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Convergence results for models of collective dynamics

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To my family.

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Contents

Abstract

In this thesis we treat models of collective behavior in networks, where agents or particles interact among each other following some specific dynamics. We focus on three specific models that we now briefly present and study their properties. In particular, we treat two different problems: the rigorous derivation of the Lighthill-Whitham-Richards model for traffic flow from the Follow-the-Leader model and the emergent behavior in cooperative systems under persistent excitation.

In Chapter 1, we deal with the Follow-the-Leader model (FtL), which is a finite-dimensional dynamical system describing the motion of N cars on a road lane, in which each car travels with a velocity that depends on its relative distance with respect to the one immediately in front. The Lighthill-Whitham-Richards (LWR) model is a hyperbolic conservation law, where the solution is a macroscopic density that typically represents the dynamics of the average spatial concentration of vehicles. With the FtL model we build a microscopic density which approximates the macroscopic one. Our main goal is to prove that the dynamics given by the FtL converges to the one given by LWR. This occurs under suitable convergence requests on the initial data. Additional stability results of the FtL model are also presented.

In Chapters 2 - 3, we study cooperative systems, which are models of interacting agents in which interaction is always attractive. The goal is to study the asymptotic behavior in time towards reaching consensus (in first-order models) or flocking (in second-order models). We provide sufficient conditions for the formation of asymptotic consensus or flocking, in the case in which dynamics are subject to communication failures between agents, if the failure satisfies a suitable persistent excitation condition. We study such phenomena for first- and second-order systems, both in the finite and infinite dimensional settings via the classical mean-field limit.

Introduction

In recent years, the study of networks with interacting agents or particles has become a significant focus in mathematics, especially when dealing with real-world situations [2, 7, 17, 47, 48, 52]. This has led to the development of models that attempt to capture and understand complex behaviors in various fields: traffic flow [13, 14, 15, 56], politics [8, 39], biology [45, 50, 57, 59], pedestrian flow [1, 5, 34, 38], linguistics [25, 26, 27], etc. Whether derived from observations in the synchronized movements of flocks of birds, the coordinated actions of fish schools, or from intricate movements within human crowds, mathematical models of collective behavior attempt to underscore the emergence of order and patterns from individual interactions. Many scales can be considered when developing a mathematical model, and we can find in the literature two main types: microscopic and macroscopic scales (see [54] for a review).

In microscopic models, also known as agent-based models or particle models, the dynamics of each agent is taken into account, by typically using an ordinary or stochastic differential equation. We then form a system of such differential equations accounting for the behavior of the whole set of agents representing the evolution in time of each agent, through interactions with all the other agents and the environment. Microscopic models provide a detailed and granular representation of individual entities and their interactions, which helps in the realistic representation of the phenomenon we are dealing with. They can also incorporate variations of individual parameters and individual heterogeneity, leading to the possibility of studying a wide range of scenarios. However, although the modelling and simulation accuracy might be appreciable, the computational demands of the simulation of such systems increase tremendously as the number of agents grows. Moreover, the immediate output of the evolution of each single agents might not be useful in practice, as the focus may instead rely on quantities like statistical averages, which then leads us to additional post-processing. This makes us therefore consider the second type of models, the macroscopic models.

In macroscopic models, also known as continuum models, the dynamics of averaged quantities such as the mean density or mean velocity of agents is taken into account by using partial differential equations. We thus focus on the solution of said partial differential equation, which represents the evolution of the mean density or velocity of agents. One prominent advantage to be immediately appreciated, as pointed out in the presentation of microscopic models in the previous paragraph, is the computational efficiency. Indeed, the reduction in detail coming from the simplification of complex systems offered by macroscopic models often results in a significant decrease in computational demands, especially for large-scale systems. Another advantage is the very fact that it puts an emphasis on system-level properties, such as average density, velocity, or concentration, which simplifies the interpretation of results and aids in extracting key information. This can result in the capture of global trends and emerging patterns, such as the dynamics of fluid flows. Finally, another important advantage is its crucial role in policy design: in fields such as urban planning, traffic engineering, and public health, macroscopic models prove valuable for designing effective policies, since they provide a broad description of system-level responses to interventions without the need to account for individual variations [54, 55]. However, some disadvantages are present as well, mainly based on the loss of individual identities of agents due to the consideration of average quantities. Indeed, we now lose individual-level details and assume homogeneity of agents, which can be a serious drawback in scenarios where it is critical to study the individual behavior of special agents. They may not be suitable at not large enough scales, where the aggregate approach might oversimplify the dynamics. The assumption of homogeneous parameters across the system may not hold in certain cases, affecting the accuracy of predictions and limiting the model's robustness, which might also render the calibration of macroscopic models more complex, see e.g. [60].

Depending on the application, we therefore might want to use either microscopic models, macroscopic models or a combination of both. The natural question would be to ask if there is a relationship between them, e.g. whether a microscopic model has a corresponding macroscopic model and if the properties are conserved. In such a case, the following question prevails: given a microscopic model, what happens when the number of agents "grows to infinity"? Naturally, as the number of agents grows, it becomes increasingly unmanageable to deal with all the equations describing the behavior of each agent. This is called the *curse of dimensionality*. This is one of the most classical problems in kinetic theory: going from particle to continuum descriptions, known as *mean-field limit*, introduced for the first time in the context of gas dynamics [12].

This thesis explores microscopic models and their corresponding macroscopic models, particularly in the context of traffic flow and interacting agents in time-varying networks. In particular, we focus on three microscopic models: the Follow-the-Leader model as well as first- and second-order cooperative models. The first model is applied to traffic flow in one dimension. Here, we consider a one-dimensional lane of N cars, that are considered as moving particles. These particles follow a nearest neighbour type interaction, and in particular each one moves with a velocity which is proportional to the distance with respect to the particle in front. The second and third models are applied to a general network of agents, which could be interpreted as people, animals, opinions, votes, etc. In these systems, agents communicate with each other and each agent seeks to agree with the agents with whom it is interacting. In the case of first-order models and second-order models, the aim is to study the formation of *consensus* and *flocking*, respectively. Once we thoroughly study the microscopic behavior of these three models, we then study the rigorous derivation of the corresponding macroscopic model in the case of the Follow-the-Leader model and the conservation of properties at the microscopic level (independent of the number of agents) in the macroscopic models in the case of first- and second-order cooperative models.

This thesis is organized as follows:

- In Chapter 1 we present a new result regarding the mean-field limit of the Follow-the-Leader, which is a microscopic model describing the motion of N cars on a road lane, in which each car travels with a velocity that depends on its relative distance with respect to the one immediately in front. We provide a convergence result for general discretization schemes, extending results given in [32]. We furthermore provide a stability result concerning the convergence of the approximating profiles due to two possibly different discretization schemes.

- In Chapter 2 we consider a class of first-order cooperative systems, i.e. where agents attract each other. We study the asymptotic formation of consensus under possible communication failures between agents. We require the communication to be sufficiently frequent, encoding it into a persistent excitation condition, and to be strong enough. We show that consensus holds both in the finite- and infinite-dimensional settings. The main novelty is the technique introduced, which is to identify and treat the "worst-case scenario", i.e. the specific combination of communication failures between agents which drives them the furthest from consensus.

- In Chapter 3 we consider a problem similar to Chapter 2, when the cooperative system is of second-order. The goal is then to study the asymptotic formation of unconditional flocking. We show here that the "worst case scenario" is the same as in the case of first-order models, and proceed to treat it and show that flocking holds under suitable assumptions. We then show that such a property is again conserved in the classical mean-field setting.

Chapter 1:
On the continuum limit of the Follow-the-Leader Model

Among traffic flow models, we find two main types: microscopic models and macroscopic models. We briefly present two models that are intimately related: the Follow-the-Leader (FtL) model and the Lighthill-Whitham-Richards (LWR) model.

Let us first briefly describe the FtL model. We consider $N+1$ cars on a one-dimensional road lane. Let $\{x_j^N(0)\}_{j=0}^N$ denote the initial positions of the cars evolving in time according to the FtL dynamics. We have a trajectory $\{x_j^N(t)\}_{j=0}^N$, where each $x_j^N(t)$ travels with a velocity depending on the distance with respect to the car immediately in front of it $x_{j+1}^N(t)$. The leader $x_N^N(t)$ has no cars in front and thus it travels with the maximum velocity $v_{\max}$. A discrete function $\rho^{E,N}$, that we denote by *Eulerian discrete density*, composed of $N \in \mathbb{N}$ regions each of mass $1/N$ is then defined by $\{x_j^N\}_{j=0}^N$ as

$$\rho^{E,N}(x) := \sum_{j=0}^{N-1} \frac{1/N}{x_{j+1}^N - x_j^N} \chi_{[x_j^N, x_{j+1}^N)}(x) \qquad x \in \mathbb{R}. \tag{1.0.1}$$

The reason why the term Eulerian is used is explained in Section 1.2. We now briefly describe the classical LWR model

$$\begin{cases} \rho_t + (f(\rho))_x = 0, \quad t > 0 \quad x \in \mathbb{R} \\ \rho(0,x) = \bar{\rho}(x) \quad x \in \mathbb{R} \end{cases} \tag{1.0.2}$$

with $\bar{\rho}$ a given initial data with compact support. The variable ρ describes a macroscopic density of cars, and the flux $f(\rho)$ at a point $x \in \mathbb{R}$ represents the number of cars passing through the given point $x \in \mathbb{R}$ per unit of time. We consider the following flux from now on:

$$f(\rho) := \rho v(\rho).$$

We also assume that the maximal admissible density is $\rho_{max} := 1$ and that $\|\bar{\rho}\|_{L^1(\mathbb{R})} = 1$.

From now on, we assume that the velocity function $v(\rho)$ satisfies the following assumptions:

$$v \in \mathrm{Lip}([0, \rho_{\max}]) \ \text{ with Lipschitz constant } L, \qquad v(\rho_{\max}) = 0, \qquad v \ \text{ is decreasing.} \tag{V1}$$

We also use the notation $v_{max} := v(0)$. In some instances, we also make use of an additional assumption on the velocity:

$$\text{the map } [0, +\infty) \ni \rho \mapsto \rho v'(\rho) \in [0, +\infty) \text{ is non-increasing.} \tag{V2}$$

Now that we have a microscopic density $\rho^{E,N}$ and a macroscopic density ρ, we would like to answer the following question: which properties of the initial data and/or of the convergence of the discretized initial data ensure convergence of the microscopic density $\rho^{E,N}$ to the macroscopic one ρ?

More precisely: let an initial configuration $\{x_j^N(0)\}_{j=0}^N$ be given, and consider the discrete approximation sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$, with $\{x_j^N(t)\}_{j=0}^N$ subject to the FtL dynamics. Does this sequence converge to the solution ρ of the Cauchy problem (1.0.2) when $N \to \infty$? In what topology and how arbitrary can the initial positioning be? Our main result answers to this question. Before introducing the result, we first present a fundamental condition on the support at initial time of $\rho^{E,N}$. This assumption on initial support condition replaces the requirement of the scheme in [32] that $x_N^N(0) - x_0^N(0) = \overline{x}_{max} - \overline{x}_{min}$, where $\overline{x}_{max}, \overline{x}_{min}$, denote the extremal points of the convex hull of the support of $\overline{\rho}$. Here instead, it may well happen that $x_0^N(0) < \overline{x}_{min}$ and $x_N^N(0) > \overline{x}_{max}$.

Definition 1.0.1 (Uniformly bounded initial support condition)**.** *We say that $\{x_j^N(t)\}_{j=0}^N$ satisfies the condition of uniformly bounded initial support if there exists a constant $K_1 > 0$ independent of N, such that for all $N \in \mathbb{N}$ there holds*

$$x_N^N(0) - x_0^N(0) < K_1. \tag{1.0.3}$$

Theorem 1.0.1. *Assume that the velocity map v satisfy* (V1)*. Let $\bar{\rho} \in L^\infty(\mathbb{R};[0,1])$ be with compact support and such that $\|\bar{\rho}\|_{L^1(\mathbb{R})} = 1$. Let $\{x_j^N(t)\}_{j=0}^N$ be solutions of the FtL system* (1.1.3) *that satisfy the uniformly bounded initial support condition* (1.0.3)*. Consider the corresponding Eulerian discrete density $\rho^{E,N} \in L^\infty([0,+\infty) \times \mathbb{R};[0,1])$ defined by* (1.0.1)*. Assume that*

$$\rho^{E,N}(0) \rightharpoonup \bar{\rho}, \tag{1.0.4}$$

and that one of the following two conditions hold:

1. *$\bar{\rho} \in BV(\mathbb{R})$ and there exists $K_2 > 0$ such that $\mathrm{TV}(\rho^{E,N}(0);\mathbb{R}) < K_2$ for all N, i.e. such that*

$$\left(\frac{1}{x_1^N(0) - x_0^N(0)} + \frac{1}{x_N^N(0) - x_{N-1}^N(0)} + \sum_{j=0}^{N-2} \left| \frac{1}{x_{j+2}^N(0) - x_{j+1}^N(0)} - \frac{1}{x_{j+1}^N(0) - x_j^N(0)} \right| \right) < N\,K_2,$$

   for all N;

2. *the velocity function v satisfies also* (V2)*.*

Then the sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges strongly to the weak entropy solution ρ of the Cauchy problem (1.0.2) *in $L^1_{\mathrm{loc}}([0,+\infty) \times \mathbb{R};[0,1])$.*

Remark 1.0.1. It is important to note that the main difference with respect to [32, Theorem 3] is the replacement of the specific discretizaton scheme for the initial data adopted in [32], with a general one satisfying the hypothesis (1.0.4). Indeed, the discretization scheme provided in [32] satisfies a stronger assumption that implies (1.0.4). Here, we show that any other discretization scheme satisfying the hypothesis (1.0.4) is valid as well. Another novelty here is the proof of the convergence of the sequence of Eulerian discrete density $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$, which is based on the convergence of the cumulative and pseudoinverse functions associated to $\rho^{E,N}$, and on the 1-Wasserstein convergence of $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$. This proof is simpler than the one presented in [32, Theorem 3], where the authors achieve the L^1-compactness of $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ relying on a generalization of the Aubin-Lions lemma.

Another contribution of this article is the following stability result with respect to the 1-Wasserstein distance W_1. It is a microscopic stability result for two different initial discretization schemes, which in turn yields a stability result with respect to the L^1 norm that is uniform in time.

Theorem 1.0.2 (Discrete Eulerian Stability Theorem). *Assume that the velocity map v satifies* (V1). *Let $\{x_j^N(t)\}_{j=0}^N$,$\{\tilde{x}_j^N(t)\}_{j=0}^N$ be solutions of the FtL system* (1.1.3) *that satisfy the condition of the uniformly bounded initial support* (1.0.3). *Consider the corresponding Eulerian discrete densities $\rho^{E,N}$, $\tilde{\rho}^{E,N} \in L^\infty(([0,+\infty) \times \mathbb{R}); [0,1])$ defined by* (1.0.1). *Then, for all $T > 0$, and for all $N \in \mathbb{N}$, there holds*

$$\sup_{t\in[0,T]} W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) \leq W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) + \\ + 2LT \sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))|, \tag{1.0.5}$$

L being the Lipschitz constant of v. Moreover, if there holds $x_N^N(0) = \tilde{x}_N^N(0)$ for all $N \in \mathbb{N}$, and

$$\lim_{N\to+\infty} \sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))| = 0, \tag{1.0.6}$$

then the following two properties are satisfied:

1. *if there exists $K_2 > 0$ such that $\mathrm{TV}\left(\rho^{E,N}(0); \mathbb{R}\right), \mathrm{TV}\left(\tilde{\rho}^{E,N}(0); \mathbb{R}\right) < K_2$ for all N, then for all $T > 0$ there holds*

$$\lim_{N\to+\infty} \sup_{t\in[0,T]} \left\| \rho^{E,N}(t) - \tilde{\rho}^{E,N}(t) \right\|_{L^1(\mathbb{R})} = 0; \tag{1.0.7}$$

2. *if the velocity v satisfies also* (V2), *then for all $T > 0$ there holds*

$$\lim_{k\to+\infty} \sup_{t\in[1/k,T]} \left\| \rho^{E,N_k}(t) - \tilde{\rho}^{E,N_k}(t) \right\|_{L^1(\mathbb{R})} = 0, \tag{1.0.8}$$

   for some subsequences $\{\rho^{E,N_k}\}_k$, $\{\tilde{\rho}^{E,N_k}\}_k$.

Remark 1.0.2. If $x_N^N(0) = \tilde{x}_N^N(0)$ for all N and there holds (1.0.6), then one can show that (see Proposition 1.4.1)

$$\lim_{N\to+\infty} W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) = 0\,, \tag{1.0.9}$$

which implies that $\rho^{E,N}(0) - \tilde{\rho}^{E,N}(0) \rightharpoonup 0$. Thus, letting $\bar{\rho}, \tilde{\rho}$ denote the weak* limit of $\{\rho^{E,N}(0)\}_N$, $\{\tilde{\rho}^{E,N}(0)\}_N$, respectively, we have $\bar{\rho} = \tilde{\rho}$. Hence, applying Theorem 1.0.1 we deduce that both sequences $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$, $\left\{\tilde{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$, converge in $L^1_{\text{loc}}([0,+\infty)\times\mathbb{R})$ to the weak entropy solution of the Cauchy problem (1.0.2), which implies

$$\lim_{N\to+\infty}\left\|\rho^{E,N}(t)-\tilde{\rho}^{E,N}(t)\right\|_{L^1(\mathbb{R})}=0 \qquad \text{for a.e. } t>0\,. \tag{1.0.10}$$

The main new property provided by Theorem 1.0.2 is the fact that, thanks to the stability estimate (1.0.5), the convergence in (1.0.10) is actually uniform in time.

Remark 1.0.3. In [44, Theorem 3.6], the authors provide a Cauchy property and rate of convergence of a Eulerian microscopic density built with a specific discretization scheme of initial data $\bar{\rho}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ satisfying $\bar{\rho}>0$ and $\int_{\mathbb{R}}|x|\bar{\rho}(x)dx<\infty$. This is given in the form of a microscopic stability between $\rho^{E,N}$ and $\rho^{E,M}$ for $M,N\in\mathbb{N}$ large enough. The main idea is that the Eulerian microscopic density is seen as a quasi-entropy solution of the conservation law. However, it is important to note that Theorem 1.0.2 treats the case of two completely different discretization schemes $\rho^{E,N}$ and $\tilde{\rho}^{E,N}$, but comparing them at the same level of N. Eventually, these are applicable to initial data $\bar{\rho}\in BV(\mathbb{R})$ with compact support in case (1), and to initial data $\bar{\rho}\in L^\infty$ with compact support in case (2). Also note that in our case we are essentially providing a result showing that weak convergence implies strong convergence. This is possible when we have a control of the total variation, as explained in the proof of both Theorem 1.0.1 and Theorem 1.0.2. Both theorems follow the same strategy, and in both of them we do have a control on the total variation. Therefore, in both theorems the key is to show that weak convergence holds. In the proof of Theorem 1.0.1 we actually conclude that such a weak convergence holds by a compactness argument, provided in [32] and further explained in Section 1.2. However, in the proof of Theorem 1.0.2 for such a weak convergence to hold we find that we need hypothesis (1.0.6) to be satisfied, which concerns only the discretization scheme. Such a hypothesis is assumed for instance in [37, (2.11)] in the case of initial data $\bar{\rho}\in BV(\mathbb{R})$ away from vacuum.

The paper is organized as follows. In Section 1.1 we recall the definition of the Follow-the-Leader dynamics and provide a stability result for it. In Section 1.2 we define the Eulerian and Lagrangian discrete densities, their cumulative functions with the corresponding pseudo-inverses and discuss their properties and interpretations. In Section 1.3 we state and prove the main result of the article, i.e. is Theorem 1.0.1. In Section 1.4 we provide the main stability result, i.e. Theorem 1.0.2.

1.1 The Follow-the-Leader model

In this section, we introduce the Follow-the-Leader (FtL) model and study its behaviour. It is a classical model for road traffic, see e.g. [19, 33] . The goal here is to investigate its stability properties with respect to the initial data. We shall first define the dynamics of the positions of vehicles $x_j(t)$, then we consoder the associated discrete density $\rho_j(t)$, and finally we introduce the inverse discrete density $y_j(t)$. For each of these quantities, we analize the dynamics and some useful properties.

We start by considering $N+1$ vehicles, of length l, with initial positions

$$\bar{x}_0^N < \cdots < \bar{x}_N^N \tag{1.1.1}$$

satisfying

$$\bar{x}_{i+1}^N - \bar{x}_i^N \geq l, \qquad l := \frac{1}{N}. \tag{1.1.2}$$

This standard condition ensures non overlapping of cars.

We now define the FtL dynamics:

Definition 1.1.1. *The FtL model is*

$$\begin{cases} \dot{x}_N^N = v_{max} \\ \dot{x}_j^N = v\left(\dfrac{l}{x_{j+1}^N - x_j^N}\right) & j = 0, ..., N-1, \\ x_j^N(0) = \bar{x}_j^N & j = 0, ..., N, \end{cases} \tag{1.1.3}$$

where the initial positions $\bar{x}_j^N$, $j = 0, ..., N$, satisfy conditions (1.1.1)-(1.1.2).

The FtL model describes the evolution of each car x_j^N that adapts its speed with respect to the distance with the car immediately in front x_{j+1}^N. As in [32], we also introduce the corresponding definition of discrete density and of its dynamics.

Definition 1.1.2. *Given $\{x_j^N(t)\}_{j=0}^N$ a solution of* (1.1.3)*, define the discrete density as*

$$\rho_j^N(t) := \frac{l}{x_{j+1}^N(t) - x_j^N(t)} \qquad j = 0, ..., N-1. \tag{1.1.4}$$

Because of (1.1.3), the discrete density satisfies the dynamics

$$\begin{cases} \dot{\rho}_{N-1}^N = -N(\rho_{N-1}^N)^2 \left(v_{\max} - v(\rho_{N-1}^N)\right) \\ \dot{\rho}_j^N = N(\rho_j^N)^2 \left(v(\rho_j^N) - v(\rho_{j+1}^N)\right) \qquad j = 0, ..., N-2 \\ \rho_j^N(0) = \bar{\rho}_j^N \qquad j = 0, ..., N-1 \end{cases} \tag{1.1.5}$$

where the initial data is

$$\bar{\rho}_j^N := \frac{l}{\bar{x}_{j+1}^N - \bar{x}_j^N} \qquad j = 0, ..., N-1.$$

We finally consider the inverse discrete density introduced in [37].

Definition 1.1.3. *Given* $\{x_j^N(t)\}_{j=0}^N$ *a solution of* (1.1.3)*, define the inverse discrete density as*

$$y_j^N(t) := \frac{x_{j+1}^N(t) - x_j^N(t)}{l} \qquad j = 0, ..., N-1. \tag{1.1.6}$$

Because of (1.1.3), the inverse discrete density satisfies the dynamics

$$\begin{cases} \dot{y}_{N-1}^N = N\left(v_{\max} - V(y_{N-1}^N)\right) \\ \dot{y}_j^N = N\left(V(y_{j+1}^N) - V(y_j^N)\right) \qquad j = 0, ..., N-2 \\ y_j^N(0) = \bar{y}_j^N := \dfrac{\bar{x}_{j+1}^N(t) - \bar{x}_j^N(t)}{l} \qquad j = 0, ..., N-1 \end{cases} \tag{1.1.7}$$

where the velocity of the inverse discrete density is defined by

$$V(y) := v\left(\frac{1}{y}\right). \tag{1.1.8}$$

Here, the first equation of (1.1.7) prescribes that the inverse discrete density of the leading particle evolves with the maximum velocity

$$V(y_N^N) = v(0) = v_{\max}, \tag{1.1.9}$$

which could be viewed as setting $''y_N^N = +\infty''$, corresponding to have an empty road in front of the leader x_N^N. As a consequence of (V1), the velocity of the inverse discrete density satisfies the conditions

$$V \in \mathrm{Lip}([1,+\infty)) \ \text{ with Lipschitz constant } L, \qquad V(1)=0, \qquad V \text{ is increasing.} \tag{V1'}$$

Remark 1.1.1 (Discrete Minimum/Maximum Principle). The solution of the FtL model (1.1.3) and the corresponding discrete density (1.1.5) satisfy a discrete minimum/maximum principle, which is the microscopic version of the well-known maximum principle enjoyed by solutions to (1.0.2) (see for example [28, Theorem 6.2.4]). Indeed, the following estimates hold:

$$\begin{aligned}
&\min_{j=0,\dots,N-1}(x_{j+1}^N(t)-x_j^N(t)) \ \geq \min_{j=0,\dots,N-1}(\bar{x}_{j+1}^N-\bar{x}_j^N) \ \geq l;\\
&\max_{j=0,\dots,N-1}(x_{j+1}^N(t)-x_j^N(t)) \leq \max_{j=0,\dots,N-1}(\bar{x}_{j+1}^N-\bar{x}_j^N) \leq \bar{x}_N^N-\bar{x}_0^N\,;\\
&\max_{j=0,\dots N-1}\rho_j^N(t) \leq \max_{j=0,\dots N-1}\bar{\rho}_j^N \leq 1.
\end{aligned} \tag{1.1.10}$$

A proof of (1.1.10) can be found in [32, Lemma 1]. Also the solution of the discrete inverse density (1.1.7) satisfies a discrete minimum principle due to (1.1.10). Indeed, it holds

$$\min_{j=0,\dots N-1} y_j^N(t) \geq \min_{j=0,\dots N-1} \bar{y}_j^N \geq 1. \tag{1.1.11}$$

In the same spirit of [37, Lemma 2.3], we now prove a stability estimate envolving two different solutions of system (1.1.5).

Proposition 1.1.1. *Consider two solutions* $\{x_j^N(t)\}_{j=0}^N$, $\{\tilde{x}_j^N(t)\}_{j=0}^N$ *of* (1.1.3), *with initial positions* $\{\bar{x}_j^N\}_{j=0}^N$, $\{\tilde{\bar{x}}_j^N\}_{j=0}^N$, *respectively. Let* $\{\rho_j^N(t)\}_{j=0}^{N-1}$, $\{\tilde{\rho}_j^N(t)\}_{j=0}^{N-1}$ *be the corresponding discrete density defined by* (1.1.4), *and let* $\{y_j^N(t)\}_{j=0}^{N-1}$, $\{\tilde{y}_j^N(t)\}_{j=0}^{N-1}$ *be the corresponding inverse discrete density defined by* (1.1.6). *Then, for all* $T>0$, *there holds*

$$\sum_{j=0}^{N-1}|\rho_j^N(T)-\tilde{\rho}_j^N(T)| \leq \sum_{j=0}^{N-1}|y_j^N(0)-\tilde{y}_j^N(0)|\,. \tag{1.1.12}$$

Proof. Throughout the proof we drop the superscript N for simplicity of notation. We will consider two solutions of (1.1.7) parametrized by two different variables t and τ, and we will use the Kruzkov's doubling of variables method to provide the contraction estimate for the inverse densities. Then we will rely on the maximum principle for the discrete densities to conclude.

With this aim, we define

$$V_j(t) \coloneqq V(y_j(t)) \qquad\qquad \tilde{V}_j(\tau) \coloneqq V(\tilde{y}_j(\tau)).$$

We then notice that, for $j = 0, ..., N-2$ it holds

$$\begin{aligned}
\frac{d}{dt}|y_j(t) - \tilde{y}_j(\tau)| &= N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_{j+1}(t) - V_j(t)) \\
\frac{d}{d\tau}|y_j(t) - \tilde{y}_j(\tau)| &= N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(\tilde{V}_j(\tau) - \tilde{V}_{j+1}(\tau)).
\end{aligned}$$

Therefore, we deduce that, for $j = 0, ..., N-2$, we have

$$\begin{aligned}
&\left(\frac{d}{dt} + \frac{d}{d\tau}\right)|y_j(t) - \tilde{y}_j(\tau)| \\
&\quad = N \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))[V_{j+1}(t) - V_j(t) - \tilde{V}_{j+1}(\tau) + \tilde{V}_j(\tau)] \\
&\quad = N \Big[- \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_j(t) - \tilde{V}_j(\tau)) + \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))(V_{j+1}(t) - \tilde{V}_{j+1}(\tau)) \\
&\qquad\quad + (V_{j+1}(t) - \tilde{V}_{j+1}(\tau))[\operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))\Big] \\
&\quad \leq N \Big[- \operatorname{sign}(y_j(t) - \tilde{y}_j(\tau))(V_j(t) - \tilde{V}_j(\tau)) + \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau))(V_{j+1}(t) - \tilde{V}_{j+1}(\tau))\Big].
\end{aligned} \tag{1.1.13}$$

The last inequality can be recovered as follows:

i) If

$$y_j(t) \geq \tilde{y}_j(\tau) \text{ and } y_{j+1}(t) \leq \tilde{y}_{j+1}(\tau), \tag{1.1.14}$$

then one has

$$V_{j+1}(t) - \tilde{V}_{j+1}(\tau) \leq 0,$$

and

$$\operatorname{sign}(y_j(t) - \tilde{y}_j(\tau)) - \operatorname{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) \geq 0.$$

ii) If

$$y_j(t) \leq \tilde{y}_j(\tau) \text{ and } y_{j+1}(t) \geq \tilde{y}_{j+1}(\tau), \tag{1.1.15}$$

then one has

$$V_{j+1}(t) - \tilde{V}_{j+1}(\tau) \geq 0,$$

and

$$\text{sign}(y_j(t) - \tilde{y}_j(\tau)) - \text{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) \leq 0.$$

iii) Otherwise, if neither (1.1.14) nor (1.1.15) are satisfied, then one has

$$\text{sign}(y_j(t) - \tilde{y}_j(\tau)) - \text{sign}(y_{j+1}(t) - \tilde{y}_{j+1}(\tau)) = 0.$$

Summing up the inequality in (1.1.13), we derive

$$\sum_{j=0}^{N-2} \left(\frac{d}{dt} + \frac{d}{d\tau} \right) |y_j(t) - \tilde{y}_j(\tau)| \leq N \,\text{sign}(y_{N-1}(t) - \tilde{y}_{N-1}(\tau))[V_{N-1}(t) - \tilde{V}_{N-1}(\tau)].$$

On the other hand , for $j = N-1$, it holds

$$\begin{aligned}
&\left(\frac{d}{dt} + \frac{d}{d\tau} \right) |y_{N-1}(t) - \tilde{y}_{N-1}(\tau)| \\
&\qquad = N \,\text{sign}(y_{N-1}(t) - \tilde{y}_{N-1}(\tau))[v_{\max} - V_{N-1}(t) - v_{\max} + \tilde{V}_{N-1}(\tau)] \\
&\qquad = N \,\text{sign}(y_{N-1}(t) - \tilde{y}_{N-1}(\tau))[\tilde{V}_{N-1}(\tau) - V_{N-1}(\tau)].
\end{aligned}$$

Therefore, we conclude that

$$\sum_{j=0}^{N-1} \left(\frac{d}{dt} + \frac{d}{d\tau} \right) |y_j(t) - \tilde{y}_j(\tau)| \leq 0. \tag{1.1.16}$$

Relying on (1.1.16), we can complete the proof with the same arguments of the proof of [37, Lemma 2.3]. Namely, multiplying (1.1.16) by a non-negative test function $\phi(t, \tau)$ with $\phi \in C_0^\infty((0, \infty) \times (0, \infty))$, and then integrating by parts, one obtains

$$\int_0^\infty \int_0^\infty (\phi_t + \phi_\tau) \sum_{j=0}^{N-1} |y_j(t) - \tilde{y}_j(\tau)| dt d\tau \geq 0. \tag{1.1.17}$$

Next, choose

$$\phi(t,\tau) = \psi\left(\frac{t+\tau}{2}\right)\eta_\epsilon(t-\tau),$$

where $\psi \in C_0^\infty((0,\infty)\times(0,\infty))$ is a non-negative function, and η_ϵ is a standard mollifier converging to the Dirac delta at the origin as $\epsilon \to 0$. Then, inserting this test function in (1.1.17) and sending $\epsilon \to 0$ we get

$$\int_0^\infty \psi'(t)\sum_{j=0}^{N-1}|y_j(t)-\tilde{y}_j(t)|dt \geq 0 \tag{1.1.18}$$

Now, taking ψ in (1.1.18) to be a smooth approximation of the characteristic function of the interval $(t_1,t_2)\subset(0,T)$ we get

$$\sum_{j=0}^{N-1}|y_j(t_2)-\tilde{y}_j(t_2)|\leq \sum_{j=1}^{N-1}|y_j(t_1)-\tilde{y}_j(t_1)|. \tag{1.1.19}$$

Then, letting $t_1 \to 0$ and $t_2 \to T$ in (1.1.19), we obtain

$$\sum_{j=0}^{N-1}|y_j(T)-\tilde{y}_j(T)|\leq \sum_{j=1}^{N-1}|y_j(0)-\tilde{y}_j(0)|. \tag{1.1.20}$$

Finally, by using (1.1.20) and the maximum principle (1.1.10), we find

$$\begin{aligned}\sum_{j=0}^{N-1}|\rho_j(T)-\tilde{\rho}_j(T)| &= \sum_{j=0}^{N-1}\rho_j(T)\tilde{\rho}_j(T)\left|y_j(T)-\tilde{y}_j(T)\right| \\ &\leq \sum_{j=0}^{N-1}\left|y_j(T)-\tilde{y}_j(T)\right| \\ &\leq \sum_{j=0}^{N-1}\left|y_j(0)-\tilde{y}_j(0)\right|,\end{aligned}$$

thus establishing (1.1.12). □

Remark 1.1.2. In [37] the authors establish the contractive estimate (1.1.20) in two settings: either they assume to have infinitely many equally spaced vehicles in front of the leading one located at x_N^N, assuming a distance M/N between two consecutive ones, for some constant $M > 1$, or they require that the location of the vehicles is periodic in an interval $[a,b]$, so that the distance between the vehicle located in x_N^N and the one located at x_1^N is $(b-x_N^N)+(x_1^N-a)$. This corresponds to define the inverse discrete density related to the leading vehicle as

$$y_N^N = \begin{cases} M & \text{in non-periodic case} \\ N(b - x_N + x_1 - a) & \text{in periodic case.} \end{cases}$$

In the non periodic setting this definition leads to prescribe the velocity

$$V(y_N^N) = v\Big(\frac{1}{M}\Big)$$

for the inverse discrete density in front of the leader. Here, instead we obtain the contractive estimate (1.1.20) prescribing in the first equation in (1.1.5) that the velocity of the inverse discrete density in front of the leader is given by (1.1.9), which is a consequence of having $\dot{x}_N^N = v_{max}$ in (1.1.3). Therefore, Proposition 1.1.1 here provides an extension of [37, Lemma 2.3], in the non-periodic setting, to the case "$M = +\infty$" corresponding to empty road ahead of the leader.

