

UNIVERSITÀ DEGLI STUDI DI PADOVA
DIPARTIMENTO DI MATEMATICA

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE
MATEMATICHE
INDIRIZZO MATEMATICA
CICLO XXXIII

STABLE HYPERSURFACES IN THE
COMPLEX PROJECTIVE SPACE
AND
ISOPERIMETRIC SETS IN THE
SUB-FINSLER HEISENBERG GROUP

Relatore:
Chiar.mo Prof.
ROBERTO MONTI

Dottorando:
ALBERTO RIGHINI

Contents

Introduction	v
1 Stable Hypersurfaces in CP^n	1
1.1 Introduction to the Isoperimetric Problem	1
1.1.1 Euclidean Case	1
1.1.2 The C^2 Uniqueness Theory	3
1.1.3 Steiner Symmetrization	4
1.1.4 Stable CMC hypersurfaces	6
1.2 Two Characterizations of the Sphere in CP^n	17
1.2.1 Stability of spheres and tubes	18
1.2.2 Geometry of the isometric immersion of CP^n into H^{n+1}	20
1.2.3 Main results	24
1.2.4 Laplacian of position and normal	26
1.2.5 Laplacian of σ_N	28
1.2.6 Trace of the second variation of the area	31
1.2.7 Proof of Theorems 1.2.13 and 1.2.14	36
1.3 The case of the Real Projective Space	38
1.3.1 Main results	39
1.3.2 Spheres and Tubes in RP^n	40
1.3.3 Immersion of RP^n into $Sym(n+1)$	41
1.3.4 Laplacian of position and normal	44
1.3.5 Laplacian of σ_N	45
1.3.6 The second variation of the area	47
1.3.7 Proof of Theorem 1.3.1	51
2 Sub-Finsler Isoperimetric Problem	53
2.1 Heisenberg Group and Pansu's Conjecture	53
2.1.1 The Sub-Riemannian Heisenberg group	53
2.1.2 Isoperimetric problem and Pansu's conjecture	56
2.1.3 Sub-Finsler Geometry	59
2.2 A Classification of C^2 Isoperimetric Sets	64
2.2.1 First variation of the isoperimetric quotient	66
2.2.2 Integration of the curvature equation	68
2.2.3 Foliation property with geodesics	71
2.2.4 Characteristic set of ϕ -critical surfaces	73
2.2.5 Classification of ϕ -isoperimetric sets of class C^2	84

2.2.6	Regularity of ϕ -bubbles	85
2.3	The Crystalline Case	90
2.3.1	The isoperimetric problem for general norms	93
2.3.2	Smooth approximation of norms in the plane	93
2.3.3	Crystalline ϕ -bubbles as limits of smooth isoperimetric sets	94
2.4	Necessary sign conditions at interface curves	94
2.4.1	Same vector field foliating S_1 and S_2	95
2.4.2	Different foliation on S_1 and S_2	97
2.4.3	Recap on minimality and geometric sign conditions	100
2.5	Multiple points and blow-up	100
Bibliography		105

Abstract

In this thesis, we study the stability of hypersurfaces in the complex projective space and the isoperimetric problem in the sub-Finsler Heisenberg group. The work is organized as follows.

Chapter 1 is dedicated to the study of stability. In Section 1.1, we briefly introduce the isoperimetric problem in the Euclidean setting, recalling some classical results. We also introduce the notion of stability and we investigate its relation with the isoperimetric problem, presenting some results in the Euclidean and in the Riemannian case. In Section 1.2, we prove the main results of this chapter, namely we characterize the sphere in $\mathbf{C}P^n$ to be the only compact stable hypersurface with constant mean curvature satisfying a certain bound on its curvatures. This result is proven by an isometric immersion of $\mathbf{C}P^n$ into the space of Hermitian matrices H^{n+1} , followed by a study of the second variation of the area for a suitable test function. In Section 1.3, we prove a non-existence result for compact oriented stable CMC hypersurfaces in the real projective space $\mathbf{R}P^n$, which satisfy a certain bound on their curvatures. This is done with the same techniques used for the $\mathbf{C}P^n$ -case.

Chapter 2 is dedicated to the study of the isoperimetric problem in the sub-Finsler Heisenberg group. In Section 2.1, we introduce the sub-Riemannian Heisenberg group and Pansu's conjecture regarding its isoperimetric sets. We also introduce the basic notions of sub-Finsler geometry and the sub-Finsler Heisenberg group, proving a representation formula for the induced perimeter and the existence of isoperimetric sets. In Section 2.2, we prove some new results on the problem. In particular, we prove a foliation property by sub-Finsler geodesics for regular surfaces with constant ϕ -curvature (where ϕ is the sub-Finsler norm on the horizontal distribution of the group). Then, we show that the candidate ϕ -isoperimetric set is $\mathbf{C}^2$ -smooth under suitable regularity conditions. Finally, we show the main result of this chapter, namely the characterization of $\mathbf{C}^2$ -smooth ϕ -isoperimetric sets when ϕ and ϕ^* are in $\mathbf{C}^2(\mathbb{R}^2 \setminus \{0\})$. In Section 2.3, we study norms ϕ which have non-differentiability points, in particular crystalline norms. By an approximation procedure, we prove a conditional minimality property for the candidate isoperimetric set. Finally, in Sections 2.4 and 2.5, we study the L^∞ norm, which is crystalline. We show how to get some necessary conditions for a set to be isoperimetric, by using a first-variation argument and by studying the blow-ups of the interface curves.

Introduction

The *isoperimetric problem* is an ancient problem that goes back to the legend of Dido and the foundation of Carthage. In its simplest formulation, it can be stated as the problem of finding the planar figure with the greatest possible area and a prescribed length of the boundary.

Although the formulation is very simple to understand and the solution (the circle) is quite easily guessable, it has taken mathematicians many centuries to arrive to a formal and general proof. In [77], [78] and [79], the swiss mathematician Jakob Steiner gave five proofs of the following isoperimetric theorem:

Theorem. Among all curves of a given length, the circle encloses the greatest area.

All the proofs, however, assume the existence of a solution. Such existence was addressed and proved by Schwarz [76] in dimension 3 and more in general by Weierstrass in [85] by the means of calculus of variations. The first complete proof of the isoperimetric problem in the euclidean setting was given by De Giorgi (his work can be found in [31]).

The isoperimetric problem is often formulated as an analytic *isoperimetric inequality*. A sharp quantitative isoperimetric inequality is proven by Fusco, Maggi and Pratelli in [40]. Figalli, Maggi and Pratelli also studied it by the means of mass transportation theory in [34].

The solution to the problem is also known in the sphere and in the hyperbolic space. For the latter, the isoperimetric inequality was proven by Schmidt in [75]. In these space forms the solution is the geodesic sphere.

The isoperimetric problem is closely related to stability. Let $M^{n+1}(c)$ be a simply-connected complete Riemannian manifold with constant sectional curvature c and let $x : M \rightarrow \bar{M}^{n+1}(c)$ be an immersion of a differentiable manifold M of dimension n . Let $D \subset M$ be a relatively compact domain with smooth boundary. We will denote by $\mathcal{A}_D(x)$ the n -area of D and $\mathcal{V}_D(x)$ the volume of D in x (see equation (1.6) for the definition). Then we consider normal, volume-preserving and boundary-fixing variations x_t of D . With Proposition 1.1.15, the authors prove the equivalence between being a critical point of a variational isoperimetric problem and being a hypersurface with constant mean curvature. The property of *stability* can be defined as follows.

Let $x : M \rightarrow M^{n+1}$ have constant mean curvature and let $D \subset M$ be a relatively compact domain with smooth boundary. We say that D is *stable* if the second variation of the area functional is positive semi-definite for every volume-preserving variation that fixes ∂D . If every such D is stable, we say that the immersion is stable.

With that definition in mind, it is clear that the boundary of an isoperimetric set, being it a minimum for the isoperimetric problem, must be stable. Then the main

result contained in [13] is a characterization of stable hypersurfaces as the round spheres in M^{n+1} (see Theorem 1.1.18).

If the sectional curvature is not constant the (geodesic) spheres are not the only stable hypersurfaces. Barbosa, Do Carmo and Eschenburg prove a characterization for the radius of stable tubes in Projective spaces (see 1.1.20). In [65], Ritorè and Ros solved this stability problem in the 3-dimensional real projective space.

The original work of this thesis consists in two parts. The first part is presented in Chapter 1, Sections 1.2 and 1.3 and it is contained in [14] and [36]. and is about the study of stable hypersurfaces in the complex projective space $\mathbf{CP}^n$. Section 1.2 starts with the introduction and definition of spheres and tubes in the complex projective space. For any fixed $[w] \in \mathbf{CP}^n$ and $0 < r < \pi/2$, the geodesic sphere centered at $[w]$ with radius r is

$$\Sigma_r = \{[z] \in \mathbf{CP}^n : |z \cdot \bar{w}| = \cos r\}.$$

For $0 < r < \pi/2$, we define the tube around $\mathbf{CP}^k$ as

$$T_r^k = \{[z] = [\mathcal{X}', z''] \in \mathbf{CP}^n : |z| = 1, z' \in \mathbf{C}^{k+1}, |z'| = \cos r\}.$$

Finally, we consider the geodesic tube in $\mathbf{CP}^n$ with radius r around $\mathbf{RP}^n$ is

$$V_r = \left\{ [z] \in \mathbf{CP}^n : \left| \sum_{j=1}^{n+1} z_j^2 \right| = \cos 2r \right\}.$$

Theorem 1.1.20 states that both Σ_r and T_r^k are stable for certain values of the radius. On the other hand, the tube V_r turns out to be not stable for every value of r .

The main result of this section and, indeed, of Chapter 1, is the following characterization of the sphere in $\mathbf{CP}^n$.

Theorem. Let $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$, be a compact stable hypersurface with constant mean curvature H . If the characteristic curvature κ of Σ satisfies

$$p(\kappa; H, n) := (2n+1)\kappa^2 - 2H\kappa - H^2 - 4(n^2 - 1) \geq 0$$

then Σ is a geodesic sphere of radius $r > 0$ with $\tan^2 r = 2n+1$.

The *characteristic curvature* of Σ at the point $[z] \in \Sigma$ is $\kappa = h(JN, JN)$, where h is the second fundamental form of Σ and J is the complex structure of $\mathbf{CP}^n$. This is interesting because in Theorem 1.1.20, Barbosa, Do Carmo and Eschenburg find an interval for stability of the geodesic spheres in $\mathbf{CP}^n$ and we find their borderline case. The proof is done through several steps, but crucial is the isometric immersion of $\mathbf{CP}^n$ into the space of Hermitian matrices H^{n+1} and the following inequality, which comes from stability.

Theorem. Let $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$, be a compact stable hypersurface with constant mean curvature H . Then we have

$$\int_{\Sigma} \left\{ |h|^2 + \frac{(H + \kappa)^2 + |h_N|^2}{2(n^2 - 1)} - \frac{H^2}{n-1} - 2n \right\} d\mu \leq 0,$$

where μ is the Riemannian hypersurface measure. By $h_N \in \mathbf{CT}\Sigma$, we denote the projection of $\nabla_{JN}^\top N$ onto $\mathbf{CT}\Sigma$, where $\nabla^\top$ is the Levi-Civita connection of $\mathbf{CP}^n$.

Finally, in the last section of the chapter, we investigate briefly what happens in the real projective space and we prove a non-existence result about stable hypersurfaces with constant mean curvature. The result is the following. Let $\Sigma \subset \mathbb{R}P^n$ be a hypersurface and let $T\Sigma$ be the real tangent space of Σ , let T_1 and T_2 be nonzero subbundles of $T\Sigma$ such that

$$T\Sigma = T_1 \oplus T_2,$$

and denote their ranks by $\alpha = \text{rank}(T_1)$ and $\beta = \text{rank}(T_2)$. Then we have $\alpha + \beta = \text{rank}(T\Sigma) = n - 1$ and $1 \leq \alpha < n - 1$. Let h_1 and h_2 be the restrictions of the second fundamental form h of Σ to T_1 and T_2 , and denote by $H_1 = \text{tr}(h_1)$ and $H_2 = \text{tr}(h_2)$ their respective traces.

Theorem. Let $n \geq 3$ be odd. There exists no compact oriented stable CMC hypersurface $\Sigma \subset \mathbf{R}P^n$ satisfying

$$p(H_1, H_2; n, \alpha) := \left(\frac{H_1^2}{\alpha} + \frac{H_2^2}{\beta} \right) (n - 2) - 2(H_1 + H_2)^2 - n(n - 1) \geq 0,$$

for the previous decomposition.

The procedure for obtaining this non-existence result is the same as in the case of the complex projective space. Other important results related to this topic can be found in [4], where the authors characterize weakly stable, compact two-sided hypersurfaces with constant mean curvature in $\mathbf{R}P^n$, and in [84], where the author classifies compact and orientable stable hypersurfaces in $\mathbf{R}P^n$.

The second part of the original work presented in this thesis we focus on the sub-Finsler Heisenberg group. It is contained in Chapter 2, Sections 2.2, 2.3, 2.4 and 2.5 and it is from [37].

Sub-Finsler geometry is a natural generalization of sub-Riemannian geometry that comes from optimal control theory, but appears naturally in many different settings. For example, Lie groups equipped with a sub-Finsler structure are studied in geometric group theory by Pansu in [61]. They also appear in the study of isometrically homogeneous spaces which, under additional assumptions, are indeed sub-Finsler manifolds (see [41], [19], [55], [15], [16] for details). In [11], the authors study the sub-Finsler geometry as a time-optimal control problem (in particular, they characterize extremal curves and calculate the metric spheres, proving their Euclidean rectifiability), in [28] this geometry is studied in three-dimensional manifolds.

Let M be a n -dimensional manifold and $\mathcal{D}$ a smooth distribution of rank k on M . Then a *sub-Finsler metric* on $\mathcal{D}$ is a smoothly varying Finsler metric ϕ on each subspace $\mathcal{D}(p) \subset T_p M$,

A *sub-Finsler manifold* is a smooth n -dimensional manifold equipped with a sub-Finsler metric on a bracket-generating distribution $\mathcal{D}$ of rank k and it is denoted by the triple $(M, \mathcal{D}, \phi)$.

A *Finsler metric* on M is a smoothly varying Minkowski norm on each tangent space $T_p M$.

Recently there has been an increasingly high interest in the study of Metric Spaces and, in particular, Carnot-Carathéodory spaces. These spaces have been formally defined by Gromov in [44]. An example of a Carnot-Carathéodory space is the Heisen-

berg group $\mathbb{H}^n$, which is the manifold $\mathbb{C}^n \times \mathbb{R}$ with the group product

$$(\xi, z) * (\xi', z') = \left(\xi + \xi', z + z' + \frac{1}{2} \operatorname{Im}(\xi, \bar{\xi}') \right),$$

where $z, z' \in \mathbb{R}$, $\xi, \xi' \in \mathbb{C}^n$ and $\langle \xi, \bar{\xi}' \rangle = \xi_1 \bar{\xi}'_1 + \dots + \xi_n \bar{\xi}'_n$. The Heisenberg group is one of the simplest Carnot groups and it has been deeply studied in the past years. In particular, Pansu proved an isoperimetric inequality in $\mathbb{H}^1$ (see [59]) and conjectured a solution to the isoperimetric problem in $\mathbb{H}^1$ (see [60]).

Many results have been proven regarding stability and the variational isoperimetric problem also in the Heisenberg group, for example in [48], Hurtado, Ritoré and Rosales classify complete, stable, area-stationary surfaces in $\mathbb{H}^1$ as either Euclidean planes or hyperbolic paraboloids. In [72] Rosales proves a classification result for stable complete surfaces with empty singular set in Sasakian sub-Riemannian 3-manifolds. Moreover, Pansu's conjecture has been proven with some additional assumptions (see [66], [57] and [56]). The isoperimetric problem is also studied in H -type groups and in Grushin spaces by Franceschi and Monti in [35]. Finally, in [25], the authors present a very detailed survey on the Heisenberg group and on the isoperimetric problem, together with an original approach to the horizontal differential geometry of submanifolds via Riemannian approximation.

If we want to approach Pansu's conjecture, it's important to note that actually the Heisenberg group appears as a horosphere in the complex hyperbolic space (see [43]). Then it would be an important step to characterize the stable hypersurfaces in the complex hyperbolic space. This has also inspired our study of the (simpler) case of the complex projective space discussed previously.

In our work study the isoperimetric problem on the Heisenberg group $\mathbb{H}^1$ equipped with a left invariant norm ϕ on the horizontal distribution. Very recently, in [62] Pozuelo and Ritoré have obtained several results on this topic, in the case when ϕ is convex and homogeneous, but not necessarily a norm.

We consider anisotropic left-invariant perimeter measures associated with ϕ . Let $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ be a norm in $\mathbb{R}^n$, $n \geq 2$. The associated *Finsler* or *anisotropic perimeter* of a Lebesgue measurable set $E \subset \mathbb{R}^n$ is defined as

$$P_\phi(E) = \sup \left\{ \int_E \operatorname{div}(V) \, dp : V \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n) \text{ with } \max_{p \in \mathbb{R}^n} \phi(V(p)) \leq 1 \right\}.$$

For a regular set E (e.g. E is bounded with Lipschitz boundary), $P_\phi(E)$ can be represented as the surface integral

$$P_\phi(E) = \int_{\partial E} \phi^*(\nu_E) \, d\mathcal{H}^{n-1},$$

where ν_E is the inner unit normal to ∂E and $\phi^* : \mathbb{R}^n \rightarrow [0, \infty)$ is the dual norm of ϕ (see (2.20) for the definition). In the case when ϕ is the Euclidean norm in $\mathbb{R}^2$, then we have the standard horizontal perimeter and we are in the sub-Riemannian case, for which Pansu conjectured that the isoperimetric sets are obtained through *left-translations* and *anisotropic dilations* of the Pansu's bubble (i.e. the bounded set whose boundary is foliated by horizontal lifts of planar circles of a given radius, passing through the origin).

In Section 2.1 we recall the definition of the sub-Riemannian Heisenberg group and we present some well known results regarding Pansu's conjecture. We also formally introduce sub-Finsler geometry, proving a representation formula for the ϕ -perimeter in terms of the dual norm ϕ^* of ϕ and proving the existence of isoperimetric sets, following the arguments of [51]. We also generalize the construction of Pansu's bubble to the sub-Finsler context and we call it ϕ -bubble E_ϕ .

Our main result is the characterization of C^2 -smooth ϕ -isoperimetric sets when ϕ and ϕ^* are in $C^2(\mathbb{R}^2 \setminus \{0\})$, suggesting that the ϕ -bubble E_ϕ is the solution to the isoperimetric problem for P_ϕ . The result reads as follows.

Theorem. Let ϕ be a norm of class $C^2(\mathbb{R}^2 \setminus \{0\})$ such that ϕ^* is of class $C^2(\mathbb{R}^2 \setminus \{0\})$. If $E \subset \mathbb{H}^1$ is a ϕ -isoperimetric set of class C^2 then we have $E = E_\phi$, up to left-translations and anisotropic dilations.

The proof is contained in Section 2.2 and it's based on the study of the *characteristic set* of isoperimetric sets, which is

$$\mathcal{C}(E) = \mathcal{C}(\partial E) = \{p \in \partial E : T_p \partial E = \mathcal{D}(p)\}.$$

In Corollary 2.2.27 we characterize the structure of $\mathcal{C}(E)$ for a C^2 -smooth ϕ -isoperimetric set $E \subset \mathbb{H}^1$, proving that $\mathcal{C}(E)$ is made of isolated points. This result is a consequence of a qualitative study of the structure of the characteristic set (see Lemma 2.2.17) and a study of characteristic curves in ϕ -critical surfaces (see Theorem 2.2.18).

Then we study the regularity of the candidate isoperimetric set E_ϕ . We prove the following result.

Theorem. Assume that ϕ is in $C^4(\mathbb{R}^2 \setminus \{0\})$ and that ϕ -circles have strictly positive curvature. Then ∂E_ϕ is an embedded surface of class C^2 .

For the definition of ϕ -circles, see (2.22).

In Section 2.3, we focus on the case when ϕ is a general, non-differentiable norm and, in particular, a crystalline norm. A norm $\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ is called crystalline if the ϕ -circle $C_\phi = C_\phi(0, 1)$ is a convex polygon centrally symmetric with respect to the origin. For a norm of this type, we are able to prove a foliation result for pieces of the boundary of an isoperimetric set which are z -graphs (see 2.3.4). Then, we show that the isoperimetric problem of a general norm ϕ can be approximated by the isoperimetric problem for the smooth norms. The result we obtain is the following.

Theorem. Assume that, for any norm ϕ of class C_+^∞ , ϕ -isoperimetric sets are of class C^2 . Then for any norm ϕ in $\mathbb{R}^2$ the ϕ -bubble $E_\phi \subset \mathbb{H}^1$ is ϕ -isoperimetric.

Here, a norm ϕ in $\mathbb{R}^2$ is said to be of class C_+^∞ if $\phi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and ϕ -circles have strictly positive curvature.

Even if the first order necessary conditions satisfied by an isoperimetric set are not sufficient to reconstruct its structure, in Sections 2.4, through the study of the first variation of the Area, we prove some necessary condition for a set to be isoperimetric in the crystalline case. These results are contained in Theorem 2.4.3 and in Lemmas 2.4.5 and 2.4.6. Finally, in Section 2.5, we retrieve more information with the study of the blow-ups of *interface curves* (see Definition 2.4.1).

Acknowledgements

I would like to thank a few people, who supported me during my PhD studies. First of all, my supervisor Roberto Monti, for his consistent support and guidance during these three years of work. Thank you for your patience and your valuable advice. Secondly, I would like to thank Valentina Franceschi and Mario Sigalotti, for our collaboration and their precious help during and after the time I spent in Paris. I would also like to thank Erika Battaglia, for our work together during my first year of studies. Special thanks to Annalisa, without whom I simply couldn't have done this. Last but not the least, thanks to my parents and my brother for encouraging and supporting me in every step of my studies.

Chapter 1

Stable Hypersurfaces in the Complex Projective Space

1.1 Introduction to the Isoperimetric Problem in Riemannian Manifolds

1.1.1 Euclidean Case

Let's start by introducing the *isoperimetric problem* in the plane. A more detailed introduction can be found in Chavel [26]. We can formulate the problem in the following ways.

1. Consider all *isoperimetric* bounded domains in $\mathbb{R}^2$ (i.e. all open connected sets with fixed given perimeter). Find the domain with the greatest area.
2. Consider all bounded domains with a fixed given area. Find the domain with minimal perimeter.
3. Prove the following analytic inequality:

$$L^2 \geq 4\pi A, \quad (1.1)$$

where A is the area of the domain and L is its perimeter.

The answer to this problem will be the disk. Inequality (1.1) is referred to as *isoperimetric inequality* and one can prove that it holds for every bounded domain of $\mathbb{R}^2$, with equality if and only if the domain is a disk.

If we consider the problem in $\mathbb{R}^n$ for any $n \geq 2$, the isoperimetric inequality becomes

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A(\mathbf{S}^{n-1})}{V(\mathbf{B}^n)^{1-1/n}}, \quad (1.2)$$

where Ω is any bounded domain in $\mathbb{R}^n$, V is the n -measure and A the $(n-1)$ -measure in $\mathbb{R}^n$, $\mathbf{B}^n$ and $\mathbf{S}^{n-1}$ are the unit disk and the unit sphere respectively.

As in the case of the plane, one wants to prove that inequality (1.2) hold for every Ω , with equality if and only if $\Omega = \mathbf{B}^n$.

If we want to extend the investigation to model spaces with constant sectional curvature, say κ , things change. Still, we have an isoperimetric inequality, meaning that all domains with the same volume have the area of their boundary minimized by disks. For $n = 2$ the isoperimetric inequality is

$$L^2 \geq 4\pi A - \kappa A^2, \quad (1.3)$$

with equality if and only if the domain in question is a disk. For the hyperbolic space this has been proven by Schmidt in [75].

Isoperimetric inequality in the plane

We will briefly discuss the isoperimetric inequality in the plane. All the argument used for the following results are from classical calculus of variations and can be found, for example, in [26, Chapter 1].

Theorem 1.1.1 (Isoperimetric inequality in $\mathbb{R}^2$). *If Ω is a relatively compact domain in $\mathbb{R}^2$, with C^1 boundary consisting of one component, then*

$$L^2(\partial\Omega) \geq 4\pi A(\Omega),$$

with equality if and only if Ω is a disk.

One could prove it by using Wirtinger's Inequality and Fourier series (proved by Hurwitz in [49], see [26] for a detailed presentation). However it can also be done in a very short way by using complex variables. This proof follows Topping's in [83].

Proof. With the usual identification $(x, y) \sim z = x + iy$, we have

$$dA = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}.$$

Then, using the fact that the winding number of $\partial\Omega$ is 1 and using Green's theorem, we have

$$\begin{aligned} 4\pi A(\Omega) &= \iint_{\Omega} 2\pi i dz \wedge d\bar{z} = \iint_{\Omega} dz \wedge d\bar{z} \int_{\partial\Omega} \frac{d\zeta}{\zeta - z} \\ &= \int_{\partial\Omega} d\zeta \iint_{\Omega} \frac{dz \wedge d\bar{z}}{\zeta - z} = \int_{\partial\Omega} d\zeta \int_{\partial\Omega} \frac{\bar{\zeta} - \bar{z}}{\zeta - z} dz \leq L^2(\partial\Omega), \end{aligned}$$

which is what we wanted to prove. $\square$

Another way to prove the isoperimetric inequality in the plane is based on a symmetry-and-convexity argument and it was done by Howards, Hutchings and Morgan in [46].

Their argument starts from the hypothesis of existence of a solution Ω to the isoperimetric problem. In this sense, the result they prove is a weak one. Then the main idea is that an isoperimetric set, i.e. a minimizer for the isoperimetric functional

$$D \rightarrow \frac{L^2(\partial D)}{A(D)},$$

then any line dividing Ω in two open sets Ω_1 and Ω_2 of equal area, is such that they have also equal bounding length. From this one can recover a symmetry property of $\partial\Omega$ and can prove that it consists on a finite number of circular arcs connected by segments of lines passing through the origin. Convexity follows from minimality and implies that $\partial\Omega$ is, in fact, a circle.

Although this is a weak result, it is interesting because it requires the problem to be defined in the extended class of relatively compact domains with *piecewise* C^1 boundary. However, a symmetry-and-convexity argument must be treated carefully if we wanted a result in higher dimension, because it is not true, for example, that passing to the convex hull of a domain decreases its isoperimetric quotient (see [26, Chapter 1, Remark 1.2.2] for details).

1.1.2 The C^2 Uniqueness Theory

One way to address the isoperimetric problem is by using differential geometry. To do so one must first assume that the boundary of the considered domains are C^2 , and then try to weaken the assumption to C^1 .

Then, with differential geometric arguments, one can prove that if a domain is just an extremal of the C^2 isoperimetric functional, then it is a disk. Here we will use the variation of the volume and area formulas that come from classical differential geometry and that are proved for generic Riemannian manifolds in Lemma 1.1.14 in Section 1.1.4. Following the presentation by Chavel, we then give Almgren's characterization of the solution to the isoperimetric problem and Alexandrov's characterization of an extremal to the isoperimetric functional.

To weaken the hypothesis on the domains to C^1 , we'll make an argument based on Stoke's theorem, following Gromov. One issue with this method is that the domain must be assumed to be convex, but that can be overcome with Steiner symmetrization.

The following results are based on Almgren [5] and follow the presentation given by Chavel.

Let Ω be a domain in $\mathbb{R}^n$ with C^2 boundary Γ . Since we always consider the normal unit vector to Γ to be outward, then if Ω is convex, the second fundamental form h would be negative semidefinite. This can be seen by simply expressing h in local coordinates.

If Ω is a solution to the isoperimetric problem, following the proof of [26, Theorem II.1.2], one can prove the following bound for the trace of the second fundamental form h

$$-H \leq n - 1.$$

Moreover, it can be proved that, if Γ is the C^2 boundary of Ω and c_{n-1} is the $(n-1)$ -dimensional surface area of $\mathbf{S}^{n-1}$, then

$$\mathcal{A}(\Gamma) \geq c_{n-1}.$$

Here we have equality if and only if Ω is a ball. This result follows from the arithmetic-geometric mean inequality in the following way.

$$\begin{aligned}
c_{n-1} &= \int_{\Sigma \cap \Gamma} \det(-G^{-1}h) dA \leq \int_{\Sigma \cap \Gamma} \left\{ \frac{\text{tr}(-G^{-1}h)}{n-1} \right\} dA \\
&= \int_{\Sigma \cap \Gamma} \left(\frac{-H}{n-1} \right)^{n-1} dA \leq \mathcal{A}(\Sigma \cap \Gamma) \leq \mathcal{A}(\Gamma),
\end{aligned}$$

where $-G^{-1}h$ is the linear map associated to the Jacobian transformation of the Gauss map of the boundary Σ of the convex hull of $\bar{\Omega}$. If we have equality, then $\Sigma = \Gamma$. Also, we have equality in the arithmetic-geometric mean inequality, which implies all points are umbilics, which implies $-G^{-1}h$ is the identity. For more details, see [26].

Now we will state Alexandrov's characterization of an extremal of the isoperimetric functional. A proof can be found in Chavel [26, Chapter II] and it is the original proof given by Alexandrov in [3]. Other proofs can be found in [63] and [71]. Also relevant are the results contained in [32] and [70].

Theorem 1.1.2 (Alexandrov). *In the Euclidean space, a bounded C^2 extremal is a finite union of round balls of the same radius.*

The result is based on PdEs techniques, the maximum principle, Hopf Lemma for elliptic operators and the moving planes method.

In [12] Barbosa and Do Carmo characterize extremals for the isoperimetric functional as hypersurfaces with constant mean curvature, for orientable, differentiable immersions in the Euclidean space.

1.1.3 Steiner Symmetrization

One way to weaken the assumption to the isoperimetric theorem from C^2 to C^1 is by the means of the Steiner symmetrization technique.

With the additional assumption on the C^1 regularity of the boundary, Steiner symmetrization allows to characterize the disk as the only case of equality.

Steiner symmetrization goes back to Steiner (see [77]).

We will overlook some general recalls on the Hausdorff metric on compact sets. Those are very well known and we refer the reader to [26, Section III.1] for a detailed presentation.

The Isoperimetric Inequality of Minkowski Area

We start with the definition of Minkowski area.

Definition 1.1.3 (Minkowski Area). *Given a compact subset $K \subset \mathbb{R}^n$, we define the Minkowski area of K as*

$$\text{Mink}(K) = \liminf_{h \downarrow 0} \frac{v_n([K]_h) - v_n(K)}{h},$$

where $[K]_h = \{x \in \mathbb{R}^n : d(x, K) \leq h\}$ denotes the h -thickening of K . Here d is the Euclidean metric on $\mathbb{R}^n$.

Here we have no assumption on the regularity of K , but when K is the closure of a domain Ω in $\mathbb{R}^n$ with C^1 boundary, then

$$\text{Mink}(K) = \mathcal{A}(\partial\Omega),$$

where $\mathcal{A}$ is the standard Riemannian $(n-1)$ -dimensional area.

Now we give the definition of *Steiner symmetrization* of a compact K with respect to a hyperplane Π .

Definition 1.1.4 (Steiner Symmetrization). *Let $\mathbf{X}$ be the collection of compact subsets of $\mathbb{R}^n$ with Hausdorff metric and Π a hyperplane in $\mathbb{R}^n$.*

For every $x \in \mathbb{R}^n$, let l^x be the line in $\mathbb{R}^n$ through x perpendicular to Π . For every $K \in \mathbf{X}$, define the Steiner symmetrization of K with respect to Π as

$$\text{st}_\Pi K = \bigcup_{w \in \Pi} \{w\} \times I^w,$$

where $I^w = [-\sigma_w, \sigma_w]$ and $\sigma_w = \frac{1}{2}v_1(l^w \cap K)$.

If K is a compact subset of $\mathbb{R}^n$ and $K \neq D_K$, then there exists a *finite number* of Steiner symmetrizations with respect to hyperplanes $\Pi_1, \dots, \Pi_k$ such that

$$r(\text{st}_{\Pi_1} \circ \dots \circ \text{st}_{\Pi_k} K) < r(K). \quad (1.4)$$

For a proof of this result see [26, Lemma III.2.1].

By using Steiner symmetrization, one can prove the *Brunn-Minkowski Inequality*, which implies the isoperimetric inequality. For a proof, see Chavel [26]. This topic is also treated by Burago and Zalgaller, in [22].

Theorem 1.1.5 (Brunn-Minkowski Inequality). *Given a compact set K in $\mathbb{R}^n$, let D be the closed n -disk such that $v_n(D) = v_n(K)$. Then*

$$v_n([D]_\varepsilon) \leq v_n([K]_\varepsilon)$$

for all $\varepsilon > 0$.

Theorem 1.1.6 (Isoperimetric Inequality). *If K is a compact subset of $\mathbb{R}^n$, then Steiner symmetrization of K does not increase its Minkowski area, that is*

$$\text{Mink}(\text{st}_\Pi K) \leq \text{Mink}(K)$$

for any hyperplane Π .

Furthermore, if the closed n -disk $D(R)$ has the same measure of K , then

$$\mathcal{A}(S^{n-1}(R)) = \text{Mink}(D(R)) \leq \text{Mink}(K).$$

If K is the closure of an open subset of Ω in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$, then

$$\mathcal{A}(S^{n-1}(R)) \leq \mathcal{A}(\partial\Omega). \quad (1.5)$$

As we stated before, this is a consequence of the previous result. Indeed the Brunn-Minkowski inequality implies

$$\frac{v_n(D(R+h)) - v_n(D(R))}{h} = \frac{v_n([D(R)]_h) - v_n(D(R))}{h} \leq \frac{v_n([K]_h) - v_n(K)}{h}$$

for all $h > 0$, which implies the claim.

This approach with Steiner symmetrization proves quite easily the isoperimetric inequality but, as it is, does not characterize the case of equality. With the following result (see [26, Theorem III.2.4] for the proof) we give this characterization.

Theorem 1.1.7. *Assume Ω has C^1 boundary and consider $K = \bar{\Omega}$ compact. One has equality in (1.5) if and only if Ω is a disk.*

The proof is based on a sharpened Brunn-Minkowski inequality, in order to include an error term that vanishes if and only if the compact set considered was a disk.

Another approach to the minimizing property of the ball is through the sequences of Steiner symmetrizations: starting from a compact set K and applying to it a sequence of symmetrizations it will converge, in Hausdorff metric, to a closed ball. We will not investigate the topic here, but it is treated in details in [26, Section III.2.2].

1.1.4 Stable CMC hypersurfaces

In this section, we will introduce the concept of *stability* of a hypersurface and we will investigate its relation with the isoperimetric problem. We will treat the Euclidean and the Riemannian case and, in particular, we will present the results obtained by Barbosa and Do Carmo in [12] and Barbosa, Do Carmo and Eschenburg in [13]. These results were the starting point of our work on the complex projective space that will be presented in Section 1.2.

Euclidean Space

A smooth function $f : M \rightarrow N$ between smooth manifolds is said to be an *immersion* if its derivative is everywhere injective. We call *embedding* an immersion which is injective and which is a homeomorphism onto its image. In particular, immersed (but not embedded) manifold can self-intersect.

Let M be an orientable, n -dimensional differentiable manifold and let $x : M \rightarrow \mathbb{R}^{n+1}$ be an immersion. Let $D \subset M$ be a relatively compact domain with smooth boundary. We will denote by $\mathcal{A}_D(x)$ the n -area of D and $\mathcal{V}_D(x)$ the volume of D in x

$$\mathcal{V}_D(x) = \frac{1}{n+1} \int_D \langle x, N \rangle dM \quad (1.6)$$

where N is a unit normal vector field along x , dM is the n -area element in the induced metric and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{n+1}$.

Let $x_t : \bar{D} \rightarrow \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$, $x_0 = x$ be a variation of D and let $\mathcal{A}_D(t) := \mathcal{A}_D(x_t)$, $\mathcal{V}_D(t) := \mathcal{V}_D(x_t)$. We will denote the variation vector of x_t by

$$\xi(p) = \left. \frac{dx_t(p)}{dt} \right|_{t=0}, \quad p \in \bar{D}. \quad (1.7)$$

We will call a variation volume-preserving, if $\mathcal{V}_D(t) = \mathcal{V}_D(0)$ for all $t \in (-\varepsilon, \varepsilon)$ and normal if for all $p \in \bar{D}$, $\xi(p) = f(p)N(p)$, where $f : M \rightarrow \mathbb{R}$ is a real function. Finally, we will say that a variation fixes the boundary if $x_t(p) = x_0(p)$ for all $p \in \partial D$ and $t \in (-\varepsilon, \varepsilon)$. In the case of the complex projective space we will consider, instead, surfaces without boundary.

The next result gives us formulas for the first variations of area and volume for variations of the above type.

Proposition 1.1.8. *Given a variation $x_t : \bar{D} \rightarrow \mathbb{R}^{n+1}$ that fixes the boundary, we have*

$$\mathcal{A}'_D(0) = - \int_D nHf dM, \quad \mathcal{V}'_D(0) = \int_D f dM,$$

where H denotes the mean curvature of x_0 and fN is the normal component of the variation vector of x_t .

This result will be proved later in the more general setting of Riemannian manifolds (see Lemma 1.1.14).

Definition 1.1.9. *Let $x : M \rightarrow \mathbb{R}^{n+1}$ be the immersion of an orientable, n -differentiable manifold M and let h be its second fundamental form at a point p of M . We define the mean curvature H of the immersion x as the trace of the second fundamental form h of x .*

It's easy to see that, by the definition of H ,

$$H = \frac{1}{n} \operatorname{div}_M(N), \tag{1.8}$$

where N is a normal vector field on M and $\operatorname{div}_M(N)$ is the trace of the linear map $X \mapsto \nabla_X N$, with ∇ the Levi-Civita connection of $\mathbb{R}^{n+1}$.

For this kind of variations, Bolza (see [21, pp. 458-459]) proved that, given any piecewise smooth function f on $\bar{D}$ with zero mean, then we can always find a volume-preserving normal variations with variation vector fN .

A first result proved in [12] concerns the (well known) equivalence of the property of having constant mean curvature H_0 and being a critical point of a variational isoperimetric problem. We will see the result directly in the more general case of space forms (see Proposition 1.1.15).

Now consider the same setting as before: take $x : M \rightarrow \mathbb{R}^{n+1}$ immersion with constant mean curvature H_0 , x_t a normal variation of $D \subset M$ that fixes the boundary, let fN be the normal component of x_t and denote by $|h|^2$ the square norm of the second fundamental form h of x .

Then we have a formula for the second variation of the area functional $\mathcal{A}_D$. A generalization of this formula will be proved in the next section (see 1.1.16).

$$\mathcal{A}''_D(0) = \int_D (-f\Delta f - |h|^2 f^2) dM. \tag{1.9}$$

Having the formula for the second variation of the area, we can define a stable immersion, in the following way.

Definition 1.1.10 (Stability). *Let $x : M \rightarrow \mathbb{R}^{n+1}$ have constant mean curvature and let $D \subset M$ be a relatively compact domain with smooth boundary. We say that D is stable if $\mathcal{A}_D''(0) \geq 0$ for every volume-preserving variation that fixes the ∂D . If every such D is stable, we say that the immersion is stable.*

Then, we can see that, with the previous definition, spheres are stable.

Let $D \subset S^n$ be a domain in S^n and assume that S^n has radius 1. Given $f \in \mathcal{F}_D$ we can extend it to a piecewise smooth function $\bar{f} : S^n \rightarrow \mathbb{R}$ by setting $\bar{f} \equiv 0$ on $S^n \setminus D$.

Denote $\mu_1(S^n)$ the first eigenvalue of the problem

$$\begin{cases} \Delta g + \mu g = 0 \\ \int_{S^n} g \, dM = 0 \end{cases}$$

It is well known that $\mu_1(S^n) = n$ and that

$$\mu_1(S^n) \leq \left(\int_{S^n} |\nabla g|^2 \, dM \right) \left(\int_{S^n} g^2 \, dM \right)^{-1}, \quad (1.10)$$

for all piecewise smooth functions $g : S^n \rightarrow \mathbb{R}$ with zero mean on S^n . Here ∇g denotes the gradient of g in the metric induced by $\mathbb{R}^{n+1}$ on S^n . By using Stokes theorem and the fact that $|h|^2 = n$, we obtain, for f , that

$$\mathcal{A}_D''(0)(f) = \int_D (|\nabla f|^2 - n f^2) \, dM.$$

Since (1.10) holds for $\bar{f}$, we conclude that

$$\mathcal{A}_D''(0)(f) = \int_D (|\nabla f|^2 - n f^2) \, dM \geq (\mu_1(S^n) - n) \int_D f^2 \, dM = 0.$$

Since f was an arbitrary function in $\mathcal{F}_D$, then, by definition, S^n is stable. Another way to prove the stability of the sphere is by means of the isoperimetric inequality (1.5).

The equivalence of critical points of isoperimetric variational problems and having constant mean curvature implies that the boundary of an *isoperimetric set* E is a hypersurface with constant mean curvature. Moreover, it is a minimum for the area $\mathcal{A}(t)$ for all compactly supported volume-preserving variations x_t . Thus, by the definition of stability, ∂E must be stable under such variations.

In the simple case of the Euclidean space $\mathbb{R}^{n+1}$, one can prove more. With the following result, Barbosa and Do Carmo (in [12]) proved that the round spheres are the only compact hypersurfaces with constant mean curvature in $\mathbb{R}^{n+1}$ that are stable.

Theorem 1.1.11 (Barbosa, do Carmo). *Let M be a compact, orientable, n -dimensional manifold and let $x : M \rightarrow \mathbb{R}^{n+1}$ be an immersion with constant mean curvature. Then x is stable if and only if $x(M) \subset \mathbb{R}^{n+1}$ is a round sphere $S^n \subset \mathbb{R}^{n+1}$.*

Regarding the proof of Theorem 1.1.11, we have seen that spheres are stable. Thus, we just need to prove the converse.

The next lemma states an algebraic property of matrices, applied to the second fundamental form h and its trace H .

Lemma 1.1.12. *Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an immersion with mean curvature H . Then*

$$|h|^2 \geq nH^2$$

and the equality holds at a point $p \in M$ if and only if p is umbilic.

For the proof, see [12]. Let $g = \langle x, N \rangle$ be the support function of x . The next lemma gives us a formula for the Laplacian of g .

Lemma 1.1.13. *Assume that the immersion x has constant mean curvature $H = H_0$. Then the support function g of x satisfies*

$$\Delta g = -nH_0 - |h|^2 g. \quad (1.11)$$

The proof consists in a rather simple computation and it's done in [12].

Proof of 1.1.11. Let M be compact. In this case, it is well known that the support function g of x satisfies Minkowski's integral formula

$$\int_M gH \, dM = - \int_M dM. \quad (1.12)$$

Since $H = H_0$ is constant, we obtain, by integrating (1.11) over M and using Stokes theorem,

$$- \int_M |h|^2 H_0 g \, dM = nH_0^2 \int_M dM. \quad (1.13)$$

Now, set $f = H_0 g + 1$. By (1.12), $\int_M f \, dM = 0$. Consider a variation of M whose normal component is fN and compute

$$\mathcal{A}_M''(0)(f) = \int_M (-f\Delta f - |h|^2 f^2) \, dM.$$

By using Lemma 1.1.13, it is easily checked that

$$\begin{aligned} -f\Delta f - |h|^2 f^2 &= -H_0(gH_0 + 1)\Delta g - |h|^2(gH_0 + 1)^2 \\ &= -gH_0^2(-nH_0 - |h|^2 g) - H_0(-nH_0 - |h|^2) - |h|^2(g^2 H_0^2 + 2gH_0 + 1) \\ &= nH_0^2 f - |h|^2 f. \end{aligned}$$

It follows that

$$\mathcal{A}_M''(0)(f) = - \int_M |h|^2 (gH_0 + 1) \, dM. \quad (1.14)$$

Finally, assume that x is stable. Then, by definition, $\mathcal{A}_M''(0)(f) \geq 0$ and we obtain, from (1.13), (1.14) and Lemma 1.1.12

$$\int_M nH_0^2 \, dM = - \int_M |h|^2 H_0 g \, dM \geq \int_M |h|^2 \, dM \geq \int_M nH_0^2 \, dM.$$

Then we have all equality and, in particular, $|h|^2 = nH_0^2$, hence all points of M are umbilic. Since M is compact, $x(M) \subset \mathbb{R}^{n+1}$ is a sphere. $\square$

A consequence of Theorem 1.1.11 is that in $\mathbb{R}^{n+1}$, even if we request our manifold M to be immersed, from the stability hypothesis follows that the immersion $x : M \rightarrow \mathbb{R}^{n+1}$ is actually an embedding.

This is interesting in view of the fact that there are many examples of compact nonspherical hypersurfaces with constant mean curvature in $\mathbb{R}^{n+1}$. Wente (see [86]) gave an example for $n = 2$, other examples can be found in the works of Abresch [1] also for $n = 2$, and Hsiang, Teng and Yu [47] for $n > 2$. In view of this last result, all these hypersurfaces are not stable and, in fact, they cannot be the boundary of isoperimetric sets.

The example presented by Wente was a compact oriented hypersurface of genus 1 in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and it was given as a counter-example to the *Hopf conjecture* in dimension 3, which states that, if we consider Σ an immersion of an oriented, closed hypersurface with constant mean curvature $H \neq 0$ in $\mathbb{R}^{n+1}$. Then Σ must be the standard embedded n -sphere.

An important result relating to this conjecture is due to Alexandrov [2], who showed that the conjecture is true if Σ is an embedded hypersurface in $\mathbb{R}^{n+1}$.

Extension to Riemannian Manifolds

In [13], Barbosa, do Carmo and Eschenburg extended the previous results to hypersurfaces of Riemannian manifolds with constant sectional curvature (also called space forms). In this section, we present their main results, which are the starting point of our work in the case of the complex projective space.

In the following, the ambient Riemannian manifold will be denoted by $\bar{M}^{n+1}$, $(n+1)$ being the dimension, and the immersed hypersurface will be denoted as M .

Let $\bar{M}^{n+1}(c)$ be a simply-connected complete Riemannian manifold with constant sectional curvature c and let $x : M \rightarrow \bar{M}^{n+1}(c)$ be an immersion of a differentiable manifold M of dimension n . Moreover, suppose that M is compact, connected, orientable and with boundary ∂M (possibly $= \emptyset$). We choose the orientation on M to be compatible with the orientation of $\bar{M}$. Let $X_1, \dots, X_n, X_{n+1}$ be an orthonormal frame in a neighbourhood $U \subset \bar{M}$ of $x(p)$, $p \in M$ such that

- $X_1, \dots, X_n$ are tangent to $x(M)$ (i.e. the frame is adapted)
- $d\bar{M}(X_1, \dots, X_{n+1}) > 0$, where $d\bar{M}$ is the volume form of $\bar{M}$ (i.e. the frame is positive).

Then if M is orientable, $N := X_{n+1}$ is a globally defined unit normal vector field and we choose it as the orientation of M .

Recall that a variation of x is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \rightarrow \bar{M}$ such that $X_t : M \rightarrow \bar{M}$ defined by $X_t(p) = X(t, p)$, $p \in M$, $t \in (-\varepsilon, \varepsilon)$ is an immersion and $X_0 = x$, $X_t|_{\partial M} = x|_{\partial M}$ for all t .

Define also the *area function* and the *volume function*, respectively

$$\mathcal{A}(t) = \int_M dM_t \quad \text{and} \quad \mathcal{V}(t) = \int_{[0,t] \times M} X^* d\bar{M},$$

where $M_t = X_t(M) = X(t, M)$ and dM_t is the volume element of M in the metric induced by X_t .

Finally, let $W(p) = (\partial X)/(\partial t)|_{t=0}$ be the variation vector field of X and set $f := \langle W, N \rangle$.

Lemma 1.1.14. *Let H be the mean curvature of the immersion x . Then we have*

$$(i) \quad \frac{dA}{dt}(0) = - \int_M n H f \, dM,$$

$$(ii) \quad \frac{dV}{dt}(0) = \int_M f \, dM.$$

Proof. For the area formula (i), let $\bar{M}^{n+1}$ be a Riemannian manifold of dimension $n+1$ and metric g and let $M \subset \bar{M}$ be a submanifold of dimension n . Let $D \subset \mathbb{R}^n$ be an open set and $F : D \times \mathbb{R} \rightarrow \bar{M}$ be a regular function such that

$$M = \{F(x, 0) \in \bar{M} \mid x \in D\}.$$

Call, for $i = 1, \dots, n$

$$\begin{aligned} X_i &= F_* \frac{\partial}{\partial x_i} \\ T &= F_* \frac{\partial}{\partial t}. \end{aligned} \tag{1.15}$$

Let N be a normal vector field on M , more precisely:

- $|N| = g(N, N)^{\frac{1}{2}} = 1$;
- $g(N, X_i) = 0$ on $M \quad \forall i = 1, \dots, n$;
- $\nabla_N N = 0$ where ∇ is the Levi-Civita connection of $\bar{M}$;
- $N = \frac{T}{|T|} = \frac{T}{g(T, T)^{\frac{1}{2}}}$.

In the following we denote by $g_{ij} := g(X_i, X_j) = \langle X_i, X_j \rangle$ and by $\gamma = \sqrt{\det(g_{ij})}$.

Now, let $f : \bar{M} \rightarrow \mathbb{R}$ be a function such that $T = fN$ and $Nf = 0$. More precisely, we take f assigned on M and continuously extended on $\bar{M}$. We also suppose $f \in C_c^\infty(M)$.

