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Some stratifications on algebraic varieties

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Abstract

In this thesis we study stratifications on algebraic varieties in two different contexts. The first one concerns the relation between the conjugacy classes in a reductive algebraic group and the irreducible representations of its associated Weyl group, with a focus on non connected algebraic groups. Let G be a non-connected reductive algebraic group over an algebraically closed field and let D be a connected component of G. The connected component of G containing the identity is denoted by G°, and W is its Weyl group. In [46] G. Lusztig defines a map from D to the set of isomorphism classes of irreducible representations of a subgroup of W depending on D. In this thesis we study the geometry of the fibers of this map, namely the Lusztig strata. In particular we prove that they are locally closed and we describe their irreducible components. In order to do that we study the stratification of G into Jordan classes as defined in [39]. In [44] another approach to study the connection between conjugacy classes in G and irreducible representations of W is established through a map Ψ from the set of unipotent conjugacy classes in D to the set of twisted conjugacy classes of W. For G° simple, we give a combinatorial description of the restriction of Ψ to the set of unipotent spherical conjugacy classes in D and we give a classification of spherical unipotent conjugacy classes of G.

The second part of the thesis is devoted to the application of the theory of Seshadri stratifications to matrix Schubert varieties, namely varieties of matrices defined by conditions on the rank of some their submatrices. These varieties were first introduced in [23] and they are useful in the study of combinatorics of determinantal ideals and Schubert polynomials [53]. The theory of Seshadri stratifications has been introduced in [16]. One of the aims of this theory is to provide a geometric setup for standard monomial theory. A Seshadri stratification of an embedded projective variety X is the datum of a suitable col-

lection of subvarieties X_τ that are smooth in codimension one, and a collection of suitable homogeneous functions f_τ on X indexed by the same finite set. In this thesis we provide a Seshadri stratification for the matrix Schubert varieties.

Riassunto

In questa tesi studiamo stratificazioni di alcune varietà algebriche in due contesti differenti. Il primo contesto riguarda la relazione tra le classi di coniugio in un gruppo algebrico riduttivo e le rappresentazioni irriducibili del suo gruppo di Weyl associato, concentrandoci sul caso dei gruppi algebrici non connessi. Sia G un gruppo algebrico riduttivo non connesso su un campo algebricamente chiuso e sia D una componente connessa di G. La componente connessa di G contenente l'identità è indicata con G°, e W è il suo gruppo di Weyl. In [46], G. Lusztig definisce una mappa da D all'insieme delle classi di isomorfismo di rappresentazioni irriducibili di un sottogruppo di W che dipende da D. In questa tesi studiamo la geometria delle fibre di questa mappa, chiamati gli strati di Lusztig. In particolare dimostriamo che sono localmente chiusi e ne descriviamo le componenti irriducibili. Per farlo, studiamo la stratificazione di G in classi di Jordan come definite in [39]. In [44] è stato anche proposto un altro approccio per studiare la connessione tra le classi di coniugio in G e le rappresentazioni irriducibili di W attraverso una mappa Ψ dall'insieme delle classi di coniugio unipotenti in D all'insieme delle classi di coniugio twistate di W. Nel caso in cui G° sia semplice, forniamo una descrizione combinatoria della restrizione di Ψ all'insieme delle classi di coniugio unipotenti sferiche in D e presentiamo anche una classificazione delle classi di coniugio unipotenti sferiche di G.

La seconda parte della tesi è dedicata all'applicazione della teoria delle stratificazioni di Seshadri alle varietà di Schubert di matrici, cioè varietà di matrici definite da condizioni sul rango di alcune loro sottomatrici. Queste varietà sono state introdotte per la prima volta in [23] e sono utili nello studio della combinatoria degli ideali determinantali e dei polinomi di Schubert [53]. La teoria delle stratificazioni di Seshadri è stata introdotta in [16]. Uno degli obiettivi di questa teoria è fornire un contesto geometrico per la teoria dei monomi standard. Una stratificazione di Seshadri di una varietà proiettiva immersa X è il dato di una collezione di opportune sottovarietà X_τ che sono lisce in codimensione uno e una collezione di opportune funzioni omogenee f_τ su X indicizzate da uno stesso insieme finito. In questa tesi forniamo una stratificazione di Seshadri per le varietà di Schubert di matrici.

Introduction

This thesis consists of two independent parts both related to stratifications of algebraic varieties of representation theoretic interest. The first one concerns non-connected algebraic groups, with focus on geometrical properties of some stratifications of a non connected algebraic group, introduced by G. Lusztig. In the second one we study a Seshadri stratification for the matrix Schubert varieties.

The first part has its roots in the deep relation between the conjugacy classes in a reductive algebraic group and the irreducible representations of its associated Weyl group stemming from the celebrated Springer correspondence [64] and its generalizations [50, 31]. This relation has been explored through different approaches, both for connected reductive algebraic groups and non-connected ones.

More precisely, let H be a connected reductive algebraic group over an algebraically closed field $\mathbb{K}$, and let W_H be the associated Weyl group. We denoted by $\mathrm{Irr}W_H$ the set of isomorphism classes of irreducible representations of W_H.

In [64] Springer established a correspondence between the unipotent conjugacy classes of H and irreducible representations of W_H.

Generalizing this correspondence, G. Lusztig in [45] defines a map

$$\mathcal{E}_C : H \longrightarrow \mathrm{Irr}\, W_H.$$

that is constant on conjugacy classes. Let $h \in H$ and let $h_s h_u$ the Jordan decomposition of h. Then $\mathcal{E}_C(h)$ is the truncated induction, defined as in [49], of the Springer representation of the centralizer of h_s in W_H associated with h_u and trivial local system.

Unlike the Springer correspondence, the image of $\mathcal{E}_C$ is independent of the characteristic of the base field $\mathbb{K}$.

In [42, 43], G. Lusztig uses a different approach to relate conjugacy classes of the group H and conjugacy classes of the Weyl group W_H. Given $\underline{W}_H$ the set of conjugacy classes in W_H and $\underline{H}$ the set of unipotent conjugacy classes in H, he defines a surjective map

$$\Upsilon_C : \underline{W}_H \longrightarrow \underline{H},$$

by mapping a conjugacy class C in W_H to a minimal unipotent conjugacy class in H with respect to Zariski closure, having non-empty intersection with the Bruhat double coset corresponding to a minimal length element in C.

The surjectivity of Υ_C leads G. Lusztig to define in [43] a right inverse Ψ_C of Υ_C. The map Ψ_C maps a unipotent class γ in H to the unique class $C \in \Upsilon_C^{-1}(\gamma)$ for which the dimension of the fixed point space of $w \in C$ in the geometric representation of W_H is minimal. In [32], Kazhdan and Lusztig define a map from the set of unipotent conjugacy classes of H to the set of conjugacy classes of W_H for $\mathbb{K} = \mathbb{C}$. They conjectured that this map was injective. The right inverse Ψ_C coincides with the map defined by Kazhdan and Lusztig [68].

In [45], the author provides a relation between the maps $\mathcal{E}_C$ and Υ_C. The fibers of $\mathcal{E}_C$ give a partition of H into finitely many strata such that each stratum is a union of conjugacy classes of fixed dimension. In [45, Section 5] these strata are defined also using the map Υ_C.

All these constructions were generalized to the case of non-connected reductive algebraic group. Such groups are relevant because they appear frequently in the study of algebraic groups, for example as centralizers of semisimple elements in non-simply connected semisimple groups. In particular, they often occur in the study of algebraic groups when one needs to apply inductive arguments related to conjugacy classes.

We sketch here the content of the first part of the thesis. Let G be a non connected algebraic group with G° the connected component containing the identity, and let W be the Weyl group of G°. We denote by D a connected component of G.

In [46], G. Lusztig, generalizing the construction of $\mathcal{E}_C$, defines a map $\mathcal{E}$ from D to the the set of isomorphism classes of irreducible representations of a subgroup of W depending on D. The definition of the map $\mathcal{E}$ is analogous to the definition of the map $\mathcal{E}_C$. The fibers of this map are called *Lusztig strata*. Since the geometrical properties of unipotent classes are strictly related to representation theoretic properties, it is natural to wonder whether Lusztig strata can be endowed with some geometric structure. This property is not immediate from the definition of strata. In this thesis we prove the following theorem, whose statement was suggested by G. Lusztig in [46].

Theorem 1. Let X be a Lusztig stratum. Then X is a locally closed subset of D.

In more recent works G. Lusztig studied the connection between strata of an algebraic group and unipotent character sheaves. In particular in [47] he defines a surjective map from the unipotent character sheaves on a connected reductive group and its set of strata. This map is generalized to the non-connected case in [48]. This gives a new parametrization of the set of isomorphism classes of character sheaves on an algebraic group in every characteristic, and has potential applications to the study of unipotent irreducible representations of finite groups of Lie type.

For more details on theory of character sheaves in an algebraic group see [38] for the connected setting, and [39] for the non-connected setting.

Once strata are seen as varieties, it is natural to study their irreducible components. It would be interesting to see how these depend on the characteristic of $\mathbb{K}$, and what distinguishes elements lying in different components of the same stratum.

In order to explore the geometrical properties of a stratum, our approach involve examining the partition of D into Jordan classes, i.e. the equivalence classes defined in [39]. Indeed, strata of D are unions of finitely many Jordan classes of elements whose G°-orbits have the same dimension [46]. We show that a stratum X is a union of the regular part of the closure of the Jordan classes it contains. This property allowed us to prove Theorem 1. Moreover, we study the sheets of a stratum, namely maximal irreducible subsets of X consisting of equidimensional G°-conjugacy classes. It emerges that the Lusztig strata are unions of the regular parts of the closures of certain Jordan classes and that the irreducible components of a stratum coincide with its sheets.

A second approach in studying the relations between representations of W and conjugacy classes of D was given in [44], where the author obtains a generalization of the map Υ_C and Ψ_C for the non-connected case. In particular it was defined a map Φ from twisted conjugacy classes in W to unipotent conjugacy classes in a fixed connected component D of G. If C is a twisted conjugacy class in W, then $\Phi(C)$ is the minimal unipotent conjugacy class in D, with respect to Zariski closure, having non-empty intersection with the Bruhat double coset in D corresponding to a minimal length element in C. It is a non-trivial result that this map is well defined. The general case is reduced to the connected case and the cases in which the connected component G° of G is simple and admits a non-trivial graph automorphism:

(a) $G = PGL_m(\mathbb{K})$, $m \geq 3$, $\mathrm{char}\mathbb{K} = 2$;

(b) $G = PSO_{2m}(\mathbb{K})$, $m \geq 4$, $\mathrm{char}\mathbb{K} = 2$;

(c) $G = PSO_8(\mathbb{K})$, $\mathrm{char}\mathbb{K} = 3$;

(d) G adjoint of type E_6 and $\mathrm{char}\mathbb{K} = 2$.

In this case $G = G^\circ \rtimes \langle\theta\rangle$, where θ is a graph-automorphism of order 2 in (a), (b), (d) and of order 3 in (c) and $D = G^\circ\theta$. The twisted action of W on itself is defined by $w_1 \cdot_\theta w = w_1 w\, \theta(w_1)^{-1}$ for $w_1, w \in W$, where θ can be seen as an automorphism of W.

The generalization of Ψ_C is a right inverse Ψ of Φ. The map Ψ is defined by taking, for a given unipotent class γ in D, the unique twisted class C in the fiber $\Phi^{-1}(\gamma)$ for which a certain invariant $\mu(C)$ reaches its minimum ([44, Theorem 1.16]). Also in this case, the fact that this map is well defined is a deep result.
In this thesis we give a different and direct combinatorial description of the restriction of the map Ψ to the set of spherical unipotent G°-conjugacy classes in D, generalizing the result in [10]. Unipotent spherical G°-conjugacy classes in D are the unipotent G°-conjugacy classes for which there is a dense orbit of a Borel subgroup B of G. In order to give our description we first classify the unipotent spherical G°-conjugacy classes in D completing the work initiated in [5, 6, 19].
Our second main result is the following.

Theorem 2. Let G° be a simple algebraic group over an algebraically closed field of characteristic 2, and let τ be a graph-automorphism of G° of order 2. If γ is a spherical unipotent G°-conjugacy class in $G^\circ\tau$, then $\Psi(\gamma) = W \cdot_\tau w_\gamma$, where $w_\gamma \in W$ be the unique element in W for which $Bw_\gamma B\theta \cap \gamma$ is dense in γ.

The first part of the thesis is structured as follows. In the first chapter we explore some properties of non-connected algebraic groups. We recall the classification of non-connected algebraic groups for which the connected component containing the identity is simply connected or of adjoint type. Subsequently we recall the notion of parabolic subgroups of a non-connected reductive group and their Levi decomposition. In Chapter 2 we study the Jordan classes on a reductive non-connected algebraic group G. We describe a procedure of induction of an orbit from a connected component of the normalizer of a Levi subgroup of G° to a G°-orbit in D. This allows us to investigate the closure and the regular locus in the closure of a Jordan class. In particular, similarly to the connected case described in [11], we show that the regular locus in the closure of a Jordan class is a union of Lusztig-Spaltestein induced G°-orbits. In Chapter 3 we introduce the Lusztig strata on G, recalling the definition in [46]. We prove that these strata are locally closed, and we describe the irreducible components of the strata. In Chapter 4 we classify the spherical unipotent conjugacy classes in a connected component D of G, in the case of G° simple, and we obtain the combinatorial description of the restriction of the map Ψ to spherical unipotent conjugacy classes in D.

The second part of the thesis originates from the seminal work of R. Chirivì, X. Fang and P. Littelmann on Seshadri stratifications, and focuses on the example of matrix Schubert varieties.

The theory of Seshadri stratifications on embedded projective varieties has been introduced in [16]. One of the aims of this theory is to provide a geometrical setup for standard monomial theory.

More precisely, let $X \subseteq \mathbb{P}(V)$ be an embedded projective variety, where V is a finite dimensional vector space on algebraically closed field $\mathbb{K}$. We denote by $\hat{X}$ the affine cone over X.

A Seshadri stratification on X consists of a collection of smooth in codimension one subvarieties $X_\tau \subseteq X$ and homogeneous functions $f_\tau \in Sym(V^*)$ indexed by a finite set $\mathcal{A}$. The finite set $\mathcal{A}$ is endowed with a partial order given by the inclusion relations between the subvarieties X_τ. Moreover $\mathcal{A}$ has the following properties

- if $\sigma < \tau$ is a cover relation in $\mathcal{A}$, namely if there is no $\sigma' \in \mathcal{A}$ such that $\sigma < \sigma' < \tau$ in $\mathcal{A}$, then X_σ is a prime divisor in X_τ;

- there exists a unique maximal element $\tau_{\max} \in \mathcal{A}$ such that $X_{\tau_{\max}} = X$.

The subvarieties X_τ and the functions f_τ for $\tau \in \mathcal{A}$ are compatible in the following sense:

- the vanishing set of the restriction of f_τ to X_τ is the union of all X_σ with $\sigma < \tau$ a cover relation;

- f_τ vanishes on X_σ for $\sigma \not\geq \tau$.

For each maximal chain $\mathfrak{C}$ in $\mathcal{A}$, in [16] a valuation $\mathcal{V}_{\mathfrak{C}}$ on the field of rational functions of $\hat{X}$ is defined in terms of the vanishing order on the subvarieties corresponding to the elements in $\mathfrak{C}$. This construction is a variant of Newton-Okounkov theory.
Refining to a total order the order on $\mathcal{A}$, the vector space $\mathbb{Q}^{\mathcal{A}}$ is endowed with a total order using the lexicographic order induced by the new order on $\mathcal{A}$. A quasi-valuation $\mathcal{V}$ with values in $\mathbb{Q}^{\mathcal{A}}$ is defined as the minimum among the valuations $\mathcal{V}_{\mathfrak{C}}$ when $\mathfrak{C}$ runs through the set of maximal chains in $\mathcal{A}$. Let Γ be the image of $\mathcal{V}$. It is a fan of monoids. More precisely, it is a union of finitely generated monoids $\Gamma_{\mathfrak{C}}$ in $\mathbb{Q}^{\mathcal{A}}$ corresponding to maximal chains in $\mathcal{A}$. It is important to stress that the monoids $\Gamma_{\mathfrak{C}}$ are finitely generated, unlike those occurring in the theory of Newton-Okounkov bodies.
As a generalization of Newton-Okounkov theory, from this quasi-valuations, a flat degeneration of X in a union of toric varieties is constructed.

The theory of Seshadri stratifications provides geometric setups for standard monomial theory on the graded homogeneous coordinate ring $\mathcal{R}$ of X. If $\Gamma_{\mathfrak{C}}$ is normal ([16, Definition 13.7]), then each element $\underline{a} \in \Gamma_{\mathfrak{C}}$ can be uniquely decomposed as a sum of indecomposable elements in the sense of [16, Definition 15.2]. From the properties of the quasi-valuation $\mathcal{V}$, for each indecomposable element $\underline{a} \in \Gamma_{\mathfrak{C}}$ there exists a regular function $x_{\underline{a}} \in R$ with $\mathcal{V}(x_{\underline{a}}) = \underline{a}$. The condition of being standard will be defined on monomials in these regular functions: a monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_k}$ with $\underline{a}_1, \ldots, \underline{a}_k \in \Gamma_{\mathfrak{C}}$ is standard if for $i = 1, \ldots, k-1$, we have that $\min \operatorname{supp}(\underline{a}_i) \geq \max \operatorname{supp}(\underline{a}_{i+1})$, where $\operatorname{supp}(\underline{a}_i)$ is the set of elements in $\mathcal{A}$ such that the corresponding entry of $\underline{a}_i$ is non-zero. This defines a standard monomial theory on R in terms of vanishing order of functions on X. Under some assumptions of regularity of Γ, from the standard monomial theory of $\mathcal{R}$ one can see that $\mathcal{R}$ has the structure of an LS-algebra as defined in [14].
When X is a Schubert variety in a flag variety these constructions provide a geometrical interpretation of the Lakshmibai-Seshadri path model [35, 36]. In particular it is pointed out the connection between LS-paths and the elements in Γ, giving an interpretation of the

LS-paths as vanishing orders of functions [15].

In this thesis we apply the theory of Seshadri stratification to matrix Schubert varieties. Matrix Schubert varieties are varieties of matrices described by conditions on the rank of their submatrices. They first appeared in the work of Fulton [23], in which he studies the degeneracy loci of flagged vector bundles. Matrix Schubert varieties have a deep relation with combinatorics of determinantal ideals and Schubert polynomials [53]. Moreover these varieties are strictly related to Schubert varieties.

Concretely, we consider the affine variety $\mathrm{Mat}_{m\times n}$ of $m \times n$ matrices with coefficients in an algebraically closed field $\mathbb{K}$. The group $\mathbb{B}_m^- \times \mathbb{B}_n^+$, where $\mathbb{B}_m^-$ (respectively $\mathbb{B}_n^+$) is the subgroup of $\mathrm{GL}_{m\times m}$ (respectively $\mathrm{GL}_{n\times n}$) consisting of lower (respectively upper) triangular matrices, acts naturally on $\mathrm{Mat}_{m\times n}$. The matrix Schubert varieties are the closures of the orbits for this action. Considering $\mathrm{Mat}_{m\times n}$ as the affine cone over $\mathbb{P}(\mathrm{Mat}_{m\times n})$, the projection of matrix Schubert varieties on $\mathbb{P}(\mathrm{Mat}_{m\times n})$ gives a collection of embedded projective subvarieties in $\mathbb{P}(\mathrm{Mat}_{m\times n})$. Using this collection of subvarieties, we define a Seshadri stratification of $\mathbb{P}(\mathrm{Mat}_{m\times n})$. When $n = 2$ we are able to describe explicitly the image of the quasi-valuation associated with the defined Seshadri stratification.

The second part of this thesis is structured as follows. In Chapter 5 we recall definitions and properties of the Seshadri stratifications from [16]. In Chapter 6 we define the matrix Schubert varieties and a Seshadri stratification on them. Finally, we describe the Seshadri stratification of $\mathbb{P}(\mathrm{Mat}_{m\times 2})$, the associated quasi-valuation, and its image.

Contents

Part I

Jordan classes and Lusztig strata

Notation

Let G be a reductive algebraic group, not necessarily connected, over an algebraically closed field $\mathbb{K}$. Let G° be the connected component of G containing the identity. If x is an element of G, and H a closed subgroup of G, then

- H° denotes the connected component of H containing the identity;
- $N_G H$ denotes the normalizer of H in G;
- $Z(H)$ denotes the center of H;
- x_s, x_u denote, respectively, the semisimple and the unipotent parts of x.

We denote by D a connected component of G. Given an element $g \in D$, then $D = G^\circ g$. So we use both notation for a connected component of G. Let W be the Weyl group of G° . We can see W in two different ways: if we fix a maximal torus T of G°, then $W = N_{G^\circ}(T)/T$. More intrinsically, let $\mathcal{B}$ be the flag manifold of G° (defined as the variety of Borel subgroups of G°). Then G° acts on $\mathcal{B} \times \mathcal{B}$ diagonally by conjugation and W is in bijection with the set of G°-orbits as follows. Let $\mathcal{O}$ be a G°-orbit in $\mathcal{B} \times \mathcal{B}$, and fix $(B_0, B_1) \in \mathcal{O}$. Using the Bruhat decomposition of G° [52, Theorem 11.17] and that all Borel subgroups of G° are G°-conjugate [52, Theorem 6.4(a)], we find in $\mathcal{O}$ a unique element of the form (B_0, wB_0w^{-1}) with $w \in N_{G^\circ}(T)$. Associating to $\mathcal{O}$ the element $w \in W$, gives the sought bijection.

The first definition is more manageable, while the second one is canonical, in the sense that no choice of a maximal torus is required. We use the first one mostly in Chapter 4. If H is a reductive subgroup of G, we denote the Weyl group of H by $W(H)$.

In Chapter 4 we fix a Borel subgroup B of G° containing T. We denote by U the unipotent radical of B. We denote by Φ the set of roots relative to T, then B determines the set of positive roots Φ^+ and the simple roots $\Delta = \{\alpha_1, ..., \alpha_n\}$. We set w_0 to be the longest element of W with respect to Δ. Let E be the Euclidean space spanned by Φ, furthermore s_α is the reflection with respect to $\alpha \in \Phi$, and s_i denotes the simple reflection associated with α_i for $i = 1, ..., n$. Then W is isomorphic to the group generated by the reflections s_i, for $i = 1, \ldots, n$. For $\alpha \in \Phi$ we put $U_\alpha = \{x_\alpha(\xi) \mid \xi \in \mathbb{K}\}$, the root subgroup corresponding to α. We use the numbering of the simple roots as in [52, Table 9.1].

If B' is a Borel subgroup of a connected algebraic group, and T' is a maximal torus contained in B', we refer to them as a *pair torus/Borel*, and we denote the pair by (T', B').

Definition 0.1.

- An automorphism of a connected algebraic group is called *quasi-semisimple* if it stabilizes a pair torus/Borel.
- An element of a connected algebraic group is called quasi-semisimple if it induces by conjugation a quasi-semisimple automorphism.

If f is an automorphism of E, then E^f denotes the subset of points in E fixed by f and if λ is an eigenvalue of f, we write $E_\lambda(f)$ for the eigenspace of f relative to λ. We use the notation in [29, 12] for algebraic groups. In particular, let $I = \{1, ..., n\}$: for $J \subseteq I$, we set $\Delta_J := \{\alpha_j \mid j \in J\}$, Φ_J is the corresponding root system, $E_J = \operatorname{Span}_{\mathbb{R}} \Delta_J$, W_J is the group generated by s_α, with $\alpha \in \Delta_J$, P_J is the standard parabolic subgroup of G° associated with J, and L_J is the standard Levi subgroup of P_J. For $z \in W$ we put $U_z = U \cap (w_0 z)^{-1} U w_0 z$. Then the unipotent radical of P_J is $U_{w_0 w_J}$, where w_J is the longest element of W_J with respect to Δ_J. Moreover $U \cap L_J = U_{w_J}$ is a maximal unipotent subgroup of L_J and $U = U_{w_J} U_{w_0 w_J}$. We put $T_J = [L_J, L_J] \cap T$. It is a maximal torus of $[L_J, L_J]$: then $B_J = T_J U_{w_J}$ is a Borel subgroup of $[L_J, L_J]$.

For $x \in G$, we consider the automorphism of G

$$\begin{aligned} c_x : G &\longrightarrow G \\ g &\mapsto x g x^{-1} \end{aligned} \tag{1}$$

We refer to c_x as the *conjugation morphism* by x. It restricts to an automorphism of G°. Let $\operatorname{Aut}(G^\circ)$ be the group of automorphisms of G°. Let Γ be the Dynkin diagram of G°.

We denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of the graph Γ. We use θ to denote a Dynkin diagram automorphism. Where there is no confusion, we still denote by θ the following objects: the permutation of I such that $\theta(i) = j$ if $\theta(\alpha_i) = \alpha_j$; the isometry of E associated with the automorphism of Γ; and the automorphism of W induced by θ. In particular, $\theta(s_\alpha) = s_{\theta(\alpha)}$ for all α in Φ. If θ is an involution (i.e. if it has order 2), we write τ instead of θ.

We consider the conjugation action of G° and G on G itself. If H is a group, for an element $x \in H$ we denote by H^x or $C_H(x)$ the centralizer of x in H.

If X is a G°-variety, then $X_{(d)} := \{x \in X \mid \dim(G^\circ \cdot x) = d\}$. An irreducible component of $X_{(d)}$ is called a *sheet* of X for the action of G°. In particular, when $X = G$, we consider the action of G° on G given by the conjugation, hence we denote by $G_{(d)}$ the set $\{g \in G \mid \dim(G^\circ \cdot g) = d\}$.
Let X be a G°-variety, let $Y \subseteq X$ and let m be the maximum integer such that $Y \cap X_{(m)} \neq \emptyset$. We indicate with Y^{reg} the set of regular elements of Y, i.e., the elements $y \in Y$ such that $\dim(G \cdot y) = m$. Furthermore with $\overline{Y}^{\mathrm{reg}}$ we indicate the set of regular elements in the Zariski closure of Y, we call this the *regular closure* of Y.

Lemma 0.2. Let X be a G-variety, and let Y be an irreducible G-subvariety of X. Then

(a) Y^{reg} is open dense in Y;

(b) $\bigcup_{d \leq n} Y_{(d)}$, is closed for every $n \in \mathbb{N}$.

Proof. By the upper semi-continuity of the dimension of the fiber of a morphism [55, Corollary 3, p.51], given the map

$$f : G \times Y \longrightarrow Y \times Y$$

$$(x, y) \mapsto (g \cdot y, y),$$

the set $S_k(f) := \{(y, y) \in Y \times Y \mid \dim f^{-1}(y, y) \geq k\}$ is closed. We observe that $f^{-1}(y, y) \cong \mathrm{Stab}_G(y)$, where $\mathrm{Stab}_G(y)$ is the stabilizer of y in G. Therefore

$$S_k(f) = \{y \in Y \mid \dim \mathrm{Stab}_G(y) \geq k\}.$$

Since $\dim \mathrm{Stab}_G(y) = \dim G - \dim G \cdot y$,

$$S_k(f) = \bigcup_{d \leq \dim G - k} Y_{(d)}.$$

Thus $\bigcup_{d\leq n} Y_{(d)}$ is closed for every $n \in \mathbb{N}$, giving (b).
Moreover, if m is the maximum integer such that $Y \cap X_{(m)} \neq \emptyset$, then

$$Y^{\text{reg}} = Y \setminus S_{\dim G - m+1}(f).$$

So Y^{reg} is a Zariski open set in Y, therefore Y^{reg} is dense in Y, giving (a). $\square$

We are frequently interested in studying the unipotent elements in an algebraic group.

Remark 0.3. When the characteristic of the base field $\mathbb{K}$ is $p > 0$, then an element $u \in G$ is unipotent if and only if u is a p-element. Indeed, let $\rho : G \longrightarrow \mathrm{GL}_r(\mathbb{K})$ be an embedding of G into $\mathrm{GL}_r(\mathbb{K})$ for some $r \in \mathbb{N}$. Let u be a p-element. Then $\rho(u)$ is a p-element in $\mathrm{GL}_r(\mathbb{K})$. Then there is $l > 0$ such that

$$0 = \rho(u)^{p^l} - \mathrm{Id} = (\rho(u) - \mathrm{Id})^{p^l},$$

so u is unipotent in G. Conversely, if u is unipotent, then there exists $k \in \mathbb{N}$ such that $(\rho(u) - \mathrm{Id})^k = 0$. Let $j \in \mathbb{N}$ be such that $p^j > k$, then $(\rho(u) - \mathrm{Id})^{p^j} = 0$, so $\rho(u)^{p^j} = \mathrm{Id}$. By [52, theorem 2.5] this result is independent of the embedding ρ.

CHAPTER 1

Disconnected algebraic groups

In this chapter we deal with the structure of non connected algebraic groups, looking at their connected components as G°-varieties, where G° acts by conjugation. Moreover, in Section 1.1, under further assumptions on G°, we show that it is not restrictive to consider the case in which G is of the form $G^\circ \rtimes \langle\theta\rangle$, with $\theta \in \mathrm{Aut}(\Gamma)$. In Section 1.2, we recall from [21] the analogue in the non-connected case of parabolic subgroups and their Levi decomposition.

Let $x \in G$ and consider the multiplication morphism

$$\begin{aligned} \mu_x : \quad & G \longrightarrow G \\ & g \mapsto gx. \end{aligned}$$

Since μ_x is a homeomorphism, then the image of connected components are connected components, so $\mu_x(G^\circ) = G^\circ x$ is a connected component of G. Let D be the connected component of G containing x, then $x \in G^\circ x \cap D$, therefore $D = G^\circ x$.

It is a classical result [65, Proposition 2.2.1] that G° is a normal subgroup of finite index. So G/G° is a finite group of order m for some $m \in \mathbb{N}$. Then G has finitely many connected components. Moreover, by [65, Proposition 2.2.1], the connected components of G coincide with the irreducible components, so they are irreducible.

Every connected component of G is G°-stable. Therefore we can restrict the study of the conjugation action of G° on a single connected component $G^\circ x$ for $x \in G$. We observe that, since G/G° has finite order, there exists $r \in \mathbb{N}$ such that $x^r \in G^\circ$. To study the G°-conjugation action on a single connected component, it is enough to consider the case

$G = \langle G^\circ, x\rangle$. We observe that, since x normalizes G°, then $\langle G^\circ, x\rangle = \bigcup_{k=0}^{r-1} G^\circ x^k$ as sets.

We give now another point of view to study the action of G° on the coset $G^\circ x$.

Definition 1.1. Let K be a group, φ an automorphism of K. Consider the φ-twisted conjugation action $(k, y) \mapsto k \cdot_\varphi y := ky\varphi(k^{-1})$ of K on itself. The orbits $K \cdot_\varphi y$ for $y \in K$ are called *φ-twisted conjugacy classes*.

Twisted conjugacy probably occurs first in Gantmacher's paper [24] on automorphisms for $k = \mathbb{C}$. Also, [66] contains relevant results.

Let us consider the semidirect product $\widetilde{K} = K \rtimes \langle\varphi\rangle$, with the product defined by $\varphi k \varphi^{-1} = \varphi(k)$ for any k in K. Then the study of the K-conjugation action on the coset $K\varphi$ is equivalent to the study of φ-twisted conjugacy in K: for y, k in K one has $ky\varphi(k^{-1}) = ky\varphi k^{-1}\varphi^{-1}$, hence $(K \cdot_\varphi y)\varphi = K \cdot y\varphi$, where $K \cdot y\varphi$ is the K-conjugacy class of $y\varphi$ (in $K\varphi$).

We consider $K = G^\circ$, $\varphi = c_x$ and $\widetilde{G} = G^\circ \rtimes \langle c_x\rangle$. Studying c_x-twisted conjugacy classes in G° is equivalent to studying G°-conjugacy classes in $\widetilde{G}$.

Lemma 1.2. Let $x \in G$ and let G° act by conjugation on $\widetilde{G}$ and G. Then

(a) the varieties $G^\circ c_x$ and $G^\circ x$ are isomorphic as G°-varieties, where the action of G° is conjugation on $\widetilde{G}$ and G respectively;

(b) if $h \in G^\circ$, then right translation by h induces a G°-equivariant isomorphism between the $(c_h \circ c_x)$-twisted conjugacy class of x and the c_x-twisted conjugacy class of xh.

Proof. The isomorphism

$$\rho : G^\circ c_x \longrightarrow G^\circ x$$

$$hc_x \mapsto hx$$

is G°-equivariant. Indeed, for $h, g_0 \in G^\circ$,

$$\rho(g_0 h c_x g_0^{-1}) = \rho(g_0 h c_x(g_0^{-1}) c_x) =$$

$$g_0 h c_x(g_0^{-1}) x = g_0 h x g_0^{-1} x^{-1} x = g_0 h x g_0^{-1} = g_0 \rho(hc_x) g_0^{-1},$$

giving (a). The proof of (b) is analogue. $\square$

1.1 Semisimple non connected algebraic groups

Through this section $G = \langle G^\circ, x\rangle$ and G° is assumed to be semisimple. By Lemma 1.2, we can choose the representative x of its coset so that c_x is easy to deal with. Now we study $\mathrm{Aut}(G^\circ)$ to detect a good choice for x.

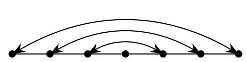
We recall that all the Borel subgroups of G° are conjugate in G° [52, Theorem 6.4 (a)] and all the maximal tori contained in a Borel subgroup are conjugate by elements of the Borel subgroup [52, Theorem 4.4 (b)]. Since $c_x \in \mathrm{Aut}(G^\circ)$, there holds $xBx^{-1} = B'$ with B' a Borel subgroup of G°. Thus there exists $h \in G^\circ$ such that $hxBx^{-1}h^{-1} = B$. Moreover $hxTx^{-1}h^{-1} = T'$ where T' is a maximal torus of B, hence there exists $b \in B$ such that $bhgTg^{-1}h^{-1}b^{-1} = T$. So the pair torus/Borel $(c_x(T), c_x(B))$ in G°, is conjugate to the pair torus/Borel (T, B) by an element $\overline{g} = bhg \in G^\circ$. Therefore the morphism $\widetilde{\theta} = c_{\overline{g}^{-1}} \circ c_x$ is an automorphism of G° fixing the pair torus/Borel (T, B).

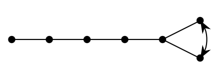
By [52, Theorem 11.11 (b)], $\widetilde{\theta}$ induces an automorphism of Γ. We assume in addition that G° is simply connected or of adjoint type. By [52, Theorem 11.12], an element $\theta \in \mathrm{Aut}(\Gamma)$ can be lifted to a quasi-semisimple automorphism $\widetilde{\theta}_1$ of G° such that $\mathrm{ord}\,\widetilde{\theta}_1 = \mathrm{ord}\,\theta$. We observe that by [52, Theorem 11.11 (b)], there exists $\overline{h} \in G^\circ$ such that $(c_{\overline{h}}(T), c_{\overline{h}}(B)) = (T, B)$ and $\widetilde{\theta}_1 = c_{\overline{h}} \circ \widetilde{\theta}$. Moreover $\overline{h} \in G^\circ$ fix T and B, hence $\overline{h} \in T$.

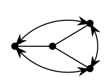
Thus, up to changing the representative of the coset $G^\circ x$, we may assume that c_x is the lift of some $\theta \in \mathrm{Aut}(\Gamma)$ with the same order of θ. Therefore, when G° is simply connected or of adjoint type, we can restrict the study of the action on a disconnected group of the form $G^\circ \rtimes \langle\theta\rangle$.

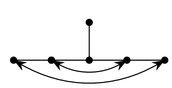
We recall the classification of the automorphisms of irreducible Dynkin diagrams.

Proposition 1.3. Let Γ be an irreducible Dynkin diagram. For $n \in \mathbb{N}$, the only non trivial automorphisms in $\mathrm{Aut}(\Gamma)$ are the following:

A_n  $\mathrm{Aut}(A_n) = \mathbb{Z}/2\mathbb{Z}$, for $n \geq 2$;

D_n  $\mathrm{Aut}(D_n) = \mathbb{Z}/2\mathbb{Z}$, for $n > 4$;

D_4  $\mathrm{Aut}(D_4) = S_3$;

E_6  $\mathrm{Aut}(E_6) = \mathbb{Z}/2\mathbb{Z}$.

Example 1.4. Let $n \geq 3$ and let $\widetilde{\tau}$ be the involution of $SL(n)$, defined as follows

$$\widetilde{\tau} : SL(n) \longrightarrow SL(n)$$

$$X \mapsto J\ {}^tX^{-1}J,$$

with $J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$, and ${}^tX^{-1}$ the transposed of the inverse of X.

Let $T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}$ be the maximal torus in $SL(n)$ consisting of diagonal matrices, and $B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \right\}$ be the Borel subgroup of upper triangular matrices, see [52, Example 6.7 (2)].

Then $\widetilde{\tau}$ preserves B and T. So $\widetilde{\tau}$ induces an automorphism τ of the Dynkin diagram of $SL(n)$, namely the non-trivial automorphism of A_n from Proposition 1.3.

1.2 Levi decomposition

In the context of connected reductive algebraic groups, the parabolic subgroups and their Levi decomposition play an important role in the study of algebraic groups and their representation theory.

There are similar notions in the non connected case.

Through this section we assume G° to be reductive. In this section we recall some results that allow us to have an analogue of the Levi decomposition of parabolic subgroups in the non connected case. For a complete description see [21, Section 1]. For T a maximal torus in a Borel subgroup B of G°, we consider the subgroups $N_G(B)$ and $N_G(B) \cap N_G(T)$ as non-connected analogues of B and T respectively. Since a Borel subgroup is the semidirect product of its unipotent radical and a maximal torus, one can see that $G^\circ \cap N_G B \cap N_G T = T$.