Finally, we recall the discrete Oleinik-type condition proved in [32, Corollary 1 of Lemma 6]:

Lemma 1.1.1 (Discrete Oleinik-type condition). *Consider a solution* $\{x_j^N(t)\}_{j=0}^N$ *of* (1.1.3)*, and let* $\{\rho_j^N(t)\}_{j=0}^{N-1}$ *be the corresponding discrete density defined by* (1.1.4)*. Assume that* v *satisfies* (V1) *and* (V2)*. Then, for any* $j = 0, \dots, N-2$*, there holds*

$$\frac{v(\rho_{j+1}^N(t)) - v(\rho_j^N(t))}{x_{j+1}^N(t) - x_j^N(t)} \leq \frac{1}{t} \qquad \forall\, t \geq 0.$$

1.2 Eulerian and Lagrangian densities

In this section, we first define two densities that approximate the solution of (1.0.2): the Eulerian discrete density and the (Dirac) empirical measure. We then define the Lagrangian discrete density and the inverse Lagrangian discrete density, which instead provide an approximation of the solution of (1.0.2) expressed in Lagrangian coordinates. After that, we define the cumulative function of the Eulerian density and its corresponding pseudo-inverse, which are used to transform the Lagrangian density into the Eulerian one and viceversa. In the end, we provide some convergence results. This section is mainly based on the analysis developed in [32].

The Eulerian discrete density can be understood as a discrete approximation of the solution of the LWR model (1.0.2). Its precise definition is given here.

Definition 1.2.1. *Given* $\{x_j^N(t)\}_{j=0}^N$ *solution of* (1.1.3)*, define the Eulerian discrete density as*

$$\rho^{E,N}(t,x) \coloneqq \sum_{j=0}^{N-1} \rho_j^N(t)\chi_{[x_i^N(t),x_{i+1}^N(t))}(x), \tag{1.2.1}$$

where ρ_j^N *are defined by* (1.1.4)*.*

Notice that the Eulerian discrete density can be seen as a quasi-entropy solution of (1.0.2), as discussed in [44]. We now define the inverse Eulerian discrete density and the (Dirac) empirical measure.

Definition 1.2.2. *Given* $\{x_j^N(t)\}_{j=0}^N$ *solution of* (1.1.3)*, define the inverse Eulerian discrete density as*

$$y^{E,N}(t,x) \coloneqq \sum_{j=0}^{N-1} y_j^N(t)\chi_{[x_i^N(t),x_{i+1}^N(t))}(x).$$

and the (Dirac) empirical measure as

$$\rho^{D,N}(t,x) \coloneqq \frac{1}{N}\sum_{j=0}^{N-1} \delta_{x_j(t)}(x), \tag{1.2.2}$$

where y_j^N *are defined by* (1.1.6)*, and* δ_x *denoted the Dirac delta at the point* x*.*

We finally define the Lagrangian discrete density and the inverse Lagrangian density. The latter can be understood as a piecewise constant approximation of the solution of the Lagrangian version of the LWR model, see [37].

Definition 1.2.3. *Given* $\{x_j^N(t)\}_{j=0}^N$ *solution of* (1.1.3)*, and letting* $l = 1/N$*, define the Lagrangian discrete density as*

$$\rho^{L,N}(t,z) \coloneqq \sum_{j=0}^{N-1} \rho_j^N(t)\chi_{[jl,(j+1)l)}(z), \tag{1.2.3}$$

and the inverse Lagrangian discrete density as

$$y^{L,N}(t,z) \coloneqq \sum_{j=0}^{N-1} y_j^N(t)\chi_{[jl,(j+1)l)}(z). \tag{1.2.4}$$

The coordinate $z \in [0,1]$ can be seen as a Lagrangian mass coordinate. As pointed out in [37], the integer part of $\frac{z}{l}$ measures how many vehicles are located to the left of z.

Notice that, while the L^1 norm of the Eulerian discrete density $\rho^{E,N}$ represents the total mass of vehicles, the L^1 norm of the inverse Lagrangian discrete density $y^{L,N}$ provides the measure of their support. Indeed, given $\{x_j^N(t)\}_{j=0}^N$ solution of (1.1.3), it holds

$$\left\|y^{L,N}(t)\right\|_{L^1([0,1])} = \sum_{j=0}^{N-1} y_j^N(t)\int_0^1 \chi_{[jl,(j+1)l)}(z)dz = \sum_{j=0}^{N-1} x_{j+1}^N(t) - x_j^N(t) = x_N^N(t) - x_0^N(t).$$

Therefore, if $\{x_j^N(t)\}_{j=0}^N$ satisfy the uniformly bounded initial support condition (1.0.3), relying on the discrete maximum principle (1.1.10) we deduce that the corresponding inverse Lagrangian discrete density $y^{L,N}(t)$ has a bound in $L^1([0,1])$ that is uniform with respect to N, for all $t > 0$.

1.2.1 Cumulative and pseudo-inverse functions

We recall that the one dimensional Wasserstein distance can be defined using the cumulative or the pseudo-inverse functions, given by Definition A.2.3, as shown in Proposition A.2.2 see e.g. [63].

Since the discrete density $\rho^{E,N}$ is a probability measure in $\mathcal{P}_c(\mathbb{R})$, if we apply Definition A.2.3 to $\rho^{E,N}$ we find that its cumulative distribution takes the form:

$$\begin{aligned} F_{\rho^{E,N}}(t,x) &= \int_{-\infty}^{x} \rho^{E,N}(t,y)dy \\ &= \sum_{j=0}^{N-1}\left[jl + \rho_j^N(t)(x - x_j(t))\right]\chi_{[x_j(t),x_{j+1}(t))}(x) + \chi_{[x_N(t),+\infty)}(x), \end{aligned} \qquad x \in \mathbb{R}, \tag{1.2.5}$$

while the corresponding pseudo-inverse takes the form:

$$X_{\rho^{E,N}}(t,z) = \sum_{j=0}^{N-1}\left[x_j^N(t) + \frac{z - jl}{\rho_j^N(t)}\right]\chi_{[jl,(j+1)l)}(z) + \left[x_N^N(t)\right]\chi_{\{1\}}(z), \quad z \in [0,1]. \tag{1.2.6}$$

Similarly, if we apply Definition A.2.3 to the (Dirac) empirical measure $\rho^{D,N}$ the cumulative distribution takes the form

$$F_{\rho^{D,N}}(t,x) = \sum_{j=0}^{N-1} [(j+1)l]\, \chi_{[x_j(t),x_{j+1}(t))}(x) + \chi_{[x_N(t),+\infty)}(x) \tag{1.2.7}$$

while the corresponding pseudo-inverse takes the form:

$$X_{\rho^{D,N}}(t,z) = \sum_{j=0}^{N-1} \left[x_j^N(t)\right] \chi_{[jl,(j+1)l)}(z) + \left[x_N^N(t)\right] \chi_{\{1\}}(z). \tag{1.2.8}$$

Notice that the cumulative distribution $F_{\rho^{E,N}}$ is 1-Lipschitz in the x-variable, and the pseudo-inverse $X_{\rho^{E,N}}$ satisfies

$$\rho^{L,N}(t,z) = \rho^{E,N}(t, X_{\rho^{E,N}}(t,z)), \quad y^{L,N}(t,z) = y^{E,N}(t, X_{\rho^{E,N}}(t,z)) \qquad \forall\, t \geq 0,\ z \in [0,1].$$

The cumulative function $F_{\rho^{E,N}}$ then satisfies

$$\rho^{L,N}(t, F_{\rho^{E,N}}(t,x)) = \rho^{E,N}(t,x), \quad y^{L,N}(t, F_{\rho^{E,N}}(t,x)) = y^{E,N}(t,x) \qquad \forall\, t \geq 0,\ x \in \mathbb{R}.$$

1.2.2 Convergence results of the cumulative and pseudo-inverse functions

We recall now some results, first given in [32], about the limits of $X_{\rho^{E,N}}$ and $X_{\rho^{D,N}}$, as well as of $F_{\rho^{E,N}}$ and $F_{\rho^{D,N}}$. The proofs are valid for any initial data $\{x_j^N(0)\}_{j=0}^N$ of system (1.1.3) that satisfies the condition of the uniformly bounded initial support (1.0.3).

Proposition 1.2.1. *Let* $\{x_j^N(t)\}_{j=0}^N$ *be solutions of* (1.1.3) *that satisfy the uniformly bounded initial support condition* (1.0.3). *Consider the corresponding Eulerian discrete density* $\rho^{E,N} \in L^\infty([0,+\infty) \times \mathbb{R}; [0,1])$ *defined by* (1.2.1) *and the (Dirac) empirical measure* $\rho^{D,N} \in L^\infty([0,+\infty); W_1(\mathcal{P}_c(\mathbb{R})))$ *defined by* (1.2.2). *Let* $F_{\rho^{E,N}}$, $X_{\rho^{E,N}}$, *and* $F_{\rho^{D,N}}$, $X_{\rho^{D,N}}$, *be the corresponding cumulative distributions and pseudo-inverses defined by* (1.2.5), (1.2.6), *and* (1.2.7), (1.2.8), *respectively. Then, the following hold:*

i) there exists a non-decreasing function $X \in L^\infty([0,+\infty)\times[0,1])$ *such that, up to a subsequence, both* $\{X_{\rho^{E,N}}\}_N$ *and* $\{X_{\rho^{D,N}}\}_N$ *converge to* X *in* $L^1_{loc}([0,+\infty)\times [0,1])$;

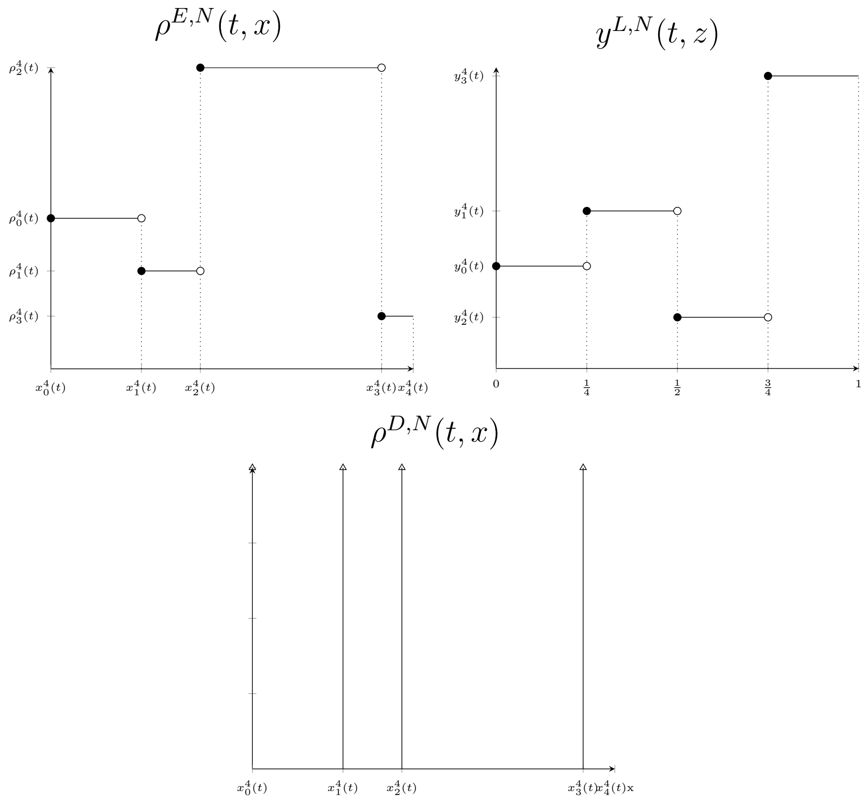

Figure 1.1: The Eulerian discrete density, the inverse Lagrangian discrete density and the (Dirac) empirical measure profiles ($N = 4$).

ii) letting $F : [0, +\infty) \times \mathbb{R} \to [0, 1]$ *be the map defined by*

$$F(t, x) \coloneqq \text{meas}\{z \in [0, 1] : X(t, z) \leq x\}, \qquad t \geq 0, \; x \in \mathbb{R}, \tag{1.2.9}$$

up to a subsequence, both $\{F_{\rho^{E,N}}\}_N$ *and* $\{F_{\rho^{D,N}}\}_N$ *converge to* F *in* $L^1_{\text{loc}}([0, +\infty) \times \mathbb{R})$.

Proof. See [32, Proposition 1, Proposition 2, Lemma 4] with $L = 1$, and $R = 1$ (due to the maximum principle (1.1.10)), using their notation. □

Remark 1.2.1. Notice that, differently from the results in [32], Proposition 1.2.1 here only states the convergence of $\left\{X_{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$, $\left\{X_{\rho^{D,N}}\right\}_{N\in\mathbb{N}}$ and $\left\{F_{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$, $\left\{F_{\rho^{D,N}}\right\}_{N\in\mathbb{N}}$ up to a subsequence, which is obtained relying on Helly's compactness theorem. In [32] the authors conclude that the whole sequences $\left\{X_{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$, $\left\{X_{\rho^{D,N}}\right\}_{N\in\mathbb{N}}$ converge, exploiting the fact that their atomization scheme for the FtL model guarantees that $X_{\rho^{D,N+1}}(t, z) \leq X_{\rho^{D,N}}(t, z)$ for all $t \geq 0$ and $z \in [0, 1]$. In turn, by the definition of the Wasserstein distance (A.2.3), the convergence of the whole sequences $\left\{X_{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$ and $\left\{X_{\rho^{D,N}}\right\}_{N\in\mathbb{N}}$ yields the convergence of $\left\{F_{\rho^{E,N}}\right\}_{N\in\mathbb{N}}$ and $\left\{F_{\rho^{D,N}}\right\}_{N\in\mathbb{N}}$.

Proposition 1.2.2. *According with Proposition 1.2.1-(ii), consider two sequences* $\{F_{\rho^{E,N}}\}_N$, $\{F_{\rho^{D,N}}\}_N$ *of cumulative distributions associated to the Eulerian discrete density* $\rho^{E,N}$, *and to the (Dirac) empirical measure* $\rho^{D,N}$, *respectively, that converge to a function* F *defined by* (1.2.9), *which is Lipschitz continuous with respect to* x. *For any* $t \geq 0$, *let* $\rho(t)$ *be the distributional derivative of* $x \mapsto F(t, x)$. *Then the following hold:*

i) $\rho(t) \in \mathcal{P}_c(\mathbb{R})$ *for all* $t \geq 0$,

ii) $0 \leq \rho(t) \leq 1$ *for almost every* $t \geq 0$ *and* $x \in \mathbb{R}$,

iii) $\{\rho^{E,N}\}_N$ *and* $\{\rho^{D,N}\}_N$ *converge to* ρ *in* $L^1_{\text{loc}}([0, +\infty); W_1(\mathcal{P}_c(\mathbb{R})))$.

Proof. See [32, Proposition 3] with $L = 1$, and $R = 1$ (due to the maximum principle (1.1.10)), using their notation. □

Remark 1.2.2. Given a map $F : [0, +\infty) \times \mathbb{R} \to [0, 1]$, letting $\rho(t)$ be the distributional derivative of $x \mapsto F(t, x)$, and assuming that $\rho(t) \in \mathcal{P}_c(\mathbb{R})$, then if we consider the cumulative distribution $F_{\rho(t)}$ as defined in (A.2.1), one has

$$F_{\rho(t)}(x) = F(t, x), \quad \text{for a.e. } x \in \mathbb{R}. \tag{1.2.10}$$

Lemma 1.2.1. *Let* $\{x_j^N(t)\}_{j=0}^N$ *be solutions of* (1.1.3)*, and consider the Lagrangian discrete density* $\rho^{L,N} \in L^\infty([0,+\infty) \times [0,1])$ *defined by* (1.2.3)*. Then, there exists* $\rho^L \in L^\infty([0,T] \times [0,1])$ *such that, up to a subsequence,* $\left\{\rho^{L,N}\right\}_{N\in\mathbb{N}}$ *converges to* ρ^L *weakly-* in* $L^\infty([0,+\infty) \times [0,1])$*.*

Proof. See [32, Lemma 5] with $L = 1$, and $R = 1$ (due to the maximum principle (1.1.10)), using their notation. □

1.3 Proof of the micro-to-macro convergence

In this section we prove the first main result of this article, i.e. Theorem 1.0.1. With this goal, we first recall standard tools for studying the Cauchy problem (1.0.2): the definition of weak solution and classical results of existence and uniqueness of entropy solutions. Then, after proving a technical lemma we present the proof of Theorem 1.0.1.

Given the Cauchy problem (1.0.2), we recall the definition of weak and entropy weak solution of (1.0.2).

Definition 1.3.1. *A function* $\rho \in L^\infty([0,+\infty) \times \mathbb{R})$ *is a weak solution to* (1.0.2) *if it holds*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} [\rho(t,x)\varphi_t(t,x) + (\rho(t,x)v(\rho(t,x)))\varphi_x(t,x)]\,\mathrm{d}t\ \mathrm{d}x + \int_{\mathbb{R}} \bar\rho(x)\varphi(0,x)\mathrm{d}x = 0 \tag{1.3.1}$$

for all $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$*.*

Definition 1.3.2. *A function* $\rho \in L^\infty([0,+\infty) \times \mathbb{R})$ *is a Kružkov's entropy solution to* (1.0.2) *if it satisfies the entropy inequality*

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}_+} [|\rho(t,x) - k|\varphi_t(t,x) + \mathrm{sign}(\rho(t,x) - k)[f(\rho(t,x)) - f(k)]\varphi_x(t,x)]dtdx \\ + \int_{\mathbb{R}} |\bar\rho(x) - k|\varphi(0,x)dx \geq 0\end{aligned} \tag{1.3.2}$$

for all $\varphi \in C_c^\infty([0,+\infty) \times \mathbb{R})$ *with* φ *non-negative, and for all constants* $k \in \mathbb{R}$*.*

We now present two well-known results about the existence and uniqueness of the weak entropy solution to the Cauchy problem (1.0.2).

Theorem 1.3.1 (Uniqueness of Kružkov's solution, [43])**.** *Assume that the flux* $f(\rho)$ *is locally Lipschitz. For any given initial data* $\bar\rho \in L^\infty$ *with compact support, there exists a unique Kružkov's entropy solution* $\rho \in L^\infty([0,+\infty)] \times \mathbb{R})$ *to* (1.0.2)*.*

Theorem 1.3.2 (Chen and Rascle's entropy solution, [20])**.** *Assume that the flux is genuinely nonlinear almost everywhere, i.e. there exists no nontrivial interval on which the flux $f(\rho)$ is affine. For a given initial data $\bar{\rho} \in L^\infty$ with compact support, there exists a unique $\rho \in L^\infty([0,+\infty) \times \mathbb{R})$ weak solution of* (1.0.2) *in the sense of Definition 1.3.1 that satisfies the entropy inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} [|\rho(t,x) - k|\varphi_t(t,x) + \operatorname{sign}(\rho(t,x) - k)[f(\rho(t,x)) - f(k)]\varphi_x(t,x)]\, dtdx \geq 0 \tag{1.3.3}$$

for all $\varphi \in C_c^\infty((0,+\infty) \times \mathbb{R})$ with φ non-negative and for all constants $k \in \mathbb{R}$. Moreover, ρ is the unique Kružkov's entropy solution to (1.0.2)

In Theorem 1.3.2 we see that, if the flux is genuinely nonlinear almost everywhere, uniqueness of entropy solution is preserved for a relaxed notion of entropy solution, which does not require that the entropy inequality (1.3.2) be satisfied at $t = 0$. This is due to the fact that the nonlinearity of the flux ensures the existence of a strong trace at $t = 0$ of a weak solution to (1.0.2) in the sense of Definition 1.3.1.

We now present the following lemma, which is used in the proof of Theorem 1.0.1.

Lemma 1.3.1. *Consider a function $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which is $1-$Lipschitz. It holds*

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{\|f\|_{L^1(\mathbb{R})}}.$$

Proof. Let $N \in \mathbb{N}$. Let $\bar{x}_N \in \mathbb{R}$ and define $M_N \coloneqq f(\bar{x}_N)$. Let

$$M_N \to \|f\|_{L^\infty(\mathbb{R})} \qquad \text{as } N \to +\infty. \tag{1.3.4}$$

Since $|f|$ is 1-Lipschitz, notice that for every $N \in \mathbb{N}$ it holds

$$|f(x)| \geq \max\{M_N - |x - \bar{x}_N|, 0\} \quad \forall\, x \in \mathbb{R}.$$

By integrating in space, for every $N \in \mathbb{N}$ it holds

$$\int_{\mathbb{R}} |f(x)| dx \geq \int_{\mathbb{R}} \max\{M_N - |x - \bar{x}_N|, 0\} dx = M_N^2.$$

By (1.3.4) we recover the result. $\square$

We are now ready to provide:

Proof of Theorem 1.0.1. Given the Eulerian discrete density $\rho^{E,N} \in L^\infty([0,+\infty)\times \mathbb{R};[0,1])$ defined by (1.0.1), to ease notation set

$$F^N := F_{\rho^{E,N}},$$

where $F_{\rho^{E,N}}$ denotes the cumulative distribution of $\rho^{E,N}$, and let $F(t,x)$ be the function defined by (1.2.9), which is equal to the cumulative distribution $F_{\rho(t)}(x)$ of its x-distributional derivative $\rho(t)$ (see Remark 1.2.2).

1. In this step we prove that $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$, up to a subsequence, is a Cauchy sequence in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$, in both cases (1) and (2) of Theorem 1.0.1, and thus it converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to some limit function $\rho \in L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$.

Recall by Propositions 1.2.1-1.2.2 that, up to a subsequence, and for every $T>0$ it holds

$$\lim_{N\to+\infty}\int_0^T W_1(\rho^{E,N}(t),\rho(t))dt = \lim_{N\to+\infty}\int_0^T \left\|F^N(t)-F(t)\right\|_{L^1(\mathbb{R})} dt = 0. \tag{1.3.5}$$

Since F^N, F^M are 1$-$Lipschitz in the x variable, then also the function $F^N - F^M$ is 1$-$Lipschitz in the x variable. Therefore, by Lemma 1.3.1 it holds

$$\left\|F^N(t)-F^M(t)\right\|_{L^\infty(\mathbb{R})} \le \sqrt{\|F^N(t)-F^M(t)\|_{L^1(\mathbb{R})}} \quad \forall\, N,M\in\mathbb{N}, \quad \forall\, t>0. \tag{1.3.6}$$

Notice that since it holds

$$\lim_{x\to\pm\infty}|F^N(t,x)-F^M(t,x)|=0, \qquad \forall N,M\in\mathbb{N}, \quad \forall\, t>0,$$

then integrating by parts we find

$$\begin{aligned}
&\int_{\mathbb{R}} (\rho^{E,N}(x) - \rho^{E,M}(x))^2 dx \\
&= \int_{\mathbb{R}} \frac{d}{dx} \left(F^N(t,x) - F^M(t,x)\right) \left(\rho^{E,N}(t,x) - \rho^{E,M}(t,x)\right) dx \\
&= -\int_{\mathbb{R}} \left(F^N(t,x) - F^M(t,x)\right) \frac{d}{dx} \left(\rho^{E,N}(t,x) - \rho^{E,M}(t,x)\right) dx \\
&\leq \left\|F^N(t) - F^M(t)\right\|_{L^\infty(\mathbb{R})} \mathrm{TV}\left(\rho^{E,N}(t) - \rho^{E,M}(t); \mathbb{R}\right) \\
&\leq \left\|F^N(t) - F^M(t)\right\|_{L^\infty(\mathbb{R})} \left[\mathrm{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t); \mathbb{R}\right)\right].
\end{aligned}$$

By Hölder inequality and by using (1.3.6), we thus get that for all $\Omega \subset \mathbb{R}$ bounded, for all $N, M \in \mathbb{N}$, and for all $t > 0$, it holds

$$\begin{aligned}
&\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\Omega)} \\
&\leq \mathrm{meas}(\Omega) \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^2(\Omega)} \\
&\leq \mathrm{meas}(\Omega) \left\|F^N(t) - F^M(t)\right\|_{L^\infty(\mathbb{R})} \left[\mathrm{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t); \mathbb{R}\right)\right] \\
&\leq \mathrm{meas}(\Omega) \sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}} \left[\mathrm{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t); \mathbb{R}\right)\right].
\end{aligned} \tag{1.3.7}$$

The further treatment of this inequality is now adressed in the two cases (1) and (2) in the two following steps.

1.1. We assume that $\mathrm{TV}(\rho^{E,N}(0); \mathbb{R}), \mathrm{TV}(\rho^{E,M}(0); \mathbb{R}) < K_2$ for all $N, M \in \mathbb{N}$ for some $K_2 > 0$ independent of N, M. Because of the BV contractivity property enjoyed by $\rho^{E,N}$ and $\rho^{E,M}$ (see [32, Proposition 5]) and relying on the hypothesis on the total variation of $\rho^{E,N}(0)$ and $\rho^{E,M}(0)$, it holds

$$\begin{aligned}
\mathrm{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t); \mathbb{R}\right) &\leq \mathrm{TV}\left(\rho^{E,N}(0); \mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(0); \mathbb{R}\right) \\
&\leq 2K_2.
\end{aligned}$$

Thus, we deduce from (1.3.7) that, for all $N, M \in \mathbb{N}$, and for all $t > 0$, it holds

$$\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\Omega)} \leq 2K_2 \,\mathrm{meas}(\Omega) \sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}}. \tag{1.3.8}$$

Notice that, by Hölder's inequality, we have

$$\begin{aligned}&\int_0^T \sqrt{\|F^N(t)-F^M(t)\|_{L^1(\mathbb{R})}}dt\\ &\le \|1\|_{L^2([0,T])}\left\|\sqrt{\|F^N(t)-F^M(t)\|_{L^1(\mathbb{R})}}\right\|_{L^2([0,T])}\\ &=\sqrt{T}\sqrt{\int_0^T \|F^N(t)-F^M(t)\|_{L^1(\mathbb{R})}\,dt}.\end{aligned} \tag{1.3.9}$$

Then, integrating (1.3.8) in the time interval $[0,T]$, and using (1.3.9), we find that for all $N,M\in\mathbb{N}$ there hold

$$\int_0^T \left\|\rho^{E,N}(t)-\rho^{E,M}(t)\right\|^2_{L^1(\Omega)}dt \tag{1.3.10}$$

$$\le 2K_2\,\mathrm{meas}(\Omega)\sqrt{T}\sqrt{\int_0^T \|F^N(t)-F^M(t)\|_{L^1(\mathbb{R})}\,dt}. \tag{1.3.11}$$

Finally, by Hölder's inequality, we derive from (1.3.10) that

$$\begin{aligned}&\int_0^T \left\|\rho^{E,N}(t)-\rho^{E,M}(t)\right\|_{L^1(\Omega)}dt\\ &\quad\le \sqrt{2K_2\,\mathrm{meas}(\Omega)}\cdot T^{\frac{3}{4}}\left(\int_0^T \left\|F^N(t)-F^M(t)\right\|_{L^1(\mathbb{R})}dt\right)^{\frac{1}{4}}.\end{aligned} \tag{1.3.12}$$

Therefore, in case (1) the convergence result (1.3.5) implies that, for every $T>0$, and for any bounded $\Omega\subset\mathbb{R}$, $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T]\times\Omega)$. Now consider an increasing sequence $T_n>0$ and a sequence of bounded domain Ω_n such that $\cup_n([0,T_n]\times\Omega_n)=[0,+\infty)\times\mathbb{R}$. Then, by a diagonal argument we can repeatedly extract subsequnces of $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ that have the Cauchy property on each domain $[0,T_n]\times\Omega_n$, and thus we construct a Cauchy subsequence in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$.