Then the area of $M_t = \{F(x, t) \in M : x \in D\}$ is

$$A(t) = \int_D \gamma \, dx \tag{1.16}$$

and

$$\begin{aligned} A'(t) &= \int_D \frac{\partial}{\partial t} \gamma \, dx = \int_D \frac{1}{2} \frac{1}{\gamma} \sum_{\sigma} (-1)^{\text{sign}(\sigma)} \frac{\partial}{\partial t} \prod_{k=1}^n g_{k, \sigma(k)} \, dx \\ &= \int_D \frac{1}{2\gamma} \sum_{\sigma} (-1)^{\text{sign}(\sigma)} \sum_{k=1}^n \prod_{h \neq k} g_{h, \sigma(h)} \frac{\partial}{\partial t} g_{k, \sigma(k)} \, dx. \end{aligned} \tag{1.17}$$

Once fixed a point $x \in D$ it is not restrictive to suppose that $X_1, \dots, X_n$ are orthogonal in $F(x, t) \in \bar{M}$. Then, using the fact that ∇ is torsion-free we get

$$\begin{aligned}
\mathcal{A}'(t) &= \int_D \frac{1}{2\gamma} \sum_{k=1}^n \prod_{h=1}^n g_{hh} \frac{\frac{\partial}{\partial t} g_{kk}}{g_{kk}} dx \\
&= \int_D \frac{1}{2} \gamma \sum_{k=1}^n \frac{2\langle \nabla_T X_k, X_k \rangle}{|X_k|^2} = \int_D \operatorname{div}_{M_t}(T) \gamma dx.
\end{aligned} \tag{1.18}$$

Hence, we just proved the following formula for the first variation of the area functional

$$A'(t) = \int_D \operatorname{div}_{M_t}(T) \gamma dx = \int_D \operatorname{div}_{M_t}(fN) \gamma dx = \int_D f \operatorname{div}_{M_t}(N) \gamma dx. \tag{1.19}$$

That concludes the first part of the proof, since the mean curvature of M satisfies (1.8).

For the second part, it's enough to notice that, once we fix a point $p \in M$ and a positive adapted orthonormal frame $X_1, \dots, X_n, X_{n+1} = N$ around $x(p) \in x(M)$, then

$$X^*(d\bar{M}) = a(t, p) dt \wedge dM,$$

where

$$\begin{aligned}
a(t, p) &= X^*(d\bar{M})\left(\frac{\partial}{\partial t}, X_1, \dots, X_n\right) = d\bar{M}\left(\frac{\partial X}{\partial t}, dX_t(X_1), \dots, dX_t(X_n)\right) \\
&= \operatorname{vol}\left(\frac{\partial X}{\partial t}, dX_t(X_1), \dots, dX_t(X_n)\right) = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle,
\end{aligned}$$

where N_t is a unit normal vector to the immersion X_t . □

A variation X_t is said to be *normal* if W is parallel to N , and *volume-preserving* if $\mathcal{V}(t) = \mathcal{V}(0)$ for all $t \in (-\varepsilon, \varepsilon)$.

We still have the existence result proved by Bolza for volume-preserving normal variations with a given variation vector.

For a given variation X_t of an immersion $x : M \rightarrow \bar{M}^{n+1}$ we define H_0 as

$$H_0 = A^{-1} \int_D H dM, \quad A = \mathcal{A}_D(0)$$

and $J : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ by $J(t) = \mathcal{A}(t) + nH_0\mathcal{V}(t)$. Then we have the following result which states the equivalence between critical points of variational isoperimetric problems of this kind and constant mean curvature hypersurfaces.

Proposition 1.1.15. *Let $x : M \rightarrow \bar{M}^{n+1}$ be an immersion. The following statements are equivalent:*

- 1 X has constant mean curvature H_0 .
- 2 For all volume-preserving variations, $\mathcal{A}'(0) = 0$.
- 3 For all (arbitrary) variations, $J'(0) = 0$.

For the proof, see [13].

As an immediate consequence of Lemma 1.1.14, we get:

$$\begin{aligned}\frac{dJ}{dt} &= \int_M (-nH_t + nH_0) f_t dM \\ J''(0) &= - \int_M \left(\frac{\partial H_t}{\partial t} \right)(0) f dM,\end{aligned}\tag{1.20}$$

where H_t is the mean curvature of X_t and $f_t = \langle \frac{\partial X}{\partial t}, N_t \rangle$, with N_t the unit normal vector of X_t .

Then, we have the following result.

Proposition 1.1.16. *Let $x : M \rightarrow \bar{M}^{n+1}$ be an immersion with constant mean curvature H and let X be a variation of x . Then $J''(0)$ depends only on f and is given by*

$$J''(0)(f) = \int_M (-f\Delta f - (\text{Ricc}(N) + |h|^2)f^2) dM.$$

Here Δ is the Laplacian in the induced metric, $|h|$ is the norm of the second fundamental form of x , and $\text{Ricc}(N)$ is the Ricci curvature of $\bar{M}$ in the direction N .

Proof. First recall that the mean curvature of M satisfies (1.8), namely

$$H = \frac{1}{n} \text{div}_M(N).$$

Then, starting from the first variation of the area of Lemma 1.1.14, for the second derivative, we have

$$\mathcal{A}''(t) = \int_D \left\{ \frac{\partial}{\partial t} \text{div}_{M_t}(T) + (\text{div}_{M_t}(T))^2 \right\} \gamma dx \tag{1.21}$$

then, in $t = 0$

$$\mathcal{A}''(t)|_{t=0} = \int_D \frac{\partial}{\partial t} \text{div}_{M_t}(T)|_{t=0} \gamma dx. \tag{1.22}$$

Since we have supposed that X_i 's are orthogonal, then we have

$$\text{div}_{M_t}(T) = \sum_{i=1}^m \frac{\langle \nabla_{X_i} T, X_i \rangle}{|X_i|^2}.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial t} \text{div}_{M_t}(T) &= \sum_{i=1}^m \frac{\left(\frac{\partial}{\partial t} \langle \nabla_{X_i} T, X_i \rangle \right) \langle X_i, X_i \rangle - 2 \langle \nabla_T X_i, X_i \rangle \langle \nabla_{X_i} T, X_i \rangle}{\langle X_i, X_i \rangle^2} \\ &= \sum_{i=1}^m \frac{\langle X_i, X_i \rangle \{ \langle \nabla_T \nabla_{X_i} T, X_i \rangle + \langle \nabla_{X_i} T, \nabla_T X_i \rangle \} - 2 \langle \nabla_{X_i} T, X_i \rangle^2}{\langle X_i, X_i \rangle^2} \\ &= \sum_{i=1}^m \left\{ \frac{\langle \nabla_T \nabla_{X_i} T, X_i \rangle}{\langle X_i, X_i \rangle} + \frac{|\nabla_{X_i} T|^2}{\langle X_i, X_i \rangle} - 2 \frac{\langle \nabla_{X_i} T, X_i \rangle^2}{\langle X_i, X_i \rangle^2} \right\}.\end{aligned}\tag{1.23}$$

Note that

$$\nabla_T T = 0 \tag{1.24}$$

and, as a consequence,

$$\langle \nabla_T \nabla_{X_i} T, X_i \rangle = \langle R(T, X_i)T, X_i \rangle, \quad (1.25)$$

where $R(\cdot, \cdot)$ is the Riemann curvature tensor.

In fact, since $T = fN$ and $Nf = 0$, one has that

$$\nabla_T T = \nabla_{fN}(fN) = f\nabla_N(fN) = f^2\nabla_N N + f(Nf)N = 0, \quad (1.26)$$

because $\nabla_N N = 0$. Moreover, using the fact that $[X_i, T] = 0$ and by the definition of tensor R , we have that

$$\langle \nabla_T \nabla_{X_i} T, X_i \rangle = \langle R(T, X_i)T, X_i \rangle + \langle \nabla_{X_i} \nabla_T T, X_i \rangle = \langle R(T, X_i)T, X_i \rangle,$$

which is what we wanted to prove.

For the last two terms of (1.23) we have the following.

$$|\nabla_{X_i} T|^2 = |\nabla_{X_i}(fN)|^2 = |(X_i f)N + f\nabla_{X_i} N|^2 = (X_i f)^2 + f^2|\nabla_{X_i} N|^2, \quad (1.27)$$

where

$$\begin{aligned} \nabla_{X_i} N &= \sum_{j=1}^m \langle \nabla_{X_i} N, X_j \rangle \frac{X_j}{|X_j|^2} \\ |\nabla_{X_i} N|^2 &= \sum_{j=1}^m \langle \nabla_{X_i} N, \frac{X_j}{|X_j|} \rangle^2. \end{aligned} \quad (1.28)$$

Hence, if we suppose also that the vector fields X_i 's are eigenvectors of the second fundamental form h , we have

$$\sum_{i=1}^m \frac{|\nabla_{X_i} N|^2}{|X_i|^2} = \sum_{i,j=1}^m \langle \nabla_{X_i} N, X_j \rangle^2 \frac{1}{|X_i|^2 |X_j|^2} = |h|^2. \quad (1.29)$$

Moreover,

$$\langle \nabla_{X_i}(fN), X_i \rangle^2 = f^2 \langle \nabla_{X_i} N, X_i \rangle^2,$$

hence

$$-2 \sum_{i=1}^m \frac{\langle \nabla_{X_i}(fN), X_i \rangle^2}{|X_i|^2} = -2f^2|h|^2. \quad (1.30)$$

Finally, putting together (1.24), (1.25), (1.29) and (1.30) in (1.23), we complete the proof. $\square$

The definition of stability is essentially the same as in 1.1.10 and we will omit it.

One can prove the following criterion for stability.

Proposition 1.1.17. *Let $\mathcal{F}$ be the set of differentiable functions $f : M \rightarrow \mathbb{R}$ with $f|_{\partial M} = 0$ and zero mean on M . Then an immersion $x : M \rightarrow \bar{M}^{n+1}$ is stable if and only if $J''(0)(f) \geq 0$ for all $f \in \mathcal{F}$.*

The proof can be found in [13].

In view of Proposition 1.1.17, we will denote $J''(0)(f)$ directly by $\mathcal{A}''(0)(f)$. Actually the two quantities are equal if the associated variation preserves the volume up to the second order.

Similarly to the Euclidean case, it can be proven that the geodesic spheres $S^n \subset \bar{M}^{n+1}(c)$ are stable (for details see [13, Section 3]).

In this context of ambient manifolds with constant sectional curvature c , Barbosa, do Carmo and Eschenburg in [13] extended Theorem 1.1.11, proving the following result.

Theorem 1.1.18 (Barbosa, Do Carmo, Eschenburg). *Assume that M is a compact manifold of dimension n without boundary and that $x : M \rightarrow \bar{M}^{n+1}(c)$ is an immersion with constant mean curvature, with $\bar{M}^{n+1}(c)$ a Riemannian manifold with constant sectional curvature c . Then x is stable if and only if $x(M) \subset \bar{M}^{n+1}(c)$ is a geodesic sphere.*

The following proof was what inspired our results on $\mathbf{CP}^n$. The first part of the proof is for the general case $c \neq 0$. However, in the second part of the proof we will focus on the positive case. The remaining case $c < 0$ can be found in [13].

Proof. The case $c = 0$ is Theorem 1.1.11. Let's suppose $c \neq 0$. Let L^{n+2} be the euclidean space $\mathbb{R}^{n+2}$ with the canonical basis $a_A = (0, \dots, 0, 1, 0, \dots, 0)$, $A = 0, \dots, n+1$, and inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle a_0, a_0 \rangle = \frac{c}{|c|}, \quad \langle a_\alpha, a_\beta \rangle = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n+1, \quad \langle a_0, a_\beta \rangle = 0.$$

Let $S^{n+1}(c) \subset L^{n+2}$ be a connected component of

$$\left\{ y \in L^{n+2} : \langle y, y \rangle = \frac{1}{c} \right\}.$$

It is well known that $S^{n+1}(c)$ with the induced metric is isometric to $\bar{M}^{n+1}(c)$.

Now let $x : M \rightarrow S^{n+1}(c)$ be an immersion with mean curvature H . Let N be a unit normal vector field along x that defines the orientation of M and fix a vector $v \in L^{n+2}$. Define functions $g : M \rightarrow \mathbb{R}$ and $f : M \rightarrow \mathbb{R}$ by

$$g(p) = \langle x(p), v \rangle, \quad f(p) = \langle N(p), v \rangle. \quad (1.31)$$

With a simple computation one can check the following result.

Lemma 1.1.19. *Let Δ be the Laplacian of M in the metric induced by x . Then*

$$(a) \quad \Delta g = -nHf - cng.$$

If, moreover, H is constant,

$$(b) \quad \Delta f = -|h|^2 f - cnHg.$$

We know that geodesics spheres are stable. For the converse, we will use the fact that, in general, $|h|^2 \geq nH^2$ holds (see Lemma 1.1.12) and equality holds at a point $p \in M$ if and only if p is umbilic. Since the only compact umbilic hypersurfaces of $\bar{M}^{n+1}(c)$ are the geodesic spheres, the proof will be complete if we show that the opposite inequality holds.

Observe first that the function $u = Hf + cg$ satisfies, by point (a) of Lemma 1.1.19, the condition

$$\int_M u dM = \int_M Hf + cg = -\frac{1}{n} \int_M \Delta g = 0,$$

since M is compact without boundary. A straightforward computation shows that

$$\begin{aligned} \mathcal{A}''(0)(u) &= - \int_M (u\Delta u + (|h|^2 + nc)u^2) dM \\ &= - \int_M (|h|^2 - nH^2)(c^2g^2 + cHfg) dM. \end{aligned}$$

In the following we'll write $J''(0)(u) = I(u)$. We will prove that $I(u) \geq 0$ implies that $|h|^2 = nH^2$, concluding the proof.

Consider the cases $c > 0$. We can assume, without loss of generality that $c = 1$. Choose v as an element of the canonical orthonormal basis a_A of L^{n+2} and let f_A and g_A be the functions in (1.31) that correspond to $v = a_A$, $A = 0, 1, \dots, n+1$. Set $u_A = Hf_A + cg_A$. Since x is stable,

$$I(u_A) = - \int_M (|h|^2 - nH^2)(g_A^2 + Hf_Ag_A) dM \geq 0.$$

Thus

$$0 \leq \sum I(u_A) = - \int_M (|h|^2 - nH^2)(\sum g_A^2 - H \sum f_Ag_A) dM.$$

Since $x(M)$ is contained in a unit sphere of $\mathbb{R}^{n+2}$, we obtain

$$\begin{aligned} \sum g_A^2 &= \sum \langle x, a_A \rangle \langle x, a_A \rangle = \langle x, x \rangle = 1, \\ \sum f_Ag_A &= \sum \langle N, a_A \rangle \langle x, a_A \rangle = \langle N, x \rangle = 0. \end{aligned}$$

It follows that

$$0 \leq - \int_M (|h|^2 - nH^2) dM,$$

and since $|h|^2 \geq nH^2$, we obtain $|h|^2 = nH^2$. $\square$

For a general Riemannian manifold with no assumption on its sectional curvature, the situation is different. Not all the (geodesic) spheres are stable and not all stable hypersurfaces are spheres. For example in projective spaces also certain tubes are stable (see Section 1.2.1 for details).

In [13] Barbosa, Do Carmo and Eschenburg prove the following result about stability of tubes and spheres in projective spaces.

Let $\bar{M} = \mathbf{K}P^{r-1}$ be the projective space over the field $\mathbf{K}$, with metric of diameter $\frac{\pi}{2}$ and curvature between 1 and 4, where $\mathbf{K}$ can be the Real space, the Complex space or the Hyperbolic space.

For $q < r$ let $U_\rho(\mathbf{K}P^{q-1})$ be the tubular neighbourhood of radius ρ around the totally geodesic subspace $\mathbf{K}P^{q-1}$ of $\mathbf{K}P^{r-1}$, and put $T_\rho(q) = \partial U_\rho(\mathbf{K}P^{q-1})$. Note that

$T_\rho(q)$ is congruent to $T_{\frac{\pi}{2}-\rho}(p)$ if $p = r - q$ and that $T_\rho(1)$ is the geodesic sphere of radius ρ . Set $d = \dim_{\mathbb{R}} \mathbf{K}$ and assume that r is even if $\mathbf{K}$ is the Real space (for orientability reasons).

Theorem 1.1.20 (Barbosa, Do Carmo, Eschenburg). *For $2 \leq q \leq r - 2$, $T_\rho(q)$ is stable in $\mathbf{KP}^{q-1}$ if and only if*

$$\frac{pd - 1}{qd + 1} \leq \tan^2 \rho \leq \frac{pd + 1}{qd - 1}.$$

For $q = 1$ ($q = r - 1$), the lower (upper) bound is not present: a sphere of radius ρ is stable if and only if $\tan^2 \rho \leq \frac{(r-1)d+1}{d-1}$.

The theorem is proved in the context of group-invariant stability (for details, see [13, Section 4]). We will recall this theorem in the specific cases of $\bar{M} = \mathbf{CP}^n$ (see Theorem 1.2.2 in Sect. 1.2.1) and $\bar{M} = \mathbf{RP}^n$ (see Theorem 1.3.3 in Sect. 1.3.2).

In general, if we are not in the Euclidean case and we are not tanking specific geometric assumptions, stability is only a necessary condition for a surface to be the boundary of an isoperimetric set (see [73] for details).

Another remarkable result is due to Ritoré and Ros. In [65] they give a complete solution of the stability problem in the 3-dimensional projective space. Their result reads as follows.

Theorem 1.1.21 (Ritoré, Ros). *Let $x : M \rightarrow \mathbf{RP}^3$ be a complete orientable CMC surface immersed into the real projective space. If the immersion is stable, then either*

- (i) *M is a compact surface with $\text{genus}(M) = 0$ and x is an embedded geodesic sphere or a twofold covering of a totally geodesic projective plane, or*
- (ii) *M is a compact surface of genus 1 and x is an embedded flat tube of radius r about a geodesic, with $\pi/6 \leq r \leq \pi/3$.*

1.2 Two Characterizations of the Sphere in $\mathbf{CP}^n$

In this section, we characterize the geodesic sphere with radius $\tan^2 r = 2n + 1$ as the unique stable compact hypersurface subject to a bound either on the *characteristic curvature*. or on the restriction of the second fundamental form to the complex tangent space.

Preliminarily, in Subsection 1.2.1 we will recall some result on the stability of spheres and tubes in the complex projective space and their geometric characterization. We'll see that in $\mathbf{CP}^n$ not all spheres are stable and not all the stable hypersurfaces are spheres.

In Section 1.2.3 we will state our main results. The idea that led to these results comes from the proof done by Barbosa, Do Carmo and Eschenburg of Theorem 1.1.18. For their result it was enough the immersion of the ambient $(n + 1)$ -space form into a euclidean space of dimension $n + 2$. In our case, the suitable space is the space of Hermitian matrices of order $(n + 1)$, this resulting in the dimension of the normal space to the "immersed" $\mathbf{CP}^n$ being much higher than 1. This fact led to very long computations, which can be found in Sections 1.2.4 and 1.2.5.

1.2.1 Stability of spheres and tubes

As we've seen in Theorem 1.1.20, Barbosa, Do Carmo and Eschenburg proved that a geodesic sphere with radius $r \in (0, \pi/2)$ in the complex projective space $\mathbf{CP}^n$, $n \geq 2$, is stable for the area functional with fixed enclosed volume if and only if $\tan^2 r \leq 2n+1$. They also computed the stability-intervals for the radius of a geodesic tube around $\mathbf{CP}^k \subset \mathbf{CP}^n$, with $1 \leq k < n$.

Let's start by defining spheres and a certain type of tubes in $\mathbf{CP}^n$. The n -dimensional complex projective space is the quotient of the unit sphere $\mathbf{S}^{2n+1} = \{z \in \mathbf{C}^{n+1} : |z| = 1\}$ by the Hopf-action of $\mathbf{S}^1$, $(e^{i\vartheta}, z) \mapsto e^{i\vartheta}z$. We denote by $[z]$ the equivalence class of $z \in \mathbf{S}^{2n+1}$. The tangent space of $\mathbf{CP}^n$ at the point $[z]$ is

$$T_{[z]}\mathbf{CP}^n = \{w \in \mathbf{C}^{n+1} : z \cdot \bar{w} = 0\},$$

where $z \cdot \bar{w} = z_1 \bar{w}_1 + \dots + z_{n+1} \bar{w}_{n+1}$ is the standard Hermitian product of $\mathbf{C}^{n+1}$. The complex structure on $T_{[z]}\mathbf{CP}^n$ is given by $Jw = iw$, the standard multiplication by i of $w \in T_{[z]}\mathbf{CP}^n \subset \mathbf{C}^{n+1}$.

The metric $\langle \zeta, w \rangle_{FS} = \operatorname{Re}(\zeta \cdot \bar{w})$, with $\zeta, w \in T_{[z]}\mathbf{CP}^n$, is the *Fubini-Study* metric of $\mathbf{CP}^n$, that makes the complex projective space a Riemannian manifold. The induced distance function $d : \mathbf{CP}^n \times \mathbf{CP}^n \rightarrow [0, \pi/2]$ is $d([z], [w]) = \arccos |z \cdot \bar{w}|$.

Let $\Sigma \subset \mathbf{CP}^n$ be a C^∞ -smooth hypersurface oriented by the unit normal N . We define the second fundamental form h of Σ with the following sign convention

$$h(X, Y) = \langle \nabla_X^\top N, Y \rangle, \quad X, Y \in T_{[z]}\Sigma.$$

The *characteristic curvature* of Σ at the point $[z] \in \Sigma$ is $\kappa = h(JN, JN)$.

For any fixed $[w] \in \mathbf{CP}^n$ and $0 < r < \pi/2$, the geodesic sphere centered at $[w]$ with radius r is

$$\Sigma_r = \{[z] \in \mathbf{CP}^n : |z \cdot \bar{w}| = \cos r\}.$$

We omit reference to the center. The curvatures of Σ_r are well-known, see e.g. [29, Example 1 page 493]. Letting $t = \tan r$, they are

$$\begin{aligned} \lambda &= \cot r = \frac{1}{t}, \quad \text{with multiplicity } 2(n-1), \\ \kappa &= 2 \cot(2r) = \frac{1}{t} - t, \quad \text{the characteristic curvature.} \end{aligned} \tag{1.32}$$

These two curvatures are constant and distinct for each value of $t > 0$. In [81], Takagi proved that this property characterizes the sphere in [81].

Proposition 1.2.1 (Takagi). *If $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$, is a connected hyperface with precisely two distinct constant curvatures, then Σ is a subset of a sphere Σ_r .*

In fact, the constancy assumption on the curvatures can be dropped, see [29].

We now discuss tubes around $\mathbf{CP}^k$. For $k = 1, \dots, n-1$, the natural inclusion $\mathbf{S}^{2k+1} = \{z \in \mathbf{S}^{2n+1} : z_{k+2} = \dots = z_{n+1} = 0\} \subset \mathbf{S}^{2n+1}$ induces the inclusion $\mathbf{CP}^k \subset \mathbf{CP}^n$. For $0 < r < \pi/2$, we define the tube

$$\begin{aligned} T_r^k &= \{[z] \in \mathbf{CP}^n : \operatorname{dist}([z], \mathbf{CP}^k) = r\} \\ &= \{[z] = [\xi', z''] \in \mathbf{CP}^n : |z| = 1, z' \in \mathbf{C}^{k+1}, |z'| = \cos r\}. \end{aligned}$$

The curvatures of T_r^k are computed in [29]. Letting $t = \tan r$, they are

$$\begin{aligned}\lambda_1 &= \cot\left(r - \frac{\pi}{2}\right) = -t, \quad \text{with multiplicity } 2k, \\ \lambda_2 &= \cot r = \frac{1}{t}, \quad \text{with multiplicity } 2\ell = 2(n-1-k), \\ \kappa &= 2 \cot(2r) = \frac{1}{t} - t, \quad \text{the characteristic curvature.}\end{aligned}\tag{1.33}$$

These three curvatures are constant and distinct for each value of $t > 0$. In particular, T_r^k has constant mean curvature. For $r + s = \pi/2$ and $k + \ell = n - 1$ the hypersurfaces T_r^k and T_s^ℓ are congruent.

The tubes T_r^k share with the sphere Σ_r the property of being stable for some value of r , as proved in Theorem 1.2.2. The following theorem is a reformulation of Theorem 1.1.20.

Theorem 1.2.2 (Barbosa, Do Carmo, Eschenburg [13]). *Let $n \geq 2$ and $k = 1, \dots, n-1$. Then:*

- 1) *The sphere Σ_r is stable if and only if $\tan^2 r \leq 2n + 1$.*
- 2) *The tube T_r^k is stable if and only if*

$$\frac{2n - 2k - 1}{2k + 3} \leq \tan^2 r \leq \frac{2n - 2k + 1}{2k + 1}.$$

Finally, we consider geodesics tubes around $\mathbf{RP}^n$. We start from the following embedding of the sphere $\mathbf{S}^n$ into $\mathbf{S}^{2n+1}$:

$$\mathbf{S}^n = \{z \in \mathbf{C}^{n+1} : |z| = 1, z = \bar{z}\} \subset \mathbf{S}^{2n+1}.$$

Passing to the quotient, the inclusion $\mathbf{S}^n \subset \mathbf{S}^{2n+1}$ gives an embedding of $\mathbf{RP}^n$ into $\mathbf{CP}^n$. The distance of $z \in \mathbf{S}^{2n+1}$ from $\mathbf{S}^n$ is

$$\text{dist}(z, \mathbf{S}^n) = \frac{1}{2} \arccos\left(\left|\sum_{j=1}^{n+1} z_j^2\right|\right),$$

and it does not depend on the equivalence class of z . The level-sets of this distance form the isoparametric family of hypersurfaces in $\mathbf{S}^{2n+1}$ studied in [58, Theorem 1]. Hence, the geodesic tube in $\mathbf{CP}^n$ with radius r around $\mathbf{RP}^n$ is

$$V_r = \left\{[z] \in \mathbf{CP}^n : \left|\sum_{j=1}^{n+1} z_j^2\right| = \cos 2r\right\}.$$

The curvatures of V_r are computed in [29] starting from the formulas in [58] for the preimage of V_r in $\mathbf{S}^{2n+1}$. Letting $t = \tan r$, they are

$$\begin{aligned}\lambda_1 &= -\cot r = -\frac{1}{t}, \quad \text{with multiplicity } n-1, \\ \lambda_2 &= -\cot\left(\frac{\pi}{2} - r\right) = t, \quad \text{with multiplicity } n-1, \\ \kappa &= 2 \cot\left(\frac{\pi}{2} - 2r\right) = \frac{4t}{1 - t^2}, \quad \text{the characteristic curvature.}\end{aligned}\tag{1.34}$$

These three curvatures are constant and distinct for each value of $t \in (0, 1)$. We have a third example of a complete constant mean curvature hypersurface in $\mathbf{CP}^n$. We will see in Lemma 1.2.27 that V_r is not stable for any $0 < r < \pi/4$.

In [82] Takagi proved that the tubes V_r and the tubes T_r^k are characterized by the fact of having precisely three *distinct and constant* curvatures.

Proposition 1.2.3 (Takagi). *Let $n \geq 3$. If $\Sigma \subset \mathbf{CP}^n$ is a connected hypersurface with three distinct constant curvatures, then Σ is a subset of some tube T_r^k or V_r .*

We shall use Proposition 1.2.3 and Theorem 1.2.2 in the proof of Theorem 1.2.14 in Subsection 1.2.7.

1.2.2 Geometry of the isometric immersion of $\mathbf{CP}^n$ into H^{n+1}

Let $H^{n+1} = \{A \in \mathfrak{gl}(n+1, \mathbf{C}) : \bar{A} = A^t\}$ be the set of $(n+1)$ -dimensional Hermitian matrices. This is a $(n+1)^2$ -dimensional real subspace of $\mathfrak{gl}(n+1, \mathbf{C})$. The standard scalar product on H^{n+1} is

$$\langle A, B \rangle := \frac{1}{2} \operatorname{tr}(AB), \quad A, B \in H^{n+1}. \quad (1.35)$$

Let $\Phi : \mathbf{CP}^n \rightarrow H^{n+1}$ be the mapping that takes the equivalence class $[z] \in \mathbf{CP}^n$ to the Hermitian matrix $A = \Phi([z]) \in H^{n+1}$ of the projection of $\mathbf{C}^{n+1}$ onto the complex line $[z]$. The matrix A satisfies $A^2 = A$ because it is a projection and $\operatorname{tr}(A) = 1$, because it projects onto a complex line. It can be checked that Φ is an isometry from $\mathbf{CP}^n$ with the Fubini-Study metric into H^{n+1} with the metric (1.35). Hence, from now on we identify the complex projective space with

$$\mathbf{CP}^n = \{A \in H^{n+1} : A^2 = A, \operatorname{tr}(A) = 1\}.$$

For details on this identification and for the proof of the following lemmas, we refer the reader to [68]. Our normalization in (1.35) of the scalar product is different from the one by Ros. Namely, the relation between the metric g used by Ross and the metric in (1.35) is $g(A, B) = 4 \langle A, B \rangle$. The isometric embedding Φ was introduced in [80].

For any $A \in \mathbf{CP}^n$, we denote by $T_A \mathbf{CP}^n$ and $T_A^\perp \mathbf{CP}^n$ the tangent space and the normal space of $\mathbf{CP}^n$ at the point $A \in H^{n+1}$, respectively.

Lemma 1.2.4. *For any $A \in \mathbf{CP}^n$, we have:*

$$T_A \mathbf{CP}^n = \{X \in H^{n+1} : XA + AX = X\}, \quad (1.36)$$

$$T_A^\perp \mathbf{CP}^n = \{Z \in H^{n+1} : AZ = ZA\}. \quad (1.37)$$

For the proof see [68]. We easily see that $A, I \in T_A^\perp \mathbf{CP}^n$, where I is the identity matrix. We call the matrix $A_0 \in \mathbf{CP}^n$

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the *origin* of $\mathbf{CP}^n$. This is the projection onto the complex line of the versor $e_0 = (1, 0, \dots, 0) \in \mathbf{C}^{n+1}$.

For $i, j \in \{0, 1, \dots, n\}$, let E_{ij} be the $(n+1) \times (n+1)$ matrix with entry 1 at the position (i, j) and with 0 elsewhere. With this notation we have $E_{00} = A_0$. Then the matrices $X_1, \dots, X_n, \widehat{X}_1, \dots, \widehat{X}_n$, where for $j = 1, \dots, n$

$$X_j = E_{j0} + E_{0j} \quad \text{and} \quad \widehat{X}_j = iE_{j0} - iE_{0j}, \quad (1.38)$$

form an orthonormal basis for the tangent space of $\mathbf{CP}^n$ at the point A_0 . The identity $\widehat{X}_j = JX_j$ can be checked using formula (1.44) below for the complex structure.

For any point $A \in \mathbf{CP}^n$ there exists a (non-unique) unitary transformation $Q \in U(n+1)$ such that $A = QA_0Q^{-1}$. The conjugation $T_Q : H^{n+1} \rightarrow H^{n+1}$, $T_Q X = QXQ^{-1}$, preserves the metric of H^{n+1} :

$$\langle T_Q X, T_Q Y \rangle = \frac{1}{2} \text{tr}(QXYQ^{-1}) = \frac{1}{2} \text{tr}(XY) = \langle X, Y \rangle, \quad X, Y \in H^{n+1}.$$

In particular, T_Q maps isometrically the tangent space $T_{A_0}\mathbf{CP}^n$ onto $T_A\mathbf{CP}^n$. We will use these isometries to reduce computations of isometric-invariant quantities to the origin A_0 .

By elementary computations based on the projection equation $A^2 = A$ and on the equation $X = AX + XA$ for tangent vectors, it is possible to check the following algebraic identities.

Lemma 1.2.5. *For any point $A \in \mathbf{CP}^n$ and for any vector fields $X, Y \in T_A\mathbf{CP}^n$, we have:*

$$AXY = XYA, \quad (1.39)$$

$$AXA = 0, \quad (1.40)$$

$$X(I - 2A) = -(I - 2A)X, \quad (1.41)$$

$$(I - 2A)^2 = I, \quad (1.42)$$

$$(I - 2A)XY = XY(I - 2A). \quad (1.43)$$

Using the isometric identification Φ , the natural complex structure on $T_{[z]}\mathbf{CP}^n \subset \mathbf{C}^{n+1}$ can be taken to $T_A\mathbf{CP}^n \subset H^{n+1}$. The resulting mapping is described in the following proposition.

Proposition 1.2.6. *For any $A \in \mathbf{CP}^n$, the mapping $J_A : T_A\mathbf{CP}^n \rightarrow T_A\mathbf{CP}^n$ defined by the formula*

$$J_A X = i(I - 2A)X, \quad (1.44)$$

satisfies the following properties:

- i) *it is an isometry;*
- ii) *it satisfies $J_A^2 = -\text{Id}$, where Id is the identity mapping;*
- iii) *it commutes with the isometries T_Q , i.e., for any $A, B \in \mathbf{CP}^n$ with $A = T_Q B$ for some $Q \in U(n+1)$, and for any $X \in T_B\mathbf{CP}^n$, we have*

$$J_A T_Q X = T_Q J_B X. \quad (1.45)$$

We compute the mean curvature of the immersion of $\mathbf{CP}^n$ into H^{n+1} . For any $A \in \mathbf{CP}^n$ we define the orthogonal projections $\pi_A^\top : H^{n+1} \rightarrow T_A \mathbf{CP}^n$ and $\pi_A^\perp : H^{n+1} \rightarrow T_A^\perp \mathbf{CP}^n$. Explicit formulas for $\pi^\top$ and $\pi^\perp$ can be expressed using the symmetric product $\pi : H^{n+1} \times H^{n+1} \rightarrow H^{n+1}$

$$\pi(X, Y) = XY + YX.$$

Notice that by (1.37) and (1.39), we have $\pi : T_A \mathbf{CP}^n \times T_A \mathbf{CP}^n \rightarrow T_A \mathbf{CP}^n$.

Lemma 1.2.7. *For any $A \in \mathbf{CP}^n$ and $X \in H^{n+1}$, we have*

$$\pi_A^\top(X) = \pi(A, X) - 2AXA, \quad (1.46)$$

$$\pi_A^\perp(X) = X - \pi(A, X) + 2AXA. \quad (1.47)$$

Proof. The mapping defined by formula (1.116) is linear and is the identity on $T_A \mathbf{CP}^n$. Indeed, for any $X \in T_A \mathbf{CP}^n$, by (1.40) and (1.36) we have $\pi_A^\top(X) = AX + XA = X$. We claim that $\pi_A^\top(X) \in T_A \mathbf{CP}^n$ for any $X \in H^{n+1}$. Indeed, we have

$$\begin{aligned} \pi_A^\top(X)A + A\pi_A^\top(X) &= AXA + XA - 2AXA + AX + AXA - 2AXA \\ &= \pi(A, X) - 2AXA = \pi_A^\top(X), \end{aligned}$$

where we used $A^2 = A$.

Then formula (1.116) defines the projection onto $T_A \mathbf{CP}^n$. Formula (1.117) follows from (1.116). $\square$

We split the standard connection ∇ of H^{n+1} into the part that is tangent to $\mathbf{CP}^n$ and the part that is normal. Namely, for $X \in \Gamma(T\mathbf{CP}^n)$, $Y \in \Gamma(TH^{n+1})$, and $A \in \mathbf{CP}^n$ we let $\nabla_X^\top Y(A) = \pi_A^\top(\nabla_X Y)$, and $\nabla_X^\perp Y(A) = \pi_A^\perp(\nabla_X Y)$. By (1.116) and (1.117), we have the formulas

$$\nabla_X^\top Y = \pi(A, \nabla_X Y) - 2A(\nabla_X Y)A, \quad (1.48)$$

$$\nabla_X^\perp Y = \nabla_X Y - \pi(A, \nabla_X Y) + 2A(\nabla_X Y)A. \quad (1.49)$$

The second fundamental form of the immersion of $\mathbf{CP}^n$ into H^{n+1} is the mapping $\sigma_A : T_A \mathbf{CP}^n \times T_A \mathbf{CP}^n \rightarrow T_A^\perp \mathbf{CP}^n$, $A \in \mathbf{CP}^n$, defined by $\sigma_A(X, Y) = \nabla_X^\perp Y(A)$. When no confusion arises, we drop the subscript A and write $\sigma = \sigma_A$.

The non-normalized mean curvature vector of the immersion is the trace of σ , i.e.,

$$\mathcal{H} = \text{tr}(\sigma) = \sum_{i=1}^{2n} \sigma(X_i, X_i), \quad (1.50)$$

where $X_1, \dots, X_{2n}$ is any orthonormal frame of $T_A \mathbf{CP}^n$.

Proposition 1.2.8. *For any $A \in \mathbf{CP}^n$ and $X, Y \in T_A \mathbf{CP}^n$ we have*

$$\sigma(X, Y) = \pi(X, Y)(I - 2A), \quad (1.51)$$

and the trace of σ is

$$\mathcal{H} = 4(I - (n+1)A), \quad A \in \mathbf{CP}^n. \quad (1.52)$$

For a proof of (1.51) and (1.52) see [68, Proposition 1.4].

Remark 1.2.9. Using (1.51) and (1.44), it is possible to check the following identity for any $X, Y, V, W \in T_A \mathbf{CP}^n$:

$$\begin{aligned} \langle \sigma(X, Y), \sigma(V, W) \rangle = & 2 \langle X, Y \rangle \langle V, W \rangle + \langle X, W \rangle \langle Y, V \rangle + \langle X, V \rangle \langle Y, W \rangle + \\ & + \langle X, JW \rangle \langle Y, JV \rangle + \langle X, JV \rangle \langle Y, JW \rangle. \end{aligned} \quad (1.53)$$

The *Weingarten endomorphism* of the immersion is the mapping $\Lambda : T_A \mathbf{CP}^n \times T_A^\perp \mathbf{CP}^n \rightarrow T_A \mathbf{CP}^n$, $A \in \mathbf{CP}^n$, defined by the formula $\Lambda(X, Z) = \Lambda_Z(X) = -\nabla_X^\perp Z$.

Proposition 1.2.10. For any $A \in \mathbf{CP}^n$, $X \in T_A \mathbf{CP}^n$, and $Z \in T_A^\perp \mathbf{CP}^n$ we have

$$\Lambda_Z(X) = (XZ - ZX)(I - 2A). \quad (1.54)$$

For the proof see [68, Proposition 1.4]. We establish some identities linking Λ and σ . Let $X_1, \dots, X_{2n}$ be an orthonormal frame for $T\mathbf{CP}^n$ and we use the notation $N = X_{2n}$. In the sequel, we also let

$$\begin{aligned} \pi_{ij} &= \pi(X_i, X_j) \quad \text{and} \quad \pi_{i,N} = \pi(X_i, N), \\ \sigma_{ij} &= \sigma(X_i, X_j) \quad \text{and} \quad \sigma_{i,N} = \sigma(X_i, N). \end{aligned}$$

The second fundamental form $\sigma(X, Y)$ is defined when X and Y are tangent sections of $\mathbf{CP}^n$. However, the right-hand side of (1.51) is defined for any $X, Y \in H^{n+1}$. In the next lemma and in the following, we will use (1.51) as the general definition of σ .

Lemma 1.2.11. Let $X_1, \dots, X_{2n-1}, N$ be an orthonormal frame of $\mathbf{CP}^n$. Then for any $i, j = 1, \dots, 2n-1$ we have

$$\Lambda_{\sigma_{j,N}}(X_i) = \pi(\pi_{j,N}, X_i) = 2\pi_{j,N}X_i - \sigma(\sigma_{ij}, N) - \sigma(X_j, \sigma_{i,N}). \quad (1.55)$$

Proof. We prove the identity on the left of (1.55). By (1.51), (1.54), (1.41), and (1.42) we get

$$\begin{aligned} \Lambda_{\sigma_{j,N}}(X_i) &= (X_i \sigma_{j,N} - \sigma_{j,N} X_i)(I - 2A) \\ &= (X_i \pi_{j,N}(I - 2A) - \pi_{j,N}(I - 2A)X_i)(I - 2A) \\ &= X_i \pi_{j,N} + \pi_{j,N} X_i = \pi(\pi_{j,N}, X_i). \end{aligned} \quad (1.56)$$

Now we check the identity on the right. Using (1.41) and (1.42) we have:

$$\begin{aligned} & 2\pi_{j,N}X_i - [\pi(\sigma_{ij}, N) + \pi(X_j, \sigma_{i,N})](I - 2A) \\ &= -(\pi_{ij}(I - 2A)N + N\pi_{ij}(I - 2A) + X_j\pi_{i,N}(I - 2A) + \\ & \quad + \pi_{i,N}(I - 2A)X_j)(I - 2A) + 2\pi_{j,N}X_i \\ &= \pi_{ij}N - N\pi_{ij} - X_j\pi_{i,N} + \pi_{i,N}X_j + 2\pi_{j,N}X_i \\ &= X_iX_jN + NX_jX_i + X_jNX_i + X_iNX_j \\ &= X_i\pi_{j,N} + \pi_{j,N}X_i \\ &= \pi(X_i, \pi_{j,N}). \end{aligned}$$

□

Lemma 1.2.12. *For any orthonormal frame $X_1, \dots, X_{2n-1}, N$ of $\mathbf{CP}^n$ we have*

$$\sum_{i=1}^{2n-1} \Lambda_{\sigma_{i,N}}(X_i) = 2(n-1)N. \quad (1.57)$$

Proof. It is enough to verify (1.57) at the point $A_0 \in \mathbf{CP}^n$, with the frame (1.38) where $N = \widehat{X}_n$. Using formula (1.54) for Λ and the identities (1.41) and (1.42), for any $i = 1, \dots, n$ and $j = 1, \dots, n-1$ we find

$$\begin{aligned} \Lambda_{\sigma_{i,N}}(X_i) &= \delta_{in}(N - 2\widehat{X}_i) + N, \\ \Lambda_{\sigma_{\widehat{j},N}}(\widehat{X}_j) &= \delta_{jn}(2\widehat{X}_j + N) + N = N. \end{aligned}$$

Summing up, we obtain (1.57). □

1.2.3 Main results

For fixed $H \in \mathbb{R}$ and $n \in \mathbb{N}$, let $p(\cdot; H, n)$ be the quadratic polynomial of the real variable $t \in \mathbb{R}$

$$p(t; H, n) = (2n+1)t^2 - 2Ht - H^2 - 4(n^2 - 1). \quad (1.58)$$

To fix the notation, in the following we denote by H the mean curvature of a hypersurface $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$. In particular we consider the non-normalized mean curvature, namely

$$H := \text{tr}(h), \quad (1.59)$$

where h is the second fundamental form of Σ .

The first result is the following theorem.

Theorem 1.2.13. *Let $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$, be a compact stable hypersurface with constant mean curvature H . If the characteristic curvature κ of Σ satisfies $p(\kappa; H, n) \geq 0$ then Σ is a geodesic sphere of radius $r > 0$ with $\tan^2 r = 2n+1$.*

Let $\mathbf{CT}\Sigma$ be the complex tangent space of a hypersurface Σ , let T_1 and T_2 be nonzero subbundles of $\mathbf{CT}\Sigma$ such that

$$\mathbf{CT}\Sigma = T_1 \oplus T_2, \quad (1.60)$$

and denote their dimensions by $\alpha = \text{rank}(T_1)$ and $\beta = \text{rank}(T_2)$. Then we have $\alpha + \beta = \text{rank}(\mathbf{CT}\Sigma) = 2(n-1)$ and $1 \leq \alpha < 2(n-1)$. Let h_1 and h_2 be the restrictions of the second fundamental form h of Σ to T_1 and T_2 , and denote by $H_1 = \text{tr}(h_1)$ and $H_2 = \text{tr}(h_2)$ their respective traces.

For fixed $H \in \mathbb{R}$, $n \in \mathbb{N}$ with $n \geq 2$, and $1 \leq \alpha < 2(n-1)$ let $p(\cdot; H, n, \alpha)$ be the quadratic convex polynomial of the variables $(s, t) \in \mathbb{R}^2$

$$p(s, t; H, n, \alpha) = \frac{s^2}{\alpha} + \frac{t^2}{\beta} + (s+t-H)^2 + \frac{(s+t-2H)^2}{2(n^2-1)} - \frac{H^2}{n-1} - 2n. \quad (1.61)$$

The second result is a refined version of Theorem 1.2.13.

Theorem 1.2.14. *Let $\Sigma \subset \mathbf{CP}^n$, $n \geq 3$, be a compact stable hypersurface with constant mean curvature H . If for the decomposition (1.60), with $1 \leq \alpha = \text{rank}(T_1) < 2(n-1)$, we have $p(H_1, H_2; H, n, \alpha) \geq 0$ then Σ is a geodesic sphere of radius $r > 0$ with $\tan^2 r = 2n+1$.*

Both Theorems 1.2.13 and 1.2.14 are a consequence of the following geometric inequality that is implied by stability. Let $\nabla^\top$ be the Levi-Civita connection of $\mathbf{CP}^n$, see Subsection 1.2.2 for the notation, and consider the covariant derivative $\nabla_{JN}^\top N \in T\Sigma$ of the normal N to Σ . We denote by $h_N \in \mathbf{CT}\Sigma$ the projection of $\nabla_{JN}^\top N$ onto $\mathbf{CT}\Sigma$. By $|h|^2$ we denote the squared norm of h .

Theorem 1.2.15. *Let $\Sigma \subset \mathbf{CP}^n$, $n \geq 2$, be a compact stable hypersurface with constant mean curvature H . Then we have*

$$\int_{\Sigma} \left\{ |h|^2 + \frac{(H + \kappa)^2 + |h_N|^2}{2(n^2 - 1)} - \frac{H^2}{n-1} - 2n \right\} d\mu \leq 0, \quad (1.62)$$

where μ is the Riemannian hypersurface measure.

The method for obtaining formula (1.62) starts from an idea contained in the proof of Theorem 1.1.18 (see also [13]) that geodesic spheres are the unique stable complete hypersurfaces in the standard sphere. We will make use of the isometric embedding of $\mathbf{CP}^n$ into H^{n+1} , the space of $(n+1) \times (n+1)$ Hermitian matrices done in the previous paragraph (see also [68] and [80]). Once the hypersurface Σ is embedded in H^{n+1} , we can consider the *position matrix* $A \in \Sigma$ and compute its tangential Laplacian, see Theorem 1.2.16,

$$\Delta A = \text{tr}(\sigma) - \sigma(N, N) - \text{tr}(h)N, \quad (1.63)$$

where Δ is the Laplace-Beltrami operator of Σ and $\sigma : T_A \mathbf{CP}^n \times T_A \mathbf{CP}^n \rightarrow T_A^\perp \mathbf{CP}^n$ is the second fundamental form of the immersion $\mathbf{CP}^n \subset H^{n+1}$. We shall make systematic use of the geometric formulas concerning σ proved by Ros in [68] and [69].

For any a smooth function $u : \Sigma \rightarrow \mathbb{R}$, the second variation of the area functional in the normal direction uN is given by the formula

$$\mathcal{A}''(u) = - \int_{\Sigma} u \mathcal{L}u \, d\mu,$$

where $\mathcal{L}u = \Delta u + (|h|^2 + \text{Ric}(N))u$ is the Jacobi operator. When u has zero mean, the deformation of Σ encloses a region with the same volume as Σ , at the infinitesimal level.

For any fixed $V \in H^{n+1}$, the function $u_V = \langle \Delta A, V \rangle$ has zero mean. In Subsection 1.2.6, we compute the trace of the quadratic form Q_Σ on H^{n+1} defined by $Q_\Sigma(V) = \mathcal{A}''(u_V)$. If Σ is stable, this trace is nonnegative and this fact is precisely inequality (1.62).

From (1.63), it is clear that in the computation for $\mathcal{L}u_V$ we need geometric formulas for $\Delta \text{tr}(\sigma)$, $\Delta \sigma(N, N)$, and ΔN . The computation for ΔN is done in Theorem 1.2.17. The formula for $\Delta \text{tr}(\sigma)$ follows easily from the formula for the trace of σ , see (1.52). The difficult task is to compute the tangential Laplacian of $\sigma(N, N)$. This is done in Subsection 1.2.5 and the resulting formula is in (1.75). In Subsection 1.2.6, we collect all these preliminary computations and we finish the proof of Theorem 1.2.15.

The proof of Theorem 1.2.13 now follows from Theorem 1.2.15 using Takagi's and Cecil-Ryan's characterization of the sphere in $\mathbf{CP}^n$ as the unique hypersurface having precisely two (constant) different curvatures. The proof of Theorem 1.2.14 uses Takagi's classification of hypersurfaces in $\mathbf{CP}^n$, $n \geq 3$, having precisely three different constant curvatures: they are either geodesics tubes around $\mathbf{CP}^k$ for some $k = 1, \dots, n-1$ or geodesics tubes around the real projective space $\mathbf{RP}^n$.

In the following, by “hypersurface” we always mean “embedded hypersurface” in $\mathbf{CP}^n$.

1.2.4 Laplacian of position and normal

Let $\Sigma \subset \mathbf{CP}^n$ be a hypersurface oriented by the unit normal N . In the following we adopt the short notation $\sigma_N = \sigma(N, N)$ and $\pi_N = \pi(N, N)$. In this subsection, we compute ΔA and ΔN , where Δ is the Laplace-Beltrami operator of Σ .

The second fundamental form of Σ is the mapping $h : T_A \Sigma \times T_A \Sigma \rightarrow \mathbb{R}$, $h(X, Y) = \langle \nabla_X^\top N, Y \rangle$, and we denote its trace by $H = \text{tr}(h)$.

