)}d\sigma\right) f_{v_{t_0,s}(t)} \\ &= \sum_{i=1}^{d} v^i_{t_0,s}(t)\mathrm{Ad}\left(\overrightarrow{\exp}\int_1^{\tau} f_{u(\sigma)}d\sigma\right) f_i \\ &= \sum_{i=1}^{d} v^i_{t_0,s}(t) g^{\tau}_i, \end{aligned}$$

where g^{τ}_i is defined via the last identity. Letting, for $\alpha \in \mathcal{I}_{h,d}$,

$$J^{\alpha}_{t_0,s} = \int_{\Sigma_h(t_0,s)} v^{\alpha_h}_{t_0,s}(\tau_h)\dots v^{\alpha_1}_{t_0,s}(\tau_1)[g^{\tau_h}_{\alpha_h},\dots,g^{\tau_1}_{\alpha_1}]d\mathcal{L}^h, \tag{2.35}$$

formula (2.28) reads

$$W_h(v_{t_0,s}) = \sum_{\alpha\in\mathcal{I}_{h,d}} J^{\alpha}_{t_0,s}.$$

With the change of variable $\vartheta_i = \frac{\tau_i - t_0}{s}$, for $i = 1,\dots,n$, the integral in (2.35) becomes

$$J^{\alpha}_{t_0,s} = s^h \int_{\Sigma_h} v^{\alpha_h}(\vartheta_h)\dots v^{\alpha_1}(\vartheta_1)[g^{s\vartheta_h+t_0}_{\alpha_h},\dots,g^{s\vartheta_1+t_0}_{\alpha_1}]d\mathcal{L}^h,$$

Since the maps

$$t \mapsto g^t_i = \mathrm{Ad}\left(\overrightarrow{\exp}\int_1^{t} f_{u(\sigma)}d\sigma\right) f_i, \quad i = 1,\dots,d,$$

are Lipschitz continuous, for every $i = 1,\dots,d$ and $j = 1,\dots,h$ we have the expansion

$$g^{s\vartheta_j+t_0}_i = g^{t_0}_i + O(s),$$

with a uniform error $O(s)$ for $\vartheta_j \in I$. So we conclude that

$$J^{\alpha}_{t_0,s} = s^h[g^{t_0}_{\alpha_h},\dots,g^{t_0}_{\alpha_1}]\int_{\Sigma_h} v^{\alpha_h}(\vartheta_h)\dots v^{\alpha_1}(\vartheta_1)d\mathcal{L}^h + O(s^{h+1}).$$

The claim (2.34) follows by summing over $\alpha \in \mathcal{I}_{h,d}$. □

Corollary 2.24. *Let $v \in V_h$, for some $h \in \mathbb{N}$, and $t_0 \in (0,1)$. For any $s \in (0, 1-t_0)$ we have $d^h_0 G(v_{t_0,s},\dots,v_{t_0,s}) = O(s^{h+1})$.*

Proof. By formula (2.33), the h-differential of G has the representation

$$d^h_0 G(w) = \sum_{k=1}^{h}\sum_{\beta\in\mathcal{I}_k,|\beta|=h} c_\beta W_\beta(v_{t_0,s},\dots,v_{t_0,s}).$$

Let $\beta \in \mathcal{I}_k$ with $|\beta| = h$. We claim that the coefficient of s^h in the expansion of $s \mapsto W_\beta(v_{t_0,s}, \dots, v_{t_0,s})$ vanishes. Indeed, consider the coordinate $j = \beta_i$. By Proposition 2.23 we have

$$\begin{aligned} W_j(v_{t_0,s}) &= s^j \sum_{\alpha \in \mathcal{I}_{j,d}} [g^{t_0}_{\alpha_j}, \dots, g^{t_0}_{\alpha_1}] \int_{\Sigma_j} v^{\alpha_j}(\vartheta_j) \dots v^{\alpha_1}(\vartheta_1) d\mathcal{L}^j + O(s^{j+1}) \\ &= O(s^{j+1}), \end{aligned}$$

because for $j \le h$ we have

$$\int_{\Sigma_j} v^{\alpha_j}(\vartheta_h) \dots v^{\alpha_1}(\vartheta_1) d\mathcal{L}^j = 0,$$

by our assumption $v \in V_h$ and by Lemma 2.18. The claim follows.

□

Let $v \in X$ and $s > 0$. Assume that the function $w_{1;t_0,s} = v_{t_0,s}$ belongs to $\ker(d_0 G)$. By Proposition 2.9 with $\ell = 1$ and $t = 1$, there exist functions $w_{j;t_0,s} \in X$, $j = 2, \dots, n-1$, such that $w_{t_0,s} = (w_{1;t_0,s}, \dots, w_{n-1;t_0,s}) \in \mathrm{dom}(\mathcal{D}^n_0 G)$.

Lemma 2.25. *If $v \in V_{n-1}$ then $\|w_{j;t_0,s}\|_X = O(s^{j+1})$ for all $j = 2, \dots, n-1$.*

Proof. The proof is by induction on $j = 2, \dots, n-1$. We start with $j = 2$. Since $(w_{1;t_0,s}, w_{2;t_0,s}) \in \mathrm{dom}(\mathcal{D}^3_0 G)$ we have $D^2_0 G(w_{1;t_0,s}, w_{2;t_0,s}) = 0$, and by (2.4) this equation reads

$$d_0 G(w_{2;t_0,s}) = -d^2_0 G(v_{t_0,s}, v_{t_0,s}) = O(s^3),$$

by Corollary 2.24. The claim follows composing with the inverse of $d_0 G$.

Now we assume that the claim holds for $j \le n-2$ and we prove it for $j = n-1$. Since $w_{t_0,s} \in \mathrm{dom}(\mathcal{D}^n_0 G)$ we have $D^{n-1}_0 G(w_{t_0,s}) = 0$ and, by (2.4), this equation reads

$$d_0 G(w_{n-1;t_0,s}) = -d^{n-1}_0 G(v_{t_0,s}, \dots, v_{t_0,s}) - \sum_{h=2}^{n-2} \sum_{\alpha \in \mathcal{I}_h, |\alpha| = n-1} \frac{(n-1)!}{\alpha! h!} d^h_0 G\big((w_{t_0,s})_\alpha\big).$$

We clearly have $d^{n-1}_0 G(v_{t_0,s}, \dots, v_{t_0,s}) = O(s^n)$, by Corollary 2.24.

We estimate the terms in the sum. When $2 \le h \le n-2$ and $\alpha \in \mathcal{I}_h$ with $|\alpha| = n-1$, the multi-index α contains at least one coordinate different from 1 and does not contain the coordinate $n-1$, and so

$$\mathrm{Card}\{j \mid \alpha_j = 1\} + \sum_{i=2}^{n-2} (i+1) \, \mathrm{Card}\{j \mid \alpha_j = i\} > |\alpha| = n-1.$$

Then, from our inductive assumption it follows that $d^h_0 G\big((w_{t_0,s})_\alpha\big) = O(s^n)$.

□

Lemma 2.26. *Let $v \in X$, $s > 0$ and $t_0 \in [0,1)$ be such that $v_{t_0,s} \in \ker(d_0 G)$ and let $w_{t_0,s} \in \mathrm{dom}(\mathcal{D}_0^n G)$ be the extension of $v_{t_0,s}$. If $v \in V_{n-1}$ then we have*

$$D_0^n G(w_{t_0,s}) = c_n s^n \int_{\Sigma_n} [g^{t_0}_{v(t_n)}, \ldots, g^{t_0}_{v(t_1)}] d\mathcal{L}^n + O(s^{n+1}). \tag{2.36}$$

Proof. By formula (2.32),

$$D_0^n G(w_{t_0,s}) = \sum_{p=1}^{n} \sum_{\alpha \in \mathcal{I}_p, |\alpha| = n} \frac{n!}{\alpha! p!} d_0^p G((w_{t_0,s})_\alpha).$$

If $\alpha \in \mathcal{I}_p$ has one entry different from 1, then $d_0^p G((w_{t_0,s})_\alpha) = O(s^{n+1})$ by Lemma 2.25. The leading term is given by $p = n$ and $\alpha \in \mathcal{I}_n$ with $\alpha = (1, \ldots, 1)$. The expansion of this term is given by formula (2.34) with $h = n$ and this yields formula (2.36).

□

2.5 Open mapping property for the extended end-point map

In this section, we study the open mapping property for the extended end-point map at critical points of corank 1.

We recall that the extended end-point map is the map $F_J : X \to M \times \mathbb{R}$ given by $F_J(u) = (F(u), J(u))$, where J is the energy functional defined in ... . We also recall that minimize the energy is equivalent to minimize the length. We can also define the extended variation map $G_J(v) = (F(u+v), J(u+v))$. Then, $0 \in X$ is a regular, singular, strictly singular critical point of G_J if and only if u is a regular, singular, strictly singular critical point of F_J, respectively.

We are interested in strictly singular critical points of F_J. In this case, $\mathrm{Im}(d_u F_J) = \mathrm{Im}(d_u F) \oplus \mathbb{R}$, that is, $\mathrm{coker}(d_u F_J)$ and $\mathrm{coker}(d_u F)$ are isomorphic and can be identified. The differential analysis of the extended map F_J can be consequently reduced to the analysis of the end-point map F. In fact, for any $h \geq 2$ we have $\mathcal{D}_u^h F_J = \mathcal{D}_u^h F\big|_{\ker(d_u F_J)}$, where the kernel $\ker(d_u F_J) = \ker(d_u F) \cap \ker(d_u J)$ is finitely complemented in X, and the restriction to $\ker(d_u F_J)$ means $\mathrm{dom}(\mathcal{D}_0^h F_J) = \{v \in \mathrm{dom}(\mathcal{D}_0^h F) \mid v_1 \in \ker(d_u J)\}$. Similarly, we have

$$\mathcal{D}_0^h G_J = \mathcal{D}_0^h G\big|_{\ker(d_0 G_J)}, \qquad h \geq 2, \tag{2.37}$$

with $\ker(d_0 G_J)$ finitely complemented in X, and

$$\mathrm{dom}(\mathcal{D}_0^h G_J) = \{v \in \mathrm{dom}(\mathcal{D}_0^h G) \mid v_1 \in \ker(d_u J)\}. \tag{2.38}$$

Finally, $0 \in X$ is a critical point for G_J of corank $\ell = 1$ if and only if u is a critical point for G of corank $\ell = 1$.

Thanks to the previous remarks, we can without loss of generality consider the situation where 0 is a corank-one critical point for G. This means that $\mathrm{coker}(d_0G)$ has dimension 1. We fix a nonzero dual vector $\lambda \in \mathrm{coker}(d_0G)^*$ such that $\langle \lambda, w\rangle = \mathrm{proj}(w)$, $w \in \mathbb{R}^m$, where proj is the projection on $\mathrm{coker}(d_0G)$.

For $n \geq 2$ and $t_0 \in [0,1)$, we consider the function $\mathcal{G}^n_{t_0} : X \to \mathbb{R}$

$$\mathcal{G}^n_{t_0}(v) = \int_{\Sigma_n} \langle \lambda, [g^{t_0}_{v(t_n)}, \dots, g^{t_0}_{v(t_1)}]\rangle d\mathcal{L}^n, \quad v \in X. \tag{2.39}$$

This is the coefficient of the leading term in the expansion of $D^n_0 G(w_{t_0,s})$ in (2.36), up to the constant c_n. Here and hereafter, vector fields are evaluated at the end-point $\bar{q}$, with notation as in the previous section.

For a multi-index $\alpha \in \mathcal{I}_{n,d}$ let us introduce the short notation

$$[g^{t_0}_\alpha] = [g^{t_0}_{\alpha_n}, \dots, g^{t_0}_{\alpha_1}],$$

where the entries $\alpha_1, \dots, \alpha_n$ appear in the bracket with reversed order, and

$$I^\alpha(v) = \int_{\Sigma_n} v^{\alpha_n}(t_n) \dots v^{\alpha_1}(t_1) d\mathcal{L}^n. \tag{2.40}$$

Then formula (2.39) reads

$$\mathcal{G}^n_{t_0}(v) = \sum_{\alpha \in \mathcal{I}_{n,d}} \langle \lambda, [g^{t_0}_\alpha]\rangle I^\alpha(v).$$

Lemma 2.27. *If $v \in V_{n-1}$ then $v^\flat \in V_{n-1}$ and $\mathcal{G}^n_{t_0}(v^\flat) = (-1)^{n-1}\mathcal{G}^n_{t_0}(v)$.*

Proof. We have $v^\flat \in V_{n-1}$ by Corollary 2.19. By Lemma 2.18 – here we use the assumption $v \in V_{n-1}$, – the integrals $I^\alpha(v)$ can be transformed in the following way:

$$\begin{aligned} I^\alpha(v) &= \int_0^1 v^{\alpha_n}(t_n)\Big(\int_{\Sigma_{n-1}(t_n,1-t_n)} v^{\alpha_{n-1}}(t_{n-1}) \dots v^{\alpha_1}(t_1) d\mathcal{L}^{n-1}\Big) dt_n \\ &= (-1)^{n-1} \int_0^1 v^{\alpha_n}(t_n)\Big(\int_{\Sigma^\flat_{n-1}(0,t_n)} v^{\alpha_{n-1}}(t_{n-1}) \dots v^{\alpha_1}(t_1) d\mathcal{L}^{n-1}\Big) dt_n \\ &= (-1)^{n-1} \int_{\Sigma^\flat_n} v^{\alpha_n}(t_n) \dots v^{\alpha_1}(t_1) d\mathcal{L}^n \\ &= (-1)^{n-1} I^\alpha(v^\flat). \end{aligned}$$

The last identity follows by the change of variable $t_i = 1 - s_i$. This proves our claim $\mathcal{G}^n_{t_0}(v^\flat) = (-1)^{n-1}\mathcal{G}^n_{t_0}(v)$.

□

In the next step, we show that if $\mathcal{G}^n_{t_0}$ is positive and additive on a suitable subspace of V_{n-1} then the extended map G_J is open at zero. As usual, $m = \dim(M)$ is the dimension of the manifold.

Theorem 2.28. *Let* 0 *be a strictly singular critical point of* G_J *with corank* 1 *and assume that* $\mathcal{D}^h_0 G = 0$ *for* $2 \le h \le n-1$ *with* $n \ge 2$*. Also, assume that there exist* $t_0 \in [0,1)$ *and* $v_1, \dots, v_k \in V_{n-1}$ *such that:*

i) $k \ge m+1$ *when* n *is even and* $k \ge m+2$ *when* n *is odd;*

ii) $\mathcal{G}^n_{t_0}(v_i) = 1$ *for* $i = 1, \dots, k$*;*

iii) $v_1, \dots, v_k$ *span a vector space* $Y \subset V_{n-1}$ *of dimension* k*;*

iv) $\mathcal{G}^n_{t_0}$ *is additive on* $v_1, \dots, v_k$*, in the sense that*

$$\mathcal{G}^n_{t_0}\Big(\sum_{i=1}^k \tau_i v_i\Big) = \sum_{i=1}^k \mathcal{G}^n_{t_0}(\tau_i v_i)$$

for any $\tau_1, \dots, \tau_k \in \mathbb{R}$*.*

Then the extended map G_J *is open at* 0*.*

Proof. The kernel $\ker(d_0G)$ has codimension $m-1$ in X and thus $\ker(d_0G_J) = \ker(d_0G) \cap \ker(d_0J)$ has codimension at most m in X.

For $s > 0$, let $L_s : X = L^1(I;\mathbb{R}^d) \to X_s : L^1([t_0, t_0+s];\mathbb{R}^d)$ be the linear isomorphism $L_s(v) = v_{t_0,s}$. Since $\ker(d_0G_J) \cap X_s$ has codimension at most m in X_s, then $L_s^{-1}(\ker(d_0G_J) \cap X_s) \subset X$ has codimension at most m and thus

$$\begin{aligned} &\dim(Y \cap L_s^{-1}(\ker(d_0G_J) \cap X_s)) \ge 1 \qquad \text{when } n \text{ is even,} \\ &\dim(Y \cap L_s^{-1}(\ker(d_0G_J) \cap X_s)) \ge 2 \qquad \text{when } n \text{ is odd.} \end{aligned} \tag{2.41}$$

We discuss the case when n is even. By iv), n-homogeneity and ii), for $\tau \in \mathbb{R}^k$, $\tau \ne 0$, we have

$$\mathcal{G}^n_{t_0}\Big(\sum_{i=1}^k \tau_i v_i\Big) = \sum_{i=1}^k \tau_i^n > 0.$$

Thus, the function $\mathcal{G}^n_{t_0}$ attains a positive minimum on the sphere $K = \{v \in Y : \|v\|_X = 1\}$: there exists $\delta > 0$ such that

$$\mathcal{G}^n_{t_0}(v) \ge \delta > 0 \quad \text{for all } v \in K. \tag{2.42}$$

By (2.41), for any $s > 0$ there exists $v \in K$ such that $v_{t_0,s} \in \ker(d_0G_J)$. This $v \in K$ depends on s. Let $w_{t_0,s}$ be the extension of $v_{t_0,s}$ given by Proposition 2.9

applied with $\ell = 1$ and $t = 1$. We have $w_{t_0,s} \in \mathrm{dom}(\mathcal{D}_0^n G_J)$ for all $s > 0$. By (2.37) and by formula (2.36)

$$\mathcal{D}_0^n G_J(w_{s,t_0}) = \mathcal{D}_0^n G(w_{s,t_0}) = s^n c_n \mathcal{G}_{t_0}^n(v) + O(s^{n+1}),$$

where $|O(s^{n+1})| \leq C_1 s^{s+1}$ for a constant $C_1 > 0$ independent of v with $\|v\|_X \leq 1$. Choosing $0 < s < \frac{1}{2}\delta/C_1 c_n$, from (2.42) we deduce that

$$\mathcal{D}_0^n G_J(w_{s,t_0}) \geq s^n \big(c_n \mathcal{G}_{t_0}^n(v) - C_1 s\big) \geq s^n \frac{\delta}{2} > 0.$$

By Lemma 2.27, for n even we have $\mathcal{G}_{t_0}^n(v^\flat) = -\mathcal{G}_{t_0}^n(v)$. Repeating the above argument starting from $v_1^\flat, \dots, v_k^\flat$, we conclude that for all $s > 0$ small enough there exists $w_{t_0,s}^\flat \in \mathrm{dom}(\mathcal{D}_0^n G_J)$ such that $\mathcal{D}_0^n G_J(w_{s,t_0}^\flat) = -1$. By Corollary 2.16 part i), we conclude that G_J is open at 0.

We pass to the case when n is odd. Let $H \subset Y$ be the set $H = \{v \in Y : \mathcal{G}_{t_0}^n(v) = 1\}$. We claim that there exists $v \in H \cap T_s^{-1}(\ker(d_0 G_J) \cap X_s)$ such that $\|v\|_X \leq C_2$ for a constant C_2 independent of $s > 0$.

We may assume that $k = m + 2$ and on Y we fix the coordinates $\tau \in \mathbb{R}^k$ with respect to the basis $v_1, \dots, v_k$, i.e., $v = \tau_1 v_1, + \cdots + \tau_k v_k \in Y$. The equation $\mathcal{G}_{t_0}^n(v) = 1$ reads

$$\tau_1^n + \cdots + \tau_k^n = 1. \tag{2.43}$$

The set $Y \cap L_s^{-1}(\ker(d_0 G_K) \cap X_s)$ contains a hyperplane of the form $b_1\tau_1 + \cdots + b_k\tau_k = 0$ where $(b_1, \dots, b_k) \neq 0$ are coefficients depending on $s > 0$. Without loss of generality we can assume that $b_k = 1$ and $|b_1|, \dots, |b_{k-1}| \leq 1$. Then last equation reads $\tau_k = -(b_1\tau_1 + \cdots + b_{k-1}\tau_{k-1})$, and equation (2.43) becomes

$$p(\tau_1, \dots, \tau_{k-1}) = -(b_1\tau_1 + \cdots + b_{k-1}\tau_{k-1})^n + \tau_1^n + \cdots + \tau_k^n = 1. \tag{2.44}$$

The polynomial p is not the zero polynomial and has homogeneous degree n, that is odd. So the equation $p(\tau_1, \dots, \tau_{k-1}) = 1$ has solutions, for any $b_1 \dots, b_{k-1}$. We are left to show that there are solutions bounded by a constant independent of $b_1, \dots, b_{k-1}$.

For $\delta \in (0, 1)$ consider the set

$$Q_\delta = \big\{b = (b_1, \dots, b_{k-1}) \in \mathbb{R}^{k-1} \mid \min\{|b_1|, \dots, |b_{k-1}|\} \geq \delta, \max\{|b_1|, \dots, |b_{k-1}|\} \leq 1\big\}.$$

When $b \in Q_\delta$, we look for a solution $\tau = (\tau_1, \dots, \tau_{k-1})$ of equation (2.44) of the form $\tau = tb/|b|$ for some $t \in \mathbb{R}$, where $|b| = \sqrt{b_1^2 + \cdots + b_{k-1}^2}$. The equation reads

$$t^n \Big[\frac{b_1^n + \cdots + b_{k-1}^n}{|b|^n} - |b|^n\Big] = 1. \tag{2.45}$$

On Q_δ, the quantity $(b_1^n + \dots + b_{k-1}^n)/|b|^n$ attains the maximum at the points where all the coordinates are δ except one that is 1, while $-|b|^n$ attains the maximum at the vertex $(\delta, \dots, \delta)$. So we get

$$\frac{b_1^n + \dots + b_{k-1}^n}{|b|^n} - |b|^n \le \frac{1 + (k-2)\delta^n}{(1 + (k-2)\delta^2)^{n/2}} - ((k-1)\delta^2)^{n/2} = \psi(\delta).$$

Since $\psi(1) = (k-1)^{1-n/2} - (k-1)^{n/2} < 0$ for $k > 1$, there exist $\varepsilon_0 > 0$ and $0 < \delta < 1$ such that $\psi(\delta) \le -\varepsilon_0$. Now $\delta = \delta(n)$ is fixed. The solution $t \in \mathbb{R}$ of equation (2.45) is negative and satisfies $|t| \le \varepsilon_0^{-1/n}$. This proves the existence of a solution $(\tau_1, \dots, \tau_{k-1}) \in \mathbb{R}^{k-1}$ of (2.44) that is uniformly bounded when $b \in Q_\delta$.

In the case that $\min\{|b_1|, \dots, |b_{k-1}|\} \le \delta$ we argue as follows. Assume for instance that $|b_1| \le \delta$. With $\tau_2 = \dots \tau_{k-1} = 0$ we have the equation $-(b_1\tau_1)^n + \tau_1^n = 1$ that has a solution $\tau_1 > 0$ bounded by

$$\tau_1 \le \Big(\frac{1}{1-\delta^n}\Big)^{1/n}.$$

Now the proof can be concluded as in the case of even n.