1.2. We assume that (V2) holds. Observe that, for any fixed $T>0$, and for all $N,M\in\mathbb{N}$, it holds

$$\mathrm{supp}(\rho^{E,N}(t)-\rho^{E,M}(t))\subset\Omega_{T,N,M}\qquad\forall\, t\in[0,T], \tag{1.3.13}$$

where

$$\Omega_{T,N,M}\coloneqq\left[\min\{x_0^N(0),x_0^M(0)\},\max\{x_N^N(0),x_M^M(0)\}+Tv_{\max}\right]. \tag{1.3.14}$$

Then, because of the condition (1.0.3) on uniformly bounded initial support, it follows that there will be some constant $K_T > 0$ such that, for all $N, M \in \mathbb{N}$, it holds

$$\operatorname{meas}\left(\operatorname{supp}(\rho^{E,N}(t)-\rho^{E,M}(t))\right) \leq \operatorname{meas}(\Omega_{T,N,M}) \leq K_T\,, \qquad \forall\, t \in [0,T]\,. \tag{1.3.15}$$

Moreover, relying on (1.3.13), and using the discrete Oleinik estimate given in Lemma 1.1.1 for $\rho^{E,N}$ and $\rho^{E,M}$ it follows that, for any fixed $T, \delta > 0$, there will be some constant $K_{\delta,T} > 0$ such that, for all $N, M \in \mathbb{N}$, it holds

$$\begin{aligned}&\sup_{t\in[\delta,T]} \left[\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right] = \\ &\quad = \sup_{t\in[\delta,T]} \left[\mathrm{TV}\left(\rho^{E,N}(t);\Omega_{T,N,M}\right) + \mathrm{TV}\left(\rho^{E,M}(t);\Omega_{T,N,M}\right)\right] \leq K_{\delta,T}\,,\end{aligned} \tag{1.3.16}$$

(see [31, Proposition 3.3] and [49]). For any given bounded $\Omega \subset \mathbb{R}$, with the same analysis in (1.3.7), (1.3.8), it thus follows that, for all $N, M \in \mathbb{N}$, and for all $t \in [\delta, T]$, it holds

$$\begin{aligned}&\left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\Omega)} \\ &\leq \operatorname{meas}(\Omega) \sup_{t\in[\delta,T]} \left[\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \mathrm{TV}\left(\rho^{E,M}(t);\mathbb{R}\right)\right] \sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}}, \\ &\leq 2K_{\delta,T} \operatorname{meas}(\Omega)\sqrt{\|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}}.\end{aligned} \tag{1.3.17}$$

Integrating in the time interval $[\delta, T]$ and using the Hölder's inequality as done in the previous step we then find that, for all $N, M \in \mathbb{N}$, there hold

$$\int_\delta^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|^2_{L^1(\Omega)} dt \tag{1.3.18}$$

$$\leq 2K_{\delta,T} \operatorname{meas}(\Omega)\sqrt{T}\sqrt{\int_\delta^T \|F^N(t) - F^M(t)\|_{L^1(\mathbb{R})}\, dt}, \tag{1.3.19}$$

and

$$\begin{aligned}&\int_\delta^T \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\Omega)} dt \leq \\ &\qquad \leq \sqrt{2K_{\delta,T} \operatorname{meas}(\Omega)} \cdot T^{\frac{3}{4}} \left(\int_\delta^T \left\|F^N(t) - F^M(t)\right\|_{L^1(\mathbb{R})} dt\right)^{\frac{1}{4}}.\end{aligned} \tag{1.3.20}$$

Observe now that, for any fixed $\epsilon > 0$, setting $\delta_\epsilon \doteq \epsilon/(2\,\mathrm{meas}(\Omega))$, we have

$$\int_0^{\delta_\epsilon} \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\Omega)} dt \leq \frac{\epsilon}{2}, \qquad \forall N, M \in \mathbb{N}\,. \tag{1.3.21}$$

On the other hand, the convergence result (1.3.5), together with (1.3.20), implies that there exists $N(\epsilon) > 0$ such that

$$\int_{\delta_\epsilon}^{T} \left\|\rho^{E,N}(t) - \rho^{E,M}(t)\right\|_{L^1(\Omega)} dt \leq \frac{\epsilon}{2}, \qquad \forall N, M \geq N(\epsilon)\,. \tag{1.3.22}$$

Therefore, combining (1.3.21)-(1.3.22) we find also in case (2) that, for every $T > 0$, and for any bounded $\Omega \subset \mathbb{R}$, $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^1([0,T]\times\Omega)$. Then we conclude as in case (1) that by diagonal procedure we can estract a Cauchy subsequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$.

Summing up, by treating the cases (1) and (2) in **1.1.** and **1.2.**, respectively, we get in both cases that the sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$, up to a subsequence, to some function function $\rho \in L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$.

2. In this step we show that the function ρ determined in the previous step is the weak entropy solution of the Cauchy problem (1.0.2), and that actually the whole sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to ρ.

Recalling that $\left\{\rho^{E,N}(0)\right\}_{N\in\mathbb{N}}$ weakly converges to $\bar{\rho}$ by hypothesis (1.0.4), and following the same procedure as in Step 1-Case 1 of the proof of [31, Theorem 2], we deduce that ρ is a weak solution to (1.0.2) in the sense of Definition 1.3.1. Furthermore, it also holds that ρ satisfies the entropy inequality (1.3.3) by applying the exact same computations as done in the part (vi) of the proof of [32, Theorem 3]. In turn, this implies that ρ is a weak entropy solution of the Cauchy problem (1.0.2), thanks to Theorem 1.3.2. By merging Step 1 and Step 2, we conclude that, up to a subsequence, $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,+\infty)\times\mathbb{R})$ to the unique weak entropy solution of (1.0.2). Since, with the same arguments, we can show that any subsequence of $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ admits a subsubsequence converging to the unique weak entropy solution of (1.0.2), it follows that the whole sequence $\left\{\rho^{E,N}\right\}_{N\in\mathbb{N}}$ converges to ρ, thus completing the proof of the theorem.

□

Remark 1.3.1. If $\bar{\rho} \in BV(\mathbb{R})$ satisfies the assumptions of Theorem 1.0.1, relying on the analysis performed in the above proof one can derive the convergence rate for the initial Eulerian discrete density $\rho^{E,N}(0)$ associated to the atomization scheme introduced in [32, (19a) and (19b)]), which is defined as follows.

Denote by $\bar{x}_{\min} < \bar{x}_{\max}$ the extremal points of the support of $\bar{\rho}$. Consider the following discretization scheme: set

$$\tilde{x}_0^N(0) := \bar{x}_{\min}$$

and recursively

$$\tilde{x}_j^N(0) := \sup\left\{x \in \mathbb{R} : \quad \int_{x_{j-1}^N(0)}^{x} \bar{\rho}(y)dy \leq \frac{1}{N}\right\}, \qquad j = 1, ..., N$$

It is possible to give a convergence rate of such discretization scheme with respect to the L^1 norm in Eulerian coordinates. It has been proved in [32, Proposition 4] that it holds

$$W_1(\tilde{\rho}^{E,N}(0), \bar{\rho}) \leq \frac{(\tilde{x}_N^N(0) - \tilde{x}_0^N(0))}{N}.$$

As shown in Step 1 in the proof, applying the inequality (1.3.7) at $t = 0$ for $\Omega = [\tilde{x}_0^N(0), \tilde{x}_N^N(0)]$ and by the condition of the uniformly bounded initial support (1.0.3), we get

$$\left\|\tilde{\rho}^{E,N}(0) - \bar{\rho}\right\|_{L^1(\mathbb{R})} \leq \sqrt{2(\tilde{x}_N^N(0) - \tilde{x}_N^N(0))\,\mathrm{TV}\,(\bar{\rho};\mathbb{R})}\,\left\|\tilde{F}^N(0) - F_{\bar{\rho}}\right\|_{L^1(\mathbb{R})}^{\frac{1}{4}} \leq \frac{C}{N^{1/4}}$$

where $C = C(K, \mathrm{TV}(\bar{\rho};\mathbb{R}))$ is independent of N, and K being the constant corresponding to the condition of the uniformly bounded initial support (1.0.3).

Remark 1.3.2. It can also be proved that the sequence of the corresponding empirical measures $\left\{\rho^{D,N}\right\}_{N\in\mathbb{N}}$ where $\rho^{D,N}$ is given by (1.2.2) also converges in $L^1_{\mathrm{loc}}([0, +\infty]; W_1)$ to the unique weak entropy solution ρ of (1.0.2). Indeed, notice that

$$\int_0^T W_1(\rho^{D,N}(t), \rho(t))dt \leq \int_0^T W_1(\rho^{D,N}(t), \rho^{E,N}(t))dt + \int_0^T W_1(\rho^{E,N}(t), \rho(t))dt. \tag{1.3.23}$$

Notice that it holds

$$\int_0^1 |X^{E,N}(z) - X^{D,N}(z)|dz = \sum_{j=0}^{N-1} y_j^N \int_{jl}^{(j+1)l} [z - jl]dz = \frac{l}{2} \sum_{j=0}^{N-1} x_{j+1}^N(t) - x_j^N(t)$$
$$= \frac{l}{2} \left(x_N^N(t) - x_0^N(t) \right).$$

Therefore it holds

$$\int_0^T \int_0^1 |X^{E,N}(t,z) - X^{D,N}(t,z)|dz \leq \frac{T(x_N^N(0) - x_0^N(0) + v_{\max} T)}{2N}$$

and, by the condition of the uniformly bounded initial support (1.0.3), it holds

$$\lim_{N\to+\infty} \int_0^T W_1(\rho^{D,N}(t), \rho^{E,N}(t))dt = 0. \tag{1.3.24}$$

By Poincaré's inequality, for some $C > 0$ independent of N and for $\Omega \subset \mathbb{R}$ compact such that $\mathrm{supp}(\rho^{E,N}(t) - \rho(t)) \subset \Omega$ for all $t \in [0,T]$, for instance by choosing $\Omega = \Omega_T$ where Ω_T is defined by (1.3.14), it holds

$$\int_0^T W_1(\rho^{E,N}(t), \rho(t))dt \leq C \left\| \rho^{E,N} - \rho \right\|_{L^1([0,T]\times\mathbb{R})}. \tag{1.3.25}$$

Now consider (1.3.24), (1.3.25) and Theorem 1.0.1. By (1.3.23), it holds

$$\lim_{N\to+\infty} \int_0^T W_1(\rho^{D,N}(t), \rho(t))dt = 0.$$

1.4 Proof of stability of Eulerian density

In this section we provide a stability result in both the Wasserstein norm and in L^1 for two different Eulerian discrete densities. In the rest of the paper, we compare two solutions $\{x_j^N(t)\}_{j=0}^N$ and $\{\tilde{x}_j^N(t)\}_{j=0}^N$ of the FtL model (1.1.3) and the corresponding Eulerian discrete densities $\rho^{E,N}$ and $\tilde{\rho}^{E,N}$ defined by (1.2.1). The goal of this section is to prove Theorem 1.0.2.

We now present three propositions. They lead to the proof of the theorem, that is postponed at the end of this section.

Proposition 1.4.1. *Let two sequences of configurations $\{x_j^N\}_{j=0}^N$,$\{\tilde{x}_j^N\}_{j=0}^N$, indexed by $N \in \mathbb{N}$, be given. Assume that $x_N^N = \tilde{x}_N^N$ for all $N \in \mathbb{N}$. Consider the corresponding Eulerian densities $\rho^{E,N}$, $\tilde{\rho}^{E,N} \in L^\infty(\mathbb{R})$ defined by* (1.2.1)*. Then it* *holds*

$$W_1(\rho^{E,N},\tilde{\rho}^{E,N}) \leq 2\sum_{j=0}^{N-1}|x_{j+1}-x_j-(\tilde{x}_{j+1}-\tilde{x}_j)|.$$

Proof. Fix $j=0,...,N-1$ and $z\in[jl,(j+1)l)$. Recalling (1.1.6), we have

$$\begin{aligned}
\left|x_j^N-\tilde{x}_j^N+(z-jl)\left(y_j^N-\tilde{y}_j^N\right)\right| &= \left|x_j^N-\tilde{x}_j^N+\frac{z-jl}{l}\left(x_{j+1}^N-\tilde{x}_{j+1}^N-(x_j^N-\tilde{x}_j^N)\right)\right|\\
&\leq \left|x_j^N-\tilde{x}_j^N\right|+\left|x_{j+1}^N-\tilde{x}_{j+1}^N-(x_j^N-\tilde{x}_j^N)\right|\\
&\leq 2\left|x_j^N-\tilde{x}_j^N-(x_{j+1}^N-\tilde{x}_{j+1}^N)\right|+\left|x_{j+1}^N-\tilde{x}_{j+1}^N\right|\\
&\leq 2\left(\sum_{k=j}^{N-1}\left|x_k^N-\tilde{x}_k^N-(x_{k+1}^N-\tilde{x}_{k+1}^N)\right|\right)+\left|x_N^N-\tilde{x}_N^N\right|,
\end{aligned}$$

where in the last inequality we repeatedly make use of the triangular inequality

$$\left|x_k^N-\tilde{x}_k^N\right| \leq \left|x_k^N-\tilde{x}_k^N-(x_{k+1}^N-\tilde{x}_{k+1}^N)\right|+\left|x_{k+1}^N-\tilde{x}_{k+1}^N\right| \qquad k=j+1,\dots,N-1$$

Therefore, by summing in j, it holds

$$\begin{aligned}
\sum_{j=0}^{N-1}\left|x_j^N-\tilde{x}_j^N+(z-jl)\left(y_j^N-\tilde{y}_j^N\right)\right| &\leq 2\left(\sum_{j=0}^{N-1}\sum_{k=j}^{N-1}\left|x_k^N-\tilde{x}_k^N-(x_{k+1}^N-\tilde{x}_{k+1}^N)\right|\right)+N\left|x_N^N-\tilde{x}_N^N\right|\\
&\leq 2N\left(\sum_{j=0}^{N-1}\left|x_j^N-\tilde{x}_j^N-(x_{j+1}^N-\tilde{x}_{j+1}^N)\right|\right)+N\left|x_N^N-\tilde{x}_N^N\right|\\
&= 2\sum_{j=0}^{N-1}\left|y_j^N-\tilde{y}_j^N\right|+N\left|x_N^N-\tilde{x}_N^N\right|.
\end{aligned}$$

We are now ready to prove the result:

$$\begin{aligned}
\int_0^1|X_{\rho^{E,N}}(z)-X_{\tilde{\rho}^{E,N}}(z)|dz &= \int_0^1\sum_{j=0}^{N-1}\left|x_j^N-\tilde{x}_j^N+(z-jl)\left(y_j^N-\tilde{y}_j^N\right)\right|\chi_{[jl,(j+1)l)}(z)dz\\
&\leq \int_0^1\left[2\sum_{j=0}^{N-1}\left|y_j^N-\tilde{y}_j^N\right|+N\left|x_N^N-\tilde{x}_N^N\right|\right]\chi_{[jl,(j+1)l)}(z)dz\\
&= \frac{2}{N}\sum_{j=0}^{N-1}\left|y_j^N-\tilde{y}_j^N\right|+\left|x_N^N-\tilde{x}_N^N\right|\\
&= 2\left\|y^{L,N}-\tilde{y}^{L,N}\right\|_{L^1([0,1])}
\end{aligned}$$

$\square$

Proposition 1.4.2. *Let* $\{x_j^N(t)\}_{j=0}^N$,$\{\tilde{x}_j^N(t)\}_{j=0}^N$ *be solutions of* (1.1.3)*, indexed by* $N \in \mathbb{N}$ *that satisfy the condition of the uniformly bounded initial support* (1.0.3)*. Consider the corresponding Eulerian discrete densities* $\rho^{E,N}$*,* $\tilde{\rho}^{E,N} \in L^\infty([0,+\infty) \times \mathbb{R})$ *defined by* (1.2.1)*. Then for all* $T > 0$ *it holds*

$$\sup_{t\in[0,T]} W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) \leq W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) + 2LT \sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))|.$$

Proof.

Step 1. Set $j = 0, \ldots, N-1$ and $z \in [jl, (j+1)l)$. In this step we find a bound for

$$\left| x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right|.$$

First, notice that for $j = N$ it holds

$$x_N^N(t) - \tilde{x}_N^N(t) = x_N^N(0) - \tilde{x}_N^N(0)$$

while for $j = 0, \ldots, N-1$ it holds

$$x_j^N(t) - \tilde{x}_j^N(t) = x_j^N(0) - \tilde{x}_j^N(0) + \int_0^t v(\rho_j^N(t)) - v(\tilde{\rho}_j^N(t))dt.$$

Thus, for $j = N-1$ and $z \in [1-l, 1)$ it holds

$$\begin{aligned}
&\left| x_{N-1}^N(t) - \tilde{x}_{N-1}^N(t) + (z-1+l)\left(\frac{1}{\rho_{N-1}^N(t)} - \frac{1}{\tilde{\rho}_{N-1}^N(t)}\right)\right| \\
&= \left| (x_{N-1}^N(t) - \tilde{x}_{N-1}^N(t))\left(1 - \frac{z-1+l}{l}\right) + \frac{z-1+l}{l}(x_N^N(t) - \tilde{x}_N^N(t))\right| \\
&= \left| \left(x_{N-1}^N(0) - \tilde{x}_{N-1}^N(0) + \int_0^t (v(\rho_{N-1}^N(s)) - v(\tilde{\rho}_{N-1}^N(s)))ds\right)\left(1 - \frac{z-1+l}{l}\right) \right. \\
&\quad \left. + \frac{z-1+l}{l}\left(x_N^N(0) - \tilde{x}_N^N(0)\right)\right|
\end{aligned}$$

and therefore it holds

$$
\begin{aligned}
&\left| x_{N-1}^N(t) - \tilde{x}_{N-1}^N(t) + (z-1+l)\left(\frac{1}{\rho_{N-1}^N(t)} - \frac{1}{\tilde{\rho}_{N-1}^N(t)}\right)\right| \\
&\leq \left| (x_{N-1}^N(0) - \tilde{x}_{N-1}^N(0))\left(1 - \frac{z-1+l}{l}\right) + \frac{z-1+l}{l}(x_N^N(0) - \tilde{x}_N^N(0))\right| \\
&\quad + \left(1 - \frac{z-1+l}{l}\right)\int_0^t |v(\rho_{N-1}^N(s)) - v(\tilde{\rho}_{N-1}^N(s))|ds \\
&\leq \left| x_{N-1}^N(0) - \tilde{x}_{N-1}^N(0) + (z-1+l)\left(\frac{1}{\rho_{N-1}^N(0)} - \frac{1}{\tilde{\rho}_{N-1}^N(0)}\right)\right| \\
&\quad + L\int_0^t \left|\rho_{N-1}^N(s) - \tilde{\rho}_{N-1}^N(s)\right| ds.
\end{aligned}
$$

where in the last inequality we have used the Lipschitz continuity of the velocity, the fact that

$$
1 - \frac{z-1+l}{l} \leq 1 \qquad \forall\, z \in [1-l, 1],
$$

and the identity

$$
\begin{aligned}
&(x_{N-1}^N - \tilde{x}_{N-1}^N)\left(1 - \frac{z-1+l}{l}\right) + \frac{z-1+l}{l}\left(x_N^N - \tilde{x}_N^N\right) \\
&\qquad\qquad = x_{N-1}^N - \tilde{x}_{N-1}^N + (z-1+l)\left(\frac{1}{\rho_{N-1}^N} - \frac{1}{\tilde{\rho}_{N-1}^N}\right).
\end{aligned}
$$

With similar computations, we have that for $j = 0, ..., N-2$ and $z \in [jl, (j+1)l)$ it holds

$$
\begin{aligned}
&\left| x_j^N(t) - \tilde{x}_j^N(t) + (z-jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \\
&= \left| (x_j^N(0) - \tilde{x}_j^N(0))\left(1 - \frac{z-jl}{l}\right) + \frac{z-jl}{l}(x_{j+1}^N(0) - \tilde{x}_{j+1}^N(0)) \right. \\
&\quad + \left(1 - \frac{z-jl}{l}\right)\left(\int_0^t (v(\rho_j^N(s)) - v(\tilde{\rho}_j^N(s)))ds\right) \\
&\quad \left. + \frac{z-jl}{l}\left(\int_0^t (v(\rho_{j+1}^N(s)) - v(\tilde{\rho}_{j+1}^N(s)))ds\right)\right|
\end{aligned}
$$

and therefore it holds

$$
\begin{aligned}
&\left|x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \\
&\leq \left|(x_j^N(0) - \tilde{x}_j^N(0))\left(1 - \frac{z - jl}{l}\right) + \frac{z - jl}{l}(x_{j+1}^N(0) - \tilde{x}_{j+1}^N(0))\right| \\
&\qquad + L\int_0^t \left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| + \left|\rho_{j+1}^N(s) - \tilde{\rho}_{j+1}^N(s)\right| ds \\
&\leq \left|x_j^N(0) - \tilde{x}_j^N(0) + (z - jl)\left(\frac{1}{\rho_j^N(0)} - \frac{1}{\tilde{\rho}_j^N(0)}\right)\right| + 2L\int_0^t \left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| ds.
\end{aligned}
$$

where in the last inequality we have used the Lipschitz continuity of the velocity, the fact that

$$
1 - \frac{z - jl}{l} \leq 1 \qquad \text{and} \qquad \frac{z - jl}{l} \leq 1 \qquad \forall\, z \in [jl, (j+1)l],
$$

and the identity

$$
(x_j^N - \tilde{x}_j^N)\left(1 - \frac{z - jl}{l}\right) + \frac{z - jl}{l}(x_{j+1}^N - \tilde{x}_{j+1}^N) = x_j^N - \tilde{x}_j^N + (z - jl)\left(\frac{1}{\rho_j^N} - \frac{1}{\tilde{\rho}_j^N}\right).
$$

Step 2. By definition of the pseudo-inverse given in (1.2.6), using the bounds found in Step 1 and using Proposition 1.1.1, it holds

$$
\begin{aligned}
&\int_0^1 |X_{\rho^{E,N}(t)}(z) - X_{\tilde{\rho}^{E,N}(t)}(z)| dz \\
&= \int_0^1 \sum_{j=0}^{N-1} \left|x_j^N(t) - \tilde{x}_j^N(t) + (z - jl)\left(\frac{1}{\rho_j^N(t)} - \frac{1}{\tilde{\rho}_j^N(t)}\right)\right| \chi_{[jl,(j+1)l)}(z) dz \\
&\leq \int_0^1 \sum_{j=0}^{N-1} \left|x_j^N(0) - \tilde{x}_j^N(0) + (z - jl)\left(\frac{1}{\rho_j^N(0)} - \frac{1}{\tilde{\rho}_j^N(0)}\right)\right| \chi_{[jl,(j+1)l)}(z) dz \\
&\qquad + 2L\int_0^1 \int_0^t \sum_{j=0}^{N-1} \left|\rho_j^N(s) - \tilde{\rho}_j^N(s)\right| \chi_{[jl,(j+1)l)}(z) ds dz.
\end{aligned}
$$

By Proposition 1.1.1 and integration, it holds

$$
\begin{aligned}
&\int_0^1 |X_{\rho^{E,N}(t)}(z) - X_{\tilde{\rho}^{E,N}(t)}(z)| dz \\
&\leq \left\|X_{\rho^{E,N}}(0) - X_{\tilde{\rho}^{E,N}}(0)\right\|_{L^1([0,1])} + \frac{2Lt}{N} \sum_{j=0}^{N-1} \left|y_j^N(0) - \tilde{y}_j^N(0)\right|.
\end{aligned}
$$

Notice that

$$\begin{aligned}
&\left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])} \\
&= \sum_{j=0}^{N-1} |y_j^N(0) - \tilde{y}_j^N(0)| \int_0^1 \chi_{[jl,(j+1)l)}(z)dz = \frac{1}{N} \sum_{j=0}^{N-1} |y_j^N(0) - \tilde{y}_j^N(0)|.
\end{aligned}$$

Therefore, it holds

$$\begin{aligned}
&\left\| X_{\rho^{E,N}}(t) - X_{\tilde{\rho}^{E,N}}(t) \right\|_{L^1([0,1])} \\
&\leq \left\| X_{\rho^{E,N}}(0) - X_{\tilde{\rho}^{E,N}}(0) \right\|_{L^1([0,1])} + 2Lt \left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])}.
\end{aligned}$$

From Definition A.2.2, we restate the inequality in terms of of the Wasserstein distance:

$$W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) \leq W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) + 2Lt \left\| y^{L,N}(0) - \tilde{y}^{L,N}(0) \right\|_{L^1([0,1])}.$$

By taking the supremum in the time interval $[0, T]$ for arbitrary $T > 0$ we have the result.

□

Proposition 1.4.3. *Assume v satifies* (V1)*. Let $\{x_j^N(t)\}_{j=0}^N$,$\{\tilde{x}_j^N(t)\}_{j=0}^N$ be solutions of* (1.1.3)*, indexed by $N \in \mathbb{N}$. Consider the corresponding Eulerian discrete densities $\rho^{E,N}$, $\tilde{\rho}^{E,N} \in L^\infty([0, +\infty) \times \mathbb{R})$ defined by* (1.2.1)*. Define*

$$\Omega_T(\rho^{E,N}, \tilde{\rho}^{E,N}) := \left[\min\{x_0^N(0), \tilde{x}_0^N(0)\}, \max\{x_N^N(0), \tilde{x}_N^N(0)\} + Tv_{\max}\right]. \quad (1.4.1)$$

Define

$$W^{\sup} := \sup_{t \in [0,T]} \left\| F^N - \tilde{F}^N \right\|_{L^1(\mathbb{R})}$$

Consider the following two cases:

1. *Assume that* $\mathrm{TV}\left(\rho^{E,N}(0); \mathbb{R}\right), \mathrm{TV}\left(\tilde{\rho}^{E,N}(0); \mathbb{R}\right) \leq K$ *for some $K > 0$ independent of N. For all $T > 0$, it holds*

$$\begin{aligned}
&\sup_{t \in [0,T]} \left\| \rho^{E,N}(t) - \tilde{\rho}^{E,N}(t) \right\|_{L^1(\mathbb{R})}^2 \\
&\leq \mathrm{meas}(\Omega_T(\rho^{E,N}, \tilde{\rho}^{E,N})) \left[\mathrm{TV}\left(\rho^{E,N}(0); \mathbb{R}\right) + \mathrm{TV}\left(\tilde{\rho}^{E,N}(0); \mathbb{R}\right)\right] \sqrt{W^{\sup}}.
\end{aligned}$$

2. *Assume that assumption* (V2) *holds. For all $\delta, T > 0$, it holds*

$$\begin{aligned}
&\sup_{t\in[\delta,T]} \left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|^2_{L^1(\mathbb{R})} \\
&\le \operatorname{meas}(\Omega_T(\rho^{E,N}, \tilde{\rho}^{E,N})) \left[\sup_{t\ge\delta} \operatorname{TV}\left(\rho^{E,N}(t); \mathbb{R}\right) + \sup_{t\ge\delta} \operatorname{TV}\left(\tilde{\rho}^{E,N}(t); \mathbb{R}\right)\right] \sqrt{W^{\sup}}.
\end{aligned}$$

Proof. To ease notation, denote

$$F^N := F_{\rho^{E,N}} \quad \text{and} \quad \tilde{F}^N := F_{\tilde{\rho}^{E,N}}.$$

Since F^N and $\tilde{F}^N$ are monotone non-decreasing and $1-$Lipschitz, then the function $F^N - \tilde{F}^N$ is $1-$Lipschitz. Therefore, by Lemma 1.3.1 it holds

$$\left\|F^N - \tilde{F}^N\right\|_{L^\infty(\mathbb{R})} \le \sqrt{\left\|F^N - \tilde{F}^N\right\|_{L^1(\mathbb{R})}} \quad \forall\, N \in \mathbb{N}. \tag{1.4.2}$$

Omitting the time variable since no dynamics are involved, notice that since it holds

$$\lim_{x\to+\infty} |F^N(x) - \tilde{F}^N(x)| = \lim_{x\to-\infty} |F^N(x) - \tilde{F}^N(x)| = 0 \qquad \forall N \in \mathbb{N},$$

then by integration by parts it holds

$$\begin{aligned}
\int_{\mathbb{R}} (\rho^{E,N}(x) - \tilde{\rho}^{E,N}(x))^2 dx &= \int_{\mathbb{R}} \frac{d}{dx}\left(F^N(x) - \tilde{F}^N(x)\right)\left(\rho^{E,N}(x) - \tilde{\rho}^{E,N}(x)\right) dx \\
&= -\int_{\mathbb{R}} \left(F^N(x) - \tilde{F}^N(x)\right) \frac{d}{dx}\left(\rho^{E,N}(x) - \tilde{\rho}^{E,N}(x)\right) dx \\
&\le \left\|F^N - \tilde{F}^N\right\|_{L^\infty(\mathbb{R})} \operatorname{TV}\left(\rho^{E,N} - \tilde{\rho}^{E,N}; \mathbb{R}\right) \\
&\le \left\|F^N - \tilde{F}^N\right\|_{L^\infty(\mathbb{R})} \left[\operatorname{TV}\left(\rho^{E,N}; \mathbb{R}\right) + \operatorname{TV}\left(\tilde{\rho}^{E,N}; \mathbb{R}\right)\right].
\end{aligned}$$

By L^p inclusion and by using (1.4.2), we thus get that for all $\Omega \subset \mathbb{R}$ bounded, for all $N, T > 0$ and for all $t \in [0, T]$ it holds

$$\begin{aligned}
\left\|\rho^{E,N} - \tilde{\rho}^{E,N}\right\|^2_{L^1(\Omega)} &\le \operatorname{meas}(\Omega) \left\|\rho^{E,N} - \tilde{\rho}^{E,N}\right\|^2_{L^2(\mathbb{R})} \\
&\le \operatorname{meas}(\Omega) \left\|F^N - \tilde{F}^N\right\|_{L^\infty(\mathbb{R})} \left[\operatorname{TV}\left(\rho^{E,N}; \mathbb{R}\right) + \operatorname{TV}\left(\tilde{\rho}^{E,N}; \mathbb{R}\right)\right] \\
&\le \operatorname{meas}(\Omega) \sqrt{\left\|F^N - \tilde{F}^N\right\|_{L^1(\mathbb{R})}} \left[\operatorname{TV}\left(\rho^{E,N}; \mathbb{R}\right) + \operatorname{TV}\left(\tilde{\rho}^{E,N}; \mathbb{R}\right)\right].
\end{aligned}$$

Now notice that for all $N, T > 0$ and for all $t \in [0, T]$ it holds

$$\mathrm{supp}(\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)) \in \Omega_T(\rho^{E,N}, \tilde{\rho}^{E,N})$$

where $\Omega_T(\rho^{E,N}, \tilde{\rho}^{E,N})$ is defined by (1.4.1). To ease notation, in the rest of the proof we simply denote it as Ω_T. Thus, for all $N, T > 0$ and for all $t \in [0, T]$ it holds

$$\left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|_{L^1(\mathbb{R})} = \left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|_{L^1(\Omega_T)}. \tag{1.4.3}$$

Therefore, by L^p inclusion it holds

$$\left\|\rho^{E,N} - \tilde{\rho}^{E,N}\right\|^2_{L^1(\mathbb{R})} \leq \mathrm{meas}(\Omega_T)\sqrt{\left\|F^N - \tilde{F}^N\right\|_{L^1(\mathbb{R})}}\left[\mathrm{TV}\left(\rho^{E,N};\mathbb{R}\right) + \mathrm{TV}\left(\tilde{\rho}^{E,N};\mathbb{R}\right)\right].$$

The further treatment of this inequality is now adressed in the two cases (1) and (2) in the two following steps.

Step 1. We assume that $\mathrm{TV}\left(\rho^{E,N}(0);\mathbb{R}\right), \mathrm{TV}\left(\tilde{\rho}^{E,N}(0);\mathbb{R}\right) \leq K$ for some $K > 0$ independent of N. We use the contractivity property in time of the total variation of $\rho^{E,N}$ and $\tilde{\rho}^{E,N}$ (see [32, Proposition 5]). Thus, for all $t \in [0, T]$, it holds

$$\begin{aligned}&\left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|^2_{L^1(\mathbb{R})}\\ &\leq \mathrm{meas}(\Omega_T)\left[\mathrm{TV}\left(\rho^{E,N}(0);\mathbb{R}\right) + \mathrm{TV}\left(\tilde{\rho}^{E,N}(0);\mathbb{R}\right)\right]\sqrt{\left\|F^N - \tilde{F}^N\right\|_{L^1(\mathbb{R})}}.\end{aligned}$$

By taking the supremum in time, the first part of the proposition holds.