Similarly to the connected case we have the following (see [66, 7.2, 7.5] for the general case).

Proposition 1.5. Let G be a linear algebraic group and let $g \in G$. Then there exists a Borel subgroup B' of G° such that $g \in N_G B'$. Moreover, if $g \in G$ is semisimple, there exists a torus $T' \subseteq B'$ such that $g \in N_G T'$.

Let P be a parabolic subgroup of G° with Levi decomposition $P = LU_P$ where L is a Levi subgroup of P and U_P is the unipotent radical of P. We refer to P and L as a *pair Levi/parabolic* subgroups of G° and we denote it by (L, P).

Similarly to the pairs torus/Borel, we consider the normalizer of P in G and the normalizer of L in G, for the pair Levi/parabolic subgroup (L, P).

Remark 1.6. We claim that $N_G L \cap U_P = \{\mathrm{Id}\}$. Indeed, we recall that the subgroup P is the semidirect product of L and its unipotent radical U_P, which is a normal subgroup in P. It holds that $N_{U_P}(L) \leq C_{U_P}(L)$, in fact if $u \in N_{U_P}(L)$ and $l \in L$, then $ulu^{-1} = l' \in L$, for some $l' \in L$. By the normality of U_P, there exists $u' \in U_P$ such that $ul = lu'$. This implies $lu' = ul = l'u$, so $l = l'$ and $u = u'$, giving commutativity. Let now T' be a maximal torus of L. Then $N_{U_P}(L) \leq C_{U_P}(L) \leq C_{U_P}(T') = U_P \cap T' = 1$. Therefore $N_G L \cap U_P = \{\mathrm{Id}\}$.

Lemma 1.7. Let (L, P) be a pair Levi/parabolic subgroups and let U_P be the unipotent radical of P. Then

(a) $N_G P \cap N_G L \cap G^\circ = L$;

(b) $(N_G P \cap N_G L)^\circ = L$.

Proof. Since P is a parabolic subgroup of G°, $N_{G^\circ} P = P$. Thus

$$N_G P \cap N_G L \cap G^\circ = P \cap N_{G^\circ} L.$$

Let $p \in P$ be such that $pLp^{-1} = L$. We show that $p \in L$. By the Levi decomposition of P, there exists $l \in L$ and $u \in U_P$ such that $p = lu$. Then $luLu^{-1}l^{-1} = L$, so

$$uLu^{-1} = l^{-1}Ll = L.$$

Thus $u \in N_{G^\circ} L \cap U_P$. By Remark 1.6, the element u is trivial, therefore $p \in L$.
Now we prove (b). The Levi subgroup L is contained in $N_G P \cap N_G L$ and L is connected. Then

$$L \subseteq (N_G P \cap N_G L)^\circ \subseteq N_G P \cap N_G L \cap G^\circ.$$

By (a), the equality holds. □

Furthermore there is an analogue of the Levi decomposition.

Proposition 1.8. [21, Proposition 1.5]. Let (L, P) be a pair Levi/parabolic subgroups of G°, let U_P be the unipotent radical of P. Then

$$N_G P = (N_G L \cap N_G P) \ltimes U_P.$$

□

Proposition 1.9. Let P be a parabolic subgroup of G° with Levi decomposition LU_P. Let D be a connected component of G such that
$N_G P \cap N_G L \cap D \neq \emptyset$, and let $h \in N_G P \cap N_G L \cap D$. Then

$$N_G P \cap N_G L \cap D = Lh, \tag{1.1}$$

and it is a connected component of $N_G P \cap N_G L$.

Proof. We observe that $D = G^\circ h$. By Lemma 1.7, the group $N_G P \cap N_G L \cap G^\circ = L$, so

$$Lh \subseteq N_G P \cap N_G L \cap G^\circ h = (N_G P \cap N_G L \cap G^\circ)h = Lh.$$

So $N_G P \cap N_G L \cap D = Lh$. By Lemma 1.7 (b) it is a connected component of $N_G P \cap N_G L$. □

Proposition 1.10. Let L be a Levi subgroup of a parabolic subgroup P of G°. Then

$$(N_G L)^\circ = L.$$

Proof. It is evident that $L \subseteq (N_G L)^\circ$. By [65, Proposition 2.2.1 (iii)], to show the other inclusion it is enough to prove that L has finite index in $N_G L$. The index of L in $N_G L$ can be expressed as the product of the index of $(N_G L)^\circ$ in $N_G L$, which is finite ([65, Proposition 2.2.1 (i)]), and the index of L in $N_{G^\circ} L$, which is also finite according to [52, Corollary 12.11], giving the claim. □

By the characterization of the unipotent radical of an algebraic group in [52, p. 41], U_P is characteristic in P, and so U_P is normalized by $N_G P$. Thus for all $x \in N_G P$ the coset xU_P is U_P-stable.

Remark 1.11. Let $s \in N_G P$ be a semisimple element. Then the automorphism $c_s : G^\circ \longrightarrow G^\circ$ such that $h \mapsto shs^{-1}$ is a semisimple automorphism of G°. Since c_s is semisimple, it is quasi-semisimple [66, 7.5]. By [21, Proposition 1.11 (i)], applied to the quasi-semisimple automorphism c_s, there exists a Levi subgroup L' of P such that $c_s(L') = L'$, so that $s \in N_G L'$.

Adapting the proof of [39, Proposition 3.15], we show the next proposition.

Proposition 1.12. Let P be a parabolic subgroup of G° with Levi decomposition $P = LU_P$. Let $h \in N_G P \cap N_G L$. Then the semisimple parts of the elements in hU_P are all U_P-conjugate.

Proof. Let $a = hu \in hU_P \subseteq N_G P$ and let $a = a_s a_u$ be its Jordan decomposition in $N_G P$. By Remark 1.11, the semisimple part a_s normalizes a Levi subgroup L' of P. Now, hU_P is U_P-stable, so since L' is U_P-conjugate to L, conjugating a by some element in U_P, we may assume $a_s \in N_G P \cap N_G L$. Let π be the projection of the semidirect product $(N_G P \cap N_G L)\, U_P$ onto $N_G P \cap N_G L$ (a homomorphism of algebraic groups). Then $h = \pi(a)$ and $h_s = \pi\left(a\right)_s = \pi(a_s)$. Since $a_s \in N_G P \cap N_G L$, we have $\pi\left(a_s\right) = a_s$ so $a_s = h_s$.

□

CHAPTER 2

Jordan classes in disconnected reductive groups

In [39] G. Lusztig introduced a finite stratification of a non connected reductive group G. We refer to the strata as Jordan classes.

These Jordan classes have good geometrical properties, they are locally closed, irreducible and smooth subvarieties of G.

Each Jordan class is a union of G°-conjugacy classes of the same dimension. Moreover the closure of a Jordan class is a union of Jordan classes.

In this chapter we give the definition of a Jordan class and we prove some of their properties. Furthermore we study the closure and the regular closure of a Jordan class.

2.1 Definition and basic properties

Let a be an element of G with Jordan decomposition $a = a_s a_u$. Following [39, 2.1], we set

$$T(a) = (Z(C_G(a_s)^\circ) \cap C_G(a_u))^\circ = (Z(C_{G^\circ}(a_s)^\circ) \cap C_{G^\circ}(a_u))^\circ$$

By [66, Corollary 9.4], $C_{G^\circ}(a_s)^\circ$ is a reductive group, so $Z(C_{G^\circ}(a_s))^\circ$ is a torus in G° commuting with a_s. Hence $T(a)$ is a closed connected subgroup of a torus, so $T(a)$ is a torus in G° commuting with a_s and a_u.

We consider the equivalence relation on G:

$$a \sim h \text{ if } \exists x \in G^\circ \text{ such that } T(xhx^{-1}) = T(a) \text{ and } xhx^{-1} \in T(a)a.$$

Definition 2.1. A *Jordan class* is an equivalence class for this relation, and we denote the Jordan class containing a by $J(a)$.

Remark 2.2. Let $h \in T(a)a$. Then

(a) $h_s = za_s$, for some $z \in T(a)$;

(b) $h_u = a_u$;

(c) $C_G(a_s)^\circ \subseteq C_G(h_s)^\circ$;

(d) If, in addition, $T(a) = T(h)$, then $C_G(a_s)^\circ = C_G(h_s)^\circ$.

Proof. Let $h \in T(a)a$. By construction $h = za$, with $z \in T(a)$ and, since $T(a)$ is a torus of G°, the element z is semisimple. By definition of $T(a)$, the element $z \in T(a)$ commutes with a_s and a_u, so za_s is semisimple and $za_s = h_s$ and $h_u = a_u$, and this shows (a) and (b).

We prove (c). Let $x \in C_G(a_s)^\circ$. Since $z \in T(a) \subseteq Z(C_G(a_s)^\circ)$, z commutes with x. Hence, using (a) and the commutativity of z with x,

$$h_s = za_s = zxa_sx^{-1} = xza_sx^{-1} = xh_sx^{-1}.$$

Thus $C_G(a_s)^\circ \subseteq C_G(h_s)^\circ$.

We prove (d). Assuming now $T(a) = T(h)$, so $a = z^{-1}h$, with $z^{-1} \in T(a) = T(h)$. Applying (a), (b) and (c) to $a \in T(h)h$, we get $C_G(h_s)^\circ \subseteq C_G(a_s)^\circ$. Therefore $C_G(a_s)^\circ = C_G(h_s)^\circ$.

$\square$

We set

$$(T(a)a)^\bullet = \{h \in T(a)a \mid T(h) = T(a)\}. \tag{2.1}$$

Lemma 2.3. With the notation above, the following holds:

(a) $(T(a)a_s)^{\mathrm{reg}} = \{h_s \in T(a)a_s \mid C_G(h_s)^\circ = C_G(a_s)^\circ\}$;

(b) $(T(a)a)^\bullet = (T(a)a_s)^{\mathrm{reg}}a_u$.

Proof. By Remark 2.2(c), we have $C_G(a_s)^\circ \subseteq C_G(h_s)^\circ$ for all $h_s \in T(a)a_s$. So $\dim(C_G(a_s)^\circ) \leq \dim(C_G(h_s)^\circ)$ for all $h \in T(a)a_s$. Therefore

$$(T(a)a_s)^{\mathrm{reg}} = \{h_s \in T(a)a_s \mid C_G(h_s)^\circ = C_G(a_s)^\circ\},$$

giving (a).
We prove (b). Let $h \in (T(a)a)^{\bullet}$. By (a) and Remark 2.2(d), we see that $h \in (T(a)a_s)^{\mathrm{reg}}a_u$. Conversely, let $y \in (T(a)a_s)^{\mathrm{reg}}a_u$. There is $z \in T(a)$ such that $y = za_sa_u$, so, by Remark 2.2(a),(b), we have $y_s = za_s$ and $y_u = a_u$. Then, by hypothesis, $y_s \in (T(a)a_s)^{\mathrm{reg}}$. So (a) gives $C_G(y_s)^\circ = C_G(a_s)^\circ$. Therefore

$$Z(C_G(y_s)^\circ) = Z(C_G(a_s)^\circ),$$

thus

$$T(y) = (Z(C_G(y_s)^\circ) \cap C_G(y_u))^\circ = (Z(C_G(a_s)^\circ) \cap C_G(a_u))^\circ = T(a).$$

and $y \in (T(a)a)^{\bullet}$. □

By Lemma 2.3 (b),

$$J(a) = G^\circ \cdot ((T(a)a_s)^{\mathrm{reg}}a_u) = G^\circ \cdot (T(a)a)^{\bullet}. \tag{2.2}$$

Remark 2.4. Let $J(a)$ be a Jordan class and let $a' \in J(a)$. By Remark 2.2(b)(d), we get $\dim(C_G(a)^\circ) = \dim(C_G(a')^\circ)$. Thus $\dim G^\circ \cdot a = \dim G^\circ \cdot a'$, so there exists $d \in \mathbb{N}$ such that $J(a) \subseteq G_{(d)}$.

We set

$$L(a) = C_{G^\circ}(T(a)).$$

Since $L(a)$ is the centralizer of the torus $T(a)$, by [52, Proposition 12.10], it is the Levi subgroup of some parabolic subgroup P of G°. Furthermore one may choose P so that $a \in N_G P$ ([39, 2.1 (a)]).

Remark 2.5. By construction, the torus $T(a) \subseteq Z(C_G(a_s)^\circ)$. Then any element of $T(a)$ commutes with any element of the centralizer $C_G(a_s)^\circ$. Therefore, by definition of $L(a)$,

$$C_G(a_s)^\circ \subseteq L(a). \tag{2.3}$$

Moreover $a \in N_G(L(a))$. In fact, if $t \in T(a)$, by construction of $T(a)$, the element t commutes with a, and also, if $l \in L(a)$, by definition of $L(a)$, the element l commutes with t, then

$$ala^{-1}tal^{-1}a^{-1} = altl^{-1}a^{-1} = t.$$

Thus $ala^{-1} \in L(a)$, that is, $a \in N_G L(a)$.

2.1.1 Isolated Jordan classes

An element $a \in G$ is called *isolated* in G if $L(a) = G^{\circ}$.

According to [39, 2.2] this is equivalent to any of the following conditions:

(i) $T(a) \subseteq Z(G^{\circ})$;

(ii) $T(a) = (Z(G^{\circ}) \cap C_G(a))^{\circ}$;

(iii) there is no proper parabolic subgroup P of G° with a Levi subgroup L such that $a \in N_G P \cap N_G L$ and $C_G(a_s)^{\circ} \subseteq L$;

(iv) there is no proper parabolic subgroup P of G° such that $a \in N_G P$ and $C_G(a_s)^{\circ} \subseteq P$.

An *isolated Jordan class* is a Jordan class in which every element or equivalently some element is isolated ([39, 3.3]). By equation (2.2), an isolated Jordan class is of the form $(Z(G^{\circ}) \cap C_G(a))^{\circ}(G^{\circ} \cdot a)$, with a isolated in G.

By [39, 2.2 (b)], the element $a \in G$ is isolated in $N_G(L(a))$, that is $L(a) = (N_G(L(a)))^{\circ}$.

Example 2.6. If $a = a_u$, then $T(a) = (Z(G^{\circ}) \cap C_G(a))^{\circ}$. In this case, $J(a)$ is isolated and it equals $J(a) = (Z(G^{\circ}) \cap C_G(a))^{\circ}(G^{\circ} \cdot a)$.

If, in addition, G° is a semisimple group, then $T(a)$ is trivial, so the Jordan class of a is the unipotent orbit $G^{\circ} \cdot a$.

We denote by $T_{N_G(L(a))}(a)$ the torus $(Z((C_{N_G(L(a))}(a_s))^{\circ}) \cap C_{N_G(L(a))}(a_u))^{\circ}$, defined as $T(a)$ replacing G by $N_G(L(a))$.

Proposition 2.7. The following properties hold:

(a) $T(a) = T_{N_G(L(a))}(a)$,

(b) $T_{N_G(L(a))}(a) = (Z(L(a))^{\circ} \cap C_{N_G(L(a))}(a))^{\circ}$.

(c) $T(a) = (Z(L(a))^{\circ} \cap C_{N_G(L(a))}(a))^{\circ}$.

Proof.

(a) This is [39, 2.1(d)].

(b) This is [39, 3.9].

(c) This easily follows from (a),(b).

□

Adapting the proof of [39, 1.22], we prove the next proposition.

Proposition 2.8. Let H be an algebraic group, let D' be a connected component of H, $h \in D'$, and $x \in \overline{H^\circ \cdot h}$. Then $x_s \in H^\circ \cdot h_s$.

Proof. Let $C = \{y \in D' \mid y_s \in H^\circ \cdot h_s\}$ and let

$$(H^\circ \cdot h)_s = \{f_s \in D' \mid f \in H^\circ \cdot h\}.$$

We observe that $(H^\circ \cdot h)_s$ is the orbit $H^\circ \cdot h_s$, so, by [39, 1.4 (e)], it is closed. Let $H \subseteq GL(n)$ be an embedding of algebraic groups for some n. Let Y be the semisimple class of $GL(n)$ containing $H^\circ \cdot h_s$ and let $Y' = \{a \in GL(n) \mid a_s \in Y\}$. Then Y' is the set in $GL(n)$ consisting out of the matrices that have characteristic polynomial equal to that of a fixed matrix in Y, so Y' is closed. We consider the morphism

$$\rho : Y' \longrightarrow Y$$

$$a \mapsto a_s.$$

Since $\rho^{-1}(H^\circ \cdot h_s) = \{g \in Y' \mid g_s \in H^\circ \cdot h_s\}$ is Zariski-closed in Y', then $C = \rho^{-1}(H^\circ \cdot h_s) \cap D'$ is closed in D'. So, since $H^\circ \cdot h \subseteq C$, we have $\overline{H^\circ \cdot h} \subseteq C$. □

2.2 Induction of orbits

In this section we recall the induction procedure of G°-orbits in G. The induction of unipotent orbits in a disconnected group is reported in [60, II, 3], generalizing [49].

Definition 2.9. Let H be a closed subgroup of G, and Y an H-variety. We define

$$G \times^H Y := G \times Y / \sim$$

where $(a, y) \sim (a', y')$ if $\exists h \in H$ such that $ah = a'$ and $h^{-1} \cdot y = y'$. We denote the elements of $G \times^H Y$ by $[(a, y)]$.

There is an action of G on $G \times^H Y$ given by

$$b * [(a,y)] = [(ba,y)] \quad \text{for } b \in G,\ [(a,y)] \in G \times^H Y.$$

Let X be a G-variety, and assume that Y is an H-stable subvariety of X. Let $Z = \left\{(aH, z) \in G/H \times G \cdot Y \mid a^{-1} \cdot z \in Y\right\} \subseteq G/H \times X$. It is a G-variety with action

$$b * (aH, z) = (baH, b \cdot z) \quad \text{for } b \in G,\ (aH, z) \in Z.$$

The next lemma follows from [58, 3.7 Lemma 1].

Lemma 2.10. Let X be a G-variety, let $H \leq G$ and Y an H-stable subvariety of X. Let $Z = \left\{(aH, z) \in G/H \times G \cdot Y \mid a^{-1} \cdot z \in Y\right\} \subseteq G/H \times X$. Then

$$\psi : G \times^H Y \longrightarrow Z$$
$$[(a,y)] \mapsto (aH, a \cdot y).$$

is a well defined G-equivariant isomorphism of varieties.

Let (L, P) be a pair Levi/parabolic subgroups of G° and let $g \in N_G P \cap N_G L$. We observe that $\overline{(L \cdot g)} U_P$ is contained in the semidirect product $(N_G P \cap N_G L) U_P$.
Let $\pi : (N_G P \cap N_G L) U_P \longrightarrow N_G P \cap N_G L$ be the projection morphism, it is surjective. We observe that $\overline{L \cdot g} U_P = \pi^{-1}(\overline{L \cdot g})$, so $\overline{L \cdot g} U_P$ is closed. Moreover $\overline{L \cdot g} U_P$ is isomorphic, as variety, to the product of $\overline{L \cdot g}$ and U_P that are both irreducible, so $\overline{L \cdot g} U_P$ is irrreducible. Furthermore $\overline{L \cdot g}$ is L-stable, so $\overline{(L \cdot g)} U_P$ is P-stable. We consider $G^\circ \times^P \overline{(L \cdot g)} U_P$.

With the next proposition we can define the induction of orbits in reductive non-connected algebraic groups. The definition that we give here is the generalization of the definition in [60, II, 3] that describes the induction for a unipotent orbit.

Proposition 2.11. Let (L, P) be a pair Levi/parabolic subgroups of G° and let $g \in N_G P \cap N_G L$, with Jordan decomposition $g = su$. Let ϕ be the G°-equivariant morphism:

$$\phi : G^\circ \times^P \overline{L \cdot g} U_P \to G \tag{2.4}$$
$$[(h,y)] \mapsto hyh^{-1},$$

where the action of G° on G is given by conjugation. Then the image of ϕ is the closure of a single orbit for the conjugation action of G°.

Proof. As we observed above, $\overline{L \cdot g}U_P$ is P-stable, irreducible and it is closed in $(N_G P \cap N_G L)U_P$. Then we can apply [30, 0.15] to $\overline{L \cdot g}U_P$, so Imϕ is irreducible and closed. Since Imϕ is also G°-stable, it is a union of G°-orbits. We claim that these orbits are finitely many. Each of these orbits is represented in $\overline{L \cdot g}U_P$ by construction. We show that $\overline{L \cdot g}U_P$ meets finitely many G°-orbits. Let $x \in \overline{L \cdot g}U_P$. By Remark 1.6, the subgroup $N_G L \cap U_P$ is trivial. Since $N_G L$ is closed in G, then $\overline{L \cdot g}$ is contained in $N_G L$, hence $\overline{L \cdot g} \cap U_P$ is trivial. Therefore there exists a unique $y \in \overline{L \cdot g}$ such that $x \in yU_P$. By Proposition 1.12 the semisimple parts x_s and y_s are U_P-conjugate, and $y_s \in \overline{L \cdot g}$. By Proposition 2.8, $y_s \in L \cdot s$, where s is the semisimple part of g. Thus the G°-orbits of the elements in $\overline{L \cdot g}U_P$ are in bijection with the unipotent orbits of $C_G(s)$, which are finitely many by [39, 1.15]. Since Imϕ is closed, G°-stable and irreducible, it is the closure of a single G°-orbit. □

The image of ϕ from equation (2.4) is the closure of a single orbit. Let D_L be the connected component of $N_G P \cap N_G L$ containing g, and let $D = G^\circ g$. We call this orbit the induced orbit from D_L to D of the orbit $L \cdot g$, and we denote it by

$$\operatorname{Ind}_{D_L}^{D}(L \cdot g). \tag{2.5}$$

By Lemma 0.2, we have that $(gU_P)^{\mathrm{reg}}$ is open in gU_P, therefore it is irreducible. Moreover, $G^\circ \cdot (gU_P)^{\mathrm{reg}}$ is the image through ϕ of $G^\circ \times^P (gU_P)^{\mathrm{reg}}$, so $G^\circ \cdot (gU_P)^{\mathrm{reg}}$ is irreducible. Since $G^\circ \cdot (gU_P)^{\mathrm{reg}}$ is contained in the image of ϕ, it is union of finitely many G°-orbits. Therefore it consists of one G°-orbit. By Lemma 0.2, the subvariety $G^\circ \cdot (gU_P)^{\mathrm{reg}}$ is open in $\overline{\operatorname{Ind}_{D_L}^{D}(L \cdot g)}$. Since $\operatorname{Ind}_{D_L}^{D}(L \cdot g)$ is open in its closure, it intersects the open subset $G^\circ \cdot (gU_P)^{\mathrm{reg}}$ which is also a single G°-orbit, thus

$$\operatorname{Ind}_{D_L}^{D}(L \cdot g) = G^\circ \cdot (gU_P)^{\mathrm{reg}}. \tag{2.6}$$

Our next goal is to describe the induced orbits in terms of unipotent induced orbits. We retain the notation of Proposition 2.11, so $g = su \in D \cap N_G P \cap N_G L$, with (L, P) a pair Levi/parabolic of G°. By [39, 1.12] $P \cap (G^s)^\circ$ is a parabolic subgroup of $(G^s)^\circ$, so it is connected. Hence $P^{s\circ} = P \cap (G^s)^\circ$ and it has Levi decomposition $(L^s)^\circ (U_P^s)$. So we can consider $\operatorname{Ind}_{(L^s)^\circ h}^{(G^s)^\circ h}((L^s)^\circ \cdot h)$ for the $(L^s)^\circ$-orbit of an element $h \in N_{G^s}(P^s)^\circ \cap N_{G^s}(L^s)^\circ$. Note that $(G^s)^\circ h$ is the connected component of G^s containing h, and $(L^s)^\circ h$ is the connected component of $N_{G^s}((P^s)^\circ) \cap N_{G^s}((L^s)^\circ)$ containing h, as observed in Proposition 1.9.

The following Lemma is a variation of [11, Lemma 4.5], with similar proof.

Lemma 2.12. Let $g \in D$ and $g \in N_G P \cap N_G L$, with (L, P) a pair Levi/parabolic subgroups of G°. Let D_L be the connected component of $N_G P \cap N_G L$ containing g. Let $g = su$ be the Jordan decomposition of g. Then

$$\mathrm{Ind}_{D_L}^{D}(L \cdot su) \cap suU_P^s \neq \emptyset.$$

Proof. Let $\rho : U_P \times suU_P^s \longrightarrow U_P \cdot (suU_P^s)$ be the dominant morphism mapping (v, x) to vxv^{-1}. By [55, I, Corollary 1], there exists an open subset V' of $U_P \cdot (suU_P^s)$ such that $\dim U_P \cdot (suU_P^s) = \dim U_P + \dim U_P^s - \dim \rho^{-1}(y')$ for all $y' \in V'$. If $y' \in V' \cap u' suU_P^s u'^{-1}$, then $y := u'^{-1} y' u' \in suU_P^s$ and, by U_P-equivariance,

$$\dim \rho^{-1}(y) = \dim \rho^{-1}(y') = \dim U_P + \dim U_P^s - \dim U_P \cdot (suU_P^s). \tag{2.7}$$

Let us consider the Levi decomposition $(P^s)^\circ = (L^s)^\circ(U_P^s)$. Observe that $su \in N_{G^s}((P^s)^\circ) \cap N_{G^s}((L^s)^\circ)$, so, by Proposition 1.12, the elements in suU_P^s have semisimple part conjugate to s by an element of U_P^s, hence their semisimple parts are all equal to s. Let $(v, x) \in \rho^{-1}(y)$. Then $vxv^{-1} = y$. Thus $y_s = s = x_s$ and $vsv^{-1} = vx_sv^{-1} = y_s = s$, so $v \in U_P^s$. Therefore

$$\rho^{-1}(y) = \{(v, x) \in U_P^s \times (suU_P^s) \mid v \cdot x = y\} =$$

$$= \{(v, v^{-1} \cdot y) \in U_P^s \times (U_P^s \cdot y) \mid v \in U_P^s\} \cong U_P^s$$

By (2.7), we have $\dim(U_P \cdot (suU_P^s)) = \dim U_P$, hence the inclusion $\overline{U_P \cdot (suU_P^s)} \subseteq suU_P$ is an equality and so $U_P \cdot (suU_P^s)$ is dense in suU_P. By Lemma 0.2, $(suU_P)^{reg}$ is open in (suU_P). Thus $U_P \cdot (suU_P^s) \cap (suU_P)^{reg} \neq \emptyset$.

□

The next proposition is the generalization to the non-connected case of [11, Proposition 4.6], showing how to reduce induction of an orbit to the induction of the unipotent part.

Proposition 2.13. Let $g \in D$ and $g \in N_G P \cap N_G L$, with (L, P) a pair Levi/parabolic subgroups of G°. Let D_L be the connected component of $N_G P \cap N_G L$ containing g. Let $g = su$ be the Jordan decomposition of g. Then

$$\mathrm{Ind}_{D_L}^{D}(L \cdot su) = G^\circ \cdot s(\mathrm{Ind}_{(L^s)^\circ u}^{(G^s)^\circ u}((L^s)^\circ \cdot u)).$$

Proof. By Lemma 2.12, $\mathrm{Ind}_{D_L}^D(L\cdot su)\cap suU_P^s\neq\emptyset$, hence $(suU_P^s)^{\mathrm{reg}}\subseteq(suU_P)^{\mathrm{reg}}$, so

$$s(\mathrm{Ind}_{(L^s)^\circ u}^{(G^s)^\circ u}((L^s)^\circ\cdot u))=(G^s)^\circ\cdot(suU_P^s)^{\mathrm{reg}}\subseteq G^\circ\cdot(suU_P)^{\mathrm{reg}}=\mathrm{Ind}_{D_L}^D(L\cdot su).$$

Thus $\mathrm{Ind}_{D_L}^D(L\cdot su)=G^\circ\cdot s(\mathrm{Ind}_{(L^s)^\circ u}^{(G^s)^\circ u}((L^s)^\circ\cdot u))$.

$\square$

Let w be a representative of $\mathrm{Ind}_{(L^s)^\circ u}^{(G^s)^\circ u}((L^s)^\circ\cdot u)$. By Proposition 2.13, $\mathrm{Ind}_{D_L}^D(L\cdot su)=G^\circ\cdot sw$. Hence

$$\begin{aligned}\operatorname{codim}_D\mathrm{Ind}_{D_L}^D(L\cdot su)&=\operatorname{codim}_D G^\circ\cdot sw=\dim((G^s)^\circ\cap G^{w\circ})=\\&=\dim(G^s)^\circ-\dim(G^s)^\circ\cdot w=\operatorname{codim}_{(G^s)^\circ u}(G^s)^\circ\cdot w.\end{aligned}\tag{2.8}$$

Proposition 2.14. With the notation as above

$$\operatorname{codim}_{D_L}(L\cdot g)=\operatorname{codim}_D\mathrm{Ind}_{D_L}^D(L\cdot g)$$

Proof. We have the following equalities

$$\begin{aligned}&\operatorname{codim}_D\mathrm{Ind}_{D_L}^D(L\cdot su)=\operatorname{codim}_{(G^s)^\circ u}\mathrm{Ind}_{(L^s)^\circ u}^{(G^s)^\circ u}((L^s)^\circ\cdot u)=\\&=\operatorname{codim}_{(L^s)^\circ u}(L^s)^\circ\cdot u=\dim(L^s)^\circ u-\dim(L^s)^\circ\cdot u=\\&=\dim(L^s)^\circ-\dim(L^s)^\circ+\dim((L^s)^\circ\cap L^{u\circ})=\\&=\dim((L^s)^\circ\cap L^{u\circ})=\operatorname{codim}_{D_L}L\cdot su.\end{aligned}$$

where the first equality is (2.8), the second is [60, Proposition 3.2], and the others follow from properties of the Jordan decomposition and the equality $\dim D_L=\dim L$. $\square$

2.3 Closure and regular closure

In [39], G. Lusztig described the closure of a Jordan class. Building on this, we describe the closure and the regular closure of a Jordan class in terms of induced orbits.

We retain the notation of Subsection 2.1 and Subsection 2.2. Moreover from now on g always denote an element of D, with Jordan decomposition $g=su$. We set $L:=L(g)$, moreover P denotes a parabolic subgroup of G° with Levi subgroup L and such that $g\in N_GL\cap N_GP$. The existence of P is ensured by [39, 2.1 (a)]. By [39, 2.2 (b)], the element g is isolated in N_GL. We consider the (isolated) N_GL-Jordan class of g, and we denote it by S. Since S is isolated, $S = (Z(L) \cap C_{N_G L}(g))^\circ L \cdot g$, and $g \in N_G P$. Then $S \subseteq N_G P$. Furthermore, since $L = (N_G P \cap N_G L)^\circ$, from the description of isolated Jordan classes, we see that S is an isolated Jordan class also in $N_G P \cap N_G L$. We set $D_L := Lg = N_G P \cap N_G L \cap D$.

We observe that, since the set $\overline{S}U_P$ is contained in the semidirect product $(N_G P \cap N_G L)U_P$, and $\overline{S}$ is L-stable, then $\overline{S}U_P$ is P-stable.
This allows us to consider the variety

$$\widetilde{X} = \left\{(xP, h) \in G^\circ/P \times G \mid x^{-1}hx \in \overline{S}U_P\right\}.$$

By [39, Lemma 3.14] the image of $\widetilde{X}$ through the projection π on the second factor of $\widetilde{X}$ is $\overline{J(g)}$, so

$$\pi(\widetilde{X}) = \bigcup_{x \in G^\circ} x\overline{S}U_P x^{-1} = G^\circ \cdot \overline{S}U_P = \overline{J(g)},$$

and by [39, Proposition 3.15] it is a union of Jordan classes. Let $\widetilde{\phi}$ be the morphism

$$\begin{aligned} \tilde{\phi} : G^\circ \times^P \overline{S}U_P &\longrightarrow G \\ [(x,y)] &\mapsto xyx^{-1}. \end{aligned} \tag{2.9}$$

By the identification in Lemma 2.10, the image of the map $\tilde{\phi}$ is $\overline{J(g)}$.

Now, we look at the structure of isolated Jordan classes.

By [39, 3.3 (a)], any isolated Jordan class of a reductive group H is a single orbit for the action of $Z(H^\circ)^\circ \times H^\circ$ on H:

$$\begin{aligned} (Z(H^\circ)^\circ \times H^\circ) \times H &\to H \\ ((z,x),y) &\mapsto xzyx^{-1}. \end{aligned} \tag{2.10}$$

We apply it to the case $H = (N_G P \cap N_G L)$, so by Proposition 1.7 (b) $H^\circ = L$ and the isolated Jordan class S in $N_G P \cap N_G L$ considered above is

$$S = Z(L)^\circ(L \cdot g).$$

For the description of the regular closure of a Jordan class, we need a description of isolated classes in $N_G L$:

Lemma 2.15. With the notation as above, the following hold:

(a) $S = Z(L)^\circ([L, L] \cdot g)$;

(b) $\overline{S} = Z(L)^\circ(\overline{L \cdot g}) = Z(L)^\circ(\overline{[L, L] \cdot g})$;

(c) $S = T(g)(L \cdot g)$;

(d) $\overline{S} = T(g)(\overline{L \cdot g})$.

Proof.

(a) Obviously $Z(L)^\circ([L, L] \cdot g) \subseteq Z(L)^\circ(L \cdot g) = S$. Conversely, the subgroup $Z(L)^\circ$ is a characteristic subgroup of L, since $g \in N_G L$, we have
$g \in N_G(Z(L)^\circ)$. In addition L is reductive, so $L = Z(L)^\circ[L, L]$. Hence, if $x \in Z(L)^\circ(L \cdot g)$, there exist $z_1, z_2 \in Z(L)^\circ$ and $l \in [L, L]$ such that $x = z_1 z_2 l g l^{-1} z_2^{-1}$. Since $z_2^{-1} \in Z(L)^\circ$ and $lgl^{-1} \in N_G(Z(L)^\circ)$,

$$x = z_3 l g l^{-1},$$

for some $z_3 \in Z(L)^\circ$. Therefore $Z(L)^\circ(L \cdot g) \subseteq Z(L)^\circ([L, L] \cdot g)$ and $S = Z(L)^\circ(L \cdot g) = Z(L)^\circ([L, L] \cdot g)$.

(b) The derived subgroup $[L, L]$ is characteristic in L. So if $g \in N_G([L, L])$, then $[L, L] \cdot g \subseteq [L, L]g$.
The morphism $m : Z(L)^\circ \times [L, L]g \longrightarrow Lg$ given by the multiplication is finite ([62, Prop 2.3.2, Theorem 2.4.9]) hence closed. Thus $Z(L)^\circ(\overline{[L, L] \cdot g}) = m(Z(L)^\circ \times \overline{[L, L] \cdot g})$ is closed in Lg, since
$Z(L)^\circ \times \overline{[L, L] \cdot g} \subseteq Z(L)^\circ \times [L, L]g$ is closed.

Clearly $\overline{[L, L] \cdot g} \subseteq \overline{Z(L)^\circ([L, L] \cdot g)} = \overline{S}$, so by $Z(L)^\circ$-stability of $\overline{S}$ we obtain

$$S = Z(L)^\circ([L, L] \cdot g) \subseteq Z(L)^\circ(\overline{[L, L] \cdot g}) \subseteq \overline{Z(L)^\circ([L, L] \cdot g)} = \overline{S}.$$

Since $Z(L)^\circ(\overline{[L, L] \cdot g})$ is closed, passing to closures gives

$$\overline{S} = Z(L)^\circ(\overline{[L, L] \cdot g}).$$

By point (a), the closure $\overline{S} = \overline{Z(L)^\circ([L, L] \cdot g)}$. Furthermore, from the $Z(L)^\circ$-stability of $\overline{S}$, we get

$$Z(L)^\circ(\overline{[L, L] \cdot g}) \subseteq Z(L)^\circ(\overline{L \cdot g}) \subseteq \overline{S} = Z(L)^\circ(\overline{[L, L] \cdot g}).$$

So all the inclusions are equalities.

(c) By [39, 1.21(d)], since $S \subseteq D_L$ and S is isolated, it is an orbit for the action (2.10) restricted to $(Z(L)^\circ \cap C_{N_G L}(g)) \times L$. By Remark 2.7 (c), it holds that $(Z(L)^\circ \cap C_{N_G L}(g))^\circ = T(g)$, so

$$S = T(g)(L \cdot g).$$

(d) By point (b), $Z(L)^\circ \overline{L \cdot g} = \overline{S}$. Hence, since $T(g) \subseteq Z(L)^\circ$,

$$T(g)\overline{L \cdot g} \subseteq Z(L)^\circ \overline{L \cdot g} = \overline{S}.$$

By [39, 1.21 (b)], for a L-stable subset X of D_L, if $T(g)X \subseteq X$ then $Z(L)^\circ X \subseteq X$. Let $X = T(g)\overline{L \cdot g}$. Obviously, $T(g)X \subseteq X$, so $Z(L)^\circ X \subseteq X$, that is $Z(L)^\circ T(g)\overline{L \cdot g} = Z(L)^\circ \overline{L \cdot g} = \overline{S} \subseteq T(g)\overline{L \cdot g}$. Thus $\overline{S} = T(g)\overline{L \cdot g}$.