$\square$

In fact, in order G_J to be open it is sufficient that $\mathcal{G}_{t_0}^n$ is positive at one element of V_{n-1}.

Theorem 2.29. *Let* 0 *be a strictly singular critical point of* G_J *with corank* 1 *and assume that* $\mathcal{D}_0^h G = 0$ *for* $2 \le h \le n-1$ *with* $n \ge 2$. *If there exist* $t_0 \in [0,1)$ *and* $v \in V_{n-1}$ *such that* $\mathcal{G}_{t_0}^n(v) \neq 0$, *then* G_J *is open at* 0.

Proof. For any $k \in \mathbb{N}$, we apply iteratively Proposition 2.21 to find 2^k functions $v_1, \dots, v_{2^k}$ with mutually disjoint support, spanning a linear space in V_{n-1} and such that

$$\mathcal{G}_{t_0}^n\Big(\sum_{i=1}^{2^k} v_i\Big) = \sum_{i=1}^{2^k} \mathcal{G}_{t_0}^n(v_i),$$

and $\mathcal{G}_{t_0}^n(v_i) = \frac{1}{2^{kn}}\mathcal{G}_{t_0}^n(v)$. The claim follows from Theorem 2.28. $\square$

If G_J is not open at 0 we have $\mathcal{G}_{t_0}^n(v) = 0$ for all $t_0 \in [0,1)$ and $v \in V_{n-1}$. Even though V_{n-1} is not a linear space, we polarize the map $T = \mathcal{G}_{t_0}^n$.

The polarization of $T : X \to \mathbb{R}$ is the multilinear map $\mathcal{T} : X^n \to \mathbb{R}$ defined in one of the two equivalent ways

$$\begin{aligned}\mathcal{T}(v_1, \dots, v_n) &= \frac{\partial^n}{\partial t_1 \dots \partial t_n} T\Big(\sum_{h=1}^n t_h v_h\Big)\Big|_{t_1 = \dots = t_n = 0} \\ &= \frac{1}{n!}\sum_{\sigma \in S_n}\sum_{\alpha \in \mathcal{I}_{n,d}} \langle \lambda, [g_\alpha^{t_0}]\rangle \int_{\Sigma_n} v_{\sigma_n}^{\alpha_n}(t_n) \dots v_{\sigma_1}^{\alpha_1}(t_1)\, d\mathcal{L}^n.\end{aligned} \tag{2.46}$$

If $Y \subset X$ is a linear subspace such that $T = 0$ on Y then $\mathcal{T} = 0$ on Y^n. This follows easily from the differential definition of polarization. Linear spaces $Y \subset V_{n-1}$ where $T = 0$ can be obtained with the following construction.

Fix $w_1, \ldots, w_n \in X$, with coordinates $w_i = (w_i^1, \ldots, w_i^d)$, and fix a selection function $s : \{1, \ldots, n\} \to \{1, \ldots, d\}$, $s(i) = s_i$. This function will be used to select (with multiplicity) which vector-fields from $f_1, \ldots, f_d$ appear in the bracket $[g_\alpha^{t_0}]$.

We can define $v_1, \ldots, v_n \in X$ setting, for $i = 1, \ldots, n$,

$$v_i = (0, \ldots, 0, u^i, 0, \ldots, 0), \quad \text{with } u^i = w_i^{s_i}, \tag{2.47}$$

where u^i is the i-th coordinate. Then we define $u \in X$ as $u = (u^1, \ldots, u^n)$. The function u depends on the selection function s, but we do not keep track of this dependence in our notation. As in (2.40), for a permutation $\sigma \in S_n$ we let

$$I^\sigma(u) = \int_{\Sigma_n} u^{\sigma_n}(t_n) \cdots u^{\sigma_1}(t_1) d\mathcal{L}^n.$$

When $u \in V_{n-1}$, where now V_{n-1} is defined as in (2.23) and (2.22) but with $d = n$, the polarization $\mathcal{T}$ takes the following form.

Lemma 2.30. *For any selection function s, if $u \in V_{n-1}$ then $v_1, \ldots, v_n$ span a linear subspace of V_{n-1} and*

$$\mathcal{T}(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u). \tag{2.48}$$

Proof. Given $\sigma \in S_n$ and $\alpha \in \mathcal{I}_{n,d}$, by the structure (2.47) of $v_1, \ldots, v_n$, there holds

$$\int_{\Sigma_n} v_{\sigma_n}^{\alpha_n}(t_n) \ldots v_{\sigma_1}^{\alpha_1}(t_1) d\mathcal{L}^n = 0$$

as soon as there exists i such that $\alpha_i \neq s(\sigma_i)$. For the surviving terms it must be $\alpha = s\sigma$ and in this case

$$\int_{\Sigma_n} v_{\sigma_n}^{\alpha_n}(t_n) \ldots v_{\sigma_1}^{\alpha_1}(t_1) d\mathcal{L}^n = \int_{\Sigma_n} w_{\sigma_n}^{s_{\sigma_n}}(t_n) \ldots w_{\sigma_1}^{s_{\sigma_1}}(t_1) d\mathcal{L}^n = I^\sigma(u).$$

The claim follows from the combinatorial definition of polarization in (2.46).

□

2.6 Generalized Jacobi identities and integrals on simplexes

In this section we fix a selection function $s : \{1, \ldots, n\} \to \{1, \ldots, d\}$ and $u = (u^1, \ldots, u^n)$, as in (2.47). For varying $\sigma \in S_n$, the brackets $[g_\sigma^{t_0}] = [g_{\sigma_n}^{t_0}, \ldots, g_{\sigma_1}^{t_0}]$

satisfy several linear relations. Using generalized Jacobi identities, we clean up formula (2.48) getting rid of these relations. Our goal is to prove the following formula, where we work with permutations $\sigma \in S_n$ fixing 1

$$S_n^1 = \{\sigma \in S_n \mid \sigma_1 = 1\}.$$

Theorem 2.31. *For any selection function s and for any $v_1, \ldots, v_n$ as in* (2.47)*, we have the identity*

$$\mathcal{T}(v_1, \ldots, v_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u). \tag{2.49}$$

The fact that in (2.49) the sum is restricted to permutations fixing 1 will be important in the next section. The proof relies upon the generalized Jacobi identities proved in [6]. For $n, j \in \mathbb{N}$ with $1 \leq j \leq n$, let us consider the sets of permutations

$$X_{nj} = \{\xi \in S_n \,|\, \xi_1 > \xi_2 > \cdots > \xi_j = 1 \text{ and } \xi_j < \xi_{j+1} < \cdots < \xi_n\},$$

and

$$X_n = \bigcup_{j=1}^{n} X_{nj}. \tag{2.50}$$

The set X_{n1} contains only the identity permutation, while X_{nn} contains only the order reversing permutation. We are denoting elements of X_n by ξ, while in [6] they are denoted by π.

Let $g_1, \ldots, g_n$ be elements of a Lie algebra. The action of a permutation $\sigma \in S_n$ on the iterated bracket $[g_n, \ldots, g_1] = [g_n, [\ldots, [g_2, g_1] \ldots]]$ is denoted by

$$\sigma[g_n, \ldots, g_1] = [g_{\sigma_n}, \ldots, g_{\sigma_1}].$$

The selection function s acts similarly, $s[g_n, \ldots, g_1] = [g_{s_n}, \ldots, g_{s_1}]$, and so in the notation used above we have $[g_{s\sigma}^{t_0}] = s[g_\sigma^{t_0}]$.

The generalized Jacobi identities of order n that we need are described in the next theorem.

Theorem 2.32. *For any Lie elements $g_1, \ldots, g_n$ and for any permutation $\sigma \in S_n$ such that $\sigma_1 \neq 1$,*

$$\Big(\sigma + \sum_{\xi \in X_n,\, \sigma\xi(1)=1} (-1)^{\xi^{-1}(1)} \sigma\xi\Big)[g_n, \ldots, g_1] = 0, \tag{2.51}$$

where X_n is the set of permutations introduced in (2.50).

Proof. The proof of formula (2.51) is contained in [6] on pages 117 and 119. $\square$

Lemma 2.33. *For $v_1, \ldots, v_n$ as in (2.47), we have the identity*

$$\mathcal{T}(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle \sum_{\xi \in X_n} c_\xi I^{\sigma\xi^{-1}}(u), \tag{2.52}$$

where $c_\xi = (-1)^{1+\xi^{-1}(1)}$.

Proof. Starting from (2.48) and using (2.51), we get

$$\begin{aligned}
n!\, \mathcal{T}(v_1, \ldots, v_n) &= \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u) + \sum_{\sigma \in S_n,\, \sigma 1 \neq 1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u) \\
&= \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u) + \sum_{\substack{\xi \in X_n,\, \sigma \in S_n \\ \sigma 1 \neq 1,\, \sigma\xi(1) = 1}} c_\xi \langle \lambda, [g_{s\sigma\xi}^{t_0}] \rangle I^\sigma(u) \\
&= \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle \Big(I^\sigma(u) + \sum_{\xi \in X_n,\, \sigma\xi^{-1}(1) \neq 1} c_\xi I^{\sigma\xi^{-1}}(u) \Big) \\
&= \sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle \sum_{\xi \in X_n} c_\xi I^{\sigma\xi^{-1}}(u).
\end{aligned}$$

In the last line, we used the fact that, when $\sigma_1 = 1$, we have $\sigma\xi^{-1}(1) = 1$ if and only if ξ is the identity.

∎

A permutation $\sigma \in S_n$ acts on the integrals $I^{\xi^{-1}}(u)$ as $\sigma I^{\xi^{-1}}(u) = I^{\sigma\xi^{-1}}(u)$. So, the sum over $\xi \in X_n$ appearing in (2.52) reads

$$\sum_{\xi \in X_n} c_\xi I^{\sigma\xi^{-1}}(u) = \sigma \Big(\sum_{\xi \in X_n} c_\xi I^{\xi^{-1}}(u) \Big),$$

where the action is extended linearly. Our next task is to compute the sum in the round brackets.

A permutation $\sigma \in S_n$ acts on the simplexes $\Sigma_n(t,s)$, with $0 \leq t < t+s \leq 1$, as

$$\sigma\Sigma_n(t,s)v = \big\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t < t_{\sigma_n} < \ldots < t_{\sigma_1} < t+s \big\}.$$

In particular, if $\bar\sigma \in S_n$ is the reversing order permutation, $\bar\sigma(i) = n - i + 1$, then $\Sigma_n^\flat(t,s) = \bar\sigma\Sigma_n(t,s)$. We also let $\Sigma_n^\sigma(t,s) = \sigma\Sigma_n(t,s)$ and $\Sigma_n^\sigma = \Sigma_n^\sigma(0,1)$.

Finally, for $k = 1, \ldots, n$ we let

$$\begin{aligned}
I_k^\flat(u) &= \int_{\Sigma_k^\flat} u^1(s_1) \ldots u^k(s_k) d\mathcal{L}^k, \\
I_{n-k}(u) &= \int_{\Sigma_{n-k}} u^{k+1}(s_{k+1}) \ldots u^n(s_n) d\mathcal{L}^{n-k}.
\end{aligned}$$

Lemma 2.34. *For any $j = 2, \dots, n$ we have the identity*

$$\sum_{\xi \in X_{nj}} I^{\xi^{-1}}(u) = I^{\flat}_{j-1}(u) I_{n-j+1}(u) - \sum_{\xi \in X_{n,j-1}} I^{\xi^{-1}}(u). \tag{2.53}$$

Proof. Fix a permutation $\xi \in X_{nj}$, so that $\xi_j = 1$. In the integral $I^{\xi^{-1}}(u)$ we perform the change of variable $t_{\xi_k} = s_k$. The integration domain Σ_n is transformed into the new domain $\Sigma_n^{\xi^{-1}} = \{0 < s_{\xi_n^{-1}} < \cdots < s_{\xi_1^{-1}} < 1\}$:

$$\begin{aligned} I^{\xi^{-1}}(u) &= \int_{\Sigma_n} u^{\xi_n^{-1}}(t_n) \dots u^{\xi_1^{-1}}(t_1) d\mathcal{L}^n \\ &= \int_{\Sigma_n} u^n(t_{\xi_n}) \dots u^1(t_{\xi_1}) d\mathcal{L}^n \\ &= \int_{\Sigma_n^{\xi^{-1}}} u^n(s_n) \dots u^1(s_1) d\mathcal{L}^n. \end{aligned} \tag{2.54}$$

We denote by $\widehat{s}_j$ the variables $(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_n)$. Since $s_{\xi^{-1}1} = s_j$, the set $\Sigma_n^{\xi^{-1}}$ is

$$\Sigma_n^{\xi^{-1}} = I \times \Sigma_{n-1;j}^{\xi^{-1}}(0, s_j),$$

where $s_j \in I$ and

$$\Sigma_{n-1;j}^{\xi^{-1}}(0, s_j) = \big\{\widehat{s}_j \in \mathbb{R}^{n-1} \mid 0 < s_{\xi_n^{-1}} < \cdots < s_{\xi_2^{-1}} < s_j\big\}.$$

Since $\xi \in X_{nj}$, here we have $s_n < \cdots < s_{j+1} < s_j$ and $s_1 < \cdots < s_{j-1} < s_j$. For varying $\xi \in X_{nj}$, we obtain all the shuffles of $s_1 < \cdots < s_{j-1}$ into $s_n < \cdots < s_{j+1}$ and thus we have

$$\bigcup_{\xi \in X_{nj}} \Sigma_{n-1;j}^{\xi^{-1}}(0, s_j) = A_{j-1}(s_j) \times B_{n-j}(s_j),$$

where $A_{j-1}(s_j) = \Sigma^{\flat}_{j-1}(0, s_j)$ and $B_{n-j}(s_j) = \Sigma_{n-j}(0, s_j)$.

Summing (2.54) over $\xi \in X_{nj}$ we get

$$\begin{aligned} \sum_{\xi \in X_{nj}} I^{\xi^{-1}}(u) &= \sum_{\xi \in X_{nj}} \int_0^1 \Big(\int_{\Sigma_{n-1}^{\xi^{-1}}(0,s_j)} \prod_{k \neq j} u^k(s_k) d\mathcal{L}^{n-1}(\widehat{s}_j) \Big) u^j(s_j) ds_j \\ &= \int_0^1 \Big(\int_{A_{j-1}(s_j) \times B_{n-j}(s_j)} \prod_{k \neq j} u^k(s_k) d\mathcal{L}^{n-1}(\widehat{s}_j) \Big) u^j(s_j) ds_j. \end{aligned} \tag{2.55}$$

The inner integral is the product of two integrals. Namely, letting

$$\begin{aligned} f(s_j) &= u^j(s_j) \int_{B_{n-j}(s_j)} u^{j+1}(s_{j+1}) \dots u^n(s_n) d\mathcal{L}^{n-j}, \\ g(s_j) &= \int_{A_{j-1}(s_j)} u^{j-1}(s_{j-1}) \dots u^1(s_1) d\mathcal{L}^{j-1}, \end{aligned}$$

formula (2.55) becomes

$$\sum_{\xi \in X_{nj}} I^{\xi^{-1}}(u) = \int_0^1 f(s_j) g(s_j) ds_j.$$

A primitive for f is the function $h(s_j) = \int_0^{s_j} f(\sigma) d\sigma$, and an integration by parts gives

$$\int_0^1 f(s_j) g(s_j) ds_j = h(1) g(1) - \int_0^1 h(s_j) g'(s_j) ds_j,$$

where the boundary term is easily computed:

$$\begin{aligned} h(1) &= \int_{B_{n-j+1}(1)} u^j(s_j) \dots u^n(s_n) d\mathcal{L}^{n-j+1} = I_{n-j+1}(u), \\ g(1) &= \int_{A_{j-1}(1)} u^{j-1}(s_{j-1}) \dots u^1(s_1) d\mathcal{L}^{j-1} = I^\flat_{j-1}(u). \end{aligned}$$

In order to compute the integral, notice that

$$g'(s_j) = u^{j-1}(s_j) \int_{A_{j-2}(s_j)} u^{j-2}(s_{j-2}) \dots u^1(s_1) d\mathcal{L}^{j-2},$$

and thus, by (2.55) but for $j-1$, we have

$$\begin{aligned} \int_0^1 h(s_j) g'(s_j) ds_j &= \int_0^1 \Big(\int_{A_{j-2}(\sigma) \times B_{n-j+1}(\sigma)} \prod_{k \neq j-1} u^k(s_k) d\mathcal{L}^{n-1}(\widehat{s}_{j-1}) \Big) u^{j-1}(\sigma) d\sigma \\ &= \sum_{\xi \in X_{n,j-1}} I^{\xi^{-1}}(u). \end{aligned}$$

□

Corollary 2.35. *For any $u \in V_{n-1}$ there holds*

$$\sum_{\xi \in X_n} c_\xi I^{\xi^{-1}}(u) = n \int_{\Sigma_n} u^1(t_1) \dots u^n(t_n) d\mathcal{L}^n.$$

Proof. When $u \in V_{n-1}$ we have $I^\flat_{j-1}(u) = I_{n-j+1}(u) = 0$ for $j = 2, \dots, n$. Taking into account the constants $c_\xi = (-1)^{1+\xi_1^{-1}}$, formula (2.53) reads

$$\sum_{\xi \in X_{nj}} c_\xi I^{\xi^{-1}}(u) = \sum_{\xi \in X_{n,j-1}} c_\xi I^{\xi^{-1}}(u).$$

Applying iteratively this identity, we obtain

$$\begin{aligned}\sum_{\xi\in X_n} c_\xi I^{\xi^{-1}}(u) &= \sum_{\xi\in X_{nn}} c_\xi I^{\xi^{-1}}(u) + \sum_{\xi\in X_{n,n-1}} c_\xi I^{\xi^{-1}}(u) + \cdots + \sum_{\xi\in X_{n1}} c_\xi I^{\xi^{-1}}(u)\\ &= 2\sum_{\xi\in X_{n,n-1}} c_\xi I^{\xi^{-1}}(u) + \sum_{\xi\in X_{n,n-2}} c_\xi I^{\xi^{-1}}(u) + \cdots + \sum_{\xi\in X_{n1}} c_\xi I^{\xi^{-1}}(u)\\ &= 3\sum_{\xi\in X_{n,n-2}} c_\xi I^{\xi^{-1}}(u) + \sum_{\xi\in X_{n,n-3}} c_\xi I^{\xi^{-1}}(u) + \cdots + \sum_{\xi\in X_{n1}} c_\xi I^{\xi^{-1}}(u)\\ &= \dots\\ &= n\sum_{\xi\in X_{n1}} c_\xi I^{\xi^{-1}}(u).\end{aligned}$$

Since X_{n1} contains only the identity permutation, the claim follows. □

This corollary concludes the proof of Theorem 2.31.

2.7 Non-singularity via trigonometric functions

We start the study of equation $\mathcal{T}(v_1,\dots,v_n)=0$ for the polarization map $\mathcal{T}$ in (2.49). We will work with functions v_i as in (2.47) of trigonometric-type.

For each permutation fixing 1, $\sigma\in S_n^1$, we introduce a real unknow x_σ. There are $(n-1)! = \mathrm{Card}(S_n^1)$ unknowns. We are interested in the linear system of equations

$$\sum_{\sigma\in S_n^1} I^\sigma(u_\tau)x_\sigma = 0, \quad \tau\in S_n^1, \tag{2.56}$$

where $I^\sigma(u_\tau)$ are regarded as coefficients depending on $u_\tau\in V_{n-1}$. In this section, we prove the following preparatory result.

Theorem 2.36. *There exist $u_\tau\in V_{n-1}$, $\tau\in S_n^1$, such that* $\det(I^\sigma(u_\tau))_{\sigma,\tau\in S_n^1}\neq 0$.

With a choice of coefficients as in Theorem 2.36, the linear system (2.56) has only the zero solution, implying $x_\sigma=0$ for all $\sigma\in S_n^1$. This fact will be used in Section 2.8.

For $z=a+ib\in\mathbb{C}$ and $k\in\mathbb{N}$ we let

$$w_{z;k}(t) = a\cos(2k\pi t) + b\sin(2k\pi t), \quad t\in I.$$

We call $w_{z;k}$ a w-type function of parameters z and k, and we call k frequence of $w_{z;k}$. We need exact computations for iterated integrals on n-simplexes of w-type functions. In particular, we are interested in the case when every linear combination with coefficients ± 1 of at most $n-1$ frequencies out of a set of n frequencies is not zero, see (2.59) below. This condition will ensure assumption $u \in V_{n-1}$ in Lemma 2.30.

Any w-type function satisfies the integration formula

$$\int_t^1 w_{z;k}(s)ds = \frac{1}{2k\pi}\big(w_{iz;k}(1) - w_{iz;k}(t)\big), \quad k \neq 0, \tag{2.57}$$

and a pair of w-type functions satisfies the multiplication formula (Werner's identities)

$$w_{z;k}w_{\zeta;h} = \frac{1}{2}\big(w_{z\zeta;k+h} + w_{\bar{z}\zeta;h-k}\big). \tag{2.58}$$

For $h \in \mathbb{N}$, we let $\mathcal{J}_h = \{1, \dots, h\}$ and

$$\mathcal{A}_h = \big\{\alpha : \mathcal{J}_h \to \{1, -1\} \mid \alpha_1 = 1\big\}.$$

Here we are denoting $\alpha(j) = \alpha_j$ and, with a slight abuse of notation, we identify $\alpha \in \mathcal{A}_h$ with $\alpha = (\alpha_1, \dots, \alpha_h) \in \{1, -1\}^h$. Letting $\mathrm{z}_h = (z_1, \dots, z_h) \in \mathbb{C}^h$, for each $\alpha \in \mathcal{A}_h$ we define the multiplicative function $p_\alpha : \mathbb{C}^h \to \mathbb{C}$

$$p_\alpha(\mathrm{z}_h) = \prod_{\ell \in \mathcal{J}_h, \alpha_\ell = 1} z_\ell \prod_{j \in \mathcal{J}_h, \alpha_j = -1} \bar{z}_j.$$

Also, letting $\mathrm{k}_h = (k_1, \dots, k_h) \in \mathbb{N}^h$, we define the additive function $s_\alpha : \mathbb{N}^h \to \mathbb{N}$

$$s_\alpha(\mathrm{k}_h) = \sum_{j=1}^h \alpha_j k_j.$$

Notice that, since $\alpha_1 = 1$, $\bar{z}_1$ never appears in $p_\alpha(\mathrm{z}_h)$ and k_1 has always positive sign in $s_\alpha(\mathrm{k}_h)$.