Step 2. We assume that (V2) holds. Fix $\delta > 0$. Due to Lemma 1.1.1 we know that

$$\sup_{t\geq\delta}\left[\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \mathrm{TV}\left(\tilde{\rho}^{E,N}(t);\mathbb{R}\right)\right]$$

is uniformly bounded with respect to N for an arbitrary $\delta > 0$, see [31, Proposition 3.3] or [32, Proposition 6]. Thus, for all $t \in [\delta, T]$, it holds

$$\begin{aligned}&\left\|\rho^{E,N}(t) - \tilde{\rho}^{E,N}(t)\right\|^2_{L^1(\mathbb{R})}\\ &\leq \mathrm{meas}(\Omega_T)\left[\sup_{t\geq\delta}\mathrm{TV}\left(\rho^{E,N}(t);\mathbb{R}\right) + \sup_{t\geq\delta}\mathrm{TV}\left(\tilde{\rho}^{E,N}(t);\mathbb{R}\right)\right]\sqrt{\left\|F^N - \tilde{F}^N\right\|_{L^1(\mathbb{R})}}.\end{aligned}$$

By taking the supremum in time in the interval $[\delta, T]$, the second part of the proposition holds.

□

We are now ready to prove the main result of this section.

Proof of Theorem 1.0.2. To ease notation, denote

$$F^N := F_{\rho^{E,N}}, \qquad \tilde{F}^N := F_{\tilde{\rho}^{E,N}}.$$

The proof is based on the concatenation of the above propositions. Notice that the first part of the theorem is proved by Proposition 1.4.1 and Proposition 1.4.2. Indeed, applying Proposition 1.4.1 for $t = 0$, it holds

$$W_1(\rho^{E,N}(0), \tilde{\rho}^{E,N}(0)) \leq 2 \sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))|. \tag{1.4.4}$$

Therefore, if it holds

$$\lim_{N \to +\infty} \sum_{j=0}^{N-1} |x_{j+1}(0) - x_j(0) - (\tilde{x}_{j+1}(0) - \tilde{x}_j(0))| = 0,$$

then, by (1.4.4) and Proposition 1.4.2, for all $T > 0$ it holds

$$\lim_{N \to +\infty} \sup_{t \in [0,T]} W_1(\rho^{E,N}(t), \tilde{\rho}^{E,N}(t)) = 0$$

and thus the first part of the theorem is proved. Now Proposition 1.4.3 shows that as a conssequence, recalling Definition (A.2.2), also the second part of the theorem holds.

□

Chapter 2: First-order cooperative systems under persistent excitation

Studying the self-organization and emergence of patterns derived from collective dynamics has gained significant prominence in applied mathematics. In particular, an increasing body of research has focused on achieving a detailed mathematical comprehension of multi-agent systems, where the goal is to understand the underlying mechanisms that drive the emergence of *consensus*. In simple terms, *consensus* means that all agents arrive to an agreement, as for instance arriving to a unanimous vote in elections or forming a unanimous opinion on a subject. Applications of such models are found in a wide variety of fields, such as in aviation, robotics, social sciences, finance and biology [6, 9, 23, 40, 41, 61, 65].

First-order cooperative models are of the following form:

$$\dot{x}_i(t) = \frac{\lambda_i}{N} \sum_{j=1}^{N} \phi(x_i(t), x_j(t)) \cdot (x_j(t) - x_i(t)) \qquad i \in \{1, \dots, N\} \tag{2.0.1}$$

Here, we consider the evolution of $N \geq 2$ agents identified as points in a configuration space, in our case the Euclidean space $\mathbb{R}^d$. The position $x_i(t) \in \mathbb{R}^d$ may represent opinion, velocity or other attributes of agent i at time t. The (nonlinear) influence function $\phi(x_i(t), x_j(t))$ is used to quantify the influence of agent j on agent i, where $i, j \in \{1, \dots, N\}$. The term λ_i is a scaling parameter. In the classical case, the function $\phi(x_i, x_j)$ is symmetric and $\lambda_i > 0$, [30, 42]. The main property that helps in the analysis of such a model is precisely the symmetry of the influence function with respect to i and j. Indeed, for general symmetric influence functions, the mean value is conserved. The system is cooperative and if the influence function is uniformly bounded in time from below by some strictly positive constant, we then have that the dynamics converges to the initial mean state. Such a symmetry in the influence function allows the use of spectral analysis and l^2-based arguments to study the variance of the system.

However, such a model presents the drawback that the dynamics of an agent is influenced by the weight of all agents. Indeed, when considering such a scaling, the internal dynamics of a small group of agents located far away from a much larger group of agents is negligible. As proposed in [46], a more realistic model has a different weighting procedure setting

$$\lambda_i = \frac{N}{\sum_{l=1}^{N} \phi(x_i, x_l)}.$$

The main feature now is that the influence of agent j on agent i is weighted by the total influence $\sum_{l=1}^{N} \phi(x_i, x_l)$ applied to agent i. However, even in the case where $\phi(x_i, x_l)$ is symmetric the resulting model becomes nonsymmetric due to such scaling, and we cannot therefore rely on standard variance-based strategies developed in the case of symmetric models. Instead of relying then on L^2 *theory* and thus studying the variance

$$B(x, x) := \frac{1}{2N} \sum_{i,j=1}^{N} ||x_i - x_j||^2$$

one may opt to consider the L^∞ *theory* and thus study the evolution of the diameter, which is defined as

$$d_X := \max_{i,j \in \{1,\dots,N\}} |x_i - x_j| \tag{2.0.2}$$

where $X = \{x_i\}_{i \in \{1,\dots,N\}}$.

In any case, from the modelling point of view in both of these types of models, each agent is expected to communicate with its neighbors through a *network topology* influenced by sensor characteristics and the environment. While the easiest scenario involves a fixed network topology over time (e.g. [51, 64]), practical situations often involve dynamic changes, due to factors like communication dropouts, security concerns, or intermittent actuation. As a result, the time-varying network topology gives rise to potential connection losses between agents, hindering the attainment of consensus. Therefore, when interactions between agents are subject to failure, it becomes crucial to investigate whether consensus can still be achieved or not. In order to consider such scenarios we then consider models of the following form:

$$\dot{x}_i(t) = \frac{\lambda}{N}\sum_{j=1}^{N} M_{ij}(t)\phi(x_i(t), x_j(t)) \cdot (x_j(t) - x_i(t)) \qquad i \in \{1, \ldots, N\}$$

where now the terms $M_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ represent the weight given to the influence of agent j on agent i. They encode the time-varying network topology and account for potential communication failures that can occur in the system (e.g., when they vanish). In order to quantify these possible lack of interactions, we recur to the condition of *persistent excitation* (PE from now on):

Definition 2.0.1 (Persistent excitation)**.** *Consider a function $M \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$. Let $T, \mu > 0$ be given. We say that the function $M(\cdot)$ satisfies the PE condition with parameters μ, T if it holds*

$$\int_t^{t+T} M(s)\, ds \geq \mu \qquad \forall t \geq 0. \tag{PE}$$

We further define the space of functions $M \in L^\infty([0, +\infty); [0, 1])$ satisfying such a condition as

$$\mathcal{M}_{T,\mu} := \left\{ M \in L^\infty([0, +\infty); [0, 1]) \quad s.t. \quad \int_t^{t+T} M(s)\, ds \geq \mu \quad \forall t \in [0, +\infty) \right\}. \tag{2.0.3}$$

Imposing the PE condition on a function $M \in \mathcal{M}_{T,\mu}$ means that such a function is not too weak on any given time interval of length T. In both this chapter and in the next one we give results where we impose the PE condition on the *scrambling coefficient* (see e.g. [58]) of $M_N := \{M_{ij}\}_{i,j\in\{1,\ldots,N\}}$, defined as follows

$$\eta(M_N) := \min_{i,j\in\{1,\ldots,N\}} \frac{1}{N}\sum_{k=1}^{N} \min\{M_{ik}, M_{jk}\}. \tag{2.0.4}$$

By letting $M_{ii} > 0$ for all $i \in \{1, \ldots, N\}$ we get the following interpretation of the scrambling coefficient: the scrambling coefficient is positive if and only if for every $i, j \in \{1, \ldots, N\}$ either i and j are interacting with each other or if they are both following a third agent $k \in \{1, \ldots, N\} \setminus \{i, j\}$.

This chapter is structured as follows. In Section 2.1 we present the class of finite-dimensional models that we consider and prove that consensus is reached under suitable PE conditions related to the weight kernels. Then, in Section 2.2

we consider the infinite dimensional setting in the classical mean-field limit and prove that consensus is reached using the results from the previous section, and then cite the result in [11] showing that consensus is reached also in the graph limit.

2.1 Consensus in the finite-dimensional particles system

In this section we first present the class of models we treat and then provide two main results. In Theorem 2.1.1 we prove that we have exponential consensus if we impose the PE condition on the scrambling coefficient of $\{M_{ij}\}_{i,j\in\{1,\ldots,N\}}$ as defined by (2.0.4). In Theorem 2.1.2 we prove that we have consensus if we impose the PE condition only on the weight kernels M_{ij} without any condition on the scrambling coefficient, and thus generalizing the previous result.

We consider the following Cauchy problem

$$\begin{cases} \dot{x}_i(t) = \frac{\lambda_i}{N} \sum_{j=1}^{N} M_{ij}(t)\phi_{ij}(t) \cdot (x_j(t) - x_i(t)) & t \geq 0, \\ x_i(0) = \bar{x}_i, \end{cases} \tag{2.1.1}$$

where $\bar{x}_i \in \mathbb{R}^n$, where

$$\phi_{ij} \coloneqq \phi(|x_j - x_i|) \qquad \forall i, j \in \{1, \ldots, N\}$$

and where

$$\lambda_i \coloneqq \begin{cases} 1 & \text{in the case where we consider equal weights} \\ \frac{N}{\sum_{l=1}^{N} \phi_{il}} & \text{in the case where we consider normalized weights} \end{cases}. \tag{2.1.2}$$

We assume that the map $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $M_{ij} : \mathbb{R}_+ \to [0, 1]$ satisfy the following hypotheses:

(H1) The function $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is locally Lipschitz continuous, positive and bounded from above by some $\phi_{\max} > 0$.

(H2) The weight kernels $M_{ij} : \mathbb{R}_+ \to [0, 1]$ are $\mathscr{L}^1$-measurable for all $i, j \in \{1, \ldots, N\}$.

We define the collection of solutions $\{x_i\}_{i=1}^{N}$ where x_i is the solution to system

(2.1.1) for all $i \in \{1, \dots, N\}$ as

$$X(t) := \{x_i(t)\}_{i=1}^N \qquad \forall t \geq 0. \tag{2.1.3}$$

We now define the concept of consensus in the first-order model (2.1.1).

Definition 2.1.1. *Let $X \in \mathbb{R}^{nN}$ be a solution of system* (2.1.1) *and the corresponding diameter as defined in* (2.0.2). *We say that the system converges to consensus if it holds*

$$\lim_{t \to +\infty} d_X(t) = 0. \tag{2.1.4}$$

By recalling that $d_X(0)$ corresponds to the diameter as defined by (2.0.2), a common assumption is given by:

(H3) For some $\phi_{\min} > 0$, it holds $\phi(x) \geq \phi_{\min} > 0$ for all $x \in [0, d_X(0)]$.

We now define here for convenience some terms that are recurrently used in the rest of the chapter. Define

$$K_{\max} := \begin{cases} \phi_{\max} & \text{in the case where } \lambda_i = 1 \\ \frac{\phi_{\max}}{\phi_{\min}} & \text{in the case where } \lambda_i = \frac{N}{\sum_{l=1}^N \phi_{kl}} \end{cases} \tag{2.1.5}$$

and

$$K_{\min} := \begin{cases} \phi_{\min} & \text{in the case where } \lambda_i = 1 \\ \frac{\phi_{\min}}{\phi_{\max}} & \text{in the case where } \lambda_i = \frac{N}{\sum_{l=1}^N \phi_{il}} \end{cases}. \tag{2.1.6}$$

2.1.1 Consensus under PE condition on the scrambling coefficient

In this section, we prove a similar theorem that has been proved in the context of graphons in [11], with the PE applied to the scrambling coefficient of $\{M_{ij}\}_{i,j \in \{1,\dots,N\}}$ as defined by (2.0.4). We follow the same methodology as in [11] to prove a slightly more general result and provide a simpler way to conclude that consensus holds. We then provide another strategy to conclude that consensus holds by treating the "worst case scenario", which is the main strategy followed in the next chapter.

Theorem 2.1.1. *Let $T, \mu > 0$ be given. Let $X \in \mathbb{R}^{nN}$ as defined by* (2.1.3) *be a solution of system* (2.1.1) *with initial data $X^0 \in \mathbb{R}^{nN}$ and the corresponding diameter $d_X(\cdot)$ defined by* (2.0.2)*. Assume that the influence function ϕ satisfies hypotheses (H1) and (H3). Assume that it holds*

$$\int_t^{t+T} \eta(M_N(s))ds \geq \mu \qquad \forall t \geq 0 \tag{2.1.7}$$

where $\eta(M_N(\cdot))$ is defined by (2.0.4) *and where $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ satisfies (H2). Then for system* (2.1.1) *it holds*

$$d_X(nT) \leq d_X(0)e^{-n\cdot K_{\min}\mu} \qquad \forall n \in \mathbb{N}$$

where $K_{\min}$ is defined by (2.1.6)*. In particular, consensus in the sense of* (2.1.4) *is reached.*

Proof. The function

$$d_X(t) = \max_{i,j\in\{1,\dots,N\}} |x_i - x_j|$$

is Lipschitz, because it is the pointwise maximum of a finite family of Lipschitz equicontinuous functions. By Rademacher's theorem, it is differentiable almost everywhere. By Dankin's theorem (Theorem A.1.2) it thus holds

$$\frac{1}{2}\frac{d}{dt}d_X^2(t) = \max_{i,j\in\Pi(t)} \left\langle \frac{d}{dt}(x_i(t) - x_j(t)), x_i(t) - x_j(t) \right\rangle$$

where $\Pi(t) \in \{1, \dots, N\} \times \{1, \dots, N\}$ represents the nonempty subset of pairs of indices for which the maximum is reached. Fix arbitrary $p, q \in \Pi(t)$. For easier notation, from now on we hide the dependence on time. Notice that for the case of system (2.1.1) with normalized weights it holds

$$\begin{aligned}\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle = &-\frac{1}{\sum_{k=1}^N \phi_{pk}} \sum_{j=1}^N M_{pj}\phi_{pj} \langle x_p - x_j, x_p - x_q \rangle \\ &- \frac{1}{\sum_{k=1}^N \phi_{qk}} \sum_{j=1}^N M_{qj}\phi_{qj} \langle x_j - x_q, x_p - x_q \rangle .\end{aligned}$$

By Lemma A.1.2 it holds

$$\langle x_p - x_j, x_p - x_q \rangle \geq 0 \quad \text{and} \quad \langle x_j - x_q, x_p - x_q \rangle \geq 0 \qquad \forall j \in \{1, \dots, N\}.$$

Therefore, by (2.1.6) it holds

$$\begin{aligned}
&\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle \\
&\leq -\frac{K_{\min}}{N} \left(\sum_{j=1}^{N} M_{pj} \langle x_p - x_j, x_p - x_q \rangle + \sum_{j=1}^{N} M_{qj} \langle x_j - x_q, x_p - x_q \rangle \right) \\
&= -\frac{K_{\min}}{N} \left(\sum_{j=1}^{N} [M_{pj} - M_{qj}] \langle x_p - x_j, x_p - x_q \rangle + \sum_{j=1}^{N} M_{qj} \langle x_p - x_q, x_p - x_q \rangle \right).
\end{aligned} \tag{2.1.8}$$

By adding and substracting $\frac{K_{\min}}{N} \sum_{j=1}^{N} \min\{M_{pj}, M_{qj}\} \langle x_j, x_p - x_q \rangle$, we get

$$\begin{aligned}
\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle \leq &-\frac{K_{\min}}{N} \sum_{j=1}^{N} M_{pj} \langle x_p, x_p - x_q \rangle + \frac{K_{\min}}{N} \sum_{j=1}^{N} M_{qj} \langle x_p, x_p - x_q \rangle \\
&+ \frac{K_{\min}}{N} \sum_{j=1}^{N} [M_{pj} - \min\{M_{pj}, M_{qj}\}] \langle x_j, x_p - x_q \rangle \\
&+ \frac{K_{\min}}{N} \sum_{j=1}^{N} [\min\{M_{pj}, M_{qj}\} - M_{qj}] \langle x_j, x_p - x_q \rangle \\
&- \frac{K_{\min}}{N} \sum_{j=1}^{N} M_{qj} \langle x_p - x_q, x_p - x_q \rangle .
\end{aligned}$$

Now, notice that

$$\sum_{j=1}^{N} [M_{pj} - \min\{M_{pj}, M_{qj}\}] \langle x_j, x_p - x_q \rangle \leq \sum_{j=1}^{N} [M_{pj} - \min\{M_{pj}, M_{qj}\}] \max_{j \in \{1,\dots,N\}} \langle x_j, x_p - x_q \rangle$$

and

$$\sum_{j=1}^{N} [\min\{M_{pj}, M_{qj}\} - M_{qj}] \langle x_j, x_p - x_q \rangle \leq \sum_{j=1}^{N} [\min\{M_{pj}, M_{qj}\} - M_{qj}] \min_{j \in \{1,\dots,N\}} \langle x_j, x_p - x_q \rangle .$$

By using Lemma A.1.2, we have

$$\max_{j \in \{1,\dots,N\}} \langle x_j, x_p - x_q \rangle = \langle x_p, x_p - x_q \rangle , \qquad \min_{j \in \{1,\dots,N\}} \langle x_j, x_p - x_q \rangle = \langle x_q, x_p - x_q \rangle . \tag{2.1.9}$$

Therefore it holds

$$\begin{aligned}\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle &\leq -\frac{K_{\min}}{N} \sum_{j=1}^{N} \min\left\{M_{pj}, M_{qj}\right\} \langle x_p - x_q, x_p - x_q \rangle \\ &\leq -\frac{K_{\min}}{N} \eta(M_N) \langle x_p - x_q, x_p - x_q \rangle\end{aligned}$$

This is valid for every pair $p, q \in \Pi(t)$ and therefore it holds

$$\frac{d}{dt} d_X^2(t) \leq -\frac{2K_{\min}}{N} \eta(M_N) d_X^2(t) \qquad \forall t \geq 0.$$

We obtain therefore the following differential inequality

$$\frac{d}{dt} d_X(t) \leq -\frac{K_{\min}}{N} \eta(M_N) d_X(t) \qquad \forall t \geq 0. \tag{2.1.10}$$

Integrating in $[0, T]$ and using (2.1.7) gives

$$d_X(T) \leq d_X(0) e^{-K_{\min} \int_0^T \eta(M_N(t)) dt} \leq d_X(0) e^{-K_{\min} \mu}$$

Consider now any interval $[nT, (n+1)T]$ for any $n \in \mathbb{N}$ and apply the same estimates to get

$$d_X(nT) \leq d_X((n+1)T) e^{-K_{\min} \mu}$$

and therefore

$$d_X(nT) \leq d_X(0) e^{-n \cdot K_{\min} \mu} \to 0 \quad \text{as } n \to +\infty. \tag{2.1.11}$$

$\square$

Remark 2.1.1. We note that the main difference between this proof and the existing proofs in the literature as in [47] lies in the fact that it does not require the adjacency operator to be stochastic. We replace this requirement by using Lemma A.1.2 in the one-dimensional setting in (2.1.9).

An alternative proof of Theorem 2.1.1 is to identify and treat the "worst case scenario", i.e. a specific profile of the M_{ij} which, informally, would make the diameter decrease as less as possible. This approach is taken in Chapter 3 in the of case of second-order models, but we include here the approach applied to first-order models.

The profile corresponding to such a scenario is given by M^*, which we now define. We then prove the result in Proposition 2.1.1.

Definition 2.1.2. *Let $T, \mu > 0$ be given. Define $M^* \in \mathcal{M}_{T,\mu}$ such that for all $t \geq 0$ and for all $n \in \mathbb{N}$ it holds*

$$M^*(t) = \begin{cases} 0 & t \in [(n-1)T,\, nT - \mu) \\ 1 & t \in [nT - \mu,\, nT). \end{cases} \tag{2.1.12}$$

Definition 2.1.3. *Let $M \in \mathcal{M}_{T,\mu}$, where $\mathcal{M}_{T,\mu}$ is defined by (2.0.3). Define $\mathcal{D}_X^M : \mathbb{R}_+ \to \mathbb{R}$ to be the solution of the following dynamical system:*

$$\begin{cases} \frac{d}{dt}\mathcal{D}_X^M(t) = -K_{\min} M(t) \mathcal{D}_X^M(t) \\ \mathcal{D}_X^M(0) = d_X^0 \end{cases} \tag{2.1.13}$$

where $K_{\min}$ is defined by (2.1.6).

Proposition 2.1.1. *Let $T, \mu > 0$ be given. Consider the space of functions $\mathcal{M}_{T,\mu}$ defined by (2.0.3). Consider $\mathcal{D}_X^M$ given by Definition 2.1.3. Let $M^* \in \mathcal{M}_{T,\mu}$ be defined by (2.1.12). Then, it holds*

$$\sup \left\{ \mathcal{D}_X^M(t) \;\middle|\; M \in \mathcal{M}_{T,\mu} \text{ and } \mathcal{D}_X^M(0) = d_X^0 \right\} = \mathcal{D}_X^{M^*}(t) \qquad \forall t \geq 0.$$

Proof. We first consider $t \in [0, T-\mu]$. Let $M \in \mathcal{M}_{T,\mu}$ be fixed. It holds

$$M(t) \geq 0 = M^*(t).$$

By integration, it also holds

$$\int_0^t M(s)\, ds \geq 0 = \int_0^t M^*(s)\, ds.$$

We now consider $t = T$. Since $M \in \mathcal{M}_{T,\mu}$, then it holds

$$\int_0^T M(s) ds \geq \mu = \int_0^T M^*(s)\, ds.$$

Let finally $t \in [T-\mu, T]$. Since $M(t) \leq 1$, it holds

$$\int_0^t M(s)\, ds = \int_0^T M(s)\, ds - \int_t^T M(s)\, ds \geq \mu - (T-t) = \int_0^t M^*(s)\, ds.$$

Consider now any interval $[nT,(n+1)T]$ for any $n \in \mathbb{N}$ and apply the same estimates to prove

$$\int_{jT}^{t} M(s)\,ds \geq \int_{jT}^{t} M^*(s)\,ds \qquad \forall t \in [nT,(n+1)T].$$

By concatenation of such estimates, it holds

$$\int_{0}^{t} M(s)\,ds \geq \int_{0}^{t} M^*(s)\,ds \qquad \forall t \geq 0.$$

By integrating the linear (time-varying) system (2.1.13) and observing that the function

$$x \mapsto e^{-K_{\min} x}$$

is monotonically decreasing, we have

$$\mathcal{D}^M(t) = d_X^0 e^{-K_{\min}\int_0^t M(s)\,ds} \leq d_X^0 e^{-K_{\min}\int_0^t M^*(s)\,ds} = d_X^{M^*}(t).$$

Since this holds for arbitrary $M \in \mathcal{M}_{T,\mu}$, then the result follows. □

Alternative consensus proof to Theorem 2.1.1. By following the proof, we find (2.1.10). We thus study the following system

$$\begin{cases} \frac{d}{dt} d_X(t) \leq -\eta(M_N(t)) \cdot K_{\min} d_X(t) & \forall t \geq 0 \\ d_X(0) = d_X^0 \end{cases}$$

where d_X^0 corresponds to the diameter of the initial data $\{\bar{x}_j\}_{j\in\{1,\dots,N\}}$. Now let $\mathcal{D}_X^\eta$ be the solution to the following differential equation

$$\begin{cases} \frac{d}{dt}\mathcal{D}_X^\eta(t) = -K_{\min}\eta(M_N(t))\mathcal{D}_X^\eta(t) & \forall t \geq 0 \\ d_X(0) = d_X^0 \end{cases} \tag{2.1.14}$$

Notice that it holds

$$\frac{d}{dt}\left(\mathcal{D}_X^\eta(t) - d_X(t)\right) \geq -K_{\min}\eta(M_N(t))\left(\mathcal{D}_X^\eta(t) - d_X(t)\right) \qquad \forall t \geq 0.$$

and therefore by using Gronwall's inequality it holds

$$\mathcal{D}_X^\eta(t) - d_X(t) \geq \left(\mathcal{D}_X^\eta(0) - d_X(0)\right) e^{-K_{\min}\int_0^t \eta(M_N(s))ds} = 0 \qquad \forall t \geq 0.$$

Therefore,

$$d_X(t) \leq \mathcal{D}_X^{\eta}(t) \qquad \forall t \geq 0. \tag{2.1.15}$$

By Proposition 2.1.1, we then have

$$d_X(t) \leq \mathcal{D}_X^{\eta}(t) \leq \mathcal{D}_X^{M^*}(t) \qquad \forall t \geq 0.$$

We now compute $\mathcal{D}_X^{M^*}(t)$. It satisfies for $j \in \mathbb{N}$

$$\frac{d}{dt}\mathcal{D}_X^{M^*}(t) = \begin{cases} 0 & t \in [0, T-\mu] \\ -K_{\min}\mathcal{D}_X^{M^*}(t) & t \in [T-\mu, T] \end{cases}$$

with initial datum

$$\mathcal{D}_X^{M^*}(0) = d_X^0.$$

It then satisfies

$$\mathcal{D}_X^{M^*}(T) = d_X(0)e^{-K_{\min}\mu}.$$

Then

$$d_X(T) \leq d_X(0)e^{-K_{\min}\mu}.$$

By recalling that the diameter of cooperative systems is non-increasing, by induction we have

$$d_X(nT+T) \leq d_X(nT) \leq d_X(0)e^{-n\cdot K_{\min}\mu}.$$

Now, notice that by (2.1.10) it holds

$$\frac{d}{dt}d_X(t) \leq 0 \qquad \forall t \geq 0.$$

Then,

$$\lim_{t\to+\infty} d_X(t) = 0.$$

□

Remark 2.1.2. We emphasize that, although it is longer than the more direct proof we have provided in Theorem 2.1.4, it shows to be an efficient way to deal with second-order models. Indeed, as seen in the next chapter, it helps us in the sense that we get an "on/off" behavior, and thus restrict ourselves to perform classical techniques of second-order models in the time intervals where we allow full communication.

2.1.2 Consensus under PE condition on the weigths

In this section we present a result in one dimension where we impose a persistent excitation condition solely on the kernel weights $\{M_{ij}\}_{i,j\in\{1,\ldots,N\}}$, unlike in the previous theorem.

Theorem 2.1.2. *Let $T, \mu > 0$ be given and let $N \in \mathbb{N}$. Let X^N as defined by* (2.1.3) *be a solution of system* (2.1.1) *with initial data $X^{N,0} \in \mathbb{R}^N$. Assume that the influence function ϕ satisfies hypotheses (H1) and (H3). Assume that $M_{ij} \in \mathcal{M}_{T,\mu}$ for all $i,j \in \{1,\ldots,N\}$, i.e.*

$$\int_t^{t+T} M_{ij}(s)ds \geq \mu \qquad \forall i,j \in \{1,\ldots,N\} \tag{2.1.16}$$

where M_{ij} satisfies hypothesis (H2) for all $i,j \in \{1,\ldots,N\}$. Then, consensus in the sense of (2.1.4) *is reached.*

Proof. In the rest of the proof the results are proved in the interval $[0,T]$. Then the same arguments follow when considering time intervals $[nT,(n+1)T]$ for $n \in \mathbb{N}$. In all of these time intervals, the agents are labeled according to their initial order at time $t = nT$ such that

$$x_1(nT) \leq x_2(nT) \leq \ldots \leq x_N(nT) \qquad n \in \mathbb{N}_0 \tag{2.1.17}$$

and we emphasize that such a label is maintained in the considered time interval $[nT,(n+1)T]$, even though the order might change.

We now informally present the strategy of the proof, and we identify the agents with their position on the configuration space $\mathbb{R}$:

- In **Step 1** we define the agents $\{\tilde{x}_j^{L,n}\}_{j\in\{1,\ldots,N\}}$ corresponding to the case where agents go to the left towards $x_1(nT)$ and $\{\tilde{x}_j^{R,n}\}_{j\in\{1,\ldots,N\}}$ corresponding to the case where agents go to the right towards $x_N(nT)$. Both depend on n as they are defined on each interval $[nT,(n+1)T]$. Informally, they correspond to the case where the agents forming the diameter x_1 and x_N become “frozen leaders”.

- In **Step 2** we show that the trajectory of agent x_j is always between the trajectory of $\tilde{x}_j^{L,n}$ and the trajectory of $\tilde{x}_j^{R,n}$. Moreover, if initially at time nT there is no consensus, then the trajectory is strictly bounded by $x_1(nT)$ and $x_N(nT)$, corresponding to the diameter at initial time nT.

- In **Step 3** we show that consensus is reached by mainly using the fact that trajectories were controlled in **Step 2**.

As it is used multiple times in the proof, notice that, given $K_{\max}$ defined by (2.1.5), $K_{\min}$ defined by (2.1.6) and the condition $M_{ij} \leq 1$, it holds

$$\frac{K_{\min}}{N}\sum_{j=1}^{N} M_{ij} \leq \frac{\lambda_k}{N}\sum_{j=1}^{N} M_{ij}\phi_{ij} \leq K_{\max} \qquad \forall i \in \{1,\dots,N\}. \tag{2.1.18}$$

Also, recall that by (2.1.10) it holds

$$\frac{d}{dt}d_X(t) \leq 0 \qquad \forall t \geq 0.$$

We now provide the complete proof.

Step 1 Let $n \in \mathbb{N}_0$. In this step we define the agents $\{\tilde{x}_i^{L,n}\}_{i\in\{1,\dots,N\}}$ and $\{\tilde{x}_i^{R,n}\}_{i\in\{1,\dots,N\}}$ in each interval $[nT,(n+1)T]$ and then provide some maximum/minimum principle results satisfied by both solutions.