□

Proposition 2.16. With the notation as above

$$\overline{J(g)} = \bigcup_{z \in T(g)} \overline{\mathrm{Ind}_{D_L}^{D}(L \cdot zg)}.$$

Proof. Let $z \in T(g)$ and let ϕ_z be the map

$$\phi_z : G^\circ \times^P \overline{L \cdot zg} U_P \longrightarrow D$$

$$[(a, x)] \mapsto axa^{-1}.$$

By Proposition 2.11, the image of ϕ_z is $\overline{\mathrm{Ind}_{D_L}^{D}(L \cdot zg)}$. By Lemma 2.15 (d)

$$G^\circ \times^P \overline{S} U_P = G^\circ \times^P \bigcup_{z \in T(g)} \overline{L \cdot zg} U_P = \bigcup_{z \in T(g)} G^\circ \times^P \overline{L \cdot zg} U_P$$

Let $\widetilde{\phi}$ be the map defined in (2.9). Then

$$\overline{J(g)} = \mathrm{Im}\widetilde{\phi} = \bigcup_{z \in T(g)} \mathrm{Im}\phi_z = \bigcup_{z \in T(g)} \overline{\mathrm{Ind}_{D_L}^{D}(L \cdot zg)}.$$

□

We study now the regular closure of $J(g)$, that is $\overline{J(g)}^{\mathrm{reg}}$. For the description of the regular closure of a Jordan class in a connected reductive algebraic group one can see [11, Proposition 4.8].

By Remark 2.4 there exists $d \in \mathbb{N}$ such that $J(g) \subseteq G_{(d)}$. Then, by Lemma 0.2, $\overline{J(g)} \subseteq \overline{G_{(d)}} \subseteq \bigcup_{k \leq d} G_{(k)}$, so $J(g) \subseteq \overline{J(g)}^{\text{reg}}$ thus

$$\overline{J(g)}^{\text{reg}} = G_{(d)} \cap \overline{J(g)}. \tag{2.11}$$

Remark 2.17. Recall from Remark 2.5 that $(G^s)^\circ \subseteq L$. Then $(L^{zs})^\circ = (G^s)^\circ$, for any $z \in Z(L)^\circ$. Indeed, $(G^s)^\circ \subseteq L \cap G^{zs} = L^{zs}$.
If $x \in (L^{zs})^\circ \subseteq L \cap G^{zs}$, then x commutes with zs and with z, so x also commutes with s. It follows that $x \in G^s$, thus $(L^{zs})^\circ \subseteq (G^s)^\circ$.

Proposition 2.18. With the notation as above

(a) $\overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} \operatorname{Ind}_{D_L}^{D}(L \cdot zg)$,

(b) $\overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} G^\circ \cdot zs \operatorname{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u)$.

Proof.

(a) Let $d \in \mathbb{N}$ be such that $J(g) \subseteq G_{(d)}$ and let $\mathcal{O}$ be a G°-orbit in $\overline{J(g)}$ of maximal dimension, i.e. $\dim \mathcal{O} = d$. By Proposition 2.16, there is $z \in T(g)$ such that $\mathcal{O} \subseteq \overline{\operatorname{Ind}_{D_L}^{D}(L \cdot zg)}$, so, by maximality, $\mathcal{O} = \operatorname{Ind}_{D_L}^{D}(L \cdot zg)$. Hence there exists an induced orbit in $\overline{J(g)}$ that has dimension d. We claim that all the induced G°-orbits from L-orbits in S have the same dimension. Since S is a Jordan class, the dimension of its L-orbits coincide. The claim follows from Proposition 2.14.

(b) Let $z \in T(g)$. By Remark 2.2(a)(b), we have the equalities $(zg)_s = zs$ and $(zg)_u = u$. Therefore, by Proposition 2.13 and point (a)

$$\overline{J(g)}^{\text{reg}} = \bigcup_{z \in T(g)} \operatorname{Ind}_{D_L}^{D}(L \cdot zg) = \bigcup_{z \in T(g)} G^\circ \cdot zs \operatorname{Ind}_{(L^{zs})^\circ u}^{(G^{zs})^\circ u}((L^{zs})^\circ \cdot u) =$$

$$= \bigcup_{z \in T(g)} G^\circ \cdot zs \operatorname{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u),$$

where the last equality follows from Remark 2.17.

$\square$

2.4 The poset of Jordan classes

Let J_1, J_2 be Jordan classes. In [41, 7.2 (c)], G. Lusztig describes equivalent conditions to $J_1 \subseteq \overline{J_2}$. In this section we describe when $J_1 \subseteq \overline{J_2}^{\mathrm{reg}}$ in terms of induced orbits.

We define a partial order on the set of Jordan classes:

$$J_1 \leq J_2 \text{ if and only if } J_1 \subseteq \overline{J_2}^{\mathrm{reg}}. \tag{2.12}$$

Let $g_i \in J_i$. Set $L_i = L(g_i)$ and let P_i be a parabolic subgroup of G° with Levi L_i and such that $g_i \in N_G P_i$. Let S_i be the isolated Jordan class of g_i in $N_G P_i \cap N_G(L_i)$, for $i \in \{1, 2\}$.

Let $g_1 = tv$ and $g_2 = su$ be the Jordan decompositions of g_1 and g_2 respectively.

Proposition 2.19. With the notation above if $J_1 \leq J_2$, then there exists $y \in G^\circ$ such that

(a) $v \in y \cdot (\mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u))$;

(b) $T(tv)t \subseteq y \cdot (T(su)s)$;

(c) $y \cdot (G^s)^\circ \subseteq (G^t)^\circ$.

Proof. Since $J_1 \leq J_2$, by Proposition 2.18,

$$J_1 \subseteq \bigcup_{z \in T(su)} G^\circ \cdot (zs\, \mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u)).$$

Then there exists $z \in T(su)$ and there exists $y \in G^\circ$ such that $tv \in y\cdot(zs\, \mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ\cdot u))$. By Lemma 2.12, there exists an element in $\mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ\cdot u)$ of the form uu' with $u' \in U_{P_2}^{zs}$. Therefore, since $\mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u)$ is a $(G^{zs})^\circ$-orbit, up to conjugation for an element in $(G^{zs})^\circ$, we may assume $tv = y \cdot zsuu'$. We observe that, since $z \in T(su)$, it is semisimple and commutes with s, so zs is semisimple. Moreover $zsuu' = uu'zs$ with uu' a unipotent element. So $t = (tv)_s = y \cdot (zsuu')_s = y \cdot zs$ and $v = y \cdot (tv)_u = y \cdot (zsuu')_u = y \cdot uu'$, giving (a).

Since $tv = y \cdot zsuu'$, by [39, 2.1], $T(tv) = y \cdot T(zsuu')$. Now we show that $T(zsuu') \subseteq T(su)$.

We observe that, since $(L_2^{zs})^\circ$ is a Levi subgroup of $(G^{zs})^\circ$, then $Z((G^{zs})^\circ) \subseteq Z((L_2^{zs})^\circ)$, so

$$T(zsuu') = (Z((G^{zs})^\circ) \cap C_G(uu'))^\circ \text{ and} \tag{2.13}$$

$$(Z((G^{zs})^\circ) \cap C_G(uu'))^\circ \subseteq (Z((L_2^{zs})^\circ) \cap C_G(uu'))^\circ = (Z((L_2^{zs})^\circ) \cap C_{L_2}(uu'))^\circ.$$

Let $l \in C_{L_2}(uu')$. Then $uu' = luu'l^{-1} = lul^{-1}lu'l^{-1}$, where $lul^{-1} \in N_G(L_2)$ and $lu'l^{-1} \in U_{P_2}$. Thus $u^{-1}lul^{-1} = u'lu'^{-1}l^{-1} \in N_G(L_2) \cap U_{P_2}$. By Remark 1.6, the subgroup $N_G(L_2) \cap U_{P_2}$ is trivial. So $lul^{-1} = u$. Thus

$$C_{L_2}(uu') \subseteq C_{L_2}(u) \subseteq C_G(u).$$

Combining (2.13) with Remark 2.17, we get

$$T(zsuu') \subseteq (Z((L_2^{zs})^\circ) \cap C_{L_2}(uu'))^\circ \subseteq (Z((G^s)^\circ) \cap C_G(u))^\circ = T(su).$$

Furthermore $z \in T(su)$, so $T(zsuu')zs \subseteq T(su)s$. Thus, using that $T(tv) = y \cdot T(zsuu')$, we get $T(tv)t = y \cdot (T(zsuu')zs) \subseteq y \cdot (T(su)s)$, proving (b).

By Remark 2.17, $y \cdot (G^s)^\circ = y \cdot (L^{zs})^\circ \subseteq y \cdot (G^{zs})^\circ = (G^t)^\circ$, giving (c).

□

Proposition 2.20. Let $d \in \mathbb{N}$. Then the sheets of $D_{(d)}$ are the regular closures of those Jordan classes in $D_{(d)}$ that are maximal with respect to the partial ordering defined in (2.12).

Proof. By Remark 2.4, we have that $D_{(d)}$ is union of Jordan classes. Since Jordan classes are finitely many, then $D_{(d)}$ is union of finitely many Jordan classes. So there exists I a finite index set such that $D_{(d)} = \bigcup_{i \in I} J_i$, where J_i is a Jordan class. Moreover

$$D_{(d)} = \overline{D_{(d)}} \cap D_{(d)} = \overline{\bigcup_{i \in I} J_i} \cap D_{(d)} = \bigcup_{i \in I} (\overline{J_i} \cap D_{(d)}).$$

By (2.11), $\overline{J_i}^{reg} = \overline{J_i} \cap D_{(d)}$, therefore $D_{(d)}$ is union of regular closures of finitely many Jordan classes. Since a Jordan class is irreducible, then its regular closure is also irreducible. We consider the partial order, defined in (2.12). Then, for any d, the maximal elements in this poset are the irreducible components of $D_{(d)}$, i.e., the sheets of G in D corresponding to the dimension d. □

2.5 Examples

Let $\mathbb{K}$ be an algebraically closed field of characteristic 2. In this section we analyse the Jordan classes in the semidirect product of $SL(n)$ with the group generated by τ, the non-inner automorphism of $SL(n)$ described in Example 1.4.

Since the order of τ is equal to $\operatorname{char}\mathbb{K}$, the element τ is a unipotent element of G.

Let B be the Borel subgroup of upper triangular matrices in $SL(n)$, let T be the torus of diagonal matrices contained in B, and let U be the unipotent radical of B. So $B = TU$.

Furthermore $\tau(T) = T$ and $\tau(B) = B$, so τ is a quasi semisimple element ([39, 1.4]). By [66, 8.2 pp 52], the group G^τ is connected and reductive. By [34, Theorem 1.2 (iii)], the reductive part of the centralizer of the unipotent element τ in $SL(n)$ is $C_{G^\circ}(\tau) = (G^\circ)^\tau = Sp(2\lfloor \frac{n}{2} \rfloor)$.

We analyse the Jordan classes in $D = G^\circ\tau$.

Remark 2.21. Let H be an algebraic group defined over a field of characteristic p, let σ be an automorphism of H°. Then, by [66, pp.51], σ is semisimple if and only if there exists an integer l coprime with p such that σ^l is a semisimple inner automorphism of H°, i.e., the action of σ^l is conjugation by a semisimple element of H°.
Let $x_0\tau \in G^\circ\tau$. Then $x_0\tau$ is semisimple if and only if there exists an odd integer l such that $(x_0\tau)^l \in G^\circ$ is semisimple. However, since the order of τ is 2, we have that $(x_0\tau)^l \in G^\circ\tau$. Therefore $G^\circ\tau$ contains no semisimple element.

Remark 2.22. By [61, Lemma 5], every element in $G^\circ\tau$ is G°-conjugate to an element $tu\tau$ with $t \in T^\tau$ and $u \in U$. We may ensure that $tu = ut$. Indeed, let $B' = T^\tau \ltimes U$ and let $h \in B'\tau$. By Remark 2.21, $h_s \in B'$ and $h_u = h'_u\tau \in B'\tau$. Observe that B' is a solvable group. Then by [29, 19.3], all maximal tori in B' are conjugate by elements in U. Hence, there exists $v \in U$ such that $t := vh_sv^{-1} \in T^\tau$. Thus $vhv^{-1} = vh_sv^{-1}vh'_u\tau(v^{-1})\tau = tvh'_u\tau(v^{-1})\tau$. Since $[t,\tau] = 1$ and $[t, vh'_u\tau v^{-1}] = 1$, setting $u := vh'_u\tau v^{-1}$ we see that every element of $G^\circ\tau$ is G°-conjugate to an element of the form $tu\tau$ with $t \in T^\tau$, $u \in U$ and $tu = ut$.

Let $x = tu\tau \in B'\tau$, with $t \in T^\tau$ and $u \in U$, with $tu = ut$, and let $k = \lfloor \frac{n}{2} \rfloor$.

An element $t \in T^\tau$ is of the form:

- if n is even, then

$$t = \operatorname{diag}(a_1, \dots, a_k, a_k^{-1}, \dots, a_1^{-1}), \text{ for } a_1, \dots, a_k \in \mathbb{K}^*, \tag{2.14}$$

- if n is odd, then

$$t = \operatorname{diag}(a_1, \dots, a_k, 1, a_k^{-1}, \dots, a_1^{-1}), \text{ for } a_1, \dots, a_k \in \mathbb{K}^*. \tag{2.15}$$

By [52, Example 6.7], T^τ is connected.
By [52, Corollary 14.10], t is regular if and only if $C_{SL(n)}(t)^\circ = T$. Thus t is regular if and only if $a_i \neq a_j$, $a_i \neq a_j^{-1}$ and $a_i \neq 1$ for all $i \neq j \in \{1, \dots, k\}$. So, if t is regular, the condition $tu = ut$ implies $u = \mathrm{Id}$, and $x = t\tau$.

We describe the Jordan class $J(t\tau)$ for t a regular semisimple element of G contained in T^τ.
By [52, Corollary 14.10], in this case $C_G(t)^\circ = T$. Therefore

$$T(t\tau) = (T \cap C_G(\tau))^\circ = (T^\tau)^\circ = T^\tau.$$

For $z \in T(t\tau)^{\mathrm{reg}}$ we have:

- $z = \mathrm{diag}(z_1, \dots, z_k, 1, z_k^{-1}, \dots, z_1^{-1})$ where $z_i \neq 0,\ z_i \neq z_j^{\pm 1}$, $\forall i, j \in \{1, \dots, k\}$, if $n = 2k + 1$;
- $z = \mathrm{diag}(z_1, \dots, z_k, z_k^{-1}, \dots, z_1^{-1})$ where $z_i \neq 0,\ z_i \neq z_j^{\pm 1}$, $\forall i, j \in \{1, \dots, k\}$, if $n = 2k$.

Then $J(t\tau) = SL(n) \cdot (T(t\tau)^{\mathrm{reg}}\tau)$.

Now we consider $x = tu\tau \in B'\tau$ with $tu = ut$, t not regular, and t as in (2.14) or (2.15). Then there exist entries of t that coincide. We can assume that the coinciding elements are among $a_1, \dots a_k$ and they are next to each other, because the other cases are conjugate to this one by a permutation matrix in $(G^\circ)^\tau$, so they produce the same Jordan class.
Suppose that there exist elements $a_1, \dots, a_{r-1} \in \mathbb{K}^*$ such that $a_i \neq a_j^{\pm 1}$ for all $i, j = 1, \dots, r$, $i \neq j$, and, up to conjugation,

$$t = \begin{pmatrix} a_1 \,\mathrm{Id}_{h_1} & & & & & & \\ & \ddots & & & & & \\ & & a_{r-1} \,\mathrm{Id}_{h_{r-1}} & & & & \\ & & & \mathrm{Id}_{h_r} & & & \\ & & & & a_{r-1}^{-1} \,\mathrm{Id}_{h_{r-1}} & & \\ & & & & & \ddots & \\ & & & & & & a_1^{-1} \,\mathrm{Id}_{h_1} \end{pmatrix},$$

with n even, $h_i \in \mathbb{N}$ and $2\sum_{i=1}^{r-1} h_i + h_r = n$, and

$$t = \begin{pmatrix} a_1 \operatorname{Id}_{h_1} & & & & & & \\ & \ddots & & & & & \\ & & a_{r-1} \operatorname{Id}_{h_{r-1}} & & & & \\ & & & \operatorname{Id}_{h_r} & & & \\ & & & & a_{r-1}^{-1} \operatorname{Id}_{h_{r-1}} & & \\ & & & & & \ddots & \\ & & & & & & a_1^{-1} \operatorname{Id}_{h_1} \end{pmatrix}, \tag{2.16}$$

with n odd, $h_i \in \mathbb{N}$ and $2\sum_{i=1}^{r-1} h_i + h_r = n$, where Id_{h_i} is the $h_i \times h_i$-identity matrix.
Let $z \in T(x)$. By [21, Theorem 1.8], the subgroup $T \cap C_G(t)^\circ$ is a maximal torus of $C_G(t)^\circ$. This implies $Z(C_G(t)^\circ)^\circ \subseteq T$. Thus, since, by construction, $T(x) \subseteq Z(C_G(t)^\circ)^\circ$ and $T(x) \subseteq C_G(u\tau)$, we have $z \in T \cap C_G(u\tau)$. Then the following equality holds:

$$u\tau = zu\tau z^{-1} = zu\tau(z^{-1})\tau,$$

where $\tau(z^{-1})$ indicates the image of z through the automorphism τ. Thus $zu\tau(z^{-1}) = u \in U$ and $z \in T$, looking at the projection from TU to T, we see that $z \in T^\tau$ and $[z, u] = 1$.

Furthermore, from the condition $z \in T(x) \subseteq Z(C_G(t)^\circ)^\circ$ it follows that there exist $z_1, \ldots, z_{r-1} \in \mathbb{K}^*$ such that

$$z = \operatorname{diag}(z_1 \operatorname{Id}_{h_1}, \ldots, z_{r-1} \operatorname{Id}_{h_{r-1}}, \operatorname{Id}_{h_r}, z_{r-1}^{-1} \operatorname{Id}_{h_{r-1}}, \ldots, z_1^{-1} \operatorname{Id}_{h_1}),$$

if n is even, where $h_i \in \mathbb{N}$ and $h_r \geq 0$ even, and

$$z = \operatorname{diag}(z_1 \operatorname{Id}_{h_1}, \ldots, z_{r-1} \operatorname{Id}_{h_{r-1}}, \operatorname{Id}_{h_r}, z_{r-1}^{-1} \operatorname{Id}_{h_{r-1}}, \ldots, z_1^{-1} \operatorname{Id}_{h_1}),$$

if n is odd, where $h_i \in \mathbb{N}$ and $h_r \geq 1$ odd.
Therefore, $z \in (T(x)t)^{\mathrm{reg}}$, for n even, if and only if

$$z_i \neq z_j^{\pm 1} \text{ if } i \neq j,\ z_i \neq 1,\ h_i \in \mathbb{N},\ h_r \geq 0 \text{ even },$$

and for n odd, if and only if

$$z_i \neq z_j^{\pm 1} \text{ if } i \neq j,\ z_i \neq 1,\ h_i \in \mathbb{N},\ h_r \geq 1 \text{ odd }.$$

Then $J(x) = SL(n) \cdot ((T(x)t)^{\mathrm{reg}} u\tau)$.

Lastly, as we saw in Remark 2.6, the class $J(u\tau)$ with $u\tau$ unipotent is the orbit $G^\circ \cdot u\tau$.

2.5.1 The case $n = 3$

In this section we analyse the case $n = 3$ and compute explicitly the poset of Jordan classes with respect to the partial order defined in Subsection 2.4.

By [61, Lemma 5], every element $x' \in SL(3)\tau$ is $SL(3)$-conjugate to an element $x = tu\tau$ with $t \in T^\tau$ and $u \in U$ such that $tu = ut$. Observe that a non-trivial element in T^τ is always regular when $n = 3$. Since if t is regular then $u = \mathrm{Id}$, we have only two possibilities for x, namely $x = t\tau$ or $x = u\tau$.

Let $x = t\tau$ with $t \in T^\tau$. Then, specializing (2.16) to $n = 3$,

$$(*) \qquad J(t\tau) = SL(3) \cdot \left(\left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \;\middle|\; a \neq 0, 1 \right\} \tau \right).$$

We analyse the isolated strata associated with $J(t\tau)$.
The torus $T(t\tau)$ is equal to T^τ, so in this case $L(t\tau) = C_{SL(3)}(T(t\tau)) = T$, $N_G T = T \rtimes \langle \tau \rangle$, $P = B$, $N_G P = B \rtimes \langle \tau \rangle$. Thus the isolated Jordan class S of $t\tau$ in $N_G L$ is

$$S = T(T \cdot t\tau) = T\tau.$$

Let $y\tau = \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \alpha^{-1}\beta^{-1} \end{pmatrix} \tau \in T\tau$. Then its Jordan decomposition is

$$(y\tau)_s = \begin{pmatrix} \alpha\sqrt{\beta} & & \\ & 1 & \\ & & \alpha^{-1}\sqrt{\beta}^{-1} \end{pmatrix} \text{ and } (y\tau)_u = \begin{pmatrix} \sqrt{\beta}^{-1} & & \\ & \beta & \\ & & \sqrt{\beta}^{-1} \end{pmatrix} \tau.$$

Thus the set $S^* = \{y\tau \in T\tau \mid C_G((y\tau)_s)^\circ \subseteq T\}$ is given by

$$\left\{ \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \alpha^{-1}\beta^{-1} \end{pmatrix} \tau \;\middle|\; \alpha, \beta \in \mathbb{K}^*,\ \alpha \neq \sqrt{\beta}^{-1} \right\}.$$

We now consider $x = u\tau$, with $u \in U$. By Remark 2.6, the torus $T(u\tau)$ is trivial, so the Jordan class of x is just the G°-orbit of $u\tau$. In this case the Jordan class is isolated.

The following Lemma lists all the possible Jordan classes $J(x)$ in $G\tau$ that are unipotent G°-orbits.

Lemma 2.23. There are only two unipotent $SL(3)$-orbits represented in $U\tau$:

- the orbit of τ,
- the orbit $SL(3) \cdot (u_1\tau)$ where $u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. Let $y\tau = x\tau(x^{-1})\tau \in G^\circ\tau$, and let $y_{i,j}$ be the entries of y. Computing the product $x\tau(x^{-1})$, we find that $y_{1,2} = y_{2,3}$, giving a necessary condition for $y\tau$ to lie in the G°-orbit of τ.

Hence, if $u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $u_1\tau \notin G^\circ \cdot \tau$.

Let $\overline{a} = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \in U$. By direct computation we see that:

- if $a_1 = a_3$, there is a $\overline{b} \in U$ such that $\overline{b}\overline{a}\tau(\overline{b}^{-1}) = \mathrm{Id}$. So $\overline{a}\tau \in G^\circ \cdot \tau$.
- if $a_1 \neq a_3$, there exists $\overline{b}' \in U$ such that $\overline{b}'\overline{a}\tau(\overline{b}'^{-1}) = u_1$. Then $\overline{a}\tau \in G^\circ \cdot u_1\tau$.

Therefore the only unipotent orbits in $G^\circ\tau$ are $G^\circ \cdot \tau$ and $G^\circ \cdot u_1\tau$.

□

The next theorem summarizes the results of this section.

Theorem 2.24. *The Jordan classes of $SL(3) \rtimes \langle\tau\rangle$ are*

- *$J(t\tau)$ as in (*) , with $t \in T^\tau$, $t \neq \mathrm{Id}$;*

- *$J(u_1\tau) = SL(3) \cdot u_1\tau$, where $u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;*

- *$J(\tau) = SL(3) \cdot \tau$.*

We analyse the poset of the Jordan classes.
By [60, II 2.21], there exists a unique minimal unipotent quasi-semisimple orbit in $G^\circ\tau$, that is $G^\circ \cdot \tau$.
Thus we have $\dim G^\circ \cdot u_1\tau > \dim G^\circ \cdot \tau$. So $\tau \notin G^\circ \cdot (\tau U)^{\mathrm{reg}}$, and the unipotent induced orbit $\mathrm{Ind}_{T\tau}^{G^\circ\tau}(T \cdot \tau)$ is $G^\circ \cdot u_1\tau$. By Proposition 2.18, $\mathrm{Ind}_{T\tau}^{G^\circ\tau}(T \cdot \tau) \subseteq \overline{J(t\tau)}^{\mathrm{reg}}$, so $J(u_1\tau) = G^\circ \cdot u_1\tau \subseteq \overline{J(t\tau)}^{\mathrm{reg}}$. Therefore

- $\overline{J(t\tau)}^{\mathrm{reg}} = J(t\tau) \cup J(u_1\tau)$ and $\overline{J(t\tau)} = J(t\tau) \cup J(u_1\tau) \cup J(\tau) = G^\circ\tau$;
- $\overline{J(u_1\tau)}^{\mathrm{reg}} = J(u_1\tau) = G^\circ \cdot u_1\tau$;
- $\overline{J(\tau)}^{\mathrm{reg}} = J(\tau) = G^\circ \cdot \tau$.

Then the only comparable elements are $J(u_1\tau) \leq J(t\tau)$.

CHAPTER 3

Lusztig strata in disconnected groups

In this chapter we introduce the notion of Lusztig strata of a disconnected reductive algebraic group given in [46].

We recall that the Weyl group W is in bijection with the set of G°-orbits on $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the flag manifold of G° and where G° acts diagonally by conjugation. Also the group G acts diagonally on $\mathcal{B} \times \mathcal{B}$: this induces an action of G/G° on W. Let D be a connected component of G. We can see D as an element of G/G°. So this defines an automorphism $[D] : W \to W$ whose fixed point set is denoted by W^D. By [40, Appendix], W^D is a Coxeter group.

Using a variant of the Springer's correspondence with trivial local system, Lusztig defined a map $\mathcal{E}$ from D to the set of irreducible representations of W^D. Lusztig's strata are defined as the fibers of $\mathcal{E}$.

The classical Springer correspondence with trivial local system is a map from the set of unipotent elements in a connected reductive algebraic group to the set of isomorphism classes of irreducible representations of its Weyl group. G. Lusztig in [45] defined a map $\mathcal{E}_C$ from a connected reductive algebraic group to the set of irreducible representations of its Weyl group, constructed in terms of truncated induction of Springer's representations with trivial local systems. When $D = G^\circ$ the map $\mathcal{E}_C$ and $\mathcal{E}$ coincide.

Unlike the Springer correspondence, both the images of $\mathcal{E}_C$ and $\mathcal{E}$ do not depend on the characteristic of the base field $\mathbb{K}$. In our situation the set of Lusztig strata of D is parametrized by a subset of the irreducible representations of the Coxeter group W^D independently of char($\mathbb{K}$) [46, 4.9, 4.10, 4.11].

The fibers of the classical Springer correspondence are the unipotent conjugacy classes in the ambient group. It is expected that also the fibers of $\mathcal{E}$ have a good geometric behavior. Indeed, G. Lusztig in [46, 1.12 (b)] stated that the strata of D should be locally closed, as in the connected case [9]. The aim of this chapter is to prove this assertion and to describe their irreducible components.

Following the line of proof for the connected case in [9], we describe Lusztig strata as unions of regular closures of Jordan classes. One can also see that a sheet of a connected component of G is the regular closure of a maximal Jordan class with respect to the partial order defined in Section 2.4. Since Jordan classes are irreducible and finitely many, we obtain that the sheets contained in a stratum coincide with the irreducible components of the stratum.

We start recalling the definition of $\mathcal{E}$, the map from D to the set of isomorphism classes of irreducible representations of the group W^D defined in [46].

$$\mathcal{E} : D \longrightarrow \mathrm{Irr}(W^D)$$
$$a \mapsto \mathcal{E}(a)$$

where $\mathcal{E}(a)$ is constructed according to the following rules

- Let $D_{un} := \{a \in D \mid a \text{ is unipotent } \}$ and let $a \in D_{un}$. Let π be the morphism defined as in [46, 1.2], we denote by $\pi_!$ the sheaf functor given by direct image with compact support. The shifted intersection cohomology complex $\pi_!\left(\mathbf{Q}_l\right)[\dim D]$ has a decomposition as follows

  $$\pi_!\left(\mathbf{Q}_l\right)[\dim D] = \oplus_\rho \rho \otimes \pi_!\left(\mathbf{Q}_l\right)[\dim D]_\rho$$

  where ρ runs through $\mathrm{Irr}\left(W^D\right)$ (for further details see [46, 1.2]). Then $\mathcal{E}(a)$ is the unique irreducible representation of W^D such that $\pi_!(\overline{\mathbb{Q}_l}[\dim D]_{\mathcal{E}(a)})_{|D_{un}}$ is (up to shift) the intersection cohomology complex of $\overline{G^\circ \cdot a}$ with coefficients in $\overline{\mathbb{Q}_l}$.

- If a_s is central in G, let D_{a_u} be the connected component of G containing a_u. We observe that $W^{D_{a_u}} = W^D$, because $D_{a_u} = G^\circ a_u$ and $D = G^\circ a_s a_u = a_s G^\circ a_u = a_s D_{a_u}$ with a_s central. Then $\mathcal{E}(a) \coloneqq \mathcal{E}(a_u)$.

- Let $a_s \notin Z(G)$. Note that a_s is central in $C_G(a_s)$, and we let D' be the connected component of $C_G(a_s)$ containing a. Let $W(C_G(a_s)^\circ)$ be the Weyl group of $C_G(a_s)^\circ$. We denote by $\mathcal{E}_{a_s}(a) \in \mathrm{Irr}(W(C_G(a_s)^\circ)^{D'})$ the image of a through the map $\mathcal{E}$ referred to the group $C_G(a_s)$. Then we set $\mathcal{E}(a) = \mathbf{j}_{W(C_G(a_s)^\circ)^{D'}}^{W(G^\circ)^D} \mathcal{E}_{a_s}(a)$, where $\mathbf{j}$ is the truncated induction as defined in [49, Section 3]. The description of $W(C_G(a_s)^\circ)^{D'}$ as a subgroup of $W(G^\circ)^D$ is given in [46, 1.6 (a)].

We observe that $\mathcal{E}(a)$ depends only on the G°-class of a.

Definition 3.1. Let $\rho \in \mathrm{Irr}(W^D)$. The fiber $\mathcal{E}^{-1}(\rho)$ is called a Lusztig stratum.

By [46, 1.16 (e)] Lusztig strata are unions of G°-orbits of the same dimension, so if X is a Lusztig stratum contained in D, then there exists $d \in \mathbb{N}$ such that $X \subset D_{(d)}$.

As observed in [46, 0.1], a Lusztig stratum is union of Jordan classes. For the convenience of the reader we recall its proof.

Proposition 3.2. If J is a Jordan class in D, and $a = a_s a_u, h = h_s h_u \in J$, then $\mathcal{E}(a) = \mathcal{E}(h)$, so strata are unions of Jordan classes.

Proof. By Remark 2.4, up to conjugation $C_G(a_s)^\circ = C_G(h_s)^\circ$. Furthermore without loss of generality we can assume that $h = za$ where the element $z \in T(a) \subseteq C_G(a_s)^\circ$, so a and h are in the same connected component of $C_G(a_s)$. Hence the connected component D_h' of $C_G(h_s)$ containing h, and the connected component D_a' of $C_G(a_s)$ containing a coincide, as well as the unipotent parts of a and h. By construction $\mathcal{E}(h) = E(a)$. □

We recall the notation that we are using from Subsection 2.1. So g denotes an element of D and its Jordan decomposition is su, L denotes $L(g)$, P is a parabolic subgroup of G° with Levi subgroup L and such that $gPg^{-1} = P$. The torus $T(su)$ and the Jordan class $J(su)$ are defined as in Subsection 2.1. Recall that we denote by $\overline{J(su)}^{\mathrm{reg}}$ the set of regular elements in the closure of $J(su)$.

By Proposition 2.18 (b), if $x \in \overline{J(su)}^{\mathrm{reg}}$, then there exists $z \in T(su)$ and $v \in \mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u)$ such that, up to G°-conjugation, $x = zsv$, where $\mathrm{Ind}_{(G^s)^\circ u}^{(G^{zs})^\circ u}((G^s)^\circ \cdot u)$ is defined as in (2.5).

To simplify notation, we set:

- $D' := s(G^s)^\circ u$ the connected component of G^s containing su,
- $D_u' := (G^s)^\circ u$ the connected component of G^s containing u,

- $\widetilde{D}$ the connected component of G^{zs} containing x
- $\widetilde{D_u} \coloneqq (G^{zs})^\circ u$, the connected component of G^{zs} containing u,
- $\widetilde{D'_u} \coloneqq (L^{zs})^\circ u$, the connected component of $N_{G^{zs}}((P^{zs})^\circ) \cap N_{G^{zs}}((L^{zs})^\circ)$ containing u.

As $v \in \mathrm{Ind}_{\widetilde{D'_u}}^{\widetilde{D_u}}((G^s)^\circ \cdot u)$, it follows that $v \in \widetilde{D_u}$. Since additionally, $x = zsv$, we have $W((G^{zs})^\circ)^{\widetilde{D}} = W((G^{zs})^\circ)^{\widetilde{D_u}}$. Observe also that, by Remark 2.17, $\widetilde{D'_u} = D'_u$. Hence, $W((G^s)^\circ)^{D'} = W((G^s)^\circ)^{\widetilde{D'_u}}$.

Now we prove the main result of this chapter. This statement was suggested by G. Lusztig in [46, 1.12 (b)].

Theorem 3.3. *Any stratum X is a union of regular closures of Jordan classes and it is locally closed.*

Proof. We first prove that if $J(su) \subseteq X$ then $\overline{J(su)}^{\mathrm{reg}} \subseteq X$. Let $x \in \overline{J(su)}^{\mathrm{reg}}$. We show that $\mathcal{E}(x) = \mathcal{E}(su)$. There exists $z \in T(su)$ and $v \in \mathrm{Ind}_{\widetilde{D'_u}}^{\widetilde{D_u}}((G^s)^\circ \cdot u)$ such that $x = zsv$.

By [46, 1.9 (iii)] $\mathcal{E}_s(u)$ is good in the sense of [45, 0.2], so we can apply $\mathbf{j}$-induction from $W((G^s)^\circ)^{\widetilde{D'_u}}$ to $W((G^{zs})^\circ)^{\widetilde{D_u}}$ obtaining an irreducible representation. By [51, Corollary 2.10],

$$\mathcal{E}_{zs}(v) = \mathbf{j}_{W((G^s)^\circ)^{\widetilde{D'_u}}}^{W((G^{zs})^\circ)^{\widetilde{D_u}}} \mathcal{E}_s(u).$$

By definition of $\mathcal{E}$,

$$\mathcal{E}(x) = \mathbf{j}_{W((G^{zs})^\circ)^{\widetilde{D}}}^{W(G^\circ)^D} \mathcal{E}_{zs}(v).$$

So putting together these relations and using transitivity of the truncated induction, we have

$$\mathcal{E}(x) = \mathbf{j}_{W((G^{zs})^\circ)^{\widetilde{D}}}^{W(G^\circ)^D} \mathcal{E}_{zs}(v) = \mathbf{j}_{W((G^{zs})^\circ)^{\widetilde{D}}}^{W(G^\circ)^D} \mathbf{j}_{W((G^s)^\circ)^{\widetilde{D'_u}}}^{W((G^{zs})^\circ)^{\widetilde{D_u}}} \mathcal{E}_s(u) = \mathbf{j}_{W((G^s)^\circ)^{\widetilde{D'_u}}}^{W(G^\circ)^D} \mathcal{E}_s(u).$$

By definition of $\mathcal{E}$, we have $\mathcal{E}(su) = \mathbf{j}_{W((G^s)^\circ)^{D'}}^{W(G^\circ)^D} \mathcal{E}_s(u)$, so $\mathcal{E}(x) = \mathcal{E}(su)$ and $x \in X$.

By Proposition 3.2, the stratum X is a finite union of Jordan classes, therefore X is a finite union of regular closures of Jordan classes. Let I be a finite set of indeces and let $X \subseteq G_{(d)}$, so d is the dimension of the G°-orbits in X. Then

$$X = \bigcup_{i \in I} \overline{J_i}^{\mathrm{reg}} = \bigcup_{i \in I} \overline{J_i} \cap G_{(d)} = \left(\overline{\bigcup_{i \in I} J_i} \right) \cap \left(\bigcup_{i > d-1} G_{(i)} \right).$$

By Lemma 0.2 $\left(\bigcup_{i>d-1} G_{(i)}\right)$ is an open subset of D. Therefore X is the intersection of a closed subset and an open subset of D, so X is locally closed.

$\square$

From the proof of the Theorem 3.3, a stratum X is a finite union of regular closures of Jordan classes.

Corollary 3.4. Let X be a stratum. Then

(a) X is a union of sheets for the G°-action on D;

(b) The sheets contained in X are its irreducible components.

Proof. By Proposition 2.20, the sheets in $D_{(d)}$ are the regular closures of maximal Jordan classes. By Theorem 3.3, we have $X = X \cap D_{(d)} = \bigcup_{i\in I} \overline{J_i}^{\text{reg}}$, with I a finite index set and J_i a Jordan classes $\forall i \in I$. Let $J_1, \dots J_k$ be the maximal elements through J_i, with $i \in I$, for the partial order defined in (2.12). Then $X = X \cap D_{(d)} = \bigcup_{i=1}^{k} \overline{J_i}^{\text{reg}}$. Therefore X is a finite union of sheets and these are the maximal irreducible subsets contained in X, i.e. its irreducible components.

$\square$

CHAPTER 4

Lusztig's map in disconnected groups

In this chapter we deal with the spherical unipotent conjugacy classes in a disconnected group G with G° simple. By Example 2.6, unipotent G°-conjugacy classes in G are Jordan classes. In particular, in Section 4.2, we classify the spherical unipotent conjugacy classes in a connected component D, i.e. the unipotent classes in D for which there is a dense orbit of a Borel subgroup B of G°. We consider $G = G^\circ \rtimes \langle\theta\rangle$, with θ a graph automorphism of G°. For our purposes, it will be enough to consider the case $D = G^\circ\theta$.

As a consequence of the classification of spherical unipotent classes in D we obtain the dimension formula: let γ be a unipotent conjugacy class in $D = G^\circ\theta$, $w_\gamma \in W$ be the unique element in W for which $Bw_\gamma B\theta \cap \gamma$ is dense in γ. Then γ is spherical if and only if $\dim\gamma = \ell(w_\gamma) + \mathrm{rk}(1 - w_\gamma\theta)$, where ℓ is the length function on W and rk is the rank in the geometric representation of W.