Finally, for $\ell, h \in \mathbb{N}$ with $\ell \leq h$ we let $\mathcal{B}_\ell^h = \{\beta : \mathcal{J}_\ell \to \mathcal{J}_h \mid \beta \text{ injective}\}$ and for $\mathrm{k}_h = (k_1, \dots, k_h) \in \mathbb{N}^h$ and $\beta \in \mathcal{B}_\ell^h$ we set $\mathrm{k}_h^\beta = (k_{\beta_1}, \dots, k_{\beta_\ell}) \in \mathbb{N}^\ell$. Here, we are using the math-roman font for vectors and italics for coordinates.

Theorem 2.37. *Let $\mathrm{k}_n = (k_1, \dots, k_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$, be a vector of frequencies such that*

$$s_\alpha(\mathrm{k}_n^\beta) \neq 0 \quad \textit{for all } \alpha \in \mathcal{A}_h \textit{ and } \beta \in \mathcal{B}_h^n, \quad 1 \leq h \leq n-1. \tag{2.59}$$

Then for any $1 \leq h \leq n-1$, for all $\mathrm{z}_h = (z_1, \dots, z_h) \in \mathbb{C}^h$ and for all $t \in I$ we have

$$\int_{\Sigma_h(t,1-t)} w_{z_h;k_h}(t_h) \dots w_{z_1;k_1}(t_1) d\mathcal{L}^h = g^h_{\mathrm{z}_h;\mathrm{k}_h}(t) - \sum_{\alpha \in \mathcal{A}_h} c^h_\alpha(\mathrm{k}_h) w_{p_\alpha(i\mathrm{z}_h);s_\alpha(\mathrm{k}_h)}(t), \tag{2.60}$$

where $c^h_\alpha(\mathrm{k}_h) \neq 0$ *and the function* $g^h_{\mathrm{z}_h;\mathrm{k}_h}$ *satisfies*

$$g^h_{\mathrm{z}_h;\mathrm{k}_h}(0) = \sum_{\alpha\in\mathcal{A}_h} c^h_\alpha(\mathrm{k}_h) w_{p_\alpha(i\mathrm{z}_h);s_\alpha(\mathrm{k}_h)}(0), \tag{2.61}$$

and

$$\int_{\Sigma_{n-h}} w_{z_n;k_n}(t_n)\dots w_{z_{h+1};k_{h+1}}(t_{h+1}) g^h_{\mathrm{z}_h;\mathrm{k}_h}(t_{h+1}) d\mathcal{L}^{n-h} = 0. \tag{2.62}$$

Proof. The proof is by induction on $n \in \mathbb{N}$, $n \geq 2$. The constants $c^h_\alpha(\mathrm{k}_h)$ are given by the formula

$$c^h_\alpha(\mathrm{k}_h) = \frac{2}{4^h\pi^h} \prod_{\ell=1}^{h} \frac{\alpha_\ell}{s_{(\alpha_1,\dots,\alpha_\ell)}(\mathrm{k}_\ell)}. \tag{2.63}$$

The formula is well-defined because $s_{(\alpha_1,\dots,\alpha_\ell)}(\mathrm{k}_\ell) \neq 0$ by assumption (2.59).

When $n = 2$ we only have $h = 1$ and $\alpha = 1$, so that $c^1_1(\mathrm{k}_1) = 1/2\pi k_1$. The integration formula in (2.57) gives (2.60) with $g^1_{\mathrm{z}_1;\mathrm{k}_1} = c^1_1(\mathrm{k}_1) w_{iz_1;k_1}(1)$, a constant. Identity (2.61) is satisfied and also identity (2.62):

$$\int_0^1 w_{z_2;k_2}(t_2) g^1_{\mathrm{z}_1;\mathrm{k}_1} dt_2 = g^1_{\mathrm{z}_1;\mathrm{k}_1} \int_0^1 w_{z_2;k_2}(t_2) dt_2 = 0,$$

because $k_2 \neq 0$, again by (2.59).

Now we assume the theorem holds for $n-1$ and we prove it for n. In particular, from (2.60) with $t = 0$ and (2.61) we have the inductive assumption

$$\int_{\Sigma_h} w_{z_h;k_h}(t_h)\dots w_{z_1;k_1}(t_1) d\mathcal{L}^h = 0, \quad h = 1,\dots,n-2. \tag{2.64}$$

We distinguish the cases $h = 1$ and $2 \leq h \leq n-1$. When $h = 1$, (2.60) is exactly the integration formula (2.57) with

$$g^1_{z_1,k_1} = \frac{1}{2k_1\pi} w_{z_1;k_1}(1).$$

The 1-periodicity of w-type functions also prove (2.61). In order to prove (2.62), we claim

$$\int_{\Sigma_{n-1}} w_{z_n;k_n}(t_n)\dots w_{z_2;k_2}(t_2) d\mathcal{L}^{n-1} = 0.$$

In fact,

$$\begin{aligned}\int_{\Sigma_{n-1}} w_{z_n;k_n}(t_n)\dots w_{z_2;k_2}(t_2) d\mathcal{L}^{n-1} = {} & g^1_{z_2,k_2} \int_{\Sigma_{n-2}} w_{z_n;k_n}(t_n)\dots w_{z_3;k_3}(t_3) d\mathcal{L}^{n-2} \\ & - \frac{1}{2k_2\pi} \int_{\Sigma_{n-2}} w_{z_n;k_n}(t_n)\dots w_{z_3;k_3}(t_3) w_{z_2;k_2}(t_3) d\mathcal{L}^{n-2}\end{aligned}$$

By the multiplication formula (2.58), in the second integral are involved the vectors of frequencies $(k_2 \pm k_3, k_4, \dots, k_n)$. Both of them satisfies (2.59), by assumption (2.59) itself. Then both the summands vanish thanks to the inductive assumption (2.64), proving our claim.

For $2 \leq h \leq n-1$, set

$$D_h(t) = \int_{\Sigma_h(t,1-t)} w_{z_h;k_h}(t_h) \dots w_{z_1;k_1}(t_1) d\mathcal{L}^h.$$

When $h \geq 2$, we use the inductive assumption (2.60) for $h-1$ and the multiplication formula (2.58) to obtain

$$\begin{aligned} D_h(t) &= \int_t^1 w_{z_h;k_h}(t_h) \Big(\int_{\Sigma_{h-1}(t_h,1-t_h)} w_{z_{h-1};k_{h-1}}(t_{h-1}) \dots w_{z_1;k_1}(t_1) d\mathcal{L}^{h-1} \Big) dt_h \\ &= \int_t^1 w_{z_h;k_h}(t_h) \Big(g^{h-1}_{\mathrm{z}_{h-1};\mathrm{k}_{h-1}} - \sum_{\alpha \in \mathcal{A}_{h-1}} c^{h-1}_\alpha(\mathrm{k}_{h-1}) w_{p_\alpha(i\mathrm{z}_{h-1});s_\alpha(\mathrm{k}_{h-1})} \Big) dt_h \\ &= \int_t^1 w_{z_h;k_h} g^{h-1}_{\mathrm{z}_{h-1};\mathrm{k}_{h-1}} dt_h - \frac{1}{2} \sum_{\alpha \in \mathcal{A}_{h-1}} c^{h-1}_\alpha(\mathrm{k}_{h-1}) \int_t^1 \big(w_{**} + w_{\dagger\dagger} \big) dt_h, \end{aligned}$$

where $** = z_h p_\alpha(i\mathrm{z}_{h-1}); s_\alpha(\mathrm{k}_{h-1}) + k_h$ and $\dagger\dagger = \bar{z}_h p_\alpha(i\mathrm{z}_{h-1}); s_\alpha(\mathrm{k}_{h-1}) - k_h$ satisfy

$$\begin{aligned} w_{**} &= -w_{ip_{(\alpha,1)}(i\mathrm{z}_h);s_{(\alpha,1)}(\mathrm{k}_h)}, \\ w_{\dagger\dagger} &= w_{ip_{(\alpha,-1)}(i\mathrm{z}_h);s_{(\alpha,-1)}(\mathrm{k}_h)}. \end{aligned}$$

By the integration formula (2.57), the function D_h equals

$$D_h = g^h_{\mathrm{z}_h,\mathrm{k}_h} - \frac{1}{4\pi} \sum_{\alpha \in \mathcal{A}_{h-1}} c^{h-1}_\alpha(\mathrm{k}_{h-1}) \Big(\frac{w_{p_{(\alpha,1)}(i\mathrm{z}_h);s_{(\alpha,1)}(\mathrm{k}_h)}}{s_{(\alpha,1)}(\mathrm{k}_h)} - \frac{w_{p_{(\alpha,-1)}(i\mathrm{z}_h);s_{(\alpha,-1)}(\mathrm{k}_h)}}{s_{(\alpha,-1)}(\mathrm{k}_h)} \Big),$$

where

$$g^h_{\mathrm{z}_h,\mathrm{k}_h}(t) = \int_t^1 w_{z_h;k_h} g^{h-1}_{\mathrm{z}_{h-1};\mathrm{k}_{h-1}} dt_h + c(\mathrm{z}_h, \mathrm{k}_h), \tag{2.65}$$

with $c(\mathrm{z}_h, \mathrm{k}_h)$ a constant that we are going to fix in a moment. Using $\mathcal{A}_h = \{(\alpha, 1) \mid \alpha \in \mathcal{A}_{h-1}\} \cup \{(\alpha, -1) \mid \alpha \in \mathcal{A}_{h-1}\}$, and the relations

$$\frac{1}{4\pi} c^{h-1}_\alpha(\mathrm{k}_{h-1}) \frac{1}{s_{(\alpha,1)}(\mathrm{k}_h)} = c^h_{(\alpha,1)}(\mathrm{k}_h) \quad \text{and} \quad \frac{1}{4\pi} c^{h-1}_\alpha(\mathrm{k}_{h-1}) \frac{1}{s_{(\alpha,-1)}(\mathrm{k}_h)} = -c^h_{(\alpha,-1)}(\mathrm{k}_h),$$

we conclude that

$$D_h(t) = g^h_{\mathrm{z}_h,\mathrm{k}_h}(t) - \sum_{\alpha \in \mathcal{A}_h} c^h_\alpha(\mathrm{k}_h) w_{p_\alpha(i\mathrm{z}_h);s_\alpha(\mathrm{k}_h)}(t).$$

This proves (2.60).

We are left with the proof of (2.61) and (2.62). The constant above is

$$c(\mathrm{z}_h, \mathrm{k}_h) = \sum_{\alpha\in\mathcal{A}_h} c_\alpha^h(\mathrm{k}_h) w_{p_\alpha(i\mathrm{z}_h);s_\alpha(\mathrm{k}_h)}(1).$$

By the 1-periodicity of $t \mapsto g^h_{\mathrm{z}_h,\mathrm{k}_h}(t)$, we have

$$g^h_{\mathrm{z}_h,\mathrm{k}_h}(0) = g^h_{\mathrm{z}_h,\mathrm{k}_h}(1) = c(\mathrm{z}_h, \mathrm{k}_h),$$

and this shows (2.61).

Finally, we check (2.62). By (2.65) it is sufficient to show that

$$\int_{\Sigma_{n-h}} w_{z_n;k_n}(t_n)\dots w_{z_{h+1};k_{h+1}}(t_{h+1})\Big(\int_{t_{h+1}}^1 w_{z_h;k_h} g^{h-1}_{\mathrm{z}_{h-1};\mathrm{k}_{h-1}} dt_h\Big) d\mathcal{L}^{n-h} = 0. \tag{2.66}$$

and

$$c(\mathrm{z}_h, \mathrm{k}_h)\int_{\Sigma_{n-h}} w_{z_n;k_n}(t_n)\dots w_{z_{h+1};k_{h+1}}(t_{h+1}) d\mathcal{L}^{n-h} = 0. \tag{2.67}$$

Identity (2.67) holds by (2.64) and identity (2.66) holds by the inductive validity of (2.62).

□

The explicit formula (2.63) for the constants $c_\alpha^h(\mathrm{k}_h)$ will be crucial in Lemma 2.40.

Corollary 2.38. *Let* $\mathrm{k}_n = (k_1, \dots, k_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$, *be a vector of frequencies satisfying* (2.59) *and assume there exists a unique* $\bar\alpha \in \mathcal{A}_n$ *of the form* $\bar\alpha = (\alpha, -1)$ *with* $\alpha \in \mathcal{A}_{n-1}$ *such that* $s_{\bar\alpha}(\mathrm{k}_n) = 0$. *Then we have*

$$\int_{\Sigma_n} w_{z_n;k_n}(t_n)\dots w_{z_1;k_1}(t_1) d\mathcal{L}^n = -\frac{1}{2} c_\alpha^{n-1}(\mathrm{k}_{n-1})\mathrm{Re}\big(\bar z_n p_\alpha(i\mathrm{z}_{n-1})\big). \tag{2.68}$$

Proof. Using formulas (2.60) and (2.62) we obtain

$$\begin{aligned}\int_{\Sigma_n} w_{z_n;k_n}\dots w_{z_1;k_1} d\mathcal{L}^n &= \int_0^1 w_{z_n;k_n}\Big(\int_{\Sigma(t_n,1-t_n)} w_{z_{n-1};k_{n-1}}\dots w_{z_1;k_1} d\mathcal{L}^{n-1}\Big) dt_n\\ &= \int_0^1 w_{z_n;k_n}\Big(g^{n-1}_{\mathrm{z}_{n-1};\mathrm{k}_{n-1}} - \sum_{\alpha\in\mathcal{A}_{n-1}} c_\alpha^{n-1}(\mathrm{k}_{n-1}) w_{p_\alpha(i\mathrm{z}_{n-1});s_\alpha(\mathrm{k}_{n-1})}\Big) dt_n\\ &= -c_\alpha^{n-1}(\mathrm{k}_{n-1})\int_0^1 w_{z_n;k_n} w_{p_\alpha(i\mathrm{z}_{n-1});s_\alpha(\mathrm{k}_{n-1})} dt_n.\end{aligned}$$

Now we use the multiplication formula (2.58). Only the one term with a resulting zero frequence contributes to the integral, and we get

$$\int_{\Sigma_n} w_{z_n;k_n}\dots w_{z_1;k_1} d\mathcal{L}^n = -\frac{1}{2} c_\alpha^{n-1}(\mathrm{k}_{n-1})\mathrm{Re}(\bar z_n p_\alpha(i\mathrm{z}_{n-1})).$$

□

Remark 2.39. Let $\mathrm{k}_n = (k_1, \dots, k_n) \in \mathbb{N}^n$ be a vector of frequencies such that

$$k_1 = \sum_{j=2}^{n} k_j \quad \text{and} \quad k_j > \sum_{\ell=j+1}^{n} k_\ell, \quad 2 \le j \le n-1. \tag{2.69}$$

Then k_n satisfies (2.59) and there exists a unique $\bar{\alpha} = (\alpha, -1) \in \mathcal{A}_n$ such that $s_{\bar{\alpha}}(\mathrm{k}_n) = 0$, and namely $\bar{\alpha} = (1, -1, \dots, -1)$.

Lemma 2.40. *There exists* $\mathrm{k}_n = (k_1, \dots, k_n) \in \mathbb{N}^n$ *as in* (2.69) *such that, with* $\bar{\alpha} = (1, -1, \dots, -1)$*, there holds*

$$|c^{n-1}_{\bar{\alpha}}(\mathrm{k}_{n-1})| > \sum_{\sigma \in S^1_n, \sigma \neq id} |c^{n-1}_{\bar{\alpha}}(k_{\sigma_1}, \dots, k_{\sigma_{n-1}})|. \tag{2.70}$$

Proof. Setting, for $\ell = 3, \dots, n$,

$$q(k_\ell, \dots, k_n) = \prod_{i=\ell}^{n} \frac{1}{k_\ell + \cdots + k_n},$$

by formula (2.63) and by the choice of k_1 in (2.69), we obtain

$$|c^{n-1}_{\bar{\alpha}}(\mathrm{k}_{n-1})| = \frac{2}{4^{n-1}\pi^{n-1}k_1} q(k_3, \dots, k_n),$$

and so inequality (2.70) is equivalent to

$$q(k_3, \dots, k_n) > \sum_{\sigma \in S^1_n, \sigma \neq id} q(k_{\sigma_3}, \dots, k_{\sigma_n}). \tag{2.71}$$

Notice that k_1 does not appear in (2.71), whereas k_2 may appear in the right hand side.

For $i = 1, \dots, n$, consider the set of permutations fixing $1, \dots, i$:

$$S^i_n = \{\sigma \in S_n \mid \sigma_1 = 1, \dots, \sigma_i = i\}.$$

We claim that there exist $k_2, \dots, k_n$ as in (2.69), such that for any $\ell = 3, \dots, n$ there holds

$$q(k_\ell, \dots, k_n) > \sum_{i=\ell-2}^{n-2} \sum_{\sigma \in S^i_n, \sigma(i+1) \neq i+1} q(k_{\sigma_\ell}, \dots, k_{\sigma_n}). \tag{2.72}$$

Claim (2.72) for $\ell = 3$ is exactly (2.71).

We prove (2.72) by induction on ℓ starting from $\ell = n$ and descending. When $\ell = n$, the sums in the right hand side of (2.72) reduce to the sum on one element, the permutation switching n and $n-1$. So, inequality (2.72) reads in this case

$$\frac{1}{k_n} = q(k_n) > q(k_{n-1}) = \frac{1}{k_{n-1}},$$

that holds as soon as $0 < k_n < k_{n-1}$.

By induction, assume that $k_\ell > \cdots > k_n$ are already fixed in such a way that (2.72) holds with $\ell + 1$ replacing ℓ. Notice that $k_{\ell-1}$ may and indeed does appear in the right hand side. We claim that there exists $k_{\ell-1} > k_\ell$ such that (2.72) holds.

We split the sum in (2.72) obtaining

$$q(k_\ell, \dots, k_n) > \sum_{i=\ell-1}^{n-2} \sum_{\sigma \in S_n^i, \sigma(i+1) \neq i+1} q(k_{\sigma_\ell}, \dots, k_{\sigma_n}) + \sum_{\sigma \in S_n^{\ell-2}, \sigma(\ell-1) \neq \ell-1} q(k_{\sigma_\ell}, \dots, k_{\sigma_n}), \tag{2.73}$$

and we consider the quantity

$$Q(k_\ell, \dots, k_n) = q(k_\ell, \dots, k_n) - \sum_{i=\ell-1}^{n-2} \sum_{\sigma \in S_n^i, \sigma(i+1) \neq i+1} q(k_{\sigma_\ell}, \dots, k_{\sigma_n})$$

A permutation $\sigma \in S_n^i$ with $i \geq \ell - 1$ fixes all the k_is with $i \leq \ell - 1$ and then we have

$$q(k_{\sigma\ell}, \dots, k_{\sigma n}) = \frac{1}{k_\ell + \cdots + k_n} q(k_{\sigma_{\ell+1}}, \dots, k_{\sigma_n}),$$

and thus

$$Q(k_\ell, \dots, k_n) = \frac{1}{k_\ell + \cdots + k_n} \Big(q(k_{\ell+1}, \dots, k_n) - \sum_{i=\ell-1}^{n-2} \sum_{\sigma \in S_n^i, \sigma(i+1) \neq i+1} q(k_{\sigma_{\ell+1}}, \dots, k_{\sigma_n}) \Big).$$

By our inductive assumption, we have $Q(k_\ell, \dots, k_n) > 0$. Notice that $Q(k_\ell, \dots, k_n)$ does not depend on $k_{\ell-1}$.

Conversely, every summand in the second sum in (2.73), i.e., every $q(k_{\sigma_\ell}, \dots, k_{\sigma_n})$, depends on $k_{\ell-1}$ and tends to 0 as $k_{\ell-1} \to \infty$. We conclude that for all large enough $k_{\ell-1} > k_\ell$ claim (2.73) holds. This ends the proof of (2.72).

□

Proof of Theorem 2.36. We claim that for each $\tau \in S_n^1$ there exists $u_\tau \in V_{n-1}$ such that the matrix $(I^\sigma(u_\tau))_{\sigma,\tau \in S_n^1}$ is strictly diagonally dominant and thus invertible.

Let $\mathrm{k}_n = (k_1, \dots, k_n) \in \mathbb{N}^n$ be a vector of frequencies as in (2.69) and satisfying the claim of Lemma 2.40 and choose complex numbers $\mathrm{z}_n = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that

$$-\frac{1}{2}\mathrm{Re}(\bar{z}_n p_\alpha(i\mathrm{z}_{n-1})) = 1,$$

where $\alpha = (1, -1, \dots, -1) \in \mathcal{A}_{n-1}$. The function $u = (w_{z_1;k_1}, \dots, w_{z_n;k_n})$ is in V_{n-1}, by Theorem 2.37, formulas (2.60) and (2.61). By formula (2.68) and by Lemma 2.40

$$|I^{id}(u)| = |c_{\bar{\alpha}}^{n-1}(\mathrm{k}_{n-1})| > \sum_{\sigma \in S_n^1, \sigma \neq id} |c_{\bar{\alpha}}^{n-1}(k_{\sigma_1}, \dots, k_{\sigma_{n-1}})| = \sum_{\sigma \in S_n^1, \sigma \neq id} |I^\sigma(u)|$$

For each $\tau \in S_n^1$, we define $u_\tau = (w_{z_{\bar{1}};k_{\bar{1}}}, \dots, w_{z_{\bar{n}};k_{\bar{n}}})$, where $\bar{\ell} = \tau^{-1}(\ell)$. As above, we have

$$|I^\tau(u_\tau)| > \sum_{\sigma \in S_n^1, \sigma \neq \tau} |I^\sigma(u_\tau)|.$$

This concludes the proof that $(I^\sigma(u_\tau))_{\sigma,\tau \in S_n^1}$ is strictly diagonally dominant.

□

2.8 Goh conditions of order n in the corank 1 case

Let $\Delta \subset TM$ be the distribution spanned point-wise by the vector fields $f_1, \dots, f_d$. For any $n \in \mathbb{N}$ and $q \in M$ we let

$$\Delta_n(q) = \mathrm{span}_\mathbb{R}\big\{[f_{s_n}, \dots, f_{s_1}](q) \mid s_1, \dots, s_n \in \{1, \dots, d\}\big\} \subset T_qM.$$

For a given $q \in M$, the annihilator of Δ_n is

$$\Delta_n^\perp(q) = \big\{\lambda \in T_q^*M \mid \lambda(v) = 0 \text{ for all } v \in \Delta_n(q)\big\}.$$

Theorem 2.41. *Let (M, Δ, g) be a sub-Riemannian manifold, $\gamma \in AC(I; M)$ be a horizontal curve with control $u \in X$, and $n \in \mathbb{N}$ be an integer with $n \geq 3$. Assume that:*

i) $\mathcal{D}_u^h F = 0$ for $h = 2, \dots, n-1$, where F is the end-point map starting from $\gamma(0)$;

ii) γ is a strictly singular length minimizing curve with corank 1.