Notation. From now on, we will omit the symbol of sum over repeated indices. The repeated index always runs from 1 to $2n-1$. In the other cases, we will write the sum.

Theorem 1.2.16. *Let $\Sigma \subset \mathbf{CP}^n$ be an oriented hypersurface with normal N . The position matrix A satisfies the equation*

$$\Delta A = \text{tr}(\sigma) - \sigma_N - \text{tr}(h)N, \quad A \in \Sigma, \quad (1.64)$$

where σ is the second fundamental form of the immersion $\mathbf{CP}^n \subset H^{n+1}$.

Proof. Without loss of generality we can assume that $A_0 \in \Sigma$ and we check formula (1.64) at the point A_0 . Let $X_1, \dots, X_{2n-1}$ be a frame of vector fields tangent to Σ given by normal coordinates at A_0 . Namely, for all $i, j = 1, \dots, 2n-1$, we have

$$\nabla_{X_i}^\Sigma X_j(A_0) = 0. \quad (1.65)$$

We are denoting by ∇^Σ the Levi-Civita connection of Σ . This is the restriction of $\nabla^\top$ to Σ , projected onto $T\Sigma$.

In the next line and in the rest of the paper, we shall use the identity

$$\nabla_X A = X.$$

With sum over repeated indices for $j = 1, \dots, 2n-1$, we have

$$\begin{aligned} \Delta A|_{A=A_0} &= \nabla_{X_j} \nabla_{X_j} A|_{A=A_0} = \nabla_{X_j} X_j(A_0) = \\ &= \nabla_{X_j}^\top X_j(A_0) + \sigma_{A_0}(X_j, X_j). \end{aligned}$$

In the last equality, we used the definition of the second fundamental form σ . Again in A_0 , by (1.65) we obtain

$$\nabla_{X_j}^\top X_j = \langle \nabla_{X_j}^\top X_j, N \rangle N = -\langle X_j, \nabla_{X_j}^\top N \rangle N = -\text{tr}(h)N.$$

Since $X_1, \dots, X_{2n-1}, N$ is an orthonormal frame of $\mathbf{CP}^n$, from the definition (1.50) of $\mathcal{H}$ we have $\sigma(X_j, X_j) = \mathcal{H} - \sigma_N$, and this ends the proof. $\square$

In the next theorem, we compute a formula for ΔN . The second fundamental form h of Σ can be identified with a linear operator on $T_A\Sigma$. The restriction of σ to $T_A\Sigma$ can be identified with a linear operator from $T_A\Sigma$ to $\text{End}(T_A\Sigma, T_A^\perp\mathbf{CP}^n)$. Hence, the composition $\sigma h = \sigma \circ h$ is a linear operator from $T_A\Sigma$ to $\text{End}(T_A\Sigma, T_A^\perp\mathbf{CP}^n)$. Namely, for any $X, Y \in T_A\Sigma$ we have $\sigma h(X)[Y] = \sigma(h(X), Y)$. We denote its trace by

$$\text{tr}(\sigma h) = \sigma h(X_i)[X_i] = \sigma(h_{ij}X_j, X_i) = h_{ij}\sigma_{ij} \in T_A^\perp\mathbf{CP}^n,$$

where $\sigma_{ij} = \sigma(X_i, X_j)$ and $h_{ij} = h(X_i, X_j)$ for any orthonormal frame $X_1, \dots, X_{2n-1}$ of $T_A\Sigma$.

Theorem 1.2.17. *Let $\Sigma \subset \mathbf{CP}^n$ be an oriented hypersurface with constant mean curvature. The normal N to the hypersurface satisfies the equation*

$$\Delta N = 2\text{tr}(\sigma h) - (|h|^2 + 2(n-1))N - \text{tr}(h)\sigma_N. \quad (1.66)$$

The proof is preceded by a number of lemmas. We are using a frame of vector fields satisfying (1.65).

Lemma 1.2.18. *Let $\Sigma \subset \mathbf{CP}^n$ be an orientable hypersurface with constant mean curvature. At the center $A_0 \in \Sigma$ of normal coordinates, the entries of the second fundamental form $h_{ij} = h(X_i, X_j)$ satisfy for each $j = 1, \dots, 2n-1$ the equations*

$$X_i h_{ij} = 0, \quad (1.67)$$

with sum over the repeated index.

Proof. By the Codazzi's equations, we have the identities for $i, j, \ell = 1, \dots, 2n-1$

$$\begin{aligned} X_j h_{i\ell} - X_i h_{j\ell} &= \langle R(X_i, X_j)N, X_\ell \rangle + \langle \nabla_{X_j}^\top N, \nabla_{X_i}^\top X_\ell \rangle - \langle \nabla_{X_i}^\top N, \nabla_{X_j}^\top X_\ell \rangle + \\ &\quad - \langle N, \nabla_{[X_i, X_j]}^\top X_\ell \rangle, \end{aligned}$$

where N is the normal to Σ and R is the Riemann curvature tensor of $\mathbf{CP}^n$. Since $\nabla^\top$ has vanishing torsion, at the point A_0 we have by (1.65)

$$[X_i, X_j] = \nabla_{X_i}^\top X_j - \nabla_{X_j}^\top X_i = 0.$$

Thus, at the point A_0 the previous identity reduces to

$$X_j h_{i\ell} - X_i h_{j\ell} = \langle R(X_i, X_j)N, X_\ell \rangle$$

Letting $i = \ell$, summing up in $i = 1, \dots, 2n-1$, and using the fact that Σ has constant mean curvature, we obtain

$$X_i h_{ij} = -\langle R(X_i, X_j)N, X_i \rangle = -\text{Ric}(X_j, N).$$

The last equality follows from the standard symmetries of the curvature operator. In fact, we have $\langle R(N, X_j)N, N \rangle = \langle R(X_j, N)N, N \rangle = -\langle R(N, X_j)N, N \rangle$ and thus $\langle R(N, X_j)N, N \rangle = 0$. The complex projective space is a Kähler manifold, and thus it is an Einstein manifold. From the orthogonality of X_j and N it follows that $\text{Ric}(X_j, N) = 0$. $\square$

Lemma 1.2.19. *Let $\Sigma \subset \mathbf{CP}^n$ be an orientable hypersurface with normal N and $H = \text{tr}(h)$. At the center of normal coordinates, we have the identity*

$$\nabla_{X_i} \sigma_{i,N} = \text{tr}(\sigma h) - 2(n-1)N - H\sigma_N. \quad (1.68)$$

Proof. By (1.51), (1.54) and the definition of second fundamental form σ we have

$$\begin{aligned} \nabla_{X_i} \sigma_{i,N} &= \nabla_{X_i} \pi_{i,N}(I - 2A) + \pi_{i,N} \nabla_{X_i}(I - 2A) \\ &= \sigma(\nabla_{X_i} X_i, N) + \sigma(X_i, \nabla_{X_i} N) - 2\pi_{i,N} X_i \\ &= -h_{ii} \sigma_N + \sigma(\sigma_{ii}, N) + h_{ij} \sigma_{ij} + \sigma(X_i, \sigma_{i,N}) - 2\pi_{i,N} X_i. \\ &= -h_{ii} \sigma_N + h_{ij} \sigma_{ij} + (*), \end{aligned}$$

where $(*) = \sigma(\sigma_{ii}, N) + \sigma(X_i, \sigma_{i,N}) - 2\pi_{i,N} X_i$. Using (1.55) in the particular case when $i = j$, we deduce that $(*) = -\Lambda_{\sigma_{i,N}}(X_i)$.

Hence, we proved that

$$\nabla_{X_i} \sigma_{i,N} = -h_{ii} \sigma_N + h_{ij} \sigma_{ij} - \Lambda_{\sigma_{i,N}}(X_i), \quad (1.69)$$

and our claim (1.68) follows from (1.57). $\square$

Remark 1.2.20. *Starting from*

$$\nabla_{X_i} \sigma_{i,N} = \nabla_{X_i}^\top \sigma_{i,N} + \nabla_{X_i}^\perp \sigma_{i,N} = -\Lambda_{\sigma_{i,N}}(X_i) + \nabla_{X_i}^\perp \sigma_{i,N},$$

and using (1.69) we obtain

$$\nabla_{X_i}^\perp \sigma_{i,N} = -H\sigma_N + h_{ij} \sigma_{ij}. \quad (1.70)$$

Proof of Theorem 1.2.17. We check formula (1.66) using normal coordinates at the point $A_0 \in \Sigma$. Using $|N|^2 = 1$ and $\nabla_{X_i}^\perp N = \sigma(X_i, N)$ we obtain

$$\Delta N = \nabla_{X_i} \nabla_{X_i} N = \nabla_{X_i} (\nabla_{X_i}^\perp N + \nabla_{X_i}^\top N) = \nabla_{X_i} \{\sigma_{i,N} + h_{ij} X_j\}. \quad (1.71)$$

From (1.67) and (1.65), we deduce that, at the point A_0 ,

$$\nabla_{X_i} (h_{ij} X_j) = h_{ij} \nabla_{X_i} X_j = h_{ij} \{\sigma_{ij} - h_{ij} N\} = -|h|^2 N + h_{ij} \sigma_{ij}.$$

By (1.68) we have

$$\nabla_{X_i} \sigma_{i,N} = -H\sigma_N + h_{ij} \sigma_{ij} - 2(n-1)N,$$

and then $\Delta N = -(|h|^2 + 2(n-1))N + 2h_{ij} \sigma_{ij} - H\sigma_N$. $\square$

1.2.5 Laplacian of σ_N

Let A be the position matrix. In the second variation of the area of a hypersurface Σ we shall use normal deformations associated with the coordinate-functions of the matrix $u = \Delta A$. In our formula (1.64) for ΔA there is the term σ_N and thus to know Δu we need $\Delta \sigma_N$. In this subsection we compute this crucial quantity.

Let $X_1, \dots, X_{2n-1}$ be an orthonormal frame of $T\Sigma$ such that $X_{n+j} = JX_j$ for all $j = 1, \dots, n-1$. Then we have $X_n = -JN$, where N is the normal to Σ . The tangent vector

$$h_N = \sum_{j=1, j \neq n}^{2n-1} h(JN, X_j) X_j \in \mathbf{CT}\Sigma$$

does not depend on the given frame. The vector h_N is the projection onto $\mathbf{CT}\Sigma$ of $\nabla_{JN}^\top N$. So, we have $h_N = 0$ if and only if $\nabla_{JN}^\top N = \kappa JN$ with $\kappa = h(JN, JN)$, i.e., JN is a principal direction with characteristic curvature κ .

Lemma 1.2.21. *Let $\Sigma \subset \mathbf{CP}^n$ be an oriented hypersurface with normal N , characteristic curvature κ , and $H = \text{tr}(h)$. Then we have*

$$h_{ij} \Lambda_{\sigma_{j,n}}(X_i) = (H - \kappa)N + Jh_N. \quad (1.72)$$

Proof. The proof is a computation based on the following relation between the second fundamental form σ and the *Weingarten* endomorphism, that is a consequence of (1.53):

$$\Lambda_{\sigma(X,Y)}(V) = 2\langle X, Y \rangle V + \langle Y, V \rangle X + \langle X, V \rangle Y + \langle JY, V \rangle JX + \langle JX, V \rangle JY, \quad (1.73)$$

for every $X, Y, V \in T_A \mathbf{CP}^n$. For details see [69, Section 1] and recall the normalization $g(A, B) = 4\langle A, B \rangle$.

Indeed, by (1.73) we get:

$$\begin{aligned} \Lambda_{\sigma(X_j, N)}(X_i) &= \delta_{ij}N - \delta_{in}JX_j, & j = 1, \dots, n; \\ \Lambda_{\sigma(JX_j, N)}(X_i) &= -\delta_{ij}JN + \delta_{in}X_j, & j = 1, \dots, n-1, \end{aligned}$$

for any $i = 1, \dots, n$, and

$$\begin{aligned} \Lambda_{\sigma(X_j, N)}(JX_i) &= \delta_{ij}JN, & j = 1, \dots, n; \\ \Lambda_{\sigma(JX_j, N)}(JX_i) &= \delta_{ij}N, & j = 1, \dots, n-1, \end{aligned}$$

for every $i = 1, \dots, n-1$. So, adding up appropriately we prove the thesis. $\square$

The following lemma is a technical computation.

Lemma 1.2.22. *Let $\Sigma \subset \mathbf{CP}^n$ be an oriented hypersurface with normal N . Letting, for any orthonormal frame of $T\Sigma$,*

$$S_1 = \sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)X_i \quad \text{and} \quad S_2 = -\pi(N, N)\Delta A,$$

we have the identity

$$S_1 + S_2 = 2\text{tr}(h)N - \text{tr}(\sigma) + 2(n-1)\sigma_N. \quad (1.74)$$

The proof is postponed to the end of the subsection. In the previous subsection we introduced the linear operator $\sigma h : T_A \Sigma \rightarrow \text{End}(T_A \Sigma, T_A^\perp \mathbf{CP}^n)$. In the same way, we define the linear operator $\sigma h^2(X)[Y] = \sigma(h^2(X), Y)$, for $X, Y \in T_A \Sigma$. Its trace is

$$\text{tr}(\sigma h^2) = \sigma h^2(X_j)[X_j] = \sigma(h_{ij}h(X_i), X_j) = h_{ij}h_{ik}\sigma_{jk} \in T_A^\perp \mathbf{CP}^n.$$

Now we are ready to prove the main formula of this section.

Theorem 1.2.23. *Let $\Sigma \subset \mathbf{CP}^n$ be an orientable hypersurface with constant mean curvature and normal N . Then we have*

$$\Delta\sigma_N = 4\kappa N + 2\text{tr}(\sigma h^2 - \sigma) - 2|h|^2\sigma_N - 4Jh_N, \quad (1.75)$$

where κ is the characteristic curvature of Σ .

Proof. We check formula (1.75) using normal coordinates at the point $A_0 \in \Sigma$. Using (1.66), (1.51) and the short notation $\pi_N = \pi(N, N)$ we have

$$\begin{aligned} \Delta\sigma_N &= \nabla_{X_i} \nabla_{X_i} \sigma_N = \nabla_{X_i} (2\pi(\nabla_{X_i} N, N)(I - 2A) - 2\pi_N X_i) \\ &= 2\{\sigma(\nabla_{X_i} \nabla_{X_i} N, N) + \sigma(\nabla_{X_i} N, \nabla_{X_i} N) - 4\pi(\nabla_{X_i} N, N)X_i - \pi_N \nabla_{X_i} X_i\} \\ &= 2\{\sigma(\Delta N, N) + 2\sigma(h_{ij}X_j + \sigma_{i,N}, h_{ik}X_k + \sigma_{i,N}) + \\ &\quad - 4\pi(h_{ij}X_j + \sigma_{i,N}, N)X_i - \pi_N(-h_{ii}N + \sigma_{ii})\} \\ &= -2(|h|^2 + 2(n-1))\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} + \\ &\quad + 4h_{ij}\{(\sigma(\sigma_{ij}, N) + \sigma(X_j, \sigma_{i,N})) - 2\pi_{j,N}X_i\} \\ &\quad + 2(\sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)X_i) - 2\pi_N \Delta A. \end{aligned}$$

We used the identity $\sigma(\sigma_N, N) = 0$. By (1.55) and (1.74), we have

$$\begin{aligned} \Delta\sigma_N &= -2(|h|^2 + 2(n-1))\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} - 4h_{ij}\Lambda_{\sigma_{j,N}}(X_i) \\ &\quad + 4(n-1)\sigma_N - 2\text{tr}(\sigma) + 4\text{tr}(h)N \\ &= 4\text{tr}(h)N - 2\text{tr}(\sigma) - 2|h|^2\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} - 4h_{ij}\Lambda_{\sigma_{j,N}}(X_i). \end{aligned}$$

By (1.72), this ends the proof. □

Proof of Lemma 1.2.22. We check the formula at the point $A_0 \in \Sigma$. Using the formulas (1.42), (1.43), and (1.41) we obtain

$$\begin{aligned} S_1 &= 2\pi_{i,N}(I - 2A)\pi_{i,N}(I - 2A)^2 - 4\pi(\sigma_{i,N}, N)X_i \\ &= 2[\pi_{i,N}^2(I - 2A) - 2(\pi_{i,N}(I - 2A)N + N\pi_{i,N}(I - 2A))X_i] \\ &= 2\{[\pi_{i,N}^2 - 2(\pi_{i,N}N - N\pi_{i,N})X_i](I - 2A)\}. \end{aligned}$$

A simple computation gives

$$\begin{aligned} \pi_{i,N}^2(I - 2A_0) &= -\delta_{in}(E_{in} + E_{ni}) + E_{nn} + E_{ii}, \\ \pi_{j,N}^2(I - 2A_0) &= E_{nn} + E_{jj}, \end{aligned}$$

and also

$$\begin{aligned} (\pi_{i,N}NX_i - N\pi_{i,N}X_i)(I - 2A_0) &= -\delta_{in}E_{ni} + E_{ii} + (1 - \delta_{in})A_0, \\ (\pi_{j,N}N\widehat{X}_j - N\pi_{j,N}\widehat{X}_j)(I - 2A_0) &= E_{jj} + A_0. \end{aligned}$$

Therefore at A_0 we have

$$\begin{aligned} S_1 &= 2 \sum_{i=1}^n [-\delta_{in}(E_{in} + E_{ni}) + E_{nn} + E_{ii} - 2(-\delta_{in}E_{ni} + E_{ii} + (1 - \delta_{in})A_0)] + \\ &\quad + 2 \sum_{i=1}^{n-1} [E_{nn} + E_{ii} - 2(E_{ii} + A_0)] \\ &= 2[2nE_{nn} - 2(I - A_0) + (4 - 4n)A_0] \\ &= 4nE_{nn} - 4I + (12 - 8n)A_0. \end{aligned}$$

Moreover, using (1.52) and $\sigma_N = 2(E_{nn} - A_0)$, we have $4I = \text{tr}(\sigma) + 4(n+1)A_0$ and $2E_{nn} = \sigma_N + 2A_0$, and hence we get

$$S_1 = 2n\sigma_N - \text{tr}(\sigma) + 8(1-n)A_0. \quad (1.76)$$

Now, we compute S_2 at A_0 . Using $\pi_N = 2N$, $\pi_N \text{tr}(\sigma) = -8nA_0 + 8E_{nn}$, and $\pi_N \sigma_N = -4A_0 + 4E_{nn}$, we get

$$\begin{aligned} S_2 &= -\pi_N(-\text{tr}(h)N + \text{tr}(\sigma) - \sigma_N) \\ &= -(-2\text{tr}(h)N - 8nA_0 + 8E_{nn} + 4A_0 - 4E_{nn}) \\ &= 2\text{tr}(h)N - 8(1-n)A_0 - 2\sigma_N. \end{aligned}$$

Adding S_1 and S_2 we get the claim. $\square$

1.2.6 Trace of the second variation of the area

Let Σ be a C^∞ hypersurface with normal N and without boundary and let $u \in C^\infty(\Sigma)$ be a function with zero mean:

$$\int_{\Sigma} u d\mu = 0, \quad (1.77)$$

where μ is the Riemannian hypersurface measure in $\mathbf{CP}^n$. For $t \in \mathbb{R}$ and $p \in \Sigma$ let $t \mapsto \gamma_p(t)$ be the curve (geodesic) in $\mathbf{CP}^n$ solving $\nabla_{\dot{\gamma}}^\top \dot{\gamma} = 0$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = u(p)N(p)$. For small t , the hypersurface $\Sigma(t; u) = \{\gamma_p(t) \in \mathbf{CP}^n : p \in \Sigma\}$ is well defined and we denote its area by $\mathcal{A}(t; u) = \mu(\Sigma(t; u))$. If we have

$$\left. \frac{d\mathcal{A}(t; u)}{dt} \right|_{t=0} = 0$$

for any $u \in C^\infty(\Sigma)$ satisfying (1.77) then Σ has constant mean curvature. If $\Sigma(t; u)$ is the boundary of a region with volume $V(t)$, then condition (1.77) implies that $V'(0) = 0$.

The second variation of the area functional is given by the formula

$$\mathcal{A}''(u) = \left. \frac{d^2 \mathcal{A}(t; u)}{dt^2} \right|_{t=0} = - \int_{\Sigma} u \mathcal{L}u d\mu, \quad (1.78)$$

where

$$\mathcal{L}u = \Delta u + (|h|^2 + \text{Ric}(N))u$$

is the Jacobi operator, see e.g. [13]. It is well known that the Ricci curvature on unit vectors is a geometric constant in $\mathbf{CP}^n$, and namely $\text{Ric}(N) = 2n + 2$.

On Σ we consider the matrix valued function $u = \Delta A$, where A is the position matrix. For any $V \in H^{n+1}$, we define the scalarization $u_V = \langle u, V \rangle$. By the divergence theorem, the function u_V satisfies the zero mean condition (1.77) because Σ is compact and has no boundary.

The mapping $Q_\Sigma : H^{n+1} \rightarrow \mathbb{R}$ defined by $Q_\Sigma(V) = \mathcal{A}''(u_V)$ is a quadratic form. If the surface Σ is stable then Q_Σ is positive semidefinite, i.e., $Q_\Sigma(V) \geq 0$ for any $V \in H^{n+1}$. It follows that $\text{tr}(Q_\Sigma) \geq 0$. In the next theorem we compute an explicit expression for this trace.

Theorem 1.2.24. *Let Σ be an oriented compact hypersurface with constant mean curvature. The trace of the quadratic form Q_Σ is*

$$\text{tr}(Q_\Sigma) = 4 \int_\Sigma \left\{ 2(n+1)H^2 + 2(n^2-1)(2n-|h|^2) - (H+\kappa)^2 - |h_N|^2 \right\} d\mu. \quad (1.79)$$

Proof. For any orthonormal basis $\mathcal{V}$ of H^{n+1} , we have

$$\begin{aligned} \text{tr}(Q_\Sigma) &= \sum_{V \in \mathcal{V}} Q_\Sigma(V) = - \int_\Sigma \sum_{V \in \mathcal{V}} u_V \mathcal{L}u_V d\mu \\ &= - \int_\Sigma \left\{ \langle u, \Delta u \rangle + (|h|^2 + 2(n+1))|u|^2 \right\} d\mu. \end{aligned}$$

We compute first the norm of u . By formula (1.64) we have:

$$\begin{aligned} |u|^2 &= |\Delta A|^2 = \langle \mathcal{H} - \sigma_N - \text{tr}(h)N, \mathcal{H} - \sigma_N - \text{tr}(h)N \rangle \\ &= H^2 + |\mathcal{H}|^2 - 2\langle \mathcal{H}, \sigma_N \rangle + |\sigma_N|^2, \end{aligned}$$

because the matrices N and $\mathcal{H} - \sigma_N$ are orthogonal. Using the identity $\sigma(X, Y) = \sigma(JX, JY)$ for any $X, Y \in T_A \mathbf{CP}^n$ (see [68, Proposition 1.6]), we obtain

$$|\mathcal{H}|^2 = \sum_{i,j=1}^{2n} \langle \sigma_{ii}, \sigma_{jj} \rangle = 4 \sum_{i,j=1}^n \langle \sigma_{ii}, \sigma_{jj} \rangle.$$

Now by (1.53) we have for any $i, j = 1, \dots, n$

$$\langle \sigma_{ii}, \sigma_{jj} \rangle = 2(1 + \delta_{ij}), \quad (1.80)$$

and hence

$$\begin{aligned} |\mathcal{H}|^2 &= 4 \sum_{i,j=1}^n 2(1 + \delta_{ij}) = 8 \left(n^2 + \sum_{i,j=1}^n \delta_{ij} \right) \\ &= 8(n^2 + n) = 8n(n+1). \end{aligned} \quad (1.81)$$

In the same way, we have

$$\langle \mathcal{H}, \sigma_N \rangle = \sum_{i=1}^{2n} \langle \sigma_{ii}, \sigma_N \rangle = 2 \sum_{i=1}^n \langle \sigma_{ii}, \sigma_N \rangle = 4 \sum_{i=1}^n (1 + \delta_{in}) = 4(n+1), \quad (1.82)$$

where we used (1.80) with $j = n$. Finally, by (1.80) with $i = j = n$, we have

$$|\sigma_N|^2 = 4. \quad (1.83)$$

Now, by (1.137), (1.82) and (1.83) we get

$$\begin{aligned} |u|^2 &= H^2 + 8n(n+1) - 8(n+1) + 4 = H^2 + 8(n+1)(n-1) + 4 \\ &= H^2 + 8n^2 - 8 + 4 = H^2 + 4(2n^2 - 1). \end{aligned} \quad (1.84)$$

By formula (1.52) we have $\Delta \mathcal{H} = -4(n+1)\Delta A = -4(n+1)u$. Hence, from formula (1.64) we find

$$\begin{aligned} \langle u, \Delta u \rangle &= \langle u, -H\Delta N - 4(n+1)u - \Delta \sigma_N \rangle \\ &= -4(n+1)|u|^2 - H\langle u, \Delta N \rangle - \langle u, \Delta \sigma_N \rangle. \end{aligned}$$

Since N is orthogonal to $\mathcal{H}$, σ_N , and σ_{ij} we have

$$\begin{aligned} \langle u, \Delta N \rangle &= \langle -\text{tr}(h)N + \mathcal{H} - \sigma_N, -(|h|^2 + 2(n-1))N + 2h_{ij}\sigma_{ij} - \text{tr}(h)\sigma_N \rangle \\ &= H(|h|^2 + 2(n-1)) + 2\langle \mathcal{H}, \text{tr}(\sigma h) \rangle - H\langle \mathcal{H}, \sigma_N \rangle \\ &\quad - 2\langle \sigma_N, \text{tr}(\sigma h) \rangle + H|\sigma_N|^2. \end{aligned}$$

By (1.53) we have

$$\langle \sigma(X_i, X_i), \sigma(X_j, X_k) \rangle = 2(\delta_{jk} + \delta_{ik}\delta_{ij}), \quad (1.85)$$

$$\langle \sigma(X_i, X_i), \sigma(X_j, JX_k) \rangle = 0, \quad (1.86)$$

for every $i, j, k = 1, \dots, n$. Hence, by (1.85) and (1.86) and also using the notation $h_{\hat{j}\hat{k}} := h(JX_j, JX_k)$, we get

$$\begin{aligned} \langle \mathcal{H}, \text{tr}(\sigma h) \rangle &= 2 \sum_{i=1}^n \sum_{j,k=1}^{2n-1} h_{jk} \langle \sigma_{ii}, \sigma_{jk} \rangle = \\ &= 4 \sum_{i,j,k=1}^n h_{jk} (\delta_{jk} + \delta_{ik}\delta_{ij}) + 4 \sum_{i=1}^n \sum_{j,k=1}^{n-1} h_{\hat{j}\hat{k}} (\delta_{jk} + \delta_{ik}\delta_{ij}) \\ &= 4nh_{jj} + 4h_{ii} = 4(n+1)H. \end{aligned} \quad (1.87)$$

Again by (1.85) and (1.86),

$$\langle \sigma_N, \text{tr}(\sigma h) \rangle = h_{ij} \langle \sigma_N, \sigma_{ij} \rangle = 2h_{ij}(\delta_{ij} + \delta_{jn}\delta_{in}) = 2H + 2\kappa. \quad (1.88)$$

Finally, using (1.82), (1.83), (1.87) and (1.88) we get

$$\begin{aligned} \langle u, \Delta N \rangle &= H(|h|^2 + 2(n-1)) + 8(n+1)H - 4(n+1)H - 4(H + \kappa) + 4H \\ &= (|h|^2 + 6n + 2)H - 4\kappa, \end{aligned}$$

and so we obtain the formula

$$\langle u, \Delta N \rangle = H(|h|^2 + 6n + 2) - 4\kappa. \quad (1.89)$$

We are left with the computation of $\langle u, \Delta\sigma_N \rangle$. By formula (1.75), also using $\langle u, h_N \rangle = 0$, we obtain

$$\begin{aligned} \langle u, \Delta\sigma_N \rangle &= \langle -\text{tr}(h)N + \mathcal{H} - \sigma_N, 4\kappa N - 2\mathcal{H} - 2|h|^2\sigma_N + 2\text{tr}(\sigma h^2) \rangle \\ &= -4\kappa H - 2|\mathcal{H}|^2 - 2(|h|^2 - 1)\langle \mathcal{H}, \sigma_N \rangle + 2\langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle + 2|h|^2|\sigma_N|^2 \\ &\quad - 2\langle \sigma_N, \text{tr}(\sigma h^2) \rangle. \end{aligned}$$

Again by (1.85) and (1.86),

$$\begin{aligned} \langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle &= 2 \sum_{i=1}^n \sum_{\ell, j, k=1}^{2n-1} h_{\ell j} h_{\ell k} \langle \sigma_{ii}, \sigma_{jk} \rangle = \\ &= 4 \sum_{i=1}^n \sum_{\ell=1}^{2n-1} \left\{ \sum_{j, k=1}^n h_{\ell j} h_{\ell k} (\delta_{jk} + \delta_{ik} \delta_{ij}) + \sum_{j, k=1}^{n-1} h_{\ell j} h_{\ell k} (\delta_{jk} + \delta_{ik} \delta_{ij}) \right\} \\ &= 4n \sum_{\ell, j=1}^{2n-1} h_{\ell j}^2 + 4 \sum_{\ell, i=1}^{2n-1} h_{\ell i}^2 = 4(n+1)|h|^2. \end{aligned}$$

So, we get

$$\langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle = 4(n+1)|h|^2. \quad (1.90)$$

Moreover, we have

$$\begin{aligned} \langle \sigma_N, \text{tr}(\sigma h^2) \rangle &= h_{ij} h_{ik} \langle \sigma_N, \sigma_{jk} \rangle = 2h_{ij} h_{ik} (\delta_{jk} + \delta_{kn} \delta_{jn}) \\ &= 2h_{ij}^2 + 2h_{in}^2 = 2|h|^2 + 2(|h_N|^2 + \kappa^2). \end{aligned} \quad (1.91)$$

Adding (1.137), (1.82), (1.83), (1.90) and (1.91), we get the identity

$$\langle u, \Delta\sigma_N \rangle = 4|h|^2 - 8(2n-1)(n+1) - 4H\kappa - 4\kappa^2 - 4|h_N|^2. \quad (1.92)$$

Now, in order to get (1.79), we just have to use the formulas (1.84), (1.89) and (1.92) and sum them up. $\square$

In the next lemmas, we test the trace formula (1.79) on geodesic spheres and on the tubes introduced in Subsection 1.2.1.

Lemma 1.2.25. *For the sphere $\Sigma_r \subset \mathbf{CP}^n$ we have $\text{tr}(Q_{\Sigma_r}) \geq 0$ if and only if $\tan^2 r \leq 2n+1$. The trace is zero if and only if $\tan^2 r = 2n+1$.*

Proof. Letting $t = \tan r$, by the formulas (1.32) we have:

$$\begin{aligned} |h|^2 &= (2n-1) \frac{1}{t^2} + t^2 - 2, \\ H^2 &= (2n-1)^2 \frac{1}{t^2} + t^2 - 2(2n-1), \\ (H + \kappa)^2 &= 4(n-1)^2 \frac{1}{t^2} + 4t^2 - 8(n-1). \end{aligned}$$

Inserting these values into the trace formula (1.79) we find

$$\text{tr}(Q_{\Sigma_r}) = -\frac{8n(n-1)}{t^2} \mu(\Sigma_r) \{t^4 - 2nt^2 - (2n+1)\}.$$

Then we have $\text{tr}(Q_{\Sigma_r}) \geq 0$ if and only if $t^4 - 2nt^2 - (2n+1) \leq 0$, which holds if and only if $t^2 \leq 2n+1$. In particular, the trace is zero precisely when $t^2 = 2n+1$. $\square$

Lemma 1.2.25 shows that the formula (1.79) detects the sharp stability interval for the radius of a geodesic sphere, see Theorem 1.2.2. In the next lemma we consider the tubes T_r^k .

Lemma 1.2.26. *Let $k = 1, \dots, n-1$. For the tubes $T_r^k \subset \mathbb{C}P^n$ we have:*

- 1) *If $n = 2k+1$ then $\text{tr}(Q_{T_r^k}) > 0$ for any $0 < r < \pi/2$.*
- 2) *If $n \neq 2k+1$ then $\text{tr}(Q_{T_r^k}) \geq 0$ if and only if $\tan^2 r \leq d(n, k)$ for a certain positive number $d(n, k)$ that satisfies*

$$d(n, k) > \frac{2n - 2k + 1}{2k + 1}. \quad (1.93)$$

Moreover, $\text{tr}(Q_{T_r^k}) = 0$ precisely when $\tan^2 r = d(n, k)$.

Proof. We use the short notation $t = \tan r$ and $m = 2k+1$. By the formulas (1.33) we have:

$$\begin{aligned} |h|^2 &= mt^2 + (2n - m)\frac{1}{t^2} - 2, \\ H^2 &= m^2t^2 + (2n - m)^2\frac{1}{t^2} - 2m(2n - m), \\ (H + \kappa)^2 &= (m + 1)^2t^2 + (2n - m + 1)^2\frac{1}{t^2} - 2(m + 1)(2n - m + 1). \end{aligned}$$

Inserting these values into the formula (1.79), we get the following expression for the trace of $Q_{T_r^k}$

$$\text{tr}(Q_{T_r^k}) = \frac{8\mu(T_r^k)}{t^2}(at^4 + bt^2 + c),$$

where a, b, c are coefficients depending on n and k , and namely

$$\begin{aligned} a &= m(n+1)(m-n+1) - \frac{1}{2}(m+1)^2, \\ b/2 &= (n+1)[\kappa^2 - 1] - m(2n-m) + \frac{1}{2}(m+1)(2n-m+1), \\ c &= (n+1)(2n-m)(n-m+1) - \frac{1}{2}(2n-m+1)^2. \end{aligned}$$

It is easy to check that $b = a + c$, which means that $t^2 = -1$ is a root of $at^4 + bt^2 + c = 0$. So we have the decomposition

$$\text{tr}(Q_{T_r^k}) = \frac{8\mu(T_r^k)}{t^2}(t^2 + 1)(at^2 + c).$$

Now there are two cases: $n = m$ and $n \neq m$. When $n = m = 2k+1$ then n must be odd and in this case it is $a = c = \frac{1}{2}(n^2 - 1) > 0$. It follows that $\text{tr}(Q_{T_r^k}) > 0$ for any $t > 0$.

In the case $n \neq m$ we have $a < 0$ and $c > 0$. We conclude that $\text{tr}(Q_{T_r^k}) \geq 0$ if and only if $t^2 \leq d(n, k) := -c/a$. After some computations, inequality (1.93) is equivalent to $m > 1$, that is $k > 0$. $\square$

Lemma 1.2.26 shows that Theorem 1.79 is not sharp in the case of the tubes T_r^k . In fact, when $\tan^2 r = d(n, k)$ then the trace of $Q_{T_r^k}$ is zero but the tube T_r^k is unstable by Theorem 1.2.2 part 2.

We finish this section proving the non-stability of the tubes V_r around $\mathbf{RP}^n$.

Lemma 1.2.27. *The tube $V_r \subset \mathbf{CP}^n$ is unstable for any $r > 0$.*

Proof. Using the formulas (1.34) we compute:

$$\begin{aligned} |h|^2 &= (n-1)\left(t^2 + \frac{1}{t^2}\right) + \frac{16t^2}{(1-t^2)^2}, \\ H^2 &= (n-1)^2 \frac{(1-t^2)^2}{t^2} + \frac{16t^2}{(1-t^2)^2} - 8(n-1), \\ (H + \kappa)^2 &= (n-1)^2 \frac{(1-t^2)^2}{t^2} + \frac{64t^2}{(1-t^2)^2} - 16(n-1). \end{aligned}$$

Inserting these values into the formula (1.79), we obtain the value for the trace of Q_{V_r}

$$\mathrm{tr}(Q_{V_r}) = -4(n-1)\mu(V_r)\left\{(n-1)\left(t^2 + \frac{1}{t^2}\right) + \frac{32nt^2}{(1-t^2)^2} + 2(5n-1)\right\},$$

and we see that $\mathrm{tr}(Q_{V_r}) < 0$ for any $t = \tan r \in (0, 1)$ and $n \geq 2$. We deduce that the surfaces V_r are not stable. $\square$

1.2.7 Proof of Theorems 1.2.13 and 1.2.14

Let $\Sigma \subset \mathbf{CP}^n$ be a compact stable oriented hypersurface.

Proof of Theorem 1.2.13. We denote by $\widehat{h}$ the restriction of the second fundamental form h of Σ to the complex tangent space $\mathbf{CT}\Sigma$ and by $\widehat{H}$ the trace of $\widehat{h}$. At any point of Σ , we have the identities

$$H = \widehat{H} + \kappa \quad \text{and} \quad |h|^2 = |\widehat{h}|^2 + 2|h_N|^2 + \kappa^2,$$

and the inequalities

$$|h|^2 \geq |\widehat{h}|^2 + \kappa^2 \quad \text{and} \quad |\widehat{h}|^2 \geq \frac{\widehat{H}^2}{2(n-1)} = \frac{(H - \kappa)^2}{2(n-1)}. \quad (1.94)$$

Inserting these inequalities and $|h_N| \geq 0$ into (1.79) we obtain

$$\begin{aligned} \mathrm{tr}(Q_\Sigma) &\leq 4 \int_\Sigma \left\{ 2(n+1)H^2 + 2(n^2-1)\left(2n - \kappa^2 - \frac{(H - \kappa)^2}{2(n-1)}\right) - (H + \kappa)^2 \right\} d\mu \\ &= -4n \int_\Sigma p(\kappa; H, n) d\mu, \end{aligned} \quad (1.95)$$

where $p(\cdot; H, n)$ is the polynomial in (1.58). By our assumption $p(\kappa; H, n) \geq 0$ on Σ , we deduce that $\mathrm{tr}(Q_\Sigma) \leq 0$. On the other hand, the stability of Σ implies that

$\text{tr}(Q_\Sigma) \geq 0$. We deduce that $\text{tr}(Q_\Sigma) = 0$ and that we have equality in (1.95). In turn, the equality in (1.95) implies that $p(\kappa; H, n) = 0$, that

$$|h|^2 = |\widehat{h}|^2 + \kappa^2 \quad \text{and} \quad |\widehat{h}|^2 = \frac{\widehat{H}^2}{2(n-1)}, \quad (1.96)$$

and also that $h_N = 0$ on Σ .

The equation $h_N = 0$ means that JN is an eigenvector of h . By Maeda's theorem [53], this implies that the characteristic curvature κ is constant. This also simply follows from the fact that κ is one of the roots of $p(\kappa; H, n) = 0$. Here we use the fact that Σ is connected.

The identity in the right-hand side of (1.96) implies that Σ is umbilical in $\mathbf{CT}\Sigma$, i.e., each unit vector in $\mathbf{CT}\Sigma$ is an eigenvector of h with eigenvalue $\lambda = \widehat{H}/2(n-1)$. Moreover, λ is constant on Σ , because $\widehat{H} = H - \kappa$ is constant.

The two constants κ and λ are different, because in $\mathbf{CP}^n$ there are no totally umbilical hypersurfaces. By Takagi's theorem, Proposition 1.2.1, Σ is a geodesic sphere: up to a suitable choice of the center of the sphere, we have $\Sigma = \Sigma_r$ for some $r \in (0, \pi/2)$. By Lemma 1.2.25 the equation $\text{tr}(Q_{\Sigma_r}) = 0$ implies that $\tan^2 r = 2n + 1$. $\square$

Finally, we prove Theorem 1.2.14. We shall use Takagi's characterization of tubes in Proposition 1.2.3 and the computations of Lemmas 1.2.26 and 1.2.27.

Proof of Theorem 1.2.14. We have the decomposition $\mathbf{CT}\Sigma = T_1 \oplus T_2$ where, at each point of Σ , T_1 and T_2 are subspaces of real dimension α and $\beta = 2(n-1) - \alpha$. We denote by h_1 and h_2 the restrictions of h to T_1 and T_2 , respectively, and we let $H_1 = \text{tr}(h_1)$ and $H_2 = \text{tr}(h_2)$.

We have the identity $H = H_1 + H_2 + \kappa$ and the inequalities

$$|h|^2 \geq |h_1|^2 + |h_2|^2 + \kappa^2, \quad |h_1|^2 \geq \frac{H_1^2}{\alpha} \quad \text{and} \quad |h_2|^2 \geq \frac{H_2^2}{\beta}. \quad (1.97)$$

Inserting these inequalities and $|h_N| \geq 0$ into (1.79), we obtain

$$\text{tr}(Q_\Sigma) \leq -8(n^2 - 1) \int_\Sigma p(H_1, H_2; H, n, \alpha) d\mu, \quad (1.98)$$

where $p(\cdot; H, n, \alpha)$ is the polynomial in (1.61). By our assumption $p(H_1, H_2; H, n, \alpha) \geq 0$ on Σ , we deduce that $\text{tr}(Q_\Sigma) \leq 0$. On the other hand, the stability of Σ implies that $\text{tr}(Q_\Sigma) \geq 0$. We deduce that $\text{tr}(Q_\Sigma) = 0$ and that we have equality in (1.98). In turn, the equality in (1.98) implies that $p(H_1, H_2; H, n, \alpha) = 0$, that

$$|h|^2 = |h_1|^2 + |h_2|^2 + \kappa^2 \quad \text{and} \quad |h_1|^2 = \frac{H_1^2}{\alpha}, \quad |h_2|^2 = \frac{H_2^2}{\beta} \quad (1.99)$$

and also that $h_N = 0$ on Σ .

The equation $h_N = 0$ means that JN is an eigenvector of h . By Maeda's theorem [53], the characteristic curvature κ is constant.

The identities in (1.99) imply that T_1 is an eigenspace of h for a curvature λ_1 , and T_2 is an eigenspace of h for a curvature λ_2 . We clearly have $H_1 = \alpha\lambda_1$ and $H_2 = \beta\lambda_2$.

From $p(H_1, H_2; H, n, \alpha) = 0$ and $H_1 + H_2 = \text{constant}$, we deduce that λ_1 and λ_2 are constant.

Now we have three cases:

- 1) $\kappa = \lambda_1 = \lambda_2$. This case is empty, because in $\mathbf{CP}^n$ there are no totally umbilical hypersurfaces.
- 2) Precisely two of the numbers κ, λ_1 , and λ_2 are equal. By Proposition 1.2.1, Σ is a geodesic sphere. Hence, it must be $\lambda_1 = \lambda_2$ and the radius of the sphere is $\tan^2 r = 2n + 1$, as explained at the end of the proof of Theorem 1.2.13.
- 3) The three numbers κ, λ_1 , and λ_2 are different. By Proposition 1.2.3, the surface Σ is either a tube around $\mathbf{CP}^k$, $\Sigma = T_r^k$ with $\alpha = 2k$ even, or a tube around $\mathbf{RP}^n$, $\Sigma = V_r$ with $\alpha = n - 1$. The latter case $\Sigma = V_r$ is excluded because V_r is unstable for any $r > 0$, by Proposition 1.2.27. We are left with the case $\Sigma = T_r^k$ with $\alpha = 2k$ and for some $r > 0$. The radius is determined by the equation $\text{tr}(Q_{T_r^k}) = 0$. However, this equation either has no solution (this happens in the case $\alpha = \beta$), or its unique solution $r > 0$ has the property that the tube T_k^r is unstable, as shown in Proposition 1.2.26.

The only possible case is that Σ is a geodesic sphere with radius $\tan^2 r = 2n + 1$. □

Even if Theorems 1.2.13 and 1.2.14 are a step in the right direction, we couldn't use the stability property in its full generality. We also focused on a particular test function to put into the second variation formula. One could also find a different test function which suits better the case study and, in particular, that gives some information regarding the stability of the tubes T_r^k around $\mathbf{CP}^k$, which we know that are stable for some values of r from Theorem 1.1.20.

1.3 The case of the Real Projective Space

In this section, we investigate the case of $\mathbf{RP}^n$ with the same tools that we used in the previous section. In particular, we prove a non-existence result about stable hypersurfaces with constant mean curvature in $\mathbf{RP}^n$ satisfying a certain bound on the curvatures. For orientability reasons, for the rest of the section n will be odd.

Other important results related to this topic can be found in [4], where the authors characterize weakly stable, compact two-sided hypersurfaces with constant mean curvature in $\mathbf{RP}^n$, and in [84], where the author classifies compact and orientable stable hypersurfaces in $\mathbf{RP}^n$.

The approach to the results contained in this section is very similar to the one seen in the case of $\mathbf{CP}^n$. For this reason many proofs will be completely analogous to others already seen, with differences just in the notations. We will omit them.

1.3.1 Main results

To fix the notation, in the following we denote by H the mean curvature of a hypersurface $\Sigma \subset \mathbf{R}P^n$. In particular we consider the non-normalized mean curvature, namely

$$H := \text{tr}(h), \quad (1.100)$$

where h is the second fundamental form of Σ .

Let $\Sigma \subset \mathbf{R}P^n$ be a hypersurface and let $T\Sigma$ be the real tangent space of Σ , let T_1 and T_2 be nonzero subbundles of $T\Sigma$ such that

$$T\Sigma = T_1 \oplus T_2, \quad (1.101)$$

and denote their dimensions by $\alpha = \text{rank}(T_1)$ and $\beta = \text{rank}(T_2)$. Then we have $\alpha + \beta = \text{rank}(T\Sigma) = n - 1$ and $1 \leq \alpha < n - 1$. Let h_1 and h_2 be the restrictions of the second fundamental form h of Σ to T_1 and T_2 , and denote by $H_1 = \text{tr}(h_1)$ and $H_2 = \text{tr}(h_2)$ their respective traces.

For $n \in \mathbb{N}$, with $n \geq 3$ odd, and $1 \leq \alpha < n - 1$ let $p(\cdot; n, \alpha)$ be the quadratic convex polynomial of the variables $(s, t) \in \mathbb{R}^2$

$$p(s, t; n, \alpha) = \left(\frac{s^2}{\alpha} + \frac{t^2}{\beta} \right) (n - 2) - 2(s + t)^2 - n(n - 1). \quad (1.102)$$

We prove the following theorem.

Theorem 1.3.1. *Let $n \geq 3$ be odd. There exists no compact oriented stable CMC hypersurface $\Sigma \subset \mathbf{R}P^n$ satisfying*

$$p(H_1, H_2; n, \alpha) \geq 0,$$

for the decomposition (1.101), with $1 \leq \alpha = \text{rank}(T_1) < n - 1$.

Theorems 1.3.1 is a consequence of the following geometric inequality which is implied by stability.

Theorem 1.3.2. *Let $\Sigma \subset \mathbf{R}P^n$, $n \geq 3$ odd, be a compact stable hypersurface with constant mean curvature H . Then we have*

$$2(n + 1) \int_{\Sigma} \left\{ 2H^2 - |h|^2(n - 2) + n(n - 1) \right\} d\mu \geq 0, \quad (1.103)$$

where μ is the Riemannian hypersurface measure.

As we did in for the $\mathbf{C}P^n$ case, we first embed isometrically $\mathbf{R}P^n$ into $\text{Sym}(n + 1)$, the space of $(n + 1) \times (n + 1)$ symmetric matrices. Once the hypersurface Σ is embedded in $\text{Sym}(n + 1)$, we can consider the *position matrix* $A \in \Sigma$ and compute its tangential Laplacian, see Theorem 1.3.11,

$$\Delta A = \text{tr}(\sigma) - \sigma(N, N) - \text{tr}(h)N, \quad (1.104)$$

where Δ is the Laplace-Beltrami operator of Σ and $\sigma : T_A \mathbf{R}P^n \times T_A \mathbf{R}P^n \rightarrow T_A^\perp \mathbf{R}P^n$ is the second fundamental form of the immersion $\mathbf{R}P^n \subset \text{Sym}(n + 1)$.

We shall make systematic use of the geometric formulas concerning σ proved by Ros in [68] and [69]. They are reviewed in Section 1.3.3.

Recall that for a smooth function $u : \Sigma \rightarrow \mathbb{R}$, the second variation of the area functional in the normal direction uN is given by the formula

$$\mathcal{A}''(u) = - \int_{\Sigma} u \mathcal{L}u \, d\mu,$$

where $\mathcal{L}u = \Delta u + (|h|^2 + \text{Ric}(N))u$ is the Jacobi operator.

For any fixed $V \in \text{Sym}(n+1)$, we consider the function $u_V = \langle \Delta A, V \rangle$, which has zero mean. In Subsection 1.3.6, we compute the trace of the quadratic form Q_{Σ} on $\text{Sym}(n+1)$ defined by $Q_{\Sigma}(V) = \mathcal{A}''(u_V)$ and, by stability, we get inequality (1.103).

1.3.2 Spheres and Tubes in $\mathbf{RP}^n$

The n -dimensional real projective space is the quotient of the unit sphere $\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ by the antipodal equivalence relation $x \sim y \Leftrightarrow y = \pm x$. We denote by $[x]$ the equivalence class of $x \in \mathbf{S}^n$. The tangent space of $\mathbf{RP}^n$ at the point $[x]$ is

$$T_{[x]}\mathbf{RP}^n = \{y \in \mathbb{R}^{n+1} : x \cdot y = 0\},$$

where $x \cdot y = x_1 y_1 + \dots + x_{n+1} y_{n+1}$ is the standard scalar product of $\mathbb{R}^{n+1}$.

The metric on $\mathbf{RP}^n$ is the round metric induced by the one on $\mathbf{S}^n$, and the induced distance function $d : \mathbf{RP}^n \times \mathbf{RP}^n \rightarrow [0, \pi/2]$ is $d([x], [y]) = \arccos |x \cdot y|$.