In Section 4.3 we obtain an analogue of the results of [10], giving a more combinatorial description of the map Υ and its right inverse Ψ defined by G. Lusztig in [43]. The map Υ is a surjective map from twisted conjugacy classes (in a certain sense) in W to unipotent conjugacy classes in a fixed connected component D of G.

4.1 Preliminaries

Definition 4.1. Let H be a connected reductive algebraic group and B' a Borel subgroup of H. A normal H-variety X is called *spherical* if B' has a dense open orbit in X.

It is well known (see [2, 67] in characteristic 0, [25, 33] in positive characteristic) that X

is spherical if and only if the set of B'-orbits in X is finite.

Definition 4.2. Let $\phi \in \mathrm{Aut}(H)$. An element $h \in H$ is called a *φ-twisted involution* if $\varphi(h) = h^{-1}$.

As we saw in Chapter 1, studying twisted conjugacy classes or H-conjugacy classes in the coset $H\varphi \subseteq H \rtimes \langle\varphi\rangle$ is equivalent. Moreover, for $x \in H$, the twisted conjugacy class $H \cdot_\varphi x$ is spherical in H if and only if $H \cdot x\varphi$ is a spherical conjugacy class in $H\varphi$. In this chapter we shall consider interchangeably φ-twisted conjugacy classes in H and conjugacy classes in $H\varphi$.

We are interested in the cases where $H = G^\circ$, with G° simple, or $H = W$ for $\varphi = \theta$, either a graph automorphism of G° or an automorphism of the Dynkin diagram of G°. We set $G := G^\circ \rtimes \langle\theta\rangle$ and $\widetilde{W} := W \rtimes \langle\theta\rangle$, viewed as a group of isometries of E. We denote the set of unipotent conjugacy classes in $G^\circ\theta$ by $\underline{G^\circ\theta}$. By $\underline{W}_\theta$ we denote the set of θ-twisted conjugacy classes in W. Let (T, B) a θ-stable pair torus/Borel subgroups of G°. From the Bruhat decomposition $G = \cup_{w\in W} BwB$, we get the decomposition $G^\circ\theta = \cup_{w\in W} BwB\theta$, and we may consider $BwB\theta$ as the double coset $Bw\theta B$ relative to $w\theta \in \widetilde{W}$. We have $\overline{Bw\theta B} = \cup_{z\leq w} Bz\theta B$.

Let γ be a conjugacy class in $G^\circ\theta$:

$$\text{there exists a unique } w_\gamma \in W \text{ such that } \overline{\gamma \cap Bw_\gamma\theta B} = \overline{\gamma}. \tag{4.1}$$

We put $C^\gamma = W \cdot_\theta w_\gamma$ in $\underline{W}_\theta$. Note that if $h \in G^\circ$ and $w \in W$, then $(G^\circ \cdot h\theta) \cap Bw\theta B \neq \emptyset$ if and only if $(G^\circ \cdot_\theta h) \cap BwB \neq \emptyset$.

Definition 4.3. A class $C \in \underline{W}_\theta$ is called *elliptic* if for any (proper) $J \subset I$ such that $\theta(J) = J$ we have $C \cap W_J = \emptyset$. □

Remark 4.4. By [28, Section 4.1] a class C is elliptic if and only if

$$\dim E^{w\theta} = \dim\ker(w\theta - \mathrm{Id}) = 0 \text{ for any/some } w \in C.$$

Let C be in $\underline{W}_\theta$. We recall the definition of the invariant $\mu(C)$ given by Lusztig in [44, p. 453]. He observes that there is a θ-stable subset J of I and an elliptic twisted conjugacy class C' in W_J such that $C' = C \cap W_J$ (see also [17, Theorem 2.3.4]): then he defines $\mu(C)$ to be the *number of θ-orbits in $I \setminus J$*. He observes that $\mu(C)$ is independent of J (see also [17, Remark 2.4.2, Proposition 2.4.1]), and that C is elliptic if and only if $\mu(C) = 0$.

We will need the following alternative characterization of μ.

Proposition 4.5. Let $C \in \underline{W}_\theta$. Then

$$\mu(C) = \dim E^{w\theta} \quad \text{for all } w \in C.$$

Proof. It is clear that $\dim E^{w\theta}$ is independent of $w \in C$, since $C\theta$ is a W-conjugacy class in $W\theta$. If C is elliptic, the result follows by the Remark 4.4. Therefore we suppose that C is not elliptic. Let J be a proper θ-stable subset of I such that $C \cap W_J$ is an elliptic θ-twisted conjugacy class in W_J and let $w \in C \cap W_J$. We recall that E_J is the $\mathrm{Span}_{\mathbb{R}}\{\alpha_j \mid j \in J\}$. Consider the basis $\{\omega_i \mid i \in I \setminus J\}$ of the orthogonal complement $E_J^\perp$ of E_J in E, where ω_i are the fundamental weights. We observe that both E_J and $E_J^\perp$ are θ-stable and W_J-stable. Moreover θ acts as a permutation on both bases Δ_J and $\{\omega_i \mid i \in I \setminus J\}$, and w acts trivially on $E_J^\perp$. Since $C \cap W_J$ is elliptic in W_J, we have $\ker(w\theta - \mathrm{Id}) \cap E_J = \{0\}$, so the kernel of $w\theta - \mathrm{Id}$ on E is the kernel of $\theta - \mathrm{Id}$ on $E_J^\perp$. If S is a θ-orbit in $I \setminus J$, and $\omega_S = \sum_{i \in S} \omega_i$ then $\{\omega_S \mid S \text{ a } \theta\text{-orbit in } I \setminus J\}$ is a basis of $\ker(w\theta - \mathrm{Id})$. Hence $\mu(C) = \dim E^{w\theta}$. $\square$

Remark 4.6. Proposition 4.5 gives an alternative proof of the fact that the definition of $\mu(C)$ is independent of J.

4.2 Classification of spherical unipotent orbits

In this section we classify the spherical unipotent G°- conjugacy classes for $G = G^\circ \rtimes \langle\theta\rangle$, with G° simple and $\theta \in \mathrm{Aut}(\Gamma)$.

Let γ be a unipotent G°-conjugacy class in G contained in G°. Then γ is a unipotent conjugacy class in G°. The classification of spherical conjugacy classes for the connected case is given in [5] for characteristic zero, in [7] for good characteristic and in [19] for bad characteristic. Thus we focus on the classification of the spherical unipotent orbits in $G \setminus G^\circ$.

Observe that θ preserves $Z(G^\circ)$, hence it induces an automorphism of the same order on the adjoint quotient $(G^\circ)_{\mathrm{ad}}$ of G°. A conjugacy class in G is spherical for the action of G° if and only if its image through the projection on $G_{\mathrm{ad}} \rtimes \langle\theta\rangle$ is spherical for the action of $(G^\circ)_{\mathrm{ad}}$. So, it is enough to consider the case of G° of adjoint type.

The only non trivial automorphisms θ are the ones of the Proposition 1.3. We set $p = \mathrm{char}(\mathbb{K})$. By 0.3, there are unipotent elements in $G \setminus G^\circ$ if and only if p is equal to the order of θ. Therefore we can restrict to study the following four cases:

(a) $G^\circ = PGL(m)$, $m \geq 3$, $p = 2$;

(b) $G^\circ = PSO(2m)$, $m \geq 4$, $p = 2$;

(c) $G^\circ = PSO(8)$, $p = 3$;

(d) $G^\circ = E_6$, of adjoint type, $p = 2$.

In the cases (a),(b),(d) the order of θ is 2, so we set $\theta := \tau$; in the case (c), the order of $\theta = 3$.

Note that in the following we focus on the classification of the unipotent spherical G°-orbits in the connected component $G^\circ\theta$. The connected component $G^\circ\theta^2 \neq G^\circ$ only for the case (c). In this case the classification easily follows from the one of $G^\circ\theta$.
Since, as observed before, the classification of spherical conjugacy classes do not depend on the isogeny type of G, in the case (a) we consider $G^\circ = SL(m)$, and in case (b) $G^\circ = SO(2m)$.

The set $\underline{G^\circ\theta}$ of unipotent G°-orbits in $G^\circ\theta$ is partially ordered by inclusion of closures. By [60, II.2.21. Corollaire], it has a unique minimal element: the G°-orbit of θ, which is the unique quasi-semisimple unipotent orbit in $G^\circ\theta$.

We recall that, by [18, Theorem 3.4], if γ is a non-trivial unipotent conjugacy class of a simple algebraic group in characteristic 2, then γ is spherical if and only if it consists of involutions. Moreover, if u is an involution in $G^\circ\tau$, then the conjugacy class of u is spherical ([18, §4]).

We also recall that, by [22, Remark 2.13], a G°-orbit $G^\circ \cdot x$ is spherical if and only if the homogeneous space $G/C_{G^\circ}(x)$ is spherical for the action given by the G°-conjugation.

In the connected case we have the following characterization of spherical conjugacy classes.

Theorem 4.7. *Let γ be a G°-conjugacy class in G°. Then γ is spherical if and only if*

$$\dim \gamma = \ell(w_\gamma) + \mathrm{rk}(1 - w_\gamma),$$

where w_γ is the unique element in W for which $\overline{Bw_\gamma B \cap \gamma} = \gamma$.

The proof of Theorem 4.7 is provided in [5, 6, 19] for the case in which the characteristic of $\mathbb{K}$ is good for G°, and in [18] for the case of bad characteristic.

In [37], the author presents a generalization of Theorem 4.7 for the non-connected case in characteristic zero, and in [8], Theorem 4.7 is extended to the non-connected case in good odd characteristics for automorphisms of order two.

Now, we deal with the remaining cases by providing a classification. We use the notation in [60] for the classification of the unipotent conjugacy classes.

4.2.1 Type A_n, $n \geq 2$, $p = 2$, τ of order 2

The unipotent spherical conjugacy classes contained in G° are listed in [18, Table 1]. Now we provide the classification for the unipotent spherical conjugacy classes in $G^\circ\tau$.

We assume $G^\circ = SL(n+1)$. For the unipotent conjugacy classes we use the notation in [60, Chapter I]. By [60, I.2.7 p. 21, I.2.10 p. 24, I.2.11 p. 25, Theorem II.8.2 p. 134] we get

n **even**

The unique minimal unipotent orbit in $G^\circ\tau$ is the orbit 1^{n+1} of τ. Since $\dim 1^{n+1} = \dim B$, this is the only possible spherical unipotent orbit in $G^\circ\tau$: it is made of involutions and is spherical by [18, §4.1]. Note that in [18, §4.1] it is stated that $C_{G^\circ}(\tau)$ is isomorphic to $SO(n+1)$, but in fact it is isomorphic to $Sp(n)$ (there is an isogeny from $SO(n+1)$ to $Sp(n)$ and these groups are isomorphic as abstract groups).

n **odd**

The unique minimal unipotent orbit in $G^\circ\tau$ is the orbit 1_0^{n+1} of τ. Moreover the orbit 1^{n+1} is the unique minimal element in $\underline{G^\circ\tau} \setminus \{G^\circ \cdot \tau\}$
(i.e. $\underline{G^\circ\tau} \setminus \{G^\circ \cdot \tau\}$ has minimum 1^{n+1}) and $\dim 1^{n+1} = \dim B$. Hence 1_0^{n+1} and 1^{n+1} are the only possible spherical unipotent orbits in $G^\circ\tau$: they are made of involutions and are spherical by [18, §4.1, 4.2].

4.2.2 Type D_n, $n \geq 4$, $p = 2$, τ of order 2

The unipotent spherical conjugacy classes contained in G° are listed in [18, Table 3,4]. Now we provide the classification for the unipotent spherical conjugacy classes in $G^\circ\tau$.

To deal with G° of type D_n we shall consider $G^\circ = SO(2n)$. Then the outer involutions of G° are obtained by conjugation with involutions of
$O(2n) \setminus SO(2n)$. Note that if $n = 4$, and G° is adjoint or simply-connected, there are other outer involutions in $\mathrm{Aut}(G^\circ)$: however, they are conjugate in $\mathrm{Aut}(G^\circ)$.

Let τ be the involution of $O(2n)$ inducing the graph automorphism of $SO(2n)$, i.e. the graph-automorphism acting trivially on
$\langle X_{\pm\alpha_i} \mid i \in \{1, \ldots, n-2\}\rangle$ and such that $x_{\alpha_{n-1}}(\xi) \leftrightarrow x_{\alpha_n}(\xi)$, $x_{-\alpha_{n-1}}(\xi) \leftrightarrow x_{-\alpha_n}(\xi)$ for $\xi \in k$.

The following result is a technical property and it holds in general, in particular without the hypothesis on G° to be of type D_n and without the hypothesis on the characteristic of the base field.

Lemma 4.8. Let γ be a G°-orbit in $G^\circ\tau$, and let γ' be a subset of γ. Let J be a τ-invariant subset of I such that τ acts trivially on $Z(L_J)^\circ$. Assume that $\gamma' \subseteq L_J\tau$ and $(B_J \cdot x)_{x\in\gamma'}$ is a family of pairwise distinct B_J-orbits. Then the family $(B \cdot x)_{x\in\gamma'}$ consists of pairwise distinct B-orbits.

Proof. Let x, y be elements of γ', and assume $B \cdot x = B \cdot y$. Then there exists $b \in B$ such that $bxb^{-1} = y$, i.e. $bx = yb$. Since $B = TU_{w_J}U_{w_0w_J}$, where $U_{w_0w_J}$ is the unipotent radical of the standard parabolic subgroup P_J, we can write $b = tu_1u_2$, where $t \in T$, $u_1 \in U_{w_J}$ and $u_2 \in U_{w_0w_J}$, so that $tu_1u_2x = ytu_1u_2$. Since $U_{w_0w_J}$ is normal in P_J, from uniqueness of expression of an element $h\tau$ of $P_J\tau$ as $h\tau = lv\tau$ with $l \in L_J$, $v \in U_{w_0w_J}$ and since P_J and its unipotent radical are τ-invariant, we get $tu_1x = ytu_1$. We may decompose $T = T_JZ(L_J)^\circ$, so $t = t_1t_2$, with $t_1 \in T_J$, $t_2 \in Z(L_J)^\circ$. Since τ acts trivially on $Z(L_J)^\circ$, we get $t_1u_1x = yt_1u_1$. But t_1u_1 lies in B_J, and we conclude that $B_J \cdot x = B_J \cdot y$. Therefore $x = y$ and we are done. $\square$

Our aim is to show that a unipotent conjugacy class $\gamma = G^\circ \cdot u$ in $G^\circ\tau$ is spherical if and only if u is an involution. By [18, §4.3], we are left to show that if the unipotent class γ in $G^\circ\tau$ does not consist of involutions, then γ is not spherical.

Assume u is a unipotent element in $G^\circ\tau$ of order greater than 4, and let v be an element of order 4 in the subgroup generated by u. Then v lies in G°, in fact, by Remark 0.3 and since u has no order 4, there exists $j > 2 \in \mathbb{N}$ such that $u^{2^j} = \mathrm{Id}$. Since v is in the subgroup generated by u, the element v is an even power of u, so, since τ has order two, we have that

$v \in G^\circ$. By [18, Proposition 3.12], the orbit $G^\circ \cdot v$ is not spherical. Since $C_{G^\circ}(u) \leq C_{G^\circ}(v)$, it follows that also $G^\circ \cdot u$ is not spherical. We are therefore left to consider the subset X of $\underline{G^\circ \tau}$ of conjugacy classes of elements of order 4. By [33, Theorem 2.2], it is enough to show that the minimal elements in X are not spherical.

For the notations of unipotent classes in $G^\circ\tau$ we follow [60, Chapter I]. From the explicit definition of the partial order on pairs (λ, ε) such that $C_{\lambda,\varepsilon} \leq C_{\mu,\phi} \iff (\lambda,\varepsilon) \leq (\mu,\phi)$ given in [60, I.2.10, I.2.11], it follows that the minimal elements in X are the classes $3^2 \oplus 2 \oplus 1^{2n-8}$ and $4 \oplus 1^{2n-4}$.

Proposition 4.9. Let $p = 2$ and let G° be of type D_n, for $n \geq 4$, let γ be the conjugacy class of a unipotent element u in $G^\circ\tau$, with $u^2 \neq 1$. Then γ is not spherical.

Proof. By the previous discussion we are left to deal with the cases where γ is $4 \oplus 1^{2n-4}$ or $3^2 \oplus 2 \oplus 1^{2n-8}$.

We consider $J = \{n-3, n-2, n-1, n\} \subseteq I$. Then J is τ-invariant and τ acts trivially on $Z(L_J)$ (which is connected). Moreover $[L_J, L_J]$ is isomorphic to D_4.

The D_4-orbits $4\oplus1^4$ and $3^2\oplus2$ in $D_4\tau$ are not spherical by dimension reasons: $\dim 4\oplus1^4 = 17$, $\dim 3^2 \oplus 2 = 19$, while the dimension of a Borel subgroup of D_4 is 16. Hence, in both cases, there is an infinite family γ' of elements of $[L_J, L_J] \cap \gamma$ such that $(B_J \cdot x)_{x\in\gamma'}$ is a family of pairwise distinct B_J-orbits. By Lemma 4.8 the infinite family $(B \cdot x)_{x\in\gamma'}$ consists of pairwise distinct B-orbits, hence γ is not spherical. □

4.2.3 Type D_4, $p = 3$, θ of order 3

The unipotent spherical conjugacy classes contained in G° are classified in [7, 3.4]. Now we provide the classification for the unipotent spherical conjugacy classes in $G^\circ\theta$.

This case may occur only if G° is simply-connected or adjoint. In $G^\circ\theta$ there are 5 G°-orbits of unipotent elements, and they form a chain $C_4 < C_3 < C_2 < C_1 < C_0$ ([60, §3, p. 30]): their dimensions are $14, 20, 22, 24, 26$ respectively ([34, Table 8]). Since $\dim B = 16$, only C_4 (the orbit of θ, whose centralizer is G_2) can be spherical, and indeed it is spherical (in every characteristic) as shown in [18, §4.5] (see also [3, Theorem 4.3], G_2 spherical in $SO(8)$).

4.2.4 Type E_6, $p = 2$, τ of order 2

The unipotent spherical conjugacy classes contained in G° are classified in [18, Table 7]. Now we provide the classification for the unipotent spherical conjugacy classes in $G^\circ\tau$.

For the unipotent classes in $G^\circ\tau$ we use the notation in [60, 10.14 p. 160]. The unique minimal unipotent orbit in $G^\circ\tau$ is the orbit of τ corresponding to the empty sub-root-system. Moreover the orbit A_1 is the unique minimal element in $\underline{G^\circ\tau} \setminus \{G^\circ \cdot \tau\}$ (i.e. $\underline{G^\circ\tau} \setminus \{G^\circ \cdot \tau\}$ has minimum A_1) and $\dim A_1 = \dim B$, [60, p. 160, 250]. Hence the orbit of τ and A_1 are the only possible spherical unipotent orbits in $G^\circ\tau$: these are made of involutions and are spherical by [18, §4.4].

This concludes the classification of spherical unipotent classes in $G^\circ\theta$:

Theorem 4.10. *Let G° be a simple algebraic group, θ a graph-automorphism of G°, γ a unipotent conjugacy class in $G^\circ\theta$:*

(i) if $p = 2$ and θ has order 2, then γ is spherical if and only if it consists of involutions;

(ii) if $G = D_4$, $p = 3$ and θ has order 3, then only the class of θ is spherical.

We recall that if γ is a conjugacy class in $G^\circ\theta$, w_γ is the unique element in W such that $\overline{\gamma \cap Bw_\gamma\theta B} = \overline{\gamma}$. We make use of the following result.

Proposition 4.11 ([18, Theorem 4.1],[37, Lemma 2.1])**.** Let γ be a conjugacy class in $G^\circ\theta$ and let $w \in W$.

(i) If $\gamma \cap Bw\theta B \neq \emptyset$, then $\dim\gamma \geq l(w) + \mathrm{rk}(1 - w\theta)$;

(ii) if $\dim\gamma = l(w_\gamma) + \mathrm{rk}(1 - w_\gamma\theta)$, then γ is spherical.

It was proved [37, Theorem 1.1] in characteristic zero, and in [8, Theorem 22] in $\mathrm{char}(\mathbb{K}) \neq 2$ for automorphisms of order 2, that γ is spherical if and only if $\dim\gamma = l(w_\gamma)+\mathrm{rk}(1-w_\gamma\theta)$ (note that the arguments used in [8, §6] only use $\mathrm{char}(\mathbb{K}) \neq 2$). From the results in [18, §4] and the classification of spherical unipotent classes in $G^\circ\theta$ we obtain the following theorem.

Theorem 4.12. *Let G° be a simple algebraic group. Let θ be a graph automorphism of G° and γ a conjugacy class in $G^\circ\theta$. Assume $\mathrm{char}(\mathbb{K}) = 0$, or θ of order 2 and $\mathrm{char}(\mathbb{K}) \neq 2$, or γ is unipotent. Then γ is spherical if and only if*

$$\dim\gamma = l(w_\gamma) + \mathrm{rk}(1 - w_\gamma\theta).$$

If we restrict to graph-automorphisms of order 2, we get the following theorem.

Theorem 4.13. *Let G° be a simple algebraic group over an algebraically closed field of characteristic 2, τ a graph-automorphism of G° of order 2. Let γ be a spherical unipotent conjugacy class in $G^\circ\tau$, and let $\gamma \cap Bw\tau B$ be non-empty. Then, $w\tau$ is an involution, i.e. w is a τ-twisted involution.*

Proof. Let u be an element of $\gamma \cap Bw\tau B$. From the classification of spherical unipotent conjugacy classes in $G^\circ\tau$ it follows that u is an involution. Thus, $u = u^{-1} \in (Bw\tau B)^{-1} = B\tau w^{-1}B = B(\tau w^{-1}\tau)\tau B$, forcing $w = \tau w^{-1}\tau$. Hence $w\tau w\tau = 1$. □

Let ν be the map defined by $\nu(\gamma) = w_\gamma$ for any conjugacy class γ in $G^\circ\theta$. It was proved in [13, Corollary 2.15] that w_γ lies in the set

$$W_{\theta,m} := \{w \in W \mid w \text{ is the unique maximal length element in } W \cdot_\theta w\}.$$

Remark 4.14. It is shown in [37, §3] that if $w \in W_{\theta,m}$, then w is a θ-twisted involution, an involution, and it commutes with θ and with w_0: it is of the form $w_0 w_J$ where J is a suitable θ-invariant subset of I. Although the general assumption in [37] is that the base field is of characteristic zero, the arguments used in [37, §3] from Lemma 3.1 until Proposition 3.7 are characteristic-free. The set J is recovered from w by the equality $J = \{i \in I \mid w\theta\alpha_i = \alpha_i\}$. Moreover, since w commutes with w_0, we have that $w_0 w_J = w_J w_0$.

For convenience of the reader, we recall the list of pairs (Φ, J) for every non-trivial θ from [37, Proposition 3.7]:

$$\begin{array}{ll}
(\Phi, \emptyset) & \text{for any } \Phi \text{ and any } \theta; \\
(A_{2n+1}, \{\alpha_1, \alpha_3, \ldots, \alpha_{2n+1}\}) & n \geq 1, \theta = -w_0; \\
(D_4, \{\alpha_2\}) & \theta^3 = 1; \\
(D_4, \{\alpha_2, \alpha_i, \theta\alpha_i\}) & \theta^2 = 1 \text{ and } \alpha_i \neq \theta\alpha_i; \\
(D_{2n}, \{\alpha_{2l}, \alpha_{2l+1}, \ldots, \alpha_{2n-1}, \alpha_{2n}\}) & n > 2,\ 1 \leq l \leq n-1 \text{ and } \theta\alpha_{2n-1} = \alpha_{2n}; \\
(D_{2n+1}, \{\alpha_{2l}, \alpha_{2l+1}, \ldots \alpha_{2n}, \alpha_{2n+1}\}) & n \geq 2,\ 1 \leq l \leq n, \text{ and } \theta = -w_0; \\
(E_6, \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}) & \theta = -w_0.
\end{array} \tag{4.2}$$

We analyze the image of the restriction of ν to the set $\underline{G}_{\theta,sph}$ of spherical unipotent conjugacy classes in $G^\circ\theta$. The case where $\theta = 1$ is dealt with in [10, Remark 2]: here we only consider the case $\theta \neq 1$.

Proposition 4.15. The restriction of ν to $\underline{G}_{\theta,sph}$ is injective. If G° is of type A_n, D_n, E_6 and θ has order 2 (hence $p=2$) we have $\nu(\underline{G}_{\theta,sph}) = W_{\theta,m}$. For G° of type D_4 and θ of order 3 (hence $p=3$) we have $\nu(\underline{G}_{\theta,sph}) = \{w_0 s_2\}$ (while $W_{\theta,m} = \{w_0 s_2, w_0\}$).

Proof. If θ has order 2, then, by [18, §4], for every $w = w_0 w_J \in W_{\theta,m}$ as in (4.2) there exists a unique spherical unipotent conjugacy class γ in $G^\circ\theta$ such that $w = w_\gamma$ (note that in [18, §4.3] for D_4 the cases $J = \{2,1,3\}, \{2,1,4\}$ are not considered, but they are obtained from the case $J = \{2,3,4\}$ by triality).

Assume θ has order 3. Then $\gamma = G^\circ \cdot \theta$ is the unique spherical unipotent class in $G^\circ\theta$ and, by [18, 4.5], $w_\gamma = w_0 s_2$. $\square$

4.3 The restriction of Ψ to spherical unipotent orbits

We recall the construction of the surjective map $\Upsilon : \underline{W_\theta} \to \underline{G^\circ\theta}$ and of its section Ψ defined in [44]. In [44, §5.11] Lusztig also considers the set D_{au} of almost unipotent bilinear forms in characteristic not 2, i.e. elements $g \in G^\circ\tau$ for G° of type A_n, for $n \geq 2$, with Jordan decomposition $g = su$, such that $s^2 = 1$ and u unipotent. The set $\underline{D_{au}}$ denotes the set of conjugacy classes in D_{au}, $\Upsilon' : \underline{W_\theta} \to \underline{D_{au}}$. We deal both with spherical classes in $\underline{G^\circ\theta}$ assuming $\operatorname{char} k = 2$ and θ of order 2 or $\operatorname{char} k = 3$ and θ of order 3, and in $\underline{D_{au}}$ assuming $\operatorname{char} k \neq 2$ and G° of type A_n for $n \geq 2$. For w in W we define $\Gamma_w = \{\gamma \in \underline{G^\circ\theta}\ (resp.\ \underline{D_{au}}) \mid \gamma \cap Bw\theta B \neq \emptyset\}$.
Let $C \in \underline{W_\theta}$, and let $C_{\min}$ be the set of all $w \in C$ of minimal length. By [44, Theorem 1.3 (a), §5.11], given $w \in C_{\min}$, Γ_w has minimum ζ (i.e. $\zeta \subseteq \overline{\zeta'}$ for all $\zeta' \in \Gamma_w$): then $\Upsilon(C) := \zeta$ (respectively $\Upsilon'(C) := \zeta$). This definition is independent of the chosen $w \in C_{\min}$ [44, 1.2 (a)]. Note that ζ is the unique class of minimal dimension in Γ_w.

The right inverse Ψ of Υ (Ψ' of Υ' respectively) constructed in [44] is defined as follows. Let γ be in $\underline{G^\circ\theta}$ (in $\underline{D_{au}}$ respectively). Then by [44, Theorem 1.16, §5.11] there exists a unique element C in $\Upsilon^{-1}(\gamma)$ (in $\Upsilon'^{-1}(\gamma)$ respectively) such that $\mu(C)$ is minimal. Then $\Psi(\gamma) := C$ ($\Psi'(\gamma) := C$ respectively).

We recall the following result:

Proposition 4.16 ([13, Proposition 2.13, 2.14])**.** Let γ be a conjugacy class in $G^\circ\theta$, let $w \in W$ be such that $\gamma \cap Bw\theta B \neq \emptyset$ and let z be a maximal length element in the θ-twisted conjugacy class of w. Then $\gamma \cap Bz\theta B \neq \emptyset$.

In the remainder of this section, unless otherwise stated, θ has order 2 and, as usual, we denote it by τ. We deal with classes in $\underline{G^\circ\tau}$, assuming therefore $p = 2$, and in $\underline{D_{au}}$, assuming $p \neq 2$.
We denote by $\underline{G^\circ\tau}_{sph}$ ($\underline{D_{au}}_{sph}$ respectively) the spherical conjugacy classes in $\underline{G^\circ\tau}$ (in $\underline{D_{au}}$ respectively). We are going to prove that for these classes Proposition 4.16 holds for every z in the τ-twisted conjugacy class of W.

Theorem 4.17. *Let γ be a spherical conjugacy class in $G^\circ\tau$, assume in addition that γ is unipotent if char$(\mathbb{K}) = 2$. If $\gamma \cap Bw\tau B \neq \emptyset$, then $w\tau$ is an involution, i.e. w is a τ-twisted involution.*

Proof. If char$(\mathbb{K}) \neq 2$ this is [8, Theorem 6] (note that in the paper the base field is of zero or good odd characteristic, but the arguments used in Section 2 only use char$(\mathbb{K}) \neq 2$). If char$(\mathbb{K}) = 2$, this is Theorem 4.13. □

Lemma 4.18. Let γ be a spherical conjugacy class in $G^\circ\tau$, assume in addition that γ is unipotent if char$(\mathbb{K}) = 2$, and let C be a τ-twisted class in W. If $\gamma \cap Bw\tau B \neq \emptyset$ for some $w \in C$, then $\gamma \cap Bz\tau B \neq \emptyset$ for every $z \in C$.

Proof. Let $z \in C$. Let σ be of minimal length, and $r \in \mathbb{N}$, such that $z = \sigma w\tau(\sigma)^{-1}$ (i.e. $z\tau = \sigma w\tau\sigma^{-1}$), $\sigma = s_{i_r}\cdots s_{i_1}$. Let $z_j = s_{i_j}\cdots s_{i_1}w\tau(s_{i_1})\cdots\tau(s_{i_j})$ for $j = 0,\ldots,r$, so that $z_0 = w$, $z_r = z$. We show by induction on j that $\gamma \cap Bz_j\tau B \neq \emptyset$ for $j = 0,\ldots,r$. If $j = 0$, the assertion follows by hypothesis. Assume that $\gamma \cap Bz_j\tau B \neq \emptyset$ for a given j and let $x \in \gamma \cap Bz_j\tau$. Then, if $\dot{s}_i$ is a representative of s_i in $N(T)$ for $i = 1,\ldots,n$,

$$\dot{s}_{i_{j+1}}x\dot{s}_{i_{j+1}}^{-1} \in s_{i_{j+1}}Bz_j\tau s_{i_{j+1}} = s_{i_{j+1}}Bz_j\tau(s_{i_{j+1}})\tau \subseteq Bz_{j+1}\tau B \cup Bz_j\tau(s_{i_{j+1}})\tau B\cdot$$

Suppose that $\dot{s}_{i_{j+1}}x\dot{s}_{i_{j+1}}^{-1}$ lies in $Bz_j\tau(s_{i_{j+1}})\tau B$. Then by Theorem 4.17, $z_j\tau$ and $z_j\tau(s_{i_{j+1}})\tau = z_j\tau s_{i_{j+1}}$ are involutions. Hence $z_j\tau$ and $s_{i_{j+1}}$ commute, contadicting minimality of length of σ. Thus $\dot{s}_{i_{j+1}}x\dot{s}_{i_{j+1}}^{-1}$ lies in $Bz_{j+1}\tau B$, and $\gamma \cap Bz_{j+1}\tau B \neq \emptyset$. □

We recall that if γ is a conjugacy class in $G^\circ\tau$, then w_γ is the unique element in W such that $\overline{\gamma \cap Bw_\gamma\tau B} = \overline{\gamma}$ and w_γ is the unique maximal length element in its τ-twisted conjugacy class, i.e. w_γ lies in $W_{\tau,m}$. Hence w_γ has the form recalled in Remark 4.14: $w_\gamma = w_J w_0$ with the following properties:

(i) $\tau(J) = J$;

(ii) $\Delta_J = \{\alpha \in \Delta \mid w_J w_0 \tau(\alpha) = \alpha\}$;

(iii) w_γ is both an involution and a τ-twisted involution.

We have the following result involving the maps Υ and Υ'.

Proposition 4.19. Let γ be a spherical unipotent conjugacy class in $G^\circ\tau$, assuming $p = 2$ (a spherical conjugacy class in D_{au}, assuming $p \neq 2$, respectively). Let C^γ be the τ-twisted conjugacy class of w_γ. Then

$$\Upsilon(C^\gamma) = \gamma \quad (\Upsilon'(C^\gamma) = \gamma \quad \textit{respectively})$$

Proof. Let $C = C^\gamma$. We have to prove that if $\sigma \in C_{\min}$ then γ is the minimal element in Γ_σ. We have $\gamma \in \Gamma_{w_\gamma}$ hence $\gamma \in \Gamma_\sigma$ by Lemma 4.18. Let $\gamma' \in \Gamma_\sigma$. By Proposition 4.16, $\gamma' \in \Gamma_{w_\gamma}$, hence $\dim \gamma' \geq l(w_\gamma) + \mathrm{rk}(1 - w_\gamma\tau)$ by Proposition 4.11 (i). Since γ is spherical, by Theorem 4.12, we have that $\dim \gamma = l(w_\gamma) + \mathrm{rk}(1 - w_\gamma\tau)$. Therefore γ is of minimal dimension in Γ_σ and, by uniqueness, $\Upsilon(C) = \gamma$ ($\Upsilon'(C) = \gamma$ respectively). $\square$

To deal with Lusztig's maps Ψ, Ψ', we use the following Lemma.

Lemma 4.20. Let $K \subseteq I$ be such that $w_K w_0 \tau$ is an involution and $w_K w_0 \tau(\alpha) = \alpha$ for all $\alpha \in \Delta_K$. Then

$$\mathrm{rk}(1 - w_0\tau) = \mathrm{rk}(1 - w_K) + \mathrm{rk}(1 - w_K w_0 \tau).$$

Proof. Since $w_K w_0 \tau$ is an involution and w_K, w_0, τ are orthogonal, we have $E_1(w_K w_0 \tau) = E_{-1}(w_K w_0 \tau)^\perp$, where, for a linear map f, $E_\lambda(f)$ is the eigenspace relative to the eigenvalue λ. By hypothesis, $\Delta_K \subseteq E_1(w_K w_0 \tau)$ and w_K, w_0 and τ are orthogonal, hence $E_{-1}(w_K w_0 \tau) \subseteq E_1(w_K)$ since w_K is a product of reflections with respect to roots in Δ_K. If a, b are commuting involutions on E, then $\dim E_{-1}(a) + \dim E_{-1}(b) = \dim E_{-1}(ab)$ if and only if $E_{-1}(a) \cap E_{-1}(b) = \{0\}$. Also, if $a^2 = 1$, then $\dim E_{-1}(a) = \mathrm{rk}(1 - a)$. The involutions $w_K w_0 \tau$ and w_K commute (since w_K and $w_0\tau$ commute), and

$$E_{-1}(w_K w_0 \tau) \cap E_{-1}(w_K) \subseteq E_1(w_K) \cap E_{-1}(w_K) = \{0\}$$

Thus $\dim E_{-1}(w_K) + \dim E_{-1}(w_K w_0 \tau) = \dim E_{-1}(w_0 \tau)$, so $\mathrm{rk}(1 - w_0\tau) =$
$\mathrm{rk}(1 - w_K) + \mathrm{rk}(1 - w_K w_0 \tau)$. $\square$

We prove the main result of this section.