Then any adjoint curve $\lambda \in AC(I; T^*M)$ satisfies

$$\lambda(t) \in \Delta_n^\perp(\gamma(t)) \quad \textit{for all } t \in I. \tag{2.74}$$

Proof. If γ is length minimizing then the extended end-point map F_J is not open at u, i.e., the extend variation map G_J is not open at 0. By Theorem 2.29 we consequently have $\mathcal{G}_{t_0}^n(v) = 0$ for all $t_0 \in I$ and for all $v \in V_{n-1}$. In order to use Theorem 2.29 we need both assumptions i) and ii). The map $\mathcal{G}_{t_0}^n$ is introduced in (2.39) and incorporates λ. The strict singularity of γ is used to translate the differential analysis from G_J to G.

We polarize the equation $\mathcal{G}_{t_0}^n(v) = 0$, as explained at the end of Section 2.5. The polarization, denoted by $\mathcal{T}$, is introduced in (2.46). We have $\mathcal{T} = 0$ on *linear* spaces contained in V_{n-1}. We fix any selection function $s : \{1, \dots, n\} \to \{1, \dots, d\}$ and we translate our claim (2.74) into the new claim

$$\langle \lambda(t), [f_{s_n}, \dots, f_{s_1}](\gamma(t)) \rangle = 0, \quad t \in I. \tag{2.75}$$

By formula (2.49) for $\mathcal{T}$ proved in Theorem 2.31, if we choose $v_1, \dots, v_n \in X$ as in (2.47) and such that the corresponding u satisfies $u \in V_{n-1}$, then the equation $\mathcal{T}(v_1, \dots, v_n) = 0$ reads

$$\sum_{\sigma \in S_n^1} \langle \lambda, [g_{s\sigma}^{t_0}] \rangle I^\sigma(u) = 0, \quad t_0 \in I. \tag{2.76}$$

We regard (2.76) as a linear equation in the unknowns $\langle \lambda, [g_{s\sigma}^{t_0}] \rangle$ with coefficients $I^\sigma(u)$.

By Theorem 2.36, for any $\tau \in S_n^1$ there exists $u_\tau \in V_{n-1}$ such that the matrix $(I^\sigma(u_\tau))_{\sigma,\tau \in S_n^1}$ is invertible. From (2.76), the definition of adjoint curve (see Definition 1.105), and (2.27) we deduce that for any $\sigma \in S_n^1$ and $t_0 \in I$

$$\begin{aligned} 0 = \langle \lambda, [g_{s\sigma}^{t_0}] \rangle &= \langle \lambda, [g_{s\sigma_n}^{t_0}, \dots, g_{s\sigma_1}^{t_0}] \rangle \\ &= \langle \lambda, [(P_{t_0}^1)_* f_{s\sigma_n}, \dots, (P_{t_0}^1)_* f_{s\sigma_1}] \rangle \\ &= \langle \lambda, (P_{t_0}^1)_* [f_{s\sigma_n}, \dots, f_{s\sigma_1}] \rangle \\ &= \langle (P_{t_0}^1)^* \lambda, [f_{s\sigma_n}, \dots, f_{s\sigma_1}](\gamma(t_0)) \rangle \\ &= \langle \lambda(t_0), [f_{s\sigma_n}, \dots, f_{s\sigma_1}](\gamma(t_0)) \rangle. \end{aligned}$$

This identity with $\sigma = id$ is (2.75).

□

Remark 2.42. We comment on assumption i), $\mathcal{D}_u^h F = 0$ for $h = 2, \dots, n-1$. This is equivalent to $\mathcal{D}_0^h G = 0$. Using formula (2.36), for $v \in V_{h-1}$,

$$0 = \mathcal{D}_0^h G(w_{t_0,s}) = \langle \lambda, D_0^h G(w_{t_0,s}) \rangle = c_h s^h \int_{\Sigma_h} \langle \lambda, [g_{v(t_h)}^{t_0}, \dots, g_{v(t_1)}^{t_0}] \rangle d\mathcal{L}^h + O(s^{h+1}).$$

Dividing by s^h and letting $s \to 0$ we deduce that $\mathcal{G}_{t_0}^h(v) = 0$. Now, as in the proof of Theorem 2.41 we conclude that $\langle \lambda(t_0), [f_{s_h}, \dots, f_{s_1}](\gamma(t_0)) \rangle = 0$, that is

$$\lambda(t) \in \Delta_h^\perp(\gamma(t)), \quad \text{for all } h = 2, \dots, n-1. \tag{2.77}$$

In particular, as a combination of (2.74) and (2.77), if γ is a strictly singular length minimizer satisfying i), then the associated adjoint curve annihilates all the brackets up to length n.

Remark 2.43. The inverse implication in Theorem 2.41 does not hold. Namely, a strictly singular curve satisfying assumption i) and (2.74) in Theorem 2.41 needs not be length-minimizing. A counterexample is given in the next section.

2.9 An example of singular extremal

On the manifold $M = \mathbb{R}^3$, with coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we consider the rank 2 distribution $\Delta^n = \text{span}\{f_1, f_2\}$, where f_1 and f_2 are the vector-fields defined by

$$f_1(x) = \frac{\partial}{\partial x_1}, \quad f_2(x) = (1 - x_1)\frac{\partial}{\partial x_2} + x_1^n \frac{\partial}{\partial x_3}. \tag{2.78}$$

The vector-field f_2 depends on the parameter $n \in \mathbb{N}$. We fix on Δ^n the metric g making f_1 and f_2 orthonormal.

In this section, we study the (local) length-minimality in $(\mathbb{R}^3, \Delta^n, g)$ of the curve $\gamma : I = [0, 1] \to \mathbb{R}^3$

$$\gamma(t) = (0, t, 0), \quad t \in I. \tag{2.79}$$

The curve γ is in fact defined for all $t \in \mathbb{R}$.

Our results rely upon the analysis of the variation map G based on the results of the previous sections. The minimality properties of γ are described in the following theorem.

Theorem 2.44. *For $n \in \mathbb{N}$, let us consider the sub-Riemannian manifold $(\mathbb{R}^3, \Delta^n, g)$ and the curve γ in (2.79).*

i) For any $n \geq 2$, γ is the unique strictly singular extremal in $(\mathbb{R}^3, \Delta^n)$ passing through the origin, up to reparameterization.

ii) If $n \geq 2$ is even, γ is locally length minimizing in $(\mathbb{R}^3, \Delta^n, g)$.

iii) If $n \geq 3$ is odd, γ is not length minimizing in $(\mathbb{R}^3, \Delta^n, g)$, not even locally.

Above, "locally length minimizing" means that short sub-arcs of γ are length minimizing for fixed end-points. Claims i) and ii) are well-known. In particular, claim ii) can be proved with a straightforward adaptation of Liu-Sussmann's argument for $n = 2$ in [19]. For $n = 3$, claim iii) is proved in [7] and here we prove the general case.

We compute the intrinsic differentials of G and we show that, for odd n, they satisfy the hypotheses of Theorem 2.15. This implies that the extended variation map G_J is open and, as a consequence, the non-minimality of γ.

We denote by γ^x the horizontal curve with control $u = (0, 1)$ and $\gamma^x(1) = x$, so that $\gamma^{\bar{q}} = \gamma$. By the formulas in (2.78) for the vector fields f_1 and f_2, we find

$$\gamma_1^x(t) = x_1, \quad \gamma_2^x(t) = (t - 1)(1 - x_1) + x_2, \quad \gamma_3^x(t) = (t - 1)x_1^n + x_3.$$

The "optimal flow" associated with γ is the 1-parameter family of diffeomorphisms $P_1^t \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$, $t \in \mathbb{R}$, defined by $P_1^t(x) = \gamma^x(t)$. For fixed $x \in \mathbb{R}^3$, the inverse of the differential of P_1^t is the map $P_1^t(x)_*^{-1} = P_t^1(x)_* : T_{\gamma^x(t)}\mathbb{R}^3 \to T_x\mathbb{R}^3$

$$P_t^1(x)_* = \begin{pmatrix} 1 & 0 & 0 \\ t-1 & 1 & 0 \\ -n(t-1)x_1^{n-1} & 0 & 1 \end{pmatrix}.$$

As explained in Section 2.4, see formula (2.33), the differential of G at 0 is

$$d_0G(v) = \int_0^1 g_{v(t)}^t(\bar{q})dt, \quad v = (v^1, v^2) \in L^2(I;\mathbb{R}^2).$$

Above, we set $g_{v(t)}^t = v^1(t)g_1^t + v^2(t)g_2^t$ where the vector fields g_1^t and g_2^t are

$$\begin{aligned} g_1^t &= P_t^1(x)_* f_1 = \frac{\partial}{\partial x_1} + (t-1)\frac{\partial}{\partial x_2} - n(t-1)x_1^{n-1}\frac{\partial}{\partial x_3}, \\ g_2^t &= P_t^1(x)_* f_2 = f_2 = (1-x_1)\frac{\partial}{\partial x_2} + x_1^n\frac{\partial}{\partial x_3}. \end{aligned}$$

So the differential is given by the formula

$$d_0G(v) = \begin{pmatrix} \displaystyle\int_0^1 v^1(t)dt \\ \displaystyle\int_0^1 \big\{(t-1)v^1(t) + v^2(t)\big\}dt \\ 0 \end{pmatrix}. \tag{2.80}$$

We deduce that a generator for $\mathrm{Im}(d_0G)^\perp$ is the covector $\lambda = (0,0,1)$, and that $v \in \ker(d_0G)$ if and only if

$$\int_0^1 v^1(t)dt = 0 \qquad \text{and} \qquad \int_0^1 (tv^1(t) + v^2(t))dt = 0.$$

In the computation of the differentials $\mathcal{D}_0^h G$, $h \geq 2$, we need the following lemma. For $y \in L^2(I;\mathbb{R})$ and $n \geq 2$, we let

$$\Gamma_y^n = \int_{\Sigma_n} y(t_1)\dots y(t_n)(t_2 - t_1)d\mathcal{L}^n.$$

Lemma 2.45. *For $n \geq 2$ and $y \in L^2(I;\mathbb{R})$ such that $\displaystyle\int_0^1 y(t)dt = 0$ we have*

$$\Gamma_y^n = \frac{1}{n!}\int_0^1 \left(\int_t^1 y(\tau)d\tau\right)^n dt.$$

Proof. We first observe that, integrating by parts, we have

$$\begin{aligned}\int_{t_2}^1 y(t_1)(t_2-t_1)dt_1 &= t_2\int_{t_2}^1 y(t_1)dt_1 - \int_{t_2}^1 t_1 y(t_1)dt_1\\ &= t_2\int_{t_2}^1 y(t_1)dt_1 - \left[s\int_s^1 y(t_1)dt_1\right]_{s=t_2} + \int_{t_2}^1\int_s^1 y(t_1)dt_1 ds\\ &= \int_{t_2}^1\int_s^1 y(t_1)dt_1 ds.\end{aligned}$$

Applying this identity to Γ_y^n and integrating by parts again, we get

$$\begin{aligned}\Gamma_y^n &= \int_0^1 y(t_n)\int_{t_n}^1\cdots\int_{t_2}^1\int_s^1 y(t_1)dt_1 ds dt_2\dots dt_n\\ &= \int_0^1 y(t_n)dt_n\int_0^1 y(t_{n-1})\int_{t_n}^1\cdots\int_{t_2}^1\int_s^1 y(t_1)dt_1dsdt_2\dots dt_{n-1}\\ &\quad -\int_0^1\left(\int_{t_n}^1 y(\tau)d\tau\right)y(t_n)\int_{t_n}^1 y(t_{n-2})\cdots\int_{t_2}^1\int_s^1 y(t_1)dt_1dsdt_2\dots dt_{n-2}dt_n\\ &= \frac{1}{2}\int_0^1\frac{d}{dt_n}\left(\int_{t_n}^1 y(\tau)d\tau\right)^2\int_{t_n}^1 y(t_{n-2})\cdots\int_{t_2}^1\int_s^1 y(t_1)dt_1dsdt_2\dots dt_{n-2}dt_n.\end{aligned}$$

In the last identity, we used our assumption $\int_0^1 y(t)dt = 0$. Now our claim follows by iterating this integration by parts argument. $\square$

Theorem 2.46. *Let $n\in\mathbb{N}$. The variation map G in $(\mathbb{R}^3,\Delta^n)$ satisfies:*

i) $\mathcal{D}_0^h G = 0$, for $h<n$;

ii) for any $v=(v_1,\dots,v_{n-1})\in\operatorname{dom}(\mathcal{D}_0^n G)$,

$$\mathcal{D}_0^n G(v) = \int_0^1\left(\int_t^1 v_1^1(\tau)d\tau\right)^n dt, \tag{2.81}$$

where v_1^1 is the first coordinate of v_1.

Proof. The Lie brackets of the vector fields g_1^t and $g_2^t = f_2$ are, at different times,

$$\begin{aligned}[][g_1^t, g_1^s] &= n(n-1)(t-s)x_1^{n-2}\frac{\partial}{\partial x_3},\\ [g_1^t, g_2^s] &= -\frac{\partial}{\partial x_2} + nx_1^{n-1}\frac{\partial}{\partial x_3}.\end{aligned}$$

Notice that the bracket in the latter line is time-independent. Then, for $3\le h\le n$ and $i_1,\dots,i_h\in\{1,2\}$, the iterated brackets of length h are

$$[g_{i_h}^{t_h},\dots,g_{i_1}^{t_1}] = \begin{cases} n\dots(n-h+1)(t_2-t_1)x_1^{n-h}\frac{\partial}{\partial x_3}, & \text{if } i_1=\dots=i_h=1,\\ n\dots(n-h+2)(t_2-t_1)x_1^{n-h+1}\frac{\partial}{\partial x_3}, & \text{if } i_2=\dots=i_h=1, \text{ and } i_1=2,\\ 0, & \text{otherwise.}\end{cases}$$

For $h < n$, the coefficient of $\partial/\partial x_3$ in the formulas above vanishes at the point $\bar{q} = (0, 1, 0)$ and thus the projection of these brackets along the covector $\lambda = (0, 0, 1)$ vanishes, for any $2 \leq h < n$,

$$\langle \lambda, [g_{i_h}^{t_h}, \dots, g_{i_1}^{t_1}](\bar{q}) \rangle = 0.$$

Using formulas (2.28) and (2.33), we deduce that for any $(v_1, \dots, v_{h-1}) \in \mathrm{dom}(\mathcal{D}_0^h G)$ we have

$$\mathcal{D}_0^h G(v_1, \dots, v_{h-1}) = \langle \lambda, D_0^h G(v_1, \dots, v_{h-1}, *)(\bar{q}) \rangle = 0,$$

proving claim i).

For $h = n$, the coefficient of $\partial/\partial x_3$ vanishes at $\bar{q}$ except for the case $i_1 = \cdots = i_n = 1$, that is

$$\begin{aligned} &\langle \lambda, [g_{i_n}^{t_n}, \dots, g_{i_1}^{t_1}](\bar{q}) \rangle = 0, \quad \text{if } i_j \neq 1 \text{ for some } j, \\ &\langle \lambda, [g_1^{t_n}, \dots, g_1^{t_1}](\bar{q}) \rangle = n!(t_2 - t_1). \end{aligned}$$

Then, for any $v = (v_1, \dots, v_{n-1}) \in \mathrm{dom}(\mathcal{D}_0^n G)$ we have

$$\begin{aligned} \mathcal{D}_0^n G(v) = \langle \lambda, D_0^n G(v_1, \dots, v_{n-1}, *) \rangle &= n! \int_{\Sigma_n} v_1^1(t_1) \dots v_1^1(t_n)(t_2 - t_1) d\mathcal{L}^n \\ &= \int_0^1 \left(\int_t^1 v_1^1(\tau) d\tau \right)^n dt. \end{aligned}$$

In the last identity we used Lemma 2.45. This proves claim ii). □

Before proving claim iii) of Theorem 2.44, we recall that $\ker(d_0 G_J) = \ker(d_0 G) \cap \ker(d_u J)$, where J is the energy functional (see Section 6). In particular, for any $v \in L^2(I; \mathbb{R}^2)$ we have

$$d_u J(v) = \int_0^1 \big(u^1(t) v^1(t) dt + u^2(t) v^2(t) \big) dt = \int_0^1 v^2(t) dt. \tag{2.82}$$

Proof of Theorem 2.44 - claim iii). Let $n \in \mathbb{N}$ be an odd integer. We claim that there exists $v = (v_1, \dots, v_{n-1}) \in \mathrm{dom}(\mathcal{D}_0^n G_J) \subset \mathrm{dom}(\mathcal{D}_0^n G)$ such that $\mathcal{D}_0^n G(v) \neq 0$. The inclusion of domains is ensured by (2.38). By Theorem 2.46 we have $\mathcal{D}_0^h G = 0$ for any $h < n$. Then from (2.37) it follows that also the extended map satisfies $\mathcal{D}_0^n G_J(v) \neq 0$ and $\mathcal{D}_0^h G_J = 0$ for $h < n$.

By Proposition 2.14 the differential $\mathcal{D}_0^n G_J$ is regular; here, we are using the fact that n is odd. By Theorem 2.15, G_J is open at zero and thus F_J is open at u. This implies that γ is not length minimizing.

So, the proof of our claim reduces to find a function $v_1 \in \ker(d_0 G_J)$ such that

$$\int_0^1 \left(\int_t^1 v_1^1(\tau) d\tau \right)^n dt \neq 0.$$

If such a control v_1 exists, then by Proposition 2.9 there also exist $v_2, \dots, v_{n-1} \in L^2(I; \mathbb{R}^2)$ such that $v = (v_1, \dots, v_{n-1}) \in \mathrm{dom}(\mathcal{D}_0^n G_J)$ and by (2.81) it follows $\mathcal{D}_0^n G(v) \neq 0$.

The condition $v_1 \in \ker(d_0 G_J)$ is equivalent to $d_0 G(v_1) = 0$ and $d_u J(v_1) = 0$. By (2.80) and (2.82) this means

$$\int_0^1 v_1^1(t) dt = \int_0^1 t v_1^1(t) dt = \int_0^1 v_1^2(t) dt = 0. \tag{2.83}$$

We choose any funtion v_1^2 with vanishing mean. Also choosing $v_1^1(t) = \chi_{[0,\frac{1}{2}]}(t) - 5\chi_{[\frac{1}{2},\frac{3}{4}]}(t) + 3\chi_{[\frac{3}{4},1]}(t)$, all the conditions in (2.83) are satisfied. Moreover, we have

$$\int_t^1 v_1^1(\tau) d\tau = -t\chi_{[0,\frac{1}{2}]}(t) + (5t-3)\chi_{[\frac{1}{2},\frac{3}{4}]}(t) - 3(t-1)\chi_{[\frac{3}{4},1]}(t),$$

and then, after a short computation,

$$\int_0^1 \left(\int_t^1 v_1^1(\tau) d\tau \right)^n dt = \frac{6}{5(n+1)4^n} \big(3^{n-1} - 2^{n-1} \big).$$

The last quantity is different from 0 for any odd $n \geq 3$, completing the proof.

$\square$

Remark 2.47. We briefly comment on claim ii) of Theorem 2.44. By formula (2.81), when n is even we have $\mathcal{D}_0^n G(v) \geq 0$ for any $v \in \mathrm{dom}(\mathcal{D}_0^n G)$. So condition i) in Proposition 2.14 cannot be satisfied. The differential $\mathcal{D}_0^n G$ is not regular in the sense of Definition 2.13 part i) and so the open mapping Theorem 2.15 does not work. Though not sufficient to prove the local minimality of γ, this is consistent with claim ii) of Theorem 2.44.

Remark 2.48. Claim iii) of Theorem 2.44 answers the question raised in Remark 2.43. By Theorem 2.46, the curve γ satisfies assumption i) of Theorem 2.41 for any $n \in \mathbb{N}$. On the other hand, the non-vanishing Lie brackets of the vector fields f_1 and f_2 are

$$[f_1, f_2] = -\frac{\partial}{\partial x_2} + n x^{n-1} \frac{\partial}{\partial x_3},$$
$$[\underbrace{f_1, \dots, f_1}_{(h-1)\text{-times}}, f_2] = n(n-1)\dots(n-h+2) x_1^{n-h+1} \frac{\partial}{\partial x_3}.$$

Then, for any $h \leq n$ we have

$$\langle \lambda, [\underbrace{f_1, \ldots, f_1}_{(h-1)\text{-times}}, f_2](\gamma(t)) \rangle = 0, \quad \text{for any } t \in I. \tag{2.84}$$

For $2 \leq h < n$, this is consistent with Remark 2.42: the vanishing of the h-th differential implies (2.84). When $h = n$, identity (2.84) is the Goh condition of order n in (2.75).

Thus, if n is odd the curve γ satisfies both assumption i) of Theorem 2.41 and condition (2.75). However, it is a strictly singular curve which is not length minimizing.

Chapter 3

New results on the nonminimality of spiral-like curves

In this chapter, we approach to the regularity problem following the strategy of studying the singularities of horizontal curves by creating a shorter competing curve with the same end points. In particular, we face-up the case of spiral-like curves. In the following it will be clarified why this is an interesting case to consider

The most elementary kind of singularity for a Lipschitz curve is of the corner-type: at a given point, the curve has a left and a right tangent that are linearly independent. In [18] and [12] it was proved that length minimizers cannot have singular points of this kind, namely

Theorem 3.1 (Hakavuori, Le Donne 2016)**.** *Let $\gamma : [0,1] \to M$ be a horizontal curve. If γ has corner-type singularity, then it cannot be length-minimizing.*

Here, we give a very basic idea of the proof when $M = \mathbb{R}^3$ is a sub-Riemannian manifold of dimension 3 and constant rank 2, as in the example of Section 1.2. Our computations in the case of the spirals are based on similar ideas.