Let $\Sigma \subset \mathbf{RP}^n$ be a C^∞ -smooth hypersurface oriented by the unit normal N . We define the second fundamental form h of Σ with the following sign convention

$$h(X, Y) = \langle \nabla_X^\top N, Y \rangle, \quad X, Y \in T_{[z]}\Sigma.$$

For any fixed $[x] \in \mathbf{RP}^n$ and $0 < r < \pi/2$, the geodesic sphere centered at $[x]$ with radius r is

$$\Sigma_r = \{[y] \in \mathbf{RP}^n : |x \cdot y| = \cos r\}.$$

We omit reference to the center. The geodesic sphere in $\mathbf{RP}^n$ is umbilic, meaning that in every point its normal curvatures are equal. Letting $t = \tan r$, they are

$$\lambda = -\cot r = -\frac{1}{t}, \quad \text{with multiplicity } n-1 \quad (1.105)$$

We now discuss tubes around $\mathbb{RP}^k$. For $k = 1, \dots, n-2$, the natural inclusion $\mathbf{S}^k = \{x \in \mathbf{S}^n : x_{k+2} = \dots = x_{n+1} = 0\} \subset \mathbf{S}^n$ induces the inclusion $\mathbb{RP}^k \subset \mathbb{RP}^n$. For $0 < r < \pi/2$, we define the tube

$$\begin{aligned} T_r^k &= \{[x] \in \mathbf{RP}^n : \text{dist}([x], \mathbb{RP}^k) = r\} \\ &= \{[x] = [\mathcal{X}', x''] \in \mathbf{RP}^n : |x| = 1, x' \in \mathbb{R}^{k+1}, |x'| = \cos r\}. \end{aligned}$$

Letting $t = \tan r$, the curvatures of T_r^k are

$$\begin{aligned} \lambda_1 &= \tan r = t, \quad \text{with multiplicity } k, \\ \lambda_2 &= -\cot r = -\frac{1}{t}, \quad \text{with multiplicity } \ell = n-1-k. \end{aligned} \quad (1.106)$$

These curvatures are constant and distinct for each value of $t > 0$. In particular, T_r^k has constant mean curvature. For $r + s = \pi/2$ and $k + \ell = n - 1$ the hypersurfaces T_r^k and T_s^ℓ are congruent.

While spheres are stable for every r , these tubes T_r^k are stable for some value of r . The following theorem is Theorem 1.1.20 for the real projective space.

Theorem 1.3.3 (Barbosa, Do Carmo, Eschenburg [13]). *Let $n \geq 3$ odd and $k = 1, \dots, n - 2$. Then:*

the tube T_r^k is stable if and only if

$$\frac{n - k - 1}{k + 2} \leq \tan^2 r \leq \frac{n - k + 1}{k}.$$

1.3.3 Immersion of $\mathbf{R}P^n$ into $Sym(n + 1)$

Let $Sym(n + 1) = \{A \in \mathfrak{gl}(n + 1, \mathbb{R}) : \bar{A} = A^t\}$ be the set of $(n + 1)$ -dimensional Symmetric matrices. The standard scalar product on $Sym(n + 1)$ is

$$\langle A, B \rangle := \frac{1}{2} \text{tr}(AB), \quad A, B \in Sym(n + 1). \quad (1.107)$$

Let $\Phi : \mathbf{R}P^n \rightarrow Sym(n + 1)$ be the mapping that takes the equivalence class $[x] \in \mathbf{R}P^n$ to the Symmetric matrix $A = \Phi([x]) \in Sym(n + 1)$ of the projection of $\mathbb{R}^{n+1}$ onto the line $[x]$. The matrix A satisfies $A^2 = A$ because it is a projection and $\text{tr}(A) = 1$, because it projects onto a line. It can be checked that Φ is an isometry from $\mathbf{R}P^n$ with its standard metric into $Sym(n + 1)$ with the metric (1.107). Hence, from now on we identify the real projective space with

$$\mathbf{R}P^n = \{A \in Sym(n + 1) : A^2 = A, \text{tr}(A) = 1\}.$$

For details on this identification and for the proof of the following lemmas, we refer the reader to [68]. Our normalization in (1.107) of the scalar product is different from the one by Ros. Namely, the relation between the metric g used by Ros and the metric in (1.107) is $g(A, B) = 4 \langle A, B \rangle$. The isometric embedding Φ was introduced in [80].

For any $A \in \mathbf{R}P^n$, we denote by $T_A \mathbf{R}P^n$ and $T_A^\perp \mathbf{R}P^n$ the tangent space and the normal space of $\mathbf{R}P^n$ at the point $A \in Sym(n + 1)$, respectively.

Lemma 1.3.4. *For any $A \in \mathbf{R}P^n$, we have:*

$$T_A \mathbf{R}P^n = \{X \in Sym(n + 1) : XA + AX = X\}, \quad (1.108)$$

$$T_A^\perp \mathbf{R}P^n = \{Z \in Sym(n + 1) : AZ = ZA\}. \quad (1.109)$$

For the proof see [68]. We easily see that $A, I \in T_A^\perp \mathbf{R}P^n$, where I is the identity matrix. We call the matrix $A_0 \in \mathbf{R}P^n$

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the *origin* of $\mathbf{R}P^n$. This is the projection onto the complex line of the versor $e_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

For $i, j \in \{0, 1, \dots, n\}$, let E_{ij} be the $(n+1) \times (n+1)$ matrix with entry 1 at the position (i, j) and with 0 elsewhere. With this notation we have $E_{00} = A_0$. Then the matrices $X_1, \dots, X_n$, where for $j = 1, \dots, n$

$$X_j = E_{j0} + E_{0j} \quad (1.110)$$

form an orthonormal basis for the tangent space of $\mathbf{RP}^n$ at the point A_0 .

For any point $A \in \mathbf{RP}^n$ there exists a (non-unique) unitary transformation $Q \in U(n+1)$ such that $A = QA_0Q^{-1}$. The conjugation $T_Q : \text{Sym}(n+1) \rightarrow \text{Sym}(n+1)$, $T_Q X = QXQ^{-1}$, preserves the metric of H^{n+1} :

$$\langle T_Q X, T_Q Y \rangle = \frac{1}{2} \text{tr}(QXYQ^{-1}) = \frac{1}{2} \text{tr}(XY) = \langle X, Y \rangle, \quad X, Y \in \text{Sym}(n+1).$$

In particular, T_Q maps isometrically the tangent space $T_{A_0}\mathbf{RP}^n$ onto $T_A\mathbf{RP}^n$. We will use these isometries to reduce computations of isometric-invariant quantities to the origin A_0 .

By elementary computations based on the projection equation $A^2 = A$ and on the equation $X = AX + XA$ for tangent vectors, it is possible to check the following algebraic identities.

Lemma 1.3.5. *For any point $A \in \mathbf{RP}^n$ and for any vector fields $X, Y \in T_A\mathbf{RP}^n$, we have:*

$$AXY = XYA, \quad (1.111)$$

$$AXA = 0, \quad (1.112)$$

$$X(I - 2A) = -(I - 2A)X, \quad (1.113)$$

$$(I - 2A)^2 = I, \quad (1.114)$$

$$(I - 2A)XY = XY(I - 2A). \quad (1.115)$$

We compute the mean curvature of the immersion of $\mathbf{RP}^n$ into $\text{Sym}(n+1)$. For any $A \in \mathbf{RP}^n$ we define the orthogonal projections $\pi_A^\top : \text{Sym}(n+1) \rightarrow T_A\mathbf{RP}^n$ and $\pi_A^\perp : \text{Sym}(n+1) \rightarrow T_A^\perp\mathbf{RP}^n$. Explicit formulas for $\pi^\top$ and $\pi^\perp$ can be expressed using the symmetric product $\pi : \text{Sym}(n+1) \times \text{Sym}(n+1) \rightarrow \text{Sym}(n+1)$

$$\pi(X, Y) = XY + YX.$$

By (1.109) and (1.111), we have $\pi : T_A\mathbf{RP}^n \times T_A\mathbf{RP}^n \rightarrow T_A^\perp\mathbf{RP}^n$.

Lemma 1.3.6. *For any $A \in \mathbf{RP}^n$ and $X \in \text{Sym}(n+1)$, we have*

$$\pi_A^\top(X) = \pi(A, X) - 2AXA, \quad (1.116)$$

$$\pi_A^\perp(X) = X - \pi(A, X) + 2AXA. \quad (1.117)$$

We split the standard connection ∇ of $\text{Sym}(n+1)$ into the part that is tangent to $\mathbf{RP}^n$ and the part that is normal. Namely, for $X \in \Gamma(T\mathbf{RP}^n)$, $Y \in \Gamma(T\text{Sym}(n+1))$, and $A \in \mathbf{RP}^n$ we let $\nabla_X^\top Y(A) = \pi_A^\top(\nabla_X Y)$, and $\nabla_X^\perp Y(A) = \pi_A^\perp(\nabla_X Y)$. By (1.46) and (1.47), we have the formulas

$$\nabla_X^\top Y = \pi(A, \nabla_X Y) - 2A(\nabla_X Y)A, \quad (1.118)$$

$$\nabla_X^\perp Y = \nabla_X Y - \pi(A, \nabla_X Y) + 2A(\nabla_X Y)A. \quad (1.119)$$

The second fundamental form of the immersion of $\mathbf{R}P^n$ into $Sym(n+1)$ is the mapping $\sigma_A : T_A \mathbf{R}P^n \times T_A \mathbf{R}P^n \rightarrow T_A^\perp \mathbf{R}P^n$, $A \in \mathbf{R}P^n$, defined by $\sigma_A(X, Y) = \nabla_X^\perp Y(A)$. When no confusion arises, we drop the subscript A and write $\sigma = \sigma_A$.

The non-normalized mean curvature vector of the immersion is the trace of σ , i.e.,

$$\mathcal{H} = \text{tr}(\sigma) = \sum_{i=1}^n \sigma(X_i, X_i), \quad (1.120)$$

where $X_1, \dots, X_n$ is any orthonormal frame of $T_A \mathbf{R}P^n$.

Proposition 1.3.7. *For any $A \in \mathbf{R}P^n$ and $X, Y \in T_A \mathbf{R}P^n$ we have*

$$\sigma(X, Y) = \pi(X, Y)(I - 2A), \quad (1.121)$$

and the trace of σ is

$$\mathcal{H} = 2(I - (n+1)A), \quad A \in \mathbf{R}P^n. \quad (1.122)$$

For a proof of (1.121) and (1.122) see [68, Proposition 1.4].

The *Weingarten endomorphism* of the immersion is the mapping $\Lambda : T_A \mathbf{R}P^n \times T_A^\perp \mathbf{R}P^n \rightarrow T_A \mathbf{R}P^n$, $A \in \mathbf{R}P^n$, defined by the formula $\Lambda(X, Z) = \Lambda_Z(X) = -\nabla_X^\perp Z$.

Proposition 1.3.8. *For any $A \in \mathbf{R}P^n$, $X \in T_A \mathbf{R}P^n$, and $Z \in T_A^\perp \mathbf{R}P^n$ we have*

$$\Lambda_Z(X) = (XZ - ZX)(I - 2A). \quad (1.123)$$

For the proof see [68, Proposition 1.4]. We establish some identities linking Λ and σ . Let $X_1, \dots, X_n$ be an orthonormal frame for $T\mathbf{R}P^n$ and we use the notation $N = X_n$. In the sequel, we also let

$$\begin{aligned} \pi_{ij} &= \pi(X_i, X_j) \quad \text{and} \quad \pi_{i,N} = \pi(X_i, N), \\ \sigma_{ij} &= \sigma(X_i, X_j) \quad \text{and} \quad \sigma_{i,N} = \sigma(X_i, N). \end{aligned}$$

The second fundamental form $\sigma(X, Y)$ is defined when X and Y are tangent sections of $\mathbf{R}P^n$. However, the right-hand side of (1.121) is defined for any $X, Y \in Sym(n+1)$. In the next lemma and in the next sections, we will use (1.121) as the general definition of σ .

Lemma 1.3.9. *Let $X_1, \dots, X_{n-1}, N$ be an orthonormal frame of $\mathbf{R}P^n$. Then for any $i, j = 1, \dots, n-1$ we have*

$$\Lambda_{\sigma_{j,N}}(X_i) = \pi(\pi_{j,N}, X_i) = 2\pi_{j,N}X_i - \sigma(\sigma_{ij}, N) - \sigma(X_j, \sigma_{i,N}). \quad (1.124)$$

Lemma 1.3.10. *For any orthonormal frame $X_1, \dots, X_{n-1}, N$ of $\mathbf{R}P^n$ we have*

$$\sum_{i=1}^{n-1} \Lambda_{\sigma_{i,N}}(X_i) = (n-1)N. \quad (1.125)$$

1.3.4 Laplacian of position and normal

Let $\Sigma \subset \mathbf{RP}^n$ be a hypersurface oriented by the unit normal N . In the following we adopt the short notation $\sigma_N = \sigma(N, N)$ and $\pi_N = \pi(N, N)$.

The second fundamental form of Σ is the mapping $h : T_A \Sigma \times T_A \Sigma \rightarrow \mathbb{R}$, $h(X, Y) = \langle \nabla_X^\top N, Y \rangle$, and we denote its trace by $H = \text{tr}(h)$.

Notation. From now on, we will omit the symbol of sum over repeated indices. The repeated index always runs from 1 to $n-1$. In the other cases, we will write the sum.

Theorem 1.3.11. *Let $\Sigma \subset \mathbf{RP}^n$ be an oriented hypersurface with normal N . The position matrix A satisfies the equation*

$$\Delta A = \text{tr}(\sigma) - \sigma_N - \text{tr}(h)N, \quad A \in \Sigma, \quad (1.126)$$

where σ is the second fundamental form of the immersion $\mathbf{RP}^n \subset \text{Sym}(n+1)$.

Proof. Without loss of generality we can assume that $A_0 \in \Sigma$ and we check formula (1.126) at the point A_0 . Let $X_1, \dots, X_{n-1}$ be a frame of vector fields tangent to Σ given by normal coordinates at A_0 . Namely, for all $i, j = 1, \dots, n-1$, we have

$$\nabla_{X_i}^\Sigma X_j(A_0) = 0. \quad (1.127)$$

We are denoting by ∇^Σ the Levi-Civita connection of Σ . This is the restriction of $\nabla^\top$ to Σ , projected onto $T\Sigma$.

In the next line and in the rest of the paper, we shall use the identity

$$\nabla_X A = X.$$

With sum over repeated indices for $j = 1, \dots, n-1$, we have

$$\begin{aligned} \Delta A|_{A=A_0} &= \nabla_{X_j} \nabla_{X_j} A|_{A=A_0} = \nabla_{X_j} X_j(A_0) = \\ &= \nabla_{X_j}^\top X_j(A_0) + \sigma_{A_0}(X_j, X_j). \end{aligned}$$

In the last equality, we used the definition of the second fundamental form σ . Again in A_0 , by (1.127) we obtain

$$\nabla_{X_j}^\top X_j = \langle \nabla_{X_j}^\top X_j, N \rangle N = -\langle X_j, \nabla_{X_j}^\top N \rangle N = -\text{tr}(h)N.$$

Since $X_1, \dots, X_{n-1}, N$ is an orthonormal frame of $\mathbf{RP}^n$, from the definition (1.120) of $\mathcal{H}$ we have $\sigma(X_j, X_j) = \mathcal{H} - \sigma_N$, and this ends the proof. $\square$

In the next theorem, we compute a formula for ΔN . The second fundamental form h of Σ can be identified with a linear operator on $T_A \Sigma$. The restriction of σ to $T_A \Sigma$ can be identified with a linear operator from $T_A \Sigma$ to $\text{End}(T_A \Sigma, T_A^\perp \mathbf{RP}^n)$. Hence, the composition $\sigma h = \sigma \circ h$ is a linear operator from $T_A \Sigma$ to $\text{End}(T_A \Sigma, T_A^\perp \mathbf{RP}^n)$. Namely, for any $X, Y \in T_A \Sigma$ we have $\sigma h(X)[Y] = \sigma(h(X), Y)$. We denote its trace by

$$\text{tr}(\sigma h) = \sigma h(X_i)[X_i] = \sigma(h_{ij} X_j, X_i) = h_{ij} \sigma_{ij} \in T_A^\perp \mathbf{RP}^n,$$

where $\sigma_{ij} = \sigma(X_i, X_j)$ and $h_{ij} = h(X_i, X_j)$ for any orthonormal frame $X_1, \dots, X_{n-1}$ of $T_A \Sigma$.

Theorem 1.3.12. *Let $\Sigma \subset \mathbf{R}P^n$ be an oriented hypersurface with constant mean curvature. The normal N to the hypersurface satisfies the equation*

$$\Delta N = 2\text{tr}(\sigma h) - (|h|^2 + n - 1)N - \text{tr}(h)\sigma_N. \quad (1.128)$$

The proof is preceded by a number of lemmas. We are using a frame of vector fields satisfying (1.127).

Lemma 1.3.13. *Let $\Sigma \subset \mathbf{R}P^n$ be an orientable hypersurface with constant mean curvature. At the center $A_0 \in \Sigma$ of normal coordinates, the entries of the second fundamental form $h_{ij} = h(X_i, X_j)$ satisfy for each $j = 1, \dots, n-1$ the equations*

$$X_i h_{ij} = 0, \quad (1.129)$$

with sum over the repeated index.

Lemma 1.3.14. *Let $\Sigma \subset \mathbf{R}P^n$ be an orientable hypersurface with normal N and $H = \text{tr}(h)$. At the center of normal coordinates, we have the identity*

$$\nabla_{X_i} \sigma_{i,N} = \text{tr}(\sigma h) - (n-1)N - H\sigma_N. \quad (1.130)$$

Remark 1.3.15. *Starting from*

$$\nabla_{X_i} \sigma_{i,N} = \nabla_{X_i}^\top \sigma_{i,N} + \nabla_{X_i}^\perp \sigma_{i,N} = -\Lambda_{\sigma_{i,N}}(X_i) + \nabla_{X_i}^\perp \sigma_{i,N},$$

and using (1.69) we obtain

$$\nabla_{X_i}^\perp \sigma_{i,N} = -H\sigma_N + h_{ij}\sigma_{ij}. \quad (1.131)$$

Proof of Theorem 1.3.12. We check formula (1.128) using normal coordinates at the point $A_0 \in \Sigma$. Using $|N|^2 = 1$ and $\nabla_{X_i}^\perp N = \sigma(X_i, N)$ we obtain

$$\Delta N = \nabla_{X_i} \nabla_{X_i} N = \nabla_{X_i} (\nabla_{X_i}^\perp N + \nabla_{X_i}^\top N) = \nabla_{X_i} \{\sigma_{i,N} + h_{ij}X_j\}. \quad (1.132)$$

From (1.129) and (1.127), we deduce that, at the point A_0 ,

$$\nabla_{X_i} (h_{ij}X_j) = h_{ij} \nabla_{X_i} X_j = h_{ij} \{\sigma_{ij} - h_{ij}N\} = -|h|^2 N + h_{ij}\sigma_{ij}.$$

By (1.130) we have

$$\nabla_{X_i} \sigma_{i,N} = -H\sigma_N + h_{ij}\sigma_{ij} - (n-1)N,$$

and then $\Delta N = -(|h|^2 + n - 1)N + 2h_{ij}\sigma_{ij} - H\sigma_N$. $\square$

1.3.5 Laplacian of σ_N

Let A be the position matrix. In the second variation of the area of a hypersurface Σ we shall use normal deformations associated with the coordinate-functions of the matrix $u = \Delta A$. In our formula (1.126) for ΔA there is the term σ_N and thus to know Δu we need $\Delta \sigma_N$.

The following lemma is a technical computation.

Lemma 1.3.16. *Let $\Sigma \subset \mathbf{RP}^n$ be an oriented hypersurface with normal N . Letting, for any orthonormal frame of $T\Sigma$,*

$$S_1 = \sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)X_i \quad \text{and} \quad S_2 = -\pi(N, N)\Delta A,$$

we have the identity

$$S_1 + S_2 = 2\text{tr}(h)N - \text{tr}(\sigma) + n\sigma_N. \quad (1.133)$$

In the previous section we introduced the linear operator $\sigma h : T_A\Sigma \rightarrow \text{End}(T_A\Sigma, T_A^\perp\mathbf{RP}^n)$. In the same way, we define the linear operator $\sigma h^2(X)[Y] = \sigma(h^2(X), Y)$, for $X, Y \in T_A\Sigma$. Its trace is

$$\text{tr}(\sigma h^2) = \sigma h^2(X_j)[X_j] = \sigma(h_{ij}h(X_i), X_j) = h_{ij}h_{ik}\sigma_{jk} \in T_A^\perp\mathbf{RP}^n.$$

Now we are ready to prove the main formula of this section.

Theorem 1.3.17. *Let $\Sigma \subset \mathbf{RP}^n$ be an orientable hypersurface with constant mean curvature and normal N . Then we have*

$$\Delta\sigma_N = 2\text{tr}(\sigma h^2 - \sigma) - 2(|h|^2 - 1)\sigma_N. \quad (1.134)$$

Proof. We check formula (1.134) using normal coordinates at the point $A_0 \in \Sigma$. Using (1.128), (1.121) and the short notation $\pi_N = \pi(N, N)$ we have

$$\begin{aligned} \Delta\sigma_N &= \nabla_{X_i}\nabla_{X_i}\sigma_N = \nabla_{X_i}(2\pi(\nabla_{X_i}N, N)(I - 2A) - 2\pi_N X_i) \\ &= 2\{\sigma(\nabla_{X_i}\nabla_{X_i}N, N) + \sigma(\nabla_{X_i}N, \nabla_{X_i}N) - 4\pi(\nabla_{X_i}N, N)X_i - \pi_N\nabla_{X_i}X_i\} \\ &= 2\{\sigma(\Delta N, N) + \sigma(h_{ij}X_j + \sigma_{i,N}, h_{ik}X_k + \sigma_{i,N}) + \\ &\quad - 4\pi(h_{ij}X_j + \sigma_{i,N}, N)X_i - \pi_N(-h_{ii}N + \sigma_{ii})\} \\ &= -2(|h|^2 + n - 1)\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} + \\ &\quad + 4h_{ij}\{(\sigma(\sigma_{ij}, N) + \sigma(X_j, \sigma_{i,N})) - 2\pi_{j,N}X_i\} \\ &\quad + 2(\sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)X_i) - 2\pi_N\Delta A. \end{aligned}$$

We used the identity $\sigma(\sigma_N, N) = 0$. By (1.124) and (1.133), we have

$$\begin{aligned} \Delta\sigma_N &= -2(|h|^2 + 2(n - 1))\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} - 4h_{ij}\Lambda_{\sigma_{j,N}}(X_i) \\ &\quad + 2n\sigma_N - 2\text{tr}(\sigma) + 4\text{tr}(h)N \\ &= 2\text{tr}(\sigma h^2 - \sigma) - 2(|h|^2 - 1)\sigma_N, \end{aligned}$$

where in the last equality we used

$$\Lambda_{\sigma_{j,N}}(X_i) = \delta_{ij}N.$$

□

1.3.6 The second variation of the area

Let Σ be a C^∞ hypersurface with normal N and without boundary and let $u \in C^\infty(\Sigma)$ be a function with zero mean. For $t \in \mathbb{R}$ and $p \in \Sigma$ let $t \mapsto \gamma_p(t)$ be the curve (geodesic) in $\mathbf{R}P^n$ solving $\nabla_{\dot{\gamma}}^\top \dot{\gamma} = 0$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = u(p)N(p)$. For small t , the hypersurface $\Sigma(t; u) = \{\gamma_p(t) \in \mathbf{R}P^n : p \in \Sigma\}$ is well defined and we denote its area by $\mathcal{A}(t; u) = \mu(\Sigma(t; u))$. If we have

$$\left. \frac{d\mathcal{A}(t; u)}{dt} \right|_{t=0} = 0$$

for any $u \in C^\infty(\Sigma)$ with zero mean then Σ has constant mean curvature. If $\Sigma(t; u)$ is the boundary of a region with volume $V(t)$, then u having zero mean implies that $V'(0) = 0$.

Recall that the second variation of the area functional is given by the formula

$$\mathcal{A}''(u) = \left. \frac{d^2 \mathcal{A}(t; u)}{dt^2} \right|_{t=0} = - \int_{\Sigma} u \mathcal{L} u d\mu, \quad (1.135)$$

where

$$\mathcal{L}u = \Delta u + (|h|^2 + \text{Ric}(N)(N))u.$$

It is well known that the Ricci curvature on unit vectors is a geometric constant in $\mathbf{R}P^n$, and namely $\text{Ric}(N)(N) = n - 1$.

We also have the same definition of stability as in 1.1.10, namely an oriented hypersurfaces Σ without boundary and with constant mean curvature is *stable* if $\mathcal{A}''(u) \geq 0$ for any $u \in C^\infty(\Sigma)$ with zero mean.

On Σ we consider the matrix valued function $u = \Delta A$, where A is the position matrix. For any $V \in \text{Sym}(n+1)$, we define the scalarization $u_V = \langle u, V \rangle$. By the divergence theorem, the function u_V satisfies the zero-mean condition because Σ is compact and has no boundary.

The mapping $Q_\Sigma : \text{Sym}(n+1) \rightarrow \mathbb{R}$ defined by $Q_\Sigma(V) = \mathcal{A}''(u_V)$ is a quadratic form. If the surface Σ is stable then Q_Σ is positive semidefinite, i.e., $Q_\Sigma(V) \geq 0$ for any $V \in \text{Sym}(n+1)$. It follows that $\text{tr}(Q_\Sigma) \geq 0$. In the next theorem we compute an explicit expression for this trace.

Theorem 1.3.18. *Let Σ be an oriented compact hypersurface with constant mean curvature. The trace of the quadratic form Q_Σ is*

$$\text{tr}(Q_\Sigma) = 2(n+1) \int_{\Sigma} \left\{ 2H^2 - (n-2)|h|^2 + n(n-1) \right\} d\mu. \quad (1.136)$$

Proof. For any orthonormal basis $\mathcal{V}$ of $\text{Sym}(n+1)$, we have

$$\begin{aligned} \text{tr}(Q_\Sigma) &= \sum_{V \in \mathcal{V}} Q_\Sigma(V) = - \int_{\Sigma} \sum_{V \in \mathcal{V}} u_V \mathcal{L} u_V d\mu \\ &= - \int_{\Sigma} \left\{ \langle u, \Delta u \rangle + (|h|^2 + n-1)|u|^2 \right\} d\mu. \end{aligned}$$

We compute first the norm of u . By formula (1.126) we have:

$$\begin{aligned} |u|^2 &= |\Delta A|^2 = \langle \mathcal{H} - \sigma_N - \text{tr}(h)N, \mathcal{H} - \sigma_N - \text{tr}(h)N \rangle \\ &= H^2 + |\mathcal{H}|^2 - 2 \langle \mathcal{H}, \sigma_N \rangle + |\sigma_N|^2, \end{aligned}$$

because the matrices N and $\mathcal{H} - \sigma_N$ are orthogonal.

By (1.122), it's easy to see that

$$|\mathcal{H}|^2 = 2n(n+1). \quad (1.137)$$

Moreover, by (1.107), (1.121) and by the definition of X_j 's we get, for every $i, j = 1, \dots, n$

$$\langle \sigma_{ii}, \sigma_{jj} \rangle = 2(1 + \delta_{ij}), \quad (1.138)$$

then, when $j = n$ we get

$$\langle \mathcal{H}, \sigma_N \rangle = \sum_{i=1}^n \langle \sigma_{ii}, \sigma_N \rangle = 2(n+1). \quad (1.139)$$

Finally, with a similar computation, one gets

$$|\sigma_N|^2 = 4. \quad (1.140)$$

Now, by (1.137), (1.139) and (1.140) we get

$$\begin{aligned} |u|^2 &= H^2 + 2n(n+1) - 4(n+1) + 4 = H^2 + 2(n+1)(n-2) + 4 \\ &= H^2 + 2n^2 - 2n - 4 + 4 = H^2 + 2n(n-1). \end{aligned} \quad (1.141)$$

By formula (1.122) we have $\Delta \mathcal{H} = -2(n+1)\Delta A = -2(n+1)u$. Hence, from formula (1.126) we find

$$\begin{aligned} \langle u, \Delta u \rangle &= \langle u, -H\Delta N - 2(n+1)u - \Delta \sigma_N \rangle \\ &= -2(n+1)|u|^2 - H\langle u, \Delta N \rangle - \langle u, \Delta \sigma_N \rangle. \end{aligned}$$

Since N is orthogonal to $\mathcal{H}, \sigma_N$, and σ_{ij} we have

$$\begin{aligned} \langle u, \Delta N \rangle &= \langle -\text{tr}(h)N + \mathcal{H} - \sigma_N, -(|h|^2 + n - 1)N + 2h_{ij}\sigma_{ij} - \text{tr}(h)\sigma_N \rangle \\ &= H(|h|^2 + n - 1) + 2\langle \mathcal{H}, \text{tr}(\sigma h) \rangle - H\langle \mathcal{H}, \sigma_N \rangle \\ &\quad - 2\langle \sigma_N, \text{tr}(\sigma h) \rangle + H|\sigma_N|^2. \end{aligned}$$

Again by (1.107), (1.121) and by the definition of X_j we have

$$\langle \sigma(X_i, X_i), \sigma(X_j, X_k) \rangle = 2(\delta_{jk} + \delta_{ik}\delta_{ij}), \quad (1.142)$$

for every $i, j, k = 1, \dots, n$. Hence, by (1.142), we get

$$\begin{aligned} \langle \mathcal{H}, \text{tr}(\sigma h) \rangle &= \sum_{i=1}^n \sum_{j,k=1}^{n-1} h_{jk} \langle \sigma_{ii}, \sigma_{jk} \rangle = \\ &= 2n \sum_{j=1}^{n-1} h_{jj} + 2 \sum_{i=1}^n \sum_{j,k=1}^{n-1} h_{jk} h_{ik} h_{ij} \\ &= 2nh_{jj} + 2h_{ii} = 2(n+1)H. \end{aligned} \quad (1.143)$$

Again by (1.142),

$$\langle \sigma_N, \text{tr}(\sigma h) \rangle = h_{ij} \langle \sigma_N, \sigma_{ij} \rangle = 2h_{ij}(\delta_{ij} + \delta_{jn}\delta_{in}) = 2H. \quad (1.144)$$

Finally, using (1.139), (1.140), (1.143) and (1.144) we get

$$\begin{aligned} \langle u, \Delta N \rangle &= H(|h|^2 + n - 1) + 4(n + 1)H - 2(n + 1)H - 4H + 4H \\ &= (|h|^2 + 3n + 1)H, \end{aligned}$$

and so we obtain the formula

$$\langle u, \Delta N \rangle = H(|h|^2 + 3n + 1). \quad (1.145)$$

We are left with the computation of $\langle u, \Delta \sigma_N \rangle$. By formula (1.134), also using $\langle u, h_N \rangle = 0$, we obtain

$$\begin{aligned} \langle u, \Delta \sigma_N \rangle &= \langle -\text{tr}(h)N + \mathcal{H} - \sigma_N, -2\mathcal{H} - 2(|h|^2 - 1)\sigma_N + 2\text{tr}(\sigma h^2) \rangle \\ &= -2|\mathcal{H}|^2 - 2(|h|^2 - 2)\langle \mathcal{H}, \sigma_N \rangle + 2\langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle + 2(|h|^2 - 1)|\sigma_N|^2 \\ &\quad - 2\langle \sigma_N, \text{tr}(\sigma h^2) \rangle. \end{aligned}$$

By (1.142),

$$\begin{aligned} \langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle &= \sum_{i=1}^n \sum_{\ell, j, k=1}^{n-1} h_{\ell j} h_{\ell k} \langle \sigma_{ii}, \sigma_{jk} \rangle = \\ &= 2 \sum_{i=1}^n \sum_{\ell, j, k=1}^{n-1} h_{\ell j} h_{\ell k} (\delta_{jk} + \delta_{ik}\delta_{ij}) \\ &= 2n \sum_{\ell, j=1}^{n-1} h_{\ell j}^2 + 2 \sum_{\ell, i=1}^{n-1} h_{\ell i}^2 = 2(n + 1)|h|^2. \end{aligned}$$

So, we get

$$\langle \mathcal{H}, \text{tr}(\sigma h^2) \rangle = 2(n + 1)|h|^2. \quad (1.146)$$

Moreover, we have

$$\begin{aligned} \langle \sigma_N, \text{tr}(\sigma h^2) \rangle &= h_{ij} h_{ik} \langle \sigma_N, \sigma_{jk} \rangle = 2h_{ij} h_{ik} (\delta_{jk} + \delta_{kn}\delta_{jn}) \\ &= 2h_{ij}^2 = 2|h|^2. \end{aligned} \quad (1.147)$$

Adding (1.137), (1.139), (1.140), (1.146) and (1.147), we get

$$\langle u, \Delta \sigma_N \rangle = 4|h|^2 - 4n(n - 1). \quad (1.148)$$

Now, in order to get (1.136), we just have to use the formulas (1.141), (1.145) and (1.148) and sum them up. □

In the next lemmas, we test the trace formula (1.136) on geodesic spheres and on the tubes introduced in Subsection 1.3.2.

Lemma 1.3.19. *For the sphere $\Sigma_r \subset \mathbf{RP}^n$ we have $\text{tr}(Q_{\Sigma_r}) > 0$ for every $r > 0$.*

Proof. Letting $t = \tan r$, by formula (1.105) we have:

$$\begin{aligned} |h|^2 &= (n-1) \frac{1}{t^2}, \\ H^2 &= (n-1)^2 \frac{1}{t^2}. \end{aligned}$$

Inserting these values into the trace formula (1.136) we find

$$\text{tr}(Q_{\Sigma_r}) = 2n(n+1)(n-1)\mu(\Sigma_r) \left\{ \frac{1}{t^2} + 1 \right\} > 0,$$

for every $t \in \mathbb{R}$. □

In the next lemma we consider the tubes T_r^k . Recall that we consider the dimension $n \geq 3$ to be odd, for orientability reasons.

Lemma 1.3.20. *Let $k = 1, \dots, n-2$. For the tubes $T_r^k \subset \mathbf{RP}^n$ we have:*

- 1) *If $n = 2k+1$ then $\text{tr}(Q_{T_r^k}) > 0$ for any $0 < r < \pi/2$.*
- 2) *If $n \neq 2k+1$ then $\text{tr}(Q_{T_r^k}) \geq 0$ if and only if $\tan^2 r \leq d(n, k)$ for a certain positive number $d(n, k)$ that satisfies*

$$\begin{aligned} d(n, k) &> \frac{n-k+1}{k} \text{ if } n > 2k+1 \\ d(n, k) &< \frac{n-k-1}{k+2} \text{ if } n < 2k+1 \end{aligned} \tag{1.149}$$

Moreover, $\text{tr}(Q_{T_r^k}) = 0$ precisely when $\tan^2 r = d(n, k)$.

Proof. We use the short notation $t = \tan r$. By the formulas (1.106) we have:

$$\begin{aligned} |h|^2 &= kt^2 + (n-k-1) \frac{1}{t^2}, \\ H^2 &= k^2 t^2 + (n-k-1)^2 \frac{1}{t^2} - 2k(n-k-1). \end{aligned}$$

Inserting these values into the formula (1.136), we get the following expression for the trace of $Q_{T_r^k}$

$$\text{tr}(Q_{T_r^k}) = \frac{2(n+1)\mu(T_r^k)}{t^2} (at^4 + bt^2 + c),$$

where a, b, c are coefficients depending on n and k , and namely

$$\begin{aligned} a &= k(2k - n + 2), \\ b &= n(n-1) - 4k(n-k-1), \\ c &= n(n-1) - k(3n-2k-2). \end{aligned}$$

It is easy to check that $b = a + c$, which means that $t^2 = -1$ is a root of $at^4 + bt^2 + c = 0$. So we have the decomposition

$$\operatorname{tr}(Q_{T_r^k}) = \frac{2(n+1)\mu(T_r^k)}{t^2}(t^2 + 1)(at^2 + c).$$

We have the following three cases: $n = 2k + 1$, $n < 2k + 1$ and $n > 2k + 1$. When $n = 2k + 1$ then, because n is odd, k is even and in this case it is $a = c = \frac{n-1}{2} > 0$. It follows that $\operatorname{tr}(Q_{T_r^k}) > 0$ for any $t > 0$.

In the case $n < 2k + 1$ we have $a > 0$ and $c < 0$ and, after some computations, inequality $d(n, k) < \frac{n-k-1}{k+2}$ is equivalent to $k < n-1$ (which is always true by hypothesis). Similarly, in the case $n > 2k + 1$ we have $a < 0$ and $c > 0$ and inequality $d(n, k) > \frac{n-k+1}{k}$ is equivalent to $k > -1$.

We conclude that $\operatorname{tr}(Q_{T_r^k}) \geq 0$ if and only if $t^2 \leq d(n, k) := -c/a$. □

Lemma 1.2.26 shows that Theorem 1.79 is not sharp in the case of the tubes T_r^k . In fact, when $\tan^2 r = d(n, k)$ then the trace of $Q_{T_r^k}$ is zero but the tube T_r^k is unstable by Theorem 1.3.3.

1.3.7 Proof of Theorem 1.3.1

Let $n \geq 3$ odd and let $\Sigma \subset \mathbf{R}P^n$ be a compact stable oriented hypersurface.

Proof of Theorem 1.3.1. We have the decomposition $\mathbf{C}T\Sigma = T_1 \oplus T_2$ where, at each point of Σ , T_1 and T_2 are subspaces of real dimension α and $\beta = n - 1 - \alpha$. We denote by h_1 and h_2 the restrictions of h to T_1 and T_2 , respectively, and we let $H_1 = \operatorname{tr}(h_1)$ and $H_2 = \operatorname{tr}(h_2)$.

We have the identity $H = H_1 + H_2$ and the inequalities

$$|h|^2 \geq |h_1|^2 + |h_2|^2, \quad |h_1|^2 \geq \frac{H_1^2}{\alpha} \quad \text{and} \quad |h_2|^2 \geq \frac{H_2^2}{\beta}. \quad (1.150)$$

Inserting these inequalities and $|h_N| \geq 0$ into (1.136), we obtain

$$\operatorname{tr}(Q_\Sigma) \leq -2(n-1) \int_\Sigma p(H_1, H_2; n, \alpha) d\mu, \quad (1.151)$$

where $p(\cdot; n, \alpha)$ is the polynomial in (1.102). By our assumption $p(H_1, H_2; n, \alpha) \geq 0$ on Σ , we deduce that $\operatorname{tr}(Q_\Sigma) \leq 0$. On the other hand, the stability of Σ implies that $\operatorname{tr}(Q_\Sigma) \geq 0$. We deduce that $\operatorname{tr}(Q_\Sigma) = 0$ and that we have equality in (1.151). In turn, the equality in (1.151) implies that $p(H_1, H_2; n, \alpha) = 0$, that

$$|h|^2 = |h_1|^2 + |h_2|^2 \quad \text{and} \quad |h_1|^2 = \frac{H_1^2}{\alpha}, \quad |h_2|^2 = \frac{H_2^2}{\beta}. \quad (1.152)$$

The identities in (1.152) imply that T_1 is an eigenspace of h for a curvature λ_1 , and T_2 is an eigenspace of h for a curvature λ_2 . We clearly have $H_1 = \alpha\lambda_1$ and $H_2 = \beta\lambda_2$. From $p(H_1, H_2; n, \alpha) = 0$ and $H_1 + H_2 = \text{constant}$, we deduce that λ_1 and λ_2 are constant.

Now we have two cases:

- 1) $\lambda_1 = \lambda_2$. This case is empty, because the only totally umbilical hypersurfaces in $\mathbf{RP}^n$ are the spheres, but in Lemma 1.3.19 we saw that $\text{tr}(Q_\Sigma) \neq 0 (> 0)$ for all the spheres.
- 2) $\lambda_1 \neq \lambda_2$. By a well known result by Cartan [23], the only compact hypersurface of $\mathbf{S}^{n+1}$ with two different principal curvatures is a CMC Clifford torus (see [4, Section 2] for a description of the geometry of these tubes). Passing to the quotient, this implies that Σ is a tube T_r^k for some $0 < k \leq n - 2$ such that $\text{tr}(Q_\Sigma) = 0$. By Lemma 1.3.20, then, $n \neq 2k + 1$ and $\tan^2 r = d(n, k) > \frac{n-k+1}{k}$ if $n > 2k + 1$ or $\tan^2 r = d(n, k) < \frac{n-k-1}{k+2}$ if $n < 2k + 1$, which means, in both cases, by Theorem 1.3.3, that T_r^k is unstable. Therefore also this case is empty.

We can conclude that there exists no compact oriented stable CMC hypersurface $\Sigma \subset \mathbf{RP}^n$ satisfying $p(H_1, H_2; n, \alpha) \geq 0$. $\square$

Chapter 2

Sub-Finsler Isoperimetric Problem in the Heisenberg Group

2.1 Introduction to the Heisenberg Group and Pansu's Conjecture

2.1.1 The Sub-Riemannian Heisenberg group

Here we will treat the $(2n + 1)$ -dimensional Heisenberg group $\mathbb{H}^n$, $n \in \mathbb{N}$, which is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the group product

$$(\xi, z) * (\xi', z') = (\xi + \xi', z + z' + \omega(\xi, \xi')), \quad (2.1)$$

where $z, z' \in \mathbb{R}$, $\xi, \xi' \in \mathbb{C}^n$.

Here $\omega : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$ is the symplectic form

$$\omega(\xi, \xi') = \frac{1}{2} \operatorname{Im} \langle \xi, \xi' \rangle \quad (2.2)$$

and $\langle \xi, \bar{\xi}' \rangle = \xi_1 \bar{\xi}'_1 + \dots + \xi_n \bar{\xi}'_n$. We will also use the notation $\xi = x + iy$.

$\mathbb{H}^n$ endowed with this product is a (noncommutative) Lie group with identity element $0 = (0, 0)$ and inverse of $p = (\xi, z)$, $p^{-1} = (-\xi, -z)$. We will denote by $L_p : \mathbb{H}^n \rightarrow \mathbb{H}^n$ the *left translation* by $p \in \mathbb{H}^n$, namely

$$L_p(q) = p * q.$$

Moreover, we will denote by $\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$ the *dilation* by any $\lambda > 0$, namely

$$\delta_\lambda(\xi, z) = (\lambda\xi, \lambda^2 z).$$

Both left translations and dilations are linear mappings in $\mathbb{H}^n = \mathbb{R}^{2n+1}$ and dilations form a 1-parameter group of automorphisms of $\mathbb{H}^n$. In the following we will denote by $|E|$ the Lebesgue measure of any Lebesgue measurable subset E of $\mathbb{H}^n$.

By (2.1), it can be easily checked that the differential dL_p is an upper triangular matrix with 1 along the principal diagonal. Then $\det dL_p = 1$ on $\mathbb{H}^n$ for any p and

$$|L_p E| = |E|$$

for any $p \in \mathbb{H}^n$ and $E \subset \mathbb{H}^n$.

For the dilations, we have that $\det \delta_\lambda = \lambda^Q$, where

$$Q = 2n + 2$$

is called *homogeneous dimension* of $\mathbb{H}^n$. Then we have

$$|\delta_\lambda E| = \lambda^Q |E|.$$

Definition 2.1.1. A C^∞ vector field $X \in \mathbb{H}^n$ is left invariant if, for any function $f \in C^\infty(\mathbb{H}^n)$ and for any $p \in \mathbb{H}^n$, the following holds:

$$X(f \circ L_p) = (Xf) \circ L_p.$$

If we denote by XI the (column) vector of the components of X , then it can be easily proven that a left invariant vector field is completely determined by its value at the origin (and by the Jacobian matrix at the origin of the left translation), namely

$$(XI)(p) = dL_p(0) \cdot (XI)(0), \quad \text{for every } p \in \mathbb{H}^n.$$

Left invariant vector fields with the Lie bracket $[\cdot, \cdot]$ form a nilpotent Lie algebra, which is spanned by the vector fields ($j = 1, \dots, n$)

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial z}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}. \quad (2.3)$$

We will call *horizontal distribution* the $2n$ -dimensional planes $\mathcal{D}(p)$ spanned by the vector fields X_j and Y_j at a point p , namely

$$\mathcal{D}(p) = \text{span}\{X_j(p), Y_j(p) : j = 1, \dots, n\}. \quad (2.4)$$

For every $j = 1, \dots, n$ we have that $[X_j, Y_j] = Z \neq 0$ and all other commutators vanish, hence the distribution is nonintegrable and it's bracket-generating of step 2.

A very general treatment on stratified Lie groups and, in particular, Carnot groups can be found, for example, in [18].

The Carnot-Carathéodory metric

Now we introduce the Carnot-Carathéodory metric and the geodesics of this metric. We need to introduce this metric to the Heisenberg group in order to discuss Pansu's conjecture in the following sections. After that, we will pass to the sub-Finsler structure on $\mathbb{H}^1$.

Definition 2.1.2 (Horizontal curve). A Lipschitz curve $\gamma : [0, 1] \rightarrow \mathbb{H}^n$ is said to be horizontal if $\dot{\gamma}(t) \in \mathcal{D}(\gamma(t))$ for a.e. $t \in [0, 1]$.

Equivalently, γ is horizontal if there exists $2n$ functions $h_j \in L^\infty([0, 1])$ such that

$$\dot{\gamma} = \sum_{j=1}^n h_j X_j(\gamma) + h_{n+j} Y_j(\gamma), \quad \text{a.e. on } [0, 1]. \quad (2.5)$$

If we identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and if we write γ in coordinates $(\gamma_1, \dots, \gamma_{2n+1})$, then $h_j = \dot{\gamma}_j$ for every $j \in \{1, \dots, 2n+1\}$. We will call the first $2n$ components of γ the *horizontal projection* of γ , κ .

Then from κ , by (2.5), we can retrieve the last component of γ , γ_{2n+1} , also called the *vertical component*, namely

$$\dot{\gamma}_{2n+1} = \frac{1}{2} \sum_{j=1}^n h_j \gamma_{n+j} - h_{n+j} \gamma_j = \frac{1}{2} \sum_{j=1}^n \dot{\kappa}_j \kappa_{n+j} - \dot{\kappa}_{n+j} \kappa_j,$$

which gives us, by integration, the following equation

$$\gamma_{2n+1}(t) = \gamma_{2n+1}(0) + \frac{1}{2} \int_0^1 \omega(k, \dot{k}) ds. \quad (2.6)$$

Equation (2.6) can be used to lift any given Lipschitz curve κ in $\mathbb{R}^{2n}$ and the resulting γ will be called the *horizontal lift* of κ .

The nonintegrability of the distribution implies that, given any pair of points $p, q \in \mathbb{H}^n$, there exists a horizontal curve γ in $\mathbb{H}^n$ joining p and q . This result is known as the Chow-Rashevskii Theorem.

Now we can fix on the distribution $\mathcal{D}(p)$ the positive quadratic form making the X_j 's and the Y_j 's orthonormal at every point $p \in \mathbb{H}^n$. Call $g(p; \cdot)$ such quadratic form. Then we can define the *length* of a horizontal curve $\gamma : [0, 1] \rightarrow \mathbb{H}^n$ as follows:

$$L(\gamma) = \int_0^1 g(\gamma; \dot{\gamma})^{\frac{1}{2}} dt = \int_0^1 |\dot{\kappa}| dt, \quad (2.7)$$

where κ is the horizontal projection of γ and $|\cdot|$ is the *Euclidean norm* in $\mathbb{R}^{2n}$. Here the Euclidean norm is specific to the sub-Riemannian structure of the problem. In Section 2.1.3 we will consider, instead of $|\cdot|$, a generic left invariant norm, which makes the Heisenberg group a *sub-Finsler manifold* (see 2.1.8).

Finally, we can define the following function, which is a distance and is called the *Carnot-Carathéodory distance* on $\mathbb{H}^n$.

$$d(p, q) := \inf \left\{ L(\gamma) : \gamma : [0, 1] \rightarrow \mathbb{H}^n \text{ is horizontal, } \gamma(0) = p, \gamma(1) = q \right\}, \quad (2.8)$$

for every $p, q \in \mathbb{H}^n$.

It can be proved that, for every compact set $K \subset \mathbb{H}^n$, there exists a positive and finite constant C_K such that

$$(C_K)^{-1} |p - q| \leq d(p, q) \leq C_K |p - q|^{\frac{1}{2}}, \quad \forall p, q \in \mathbb{H}^n \quad (2.9)$$

Moreover, it can be easily proved that the distance d is left invariant and 1-homogeneous.

Now that we have a metric, we can define the Carnot-Carathéodory ball centered at $p \in \mathbb{H}^n$ with radius $r > 0$ as

$$B_r(p) := \{q \in \mathbb{H}^n : d(p, q) < r\}. \quad (2.10)$$

The metric space $(\mathbb{H}^n, d)$ is complete and locally compact and it is also a length space. Moreover it can be proved to be a *geodesic space*, meaning that for every

$p, q \in \mathbb{H}^n$, there exists a horizontal curve joining the two which realizes $d(p, q)$. Such a curve is called a *geodesic* or *length minimizing curve*.

A more general estimate holds for any homogeneous Carnot group $\mathbb{G}$ with d any homogeneous norm on $\mathbb{G}$, with the $\frac{1}{2}$ replaced with $\frac{1}{r}$, r being the step of the group. For the details and a treatment of the existence of geodesic in this general case, see [18, Appendix C].

2.1.2 Isoperimetric problem and Pansu's conjecture

In this section, we take a look at the isoperimetric problem in the Heisenberg group. We will start by recalling some basic facts on the isoperimetric sets.