For $t \in [nT,(n+1)T]$, define $\{\tilde{x}_i^{L,n}\}_{i\in\{1,\dots,N\}}$ solution to

$$\begin{cases} \dot{\tilde{x}}_i^{L,n}(t) = K_{\max}(x_1(nT) - \tilde{x}_i^{L,n}(t)), \\ \tilde{x}_i^{L,n}(0) = x_i(nT). \end{cases} \qquad i = 1,\dots,N \tag{2.1.19}$$

and define $\{\tilde{x}_i^{R,n}\}_{i\in\{1,\dots,N\}}$ solution to

$$\begin{cases} \dot{\tilde{x}}_i^{R,n}(t) = K_{\max}(x_N(nT) - \tilde{x}_i^{R,n}(t)), \\ \tilde{x}_i^{R,n}(0) = x_i(nT) \end{cases} \qquad i = 1,\dots,N. \tag{2.1.20}$$

where the initial data $\{x_i(nT)\}_{i\in\{1,\dots,N\}}$ corresponds to the position of the original agents $\{x_i(t)\}_{i\in\{1,\dots,N\}}$ at time $t = nT$. In particular, notice that

$$\tilde{x}_1^{L,n}(t) = x_1(nT) \quad \text{and} \quad \tilde{x}_N^{R,n}(t) = x_N(nT) \qquad \forall t \in [nT,(n+1)T]. \tag{2.1.21}$$

In this step, we prove that their orders are maintained in the interval of time $[nT,(n+1)T]$, i.e. it holds

$$\begin{cases} \tilde{x}_1^{L,n}(t) \le \tilde{x}_2^{L,n}(t) \le \ldots \le \tilde{x}_N^{L,n}(t) \\ \tilde{x}_1^{R,n}(t) \le \tilde{x}_2^{R,n}(t) \le \ldots \le \tilde{x}_N^{R,n}(t) \end{cases} \qquad \forall t \in [nT,(n+1)T]. \tag{2.1.22}$$

Furthermore, we prove the following results:

i) for all $i \in \{1,\dots,N\}$ it holds for all $t \in [nT,(n+1)T]$

$$x_1(nT) \le \tilde{x}_i^{L,n}(t) \le x_N(nT) \in [nT,(n+1)T] \tag{2.1.23}$$

ii) for all $i \in \{1,\dots,N\}$ it holds for all $t \in [nT,(n+1)T]$

$$\tilde{x}_i^{L,n}(t) \ge \tilde{x}_i^{L,n}((n+1)T) \quad \text{and} \quad \tilde{x}_i^{R,n}(t) \le \tilde{x}_i^{R,n}((n+1)T). \tag{2.1.24}$$

Without loss of generality due to the relabeling process (2.1.17), we focus on the time interval $[0,T]$. Notice that for all $k \in \{1,\dots,N-1\}$ we have that for $t \in [0,T]$ it holds

$$\begin{aligned} \frac{d}{dt}\left(\tilde{x}_{k+1}^{L,0}(t) - \tilde{x}_k^{L,0}(t)\right) &= K_{\max}(\bar{x}_1 - \tilde{x}_{k+1}^{L,0}(t)) - K_{\max}(\bar{x}_1 - \tilde{x}_k^{L,0}(t)) \\ &= -K_{\max}(\tilde{x}_{k+1}^{L,0}(t) - \tilde{x}_0^{L,n}(t)). \end{aligned}$$

By Gronwall's inequality it thus holds for all $t \in [0,T]$

$$\tilde{x}_{k+1}^{L,0}(t) - \tilde{x}_k^{L,0}(t) = (\bar{x}_{k+1} - \bar{x}_k)e^{-K_{\max}t} \qquad \forall k \in \{1,\dots,N-1\}$$

and by the initial ordering (2.1.17) it holds

$$\tilde{x}_1^{L,0}(t) \le \tilde{x}_2^{L,0}(t) \le \ldots \le \tilde{x}_N^{L,0}(t) \qquad \forall t \in [0,T] \tag{2.1.25}$$

and we have thus proved the first part in (2.1.22). The second part is completely equivalent. Now notice that for all $i \in \{1, \dots, N\}$ it holds

$$\frac{d}{dt}\left(\tilde{x}_i^{L,0}(t) - \bar{x}_1\right) = -K_{\max}(\tilde{x}_i^{L,0}(t) - \bar{x}_1) \tag{2.1.26}$$

and therefore by Gronwall's inequality and by (2.1.17) it holds

$$\tilde{x}_i^{L,0}(t) - \bar{x}_1 = (\bar{x}_i - \bar{x}_1)e^{-K_{\max}t} \geq 0 \qquad \forall i \in \{1, \dots, N\} \tag{2.1.27}$$

and thus proving the first inequality in (2.1.23). The second inequality is completely equivalent. Now, notice therefore that by (2.1.23) it holds

$$\frac{d}{dt}\tilde{x}_i^{L,0}(t) \leq 0 \quad \forall t \in [0, T] \quad \forall i \in \{1, \dots, N\}$$

which implies that

$$\tilde{x}_i^{L,0}(t) \geq \tilde{x}_i^{L,0}(T) \quad \forall i \in \{1, \dots, N\}$$

and we have thus proved the first part of (2.1.24). The second inequality is completely equivalent.

Step 2 Let $n \in \mathbb{N}_0$. In this step we prove that for all $i \in \{1, \dots, N\}$ it holds

$$\tilde{x}_i^{L,n}(t) \leq x_i(t) \leq \tilde{x}_i^{R,n}(t) \qquad \forall t \in [nT, (n+1)T]. \tag{2.1.28}$$

Without loss of generality due to the relabeling process (2.1.17), we restrict ourselves to the time interval $[0, T]$. Before proceeding to the proof of the statement above, we recall that by the dissipative property of the original solution $\{x_i\}_{i \in \{1,\dots,N\}}$ it holds

$$\bar{x}_1 \leq x_i(t) \leq \bar{x}_N \qquad \forall t \in [0, T] \quad \forall i \in \{1, \dots, N\}. \tag{2.1.29}$$

We now prove that it holds

$$\tilde{x}_i^{L,0}(t) \leq x_i(t) \quad \forall t \in [0, T] \quad \forall i \in \{1, \dots, N\}. \tag{2.1.30}$$

We have that

$$\frac{d}{dt}\left(x_i(t) - \tilde{x}_i^{L,0}(t)\right) = \frac{\lambda_i}{N}\sum_{j=1}^{N} M_{ij}(t)\phi_{ij}(t)(x_j(t) - x_i(t)) - K_{\max}(\bar{x}_1 - \tilde{x}_i^{L,0}(t)).$$

By using (2.1.18) and (2.1.29) then for all $i \in \{1, \ldots, N\}$ and for $t \in [0, T]$ it holds

$$
\begin{aligned}
&\frac{d}{dt}\left(x_i(t) - \tilde{x}_i^{L,0}(t)\right) \\
&= \frac{\lambda_i}{N}\sum_{j=1}^N M_{ij}(t)\phi_{ij}(t)(x_j(t) - \bar{x}_1) + \left(\frac{\lambda_i}{N}\sum_{j=1}^N M_{ij}(t)\phi_{ij}(t)\right)(\bar{x}_1 - x_i(t)) \\
&\quad - K_{\max}(\bar{x}_1 - \tilde{x}_i^{L,0}(t)) \\
&\geq 0 + K_{\max}(\bar{x}_1 - x_i(t)) - K_{\max}(\bar{x}_1 - \tilde{x}_i^{L,0}(t)) \\
&= -K_{\max}(x_i(t) - \tilde{x}_i^{L,0}(t)).
\end{aligned}
\tag{2.1.31}
$$

By Gronwall's inequality it thus holds

$$
x_i(t) - \tilde{x}_i^{L,0}(t) \geq (\bar{x}_i - \tilde{x}_i^{L,0}(0))e^{-K_{\max}t} = 0 \quad \forall t \in [0, T] \quad \forall i \in \{1, \ldots, N\}.
$$

The first inequality in (2.1.30) is proved. The proof of the second inequality is completely equivalent.

Step 3 In this step, we show that it holds

$$
\lim_{t \to +\infty} d_X(t) = 0. \tag{2.1.32}
$$

Observe that the set of permutations of N indices is finite while the set of discrete times $\{T_n\}_{n \in \mathbb{N}}$ is infinite. Therefore, we can choose a subsequence, which we we still denote by $\{T_n\}_{n \in \mathbb{N}}$ such that $T_{n+1} \geq T_n + T$ for all $n \in \mathbb{N}$, and reorder the indices only once such that

$$
x_1(T_n) \leq x_2(T_n) \leq \ldots \leq x_N(T_n) \qquad \forall n \in \mathbb{N}. \tag{2.1.33}
$$

From now on, we only treat such a subsequence. Notice that by the contractiviy property of the support, we have that $x_i(t)$ is bounded both from above and below for all $i \in \{1, \ldots, N\}$. Therefore, for all $i \in \{1, \ldots, N\}$, there exists $x_i^* \in [\bar{x}_1, \bar{x}_N]$ such that by choosing again a subsequence $\{T_n\}_{n \in \mathbb{N}}$ that we do not relabel, it holds

$$
\lim_{n \to +\infty} x_i(T_n) = x_i^* \qquad \forall i \in \{1, \ldots, N\}. \tag{2.1.34}
$$

We furthermore have that

$$x_1^* \leq x_2^* \leq \ldots \leq x_N^*, \tag{2.1.35}$$

due to the ordering at the discrete times $\{T_n\}_{n\in\mathbb{N}}$ given by (2.1.33).

Consider the time intervals $[T_n, T_n + T]$, $n \in \mathbb{N}$. Notice that for an infinite number of indices $n \in \mathbb{N}$ we have that

i) either $x_1(T_n + T) < \tilde{x}_2^{L,n}(T_n + T)$

ii) or $x_1(T_n + T) \geq \tilde{x}_2^{L,n}(T_n + T)$.

By choosing a further subsequence, still denoted by $\{T_n\}_{n\in\mathbb{N}}$, we now consider the following two cases:

i) either $x_1(T_n + T) < \tilde{x}_2^{L,n}(T_n + T)$ for all $n \in \mathbb{N}$

ii) or $x_1(T_n + T) \geq \tilde{x}_2^{L,n}(T_n + T)$ for all $n \in \mathbb{N}$

and in particular notice that at least one of them holds.

Step 3.1 Consider the time interval $[T_n, T_n+T]$. Assume that condition i) holds, i.e.

$$x_1(T_n + T) < \tilde{x}_2^{L,n}(T_n + T) \qquad \forall n \in \mathbb{N}. \tag{2.1.36}$$

The goal is to prove that (2.1.32) holds. We have two cases.

Step 3.1.1 The first case corresponds to $x_N^* = x_1^*$. Then, by construction it holds

$$\lim_{n\to+\infty} d_X(T_n) = \lim_{n\to+\infty} x_N(T_n) - x_1(T_n) = 0.$$

Since the function $d_X(\cdot)$ is non-increasing, in this case (2.1.32) holds.

Step 3.1.2 We now focus on the second case, that is $x_N^* > x_1^*$. By the choice of the subsequence $\{T_n\}_{n\in\mathbb{N}}$ we have that for some $k \in \mathbb{N}$ sufficiently big there exists $\epsilon > 0$ such that it holds

$$x_N^* - x_1^* > \epsilon \tag{2.1.37}$$

and

$$x_N(T_n) > x_N^* + \epsilon, \quad x_1(T_n) < x_1^* - \epsilon \qquad \forall n \geq k.$$

We now focus on the time interval $[T_k, T_k + T]$. Observe that for all $t \in [T_k, T_k + T]$ and for all $j \in \{2, \dots, N\}$ it holds

$$\tilde{x}_2^{L,k}(T_k + T) \overset{(2.1.22)}{\leq} \tilde{x}_j^{L,k}(T_k + T) \overset{(2.1.24)}{\leq} \tilde{x}_j^{L,k}(t) \overset{(2.1.28)}{\leq} x_j(t)$$

and therefore by (2.1.36) it holds

$$x_1(t) < x_j(t) \qquad \forall t \in [T_k, T_k + T] \quad \forall j \in \{2, \dots, N\}. \tag{2.1.38}$$

Therefore, notice that in this case the order of $x_1(\cdot)$ is conserved in $[T_k, T_k + T]$. Moreover, by the contractivity of the support and by the chosen sequence $\{T_n\}_{n\in\mathbb{N}}$, recalling that $T_{k+1} \geq T_k + T$, it thus holds

$$x_1(T_n) \geq x_1(T_n + T) \qquad \forall n > k.$$

Since the support is non-decreasing in time, we therefore have that

$$x_1^* \geq x_1(T_n)$$

and in particular it thus holds

$$x_1^* \geq x_1(t) \qquad \forall t \in [T_k, T_k + T]. \tag{2.1.39}$$

By (2.1.24) and (2.1.28) we therefore have that for $t \in [T_k, T_k + T]$ it holds

$$\begin{aligned}
\dot{x}_1 &= \frac{\lambda_1}{N} \sum_{j=2}^{N} M_{1j}(t)\phi_{1j}(t)(x_j(t) - \tilde{x}_j^{L,k}(T) + \tilde{x}_j^{L,k}(T) - x_1(t)) \\
&\geq \frac{\lambda_1}{N} \sum_{j=2}^{N} M_{1j}(t)\phi_{1j}(t)(\tilde{x}_j^{L,k}(T) - x_1(t) - x_1^* + x_1^* - x_1(T_k) + x_1(T_k)) \\
&= \frac{\lambda_1}{N} \sum_{j=2}^{N} M_{1j}(t)\phi_{1j}(t)[(\tilde{x}_j^{L,k}(T) - x_1(T_k)) - (x_1^* - x_1(T_k)) + (x_1^* - x_1(t))]
\end{aligned}$$

By (2.1.27) and (2.1.39) we thus have that for $t \in [T_k, T_k + T]$ it holds

$$\dot{x}_1 \geq \frac{K_{\min}}{N} \sum_{j=2}^{N} M_{1j}(t)[(x_j(T_k) - x_1(T_k))e^{-K_{\max}T} - (x_1^* - x_1(T_k))]$$

By integrating in $[T_k, T_k + T]$ and using the PE condition (2.1.16) and by recalling the bounds (2.1.18), it thus holds

$$\begin{aligned} & x_1(T_k + T) \\ & \geq x_1(T_k) - K_{\max}T(x_1^* - x_1(T_k)) + \frac{K_{\min}\mu e^{-K_{\max}T}}{N} \sum_{j=2}^{N}(x_j(T_k) - x_1(T_k)) \\ & \geq x_1(T_k) - K_{\max}T(x_1^* - x_1(T_k)) + \frac{K_{\min}\mu e^{-K_{\max}T}}{N}(x_N(T_k) - x_1(T_k)) \end{aligned}$$

Notice that by the contractivity property of the support, it holds

$$x_N(T_k) - x_1(T_k) \geq x_N^* - x_1^*. \tag{2.1.40}$$

Define

$$\eta_1 \coloneqq \frac{1}{2} \min \left\{ \frac{1}{K_{\max}T} \frac{K_{\min}\mu e^{-K_{\max}T}}{N}, \frac{\epsilon}{x_N^* - x_1^*}, 1 \right\}. \tag{2.1.41}$$

By i), this implies that

$$x_1^* - x_1(T_n) \leq \eta_1(x_N^* - x_1^*) \qquad \forall n \geq k. \tag{2.1.42}$$

We also define

$$\mu_{\min} \coloneqq \frac{2\eta_1 K_{\max} T N}{K_{\min} e^{-K_{\max}T}}.$$

By (2.1.40) and (2.1.42), it holds

$$\begin{aligned} x_1(T_k + T) & \geq x_1(T_k) + \left[\frac{K_{\min}\mu e^{-K_{\max}T}}{N} - K_{\max}T\eta_1 \right] (x_N^* - x_1^*) \\ & \geq x_1(T_k) + \frac{K_{\min}\mu_{\min} e^{-K_{\max}T}}{2N}(x_N^* - x_1^*) \end{aligned}$$

where in the last inequality we have used the definition of η_1 in

(2.1.41). Therefore, by using (2.1.37), it holds

$$x_1(T_k) + \frac{K_{\min}\mu e^{-K_{\max}T}}{2N}\epsilon \le x_1(T_k+T) \le x_i(T_k+T) \quad \forall i \in \{2,\dots,N\}$$

where we used (2.1.38). By contractivity of the support, it thus holds

$$x_i(t) \ge x_1(T_k) + \frac{K_{\min}\mu e^{-K_{\max}T}}{2N}\epsilon \qquad \forall t \ge T_k + T \quad \forall i \in \{1,\dots,N\}.$$

In particular, by considering our chosen subsequence and by recalling that $T_{n+1} \ge T_n + T$ by construction, it thus holds

$$x_1(T_{n+1}) \ge x_1(T_n) + \frac{K_{\min}\mu e^{-K_{\max}T}}{2N}\epsilon \qquad \forall n \ge k$$

which by induction implies that

$$\lim_{n\to+\infty} x_1(T_n) = +\infty$$

which is a contradiction, and therefore (2.1.37) cannot hold. Thus (2.1.32) holds.

Step 3.2 We now the second possibility. Assume that condition ii) hodlds, i.e.

$$x_1(T_n+T) \ge \tilde{x}_2^{L,n}(T_n+T) \qquad \forall n \in \mathbb{N}. \tag{2.1.43}$$

The goal is to prove that (2.1.32) holds. We prove it by contradiction. In a first step, we prove that, if there is no consensus, then at least two particles realize the minimum and two realize the maximum, at the limit. In a second step, we prove that this raises a contradiction.

Step 3.2.1 Assume that the diameter does not converge to zero. We prove that there exists indices $j,k,l,m \in \{1,\dots,N\}$, labeled as $1, 2, N-1, N$, such that

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} x_2(t) = \lim_{t\to+\infty} \min_{i\in\{1,\dots,N\}} \{x_i(t)\} \tag{2.1.44}$$

and

$$\lim_{t\to+\infty} x_N(t) = \lim_{t\to+\infty} x_{N-1}(t) = \lim_{t\to+\infty} \max_{i\in\{1,\dots,N\}} \{x_i(t)\}. \tag{2.1.45}$$

First of all, remark that the contractivity of the support implies that the function $\psi(t) \coloneqq \min_{i\in\{1,\dots,N\}}\{x_i(t)\}$ is continuous, non-decreasing and bounded from above. It therefore admits a finite limit. Notice that by the choice of the subsequence $\{T_n\}_{n\in\mathbb{N}}$ it holds $\psi(T_n) = x_1(T_n)$, and therefore

$$\lim_{n\to+\infty} \psi(T_n) = x_1^*.$$

Therefore, to prove (2.1.44), we simply need to prove that

$$x_1^* = x_2^*. \tag{2.1.46}$$

Assume by contradiction that

$$x_1^* < x_2^*. \tag{2.1.47}$$

Then, by the choice of the subsequence $\{T_n\}_{n\in\mathbb{N}}$ we have that for some $k \in \mathbb{N}$ sufficiently big there exists $\epsilon > 0$ such that it holds

$$x_1(T_k) \in [x_1^* - \epsilon, x_1^* + \epsilon], \quad x_2(T_k) \in [x_2^* - \epsilon, x_2^* + \epsilon] \tag{2.1.48}$$

satisfying

$$x_2(T_k) - x_1(T_k) \geq \frac{4\epsilon}{e^{-K_{\max}T}}. \tag{2.1.49}$$

Now, by recalling (2.1.27), it holds

$$\tilde{x}_2^{L,k}(T_k + T) = x_1(T_k) + (x_2(T_k) - x_1(T_k))e^{-K_{\max}T}.$$

By (2.1.48) and (2.1.49) it thus holds

$$\begin{aligned}
\tilde{x}_2^{L,k}(T_k + T) &\geq x_1^* - \epsilon + (x_2^* - \epsilon - x_1^* - \epsilon)e^{-K_{\max}T} \\
&> x_1^* - \epsilon + (x_2^* - x_1^*)e^{-K_{\max}T} - 2\epsilon e^{-K_{\max}T} \\
&\geq x_1^* - \epsilon + 4\epsilon - 2\epsilon e^{-K_{\max}T} \\
&\geq x_1^* + \epsilon.
\end{aligned}$$

By (2.1.43), it then holds

$$x_1(T_k + T) \geq \tilde{x}_2^{L,k}(T_k + T) \geq x_1^* + \epsilon. \tag{2.1.50}$$

Now notice that for all $i \in \{2, \ldots, N\}$ it holds

$$x_i(T_k + T) \overset{(2.1.28)}{\geq} \tilde{x}_i^{L,k}(T_k + T) \overset{(2.1.22)}{\geq} x_2^{L,k}(T_k + T). \tag{2.1.51}$$

Therefore, by (2.1.50) and (2.1.51) we thus have that

$$x_i(T_k + T) \geq x_1^* + \epsilon \qquad \forall i \in \{1, \ldots, N\}.$$

By contraction of the support, we have that

$$x_i(t) \geq x_1^* + \epsilon \qquad \forall t \geq T_k + T \quad \forall i \in \{1, \ldots, N\}$$

and in particular, by considering our chosen subsequence, it thus holds

$$x_1(T_n) \geq x_1^* + \epsilon \qquad \forall n \geq k,$$

which implies

$$\lim_{n \to +\infty} x_1(T_n) > x_1^*$$

which is a contradiction with (2.1.34). Therefore, (2.1.47) does not hold and, by the ordering (2.1.35), it holds (2.1.46). The proof of (2.1.45) follows the exact same argument.

Step 3.2.2 We now prove that (2.1.44), (2.1.45) implies $x_N^* = x_1^*$. By contradiction, and recalling that $x_1^* \leq x_N^*$ by construction, assume that there exists $\epsilon > 0$ such that it holds

$$x_N^* - x_1^* > \epsilon. \tag{2.1.52}$$

By eventually reducing $\epsilon > 0$ and using (2.1.44), (2.1.45), there exists $T_k > 0$ such that for all $t > T_k$ it holds

$$x_1(T_k), x_2(T_k) \in [x_1^* - \epsilon, x_1^* + \epsilon], \quad x_N(T_k), x_{N-1}(T_k) \in [x_N^* - \epsilon, x_N^* + \epsilon]. \tag{2.1.53}$$

We now follow the same lines of the argument presented to prove (2.1.44).
For $t \in [T_k, T_k + T]$ we have that

$$\begin{aligned}\dot{x}_1(t) &= \frac{\lambda_1}{N}\sum_{j=2}^{N} M_{1j}(t)\phi_{1j}(t)(x_j(t) - x_1(t)) \\ &= \frac{\lambda_1}{N}\sum_{\{j:\, x_j(t)\le x_1(t)\}} M_{1j}(t)\phi_{1j}(t)(x_j(t) - x_1(t)) \\ &\quad + \frac{\lambda_1}{N}\sum_{\{j:\, x_j(t)> x_1(t)\}} M_{1j}(t)\phi_{1j}(t)(x_j(t) - x_1(t)).\end{aligned}$$

Therefore, by (2.1.53) we have that

$$x_1^* + \epsilon \ge x_1(t) \quad \forall t \in [T_k, T_k + T].$$

It holds

$$\begin{aligned}\dot{x}_1(t) \ge{}& \frac{\lambda_1}{N}\sum_{\{j:\, x_j(t)\le x_1(t)\}} M_{1j}(t)\phi_{1j}(t)(x_1(T_k) - x_1^* - \epsilon) \\ &+ \frac{\lambda_1}{N} M_{1N}(t)\phi_{1N}(t)(x_N(t) - x_1(t)) \\ &+ \frac{\lambda_1}{N} M_{1(N-1)}(t)\phi_{1(N-1)}(t)(x_{N-1}(t) - x_1(t)).\end{aligned}$$

By recalling the bounds (2.1.18), it holds

$$\begin{aligned}\dot{x}_1(t) \ge{}& K_{\max}(x_1(T_k) - x_1^* - \epsilon) \\ &+ \frac{K_{\min}}{N} M_{1N}(t)(x_N(t) - x_1(t)) \\ &+ \frac{K_{\min}}{N} M_{1(N-1)}(t)(x_{N-1}(t) - x_1(t)).\end{aligned}$$

By using (2.1.53), it therefore holds

$$\dot{x}_1(t) \geq -3\epsilon K_{\max} + \frac{K_{\min}}{N} M_{1N}(t)(x_N^* - x_1^* - 2\epsilon) + \frac{K_{\min}}{N} M_{1(N-1)}(t)(x_N^* - x_1^* - 2\epsilon).$$

By integrating in $[T_k, T_k + T]$ and using the PE condition (2.1.16) it then holds

$$x_1(T_k + T) \geq x_1(T_k) + \frac{2\mu K_{\min}}{N}(x_N^* - x_1^*) - \epsilon\left(3K_{\max}T + \frac{\mu K_{\min}}{N}4\right). \tag{2.1.54}$$

I now choose $\epsilon > 0$ such that

$$\frac{2\mu K_{\min}}{N}(x_N^* - x_1^*) - \epsilon\left(3K_{\max}T + \frac{\mu K_{\min}}{N}4\right) > 2\epsilon,$$

namely

$$\epsilon < \frac{2\mu K_{\min}(x_N^* - x_1^*)}{N\left(2 + 3K_{\max}T + \frac{\mu K_{\min}}{N}4\right)}.$$

It then holds by (2.1.54)

$$x_1(T_k + T) - x_1(T_k) > 2\epsilon$$

which is a contradiction with (2.1.53). Therefore, (2.1.52) cannot hold and we thus have that (2.1.32) holds. This raises a contradiction with the fact that (2.1.32) does not hold, as required at the beginning of **Step 3.2**.

We have then proved that each of the conditions i) and ii) ensure

$$\lim_{t \to +\infty} d_X(t) = 0.$$

□

Remark 2.1.3. The main usefulness of the one-dimensional setting is that order exists, which is not the case in the multi-dimensional setting. Notice that the hypothesis of asking for the kernel weights to be persistently excited themselves is weaker than imposing on the their scrambling coefficient. Indeed, at least in one dimension, we can then see that we can choose M_{ij} such that the scrambling coefficients are null but we still get to consensus. An even weaker hypothesis would be to set the PE condition on the *in-degree function*:

$$\int_t^{t+T} \frac{1}{N} \sum_{j=1}^{N} M_{ij}(s)ds \geq \mu \qquad \forall i \in \{1, \ldots, N\}. \tag{2.1.55}$$

2.2 Consensus in infinite dimension

In this section we consider the infinite dimensional setting. We prove that consensus holds also in the classical mean-field setting.

When considering the classical mean-field limit, we can only account for systems with dynamics which are invariant under permutations of the labels of the agents. Indeed, it requires all particles to be indistinguishable since it describes the population by its density. This means that a classical mean-field limit can only make sense in the setting where

$$M_{ij}(t) = M(t) \qquad \forall i, j \in \{1, \ldots, N\} \tag{2.2.1}$$

and we therefore consider the finite particle systems

$$\begin{cases} \dot{x}_i(t) = \frac{\lambda_i}{N} \sum_{j=1}^{N} M(t)\phi(|x_j(t) - x_i(t)|)\,(x_j(t) - x_i(t)) & t \geq 0, \\ x_i(0) = x_i^0, \end{cases} \tag{2.2.2}$$

where the influence function ϕ satisfies hypotheses (H1) and (H3), λ_i is defined by (2.1.2) and $M \in \mathcal{M}_{T,\mu}$, defined by (2.0.3). Notice that in this case we then have the following theorem

Theorem 2.2.1. *Let $T, \mu > 0$ be given. Let $X \in \mathbb{R}^{nN}$ as defined by* (2.1.3) *be a solution of system* (2.2.2) *with initial data $X^0 \in \mathbb{R}^{nN}$. Define the diameter $d_X(\cdot)$ by* (2.0.2)*. Assume that the influence function ϕ satisfies hypotheses (H1) and (H3). Assume that $M \in \mathcal{M}_{T,\mu}$ where $\mathcal{M}_{T,\mu}$ is defined by* (2.0.3)*. Then for system* (2.2.2) *it holds*

$$d_X(nT) \leq d_X(0)e^{-n \cdot K_{\min}\mu} \qquad \forall n \in \mathbb{N}$$

where $K_{\min}$ is defined by (2.1.6)*. In particular, consensus in the sense of* (2.1.4) *is reached.*

Proof. Consider system (2.1.1). In the case where $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ satisfies (2.2.1) we have that it holds

$$\frac{1}{N}\min_{i,j\in\{1,\dots,N\}}\sum_{k=1}^{N}\min\{M_{ik}(s),M_{jk}(s)\}=M(t)$$

and by using Theorem 2.1.1 the result follows by noticing that the solution of (2.1.1) with $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ satisfying (2.2.1) is exactly the solution of (2.2.2). □

We now consider the mean-field setting corresponding to (2.2.2). Define the diameter for a compactly supported measure $\nu\in\mathcal{P}_1(\mathbb{R}^d)$ as

$$d_X[\nu]:=\operatorname{diam}(\operatorname{supp}\nu). \tag{2.2.3}$$

Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$. Then, the continuum model corresponding to (2.2.2) is

$$\begin{cases}\partial_t\mu_t+\nabla_x\left(V[t,\mu_t]\mu_t\right)=0 & \forall x\in\mathbb{R}^d,\quad t>0\\ \mu_0=\bar{\mu}\end{cases} \tag{2.2.4}$$

where the initial datum $\bar{\mu}\in\mathcal{M}(\mathbb{R}^d)$. In the case we are treating the particle system (2.2.2) with equal weights, i.e. λ_i constant, the non-local vector-field is given by

$$V[t,\mu_t](x)=\int_{\mathbb{R}^d}M(t)\phi(x,y)(y-x)d\mu_t(y)\qquad\forall x\in\mathbb{R}^d. \tag{2.2.5}$$

In the case we are treating the particle system (2.2.2) with normalized weights, i.e.

$$\lambda_i=\frac{N}{\sum_{l=1}^{N}\phi(x_i,x_l)}.$$

we define the non-local vector field by

$$V[t,\mu_t](x):=\frac{\int_{\mathbb{R}^d}M(t)\phi(x,y)(y-x)d\mu_t(y)}{\int_{\mathbb{R}^d}\phi(x,y)d\mu_t(y)}\qquad\forall x\in\mathbb{R}^d. \tag{2.2.6}$$

Definition 2.2.1. *Let $T>0$. A measure $\mu_t\in C([0,T];\mathcal{M}(\mathbb{R}^d))$ is a measure-valued solution of* (2.2.4) *in the time interval $[0,T]$ with initial datum $\bar{\mu}\in\mathcal{M}(\mathbb{R}^d)$*

if it holds for all $\phi \in C_c^\infty(\mathbb{R}^d \times T)$

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \phi + V[t, \mu_t](x) \cdot \nabla_v \phi)\, d\mu_t(v) dt + \int_{\mathcal{R}^d} \phi(v, 0) d\bar{\mu}(v) = 0. \tag{2.2.7}$$

We now state an existence-uniqueness theorem and a stability theorem of (2.2.4). Such theorems have been proved in [21] for the case $M \equiv 1$, i.e. full communication between agents at all times. However, they still hold true in the case where we have a uniform multiplicative persistently excited term as shown in the next lemma, which is basically [21, Lemma 3.4] applied to our case.

Lemma 2.2.1. *Let* $\mu_t \in C([0,T]; \mathcal{M}(\mathbb{R}^d))$ *have uniform compact support, i.e.*

$$\operatorname{supp} \mu_t \subset B^d(0, R), \quad \forall t \in [0, T]$$

where $B^d(0, R)$ *stands for a d-dimensional ball centered at the origin with radius* $R > 0$. *Let* $M \in \mathcal{M}_{T,\mu}$ *be given, where* $\mathcal{M}_{T,\mu}$ *is defined by* (2.0.3). *Then there exists a constant* $K > 0$ *such that*

$$|V[t, \mu_t](x) - V[t, \mu_t](\tilde{x})| \leq K|x - \tilde{x}|$$

for all $x, \tilde{x} \in B^d(0, R)$ *and for all* $t \in [0, T]$. *Moreover, there exists a constant* $C > 0$ *such that*

$$|V[t, \mu_t](x)| \leq C$$

for all $x \in B^d(0, R)$ *and for all* $t \in [0, T]$.