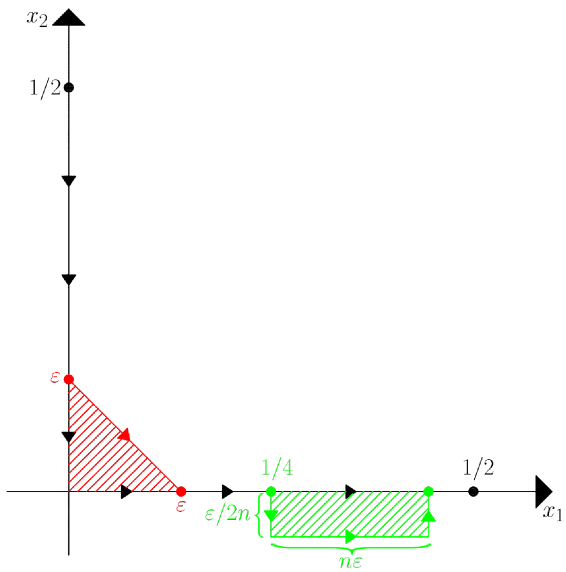
In this case, a horizontal curve is written $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, where the third coordinate of γ is the area of the graph of its horizontal projection $\kappa(t) = (\gamma_1(t), \gamma_2(t))$.

Suppose that γ has a corner, namely suppose for example that κ is a path walking through the coordinate axes, with corner singularity at the origin.

One builds a competitor curve $\bar{\gamma}$ depending on a fixed positive parameter ε and on an integer n, modifying the horizontal projection κ. This modification on κ also modifies the third coordinate of γ. Since the sub-Riemannian length of γ is the Euclidean length of κ, such a modification of γ is made through the following two steps

i) First, one cuts the original curve near the singularity, as in the red path of the figure below. This implies a gain of length for $\bar{\gamma}$, but it modifies the end point.

ii) In order to restore the end point, one has to correct the curve in such a way to have a positive gain of length. This step is realized by the green path in the figure below.



$$\kappa(t) = \begin{cases} (0, -t), & t \in [-1, 2, 0), \\ (t, 0), & t \in [0, 1/2]. \end{cases}$$

In this case, we have for n large enough a gain of length given by

$$\ell(\gamma) - \ell(\bar{\gamma}) = (2 - \sqrt{2} - \frac{1}{n})\varepsilon > 0.$$

This prove that the curve γ cannot be a length minimizer, because of the presence of a corner. When M has dimension n and rank m, one needs $n-m$ integer parameters $k_1, \ldots, k_{n-m}$ and $n-m$ associated correction paths to restore the end point, solving a system of linear equations.

The result of Theorem 3.1 has been improved in [22]: here, the authors define the tangent cone of a horizontal path as its blow-up (for the precise definition of the tangent cone, we refer the reader to the parer), then they prove the following statement.

Theorem 3.2 (Monti, Pigati, Vittone 2018). *Let $\gamma : [0,1] \to M$ be a horizontal length-minimizing curve. Then, for any $t \in [0,1]$, there is at least one horizontal line in the tangent cone of γ at the time t.*

The uniqueness of this tangent line for length minimizers is an open problem. Indeed, there exist other types of singularities related to the non-uniqueness of the tangent. In particular, there exist spiral-like curves whose tangent cone at the center contains many and in fact all tangent lines. For this reason, the study of spiral-like curves is very interesting, and also nontrivial: these curves may appear as Pontryagin extremals in sub-Riemannian geometry and theorem 3.2 is not enough to prove the nonminimality of spiral-like extremals.

Let (M, Δ, g) be an n-dimensional, $n \geq 3$, analytic sub-Riemannian manifold where $\Delta \subset TM$ has constant rank 2. Our notion of horizontal spiral in a sub-Riemannian manifold of rank 2 is fixed in Definition 3.3 below. The aim of this chapter is to show that spirals are not length-minimizing when the horizontal distribution Δ satisfies the following commutativity condition. Fix two vector fields $f_1, f_2 \in \Delta$ that are linearly independent at some point $p \in M$. For $k \in \mathbb{N}$ and for a multi-index $J = (j_1, \ldots, j_k)$, with $j_i \in \{1, 2\}$, we denote by $f_J = [f_{j_1}, [\ldots, [f_{j_{k-1}}, f_{j_k}] \cdots]]$ the iterated commutator associated with J. We define its length as the length of the multi-index J, i.e., $\mathrm{len}(f_J) = \mathrm{len}(J) = k$. Then, our commutativity assumption is that, in a neighborhood of the point p,

$$[f_I, f_J] = 0 \quad \text{for all multi-indices with } \mathrm{len}(I), \mathrm{len}(J) \geq 2. \tag{3.1}$$

This condition is not intrinsic and depends on the basis f_1, f_2 of the distribution Δ.

After introducing exponential coordinates of the second type, the vector fields f_1, f_2 can be assumed to be of the form (3.5) below, and the point p will be the center of the spiral. In coordinates we have $\gamma = (\gamma_1, \ldots, \gamma_n)$ and, by (3.5), the γ_j's satisfy for $j = 3, \ldots, n$ the following integral identities

$$\gamma_j(t) = \gamma_j(0) + \int_0^t a_j(\gamma(s))\dot{\gamma}_2(s)ds, \quad t \in [0, 1]. \tag{3.2}$$

When $\gamma(0)$ and γ_1, γ_2 are given, these formulas determine in a unique way the whole horizontal curve γ. We call $\kappa \in AC([0, 1]; \mathbb{R}^2)$, $\kappa = (\gamma_1, \gamma_2)$, the horizontal coordinates of γ.

Definition 3.3 (Spiral)**.** We say that a horizontal curve $\gamma \in AC([0, 1]; M)$ is a *spiral* if, in exponential coordinates of the second type centered at $\gamma(0)$, the horizontal coordinates $\kappa \in AC([0, 1]; \mathbb{R}^2)$ are of the form

$$\kappa(t) = t e^{i\varphi(t)}, \quad t \in]0, 1], \tag{3.3}$$

where $\varphi \in C^1(]0, 1]; \mathbb{R}^+)$ is a function, called *phase* of the spiral, such that $|\varphi(t)| \to \infty$ and $|\dot{\varphi}(t)| \to \infty$ as $t \to 0^+$. The point $\gamma(0)$ is called center of the spiral.

A priori, Definition 3.3 depends on the basis f_1, f_2 of Δ, see however our comments about its intrinsic nature in Remark 3.13. Without loss of generality, we shall focus our attention on spirals that are oriented clock-wise, i.e., with a phase satisfying

$\varphi(t) \to \infty$ and $\dot\varphi(t) \to -\infty$ as $t \to 0^+$. Such a phase is decreasing near 0. Notice that if $\varphi(t) \to \infty$ and $\dot\varphi(t)$ has a limit as $t \to 0^+$ then this limit must be $-\infty$.

Our main result in this chapter is the following

Theorem 3.4. *Let (M, Δ, g) be an analytic sub-Riemmanian manifold of rank 2 satisfying* (3.1)*. Any horizontal spiral $\gamma \in AC([0,1]; M)$ is not length-minimizing near its center.*

The nonminimality of spirals combined with the necessary conditions given by Pontryagin Maximum Principle is likely to give new regularity results on classes of sub-Riemannian manifolds, in the spirit of [4]. We think, however, that the main interest of Theorem 3.4 is in the deeper understanding that it provides on the loss of minimality caused by singularities.

The proof of Theorem 3.4 consists in constructing a competing curve shorter than the spiral. The construction uses exponential coordinates of the second type.

3.1 Exponential coordinates at the center of the spiral

In this section, we introduce in M exponential coordinates of the second type centered at a point $p \in M$, that will be the center of the spiral.

Let $f_1, f_2 \in \Delta$ be linearly independent at p. Since the distribution Δ is bracket-generating we can find vector-fields $f_3, \dots, f_n$, with $n = \dim(M)$, such that each f_i is an iterated commutator of f_1, f_2 with length $w_i = \mathrm{len}(f_i)$, $i = 3, \dots, n$, and such that $f_1, \dots, f_n$ at p are a basis for T_pM. By continuity, there exists an open neighborhood U of p such that $f_1(q), \dots, f_n(q)$ form a basis for T_qM, for any $q \in U$. We call $f_1, \dots, f_n$ a stratified basis of vector-fields in M.

Let $\varphi \in C^\infty(U; \mathbb{R}^n)$ be a chart such that $\varphi(p) = 0$ and $\varphi(U) = V$, with $V \subset \mathbb{R}^n$ open neighborhood of $0 \in \mathbb{R}^n$. Then $\widetilde{f_1} = \varphi_* f_1, \dots, \widetilde{f_n} = \varphi_* f_n$ is a system of point-wise linearly independent vector fields in $V \subset \mathbb{R}^n$. Since our problem has a local nature, we can without loss of generality assume that $M = V = \mathbb{R}^n$ and $p = 0$.

After these identifications, we have a stratified basis of vector-fields $f_1, \dots, f_n$ in $\mathbb{R}^n$. We say that $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ are exponential coordinates of the second type associated with the vector fields $f_1, \dots, f_n$ if we have

$$x = P^{f_1}_{x_1} \circ \cdots \circ P^{f_n}_{x_n}(0), \quad x \in \mathbb{R}^n. \tag{3.4}$$

Here, as usual for all this thesis, we are using the notation $P^X_s = \exp(sX)$, $s \in \mathbb{R}$, to denote the flow of a vector-field X. From now on, we assume without loss of generality that $f_1, \dots, f_n$ are complete and induce exponential coordinates of the second type.

We define the homogeneous degree of the coordinate x_i of $\mathbb{R}^n$ as $w_i = \mathrm{len}(f_i)$. We introduce the 1-parameter group of dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda > 0$,

$$\delta_\lambda(x) = (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n), \qquad x \in \mathbb{R}^n,$$

and we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is δ-homogeneous of degree $w \in \mathbb{N}$ if $f(\delta_\lambda(x)) = \lambda^w f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. An example of δ-homogeneous function of degree 1 is the pseudo-norm

$$\|x\| = \sum_{j=1}^{n} |x_i|^{1/w_i}, \quad x \in \mathbb{R}^n.$$

The following theorem is proved in [13] in the case of general rank. A more modern approach to nilpotentization can be found in [5] and [14].

Theorem 3.5. *Let $\Delta = \mathrm{span}\{f_1, f_2\} \subset TM$ be an analytic distribution of rank 2. In exponential coordinates of the second type around a point $p \in M$ identified with $0 \in \mathbb{R}^n$, the vector fields f_1 and f_2 have the form*

$$\begin{aligned} f_1(x) &= \partial_{x_1}, \\ f_2(x) &= \partial_{x_2} + \sum_{j=3}^{n} a_j(x) \partial_{x_j}, \end{aligned} \tag{3.5}$$

for $x \in U$, where U is a neighborhood of 0. The analytic functions $a_j \in C^\infty(U)$, $j = 3, \dots, n$, have the structure $a_j = p_j + r_j$, where:

(i) p_j are δ-homogeneous polynomials of degree $w_j - 1$ such that $p_j(0, x_2, \dots, x_n) = 0$;

(ii) $r_j \in C^\infty(U)$ are analytic functions such that, for some constants $C_1, C_2 > 0$ and for $x \in U$,

$$|r_j(x)| \le C_1 \|x\|^{w_j} \quad \textit{and} \quad |\partial_{x_i} r_j(x)| \le C_2 \|x\|^{w_j - w_i}.$$

Proof. The proof that $a_j = p_j + r_j$ where p_j are polynomials as in (i) and the remainders r_j are real-analytic functions such that $r_j(0) = 0$ can be found in [13]. The proof of (ii) is also implicitly contained in [13]. Here, we add some details. The Taylor series of r_j has the form

$$r_j(x) = \sum_{\ell = w_j}^{\infty} r_{j\ell}(x) = \sum_{\ell = w_j}^{\infty} \sum_{\alpha \in \mathcal{A}_\ell} c_{\alpha\ell} x^\alpha,$$

where $\mathcal{A}_\ell = \{\alpha \in \mathbb{N}^n : \alpha_1 w_1 + \ldots + \alpha_n w_n = \ell\}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $c_{\alpha\ell} \in \mathbb{R}$ are constants. Here and in the following, $\mathbb{N} = \{0, 1, 2, \ldots\}$. The series converges absolutely in a small homogeneous cube $Q_\delta = \{x \in \mathbb{R}^n : \|x\| \leq \delta\}$ for some $\delta > 0$, and in particular

$$\sum_{\ell=w_j}^{\infty} \delta^\ell \sum_{\alpha \in \mathcal{A}_\ell} |c_{\alpha\ell}| < \infty.$$

Using the inequality $|x^\alpha| \leq \|x\|^\ell$ for $\alpha \in \mathcal{A}_\ell$, for $x \in Q_\delta$ we get

$$|r_j(x)| \leq C_1 \|x\|^{w_j}, \quad \text{with } C_1 = \sum_{\ell=w_j}^{\infty} \delta^{\ell - w_j} \sum_{\alpha \in \mathcal{A}_\ell} |c_\alpha| < \infty.$$

The estimate for the derivatives of r_j is analogous. Indeed, we have

$$\partial_{x_i} r_j(x) = \sum_{\ell=w_j}^{\infty} \sum_{\alpha \in \mathcal{A}_\ell} \alpha_i c_{\alpha\ell} x^{\alpha - \mathrm{e}_i},$$

where $\alpha - \mathrm{e}_i \in \mathcal{A}_{\ell - w_i}$ whenever $\alpha \in \mathcal{A}_\ell$. Above, $\mathrm{e}_i = (0, \ldots, 1, \ldots, 0)$ with 1 at position i is the canonical ith versor of $\mathbb{R}^n$. Thus the leading term in the series has homogeneous degree $w_j - w_i$ and repeating the argument above we get the estimate $|\partial_{x_i} r_j(x)| \leq C_2 \|x\|^{w_j - w_i}$ for $x \in Q_\delta$. □

Remark 3.6. Let $\Delta = \operatorname{span}\{f_1, f_2\}$ be as in Theorem 3.5. In exponential coordinates of the second type, we consider the vector fields

$$\hat{f}_1(x) = f_1(x) = \partial_{x_1},$$
$$\hat{f}_2(x) = \partial_{x_2} + \sum_{j=3}^{n} p_j(x) \partial_{x_j}.$$

It is implicitly proved in [13] that the Lie algebra generated by $\hat{\Delta} = \operatorname{span}\{\hat{f}_1, \hat{f}_2\}$ is nilpotent. In particular, when $(\mathbb{R}^n, \Delta)$ is an equiregular sub-Riemannian structure, then $(\mathbb{R}^n, \hat{\Delta})$ is a Carnot group.

Definition 3.7. Let $\Delta = \operatorname{span}\{f_1, f_2\}$ be as in Theorem 3.5 and let $\hat{\Delta} = \operatorname{span}\{\hat{f}_1, \hat{f}_2\}$ be as in Remark 3.6. Then $\hat{f}_2$, $\hat{\Delta}$ and $(\mathbb{R}^n, \hat{\Delta})$ are called, respectively, the nilpotent approximations of f_2, Δ and $(\mathbb{R}^n, \Delta)$.

When the distribution Δ satisfies the commutativity assumption (3.1) the coefficients a_j appearing in the vector-field f_2 in (3.5) enjoy additional properties. In the next theorem, the specific structure of exponential coordinates of the second type will be very helpful in the computation of various derivatives. In particular, in Lemma 3.9 we need a nontrivial formula from [13, Appendix A], given in such coordinates.

Theorem 3.8. *Let $\Delta \subset TM$ be an analytic distribution of rank 2. Then the functions $a_3, \dots, a_n$ of Theorem 3.5 depend only on the variables x_1 and x_2 if and only if Δ satisfies* (3.1).

Proof. If the functions $a_3, \dots, a_n$ of Theorem 3.5 depend only on the variables x_1 and x_2, for every $I = (i_1, \dots, i_k)$ with $\mathrm{len}(I) \geq 2$, we have

$$f_I = \sum_{i=3}^{n} a_{i,I}(x)\partial_{x_i},$$

where $a_{i,I}(x) = a_{i,I}(x_1, x_2)$ and so $[f_I, f_J] = 0$ for every I, J with $\mathrm{len}(I) \geq 2$.

Suppose now that Δ satisfies (3.1). Let $\Gamma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the map $\Gamma(t, x) = P_t^{f_2}(x)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Here, we are using the exponential coordinates (3.4). In the following we omit the composition sign $\circ$. Defining $\Theta : \mathbb{R}^3 \times \mathbb{R}^n \to \mathbb{R}^n$ as the map $\Theta_{t,x_1,x_2}(p) = P^{f_2}_{-(x_2+t)} P^{f_1}_{-x_1} P^{f_2}_t P^{f_1}_{x_1} P^{f_2}_{x_2}(p)$, we have

$$\Gamma(t, x) = P^{f_1}_{x_1} P^{f_2}_{x_2+t} \Theta_{t,x_1,x_2} P^{f_3}_{x_3} \dots P^{f_n}_{x_n}(0).$$

We claim that there exists a $C > 0$ independent of t such that, for $t \to 0$,

$$|\Theta_{t,x_1,x_2} P^{f_j}_s - P^{f_j}_s \Theta_{t,x_1,x_2}| \leq Ct^2. \tag{3.6}$$

We will prove claim (3.6) in Lemma 3.9 below. From (3.6) it follows that there exist mappings $R_t \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\Gamma(t, x) = P^{f_1}_{x_1} P^{f_2}_{x_2+t} P^{f_3}_{x_3} \dots P^{f_n}_{x_n} \Theta_{t,x_1,x_2}(0) + R_t(x), \tag{3.7}$$

and such that $|R_t| \leq Ct^2$ for $t \to 0$.

By the structure (3.5) of the vector fields f_1 and f_2 and since Θ_{t,x_1,x_2} is the composition of C^∞ maps, there exist C^∞ functions $f_j = f_j(t, x_1, x_2)$ such that

$$\Theta_{t,x_1,x_2}(0) = \big(0, 0, f_3(t, x_1, x_2), \dots, f_n(t, x_1, x_2)\big) = \exp\Big(\sum_{j=3}^{n} f_j(t, x_1 x_2) f_j\Big)(0). \tag{3.8}$$

By (3.1), from (3.7) and (3.8) we obtain

$$\begin{aligned}\Gamma(t, x) &= P^{f_1}_{x_1} P^{f_2}_{x_2+t} \exp\Big(\sum_{i=3}^{n} (x_j + f_j(t, x_1, x_2)) f_j\Big)(0) + R_t(x) \\ &= \big(x_1, x_2 + t, x_3 + f_3(t, x_1, x_2), \dots, x_n + f_n(t, x_1, x_2)\big) + R_t(x),\end{aligned}$$

and we conclude that

$$f_2(x) = \frac{d}{dt}\Gamma(x, t)\Big|_{t=0} = \partial_2 + \sum_{j=3}^{n} \frac{d}{dt} f_j(t, x_1, x_2)\Big|_{t=0} \partial_j.$$

Thus the coefficients $a_j(x_1, x_2) = \frac{d}{dt} f_j(t, x_1, x_2)|_{t=0}$, $j = 3, \dots, n$, depend only on the first two variables, completing the proof. $\square$

In the following lemma, we prove our claim (3.6).

Lemma 3.9. *Let $\Delta \subset TM$ be an analytic distribution satisfying* (3.1)*. Then for any $j = 3, \dots, n$ the claim in* (3.6) *holds.*

Proof. Let $f = f_j$ for any $j = 3, \dots, n$ and define the map $T^f_{t,x_1,x_2;s} = \Theta_{t,x_1,x_2} P^f_s - P^f_s \Theta_{t,x_1,x_2}$. For $t = 0$ the map Θ_{0,x_1,x_2} is the identity and thus $T^f_{0,x_1,x_2;s} = 0$. So, claim (3.6) follows as soon as we show that

$$\dot{T}^f_{0,x_1,x_2;s} = \frac{\partial}{\partial t}\Big|_{t=0} T^f_{t,x_1,x_2;s} = 0,$$

for any $s \in \mathbb{R}$ and for all $x_1, x_2 \in \mathbb{R}$.

We first compute the derivative of Θ_{t,x_1,x_2} with respect to t. Letting $\Psi_{t,x_1} = P^{f_1}_{-x_1} P^{f_2}_t P^{f_1}_{x_1}$ we have $\Theta_{t,x_1,x_2} = P^{f_2}_{-(x_2+t)} \Psi_{t,x_1} P^{f_2}_{x_2}$, and, thanks to [13, Appendix A], the derivative of Ψ_{t,x_1} at $t = 0$ is

$$\dot{\Psi}_{0,x_1} = \sum_{\nu=0}^{\infty} c_{\nu,x_1} W_\nu,$$

where $W_\nu = [f_1, [\cdots, [f_1, f_2] \cdots]]$ with f_1 appearing ν times and $c_{\nu,x_1} = (-1)^\nu x_1^\nu / \nu!$. In particular, we have $c_{0,x_1} = 1$. Then the derivative of Θ_{t,x_1,x_2} at $t = 0$ is

$$\begin{aligned} \dot{\Theta}_{0,x_1,x_2} &= -f_2 + dP^{f_2}_{-x_2}\big(\dot{\Psi}_{0,x_1}(P^{f_2}_{x_2})\big) \\ &= -f_2 + \sum_{\nu=0}^{\infty} c_{\nu,x_1} dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2})\big) \\ &= \sum_{\nu=1}^{\infty} c_{\nu,x_1} dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2})\big), \end{aligned}$$

because the term in the sum with $\nu = 0$ is $dP^{f_2}_{-x_2}\big(f_2(P^{f_2}_{x_2})\big) = f_2$. Inserting this formula for $\dot{\Theta}_{0,x_1,x_2}$ into

$$\dot{T}^f_{0,x_1,x_2;s} = \dot{\Theta}_{0,x_1,x_2}(P^f_s) - dP^f_s(\dot{\Theta}_{0,x_1,x_2}),$$

we obtain

$$\begin{aligned} \dot{T}^f_{0,x_1,x_2;s} =& \sum_{\nu=1}^{\infty} c_{\nu,x_1} dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2} P^f_s)\big) - dP^f_s \sum_{\nu=1}^{\infty} c_{\nu,x_1} dP^{f_2}_{-x_2}\big(W_\nu\big(P^{f_2}_{x_2}\big)\big) \\ =& dP^f_s \sum_{\nu=1}^{\infty} c_{\nu,x_1} \Big(dP^f_{-s} dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2} P^f_s)\big) - dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2})\big)\Big). \end{aligned}$$

In order to prove that $\dot{T}^f_{0,x_1,x_2;s}$ vanishes for all x_1, x_2 and s, we have to show that

$$g(x_2, s) := dP^f_{-s} dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2} P^f_s)\big) - dP^{f_2}_{-x_2}\big(W_\nu(P^{f_2}_{x_2})\big) = 0, \tag{3.9}$$

for any $\nu \geq 1$ and for any x_2 and s. From $P_0^f = \mathrm{id}$ it follows that $g(x_2, 0) = 0$. Then, our claim (3.9) is implied by

$$h(x_2, s) := \frac{\partial}{\partial s} g(x_2, s) = 0. \tag{3.10}$$

Actually, this is a Lie derivative and, namely,

$$h(x_2, s) = -dP_{-s}^f \big[f, dP_{-x_2}^{f_2}\big(W_\nu(P_{x_2}^{f_2})\big)\big].$$

Notice that $h(0, s) = -dP_{-s}^f[f, W_\nu] = 0$ by our assumption (3.1). In a similar way, for any $k \in \mathbb{N}$ we have

$$\frac{\partial^k}{\partial x_2^k} h(0, s) = (-1)^{k+1} dP_{-s}^f[f, [f_2, \cdots [f_2, W_\nu] \cdots]] = 0,$$

with f_2 appearing k times. Since the function $x_2 \mapsto h(x_2, s)$ is analytic our claim (3.10) follows. □

We conclude this sections with some general remarks.