Theorem 4.21. *Let G° be a simple algebraic group, τ a graph-automorphism of G° of order 2. Let γ be a spherical unipotent conjugacy class in $G^\circ\tau$, assuming $p = 2$ (a spherical conjugacy class in D_{au}, assuming $p \neq 2$, respectively). Let C^γ be the τ-twisted conjugacy class of w_γ. Then*

$$\Psi(\gamma) = C^\gamma \quad (\Psi'(\gamma) = C^\gamma \quad \textit{respectively})$$

Proof. By Proposition 4.19, we have $C^\gamma \in \Upsilon^{-1}(\gamma)$ ($C^\gamma \in \Upsilon'^{-1}(\gamma)$ respectively). We need to show that C^γ is the unique class C in $\Upsilon^{-1}(\gamma)$ (in $\Upsilon'^{-1}(\gamma)$) such that $\mu(C)$ is minimal. Let C be in $\Upsilon^{-1}(\gamma)$ (in $\Upsilon'^{-1}(\gamma)$). Assume $C \neq C^\gamma$. By Proposition 4.16, $\gamma \cap Bz\tau B \neq \emptyset$ for every element z of maximal length in C. We prove that $\dim E^{z\tau} > \dim E^{w_\gamma\tau}$. Since γ is spherical, z is a τ-twisted involution by Theorem 4.17. Thus, by [63, Proposition 3.5, Lemma 3.2], $z = w_K w_0$ for a certain $K \subseteq I$ such that $\Delta_K = \{\alpha \in \Delta \mid z\tau\alpha = \alpha\}$. In particular $(w_0\tau)_{|\Delta_K} = w_{K_{|\Delta_K}}$. The same reasoning applies to w_γ: we have that $w_\gamma = w_J w_0$ for a certain $J \subseteq I$ such that $\Delta_J = \{\alpha \in \Delta \mid w_\gamma\tau\alpha = \alpha\}$. In particular $(w_0\tau)_{|\Delta_J} = w_{J_{|\Delta_J}}$. Moreover $z \leq w_\gamma$, i.e. $w_J \leq w_K$, hence $J \subseteq K$, and

$$w_{K_{|\Delta_J}} = (w_0\tau)_{|\Delta_J} = w_{J_{|\Delta_J}}.$$

Applying [10, Lemma 2.7] to Φ_K and $w_K w_J$, we get

$$\mathrm{rk}(1 - w_K) = \mathrm{rk}(1 - w_K w_J) + \mathrm{rk}(1 - w_J).$$

If $\mathrm{rk}(1 - w_K w_J) = 0$, then $1 - w_K w_J = 0$, i.e. $w_K = w_J$, $K = J$, $z = w_\gamma$, contrary to the assumption $C \neq C^\gamma$. Hence $\mathrm{rk}(1 - w_K) > \mathrm{rk}(1 - w_J)$. Thus

$$\mathrm{rk}(1 - w_J w_0\tau) = \mathrm{rk}(1 - w_0\tau) - \mathrm{rk}(1 - w_J) >$$

$$> \mathrm{rk}(1 - w_0\tau) - \mathrm{rk}(1 - w_K) = \mathrm{rk}(1 - w_K w_0\tau),$$

where the first and the last equality follow from Lemma 4.20. Thus $\dim E^{z\tau} > \dim E^{w_\gamma\tau}$. By Proposition 4.5, $\mu(C) > \mu(C^\gamma)$ and we are done. □

Remark 4.22. Let $\widetilde{\nu}$ be the map defined by $\widetilde{\nu}(\gamma) = W \cdot_\tau w_\gamma$ for any conjugacy class γ in $G^\circ\tau$. The maps $\widetilde{\nu}$ and Ψ do not coincide on the full set $\underline{G^\circ\tau}$ of unipotent conjugacy classes in $G^\circ\tau$ since Ψ is necessarily injective whereas $\widetilde{\nu}$ is not. Indeed, let γ be the (unique) maximal unipotent conjugacy class in $G^\circ\tau$. Then $w_\gamma = w_0$. Let γ_{reg} be the (unique) regular unipotent conjugacy class in $G^\circ\tau$. Then one always has $\overline{\gamma} \subset \overline{\gamma_{reg}}$, a proper inclusion, $w_0 = w_\gamma \leq w_{\gamma_{reg}} \leq w_0$, so that $\widetilde{\nu}(\gamma) = \widetilde{\nu}(\gamma_{reg}) = W \cdot_\tau w_0$.

Remark 4.23. If G° is of type D_4 and θ has order 3, then Lemma 4.18 does not hold. In fact let γ_{14} be the conjugacy class of θ in $G^\circ\theta$. Then $\gamma_{14} \cap Bw_0s_2\theta B \neq \emptyset$. Let $C = W \cdot_\theta w_0s_2$. Then $C_{\min} = \{s_1, s_3, s_4\}$ but clearly $\gamma_{14} \cap Bs_i\theta B = \emptyset$ for $i = 1, 3, 4$. There is a unique θ-twisted conjugacy class in the fiber of Υ over γ_{14}, namely $W \cdot_\theta 1$ (denoted by $\tilde{A}_2$ in [44, p. 465]). Then $\Psi(\gamma_{14}) = \tilde{A}_2 \neq C$, hence Theorem 4.21 does not hold.

We conclude this section by emphasizing a property of $W_{\theta,m}$. This had been previously proved by X. He in [27, Corollary 4.5] (note that He's statement refers to minimal length elements, but multiplying by w_0 one may obtain the result for maximal length elements, by a different twist).

Theorem 4.24. *The set $W_{\theta,m}$ coincides with the set of elements w in W such that w is the maximum element its θ-twisted conjugacy class with respect to the Bruhat order.*

Proof. Let C be a θ-twisted conjugacy class. If w is the maximum in C with respect to the Bruhat order, then clearly w is the unique maximal length element in C, i.e. w lies in $W_{\theta,m}$. Conversely, assume that w is in $W_{\theta,m}$ and let $z \in C$. If θ has order 2, by Proposition 4.15 we have $w = w_\gamma$ for some spherical unipotent conjugacy class γ in $G^\circ\theta$. By Lemma 4.18, $\gamma \cap Bz\theta B \neq \emptyset$, hence $z \leq w$. If θ is of order 3 (in type D_4), then either $w = w_0$, or $w = w_0s_2$. If $w = w_0$, then $z \leq w$. For $w = w_0s_2$, we checked by direct calculation that $z \leq w_0s_2$. □

4.4 Final remarks

Remark 4.25. In a private communication Lusztig has informed us that there is a misprint in the value of Υ for the τ-twisted class $2A_1$ in the Weyl group of E_6 at the end of Section 2.3 in [44]: $\Upsilon(2A_1)$ is not γ_{52}. In fact, if $\gamma_{52} \cap Bw\tau B \neq \emptyset$ for some $w \in 2A_1$, then by Lemma 4.18, $\gamma_{52} \cap Bw'\tau B \neq \emptyset$ for every $w' \in 2A_1$. In $2A_1$ there is the element $w' = s_{\beta_1}s_{\beta_2}$ of length 30, where β_1 is the highest root in E_6 and β_2 is the highest root in the Levi subgroup A_5 of E_6. Then, by Proposition 4.11, we would have $\dim \gamma_{52} = 26 \geq 30$, a contradiction. Let n_{β_1} and n_{β_2} be lifts in $N_{G^\circ}T$ of s_{β_1} and s_{β_2} respectively. The element $n_{\beta_1}n_{\beta_2}\tau$ is an involution in $Bw'\tau B$: it cannot lie in γ_{52}, hence it lies in γ_{36}. Therefore $\gamma_{36} \cap Bw'\tau B \neq \emptyset$. If $w_{\min}$ is a minimal length element in $2A_1$, then $\gamma_{36} \cap Bw_{\min}\tau B \neq \emptyset$ and $\Upsilon(2A_1) = \gamma_{36}$.

In the connected case the fibre of Υ over the minimal unipotent class $\{1\}$ in G° always has a single element, namely the conjugacy class of 1 in W. This is not the case in general in the disconnected case. Let G° be of type A_n, $n \geq 2$, n even. Then there is a unique conjugacy class γ of involutions in $G^\circ\tau$, the class of τ, and this is the unique minimal unipotent class in $G^\circ\tau$. In W there is a unique τ-twisted conjugacy class with a unique maximal length element, the class of w_0, so that $Bw_0\tau B$ intersects every class in $G^\circ\tau$. In particular we have $\gamma \cap Bw_0\tau B \neq \emptyset$ (see also [18, §4.1]), hence $\varphi(W \cdot_\tau w_0) = \gamma$. On the other hand $\varphi(W \cdot_\tau 1)$ is always the minimal unipotent class in $G^\circ\tau$, since $\tau \in \tau B$. Therefore $\varphi(W \cdot_\tau w_0) = \varphi(W \cdot_\tau 1) = \gamma$ but $W \cdot_\tau 1 \neq W \cdot_\tau w_0$ since $W \cdot_\tau w_0 = \{w_0\}$. In fact in this case the fibre of φ over γ consists of all $\frac{n}{2} + 1$ classes of τ-twisted involutions of W. However one has $\Psi(\gamma) = W \cdot_\tau w_0$ since $\mu(W \cdot_\tau w_0) = 0$.

Remark 4.26. In [59] Spaltestein introduces a partial order $\leq$ on the set of conjugacy classes in W by means of the loop group of G°, where G° is over $\mathbb{C}$. A priori this partial order depends on the Lie algebra of G°. It is shown in [68, Corollary 4.6] that in fact $\leq$ is independent of G°. Moreover, over the complex numbers, in [68, Corollary 11.1] it is proved that

$\Upsilon : (\underline{W}, \leq) \to (\underline{G^\circ}, \leq)$ is order-preserving;

$\Psi : (\underline{G^\circ}, \leq) \to (\mathrm{Im}\Psi_\circ, \leq)$ is an isomorphism of posets.

In [27, §4.7], He introduces a partial order $\leq_{He}$ on the set $\underline{W}_{ell}$ of elliptic conjugacy classes of W as follows: $C \leq_{He} C'$ if and only if there exist minimal length elements $w \in C$ and $w' \in C'$ such that $w \leq w'$ under the Bruhat order. In any characteristic, the map Υ restricted to $\underline{W}_{ell}$ is injective: let $\underline{G}_{bas}$ be its image (the subscript bas stands for "basic", a terminology introduced by Lusztig in [42, §4.1] and used in [68]).

In [1, Theorem 1.1] the authors prove that $\Upsilon : (\underline{W}_{ell}, \leq_{He}) \to (\underline{G}_{bas}, \leq)$ is order-reversing. Therefore the partial orders $\leq_{He}$ and $\leq$ on $\underline{W}_{ell}$ are opposite to each other.

On the other hand, in [10, Remark 2] it is observed that we may endow the set $\underline{W}_m$ (of conjugacy classes in W with a unique maximal length element) with a partial order, as follows: $C \leq_m C'$ if and only if $w \leq w'$ in the Bruhat order, where w (w' respectively) is the maximal length element in C (in C'). In any characteristic p, the map $\nu : \underline{G}_{sph} \to \underline{W}_m$, $\gamma \mapsto W \cdot w_\gamma$ is a poset isomorphism onto its image, and there exists p for which ν is also surjective. Therefore the partial orders $\leq_m$ and $\leq$ on $\underline{W}_m$ are the same.

To our knowledge no partial order on $\underline{W}_\theta$, on the lines of [59], has been defined. If such an order exists, at least when θ has order 2, it may coincide with the opposite order on $\underline{W}_{\theta,ell}$ of elliptic θ-twisted classes in W, as defined in [27, §4.7] and with the corresponding order on $\underline{W}_{\theta,m}$ of θ-twisted classes in W with a unique maximal length element.

Part II

Seshadri stratifications

Notations and preliminaries

The reader can find the unexplained notations in [26].

We fix an algebraically closed field $\mathbb{K}$.
A projective variety $Y \subseteq \mathbb{P}(V)$ for V a finite dimensional $\mathbb{K}$-vector space is called *embedded projective variety*.
We denote by V^* the dual space of the vector space V.

Let $\mathbb{A}^{t+1}_{\mathbb{K}}$ be the affine space of dimension $t+1$ over $\mathbb{K}$. We consider the projection map

$$\varphi : \mathbb{A}^{t+1}_{\mathbb{K}} \setminus \{0\} \longrightarrow \mathbb{P}(V).$$

Given a projective variety $Y \subseteq \mathbb{P}(V)$, then $\varphi^{-1}(Y) \cup \{0\}$ is the *affine cone* in V over Y, and we denote it by $\hat{Y}$. The ring of rational functions on $\hat{Y}$ is denoted by $\mathbb{K}[\hat{Y}]$, and its quotient field is denoted by $\mathbb{K}(\hat{Y})$.
The sheaf of functions of Y is denoted by $\mathcal{O}_Y$. Let $Z \subset Y$ be a prime divisor of Y, that is, an irreducible subvariety of codimension one. Let η be the generic point of Z. By $\mathcal{O}_{Y,Z}$ we denote the stalk of the sheaf $\mathcal{O}_Y$ at the point η. We observe that $\mathcal{O}_{Y,Z}$ is a local ring with Krull dimension one. Let $f \in \mathrm{Sym}\,(V^*)$ be a homogeneous polynomial, the vanishing set of f is a hypersuperface and we denote it by $\mathcal{H}_f := \{[v] \in \mathbb{P}(V) \mid f(v) = 0\}$.

Let $\mathcal{A}$ be a finite set with a partial order $\leq$, in the following we call the structure $(\mathcal{A}, \leq)$ a *poset*. For every $\tau \in \mathcal{A}$ we set $\mathcal{A}_\tau := \{\sigma \in \mathcal{A} \mid \sigma \leq \tau\}$. So, for every $\tau \in \mathcal{A}$, the finite set $\mathcal{A}_\tau$ is a poset inheriting the partial order from $\mathcal{A}$.
Let $\tau, \sigma \in \mathcal{A}$, if $\sigma < \tau$ and there is no element $\sigma' \in \mathcal{A}$ such that $\sigma < \sigma' < \tau$, we say that $\sigma < \tau$ is a cover relation and we denote it by $\sigma \prec \tau$. A *chain* $\mathfrak{C}$ in $\mathcal{A}$ is a decreasing sequence

$(\tau_1, \ldots, \tau_h)$ such that $\tau_i \in \mathcal{A}$ for all $i = 1, \ldots, h$ and $\tau_i \succ \tau_{i+1}$ for all $i = 1, \ldots, h-1$ for some $h \in \mathbb{N}$. If τ_1 is a maximal element in $\mathcal{A}$ and τ_h is a minimal element in $\mathcal{A}$ we say that $\mathfrak{C}$ is a *maximal chain* in $\mathcal{A}$.

Quasi-valuations

Definition 1. *Let $\mathcal{R}$ be a $\mathbb{K}$-algebra. A quasi-valuation on $\mathcal{R}$ with values in a totally ordered abelian group $\mathbb{G}$ is a map $\nu : \mathcal{R}\backslash\{0\} \to \mathbb{G}$ satisfying the following conditions:*

(1) $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ *for all* $x, y \in \mathcal{R}\backslash\{0\}$ *with* $x + y \neq 0$*;*

(2) $\nu(\lambda x) = \nu(x)$ *for all* $x \in \mathcal{R}\backslash\{0\}$ *and* $\lambda \in \mathbb{K}^*$*;*

(3) $\nu(xy) \geq \nu(x) + \nu(y)$ *for all* $x, y \in \mathcal{R}\backslash\{0\}$ *with* $xy \neq 0$.

The map ν is called a valuation if the inequality in (3) can be replaced by an equality:

(c') $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \mathcal{R}\backslash\{0\}$ with $xy \neq 0$.

Lemma 2. *[16, Lemma 3.4] Let $\nu, \nu_1, \ldots, \nu_k : \mathcal{R}\backslash\{0\} \to \mathbb{G}$ be quasi-valuations and let $x, y \in \mathcal{R}\backslash\{0\}$.*

(a) If $\nu(x) \neq \nu(y)$*, then* $\nu(x+y) = \min\{\nu(x), \nu(y)\}$.

(b) If $x + y \neq 0$ *and* $\nu(x+y) > \nu(x)$*, then* $\nu(x) = \nu(y)$.

(c) The map $\mathcal{R}\backslash\{0\} \to \mathbb{G}, x \mapsto \min\{\nu_j(x) \mid j = 1, \ldots, k\}$ *defines a quasi-valuation on* $\mathcal{R}$.

Natural examples of valuations arise from vanishing orders of functions.

Example 3. *Consider Y an irreducible variety, and $Z \subset Y$ a prime divisor of Y. Let $f \in \mathcal{O}_{Y,Z}$, such that f is non-zero. The vanishing order of f on Z is defined as the length of $\mathcal{O}_{Y,Z}/(f)$, we denote it by $\mathrm{ord}_Z(f)$. This length is finite, and it is additive with respect to multiplication, that is, $\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$.*
Let g and h be homogeneous polynomials in $\mathrm{Sym}(V^*)$ of same degree. The order of vanishing of $\frac{g}{h}$ on Z is defined to be $\mathrm{ord}_Z(g) - \mathrm{ord}_Z(h)$. With this definition, the order of vanishing is a function $\mathrm{ord}_Z : \mathbb{K}(\hat{Y}) \setminus \{0\} \to \mathbb{Z}$. If Y is smooth in codimension one [26, p.130], then the local ring $\mathcal{O}_{Y,Z}$ is a discrete valuation ring, and the function ord_Z is the corresponding valuation.

CHAPTER 5

Seshadri stratifications

All the definitions and results in this chapter are to be found in [16]. We recall them here for the reader's convenience.

Let $X \subseteq \mathbb{P}(V)$ be an embedded projective variety with graded homogeneous coordinate ring $\mathcal{R} := \mathbb{K}[X]$. We consider a collection of projective subvarieties X_τ in X, indexed by a finite set $\mathcal{A}$ and the partial order on $\mathcal{A}$ given by the inclusion of the subvarieties X_τ, namely for $\tau, \sigma \in \mathcal{A}$

$$\sigma \leq \tau \iff X_\sigma \subseteq X_\tau. \tag{5.1}$$

We assume that there exists a unique maximal element $\tau_{\max}$ in $\mathcal{A}$ with $X_{\tau_{\max}} = X$. For each $\tau \in \mathcal{A}$, we fix a homogeneous function $f_\tau \in \operatorname{Sym}(V^*)$ of positive degree.

Definition 5.1. The data $(X_\tau, f_\tau)_{\tau \in \mathcal{A}}$ is called a *Seshadri stratification* if the following conditions are satisfied:

(S1) the projective varieties X_τ, for $\tau \in \mathcal{A}$, are smooth in codimension one and if $\sigma \prec \tau$ is a covering relation in $\mathcal{A}$, then $X_\sigma \subset X_\tau$ is a codimension one subvariety;

(S2) for any $\tau, \sigma \in \mathcal{A}$ such that $\sigma \not\geq \tau$, the function f_τ vanishes on X_σ;

(S3) for $\tau \in \mathcal{A}$, the set-theoretical intersection satisfies

$$\mathcal{H}_{f_\tau} \cap X_\tau = \bigcup_{\sigma \prec \tau} X_\sigma.$$

In a Seshadri stratification, the functions f_τ are called *extremal functions*.

If $\tau \in \mathcal{A}$ is a minimal element, then there is no $\sigma \in \mathcal{A}$ such that $\sigma \prec \tau$. So, by (3), $\mathcal{H}_{f_\tau} \cap X_\tau = \emptyset$.

With the next lemma, we show two useful properties of a Seshadri stratification.

Lemma 5.2. Let $(X_\tau, f_\tau)_{\tau \in \mathcal{A}}$ be a Seshadri stratification of X. Let $\tau, \tau' \in \mathcal{A}$. Then

(1) the restriction of the function f_τ to $X_{\tau'}$ is not identically zero on $X_{\tau'}$ if and only if $\tau' \geq \tau$,

(2) all maximal chains in $\mathcal{A}$ have the same length, which coincides with $\dim X$.

Proof.

(1) If $f_{\tau|X_{\tau'}} \not\equiv 0$, then, by (S2), we have that $\tau' \geq \tau$. '

In order to prove the converse, by (S1) it is enough to prove that the restriction of f_τ to X_τ is non-zero because if $\tau' \geq \tau$ then $X_\tau \subseteq X_{\tau'}$ and so the restriction of f_τ to $X_{\tau'}$.

we show, first, that f_τ restricted to X_τ is not identically zero on X_τ.

Let $\tau \in \mathcal{A}$. If X_τ is a point, then $\bigcup_{\sigma \prec \tau} X_\sigma = \emptyset$. Hence, by (S3), the intersection $\mathcal{H}_{f_\tau} \cap X_\tau = \emptyset$, therefore f_τ does not vanish on X_τ. Assume $\dim X_\tau > 0$ and let $\sigma \in \mathcal{A}$ be such that $\sigma \prec \tau$. By (S1) the subvariety X_σ has codimension one in X_τ. Hence, by the finiteness of $\mathcal{A}$, the union $\bigcup_{\sigma \prec \tau} X_\sigma$ is a proper subvariety of X_τ. Thus f_τ does not identically vanish on X_τ.

(2) Let $\tau \in \mathcal{A}$. If $\dim X_\tau > 0$, then, since f_τ is a homogeneous function of positive degree, the intersection $\mathcal{H}_{f_\tau} \cap X_\tau$ is not empty. Hence, by (S3), the union $\bigcup_{\sigma \prec \tau} X_\sigma$ is not empty. Thus there exists $\sigma \in \mathcal{A}$ such that X_σ is a subvariety of X_τ of codimension one. This gives (2).

$\square$

Remark 5.3. It follows from Lemma 5.2(2) that, if $\tau_0 \in \mathcal{A}$ is a minimal element, then X_{τ_0} is 0-dimensional variety.

Remark 5.4. It is useful to extend the stratification of $X \subseteq \mathbb{P}(V)$ to the affine cone $\hat{X}$. For a minimal element $\tau \in \mathcal{A}$, since $\dim X_\tau = 0$, the affine cone $\hat{X}_\tau$ is $\mathbb{A}^1$. We set $\hat{\mathcal{A}} := \mathcal{A} \cup \{\tau_{-1}\}$ with $\hat{X}_{\tau_{-1}} := \{0\} \subset V$. Since the variety $\hat{X}_{\tau_{-1}}$ is contained in the affine cone $\hat{X}_\tau$ for any minimal element $\tau \in \mathcal{A}$, the set $\hat{\mathcal{A}}$ inherits a poset structure by requiring τ_{-1} to be the unique minimal element.

By Lemma 5.2 (2), we can give the following definition.

Definition 5.5. Let $\tau \in \mathcal{A}$. The length $\ell(\tau)$ of τ is the length of a (then any) maximal chain in $\mathcal{A}_\tau$.

By Lemma 5.2 (2),

$$\ell(\tau) = \dim X_\tau.$$

Remark 5.6. A Seshadri stratification $(X_\tau, f_\tau)_{\tau \in \mathcal{A}}$ for X induces a Seshadri stratification on every strata X_τ. Indeed, $(X_\sigma, f_\sigma)_{\sigma \in \mathcal{A}_\tau}$, satisfies the conditions (S1)-(S3), and hence defines a Seshadri stratification for X_τ.

Proposition 5.7. [16, Proposition 2.11] Every embedded projective variety $X \subseteq \mathbb{P}(V)$ that is smooth in codimension one admits a Seshadri stratification.

Example 5.8. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{K}^4$. The wedge products $e_i \wedge e_j$, with $1 \leq i < j \leq 4$, form a basis of $\bigwedge^2 \mathbb{K}^4$. Let $\{x_{i,j} \in (\Lambda^2(\mathbb{K}^4))^* \mid 1 \leq i < j \leq 4\}$ be the dual basis of $\{e_i \wedge e_j \mid 1 \leq i < j \leq 4\}$, with $x_{i,j}$ the dual of $e_i \wedge e_j$. The elements $x_{i,j}$ of the dual basis, with $1 \leq i < j \leq 4$, are called *Plücker coordinates*.
Let $I_{2,4} = \{(i_1, i_2) \in \mathbb{N}^2 \mid 1 \leq i_1 < i_2 \leq 4\}$. We endow $I_{2,4}$ with a partial order as follows:

$$(i,j) \leq (k,\ell) \iff i \leq k \text{ and } j \leq \ell. \tag{5.2}$$

Let $X := \mathrm{Gr}_2\mathbb{K}^4$ be the Grassmann variety of 2-dimensional subspaces of $\mathbb{K}^4$. There exists an embedding, namely the Plücker embedding, of X in $\mathbb{P}\left(\bigwedge^2 \mathbb{K}^4\right)$ (see, for example, [57, Chapter 13]).
For any $(i,j) \in I_{2,4}$ there is a variety $X_{i,j} \subseteq \mathrm{Gr}_2\mathbb{K}^4$, called Schubert variety, set theoretically $X_{i,j}$ is defined by

$$X_{i,j} := \left\{[v] \in \mathrm{Gr}_2\,\mathbb{K}^4 \mid x_{k,\ell}([v]) = 0 \ \forall (k,l) \in I_{2,4} \text{ such that } (k,l) \not\leq (i,j)\right\}.$$

The partial order defined in (5.2) corresponds to the partial order given by the inclusion of Schubert varieties.
Let $f_{i,j} := x_{i,j} \in \mathbb{K}[X]$, with $(i,j) \in I_{2,4}$ be a Plücker coordinate.

The data $(X_{i,j}, f_{i,j})_{I_{2,4}}$ form a Seshadri stratification of X.

5.1 Valuations on divisors

Let $\mathcal{R}_\tau := \mathbb{K}[X_\tau]$ be the homogeneous coordinate ring of X_τ with respect to the embedding $X_\tau \subseteq X \subseteq \mathbb{P}(V)$. In the following we often consider $\mathcal{R}_\tau$ as the coordinate ring of the affine cone $\hat{X}_\tau \subseteq V$ over X_τ.

If $\sigma \prec \tau$ in $\hat{\mathcal{A}}$, then $\hat{X}_\sigma \subseteq \hat{X}_\tau$ is a prime divisor in $\hat{X}_\tau$. As in Example 3, the local ring $\mathcal{O}_{\hat{X}_\tau,\hat{X}_\sigma}$ is a discrete valuation ring because $\hat{X}_\tau$ is smooth in codimension one by (S1). Let $\nu_{\tau,\sigma}$ be the associated valuation. We refer to the value $\nu_{\tau,\sigma}(f)$ for $f \in \mathcal{R}_\tau \setminus \{0\}$ as the *vanishing multiplicity* of f in the divisor $\hat{X}_\sigma$:

$$\nu_{\tau,\sigma} : \mathcal{R}_\tau \setminus \{0\} \to \mathbb{Z}. \tag{5.3}$$

The valuation $\nu_{\tau,\sigma}$ can be naturally extended to a valuation on the quotient field of $\mathcal{R}_\tau$, namely $\mathbb{K}(\hat{X}_\tau)$, as follows. If $\frac{g}{h} \in \mathbb{K}(\hat{X}_\tau)$ and it is different from 0, then $g, h \in \mathcal{R}_\tau \setminus \{0\}$. We set

$$\nu_{\tau,\sigma}\left(\frac{g}{h}\right) := \nu_{\tau,\sigma}(g) - \nu_{\tau,\sigma}(h). \tag{5.4}$$

Remark 5.9. Let τ_0 be a minimal element in $\mathcal{A}$. Then, since X_{τ_0} is a 0-dimensional projective variety, the ring $\mathcal{R}_{\tau_0}$ is a polynomial ring. Let τ_{-1} be a minimal element in $\hat{\mathcal{A}}$. We define $\nu_{\tau_0,\tau_{-1}}$ to be the vanishing multiplicity of a polynomial in $\mathcal{R}_{\tau_0}$ in $\{0\}$. In particular $\nu_{\tau_0,\tau_{-1}}(f_{\tau_0})$ is just the degree of f_{τ_0}.

Definition 5.10. Let $(X_\tau, f_\tau)_\mathcal{A}$ be a Seshadri stratification of X. Let $\sigma, \tau \in \mathcal{A}$ be such that $\sigma \prec \tau$. We call the value $\nu_{\tau,\sigma}(f_\tau)$ the *bond* between τ and σ, and we denote it by $\mathfrak{b}_{\tau,\sigma}$.

By the property (S2), the bond $\mathfrak{b}_{\tau,\sigma}$ is a positive integer.

5.2 Hasse diagrams with bonds

Let $(X_\tau, f_\tau)_\mathcal{A}$ be a Seshadri stratification of X. In this section we associate an edge-coloured directed graph, that we call the *Hasse diagram*, to $(X_\tau, f_\tau)_\mathcal{A}$.

The Hasse diagram $\mathcal{G}_\mathcal{A}$ associated with the Seshadri stratification $(X_\tau, f_\tau)_\mathcal{A}$ is a graph in which the vertices are the elements $\tau \in \mathcal{A}$ and for any cover $\sigma \prec \tau$ there is a directed edge colored with the bond $\mathfrak{b}_{\tau,\sigma}$ as follows

$$\sigma \xrightarrow{\mathfrak{b}_{\tau,\sigma}} \tau$$

Remark 5.11. We extend the construction to the poset $\hat{\mathcal{A}}$ (defined in Remark 5.4). For a minimal element $\tau_0 \in \mathcal{A}$, the bond $\mathfrak{b}_{\tau_0,\tau_{-1}}$ is defined to be the vanishing multiplicity of f_{τ_0} at $\hat{X}_{\tau_{-1}} = \{0\}$, which coincides with the degree of f_{τ_0}.

Example 5.12. We consider the Seshadri stratification on $\mathrm{Gr}_2\mathbb{K}^4$ defined in Example 5.8. The associated Hasse diagram is the following.

$$\begin{tikzcd}
& & (2,3) \arrow[dl, "1"'] & & \\
(3,4) & (2,4) \arrow[l, "1"'] & & (1,3) \arrow[ul, "1"'] \arrow[dl, "1"] & (1,2) \arrow[l, "1"'] \\
& & (1,4) \arrow[ul, "1"] & &
\end{tikzcd}$$

5.3 Valuations on maximal chains

Let $\mathfrak{C} = (\tau_r, \tau_1, \cdots, \tau_0)$ be a maximal chain in $\mathcal{A}$. Clearly the elements in $\mathfrak{C}$ are totally ordered. We consider the vector space $\mathbb{Q}^{\mathfrak{C}} = \langle e_{\tau_j} \mid \tau_j \in \mathfrak{C} \rangle$, where e_{τ_j} is a vector of the standard basis. We endow $\mathbb{Q}^{\mathfrak{C}}$ with the lexicographic order defined as follows. Let $\underline{a} = \sum_{j=0}^{r} a_j e_{\tau_j}$ and $\underline{a'} = \sum_{j=0}^{r} a'_j e_{\tau_j} \in \mathbb{Q}^{\mathfrak{C}}$,

$$(5.5) \qquad \underline{a'} > \underline{a} \iff \exists\, 0 \leq j \leq r \text{ such that } \begin{cases} a'_r = a_r, a'_{r-1} = a_{r-1}, \dots, a'_{j+1} = a_{j+1} \\ a'_j > a_j \end{cases}.$$

Let N be the least common multiple of all bonds appearing in the Hasse diagram $\mathcal{G}_{\mathcal{A}}$.

To simplify the notation, we set $X_j := X_{\tau_j}$, $f_j := f_{\tau_j}$, $\nu_{j,j-1} := \nu_{\tau_j,\tau_{j-1}}$ and $\mathfrak{b}_j := \mathfrak{b}_{\tau_j,\tau_{j-1}}$.

Let $g_r := g \in \mathbb{K}(\hat{X}_{\tau_r})$ be a non-zero rational function. We define

$$g_{r-1} := \left(\frac{g_r^N}{f_r^{N\frac{\nu_{r,r-1}(g_r)}{\mathfrak{b}_r}}} \right)_{|X_{r-1}}$$

By [16, Lemma 4.1] the function g_{r-1} is a well-defined non-zero rational function in $\mathbb{K}(\hat{X}_{r-1})$. Iterating this procedure we set

$$g_{j-1} := \left(\frac{g_j^N}{f_j^{N\frac{\nu_{j,j-1}(g_j)}{\mathfrak{b}_j}}} \right)_{|\mathbb{K}(\hat{X}_{j-1})}, \text{ for } j = r, r-1, \dots, 1.$$

In this way we obtain a sequence of rational functions $g_{\mathfrak{C}} := (g_r, g_{r-1}, \dots, g_0)$ with $g_j \in \mathbb{K}(\hat{X}_{\tau_j}) \setminus \{0\}$ for $j = 0, \dots, r$. We define a map

$$\mathcal{V}_{\mathfrak{C}} : \mathcal{R} \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}} \tag{5.6}$$

$$g \mapsto \frac{\nu_r(g_r)}{\mathfrak{b}_r} e_{\tau_r} + \frac{1}{N} \frac{\nu_{r-1}(g_{r-1})}{b_{r-1}} e_{\tau_{r-1}} + \dots + \frac{1}{N^r} \frac{\nu_0(g_0)}{\mathfrak{b}_0} e_{\tau_0}.$$

We can extend the definition of $\mathcal{V}_{\mathfrak{C}}$ on $\mathbb{K}(\hat{X}) \setminus \{0\}$ as follows

$$\mathcal{V}_{\mathfrak{C}} : \mathbb{K}(\hat{X}) \setminus \{0\} \longrightarrow \mathbb{Q}^{\mathfrak{C}} \tag{5.7}$$

$$\frac{f}{g} \mapsto \mathcal{V}_{\mathfrak{C}}(f) - \mathcal{V}_{\mathfrak{C}}(g).$$

By [16, Proposition 6.10], the map $\mathcal{V}_{\mathfrak{C}}$ is a valuation.

5.4 Quasi-valuations

We define, now, a quasi-valuation associated with the Seshadri stratification $(X_\tau, f_\tau)_{\mathcal{A}}$ of X. We fix a total order $\leq_t$ on $\mathcal{A}$ refining the partial order of $\mathcal{A}$. We consider the vector space $\mathbb{Q}^{\mathcal{A}} = \langle e_\tau \mid \tau \in \mathcal{A} \rangle$. We endow $\mathbb{Q}^{\mathcal{A}}$ with a total order taking the lexicographic order defined as in equation (5.5).

In the following for any maximal chain $\mathfrak{C}$ we consider the vector space $\mathbb{Q}^{\mathfrak{C}}$, spanned by e_τ, with $\tau \in \mathfrak{C}$, as a subspace of $\mathbb{Q}^{\mathcal{A}}$. In this way, it makes sense to write

$$\mathcal{V}_{\mathfrak{C}}(g) \in \mathbb{Q}^{\mathcal{A}}$$

for a regular function $g \in \mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$.

By Lemma 2(c), the minimum over a finite list of valuations is a quasi-valuation. So we can give the following definition.

Definition 5.13.

(1) We define the *quasi-valuation*

$$\mathcal{V} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^{\mathcal{A}}$$

$$\mathcal{V}(g) := \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \text{ a maximal chain in } \mathcal{A}\}.$$

(2) For $\underline{a} = (a_\tau)_{\tau\in\mathcal{A}} \in \mathbb{Q}^{\mathcal{A}}$, the *support* of $\underline{a}$ is defined by

$$\operatorname{supp}(\underline{a}) := \{\tau \in \mathcal{A} \mid a_\tau \neq 0\}.$$

In the next proposition we recall some useful properties of the quasi-valuation $\mathcal{V}$.

Proposition 5.14. With notation as above, we have the following properties.

(1) For any $\tau \in \mathcal{A}$, we have $\mathcal{V}(f_\tau) = e_\tau$, for any choice of the total order $\leq_t$;

(2) For all $g \in \mathcal{R} \setminus \{0\}$, we have $\mathcal{V}(g) \in \mathbb{Q}^{\mathcal{A}}_{\geq 0}$;

(3) For $g, h \in \mathcal{R} \setminus \{0\}$, we have $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$ if and only if

$$\{\mathfrak{C} \text{ maximal chain in } \mathcal{A} \mid \mathcal{V}_{\mathfrak{C}}(g) = \mathcal{V}(g)\} \cap \{\mathfrak{C} \text{ maximal chain in } \mathcal{A} \mid \mathcal{V}_{\mathfrak{C}}(h) = \mathcal{V}(h)\} \neq \emptyset.$$

Proof. The proof of this properties is given in [16], in particular (1) is Lemma 8.3 in *loc.cit*, (2) is Proposition 8.6 in *loc.cit* and (3) is Proposition 8.9 in *loc.cit*. □

Let $\Gamma := \{\mathcal{V}(g) \mid g \in \mathbb{K}[\hat{X}] \setminus \{0\}\} \subseteq \mathbb{Q}^{\mathcal{A}}$ be the image of the quasi-valuation.
Let $\mathfrak{C}$ be a maximal chain in $\mathcal{A}$. We set

$$\Gamma_{\mathfrak{C}} := \{\underline{a} \in \Gamma \mid \operatorname{supp} \underline{a} \subseteq \mathfrak{C}\}. \tag{5.8}$$

By [16, Corollary 3.5] each $\Gamma_{\mathfrak{C}}$ is a monoid. By definition, we have

$$\Gamma = \bigcup_{\mathfrak{C}} \Gamma_{\mathfrak{C}}, \tag{5.9}$$

where $\mathfrak{C}$ runs over all the maximal chains in $\mathcal{A}$.

A finite union of monoids is called a *fan of monoids*. In particular we refer to Γ as the fan of monoids associated with the quasi-valuation $\mathcal{V}$.

5.5 The LS-fan of monoids

We fix a maximal chain $\mathfrak{C} = (\tau_r, \ldots, \tau_0)$ in $\mathcal{A}$. We simplify the notation by writing $\mathfrak{b}_i$ instead of $\mathfrak{b}_{\tau_i,\tau_{i-1}}$ for the bonds and f_i instead of f_{τ_i} for the extremal functions.

Definition 5.15. The lattice $\mathrm{LS}_{\mathfrak{C}}$ defined as follows

$$\mathrm{LS}_{\mathfrak{C}} := \left\{ \underline{a} = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}^{\mathfrak{C}} \;\middle|\; \begin{array}{r} \mathfrak{b}_r a_r \in \mathbb{Z} \\ \mathfrak{b}_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\ \dots \\ \mathfrak{b}_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\ \mathfrak{b}_0(a_0 + a_1 + \ldots + a_r) \in \mathbb{Z} \end{array} \right\}, \tag{5.10}$$

is called the *Lakshmibai-Seshadri lattice* (shortly *LS-lattice*) associated with $\mathfrak{C}$.

We denote by $\mathrm{LS}^+_{\mathfrak{C}}$ the monoid obtained as $\mathrm{LS}_{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}$.

Definition 5.16. We set

$$\mathrm{LS}^+ := \bigcup_{\mathfrak{C}} \mathrm{LS}^+_{\mathfrak{C}},$$

where $\mathfrak{C}$ runs over all maximal chains in $\mathcal{A}$. The union LS^+ is called a fan of monoids of *Lakshmibai-Seshadri type* (shortly: the fan of monoids of LS-type).

Definition 5.17. Let Γ be the image of the quasi valuation $\mathcal{V}$. We say that Γ is of *LS-type* if

$$\Gamma = \bigcup_{\mathfrak{C} \text{ a maximal chain in } \mathcal{A}} \mathrm{LS}^+_{\mathfrak{C}}.$$

An important application of the theory of Seshadri stratification is to describe a Seshadri stratification and the corresponding fan of monoids of a Schubert variety. In [15], it was proven that the fan of monoids of a Seshadri stratification of a Schubert variety is of LS-type. To obtain this result, a bijection between Γ and the set of LS-path is established (we refer to [35] for the definition of LS-path). Thanks to this description of Γ one can interpret LS-paths in terms of vanishing order of homogeneous functions on a Schubert variety.
Since the equations describing a fan of monoids of LS-type are explicit, it easy to study its geometry. Thus, it could be interesting to inquire whether a given Seshadri stratification on an embedded projective variety is of LS-type or not.