For the results in this section we follow the presentation by R. Monti in [6].

In the following we denote by $P(E; A)$ the Heisenberg perimeter of $E \subset \mathbb{H}^n$ in an open set $A \subset \mathbb{H}^1$, which is defined by

$$P(E; A) = \sup \left\{ \int_E \operatorname{div}(V) dp : V \in C_c^1(A; \mathbb{R}^2) \text{ with } \|V\|_\infty \leq 1 \right\}.$$

When $A = \mathbb{H}^n$, we will also write $P(E)$. The divergence $\operatorname{div}(V) = \sum_{j=1}^n (X_j a_j + Y_j b_j)$ where a_j and b_j are the coordinates of the smooth horizontal vector field V with respect to the basis of the horizontal distribution.

For our purpose, we shall try to minimize the *isoperimetric quotient*

$$I(E) = \frac{P(E; \mathbb{H}^1)}{|E|^{(Q-1)/Q}} \quad (2.11)$$

among all measurable sets $E \subset \mathbb{H}^n$ with positive and finite measure, namely we will try to find

$$C_{isop} = \inf \{ I(E) : E \subset \mathbb{H}^n \text{ measurable with } 0 < |E| < \infty \}. \quad (2.12)$$

Such infimum is actually the infimum of perimeter for fixed volume, since any left translation of an *isoperimetric set* (namely a set realizing the infimum) is also an isoperimetric set and the same holds for dilations of isoperimetric sets. Hence we can write

$$C_{isop} = \inf \{ P(E; \mathbb{H}^n) : E \subset \mathbb{H}^n \text{ measurable with } |E| = 1 \}. \quad (2.13)$$

Moreover, the isoperimetric inequality

$$P(E; \mathbb{H}^n) \geq C_{isop} |E|^{\frac{Q-1}{Q}} \quad (2.14)$$

holds for every admissible set E and the constant C_{isop} is actually the sharp constant. We will not give any details of the proof of the isoperimetric inequality. As we've seen in the Euclidean case, many methods can be used to obtain it. For an inequality with a non-sharp positive constant, see for example [59], [60], [38], [42].

The existence of isoperimetric sets theorem is due to Leonardi and Rigot (see [51]). Their result is based on a concentration-compactness argument and it reads as follows.

Theorem 2.1.3 (Leonardi, Rigot). *Let $n > 1$. There exists a measurable set $E \subset \mathbb{H}^n$ with $|E| = 1$ realizing the minimum in (2.13).*

In [60], Pansu conjectured a solution to the isoperimetric problem in the first Heisenberg group. Even if the conjecture has not been proven yet in its full generality, there are many results proving its validity under some additional regularity assumptions. Here we will first state the conjecture, then we will state these results and we will prove some of them.

Conjecture (Pansu). *Up to a null set, a left translation and a dilation, the isoperimetric set in $\mathbb{H}^1$ is the set*

$$E_{isop} = \{(\xi, z) \in \mathbb{H}^1 : |z| < \arccos |\xi| + |\xi| \sqrt{1 - |\xi|^2}, |\xi| < 1\}.$$

Some properties of the isoperimetric set, which are obtained studying its profile function, namely the one that gives the radial value of the function whose graph is the bottom part of E_{isop} , are the following.

1. The boundary ∂E_{isop} is of class C^2 but not of class C^3 ;
2. The set E_{isop} is convex;
3. The set E_{isop} is axially symmetric.

The first result we want to discuss is due to Ritoré and Rosales, who proved, in [66], that 2.1.2 holds true in the class of bounded sets with C^2 boundary. Their result is the following.

Theorem 2.1.4 (Ritoré-Rosales). *Let $E \subset \mathbb{H}^1$ be a bounded isoperimetric set with boundary ∂E of class C^2 . Then we have $E = E_{isop}$, up to dilation and left translation.*

The proof is based on a first variation argument and on the study of the characteristic set of the isoperimetric set E . This proof is important for us, since it was inspirational for our study in the sub-Finsler case.

The next result we present is due to Monti and Rickly and it is a proof of Pansu's conjecture in the class of convex sets. We will report a sketch of the proof. For all the details, see [57].

Theorem 2.1.5 (Monti, Rickly). *Let $E \subset \mathbb{H}^1$ be a convex (in the standard way in $\mathbb{R}^3$) and open isoperimetric set. Then, up to a left translation and a dilation, $E = E_{isop}$.*

Sketch of the proof. Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set. Then

$$E = \{(\xi, z) \in \mathbb{H}^1 : \xi \in D, f(\xi) < z < g(\xi)\}, \quad (2.15)$$

where $D \subset \mathbb{C}$ is a bounded convex open set and $-g, f$ are convex real functions on D . In a similar way as in the proof of 2.1.4 one can recover the following equation.

$$\frac{P(E)}{|E|^{3/4}} = I(E) \leq I(E_\varepsilon) = \frac{P(E_\varepsilon)}{|E_\varepsilon|^{3/4}} = \frac{p(\varepsilon)}{v(\varepsilon)^{3/4}} =: \psi(\varepsilon). \quad (2.16)$$

However, here the vector $\nabla f(\xi) + 2\xi^\perp$ is only in $L^\infty(D) \cap BV_{loc}(D)$. We want to prove that integral curves of $N_f(\xi)^\perp = \frac{\nabla f(\xi) + 2\xi^\perp}{|\nabla f(\xi) + 2\xi^\perp|}$ are circles with curvature H .

- Step 1. Prove a $W_{loc}^{1,1}$ -regularity for solutions of (2.16).
- Step 2. Analysis of the flow of $v(\xi) := 2\xi - \nabla f^\perp(\xi)$, which is, by the convexity of f a BV_{loc} vector. In particular, one proves that, for any compact $K \subset D$, there exists $r > 0$ and a curve γ_ξ defined in $[-r, r]$ which is an integral curve of v passing through ξ at time 0.
- Step 3. Since, in fact, $v/|v|$ is in $W_{loc}^{1,1}(D; \mathbb{R}^2)$, one can prove that the curve γ_ξ is twice differentiable weakly. With this regularity, equation (2.16) says that the curvature of γ_ξ is the constant H . The curve γ_ξ is actually an arc of circle with radius $1/H$.

□

Let $\mathcal{S}$ be the family of measurable subset E of $\mathbb{H}^n$ with positive finite measure that are axially symmetric, namely

$$(\xi, z) \in E \rightarrow (\xi', z) \in E \text{ for all } |\xi'| = |\xi|.$$

An *axially symmetric isoperimetric set* is a set realizing the infimum

$$C_{isop}^{\mathcal{S}} = \inf \{I(E) : E \in \mathcal{S}\}. \quad (2.17)$$

Proving conjecture 2.1.2 in this class of sets results in proving that the solution to the axially symmetric isoperimetric problem is the following set:

$$E_{isop} = \left\{ (\xi, z) \in \mathbb{H}^n : |z| < \arccos |\xi| + |\xi| \sqrt{1 - |\xi|^2}, |\xi| < 1 \right\}. \quad (2.18)$$

The following result is due to Monti and can be found in [56].

Theorem 2.1.6. *The infimum in (2.17) is attained and any axially symmetric isoperimetric set coincides with the set E_{isop} in (2.18), up to a dilation, a vertical translation and a Lebesgue negligible set.*

A scheme of the proof can be found in [6]. For a detailed proof we remind the reader to [56].

The last theorem regarding Pansu's conjecture we present is a result in the class of sets having one circular horizontal section and are contained in a vertical cylinder.

Let $B = \{(\xi, 0) \in \mathbb{H}^n : |\xi| < 1\} = \{|\xi| < 1\} \subset \mathbb{C}^n$ and $C = \{(\xi, z) \in \mathbb{H}^n : |\xi| < 1, z \in \mathbb{R}\}$.

Theorem 2.1.7 (Ritoré). *Let $E \subset \mathbb{H}^n$, $n \geq 1$, be a bounded open set with finite perimeter such that:*

- i) $B \subset E \subset C$;
- ii) $|E| = |E|_{isop}$, where E_{isop} is the set in (2.18).

Then we have $P(E_{isop}; \mathbb{H}^n) \leq P(E; \mathbb{H}^n)$.

The proof is done by Ritoré in [64] with a calibration argument. The same calibration argument is also used in [62] by the authors to prove a minimization property satisfied by the balls for a left invariant norm (but not necessarily symmetric) in $\mathbb{H}^1$.

2.1.3 Sub-Finsler Geometry

In this section, we introduce the basic notions of sub-Finsler geometry and the case study which will be discussed in the next sections.

We'll start by introducing a very general notion of sub-Finsler structure: we consider, on the tangent space at any point of a manifold, a subspace endowed with a left invariant norm ϕ , without any particular assumptions on ϕ . Then we will consider the anisotropic left invariant perimeter measures associated to ϕ (we will call it ϕ -perimeter) and we will conclude the preliminaries by studying the isoperimetric problem in the sub-Finsler Heisenberg group, proving a representation formula for the ϕ -perimeter in terms of the dual norm ϕ^* of ϕ and proving the existence of isoperimetric sets.

The following definition of a sub-Finsler manifold is taken by [28].

Definition 2.1.8 (Sub-Finsler manifold). *Let M be a n -dimensional manifold and $\mathcal{D}$ a smooth distribution of rank k on M . Then a sub-Finsler metric on $\mathcal{D}$ is a smoothly varying Finsler metric ϕ on each subspace $\mathcal{D}(p) \subset T_p M$,*

A sub-Finsler manifold is a smooth n -dimensional manifold equipped with a sub-Finsler metric on a bracket-generating distribution $\mathcal{D}$ of rank k and it is denoted by the triple $(M, \mathcal{D}, \phi)$.

We wish to recall that a *Finsler metric* on M is a smoothly varying Minkowski norm on each tangent space $T_p M$ and a distribution is *bracket-generating* if all iterated brackets among its sections generate, at each point, the whole tangent space to the manifold.

We will now introduce the case study of the second part of this section, which is the first sub-Finsler Heisenberg group $\mathbb{H}^1$. We defined the group law of the group in (2.1).

Then, if we lift horizontally an absolutely continuous curve $\xi : I \rightarrow \mathbb{R}^2$ to a horizontal curve $\gamma = (\xi, z)$, equation (2.6) becomes

$$\dot{z} = \omega(\xi, \dot{\xi}) \quad (2.19)$$

in its differential formulation.

Let $\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ be a norm in the plane. Given a linear function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we associate it to a vector $\xi \in \mathbb{R}^2$ through the standard inner product, namely $f(x) = \langle x, \xi \rangle$. Then the dual norm $\phi^*(f)$ is defined as

$$\phi^*(f) := \sup\{|f(x)| : x \in \mathbb{R}^2, \phi(x) \leq 1\}. \quad (2.20)$$

The associated anisotropic perimeter measure in $\mathbb{H}^1$ is introduced in Definition 2.1.9 and takes into account only horizontal directions. For a regular set $E \subset \mathbb{R}^3$ it can be represented as

$$\mathcal{P}_\phi(E) = \int_{\partial E} \phi^*(N_E) d\mathcal{H}^2,$$

where N_E is obtained by projecting the inner unit normal ν_E onto the horizontal distribution. A set $E \subset \mathbb{H}^1$ is said to be ϕ -isoperimetric if there exists $m > 0$ such that E minimizes

$$\inf \{ \mathcal{P}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable, } \mathcal{L}^3(E) = m \}. \quad (2.21)$$

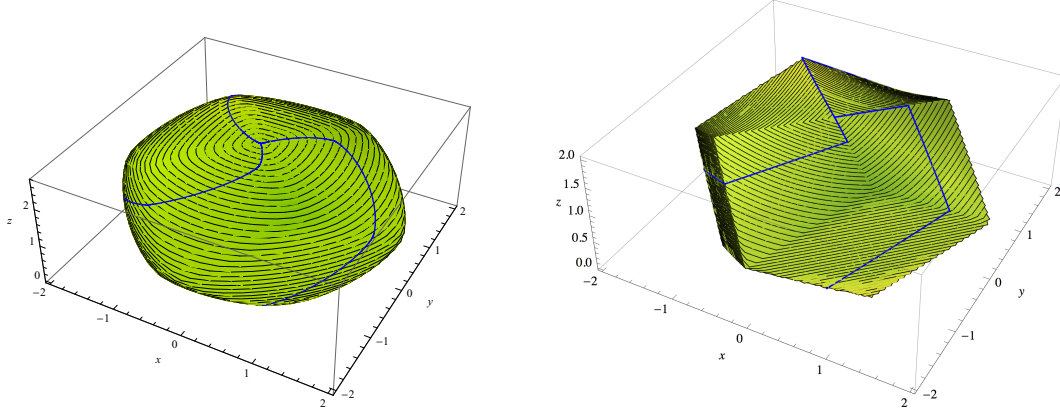


Figure 2.1: The ℓ^p -bubbles with $p = 3$ (left) and $p = 100$ (right). In blue we outlined three horizontal lifts of ℓ^p -circles foliating the ℓ^p -bubble. Images by M. Sigalotti.

If $\phi = |\cdot|$ is the Euclidean norm in $\mathbb{R}^2$, then $\mathcal{P}_\phi$ corresponds to the standard horizontal perimeter in $\mathbb{H}^1$, introduced and studied in [24, 42, 39] and we have the sub-Riemannian structure on $\mathbb{H}^1$ discussed in Section 2.1.2.

Pansu's conjecture, as we've seen in Section 2.1.2, is still an open problem in its full generality even in the sub-Riemannian case. Very little is known when ϕ is a generic norm in $\mathbb{R}^2$, although J. Pozuelo and M. Ritoré have recently obtained several results on the problem, considering also the case where ϕ is convex and homogeneous, but not necessarily a norm (see [62]).

The construction of the Pansu's bubble can be generalized to the sub-Finsler context in the following way. We call ϕ -circle of radius $r > 0$ and center $\xi_0 \in \mathbb{R}^2$ the set

$$C_\phi(\xi_0, r) = \{\xi \in \mathbb{R}^2 : \phi(\xi - \xi_0) = r\}, \quad (2.22)$$

and we call ϕ -bubble the bounded set E_ϕ whose boundary is foliated by horizontal lifts of ϕ -circles in the plane of a given radius, passing through the origin.

In Figure 2.1 we represent two ϕ -bubbles, corresponding to $\phi = \ell^p$, with $\ell^p(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}}$, in the cases $p = 3$ and $p = 100$. The latter can be seen as an approximation of the crystalline case.

Sub-Finsler perimeter

In this paragraph, we introduce the notion of ϕ -perimeter in $\mathbb{H}^1$ for a norm ϕ in $\mathbb{R}^2$. We start by fixing the notation relative to horizontal vector fields and sub-Finsler norms in $\mathbb{H}^1$.

A smooth horizontal vector field is a vector field V on $\mathbb{R}^3$ that can be written as $V = aX + bY$ where $a, b \in C^\infty(\mathbb{H}^1)$. When $A \subset \mathbb{H}^1$ is an open set and $a, b \in C_c^\infty(A)$ have compact support in A we shall write $V \in \mathcal{D}_c(A)$. We fix on $\mathcal{D}(\mathbb{H}^1)$ the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ that makes X, Y pointwise orthonormal. Then each fiber $\mathcal{D}(p)$ can be identified with the Euclidean plane $\mathbb{R}^2$.

Let $\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ be a norm. We fix on $\mathcal{D}(\mathbb{H}^1)$ the left-invariant norm associated with ϕ . Namely, with a slight abuse of notation, for any $p \in \mathbb{H}^1$ and with $a, b \in \mathbb{R}$ we

define

$$\phi(aX(p) + bY(p)) = \phi((a, b)).$$

Since the Haar measure of $\mathbb{H}^1$ is the Lebesgue measure of $\mathbb{R}^3$, the divergence in $\mathbb{H}^1$ is the standard divergence. Therefore, for a smooth horizontal vector field $V = aX + bY$ we have $\operatorname{div}(V) = Xa + Yb$.

Definition 2.1.9. *The ϕ -perimeter of a Lebesgue measurable set $E \subset \mathbb{H}^1$ in an open set $A \subset \mathbb{H}^1$ is*

$$\mathcal{P}_\phi(E; A) = \sup \left\{ \int_E \operatorname{div}(V) dp : V \in \mathcal{D}_c(A) \text{ with } \max_{\xi \in A} \phi(V(\xi)) \leq 1 \right\}. \quad (2.23)$$

When $\mathcal{P}_\phi(E; A) < \infty$ we say that E has finite perimeter in A . When $A = \mathbb{H}^1$, we let $\mathcal{P}_\phi(E) = \mathcal{P}_\phi(E; \mathbb{H}^1)$.

Since all the left-invariant norms in the horizontal distribution are equivalent, we have $\mathcal{P}_\phi(E) < \infty$ if and only if the set E has finite horizontal perimeter in the sense of [24, 39, 42].

For regular sets, we can represent $\mathcal{P}_\phi(E)$ integrating on ∂E a kernel related to the normal. Let ν_E be the Euclidean unit inner normal to ∂E . We define the horizontal vector field $N_E : \partial E \rightarrow \mathcal{D}(\mathbb{H}^1)$ by

$$N_E = \langle \nu_E, X \rangle X + \langle \nu_E, Y \rangle Y, \quad (2.24)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$.

Proposition 2.1.10 (Representation formula). *Let $E \subset \mathbb{H}^1$ be a set with Lipschitz boundary. Then for every open set $A \subset \mathbb{H}^1$ we have*

$$\mathcal{P}_\phi(E; A) = \int_{\partial E \cap A} \phi^*(N_E) d\mathcal{H}^2 \quad (2.25)$$

where $\mathcal{H}^2$ is the standard 2-Hausdorff measure in $\mathbb{R}^3$.

Proof. Let $V \in \mathcal{D}_c(A)$ be such that $\phi(V) \leq 1$. By the standard divergence theorem and by the definition of dual norm, we have

$$\begin{aligned} \int_E \operatorname{div}(V) d\xi &= - \int_{\partial E} \langle V, N_E \rangle_{\mathcal{D}} d\mathcal{H}^2 \leq \int_{\partial E \cap A} \phi(V) \phi^*(N_E) d\mathcal{H}^2 \\ &\leq \int_{\partial E \cap A} \phi^*(N_E) d\mathcal{H}^2. \end{aligned}$$

By taking the supremum over all admissible V we then obtain

$$\mathcal{P}_\phi(E; A) \leq \int_{\partial E \cap A} \phi^*(N_E) d\mathcal{H}^2. \quad (2.26)$$

To get the opposite inequality it is sufficient to prove that for every $\varepsilon > 0$ there exists $V \in \mathcal{D}_c(A)$ such that $\phi(V) \leq 1$ and

$$- \int_{\partial E} \langle V, N_E \rangle_{\mathcal{D}} d\mathcal{H}^2 \geq \int_{\partial E \cap A} \phi^*(N_E) d\mathcal{H}^2 - \varepsilon. \quad (2.27)$$

Here, without loss of generality, we assume that A is bounded. We will construct such a V with continuous coefficients and with compact support in A . The smooth case $V \in \mathcal{D}_c(A)$ will follow by a standard regularization argument.

Let us define the sets

$$\mathcal{N} = \{p \in \partial E \cap A : N_E(p) \text{ is defined}\}, \quad \mathcal{Z} = \{p \in \mathcal{N} : N_E(p) = 0\}. \quad (2.28)$$

From the results of [10] it follows that $\mathcal{Z}$ has vanishing $\mathcal{H}^2$ -measure. For any $p \in \mathcal{N} \setminus \mathcal{Z}$ we take $V \in \mathcal{D}(p)$ such that $\phi(V) = 1$ and

$$\langle V, N_E \rangle_{\mathcal{D}} = \phi^*(N_E).$$

In general, this choice is not unique. However, there is a selection $p \mapsto V(p)$ that is measurable (this follows since the coordinates are measurable, see for instance [9, Theorem 8.1.3]). We extend V to $\mathcal{Z}$ letting $V = 0$ here. This extension is still measurable.

Since $\partial E \cap A$ has finite $\mathcal{H}^2$ -measure, by Lusin's theorem there exists a compact set $K_\varepsilon \subset \partial E \cap A$ such that $\mathcal{H}^2((\partial E \cap A) \setminus K_\varepsilon) < \varepsilon$ and the restriction of V to K_ε is continuous. Now, by Tietze–Uryshon theorem we extend V from K_ε to A in such a way that the extended map, still denoted by V , is continuous with compact support in A and satisfies $\phi(V) \leq 1$ everywhere.

Our construction yields the following

$$\begin{aligned} \int_{\partial E \cap A} \phi^*(N_E) d\mathcal{H}^2 &= \int_{K_\varepsilon} \langle V, N_E \rangle_{\mathcal{D}} d\mathcal{H}^2 + \int_{(\partial E \cap A) \setminus K_\varepsilon} \phi^*(N_E) d\mathcal{H}^2 \\ &= \int_{\partial E \cap A} \langle V, N_E \rangle_{\mathcal{D}} d\mathcal{H}^2 - \int_{(\partial E \cap A) \setminus K_\varepsilon} (\langle V, N_E \rangle_{\mathcal{D}} - \phi^*(N_E)) d\mathcal{H}^2 \\ &\leq \int_{\partial E \cap A} \langle V, N_E \rangle_{\mathcal{D}} d\mathcal{H}^2 + C\varepsilon. \end{aligned}$$

In the last inequality we used the fact that $\langle V, N_E \rangle_{\mathcal{D}} - \phi^*(N_E)$ is bounded and $\mathcal{H}^2((\partial E \cap A) \setminus K_\varepsilon) < \varepsilon$. The claim follows. $\square$

Existence of isoperimetric sets

For a measurable set $E \subset \mathbb{H}^1$ with positive and finite measure and a given norm ϕ on $\mathbb{R}^2$ we define the ϕ -isoperimetric quotient as

$$\text{Isop}_\phi(E) = \frac{\mathcal{P}_\phi(E)}{\mathcal{L}^3(E)^{\frac{3}{4}}},$$

where $\mathcal{L}^3$ denotes the Lebesgue measure of $\mathbb{R}^3$.

The isoperimetric quotient is invariant under left-translation (w.r.t. the operation in (2.1)), i.e., $\text{Isop}_\phi(p * E) = \text{Isop}_\phi(E)$ for any $p \in \mathbb{H}^1$ and $E \subset \mathbb{H}^1$ admissible, and it is 0-homogeneous with respect to the one-parameter family of automorphisms

$$\delta_\lambda(\xi, z) = (\lambda\xi, \lambda^2 z), \quad (2.29)$$

i.e., $\text{Isop}_\phi(\lambda E) = \text{Isop}_\phi(E)$, where $\lambda E = \delta_\lambda(E)$.

The isoperimetric problem (2.21) is then equivalent to minimizing the isoperimetric quotient among all admissible sets. Namely, given $m \in (0, \infty)$, any isoperimetric set $E \subset \mathbb{H}^1$ with $\mathcal{L}^3(E) = m$ is a solution to

$$C_I = \inf \left\{ \text{Isop}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable, } 0 < \mathcal{L}^3(E) < \infty \right\}, \quad (2.30)$$

and, *vice versa*, any solution $E \subset \mathbb{H}^1$ to (2.30) solves (2.21) within its volume class, i.e., with $m = \mathcal{L}^3(E)$. In particular, we have

$$C_I = \inf \left\{ \mathcal{P}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable, } \mathcal{L}^3(E) = 1 \right\}. \quad (2.31)$$

The constant C_I depends on ϕ .

Since $\mathcal{P}_\phi$ is equivalent to the standard horizontal perimeter, the isoperimetric inequality in [42] implies that $C_I > 0$ and the validity of the following inequality for any measurable set E with finite measure:

$$\mathcal{P}_\phi(E) \geq C_I \mathcal{L}^3(E)^{\frac{3}{4}}. \quad (2.32)$$

The constant C_I is the largest one making true the above inequality and isoperimetric sets are precisely those for which (2.32) is an equality.

Theorem 2.1.11 (Existence of isoperimetric sets). *Let ϕ be any norm on $\mathbb{R}^2$. There exists a set $E \subset \mathbb{H}^1$ with non-zero and finite ϕ -perimeter such that*

$$\mathcal{P}_\phi(E) = C_I \mathcal{L}^3(E)^{\frac{3}{4}}. \quad (2.33)$$

Theorem 2.1.11 follows by applying the strategy of [51, Section 4]. In the sequel we denote the left-invariant homogeneous ball centered at $p \in \mathbb{H}^1$ of radius $r > 0$ by $B(p, r)$.

Proof of Theorem 2.1.11. We give a sketch of the proof. By perimeter and volume homogeneity with respect to $\{\delta_\lambda\}_{\lambda \in \mathbb{R}}$ it is enough to prove the existence of a minimizing set in the class of volume $\mathcal{L}^3(E) = 1$. Let $\{E_k\}_{k \in \mathbb{N}}$ be a minimizing sequence for (2.31) such that for $k \in \mathbb{N}$ we have

$$\mathcal{L}^3(E_k) = 1, \quad \mathcal{P}_\phi(E_k) \leq C_I \left(1 + \frac{1}{k}\right).$$

Assume that there exists $m_0 \in (0, 1/2)$ such that for any $k \in \mathbb{N}$ there exists $p_k \in \mathbb{H}^1$ satisfying

$$\mathcal{L}^3(E_k \cap B(p_k, 1)) \geq m_0. \quad (2.34)$$

Then, the translated sequence $\{-p_k \star E_k\}_{k \in \mathbb{N}}$, still denoted $\{E_k\}_{k \in \mathbb{N}}$, is also minimizing for (2.31) and satisfies $\mathcal{L}^3(E_k \cap B(0, 1)) \geq m_0$.

Since $\mathcal{P}_\phi$ is equivalent to the standard horizontal perimeter, we have a compactness theorem for sets of finite ϕ -perimeter as in [42, Theorem 1.28]. Then, we can extract a sub-sequence, still denoted $\{E_k\}_{k \in \mathbb{N}}$, converging in the $L^1_{\text{loc}}(\mathbb{H}^1)$ sense to a set $E \subset \mathbb{H}^1$ of finite ϕ -perimeter. The lower semi-continuity of $\mathcal{P}_\phi$ therefore implies

$$\mathcal{P}_\phi(E) \leq \liminf_{k \rightarrow \infty} \mathcal{P}_\phi(E_k) \leq C_I. \quad (2.35)$$

Moreover, we have

$$\begin{aligned}\mathcal{L}^3(E) &\leq \liminf_{k \rightarrow \infty} \mathcal{L}^3(E_k) = 1 \quad \text{and} \\ \mathcal{L}^3(E \cap B(0, 1)) &= \lim_{k \rightarrow \infty} \mathcal{L}^3(E_k \cap B(0, 1)) \geq m_0.\end{aligned}\tag{2.36}$$

To prove (2.33) we are left to show that $\mathcal{L}^3(E) = 1$, which follows by applying a sub-Finsler version of [51, Lemma 4.2], ensuring existence of a radius $R > 0$ such that $\mathcal{L}^3(E \cap B(0, R)) = 1$. This is based on (2.36) and on a canonical relation between perimeter and derivative of volume in balls with respect to the radius, which is valid in quite general metric structures, including sub-Finsler ones, see [7, Lemma 3.5].

We conclude by justifying the assumption (2.34). This follows by a sub-Finsler version of [51, Lemma 4.1]. Using once more the equivalence of $\mathcal{P}_\phi$ with the standard horizontal perimeter, we deduce from [42, Theorem 1.18] the validity of the following *relative isoperimetric inequality* holding for a constant $C > 0$ and any measurable set E with finite measure

$$\min \left\{ \mathcal{L}^3(B \cap E)^{\frac{3}{4}}, \mathcal{L}^3(B \setminus E)^{\frac{3}{4}} \right\} \leq C \mathcal{P}_\phi(E, \lambda B), \tag{2.37}$$

where $\lambda \geq 1$ is a constant depending only on ϕ , and B is any left-invariant homogeneous ball. Together with the fact that the family $\{B(p, \lambda) : p \in \mathbb{H}^1\}$ has bounded overlaps, we can reproduce the argument of [51, Lemma 4.1] and prove the claim. $\square$

Remark 2.1.12. *Following the arguments of [51, Lemma 4.2], one also shows that any isoperimetric set is equivalent to a bounded and connected one (i.e., it is bounded and connected up to sets of zero Lebesgue measure).*

2.2 A Classification of C^2 Isoperimetric Sets

In this section, we will present some new results obtained in a joint work with V. Franceschi, R. Monti and M. Sigalotti (see [37]): assuming some regularity on ϕ and on its dual norm ϕ^* , we'll deduce a foliation property by sub-Finsler geodesics of regular surfaces with constant ϕ -curvature. Then we'll study the characteristic set of regular surfaces and characterize C^2 -smooth ϕ -isoperimetric sets. We will also show that, under suitable regularity properties on ϕ , the candidate ϕ -isoperimetric set obtained in analogy with the sub-Riemannian Pansu's bubble is indeed C^2 -smooth.

The main result is the characterization of C^2 -smooth ϕ -isoperimetric sets when ϕ and ϕ^* are in $C^2(\mathbb{R}^2 \setminus \{0\})$. This result suggests that the ϕ -bubble is the solution to the isoperimetric problem for $\mathcal{P}_\phi$.

Here and in the following, if $\phi \in C^k(\mathbb{R}^2 \setminus \{0\})$ we say that ϕ is of class C^k , for $k \in \mathbb{N}$.

Theorem 2.2.1. *Let ϕ be a norm of class C^2 such that ϕ^* is of class C^2 . If $E \subset \mathbb{H}^1$ is a ϕ -isoperimetric set of class C^2 then we have $E = E_\phi$, up to left-translations and anisotropic dilations.*

The proof of Theorem 2.2.1 is presented in Section 2.2.5 and is based on a fine study of the *characteristic set* of isoperimetric sets. The characteristic set of a set $E \subset \mathbb{H}^1$ of class C^1 (equivalently, of its boundary ∂E) is

$$\mathcal{C}(E) = \mathcal{C}(\partial E) = \{p \in \partial E : T_p \partial E = \mathcal{D}(p)\}. \quad (2.38)$$

In Section 2.2.4 we characterize the structure of $\mathcal{C}(E)$ for a C^2 -smooth ϕ -isoperimetric set $E \subset \mathbb{H}^1$, proving that $\mathcal{C}(E)$ is made of isolated points. For the more general case of ϕ -critical surfaces we obtain the following result, that we prove by adapting to the sub-Finsler case the theory of Jacobi fields of [66]. Any ϕ -critical surface has constant ϕ -curvature and the definition is presented in Section 2.2.4.

Theorem 2.2.2. *Let ϕ and ϕ^* be of class C^2 and let $\Sigma \subset \mathbb{H}^1$ be a complete and oriented surface of class C^2 . If Σ is ϕ -critical with non-vanishing ϕ -curvature then $\mathcal{C}(\Sigma)$ consists of isolated points and C^2 curves that are either horizontal lines or horizontal lifts of simple closed curves.*

The simple closed curves of Theorem 2.2.2 are described by a suitable ordinary differential equation. We expect that these curves are $\phi^\dagger$ -circles, where $\phi^\dagger$ is the norm defined as

$$\phi^\dagger(\xi) = \phi^*(\xi^\perp), \quad \xi \in \mathbb{R}^2.$$

Here and hereafter, $\perp: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denotes the perp-operator $\perp(\xi) = \xi^\perp$, with

$$\xi^\perp = (x, y)^\perp = (-y, x), \quad \xi = (x, y) \in \mathbb{R}^2.$$

Theorem 2.2.1 then follows by combining the results of Sections 2.2.1, 2.2.2, and 2.2.4. In particular, starting from a first variation analysis, we establish a foliation property outside the characteristic set for C^2 -smooth ϕ -isoperimetric sets (and more generally for constant ϕ -curvature surfaces). Theorem 2.2.2 is a key step for concluding the proof.

We also identify an explicit relation between ϕ -isoperimetric sets and geodesics in the ambient space. In Section 2.2.3, we show that, outside the characteristic set, ϕ -isoperimetric sets are foliated by *sub-Finsler geodesics* in $\mathbb{H}^1$ relative to the norm $\phi^\dagger$. We refer to Corollary 2.2.16 for a statement of the result. Notice that when $\phi = |\cdot|$ is the Euclidean norm, $\phi^\dagger$ reduces to $|\cdot|$, and we recover the foliation by sub-Riemannian geodesics of C^2 -smooth $|\cdot|$ -isoperimetric sets.

The *regularity* of the candidate isoperimetric sets E_ϕ is a major issue that we treat in Section 2.2.5. While it is rather easy to check that ϕ -Pansu bubbles have the same regularity as ϕ outside the characteristic set (at least if ϕ -circles are strictly convex, see Lemma 2.2.29), it is not clear what regularity is inherited from ϕ at characteristic points. In Section 2.2.6 we prove the following.

Theorem 2.2.3. *Assume that ϕ is of class C^4 and that ϕ -circles have strictly positive curvature. Then ∂E_ϕ is an embedded surface of class C^2 .*

Geodesics of sub-Finsler structures on the Heisenberg group and other Carnot groups have been studied in several papers (see, in particular, [8, 11, 17, 52, 74]).

2.2.1 First variation of the isoperimetric quotient

In this paragraph we derive a first order necessary condition for ϕ -isoperimetric sets, both when ϕ is regular or not.

Notation

We now introduce some notation that will be used throughout the section. Let $E, A \subset \mathbb{H}^1$ be sets such that E is closed, A is open and there exists a function $g \in C^1(A)$, called *defining function for $\partial E \cap A$* , such that $\partial E \cap A = \{p \in A : g(p) = 0\}$ and $\nabla g(p) \neq 0$ for every $p \in \partial E \cap A$.

We say that $E \cap A$ is a *z -subgraph* if there exists an open set $D \subset \mathbb{R}^2$ and a function $f \in C^1(D)$, called *graph function for $\partial E \cap A$* , such that

$$E \cap A = \{(\xi, z) \in A : \xi \in D \text{ and } z \leq f(\xi)\}.$$

In this case, $g(\xi, z) = f(\xi) - z$ is a defining function for $\partial E \cap A$.

The definition of *z -epigraph* is analogous and all results given below for z -subgraphs have a straightforward counterpart for z -epigraphs. In a similar way, one can also define *x -subgraphs*, *y -subgraphs*, *x -epigraphs*, and *y -epigraphs*.

Given a function $g \in C^1(A)$, we denote by $\mathcal{G} = (Xg)X + (Yg)Y$ the *horizontal gradient* of g and we define the *projected horizontal gradient* as

$$G = (Xg, Yg) \in \mathbb{R}^2. \quad (2.39)$$

If $\partial E \cap A$ is a z -graph with graph function $f \in C^1(D)$, we define $F : D \rightarrow \mathbb{R}^2$ by

$$F(\xi) = G(\xi, f(\xi)) = \nabla f(\xi) - \frac{1}{2}\xi^\perp, \quad (2.40)$$

and

$$\mathcal{C}(f) = \{\xi \in D : F(\xi) = 0\}. \quad (2.41)$$

Hence $\mathcal{C}(E) \cap A = \{(\xi, f(\xi)) : \xi \in \mathcal{C}(f)\}$, where $\mathcal{C}(E)$ is the characteristic set of E , defined in (2.38). The set $\mathcal{C}(f)$ has zero Lebesgue measure in D .

If $E \cap A$ is the z -subgraph of a function $f \in C^1(D)$, by the representation formula (2.25) we have

$$\mathcal{P}_\phi(E; A) = \int_D \phi^*(F(\xi)) d\xi.$$

When the dual norm ϕ^* is of class C^1 , starting from a graph function $f \in C^1(D)$ we define the vector field $\mathcal{X}f : D \rightarrow \mathbb{R}^2$ by

$$\mathcal{X}f(\xi) = \nabla \phi^*(F(\xi)), \quad \xi \in D.$$

The geometric meaning of the vector field $\mathcal{X}f$ will be clarified in the next section, see (2.50).

Remark 2.2.4. *At any point $\xi \in D$ such that $F(\xi) \neq 0$ the vector field $\mathcal{X}f$ satisfies*

$$\phi(\mathcal{X}f(\xi)) = 1, \quad (2.42)$$

since the gradient of ϕ^ at any nonzero point has norm ϕ equal to one (even when ϕ^* is not regular, by replacing the gradient by any element of the subgradient; see, for instance, [50, Example 3.6.5]).*

Regular norms

Proposition 2.2.5 (First variation for isoperimetric sets). *Let ϕ be a norm such that ϕ^* is of class C^1 . Let $E \subset \mathbb{H}^1$ be a ϕ -isoperimetric set such that, for some open set $A \subset \mathbb{H}^1$, $E \cap A$ is a z -subgraph of class C^1 . Then the graph function $f \in C^1(D)$ satisfies in the weak sense the partial differential equation*

$$\operatorname{div}(\mathcal{X}f) = -\frac{3}{4} \frac{\mathcal{P}_\phi(E)}{\mathcal{L}^3(E)} \quad \text{in } D. \quad (2.43)$$

Proof. For small $\varepsilon \in \mathbb{R}$ and $\varphi \in C_c^\infty(D)$ let $E_\varepsilon \subset \mathbb{H}^1$ be the set such that

$$E_\varepsilon \cap A = \{(\xi, z) \in A : z \leq f(\xi) + \varepsilon\varphi(\xi), \xi \in D\},$$

and $E_\varepsilon \setminus A = E \setminus A$. Starting from the representation formula

$$\mathcal{P}_\phi(E_\varepsilon; A) = \int_{\partial E_\varepsilon \cap A} \phi^*(N_{E_\varepsilon}) d\mathcal{H}^2 = \int_D \phi^*(F + \varepsilon(X\varphi, Y\varphi)) d\xi, \quad (2.44)$$

we compute the derivative

$$\mathcal{P}'_\phi = \left. \frac{d}{d\varepsilon} \mathcal{P}_\phi(E_\varepsilon; A) \right|_{\varepsilon=0} = \int_D \langle \mathcal{X}f, (X\varphi, Y\varphi) \rangle d\xi = \int_D \langle \mathcal{X}f, \nabla\varphi \rangle d\xi. \quad (2.45)$$

On the other hand, the derivative of the volume is

$$\mathcal{V}' = \left. \frac{d}{d\varepsilon} \mathcal{L}^3(E_\varepsilon) \right|_{\varepsilon=0} = \int_D \varphi d\xi.$$

Inserting these formulas into

$$0 = \left. \frac{d}{d\varepsilon} \frac{\mathcal{P}_\phi(E_\varepsilon)^4}{\mathcal{L}^3(E_\varepsilon)^3} \right|_{\varepsilon=0} = \frac{\mathcal{P}_\phi(E)^3}{\mathcal{L}^3(E)^4} (4\mathcal{P}'_\phi \mathcal{L}^3(E) - 3\mathcal{V}' \mathcal{P}_\phi(E)),$$

we obtain

$$\int_D \langle \mathcal{X}f, \nabla\varphi \rangle d\xi = \frac{3}{4} \frac{\mathcal{P}_\phi(E)}{\mathcal{L}^3(E)} \int_D \varphi d\xi$$

for any test function $\varphi \in C_c^\infty(D)$. This is our claim. $\square$

Proposition 2.2.5 still holds if we only have $f \in \operatorname{Lip}(D)$. If ϕ^* is of class C^2 and $f \in C^2(D)$ then we have $\mathcal{X}f \in C^1(D \setminus \mathcal{C}(f); \mathbb{R}^2)$. So equation (2.43) is satisfied pointwise in $D \setminus \mathcal{C}(f)$ in the strong sense.

Definition 2.2.6. Let $f \in C^2(D)$. We call the function $H_\phi : D \setminus \mathcal{C}(f) \rightarrow \mathbb{R}$

$$H_\phi(\xi) = \operatorname{div}(\mathcal{X}f(\xi)), \quad \xi \in D \setminus \mathcal{C}(f), \quad (2.46)$$

the ϕ -curvature of the graph $\operatorname{gr}(f)$. We say that $\operatorname{gr}(f)$ has constant ϕ -curvature if there exists $h \in \mathbb{R}$ such that $H_\phi = h$ on $D \setminus \mathcal{C}(f)$. Finally, we say that $\operatorname{gr}(f)$ is ϕ -critical if there exists $h \in \mathbb{R}$ such that

$$\int_D \langle \mathcal{X}f, \nabla\varphi \rangle d\xi = -h \int_D \varphi d\xi \quad (2.47)$$

is satisfied for every $\varphi \in C_c^\infty(D)$.

Proposition 2.2.5 then asserts that the part of the boundary of a ϕ -isoperimetric set of class C^2 that can be represented as a z -graph is ϕ -critical and in particular it has constant ϕ -curvature at noncharacteristic points.

Remark 2.2.7. *Let us discuss how the proof of Proposition 2.2.5 can be adapted to the case where $E \cap A$ is a x -subgraph of class C^2 . The case of y -subgraphs is analogous. We have a defining function for $\partial E \cap A$ of the type $g(x, y, z) = f(y, z) - x$ with $f \in C^2(D)$. The projected horizontal gradient in (2.39) reads*

$$G(y, z) = \left(-1 - \frac{1}{2}y f_z, f_y + \frac{1}{2}f f_z \right).$$

For $\varepsilon \in \mathbb{R}$ and $\varphi \in C_c^\infty(D)$ let E_ε be the x -subgraph in A of $f + \varepsilon\varphi$. Then the derivative of the ϕ -perimeter of E_ε is

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \mathcal{P}_\phi(E_\varepsilon; A) \right|_{\varepsilon=0} &= \int_D \langle \nabla \phi^*(G), (-y\varphi_z/2, \varphi_y + (\varphi f)_z/2) \rangle dydz \\ &= - \int_D \varphi(y, z) \mathcal{L}f(y, z) dydz, \end{aligned}$$

where $\mathcal{L} : C^2(D) \rightarrow C(D)$ is the partial differential operator

$$\mathcal{L}f = \left(\frac{\partial}{\partial y} + \frac{f}{2} \frac{\partial}{\partial z} \right) \phi_b^*(G) - \frac{y}{2} \frac{\partial}{\partial z} \phi_a^*(G), \quad (2.48)$$

with $\nabla \phi^* = (\phi_a^*, \phi_b^*)$.

The statement for x -graphs is then that if $E \subset \mathbb{H}^1$ is ϕ -isoperimetric and $E \cap A$ is a x -subgraph with graph function $f \in C^2(D)$, then

$$\mathcal{L}f = -\frac{3}{4} \frac{\mathcal{P}_\phi(E)}{\mathcal{L}^3(E)} \quad \text{in } D.$$

When we only have $f \in \text{Lip}(D)$, $\mathcal{L}f$ is well-defined in the distributional sense.

2.2.2 Integration of the curvature equation

Throughout this section ϕ^* is a norm of class C^2 , unless explicitly mentioned otherwise.

Let $A \subset \mathbb{H}^1$ be open and $g \in C^{1,1}(A)$ be such that $\nabla g(p) \neq 0$ for every p in $\Sigma = \{p \in A : g(p) = 0\}$. The projected horizontal gradient $G : A \rightarrow \mathbb{R}^2$ introduced in (2.39) is Lipschitz continuous. Assume that Σ has no characteristic points, that is, $G(p) \neq 0$ for every $p \in \Sigma$. We use the coordinates $G = (a, b)$ with $a, b \in \text{Lip}(A)$ and we consider $G^\perp = (-b, a)$. The horizontal vector field $\mathcal{G}^\perp = -bX + aY$ is tangent to Σ .

Definition 2.2.8. *A curve $\gamma \in C^1(I; \Sigma)$ is said to be a Legendre curve of Σ if $\dot{\gamma}(t) = \mathcal{G}^\perp(\gamma(t))$ for all $t \in I$.*

In coordinates, a curve $\gamma = (\xi, z)$ in Σ is a Legendre curve if and only if

$$\dot{\xi} = G^\perp(\gamma) \quad \text{and} \quad \dot{z} = \omega(\xi, \dot{\xi}).$$

Since $\mathcal{G}^\perp$ is Lipschitz continuous, the graph Σ is foliated by Legendre curves: for any $p \in \Sigma$ there exists a unique (maximal) Legendre curve passing through p .

Consider now the case where Σ is a z -graph with graph function $f \in C^{1,1}(D)$, where D is an open subset of $\mathbb{R}^2$. Then $G(\xi, f(\xi)) = F(\xi)$, where F is defined as in (2.40), and a Legendre curve $\gamma = (\xi, z)$ satisfies

$$\dot{\xi} = F^\perp(\xi) \quad \text{and} \quad \dot{z} = \omega(\xi, \dot{\xi}). \quad (2.49)$$

The domain D is foliated by integral curves of $F^\perp$. On D we define the vector field $\mathcal{N} \in \text{Lip}(D; \mathbb{R}^2)$ by

$$\mathcal{N}(\xi) = \mathcal{X}f(\xi) = \nabla \phi^*(F(\xi)), \quad \xi \in D. \quad (2.50)$$

We know that $\phi(\mathcal{N}) = 1$, by (2.42). We may call $\mathcal{N}$ the ϕ -normal to the foliation of D by integral curves of $F^\perp$. We denote by $H_\phi = \text{div}(\mathcal{N})$ the divergence of $\mathcal{N}$.

Theorem 2.2.9. *Let ϕ^* be of class C^2 . Let Σ be the z -graph of a function $f \in C^2(D)$ with $\mathcal{C}(f) = \emptyset$. Then any Legendre curve $\gamma \in C^1(I; \Sigma)$, with $\gamma = (\xi, z)$, satisfies*

$$\frac{d}{dt}\mathcal{N}(\xi) = H_\phi(\xi)\dot{\xi} \quad \text{and} \quad \dot{z} = \omega(\xi, \dot{\xi}). \quad (2.51)$$

Proof. The second equality in (2.51) is part of the definition of a Legendre curve. We prove the first equality.

We identify $\mathcal{N}(\xi)$ and $\dot{\xi} = F^\perp(\xi)$ with column vectors and we denote by Jg the Jacobian matrix of a differentiable mapping g . By the chain rule, using the coordinates $F = (a, b)$ and $\dot{\xi} = (-b(\xi), a(\xi))$ we obtain

$$\frac{d}{dt}\mathcal{N}(\xi) = \mathcal{H}\phi^*(F(\xi))JF(\xi)\dot{\xi} = \begin{pmatrix} -ba_x\phi_{aa}^* - bb_x\phi_{ab}^* + aa_y\phi_{aa}^* + ab_y\phi_{ab}^* \\ -ba_x\phi_{ab}^* - bb_x\phi_{bb}^* + aa_y\phi_{ab}^* + ab_y\phi_{bb}^* \end{pmatrix}, \quad (2.52)$$

where $\mathcal{H}\phi^*$ is the Hessian matrix of ϕ^* and the second order derivatives of ϕ^* are evaluated at $F(\xi)$. Since ϕ^* is of class C^2 , we identified $\phi_{ab}^* = \phi_{ba}^*$. By Euler's homogeneous function theorem, since $\nabla \phi^*$ is 0-positively homogeneous there holds $\langle \nabla \phi_a^*(F), F \rangle = 0$ and $\langle \nabla \phi_b^*(F), F \rangle = 0$. These formulas read

$$a\phi_{aa}^* + b\phi_{ab}^* = 0 \quad \text{and} \quad a\phi_{ab}^* + b\phi_{bb}^* = 0.$$

Plugging these relations into (2.52), we obtain

$$\frac{d}{dt}\mathcal{N}(\xi) = (a_x\phi_{aa}^* + b_x\phi_{ab}^* + a_y\phi_{ab}^* + b_y\phi_{bb}^*)\dot{\xi}. \quad (2.53)$$

On the other hand, we have

$$\text{div}(\mathcal{N}) = \text{div}(\mathcal{X}f) = a_x\phi_{aa}^* + b_x\phi_{ab}^* + a_y\phi_{ab}^* + b_y\phi_{bb}^*,$$

so that (2.53) yields the claim. $\square$

Remark 2.2.10. *An analogue of Theorem 2.2.9 holds true for x -graphs. Let Σ be a x -graph Σ without characteristic points and with defining function $g(x, y, z) = f(y, z) - x$ for some f of class C^2 . Let $\gamma \in C^1(I; \Sigma)$ be a Legendre curve with coordinates $\gamma(t) =$*

$(f(\zeta(t)), \zeta(t))$ for $t \in I$ and consider the vector $\mathcal{N}(y, z) = \nabla \phi^*(G(y, z))$. Following the same steps as in the proof of Theorem 2.2.9, one gets

$$\frac{d}{dt} \mathcal{N}(\zeta) = \mathcal{L}f(\zeta)G^\perp(\zeta) \quad \text{on } I.$$

Hence, the conclusion of Theorem 2.2.9 holds with $H_\phi = \operatorname{div}(\mathcal{X}f)$ replaced by the quantity $\mathcal{L}f$ defined in Remark 2.2.7. Notice that H_ϕ and $\mathcal{L}f$ coincide on surfaces that are both x -graphs and z -graphs.

An analogous remark can be made for y -graphs.