Proof. For any $x, \tilde{x} \in B^d(0, R)$ and by recalling that $M(\cdot) \leq 1$, for the vector field given by (2.2.5) it holds

$$\begin{aligned}
&|V[t, \mu_t](x) - V[t, \mu_t](\tilde{x})| \\
&= \left| \int_{\mathbb{R}^d} M(t)\phi(x, y)(y - x) d\mu_t(y) - \int_{\mathbb{R}^d} M(t)\phi(x, y)(y - \tilde{x}) d\mu_t(y) \right| \\
&= |M(t)| \cdot \left| \int_{\mathbb{R}^d} \phi(x, y)(y - x) d\mu_t(y) - \int_{\mathbb{R}^d} \phi(x, y)(y - \tilde{x}) d\mu_t(y) \right|
\end{aligned}$$

and for the vector field given by (2.2.6) it holds

$$
\begin{aligned}
&|V[t,\mu_t](x) - V[t,\mu_t](\tilde{x})| \\
&= \left| \frac{\int_{\mathbb{R}^d} M(t)\phi(x,y)(y-x)d\mu_t(y)}{\int_{\mathbb{R}^d} \phi(x,y)d\mu_t(y)} - \frac{\int_{\mathbb{R}^d} M(t)\phi(\tilde{x},y)(y-\tilde{x})d\mu_t(y)}{\int_{\mathbb{R}^d} \phi(\tilde{x},y)d\mu_t(y)} \right| \\
&\leq \left| \frac{\int_{\mathbb{R}^d} \phi(x,y)(y-x)d\mu_t(y)}{\int_{\mathbb{R}^d} \phi(x,y)d\mu_t(y)} - \frac{\int_{\mathbb{R}^d} \phi(\tilde{x},y)(y-\tilde{x})d\mu_t(y)}{\int_{\mathbb{R}^d} \phi(\tilde{x},y)d\mu_t(y)} \right|.
\end{aligned}
$$

The rest of the computations follows exactly as in the proof of [21, Lemma 3.4]. □

By such a lemma and by [16, Theorem 3.10] we have local-in-time existence and uniqueness of a measure-valued solution in the sense of (2.2.7). Furthermore, this solution exists, given that it remains compactly supported in position, which is what is proved in the next theorem, where the authors estimate the growth of the support of μ_t in order to guarantee global-in-time existence and uniqueness.

Theorem 2.2.2 ([21, Theorem 3.5]). *Consider the continuum model* (2.2.4) *with* $\bar{\mu} \in \mathcal{M}(\mathbb{R}^d)$ *and suppose that there exists a constant* $R > 0$ *such that*

$$\operatorname{supp} \bar{\mu} \subset B^d(0,R)$$

where $B^d(0,R)$ *stands for a d-dimensional ball centered at the origin with radius* $R > 0$. *Then, there exists a unique measure-valued solution* $\mu_t \in C([0,+\infty); \mathcal{M}(\mathbb{R}^d))$ *of* (2.2.4) *in the sense of* (2.2.7). *Moreover,* μ_t *is uniformly compactly supported and we have*

$$\mu_t = X(t;\cdot)_{\#}\bar{\mu}$$

where $X(t;\cdot)$ *is the flow generated by* $V[t,\mu_t]$.

Theorem 2.2.3 ([21, Theorem 3.6]). *Let* $\mu_t^1, \mu_t^2 \in C([0,T]; \mathcal{M}(\mathbb{R}^d))$ *be two weak solutions of* (2.2.4) *subject to uniformly compactly supported initial data* $\bar{\mu}^1, \bar{\mu}^2$, *respectively. Define*

$$R_{i,X}^T := \max_{0 \leq t \leq T} \max_{x \in \operatorname{supp} \mu_t} |x|, \qquad i = 1,2.$$

Then, there exists a constant $C > 0$, *depending only on* $\phi, T, R_{i,X}^T$, *such that*

$$W_p(\mu_t^1, \mu_t^2) \leq C W_p(\bar{\mu}^1, \bar{\mu}^2) \qquad \forall t \in [0,T], \quad \forall p \in [1,+\infty].$$

This stability result provides a rigorous passage from the particle system (2.2.2) to (2.2.4), where the nonlocal vector field is given by (2.2.5) or (2.2.6), depending on the choice of the weighting procedure.

Notice that the result of Theorem 2.1.1 is independent of N, and is valid for system (2.1.1). Then, by following the lines of [21, Theorem 3.7] for the case of no delay ($\tau = 0$), we extend the result of [24, Theorem 4.1] by adding the case of (2.2.6) and by considering the more general class of weight kernel $M \in \mathcal{M}_{T,\mu}$:

Theorem 2.2.4. *Let $\mu_t \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ be a measure-valued solution to* (2.2.4) *with compactly supported initial data $\bar{\mu} \in \mathcal{P}_1(\mathbb{R}^d)$ with the vector field V either given by* (2.2.5) *or* (2.2.6). *Define $d_X[\cdot]$ by* (2.2.3). *Assume that the influence function ϕ satisfies hypotheses (H1) and (H3), for $d_X[\bar{\mu}]$. It then holds*

$$d_X[\mu_t] \le 2d_X[\bar{\mu}]e^{-n\cdot K_{\min}\mu} \qquad \forall n \in \mathbb{N}.$$

Proof. Let $N \in \mathbb{N}$. Define the family of $N-$particle approximations of $\bar{\mu}$, namely $\{\bar{\mu}^N\}_N$ defined as

$$\bar{\mu}^N := \frac{1}{N}\sum_{i=1}^{N} \delta(x - \bar{x}_i^N)$$

where $\{\bar{x}_i^N\}_{i\in\{1,\ldots,N\}}$ are chosen such that

$$\operatorname{supp} \bar{\mu}^N \subseteq B(0, d_X[\bar{\mu}]) \quad \forall N \in \mathbb{N} \tag{2.2.8}$$

and

$$\lim_{N\to+\infty} W_p(\bar{\mu}, \bar{\mu}^N) = 0. \tag{2.2.9}$$

Now let $\{x_i^N\}_{i\in\{1,\ldots,N\}}$, with $x_i^N \in \mathbb{R}^d$, denote the solution to the finite dimensional system (2.2.2) with initial condition $\{\bar{x}_i^N\}_{i\in\{1,\ldots,N\}}$. Define

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i^N(t)) \qquad \forall t \in [0,T].$$

Then, we have that μ_t^N is a measure-valued solution to the kinetic model (2.2.4) in the sense of (2.2.7). Now notice that if $\mu_t \in C([0,T];\mathcal{M}(R))$ is a weak solution to (2.2.4) with initial datun $\bar{\mu}$, then according to Theorem 2.2.3 there exists a constant $C > 0$ independent of N such that

$$W_p(\mu_t^N, \mu_t) \le C W_p(\bar{\mu}^N, \bar{\mu}) \qquad \forall t \in [0, T], \quad \forall p \in [1, +\infty].$$

Therefore, by (2.2.9) it holds

$$\lim_{N \to +\infty} W_p(\mu_t^N, \mu_t) = 0 \qquad \forall t \in [0, T], \quad \forall p \in [1, +\infty]. \tag{2.2.10}$$

Define

$$d_X^N(t) := \operatorname{diam}(\operatorname{supp} \mu_t^N) \quad t \ge 0$$

First, notice that by (2.2.8) we have that it holds

$$d_X^N(0) = \operatorname{diam}(\operatorname{supp} \bar{\mu}^N) \le 2 d_X[\bar{\mu}]. \tag{2.2.11}$$

By using Theorem 2.2.1 and (2.2.11) we thus have that for all $N \in \mathbb{N}$ it holds

$$\operatorname{supp} \mu_t^N \subset B(0, d_X^N(t)) \subset B(0, d_X^N(0) e^{-n \cdot K_{\min} u}) \subseteq B(0, d_X[\bar{\mu}] e^{-n \cdot K_{\min} u}). \tag{2.2.12}$$

where we recall that $d_X(\cdot)$ is defined by (2.0.2).

We have that (2.2.10) implies that $\mu_t^N \to \mu_t$ weakly-* as measures as $N \to +\infty$ for all $t \in [0, T]$, i.e.

$$\lim_{N \to +\infty} \int_{\mathbb{R}^d} f(x) d\mu_t^N(x) = \int_{\mathbb{R}^d} f(x) d\mu_t(x) \qquad \forall f \in C_c(\mathbb{R}^d). \tag{2.2.13}$$

We now prove that

$$\operatorname{supp} \mu_t \subset B(0, d_X[\bar{\mu}] e^{-n \cdot K_{\min} u}). \tag{2.2.14}$$

First, observe (2.2.13). Take now $X \subset \mathbb{R}^d$ closed, on $\mathbb{R}^d \setminus B(0, d_X[\bar{\mu}] e^{-n \cdot K_{\min} u})$ and observe that (2.2.13) implies

$$\lim_{N \to +\infty} \int_{\mathbb{R}^d} f(x) d\mu_t^N(x) = \int_{\mathbb{R}^d} f(x) d\mu_t(x) = 0 \qquad \forall f \in C_c(X).$$

This implies

$$\operatorname{supp} \mu_t \subset \mathbb{R}^d \setminus X.$$

Since X is arbitrary, then (2.2.14) holds. Therefore, we have that

$$d_X[\mu_t] \leq 2d_X[\bar{\mu}]e^{-n\cdot K_{\min}\mu} \qquad \forall n \in \mathbb{N}.$$

□

Another way of considering the mean-field limit model is by recurring to the graph limit model, also known as graphon, as explained in [3, 11]. The main difference between the classical mean-field model and the graphon model is the *indistinguishability* property: in a graphon the agents still have a labeling given by a generalized index $i \in [0,1]$ which allows to discriminate between agents, while in the mean-field limit such a labeling is lost. Thus, when considering the mean-field limit, there is an irreversible loss of information on the identification of agents. In particular, this means that if the influence function depends explicitely on the agents labels, then we cannot use to the classical mean-field limit framework. Also, there is a possibility to retrieve the classical mean-field limit from the graph-limit (see [3, 53]).

In the rest of this chapter, we now present the graphon limit of system (2.1.1). Consider the following piecewise-constant function

$$x^N(t,i) \coloneqq \sum_{k=1}^{N} \chi_{\left[\frac{k-1}{N},\frac{k}{N}\right)}(i)x_k(t) \tag{2.2.15}$$

and

$$M^N(t,i,j) \coloneqq \sum_{k=1}^{N}\sum_{l=1}^{N} \chi_{\left[\frac{k-1}{N},\frac{k}{N}\right)}(i)\chi_{\left[\frac{l-1}{N},\frac{l}{N}\right)}(j)M_{ij}(t) \tag{2.2.16}$$

defined for $i,j \in I$ where

$$I \coloneqq [0,1]. \tag{2.2.17}$$

The following holds

Proposition 2.2.1 ([11, Proposition 2.8]). *Let $N \geq 1$ and $X(t)$ be a solution of* (2.1.1). *Then, the curve $x^N \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, L^2(I,\mathbb{R}^d))$ defined by* (2.2.15) *is a solution of the Cauchy problem*

$$\begin{cases} \partial_t x^N(t,i) = \int_I M^N(t,i,j)\phi\left(\left|x^N(t,i) - x^N(t,j)\right|\right)\left(x^N(t,j) - x^N(t,i)\right)\mathrm{d}j, \\ x^N(0,i) = x^0(i), \end{cases} \tag{2.2.18}$$

for almost every $i \in I$, where $M^N(t,i,j)$ is defined by (2.2.16)*.*

Proof. See [11]. $\square$

Remark 2.2.1. As noted in [11, Remark 2.9], every finite-dimensional multi-agent system of the form (2.1.1) can be recast in the context of graphons by using (2.2.15) and (2.2.16). Therefore, every result in the context of graphons has an equivalent formulation in the finite-dimensional setting through these piecewise-constant functions.

We can therefore consider (2.2.18) as an infinite-dimensional integro-differential equation and thus consider the graphon model

$$\begin{cases} \partial_t x(t,i) = \int_I M(t,i,j)\phi\left(|x(t,i)-x(t,j)|\right)\left(x(t,j)-x(t,i)\right)dj, \\ x(0) = x^0. \end{cases} \tag{2.2.19}$$

Theorem 2.2.5 ([11, Proposition 2.10])**.** *Let $x^0 \in L^2(I,\mathbb{R}^d)$ and assume that the weight kernel $M : \mathbb{R}_+ \times I \times I \to [0,1]$ is $\mathscr{L}^1 \times \mathscr{L}^1_{\llcorner I} \times \mathscr{L}^1_{\llcorner I}$-measurable and the influence function ϕ satisfies (H1), where $\mathscr{L}^1_{\llcorner I}$ stands for the Lebesgue measure restricted to I. Then, there exists a unique solution $x \in Lip_{\mathrm{loc}}(\mathbb{R}_+, L^2(I,\mathbb{R}^d))$ to the Cauchy problem* (2.2.19)*. If it furthemore holds that $x^0 \in L^\infty(I,\mathbb{R})$ then it holds*

$$\|x(t)\|_{L^\infty(I)} \leq \left\|x^0\right\|_{L^\infty(I)} \qquad \forall t \geq 0.$$

Just as in the finite-dimensional settings, we have the notion of diameter.

Definition 2.2.2. *Let $x \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, L^2(I,\mathbb{R}^d))$ be solution of* (2.2.19)*. Define the diameter as*

$$D_X(t) \coloneqq \operatorname*{ess\,supp}_{i,j\in I}|x(t,i)-x(t,j)| \qquad t \geq 0. \tag{2.2.20}$$

We say that the system converges to consensus if it holds

$$\lim_{t\to+\infty} D_X(t) = 0. \tag{2.2.21}$$

We mention the following theorem which is mainly a result given by [11, Theorem 3.3], and therefore only briefly describe the proof.

Theorem 2.2.6. *Let $T,\mu > 0$ be given. Let I be defined by* (2.2.17)*. Let $x \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, L^2(I,\mathbb{R}^d))$ be the solution to* (2.2.19) *with equal weights with initial* *datum* $x^0 \in L^\infty(I, \mathbb{R}^d)$. *Define the diameter* $d_X(\cdot)$ *by* (2.2.20). *Assume that the influence function* ϕ *satisfies hypotheses (H1) and*

$$\phi(x) \geq \phi_{\min} > 0 \quad \forall x \in [0, D_X(0)] \quad \textit{for some} \quad \phi_{\min} > 0. \tag{2.2.22}$$

Assume moreover that

$$\int_t^{t+T} \inf_{i,j\in I} \int_I \min\{M(s,i,k), M(s,j,k)\} dk\, ds \geq \mu \qquad \forall t \geq 0$$

where $M(\cdot, i, j)$ *satisfies (H2) for all* $i, j \in I$. *It holds*

$$D_X(nT) \leq D_X(0) e^{-n\cdot\phi_{\min}\mu} \qquad \forall n \in \mathbb{N}$$

In particular, by sending n *to* $+\infty$ *we get consensus in the sense of* (2.2.21).

Proof. As done in [11, Theorem 3.3], let $x \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+, L^2(I, \mathbb{R}^d))$ be a solution of (2.2.19) and fix $\epsilon > 0$. By Scorza-Dragoni's theorem, there exists a compact set $I_\epsilon \subset I$ with $\mathrm{meas}(I \backslash I_\epsilon) < \epsilon$ such that $x : \mathbb{R}_+ \times I_\epsilon \to \mathbb{R}$ is a continuous map. Consider then the restricted diameter

$$D_X^\epsilon(t) = \max_{i,j\in I_\epsilon} |x(t,i) - x(t,j)|.$$

The function $D_X^\epsilon(t)$ is Lipschitz. By Rademacher's theorem it is differentiable almost exerywhere and thus by Danskin's theorem (Theorem A.1.2) it holds

$$\frac{1}{2}\frac{d}{dt} D_X^2(t) = \max_{i,j\in\Pi^\epsilon(t)} \left\langle \frac{\partial}{\partial t}(x(t,i) - x(t,j)), x(t,i) - x(t,j) \right\rangle$$

where $\Pi^\epsilon(t) \in I_\epsilon \times I_\epsilon$ represents the nonempty subset of indices for which the maximum is reached. By using Lemma A.1.2, after some computations we arrive to

$$\begin{aligned}
&\frac{d}{dt} \left(D_X^\epsilon(t)\right)^2 \\
&\leq -2 \left(\inf_{i,j\in I} \int_I \min\{M(s,i,k), M(s,j,k)\} dk \right) \cdot \phi_{\min} \left(D_X^\epsilon(t)\right)^2 + 6\epsilon\phi_{\max} \left(D_X(0)\right)^2
\end{aligned}$$

and by using Gronwall's lemma and Lemma A.1.1, we get the pointwise convergence

$$D_X^\epsilon(t) \to D_X(t).$$

By passing to the limit, it holds

$$D_X(t) \leq D_X(0) e^{-\phi_{\min} \int_0^t \inf_{i,j\in I} \int_I \min\{M(s,i,k), M(s,j,k)\} dk ds}.$$

As in Theorem 2.1.1, by first considering the time interval $[0, T]$ and then subsequently all the other time intervals $[nT, (n+1)T]$ for $n \in \mathbb{N}$ the estimate holds. □

Remark 2.2.2. The added novelty of the proof lies in the use of the combination of Scorza-Dragoni's theorem (Theorem A.1.1) and Danskin's theorem (Theorem A.1.2), which allows to use in the graphon setting the tools of the finite-dimensional setting. We can also prove in an entirely similar way to Proposition 2.1.1 that the "worst case scenario" would correspond to setting for almost all i, j

$$M(t, i, j) = M^*$$

and then study this case.

Chapter 3: Second-order cooperative systems under persistent excitation

In the previous chapter we have addressed first-order models, where we were interested in reaching consensus. However, in some cases the dynamics that are exhibited are of second-order, in the sense that agents share further information like “accelaration” or any higher-order information. In these cases, we are therefore faced with a higher level of complexity. In second-order models, an agent does not only look at the *velocity* of another agent, but also takes into account its *position*. The goal now is to not only study consensus of the velocities, but also to see whether we can achieve *flocking*. In simple terms, *flocking* means that all agents arrive to a consensus in the velocity variable and remain “not too far away from each other” in the position. Similarly to first-order models, applications of such models are found in a wide variety of fields, such as in robotics [4, 6, 23, 62]. Interestingly enough, the Cucker-Smale model, which is one of the models studied in this article, was originally intented to model language evolution, [25].

Second-order cooperative models are of the following form:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda_i}{N} \sum_{j=1}^{N} \phi(x_i(t), x_j(t)) \cdot (v_j(t) - v_i(t)) \end{cases} \qquad i \in \{1, \dots, N\}. \tag{3.0.1}$$

Here, we consider the evolution of $N \geq 2$ agents identified as points in a configuration space, in our case the Euclidean space $\mathbb{R}^{2d}$. For each agent i we now have that the pair $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ represents at time t its position and velocity in the *phase space*. The (nonlinear) influence function $\phi(x_i(t), x_j(t))$ is used to quantify the influence of agent j on agent i, where $i, j \in \{1, \dots, N\}$ based on their positions. The term λ_i is a scaling parameter. Just as in the previous chapter, we consider λ_i given by (2.1.2), that we recall here for convenience:

$$\lambda_i := \begin{cases} 1 & \text{in the case where we consider equal weights} \\ \frac{N}{\sum_{l=1}^{N} \phi(x_i, x_l)} & \text{in the case where we consider normalized weights} \end{cases}. \tag{3.0.2}$$

In the case of the Cucker-Smale model [25], the influence function was chosen as

$$\phi(x_i, x_j) = \frac{1}{(1 + |x_i - x_j|)^{\beta}} \qquad \beta \geq 0. \tag{3.0.3}$$

In the case where $\lambda_i = 1$, then we recover the classical Cucker-Smale model. A crucial property in the Cucker-Smale model is the symmetry of the influence function ϕ, i.e. $\phi_{ij} = \phi_{ji}$ for all $i, j \in \{1, \ldots, N\}$. Analogously to the case of first-order models, the average velocity is conserved. For $\beta \in [0, 1]$ the Cucker-Smale dynamics converges to the initial mean velocity just as in the case of first-order models and moreover *unconditional flocking* holds, i.e. flocking holds without any restriction on the initial data. For the case $\beta > 1$ we have instead conditional flocking (see [35, Proposition 4.1, Proposition 4.3]). For a general influence function ϕ we have the following result:

Proposition 3.0.1 ([35]). *Let $\{x_i, v_i\}_{i\in\{1,\ldots,N\}}$ be a solution to* (3.0.1) *where ϕ is a non-negative function. Define*

$$\Gamma(t) := \sum_{i,j=1}^{N} |x_i(t) - x_j(t)|^2, \quad \Lambda(t) := \sum_{i,j=1}^{N} |v_i(t) - v_j(t)|^2.$$

If it holds

$$\Lambda(0) < \int_{\Gamma(0)}^{+\infty} \phi(x)dx$$

then flocking holds.

In the case where $\lambda_i = \frac{N}{\sum_{l=1}^{N} \phi(x_i, x_l)}$ we recover the Motsch-Tadmor model. It came as a response to the drawback in the classical Cucker-Smale model regarding the normalization of the interactions by the total number of agents N, which is inadequate for far-from-equilibrium scenarios, as explained in the previous chapter. For this reason, in [46] the authors introduce the normalized weighting approach, which was indeed originally intended for second-order models.

Just as in the previous chapter, we would like to study the case where we might have communication failures between agents. Similarly, we impose a PE condition on the scrambling coefficient of $M_N := \{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ as done in the previous chapter, which definition we recall here for convenience:

$$\eta(M_N) := \min_{i,j\in\{1,\dots,N\}} \frac{1}{N}\sum_{k=1}^{N} \min\{M_{ik}, M_{jk}\}. \tag{3.0.4}$$

As one can expect, the added difficulty now is dealing with the higher complexity of the model given by the intertwining of the position and velocity variables. We thus study second-order models of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda}{N}\sum_{j=1}^{N} M_{ij}(t)\phi(x_i(t), x_j(t)) \cdot (v_j(t) - v_i(t)) \end{cases} \qquad i \in \{1,\dots,N\}.$$

This chapter is organized as follows. In Section 3.1, we first present the class of finite-dimensional models we consider and then we show that we get unconditional flocking for suitable conditions on the influence function and PE condition on the weight kernels as in Theorem 2.1.1. Then, in Section 2.2 we consider the classical mean-field setting of a particular class of finite-dimensional models and show that flocking holds as well, under the same conditions as in the case of finite-dimensional models.

3.1 Flocking in the finite-dimensional particles system

In this section we first present the class of models we treat and then provide our main results. In Proposition 3.1.2 we identify the "worst case scenario", which we then treat in Theorem 3.1.1 to show that unconditional flocking holds by imposing the PE condition on the scrambling coefficient of $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ as defined by (2.0.4), similarly as in case of first-order models.

We consider the following Cauchy problem

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda_i}{N}\sum_{j=1}^{N} M_{ij}(t)\phi_{ij}(t) \cdot (v_j(t) - v_i(t)) \\ (x_i(0), v_i(0)) = (\bar{x}_i, \bar{v}_i) \end{cases} \qquad i \in \{1,\dots,N\} \tag{3.1.1}$$

where $(\bar{x}_i, \bar{v}_i) \in \mathbb{R}^{2d}$, where λ_i defined by (3.0.2) and where

$$\phi_{ij}(t) \coloneqq \phi(x_i(t), x_j(t)), \quad \forall i, j \in \{1, \ldots, N\} \quad t \geq 0$$

where $\phi(x_i, x_j)$ is defined by (3.0.3) for all $i, j \in \{1, \ldots, N\}$. Just as in the case of first-order models, we assume that M_{ij} satisfies hypothesis (H2) for all $i, j \in \{1, \ldots, N\}$.

We define the collection of solutions $\{(x_i, v_i)\}_{i=1}^N$ where (x_i, v_i) is the solution to system (3.1.1) for all $i \in \{1, \ldots, N\}$ as

$$\Big(X^N(t), V^N(t)\Big) \coloneqq \{x_i(t), v_i(t)\}_{i=1}^N. \tag{3.1.2}$$

We now define the concept of diameter in the position and velocity variables, respectively, as well as the concept of flocking.

Definition 3.1.1. *Let $(X^N(t), V^N(t))$ be a solution of system (3.1.1). Define the diameter in the position variable as*

$$d_X(t) \coloneqq \max_{i,j \in \{1,\ldots,N\}} |x_i(t) - x_j(t)| \qquad t \geq 0 \tag{3.1.3}$$

and the diameter in the velocity variable as

$$d_V(t) \coloneqq \max_{i,j \in \{1,\ldots,N\}} |v_i(t) - v_j(t)| \qquad t \geq 0. \tag{3.1.4}$$

We say that the system converges to flocking if both conditions hold:

- $\sup_{t \in [0,+\infty)} d_X(t) < +\infty$,
- $\lim_{t \to +\infty} d_V(t) = 0$.

We say that unconditional flocking holds if flocking holds without any restriction on the initial data.

The goal of this section is to prove the following theorem.

Theorem 3.1.1. *Let $\beta \geq 0$ and $T, \mu > 0$ be given. Let $(X^N(t), V^N(t))$ defined by (3.1.2) be a solution of system (3.1.1), with ϕ given in (3.0.3), with the corresponding diameters $d_X(\cdot)$ and $d_V(\cdot)$ defined by (3.1.3) and (3.1.4), respectively. Assume that*

$$\int_t^{t+T} \eta(M_N(s))ds \geq \mu \qquad \forall t \geq 0$$

where $\eta(M_N(\cdot))$ *is defined by* (3.0.4) *and where* M_{ij} *satisfies (H2) for all* $i, j \in \{1, \ldots, N\}$*. If* $\beta \in [0, 1)$*, it then holds*

$$d_X(nT) \le \left[(1 + 2d_X(0))^{1-\beta} + (1-\beta)\left(K(d_V(0), d_X(0), T, \mu)\right)\right]^{\frac{1}{1-\beta}} \tag{3.1.5}$$

for all $n \in \mathbb{N}$ *where*

$$K(d_V(0), d_X(0), T, \mu) := 2d_V(0) + (T-\mu)\frac{2d_V(0)}{(1+2d_X(0))^\beta}\sum_{n=1}^{+\infty}\left(e^{-\frac{\mu}{(1+d_X(0)+d_V(0)T)^\beta}}\right)^{n^{1-\beta}}.$$

In particular, unconditional flocking occurs.

We now present two propositions, where the second one is analogous to Proposition 2.1.1, which aims to show that indeed the "worst case scenario" corresponds to the dynamics with kernel M^* defined by (2.1.12). After presenting these two propositions, we provide the proof of the theorem.

Proposition 3.1.1. *Let* $\left(X^N(t), V^N(t)\right)$ *as defined by* (3.1.2) *be a solution of system* (3.1.1)*. Define the diameters* $d_X(\cdot)$ *by* (3.1.3) *and* $d_V(\cdot)$ *by* (3.1.4)*. Let* M_{ij} *satisfy hypothesis (H2) for all* $i, j \in \{1, \ldots, N\}$*. Then for all* $t \ge 0$ *it holds*

$$\begin{cases} \frac{d}{dt}d_X(t) \le d_V(t) \\ \frac{d}{dt}d_V(t) \le -\phi(d_X(t))\eta(M_N(t))d_V(t) \end{cases} \tag{3.1.6}$$

where $\eta(M_N(\cdot))$ *is defined by* (3.0.4)*.*

Proof. The functions

$$d_V(t) = \max_{i,j\in\{1,\ldots,N\}}|v_i(t) - v_j(t)|, \quad d_X(t) = \max_{i,j\in\{1,\ldots,N\}}|x_i(t) - x_j(t)|$$

are Lipschitz, because they are the pointwise maximum of a finite family of Lipschitz equicontinuous functions. By Rademacher's theorem, they are differentiable almost everywhere. By Theorem A.1.2 it thus holds

$$\frac{1}{2}\frac{d}{dt}d_X^2(t) = \max_{i,j\in\Pi_x(t)}\left\langle\frac{d}{dt}(x_i(t) - x_j(t)), x_i(t) - x_j(t)\right\rangle$$

and

$$\frac{1}{2}\frac{d}{dt}d_V^2(t) = \max_{i,j\in\Pi_v(t)}\left\langle\frac{d}{dt}(v_i(t) - v_j(t)), v_i(t) - v_j(t)\right\rangle$$

where $\Pi_x(t) \in \{1,\dots,N\} \times \{1,\dots,N\}$ and $\Pi_v(t) \in \{1,\dots,N\} \times \{1,\dots,N\}$ represent the nonempty subset of pairs of indices for which the maximum is reached in the position phase space and in the velocity phase space, respectively.

Let us first find the differential inequality for $d_X(\cdot)$. Fix arbitrary $p_x, q_x \in \Pi_x(t)$ and $p_v, q_v \in \Pi_v(t)$. It holds

$$\begin{aligned}\left\langle \frac{d}{dt}(x_{p_x}(t) - x_{q_x}(t)), x_{p_x}(t) - x_{q_x}(t) \right\rangle &= \langle v_{p_x}(t) - v_{q_x}(t), x_{p_x}(t) - x_{q_x}(t)\rangle \\ &\leq |v_{p_v}(t) - v_{q_v}(t)|\, |x_{p_x}(t) - x_{q_x}(t)|\,.\end{aligned}$$

This is valid for all $p_x, q_x \in \Pi_x(t)$ and $p_v, q_v \in \Pi_v(t)$, and therefore

$$\frac{d}{dt} d_X^2(t) \leq 2 d_V(t) d_X(t)$$

which then implies

$$\frac{d}{dt} d_X(t) \leq d_V(t).$$

Let us now find the differential inequality for $d_V(\cdot)$. By following the same steps as in the proof of Theorem 2.1.1, fix arbitrary $p_v, q_v \in \Pi_v(t)$. For easier notation, from now on we hide the dependence on time. Notice that for the case of system (3.1.1) with normalized weights it holds

$$\begin{aligned}\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle &= -\frac{1}{\sum_{k=1}^N \phi_{pk}} \sum_{j=1}^N M_{pj}\phi_{pj} \langle x_p - x_j, x_p - x_q\rangle \\ &\quad - \frac{1}{\sum_{k=1}^N \phi_{qk}} \sum_{j=1}^N M_{qj}\phi_{qj} \langle x_j - x_q, x_p - x_q\rangle\,.\end{aligned}$$

By (3.0.3), it holds

$$\frac{1}{\sum_{k=1}^N \phi_{qk}} \geq \frac{1}{N}.$$

Therefore, for both cases of equal and normalized weights it holds

$$\begin{aligned}&\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle \\ &\leq -\frac{1}{N}\left(\sum_{j=1}^N M_{pj}\phi_{pj} \langle x_p - x_j, x_p - x_q\rangle + \sum_{j=1}^N M_{qj}\phi_{qj} \langle x_j - x_q, x_p - x_q\rangle \right).\end{aligned}$$

Notice that since $\phi(\cdot)$ is monotone non-increasing, it holds

$$\phi(|x_i(t) - x_j(t)|) \geq \phi(d_X(t)) \qquad \forall i, j \in \{1, \dots, N\} \quad \forall t \geq 0$$

and therefore it holds

$$\begin{aligned}
&\left\langle \frac{d}{dt}(x_p - x_q), x_p - x_q \right\rangle \\
&\leq -\frac{\phi(d_X(t))}{N} \left(\sum_{j=1}^{N} M_{pj} \langle x_p - x_j, x_p - x_q \rangle + \sum_{j=1}^{N} M_{qj} \langle x_j - x_q, x_p - x_q \rangle \right).
\end{aligned}$$

By doing the exact same computations as done in the proof of Theorem 2.1.1, it holds

$$\frac{d}{dt} d_V(t) \leq -\phi(d_X(t))\eta(M_N(t)) d_V(t).$$

□

Definition 3.1.2. *Consider the space of functions $\mathcal{M}_{T,\mu}$ defined by* (2.0.3)*, i.e.*

$$\mathcal{M}_{T,\mu} \coloneqq \left\{ M \in L^\infty([0, +\infty); [0, 1]) \quad s.t. \quad \int_t^{t+T} M(s)\, ds \geq \mu \quad \forall t \in [0, +\infty) \right\}.$$

Let $M \in \mathcal{M}_{T,\mu}$. Define $(\mathcal{D}_X^M, \mathcal{D}_V^M) : \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}$ to be the solution of the following dynamical system:

$$\begin{cases}
\frac{d}{dt}\mathcal{D}_X^M(t) = \mathcal{D}_V^M(t) \\
\frac{d}{dt}\mathcal{D}_V^M(t) = -M(t)\phi(\mathcal{D}_X^M(t))\mathcal{D}_V^M(t) \\
\left(\mathcal{D}_X^M(0), \mathcal{D}_V^M(0)\right) = (\mathcal{D}_X(0), \mathcal{D}_V(0))
\end{cases} \tag{3.1.7}$$

where $\phi(\cdot)$ is monotone non-increasing.