CHAPTER 6

Matrix Schubert Varieties

6.1 Preliminaries

We denote by $\mathrm{Mat}_{m\times n}$ the $\mathbb{K}$-vector space of matrices with m rows and n columns.

Let $A \in \mathrm{Mat}_{m\times n}$, and let $p \in \{1, \dots, m\}$ and $q \in \{1, \dots, n\}$. We denote by $A_{p,q}$ the (p, q)-entry in A. We denote by $A_{\{p,q\}}$ the submatrix of A of dimension $p \times q$ anchored in the top left corner, namely containing $A_{1,1}$ and $A_{p,q}$.

Definition 6.1. An element $\tau \in \mathrm{Mat}_{m\times n}$ is called a *partial permutation* if its entries are all equal to 0 except for at most one entry equal to 1 in each row and column. We denote by $\mathcal{A}_{m,n}$ the set of partial permutations in $\mathrm{Mat}_{m\times n}$.

Example 6.2. The matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

is a partial permutation in $\mathrm{Mat}_{5\times 7}$

Definition 6.3. Let $\tau \in \mathcal{A}_{m,n}$. The *matrix Schubert variety* $\hat{X}_\tau \subseteq \mathrm{Mat}_{m\times n}$ associated with τ is the subvariety

$$\hat{X}_\tau := \{A \in \mathrm{Mat}_{m\times n} \,|\, \mathrm{rk}\, A_{\{p,q\}} \le \mathrm{rk}\, \tau_{\{p,q\}} \text{ for all } p \in \{1, \dots, m\} \text{ and } q \in \{1, \dots n\}\}. \tag{6.1}$$

Example 6.4. Suppose $n = m$. Then the matrix Schubert variety associated with the identity is the variety $\mathrm{Mat}_{m\times n}$ itself.

Definition 6.5. Let $A \in \mathrm{Mat}_{m\times n}$. We call the *rank matrix* associated with A the matrix $R(A) \in \mathrm{Mat}_{m\times n}$ such that the entry $R(A)_{p,q}$ is equal to the rank of $A_{\{p,q\}}$.

Remark 6.6. Let $\tau \in \mathcal{A}_{m,n}$. Then for $p \in \{1, \dots m\}$ and $q \in \{1, \dots, n\}$, the (p, q)-entry of $R(\tau)$ is equal to the number of occurrences of 1 in the submatrix $\tau_{\{p,q\}}$.

Let $\mathbb{B}_m^-$ (respectively $\mathbb{B}_n^+$) be the subgroup of $\mathrm{GL}_m(\mathbb{K})$ (respectively $\mathrm{GL}_n(\mathbb{K})$) consisting of lower (respectively upper) triangular matrices, where $\mathrm{GL}_m(\mathbb{K})$ (respectively $\mathrm{GL}_n(\mathbb{K})$) is the general linear group of $m \times m$-matrices (respectively $n \times n$-matrices) with coefficients in $\mathbb{K}$. The natural action of $\mathrm{GL}_m(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$ on $\mathrm{Mat}_{m\times n}$ given by

$$\mathrm{GL}_m(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}) \times \mathrm{Mat}_{m\times n} \longrightarrow \mathrm{Mat}_{m\times n}$$

$$(B_1, B_2, A) \mapsto B_1 A B_2^{-1},$$

restricts to an action of the product $\mathbb{B}_m^- \times \mathbb{B}_n^+$.

Let $\tau \in \mathcal{A}_{m,n}$. We denote by $\mathcal{O}_\tau$ the $\mathbb{B}_m^- \times \mathbb{B}_n^+$-orbit of τ.

By [54, Theorem 15.31], the matrix Schubert variety $\hat{X}_\tau$ coincides with the Zariski closure of $\mathcal{O}_\tau$, and by [54, Proposition 15.27] the $\mathbb{B}_m^- \times \mathbb{B}_n^+$-orbits in $\mathrm{Mat}_{m\times n}$ are parametrized by $\mathcal{A}_{m,n}$, that is

$$\hat{X}_\tau = \overline{\mathcal{O}}_\tau, \text{ for } \tau \in \mathcal{A}_{m,n}. \tag{6.2}$$

We assume that $n \geq m$. Let ϕ be the map

$$\begin{aligned} \phi : \mathrm{Mat}_{m\times n} &\longrightarrow \mathrm{Mat}_{n\times n} \\ A &\longmapsto \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ & & & & \\ & & A & & \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n-m \\ \\ \\ \end{matrix}, \end{aligned} \tag{6.3}$$

Then ϕ is an injective closed morphism of varieties.

Consider the embedding of $\mathrm{GL}(\mathbb{K})_m$ into $\mathrm{GL}(\mathbb{K})_n$, given by

$$\iota : \mathrm{GL}(\mathbb{K})_m \hookrightarrow \mathrm{GL}(\mathbb{K})_n$$
$$B \mapsto \begin{pmatrix} \mathrm{Id}_{(n-m)\times(n-m)} & 0 \\ 0 & B \end{pmatrix}$$

Then ϕ is a $\mathrm{GL}(\mathbb{K})_m \times \mathrm{GL}(\mathbb{K})_n$-equivariant embedding, where $\mathrm{GL}(\mathbb{K})_m$ acts on $\mathrm{Mat}_{n\times n}$ by left multiplication via ι.

Let $\tau \in \mathcal{A}_{m,n}$. By construction of the morphism ϕ, the image $\phi(\tau)$ is a partial permutation in $\mathcal{A}_{n,n}$, so

$$\phi(\mathcal{A}_{m,n}) \subset \mathcal{A}_{n,n}.$$

Hence it makes sense to consider the matrix Schubert variety $\hat{X}_{\phi(\tau)}$.
With the following lemma we establish a relationship between the matrix Schubert varieties in $\mathrm{Mat}_{m\times n}$ and the matrix Schubert varieties in $\mathrm{Mat}_{n\times n}$.

Lemma 6.7. Let $n \geq m$ and let $\phi : \mathrm{Mat}_{m\times n} \longrightarrow \mathrm{Mat}_{n\times n}$ be the morphism defined in (6.3). Then

(1) $\phi(\hat{X}_\tau) = \hat{X}_{\phi(\tau)}$, for any $\tau \in \mathcal{A}_{m,n}$

(2) $\hat{X}_\tau \cong \hat{X}_{\phi(\tau)}$ through the restriction of the morphism ϕ to $\hat{X}_\tau$, for any $\tau \in \mathcal{A}_{m,n}$

(3) If $\sigma, \tau \in \mathcal{A}_{m,n}$, then $\hat{X}_\sigma \subseteq \hat{X}_\tau$ if and only if $\hat{X}_{\phi(\sigma)} \subseteq \hat{X}_{\phi(\tau)}$.

Proof.

(1) Let $\tau \in \mathcal{A}_{m,n}$. We consider the $\mathbb{B}_m^- \times \mathbb{B}_n^+$-orbit $\mathcal{O}_\tau$ and the $\mathbb{B}_n^- \times \mathbb{B}_n^+$-orbit $\mathcal{O}_{\phi(\tau)}$. We claim that

$$\phi(\mathcal{O}_\tau) = \mathcal{O}_{\phi(\tau)}.$$

Let $B_1 \in \mathbb{B}_m^-$ and $B_2 \in \mathbb{B}_n^+$. We set

$$B_1' = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\quad}^{n-m} & \overbrace{\qquad\quad}^{m} \end{array} \\ \left(\begin{array}{c|c} Id & 0 \\ \hline 0 & B_1 \end{array}\right) \end{array} \in \mathbb{B}_n^-.$$

Then

$$\phi(B_1\tau B_2) = B_1'\phi(\tau)B_2.$$

Thus $\phi(\mathcal{O}_\tau) \subseteq \mathcal{O}_{\phi(\tau)}$. Conversely, let $A \in \mathcal{O}_{\phi(\tau)}$. Then there exist $C_1 \in \mathbb{B}_n^-$ and $C_2 \in \mathbb{B}_n^+$ such that $A = C_1\phi(\tau)C_2$. Let $B_1 \in \mathbb{B}_m^-$ be the submatrix of C_1 obtained by taking the last m rows and the last m columns, that is

$$C_1 = \left(\begin{array}{c|c} \multicolumn{1}{c}{\overbrace{\qquad\ast\qquad}^{n-m}} & \multicolumn{1}{c}{\overbrace{\qquad 0\qquad}^{m}} \\ \hline \ast & B_1 \end{array}\right)$$

We have that

$$\phi(B_1\tau C_2) = C_1\phi(\tau)C_2 = A.$$

Hence $\mathcal{O}_{\phi(\tau)} \subseteq \phi(\mathcal{O}_\tau)$, giving the claim.
Therefore, since ϕ is a closed morphism,

$$\phi(\hat{X}_\tau) = \phi(\overline{\mathcal{O}_\tau}) = \overline{\phi(\mathcal{O}_\tau)} = \overline{\mathcal{O}_{\phi(\tau)}} = \hat{X}_{\phi(\tau)}.$$

(2) Since ϕ is an injective morphism, then

$$\phi_{|\hat{X}_\tau} : \hat{X}_\tau \longrightarrow \phi(\hat{X}_\tau)$$

is an isomorphism of varieties. Hence, by (1), we have that $\hat{X}_\tau \cong \hat{X}_{\phi(\tau)}$.

(3) Since ϕ is injective, we have

$$\hat{X}_\sigma \subseteq \hat{X}_\tau \iff \phi(\hat{X}_\sigma) \subseteq \phi(\hat{X}_\tau).$$

Therefore, by (1),

$$\hat{X}_\sigma \subseteq \hat{X}_\tau \iff \hat{X}_{\phi(\sigma)} \subseteq \hat{X}_{\phi(\tau)}$$

□

6.2 Parametrization

We provide a parametrization for the partial permutations in $\mathrm{Mat}_{m\times n}$, and consequently, for the matrix Schubert varieties in $\mathrm{Mat}_{m\times n}$. In [4], the authors obtained the parametrization of the matrix Schubert varieties in $\mathrm{Mat}_{n\times n}$. The parametrization that we present here is equivalent to the one in [4]. In view of the relation between the Schubert varieties in $\mathrm{Mat}_{m\times n}$ and in $\mathrm{Mat}_{n\times n}$ as established in Lemma 6.7, the equivalence of the two parameterizations is in line with our expectations.

For the sake of completeness, we include here the parametrization for the matrix Schubert variety in $\mathrm{Mat}_{m\times n}$.

Let τ be a partial permutation in $\mathrm{Mat}_{m\times n}$. We denote by $\underline{\tau}_1,\dots,\underline{\tau}_n$ the columns of τ. We associate to τ a vector $\underline{a}=(a_1,\dots,a_n)\in\mathbb{N}^n$ in the following way:

$$\begin{cases} a_i = k \text{ if } \underline{\tau}_i = e_k; \\ a_i = m+k \text{ if } \underline{\tau}_i = 0 \text{ and } |\{j\le i \mid \underline{\tau}_j = 0\}| = k, \text{ with } k\ge 0. \end{cases} \tag{6.4}$$

Example 6.8. If

$$\tau = \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix} \in \mathrm{Mat}_{5\times 7},$$

then the associated vector in $\mathbb{N}^7$ is

$$\underline{a}_\tau = (6,5,7,8,4,1,2).$$

Let $\mathcal{A}^n_m\subset\mathbb{N}^n$ be the set of vectors $\underline{a}\in\mathbb{N}^n$ such that

- $a_i\le m+n$ for all $i\in\{1,\dots,n\}$;
- $a_i\ne a_j$ for all $i,j\in\{1,\dots,n\}$ such that $i\ne j$.

Conversely, we associate to every element of $\mathcal{A}^n_m$ a partial permutation τ in $\mathrm{Mat}_{m\times n}$ in the following way

$$\begin{cases} \tau_{a_i,i} = 1 \text{ if } a_i\le m \\ \tau_{a_i,i} = 0 \text{ if } a_i > m \\ \tau_{k,l} = 0 \text{ for all pairs } (k,l)\ne(a_i,i) \text{ for any } i\in\{1,\dots,n\} \end{cases} \tag{6.5}$$

Example 6.9. Let $\underline{a} = (5,6,4,7) \in \mathcal{A}_5^4$. The corresponding partial permutation is

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{5\times 4}$$

Let $\tau \in \mathcal{A}_{m,n} \subseteq \mathrm{Mat}_{m\times n}$ and let $\underline{a} \in \mathcal{A}_m^n \subset \mathbb{N}^n$. The assignment in (6.4) gives a bijection

$$\psi : \mathcal{A}_{m,n} \longrightarrow \mathcal{A}_m^n,$$

whose inverse ψ^{-1} is defined as in (6.5).

For an element $\underline{a}_\tau \in \mathcal{A}_m^n$ associated with $\tau \in \mathcal{A}_{m,n}$, we denote by $R(\underline{a})$ the rank matrix $R(\tau)$.

Let $q \in \{1,\dots,n\}$ and let $\underline{a} \in \mathcal{A}_m^n$. We denote by $\underline{a}(q)$ the truncation of $\underline{a}$ at the q-entry, i.e.

$$\underline{a}(q) = (a_1,\dots,a_q).$$

Remark 6.10. By Remark 6.6, we have

$$R(\underline{a})_{p,q} = |\{j \in \mathbb{N} \mid j \le q \text{ and } a_j \le p\}| = |\{j \in \{1,\dots,q\} \mid \underline{a}(q)_j \le p\}|.$$

Indeed, $R(\underline{a})_{p,q}$ represents the count of non-zero entries in $\tau_{\{p,q\}}$, which corresponds to the number of non-zero columns where the non-zero entry appears in a row with index less than p.

Example 6.11. Let

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{5\times 7},$$

with associated vector $\underline{a}_\tau = (6,5,7,8,4,1,2)$ and let $p = 5$ and $q = 5$. Then $R(\underline{a}_\tau)_{5,5} = 2$, and 2 is the number of entries in the truncated vector $\underline{a}_\tau(5) = (6,5,7,8,4)$ that are less than or equal to 5.

We denote by $\widetilde{\underline{a}(q)}$ the vector obtained by reordering the entries of $\underline{a}(q)$ in increasing order.

Lemma 6.12. Let $\underline{a} \in \mathcal{A}^n_m$. Let $p \in \{1, \dots, m\}$ and let $q \in \{1, \dots, n\}$. Then

(1) $R(\underline{a})_{p,q} \geq k \iff \widetilde{\underline{a}(q)}_k \leq p$;

(2) $R(\underline{a})_{p,q} = k \iff \widetilde{\underline{a}(q)}_k \leq p$ and $\widetilde{\underline{a}(q)}_{k+1} > p$.

Proof.

(1) We assume that $R(\underline{a})_{p,q} \geq k$. Then, by Remark 6.10, $|\{j \in \{1, \dots, q\} \mid \underline{a}(q)_j \leq p\}| \geq k$. Therefore in the truncated vector $\underline{a}(q)$ there are at least k entries that are less than or equal to p. So, reordering the entries of the vector $\underline{a}(q)$, at least the first k entries are less than or equal to p, that is
$$\widetilde{\underline{a}(q)}_k \leq p.$$
Conversely, if $\widetilde{\underline{a}(q)}_k \leq p$, then $\widetilde{\underline{a}}(q)_1 < \widetilde{\underline{a}(q)}_2 < \dots \widetilde{\underline{a}(q)}_k \leq p$. Thus $\underline{a}(q)$ has at least k entries less than or equal to p. So $|\{j \in \{1, \dots, q\} \mid \underline{a}(q)_j \leq p\}| \geq k$. Hence Remark 6.10 gives $R(\underline{a})_{p,q} \geq k$.

(2) We suppose that $R(\underline{a})_{p,q} = k$. By Remark 6.10, the cardinality of the set
$$\{j \in \{1, \dots, q\} \mid \underline{a}(q)_j \leq p\}$$
is k. Reordering the entries of $\underline{a}(q)$, we obtain $\widetilde{\underline{a}(q)}_1 < \widetilde{\underline{a}(q)}_2 <, \dots, < \widetilde{\underline{a}(q)}_k < p$, and $p < \widetilde{\underline{a}(q)}_{k+1} < \dots < \widetilde{\underline{a}(q)}_q$.
We suppose now that $\widetilde{\underline{a}(q)}_k \leq p$ and $\widetilde{\underline{a}(q)}_{k+1} > p$. Then
$$k = |\{j \in \{1, \dots, q\} \mid \underline{a}(q)_j \leq p\}| = R(\underline{a})_{p,q}.$$

□

6.3 A dimension formula

In this section we recall the dimension formula for a matrix Schubert variety in [54, 15.2] and we use it to prove a dimension formula in terms of the vectors in $\mathcal{A}^n_m$.

Definition 6.13. [54, Definition 15.13] Let $\tau \in \mathcal{A}_{m,n}$. We define the *diagram* $D(\tau)$ of τ to be the diagram consisting of all entries in τ that are not to the right of a 1 nor below a 1 nor equal to 1.

Example 6.14. For

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{0} & \boxed{0} & \boxed{0} \\ 0 & 0 & \boxed{0} & 1 & 0 \end{pmatrix},$$

the squared entries correspond to the diagram $D(\tau)$.

By [54, Proposition 15.30], we have

$$\dim \hat{X}(\tau) = mn - |D(\tau)|. \tag{6.6}$$

Let $\underline{a}_\tau = (a_1, \dots, a_n) = \psi(\tau)$, where $\tau \in \mathcal{A}_{m,n}$. We set

$$a_i^* = \begin{cases} n + m - a_i - i + 1 \text{ if } a_i \leq m \\ 0 \text{ if } a_i > m \end{cases}. \tag{6.7}$$

Let $\tau \in \mathcal{A}_{m,n}$. We call the set of entries below and to the right of $\tau_{i,j}$ the (i,j)-*hook* in τ. Additionally, we denote i as the row-index and j as the column-index of the (i,j)-hook. When $a_i^* \neq 0$, it corresponds to the cardinality of the (a_i, i)-hook.

Example 6.15. Let $\tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\psi(\tau) := \underline{a}_\tau = (4, 5, 1, 6)$. The number a_3^* is equal to 4, that is the cardinality of the $(1,3)$-hook, i.e.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We define the *inversion set* of $\underline{a}_\tau \in \mathcal{A}_m^n$ as

$$\operatorname{inv}(\underline{a}_\tau) = \{(k,l) \in \{1, \dots, m\} \times \{1, \dots, n\} \mid k < l,\ m \geq a_k > a_l\}.$$

Remark 6.16. Two hooks in τ with column-indexes k and l, such that $k < l$, intersect if and only if the pair $(k, l) \in \operatorname{inv}(\underline{a}_\tau)$.

Example 6.17. Let $\underline{a}_\tau = (2, 4, 1, 5) \in \mathcal{A}_3^4$ and let $\tau = \psi^{-1}(\underline{a}_\tau) \in \mathcal{A}_{m,n}$. Then $(1, 3) \in \operatorname{inv}(\underline{a}_\tau)$. Indeed the $(1, 3)$-hook intersects the $(2, 1)$-hook,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition 6.18. We define the *length* of $\underline{a}_\tau$ as follows:

$$\ell(\underline{a}_\tau) = \sum_{i=1}^{m} a_i^* - |\operatorname{inv}(\underline{a}_\tau)| \tag{6.8}$$

If $\underline{a}_\tau = \psi(\tau) \in \mathcal{A}_m^n$, we denote the length of $\underline{a}_\tau$ also with $\ell(\tau)$.

Proposition 6.19. Let $\tau \in \mathcal{A}_{m,n}$ and let $\underline{a}_\tau = \psi(\tau)$. Then

$$\dim \hat{X}(\tau) = \ell(\underline{a}_\tau).$$

Proof. By (6.6), it is enough to show that $\ell(\underline{a}_\tau) = mn - |D(\tau)|$. If $a_i^* \neq 0$, it is the number of entries in τ below and to the right of the (a_i, i)-entry of τ.

Therefore, $\sum_{i=1}^m a_i^* = mn - |D(\tau)| + \alpha$, where α is the number of entries in τ not belonging to $D(\tau)$ but positioned to the right of a non-zero entry and also beneath a non-zero entry (so in the sum $\sum_{i=1}^m a_i^*$ entries of this type are counted twice).

We claim that $\alpha = |\operatorname{inv}(\underline{a}_\tau)|$.

An entry (a_i, j) is to the right of a 1 at position (a_i, i) and beneath a 1 at position (a_j, j) if and only if $m \geq a_i > a_j$ and $j > i$, that is, if and only if $(i, j) \in \operatorname{inv}(\underline{a}_\tau)$.

□

6.4 Partial order

We define a partial order on $\mathcal{A}_{m,n}$ and so on the matrix Schubert varieties.

Definition 6.20. Let $\sigma, \tau \in \mathcal{A}_{m,n}$. We say that

$$\sigma \leq \tau \text{ if } \hat{X}_\sigma \subseteq \hat{X}_\tau,$$

and we write $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$.

Lemma 6.21. Let $\sigma, \tau \in \mathcal{A}_{m,n}$. Then

$$\sigma \leq \tau \iff R(\sigma)_{p,q} \leq R(\tau)_{p,q} \text{ for all } p \in \{1,\dots,m\} \text{ and } q \in \{1,\dots,n\}.$$

Proof. If $\sigma \leq \tau$, then $\sigma \in \hat{X}_\tau$. So, by (6.1), for all $p \in \{1,\dots,m\}$ and $q \in \{1,\dots,n\}$,

$$R(\sigma)_{p,q} = \mathrm{rk}(\sigma_{\{p,q\}}) \leq \mathrm{rk}(\tau_{\{p,q\}}) = R(\tau)_{p,q}.$$

On the other hand, if $A \in \hat{X}_\sigma$, then $\mathrm{rk}(A_{\{p,q\}}) \leq \mathrm{rk}(\sigma_{\{p,q\}}) \leq \mathrm{rk}(\tau_{\{p,q\}})$, that means that $A \in \hat{X}_\tau$, giving the claim. □

We transport this partial order on $\mathcal{A}_m^n$. Adapting the definition of Deodhar order in [4], we define a partial order on $\mathcal{A}_m^n$, and then we show that this partial order corresponds to the order given in Definition 6.20 through the bijection ψ. The description of this partial order for $m = n$ has appeared in [4].

Definition 6.22. Let $\underline{a}, \underline{b} \in \mathcal{A}_m^n$. We say that $\underline{b} \leq_D \underline{a}$ if $\forall q \in \{1,\dots,n\}$ and $\forall k \in \{1,\dots,q\}$

$$\widetilde{\underline{a}(q)}_k \leq \widetilde{\underline{b}(q)}_k.$$

Example 6.23. Let $\underline{a} = (4,3,5)$ and $\underline{b} = (8,5,9)$ in $\mathcal{A}_6^3$. Then $\underline{a} \geq_D \underline{b}$. Indeed $\widetilde{\underline{a}(q)}_k < \widetilde{\underline{b}(q)}_k$ for all $q \in \{1,2,3\}$ and for all $k \in \{1,\dots,q\}$:

$$\begin{array}{rcl} \widetilde{\underline{a}(3)} & = & (3,4,5) \\ \widetilde{\underline{a}(2)} & = & (3,4) \\ \widetilde{\underline{a}(1)} & = & (4) \end{array} \quad , \quad \begin{array}{rcl} \widetilde{\underline{b}(3)} & = & (5,8,9) \\ \widetilde{\underline{b}(2)} & = & (5,8) \\ \widetilde{\underline{b}(1)} & = & (8) \end{array}$$

Proposition 6.24. Let $\tau, \sigma \in \mathcal{A}_{m,n}$ and let $\underline{a} = \psi(\tau)$, $\underline{b} = \psi(\sigma) \in \mathcal{A}_m^n$. Then

$$\tau \geq \sigma \iff \underline{a} \geq_D \underline{b}$$

Proof. Suppose that $\tau \geq \sigma$. Then $R(\underline{a})_{p,q} \geq R(\underline{b})_{p,q}$, for all $p \in \{1,\dots,m\}$ and $q \in \{1,\dots,n\}$. Let $q \in \{1,\dots,m\}$, let $p \in \{1,\dots,n\}$ and let k be such that $\widetilde{\underline{b}(q)}_k = p$. If there is no such a k, we set $k = 0$. By Lemma 6.12(1), the entry $R(\underline{b})_{p,q} \geq k$. Then

$$R(\underline{a})_{p,q} \geq R(\underline{b})_{p,q} \geq k.$$

Thus, using again Lemma 6.12(1),

$$\widetilde{\underline{a}(q)}_k \leq p = \widetilde{\underline{b}(q)}_k,$$

and so $\underline{a} \geq_D \underline{b}$.
Suppose now that $\underline{a} \geq_D \underline{b}$ and let $k \in \mathbb{N}$ be such that $R(\underline{b})_{p,q} = k$. By Lemma 6.12, $\widetilde{\underline{b}(q)}_k \leq p$. Then, by hypothesis, $\widetilde{\underline{a}(q)}_k \leq \widetilde{\underline{b}(q)}_k \leq p$. By Lemma 6.12(1), we have that $R(\underline{a})_{p,q} \geq k = R(\underline{b})_{p,q}$. □

We give, now, another interpretation of this partial order in term of row-semistandard Young tableaux.

Definition 6.25. A *Young diagram* is a finite collection of left-justified rows of boxes where any row is not longer than the row on top of it.

A *Young tableau* is obtained by filling in the boxes of the Young diagram with numbers in $\mathbb{N}$. We say that a Young tableau $\mathcal{T}$ is *row-semistandard* if the filling of each row is strictly increasing and the filling of each column is non-decreasing.

Example 6.26.

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 7 & 8 \\ \hline 1 & 4 & 5 & 6 & 7 & 8 \\ \hline 4 & 5 & 6 & 7 & 8 \\ \cline{1-5} 5 & 6 & 7 & 8 \\ \cline{1-4} 5 \\ \cline{1-1} \end{array}$$

Let $\mathcal{T}$ be a Young tableau. If the Young diagram has t rows and if, for each $i = 1, \ldots, t$, the i-th row has n_i boxes, we say that $\mathcal{T}$ is of *shape* $(n_1, \ldots, n_t)$.

Example 6.27. The Young tableau

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 7 & 8 \\ \hline 1 & 4 & 5 & 6 & 7 & 8 \\ \hline 4 & 5 & 6 & 7 & 8 \\ \cline{1-5} 5 & 6 & 7 & 8 \\ \cline{1-4} 5 \\ \cline{1-1} \end{array}$$

is of shape $(6, 6, 5, 4, 1)$.

Let $\mathcal{T}$ be a Young tableau of shape $(n_1, \ldots, n_t)$. When $i, j \in \mathbb{N}$ are such that (i, j) is a box in the Young tableau $\mathcal{T}$, we say that the pair (i, j) is suitable for $\mathcal{T}$. We denote by $[\mathcal{T}]$ the set of the boxes (i, j) in $\mathcal{T}$ for suitable pairs (i, j). We denote by $\mathcal{T}(i, j)$ the number in the box of $\mathcal{T}$ in the i-row and the j-column.

Definition 6.28. Let $\underline{a} \in \mathcal{A}_m^n$. The Young tableau $\mathcal{T}_{\underline{a}}$ associated with the vector $\underline{a}$ is the row-semistandard Young tableau of shape $(n, n-1, \ldots, 2, 1)$ obtained by writing the reordered truncated vectors $\widetilde{\underline{a}(n)}$, $\widetilde{\underline{a}(n-1)}$, and so on, on the top of each other. If τ is a partial permutation and $\underline{a} = \psi(\tau)$, we write $\mathcal{T}_\tau$ for the Young tableau associated with $\underline{a}$.

Example 6.29. For τ as in Example 6.2 we get $\underline{a} = (6, 5, 7, 8, 4, 1, 2)$ and

$$\mathcal{T}_\tau = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 4 & 5 & 6 & 7 & 8 \\ \cline{1-6} 4 & 5 & 6 & 7 & 8 \\ \cline{1-5} 5 & 6 & 7 & 8 \\ \cline{1-4} 5 & 6 & 7 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \tag{6.9}$$

Remark 6.30. Let $\tau \in \mathcal{A}_{m,n}$. Then the row-semistandard Young tableau $\mathcal{T}_\tau$ associated with τ has the following properties:

(1) $\mathcal{T}_\tau$ has shape $(n, n-1, \ldots, 1)$, so the i-th row has length equal to $n - i + 1$;

(2) $\{\mathcal{T}(i,j) \mid j \in \{1, \ldots, n-i+1\}\} \subset \{\mathcal{T}(i-1,j) \mid j \in \{1, \ldots, n-i+2\}\}$ for all $i \in \{2, \ldots, n\}$;

We define a partial order on the set of all row-semistandard tableaux of fixed shape.

Definition 6.31. Let $\mathcal{T}$ and $\mathcal{S}$ be two row-semistandard Young tableaux of same shape. We say that $\mathcal{T} \geq \mathcal{S}$ *boxwise* if $\mathcal{T}(i,j) \geq \mathcal{S}(i,j)$ for every suitable (i,j).

Lemma 6.32. Let $\underline{a}, \underline{b} \in \mathcal{A}_m^n$. Then

$$\underline{a} \leq_D \underline{b} \iff \mathcal{T}_{\underline{a}} \geq T_{\underline{b}} \text{ boxwise} .$$

Proof. The statement follows from construction of the tableaux $\mathcal{T}_{\underline{a}}, \mathcal{T}_{\underline{b}}$ for $\underline{a}, \underline{b} \in \mathcal{A}_m^n$ and Definition 6.31. □

In [20] the authors define a partial order on the set of Young tableaux. We recall it in the following definition.

Definition 6.33. Let $\mathcal{T}$ and $\mathcal{S}$ be two tableaux of the same shape. We define the *De Concini-Eisenbud-Procesi partial order*, setting

$$\mathcal{T} \leq_{DCPE} \mathcal{S} \text{ if for every } r, s \in \mathbb{N}$$

$$|\{(i,j) \in [\mathcal{T}] \mid i \leq r,\ \mathcal{T}(i,j) \leq s\}| \leq |\{(i,j) \in [\mathcal{S}] \mid i \leq r,\ \mathcal{S}(i,j) \leq s\}|$$

Lemma 6.34. Let $\mathcal{T}$ and $\mathcal{S}$ be row-semistandard Young tableaux with same shape. Then

$$\mathcal{T} \geq \mathcal{S} \text{ boxwise} \implies \mathcal{T} \leq_{DCPE} \mathcal{S}$$

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ have the same shape, then $[\mathcal{S}] = [\mathcal{T}]$. Assume $\mathcal{T} \geq \mathcal{S}$ boxwise. Then $\mathcal{T}(i,j) \geq \mathcal{S}(i,j)$ for every $(i,j) \in [\mathcal{T}]$, so, for $r, s \in \mathbb{N}$

$$\{(i,j) \in [\mathcal{T}] \mid i \leq r,\ \mathcal{T}(i,j) \leq s\} \subseteq \{(i,j) \in [\mathcal{S}] \mid i \leq r,\ \mathcal{S}(i,j) \leq s\}.$$

Therefore

$$|\{(i,j) \in |\mathcal{T}| \mid i \leq r,\ \mathcal{T}(i,j) \leq s\}| \leq |\{(i,j) \in [\mathcal{S}] \mid i \leq r,\ \mathcal{S}(i,j) \leq s\}|.$$

$\square$

The converse of Lemma 6.34 does not hold in general.

Example 6.35. Let $\mathcal{T}$ and $\mathcal{S}$ be the Young tableaux associated to $\underline{a}_\tau = (2,3,4)$ and $\underline{a}_\sigma = (3,2,1) \in \mathcal{A}_3^3$,

$$\mathcal{T} = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ , \quad \mathcal{S} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ .$$

We have $\mathcal{T} \leq_{DCPE} \mathcal{S}$ but $\mathcal{T} \not\geq \mathcal{S}$ boxwise. Indeed $\mathcal{T}(1,1) > \mathcal{S}(1,1)$ and $\mathcal{T}(3,1) < \mathcal{S}(3,1)$.

Lemma 6.36. Let $\tau, \sigma \in \mathcal{A}_{m,n}$ and assume $\mathcal{T}_\tau \leq_{DCPE} \mathcal{T}_\sigma$. Then $\mathcal{T}_\tau(1,j) \geq \mathcal{T}_\sigma(1,j)$ for $j \in \{1, \ldots, n\}$ and $\mathcal{T}_\tau(2,j) \geq \mathcal{T}_\sigma(2,j)$ for $j \in \{1, \ldots, n-1\}$.

Proof. For the first row we need only the hypothesis that $\mathcal{T}_\sigma$ and $\mathcal{T}_\tau$ are row-semistandard, and this is proved in [20, Lemma 1.5 (b)]. So $\mathcal{T}_\tau(1,j) \geq \mathcal{T}_\sigma(1,j)$ for $j \in \{1, \ldots, n\}$. We prove the estimate for the second row, namely we prove that $\mathcal{T}_\tau(2,j) \geq \mathcal{T}_\sigma(2,j)$ for $j \in \{1, \ldots, n-1\}$. Let $x := \mathcal{T}_\tau(2,j)$ and $y := \mathcal{T}_\sigma(2,j)$. Since $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ are row-semistandard Young tableaux constructed as in Definition 6.28, we can apply Remark 6.30. Thus, by

Remark 6.30 (2), there exist j_1 and j_2 such that $\mathcal{T}_\tau(1, j_1) = x$ and $\mathcal{T}_\sigma(1, j_2) = y$. Combining (1) and (2) of Remark 6.30, we get

$$\text{either } j_1 = j \text{ or } j+1 \text{ and either } j_2 = j \text{ or } j+1.$$

Since $\mathcal{T}_\tau(1, j) \geq \mathcal{T}_\sigma(1, j)$, the only non trivial case is $j_1 = j$ and $j_2 = j+1$. If, we had $y > x$ then we would have

$$2j - 1 \geq |\{(k, l) \mid k \leq 2, T_\sigma(k, l) \leq x\}| \geq |\{(k, l) \mid k \leq 2, \ \mathcal{T}_\tau(k, l) \leq x\}| = 2j,$$

a contradiction. Hence $y \leq x$. □

Remark 6.37. Let τ and σ be partial permutations in $\mathrm{Mat}_{m\times 2}$ and let $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$ be the associated Young tableaux. Then, by Lemma 6.36, also the converse in Lemma 6.34 holds for $\mathcal{T}_\tau$ and $\mathcal{T}_\sigma$.

6.4.1 Covers of the partial order

Our next aim is to describe the covers in the partial order of $\mathcal{A}_{m,n}$.
By [4, Section 5], if $\sigma \prec \tau$, then $\hat{X}_\sigma$ is a subvariety of codimension one in $\hat{X}_\tau$.

In the next Proposition we describe the cover $\sigma \prec \tau$, where σ is obtained from τ by swapping two non-zero columns, or by swapping a non-zero column with a 0-column. Let $\underline{c} = \psi(\tau)$ and $\underline{b} = \psi(\sigma)$. Swapping two non-zero columns means that $\underline{b}$ is obtained from $\underline{c}$ by swapping two entries that are less than or equal to m. Swapping a non-zero column with a 0-column means that $\underline{b}$ is obtained from $\underline{c}$ by swapping an entry less than or equal to m with an entry greater than m.

Proposition 6.38. Let $\tau \in \mathcal{A}_{m,n}$ and let $\underline{c} = \psi(\tau)$. We consider $\sigma \in \mathcal{A}_{m,n}$, such that, for $\underline{b} = \psi(\sigma)$ we have

- $b_k = c_k$ for every $k \neq i, j$;
- $b_i = c_j$ and $b_j = c_i$, and $c_i \leq m$, with $i < j$.

Then

(1) $\sigma < \tau \iff c_i < c_j$;

(2) assuming $c_i < c_j$, then

$$\ell(\tau) = \ell(\sigma) + 1 \iff \text{there is no } c_k \text{ such that } c_i < c_k < c_j \text{ and } i < k < j.$$

Proof. First we show (1). Assume $\sigma < \tau$. By Lemma 6.21 we have that $R(\sigma)_{p,q} \leq R(\tau)_{p,q}$ for all $p \in \{1, \dots, m\}$ and $q \in \{1, \dots, n\}$. If $c_i > c_j$ then $\tau_{k,i} = 0$ for all $1 \leq k \leq c_j$, while $\sigma_{c_j,i} = 1$, so $R(\tau)_{c_j,i} < R(\sigma)_{c_j,i}$, giving a contradiction. Conversely, if $c_i \leq c_j$, then $R(\sigma)_{p,q} \leq R(\tau)_{p,q}$ for all $p \in \{1, \dots, m\}$ and $q \in \{1, \dots, n\}$. Moreover if $c_i \leq p < c_j$ or $i \leq q < j$, then $R(\sigma)_{p,q} < R(\tau)_{p,q}$. Thus $\sigma < \tau$.

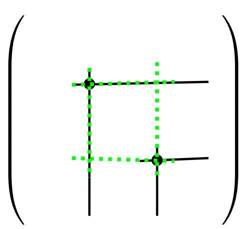
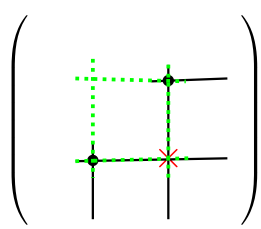
We now show (2). We need to distinguish two cases: $c_j \leq m$ or $c_j > m$.

($c_j \leq m$) In this case σ is obtained from τ by swapping two non-zero columns.