Corollary 2.2.11. *Let ϕ^* be of class C^2 . Let Σ be the z -graph of a function $f \in C^2(D)$ with $\mathcal{C}(f) = \emptyset$. If Σ has constant ϕ -curvature $h \neq 0$ then it is foliated by Legendre curves that are horizontal lift of ϕ -circles in D with radius $1/|h|$, followed in clockwise sense if $h > 0$ and in anti-clockwise sense if $h < 0$.*

Proof. Having constant ϕ -curvature h means that

$$\operatorname{div}(\mathcal{N}) = \operatorname{div}(\mathcal{X}f) = h \quad \text{in } D.$$

By Theorem 2.2.9, for any Legendre curve $\gamma = (\xi, z)$ we have

$$\frac{d}{dt} \mathcal{N}(\xi) - H(\xi)\dot{\xi} = 0.$$

We may then integrate this equation and deduce that there exists $\xi_0 \in \mathbb{R}^2$ such that along ξ we have

$$\mathcal{N}(\xi) - h\xi = -h\xi_0. \quad (2.54)$$

From (2.42) and (2.50) we conclude that

$$|h|\phi(\xi - \xi_0) = \phi(h(\xi - \xi_0)) = \phi(\mathcal{N}) = 1.$$

Finally, notice that $\langle \mathcal{N}(\xi), F(\xi) \rangle > 0$ if $F(\xi) \neq 0$, so that $t \mapsto F(\xi(t))$ rotates clockwise if $h > 0$ and anti-clockwise if $h < 0$, according to (2.51). Hence, $t \mapsto F(\xi(t))^\perp$ and $t \mapsto \xi(t)$ also rotate clockwise if $h > 0$, and anti-clockwise if $h < 0$. $\square$

Let us discuss an extension of Corollary 2.2.11 to the case in which we replace the assumption that ϕ^* is C^2 by the weaker assumption that ϕ^* is *piecewise* C^2 , in the following sense: there exists $k \in \mathbb{N}$ and $A_1, \dots, A_k \in \mathbb{R}^2$ such that ϕ^* is C^2 on $\mathbb{R}^2 \setminus \cup_{j=1}^k \operatorname{span}(A_j)$.

A relevant case where this assumption holds true is when ϕ is the ℓ^p norm

$$\ell^p(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}}, \quad x, y \in \mathbb{R},$$

with $p > 2$. Indeed, the dual norm $(\ell^p)^*$ coincides with the norm ℓ^q , with $q = p/(p-1) < 2$, which is C^2 out of the coordinate axes, but not on the whole punctured plane $\mathbb{R}^2 \setminus \{0\}$. We can prove the following.

Corollary 2.2.12. *Let ϕ^* be piecewise C^2 . Let Σ be the z -graph of a function $f \in C^2(D)$ with $\mathcal{C}(f) = \emptyset$. If Σ has constant ϕ -curvature $h \neq 0$ then it is foliated by Legendre curves that are horizontal lifts of ϕ -circles in D with radius $1/|h|$, followed in clockwise sense if $h > 0$ and in anti-clockwise sense if $h < 0$.*

Proof. Under the assumptions of the corollary, the projected horizontal gradient is C^1 on D and Legendre curves can be introduced as in Definition 2.2.8.

Consider any Legendre curve $\gamma = (\xi, z)$ on Σ . Let us denote by $I \subset \mathbb{R}$ the maximal interval of definition of γ and by J the open subset of I defined as follows: $t \in J$ if and only if $F(\xi(t))$ is in the region where ϕ^* is C^2 . For the restriction of γ to a connected component J_0 of J , Theorem 2.2.9 can be recovered. In particular, since Σ has constant ϕ -curvature $h \neq 0$, then $\gamma|_{J_0}$ is the lift of a ϕ -circle of radius $1/|h|$, followed clockwise or anti-clockwise depending on the sign of h . If $t \in I \setminus J$, then $F(\xi(t))$ belongs to one of the lines $\text{span}(A_1), \dots, \text{span}(A_k)$ on which ϕ^* may lose the C^2 regularity. Notice that the restriction of ξ to a connected component of J compactly contained in I follows an arc of ϕ -circle connecting two lines of the type $\text{span}(A_j)$. In particular, it cannot have an arbitrarily small length.

If $I \setminus J$ is made of isolated points, then $\gamma : I \rightarrow \Sigma$ is the lift of a ϕ -circle of radius $1/|h|$. Indeed, an arc of ϕ -circle of prescribed radius followed in a prescribed sense is only determined by its initial point and its tangent line there. Since γ is an arbitrary Legendre curve on Σ , the proof is complete if we show that $I \setminus J$ does not contain intervals of positive length.

Assume by contradiction that $[t_0, t_1]$ is contained in J with $t_0 < t_1$. Then $F(\xi(t))$ is constantly equal to some $A \in \mathbb{R}^2$ for $t \in [t_0, t_1]$. Let $\delta > 0$ and $\kappa : (-\delta, \delta) \rightarrow \Sigma$ be a C^1 curve such that $\kappa(0) = \gamma(t_0)$ and $\kappa'(0)$ is not proportional to $\gamma'(t_0)$. Write $\kappa(s) = (\xi_s, z_s)$ and notice that $F(\xi_s)$ converges to A as $s \rightarrow 0$. Consider for each $s \in (-\delta, \delta)$ the Legendre curve γ_s such that $\gamma_s(t_0) = \kappa(s)$. Then γ_s converges to γ and $F \circ \gamma_s$ converges to $F \circ \gamma$, uniformly on $[t_0, t_1]$, as $s \rightarrow 0$. Hence, for $\varepsilon > 0$ and $|s|$ small enough, the restriction of γ_s to $(t_0 + \varepsilon, t_1 - \varepsilon)$ cannot contain the lift of any arc of ϕ -circle of radius $1/|h|$. This implies that there exists a nonempty open region of Σ of the form $\{\gamma_s(t) : t \in (t_0 + \varepsilon, t_1 - \varepsilon), |s| < \bar{\delta}\}$ on which $F(\xi) = A$, contradicting the assumption that Σ has constant nonzero ϕ -curvature. $\square$

2.2.3 Foliation property with geodesics

In this section, we prove that the Legendre foliation of a surface (a z -graph) with constant ϕ -curvature consists of length minimizing curves in the ambient space (geodesics) relative to the norm $\phi^\dagger$ in $\mathbb{R}^2$ defined by

$$\phi^\dagger(\xi) = \phi^*(\xi^\perp), \quad \xi \in \mathbb{R}^2.$$

We consider a general norm ψ in $\mathbb{R}^2$ and for $T \geq 0$ we introduce the class of curves

$$\mathcal{A}_T = \left\{ \gamma = (\xi, z) \in \text{AC}([0, T]; \mathbb{H}^1) : \dot{z} = \omega(\xi, \dot{\xi}) \text{ and } \psi(\dot{\xi}) \leq 1 \text{ a.e.} \right\},$$

where ω is the symplectic form introduced in (2.2). In the sequel, we denote by $u = \dot{\xi} \in L^1([0, T]; \mathbb{R}^2)$ the *control* of γ . For given points $p_0, p_1 \in \mathbb{H}^1$ we consider the *optimal time problem*

$$\inf \left\{ T \geq 0 : \text{there exists } \gamma \in \mathcal{A}_T \text{ such that } \gamma(0) = p_0 \text{ and } \gamma(T) = p_1 \right\}. \quad (2.55)$$

We call a curve γ realizing the minimum in (2.55) a ψ -time minimizer between p_0 and p_1 . In this case, we call the pair (γ, u) with $u = \dot{\xi}$ an *optimal pair*. A ψ -time minimizer

is always parameterized by ψ -arclength, i.e., $\psi(u) = 1$. So, ψ -time minimizers are ψ -length minimizers parameterized by ψ -arclength.

An optimal pair (γ, u) satisfies the necessary conditions given by Pontryagin's Maximum Principle. As observed in [17], it necessarily is a *normal extremal*, whose definition is recalled below. The Hamiltonian associated with the optimal time problem (2.55) is $\mathfrak{H} : \mathbb{H}^1 \times \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\mathfrak{H}(p, \lambda, u) = \left(\lambda_x - \frac{y}{2} \lambda_z \right) u_1 + \left(\lambda_y + \frac{x}{2} \lambda_z \right) u_2 = \langle \lambda_\xi + \frac{1}{2} \lambda_z \xi^\perp, u \rangle,$$

where $\lambda = (\lambda_\xi, \lambda_z) \in \mathbb{R}^2 \times \mathbb{R}$.

Definition 2.2.13. *The pair $(\gamma, u) \in AC([0, T]; \mathbb{H}^1) \times L^1([0, T]; \mathbb{R}^2)$ is a normal extremal if there exists a nowhere vanishing curve $\lambda \in AC([0, T]; \mathbb{R}^3)$ such that (γ, λ) solves a.e. the Hamiltonian system*

$$\begin{cases} \dot{\gamma} = \mathfrak{H}_\lambda(\gamma, \lambda, u) \\ \dot{\lambda} = -\mathfrak{H}_p(\gamma, \lambda, u), \end{cases}$$

and for every $t \in [0, T]$ we have

$$1 = \mathfrak{H}(\gamma(t), \lambda(t), u(t)) = \max_{\psi(u) \leq 1} \mathfrak{H}(\gamma(t), \lambda(t), u). \quad (2.56)$$

In the coordinates $\gamma = (\xi, z)$ and $\lambda = (\lambda_\xi, \lambda_z)$, the Hamiltonian system reads

$$\begin{cases} \dot{\xi} = u, & \dot{\lambda}_\xi = \frac{1}{2} \lambda_z u^\perp, \\ \dot{z} = \omega(\xi, u), & \dot{\lambda}_z = 0. \end{cases} \quad (2.57)$$

Theorem 2.2.14. *Let ψ be of class C^1 and let $\gamma = (\xi, z) \in AC([0, T]; \mathbb{H}^1)$ be a horizontal curve. The following statements (i) and (ii) are equivalent:*

(i) γ is a local ψ -length minimizer parametrized by ψ -arclength;

(ii) the pair (γ, u) with $u = \dot{\xi}$ is a normal extremal.

Moreover, if ψ is of class C^2 then each of (i) and (ii) is equivalent to

(iii) γ is of class C^2 and parameterized by ψ -arclength, and there is $\lambda_0 \in \mathbb{R}$ such that

$$\mathcal{H}\psi(\dot{\xi})\ddot{\xi} = \lambda_0 \dot{\xi}^\perp, \quad (2.58)$$

where $\mathcal{H}\psi$ is the Hessian matrix of ψ .

Proof. The equivalence between (i) and (ii) is [17, Theorem 1].

Let us show that (ii) implies (iii). We set

$$\mathcal{M}(t) = \lambda_\xi(t) + \frac{1}{2} \lambda_z(t) \xi(t)^\perp, \quad t \in [0, T], \quad (2.59)$$

where $\lambda = (\lambda_\xi, \lambda_z)$ is the curve given by the definition of extremal. Then the maximality condition in (2.56) for normal extremals reads

$$1 = \langle \mathcal{M}(t), u(t) \rangle = \max_{\psi(u) \leq 1} \langle \mathcal{M}(t), u \rangle = \psi^*(\mathcal{M}(t)). \quad (2.60)$$

This is equivalent to the identity

$$\mathcal{M}(t) = \nabla \psi(u(t)). \quad (2.61)$$

When ψ is of class C^2 , from (2.61), (2.59), and (2.57) we obtain the differential equation for $u = \dot{\xi}$

$$\mathcal{H}\psi(u)\dot{u} = \dot{\mathcal{M}} = \dot{\lambda}_\xi + \frac{1}{2}\dot{\lambda}_z\xi + \frac{1}{2}\lambda_z u^\perp = \lambda_z u^\perp. \quad (2.62)$$

This is (2.58) with $\lambda_0 := \lambda_z$.

Now we show that (ii) is implied by (iii). Consistently with (2.61), we define $\mathcal{M}(t) = \nabla \psi(u(t))$, for $t \in [0, T]$. Then $\psi^*(\mathcal{M}) = 1$.

We define the curve $\lambda = (\lambda_\xi, \lambda_z)$ letting $\lambda_z = \lambda_0$ and $\lambda_\xi = \mathcal{M} - \frac{1}{2}\lambda_z \xi^\perp$. When ψ is of class C^2 , we obtain

$$\dot{\lambda}_\xi = \dot{\mathcal{M}} - \frac{1}{2}\lambda_z \dot{\xi}^\perp = \mathcal{H}\psi(\dot{\xi})\ddot{\xi} - \frac{1}{2}\lambda_z \dot{\xi}^\perp = \frac{1}{2}\lambda_z u^\perp.$$

Hence, all equations in (2.57) are satisfied, showing that the pair (γ, u) is a normal extremal. This proves that (iii) implies (ii). $\square$

Remark 2.2.15. When $\lambda_0 \neq 0$, equation (2.58) can be integrated in the following way. Using (2.62), the equation is equivalent to $\dot{\mathcal{M}} = \lambda_0 \dot{\xi}^\perp$, that implies $\mathcal{M} = \lambda_0(\xi^\perp - \xi_0^\perp)$ for some constant $\xi_0 \in \mathbb{R}^2$. So from (2.60) we deduce that $|\lambda_0|\psi^*(\xi^\perp - \xi_0^\perp) = 1$. If we choose $\psi = \phi^\dagger$ then we have $\psi^*(\xi^\perp) = \phi(\xi)$. So the previous equation becomes the equation for a ϕ -circle

$$\phi(\xi - \xi_0) = 1/|\lambda_0|.$$

Corollary 2.2.16. Let ϕ be a norm with dual norm ϕ^* of class piecewise C^2 and let $f \in C^2(D)$ be such that $\mathcal{C}(f) = \emptyset$. If $\text{gr}(f)$ has constant ϕ -curvature, then it is foliated by geodesics of $\mathbb{H}^1$ relative to the norm $\phi^\dagger$.

The proof is Corollary 2.2.12, combined with Remark 2.2.15 and Theorem 2.2.14.

2.2.4 Characteristic set of ϕ -critical surfaces

Here we study the characteristic set of ϕ -critical surfaces and then apply the results to ϕ -isoperimetric sets. For a C^2 surface $\Sigma \subset \mathbb{H}^1$, the characteristic set is

$$\mathcal{C}(\Sigma) = \{p \in \partial E : T_p \Sigma = \mathcal{D}(p)\}. \quad (2.63)$$

Note that any C^2 surface $\Sigma \subset \mathbb{H}^1$ is a z -graph around any of its characteristic points $p \in \mathcal{C}(\Sigma)$.

When Σ is oriented, the ϕ -curvature H_ϕ of Σ can be defined in a globally coherent way. When Σ is a z -graph at the point $p = (\xi, z) = (x, y, z) \in \Sigma$ we let $H_\phi(p) = \operatorname{div}(\mathcal{X}f)(\xi)$ where f is a z -graph function; when Σ is a x -graph, we let $H_\phi(p) = \mathcal{L}f(y, z)$, where now f is a x -graph function and $\mathcal{L}f$ is defined in (2.48); when Σ is a y -graph we proceed analogously.

We say that Σ is ϕ -critical if it is closed, has constant ϕ -curvature and it is ϕ -critical in a neighbourhood of any characteristic point.

Our goal is to prove Theorem 2.2.2. The proof is obtained combining Lemma 2.2.17 and Theorem 2.2.18 below.

In this section, ϕ and ϕ^* are two norms of class C^2 . We will omit to mention this assumptions in the various statements.

Qualitative structure of the characteristic set

Lemma 2.2.17. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 surface with constant ϕ -curvature. Then $\mathcal{C}(\Sigma)$ consists of isolated points and C^1 curves. Moreover, for every isolated point $p_0 = (\xi_0, z_0) \in \mathcal{C}(\Sigma)$ and every f such that $p_0 \in \operatorname{gr}(f) \subset \Sigma$, we have $\operatorname{rank}(JF(\xi_0)) = 2$, where F is the projected horizontal gradient introduced in (2.40).*

Proof. We let $\mathcal{C}(f)$ be as in (2.41). For any $\xi_0 \in \mathcal{C}(f)$, the Jacobian matrix $JF(\xi_0)$ has rank 1 or 2. Indeed, an explicit calculation shows that $JF(\xi_0) \neq 0$ for all $\xi_0 \in D$. If $\operatorname{rank}(JF(\xi_0)) = 2$ then ξ_0 is an isolated point of $\mathcal{C}(f)$.

We study the case $\operatorname{rank}(JF(\xi_0)) = 1$. We claim that in this case $\mathcal{C}(f)$ is a curve of class C^1 in a neighbourhood of ξ_0 . The argument that we use here is inspired by [27].

For $b \in \mathbb{R}^2$ we define $F_b : D \rightarrow \mathbb{R}$, $F_b = \langle F, b \rangle$. When $b \notin \ker(JF(\xi_0))$, the equation $F_b = 0$ defines a C^1 curve Γ_b near and through ξ_0 . We have $\mathcal{C}(f) \subset \Gamma_b$. Since $\nabla F_b(\xi_0)$ is in the image of $JF(\xi_0)$, which is a line independent of b , the normal direction to Γ_b at ξ_0 does not depend on b . We choose one of the two unit normals and we call it $N \in \mathbb{R}^2$.

We claim that there exists $a, b \in \mathbb{S}^1$, where $\mathbb{S}^1 = \{w \in \mathbb{R}^2 : |w| = 1\}$, such that

$$a \notin \{b, -b\}, \quad a, b \notin \ker(JF(\xi_0)), \quad |\langle \nabla \phi^*(b^\perp), N \rangle| \neq |\langle \nabla \phi^*(a^\perp), N \rangle|. \quad (2.64)$$

To prove the claim pick $b \in \mathbb{S}^1 \setminus \ker(JF(\xi_0))$ (this is possible since $\operatorname{rank}(JF(\xi_0)) \neq 0$), and define the set

$$K_b := \{v \in C_\phi : |\langle v, N \rangle| = |\langle \nabla \phi^*(b^\perp), N \rangle|\}.$$

Since the map $\nabla \phi^* : \mathbb{S}^1 \rightarrow C_\phi$ is continuous, the set $(\nabla \phi^*)^{-1}(K_b) \subset \mathbb{S}^1$ is closed in $\mathbb{S}^1$. Moreover, $\nabla \phi^* : \mathbb{S}^1 \rightarrow C_\phi$ is surjective, since for every $w \in C_\phi$ and every v in the subgradient of ϕ^* at w , we have $w = \nabla \phi^*(v)$ (see, e.g., [67, Theorem 23.5]). As a consequence, $(\nabla \phi^*)^{-1}(K_b) \neq \mathbb{S}^1$, since otherwise we would have $K_b = C_\phi$, which is impossible. The set

$$\Upsilon = \ker(JF(\xi_0))^\perp \cup (\nabla \phi^*)^{-1}(K_b) \cup \{b^\perp, -b^\perp\}$$

is therefore a proper closed subset of $\mathbb{S}^1$, and the claim follows by choosing $a^\perp \in \mathbb{S}^1 \setminus \Upsilon$.

Fix $a, b \in \mathbb{S}^1$ such that (2.64) holds and, for $\alpha \in (0, 1)$, let $C_\alpha := \{v \in \mathbb{R}^2 : |\langle N, v \rangle| < |v| \sin \alpha\}$ be the cone centered at ξ_0 with axis parallel to $N^\perp$ and aperture 2α . Since

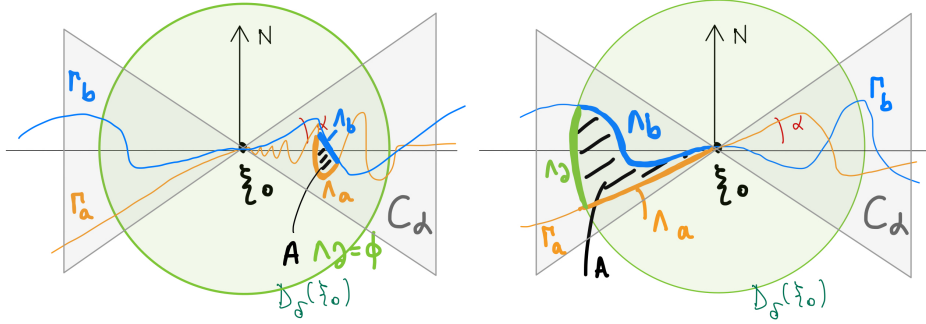


Figure 2.2: The cone C_α and the region A . On the left, A does not touch $\partial\{|\xi - \xi_0| < \delta\}$, while it does on the right. We can always restrict our attention to the case on the left when ξ_0 is a density point of $\mathcal{C}(f)$. Images by V. Franceschi.

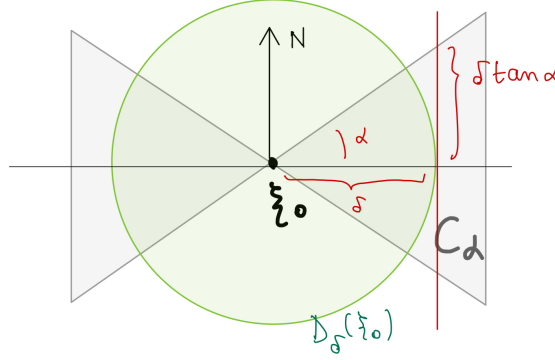


Figure 2.3: Proportions in $C_{\alpha,\delta}$.

Γ_a, Γ_b are C^1 , there exists $\delta \in (0, 1)$ such that

$$\{\xi \in \Gamma_a \cup \Gamma_b : |\xi - \xi_0| < \delta\} \subset C_{\alpha,\delta}, \quad (2.65)$$

where we set $C_{\alpha,\delta} = \{\xi \in C_\alpha : |\xi - \xi_0| < \delta\}$.

Let us assume by contradiction that $\mathcal{C}(f)$ is not a C^1 curve near ξ_0 . Then there exists a nonempty connected component A of $C_{\alpha,\delta} \setminus (\Gamma_a \cup \Gamma_b)$ such that, letting

$$\Lambda_a = \Gamma_a \cap \partial A, \quad \Lambda_b := \Gamma_b \cap \partial A, \quad \Lambda_\partial := \partial\{|\xi - \xi_0| < \delta\} \cap \partial A,$$

we have

$$\Lambda_a \neq \emptyset, \quad \Lambda_b \neq \emptyset, \quad \partial A = \Lambda_a \cup \Lambda_b \cup \Lambda_\partial, \quad \#(\Lambda_a \cap \Lambda_b) \leq 2. \quad (2.66)$$

See Figure 2.2. Notice that A , Λ_a , Λ_b , and Λ_∂ depend on δ . By (2.65) (see also Figure 2.3), we have

$$\mathcal{L}^2(A) \leq \delta^2 \tan(\alpha). \quad (2.67)$$

By (2.66) and since $\mathcal{C}(f) \subset \Lambda_a \cap \Lambda_b$, for $\xi \in \text{int}(\Lambda_a) \cup \text{int}(\Lambda_b)$ we have $F(\xi) \neq 0$, where we endow Λ_a and Λ_b with their relative topologies. We deduce that $F(\xi) = c_a(\xi)a^\perp$ with $c_a(\xi) \neq 0$ for $\xi \in \text{int}(\Lambda_a)$ and $F(\xi) = c_b(\xi)b^\perp$ with $c_b(\xi) \neq 0$ for $\xi \in \text{int}(\Lambda_b)$. Using

the fact that $\nabla\phi^*$ is positively 0-homogeneous it then follows that the vector field $\mathcal{N} : D \setminus \mathcal{C}(f) \rightarrow \mathbb{R}^2$, $\mathcal{N}(\xi) = \nabla\phi^*(F(\xi))$, is constant along Λ_a and Λ_b . Namely,

$$\begin{aligned}\mathcal{N}(\xi) &= \text{sgn}(c_a)\nabla\phi^*(a^\perp) =: \mathcal{N}_a, & \xi \in \text{int}(\Lambda_a), \\ \mathcal{N}(\xi) &= \text{sgn}(c_b)\nabla\phi^*(b^\perp) =: \mathcal{N}_b, & \xi \in \text{int}(\Lambda_b).\end{aligned}$$

By assumption, and since $\phi^* \in C^2$, there exists a constant $h \in \mathbb{R}$ such that

$$\text{div}(\mathcal{N}(\xi)) = h, \quad x \in D \setminus \mathcal{C}(f),$$

in the strong sense. Then by the divergence theorem, and since $A \cap \mathcal{C}(f) = \emptyset$, we have

$$h\mathcal{L}^2(A) = \int_A \text{div}(\mathcal{N})dxdy = \int_{\Lambda_a} \langle \mathcal{N}_a, N_a \rangle d\mathcal{H}^1 + \int_{\Lambda_b} \langle \mathcal{N}_b, N_b \rangle d\mathcal{H}^1 + \int_{\Lambda_\partial} \langle \mathcal{N}, N_\partial \rangle d\mathcal{H}^1,$$

where N_a , N_b , and N_∂ are, respectively, the normals to Λ_a , Λ_b , and Λ_∂ , exterior with respect to A . For $\alpha \rightarrow 0^+$ we have

$$\begin{aligned}\int_{\Lambda_a} N_a \mathcal{H}^1 &= \delta(-N + o(1)), \\ \int_{\Lambda_b} N_b \mathcal{H}^1 &= \delta(N + o(1)), \\ \left| \int_{\Lambda_\partial} \langle \mathcal{N}, N_\partial \rangle d\mathcal{H}^1 \right| &\leq C\delta\alpha,\end{aligned}$$

where $o(1) \rightarrow 0$ as $\alpha \rightarrow 0^+$ and $C > 0$ denotes a suitable constant. Now from (2.67) we deduce that

$$|\delta \tan(\alpha)h| \geq |\langle \mathcal{N}_b - \mathcal{N}_a, N \rangle + o(1)| - C\alpha,$$

that implies $\langle \mathcal{N}_b - \mathcal{N}_a, N \rangle = 0$ in contradiction with (2.64).

This proves that $\mathcal{C}(f)$ is a C^1 curve around any point ξ_0 with $\text{rank}(JF(\xi_0)) = 1$. $\square$

Characteristic curves in ϕ -critical surfaces

Given a surface $\Sigma \subset \mathbb{H}^1$, we call a *characteristic curve on Σ* any (nontrivial) curve $\Gamma \subset \mathcal{C}(\Sigma)$. In this section we prove the following result.

Theorem 2.2.18. *Let Σ be a complete and oriented surface of class C^2 . If Σ is ϕ -critical with non-vanishing ϕ -curvature $h \neq 0$ then any characteristic curve on Σ is either a horizontal line or the horizontal lift of a simple closed curve.*

For a characteristic curve Γ in Σ we denote its coordinates by $\Gamma = (\Xi, \zeta) \in \mathbb{R}^2 \times \mathbb{R}$. For any $p_0 = (\xi_0, z_0)$ on Γ , let $\delta > 0$ be small enough to have

$$\{\xi \in \mathbb{R}^2 : |\xi - \xi_0| < \delta\} \setminus \text{supp}(\Xi) = B^+ \cup B^-, \quad (2.68)$$

where $B^+, B^- \subset \mathbb{R}^2$ are disjoint open connected sets. The ϕ -normal $\mathcal{N}$ in (2.50) is well-defined in $B^+ \cup B^-$.

Lemma 2.2.19. *Let Σ be a C^2 surface with constant ϕ -curvature. With the above notation, the following limits exist*

$$\mathcal{N}^\pm(\xi_0) := \lim_{B^\pm \ni \xi \rightarrow \xi_0} \mathcal{N}(\xi) \quad (2.69)$$

and satisfy $\mathcal{N}^+(\xi_0) = -\mathcal{N}^-(\xi_0)$.

Proof. This is a straightforward corollary of [27, Proposition 3.5]. $\square$

Proposition 2.2.20. *Let Σ be a ϕ -critical surface of class C^2 and let $\Gamma = (\Xi, \zeta)$ be a characteristic curve on Σ . Then for every $p_0 = (\xi_0, z_0)$ in Γ we have*

$$\mathcal{N}^\pm(\xi_0) \in T_{\xi_0} \Xi, \quad (2.70)$$

where $\mathcal{N}^\pm$ is defined as in Lemma 2.2.19.

Proof. Let $f \in C^2(D)$ be a graph function for Σ with $\xi_0 \in D \subset \mathbb{R}^2$. Without loss of generality we assume $D = \{|\xi - \xi_0| < \delta\}$ and let $D^\pm := D \cap B^\pm$, where $B^\pm$ are as in (2.68). Let $h \in \mathbb{R}$ be the ϕ -curvature of Σ . Since Σ is ϕ -critical, for any $\varphi \in C_c^\infty(D)$ we have

$$\int_D \langle \mathcal{X}f, \nabla \varphi \rangle d\xi = - \int_D h \varphi d\xi$$

and $\operatorname{div}(\mathcal{X}f) = h$ pointwise in $D^+ \cup D^-$. Then, denoting by N_Ξ the normal to Ξ pointing towards D^- , by the divergence theorem we have

$$\begin{aligned} \int_D h \varphi d\xi &= \int_{D^+} \operatorname{div}(\mathcal{X}f) \varphi d\xi + \int_{D^-} \operatorname{div}(\mathcal{X}f) \varphi d\xi \\ &= - \int_{D^+ \cup D^-} \langle \mathcal{X}f, \nabla \varphi \rangle d\xi + \int_\Xi \varphi \langle \mathcal{N}^+, N_\Xi \rangle d\mathcal{H}^1 - \int_\Xi \varphi \langle \mathcal{N}^-, N_\Xi \rangle d\mathcal{H}^1 \\ &= \int_D h \varphi d\xi + \int_\Xi \varphi \langle \mathcal{N}^+ - \mathcal{N}^-, N_\Xi \rangle d\mathcal{H}^1. \end{aligned}$$

By Lemma 2.2.19, this implies that

$$\int_\Xi \varphi \langle \mathcal{N}^+, N_\Xi \rangle d\mathcal{H}^1 = 0$$

and since φ is arbitrary, this yields the claim. $\square$

Remark 2.2.21. *Under the assumptions of the previous proposition, the characteristic curves $\Gamma = (\Xi, \zeta)$ of ∂E are of class C^2 . This can be proved exactly as in Proposition 4.20 of [66] using condition (2.70). In particular, Ξ is of class C^2 .*

Parametrization of constant ϕ -curvature surfaces around characteristic curves

In this section, we study a ϕ -critical surface Σ of class C^2 having constant ϕ -curvature $h \neq 0$ near a characteristic curve. Without loss of generality we assume $h > 0$.

We assume ϕ to be normalized in such a way that $\phi(1, 0) = 1$ and we fix a parametrization $\mu : [0, M] \rightarrow \mathbb{R}^2$ of C_ϕ such that $\phi^\dagger(\dot{\mu}) = 1$, $\mu([0, M]) = C_\phi$, with

initial and end-point $\mu(0) = \mu(M)$. We choose the clockwise orientation and we extend μ to the whole $\mathbb{R}$ by M -periodicity. We have $\mu \in C^2(\mathbb{R}; \mathbb{R}^2)$ and

$$\mu(\tau) = \nabla \phi^*(\dot{\mu}(\tau)^\perp), \quad \text{for all } \tau \in \mathbb{R}. \quad (2.71)$$

In fact, letting $\mathcal{N}(t) = \nabla \phi^*(\dot{\mu}(t)^\perp)$, we have $\dot{\mathcal{N}} = \dot{\mu}$ as in (2.51). Equation (2.71) then follows by integration using the fact that 0 is the center of C_ϕ .

Let $\Gamma = (\Xi, \zeta) \in C^2(I; \Sigma)$ be a characteristic curve parameterized in such a way that

$$\phi(\dot{\Xi}) = 1 \quad \text{on } I. \quad (2.72)$$

Locally, Γ disconnects Σ and there are no other characteristic points of Σ close to Γ , by Lemma 2.2.17.

According to Corollary 2.2.11, $\Sigma \setminus \mathcal{C}(\Sigma)$ admits near Γ a Legendre foliation made of horizontal lifts of ϕ -circles of radius $1/h$, followed in the clockwise sense. Hence, given a point $(\xi_0, z_0) \in \Sigma \setminus \mathcal{C}(\Sigma)$ near Γ , there exists $c \in \mathbb{R}^2$ and $\tau \in [0, M]$ such that the horizontal lift of

$$\xi(s) = c + h^{-1}\mu(\tau + hs)$$

passing through (ξ_0, z_0) at $s = 0$ stays in Σ until it meets a characteristic point. Here, c is the center of the ϕ -circle. Notice that $\nabla \phi^*(\dot{\xi}(s)^\perp) = \mathcal{N}(\xi(s))$, so that, by Lemma 2.2.19 and (2.70), $\nabla \phi^*(\dot{\xi}(0)^\perp)$ converges to a vector collinear to $\dot{\Xi}(t)$ as ξ_0 approaches $\Xi(t)$ for some $t \in I$. By (2.42) and (2.72), $\nabla \phi^*(\dot{\xi}(0)^\perp)$ converges either to $\dot{\Xi}(t)$ or to $-\dot{\Xi}(t)$ as ξ_0 approaches $\Xi(t)$. Since Ξ locally disconnects the plane, we can fix a side from where ξ_0 approaches Ξ and, up to reversing the parameterization of Γ , we can assume that $\nabla \phi^*(\dot{\xi}(0)^\perp)$ converges to $\dot{\Xi}(t)$ as ξ_0 converges to $\Xi(t)$. Thanks to (2.71) and since $\dot{\xi}(0) = \dot{\mu}(\tau)$, we deduce that $\mu(\tau) = \nabla \phi^*(\dot{\xi}(0)^\perp)$ converges to $\dot{\Xi}(t)$ as $\xi_0 \rightarrow \Xi(t)$. In particular, the limit direction of $\xi(0)$ as $\xi_0 \rightarrow \Xi(t)$ is transversal to Ξ .

By local compactness of the set of ϕ -circles with radius $1/h$, the horizontal lift passing through $\Gamma(t)$ at $s = 0$ of a curve $c + h^{-1}\mu(\tau + hs)$ with $\mu(\tau) = \dot{\Xi}(t)$ is a Legendre curve contained in Σ , for s either in a positive or a negative neighbourhood of 0. To fix the notations, we assume that s is in a positive neighbourhood of 0, the computations being equivalent in the other case. Moreover, there is no other Legendre curve having $\Gamma(t)$ in its closure and whose projection on the xy -plane stays in the chosen side of Ξ , since $\tau \in [0, M)$ and $c \in \mathbb{R}^2$ are uniquely determined by

$$\mu(\tau) = \dot{\Xi}(t), \quad c = \Xi(t) - h^{-1}\mu(\tau) = \Xi(t) - h^{-1}\dot{\Xi}(t).$$

It is then possible to parameterize locally near Γ one of the two connected components of $\Sigma \setminus \Gamma$ by Legendre curves using the function

$$(t, s) \mapsto \gamma(t, s) = (\xi(t, s), z(t, s)) \quad (2.73)$$

where

$$\xi(t, s) = h^{-1}\mu(\tau(t) + hs) + \Xi(t) - h^{-1}\dot{\Xi}(t), \quad t \in I, \quad s > 0, \quad (2.74)$$

with τ uniquely defined via the equation

$$\mu(\tau(t)) = \dot{\Xi}(t), \quad t \in I, \quad (2.75)$$

and z defined by

$$z(t, s) = \zeta(t) + \int_0^s \omega(\xi(t, \sigma), \xi_s(t, \sigma)) d\sigma. \quad (2.76)$$

As discussed above, we have

$$\nabla \phi^*(\xi_s(t, 0)^\perp) = \dot{\Xi}(t), \quad (2.77)$$

$$\phi^\dagger(\xi_s) = 1. \quad (2.78)$$

For $t \in I$, we define the *characteristic time* $s(t)$ as the first positive time $s > 0$ such $\gamma(t, s(t)) \in \mathcal{C}(\Sigma)$. We will prove later that such a $s(t)$ exists. Finally, we let $S := \{(t, s) : t \in I, 0 \leq s \leq s(t)\}$ and we consider the surface $\gamma(S) \subset \Sigma$.

Lemma 2.2.22. *We have $\gamma \in C^1(S; \Sigma)$ with $\gamma(\cdot, 0) = \Gamma$. Moreover, the second order derivatives $\gamma_{ss}, \gamma_{ts}, \gamma_{st}$ are well-defined and*

$$\gamma_{ts} = \gamma_{st}. \quad (2.79)$$

Proof. By (2.74) and (2.76), we see that γ_{ss} exists and that $\xi_{ts} = \xi_{st}$. Moreover,

$$\begin{aligned} z_{st} &= \omega(\xi_t(t, \cdot), \xi_s(t, \cdot)) + \omega(\xi(t, \cdot), \xi_{st}(t, \cdot)) \\ &= \omega(\xi_t(t, \cdot), \xi_s(t, \cdot)) + \omega(\xi(t, \cdot), \xi_{ts}(t, \cdot)) = z_{ts}. \end{aligned}$$

□

On the surface $\gamma(S)$ we consider the vector field

$$V(t, s) := \gamma_t(t, s) = (\xi_t(t, s), z_t(t, s)) \in \mathbb{R}^3. \quad (2.80)$$

It plays the role of the Jacobi vector field V in [66, Lemma 6.2]. The characteristic time $s(t)$ is precisely the first positive time such that $\langle V(s(t), t), Z \rangle_{\mathcal{D}} = 0$. Here, with a slight abuse of notation, $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ denotes the scalar product that makes X, Y, Z orthonormal. The following computation is crucial in what follows. We recall that we are assuming the ϕ -curvature to be a constant $h \neq 0$.

Lemma 2.2.23. *We have the identity*

$$\langle V(t, s), Z \rangle_{\mathcal{D}} = 2[h^{-2}\omega(\ddot{\Xi}, \dot{\Xi}) + \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, h^{-1}\mu(\tau + hs))].$$

Proof. First notice that

$$\langle V, Z \rangle_{\mathcal{D}} = z_t + \omega(\xi_t, \xi), \quad (2.81)$$

where

$$z_t(t, s) = z_t(t, 0) + \int_0^s \omega(\xi_t(t, \sigma), \xi_s(t, \sigma)) d\sigma + \int_0^s \omega(\xi(t, \sigma), \xi_{st}(t, \sigma)) d\sigma.$$

Using (2.74), (2.75), and the skew-symmetry of ω , the above implies

$$\begin{aligned}
z_t(\cdot, s) &= \omega(\Xi, \dot{\Xi}) + \int_0^s \omega(\dot{\Xi} - h^{-1}\ddot{\Xi} + h^{-1}\dot{\tau}\dot{\mu}(\tau + h\sigma), \dot{\mu}(\tau + h\sigma)) d\sigma \\
&\quad + \int_0^s \omega(\Xi - h^{-1}\dot{\Xi} + h^{-1}\mu(\tau + h\sigma), \dot{\tau}\dot{\mu}(\tau + h\sigma)) d\sigma \\
&= \omega(\Xi, \dot{\Xi}) + h^{-1}\omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, \mu(\tau + hs) - \mu(\tau)) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot{\Xi}, \dot{\tau}\dot{\mu}(\tau + hs) - \dot{\tau}\dot{\mu}(\tau)) \\
&\quad + h^{-2}\omega(\mu(\tau + hs), \dot{\tau}\dot{\mu}(\tau + hs)) - h^{-2}\omega(\mu(\tau), \dot{\tau}\dot{\mu}(\tau)) \\
&= \omega(\Xi, \dot{\Xi}) + h^{-1}\omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, \mu(\tau + hs)) - h^{-1}\omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, \dot{\Xi}) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot{\Xi}, \dot{\tau}\dot{\mu}(\tau + hs)) - h^{-1}\omega(\Xi - h^{-1}\dot{\Xi}, \dot{\Xi}) \\
&\quad + h^{-2}\omega(\mu(\tau + hs), \dot{\tau}\dot{\mu}(\tau + hs)) - h^{-2}\omega(\dot{\Xi}, \dot{\Xi}) \\
&= \omega(\Xi, \dot{\Xi}) - h^{-1}\omega(\Xi, \ddot{\Xi}) + h^{-2}\omega(\ddot{\Xi}, \dot{\Xi}) + \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, h^{-1}\mu(\tau + hs)) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot{\Xi} + h^{-1}\mu(\tau + hs), \dot{\tau}\dot{\mu}(\tau + hs)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\omega(\xi_t, \xi) &= \omega(\dot{\Xi} - h^{-1}\ddot{\Xi} + h^{-1}\dot{\tau}\dot{\mu}(\tau + hs), \Xi - h^{-1}\dot{\Xi} + h^{-1}\mu(\tau + hs)) \\
&= h^{-1}\omega(\dot{\tau}\dot{\mu}(\tau + hs), \Xi - h^{-1}\dot{\Xi} + h^{-1}\mu(\tau + hs)) + \omega(\dot{\Xi}, \Xi) \\
&\quad - h^{-1}\omega(\ddot{\Xi}, \Xi) + \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, h^{-1}\mu(\tau + hs)) + h^{-2}\omega(\ddot{\Xi}, \dot{\Xi}).
\end{aligned}$$

Summing up, we obtain the claim. $\square$

We show next that for every $t \in I$, the Legendre curve $s \mapsto \gamma(t, s)$ meets a characteristic point before that $\xi(t, s)$ comes back to the point $\xi(t, 0) = \Xi(t)$, i.e., $hs(t) < M$.

Lemma 2.2.24. *For any $t \in I$, there exists $s(t) \in (0, M/h)$ such that $\langle V(t, s(t)), Z \rangle_{\mathcal{D}} = 0$.*

Proof. For fixed t , consider the function $\vartheta : [0, M] \rightarrow \mathbb{R}$, defined by

$$\vartheta(s) = \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, h^{-1}\mu(\tau + hs)).$$

By Lemma 2.2.23, we have that $\langle V(t, s), Z \rangle_{\mathcal{D}} = 0$ if and only if $\vartheta(s) = b$ with $b := h^{-2}\omega(\ddot{\Xi}, \dot{\Xi})$. The equation $\vartheta(s) = b$ is certainly satisfied for $hs = nM$, $n \in \mathbb{N}$. This follows by the M -periodicity of μ and the fact that $V(t, 0) = \dot{\Gamma}(t)$ is horizontal.

It is enough to consider the case $b \geq 0$, the case $b < 0$ being analogous. By (2.75) we have

$$\dot{\vartheta}(0) = \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, \dot{\mu}(\tau)) = \omega(\mu(\tau), \dot{\mu}(\tau)).$$

By the fact that C_ϕ is a convex curve around 0, it follows that $\dot{\vartheta}(0) \neq 0$.

If $\dot{\vartheta}(0) > 0$ there exists $s^* \in (0, M/(2h))$ such that $\vartheta(s^*) > \vartheta(0) = b$. In this case, by symmetry of C_ϕ we have $\mu(\tau + h(s^* + M/(2h))) = -\mu(\tau + hs^*)$, thus implying $\vartheta(s^* + M/(2h)) = -\vartheta(s^*) < -b \leq 0$. By continuity of ϑ , we deduce the existence of $\bar{s} \in (0, M/h)$ satisfying $\vartheta(\bar{s}) = b$. We argue in the same way in the case $\dot{\vartheta}(0) < 0$. $\square$

We now determine a quantity that remains constant along the Legendre curves $s \mapsto \gamma(t, s)$.

Proposition 2.2.25. *For any $t \in I$ and for all $s \in [0, s(t)]$ we have*

$$\langle V(t, s), Z \rangle_{\mathcal{D}} + h \langle \nabla \phi^\dagger(\xi_s(t, s)), \xi_t(t, s) \rangle = 0. \quad (2.82)$$

Proof. By (2.81), (2.76) and (2.79), we have

$$\begin{aligned} \frac{\partial}{\partial s} \langle V, Z \rangle_{\mathcal{D}} &= z_{ts} + \omega(\xi_{ts}, \xi) + \omega(\xi_t, \xi_s) = \frac{\partial}{\partial t} \omega(\xi, \xi_s) + \omega(\xi_{st}, \xi) + \omega(\xi_t, \xi_s) \\ &= \omega(\xi_t, \xi_s) + \omega(\xi, \xi_{st}) + \omega(\xi_{st}, \xi) + \omega(\xi_t, \xi_s) = 2\omega(\xi_t, \xi_s). \end{aligned} \quad (2.83)$$

We claim that

$$h \frac{\partial}{\partial s} (\langle \nabla \phi^\dagger(\xi_s), \xi_t \rangle) = 2\omega(\xi_s, \xi_t). \quad (2.84)$$

Indeed, by Theorem 2.2.14 and Remark 2.2.15, we have

$$\frac{\partial}{\partial s} \nabla \phi^\dagger(\xi_s) = \mathcal{H} \phi^\dagger(\xi_s) \xi_{ss} = \frac{1}{h} \xi_s^\perp, \quad (2.85)$$

and therefore

$$\frac{\partial}{\partial s} \langle \nabla \phi^\dagger(\xi_s(t, s)), \xi_t(t, s) \rangle = \frac{1}{h} \langle \xi_s^\perp, \xi_t \rangle + \langle \nabla \phi^\dagger(\xi_s), \xi_{st} \rangle$$

On differentiating (2.78) w.r.t. t we see that $\langle \nabla \phi^\dagger(\xi_s), \xi_{st} \rangle = 0$. This is (2.84).

Summing up (2.83) and (2.84), we deduce that the function $\Lambda_t(s) = \langle V(t, s), Z \rangle_{\mathcal{D}} + h \langle \nabla \phi^\dagger(\xi_s(t, s)), \xi_t(t, s) \rangle$ is constant. To conclude the proof it is enough to check that $\Lambda_t(0) = 0$. On the one hand, we have $\langle V(t, 0), Z \rangle_{\mathcal{D}} = \langle \dot{\Gamma}(t), Z \rangle_{\mathcal{D}} = 0$, since Γ is horizontal. On the other hand, since $\nabla \phi^\dagger(v) = -\nabla \phi^*(v^\perp)^\perp$ for any $v \neq 0$, using (2.77) we finally obtain

$$\langle \nabla \phi^\dagger(\xi_s(t, 0)), \xi_t(t, 0) \rangle = -\langle \nabla \phi^*(\xi_s(t, 0)^\perp)^\perp, \dot{\Xi}(t) \rangle = 0. \quad \square$$

Since the set $\Gamma_1 := \{\gamma(t, s(t)) : t \in I\}$ is made of characteristic points, it is either an isolated point or a nontrivial characteristic curve (Lemma 2.2.17). We will see in the proof of Theorem 2.2.1, contained in Section 2.2.5, that if Γ_1 were an isolated characteristic point, then the same would be true for Γ . We stress that the argument leading to such a conclusion does not rely on the characterization of Γ provided in this section. We then have that $\Gamma_1 := \{\gamma(t, s(t)) : t \in I\}$ is a nontrivial characteristic curve.

Proposition 2.2.26. *The function $t \mapsto s(t)$ is constant.*

Proof. Let $t \in I$. Since $\langle V(t, s(t)), Z \rangle_{\mathcal{D}} = 0$, the point $\gamma(t, s(t))$ is characteristic for Σ . Then, by Lemma 2.2.17 and Remark 2.2.21, Γ_1 is a C^2 characteristic curve. By the implicit function theorem, the function $t \mapsto s(t)$ is C^1 -smooth and for $t \in I$ we have

$$\dot{\Gamma}_1(t) = V(t, s(t)) + \dot{s}(t) \gamma_s(t, s(t)).$$

The curve Ξ_1 obtained by projecting Γ_1 on the xy -plane then satisfies

$$\dot{\Xi}_1(t) = \xi_t(t, s(t)) + \dot{s}(t) \xi_s(t, s(t)).$$

Since $\gamma(t, s(t)) \in \mathcal{C}(\Sigma)$, by Proposition 2.2.20, and using the fact that $\nabla\phi^\dagger(v) = -\nabla\phi^*(v^\perp)^\perp$ for any $v \neq 0$ we have

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \dot{\Xi}_1(t) \rangle = -\langle \nabla\phi^*(\xi_s(t, s(t))^\perp)^\perp, \dot{\Xi}_1(t) \rangle = 0.$$

Therefore we obtain

$$0 = \langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_t(t, s(t)) \rangle + \dot{s}(t) \langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_s(t, s(t)) \rangle, \quad (2.86)$$

where, by Proposition 2.2.25,

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_t(t, s(t)) \rangle = 0,$$

and moreover, by (2.78),

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_s(t, s(t)) \rangle = \phi^\dagger(\xi_s(t, s(t))) = 1.$$

Equation (2.86) thus implies $\dot{s} = 0$, which concludes the proof. $\square$

We are now ready to prove Theorem 2.2.18.

Proof of Theorem 2.2.18. Without loss of generality we assume $h > 0$. By Remark 2.2.21, Γ is of class C^2 and we denote by I an interval of parametrization of $\Gamma = (\Xi, \zeta)$ satisfying (2.72). We consider the parametrization γ given by Lemma 2.2.22. By Proposition 2.2.26 the characteristic time $s(t)$ is constant on I and we let $s(t) = \bar{s} \in \mathbb{R}$. Since $\langle V(t, \bar{s}), Z \rangle_{\mathcal{D}} = 0$, by Lemma 2.2.23 we thus have

$$h^{-2}\omega(\ddot{\Xi}(t), \dot{\Xi}(t)) + \omega(\dot{\Xi}(t) - h^{-1}\ddot{\Xi}(t), h^{-1}\mu(\tau(t) + h\bar{s})) = 0.$$

Using (2.75), the last equation reads

$$\dot{\tau}\omega(\dot{\mu}(\tau), \mu(\tau) - \mu(\tau + h\bar{s})) = h\omega(\mu(\tau + h\bar{s}), \mu(\tau)). \quad (2.87)$$

If the right-hand side is 0 at some $t \in I$, then $\mu(\tau(t))$ and $\mu(\tau(t) + h\bar{s})$ are parallel by definition of ω (cf. (2.2)). Since $h\bar{s} \in (0, M)$ by Lemma 2.2.24, the only possible choice is $h\bar{s} = M/2$. Plugging such choice into the left-hand side and using the fact that $\mu(\tau + M/2) = -\mu(\tau)$, we obtain

$$2\dot{\tau}\omega(\dot{\mu}(\tau), \mu(\tau)) = 0 \quad \text{on } I.$$

This implies that $\dot{\tau} = 0$ on I and therefore that τ is constant on I . By (2.75) we deduce that $\dot{\Xi}$ is constant on I implying that Ξ is a straight line.