Proposition 3.1.2. *Let $T, \mu > 0$ be given. Consider the space of functions $\mathcal{M}_{T,\mu}$ defined by* (2.0.3)*. Consider $(\mathcal{D}_X^M, \mathcal{D}_V^M)$ given by Definition 3.1.2. Let M^* be defined by* (2.1.12)*, i.e. for all $t \geq 0$ and for all $n \in \mathbb{N}$*

$$M^*(t) = \begin{cases} 0 & t \in [(n-1)T,\, nT - \mu) \\ 1 & t \in [nT - \mu,\, nT) \end{cases}.$$

Then, for all $t \geq 0$ it holds

$$\sup\left\{\mathcal{D}_V^M(t) \mid M \in \mathcal{M}_{T,\mu} \textit{ and } \mathcal{D}_V^M(0) = d_V^0\right\} = \mathcal{D}_V^{M^*}(t) \qquad \forall t \geq 0.$$

and

$$\sup\left\{\mathcal{D}_X^M(t) \mid M \in \mathcal{M}_{T,\mu} \textit{ and } \mathcal{D}_X^M(0) = d_X^0\right\} = \mathcal{D}_X^{M^*}(t) \qquad \forall t \geq 0.$$

Proof. In the first step, we provide the first time discretization part, which represents the base case. In the second step, we first present the time discretization process along with the recursive argument. The goal is to first show that it holds

$$\mathcal{D}_V^{M^*}(t) - \mathcal{D}_V^M(t) \geq 0 \qquad \forall t \in [0, T]$$

which actually implies, as proven later,

$$\mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) \geq 0 \qquad \forall t \in [0, T].$$

In the third step we explain and handle the two possible cases that might arise due to our discretization process. In the fourth step we then show the final result.

Notice that by (3.1.7), by Gronwall's inequality we have that for all $M \in \mathcal{M}_{T,\mu}$, and thus including M^*, it holds

$$\mathcal{D}_V^M(t) = \mathcal{D}_V(0)e^{-\int_0^t M(s)\phi(\mathcal{D}_X^M(s))ds} > 0 \qquad \forall t \geq 0.$$

Therefore, we have that for all $M \in \mathcal{M}_{T,\mu}$, and thus including M^*, it holds

$$\frac{d}{dt}\mathcal{D}_X^M(t) > 0, \quad \frac{d}{dt}\mathcal{D}_V^M(t) < 0 \qquad \forall t \geq 0.$$

Step 1 In this step we show the base case. For any $M \in \mathcal{M}_{T,\mu}$, choose any η_1 satisfying

$$\max\{0, \mu - (T - \mu)\} \leq \eta_1 \leq \mu \tag{3.1.8}$$

such that it holds

$$\int_0^{T-\mu} M(s) \geq \mu - \eta_1, \quad \int_{T-\mu}^{T} M(s) \geq \eta_1. \tag{3.1.9}$$

Observe that the bounds in (3.1.8) are derived from (3.1.9) by using the fact that $M \in [0,1]$.

In this step, we prove that it holds

$$\begin{cases} \mathcal{D}_V^M(T-\mu) \leq \mathcal{D}_V^{M^*}(T-\eta_1), \\ \mathcal{D}_X^M(t) \leq \mathcal{D}_X^{M^*}(t) & \forall t \in [0, T-\eta_1]. \end{cases} \tag{3.1.10}$$

We first focus on the interval $[0, T-\mu]$. It holds

$$\mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) = \mathcal{D}_V(0)t - \mathcal{D}_V(0)\int_0^t e^{-\int_0^t M(s)\phi(\mathcal{D}_X^M(s))ds}dt \geq 0 \quad \forall t \in [0, T-\mu]$$

and therefore

$$\mathcal{D}_X^M(t) \leq \mathcal{D}_X^{M^*}(t) \qquad \forall t \in [0, T-\mu]. \tag{3.1.11}$$

Since $\mathcal{D}_X^{M^*}$ is monotone increasing, it holds

$$\mathcal{D}_V^{M^*}(T-\eta_1) = \mathcal{D}_V(0)e^{-\int_{T-\mu}^{T-\eta_1}\phi(\mathcal{D}_X^{M^*}(s))ds} \geq \mathcal{D}_V(0)e^{-(\mu-\eta_1)\phi(\mathcal{D}_X^{M^*}(T-\mu))}.$$

On the other hand, since $\mathcal{D}_X^M$ is monotone increasing, by (3.1.9) and (3.1.11) it holds

$$\begin{aligned} \mathcal{D}_V^M(T-\mu) = \mathcal{D}_V(0)e^{-\int_0^{T-\mu}M(s)\phi(\mathcal{D}_X^M(s))ds} &\leq \mathcal{D}_V(0)e^{-\phi(\mathcal{D}_X^{M^*}(T-\mu))\int_0^{T-\mu}M(s)ds} \\ &\leq \mathcal{D}_V(0)e^{-(\mu-\eta_1)\phi(\mathcal{D}_X^{M^*}(T-\mu))}. \end{aligned}$$

Thus, we get that

$$\mathcal{D}_V^M(T-\mu) \leq \mathcal{D}_V^{M^*}(T-\eta_1). \tag{3.1.12}$$

We now study the difference between $\mathcal{D}_X^{M^*}$ and $\mathcal{D}_X^{M^*}$ in $[T-\mu, T-\eta_1]$. By recalling that both $\mathcal{D}_V^{M^*}$ and $\mathcal{D}_V^M$ are monotone decreasing, for $t \in$

$[T-\mu, T-\eta_1]$ it holds

$$\begin{aligned} \mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) &= \mathcal{D}_X^{M^*}(T-\mu) - \mathcal{D}_X^M(T-\mu) + \int_{T-\mu}^t \mathcal{D}_V^{M^*}(s) - \mathcal{D}_V^M(s)ds \\ &\geq 0 + \left(\mathcal{D}_V^{M^*}(T-\eta_1) - \mathcal{D}_V^M(T-u)\right)(t-(T-\mu)) \\ &\geq 0 \end{aligned} \tag{3.1.13}$$

where we have used (3.1.11) and (3.1.12).

By merging (3.1.11), (3.1.12) and (3.1.13), we recover (3.1.10).

Notice that in particular it holds

$$\mathcal{D}_V^M(t) = \mathcal{D}_V(0)e^{-\int_0^t M(s)\phi(\mathcal{D}_X^M(s))ds} \leq \mathcal{D}_V(0) = \mathcal{D}_V^{M^*}(t) \qquad \forall t \in [0, T-\mu].$$

and

$$\mathcal{D}_V^M(t) - \mathcal{D}_V^{M^*}(t) \leq \mathcal{D}_V^M(T-\mu) - \mathcal{D}_V^{M^*}(T-\eta_1) \leq 0 \qquad \forall t \in [T-\mu, T-\eta_1].$$

Therefore it holds

$$\mathcal{D}_V^M(t) \leq \mathcal{D}_V^{M^*}(t) \qquad \forall t \in [0, T-\eta_1].$$

Step 2 In this step, we show the recursive case. We now present the more general decomposition of the time interval: let $N \in \mathbb{N}$ and choose any sequence $\{\eta_i\}_{i=1}^N$ such that

$$\begin{cases} \eta_1 \coloneqq \max\{0, \mu - (T-\mu)\} \\ \eta_2 \coloneqq \max\{0, \eta_1 - (\mu - \eta_1)\} \\ \eta_{i+2} \coloneqq \max\{0, \eta_{i+1} - (\eta_i - \eta_{i+1})\} \quad \forall i \in \{1, \dots, N-2\}. \end{cases} \tag{3.1.14}$$

First, set

$$\int_0^{T-\mu} M(s) \geq \mu - \eta_1, \quad \int_{T-\mu}^{T-\eta_1} M(s) \geq \eta_1 - \eta_2. \tag{3.1.15}$$

Second, set for $i \in \{1, \dots, N-2\}$

$$\int_{T-\eta_i}^{T-\eta_{i+1}} M(s) \geq \eta_{i+1} - \eta_{i+2} \tag{3.1.16}$$

and

$$\int_{T-\eta_{N-1}}^{T} M(s) \geq \eta_N.$$

In the next steps, we first prove that the sequence $\{\eta_i\}_{i\in\mathbb{N}}$, then show the recursive argument and finally conclude with the result.

Step 2.1 In this step, we prove that

$$\lim_{i\to+\infty} \eta_i = 0. \tag{3.1.17}$$

We first prove that there exists an $i^* \in \mathbb{N}$ such that

$$\eta_{i^*} = 0. \tag{3.1.18}$$

By contradiction, assume that (3.1.18) does not hold. Then it holds by induction

$$\begin{cases} \eta_1 = 2\mu - T \\ \eta_2 = 2\eta_1 - \mu = 2(\mu - T) - T \\ \eta_{i+2} = (i+1)(\mu - T) - T \quad \forall i \in \mathbb{N}. \end{cases}$$

Since $\mu < T$, it then holds

$$\eta_i \to -\infty$$

which is a contradiction with (3.1.14). Therefore, (3.1.18) holds. This implies by construction that

$$\eta_i = 0 \quad \forall i \geq i^*$$

and thus proving (3.1.17).

Step 2.2 In this step, we present the recursive argument: assume that

$$\eta_{i+1} < \eta_i \qquad \forall i \in \{1, \ldots, N\}.$$

Assume that

$$\mathcal{D}_X^M(T - \eta_i) \leq \mathcal{D}_X^{M^*}(T - \eta_i) \tag{3.1.19}$$

and

$$\mathcal{D}_V^M(T-\eta_{i-1}) \leq \mathcal{D}_V^{M^*}(T-\eta_i). \tag{3.1.20}$$

We want to prove that it holds

$$\begin{cases} \mathcal{D}_V^M(T-\eta_i) \leq \mathcal{D}_V^{M^*}(T-\eta_{i+1}), \\ \mathcal{D}_X^M(t) \leq \mathcal{D}_X^{M^*}(t) & \forall t \in [T-\eta_i, T-\eta_{i+1}]. \end{cases} \tag{3.1.21}$$

Since $\mathcal{D}_X^{M^*}$ is monotone increasing, it holds

$$\begin{aligned} \mathcal{D}_V^{M^*}(T-\eta_{i+1}) &= \mathcal{D}_V^{M^*}(T-\eta_i) e^{-\int_{T-\eta_i}^{T-\eta_{i+1}} \phi(\mathcal{D}_X^{M^*}(s))ds} \\ &\geq \mathcal{D}_V^{M^*}(T-\eta_i) e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^{M^*}(T-\eta_i))}. \end{aligned}$$

On the other hand, since $\mathcal{D}_X^M$ is monotone increasing and by (3.1.16) it holds

$$\begin{aligned} \mathcal{D}_V^M(T-\eta_i) &= \mathcal{D}_V^M(T-\eta_{i-1}) e^{-\int_{T-\eta_{i-1}}^{T-\eta_i} M(s)\phi(\mathcal{D}_X^M(s))ds} \\ &\leq \mathcal{D}_V^M(T-\eta_{i-1}) e^{-\phi(\mathcal{D}_X^M(T-\eta_i))\int_{T-\eta_{i-1}}^{T-\eta_i} M(s)ds} \\ &\leq \mathcal{D}_V^M(T-\eta_{i-1}) e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^M(T-\eta_i))}. \end{aligned}$$

Therefore, we have that it holds

$$\begin{aligned} &\mathcal{D}_V^{M^*}(T-\eta_{i+1}) - \mathcal{D}_V^M(T-\eta_i) \\ &\geq \mathcal{D}_V^{M^*}(T-\eta_i) e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^{M^*}(T-\eta_i))} - \mathcal{D}_V^M(T-\eta_{i-1}) e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^M(T-\eta_i))} \\ &= A_1 + A_2 \end{aligned}$$

where

$$A_1 := \left(e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^{M^*}(T-\eta_i))} - e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^M(T-\eta_i))}\right) \mathcal{D}_V^M(T-\eta_{i-1}).$$

and

$$A_2 := \left(\mathcal{D}_V^{M^*}(T-\eta_i) - \mathcal{D}_V^M(T-\eta_{i-1})\right) e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^{M^*}(T-\eta_i))}.$$

By (3.1.19), it holds

$$e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^M(T-\eta_i))} \le e^{-(\eta_i-\eta_{i+1})\phi(\mathcal{D}_X^{M^*}(T-\eta_i))}$$

which implies that $A_1 \ge 0$. By (3.1.20), we have that $A_2 \ge 0$. Therefore, it holds

$$\mathcal{D}_V^M(T-\eta_i) \le \mathcal{D}_V^{M^*}(T-\eta_{i+1}). \tag{3.1.22}$$

Just as before, we now study the difference between $\mathcal{D}_X^{M^*}$ and $\mathcal{D}_X^M$ in $[T-\eta_i, T-\eta_{i+1}]$. For $t \in [T-\eta_i, T-\eta_{i+1}]$ it holds

$$\begin{aligned}
\mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) &= \mathcal{D}_X^{M^*}(T-\eta_i) - \mathcal{D}_X^M(T-\eta_i) + \int_{T-\eta_i}^t \mathcal{D}_V^{M^*}(s) - \mathcal{D}_V^M(s)ds \\
&\ge 0 + \left(\mathcal{D}_V^{M^*}(T-\eta_{i+1}) - \mathcal{D}_V^M(T-\eta_i)\right)(t-(T-\eta_i)) \\
&\ge 0
\end{aligned} \tag{3.1.23}$$

where we have used (3.1.19) and (3.1.22). It therefore holds

$$\mathcal{D}_X^M(t) \le \mathcal{D}_X^{M^*}(t) \qquad \forall t \in [T-\eta_i, T-\eta_{i+1}]. \tag{3.1.24}$$

By (3.1.22), (3.1.23) and (3.1.24) we recover (3.1.21). Note that by defining

$$\eta_0 := \mu$$

we have already proved the base case of the recursive argument in **Step 1**.

Step 2.3 In this step, we show the result.

We have proved in the previous step by recursion that if we consider a sequence $\{\eta_i\}_{i=1}^N$ such that it holds

$$\eta_{i+1} < \eta_i \qquad \forall i \in \{1, \dots, N-1\}$$

then it holds

$$\mathcal{D}_V^{M^*}(T-\eta_{i+1}) - \mathcal{D}_V^M(T-\eta_i) \ge 0 \qquad \forall i \in \{1, \dots, N-1\}. \tag{3.1.25}$$

Therefore, for all $t \in [T - \eta_i, T - \eta_{i+1}]$ and for all $i \in \{1, \dots, N\}$ it holds

$$\mathcal{D}_V^{M^*}(t) - \mathcal{D}_V^M(t) \geq \mathcal{D}_V^{M^*}(T - \eta_{i+1}) - \mathcal{D}_V^M(T - \eta_i) \geq 0$$

and by induction on the time intervals $[T - \eta_i, T - \eta_{i+1}]$ it thus holds

$$\mathcal{D}_V^{M^*}(t) - \mathcal{D}_V^M(t) \geq 0 \qquad \forall t \in [0, T - \eta_N] \tag{3.1.26}$$

which implies

$$\mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) = \int_0^t \mathcal{D}_V^{M^*}(s) - \mathcal{D}_V^M(s) ds \geq 0 \quad \forall t \in [0, T - \eta_N]$$

and therefore

$$\mathcal{D}_X^{M^*}(t) - \mathcal{D}_X^M(t) \geq 0 \qquad \forall t \in [0, T - \eta_N]. \tag{3.1.27}$$

By (3.1.17), we have that

$$\lim_{N \to +\infty} \eta_N = 0.$$

Therefore, for all $M \in \mathcal{M}_{T,\mu}$ it holds

$$\mathcal{D}_X^M(t) < \mathcal{D}_X^{M^*}(t), \quad \mathcal{D}_V^M(t) < \mathcal{D}_V^{M^*}(t) \qquad \forall t \in [0, T].$$

Consider now the time intervals $[nT, (n+1)T]$ for $n \in \mathbb{N}$. By applying the same argument as done in $[0, T]$, we get

$$\mathcal{D}_X^M(t) < \mathcal{D}_X^{M^*}(t), \quad \mathcal{D}_V^M(t) < \mathcal{D}_V^{M^*}(t) \qquad \forall t \in [nT, (n+1)T] \qquad \forall n \in \mathbb{N}.$$

Therefore, by induction it holds

$$\mathcal{D}_X^M(t) < \mathcal{D}_X^{M^*}(t), \quad \mathcal{D}_V^M(t) < \mathcal{D}_V^{M^*}(t) \qquad \forall t \geq 0$$

and thus the result follows.

□

We first state an easy proposition:

Proposition 3.1.3. *For an interval* $[0, T]$ *where* $T > 0$*, if*

$$d_V(t) < \mathcal{D}_V^\eta(t) \quad \forall t \in [0, T]$$

then $d_X(0) < \mathcal{D}_X^\eta(0)$ *implies*

$$d_X(t) < \mathcal{D}_X^\eta(t) \quad \forall t \in [0, T]. \tag{3.1.28}$$

Proof. Indeed,

$$\mathcal{D}_X^\eta(t) - d_X(t) = \mathcal{D}_X^\eta(0) - d_X(0) + \int_0^t (\mathcal{D}_V^\eta(s) - d_V(s))ds > 0 + 0 \qquad \forall t \in [0, T].$$

□

Proof of Theorem 3.1.1. In this proof we only focus on the case $\beta \in (0, 1)$, since the case $\beta = 0$ is trivial as it corresponds to the same setting as in the first-order model where $K_{\min} = 1$.

Step 1. By Proposition 3.1.1, we study the following system

$$\begin{cases} \frac{d}{dt} d_X(t) \leq d_V(t) \\ \frac{d}{dt} d_V(t) \leq -\phi(d_X(t))\eta(M_N(t))d_V(t) \\ (d_X(0), d_V(0)) = (d_X^0, d_V^0) \end{cases}.$$

where (d_X^0, d_V^0) corresponds to the diameter of the initial data $\{(\bar{x}_i, \bar{v}_i)\}_{i \in \{1,\ldots,N\}}$.

Let $(\mathcal{D}_X^\eta, \mathcal{D}_V^\eta)$ be the solution of the following system

$$\begin{cases} \frac{d}{dt} \mathcal{D}_X^\eta(t) = \mathcal{D}_V^\eta(t) \\ \frac{d}{dt} \mathcal{D}_V^\eta(t) = -\phi(\mathcal{D}_X^\eta(t))\eta(M_N(t))\mathcal{D}_V^\eta(t) \\ (\mathcal{D}_X^\eta(0), \mathcal{D}_V^\eta(0)) = (2d_X^0, 2d_V^0) \end{cases}.$$

In this step, we prove that

$$d_X(t) < \mathcal{D}_X^\eta(t), \quad d_V(t) < \mathcal{D}_V^\eta(t) \qquad \forall t \geq 0. \tag{3.1.29}$$

Now, notice that it holds

$$\frac{d}{dt}\left(\mathcal{D}_V^\eta(t) - d_V(t)\right) \geq -\eta(M_N(t))\left[\phi(\mathcal{D}_X^\eta(t))\mathcal{D}_V^\eta(t) - \phi(d_X(t))d_V(t)\right] \qquad \forall t \geq 0 \tag{3.1.30}$$

Notice that in particular, by (3.1.30) applied at $t = 0$, we have that

$$\frac{d}{dt}\left(\mathcal{D}_V^\eta(t) - d_V(t)\right)|_{t=0} \geq -\eta(M_N(t))\left[\phi(\mathcal{D}_X^\eta(0))\mathcal{D}_V^\eta(0) - \phi(d_X(0))d_V(0)\right] > 0.$$

Therefore, by continuity of the solution, there exists a time $T^* > 0$ such that

$$d_V(t) < \mathcal{D}_V^\eta(t) \quad \forall t \in (0, T^*] \tag{3.1.31}$$

which implies

$$d_X(t) < \mathcal{D}_X^\eta(t) \quad \forall t \in (0, T^*]. \tag{3.1.32}$$

We now prove (3.1.29). By contradiction, assume that (3.1.29) does not hold, thus that there exists some time $T > 0$ such that

$$d_X(T) \geq \mathcal{D}_X^\eta(T) \quad \text{or} \quad d_V(T) \geq \mathcal{D}_V^\eta(T).$$

By Proposition 3.1.3, the first case implies that there exists a time $T' \in [0, T]$ such that $d_V(T') \geq \mathcal{D}_V^\eta(T')$. Then, with no loss of generality, we assume that

$$d_V(T) \geq \mathcal{D}_V^\eta(T).$$

By continuity of the solution, this implies that there exists a time $t^* \in [T^*, T]$ defined as

$$t^* := \inf\{t \in [T^*, T] \quad \text{s.t.} \quad d_V(t) = \mathcal{D}_V^\eta(t)\}.$$

By (3.1.3), (3.1.32) and by continuity, it holds

$$d_X(t) < \mathcal{D}_X^\eta(t) \quad \forall t \in [0, t^*]$$

It then holds

$$\begin{aligned}
\mathcal{D}_V^\eta(t^*) - d_V(t^*) &\geq \mathcal{D}_V^\eta(0)e^{-\int_0^{t^*} \eta(M_N(s))\phi(\mathcal{D}_X(s))ds} - d_V(0)e^{-\int_0^{t^*} \eta(M_N(s))\phi(d_X(s))ds} \\
&= (\mathcal{D}_V^\eta(0) - d_V(0))e^{-\int_0^{t^*} \eta(M_N(s))\phi(\mathcal{D}_X(s))ds} \\
&\quad + d_V(0)\left[e^{-\int_0^{t^*} \eta(M_N(s))\phi(\mathcal{D}_X(s))ds} - e^{-\int_0^{t^*} \eta(M_N(s))\phi(d_X(s))ds}\right] \\
&> 0
\end{aligned}$$

which is a contradiction with the definition of t^*. Therefore, (3.1.29) holds.

Step 2. By (3.1.29) and by Proposition 3.1.2, we then have

$$d_X(t) \leq \mathcal{D}_X^\eta(t) \leq \mathcal{D}_X^{M^*}(t), \quad d_V(t) \leq \mathcal{D}_V^\eta(t) \leq \mathcal{D}_V^{M^*}(t) \qquad \forall t \geq 0. \quad (3.1.33)$$

where M^* is defined by (2.1.12).

From now on, we focus on $(\mathcal{D}_V^{M^*}(t), \mathcal{D}_V^{M^*}(t))$.

In the rest of the proof we only prove that (3.1.1) holds. This implies that (3.1.1) holds. Indeed, notice that if (3.1.1) holds, then by repeating the same argument as in the proof of Theorem 2.1.1 one can prove that consensus in the velocity variable holds.

Let T_n be defined by

$$T_n := nT \quad \forall n \in \mathbb{N}_0.$$

In this step, we prove by induction that for all $n \in \mathbb{N}$ it holds

$$\int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_n)} \frac{1}{(1+r)^\beta} dr \leq \mathcal{D}_V^{M^*}(0) - \mathcal{D}_V^{M^*}(T_n) + (T-\mu)\sum_{k=0}^{n-1} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta}. \quad (3.1.34)$$

We start with the base case. Consider the time interval $[0, T]$. Therefore, if $t \in [0, T-\mu)$ then we have

$$\begin{cases} \frac{d}{dt}\mathcal{D}_X^{M^*}(t) = \mathcal{D}_V^{M^*}(t) \\ \frac{d}{dt}\mathcal{D}_V^{M^*}(t) = 0 \end{cases}$$

with initial condition $(\mathcal{D}_X^{M^*}(0), \mathcal{D}_V^{M^*}(0))$. It then holds

$$\begin{cases} \mathcal{D}_X^{M^*}(T-\mu) = \mathcal{D}_X^{M^*}(0) + \mathcal{D}_V^{M^*}(0)(T-\mu) \\ \mathcal{D}_V^{M^*}(T-\mu) = \mathcal{D}_V^{M^*}(0). \end{cases} \tag{3.1.35}$$

If $t \in [T-\mu, T)$ then we have

$$\begin{cases} \frac{d}{dt}\mathcal{D}_X^{M^*}(t) = \mathcal{D}_V^{M^*}(t) \\ \frac{d}{dt}\mathcal{D}_V^{M^*}(t) = -\frac{1}{(1+\mathcal{D}_X^{M^*}(t))^\beta}\mathcal{D}_V^{M^*}(t) \end{cases}$$

with initial condition $(\mathcal{D}_X^{M^*}(T-\mu), \mathcal{D}_V^{M^*}(T-\mu))$. The solution is given by

$$\mathcal{D}_V^{M^*}(T) - \mathcal{D}_V^{M^*}(0) = \frac{(1+\mathcal{D}_X^{M^*}(T-\mu))^{1-\beta}}{1-\beta} - \frac{(1+\mathcal{D}_X^{M^*}(T))^{1-\beta}}{1-\beta}.$$

By (3.1.35), it thus holds

$$\frac{(1+\mathcal{D}_X^{M^*}(T))^{1-\beta}}{1-\beta} - \frac{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)(T-\mu))^{1-\beta}}{1-\beta} = \mathcal{D}_V^{M^*}(0) - \mathcal{D}_V^{M^*}(T). \tag{3.1.36}$$

Now notice that the left-hand side satisfies

$$\begin{aligned} &\frac{(1+\mathcal{D}_X^{M^*}(T))^{1-\beta}}{1-\beta} - \frac{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)(T-\mu))^{1-\beta}}{1-\beta} \\ &= \int_{\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)(T-\mu)}^{\mathcal{D}_X^{M^*}(T)} \frac{1}{(1+r)^\beta} dr. \end{aligned}$$

By (3.1.36), we have that

$$\begin{aligned} \int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T)} \frac{1}{(1+r)^\beta} dr &= \mathcal{D}_V^{M^*}(T-\mu) - \mathcal{D}_V^{M^*}(T) + \int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)(T-\mu)} \frac{1}{(1+r)^\beta} dr \\ &\le \mathcal{D}_V^{M^*}(T-\mu) - \mathcal{D}_V^{M^*}(T) + (T-\mu)\frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta}. \end{aligned}$$

We now proceed with the induction step. Consider now the time interval $[0, T_n]$ for $n \ge 2$. Assume that

$$\int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_{n-1})} \frac{1}{(1+r)^\beta} dr \le \mathcal{D}_V^{M^*}(0) - \mathcal{D}_V^{M^*}(T_{n-1}) + (T-\mu)\sum_{k=0}^{n-2} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta}. \tag{3.1.37}$$

The goal now is to prove that

$$\int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_n)} \frac{1}{(1+r)^\beta} dr \leq \mathcal{D}_V^{M^*}(0) - \mathcal{D}_V^{M^*}(T_n) + (T-\mu) \sum_{k=0}^{n-1} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta}.$$

By doing the same computations done in the time interval $[T-\mu, T]$ but now in $[T_n - \mu, T_n]$ we get

$$\begin{aligned} &\frac{(1+\mathcal{D}_X^{M^*}(T_n))^{1-\beta}}{1-\beta} - \frac{(1+\mathcal{D}_X^{M^*}(T_{n-1}) + \mathcal{D}_V^{M^*}(T_{n-1})(T-\mu))^{1-\beta}}{1-\beta} \\ &= \mathcal{D}_V^{M^*}(T_{n-1}) - \mathcal{D}_V^{M^*}(T_n). \end{aligned} \tag{3.1.38}$$

Now notice that

$$\begin{aligned} &\frac{(1+\mathcal{D}_X^{M^*}(T_n))^{1-\beta}}{1-\beta} - \frac{(1+\mathcal{D}_X^{M^*}(T_{n-1}) + \mathcal{D}_V^{M^*}(T_{n-1})(T-\mu))^{1-\beta}}{1-\beta} \\ &= \int_{\mathcal{D}_X^{M^*}(T_{n-1})+\mathcal{D}_V^{M^*}(T_{n-1})(T-\mu)}^{\mathcal{D}_X^{M^*}(T_n)} \frac{1}{(1+r)^\beta} dr. \end{aligned}$$

and by recalling (3.1.38) it therefore holds

$$\begin{aligned} &\int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_n)} \frac{1}{(1+r)^\beta} dr \\ &= \mathcal{D}_V^{M^*}(T_{n-1}) - \mathcal{D}_V^{M^*}(T_n) + \int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_{n-1})} \frac{1}{(1+r)^\beta} dr \\ &\quad + \int_{\mathcal{D}_X^{M^*}(T_{n-1})}^{\mathcal{D}_X^{M^*}(T_{n-1})\mathcal{D}_V^{M^*}(T_{n-1})(T-\mu)} \frac{1}{(1+r)^\beta} dr \\ &\leq \mathcal{D}_V^{M^*}(T_{n-1}) - \mathcal{D}_V^{M^*}(T_n) + \int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_{n-1})} \frac{1}{(1+r)^\beta} dr + (T-\mu) \frac{\mathcal{D}_V^{M^*}(T_{n-1})}{(1+\mathcal{D}_X^{M^*}(T_{n-1}))^\beta} \\ &\leq \mathcal{D}_V^{M^*}(0) - \mathcal{D}_V^{M^*}(T_n) + (T-\mu) \sum_{k=0}^{n-1} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta} \end{aligned}$$

where in the last inequality we have used the induction step (3.1.37). Therefore, by induction we have that (3.1.34) holds.