By the hypothesis on $\underline{c}$ and $\underline{b}$, we have that $\sum_{k=1}^{n} c_k^* = \sum_{b=1}^{n} b_k^*$. Then, by the dimension formula (6.8),

$$l(\tau) = l(\sigma) + 1 \iff |\operatorname{inv}(\underline{b})| - |\operatorname{inv}(\underline{c})| = 1.$$

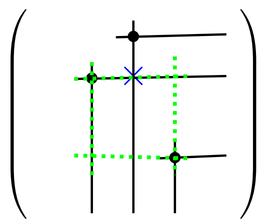
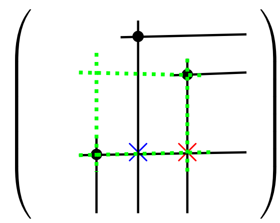
Since $c_i < c_j$ then $(i,j) \in \operatorname{inv}(\underline{b}) \setminus \operatorname{inv}(\underline{c})$. In the following picture the black lines correspond to the (c_i, i)-hook and the (c_j, j)-hook in τ, and the (c_j, i)-hook and the (c_i, j)-hook in σ. The red cross in picture (6.10) represents the element $(i,j) \in \operatorname{inv}(\underline{b}) \setminus \operatorname{inv}(\underline{c})$.

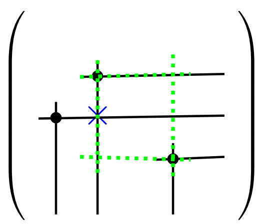
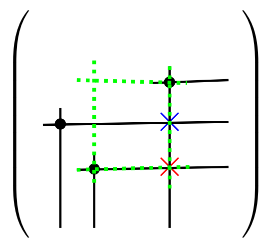
(6.10) $\tau =$  $\sigma =$ 

In τ a non-zero entry outside of the green rectangle in the picture (6.10), corresponds in the vector $\underline{c}$ to

$$\begin{aligned} &c_k \text{ with } i < k < j \text{ and such that } c_k < c_i \ \vee \ c_k > c_j, \\ &\qquad\qquad\qquad\qquad \text{or} \\ &\qquad\qquad c_k \text{ with } k < i \ \vee \ k > j. \end{aligned} \tag{6.11}$$

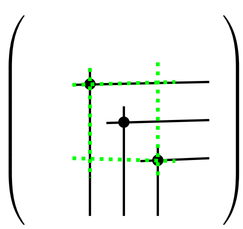
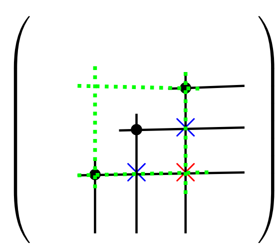
From the following pictures one can see that if c_k is as in (6.11), then the (c_k, k)-hook intersects the hooks in picture (6.10) the same number of times for τ and for σ.

$\tau =$  $\sigma =$ 

$\tau =$  $\sigma =$ 

Therefore an entry outside of the green rectangle does not contribute to the difference between the cardinalities of $\mathrm{inv}(\underline{c})$ and $\mathrm{inv}(\underline{b})$.

A non-zero entry inside of the green rectangle corresponds in $\underline{c}$ to c_k with $i < k < j$ and such that $c_i < c_k < c_j$. The (c_k, k)-hook for such a c_k intersects the (c_j, i)-hook and the (c_i, j)-hook in σ, but not in τ.

$\tau =$  $\sigma =$ 

This means that, if there exists such a c_k, then $|\,\mathrm{inv}(\underline{b})| \geq |\,\mathrm{inv}(\underline{c})| + 2$. Conversely, since the non-zero entry outside the green rectangle do not contribute to the difference between the cardinalities of $\mathrm{inv}(\underline{c})$ and $\mathrm{inv}(\underline{b})$, if $|\,\mathrm{inv}(\underline{b})| \geq |\,\mathrm{inv}(\underline{c})| + 2$, then there must be a non-zero entry within the green rectangle.

Therefore $\ell(\tau) = \ell(\sigma) + 1$ if and only if there is no non-zero entry within the green rectangle, that means that there is no c_k with $i < k < j$ and such that $c_i < c_k < c_j$.

$(c_j > m)$ In this case σ is obtained from τ by swapping a 0-column with a non-zero column. In this case the condition in (2):

$$\text{there is no } c_k \text{ such that } c_i < c_k < c_j \text{ with } i < k < j$$

is equivalent to

$$c_{i+1}, c_{i+2}, \ldots, c_{j-1} < c_i.$$

Indeed, by construction of $\underline{c}$, the elements greater than m appear in increasing order in $\underline{c}$, so there is no $c_k > c_j > m$ with $k < j$.

By the definition in (6.7),

$$b_i^* = c_j^* = 0,$$
$$b_j^* = n + m - b_j - j + 1 = n + m - c_i - j + 1 = c_i^* - (j - i).$$

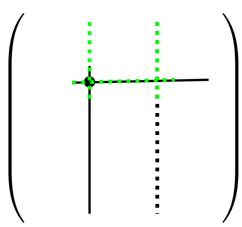
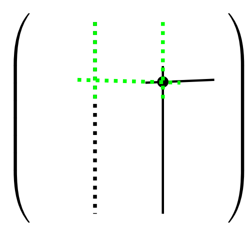
Therefore, by the dimension formula (6.8),

$$\ell(\tau) - \ell(\sigma) = |\operatorname{inv}(\underline{b})| + j - |\operatorname{inv}(\underline{c})| - i,$$

then

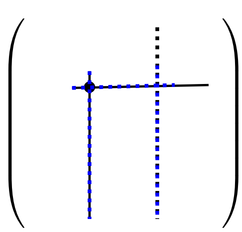
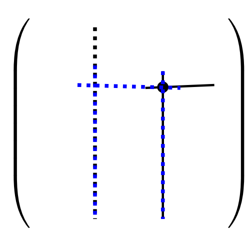
$$\ell(\tau) = \ell(\sigma) + 1 \iff |\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = j - i - 1.$$

We are in the following situation

$\tau =$  $\sigma =$ 

A non-zero entry in the green rectangle corresponds in $\underline{c}$ to the occurrence of a c_k with $i < k < j$ and such that $c_k < c_i$. One can see that these entries are the only ones that produce an element in $\operatorname{inv}(\underline{c}) \setminus \operatorname{inv}(\underline{b})$.

Now we look at the element in $\operatorname{inv}(\underline{b}) \setminus \operatorname{inv}(\underline{c})$.

$\tau =$  $\sigma =$ 

A non-zero entry in the blue rectangle corresponds in $\underline{c}$ to the occurrence of a c_k with $i < k < j$ and such that $c_k > c_i$. One can see that these entries are the only ones that produce an element in $\operatorname{inv}(\underline{b}) \setminus \operatorname{inv}(\underline{c})$.

Since $|\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = |\operatorname{inv}(\underline{c}) \setminus \operatorname{inv}(\underline{b})| - |\operatorname{inv}(\underline{b}) \setminus \operatorname{inv}(\underline{c})|$, we have

$$|\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k < c_i\}| - |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k > c_i\}|.$$

We observe that $|\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k < c_i\}| \leq j - i - 1$. Then, since $|\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k > c_i\}|$ is a non negative quantity, we have that

$$|\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k < c_i\}| - |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k > c_i\}| = j - i - 1$$
$$\text{if and only if}$$
$$|\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k < c_i\}| = j - i - 1 \text{ and } |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k > c_i\}| = 0.$$

Therefore

$$\begin{aligned} |\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = j - i - 1 &\iff \\ \begin{cases} |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k < c_i\}| = j - i - 1 \\ |\{k \in \mathbb{N} \mid i < k < j \text{ and } c_k > c_i\}| = 0. \end{cases} &\iff \\ c_k < c_i \text{ for all } i < k < j, &\iff \\ c_{i+1}, \ldots, c_{j-1} < c_i. \end{aligned}$$

□

Example 6.39. Let $\tau \in \mathcal{A}_{5,7}$ be the matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_\tau = (6, 5, 7, 8, 2, 1, 4)$$

and let $\sigma \in \mathcal{A}_{5,7}$ be the matrix

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_\sigma = (6,5,7,8,4,1,2).$$

The relation $\tau \succ \sigma$ is a cover relation in which σ differs from τ by swapping of columns different from 0. The vector $\underline{a}_\sigma$, in fact, is obtained from τ by swapping the red entries.

If instead we consider

$$\sigma' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_{\sigma'} = (6,7,5,8,2,1,4), \text{ then}$$

the relation $\tau \succ \sigma'$ is a cover relation in which σ' differs from τ by moving a 0-column to the left. The vector $\underline{a}_\sigma$, in fact, is obtained from τ by swapping the blue entries.

In the next Proposition we describe the cover relation $\sigma \prec \tau$ where σ is obtained from τ by swapping a zero row with nonzero row above it. Let $\underline{b} = \psi(\sigma)$ and $\underline{c} = \psi(\sigma)$. Moving a 0-row upwards means that $\underline{b}$ is obtained from $\underline{c}$ by changing an entry $c_i \leq m$ with $c_i + s \leq m$ for some $s \in \mathbb{N}$ such that $c_i + s \neq c_k$ for all $k \in \{1, \dots, n\}$.

Proposition 6.40. Let $\tau \in \mathcal{A}_{m,n}$ and let $\underline{c} = \psi(\tau)$. Suppose that there exists $i \in \{1, \dots, n\}$ and $s \in \mathbb{N}$ such that $c_i + s \neq c_j$ and $c_i + s \leq m$ for every $j \in \{1, \dots, n\}$. We consider $\sigma \in \mathcal{A}_{m,n}$ such that, for $\underline{b} = \psi(\sigma)$

- $b_k = c_k$ for every $k \neq i$;
- $b_i = c_i + s$.

Then $\tau > \sigma$. Moreover

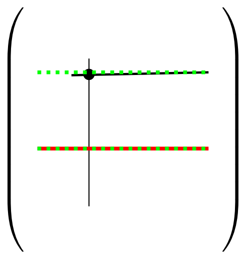
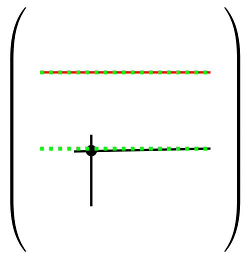
$$\ell(\tau) = \ell(\sigma) + 1 \text{ if and only if } \{c_i + 1, \dots, c_i + s - 1\} \subset \{c_1, \dots, c_{i-1}\}.$$

Proof. Since σ is obtained from τ by moving a 0-row upwards, using Remark 6.10, we see that $R(\sigma)_{p,q} \leq R(\tau)_{p,q}$ for every $p \in \{1, \ldots, m\}$ and $q \in \{1, \ldots, n\}$. Furthermore $R(\sigma)_{p,q} < R(\tau)_{p,q}$ for all $p \in \{c_i, c_i+1, \ldots, c_i+s-1\}$ and $q \in \{i, \ldots, n\}$. Thus $\sigma < \tau$.

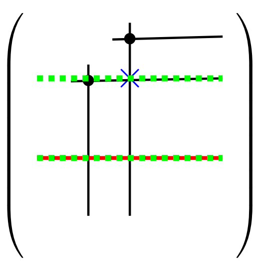
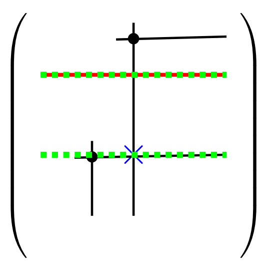
By the dimension formula (6.8), we have $\ell(\tau) - \ell(\sigma) = |\operatorname{inv}(\underline{b})| - |\operatorname{inv}(\underline{c})| + s$. Then

$$\ell(\tau) = \ell(\sigma) + 1 \iff |\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = s - 1.$$

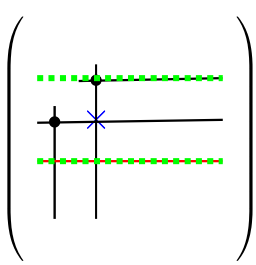
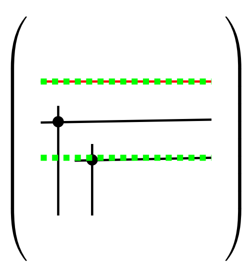
The following picture points out the (c_i, i)-hook in τ and the $(c_i + s, i)$-hook in σ. The red line in τ represents the $(c_i + s)$-th row, the red line in σ represents the c_i-th row:

(6.12) $\tau =$  $\sigma =$ 

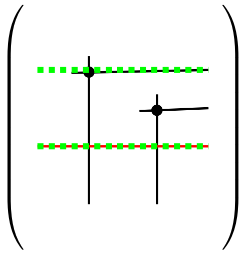
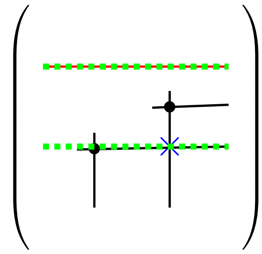
From the following picture one can see that the non-zero entry outside of the green rectangle in (6.12) does not contribute to the difference between the cardinalities $|\operatorname{inv}(\underline{c})|$ and $|\operatorname{inv}(\underline{b})|$

$\tau =$  $\sigma =$  .

If $\tau_{c_k,k} \neq 0$ with $k < i$ and $c_i < c_k < c_i + s$, then the (c_k, k)-hook intersects the (c_i, i)-hook in τ but not in σ. The following picture describes this situation.

$\tau =$  $\sigma =$ 

Moreover, if $\sigma_{c_k,k} \neq 0$ with $i < k$ and $c_i < c_k < c_i + s$, then the (c_k, k)-hook in σ intersects the $(c_i + s, i)$-hook in σ, but not in τ, as the following picture shows

$\tau =$  $\sigma =$ 

Therefore

$$|\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| =$$
$$= |\{k \in \mathbb{N} \mid 1 \leq k < i \text{ and } c_i < c_k < c_i + s\}| - |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}|.$$

We have that $|\{k \in \mathbb{N} \mid 1 \leq k < i \text{ and } c_i < c_k < c_i + s\}| \leq s - 1$.

Hence
(6.13)
$$\begin{aligned} &|\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| = \\ &= |\{k \in \mathbb{N} \mid 1 \leq k < i \text{ and } c_i < c_k < c_i + s\}| - |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}| \leq \\ &\leq s - 1 - |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}| \leq \\ &\leq s - 1. \end{aligned}$$

Thus
$$s - 1 = |\operatorname{inv}(\underline{c})| - |\operatorname{inv}(\underline{b})| \iff \text{ in (6.13) the equalities hold.}$$

Therefore
(6.14)
$$\begin{aligned} &|\{k \in \mathbb{N} \mid 1 \leq k < i \text{ and } c_i < c_k < c_i + s\}| - |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}| = \\ &= s - 1 - |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}| = s - 1. \end{aligned}$$

The equations in (6.14) hold if and only if

$$\begin{cases} |\{k \in \mathbb{N} \mid i < k \leq n \text{ and } c_i < c_k < c_i + s\}| = 0 \\ |\{k \in \mathbb{N} \mid k < i \text{ and } c_i < c_k < c_i + s\}| = s - 1 \end{cases} \iff \{c_i{+}1, \ldots, c_i{+}s{-}1\} \subset \{c_1, \ldots, c_{i-1}\}.$$

$\square$

Example 6.41. Let $\tau \in \mathcal{A}_{m,n}$,

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_\tau = (6, 5, 7, 3, 2, 1, 8).$$

Let $\sigma \in \mathcal{A}_{m,n}$,

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_\sigma = (6, 5, 7, 3, 4, 1, 8)$$

The relation $\tau > \sigma$ is a cover relation in which σ differs from τ by swapping a 0-row with a non-0 row that appears above it. The vector $\underline{a}_\sigma$, in fact, is obtained from $\underline{a}_\tau$ by setting the red entry to 4.

In the next proposition we describe the cover relation $\sigma \prec \tau$ where σ is obtained from τ by setting to 0 a non-zero entry of τ.

Proposition 6.42. Let $\tau \in \mathcal{A}_{m,n}$ and let $\underline{c} = \psi(\tau)$. Assume that there exists $i \in \{1, \ldots, n\}$ such that $\tau_{c_i,i} = 1$ (that is $c_i \leq m$). Let σ be the partial permutation obtained from τ by replacing the entry $\tau_{c_i,i}$ by 0 and let $\underline{b} = \psi(\sigma)$. The relation $\sigma < \tau$ is a cover relation if and only if $\underline{c}$ has the following properties:

(1) $\{k \in \mathbb{N} \mid c_i \leq k \leq m\} \subseteq \{c_1, \ldots, c_i\}$;

(2) $c_{i+1}, \ldots, c_n \leq c_i - 1$.

Proof. Since σ is obtained from τ by setting a non-zero entry to 0, we have that $R(\sigma)_{p,q} \leq R(\tau)_{p,q}$ for all $p \in \{1, \ldots, m\}$ and $q \in \{1, \ldots, n\}$. Moreover if $p \geq c_i$ and $q \geq i$ then $R(\sigma)_{p,q} < R(\tau)_{p,q}$. Thus $\sigma < \tau$.

By the dimension formula (6.8),

$$\ell(\tau)-\ell(\sigma)=n+m-c_i-i+1-|\operatorname{inv}(\underline{c})|+|\operatorname{inv}(\underline{b})|.$$

Thus

$$\ell(\tau)-\ell(\sigma)=1 \iff |\operatorname{inv}(\underline{c})|-|\operatorname{inv}(\underline{b})|=n+m-c_i-i.$$

By the definition of the sets $\operatorname{inv}(\underline{c}),\operatorname{inv}(\underline{b})$ and of $\underline{c}$ and $\underline{b}$, we have $\operatorname{inv}(\underline{b})\subset\operatorname{inv}(\underline{c})$ and

$$\operatorname{inv}(\underline{c})\setminus\operatorname{inv}(\underline{b})=$$

$$=(\{(k,i)\in\mathbb{N}\times\{i\}|\ k<i\}\cup\{(i,l)\in\{i\}\times\mathbb{N}\mid i<l\leq n\})\cap\operatorname{inv}(\underline{c}).$$

We observe that $|\{(i,l)\in\{i\}\times\mathbb{N}\mid i<l\leq n\}\cap\operatorname{inv}(\underline{c})|\leq n-i$. Moreover if $(\overline{k},i)\in\{(k,i)\in\mathbb{N}\times\{i\}|\ k<i\}\cap\operatorname{inv}(\underline{c})$ then $c_i<c_{\overline{k}}\leq m$. Thus there are at most $m-c_i$ choices for $c_{\overline{k}}$, so $|\{(k,i)\in\mathbb{N}\times\{i\}\mid k<i\}\cap\operatorname{inv}(\underline{c})|\leq m-c_i$.

Therefore

$$|\operatorname{inv}(\underline{c})|-|\operatorname{inv}(\underline{b})|=|\operatorname{inv}(\underline{c})\setminus\operatorname{inv}(\underline{b})|=n+m-c_i-i \iff$$

$$\begin{gathered}|\{(i,l)\in\{i\}\times\mathbb{N}\mid i<l\leq n\}\cap\operatorname{inv}(\underline{c})|=n-i\\ \text{and}\\ |(\{(k,i)\in\mathbb{N}\times\{i\}|\ k<i\}\cap\operatorname{inv}(\underline{c})|=m-c_i.\end{gathered}$$

Furthermore

$$|\{(i,l)\in\mathbb{N}\times\{i\}\mid i<l\leq n\}\cap\operatorname{inv}(\underline{c})|=n-i \iff c_k<c_i \text{ for all } k\in\{i+1,\dots,j-1\}$$

that is (2); and

$$|\{(k,i)\in\mathbb{N}\times\{i\}|\ k<i\}\cap\operatorname{inv}(\underline{c})|=m-c_i$$

that is equivalent to (1). □

Example 6.43. Let $\tau,\sigma\in\mathcal{A}_{5,7}$ be

$$\tau=\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&0\\0&1&0&0&0&0&0\end{pmatrix} \text{ with associated vector } \underline{a}_\tau=(6,5,7,3,2,1,8),$$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with associated vector } \underline{a}_\sigma = (6,7,8,3,2,1,9).$$

The relation $\tau \succ \sigma$ is a covering relation in which σ differs from τ by setting $\sigma_{5,2} = 0$.

In [56, Theorem 3.7], the authors obtain all the covering relations in the poset of square partial permutations, namely $\mathcal{A}_{n,n}$. By Lemma 6.7(3), the poset $\mathcal{A}_{m,n}$ can be seen as a subposet of $\mathcal{A}_{n,n}$. Thus we can apply [56, Theorem 3.7] also to $\mathcal{A}_{m,n}$. For convenience of the reader we rewrite the theorem here in our setting.

Theorem 6.44. *Let $\tau, \sigma \in \mathcal{A}_{m,n}$ be such that $\sigma \prec \tau$. Let $\underline{a}_\sigma = \psi(\sigma)$ and $\underline{a}_\tau = \psi(\tau)$. Then one of the following holds:*

(1) $\underline{a}_\sigma$ is obtained from $\underline{a}_\tau$ by swapping two entries;

(2) σ is obtained from τ by swapping a 0-row with a non-zero row;

(3) σ is obtained from τ by setting a non-zero entry to 0.

We aim, now, to show that the covering relations described in Propositions 6.38, 6.40 and 6.42 are all the ones occurring in the poset $\mathcal{A}_{m\times n}$. To achive this, we need that for $\sigma, \tau \in \mathcal{A}_{m\times n}$ such that $\sigma \prec \tau$, the hypothesis of one of the previous Propositions are satisfied. This is ensured by Theorem 6.44.

Therefore, by Theorem 6.44, the covering relations described in Propositions 6.38, 6.40 and 6.42 are all the covering relations appearing in the poset $\mathcal{A}_{m,n}$.

6.5 Seshadri stratification on Matrix Schubert Varieties

In this section we give a Seshadri stratification for the projective variety $\mathbb{P}(\mathrm{Mat}_{m\times n})$. We denote $\mathbb{P}(\mathrm{Mat}_{m\times n})$ by X.

6.5.1 Strata of the Seshadri stratification

We observe that $\mathrm{Mat}_{m\times n}$ is the affine cone on $\mathbb{P}(\mathrm{Mat}_{m\times n})$. We consider the projection morphism

$$\pi : \mathrm{Mat}_{m\times n} \setminus \{0\} \longrightarrow \mathbb{P}(\mathrm{Mat}_{m\times n}).$$

Let $\tau \in \mathcal{A}_{m,n}$. Since matrix Schubert varieties are cones (that is they are closed for homothety), we have $\pi^{-1}(\pi(\hat{X}_\tau \setminus \{0\})) \cup \{0\} = \hat{X}_\tau$. Therefore $\hat{X}_\tau$ is the affine cone on the projective variety $\pi(\hat{X}_\tau)$. We denote $\pi(\hat{X}_\tau)$ by X_τ and we call it the *projective matrix Schubert variety* associated with τ.

In the following sections we define a Seshadri stratification on X.
Since the matrix Schubert varieties are the affine cones of the projective matrix Schubert variety, the containment relations between projective matrix Schubert varieties correspond to the partial order of $\mathcal{A}_{m\times n} \setminus \{0\}$. Therefore the stratification of X is given by the subvarieties X_τ with $\tau \in \mathcal{A}_{m\times n} \setminus \{0\}$. The poset $\mathcal{A}_{m\times n}$ extends the stratification of X on the affine cone $\mathrm{Mat}_{m\times n}$ as in Remark 5.4.

To be strata of a Seshadri stratification, the subvariety X_τ needs to be smooth in codimension one for any $\tau \in \mathcal{A}_{m,n} \setminus \{0\}$. To check if X_τ is smooth in codimension one, it is enough to check if the affine cone $\hat{X}_\tau$ is smooth in codimension one.

Proposition 6.45. Let $\tau \in \mathcal{A}_{m,n}$. Then the matrix Schubert variety $\hat{X}_\tau$ is smooth in codimension one.

Proof. By [53, Theorem 2.4.3], a local property that is, a property that holds for any open subset of a covering of the variety, holds for matrix Schubert varieties if and only if it holds for ordinary Schubert varieties. By [15, Corollary 3.5] Schubert varieties are smooth in codimension one. To be smooth in codimension one is a local property, hence also matrix Schubert varieties are smooth in codimension one. □

6.5.2 Extremal functions

In order to define a Seshadri stratification on X, we need to define the extremal functions of the stratifications. So we define an homogeneous function $f_\tau \in K[\hat{X}_\tau] \setminus \{0\}$ for every $\tau \in \mathcal{A}_{m\times n} \setminus \{0\}$ that satisfies properties (S2) and (S3). The function f_τ satisfies (S2) and (S3) if and only if it satisfies the following properties:

(S2') for any $\tau, \sigma \in \mathcal{A}_{m\times n} \setminus \{0\}$ such that $\sigma \not\geq \tau$, the function f_τ vanishes on $\hat{X}_\sigma$;

(S3') for $\tau \in \mathcal{A}_{m\times n}$, the set-theoretical intersection satisfies

$$\mathcal{H}_{f_\tau} \cap \hat{X}_\tau = \bigcup_{\sigma \prec \tau} \hat{X}_\sigma.$$

Thus, in this Section, we define the functions f_τ, with $\tau \in \mathcal{A}_{m\times n}$, and we prove that they satisfy (S2') and (S3').

Let $\tau \in \mathcal{A}_{m,n}$ and let $\hat{X}_\tau$ be the associated matrix Schubert variety. We define inductively the function f_τ associated with τ as a product of minors.

First we introduce some notation. Let $\underline{i} = (i_1, \ldots, i_d)$ and $\underline{j} = (j_1, \ldots, j_d)$ be strictly increasing sequences of length d for some $d \in \mathbb{N}$.

For $A \in \mathrm{Mat}_{m\times n}$, we denote by $A_{\underline{i},\underline{j}}$ the $d \times d$-submatrix consisting of the entries in A in the rows $i_1, \ldots, i_d$ and columns $j_1, \ldots, j_d$. We write $\mathfrak{m}_{\underline{i},\underline{j}}$ for the minor

$$\mathrm{Mat}_{m\times n} \to \mathbb{K}, \quad A \mapsto \mathfrak{m}_{\underline{i},\underline{j}}(A) := \det(A_{\underline{i},\underline{j}}).$$

Let $\tau \in \mathrm{Mat}_{m\times n}$ be a nonzero partial permutation, let $\underline{i} = (i_1, \ldots, i_d)$ be the sequence of indexes of those rows and $\underline{j} = (j_1, \ldots, j_d)$ the sequence of indexes of those columns where a non-zero entry occurs.

Step 1 : Attach to τ the minor $\mathfrak{m}_{\underline{i},\underline{j}}$ and let $\tau' \in \mathrm{Mat}_{m\times n}$ be the partial permutation in $\mathcal{A}_{m,n}$ obtained from τ by setting to zero the non-zero entries in the i_d-th row and in the j_d-th column. Note that if $\tau_{i_d,j_d} \neq 0$, then the non-zero entries in the i_d-th row and in the j_d-th column coincide, so to obtain τ' we set only τ_{i_d,j_d} to 0.

Step 2 : Let $\underline{i'} = (i'_1, \ldots, i'_{d'})$ and $\underline{j'} = (j'_1, \ldots, j'_{d'})$ be the sequences of indexes of those rows and columns, respectively, where a non-zero entry occurs in τ'. Attach to τ the product of minors $\mathfrak{m}_{\underline{i},\underline{j}}\mathfrak{m}_{\underline{i'},\underline{j'}}$ and let $\tau'' \in \mathrm{Mat}_{m\times n}$ be the partial permutation obtained form τ' by setting to zero the non-zero entries in the $i'_{d'}$-th row respectively $j'_{d'}$-th column.

Step 3 : We repeat this procedure until the new partial permutation is a zero matrix and we associate to τ a function f_τ which is a product of minors: $f_\tau = \mathfrak{m}_{\underline{i},\underline{j}}\mathfrak{m}_{\underline{i'},\underline{j'}}\mathfrak{m}_{\underline{i''},\underline{j''}}\cdots$.

We can use the notation of bi-tableaux, namely pairs of Young tableaux, to easily look at these minors. We write then

$$\boxed{i_1}\boxed{i_2}\boxed{\ldots}\boxed{i_d}\,,\,\boxed{j_1}\boxed{j_2}\boxed{\ldots}\boxed{j_d}$$

for the minor $\mathfrak{m}_{\underline{i},\underline{j}}$. So the right side of this double tableau indicates the columns, the left side the rows.

In the notation of double tableaux it is

$$
f_\tau = \mathcal{R}_\tau, \mathcal{C}_\tau = \begin{array}{cccccc}
i_1 & i_2 & i_3 & \dots & \dots & i_d \\
 & & i'_1 & i'_2 & \dots & i'_{d'} \\
 & & & \dots & \dots & \dots \\
 & & & & & \dots
\end{array} , \begin{array}{cccccc}
j_1 & j_2 & j_3 & \dots & \dots & j_d \\
j'_1 & j'_2 & \dots & j'_{d'} & & \\
\dots & \dots & \dots & & & \\
\dots & & & & &
\end{array} , \tag{6.15}
$$

where we denote by $\mathcal{R}_\tau$ the Young tableau on the left and we denoted by $\mathcal{C}_\tau$ the Young tableau on the right.

Example 6.46. Let $\tau \in \mathrm{Mat}_{5\times 7}$ be the following matrix

$$
\tau = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

The sequences $\underline{i} = (1, 2, 4, 5)$ and $\underline{j} = (2, 5, 6, 7)$ are sequences of indexes of those rows, respectively columns, where a non-zero entry occurs in τ. So the first minor occurring in f_τ is $\mathfrak{m}_{\underline{i},\underline{j}}$. Now we obtain a matrix τ' from τ setting to zero $\tau_{5,2}$ and $\tau_{4,7}$. Then

$$
\tau' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Then the sequences $\underline{i}' = (1, 2)$ and $\underline{j}' = (2, 5)$ are sequences of indexes of those rows, respectively columns, where a non-zero entry occurs in τ'. Thus the second minor occurring in f_τ is $\mathfrak{m}_{\underline{i}',\underline{j}'}$.

Setting to zero the entries $\tau'_{2,4}$ and $\tau'_{1,5}$ we obtain the zero-matrix, then there are no more minors occurring in f_τ.

Therefore the bi-tableau associated with f_τ is

$$
\mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 4 & 5 \\
\hline
\multicolumn{1}{c}{} & & 1 & 2 \\
\cline{3-4}
\end{array} , \begin{array}{|c|c|c|c|}
\hline
2 & 5 & 6 & 7 \\
\hline
5 & 6 & \multicolumn{2}{c}{} \\
\cline{1-2}
\end{array}
$$

Now we show that the functions f_τ for $\tau \in \mathcal{A}_{m,n} \setminus \{0\}$ satisfy (S2') and (S3'). In order to do that, we need the following proposition that allows us to look only at the central point τ in $\hat{X}_\tau$ to verify where such a function vanishes on $\hat{X}_\tau$.

Proposition 6.47. Let $\underline{i} = (i_1, \dots, i_k)$ be the k-uple of the indexes of the first k non-zero rows of τ, and let $\underline{j} = (j_1, \dots, j_k)$ be the k-uple of indexes of the first k non-zero columns in τ. Let $B_1 \in \mathbb{B}_m^-$ and $B_2 \in \mathbb{B}_n^+$. Then

$$\mathfrak{m}_{\underline{i},\underline{j}}(B_1 \tau B_2) = \alpha(B_1, B_2)\mathfrak{m}_{\underline{i},\underline{j}}(\tau),$$

for some homomorphism $\alpha : \mathbb{B}_m^- \times \mathbb{B}_n^+ \to \mathbb{C}^*$.

Proof. Let $B_1 = T_1U_1$ and $B_2 = T_2U_2$ where T_1, T_2 are diagonal matrices and U_1, U_2 are unipotent matrices.
Since T_1 (respectively T_2) is a diagonal matrix, the left (respectively right) multiplication by T_1 (respectively T_2) just multiplies the rows (respectively columns) of τ with a nonzero factor. Therefore

$$\mathfrak{m}_{\underline{i},\underline{j}}(T_1 \tau T_2) = \nu_{\underline{i}}(T_1)\mu_{\underline{j}}(T_2)\mathfrak{m}_{\underline{i},\underline{j}}(\tau)$$

where $\nu_{\underline{i}}(T_1)$ is the product of the entries on the diagonal in T_1 corresponding to the rows in $\underline{i}$ and $\mu_{\underline{j}}(T_2)$ is the product of the entries on the diagonal of T_2 corresponding to the columns in $\underline{j}$.

We claim that

$$\mathfrak{m}_{\underline{i},\underline{j}}(U_1 T_1 \tau T_2 U_2) = \mathfrak{m}_{\underline{i},\underline{j}}(T_1 \tau T_2).$$

Since U_2 is an upper triangular matrix, right multiplication just adds to a j-th-column of $T_1\tau T_2$ scalar multiples of columns with index strictly smaller than j. Now by definition of $\mathfrak{m}_{\underline{i},\underline{j}}$, either the index j does not occur in $\underline{j}$, or if it occurs, say $j = j_s$, then the indexes $j_1, \dots, j_{s-1}$ are exactly the indexes of the nonzero columns of τ to the left of the j-th column. These are the same as the indexes for the nonzero columns of $T_1\tau T_2$ to the left of the j-th column. This implies $T_1\tau T_2 U_2$ is obtained from $T_1 \tau T_2$ by adding to the j-th column multiples of the columns that occurs to its left. In particular the submatrix $(T_1\tau T_2 U_2)_{\underline{i},\underline{j}}$ is obtained from the submatrix $(T_1\tau T_2)_{\underline{i},\underline{j}}$ by adding to the j-th column scalar multiples of the column that occurs to its left. Therefore $(T_1\tau T_2U_2)_{\underline{i},\underline{j}}$ and $(T_1\tau T_2)_{\underline{i},\underline{j}}$ have the same determinant. So

$$\mathfrak{m}_{\underline{i},\underline{j}}(T_1 \tau T_2 U_2) = \mathfrak{m}_{\underline{i},\underline{j}}(T_1 \tau T_2).$$

Analogously, since U_1 is a lower triangular matrix, $U_1(T_1\tau T_2U_2)$ is obtained from $T_1\tau T_2U_2$ by adding to the i-th row of $T_1\tau T_2U_2$ multiples of the nonzero rows of $T_1\tau T_2U_2$ with index strictly smaller than i. In particular the submatrix $(U_1T_1\tau T_2U_2)_{\underline{i},\underline{j}}$ is obtained from the submatrix $(U_1T_1\tau T_2U_2)_{\underline{i},\underline{j}}$ by adding to the i-th row scalar multiples of rows with index strictly smaller than i. Therefore $(U_1T_1\tau T_2U_2)_{\underline{i},\underline{j}}$ and $(U_1T_1\tau T_2U_2)_{\underline{i},\underline{j}}$ have the same determinant, so

$$\mathfrak{m}_{\underline{i},\underline{j}}(U_1T_1\tau T_2U_2) = \mathfrak{m}_{\underline{i},\underline{j}}(T_1\tau T_2U_2).$$

It follows:

$$\mathfrak{m}_{\underline{i},\underline{j}}(B_1\tau B_2) = \mathfrak{m}_{\underline{i},\underline{j}}(U_1T_1\tau T_2U_2) = \alpha\mathfrak{m}_{\underline{i},\underline{j}}(\tau)$$

with $\alpha = \nu_{\underline{i}}(T_1)\mu_{\underline{j}}(T_2) \in \mathbb{K}^*$.

□

Corollary 6.48. Let $\tau \in \mathcal{A}_{m,n}$. Let $B_1 \in \mathbb{B}_m^-$ and $B_2 \in \mathbb{B}_n^+$. Then there exist $\alpha_{B_1,B_2} \in \mathbb{K}^*$ such that

$$f_\tau(B_1\tau B_2) = \alpha(B_1, B_2)f_\tau(\tau),$$

for some homomorphism $\alpha : \mathbb{B}_m^- \times \mathbb{B}_n^+ \to \mathbb{C}^*$.

Proof. By definition of f_τ, it is the product of minors as in the hypothesis of Proposition 6.47. So the result easily follows from Proposition 6.47. □

Lemma 6.49. Let $\underline{i} = (i_1, \dots, i_r)$ and $\underline{j} = (j_1, \dots, j_r)$ be strictly increasing sequences of length r. Let $A \in \mathrm{Mat}_{m\times n}$. Then

$$\mathfrak{m}_{\underline{i},\underline{j}}(A) \neq 0 \implies R(A)_{i_r,j_h} \geq h \text{ and } R(A)_{i_h,j_r} \geq h \text{ for all } h = 1, \dots, r.$$

Proof. We assume that $\mathfrak{m}(A)_{\underline{i},\underline{j}} \neq 0$. We denote by $A_{\underline{i},\underline{j}}$ the submatrix of A obtained by taking the rows and the columns indexed by $\underline{i}$ and $\underline{j}$. Let $h \in \{1, \dots, r\}$. Since $\mathfrak{m}(A)_{\underline{i},\underline{j}} \neq 0$, the first h columns in $A_{\underline{i},\underline{j}}$ are linearly independent, so the submatrix of $A_{\underline{i},\underline{j}}$ obtained by taking the first h columns has rank equal to h. The submatrix of $A_{\underline{i},\underline{j}}$ obtained by taking the first h columns is a submatrix of $A_{\{i_r,j_h\}}$. Then the matrix $A_{\{i_r,j_h\}}$ has a $h \times h$ submatrix with non-zero determinant. Thus $R(A)_{i_r,j_h} \geq h$. Similarly one can obtain that $R(A)_{i_h,j_r} \geq h$. □

Theorem 6.50. *Let τ be a partial permutation in* $\mathrm{Mat}_{m\times n}$. *Then*

(1) if $\sigma \not\geq \tau$, then $f_{\tau|\hat{X}_\sigma} \equiv 0$;

(2) $\mathcal{H}_{f_\tau} \cap \hat{X}_\tau = \bigcup_{\sigma \prec \tau} \hat{X}_\sigma$.

Therefore the function f_τ *is an extremal function for the stratification given by* $(X_\tau, f_\tau)_{\tau \in \mathcal{A}_{m,n} \setminus \{0\}}$.