We are now left to consider the case $h\bar{s} \in (0, M)$, $h\bar{s} \neq M/2$, so that $\omega(\mu(\tau(t) + h\bar{s}), \mu(\tau(t))) \neq 0$ for every $t \in I$. Equation (2.87) reads

$$\dot{\tau} = f(\tau) \quad \text{with} \quad f(\tau) := \frac{h\omega(\mu(\tau + h\bar{s}), \mu(\tau))}{\omega(\dot{\mu}(\tau), \mu(\tau) - \mu(\tau + h\bar{s}))}.$$

For the sake of simplicity, assume $0 \in I$. Notice that f is $M/2$ -periodic and of class C^1 as a function of τ . Hence, given $\tau_0 \in \mathbb{R}$ satisfying $\mu(\tau_0) = \dot{\Xi}(0)$, there is a unique maximal solution τ to the differential equation with the initial condition $\tau(0) = \tau_0$.

Since $h\bar{s} \in (0, M)$, $h\bar{s} \neq M/2$, we have $f(\tau) \neq 0$, yielding that $\dot{\tau}$ has constant sign. To fix the ideas, assume that $\text{sign}(\dot{\tau}) = 1$. Then, there exists $T_0 > 0$ such that $\tau(T_0) = \tau_0 + M/2$. We claim that

$$\tau(t + T_0) = \tau(t) + \frac{M}{2} \quad \text{for all } t \in \mathbb{R}. \quad (2.88)$$

This follows from the fact that $\tau_1(t) := \tau(T_0 + t)$ and $\tau_2(t) := \tau(t) + M/2$ for $t \in \mathbb{R}$ solve the same Cauchy problem $\dot{\tau}(t) = f(\tau)$, $\tau(0) = \tau_0 + M/2$. Then, by (2.88), M -periodicity of μ , and (2.75), we have for every $t \in \mathbb{R}$

$$\dot{\Xi}(t + 2T_0) = \mu(\tau(t + 2T_0)) = \mu(\tau(t) + M) = \mu(\tau(t)) = \dot{\Xi}(t),$$

i.e., $\dot{\Xi}$ is $2T_0$ -periodic. This implies that Ξ is also $2T_0$ -periodic. Indeed, for $t \in \mathbb{R}$ we have

$$\begin{aligned} \Xi(t + 2T_0) - \Xi(t) &= \int_t^{t+2T_0} \dot{\Xi}(\sigma) d\sigma = \int_t^{t+2T_0} \mu(\tau(\sigma)) d\sigma \\ &= \int_t^{t+T_0} \mu(\tau(\sigma)) d\sigma + \int_t^{t+T_0} \mu(\tau(\sigma + T_0)) d\sigma \\ &= \int_t^{t+T_0} \mu(\tau(\sigma)) d\sigma - \int_t^{t+T_0} \mu(\tau(\sigma)) d\sigma = 0, \end{aligned}$$

where we have used again the symmetry of C_ϕ and (2.88).

We are left to show that $\Xi(\bar{\sigma}) \neq \Xi(\bar{t})$ for any $0 \leq \bar{\sigma} < \bar{t} < 2T_0$. Assume that $\Xi(\bar{\sigma}) = \Xi(\bar{t})$ for some $0 \leq \bar{\sigma} < \bar{t} \leq 2T_0$. Then we have $0 = \int_{\bar{\sigma}}^{\bar{t}} \dot{\Xi}(t) dt = \int_{\bar{\sigma}}^{\bar{t}} \mu(\tau(t)) dt$. Now, letting $v := \mu(\tau(\bar{\sigma}))$, by the symmetry of C_ϕ the function

$$\sigma \mapsto \int_{\bar{\sigma}}^{\sigma} \langle \mu(\tau(t), v) \rangle dt$$

is monotone increasing for $\sigma \in [\bar{\sigma}, \bar{\sigma} + T_0]$ and decreasing for $\sigma \in [\bar{\sigma} + T_0, \bar{\sigma} + 2T_0]$. Hence, the equation $\int_{\bar{\sigma}}^{\bar{t}} \mu(\tau(t)) dt = 0$ implies $\bar{\sigma} = 0$ and $\bar{t} = 2T_0$. $\square$

Characteristic set of isoperimetric sets

In this section, we apply the previous results to the study of the characteristic set of ϕ -isoperimetric sets. As a corollary of Theorem 2.2.18 we have the following

Corollary 2.2.27. *Let ϕ^* be of class C^2 and let $E \subset \mathbb{H}^1$ be a ϕ -isoperimetric set of class C^2 . Then $\mathcal{C}(E)$ consists of isolated points. Moreover, for every $p_0 = (\xi_0, z_0) \in \mathcal{C}(E)$ and every f such that $p_0 \in \text{gr}(f) \subset \partial E$, we have $\text{rank}(JF(\xi_0)) = 2$.*

Proof. By Remark 2.1.12, we know that ∂E is bounded. Therefore we exclude the possibility that $\mathcal{C}(\partial E)$ contains complete (unbounded) lifts of simple curves. $\square$

Lemma 2.2.28. *Let ϕ^* be of class C^2 and $E \subset \mathbb{H}^1$ be a ϕ -isoperimetric set of class C^2 . Let $p_0 \in \mathcal{C}(E)$. There exists $r > 0$ such that for $p \in \partial E \cap B(p_0, r)$, $p \neq p_0$, the maximal horizontal lift of the ϕ -circle in ∂E through p meets p_0 .*

Proof. The surface $\partial E \cap B(p_0, r)$ is the z -graph of $f \in C^2(D)$ and $p_0 = (\xi_0, f(\xi_0))$ with $\mathcal{C}(f) \cap \{|\xi - \xi_0| < r\} = \{\xi_0\}$. Let $\Theta_\xi \subset D$ be the maximal ϕ -circle (integral curve of $F^\perp$) passing through $\xi \in D \setminus \{\xi_0\}$. Notice that the radius of Θ_ξ does not depend on ξ , as it follows from Corollary 2.2.11. If $\xi_0 \notin \Theta_\xi$, then the normal vector $\mathcal{N}_\xi = \nabla \phi^*(F)$ is continuously defined on Θ_ξ .

Assume that there exists a sequence of such ξ with $\xi \rightarrow \xi_0$. By an elementary compactness argument it follows that there exists a ϕ -circle Θ passing through ξ_0 and there exists a normal $\mathcal{N}$ that is continuously defined along Θ and, in particular, through ξ_0 . Outside ξ_0 we have $\mathcal{N} = \nabla \phi^*(F)$.

Let $b \in \mathbb{R}^2$ the unit vector tangent to Θ at ξ_0 . Then we have

$$F(\xi_0 + tb) = F(\xi_0) + tJF(\xi_0)b + o(t) = tJF(\xi_0)b + o(t),$$

with $JF(\xi_0)b \neq 0$, because $JF(\xi_0)$ has rank 2 by Lemma 2.2.17. Since $\nabla \phi(-v) = -\nabla \phi(v)$, for $v \in \mathbb{R}^2 \setminus \{0\}$, it follows that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \nabla \phi^*(F(\xi_0 + tb)) &= \nabla \phi^*(JF(\xi_0)b), \\ \lim_{t \rightarrow 0^-} \nabla \phi^*(F(\xi_0 + tb)) &= -\nabla \phi^*(JF(\xi_0)b). \end{aligned}$$

This contradicts the continuity of $\mathcal{N}$ along Θ at ξ_0 . □

2.2.5 Classification of ϕ -isoperimetric sets of class C^2

Now we can finally prove Theorem 2.2.1, but first let's give a more precise construction of ϕ -bubbles.

Let ϕ be a norm in $\mathbb{R}^2$ that we normalize by $\phi(1, 0) = 1$. For $\xi_0 \in \mathbb{R}^2$ and $r > 0$, ϕ -circles are defined in (2.22) and we let the ϕ -disk of radius r and center ξ_0 be

$$D_\phi(\xi_0, r) = \{\xi \in \mathbb{R}^2 : \phi(\xi - \xi_0) < r\}.$$

We also let $C_\phi(r) = C_\phi(0, r)$, $C_\phi = C_\phi(1)$ and $D_\phi(r) = D_\phi(0, r)$, $D_\phi = D_\phi(1)$.

The circle C_ϕ is a Lipschitz curve and we denote by $L = L_\phi > 0$ its Euclidean length. We parametrize C_ϕ by arc-length through $\kappa \in \text{Lip}([0, L]; \mathbb{R}^2)$ such that $\kappa([0, L]) = C_\phi$ with initial and end-point $\kappa(0) = \kappa(L) = (-1, 0)$. We choose the anti-clockwise orientation and we extend κ to the whole $\mathbb{R}$ by L -periodicity. Then we have $\kappa \in \text{Lip}(\mathbb{R}; \mathbb{R}^2)$.

The map $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\xi(t, \tau) = \kappa(t) + \kappa(\tau)$, is in $\text{Lip}(\mathbb{R}^2; \mathbb{R}^2)$. We restrict ξ to the domain

$$D = \{(t, \tau) \in \mathbb{R}^2 : \tau \in [0, L], t \in [\tau + L/2, \tau + 3L/2]\}.$$

Notice that $\xi(\tau + L/2, \tau) = \xi(\tau + 3L/2, \tau) = 0$ for any $\tau \in [0, L]$. We define the function $z \in \text{Lip}(D)$,

$$z(t, \tau) = \int_{\tau+L/2}^t \omega(\xi(s, \tau), \xi_s(s, \tau)) ds. \quad (2.89)$$

The map $\Phi : D \rightarrow \mathbb{R}^3$ defined by $\Phi = (\xi, z)$ is Lipschitz continuous. Moreover, Φ is C^k if ϕ is C^k .

We define the Lipschitz surface $\Sigma_\phi = \Phi(D) \subset \mathbb{R}^3$ and call $S = \Phi(\tau + L/2, \tau) = 0 \in \Sigma_\phi$ the south pole of Σ_ϕ and $N = \Phi(\tau + 3L/2, \tau) = (0, 0, z(\tau + 3L/2, \tau))$ the north pole.

We call the bounded region $E_\phi \subset \mathbb{R}^3$ enclosed by Σ_ϕ the ϕ -bubble. E_ϕ is a topological ball and it is the candidate solution to the ϕ -isoperimetric problem. When ϕ is the Euclidean norm in the plane, the set E_ϕ is the well-known Pansu's ball.

Proof of Theorem 2.2.1. The set E is bounded and connected, by Remark 2.1.12. We may also assume that it is open. It follows from Corollary 2.2.11 (and from the analogous result for x -graphs and y -graphs based on Remark 2.2.10) that, out of the characteristic set $\mathcal{C}(E)$, the surface ∂E is foliated by horizontal lifts of ϕ -circles. Then $\mathcal{C}(E)$ contains at least one point, since otherwise, ∂E would contain an unbounded curve, contradicting the boundedness of E .

Let $f \in C^2(D)$, with $D \subset \mathbb{R}^2$ open, be a maximal function such that $\text{gr}(f) \subset \partial E$ and $\mathcal{C}(f) \neq \emptyset$. We may assume that $0 \in \mathcal{C}(f)$, $f(0) = 0$ and that E lies above the graph of f near 0. Around the characteristic point 0, the function f must have the structure described in Lemma 2.2.28. It follows that, up to a dilation, we have $\text{gr}(f) \subset \partial E_\phi$.

The maximal domain for f must be $D = D_\phi(2)$. Otherwise, at each point $\xi \in \partial D \setminus \partial D_\phi(2)$ the space $T_{(\xi, f(\xi))} \partial E = T_{(\xi, f(\xi))} \partial E_\phi$ is not vertical, contradicting the maximality of D . This shows that the graph of f is the 'lower hemisphere' of ∂E_ϕ .

Up to extending f by continuity to ∂D , we have $(\xi, f(\xi)) \notin \mathcal{C}(E)$ for each $\xi \in \partial D$. Hence there exists a ϕ -circle passing through 0 whose horizontal lift stays in ∂E and passes through $(\xi, f(\xi))$. The collection of all the maximal extensions of such horizontal lifts completes the upper hemisphere of ∂E_ϕ , thus implying that $\partial E_\phi \subset \partial E$. Moreover, since ∂E is C^2 , we deduce that ∂E_ϕ is a connected component of ∂E .

In conclusion we have proved that ∂E is the finite union of boundaries of ϕ -bubbles having the same curvature. By connectedness of E this concludes the proof. $\square$

In general, ϕ -bubbles are not of class C^2 and not even of class C^1 , e.g., in the case of a crystalline norm. Even when ϕ is regular, there may be a loss of regularity at the poles of E_ϕ .

2.2.6 Regularity of ϕ -bubbles

Here we will address the problem of the regularity of the ϕ -bubbles. We first show that ϕ -bubbles have the same regularity as ϕ outside the poles.

Lemma 2.2.29. *If ϕ is strictly convex and of class C^k , for some $k \geq 1$, then the set $\Sigma_\phi \setminus \{S, N\}$ is an embedded surface of class C^k .*

Proof. If the Jacobian of Φ has rank 2 at the point $(t, \tau) \in D$, then Σ_ϕ is an embedded surface of class C^k around the point $\Phi(t, \tau)$. A sufficient condition for this is $\det J\xi(t, \tau) \neq 0$. The Jacobian of $\xi : D \rightarrow \mathbb{R}^2$ satisfies

$$\det J\xi(t, \tau) = 0 \quad \text{if and only if} \quad \dot{\kappa}(t) = \pm \dot{\kappa}(\tau).$$

The case $\dot{\kappa}(t) = -\dot{\kappa}(\tau)$ is equivalent to $\kappa(t) = -\kappa(\tau)$, by the strict convexity of the norm. This is, in turn, equivalent to $t = \tau + L/2$ or $t = \tau + 3L/2$. In the former case we have $\Phi(t, \tau) = S$, in the latter $\Phi(t, \tau) = N$.

We are left to consider the case $\dot{\kappa}(t) = \dot{\kappa}(\tau)$. By strict convexity of ϕ , this implies $\kappa(t) = \kappa(\tau)$, that is equivalent to $t = \tau + L$. In this case, we have $\xi(t, \tau) = 2\kappa(\tau) \in C_\phi(2)$. The point $\Phi(t, \tau)$ is on the 'equator' of Σ_ϕ .

We study the regularity of Σ_ϕ at points $\Phi(\tau + L, \tau)$. The height $z(\tau + L, \tau)$ does not depend on τ because it is half the area of the disk D_ϕ . It follows that $0 = \partial_\tau(z(\tau + L, \tau)) = z_t(\tau + L, \tau) + z_\tau(\tau + L, \tau)$ and this implies that

$$z_t(\tau + L, \tau) \neq z_\tau(\tau + L, \tau), \quad (2.90)$$

as soon as we prove that the left-hand side does not vanish. Indeed, differentiating (2.89) we obtain

$$z_t(\tau + L, \tau) = 2\omega(\kappa(\tau), \dot{\kappa}(\tau)) \neq 0,$$

because $\kappa(\tau)$ and $\dot{\kappa}(\tau)$ are not proportional.

From $\dot{\kappa}(\tau + L) = \dot{\kappa}(\tau) \neq 0$ and (2.90), we deduce that the Jacobian matrix $J\Phi(\tau + L, \tau)$ has rank 2. This shows that Σ_ϕ is of class C^k also around the ‘equator’. $\square$

The regularity of Σ_ϕ at the poles is much more subtle. We study the problem in Theorem 2.2.3, whose proof is presented below.

Proof of Theorem 2.2.3. We study the regularity at the south pole. By Lemma 2.2.29 there exists a function $f \in C^2(D_\phi(2) \setminus \{0\})$ such that the graph of f is the lower hemisphere of Σ_ϕ without the south pole. We shall show that f can be extended to a function $f \in C^2(D_\phi(2))$ satisfying $\nabla f(0) = 0$ and $\mathcal{H}f(0) = 0$. Here and in the sequel, we denote by $\mathcal{H}f$ the Hessian matrix of f . Differentiating the identity

$$z(t, \tau) = f(\xi(t, \tau)), \quad \tau \in [0, L], \quad t \in (\tau + L/2, \tau + L),$$

we find the identities

$$z_t(t, \tau) = \langle \nabla f, \dot{\kappa}(t) \rangle, \quad (2.91)$$

$$z_\tau(t, \tau) = \langle \nabla f, \dot{\kappa}(\tau) \rangle, \quad (2.92)$$

$$z_{tt}(t, \tau) = \langle \mathcal{H}f \dot{\kappa}(t), \dot{\kappa}(t) \rangle + \langle \nabla f, \ddot{\kappa}(t) \rangle, \quad (2.93)$$

$$z_{\tau\tau}(t, \tau) = \langle \mathcal{H}f \dot{\kappa}(\tau), \dot{\kappa}(\tau) \rangle + \langle \nabla f, \ddot{\kappa}(\tau) \rangle, \quad (2.94)$$

$$z_{t\tau}(t, \tau) = z_{\tau t}(t, \tau) = \langle \mathcal{H}f \dot{\kappa}(t), \dot{\kappa}(\tau) \rangle, \quad (2.95)$$

where, above and in the following, $\mathcal{H}f$ and ∇f are evaluated at $\xi(t, \tau)$.

On the other hand, from (2.89) we compute the derivatives

$$z_t(t, \tau) = \omega(\kappa(t) + \kappa(\tau), \dot{\kappa}(t)), \quad (2.96)$$

$$z_\tau(t, \tau) = \omega(\dot{\kappa}(\tau), \kappa(t) + \kappa(\tau)), \quad (2.97)$$

$$z_{tt}(t, \tau) = \omega(\kappa(t) + \kappa(\tau), \ddot{\kappa}(t)), \quad (2.98)$$

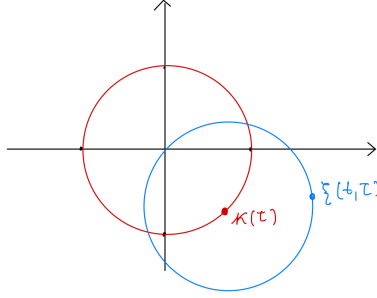
$$z_{\tau\tau}(t, \tau) = \omega(\ddot{\kappa}(\tau), \kappa(t) + \kappa(\tau)), \quad (2.99)$$

$$z_{t\tau}(t, \tau) = \omega(\dot{\kappa}(\tau), \dot{\kappa}(t)). \quad (2.100)$$

In formulas (2.91)–(2.100), we will replace $\kappa(\tau)$, $\dot{\kappa}(\tau)$, and $\ddot{\kappa}(\tau)$ with their Taylor expansions at the point $t - L/2$.

By assumption, the arc-length parameterization of the circle C_ϕ satisfies $\kappa \in C^4(\mathbb{R}; \mathbb{R}^2)$ and

$$\ddot{\kappa}(t) = \lambda(t)\dot{\kappa}(t)^\perp, \quad t \in [0, L], \quad (2.101)$$

Figure 2.4: Parametrization $\xi(t, \tau)$.

for a function (the curvature) $\lambda \in C^2(\mathbb{R})$ that is L -periodic and strictly positive. So there exists $0 < \lambda_0 \leq \Lambda_0 < \infty$ such that

$$0 < \lambda_0 \leq \lambda \leq \Lambda_0, \quad |\dot{\lambda}| \leq \Lambda_0, \quad |\ddot{\lambda}| \leq \Lambda_0.$$

The third and fourth derivatives of κ have the representation:

$$\kappa^{(3)} = \dot{\lambda}\dot{\kappa}^\perp - \lambda^2\ddot{\kappa} \quad \text{and} \quad \kappa^{(4)} = (\ddot{\lambda} - \lambda^3)\dot{\kappa}^\perp - 3\lambda\dot{\lambda}\dot{\kappa}. \quad (2.102)$$

In the following, we let $\delta = t - \tau - L/2 > 0$. The third order Taylor expansion for $\kappa(\tau)$ at $t - L/2$ is

$$\begin{aligned} \kappa(\tau) &= \kappa(t - L/2) - \delta\dot{\kappa}(t - L/2) + \frac{\delta^2}{2}\ddot{\kappa}(t - L/2) - \frac{\delta^3}{6}\kappa^{(3)}(t - L/2) + o(\delta^3) \\ &= -\kappa(t) + \delta\dot{\kappa}(t) - \frac{\delta^2}{2}\ddot{\kappa}(t) + \frac{\delta^3}{6}\kappa^{(3)}(t) + o(\delta^3) \\ &= -\kappa(t) + \delta\dot{\kappa}(t) - \frac{\delta^2}{2}\lambda(t)\dot{\kappa}(t)^\perp + \frac{\delta^3}{6}(\dot{\lambda}(t)\dot{\kappa}(t)^\perp - \lambda(t)^2\ddot{\kappa}(t)) + o(\delta^3). \end{aligned} \quad (2.103)$$

Hereafter, when not explicit, the functions κ and λ and their derivatives are evaluated at t . The little- o remainders are uniform with respect to the base point $t - L/2$. By a similar computation, using (2.102) we also obtain

$$\dot{\kappa}(\tau) = -\dot{\kappa} + \delta\lambda\dot{\kappa}^\perp - \frac{\delta^2}{2}(\dot{\lambda}\dot{\kappa}^\perp - \lambda^2\ddot{\kappa}) + \frac{\delta^3}{6}(\ddot{\lambda} - \lambda^3)\dot{\kappa}^\perp - \frac{\delta^3}{2}\lambda\dot{\lambda}\dot{\kappa} + o(\delta^3), \quad (2.104)$$

$$\ddot{\kappa}(\tau) = -\lambda\dot{\kappa}^\perp + \delta(\dot{\lambda}\dot{\kappa}^\perp - \lambda^2\ddot{\kappa}) - \frac{\delta^2}{2}((\ddot{\lambda} - \lambda^3)\dot{\kappa}^\perp - 3\lambda\dot{\lambda}\dot{\kappa}) + o(\delta^2). \quad (2.105)$$

We are ready to start the proof. We will use the identities

$$\omega(\dot{\kappa}, \dot{\kappa}^\perp) = -\omega(\dot{\kappa}^\perp, \dot{\kappa}) = \frac{1}{2}. \quad (2.106)$$

Recall our notation $\delta = t - \tau - L/2$.

Step 1. We claim that there exists $C > 0$ such that

$$|\nabla f(\xi(t, \tau))| \leq C\delta^2 \quad \text{for all } \tau \in [0, L], \quad t \in (\tau + L/2, \tau + L). \quad (2.107)$$

This estimate implies that f can be extended to a function $f \in C^1(D_\phi(2))$ satisfying $\nabla f(0) = 0$.

Inserting (2.103) and (2.104) into (2.96) and (2.97) yields

$$z_t(t, \tau) = \omega\left(\left(-\frac{\delta^2}{2}\lambda + \frac{\delta^3}{6}\dot{\lambda}\right)\dot{\kappa}(t)^\perp, \dot{\kappa}(t)\right) = \frac{\delta^2}{4}\lambda - \frac{\delta^3}{12}\dot{\lambda} + o(\delta^3), \quad (2.108)$$

$$\begin{aligned} z_\tau(t, \tau) &= \omega\left(\left(-1 + \frac{\delta^2}{2}\lambda^2\right)\dot{\kappa}, \left(-\frac{\delta^2}{2}\lambda + \frac{\delta^3}{6}\dot{\lambda}\right)\dot{\kappa}^\perp\right) + \omega\left(\left(\delta\lambda - \frac{\delta^2}{2}\dot{\lambda}\right)\dot{\kappa}^\perp, \left(\delta - \frac{\delta^3}{6}\lambda^2\right)\dot{\kappa}\right) + o(\delta^3) \\ &= -\frac{\delta^2}{4}\lambda + \frac{\delta^3}{6}\dot{\lambda} + o(\delta^3). \end{aligned} \quad (2.109)$$

Now, plugging (2.104) and (2.109) into (2.92) and then using (2.91) and (2.108) we obtain

$$\begin{aligned} -\frac{\delta^2}{4}\lambda + \frac{\delta^3}{6}\dot{\lambda} &= \langle \nabla f, \dot{\kappa}(t) \rangle \left(-1 + \frac{\delta^2}{2}\lambda^2 - \frac{\delta^3}{2}\lambda\dot{\lambda}\right) \\ &\quad + \langle \nabla f, \dot{\kappa}(t)^\perp \rangle \left(\delta\lambda - \frac{\delta^2}{2}\dot{\lambda} + \frac{\delta^3}{6}(\ddot{\lambda} - \lambda^3)\right) + o(\delta^3) \\ &= -\frac{\delta^2}{4}\lambda + \frac{\delta^3}{12}\dot{\lambda} + \langle \nabla f, \dot{\kappa}(t)^\perp \rangle \left(\delta\lambda - \frac{\delta^2}{2}\dot{\lambda} + \frac{\delta^3}{6}(\ddot{\lambda} - \lambda^3)\right) + o(\delta^3). \end{aligned}$$

Dividing the last equation by $\lambda\delta > 0$, we get

$$\langle \nabla f, \dot{\kappa}^\perp \rangle = \frac{\delta^2}{12} \frac{\dot{\lambda}}{\lambda} + o(\delta^2). \quad (2.110)$$

Thus, there exists $C > 0$ such that

$$|\langle \nabla f, \dot{\kappa}^\perp \rangle| \leq C\delta^2.$$

On the other hand, by (2.91) and (2.108), possibly changing $C > 0$ we also obtain

$$|\langle \nabla f, \dot{\kappa} \rangle| = |z_t(t, \tau)| \leq C\delta^2,$$

thus yielding (2.107).

Step 2. We claim that the norm of the Hessian matrix $\mathcal{H}f$ satisfies

$$|\mathcal{H}f(\xi(t, \tau))| = o(1) \quad \text{for } \tau \in [0, L], \quad t \in (\tau + L/2, \tau + L), \quad (2.111)$$

where $o(1) \rightarrow 0$ as $\delta = t - \tau - L/2 \rightarrow 0$. This implies that f can be extended to a function $f \in C^2(D_\phi(2))$ satisfying $\mathcal{H}f(0) = 0$.

Plugging (2.98) and (2.103) into (2.93), and then using (2.101), (2.106), and (2.110) yields

$$\begin{aligned} \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa} \rangle &= z_{tt}(t, \tau) - \langle \nabla f, \ddot{\kappa} \rangle = \omega(\kappa(t) + \kappa(\tau), \ddot{\kappa}) - \langle \nabla f, \ddot{\kappa} \rangle \\ &= \omega\left(\left(\delta - \frac{\delta^3}{6}\lambda^2\right)\dot{\kappa} + \left(-\frac{\delta^2}{2}\lambda + \frac{\delta^3}{6}\dot{\lambda}\right)\dot{\kappa}^\perp, \lambda\dot{\kappa}^\perp\right) - \langle \nabla f, \lambda\dot{\kappa}^\perp \rangle + o(\delta^3) \\ &= \frac{\delta}{2}\lambda - \frac{\delta^2}{12}\dot{\lambda} + o(\delta^2). \end{aligned} \quad (2.112)$$

On the other hand, plugging (2.104) into (2.95), and then using (2.112) we get

$$\begin{aligned} z_{t\tau}(t, \tau) &= \langle \mathcal{H}f \dot{\kappa}(t), \dot{\kappa}(t) \rangle \left(-1 + \frac{\delta^2}{2} \lambda^2 \right) + \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa}^\perp \rangle \left(\delta \lambda - \frac{\delta}{2} \dot{\lambda} \right) + o(\delta^2) \\ &= -\frac{\delta}{2} \lambda + \frac{\delta^2}{12} \dot{\lambda} + \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa}^\perp \rangle \left(\delta \lambda - \frac{\delta}{2} \dot{\lambda} \right) + o(\delta^2), \end{aligned}$$

while from (2.100), (2.104) and (2.106) we get

$$z_{t\tau}(t, \tau) = -\frac{1}{2} \left(\delta \lambda - \frac{\delta^2}{2} \dot{\lambda} \right) + o(\delta^2).$$

Therefore we obtain the identity

$$\begin{aligned} \left(\delta \lambda - \frac{\delta^2}{2} \dot{\lambda} \right) \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa}^\perp \rangle &= z_{t\tau}(t, \tau) + \frac{\delta}{2} \lambda - \frac{\delta^2}{12} \dot{\lambda} + o(\delta^2) \\ &= -\frac{1}{2} \left(\delta \lambda - \frac{\delta^2}{2} \dot{\lambda} \right) + \frac{\delta}{2} \lambda - \frac{\delta^2}{12} \dot{\lambda} + o(\delta^2) \\ &= \frac{\delta^2}{6} \dot{\lambda} + o(\delta^2), \end{aligned}$$

and dividing by $\lambda \delta > 0$ we get

$$\langle \mathcal{H}f \dot{\kappa}, \dot{\kappa}^\perp \rangle = \frac{\delta}{6} \frac{\dot{\lambda}}{\lambda} + o(\delta). \quad (2.113)$$

By symmetry of the Hessian matrix, we also have

$$\langle \mathcal{H}f \dot{\kappa}^\perp, \dot{\kappa} \rangle = \frac{\delta}{6} \frac{\dot{\lambda}}{\lambda} + o(\delta). \quad (2.114)$$

We are left to estimate $\langle \mathcal{H}f \dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle$. By (2.105), (2.110), (2.91), and (2.108) we obtain

$$\begin{aligned} \langle \nabla f, \ddot{\kappa}(\tau) \rangle &= (-\lambda + \delta \dot{\lambda} - \frac{\delta^2}{2} (\ddot{\lambda} - \lambda^3)) \langle \nabla f, \dot{\kappa}^\perp \rangle + (-\delta \lambda^2 + \frac{3}{2} \delta^2 \lambda \dot{\lambda}) \langle \nabla f, \dot{\kappa} \rangle + o(\delta^2) \\ &= (-\lambda + \delta \dot{\lambda} - \frac{\delta^2}{2} (\ddot{\lambda} - \lambda^3)) \frac{\delta^2}{12} \frac{\dot{\lambda}}{\lambda} + (-\delta \lambda^2 + \frac{3}{2} \delta^2 \lambda \dot{\lambda}) \frac{\delta^2}{4} \lambda + o(\delta^2) \\ &= -\frac{\delta^2}{12} \dot{\lambda} + o(\delta^2). \end{aligned} \quad (2.115)$$

On the other hand, by (2.104), (2.112), (2.113), (2.114) we have

$$\begin{aligned} \langle \mathcal{H}f \dot{\kappa}(\tau), \dot{\kappa}(\tau) \rangle &= (-1 + \frac{\delta^2}{2} \lambda^2)^2 \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa} \rangle + (\delta \lambda - \frac{\delta^2}{2} \dot{\lambda})^2 \langle \mathcal{H}f \dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle \\ &\quad + 2(-1 + \frac{\delta^2}{2} \lambda^2) (\delta \lambda - \frac{\delta^2}{2} \dot{\lambda}) \langle \mathcal{H}f \dot{\kappa}, \dot{\kappa}^\perp \rangle + o(\delta^2) \\ &= (-1 + \frac{\delta^2}{2} \lambda^2)^2 (\frac{\delta}{2} \lambda - \frac{\delta^2}{12} \dot{\lambda} + o(\delta^2)) + (\delta \lambda - \frac{\delta^2}{2} \dot{\lambda})^2 \langle \mathcal{H}f \dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle \\ &\quad + 2(-1 + \frac{\delta^2}{2} \lambda^2) (\delta \lambda - \frac{\delta^2}{2} \dot{\lambda}) (\frac{\delta}{6} \frac{\dot{\lambda}}{\lambda} + o(\delta)) \\ &= \frac{\delta}{2} \lambda - \frac{5}{12} \delta^2 \dot{\lambda} + \delta^2 \lambda^2 \langle \mathcal{H}f \dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle + o(\delta^2). \end{aligned} \quad (2.116)$$

Plugging (2.115) and (2.116) into (2.94) we get

$$\begin{aligned} z_{\tau\tau}(t, \tau) &= \frac{\delta}{2}\lambda - \frac{5}{12}\delta^2\dot{\lambda} + \delta^2\lambda^2\langle \mathcal{H}f\dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle - \frac{\delta^2}{12}\dot{\lambda} + o(\delta^2) \\ &= \frac{\delta}{2}\lambda - \frac{\delta^2}{2}\dot{\lambda} + \delta^2\lambda^2\langle \mathcal{H}f\dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle + o(\delta^2). \end{aligned} \quad (2.117)$$

Moreover, plugging (2.103) and (2.105) into (2.99), and using (2.106), we get

$$z_{\tau\tau}(t, \tau) = \omega\left(-\delta\lambda^2\dot{\kappa} + (-\lambda + \delta\dot{\lambda})\dot{\kappa}^\perp, \delta\dot{\kappa} - \frac{\delta^2}{2}\lambda\dot{\kappa}^\perp\right) + o(\delta^2) = -\frac{\delta}{2}(-\lambda + \delta\dot{\lambda}) + o(\delta^2). \quad (2.118)$$

Comparing (2.117) and (2.118) we therefore obtain

$$\delta^2\lambda^2\langle \mathcal{H}f\dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle = -\frac{\delta}{2}\lambda + \frac{\delta^2}{2}\dot{\lambda} - \frac{\delta}{2}(-\lambda + \delta\dot{\lambda}) + o(\delta^2) = o(\delta^2).$$

This yields $\langle \mathcal{H}f\dot{\kappa}^\perp, \dot{\kappa}^\perp \rangle = o(1)$ as $\delta \rightarrow 0$. Together with (2.112), (2.113), and (2.114) this implies (2.111) and concludes the proof of the theorem. $\square$

2.3 The Crystalline Case

In this section, we focus on a norm ϕ having non-differentiability points, and, in particular, on the case where it is crystalline. This general case will be approached by an approximation procedure and we will prove a conditional minimality property for the candidate solutions.

Recall that the dual norm ϕ^* to a non-differentiable one is not strictly convex, so that $\nabla\phi^*$ is constant on subsets of $\mathbb{R}^2$ having nonempty interior.

In the case where ϕ or ϕ^* are not differentiable, Theorems 2.2.1 and 2.2.3 cannot be applied in a direct way. In Corollary 2.2.12 we showed that the foliation property by horizontal lifts of ϕ -circles outside the characteristic set can be recovered when ϕ^* is only *piecewise* C^2 , thus allowing to cover the case $\phi = \ell^p$ for $p > 2$. For general non-differentiable norms, our next result is conditioned to the validity of the following conjecture.

Conjecture 2.3.1. *For any norm ϕ of class C_+^∞ , ϕ -isoperimetric sets are of class C^2 .*

Here, a norm ϕ in $\mathbb{R}^2$ is said to be of class C_+^∞ if $\phi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and ϕ -circles have strictly positive curvature. The proof of the following result is presented in Section 2.3.1.

Theorem 2.3.2. *Assume that Conjecture 2.3.1 holds true. Then for any norm ϕ in $\mathbb{R}^2$ the ϕ -bubble $E_\phi \subset \mathbb{H}^1$ is ϕ -isoperimetric.*

Of particular interest is the case of a *crystalline norm*.

Definition 2.3.3 (Crystalline norm). *A norm $\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ is called crystalline if the ϕ -circle $C_\phi = C_\phi(0, 1)$ is a convex polygon centrally symmetric with respect to the origin. Let $v_1, \dots, v_{2N} \in C_\phi$ be the ordered vertices of this polygon, and denote by $e_i = v_i - v_{i-1}$, $i = 1, \dots, 2N$, the edges of C_ϕ , where $v_0 = v_{2N}$.*

We consider the left-invariant vector fields

$$X_i := e_{i,1}X + e_{i,2}Y, \quad i = 1, \dots, 2N, \quad (2.119)$$

where $e_i = (e_{i,1}, e_{i,2})$, and we notice that $X_{i+N} = -X_i$ for $i = 1, \dots, N$. By a first variation argument, we deduce a foliation property for ϕ -isoperimetric sets by integral curves of the X_i .

Theorem 2.3.4. *Let $E \subset \mathbb{H}^1$ be ϕ -isoperimetric for a crystalline norm ϕ . Let $A \subset \mathbb{H}^1$ be an open set such that $\partial E \cap A$ is a connected z -graph of class C^2 . Then there exists $i = 1, \dots, N$ such that $\partial E \cap A$ is foliated by integral curves of X_i .*

Unfortunately, Theorem 2.3.4 does not provide enough information in order to establish the global foliation property by $\phi^\dagger$ -geodesics in the crystalline case.

Lemma 2.3.5. *Let O be a subset of $\mathbb{R}^2$ where $\nabla \phi^*$ exists and is constant. Let $E \subset \mathbb{H}^1$ be such that $E \cap A = \{(\xi, z) \in A : z \leq f(\xi), \xi \in D\}$ for some open set $A \subset \mathbb{H}^1$ and $f \in \text{Lip}(D)$. If $F(\xi) \in O$ for almost every $\xi \in D$ then E is not ϕ -isoperimetric.*

Proof. As in the proof of Proposition 2.2.5, consider $\varphi \in C_c^\infty(D)$ and, for $\varepsilon \in \mathbb{R}$ small, let $E_\varepsilon \subset \mathbb{H}^1$ be the set such that

$$E_\varepsilon \cap A = \{(\xi, z) \in A : z \leq f(\xi) + \varepsilon \varphi(\xi), \xi \in D\},$$

and $E_\varepsilon \setminus A = E \setminus A$. Then, as in (2.45),

$$\mathcal{P}_\phi' = \left. \frac{d}{d\varepsilon} \mathcal{P}_\phi(E_\varepsilon; A) \right|_{\varepsilon=0} = \int_D \langle \nabla \phi^*(F), \nabla \varphi \rangle d\xi.$$

By hypothesis, $\nabla \phi^*(F)$ is constant on D , so that $\mathcal{P}_\phi' = 0$.

Now, choosing $\varphi \neq 0$ with constant sign, we deduce that

$$\left. \frac{d}{d\varepsilon} \frac{\mathcal{P}_\phi(E_\varepsilon)^4}{\mathcal{L}^3(E_\varepsilon)^3} \right|_{\varepsilon=0} = -3 \frac{\mathcal{P}_\phi(E)^4}{\mathcal{L}^3(E)^4} \int_D \varphi(\xi) d\xi \neq 0,$$

contradicting the extremality of E for the isoperimetric quotient. $\square$

We are ready for the proof of Theorem 2.3.4.

Let ϕ be a crystalline norm and denote by $v_1, \dots, v_{2N} \in \mathbb{R}^2$ the ordered vertices of the polygon $C_\phi = C_\phi(0, 1)$. Notice that $v_{i+N} = -v_i$ for $i = 1, \dots, N$. The dual norm ϕ^* is also crystalline and the vertices of $C_{\phi^*}(0, 1)$ are in one-to-one correspondence with the edges $e_i = v_i - v_{i-1}$ of $C_\phi(0, 1)$ (with $v_0 = v_{2N}$). Namely, $C_{\phi^*}(0, 1)$ is the convex hull of $v_1^*, \dots, v_{2N}^*$ where, for $i = 1, \dots, 2N$, the vertex v_i^* is the unique vector of $\mathbb{R}^2$ such that

$$\langle v_i^*, e_i \rangle = 0 \quad (2.120)$$

and $\langle v_i^*, v_i \rangle = \langle v_i^*, v_{i-1} \rangle = 1$. In particular, $v_{i+N}^* = -v_i^*$ for $i = 1, \dots, N$.

Along the lines $L_i = \mathbb{R}v_i^*$, the norm ϕ^* is not differentiable. In the positive convex cone bounded by $\mathbb{R}^+v_i^*$ and $\mathbb{R}^+v_{i+1}^*$ the gradient $\nabla \phi^*$ exists and is constant, and we have $\nabla \phi^* = v_i$. For piecewise C^1 -smooth ϕ -isoperimetric sets the projected horizontal gradient F takes values in $L_1 \cup \dots \cup L_N$, by Lemma 2.3.5.

Proof of Theorem 2.3.4. Let $f \in C^2(D)$ be the graph function of $\partial E \cap A$. For $i = 1, \dots, N$, we let

$$D_i = \{\xi \in D : F(\xi) \in L_i = \mathbb{R}v_i^*\}.$$

If $\xi \in D_i$ then by (2.120) we have

$$F(\xi)^\perp \in \mathbb{R}(v_i^*)^\perp = \mathbb{R}e_i.$$

This implies that the vector field X_i in (2.119) is tangent to $\partial E \cap A$ at the point $(\xi, f(\xi))$.

We are going to prove the theorem by showing that $D = D_i$ for some $i \in \{1, \dots, N\}$. Notice that, for $i, j \in \{1, \dots, N\}$ and $i \neq j$, v_i and v_j are linearly independent. By Lemma 2.3.5 we have that $D = \cup_{i=1}^N D_i$. We claim, moreover, that

$$\overline{D} = \cup_{i=1}^N \overline{\text{int } D_i}. \quad (2.121)$$

In order to check the claim, pick $\xi \in D$ and assume by contradiction that $\xi \notin \overline{\text{int } D_i}$ for $i = 1, \dots, N$. Let i_1 be such that $\xi \in D_{i_1}$. Since $\xi \notin \text{int } D_{i_1}$, for every $\varepsilon > 0$ the set $D \setminus D_{i_1}$ intersects the disc of radius ε centered at ξ . Hence, there exists $i_2 \neq i_1$, and a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $D_{i_2} \setminus D_{i_1}$ converging to ξ . Now, either $\xi_n \in \text{int } D_{i_2}$ for infinitely many n or $\xi_n \notin \text{int } D_{i_2}$ for n large enough. In the first case $\xi \in \text{int } D_{i_2}$, leading to a contradiction. In the second case, we repeat the reasoning leading to $(\xi_n)_{n \in \mathbb{N}}$, replacing D_{i_1} by D_{i_2} and ξ by ξ_n for every $n \in \mathbb{N}$, and, by a diagonal argument, we obtain $i_3 \neq i_1, i_2$, and a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $D_{i_3} \setminus (D_{i_1} \cup D_{i_2})$ converging to ξ . Repeating the argument finitely many times, we end up with $i_N \in \{1, \dots, N\}$ and a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $D_{i_N} \setminus (\cup_{j=1}^{N-1} D_{i_j})$ converging to ξ with $D = D_{i_1} \cup \dots \cup D_{i_N}$. Since $D_{i_N} \setminus (\cup_{j=1}^{N-1} D_{i_j}) = D \setminus (\cup_{j=1}^{N-1} D_{i_j})$ is open, we deduce that $\xi \in \overline{\text{int } D_{i_N}}$. This concludes the contradiction argument, proving (2.121).

Let v_i and v_j be linearly independent. We claim that

$$\overline{\text{int}(D_i)} \cap \overline{\text{int}(D_j)} = \emptyset. \quad (2.122)$$

Consider the vector field X' on $D \times \mathbb{R}$ defined by $X'(\xi, z) = (e_i, e_{i,1}f_x(\xi) + e_{i,2}f_y(\xi))$. Then X' is C^1 and both X' and X_j are tangent to $\partial E \cap A$ in a neighbourhood of any point of S_j , where

$$S_k = \{(\xi, f(\xi)) : \xi \in \text{int}(D_k)\}, \quad k = 1, \dots, N.$$

Hence $[X', X_j] \in T_\xi(\partial E \cap A)$ for every $\xi \in S_j$. On the other hand, X' coincides with X_i on $S_i \times \mathbb{R}$, and therefore $[X', X_j] = [X_i, X_j] = c_{ij}Z$ on S_i , with $c_{ij} \in \mathbb{R} \setminus \{0\}$. Assume by contradiction that $\overline{S_i} \cap \overline{S_j}$ contains at least one point ξ . By continuity of $[X', X_j]$, we deduce from the above reasoning that $Z(\xi) \in T_\xi(\partial E \cap A)$. The contradiction comes from the remark that, by definition of S_i and S_j , also $X_i(\xi)$ and $X_j(\xi)$ are in $T_\xi(\partial E \cap A)$. We proved (2.122).

We deduce from (2.121) and (2.122) that $\{\overline{\text{int}(D_1)}, \dots, \overline{\text{int}(D_N)}\}$ is an open disjoint cover of $\overline{D}$. We conclude by connectedness of D . $\square$

2.3.1 The isoperimetric problem for general norms

In the case of crystalline norms, the first order necessary conditions satisfied by an isoperimetric set are not sufficient to reconstruct its structure, even assuming sufficient regularity. In this section, we show that the ϕ -isoperimetric problem for a general norm – in particular for a crystalline norm – can be approximated by the isoperimetric problem for smooth norms.

By Theorem 2.2.3, we know that if ϕ is of class C_+^∞ then the ϕ -bubble E_ϕ is of class C^2 . In this section, we show that the validity of Conjecture 2.3.1 implies the ϕ -isoperimetric property for the ϕ -bubble of any (crystalline) norm.

2.3.2 Smooth approximation of norms in the plane

We start with the mollification of a norm.

Proposition 2.3.6. *Let ϕ be a norm in $\mathbb{R}^2$. Then, for any $\varepsilon > 0$ there exists a norm ϕ_ε of class C_+^∞ with dual norm of class C^∞ , such that for all $\xi \in \mathbb{R}^2$ we have*

$$(1 - \eta(\varepsilon))\phi_\varepsilon(\xi) \leq \phi(\xi) \leq (1 + \eta(\varepsilon))\phi_\varepsilon(\xi), \quad (2.123)$$

and $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Proof. For $\varepsilon > 0$, we introduce the smooth mollifiers $\varrho_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$, supported in $[-\varepsilon\pi, \varepsilon\pi]$ defined by

$$\varrho_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(\frac{\pi^2 \varepsilon^2}{t^2 - \pi^2 \varepsilon^2}\right) & \text{if } |t| < \pi, \\ 0 & \text{if } |t| \geq \pi, \end{cases}$$

where c_ε is chosen in such a way that $\int_{\mathbb{R}} \varrho_\varepsilon(t) dt = 1$. Following [30, 54], we define the function $\psi_\varepsilon : \mathbb{R}^2 \rightarrow [0, \infty)$ letting

$$\psi_\varepsilon(\xi) := \int_{\mathbb{R}} \varrho_\varepsilon(t) \phi(R_t \xi) dt,$$

where R_t denotes the anti-clockwise rotation matrix of angle t . The function ψ_ε is a C^∞ norm. On the circle $\mathbb{S}^1 = \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$, the norms ψ_ε converge uniformly to ϕ as $\varepsilon \rightarrow 0^+$. So our claim (2.123) with $\eta(\varepsilon) \rightarrow 0$ holds with ψ_ε replacing ϕ_ε , by the positive 1-homogeneity of norms.

We let $\phi_\varepsilon : \mathbb{R}^2 \rightarrow [0, \infty)$ be defined by

$$\phi_\varepsilon(\xi) := \sqrt{\psi_\varepsilon(\xi)^2 + \varepsilon|\xi|^2}, \quad \xi \in \mathbb{R}^2.$$

This is a C^∞ norm in $\mathbb{R}^2$ and (2.123) is satisfied with $\eta(\varepsilon) \rightarrow 0$. The unit ϕ_ε -circle centered at the origin is the 0-level set of the function

$$F_\varepsilon(\xi) = \psi_\varepsilon^2(\xi) + \varepsilon|\xi|^2 - 1, \quad \xi \in \mathbb{R}^2.$$

Since the Hessian matrix of the squared Euclidean norm is proportional to the identity matrix I_2 and ψ_ε^2 is convex, we have that $\mathcal{H}F_\varepsilon \geq 2\varepsilon I_2$ in the sense of matrices. Then the curvature λ_ε of a unit ϕ_ε -circle satisfies

$$\lambda_\varepsilon = \frac{\langle \mathcal{H}F_\varepsilon \nabla F_\varepsilon^\perp, \nabla F_\varepsilon^\perp \rangle}{|\nabla F_\varepsilon|^3} \geq \frac{2\varepsilon}{|\nabla F_\varepsilon|} > 0.$$

The proof that the dual norm of a norm of class C_+^∞ is itself of class C^∞ is standard and we omit it. $\square$

2.3.3 Crystalline ϕ -bubbles as limits of smooth isoperimetric sets

Let ϕ be any norm in $\mathbb{R}^2$ and let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be the smooth approximating norms found in Proposition 2.3.6.

Given a Lebesgue measurable set $F \subset \mathbb{R}^2$, from (2.123) and from the definition of perimeter (Definition 2.1.9), we have

$$(1 - \eta(\varepsilon))\mathcal{P}_\phi(F) \leq \mathcal{P}_{\phi_\varepsilon}(F) \leq (1 + \eta(\varepsilon))\mathcal{P}_\phi(F). \quad (2.124)$$

The ϕ_ε -circles C_{ϕ_ε} converge in Hausdorff distance to the circle C_ϕ . This implies that the ϕ_ε -bubbles E_{ϕ_ε} converge in the Hausdorff distance to the limit bubble E_ϕ . This, in turn, implies the convergence in $L^1(\mathbb{H}^1)$, namely,

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{L}^3(E_{\phi_\varepsilon} \Delta E_\phi) = 0, \quad (2.125)$$

where Δ denotes the symmetric difference of sets.

Proof of Theorem 2.3.2. Let $F \subset \mathbb{H}^1$ be any Lebesgue measurable set with $0 < \mathcal{L}^3(F) < \infty$. Assuming the validity of Conjecture 2.2.16, E_{ϕ_ε} is isoperimetric for any $\varepsilon > 0$. So using twice (2.124) we find

$$\text{Isop}_\phi(F) \geq \frac{\text{Isop}_{\phi_\varepsilon}(F)}{1 + \eta(\varepsilon)} \geq \frac{\text{Isop}_{\phi_\varepsilon}(E_{\phi_\varepsilon})}{1 + \eta(\varepsilon)} \geq \frac{1 - \eta(\varepsilon)}{1 + \eta(\varepsilon)} \text{Isop}_\phi(E_{\phi_\varepsilon}).$$

By the lower semicontinuity of the perimeter with respect to the L^1 convergence and from (2.125), we deduce that

$$\liminf_{\varepsilon \rightarrow 0^+} \text{Isop}_\phi(E_{\phi_\varepsilon}) \geq \text{Isop}_\phi(E_\phi),$$

and using the fact that $\eta(\varepsilon) \rightarrow 0$ we conclude that $\text{Isop}_\phi(F) \geq \text{Isop}_\phi(E_\phi)$. $\square$

2.4 Necessary sign conditions at interface curves

In Section 2.2 we proved a foliation property for a z -graph with constant ϕ -curvature consists of geodesics in the ambient space relative to the dual norm of ϕ in $\mathbb{R}^2$. It was proven in Theorem 2.2.14, by the means of Pontryagin's Maximum Principle.