Remark 3.10. Every Carnot group $\mathbb{G}$ of step less or equal than 3 satisfies (3.1) because, due to the stratification of the Lie algebra, every bracket of length greater or equal than 4 vanish. If $\mathbb{G}$ has rank 2, then it satisfies (3.1) also if it has step 4, but this is false in general.

Remark 3.11. This remark answer the following question asked by one of the referee of this thesis:
"If (M, Δ) has constant rank 2, step 3, dimension 5 and satisfies (3.1)*, is it necessarily a Carnot group?"*
The answer is negative and here we provide a counterexample.

Consider in $\mathbb{R}^5$ the distribution $\Delta = \mathrm{span}\{f_1, f_2\}$ with

$$f_1(x) = \partial_{x_1}, \quad f_2(x) = \partial_{x_2} + (x_1 + r_3(x_1))\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4} + x_1 x_2 \partial_{x_5}.$$

Here we are using the notation $a_j(x) = p_j(x) + r_j(x)$, $j = 3, 4, 5$, of Theorem 3.5, assuming $r_4 = r_5 = 0$ and r_3 depending only on the variable x_1.

By Theorem 3.8, Δ satisfies (3.1) and, since $r_3 \neq 0$, Δ cannot be the distribution of a Carnot group. Moreover we have

$$\begin{aligned} f_3 &= [f_1, f_2] = (1 + r_3'(x_1))\partial_{x_3} + x_1 \partial_{x_4} + x_2 \partial_{x_5}, \\ f_4 &= [f_1, f_3] = r_3''(x_1)\partial_{x_3} + \partial_{x_4}, \\ f_5 &= \partial_{x_5}. \end{aligned}$$

The Hörmander condition on $\mathbb{R}^5$ is satisfied by Δ if

$$\det\begin{pmatrix} 1+r_3'(x_1) & r_3''(x_1) \\ x_1 & 1 \end{pmatrix} \neq 0.$$

In partiular, it is sufficient to have

$$r_3'(x_1) - x_1 r_3''(x_1) \geq -\frac{1}{2},$$

where r_3 has to satisfy (ii) of Theorem 3.5. For instance, we can take $r_3(x_1) = -\frac{1}{3}x_1^3$.

Remark 3.12. By Theorem 3.8, we can assume that $a_j(x) = a_j(x_1, x_2)$ are functions of the variables x_1, x_2. In this case, formula (3.2) for the coordinates of a horizontal curve $\gamma \in AC([0,1]; M)$ reads, for $j = 3, \dots, n$,

$$\gamma_j(t) = \gamma_j(0) + \int_0^t a_j(\gamma_1(s), \gamma_2(s))\dot{\gamma}_2(s)ds, \quad t \in [0,1]. \tag{3.11}$$

Remark 3.13. Definition 3.3 of horizontal spiral is stable with respect to change of coordinates in the following sense.

After fixing exponential coordinates, we have that $0 \in \mathbb{R}^n$ is the center of the spiral $\gamma : [0,1] \to \mathbb{R}^n$, with horizontal projection $\mathrm{k}(t)$ as in Definition 3.3.

We consider a diffeomorphism $\widetilde{F} \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $\widetilde{F}(0) = 0$. In the new coordinates, our spiral γ is $\zeta(t) = \widetilde{F}(\gamma(t))$. We define the horizontal coordinates of ζ in the following way: the set $d_0\widetilde{F}(\Delta(0))$, where $d_0\widetilde{F}$ is the differential of $\widetilde{F}$ at 0, is a 2-dimensional subspace of $\mathbb{R}^n = \mathrm{Im}(d_0\widetilde{F})$; denoting by $\pi : \mathbb{R}^n \to d_0\widetilde{F}(\Delta(0))$ the orthogonal projection, we define the horizontal coordinates of ζ as $\xi(t) = \pi(\widetilde{F}(\gamma(t))$.

We claim that ξ, in the plane $d_0\widetilde{F}(\Delta(0))$, is of the form (3.3), with a phase ω satisfying $|\omega| \to \infty$ and $|\dot{\omega}| \to \infty$. In particular, these properties of ξ are stable up to isometries of the plane. Then, we can assume that $\xi(t) = (F_1(\gamma(t)), F_2(\gamma(t)))$, with $F_i : \mathbb{R}^n \to \mathbb{R}$ of class C^∞, for $i = 1, 2$. In this setting, we will show that $|\dot{\omega}| \to \infty$.

The function $s(t) = |\xi(t)| = |(F_1(\gamma(t)), F_2(\gamma(t)))|$ satisfies

$$0 < c_0 \leq \dot{s}(t) \leq c_1 < \infty, \quad t \in (0,1]. \tag{3.12}$$

Define the function $\omega \in C^1((0,1])$ letting $\xi(t) = s(t)\mathrm{e}^{i\omega(s(t))}$. Then differentiating the identity obtained inverting

$$\tan\big(\omega(s(t))\big) = \frac{F_2(\gamma(t))}{F_1(\gamma(t))}, \quad t \in (0,1],$$

we obtain

$$\dot{s}(t)\dot{\omega}(s(t)) = \frac{1}{s(t)^2}\langle P(\gamma(t)), \dot{\gamma}(t)\rangle, \qquad t \in (0,1], \tag{3.13}$$

where the function $P(x) = F_1(x)\nabla F_2(x) - F_2(x)\nabla F_1(x)$ has the Taylor development as $x \to 0$

$$P(x) = \langle \nabla F_1(0), x\rangle \nabla F_2(0) - \langle \nabla F_2(0), x\rangle \nabla F_1(0) + O(|x|^2).$$

Observe that from (3.11) it follows that $|\dot{\gamma}_j(t)| = O(t)$ for $j \geq 3$. Denoting by $\bar{\nabla}$ the gradient in the first two variables, we deduce that as $t \to 0^+$ we have

$$\langle P(\gamma), \dot{\gamma}\rangle = \langle F_1(\gamma)\bar{\nabla} F_2(\gamma) - F_2(\gamma)\bar{\nabla} F_1(\gamma), \dot{\kappa}\rangle + O(t^2) \tag{3.14}$$

with

$$F_1(\gamma)\bar{\nabla} F_2(\gamma) - F_2(\gamma)\bar{\nabla} F_1(\gamma) = \langle \bar{\nabla} F_1(0), \kappa\rangle \bar{\nabla} F_2(0) - \langle \bar{\nabla} F_2(0), \kappa\rangle \bar{\nabla} F_1(0) + O(t^2).$$

Inserting the last identity and $\dot{\kappa} = \mathrm{e}^{i\varphi} + it\dot{\varphi}\mathrm{e}^{i\varphi}$ into (3.14), after some computations we obtain

$$\langle P(\gamma), \dot{\gamma}\rangle = \dot{\varphi} t^2 \det(d_0\bar{F}(0)) + O(t^2),$$

where $\det(d_0\bar{F}(0)) \neq 0$ is the determinant Jacobian at $x_1 = x_2 = 0$ of the mapping $(x_1, x_2) \mapsto (F_1(x_1, x_2, 0), F_2(x_1, x_2, 0))$. Now the claim $|\dot{\omega}(s)| \to \infty$ as $s \to 0^+$ easily follows from (3.12), (3.13) and from $|\dot{\varphi}(t)| \to \infty$ as $t \to 0^+$.

Example 3.14. An interesting example of horizontal spiral is the double-logarithm spiral, the horizontal lift of the curve κ in the plane of the form (3.3) with phase $\varphi(t) = \log(-\log t)$, $t \in (0, 1/2]$. In this case, we have

$$\dot{\varphi}(t) = \frac{1}{t \log t}, \quad t \in (0, 1/2],$$

and clearly $\varphi(t) \to \infty$ and $\dot{\varphi}(t) \to -\infty$ as $t \to 0^+$. In fact, we also have $t\dot{\varphi} \in L^\infty(0, 1/2)$, which means that κ and thus γ is Lipschitz continuous. This spiral has the following additional properties:

i) for any $v \in \mathbb{R}^2$ with $|v| = 1$ there exists an infinitesimal sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that $\kappa(\lambda_n t)/\lambda_n \to tv$ locally uniformly, as $n \to \infty$;

ii) for any infinitesimal sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}$ there exists a subsequence and a $v \in \mathbb{R}^2$ with $|v| = 1$ such that $\kappa(\lambda_{n_k} t)/\lambda_{n_k} \to tv$ as $k \to \infty$, locally uniformly.

This means that the tangent cone of κ at $t = 0$ consists of all half-lines in $\mathbb{R}^2$ emanating from 0.

3.2 Cut and correction devices

In this section, we begin the construction of the competing curve. Let γ be a spiral with horizontal coordinates κ as in (3.3). We can assume that φ is decreasing and that $\varphi(1) = 1$ and we denote by $\psi : [1, \infty) \to (0, 1]$ the inverse function of φ. For $k \in \mathbb{N}$ and $\eta \in [0, 2\pi)$ we define $t_{k\eta} \in (0, 1]$ as the unique solution to the equation $\varphi(t_{k\eta}) = 2\pi k + \eta$, i.e., we let $t_{k\eta} = \psi(2\pi k + \eta)$. The times

$$t_k = t_{k0} = \psi(2\pi k), \quad k \in \mathbb{N},$$

will play a special role in our construction. The points $\kappa(t_k)$ are in the positive x_1-axis.

For a fixed $k \in \mathbb{N}$, we cut the curve k in the interval $[t_{k+1}, t_k]$ following the line segment joining $\mathrm{k}(t_{k+1})$ to $\mathrm{k}(t_k)$ instead of the path k, while we leave unchanged the remaining part of the path. We call this new curve $\mathrm{k}_k^{\mathrm{cut}}$ and, namely, we let

$$\begin{aligned} \mathrm{k}_k^{\mathrm{cut}}(t) &= \kappa(t) \quad \text{for} \quad t \in [0, t_{k+1}] \cup [t_k, 1], \\ \mathrm{k}_k^{\mathrm{cut}}(t) &= (t, 0) \quad \text{for} \quad t \in [t_{k+1}, t_k]. \end{aligned}$$

We denote by $\gamma_k^{\mathrm{cut}} \in AC([0, 1]; M)$ the horizontal curve with horizontal coordinates $\mathrm{k}_k^{\mathrm{cut}}$ and such that $\gamma_k^{\mathrm{cut}}(0) = \gamma(0)$. For $t \in [0, t_{k+1}]$, we have $\gamma_k^{\mathrm{cut}}(t) = \gamma(t)$. To correct the errors produced by the cut on the end-point, we modify the curve $\mathrm{k}_k^{\mathrm{cut}}$ using a certain number of devices. The construction is made by induction.

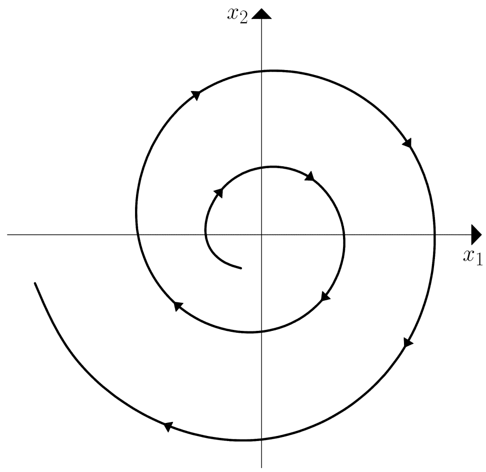
We start with the base construction. Let $\mathscr{E} = (h, \eta, \varepsilon)$ be a triple such that $h \in \mathbb{N}$, $0 < \eta < \pi/4$, and $\varepsilon \in \mathbb{R}$. Starting from a curve $\kappa : [0, 1] \to \mathbb{R}^2$, we define the curve $\mathrm{D}(\kappa; \mathscr{E}) : [0, 1 + 2|\varepsilon|] \to \mathbb{R}^2$ in the following way:

$$\mathrm{D}(\kappa; \mathscr{E})(t) = \begin{cases} \kappa(t) & t \in [0, t_{h\eta}] \\ \kappa(t_{h\eta}) + (\mathrm{sgn}(\varepsilon)(t - t_{h\eta}), 0) & t \in [t_{h\eta}, t_{h\eta} + |\varepsilon|] \\ \kappa(t - |\varepsilon|) + (\varepsilon, 0) & t \in [t_{h\eta} + |\varepsilon|, t_h + |\varepsilon|] \\ \kappa(t_h) + (2\varepsilon + \mathrm{sgn}(\varepsilon)(t_h - t), 0) & t \in [t_h + |\varepsilon|, t_h + 2|\varepsilon|] \\ \kappa(t - 2|\varepsilon|) & t \in [t_h + 2|\varepsilon|, 1 + 2|\varepsilon|]. \end{cases} \tag{3.15}$$

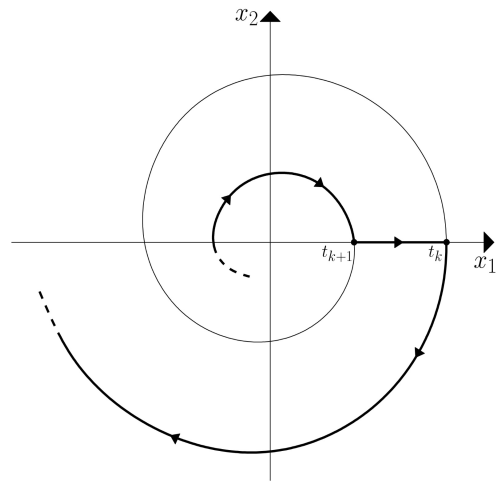
We denote by $\mathrm{D}(\gamma; \mathscr{E})$ the horizontal curve with horizontal coordinates $\mathrm{D}(\mathrm{k}; \mathscr{E})$. We let $\dot{\mathrm{D}}(\gamma; \mathscr{E}) = \frac{d}{dt}\mathrm{D}(\gamma; \mathscr{E})$ and we indicate by $\mathrm{D}_i(\gamma; \mathscr{E})$ the i-th coordinate of the corrected curve in exponential coordinates.

In the lifting formula (3.11), the intervals where $\dot{\gamma}_2 = 0$ do not contribute to the integral. For this reason, in (3.15) we may cancel the second and fourth lines, where $\dot{\mathrm{D}}_2(\gamma; \mathscr{E}) = 0$, and then reparameterize the curve on $[0, 1]$. Namely, we define the discontinuous curve $\overline{\mathrm{D}}(\mathrm{k}; \mathscr{E}) : [0, 1] \to \mathbb{R}^2$ as

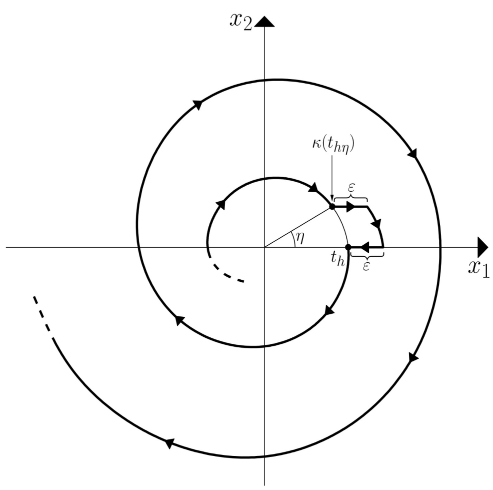
$$\overline{\mathrm{D}}(\mathrm{k}; \mathscr{E})(t) = \begin{cases} \kappa(t) & t \in [0, t_{h\eta}] \\ \kappa(t) + (\varepsilon, 0) & t \in (t_{h\eta}, t_h) \\ \kappa(t) & t \in [t_h, 1]. \end{cases}$$



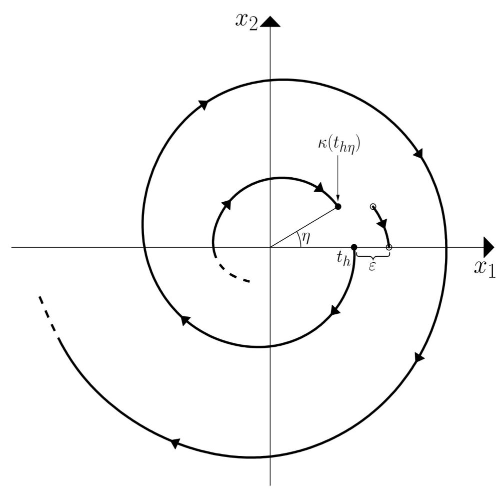
The curve $\kappa(t)$.



The curve $\kappa^{\mathrm{cut}}(t)$.



The curve $\mathrm{D}(\kappa;\mathscr{E})(t)$.



The curve $\overline{\mathrm{D}}(\kappa;\mathscr{E})(t)$.

The "formal" i-th coordinate of the curve $\overline{\mathrm{D}}(\mathrm{k};\mathscr{E})$ is given by

$$\overline{\mathrm{D}}_i(\gamma;\mathscr{E})(t) = \int_0^t a_i(\overline{\mathrm{D}}(\kappa;\mathscr{E})(s))\dot{\kappa}_2(s)ds.$$

The following identities with $\varepsilon > 0$ can be checked by an elementary computation

$$\overline{\mathrm{D}}(\gamma;\mathscr{E})(t) = \begin{cases} \mathrm{D}(\gamma;\mathscr{E})(t) & t \in [0, t_{h\eta}] \\ \mathrm{D}(\gamma;\mathscr{E})(t+\varepsilon) & t \in (t_{h\eta}, t_h) \\ \mathrm{D}(\gamma;\mathscr{E})(t+2\varepsilon) & t \in [t_h, 1]. \end{cases} \tag{3.16}$$

When $\varepsilon < 0$ there are similar identities. With this notation, the final error produced on the i-th coordinate by the correction device $\mathscr{E}$ is

$$\gamma_i(1) - \mathrm{D}_i(\gamma;\mathscr{E})(1+2|\varepsilon|) = \int_0^1 \left\{a_i(\kappa(s)) - a_i(\overline{\mathrm{D}}(\kappa;\mathscr{E})(s))\right\}\dot{\kappa}_2(s)ds. \tag{3.17}$$

The proof of this formula is elementary and can be omitted.

We will iterate the above construction a certain number of times depending on a collections of triples $\mathscr{E}$. We first fix the number of triples and iterations.

For $i = 3, \ldots, n$, let $\mathcal{B}_i = \{(\alpha, \beta) \in \mathbb{N}^2 : \alpha + \beta = w_i - 2\}$, where $w_i \geq 2$ is the homogeneous degree of the coordinate x_i. Then, the polynomials p_i given by Theorem 3.5 and Theorem 3.8 are of the form

$$p_i(x_1, x_2) = \sum_{(\alpha,\beta)\in\mathcal{B}_i} c_{\alpha\beta}\, x_1^{\alpha+1} x_2^{\beta}, \tag{3.18}$$

for suitable constants $c_{\alpha\beta} \in \mathbb{R}$. We set

$$\ell = \sum_{i=3}^{n} \mathrm{Card}(\mathcal{B}_i),$$

and we consider an $(\ell - 2)$-tuple of triples $\bar{\mathscr{E}} = (\mathscr{E}_3, \ldots, \mathscr{E}_\ell)$ such that $h_\ell < h_{\ell-1} < \ldots < h_3 < k$. Each triple is used to correct one monomial.

Without loss of generality, we simplify the construction in the following way. In the sum (3.18), we can assume that $c_{\alpha\beta} = 0$ for all $(\alpha, \beta) \in \mathcal{B}_i$ but one. Namely, we can assume that

$$p_i(x_1, x_2) = x_1^{\alpha_i+1} x_2^{\beta_i} \quad \text{with} \quad \alpha_i + \beta_i = w_i - 2, \tag{3.19}$$

and with $c_{\alpha_i\beta_i} = 1$. In this case, we have $\ell = n$ and we will use $n - 2$ devices associated with the triples $\mathscr{E}_3, \ldots, \mathscr{E}_n$ to correct the coordinates $i = 3, \ldots, n$. By the bracket generating property of the vector fields f_1 and f_2 and by the stratified basis property for $f_1, \ldots, f_n$, the pairs (α_i, β_i) satisfy the following condition

$$(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j) \quad \text{for} \quad i \neq j. \tag{3.20}$$

In the general case (3.18), we use a larger number $\ell \geq n$ of devices, one for each monomial $x_1^{\alpha+1} x_2^{\beta}$ appearing in $p_3(x_1, x_2), \ldots, p_n(x_1, x_2)$, and we correct the error produced by the cut on each monomial. The argument showing the nonminimality of the spiral will be the same. So, from now on in the rest of the Chapter we will assume that the polynomials p_i are of the form (3.19) with (3.20).

Now we clarify the inductive step of our construction. Let $\mathscr{E}_3 = (h_3, \eta_3, \varepsilon_3)$ be a triple such that $h_3 < k$. We define the curve $\kappa^{(3)} = \mathrm{D}(\mathrm{k}_k^{\mathrm{cut}}; \mathscr{E}_3)$. Given a triple $\mathscr{E}_4 = (h_4, \eta_4, \varepsilon_4)$ with $h_4 < h_3$ we then define $\kappa^{(4)} = \mathrm{D}(\mathrm{k}^{(3)}; \mathscr{E}_4)$. By induction on $\ell \in \mathbb{N}$, given a triple $\mathscr{E}_\ell = (h_\ell, \eta_\ell, \varepsilon_\ell)$ with $h_\ell < h_{\ell-1}$, we define $\kappa^{(\ell)} = \mathrm{D}(\mathrm{k}^{(\ell-1)}; \mathscr{E}_\ell)$. When $\ell = n$ we stop.