Proof. First we show (1). Let $\sigma \not\geq \tau$, and let $\underline{b} = \psi(\sigma)$ and let $\underline{c} = \psi(\tau)$. By Proposition 6.24,

$$\sigma \not\geq \tau \iff \exists q \in \{1, \dots, n\} \text{ and } k \in \{1, \dots, q\} \text{ such that } \widetilde{\underline{c}(q)}_k < \widetilde{\underline{b}(q)}_k.$$

Let $f_\tau = \mathfrak{m}_{\underline{i},\underline{j}} \mathfrak{m}_{\underline{i'},\underline{j'}} \cdots$ be the function associated with $\hat{X}_\tau$ defined as in (6.15).

Let $\underline{i} = (i_1, \dots, i_r)$ and $\underline{j} = (j_1, \dots, j_r)$ be the r-uples of the indexes of the rows, respectively of the columns, of the maximal rank submatrix of τ, i.e. the indexes appearing in $\mathfrak{m}_{\underline{i},\underline{j}}$.
Let $q \in \{1, \dots, n\}$ be the maximum such that there exists $k \in \{1, \dots, q\}$ such that $\widetilde{\underline{c}(q)}_k < \widetilde{\underline{b}(q)}_k$. Let k be the minimum in $\{1, \dots, q\}$ such that $\widetilde{\underline{c}(q)}_k < \widetilde{\underline{b}(q)}_k$.
Since k is the minimum, one can see that there exists $j \in \{1, \dots, r\}$ such that $\widetilde{\underline{c}(q)}_k = i_j$, that is $\widetilde{c(q)}_k$ is an index of a non-zero row. Therefore $k = j$ and $\widetilde{\underline{b}(q)}_j > i_j$.

First we assume $q \geq j_r$. Therefore, by Lemma 6.12(6.12), we have $R(\sigma)_{i_j,q} < j$. Since $q \geq j_r$, we have the following inequalities

$$R(\sigma)_{i_j,j_r} \leq R(\sigma)_{i_j,q} < j.$$

Let $A \in \hat{X}_\sigma$, by definition of $\hat{X}_\sigma$, we have that $R(A)_{i_j,j_r} \leq R(\sigma)_{i_j,j_r}$, so, by Lemma 6.49, $\mathfrak{m}_{\underline{i},\underline{j}}(A) = 0$.
Therefore $f_\tau \equiv 0$ in $\hat{X}_\sigma$.

We assume now that $j_{r-1} \leq q < j_r$. If $\widetilde{\underline{c}(q)}_k = i_r$, then $k = r-1$. Thus $\widetilde{\underline{b}(q)}_{r-1} > i_r$. Then, by Lemma 6.12 (1), $R(\sigma)_{i_r,q} < r-1$. Since $q \geq j_{r-1}$, we have the following inequalities

$$R(\sigma)_{i_r,j_{r-1}} \leq R(\sigma)_{i_r,q} < r-1.$$

By definition of $\hat{X}_\sigma$, we have $R(A)_{i_r,j_{r-1}} \leq R(\sigma)_{i_r,j_{r-1}}$ for all $A \in \hat{X}_\sigma$. Therefore, as before, Lemma 6.49 implies that $f_\tau \equiv 0$ in $\hat{X}_\sigma$.

If, instead, $\widetilde{\underline{c}(q)}_k \neq i_r$, then we proceed as for $q \geq j_r$, using the minors $\mathfrak{m}_{\underline{i'},\underline{j'}}$.

We continue analogously this procedure for $q < j_{r-1}$ using the minors $\mathfrak{m}_{\underline{i'},\underline{j'}} \mathfrak{m}_{\underline{i''},\underline{j''}}, \dots$.

Now we show (2). By construction $f_\tau(\tau) \neq 0$. Therefore, by Corollary 6.48, the function f_τ takes non-zero values on any point in the orbit $\mathcal{O}_\tau$. Thus $f_{\tau_{|\hat{X}_\tau}}$ is zero only on the boundary of $\hat{X}_\tau$, namely

$$\mathcal{H}_{f_\tau} \cap \hat{X}_\tau \subseteq \hat{X}_\tau \setminus \mathcal{O}_\tau = \bigcup_{\sigma \prec \tau} \hat{X}_\sigma.$$

So $\mathcal{H}_{f_\tau} \cap \hat{X}_\tau \subseteq \bigcup_{\sigma \prec \tau} \hat{X}_\sigma$. The other inclusion follows from (1).

□

6.5.3 Bonds

In this section we calculate the bonds of the extremal functions f_τ, for any $\tau \in \mathcal{A}_{m,n}$ as in Definition 5.10.

In the next proposition we calculate the bond $\mathfrak{b}_{\tau,\sigma}$, where $\sigma \prec \tau \in \mathcal{A}_{m,n}$ is a covering relation as in Proposition 6.38.

Proposition 6.51. Let τ and σ be partial permutations in $\mathrm{Mat}_{m\times n}$ and let $\underline{c} = \psi(\tau)$ and $\underline{b} = \psi(\sigma)$. Suppose that $b_i = c_j$ and $b_j = c_i$, with $c_i \leq m$ for some $i < j$ and suppose that $b_k = c_k$ for every $k \neq i, j$. Assume that

(1) $c_i < c_j$;

(2) there is no c_k such that $c_i < c_k < c_j$ and $i < k < j$.

Let $f_\tau = \mathcal{R}_\tau | \mathcal{C}_\tau$ be the bi-tableau associated with τ. Let s_i be the number of times that c_i appears in $\mathcal{R}_\tau$ and let s_j be the number of times that c_j appears in $\mathcal{R}_\tau$.

Then the bond between τ and σ is $\mathfrak{b}_{\tau,\sigma} = s_i - s_j$.

Proof. We analyzes the cases $c_j \leq m$ and $c_j > m$ separately.

($c_j \leq m$) In this case the (c_i, i)-entry and the (c_j, j)-entry of τ are different from zero; moreover σ is obtained from τ by interchanging the i-column with the j-column, so the (c_j, i)-entry and the (c_i, j)-entry of σ are different from zero, and all the other entries are as in τ.

Then all the minors occurring in the expression of f_τ and involving both the c_i-th and the c_j-th rows and the i-th and the j-th columns do not vanish on σ. By Corollary 6.48, they do not vanish on any element of $\mathcal{O}_\sigma$, so they do not contribute to the vanishing order of f_τ on $\hat{X}_\sigma$. Furthermore all the minors occurring in the expression of f_τ that do not involve both the c_i-th and the c_j-th rows simultaneously or the i-th and the j-th columns simultaneously are different from 0 on σ. Hence evaluation of these minors is non-zero on any element of $\mathcal{O}_\sigma$, so they do not contribute to the vanishing order of f_τ on $\hat{X}_\sigma$.

We observe that, since $c_i < c_j$ and $i < j$, all the minors occurring in the expression of f_τ involving c_j, involve also c_i. Then, looking at the bi-tableau $\mathcal{R}_\tau|\mathcal{C}_\tau$, we see that $s_i > s_j$. Moreover, in the expression of f_τ there is a minor involving the c_i-th row and the i-th column and not involving the c_j-th row and the j-th column. The non trivial entry in the c_i-th row of σ occurs in the j-th column. Hence, if a minor occurring in the expression of f_τ involves the c_i-th row but not the j-th column, then this minor evaluated on σ is zero with multiplicity one. Thus all the minors occurring in the expression of f_τ and involving the c_i-th row and not the c_j-th row (or else involving the i-th column and not the j-th column) vanish on σ with multiplicity 1. The number of these minors is exactly $s_i - s_j$. Therefore the vanishing order of f_τ on $\hat{X}_\sigma$ is $s_i - s_j$.

$(c_j > m)$ In this case σ is obtained from τ by interchanging the j-th column, which is a 0-column, with the i-th column, which is different from 0. Let $\underline{k} = (k_1, \ldots, k_r)$ and $\underline{l} = (l_1, \ldots, l_r)$ be increasing sequences of integers indexing the non-zero rows and columns of τ. Then there exists $h \in \{1, \ldots, r\}$ such that $l_h = i$. Thus $R(\tau)_{m,l_h} = h$. Since the i-th column in σ is zero and the columns on the left of the i-th column are equal to the corresponding columns in τ, then $R(\sigma)_{m,l_h} < R(\tau)_{m,l_h} = h$.
We consider a minor $\mathfrak{m}_{\underline{k}',\underline{l}'}$ occurring in the expression of f_τ involving the i-th column (respectively the c_i-th row). Let $\underline{k}' = (k'_1, \ldots, k'_{r'})$ and $\underline{l}' = (l'_1, \ldots, l'_{r'})$, then $r' \geq h$. By construction of f_τ, we have $l'_h = l_h = i$. Let $A \in \hat{X}_\sigma$. Then

$$R(A)_{k'_{r'},l_h} \leq R(\sigma)_{k'_{r'},l_h} \leq R(\sigma)_{m,l_h} < h.$$

Therefore, by Lemma 6.49, the minor $\mathfrak{m}_{\underline{k}',\underline{l}'}(A) = 0$, for all $A \in \hat{X}_\sigma$. Therefore each of the minors occurring in the expression of f_τ involving the i-th column and (respectively the c_i-th row) vanishes on $\hat{X}_\sigma$ with multiplicity 1.

The number of minors involving the i-th column (so the c_i-th row) is exactly s_i. Moreover, by definition of f_τ, since $c_j > m$, every minor in f_τ does not contain the

c_j-th row (and then the j-column). Then, in this case, $s_j = 0$. By hypothesis on σ and τ we have that $\sigma_{k,l} = \tau_{k,l}$ for every pair (k,l) different from the pairs (c_i, i) and (c_i, j). Hence evaluation on σ of any minor not involving the c_i-th row and the i-th column is different from 0. Therefore the bond $\mathfrak{b}_{\tau,\sigma} = s_i$.

□

Example 6.52. Let $\tau, \sigma \in \mathcal{A}_{m,n}$ be given by the following matrices:

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have that $\sigma \prec \tau$, and this is a cover relation as in Proposition 6.38.
The bi-tableau associated with f_τ is

$$\mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{3-4} \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & 5 & 6 & 7 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

The bond $\mathfrak{b}_{\tau,\sigma}$ is equal to the difference between the number of occurrences of 2 in $\mathcal{R}_\tau$ and the number of occurrences of 4 in $\mathcal{R}_\tau$, so $\mathfrak{b}_{\tau,\sigma} = 1$.

Let σ' be obtained from τ by swapping the second and the third columns. Then $\sigma' \prec \tau$, and this is a cover relation as in Proposition 6.38. Then σ' is the matrix

$$\sigma' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The bi-tableau associated with f_τ is as before

$$\mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{3-4} \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & 5 & 6 & 7 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

The bond $\mathfrak{b}_{\tau,\sigma'}$ is equal to the number of occurrences of 5 in $\mathcal{R}_\tau$, so $\mathfrak{b}_{\tau,\sigma'} = 1$.

In the next proposition we calculate the bond $\mathfrak{b}_{\tau,\sigma}$, where $\sigma, \tau \in \mathcal{A}_{m,n}$ with $\sigma \prec \tau$ a cover relation as in Proposition 6.40.

Proposition 6.53. Let τ and σ be partial permutations in $\mathrm{Mat}_{m\times n}$, and let $\underline{c} = \psi(\tau)$ and $\underline{b} = \psi(\sigma)$. Suppose that there exists $i \in \{1, \ldots, n\}$ and $s \in \mathbb{N}$ such that

- $c_k = b_k$ for every $k \neq i$ and that $c_j \neq c_i + s$ for every $j \in \{1, \ldots, n\}$ and $c_i + s \leq m$;
- $b_i = c_i + s$;
- $\{c_i + 1, \ldots, c_i + s - 1\} \subset \{c_1, \ldots, c_{i-1}\}$.

Let $f_\tau = \mathcal{R}_\tau|\mathcal{C}_\tau$ be the extremal function associated with τ. Let s_i be the number of occurrences of c_i in $\mathcal{R}_\tau$. Then $\mathfrak{b}_{\tau,\sigma} = s_i$.

Proof. We observe that σ is obtained from τ by swapping the non-trivial c_i-th row with the trivial (c_i+s)-th row. So in σ the c_i-th row is 0 and the (c_i+s)-th row is different from 0. Let $\underline{k} = (k_1, \ldots, k_r)$ and $\underline{l} = (l_1, \ldots, l_r)$ be increasing sequences of integers indexing the non-zero rows and columns in τ. Then there exists $h \in \{1, \ldots, r\}$ such that $k_h = c_i$. Thus $R(\tau)_{k_h,n} = h$. Since the c_i-th row in σ is zero and the rows above the c_i-th row are equal to the corresponding ones in τ, then $R(\sigma)_{k_h,n} < R(\tau)_{k_h,n} = h$.
We consider a minor $\mathfrak{m}_{\underline{k}',\underline{l}'}$ occurring in the expression of f_τ involving the c_i-th row (respectively the i-th column). Let $\underline{k}' = (k'_1, \ldots, k'_{r'})$ and $\underline{l}' = (l'_1, \ldots, l'_{r'})$, then $r' \geq h$. By construction of f_τ, we have $k'_h = k_h = c_i$. Let $A \in \hat{X}_\sigma$. Then

$$R(A)_{k_h,l'_{r'}} \leq R(\sigma)_{k_h,l'_{r'}} \leq R(\sigma)_{k_h,n} < h.$$

Therefore, by Lemma 6.49, the minor $\mathfrak{m}_{\underline{k}',\underline{l}'}(A) = 0$, for all $A \in \hat{X}_\sigma$. Then all the minors occurring in the expression of f_τ and involving the c_i-th row vanish on σ, whence on $\hat{X}_\sigma$, with multiplicity 1. Since all the other entries of τ and σ are equal, evaluation on σ of the remaining minors occurring in the expression of f_τ is different from 0. Thus the bond $\mathfrak{b}_{\tau,\sigma}$ is the number of minors occurring in the expression of f_τ and involving the c_i-th row, namely $\mathfrak{b}_{\tau,\sigma} = s_i$. $\square$

Example 6.54. Let $\tau, \sigma \in \mathcal{A}_{5,7}$ be the following matrices:

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

The partial permutation σ is obtained from τ by swapping the second and the fourth rows. So $\sigma \prec \tau$ is a cover relation as in Proposition 6.40.

The bi-tableau of the extremal function f_τ is

$$\mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 3 \\ \cline{3-4} \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & 4 & 5 & 6 \\ \hline 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

The bond $\mathfrak{b}_{\tau,\sigma}$ is equal to 2, that is the number of occurrences of 2 in $\mathcal{R}_\tau$ (and consequently the number of occurrences of 5 in $\mathcal{C}_\tau$).

In the next proposition we calculate the bond $\mathfrak{b}_{\tau,\sigma}$, where $\sigma \prec \tau \in \mathcal{A}_{m,n}$ is a covering relation as in Proposition 6.42.

Proposition 6.55. Let $\tau \in \mathcal{A}_{m,n}$ and let $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$, be such that $\tau_{k,l} = 1$. Let σ be the partial permutation obtained from τ by replacing the entry $\tau_{k,l}$ by 0. Let $\underline{c} = \psi(\tau)$. Suppose that

1. $\{k, k+1, \ldots, m\} \subseteq \{c_1, \ldots, c_l\}$;
2. $c_{l+1}, c_{l+2}, \ldots, c_n \leq k-1$.

Then, the extremal function $f_\tau|_{\hat{X}_\tau}$ vanishes on the divisor $\hat{X}_\sigma$ with multiplicity $\mathfrak{b}_{\tau,\sigma} = \min\{m+1-k, n+1-l\}$.

Proof. Recall the construction of the extremal functions in Section 6.5.2. It is a product of minors: $f_\tau = \mathfrak{m}_{\underline{i},\underline{j}} \mathfrak{m}_{\underline{i}',\underline{j}'} \cdots$.
By construction, the largest minor $\mathfrak{m}_{\underline{i},\underline{j}}$ involves all the non-zero rows and respectively non-zero columns of τ, in particular it involves the pair (k, l). The last minor occurring in the expression of f_τ involving the pair (k, l) is the p-th, where $p = \min\{m+1-k, n+1-l\}$.

Note that for $t > p$ the t-th minor occurring in f_τ is also a factor of f_σ (by construction). In particular, these factors do not vanish identically on $\hat{X}_\sigma$ and hence f_τ vanishes on $\hat{X}_\sigma$ with multiplicity $p = \min\{m+1-k, n+1-l\}$, that is $\mathfrak{b}_{\tau,\sigma} = \min\{m+1-k, n+1-l\}$. $\square$

Example 6.56. Let $\tau, \sigma \in \mathcal{A}_{m,n}$,

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Let $k = 4$ and $l = 5$, then $\tau_{k,l} = 1$ and $\sigma_{k,l} = 0$. The bi-tableaux of f_τ is

$$\mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 4 \\ \cline{3-4} \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & 5 & 6 & 7 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

and the bond between σ and τ is equal to $\mathfrak{b}_{\tau,\sigma} = 2$ because:

$$\min\{m + 1 - k, n + 1 - l\} = \min\{5 + 1 - 4, 7 + 1 - 5\} = \min\{2, 3\} = 2.$$

6.6 The case $m \times 2$

In this section we exhibit a Seshadri stratification for the matrix Schubert varieties in $\mathrm{Mat}_{m \times 2}$. In this case we are able to calculate the image of the quasi-valuation associated with this Seshadri stratification. We prove that the fan of monoids associated with the provided Seshadri stratification is of LS-type.

As in Section 6.2 we can parametrize partial permutations in $\mathcal{A}_{m,2}$ as vectors in $\mathbb{N}^2$. Let $\tau \in \mathcal{A}_{m,2}$ and let $\underline{a}_\tau = (i, j) \in \mathcal{A}_m^2$ be the vector associated with τ.
Since $\mathcal{A}_{m,2}$ and $\mathcal{A}_m^2$ are in bijection, for simplicity we will write, through this section, $\tau = (i, j)$ and we write $f_{(i,j)}$ for the extremal function of $\hat{X}_\tau$.

We observe that if $i, j > m$, the matrix τ is the 0 matrix. So if $\tau \neq 0$ we fall in one of the following cases:

1. if $i < j \leq m$, then the maximal rank submatrix of τ is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. if $m \geq i > j$, then the maximal rank submatrix of τ is of the form
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. if $i = m+1$, then there exists only one entry in τ different from 0 and it is on the second column;

4. if $j = m+1$, then there exists only one entry in τ different from 0 and it is on the first column.

We look at the extremal function associated with $\tau \neq 0$. We denote by $x_{i,j}$ the (i,j)-th coordinate function in $\mathbb{K}[\mathrm{Mat}_{m\times 2}]$ defined as follows

$$x_{i,j} : \mathrm{Mat}_{m\times n} \longrightarrow \mathbb{K}$$
$$A = (a_{i,j}) \mapsto a_{i,j}.$$

Let $\mathfrak{m}_{i,j}$ be the function in $\mathbb{K}[\mathrm{Mat}_{m\times 2}]$ associating to a matrix A the determinant of the submatrix obtained by taking the rows i, j and the columns $1, 2$. The extremal functions in the Seshadri stratifications of $\mathrm{Mat}_{m\times 2}$ are the following:

1. if $i < j \leq m$, then
$$f_\tau = \mathfrak{m}_{i,j} x_{i,1} = (x_{i,1}x_{j,2} - x_{j,1}x_{i,2})x_{i,1}$$

2. if $m \geq i > j$, then
$$f_\tau = \mathfrak{m}_{i,j} = x_{i,1}x_{j,2} - x_{j,1}x_{i,2}$$

3. if $i = m+1$ and $j \leq m$, then
$$f_\tau = x_{j,2}$$

4. if $j = m+1$ and $i \leq m$, then
$$f_\tau = x_{i,1}.$$

We look at the cover relations in this case.

Proposition 6.57. Let $\tau = (i,j)$ be a partial permutation in $\mathrm{Mat}_{m\times 2}$, and let σ be a partial permutation in $\mathrm{Mat}_{m\times 2}$ such that $\sigma \prec \tau$. We distinguish two cases:

1. if $i < j$, then σ is one of the following

- $\sigma = (i+1, j)$, which occurs only if $j > i+1$
- $\sigma = (j, i)$
- $\sigma = (i, j+1)$

2. if $i > j$, then σ is one of the following

- $\sigma = (i, i+1)$, which occurs if $i = j+1$
- $\sigma = (i, j+1)$, which occurs if $i > j+1$
- $\sigma = (i+1, j)$

Proof. The result follows from the descriptions of the covers in Section 6.4.1. $\square$

Since the extremal functions are very explicit here we can easily look at the bonds in the Hasse diagram. The next proposition can be proved by direct calculation.

Proposition 6.58. Let $\tau, \sigma \in \mathcal{A}_{m,2}$ and let $\sigma \prec \tau$ be a cover. Let $\underline{a}_\tau = (i, j) = \psi(\tau)$ and let $\underline{a}_\sigma = \psi(\sigma)$.

1. If $i < j \leq m$, we have the following cases:

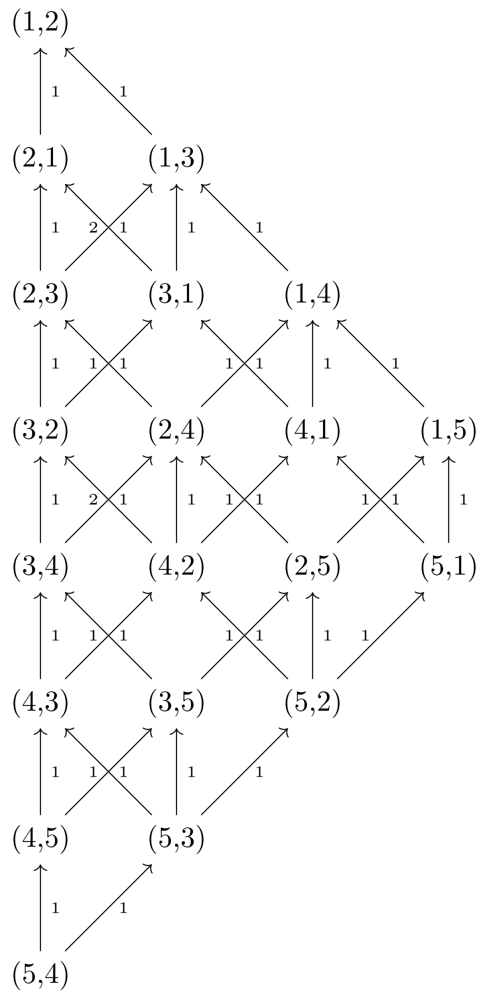
   (a) if $j > i+1$ and $\underline{a}_\sigma = (i+1, j)$, then $\mathfrak{b}_{\tau,\sigma} = 2$

   (b) if $\underline{a}_\sigma = (j, i)$, then $\mathfrak{b}_{\tau,\sigma} = 1$

   (c) if $\underline{a}_\sigma = (i, j+1)$, then $\mathfrak{b}_{\tau,\sigma} = 1$

2. If $i > j$, then $\mathfrak{b}_{\tau,\sigma} = 1$ for every $\sigma \prec \tau$.

Example 6.59. The following is the Hasse diagram for the case matrix Schubert varieties in $\mathrm{Mat}_{4\times 2}$.



6.6.1 Image of the quasi-valuation

We choose a total order on $\mathcal{A}_{m,2}$ extending the partial order on $\mathcal{A}_{m,2}$. We consider the vector space $\mathbb{Q}^{\mathcal{A}_{m,2}}$ of sequences with coefficients in $\mathbb{Q}$ and length equal to the cardinality of $\mathcal{A}_{m,2}$. We put the lexicographic order on $\mathbb{Q}^{\mathcal{A}_{m,2}}$ as in Section 5.4. Let $\tau \in \mathcal{A}_{m,2}$. We denote by e_τ the vector in $\mathbb{Q}^{\mathcal{A}_{m,2}}$ with all entries equal to 0 except from the one corresponding to τ, which is equal to 1.

We define the quasi-valuation $\mathcal{V}$ as in Definition 5.13. Recall that $\mathcal{V}$ is obtained as the minimum of the valuations $\mathcal{V}_{\mathfrak{C}}$, defined as in (5.6), where $\mathfrak{C}$ varies in the set of maximal chains in $\mathcal{A}_{m,2}$. We denote by $\mathbb{Q}^{\mathfrak{C}}$ the vector space spanned by e_τ for $\tau \in \mathfrak{C}$. Observe that $\mathcal{V}_{\mathfrak{C}}$ has values in $\mathbb{Q}^{\mathfrak{C}}$. In order to define $\mathcal{V}$, we consider $\mathbb{Q}^{\mathfrak{C}}$ as a subspace of $\mathbb{Q}^{\mathcal{A}_{m,2}}$. Let $r+1$ be the length of some/any maximal chain in $\mathcal{A}_{m,2}$.

In this section we study the image Γ of $\mathcal{V}$. In particular, we prove that Γ is of LS-type, namely

$$\Gamma = \bigcup_{\mathfrak{C} \text{ maximal chain in } \mathcal{A}_{m,2}} \mathrm{LS}^+_{\mathfrak{C}}$$

where $\mathrm{LS}^+_{\mathfrak{C}}$ is as in Definition 5.15.

Since, by (5.9), the image Γ is the union of the monoids $\Gamma_{\mathfrak{C}} \subseteq \mathbb{Q}^{\mathfrak{C}}$, with $\mathfrak{C}$ a maximal chain in $\mathcal{A}_{m,2}$, our strategy is to prove that $\Gamma_{\mathfrak{C}} = \mathrm{LS}^+_{\mathfrak{C}}$.

Lemma 6.60. Let $\sigma, \tau \in \mathcal{A}_{m,2}$ be such that $\sigma = (i,j)$ and $\tau = (i+k, j)$. Assume that $j > i+k \geq i$. Let $g_{\sigma,\tau} := x_{i1}\mathfrak{m}_{i+k,j}$. Then

$$\mathcal{V}(g_{\sigma,\tau}) = \frac{1}{2}e_\sigma + \frac{1}{2}e_\tau.$$

Proof. By direct calculation we have

$$g^2_{\sigma,\tau} = f_\sigma f_\tau - f_{(i,i+k)} f_{(j,i+k)} f_{(j,m+1)}.$$

We observe that σ and τ are comparable, hence by Proposition 5.14(3)

$$\mathcal{V}(f_\sigma f_\tau) = \mathcal{V}(f_\sigma) + \mathcal{V}(f_\tau) = e_\sigma + e_\tau.$$

Moreover $(i, i+k) > (j, i+k) > (j, m+1)$. Therefore, by Proposition 5.14(3),

$$\mathcal{V}(-f_{(i,i+k)} f_{(j,i+k)} f_{(j,m+1)}) = \mathcal{V}(f_{(i,i+k)}) + \mathcal{V}(f_{(j,i+k)}) + \mathcal{V}(f_{(j,m+1)}) =$$

$$= e_{(i,i+k)} + e_{(j,i+k)} + e_{(j,m+1)}.$$

Since $(i, i+k) > (i,j)$, it holds that $\mathcal{V}(f_{(i,i+k)} f_{(j,i+k)} f_{(j,m+1)}) > \mathcal{V}(f_\sigma f_\tau)$. By Lemma 2 (b), we have

$$\mathcal{V}(g^2_{\sigma,\tau}) = \mathcal{V}(f_\sigma f_\tau) = e_\sigma + e_\tau.$$

Thus

$$\mathcal{V}(g_{\sigma,\tau}) = \frac{1}{2}e_\sigma + \frac{1}{2}e_\tau.$$

$\square$

Let $\mathfrak{C} = (\sigma_r, \dots, \sigma_0)$ be a maximal chain in $\mathcal{A}_{m,2}$. For simplicity we denote by $\mathfrak{b}_k$ the bond between σ_k and σ_{k-1}, by f_k the extremal function f_{σ_k} and by e_k the vector e_{σ_k} for $k \in \{0, \dots, r\}$. The valuation $\mathcal{V}_{\mathfrak{C}}$, defined as in (5.6), takes values in $\mathbb{Q}^{\mathfrak{C}} \cong \mathbb{Q}^{r+1}$.

Definition 6.61. We define the lattice

$$\mathrm{LS}_{\mathfrak{C}} = \left\{ \begin{pmatrix} u_r \\ \vdots \\ u_0 \end{pmatrix} \in \mathbb{Q}^{\mathfrak{C}} \;\middle|\; \begin{matrix} \mathfrak{b}_r u_r \in \mathbb{Z} \\ \mathfrak{b}_{r-1}(u_r + u_{r-1}) \in \mathbb{Z} \\ \vdots \\ \mathfrak{b}_0(u_0 + \dots + u_r) \in \mathbb{Z} \end{matrix} \right\}.$$

If $u \in \mathbb{Q}^{\mathfrak{C}}$, then $u \in \mathrm{LS}_{\mathfrak{C}}$ if and only if $B_{\mathfrak{C}} \cdot u \in \mathbb{Z}^{r+1}$ for

$$B_{\mathfrak{C}} = \begin{pmatrix} \mathfrak{b}_r & & & \\ \mathfrak{b}_{r-1} & \mathfrak{b}_{r-1} & & \\ \vdots & \ddots & \ddots & \\ \mathfrak{b}_0 & \cdots & \cdots & \mathfrak{b}_0 \end{pmatrix}. \tag{6.16}$$

Furthermore we denote by $\mathrm{LS}^+_{\mathfrak{C}}$ the monoid $\mathrm{LS}_{\mathfrak{C}} \cap \mathbb{Q}^{r+1}_{\geq 0}$.

We observe that $\mathfrak{b}_0 = 1$. Indeed it is the degree of the last extremal function in the maximal chain, that is $f_{(m,m+1)} = x_{m,2}$ for every maximal chain.

Let $\mathfrak{C}_1, \dots, \mathfrak{C}_d$ be subchains of $\mathfrak{C}$ such that every bond in $\mathfrak{C}_k$ is equal to 2. Let $G_1 = \{e_k \mid k = 0, \dots, r\}$ and $G_2 = \bigcup_{t=1}^{d} \left\{ \frac{1}{2} e_k + \frac{1}{2} e_j \mid \sigma_k, \sigma_j \text{ elements of the chain } \mathfrak{C}_t \right\}$.

Lemma 6.62. With the notation above the set

$$G = G_1 \cup G_2$$

is a generating set for the monoid $\mathrm{LS}^+_{\mathfrak{C}}$.

Proof. Let $u = (u_r, \dots, u_0) \in \mathrm{LS}^+_{\mathfrak{C}}$. Let J be the subset of $\{0, \dots, r\}$ satisfying

$$u_k \in \mathbb{N}_0 \quad \text{for all } k \in J$$
$$u_k \in \mathbb{Q}_{\geq 0} \setminus \mathbb{N} \text{ otherwise}$$

We consider the vector $\overline{u} = u - \sum_{k \in J} u_k e_k$. We observe that $\overline{u} \in \mathrm{LS}^+_{\mathfrak{C}}$ and $\overline{u}_l \in \frac{1}{2}\mathbb{N} \cup \{0\}$ for all $l \in \{r, \dots, 0\}$. From the last condition in the definition of $\mathrm{LS}^+_{\mathfrak{C}}$, we get that the number of entries different from 0 in $\overline{u}$ has to be even. Let $i, j \in \{0, \dots, r+1\}$ the least two indexes such that $u_i, u_j \neq 0$, so $u_i, u_j \in \frac{1}{2}\mathbb{N}$. The condition of the definition of $\mathrm{LS}_{\mathfrak{C}}^{+}$ implies that the chain joining σ_i and σ_j is equal to $\mathfrak{C}_t$ for some $t \in \{1, \dots, d\}$. Then $\overline{u} = \alpha_i \frac{1}{2} e_i + \alpha_j \frac{1}{2} e_j + \overline{u}'$, with $\alpha_i, \alpha_j \in \mathbb{N}$, and where $\overline{u}'$ is the vector obtained from $\overline{u}$ by setting to 0 the i-th and the j-th entries. Now we repeat the procedure for $\overline{u}'$, considering the least two non-zero entries of $\overline{u}'$. Since the number of non-zero entries in $\overline{u}$ is even, continuing this procedure, we get $\overline{u} \in \{\sum_{l=1}^{s} \beta_l e_l' \in \mathbb{Q}^{\mathfrak{C}} \mid \beta_l \in \mathbb{N} \text{ and } e_l' \in G_2\}$. Therefore $u \in \{\sum_{l=1}^{s} \beta_l v_l \in \mathbb{Q}^{\mathfrak{C}} \mid \beta_l \in \mathbb{N} \text{ and } v_l \in G\}$. □

Proposition 6.63. With notation as above

$$\Gamma_{\mathfrak{C}} = \mathrm{LS}_{\mathfrak{C}}^{+}$$

Proof. Let $\mathbb{K}(\hat{X})$ be the field of rational functions on $\mathrm{Mat}_{m \times 2}$. We consider the valuation map

$$\mathcal{V}_{\mathfrak{C}} : \mathbb{K}(\hat{X}) \longrightarrow \mathbb{Q}^{r+1}.$$

Let $I \in \{1, \dots r\}$ be the set of indexes such that

$$\mathfrak{b}_h = 2 \text{ if } h \in I$$
$$\mathfrak{b}_h = 1 \text{ otherwise.}$$

The bond $\mathfrak{b}_h$ is equal to 2 if and only if $\sigma_h = (i, j)$ and $\sigma_{h-1} = (i+1, j)$, with $j > i+1$. We set $g_{h,h-1} = x_{1i}\mathfrak{m}_{i+1,j}$. For $h \in \{r, \dots, 1\}$, we define

$$F_h = \begin{cases} \frac{f_h}{f_{h-1}} \text{ if } h \notin I \\ \frac{g_{h,h-1}}{f_{h-1}} \text{ if } h \in I \end{cases},$$

and $F_0 = f_0$. We observe that

$$\mathcal{V}_{\mathfrak{C}}(F_h) = \frac{1}{\mathfrak{b}_h} e_h - \frac{1}{\mathfrak{b}_h} e_{h-1} \text{ and } \mathcal{V}_{\mathfrak{C}}(F_0) = e_0.$$

Let $M = (\mathcal{V}_{\mathfrak{C}}(F_h))_{h=0,\dots,r}$ be the matrix whose columns are given by $\mathcal{V}_{\mathfrak{C}}(F_h)$. We observe that M is invertible and that $M^{-1} = B_{\mathfrak{C}}$, where $B_{\mathfrak{C}}$ is the matrix in (6.16).

Let $\mathrm{L}_{\mathcal{V}_{\mathfrak{C}}}$ be the lattice generated by the image of $\mathcal{V}_{\mathfrak{C}}$. By [16, Proposition 6.13, Proposition 6.14]

$$\mathrm{L}_{\mathcal{V}_{\mathfrak{C}}} = < \mathcal{V}_{\mathfrak{C}}(F_h) \mid h \in \{0, \dots, r\} >_{\mathbb{Z}} = \mathrm{LS}_{\mathfrak{C}}$$

Thus, Proposition 5.14(2) gives $\Gamma_{\mathfrak{C}} \subset \mathrm{L}_{\mathcal{V}_{\mathfrak{C}}} \cap \mathbb{Q}_{\geq 0}^{r+1} = \mathrm{LS}_{\mathfrak{C}}^{+}$.
Let G_1, G_2 be as in Lemma 6.62 and let $e_h \in G_1$. Then $e_h = \mathcal{V}(f_h) = \mathcal{V}_{\mathfrak{C}}(f_h) \in \Gamma_{\mathfrak{C}}$. Let $u = \frac{1}{2}e_h + \frac{1}{2}e_k \in G_2$. By Lemma 6.60, we have that $u = \mathcal{V}(g_{h,k}) \in \Gamma_{\mathfrak{C}}$. Since $G_1 \cup G_2$ is a generating set of $\mathrm{LS}_{\mathfrak{C}}^{+}$, we conclude $\mathrm{LS}_{\mathfrak{C}}^{+} \subseteq \Gamma_{\mathfrak{C}}$. $\square$

Therefore the image Γ of $\mathcal{V}$ is of LS-type (Definition 5.17). Indeed, by (5.9) and Proposition 6.63,

$$\Gamma = \bigcup_{\mathfrak{C}} \Gamma_{\mathfrak{C}} = \bigcup_{\mathfrak{C}} \mathrm{LS}_{\mathfrak{C}}^{+},$$

where $\mathfrak{C}$ varies in the set of maximal chains in $\mathcal{A}_{m,2}$.

6.7 Γ is not of LS-type in general

Proposition 6.63 does not hold for arbitrary choices of m and n. In this section we give a counterexample.

Let $m = n = 4$. Let

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{A}_{4,4}$$

We have $\tau > \sigma$ and $l(\tau) = l(\sigma) + 1$, so this is a cover. The extremal functions are

$$f_\tau = \mathcal{R}_\tau | \mathcal{C}_\tau = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 3 \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{4-4} \end{array}\,, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$f_\sigma = \mathcal{R}_\sigma | \mathcal{C}_\sigma = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{3-3} \end{array}\,, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Since the minors $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}$ are both equal to zero on the orbit of σ, then $\mathfrak{b}_{\tau,\sigma} = 2$. Let $\mathfrak{C}$ be a maximal chain in $\mathcal{A}_{m,n}$ containing τ and σ. The monoid $\mathrm{LS}_{\mathfrak{C}}^{+}$ contains the vector $u = (0, \ldots, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ where the non-zero entries correspond to τ and σ. We claim that $u \notin \Gamma_{\mathfrak{C}}$. Indeed, if $u \in \Gamma_{\mathfrak{C}}$, then there would exist a non-zero function $g \in \mathbb{K}[\mathrm{Mat}_{m\times 2}]$ such that

$$g^2_{|\mathcal{O}_\tau} = (f_\tau f_\sigma)_{|\mathcal{O}_\tau}.$$

On the other hand $\deg(f_\tau f_\sigma)$ is odd and $\deg(g^2)$ is even, hence there is no such a function g and $\Gamma_{\mathfrak{C}} \neq \mathrm{LS}^+_{\mathfrak{C}}$. Therefore Γ is not of LS-type.

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