In the case of a crystalline norm, we've seen in Theorem 2.3.4 that any z -graph, piece of the boundary of an isoperimetric set is foliated by integral curves of the vector fields defined in (2.119).

In this section, we consider a particular crystalline norm, namely the L^∞ norm, for which the dual norm is the L^1 one. We want to show how to get some necessary conditions for a set to be isoperimetric, by using a first-variation argument and by studying the blow-ups of the *interface curves* (see Definition 2.4.1 below). We still want to stress that these first-order necessary conditions are not enough to fully reconstruct the structure of a crystalline isoperimetric set. For that purpose one needs to proceed by approximation as in the previous paragraph.

Throughout this section we will use the following notation. Let E be a piecewise C^1 isoperimetric set, and $S = \{(x, y, u(x, y)), (x, y) \in \Omega \subset \mathbb{R}^2\} \subset \partial E$ be a z -graph, with $u : \Omega \rightarrow [0, \infty)$, $\Omega \subset \mathbb{R}^2$ open connected. Assume that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ where Ω_1, Ω_2 are open connected and $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \cap \Omega$ is a planar curve. Assume also that $S_1 := u(\Omega_1)$ and $S_2 := u(\Omega_2)$ are C^1 and call $\gamma = \{(x, y, u(x, y)) : (x, y) \in \Gamma\}$ the *interface curve*. Without loss of generality we always consider Ω to be the ε -by- ε open square centered at the origin.

Definition 2.4.1 (Interface curve). *Let E be a piecewise C^1 set and $\gamma \subset E$ a curve. We say that γ is an interface curve if there exists E_1, E_2 C^1 subsets of E such that $\gamma = \bar{E}_1 \cap \bar{E}_2$ and for every curve η , with $\eta \cap \gamma = \{p\}$, $p \in E_1 \cap E_2$ such that does not exist $\lambda \in \mathbb{R}$ with $\dot{\gamma}(p) = \lambda \dot{\eta}(p)$, then*

$$\lim_{E_1 \ni q \rightarrow p} \dot{\eta}(q) \neq \lim_{E_2 \ni q \rightarrow p} \dot{\eta}(q).$$

2.4.1 Same vector field foliating S_1 and S_2

We start by investigating the case when S_1 and S_2 are foliated by the same vector field. Our results are specifically about the foliation by X , but, in the same way, one can study the other foliation (by Y) on S_1 and S_2 .

Lemma 2.4.2. *If S_1 and S_2 are foliated by X , namely $\langle X, N \rangle = 0$ for all $(x, y) \in \Omega_1 \cup \Omega_2$, then the interface curve γ must be an integral curve of X .*

Proof. Suppose that γ is not an integral curve of X . Since $\gamma = \bar{S}_1 \cap \bar{S}_2$ and both S_1 and S_2 are foliated by X , then for every point $p = (p_1, p_2, p_3) \in \gamma$ there exists an integral curve of X starting from p and lying in S_1 , say I_1 , and an integral curve of X starting from p and lying in S_2 , say I_2 . By the uniqueness of solution of the Cauchy Problem

$$\dot{s}(t) = X(s(t)), \quad s(0) = p$$

we have that I_1 must be the extension of I_2 . If we call I the integral curve of X through p , then $I(t) = (p_1 + t, p_2, p_3 - \frac{p_2 t}{2})$. We can suppose that for $t < 0$, $I(t) \in S_1$ and for $t > 0$, $I(t) \in S_2$. Obviously

$$\lim_{t \rightarrow 0^-} I(t) = \lim_{t \rightarrow 0^+} I(t),$$

but also

$$\lim_{t \rightarrow 0^-} I'(t) = \lim_{t \rightarrow 0^+} I'(t),$$

namely we have at least C^1 regularity on γ that is impossible because γ is an interface curve by hypothesis. $\square$

Theorem 2.4.3. *Assume that S_1 and S_2 are foliated by X , namely $\langle X, N \rangle = 0$ for all $(x, y) \in \Omega_1 \cup \Omega_2$. Given $p \in \gamma$ assume that there exists a neighbourhood $U \subset S_1 \cup S_2$ of p such that $\langle Y, N \rangle \neq 0$ on $U \setminus \gamma$. Then $\langle Y, N \rangle$ must have the same sign on $U \cap S_1$ and $U \cap S_2$.*

Proof. Note that

$$\begin{aligned}\langle Y, N \rangle(x, y, u(x, y)) &= \left\langle \left(0, 1, \frac{x}{2}\right), \frac{1}{\sqrt{1 + |\nabla u|^2}}(u_x, u_y, -1) \right\rangle = \frac{1}{\sqrt{1 + |\nabla u|^2}}(u_y - \frac{x}{2}), \\ \langle X, N \rangle(x, y, u(x, y)) &= \frac{1}{\sqrt{1 + |\nabla u|^2}}(u_x + \frac{y}{2}).\end{aligned}\tag{2.126}$$

Let $\vartheta \in C_c^\infty(\Omega)$ and consider $S_\varepsilon = (u + \varepsilon\vartheta)(\Omega)$. It's not restrictive to suppose that Ω is a δ -by- δ square, with $\delta \ll 1$ (we will choose δ later). For $\varepsilon > 0$, by the representation formula, we have that

$$\begin{aligned}A(u + \varepsilon\vartheta) &= \int_{\Omega} |u_x + \varepsilon\vartheta_x + \frac{y}{2}| + |u_y + \varepsilon\vartheta_y - \frac{x}{2}| dx dy \\ &= \int_{\Omega_2} \varepsilon|\vartheta_x| + |u_y + \varepsilon\vartheta_y - \frac{x}{2}| + \int_{\Omega_1} |u_y + \varepsilon\vartheta_y - \frac{x}{2}| + \varepsilon|\vartheta_x|.\end{aligned}$$

Taking the derivative in $\varepsilon = 0^+$, we get

$$\frac{d}{d\varepsilon} A(u + \varepsilon\vartheta)|_{\varepsilon=0^+} = \int_{\Omega_2} |\vartheta_x| + \frac{u_y - \frac{x}{2}}{|u_y - \frac{x}{2}|} \vartheta_y + \int_{\Omega_1} |\vartheta_x| + \frac{u_y - \frac{x}{2}}{|u_y - \frac{x}{2}|} \vartheta_y \tag{2.127}$$

The next assumption is legit if we have δ small enough, namely if $u_y - \frac{x}{2} \neq 0$ in $\Omega = [-\delta, \delta] \times [-\delta, \delta]$. Assume by contradiction that $\langle Y, N \rangle$ has different signs on Ω_1 and Ω_2 . For instance let $\langle Y, N \rangle$ have sign -1 on Ω_2 and $+1$ on Ω_1 , i.e.,

$$\frac{u_y - \frac{x}{2}}{|u_y - \frac{x}{2}|} = -1 \text{ on } \Omega_2 \text{ and } \frac{u_y - \frac{x}{2}}{|u_y - \frac{x}{2}|} = +1 \text{ on } \Omega_1.$$

Then (2.127) reads

$$\frac{d}{d\varepsilon} A(u + \varepsilon\vartheta)|_{\varepsilon=0^+} = \int_{\Omega} |\vartheta_x| - \int_{\Omega_2} \vartheta_y + \int_{\Omega_1} \vartheta_y.$$

Since S_1 and S_2 are foliated both by X , then, by Lemma 2.4.2 γ must be an integral curve of X . Then it is supported on the x -axis. Since E is isoperimetric (and we are only considering $\varepsilon > 0$) we have that

$$\begin{aligned}0 &\leq \frac{d}{d\varepsilon} A(u + \varepsilon\vartheta)|_{\varepsilon=0^+} = \int_{\Omega} |\vartheta_x| - \int_{-\delta}^{\delta} \int_{-\delta}^0 \vartheta_y dy dx + \int_{-\delta}^{\delta} \int_0^{\delta} \vartheta_y dy dx = \\ &= \int_{\Omega} |\vartheta_x| - 2 \int_{-\delta}^{\delta} \vartheta(x, 0) dx.\end{aligned}\tag{2.128}$$

Now letting ϑ be such that $\vartheta(x, y) = h(x)\psi(y)$ with $\int \psi = 0$, $\text{supp}(\psi) \subset [-\tilde{\delta}, \tilde{\delta}]$, $\psi(0) = 1$, $\int |h_x| = 2$ and h has compact support, we get

$$\begin{aligned}\int_{\Omega} |\vartheta_x| &= \int_{-1}^1 |\psi(y)| ty \cdot \int_{-1}^1 |h(x)| dx = 2 \int_{-1}^1 |\psi(y)| dy \leq 8\tilde{\delta}, \\ \int_{-1}^1 \vartheta(x, 0) dx &= \int_{-1}^1 h(x)\psi(0) dx = c,\end{aligned}$$

where the last constant does not depend on $\tilde{\delta}$. So if $\tilde{\delta}$ is small enough, we have a contradiction on (2.128). $\square$

2.4.2 Different foliation on S_1 and S_2

We pass to the case where S_1 and S_2 are foliated one by X and the other by Y . With a little abuse of notation, we adopt the notation S_X and S_Y to denote which S_i , $i = 1, 2$ is foliated by X and which one is foliated by Y . In particular, if S_i , $i = 1, 2$ is foliated by X (resp. Y), we denote $\Omega_i = \Omega_X$ (resp. Ω_Y), and $S_i = S_X$ (resp. $S_i = S_Y$). Without loss of generality we assume $\Omega = [-\varepsilon, \varepsilon]^2$ for a suitable $\varepsilon > 0$.

Furthermore, we will assume that in S_1 and in S_2 there aren't any characteristic points. This assumption will hold until the end of the section.

We first prove a geometric lemma that *does not rely on minimality*.

Lemma 2.4.4 (Geometric lemma). *Assume $\Gamma = \{(x, \varphi(x)) : -\varepsilon < x < \varepsilon\}$ for a function $\varphi : (-\varepsilon, \varepsilon) \rightarrow (-\varepsilon, \varepsilon)$ such that $\varphi(0) = 0$. Then*

- if $\varphi'(0) > 0$, then $\text{sgn}_{\Omega_X}(\langle N, Y \rangle) = \text{sgn}_{\Omega_Y}(\langle N, X \rangle)$;
- if $\varphi'(0) < 0$, then $\text{sgn}_{\Omega_X}(\langle N, Y \rangle) \neq \text{sgn}_{\Omega_Y}(\langle N, X \rangle)$.

Proof. Let's start from the case $\varphi'(0) > 0$. We know that $\langle N, Y \rangle = \partial_y u - \frac{x}{2} = 0$ on Ω_Y and $\langle N, X \rangle = \partial_x u + \frac{y}{2} = 0$ on Ω_X . Then we can find a primitive of $\partial_x u + \frac{y}{2}$ on Ω_X by integrating with respect to x on Ω_X and it must be a constant w.r.t. x , thus we have

$$u(x, y) + \frac{xy}{2} + f_1(y) = 0 \quad \forall (x, y) \in \Omega_X.$$

In the same way we have

$$u(x, y) - \frac{xy}{2} + f_2(x) = 0 \quad \forall (x, y) \in \Omega_Y.$$

Then on the curve $(x, \varphi(x))$ we have

$$\frac{x\varphi(x)}{2} - f_2(x) = u(x, \varphi(x)) = -\frac{x\varphi(x)}{2} - f_1(\varphi(x))$$

and that implies

$$x\varphi(x) = f_2(x) - f_1(\varphi(x)). \quad (2.129)$$

Moreover on Ω_X we have that $S_X = \{(x, y, -\frac{xy}{2} - f_1(y))\}$ and the normal to S_X is proportional to $(-\frac{y}{2}, -\frac{x}{2} - f_1'(y), -1)$. Then

$$\text{sgn}_{\Omega_X}(\langle N, Y \rangle) = \text{sgn}(-f_1'(y) - x)$$

and, since the normal to S_Y is proportional to $(\frac{y}{2} - f_2'(x), \frac{x}{2}, -1)$

$$\text{sgn}_{\Omega_Y}(\langle N, X \rangle) = \text{sgn}(-f_2'(x) + y).$$

When $(x, y) \rightarrow (x, \varphi(x))$ on the proper side, since we have assumed the absence of characteristic points, we get

$$\begin{aligned} \text{sgn}_{\Omega_X}(\langle N, Y \rangle) &= -\text{sgn}(f_1'(0)) \\ \text{sgn}_{\Omega_Y}(\langle N, X \rangle) &= -\text{sgn}(f_2'(0)). \end{aligned}$$

On the other hand, differentiating in (2.129) and evaluating at $x = 0$ we get

$$f'_2(0) = \varphi'(0)f'_1(0).$$

Since, by assumption $\varphi'(0) > 0$, we get that

$$\text{sgn}_{\Omega_X}(\langle N, Y \rangle) = \text{sgn}_{\Omega_Y}(\langle N, X \rangle).$$

For the second part of the statement, if $\varphi'(0) < 0$, we get the opposite, namely

$$\text{sgn}_{\Omega_X}(\langle N, Y \rangle) \neq \text{sgn}_{\Omega_Y}(\langle N, X \rangle).$$

□

In view of the previous result, we want to study the cases where $\varphi'(0) > 0$ and $\text{sgn}_{\Omega_X}(\langle N, Y \rangle) = \text{sgn}_{\Omega_Y}(\langle N, X \rangle)$ and the case where $\varphi'(0) < 0$ and $\text{sgn}_{\Omega_X}(\langle N, Y \rangle) \neq \text{sgn}_{\Omega_Y}(\langle N, X \rangle)$. We aim at finding new information by using the minimality of E , which didn't play any role in the previous lemma. In order to do that, we need to specify if Ω_X is in the epigraph or in the subgraph of φ , as in the following notation.

We assume that $\Omega = \Omega_+ \cup \Omega_- \cup \Gamma = (-\varepsilon, \varepsilon)^2$, where

$$\Gamma := \{(x, \varphi(x)) : -\varepsilon < x < \varepsilon\}, \quad (2.130)$$

$\Omega_+ := \{(x, y) \in \Omega : y > \varphi(x)\}$, $\Omega_- := \{(x, y) \in \Omega : y < \varphi(x)\}$, where the curve $\varphi : (-\varepsilon, \varepsilon) \rightarrow (-\varepsilon, \varepsilon)$ is assumed to be *invertible*. We also let $S^\pm = u(\Omega^\pm)$.

Lemma 2.4.5. *Let S^- and S^+ as before and suppose S^- foliated by X , S^+ foliated by Y and $\varphi'(0) > 0$. Then $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = \text{sgn}_{\Omega^+}(\langle N, X \rangle) = 1$.*

Proof. By continuity of φ , is not restrictive to suppose that $\varphi'(x) > 0$ in $(-\varepsilon, \varepsilon)$. By Lemma 2.4.4 $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = \text{sgn}_{\Omega^+}(\langle N, X \rangle)$.

Assume by contradiction that they are equal to -1 . We show that in this case there exists a variation $\vartheta \in C_c^\infty(\Omega)$ preserving the perimeter and increasing the volume.

By the representation formula and exploiting the foliations on Ω^- and Ω^+ (namely $\langle X, N \rangle = 0$ on Ω^- and $\langle Y, N \rangle = 0$ on Ω^+), we get, for every $\vartheta \in C_c^\infty(\Omega)$ and for $\varepsilon > 0$,

$$\frac{d}{d\varepsilon} A(u + \varepsilon\vartheta)|_{\varepsilon=0^+} = \int_{\Omega^-} |\vartheta_x| - \int_{\Omega^+} \vartheta_x + \int_{\Omega^+} |\vartheta_y| - \int_{\Omega^-} \vartheta_y. \quad (2.131)$$

Let's consider the first two integrals. We have

$$\int_{\Omega^+} \vartheta_x = \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varphi^{-1}(y)} \vartheta_x dx dy = \int_{-\varepsilon}^{\varepsilon} \vartheta(\varphi^{-1}(y), y) dy,$$

implying

$$\int_{\Omega^-} |\vartheta_x| \geq \left| \int_{\Omega^-} \vartheta_x \right| = \left| \int_{-\varepsilon}^{\varepsilon} \vartheta(\varphi^{-1}(y), y) dy \right| \geq \int_{-\varepsilon}^{\varepsilon} \vartheta(\varphi^{-1}(y), y) dy = \int_{\Omega^+} \vartheta_x. \quad (2.132)$$

With an analogous computation on ϑ_y we get

$$\int_{\Omega^-} \vartheta_y = \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varphi(x)} \vartheta_y dy dx = \int_{-\varepsilon}^{\varepsilon} \vartheta(x, \varphi(x)) dx,$$

implying

$$\int_{\Omega^+} |\vartheta_y| \geq \left| \int_{\Omega^+} \vartheta_y \right| = \left| \int_{-\varepsilon}^{\varepsilon} \vartheta(x, \varphi(x)) dx \right| \geq \int_{-\varepsilon}^{\varepsilon} \vartheta(x, \varphi(x)) dx = \int_{\Omega^-} \vartheta_y. \quad (2.133)$$

Then, by (2.131), in order to have $\frac{d}{d\varepsilon} A(u + \varepsilon \vartheta)|_{\varepsilon=0^+} = 0$ we need to characterize equality case in (2.132), (2.133), which implies the following

- ϑ_x has constant sign on Ω^- (from the first inequality in (2.132))
- ϑ_y has constant sign on Ω^+ (from the first inequality in (2.133))
- $\vartheta(x, \varphi(x)) \geq 0$ for every $x \in (-\varepsilon, \varepsilon)$

Since ϑ has compact support in Ω (then it is zero on the boundary), we deduce that $\vartheta_x \leq 0$ in Ω^- and $\vartheta_y \leq 0$ in Ω^+ implying

$$\vartheta \geq 0 \text{ in } \Omega,$$

that yields $\int_{\Omega} \vartheta \geq 0$.

We have then two cases: either $\int_{\Omega} \vartheta = 0$ in Ω and we have the trivial variation, or $\int_{\Omega} \vartheta > 0$, then we have a variation which preserves the Area but increases the Volume and that contradicts the minimality of S . $\square$

Let's suppose now that $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = \text{sgn}_{\Omega^+}(\langle N, X \rangle) = 1$. Then, with an analogous argument one can prove that the only possible non trivial variation ϑ preserving the Area is such that

$$\vartheta_x \geq 0 \text{ in } \Omega^- \text{ and } \vartheta_y \geq 0 \text{ in } \Omega^+ \text{ and } \vartheta(x, \varphi(x)) \leq 0 \forall x$$

but that doesn't give any contradiction to minimality.

Let's consider the other type of foliation, namely we suppose that S^- foliated by Y and S^+ foliated by X . In this case we have the opposite to Lemma 2.4.5: if we suppose $\text{sgn}_{\Omega^-}(\langle N, X \rangle) = \text{sgn}_{\Omega^+}(\langle N, Y \rangle) = 1$ then we get that the only possible non trivial variation preserving the Area also increases the Volume, contradicting minimality. On the other hand, if we suppose $\text{sgn}_{\Omega^-}(\langle N, X \rangle) = \text{sgn}_{\Omega^+}(\langle N, Y \rangle) = -1$, then the only possible non trivial variation preserving the Area is such that

$$\vartheta_x \leq 0 \text{ in } \Omega^+ \text{ and } \vartheta_y \leq 0 \text{ in } \Omega^- \text{ and } \vartheta(x, \varphi(x)) \leq 0 \forall x$$

and that doesn't give any contradiction to minimality.

For the remaining cases, when $\varphi'(0) < 0$, we have the following results.

Lemma 2.4.6. *Let S^- and S^+ as before and suppose S^- foliated by X , S^+ foliated by Y and $\varphi'(0) < 0$. Then it's not possible that $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = -1$ and $\text{sgn}_{\Omega^+}(\langle N, X \rangle) = +1$.*

Proof. With the same argument as in Lemma 2.4.5, one can prove that the variation ϑ satisfies $\vartheta_x \geq 0$ in Ω^- , $\vartheta_y \leq 0$ in Ω^+ and $\vartheta(x, \varphi(x)) \geq 0$, which implies that if ϑ preserves the Area it also increases the Volume, contradicting minimality. $\square$

If we suppose that $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = +1$ and $\text{sgn}_{\Omega^+}(\langle N, X \rangle) = -1$, then, with an analogous argument one can prove that the only possible non trivial variation ϑ preserving the Area is such that

$$\vartheta_x \leq 0 \text{ in } \Omega^- \text{ and } \vartheta_y \geq 0 \text{ in } \Omega^+ \text{ and } \vartheta(x, \varphi(x)) \leq 0 \text{ for all } x$$

but that doesn't give any contradiction to minimality.

Let's consider, now, the other type of foliation, namely we suppose that S^- foliated by Y and S^+ foliated by X . In this case we have the opposite to Lemma 2.4.6: if we suppose $\text{sgn}_{\Omega^-}(\langle N, X \rangle) = -1$ and $\text{sgn}_{\Omega^+}(\langle N, Y \rangle) = +1$ then we get that the only possible non trivial variation preserving the Area also increases the Volume, contradicting minimality.

On the other hand, if we suppose $\text{sgn}_{\Omega^-}(\langle N, X \rangle) = +1$ and $\text{sgn}_{\Omega^+}(\langle N, Y \rangle) = -1$, then the only possible non trivial variation preserving the Area is such that

$$\vartheta_x \geq 0 \text{ in } \Omega^+ \text{ and } \vartheta_y \leq 0 \text{ in } \Omega^- \text{ and } \vartheta(x, \varphi(x)) \leq 0 \text{ for all } x$$

and that doesn't give any contradiction to minimality.

2.4.3 Recap on minimality and geometric sign conditions

Let E be a piecewise C^1 isoperimetric set, and $S = \{(x, y, u(x, y)), (x, y) \in \Omega \subset \mathbb{R}^2\} \subset \partial E$ be a z -graph, with $u : \Omega \rightarrow [0, \infty)$, $\Omega \subset \mathbb{R}^2$ open connected. Assume that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$ where Ω^+, Ω^- are open connected and $\Gamma := \Omega^+ \cap \Omega^- \cap \Omega$ is a planar curve. Assume also that $S^+ := u(\Omega^+)$ and $S^- := u(\Omega^-)$ are C^1 and call $\gamma = \{(x, y, u(x, y)) : (x, y) \in \Gamma\}$ the *interface curve*, with $\Gamma = \{(x, \varphi(x))\}$. With the following remark, we summarise the result we obtained in this section.

Remark 2.4.7. *Let N be the inward unit normal at ∂E and assume that there are no characteristic points on the smooth pieces of ∂E . Then, the following cases might occur:*

- if S^+ is foliated by X , S^- is foliated by Y and $\varphi'(0) > 0$, then $\text{sgn}_{\Omega^+}(\langle N, Y \rangle) = \text{sgn}_{\Omega^-}(\langle N, X \rangle) = -1$;
- if S^- is foliated by X , S^+ is foliated by Y and $\varphi'(0) > 0$, then $\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = \text{sgn}_{\Omega^+}(\langle N, X \rangle) = +1$;
- if S^+ is foliated by X , S^- is foliated by Y and $\varphi'(0) < 0$, then $\text{sgn}_{\Omega^-}(\langle N, X \rangle) = -\text{sgn}_{\Omega^+}(\langle N, Y \rangle) = +1$;
- if S^- is foliated by X , S^+ is foliated by Y and $\varphi'(0) < 0$, then $\text{sgn}_{\Omega^+}(\langle N, X \rangle) = -\text{sgn}_{\Omega^-}(\langle N, Y \rangle) = -1$;

2.5 Multiple points and blow-up

When studying the blow-ups of interface curves, one can recover more information on the geometry of an isoperimetric set around a multiple point, namely a point where two or more interface curves meet.

The first lemma we prove is a simple computation which justifies the definition we give of a blow-up.

Lemma 2.5.1. *Let $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^3$ be a C^2 curve such that $0 \in [\alpha, \beta]$, $\gamma(0) = 0$. If γ is horizontal, then $\dot{\gamma}_3(0) = \ddot{\gamma}_3(0) = 0$. If $\dot{\gamma}(0) \notin \text{span}\{X(0), Y(0)\}$, $\dot{\gamma}_3(0) \neq 0$.*

Proof. Assume that $\gamma \in C^2$ is horizontal. Then we have

$$\dot{\gamma}_3(t) = \frac{1}{2}(-\dot{\gamma}_1(t)\gamma_2(t) + \dot{\gamma}_2(t)\gamma_1(t)), \quad \forall t \in [\alpha, \beta],$$

which implies

$$\ddot{\gamma}_3(t) = \frac{1}{2}(-\ddot{\gamma}_1(t)\gamma_2(t) + \ddot{\gamma}_2(t)\gamma_1(t)), \quad \forall t \in [\alpha, \beta].$$

In particular, since $\gamma(0) = (0, 0, 0)$, then $\dot{\gamma}_3(0) = \ddot{\gamma}_3(0) = 0$.

If $\dot{\gamma}(0) \notin \text{span}\{X(0) = \partial_x, Y(0) = \partial_y\}$, the statement follows by writing $\dot{\gamma}(0) = \dot{\gamma}_1(0)X(0) + \dot{\gamma}_2(0)Y(0) + \dot{\gamma}_3(0)\partial_z$. $\square$

Definition 2.5.2 (Blow-up at 0 of C^2 curves). *Let $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^3$ be a C^2 curve such that $0 \in [\alpha, \beta]$, $\gamma(0) = 0$.*

If γ is horizontal, we define

$$\gamma^B : \mathbb{R} \rightarrow \mathbb{R}^3, \quad \gamma^B(\tau) = (\dot{\gamma}_1(0)\tau, \dot{\gamma}_2(0)\tau, 0), \quad \tau \in \mathbb{R}. \quad (2.134)$$

If $\dot{\gamma}(0) \notin \text{span}\{X(0), Y(0)\}$ we define

$$\gamma^B : \mathbb{R} \rightarrow \mathbb{R}^3, \quad \gamma^B(\tau) = (0, 0, \dot{\gamma}_3(0)\tau), \quad \tau \in \mathbb{R}. \quad (2.135)$$

To fix the notation we will use in the rest of the section, we define, now, the surface issued from a curve.

Definition 2.5.3 (Surface S_X (resp. S_Y) issued from a curve). *Let $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^3$ be a C^2 curve such that $0 \in [\alpha, \beta]$, $\gamma(0) = 0$. We let $S_X(\gamma)$ (resp. S_Y) be the C^1 surface foliated by X (resp. Y) issued from γ .*

The next proposition states the convergence of the blow-up of the surface issued from a curve γ to the surface issued from the blow-up of γ .

Proposition 2.5.4. *Let $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^3$ be a C^2 curve such that $0 \in [\alpha, \beta]$, $\gamma(0) = 0$. Assume that either γ is horizontal or $\dot{\gamma}(0) \notin \text{span}\{X(0), Y(0)\}$. Then $\delta_{\frac{1}{r}} S_X(\gamma)$ converges uniformly on compact sets to $S_X(\gamma^B)$ as $r \rightarrow 0$.*

Proof. Assume that γ is horizontal. By Lemma 2.5.1 we get

$$\gamma(t) = (\dot{\gamma}_1(0)t + o(t), \dot{\gamma}_2(0)t + o(t), o(t^2)) \quad \text{as } t \rightarrow 0 \quad (2.136)$$

Then, given $\tau \in \mathbb{R}$ we have

$$\delta_{\frac{1}{r}}(\gamma(r\tau)) = (\dot{\gamma}_1(0)\tau + \tau o(1), \dot{\gamma}_2(0)\tau + \tau o(1), \tau^2 o(1)), \quad \text{as } r \rightarrow 0^+, \quad (2.137)$$

yielding,

$$\lim_{r \rightarrow 0^+} \delta_{1/r} \gamma(r\tau) = \gamma^B(\tau), \quad (2.138)$$

locally uniformly w.r.t. $\tau \in \mathbb{R}$.

Similarly, if $\dot{\gamma}(0) \notin \text{span}\{X(0), Y(0)\}$, by Lemma 2.5.1 and a Taylor development of $\gamma(t)$ around $t = 0$, the same argument yields

$$\lim_{r \rightarrow 0^+} \delta_{1/r} \gamma(r^2\tau) = \gamma^B(\tau), \quad (2.139)$$

locally uniformly with respect to $\tau \in \mathbb{R}$.

To conclude, notice that we have $(\delta_{1/r})_* X = \frac{1}{r} X$. Hence, in both cases we deduce

$$\delta_{\frac{1}{r}}(e^{rsX}(\gamma(r^\alpha\tau))) \rightarrow e^{sX}(\gamma^B(\tau)), \quad r \rightarrow 0^+$$

locally uniformly with respect to $(\tau, s) \in \mathbb{R} \times [0, \infty)$, where $\alpha = 1$ if γ is horizontal and $\alpha = 2$ otherwise. This yields the statement. $\square$

From now on, we assume that, given an isoperimetric set E , there are no smooth characteristic points in ∂E .

Remark 2.5.5. Let $S = \{x \in \mathbb{R}^n : V(x) \leq 0\}$, for a certain $V : \mathbb{R}^n \rightarrow \mathbb{R}$, be a surface with normal N_S , X a vector field on S and Φ a diffeomorphism. Then

$$\langle N_{\Phi(S)}, \Phi_* X \rangle = \langle N_S, X \rangle.$$

Proof. If $S = \{x \in \mathbb{R}^n : V(x) \leq 0\}$, then $\Phi(S) = \{y \in \mathbb{R}^n : V \circ \Phi^{-1}(y) \leq 0\}$ and $\Phi_* X = D\Phi X$. The normal direction to S is determined by ∇V and the normal direction to ΦS is determined by $\nabla(V \circ \Phi^{-1})$. Moreover,

$$\nabla(V \circ \Phi^{-1}) = \left(\sum_{j=1}^n \partial_j V \partial_i \Phi_j^{-1} \right)_{i=1, \dots, n} = \nabla V \cdot D(\Phi^{-1}) = \nabla V \cdot (D\Phi)^{-1}.$$

Then

$$\langle N_{\Phi(S)}, \Phi_* X \rangle = \left(\nabla V \cdot (D\Phi)^{-1} \right) \cdot (D\Phi X) = \nabla V \cdot X.$$

$\square$

Now we consider every case of Remark 2.4.7 and we want to see what happens after blowing-up.

Theorem 2.5.6. With the same hypothesis of observation 2.4.7, if the interface curve is not horizontal, then the blow-up of the set lies in the convex part enclosed by the blow-ups of S^+ and S^- .

Proof. We will prove the statement only in one case, the others will be analogous. Suppose $S^+ = S_X$, $S^- = S_Y$, with interface curve $\gamma = \{(x, y, u(x, y)), (x, y) \in \Gamma\}$, where $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the function defining the set, as in Remark 2.4.7. We also suppose that $\Gamma = \{(x, \varphi(x))\}$ and $\varphi'(0) > 0$. Because γ is not horizontal, by Lemma 2.5.1, $\dot{\gamma}_3(0) \neq 0$, then by Proposition 2.5.4 its blow-up is

$$\gamma^B(\tau) = (0, 0, \dot{\gamma}_3(0)\tau).$$

Again by Proposition 2.5.4, recalling Definition 2.5.3, we know that the blow-up of $S^+ = S_X(\gamma)$ and $S^- = S_Y(\gamma)$ are $S_X(\gamma^B)$ and $S_Y(\gamma^B)$ respectively. Explicitly:

$$\begin{aligned} S_X(\gamma^B) &= \{y = 0, x \leq 0, \text{sign}(z) = \text{sign}(\dot{\gamma}_3(0))\} \\ S_Y(\gamma^B) &= \{x = 0, y \leq 0, \text{sign}(z) = \text{sign}(\dot{\gamma}_3(0))\}. \end{aligned}$$

Their *inward* normal vectors are

$$\begin{aligned} N_{S_X(\gamma^B)} &= \pm \nu_X = (0, \pm 1, 0) \\ N_{S_Y(\gamma^B)} &= \pm \nu_Y = (\pm 1, 0, 0). \end{aligned} \tag{2.140}$$

Because $X^B = X$, $Y_B = Y$, remarks 2.5.5 and 2.4.7 yield

$$\begin{aligned} \text{sign}\langle N_{S_X(\gamma^B)}, Y \rangle &= \text{sign}\langle N_{S_X(\gamma)}, Y \rangle = -1 \\ \text{sign}\langle N_{S_Y(\gamma^B)}, X \rangle &= \text{sign}\langle N_{S_Y(\gamma)}, X \rangle = -1. \end{aligned}$$

Then (2.140) becomes

$$\begin{aligned} N_{S_X(\gamma^B)} &= (0, -1, 0) \\ N_{S_Y(\gamma^B)} &= (-1, 0, 0) \end{aligned} \tag{2.141}$$

and this yields the statement. $\square$

Definition 2.5.7. Given γ_1 and γ_2 interface curves, we say that γ_1 is adjacent to γ_2 if the surface enclosed by γ_1 and γ_2 is smooth and foliated by X or Y .

Proposition 2.5.8. Let γ be a horizontal interface curve adjacent to a vertical one η . Then either $\gamma_1^B \equiv 0$ or $\gamma_2^B \equiv 0$.

Proof. By Proposition 2.5.4 we know that the blow-ups of γ and η are

$$\begin{aligned} \gamma^B(\tau) &= (\dot{\gamma}_1(0)\tau, \dot{\gamma}_2(0)\tau, 0) \\ \eta^B(\tau) &= (0, 0, \dot{\eta}_3(0)\tau). \end{aligned} \tag{2.142}$$

It is not restrictive to suppose $\dot{\eta}_3(0) > 0$. We also know that the foliation property is preserved. Suppose that the surface S enclosed by γ and η is foliated by X , namely

$$S = S_X(\gamma) = S_X(\eta).$$

Again by Proposition 2.5.4, recalling definition 2.5.3, we have

$$S_X^B = S_X(\gamma^B) = S_X(\eta^B), \tag{2.143}$$

but, explicitly

$$\begin{aligned} S_X(\eta^B) &= \{(t, 0, z) \in \mathbb{R}^3 : t \geq 0, z \geq 0\} \\ S_X(\gamma^B) &= \{(\dot{\gamma}_1(0)\tau + t, \dot{\gamma}_2(0)\tau, -\frac{\dot{\gamma}_1(0)}{2}\tau t), t \geq 0, \tau \geq 0\}. \end{aligned}$$

Now (2.143) implies that $S_X(\eta^B) \cap \{z = 0\} = \gamma^B$, thus

$$\dot{\gamma}_2(0)\tau = 0 \quad \forall \tau > 0 \Rightarrow \dot{\gamma}_2(0) = 0,$$

namely $\gamma_2^B \equiv 0$. Finally, if we consider the foliation by Y instead of X , with the same argument, we prove $\gamma_1^B \equiv 0$. $\square$

Proposition 2.5.9. *It is not possible that the blow-ups of two horizontal interface curves lie in the same quarter.*

Proof. Let γ and η be two horizontal interface curves. By Proposition 2.5.4, their blow-ups are

$$\begin{aligned}\gamma^B : \mathbb{R} &\rightarrow \mathbb{R}^3, \quad \gamma^B(\tau) = (\dot{\gamma}_1(0)\tau, \dot{\gamma}_2(0)\tau, 0) \\ \eta^B : \mathbb{R} &\rightarrow \mathbb{R}^3, \quad \eta^B(\tau) = (\dot{\eta}_1(0)\tau, \dot{\eta}_2(0)\tau, 0).\end{aligned}$$

Call $\alpha = \frac{\dot{\gamma}_2(0)}{\dot{\gamma}_1(0)}$ and $\beta = \frac{\dot{\eta}_2(0)}{\dot{\eta}_1(0)}$ their slopes. It's not restrictive to suppose $\alpha < \beta$ and $\alpha, \beta \in]0, \frac{\pi}{2}[$. It is also not restrictive to suppose that the surface S enclosed by γ and η is foliated by X , namely

$$S = S_X(\gamma) = S_X(\eta).$$

Then, because $X^B = X$, also the blow-up of S , call it S^B is foliated by X and, by Proposition 2.5.4, we have

$$S^B = S_X(\gamma^B) = S_X(\eta^B). \quad (2.144)$$

Now, if we consider $S_X(\eta^B)$ we have

$$S_X(\eta^B) = \{(x+t, \beta x, -\frac{\beta xt}{2}) : x \geq 0, t \geq 0\}$$

and, by (2.144) there must exist (x, t) such that

$$(x+t, \beta x, -\frac{\beta xt}{2}) \in \gamma^B$$

but that is impossible because $-\frac{\beta xt}{2} \neq 0 \forall x, t \neq 0$ and for all $p \in \gamma^B$, $p_3(\tau) = 0 \forall \tau > 0$. $\square$

We want to stress that the results we presented in this section and in the previous one are partial. In fact, these were the first results we obtained with our study of the isoperimetric problem on the Heisenberg group equipped with a sub-Finsler structure. We then moved on to a more general approach to the problem that led to the more complete and interesting result presented in the first three sections of this chapter.

Bibliography

- [1] Abresch, U., *Constant mean curvature tori in terms of elliptic functions*. Journal für die reine und angewandte Mathematik **374** (1987), 169-192.
- [2] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large*, V. Vestnik, Leningrad Univ. 13, No. **19** (1958), 5-8. Amer. Math. Soc. Trans. (Series 2) 21, 412-416.
- [3] A. D. Alexandrov, *A characteristic property of spheres*. Ann. Math. Pura Appl. **58** (1962), 303-315.
- [4] L. Alías, A. Brasil and O. Perdomo (2006). *Stable constant mean curvature hypersurfaces in the real projective space*. Manuscripta Mathematica. 121. 329-338. 10.1007/s00229-006-0038-2.
- [5] F. Almgren, *Optimal isoperimetric inequalities*. Indiana Univ. Math. J. **35** (1986), 451-547.
- [6] L. Ambrosio (editor), *Geometric Measure Theory and Real Analysis*, 2014. ISBN 978-88-7642-522-6, e-ISBN 978-88-7642-523-3.
- [7] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*. Adv. Math., 159(1):51–67, 2001.
- [8] A. A. Ardentov, E. Le Donne, and Y. L. Sachkov. *Sub-Finsler geodesics on the Cartan group*. Regul. Chaotic Dyn., **24** (1): 36–60, 2019.
- [9] J.-P. Aubin and H. Frankowska, *Set-valued analysis*. Modern Birkhauser Classics. Birkhauser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition [MR1048347].
- [10] Z. M. Balogh, *Size of characteristic sets and functions with prescribed gradient*. J. Reine Angew. Math., 564:63–83, 2003.
- [11] D. Barilari, U. Boscain, E. Le Donne, and M. Sigalotti, *Sub-Finsler structures from the time-optimal control viewpoint for some nilpotent distributions*, J. Dyn. Control Syst. **23** (3), 547–575 (2017).
- [12] J. Lucas Barbosa and Manfredo do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), no. 3, 339–353. MR 731682.

- [13] J. Lucas Barbosa, Manfredo do Carmo, and Jost Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), no. 1, 123–138. MR 917854.
- [14] E. Battaglia, R. Monti and A. Righini, *Stable hypersurfaces in the complex projective space*. Annali di Matematica **199**, 231–251 (2020). <https://doi.org/10.1007/s10231-019-00875-4>
- [15] V. N. Berestovskii, *Homogeneous manifolds with an intrinsic metric. II*, Sibirsk. Mat. Zh. **30** (1989), no. 2, 14–28, 225.
- [16] V. N. Berestovskii, *The structure of locally compact homogeneous spaces with an intrinsic metric*, Sibirsk. Mat. Zh. **30** (1989), no. 1, 23–34.
- [17] V. N. Berestovskii. *Geodesics of nonholonomic left-invariant inner metrics on the Heisenberg group and isoperimetrics of the Minkowski plane*. Sibirsk. Mat. Zh., **35** (1): 3–11, i, 1994.
- [18] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [19] S. Bochner, D. Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. (2) **47** (1946), 639–653.
- [20] Böhme R., Tomi, F., *Zur Struktur der Lösungsmenge des Plateauproblems*. Math Z 133, 1–29 (1973).
- [21] O. Bolza, *Vorlesungen fiber Variationsrechnung*, Berlin-Leipzig: Teubner 1909.
- [22] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*. 1988. DOI 10.1007/978-3-662-07441-1. Original Russian edition published by Nauka, Leningrad, 1980.
- [23] E. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*. Ann. Math. **17**, 177–191 (1938)
- [24] L. Capogna, D. Danielli, and N. Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*. Comm. Anal. Geom., 2(2):203–215, 1994.
- [25] L. Capogna, D. Danielli, S. Pauls and J. Tyson, Jeremy. *An Introduction to the Heisenberg Group and the sub-Riemannian Isoperimetric Problem*. 2007. DOI 10.1007/978-3-7643-8133-2
- [26] I. Chavel, *Isoperimetric inequalities. Differential geometric and analytic perspectives*, Cambridge Tracts in Mathematics. **145**. Cambridge: Cambridge University Press. xii, 268 p. , 2001.
- [27] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang. *Minimal surfaces in pseudohermitian geometry*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **4** (1): 129–177, 2005.

- [28] J. N. Clelland and C. G. Moseley, *Sub-Finsler geometry in dimension three*, Differential Geom. Appl. **24** (2006), no. 6, 628–651.
- [29] Thomas E. Cecil and Patrick J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 481–499. MR 637703.
- [30] W. P. Dayawansa and C. F. Martin, *A converse Lyapunov theorem for a class of dynamical systems which undergo switching*. IEEE Trans. Automat. Control, **44** (4): 751–760, 1999.
- [31] E. De Giorgi, *Selected papers*. Springer, The Netherlands, 2006.
- [32] M. Delgadino and F. Maggi, *Alexandrov’s theorem revisited*. Analysis and PDE, 2017.
- [33] H. Federer, *Geometric Measure Theory*. NewYork: Springer-Verlag. (1969).
- [34] A. Figalli, F. Maggi, A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*. Invent math (2010) **182**: 167–211 DOI 10.1007/s00222-010-0261-z
- [35] V. Franceschi, R. Monti *Isoperimetric Problem in H-type groups and Grushin spaces*. Rev. Mat. Iberoam. **32** (2016) 1227–1258.
- [36] V. Franceschi, R. Monti, A. Righini, *Stable hypersurfaces in the real projective space* (in preparation)
- [37] V. Franceschi, R. Monti, A. Righini, M. Sigalotti, *The isoperimetric problem for regular and crystalline norms in $\mathbb{H}^1$* (submitted)
- [38] B. Franchi, S. Gallot and R. L. Wheeden, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300** (1994), 557–571.
- [39] B. Franchi, R. Serapioni, and F. Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*. Houston J. Math., 22(4):859–890, 1996.
- [40] N. Fusco, F. Maggi, A. Pratelli, *The sharp quantitative isoperimetric inequality*. Annals of Mathematics, **168** (2008), 941–980
- [41] A. M. Gleason, *Groups without small subgroups*, Ann. of Math. (2) **56** (1952), 193–212.
- [42] N. Garofalo and D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math **49**(1996), 1081–1144.
- [43] W. Goldman, *Complex Hyperbolic Geometry*. Oxford University Press, Oxford, 1999.

- [44] M. Gromov, *Structures métriques pour les variétés riemanniennes*. Edited by J. La- fontaine and P. Pansu. Textes Mathématiques, 1, CEDIC, Paris, 1981. iv+152 pp.
- [45] M. Gromov, *Isoperimetric inequalities in Riemannian manifolds*. In *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes Math. **1200** (1986), Appendix I, 114-129. Berlin: Springer-Verlag.
- [46] Howards H., Hutchings M., Morgan F., *The isoperimetric problem on surfaces*. Amer. Math. Monthly **106**, 430-439 (1999).
- [47] Hsiang, W.Y., Teng, Z.H., Yu, W., *New examples of constant mean curvature immersions of $(2k - 1)$ -spheres into Euclidean $2k$ -space*. Ann. Math. **117**, 609-625 (1983).
- [48] A. Hurtado, M. Ritoré, C. Rosales, *The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$* , Adv. Math. **224** (2010), 561–600.
- [49] Hurwitz A., *Sur le problème des isopérimètres*. C. R. Acad. Sci. Paris **132**, 401-403 (1901).
- [50] K. Lange. *MM optimization algorithms*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2016.
- [51] G. P. Leonardi and S. Rigot, *Isoperimetric sets on Carnot groups*, Houston J. Math. **29** (2003), 609-637.
- [52] L. V. Lokutsievskii. *Convex trigonometry with applications to sub-Finsler geometry*. Mat. Sb., **210** (8): 120–148, 2019.
- [53] Yoshiaki Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976), no. 3, 529–540. MR 0407772.
- [54] P. Mason, U. Boschain, and Y. Chitour. *Common polynomial Lyapunov functions for linear switched systems*. SIAM J. Control Optim., **45** (1): 226–245, 2006.
- [55] D. Montgomery, L. Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974, Reprint of the 1955 original.
- [56] R. Monti, *Heisenberg isoperimetric problem. The axial case*, Adv. Calc. Var. **1** (2008), no. 1, 93-121.
- [57] R. Monti and M. Rickly, *Convex isoperimetric sets in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), no. 2, 391-415.
- [58] Katsumi Nomizu, *Some results in E. Cartan's theory of isoparametric families of hypersurfaces*, Bull. Amer. Math. Soc. **79** (1973), 1184–1188. MR 0326625.
- [59] P. Pansu, *Une inégalité isopérimétrique sur le group de Heisenberg*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 127-130.

- [60] P. Pansu, *An isoperimetric inequality on the Heisenberg group. Conference on differential geometry on homogeneous spaces (Turin, 1983)*, Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 159-174.
- [61] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1-60.
- [62] J. Pozuelo and M. Ritoré. *Pansu-Wulff shapes in H^1* . arXiv e-prints, page arXiv:2007.04683, July 2020.
- [63] R. C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*. Indiana Univ. Math. J. **26**, 459-472 (1977).
- [64] M. Ritoré, *A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$* . Calc. Var. Partial Differential Equations **44** (2012), no. 1-2, 47-60.
- [65] M. Ritoré, A. Ros, *Stable constant mean curvature tori and the isoperimetric problem in three space forms*, Commentarii Mathematici Helvetici **67**, 293-305 (1992).
- [66] M. Ritoré and C. Rosales, *Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$* , Adv. Math. **219**(2008), no. 2, 633-671.
- [67] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.
- [68] Antonio Ros, *Spectral geometry of CR-minimal submanifolds in the complex projective space*, Kodai Math. J. **6** (1983), no. 1, 88-99. MR 698330.
- [69] Antonio Ros, *On spectral geometry of Kähler submanifolds*, J. Math. Soc. Japan **36** (1984), no. 3, 433-448. MR 746704.
- [70] Antonio Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberoamericana, Volume 3, Issue 3, 1987, pp. 447-453, 1987.
- [71] A. Ros, *Compact surfaces with constant scalar curvature and a congruence theorem*. J. Diff. Geom. **27**, 215-220 (1988).
- [72] C. Rosales, *Complete stable CMC surfaces with empty singular set in Sasakian sub-Riemannian 3-manifolds*. Calc. Var. Partial Differential Equations **43**, no. 3-4 (2012), 311-345.
- [73] M. Ross, *Stability properties of complete two-dimensional minimal surfaces in Euclidean space*, PhD Thesis, Berkeley, 1989.
- [74] Y. L. Sachkov. *Periodic Controls in Step 2 Strictly Convex Sub-Finsler Problems*. Regul. Chaotic Dyn., **25** (1): 33-39, 2020.

- [75] E. Schmidt, *Über die isoperimetrische Aufgabe im n -dimensionalen Raum konstanter negativer Krümmung. I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n -dimensionalen hyperbolischen Raum.* Math. Z. **46** (1940), 204-2
- [76] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen.* Springer-Verlag, Berlin, 1890.
- [77] J. Steiner (1838), *Einfache Beweise der isoperimetrischen Hauptsätze.* J. Reine Angew. Math. **18**, 281-296. Reprinted, *Gesammelte Werke.* Bronx, NY: Chelsea (1971) (reprint of 1881-1882 edn.), Vol. II, 75-91.
- [78] J. Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général. premier mémoire,* J. Reine Angew. Math. **24** (1842) 93-162.
- [79] J. Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général. second mémoire,* J. Reine Angew. Math. **24** (1842) 189-250.
- [80] Shin-Sheng Tai, *Minimum imbeddings of compact symmetric spaces of rank one,* J. Differential Geometry **2** (1968), 55-66. MR 0231395.
- [81] Ryoichi Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures,* J. Math. Soc. Japan **27** (1975), 43-53. MR 0355906.
- [82] Ryoichi Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures. II,* J. Math. Soc. Japan **27** (1975), no. 4, 507-516. MR 0400120.
- [83] Topping P., *The optimal constant in Wente's L^∞ estimate.* Comment. Mat. Helv. **139**, 316-328 (1997).
- [84] Viana, Celso. (2019). *Isoperimetry and volume preserving stability in real projective spaces.* arXiv:1907.09445.
- [85] K. Weierstrass, *Mathematische Werke,* vol. 7, Mayer and Muller, Berlin, 1927.
- [86] Wente H., *Counter-example to the Hopf conjecture.* Pac. J. Math. **121**, 193-244 (1986).