We define the planar curve $\mathrm{D}(\mathrm{k}; k, \bar{\mathscr{E}}) \in AC([0, 1 + 2\bar{\varepsilon}]; \mathbb{R}^2)$ as $\mathrm{D}(\mathrm{k}; k, \bar{\mathscr{E}}) = \kappa^{(n)}$ according to the inductive construction explained above, where $\bar{\varepsilon} = |\varepsilon_3| + \ldots + |\varepsilon_n|$.

Then we call $\mathrm{D}(\gamma; k, \bar{\mathscr{E}}) \in AC([0, 1+2\bar{\varepsilon}]; M)$, the horizontal lift of $\mathrm{D}(\mathrm{k}; k, \bar{\mathscr{E}})$ with $\mathrm{D}(\gamma; k, \mathscr{E})(0) = \gamma(0)$, the modified curve of γ associated with $\bar{\mathscr{E}}$ and with cut of parameter $k \in \mathbb{N}$. There is a last adjustment to do. In $[0, 1+2\bar{\varepsilon}]$ there are $2(n-2)$ subintervals where $\dot{\kappa}_2^{(n)} = 0$. On each of these intervals the coordinates $\mathrm{D}_j(\gamma; k, \bar{\mathscr{E}})$ are constant. According to the procedure explained in (3.15)–(3.16), we erase these intervals and we parameterize the resulting curve on $[0, 1]$. We denote this curve by $\bar{\gamma} = \overline{\mathrm{D}}(\gamma; k, \bar{\mathscr{E}})$.

Definition 3.15 (Adjusted modification of γ)**.** We call the curve $\bar{\gamma} = \overline{\mathrm{D}}(\gamma; k, \bar{\mathscr{E}}) : [0, 1] \to M$ the adjusted modification of γ relative to the collections of devices $\bar{\mathscr{E}} = (\mathscr{E}_3, \dots, \mathscr{E}_n)$ and with cut of parameter k.

Our next task is to compute the error produced by cut and devices on the end-point of the spiral. For $i = 3, \dots, n$ and for $t \in [0, 1]$ we let

$$\Delta_i^\gamma(t) = a_i(\mathrm{k}(t))\dot{\mathrm{k}}_2(t) - a_i(\bar{\mathrm{k}}(t))\dot{\bar{\mathrm{k}}}_2(t).$$

When $t < t_{k+1}$ or $t > t_k$ we have $\dot{\kappa}_2 = \dot{\bar{\mathrm{k}}}_2$ and so the definition above reads

$$\Delta_i^\gamma(t) = \big(a_i(\mathrm{k}(t)) - a_i(\bar{\mathrm{k}}(t))\big)\dot{\kappa}_2(t).$$

By the recursive application of the argument used to obtain (3.17), we get the following formula for the error at the final time $\bar{t} = t_{h_n}$:

$$\begin{aligned} E_i^{k,\bar{\mathscr{E}}} &= \gamma_i(\bar{t}) - \bar{\gamma}_i(\bar{t}) = \int_{t_{k+1}}^{\bar{t}} \Delta_i^\gamma(t)dt \\ &= \int_{F_k} \Delta_i^\gamma(t)dt + \sum_{j=3}^{n} \Big(\int_{A_j} \Delta_i^\gamma(t)dt + \int_{B_j} \Delta_i^\gamma(t)dt \Big). \end{aligned} \tag{3.21}$$

In (3.21) and in the following, we use the following notation for the intervals:

$$F_k = [t_{k+1}, t_k], \quad A_j = [t_{h_{j-1}}, t_{h_j\eta_j}], \quad B_j = [t_{h_j\eta_j}, t_{h_j}],$$

with $t_{h_2} = t_k$. We used also the fact that on $[0, t_{k+1}]$ we have $\gamma = \bar{\gamma}$.

On the interval F_k we have $\dot{\bar{\mathrm{k}}}_2 = 0$ and thus

$$\int_{F_k} \Delta_i^\gamma \, dt = \int_{F_k} \big\{p_i(\mathrm{k}) + r_i(\mathrm{k})\big\}\dot{\mathrm{k}}_2 dt. \tag{3.22}$$

On the intervals A_j we have $\kappa = \bar{\kappa}$ and thus

$$\int_{A_j} \Delta_i^\gamma dt = 0, \tag{3.23}$$

because the functions a_i depend only on k. Finally, on the intervals B_j we have $\bar{\mathrm{k}}_1 = \mathrm{k}_1 + \varepsilon_j$ and $\kappa_2 = \bar{\kappa}_2$ and thus

$$\int_{B_j} \Delta_i^\gamma \, dt = \int_{B_j} \{p_i(\mathrm{k}) - p_i(\mathrm{k} + (\varepsilon_j, 0))\}\dot{\kappa}_2 dt + \int_{B_j} \{r_i(\mathrm{k}) - r_i(\mathrm{k} + (\varepsilon_j, 0))\}\dot{\kappa}_2 dt. \quad (3.24)$$

Our goal is to find $k \in \mathbb{N}$ and devices $\bar{\mathscr{E}}$ such that $E_i^{k,\bar{\mathscr{E}}} = 0$ for all $i = 3, \dots, n$ and such that the modified curve $\mathrm{D}(\gamma; k, \bar{\mathscr{E}})$ is shorter than γ.

3.3 Effect of cut and devices on monomials and remainders

Let γ be a horizontal spiral with horizontal coordinates $\mathrm{k} \in AC([0,1]; \mathbb{R}^2)$ of the form (3.3). We prove some estimates about the integrals of the polynomials (3.19) along the curve k. These estimates are preliminary to the study of the errors introduced in (3.21).

For $\alpha, \beta \in \mathbb{N}$, we associate with the monomial $p_{\alpha\beta}(x_1, x_2) = x_1^{\alpha+1} x_2^\beta$ the function $\gamma_{\alpha\beta}$ defined for $t \in [0,1]$ by

$$\gamma_{\alpha\beta}(t) = \int_{\mathrm{k}|_{[0,t]}} p_{\alpha\beta}(x_1, x_2) dx_2 = \int_0^t p_{\alpha\beta}(\mathrm{k}(s)) \dot{\mathrm{k}}_2(s) ds.$$

When $p_i = p_{\alpha\beta}$, the function $\gamma_{\alpha\beta}$ is the leading term in the i-th coordinate of γ in exponential coordinates. In this case, the problem of estimating $\gamma_i(t)$ reduces to the estimate of integrals of the form

$$I_{\omega\eta}^{\alpha\beta} = \int_{t_\eta}^{t_\omega} \mathrm{k}_1(t)^{\alpha+1} \mathrm{k}_2(t)^\beta \dot{\mathrm{k}}_2(t) dt, \quad (3.25)$$

where $\omega \leq \eta$ are angles, $t_\omega = \psi(\omega)$ and $t_\eta = \psi(\eta)$. For $\alpha, \beta \in \mathbb{N}$, $h \in \mathbb{N}$ and $\eta \in (0, \pi/4)$ we also let

$$\jmath_{h\eta}^{\alpha\beta} = \eta^\beta \int_{2h\pi}^{2h\pi+\eta} t_\vartheta^{\alpha+\beta+2} \, d\vartheta = \int_{t_{h\eta}}^{t_h} t^{\alpha+\beta+2} |\dot{\varphi}(t)| dt, \quad (3.26)$$

where in the second equality we set $\vartheta = \varphi(t)$.

Proposition 3.16. *There exist constants $0 < c_{\alpha\beta} < C_{\alpha\beta}$ depending on $\alpha, \beta \in \mathbb{N}$ such that for all $h \in \mathbb{N}$ and $\eta \in (0, \pi/4)$ we have*

$$c_{\alpha\beta} \jmath_{h\eta}^{\alpha\beta} \leq |I_{2h\pi, 2h\pi+\eta}^{\alpha\beta}| \leq C_{\alpha\beta} \jmath_{h\eta}^{\alpha\beta}. \quad (3.27)$$

Before proving this proposition, we notice that the integrals $I^{\alpha\beta}_{\omega\eta}$ in (3.25) are related to the integrals

$$J^{\alpha\beta}_{\omega\eta} = \int_{\omega}^{\eta} t_{\vartheta}^{\alpha+\beta+2} \cos^{\alpha}(\vartheta) \sin^{\beta}(\vartheta) d\vartheta. \tag{3.28}$$

Lemma 3.17. *For any $\alpha, \beta \in \mathbb{N}$ and $\omega \leq \eta$ we have the identity*

$$(\alpha+\beta+2) I^{\alpha\beta}_{\omega\eta} = t_{\omega}^{\alpha+\beta+2} D^{\alpha\beta}_{\omega} - t_{\eta}^{\alpha+\beta+2} D^{\alpha\beta}_{\eta} - (\alpha+1) J^{\alpha\beta}_{\omega\eta}, \tag{3.29}$$

where we set $D^{\alpha\beta}_{\omega} = \cos^{\alpha+1}(\omega)\sin^{\beta+1}(\omega)$.

Proof. Inserting into $I^{\alpha\beta}_{\omega\eta}$ the identities $\mathrm{k}_1(t) = t\cos(\varphi(t))$, $\mathrm{k}_2(t) = t\sin(\varphi(t))$, and $\dot{\mathrm{k}}_2(t) = \sin(\varphi(t)) + t\cos(\varphi(t))\dot{\varphi}(t)$ we get

$$I^{\alpha\beta}_{\omega\eta} = \int_{t_\eta}^{t_\omega} t^{\alpha+\beta+1} D^{\alpha\beta}_{\varphi(t)} dt + \int_{t_\eta}^{t_\omega} t^{\alpha+\beta+2} \cos^{\alpha+2}(\varphi(t)) \sin^{\beta}(\varphi(t)) \dot{\varphi}(t) dt,$$

and, integrating by parts in the first integral, this identity reads

$$\begin{aligned} I^{\alpha\beta}_{\omega\eta} = & \left[\frac{t^{\alpha+\beta+2} D^{\alpha\beta}_{\varphi(t)}}{\alpha+\beta+2} \right]_{t_\eta}^{t_\omega} + \frac{\alpha+1}{\alpha+\beta+2} \int_{t_\eta}^{t_\omega} t^{\alpha+\beta+2} \cos^{\alpha}(\varphi(t)) \sin^{\beta+2}(\varphi(t)) \dot{\varphi}(t) dt \\ & - \frac{\beta+1}{\alpha+\beta+2} \int_{t_\eta}^{t_\omega} t^{\alpha+\beta+2} \cos^{\alpha+2}(\varphi(t)) \sin^{\beta}(\varphi(t)) \dot{\varphi}(t) dt \\ & + \int_{t_\eta}^{t_\omega} t^{\alpha+\beta+2} \cos^{\alpha+2}(\varphi(t)) \sin^{\beta}(\varphi(t)) \dot{\varphi}(t) dt. \end{aligned}$$

Grouping the trigonometric terms and then performing the change of variable $\varphi(t) = \vartheta$, we get

$$I^{\alpha\beta}_{\omega\eta} = \left[\frac{t_{\vartheta}^{\alpha+\beta+2} D^{\alpha\beta}_{\vartheta}}{\alpha+\beta+2} \right]_{\eta}^{\omega} + \frac{\alpha+1}{\alpha+\beta+2} \int_{\eta}^{\omega} t_{\vartheta}^{\alpha+\beta+2} \cos^{\alpha}(\vartheta) \sin^{\beta}(\vartheta) d\vartheta.$$

This is our claim. □

Proof of Proposition 3.16. From (3.29) with $D^{\alpha\beta}_{2h\pi} = 0$ we obtain

$$(\alpha+\beta+2) |I^{\alpha\beta}_{2h\pi, 2h\pi+\eta}| = t_{2h\pi+\eta}^{\alpha+\beta+2} D^{\alpha\beta}_{\eta} + (\alpha+1) J^{\alpha\beta}_{2h\pi, 2h\pi+\eta},$$

where $c_{\alpha\beta}\eta^{\beta+1} \leq D^{\alpha\beta}_{\eta} \leq \eta^{\beta+1}$, because $\eta \in (0, \pi/4)$, and

$$c_{\alpha\beta}\eta^{\beta+1} t_{2h\pi+\eta}^{\alpha+\beta+2} \leq c_{\alpha\beta}\eta^{\beta} \int_{2h\pi}^{2h\pi+\eta} t_{\vartheta}^{\alpha+\beta+2} d\vartheta \leq J^{\alpha\beta}_{2h\pi, 2h\pi+\eta} \leq \eta^{\beta} \int_{2h\pi}^{2h\pi+\eta} t_{\vartheta}^{\alpha+\beta+2} d\vartheta.$$

The claim follows. □

Remark 3.18. We will use the estimates (3.27) in the proof of the solvability of the end-point equations. In particular, the computations above are possible thanks to the structure of the monomials p_i: here, their dependence only on the variables x_1 and x_2, ensured by (3.1), is crucial. When the coefficients a_i depend on all the variables $x_1, \dots, x_n$, repeating the same computations seems difficult. Indeed, in the integrals (3.25) and (3.28) there are also the coordinates $\gamma_3, \dots, \gamma_n$. Then, the new identity (3.29) becomes more complicated because other addends appear after the integration by parts, owing to the derivatives of $\gamma_3, \dots, \gamma_n$. Now, by the presence of these new terms the estimates from below in (3.27) are difficult, while the estimates from above remain possible.

We denote by k_ε the rigid translation by $\varepsilon \in \mathbb{R}$ in the x_1 direction of the curve k. Namely, we let $\mathrm{k}_{\varepsilon,1} = \mathrm{k}_1 + \varepsilon$ and $\mathrm{k}_{\varepsilon,2} = \mathrm{k}_2$. Recall the notation $t_h = \psi(2\pi h)$ and $t_{h\eta} = \psi(2\pi h + \eta)$, for $h \in \mathbb{N}$ and $\eta > 0$. In particular, when we take ε_j, h_j and η_j related to the j-th correction-device, we have $\mathrm{k}_{\varepsilon_j}|_{B_j} = \bar{\mathrm{k}}|_{B_j}$.

In the study of the polynomial part of integrals in (3.24) we need estimates for the quantities

$$\Delta^{\alpha\beta}_{h\eta\varepsilon} = \int_{\mathrm{k}_\varepsilon|_{[t_{h\eta}, t_h]}} p_{\alpha\beta}(x_1, x_2)dx_2 - \int_{\mathrm{k}|_{[t_{h\eta}, t_h]}} p_{\alpha\beta}(x_1, x_2)dx_2.$$

Lemma 3.19. *We have*

$$\Delta^{\alpha\beta}_{h\eta\varepsilon} = (\alpha+1)\varepsilon I^{\alpha-1,\beta}_{2h\pi, 2h\pi+\eta} + O(\varepsilon^2), \tag{3.30}$$

where $O(\varepsilon^2)/\varepsilon^2$ is bounded as $\varepsilon \to 0$.

Proof. The proof is an elementary computation:

$$\begin{aligned}
\Delta^{\alpha\beta}_{h\eta\varepsilon} &= \int_{t_{h\eta}}^{t_h} \dot{\mathrm{k}}_2(t)\mathrm{k}_2(t)^\beta \left[(\mathrm{k}_1(t)+\varepsilon)^{\alpha+1} - \mathrm{k}_1(t)^{\alpha+1}\right] dt \\
&= \sum_{i=0}^{\alpha} \binom{\alpha+1}{i} \varepsilon^{\alpha+1-i} \int_{t_{h\eta}}^{t_h} \dot{\mathrm{k}}_2(t)\mathrm{k}_1(t)^i \mathrm{k}_2(t)^\beta dt \\
&= \sum_{i=0}^{\alpha} \binom{\alpha+1}{i} \varepsilon^{\alpha+1-i} I^{i-1,\beta}_{2h\pi, 2h\pi+\eta} \\
&= (\alpha+1)\varepsilon I^{\alpha-1,\beta}_{2h\pi, 2h\pi+\eta} + O(\varepsilon^2).
\end{aligned}$$

□

We estimate the terms in (3.22). The quantities Δ^γ_i are introduced in (3.30).

Lemma 3.20. *Let γ be a horizontal spiral with phase φ. For all $i = 3, \dots, n$ and for all $k \in \mathbb{N}$ large enough we have*

$$\Big| \int_{F_k} \Delta_i^\gamma dt \Big| \le \int_{F_k} t^{\alpha_i+\beta_i+2} |\dot\varphi| dt. \tag{3.31}$$

Proof. By (3.29) with vanishing boundary contributions, we obtain

$$\begin{aligned} \Big| \int_{F_k} p_i(\kappa)\dot\kappa_2 dt \Big| &= |I^{\alpha_i\beta_i}_{2k\pi,2(k+1)\pi}| = \frac{\alpha_i+1}{\alpha_i+\beta_i+2} |J^{\alpha_i\beta_i}_{2k\pi,2(k+1)\pi}| \\ &\le \frac{\alpha_i+1}{\alpha_i+\beta_i+2} \int_{F_k} t^{\alpha_i+\beta_i+2} |\dot\varphi| dt, \end{aligned}$$

so we are left with the estimate of the integral of r_i. Using $\kappa_2 = t\sin(\varphi(t))$ we get

$$\begin{aligned} \int_{F_k} r_i(\kappa)\dot\kappa_2 dt &= \int_{F_k} r_i(\kappa)(\sin(\varphi) + t\cos(\varphi)\dot\varphi) dt \\ &= \int_{F_k} (tr_i(\kappa) - R_i)\cos(\varphi)\dot\varphi dt, \end{aligned}$$

where we let

$$R_i(t) = \int_{t_{k+1}}^t r_i(\kappa) ds.$$

From (3.3), we have $|\kappa(t)| \le t$ for all $t \in [0,1]$. By part (ii) of Theorem 3.5 we have $|r_i(x)| \le C\|x\|^{w_i}$ for all $x \in \mathbb{R}^n$ near 0, with $w_i = \alpha_i + \beta_i + 2$. It follows that $|r_i(\kappa(t))| \le Ct^{w_i}$ for all $t \in [0,1]$, and $|R_i(t)| \le Ct^{w_i+1}$. We deduce that

$$\Big| \int_{F_k} r_i(\kappa)\dot\kappa_2 dt \Big| \le C \int_{F_k} t^{\alpha_i+\beta_i+3} |\dot\varphi| dt,$$

and the claim follows. $\square$

Now we study the integrals in (3.24). Let us introduce the following notation

$$\Delta^\gamma_{r_i} = \big(r_i(\mathrm{k}) - r_i(\bar{\mathrm{k}})\big)\dot\kappa_2.$$

Lemma 3.21. *Let γ be a horizontal spiral with phase φ. Then for any $j = 3, \dots, n$ and for $|\varepsilon_j| < t_{h_j\eta_j}$, we have*

$$\Big| \int_{B_j} \Delta^\gamma_{r_i}(t) dt \Big| \le C|\varepsilon_j| \int_{B_j} t^{w_i} |\dot\varphi(t)| dt,$$

where $C > 0$ is constant.

Proof. For $t \in B_j$ we have $\mathrm{k}_2(t) = \bar{\mathrm{k}}_2(t)$ and $\bar{\mathrm{k}}_1(t) = \mathrm{k}_1(t) + \varepsilon_j$. By Lagrange Theorem it follows that

$$\delta^\gamma_{r_i} := r_i(\mathrm{k}) - r_i(\bar{\mathrm{k}}) = \varepsilon_j \partial_1 r_i(\mathrm{k}^*(t)),$$

where $\mathrm{k}^*(t) = (\mathrm{k}_1^*(t), \mathrm{k}_2(t))$ and $\mathrm{k}_1^*(t) = \mathrm{k}_1(t) + \sigma_j$, $0 < \sigma_j < \varepsilon_j$. By Theorem 3.5 we have $|\partial_1 r_i(x)| \leq C\|x\|^{w_i - 1}$ and so, also using $\sigma_j < \varepsilon_j < t$,

$$|\partial_1 r_i(\mathrm{k}^*(t))| \leq C\|\mathrm{k}^*(t)\|^{w_i-1} = C\Big(|\mathrm{k}_1(t) + \sigma_j| + |\mathrm{k}_2(t)|\Big)^{w_i-1} \leq Ct^{w_i-1}.$$

This implies $|\delta^\gamma_{r_i}(t)| \leq C|\varepsilon_j| t^{w_i-1}$.

Now, the integral we have to study is

$$\int_{B_j} \Delta^\gamma_{r_i} dt = \int_{B_j} \delta^\gamma_{r_i} \dot{\kappa}_2 dt = \int_{B_j} \delta^\gamma_{r_i} \sin\varphi dt + \int_{B_j} \delta^\gamma_{r_i} t\dot{\varphi}\cos\varphi dt.$$

We integrate by parts the integral without $\dot{\varphi}$, getting

$$\int_{B_j} \delta^\gamma_{r_i} \sin\varphi dt = \Big[\sin\varphi(t) \int_{t_{h_j\eta_j}}^t \delta^\gamma_{r_i} ds\Big]_{t=t_{h_j\eta_j}}^{t=t_{h_j}} - \int_{B_j} \Big\{\dot{\varphi}\cos\varphi \int_{t_{h_j\eta_j}}^t \delta^\gamma_{r_i} ds\Big\} dt.$$

Since the boundary term is 0, we obtain

$$\int_{B_j} \delta^\gamma_{r_i} \dot{\kappa}_2 dt = \int_{B_j} \Big\{t\delta^\gamma_{r_i} - \int_{t_{h_j\eta_j}}^t \delta^\gamma_{r_i} ds\Big\} \dot{\varphi}\cos\varphi dt,$$

and thus

$$\Big|\int_{B_j} \delta^\gamma_{r_i} \dot{\kappa}_2 dt\Big| \leq \int_{B_j} \Big\{t|\delta^\gamma_{r_i}| + \int_{t_{h_j\eta_j}}^t |\delta^\gamma_{r_i}| ds\Big\} |\dot{\varphi}| dt \leq C|\varepsilon_j| \int_{B_j} t^{w_i} |\dot{\varphi}| dt.$$

□

Remark 3.22. We stress again the fact that, when the coefficients a_i depend on all the variables $x_1, \ldots, x_n$, the computations above become less clear. As a matter of fact, there is a non-commutative effect of the devices due to the varying coordinates $\gamma_3, \ldots, \gamma_n$ that modifies the coefficients of the parameters ε_j.

3.4 Solution to the end-point equations

In this section we solve the system of equations $E_i^{k,\bar{\varepsilon}} = 0$, $i = 3, \ldots, n$. The homogeneous polynomials p_j are of the form $p_j(x_1, x_2) = x_1^{\alpha_j+1} x_2^{\beta_j}$, as in (3.19).