Step 3. In this step we prove that

$$\sum_{n=1}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_n)}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} \leq \frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta} \sum_{n=1}^{+\infty} \left(e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)T)^\beta}} \right)^{n^{1-\beta}}. \tag{3.1.39}$$

Let $t \in [T_n - \mu, T_n]$. It holds

$$\begin{aligned}\frac{d}{dt}\left[\frac{\mathcal{D}_V^{M^*}(t)}{(1+\mathcal{D}_X^{M^*}(t))^\beta}\right] &= -\frac{M(t)}{(1+\mathcal{D}_X^{M^*}(t))^\beta}\frac{\mathcal{D}_V^{M^*}(t)}{(1+\mathcal{D}_X^{M^*}(t))^\beta} - \frac{\beta d_V^2(t)}{(1+\mathcal{D}_X^{M^*}(t))^{\beta+1}}\\ &\leq -\frac{M(t)}{(1+\mathcal{D}_X^{M^*}(t))^\beta}\frac{\mathcal{D}_V^{M^*}(t)}{(1+\mathcal{D}_X^{M^*}(t))^\beta}.\end{aligned}$$

For $n=1$, i.e. at time $t=T$, and by recalling that $\mathcal{D}_X^{M^*}$ is increasing, it holds

$$\begin{aligned}\frac{\mathcal{D}_V^{M^*}(T)}{(1+\mathcal{D}_X^{M^*}(T))^\beta} &\leq \frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta}e^{-\frac{1}{(1+\mathcal{D}_X^{M^*}(T))^\beta}\int_0^T M(t)dt}\\ &\leq \frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta}e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(T))^\beta}}.\end{aligned}$$

For all $n>1$, it similarly holds

$$\frac{\mathcal{D}_V^{M^*}(T_n)}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} \leq \frac{\mathcal{D}_V^{M^*}(T_n-\mu)}{(1+\mathcal{D}_X^{M^*}(T_n-\mu))^\beta}e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}}.$$

Now since it holds for all $n \in \mathbb{N}$

$$\mathcal{D}_V^{M^*}(T_n-\mu) = \mathcal{D}_V^{M^*}(T_{n-1}) \quad \text{and} \quad \mathcal{D}_X^{M^*}(T_{n-1}) \leq \mathcal{D}_X^{M^*}(T_n-\mu)$$

then it holds for all $n \in \mathbb{N}$

$$\frac{\mathcal{D}_V^{M^*}(T_n)}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} \leq \frac{\mathcal{D}_V^{M^*}(T_{n-1})}{(1+\mathcal{D}_X^{M^*}(T_{n-1}))^\beta}e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}} \tag{3.1.40}$$

For $n>1$, since it holds

$$\mathcal{D}_X^{M^*}(T_k) \leq \mathcal{D}_X^{M^*}(T_n) \qquad \forall k \in \{0,\dots,n-1\},$$

then, recalling (3.1.40), by iteration it holds

$$\frac{\mathcal{D}_V^{M^*}(T_n)}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} \leq \frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta}e^{-\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}}.$$

Therefore, it holds

$$\sum_{n=1}^{+\infty}\frac{\mathcal{D}_V^{M^*}(T_n)}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} \leq \frac{\mathcal{D}_V^{M^*}(0)}{(1+\mathcal{D}_X^{M^*}(0))^\beta}\sum_{n=1}^{+\infty}e^{-\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}}.$$

The goal now is to show that

$$\sum_{n=1}^{+\infty} e^{-\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}} \leq \sum_{n=1}^{+\infty} \left(e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)T)^\beta}} \right)^{n^{1-\beta}}. \tag{3.1.41}$$

Notice that for all $n \in \mathbb{N}$ it holds

$$\mathcal{D}_X^{M^*}(T_n) \leq \mathcal{D}_X^{M^*}(0) + nT \cdot \mathcal{D}_V^{M^*}(0)$$

It thus holds for all $n \in \mathbb{N}$

$$\begin{aligned} -\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta} &\leq -\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(0)+n\cdot\mathcal{D}_V^{M^*}(0)T)^\beta} \\ &\leq -\frac{\mu}{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)T)^\beta} \cdot n^{1-\beta} \end{aligned}$$

and therefore it holds

$$\sum_{n=1}^{+\infty} e^{-\frac{n\mu}{(1+\mathcal{D}_X^{M^*}(T_n))^\beta}} \leq \sum_{n=1}^{+\infty} \left(e^{-\frac{\mu}{(1+\mathcal{D}_X^{M^*}(0)+\mathcal{D}_V^{M^*}(0)T)^\beta}} \right)^{n^{1-\beta}}$$

thus proving (3.1.39).

Step 4. In this step we show the result. Now notice that by (3.1.34), for all $n \in \mathbb{N}$ it holds

$$\int_{\mathcal{D}_X^{M^*}(0)}^{\mathcal{D}_X^{M^*}(T_n)} \frac{1}{(1+r)^\beta} dr \leq \mathcal{D}_V^{M^*}(0) + (T-\mu) \sum_{k=0}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta}.$$

Therefore, it holds

$$\frac{(1+\mathcal{D}_X^{M^*}(T_n))^{1-\beta}}{1-\beta} \leq \frac{(1+\mathcal{D}_X^{M^*}(0))^{1-\beta}}{1-\beta} + \mathcal{D}_V^{M^*}(0) + (T-\mu) \sum_{k=0}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta}$$

and thus

$$\mathcal{D}_X^{M^*}(T_n) \leq \left[(1+\mathcal{D}_X^{M^*}(0))^{1-\beta} + (1-\beta) \left(\mathcal{D}_V^{M^*}(0) + (T-\mu) \sum_{k=0}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta} \right) \right]^{\frac{1}{1-\beta}}.$$

By (3.1.33), it thus holds

$$d_X(T_n) \leq \left[(1+\mathcal{D}_X^{M^*}(0))^{1-\beta} + (1-\beta) \left(\mathcal{D}_V^{M^*}(0) + (T-\mu) \sum_{k=0}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^\beta} \right) \right]^{\frac{1}{1-\beta}}.$$

Since (3.1.39) holds and by recalling that $(\mathcal{D}_X^{M^*}(0), \mathcal{D}_V^{M^*}(0)) = (2d_X(0), 2d_V(0))$, we thus get (3.1.5). In particular, it holds

$$\lim_{n\to+\infty} d_X(T_n)$$
$$\leq \left[(1+\mathcal{D}_X^{M^*}(0))^{1-\beta} + (1-\beta)\left(\mathcal{D}_V^{M^*}(0) + (T-\mu)\sum_{k=0}^{+\infty} \frac{\mathcal{D}_V^{M^*}(T_k)}{(1+\mathcal{D}_X^{M^*}(T_k))^{\beta}}\right)\right]^{\frac{1}{1-\beta}}$$

and therefore we get unconditional flocking in the sense of (3.1.1).

□

Remark 3.1.1. The case $\beta = 1$ is more subtle. If we attempt to use the same arguments as presented in the proof, then by following the same arguments we get

$$\int_{d_X(0)}^{d_X(T_n)} \frac{1}{1+r} dr \leq d_V(0) - d_V(T_n) + (T-\mu)\sum_{k=0}^{T_n} \frac{d_V(T_k)}{1+d_X(T_k)}$$

and again unconditional flocking would hold if the sum is finite. In this special case, notice that for $t \in [T_k - \mu, T_k]$ we get the following differential equation

$$\frac{d}{dt}\left[\frac{d_V(t)}{1+d_X(t)}\right] = -\frac{d_V(t)}{(1+d_X(t))^2} - \left(\frac{d_V(t)}{1+d_X(t)}\right)^2$$

One attempt to solve it would be to see that

$$\frac{d}{dt}\left[\frac{d_V(t)}{1+d_X(t)}\right] = -\left(\frac{1}{d_V(t)}+1\right)\left(\frac{d_V(t)}{1+d_X(t)}\right)^2$$

and by solving it in $[T_k - \mu, T_k]$ it holds

$$\frac{d_V(T_k)}{1+d_X(T_k)} = \frac{1}{\frac{1}{\frac{d_V(T_k-\mu)}{1+d_X(T_k-\mu)}} + \int_{T-\mu}^{T}\left(\frac{1}{d_V(s)}+1\right)ds}$$

and thus by recursion it holds

$$\frac{d_V(T_k)}{1+d_X(T_k)} = \frac{1}{\frac{1}{\frac{d_V(0)}{1+d_X(0)}} + \sum_{j=1}^{k}\int_{T_j-\mu}^{T_j}\left(\frac{1}{d_V(s)}+1\right)ds}$$

Therefore, if one can show for instance that there exists some $\alpha > 1$ such that

$$\sum_{j=1}^{k} \int_{T_j-\mu}^{T_j} \left(\frac{1}{d_V(s)} + 1 \right) ds \sim k^{\alpha}$$

then we would get that the sum converges and thus unconditional flocking holds as well.

3.2 Flocking in the mean-field model for a uniform PE

In this section we consider the infinite dimensional setting and prove that unconditional flocking holds also in the classical mean-field setting. Just as motivated in the mean-field setting in the case of first-order models in Section 2.2, we now consider the case such that it holds

$$M_{ij}(t) = M(t) \qquad \forall i, j \in \{1, \ldots, N\}. \tag{3.2.1}$$

Consider therefore the particle systems

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{\lambda_i}{N} \sum_{j=1}^{N} M(t)\phi_{ij}(t) \cdot (v_j(t) - v_i(t)) \end{cases} \qquad i \in \{1, \ldots, N\} \tag{3.2.2}$$

where

$$\phi_{ij}(t) := \frac{1}{(1 + |x_i(t) - x_j(t)|)^{\beta}} \qquad \beta \geq 0, \quad t \geq 0.$$

and where $M \in \mathcal{M}_{T,\mu}$.

Theorem 3.2.1. *Let $\beta \geq 0$ and $T, \mu > 0$ be given. Let $(X^N(t), V^N(t))$ defined by* (3.1.2) *be a solution of system* (3.1.1)*. Define the diameters $d_X(\cdot)$ by* (3.1.3) *and $d_V(\cdot)$ by* (3.1.4)*. Assume that $M \in \mathcal{M}_{T,\mu}$ where $\mathcal{M}_{T,\mu}$ is defined by* (2.0.3)*. If $\beta \in [0, 1)$, it then holds*

$$d_X(nT) \leq \left[(1 + d_X(0))^{1-\beta} + (1 - \beta) \left(K(d_V(0), d_X(0), T, \mu) \right) \right]^{\frac{1}{1-\beta}} \tag{3.2.3}$$

for all $n \in \mathbb{N}$ where

$$K(d_V(0), d_X(0), T, \mu) := d_V(0) + (T-\mu)\frac{d_V(0)}{(1+d_X(0))^\beta}\sum_{j=1}^{+\infty}\left(e^{-\frac{\mu}{(1+d_X(0)+d_V(0)T)^\beta}}\right)^{n^{1-\beta}}.$$

In particular, unconditional flocking occurs for $\beta \in [0,1)$.

Proof. Consider system (3.1.1). In the case where $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ satisfies (3.2.1) we have that it holds

$$\frac{1}{N}\min_{i,j\in\{1,\dots,N\}}\sum_{k=1}^{N}\min\{M_{ik}(s), M_{jk}(s)\} = M(t)$$

and by using Theorem 3.1.1 the result follows by noticing that the solution of (3.1.1) with $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ satisfying (3.2.1) is exactly the solution of (3.2.2). $\square$

In this section we consider flocking formation in the mean-field limit of such particle systems.

Define the diameter for a compactly supported measure $\nu \in \mathcal{P}_1(\mathbb{R}^{2d})$ as

$$d_X[\nu] := \mathrm{diam}(\mathrm{supp}_x\,\nu), \quad d_V[\nu] := \mathrm{diam}(\mathrm{supp}_v\,\nu)$$

where $\mathrm{supp}_x f$ and $\mathrm{supp}_v f$ denote the x-projection and the $v-$projection, respectively, of supp_f.

Let $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. Then, the continuum model corresponding to (3.2.2) and (3.2.2) is

$$\begin{cases} \partial_t\mu_t + v\cdot\nabla_x\mu_t + \nabla_v\cdot(V[t,\mu_t]\mu_t) = 0, & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d, \quad t>0 \\ \mu_0 = \bar{\mu} \end{cases} \tag{3.2.4}$$

where the initial datum $\bar{\mu} \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$. In the case we are treating the particle system (3.1.1) with equal weights, i.e. λ_i constant, and the non-local vector-field is given by

$$V[t,\mu_t](x,v) = \int_{\mathbb{R}^{2d}} M(t)\phi(|x-y|)(w-v)d\mu_t(y,w) \qquad \forall (x,v)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{3.2.5}$$

In the case we are treating the particle system (3.1.1) with normalized weights, i.e.

$$\lambda_i = \frac{N}{\sum_{l=1}^{N} \phi(x_i, x_l)}.$$

we define the non-local vector field

$$V[t, \mu_t](x, v) = \frac{\int_{\mathbb{R}^{2d}} M(t)\phi(|x-y|)(w-v)d\mu_t(y, w)}{\int_{\mathbb{R}^{2d}} \phi(|x-y|)d\mu_t(y, w)} \qquad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{3.2.6}$$

Definition 3.2.1. *Let $T > 0$. A measure $\mu_t \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{2d}))$ is a measure-valued solution of* (3.2.4) *in the time interval $[0,T]$ with initial datum $\bar{\mu} \in \mathcal{P}_1(\mathbb{R}^{2d})$ if it holds*

$$\int_0^T \int_{\mathbb{R}^{2d}} (\partial_t \phi + v \cdot \nabla_x \phi + V[t, \mu_t] \cdot \nabla_v \phi)\, d\mu_t(x, v)dt + \int_{\mathbb{R}^{2d}} \phi(x, v, 0)d\bar{\mu}(x, v) = 0$$

for all $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d \times T)$.

We now state an existence-uniqueness theorem and a stability theorem of (3.2.4). Such theorems have been proved in [22] for the case $M \equiv 1$, i.e. full communication between agents at all times. However, similarly to the first-order mean-field model, they still hold true in the case where we have a uniform multiplicative persistently excited term depending only on time. Similarly to Lemma 2.2.1, there is [22, Lemma 3.1] which is its equivalent in the second-order model setting, which again is satisfied in our case by noting that $M(\cdot) \leq 1$ and then following the exact same calculations, as has been shown in Lemma 2.2.1 in the context of first-order models. We thus have the following theorem:

Theorem 3.2.2 ([22, Theorem 3.1])**.** *Consider the continuum model* (3.2.4) *with $\bar{\mu} \in \mathcal{P}_1(\mathbb{R}^{2d})$ and suppose that there exists a constant $R > 0$ such that*

$$\operatorname{supp} \bar{\mu} \subset B^{2n}(0, R)$$

where $B^{2n}(0,R)$ stands for a $2d$-dimensional ball centered at the origin with radius $R > 0$. Then, for $T > 0$ there exists a unique measure-valued solution $\mu_t \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{2d}))$ of (3.2.4) *in the sense of* (3.2.1)*. Moreover, μ_t is uniformly compactly supported and we have*

$$\mu_t = (X_1(t; \cdot, \cdot), X_2(t; \cdot, \cdot))_{\#}\, \bar{\mu}$$

where $(X_1(t;\cdot,\cdot), X_2(t;\cdot,\cdot))$ *is the flow generated by* $(v, V[t,\mu_t])$.

Theorem 3.2.3 ([22, Theorem 3.2]). *Let* $\mu_t^1, \mu_t^2 \in C([0,T);\mathcal{P}_1(\mathbb{R}^{2d}))$ *be two weak solutions of* (3.2.4) *subject to uniformly compactly supported initial data* $\bar{\mu}^1, \bar{\mu}^2 \in \mathcal{P}_1(\mathbb{R}^{2d})$, *respectively. Define*

$$R_{i,X}^T := \max_{0\le t\le T} \max_{x\in \operatorname{supp}\mu_t} |x|, \quad R_{i,V}^T := \max_{0\le t\le T} \max_{v\in \operatorname{supp}\mu_t} |v| \qquad i=1,2.$$

Then, there exists a constant $C>0$, *depending only on* $\phi, T, R_{i,X}^T, R_{i,V}^T$, *such that*

$$W_1(\mu_t^1,\mu_t^2) \le C W_1(\bar{\mu}^1,\bar{\mu}^2) \qquad \forall t\in[0,T).$$

This stability result provides a rigorous passage from the particle system (3.2.2) to (3.2.4) where the nonlocal vector field is given by (3.2.5) or (3.2.6) depending on the choice of the weighting procedure.

Notice that the result of Theorem 3.1.1 is independent of N. Then, by following the lines of [22, Theorem 3.3] for the case of no delay ($\tau = 0$ using the authors notation), we thus get the following result:

Theorem 3.2.4. *Let* $T,\mu>0$ *be given. Let* $M\in\mathcal{M}_{T,\mu}$ *where* $\mathcal{M}_{T,\mu}$ *is defined by* (2.0.3). *Let* $\mu_t \in C([0,T];\mathcal{P}_1(\mathbb{R}^{2d}))$ *be a measure-valued solution to* (3.2.4) *with compactly supported initial data* $\bar{\mu}\in\mathcal{P}_1(\mathbb{R}^{2d})$ *with the vector field* V *either given by* (3.2.6) *or* (3.2.5). *If* $\beta\in[0,1)$, *it then holds it then holds*

$$d_X[\mu_{nT}] \le \left[(1+d_X[\bar{\mu}])^{1-\beta} + (1-\beta)\left(K_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu)\right)\right]^{\frac{1}{1-\beta}} \tag{3.2.7}$$

for all $n\in\mathbb{N}$ *where*

$$K_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T,\mu) := d_V[\bar{\mu}] + (T-\mu) d_V[\bar{\mu}] \sum_{n=1}^{+\infty} \left(e^{-\frac{\mu}{(1+d_X[\bar{\mu}]+d_V[\bar{\mu}]T)^\beta}}\right)^{n^{1-\beta}}.$$

then unconditional flocking occurs.

Proof. Let $N\in\mathbb{N}$. Define the family of $N-$particle approximations of $\bar{\mu}$, namely $\{\bar{\mu}^N\}_N$ defined as

$$\bar{\mu}^N := \frac{1}{N}\sum_{i=1}^N \delta(x-\bar{x}_i^N)\otimes\delta(v-\bar{v}_i^N)$$

where $\{\bar{x}_i^N, \bar{v}_i^N\}_{i\in\{1,\dots,N\}}$, with $x_i^N, v_i^N \in \mathbb{R}^d$, are chosen such that

$$\operatorname{supp} \bar{\mu}^N \subseteq B(0, d_X[\bar{\mu}]) \times B(0, d_V[\bar{\mu}]) \quad \forall N \in \mathbb{N} \tag{3.2.8}$$

and

$$\lim_{N\to+\infty} W_p(\bar{\mu}, \bar{\mu}^N) = 0. \tag{3.2.9}$$

Now let $\{x_i^N, v_i^N\}_{i\in\{1,\dots,N\}}$ denote the solution to the finite dimensional system (3.2.2) or (3.2.2) with initial condition $\{\bar{x}_i^N, \bar{v}_i^N\}_{i\in\{1,\dots,N\}}$. Define

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta(x - x_i^N(t)) \otimes \delta(x - v_i^N(t)) \qquad \forall t \in [0,T].$$

Then, we have that μ_t^N is a measure-valued solution to the kinetic model (3.2.4) in the sense of (3.2.1). Now notice that if $\mu_t \in C([0,T];\mathcal{M}(R^{2n}))$ is a weak solution to (3.2.4) with initial datum $\bar{\mu}$, then according to Theorem 3.2.3 there exists a constant $C > 0$ independent of N such that

$$W_1(\mu_t^N, \mu_t) \leq C W_1(\bar{\mu}^N, \bar{\mu}) \qquad \forall t \in [0,T), \quad \forall p \in [1,+\infty].$$

Therefore, by (3.2.9) it holds

$$\lim_{N\to+\infty} W_p(\mu_t^N, \mu_t) = 0 \qquad \forall t \in [0,T), \quad \forall p \in [1,+\infty].$$

By following the same argument as done in Theorem 2.2.4, we have that by Theorem 3.2.1 it holds

$$\operatorname{supp}_x \mu_t^N \subset B(0, d_X^N(t)) \subset B(0, C(d_V^N(0), d_X^N(0), T, \mu, \beta)) \quad \forall t \geq 0$$

where

$$d_X^N(t) := \operatorname{diam}(\operatorname{supp}_x \mu_t^N) \quad t \geq 0$$

and where $C(d_V^N(0), d_X^N(0), T, \mu, \beta)$ is the left-hand side of (3.2.3). Now notice that by (3.2.8) it holds

$$C(d_V^N(0), d_X^N(0), T, \mu, \beta) \leq C_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu, \beta)$$

where $C_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu, \beta)$ is the left-hand side of (3.2.7). Therefore it holds

$$\operatorname{supp}_x \mu_t^N \subset B(0, d_X^N(t)) \subset B(0, C_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu, \beta)) \quad \forall t \geq 0$$

Again, since convergence in Wasserstein implies that $\mu_t^N \to \mu_t$ weakly-* as measures as $N \to +\infty$, then we know that the support of a measure is stable under weak-* limits, as shown in the proof of Theorem 2.2.4. This implies that

$$\operatorname{supp}_x \mu_t \subset B(0, C_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu, \beta)) \quad \forall t \geq 0$$

and therefore

$$d_X(\mu_t) \leq C_{MF}(d_V[\bar{\mu}], d_X[\bar{\mu}], T, \mu, \beta) \qquad \forall t \in [0, T]$$

and since T is arbitrary, then the result follows. $\square$

Conclusions

We now present the results we have achieved in each chapter along with possible future directions.

- In the first chapter, we have seen in Theorem 1.0.1 that letting the initial approximation profile issued by some discretization scheme converge weakly to the compactly supported initial data, then the microscopic solution built from the Follow-the-Leader model converges to the entropy solution of the macroscopic Lighthill-Whitham-Richards model. Moreover, in Theorem 1.0.2 we have provided a stability result regarding the solution issued by two different discretization schemes.

  A future work could be devoted to the study of the mean-field limit in networks and address optimal control problems.

- In the second chapter, we have seen in Theorem 2.1.1 that consensus holds under a PE condition on the scrambling coefficient of the weights $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ for an influence function bounded from below for system (2.1.1). We have seen that in case the indistinguishability property (2.2.1) holds, then we can use the result of Theorem 2.1.1 to conclude that consensus holds also in the mean-field setting as shown in Theorem 2.2.4. A new result in the finite-dimensional setting is given by Theorem 2.1.2, where we simply impose the PE condition on the weights $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$ only, independently of the state of the dynamics, and guarantee that consensus holds.

  A future work is devoted to the generalization of the result in the multi-dimensional setting. Another open problem is to see whether the one-dimensional result holds if we impose a PE condition only on the mean of $\{M_{ij}\}_{j\in\{1,\dots,N\}}$ for all $i \in \{1, \dots, N\}$. Another task is to consider the graph-limit setting and try to use the same logic as in the finite-dimensional setting by using Scorza-Dragoni's theorem (Theorem A.1.1) and Danskin's theorem (Theorem A.1.2) in the spirit of [11]. Since the main goal is to treat the case of communicatino failures, it would be interesting to see whether we can relate the point of view of imposing a PE condition with the point of view of letting the communication failures follow some stochastic process.

- In the third chapter, we have seen in Theorem 3.1.1 that unconditional flocking holds for $\beta \in [0, 1)$ in the case where we impose a PE condition on the scrambling coefficient of the weights $\{M_{ij}\}_{i,j\in\{1,\dots,N\}}$. The main tool was the identification and treatment of the "worst-case scenario" in the sense of Proposition 3.1.2. Then, in Theorem 3.2.1 we have seen that we can use the result by Theorem 3.1.1 to conclude that unconditional flocking holds under the same assumptions in the mean-field setting as well.

  A future work could be to first investigate the case of unconditional flocking $\beta = 1$ as explained in Remark 3.1.1 and then conditional flocking for the case $\beta > 1$. Another future work could be to consider the graph limit of the finite-dimensional model, prove that its "worst-case scenario" is the same and, again, by using Scorza-Dragoni's theorem (Theorem A.1.1) and Danskin's theorem (Theorem A.1.2) in the spirit of [11] conclude that unconditional flocking holds for $\beta \in [0, 1)$ and again treat as well the case $\beta \geq 1$.

Appendix A: Preliminaries

A.1 Measure theory and optimization

In this section, we present some general tools already applied to opinion dynamics (see e.g. [11]). We now present simplified versions of Scorza-Dragoni and Danskin theorems.

Theorem A.1.1 (Scorza-Dragoni, [10]). *Consider a complete separable metric space* $(\mathscr{S}, d_{\mathscr{S}})$*. Let* $\Omega \subset \mathbb{R}^d$ *be a Borel set and* $f : \mathbb{R}_+ \times \Omega \to \mathscr{S}$ *be such that* $x \in \Omega \mapsto f(t,x) \in \mathscr{S}$ *is* $\mathscr{L}^d$*-measurable for each* $t \geq 0$*, and* $t \in \mathbb{R}_+ \mapsto f(t,x) \in \mathscr{S}$ *is continuous for* $\mathscr{L}^d$*-almost every* $x \in \Omega$*. Then for every* $\varepsilon > 0$*, there exists a compact set* $\Omega_\varepsilon \subset \Omega$ *satisfying* $\mathscr{L}^d(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ *and such that the restricted map* $f : \mathbb{R}_+ \times \Omega_\varepsilon \to \mathscr{S}$ *is continuous.*

Theorem A.1.2 (Danskin, [29]). *Let* $\Omega \subset \mathbb{R}^d$ *be a compact set and* $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ *be a continuous function such that* $t \in \mathbb{R}_+ \mapsto f(t,x) \in \mathbb{R}$ *is differentiable for all* $x \in \Omega$*. Then, the application* $g : t \in \mathbb{R}_+ \mapsto \max_{x \in \Omega} f(t,x) \in \mathbb{R}$ *is differentiable* $\mathscr{L}^1$*-almost everywhere, with*

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \max_{x \in \hat{\Omega}(t)} \partial_t f(t,x)$$

for $\mathscr{L}^1$*-almost every* $t \geq 0$*, where we introduced the notation* $\hat{\Omega}(t) := \underset{x \in \Omega}{argmax} f(t,x)$*.*

Lemma A.1.1 ([11, Lemma 2.3]). *Let* $\Omega \subset \mathbb{R}^d$ *be a compact set and* $f \in L^\infty(\Omega, \mathbb{R}^d)$*. Then, for every* $\delta > 0$*, there exists* $\epsilon > 0$ *such that*

$$\|f\|_{L^\infty(\Omega)} - \delta \leq \|f\|_{L^\infty(\Omega_\epsilon)} \leq \|f\|_{L^\infty(\Omega)}$$

for all measurable sets $\Omega_\epsilon \subset \Omega$ *satisfying* $\mathrm{meas}(\Omega \backslash \Omega_\epsilon) < \epsilon$. *In particular, it holds*

$$\lim_{\epsilon \to 0} \|f\|_{L^\infty(\Omega_\epsilon)} = \|f\|_{L^\infty(\Omega)}$$

for every family of sets $(\Omega_\epsilon)_{\epsilon > 0} \subset \mathcal{P}(\Omega)$ *satisfying these properties.*

We now present a scalar product inequality which is used in the graphon framework.

Lemma A.1.2 ([11, Lemma 3.4])**.** *Let* $J \in [0,1]$ *be a closed set,* $x \in C(J, \mathbb{R}^d)$ *and* $i, j \in J$ *be a pair of indices such that*

$$\max_{k,l \in J} |x(k) - x(l)| = |x(i) - x(j)|.$$

It then holds

$$\max_{k \in J} \langle x(k), x(i) - x(j) \rangle = \langle x(i), x(i) - x(j) \rangle$$

and

$$\min_{k \in J} \langle x(j), x(i) - x(j) \rangle = \langle x(j), x(i) - x(j) \rangle.$$

Remark A.1.1. In particular, Lemma A.1.2 is applied in the finite-dimensional framework, where indices belong to the closed set $\{1, \dots, N\}$, and in the graph limit framework, where indices belong to $[0,1]$.

A.2 Optimal transportation

We now present main definitions in optimal transportation.

Definition A.2.1. *Let* μ *be a Borel measure on* $\mathbb{R}^d$ *and let* $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ *be a measurable map. Define the push-forward of* μ *through* T *as the measure given by*

$$T_\# \mu(B) \coloneqq \mu\left(T^{-1}(B)\right)$$

for all Borel sets $B \in \mathbb{R}^d$.

Definition A.2.2. *Let* $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d)$ *be two probability measures on* $\mathbb{R}^d$. *Let* $\Pi(\mu_1, \mu_2)$ *be the set of all probability measures on* $\mathbb{R}^d \times \mathbb{R}^d$ *with marginals* μ_1 *and*

μ_2, *i.e.*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)d\pi(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)d_{\mu_1}(x), \quad \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)d\pi(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)d_{\mu_2}(x)$$

for all continuous and bounded functions $\phi \in C_b\left(\mathbb{R}^d\right)$

Then, we define the Wasserstein distance of order $1 \leq p < +\infty$ *between* μ_1 *and* μ_2 *as*

$$W_p(\mu_1,\mu_2) := \inf_{\pi\in\Pi(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\pi(x,y)\right)$$

and for $p = \infty$ *as*

$$W_\infty(\mu_1,\mu_2) := \inf_{\pi\in\Pi(\mu_1,\mu_2)} \left(\sup_{(x,y)\in\mathrm{supp}(\pi)} |x-y|^p\right).$$

Proposition A.2.1. *The set of probability measures with finite moments of order* $p \in [1,+\infty)$, *denoted by* $\mathcal{P}_p\left(\mathbb{R}^d\right)$, *endowed with the* p*-Wasserstein distance* W_p *is a complete metric space.*

Proof. See [63]. □

We now define the cumulative distribution of a function and the corresponding pseudo-inverse.

Definition A.2.3. *Consider the space of probabilities*

$$\mathcal{P}_c(\mathbb{R}) := \{\rho \text{ Radon measure on } \mathbb{R} \text{ with compact support such that } \rho \geq 0, \quad \rho(\mathbb{R}) = 1\}.$$

Given $\rho \in \mathcal{P}_c(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *define the cumulative distribution* $F_\rho : \mathbb{R} \mapsto [0,1]$ *as :*

$$F_\rho(x) := \rho((-\infty,x]), \qquad x \in \mathbb{R}, \tag{A.2.1}$$

and its associated pseudo-inverse $X_\rho : [0,1] \mapsto \mathbb{R}$ *as*

$$X_\rho(z) := \inf\{x \in \mathbb{R} \quad | \quad F_\rho(x) \geq z\}, \qquad z \in [0,1]. \tag{A.2.2}$$

Observe that F_ρ is non-decreasing and right-continuous. We recall that the one dimensional Wasserstein distance can be defined using the cumulative or the pseudo-inverse functions, see e.g. [63].

Proposition A.2.2. *The one-dimensional 1-Wasserstein distance between $\rho, \tilde{\rho} \in \mathcal{P}_c(\mathbb{R})$ is*

$$W_1(\rho, \tilde{\rho}) = \|F_\rho - F_{\tilde{\rho}}\|_{L^1(\mathbb{R})} = \|X_\rho - X_{\tilde{\rho}}\|_{L^1([0,1])}\,. \tag{A.2.3}$$

A.3 Ordinary differential equations

Given an ordinary differential equation, the right-hand side of the dynamics may present discontinuities, and we thus have to consider a more generalized notion of solution, see [18]. Throughout this paper we consider the notion of Carathéodory solution whenever the dynamics studied contain a measurable function on the right-hand side. We now recall some definitions and theorems, see e.g. [36].

Definition A.3.1 (Carathéodory solution)**.** *Consider a non-autonomous ODE*

$$\dot{x}(t) = f(t, x(t)) \tag{A.3.1}$$

where $x \in \mathbb{R}^d$ and $g : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a measurable and locally bounded function defined at every point. A Carathéodory solution is an absolutely continuous function $x : [0, T] \mapsto \mathbb{R}^d$ which satisfies (A.3.1) *at almost every time $t \in [0, T]$.*

Definition A.3.2 (Carathéodory conditions)**.** *Supposed D is an open subset in $\mathbb{R}^{d+1}$. We say that $f : D \mapsto \mathbb{R}^d$ satisfies the Carathéodory conditions on D if f is measurable in t for each fixed x, continuous in x for each fixed t and for each compact set U of D, there is an integrable function $m_U(t)$ such that*

$$|f(t, x)| \leq m_U(t), \quad (t, x) \in U.$$

Theorem A.3.1. *If D is an open set in $\mathbb{R}^{d+1}$ and f satisfies the Carathedoroy conditions on D, then, for any (t_0, x_0) in D, there is a solution of* (A.3.1) *through (t_0, x_0).*

Theorem A.3.2. *Supposed D is an open subset in $\mathbb{R}^{d+1}$, f satisfies the Carathéodory conditions on D and for each compact set U in D, there is an integrable function $k_U(t)$ such that*

$$|f(t, x) - f(t, y)| \leq k_U(t)|x - y|, \quad (t, x) \in U, \quad (t, y) \in U.$$

Then, for any (t_0, x_0) in U there exists a unique solution $x(t, t_0, x_0)$ of (A.3.1) *passing through* (t_0, x_0)*. The domain* E *in* $\mathbb{R}^{d+2}$ *of definition of the function* $x(t, t_0, x_0)$ *is open and* $x(t, t_0, x_0)$ *is continuous in* E*.*

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