The quantities (3.22), (3.23) and (3.24) are, respectively,

$$\begin{aligned}
&\int_{F_k} \Delta_i^\gamma dt = I_k^{\alpha_i\beta_i} + \int_{F_k} r_i(\mathrm{k}(t))dt,\\
&\int_{A_j} \Delta_i^\gamma dt = 0,\\
&\int_{B_j} \Delta_i^\gamma dt = -\Delta_{h_j\eta_j\varepsilon_j}^{\alpha_i\beta_i} + \int_{B_j} \Delta_{r_i}^\gamma dt,
\end{aligned}$$

where we used the short-notation $I_k^{\alpha_i\beta_i} = I_{2\pi k, 2\pi(k+1)}^{\alpha_i\beta_i}$. So the end-point equations $E_i^{k,\bar{\mathcal{E}}} = 0$ read

$$f_i(\varepsilon) = b_i, \quad i = 3, \dots, n. \tag{3.32}$$

with

$$f_i(\varepsilon) = \sum_{j=3}^n \Big(\Delta_{h_j\eta_j\varepsilon_j}^{\alpha_i\beta_i} - \int_{B_j} \Delta_{r_i}^\gamma dt \Big) \quad \text{and} \quad b_i = \int_{F_k} \Delta_i^\gamma dt.$$

We will regard k, h_j, and η_j as parameters and we will solve the system of equations (3.32) in the unknowns $\varepsilon = (\varepsilon_3, \dots, \varepsilon_n)$. The functions $f_i : \mathbb{R}^{n-2} \to \mathbb{R}$ are analytic and the data b_i are estimated from above by (3.31):

$$|b_i| \le \int_{F_k} t^{w_i} |\dot\varphi| dt.$$

Theorem 3.23. *There exist real parameters $\eta_3, \dots, \eta_n > 0$ and integers $h_3 > \dots > h_n$ such that for all $k \in \mathbb{N}$ large enough the system of equations* (3.32) *has a unique solution $\varepsilon = (\varepsilon_3, \dots, \varepsilon_n)$ satisfying*

$$|\varepsilon| \ \le C \sum_{i=3}^n |b_i|, \tag{3.33}$$

for a constant $C > 0$ independent of k.

Proof. We will use the inverse function theorem. Let $A = \big(a_{ij}^h\big)_{i,j=3,\dots,n} \in M_{n-2}(\mathbb{R})$ be the Jacobian matrix of $f = (f_3, \dots, f_n)$ in the variables $\varepsilon = (\varepsilon_3, \dots, \varepsilon_n)$ computed at $\varepsilon = 0$. By (3.30) and Lemma 3.21 we have

$$a_{ij}^h = \frac{\partial f_i(0)}{\partial \varepsilon_j} = (\alpha_i + 1) I_{h_j\eta_j}^{\alpha_i - 1, \beta_i} + o(I_{h_j\eta_j}^{\alpha_i - 1, \beta_i}). \tag{3.34}$$

Here, we are using the fact that for $h_j \to \infty$ we have

$$\int_{B_j} t^{w_i} |\dot\varphi| dt = o\Big(\int_{B_j} t^{w_i - 1} |\dot\varphi| dt \Big).$$

The proof of Theorem 3.23 will be complete if we show that the matrix A is invertible.

We claim that there exist real parameters $\eta_3, \dots, \eta_n > 0$ and positive integers $h_3 > \ldots > h_n$ such that

$$\det(A) \neq 0. \tag{3.35}$$

The proof is by induction on n. When $n = 3$, the matrix A boils down to the real number a_{33}. From (3.34) and (3.27) we deduce that for any $\eta_3 \in (0, \pi/4)$ we have

$$|a_{33}| \geq \frac{1}{2}(\alpha_3 + 1)|I^{\alpha_3-1,\beta_3}_{h_3\eta_3}| \geq c_{\alpha\beta} j^{\alpha_3-1,\beta_3}_{h_3\eta_3} > 0.$$

We can choose $h_3 \in \mathbb{N}$ as large as we wish.

Now we prove the inductive step. We assume that (3.35) holds when A is a $(n-3) \times (n-3)$ matrix, $n \geq 4$. We develop $\det(A)$ with respect to the first column using Laplace formula:

$$\det(A) = \sum_{i=3}^{n} (-1)^{i+1} a_{i3} P_i,$$

where

$$P_i = P_i(a_{43}, \dots, a_{4n}, \dots, \hat{a}_{i3}, \dots, \hat{a}_{in}, \dots, a_{n3}, \dots, a_{nn})$$

are the determinants of the minors. By the inductive assumption, there exist $\eta_4, \dots, \eta_n \in (0, \pi/4)$ and integers $h_4 > \cdots > h_n$ such that $|P_i| > 0$. By (3.27), for any $\eta_3 \in (0, \pi/4)$ we have the estimates

$$c_0 j^{\alpha_i-1,\beta_i}_{h_3\eta_3} \leq |a_{i3}| \leq C_0 j^{\alpha_i-1,\beta_i}_{h_3\eta_3}, \tag{3.36}$$

for absolute constants $0 < c_0 < C_0$. The leading (larger) $|a_{i3}|$ can be found in the following way. On the set $\mathcal{A} = \{(\alpha_i, \beta_i) \in \mathbb{N} \times \mathbb{N} : i = 3, \dots, n\}$ we introduce the order $(\alpha, \beta) < (\alpha', \beta')$ defined by the conditions $\alpha + \beta < \alpha' + \beta'$, or $\alpha + \beta = \alpha' + \beta'$ and $\beta < \beta'$. We denote by $(\alpha_\iota, \beta_\iota) \in \mathcal{A}$, for some $\iota = 3, \dots, n$, the minimal element with respect to this order relation.

We claim that, given $\varepsilon_0 > 0$, for all $h_3 > h_4$ large enough and for some $0 < \eta_3 < \pi/4$ the following inequalities hold:

$$|a_{i3}||P_i| \leq \varepsilon_0 |a_{\iota 3} P_\iota|, \quad \text{for} \quad i \neq \iota. \tag{3.37}$$

In the case when $i = 3, \dots, n$ is such that $\alpha_i + \beta_i = \alpha_\iota + \beta_\iota$, then we have $\beta_i > \beta_\iota$. By (3.36) and (3.26), inequality (3.37) is implied by $\eta_3^{\beta_i - \beta_\iota} |P_i| \leq \varepsilon_0 |P_\iota|$, possibly for a smaller ε_0. So we fix $\eta_3 \in (0, \pi/4)$ independently from h_3 such that

$$0 < \eta_3 \leq \min\Big\{ \Big(\frac{\varepsilon_0 |P_\iota|}{|P_i|} \Big)^{1/(\beta_i - \beta_\iota)} : i \neq \iota \Big\}.$$

In the case when $i = 3, \dots, n$ is such that $\alpha_i + \beta_i > \alpha_\iota + \beta_\iota$, inequality (3.37) is implied by

$$\int_{B_3} t^{\alpha_i+\beta_i} |\dot\varphi(t)| dt \le \varepsilon_0 \eta_3^{\beta_\iota - \beta_i} \frac{|P_\iota|}{|P_i|} \int_{B_3} t^{\alpha_\iota+\beta_\iota} |\dot\varphi(t)| dt.$$

This holds for all $h_3 \in \mathbb{N}$ large enough.

Now we can estimate from below the determinant of A using (3.37). We have

$$|\det(A)| \ge |a_{\iota 3} P_\iota| - \sum_{i \ne \iota} |a_{i3}||P_i| \ge \frac{1}{2} |a_{\iota 3} P_\iota|$$

and the last inequality holds for all $h_3 \in \mathbb{N}$ large enough, after fixing $\eta_3 > 0$. This ends the proof of the theorem. □

3.5 Nonminimality of the spiral

In this section we prove Theorem 3.4. Let $\gamma \in AC([0,1]; M)$ be a horizontal spiral of the form (3.3). We work in exponential coordinates of the second type centered at $\gamma(0)$.

We fix on Δ the metric g making orthonormal the vector fields f_1 and f_2 spanning Δ. This is without loss of generality, because any other metric is equivalent to this one in a neighborhood of the center of the spiral. With this choice, the length of γ is the standard length of its horizontal coordinates and for a spiral as in (3.3) we have

$$L(\gamma) = \int_0^1 |\dot\kappa(t)| dt = \int_0^1 \sqrt{1 + t^2 \dot\varphi(t)^2} dt.$$

In particular, γ is rectifiable precisely when $t\dot\varphi \in L^1(0,1)$, and k is a Lipschitz curve in the plane precisely when $t\dot\varphi \in L^\infty(0,1)$.

For $k \in \mathbb{N}$ and $\bar{\mathscr{E}} = (\mathscr{E}_3, \dots, \mathscr{E}_n)$, we denote by $\mathrm{D}(\gamma; k, \bar{\mathscr{E}})$ the curve constructed in Section 3.2. The devices $\mathscr{E}_j = (h_j, \eta_j, \varepsilon_j)$ are chosen in such a way that the parameters h_j, η_j are fixed as in Theorem 3.23 and $\varepsilon_3, \dots, \varepsilon_n$ are the unique solutions to the system (3.32), for k large enough. In this way the curves γ and $\mathrm{D}(\gamma; k, \bar{\mathscr{E}})(1)$ have the same initial and end-point.

We claim that for $k \in \mathbb{N}$ large enough the length of $\mathrm{D}(\gamma; k, \bar{\mathscr{E}})$ is less than the length of γ. We denote by $\Delta L(k) = L(\gamma) - L(\mathrm{D}(\gamma; k, \bar{\mathscr{E}}))$ the difference of length and, namely,

$$\begin{aligned} \Delta L(k) &= \int_{F_k} \sqrt{1 + t^2 \dot\varphi(t)^2} dt - \Big(t_k - t_{k+1} + 2 \sum_{j=3}^n |\varepsilon_j| \Big) \\ &= \int_{F_k} \frac{t^2 \dot\varphi(t)^2}{\sqrt{1 + t^2 \dot\varphi(t)^2} + 1} dt - 2 \sum_{j=3}^n |\varepsilon_j|. \end{aligned} \tag{3.38}$$

By (3.33), there exists a constant $C_1 > 0$ independent of k such that the solution $\varepsilon = (\varepsilon_3, \ldots, \varepsilon_n)$ to the end-point equations (3.32) satisfies

$$|\varepsilon| \leq C_1 \sum_{i=3}^{n} |I_k^{\alpha_i \beta_i}| \leq C_2 \sum_{i=3}^{n} \int_{F_k} t^{w_i} |\dot{\varphi}(t)| dt \leq C_3 \int_{F_k} t^2 |\dot{\varphi}(t)| dt. \tag{3.39}$$

We used (3.27) and the fact that $w_i \geq 2$. The new constants C_2, C_3 do not depend on k.

By (3.38) and (3.39), the inequality $\Delta L(k) > 0$ is implied by

$$\int_{F_k} \frac{t^2 \dot{\varphi}(t)^2}{\sqrt{1 + t^2 \dot{\varphi}(t)^2} + 1} dt > C_4 \int_{F_k} t^2 |\dot{\varphi}(t)| dt, \tag{3.40}$$

where C_4 is a large constant independent of k. For any $k \in \mathbb{N}$, we split the interval $F_k = F_k^+ \cup F_k^-$ where

$$F_k^+ = \{t \in F_k : |t\dot{\varphi}(t)| \geq 1\} \quad \text{and} \quad F_k^- = \{t \in F_k : |t\dot{\varphi}(t)| < 1\}.$$

On the set F_k^+ we have

$$\int_{F_k^+} \frac{t^2 \dot{\varphi}(t)^2}{\sqrt{1 + t^2 \dot{\varphi}(t)^2} + 1} dt \geq \frac{1}{3} \int_{F_k^+} t |\dot{\varphi}(t)| dt \geq C_4 \int_{F_k^+} t^2 |\dot{\varphi}(t)| dt, \tag{3.41}$$

where the last inequality holds for all $k \in \mathbb{N}$ large enough, and namely as soon as $3C_4 t_k < 1$. On the set F_k^- we have

$$\int_{F_k^-} \frac{t^2 \dot{\varphi}(t)^2}{\sqrt{1 + t^2 \dot{\varphi}(t)^2} + 1} dt \geq \frac{1}{3} \int_{F_k^-} t^2 |\dot{\varphi}(t)|^2 dt \geq C_4 \int_{F_k^-} t^2 |\dot{\varphi}(t)| dt, \tag{3.42}$$

where the last inequality holds for all $k \in \mathbb{N}$ large enough, by our assumption on the spiral

$$\lim_{t \to 0^+} |\dot{\varphi}(t)| = \infty.$$

Now (3.41) and (3.42) imply (3.40) and thus $\Delta L(k) > 0$. This ends the proof of Theorem 3.4.

Chapter 4

Some open problems

In this final very short chapter, we list some open questions concerning the regularity problem of sub-Riemannian geodesics coming from the arguments developed in this thesis. This is also an overview of possible future research directions.

Problem 1. An interesting question is the possibility of dropping the assumptions of vanishing lower differentials. Instead, for a C^∞ map F with critical point 0, our Definition 2.13 of regular n-differential requires that $\mathcal{D}_0^h F = 0$ for $h = 2, \dots, n-1$. The assumption is restrictive but natural: it is made to have a dominant term of order n in the Taylor expansion of F.

A more general definition of regular n-th differentials when lower ones do not vanish has to preserve this property for the Taylor expansion of F. In other words, the correct definition of regular differentials should provide a statement similar to the following, which is in fact our conjecture:

"Let $n \in \mathbb{N}$ be such that $\mathcal{D}_0^n F$ is regular at the critical point 0. Then the dominant term in the Taylor expansion of F at 0 is of order n."

We believe that, with such a definition of regularity, an open mapping property similar to our Theorem 2.15 is preserved, i.e.,

"If there exists $n \in \mathbb{N}$ such that $\mathcal{D}_0^n F$ is regular at the critical point 0, then the map F is open at 0"

The open mapping theorem is at the basis of our sub-Riemannian theory: we believe that its application to the end-point map should provide, as in Chapter 2, Goh conditions of order n in a more general setting.

Problem 2. Classical and also our higher order Goh conditions are just necessary for abnormal curves to be length minimizers. Instead, in [2, Chapter 12], there are proved second order sufficient conditions for the minimality of abnormal curves in rank-2 sub-Riemannian manifolds. In particular, the key result is the following

Theorem 4.1. *Let M be a rank-2 sub-Riemannian manifold and let $\gamma : I \to M$ be an abnormal trajectory with control $\bar{u}$. Suppose that there exists a positive constant $C > 0$ such that the second differential satisfies*

$$\mathcal{D}^2_{\bar{u}}E(u) \geq C\|v\|^2_{L^2}, \quad u \in L^2(I, \mathbb{R}^2), \tag{4.1}$$

then γ is locally length minimizer in the L^2 topology of controls. Above, in (4.1) *we set $v(t) = \int_0^t u(\tau)d\tau$.*

In other words, a local minimality is implied by a "coercivity" of the Hessian. The proof of this Theorem uses classical order 2 Goh conditions.

Our first question is to generalize, under the assumption of vanishing lower differentials, Theorem 4.1 to the following statement

"Let M be a rank-2 sub-Riemannian manifold and let $\gamma : I \to M$ be an abnormal trajectory with control $\bar{u}$. Suppose that there exist an even $p \in \mathbb{N}$ such that $\mathcal{D}^2_{\bar{u}}E = \cdots = \mathcal{D}^{p-1}_{\bar{u}}E = 0$ and a positive constant $C > 0$ such that

$$\mathcal{D}^p_{\bar{u}}E(u) \geq C\|v\|^p_{L^p}, \quad u \in L^2(I, \mathbb{R}^2), \tag{4.2}$$

then γ is locally length minimizer in the L^p topology of controls. Above, in (4.2) *we set $v(t) = \int_0^t u(\tau)d\tau$. "*

A second more difficult question is to investigate in the general validity of Theorem 4.1 and its higher order version in sub-Riemannian manifolds of rank greater than 2.

A direct application of this theory can be found in the example of Section 2.9. We consider the curve $\gamma(t) = (0, t, 0)$, $t > 0$ when $M = \mathbb{R}^3 = (x_1, x_2, x_3)$ and Δ is spanned, for $n \in \mathbb{N}$, by the vector fields

$$f_1 = \frac{\partial}{\partial x_1}, \quad f_2 = (1 - x_1)\frac{\partial}{\partial x_2} + x_1^n \frac{\partial}{\partial x_3}.$$

As we said in Chapter 2, it was known (see [19]) that γ is length minimizer for any even n, while we proved that γ cannot be minimal when n is odd.

Our theorems do not instead apply when n is even, rightly since γ is minimal in this case. We believe that higher order sufficient conditions will provide an alternative proof of the minimality of γ, based on differential intrinsic properties of the end-point map. We also believe that a deeper understanding of this example is strictly related with Problem 3 below.

Problem 3. The problem presented here is actually an ongoing research project with Ludovic Rifford and Mario Sigalotti.

As explained in the introductory part, we do not know whether abnormal minimizers are more than Lipschitz regular, or whether there is a maximum regularity bound. So, we are interesting in proving the existence of abnormal minimizers (at least one) of class C^1 but not C^2.

When M is analytic of dimension 3, Rifford et al. recently proved the C^1 regularity for length minimizers in [10]. This setting is very interesting for our purposes because, when $\dim M = 3$ abnormal curves live on the Martinet surface

$$\Sigma = \{x \in M : \Delta(x) + [\Delta, \Delta](x) \neq T_x M\}.$$

Then we have 2 strong necessary condition for abnormal curves to be minimizer. In particular, several candidates to reach our claim can be found if Σ is of class C^1 but not C^2: every abnormal curve passing through the singularities of Σ could be a good candidate to prove our claim.

It is not difficult to build many examples in order to have the situation described above. Indeed, it is sufficient to consider $M = \mathbb{R}^3 = (x_1, x_2, x_3)$ and $\Delta = \text{span}\{f_1, f_2\}$, where

$$f_1 = \partial_{x_1}, \quad f_2 = (1 + x_1 \Phi(x))\partial_{x_2} + a(x)\partial_{x_3},$$

and a, Φ have to be chosen in such a way that Σ is of class C^1 but not C^2.

For instance, when $\Phi = -1$ and $a = x_1^n$, we are in the previous case of Problem 2. In this case Σ is the plane $\{x_1 = 0\}$ and we know that the curve $\gamma(t) = (0, t, 0)$ is an abnormal minimizer when n is even. In spite of this curve is smooth, a deeper study of its minimality will be useful.

When $\Phi = 0$ it is sufficient to choose a such that Σ, which in this case is described by the equation $\partial_{x_1} a = 0$, is C^1 but not C^2.

Problem 4. Generalize the results of Chapter 3 dropping the following commutativity assumption

$$\big[[f_{i_1}, \ldots, [f_{i_{s-1}}, f_{i_s}] \ldots], [f_{j_1}, \ldots, [f_{j_{s-1}}, f_{j_s}] \ldots]\big] = 0, \tag{4.3}$$

where $s, p \geq 2$, $i_1, \ldots, i_s, j_1, \ldots, j_p \in \{1, 2\}$ and

$$f_1 = \partial_{x_1}, \quad f_2 = \partial_{x_2} + \sum_{j=3}^{n} a_j(x)\partial_{x_j}.$$

In spite of assumption (4.3) is algebraically strong, it has no geometric meaning: it is a technical assumption to ensure $a_j(x) = a_j(x_1, x_2)$. Thanks to this property of the coefficients a_j, in Section 3.3 we proved that

$$c_{\alpha_1\alpha_2} \int_{2h\pi+\eta}^{2h\pi} t^{\alpha_1+\alpha_2+2} |\dot{\varphi}(t)| dt \leq |I_{h,\eta}^{\alpha_1\alpha_2}| \leq C_{\alpha_1\alpha_2} \int_{2h\pi+\eta}^{2h\pi} t^{\alpha_1+\alpha_2+2} |\dot{\varphi}(t)| dt \tag{4.4}$$

for some constants $c_{\alpha_1\alpha_2}, C_{\alpha_1\alpha_2} > 0$. Here in (4.4) $h \in \mathbb{N}$, $\eta \in (0, \pi/4)$, $\dot{\varphi}$ is the phase of the spiral γ and

$$I_{h,\eta}^{\alpha_1\alpha_2} = \int_{2h\pi+\eta}^{2h\pi} \gamma_1(t)^{\alpha+1} \gamma_2(t)^{\beta} \dot{\gamma}_2(t) dt. \tag{4.5}$$

Moreover, when the a_j depend only on x_1 and x_2, integrals like in (4.5) describe all the nonhorizontal coordinates $\gamma_3, \dots, \gamma_n$ of the spiral γ.

Without assumption (4.3), we have

$$\gamma_{j+1}(t) = \int_0^t \gamma_1(\tau)^{\alpha_1+1} \gamma_2(\tau)^{\alpha_2} \dots \gamma_j(\tau)^{\alpha_j} \dot{\gamma}_2(\tau) d\tau,$$

and our open problem is to prove that similar estimates as in (4.4) hold also for integrals of the following type

$$I_{h,\eta}^{\alpha_1\dots\alpha_j} = \int_{2h\pi+\eta}^{2h\pi} \gamma_1(\tau)^{\alpha_1+1} \gamma_2(\tau)^{\alpha_2} \dots \gamma_{j-1}(\tau)^{\alpha_j} \dot{\gamma}_2(\tau) d\tau.$$

Our conjecture is the following statement:

“There exist constants $0 < c_{\alpha_1\dots\alpha_j} < C_{\alpha_1\dots\alpha_j}$ depending on $\alpha_1, \dots, \alpha_j \in \mathbb{N}$ such that for all $h \in \mathbb{N}$ and $\eta \in (0, \pi/4)$ we have

$$c_{\alpha_1\dots\alpha_j} \int_{2h\pi+\eta}^{2h\pi} t^{2+\sum_{i=1}^j w_i\alpha_i} |\dot{\varphi}(t)| dt \leq |I_{h,\eta}^{\alpha_1\dots\alpha_j}| \leq C_{\alpha_1\dots\alpha_j} \int_{2h\pi+\eta}^{2h\pi} t^{2+\sum_{i=1}^j w_i\alpha_i} |\dot{\varphi}(t)| dt,$$

where w_i is the sub-Riemannian weight of the coordinate x_i.”

The difficult part in the proof of this claim is the presence of nonhorizontal coordinates of γ in $I_{h,\eta}^{\alpha_1\dots\alpha_j}$: they generate multiple integrations of the type (4.5) one inside the other. This makes very hard the proof for the estimates from below, while for the estimates from above the computations are similar to Section 3.3. Once reached this claim, it should be easy to prove a more general theorem of nonminimality of spirals without assumption (4.3).

In our opinion, besides providing a more general result for the non-minimality of spirals, these kind of computations would be a model to study and understand the behavior of nonhorizontal coordinates in a general setting.

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