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Structural properties of solutions, approximation and control for conservation laws with discontinuous flux and bioinspired PDE models

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A mio nonno Nuccio
e a mia zia Tiziana

Vá dalla formica, o pigro,
guarda le sue abitudini e diventa saggio.
Essa non ha né capo,
né sorvegliante, né padrone,
eppure d'estate si provvede il vitto,
al tempo della mietitura accumula il cibo.
Fino a quando, pigro, te ne starai a dormire?
Quando ti scuoterai dal sonno?

Libro dei proverbi

"So don't be frightened, dear friend, if a sadness confronts you larger than any you have ever known, casting its shadow over all you do. You must think that something is happening within you, and remember that life has not forgotten you; it holds you in its hand and will not let you fall. Why would you want to exclude from your life any uneasiness, any pain, any depression, since you don't know what work they are accomplishing within you?"

Letters to a Young Poet - Rainer Maria Rilke

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English Abstract

In this thesis we study theoretical and control type properties of three different classes of PDE: Scalar Conservation Laws with flux discontinuous, respectively, in the space and in the conserved quantity, and a Partial Integro-Differential Equation. In the first chapter we analyze the set of attainable profiles, at fixed time (with initial datum regarded as a control), by solutions to conservation laws with flux having a single discontinuity in the space and strictly convex or strictly concave behaviour outside the discontinuity. This analysis yields compactness properties of such a set which is instrumental to study many variational problems involving the profiles of the solutions. In the second chapter we examine a class of conservation laws with flux discontinuous in the conserved quantity that emerges in a model of industrial conveyor belt and in supply chains. We first introduce an appropriate notion of pair of entropic solution-flux, we provide existence of entropic solution-fluxes by front-tracking and we show a Kruzkhov's type stability of such solutions. Next, we analyze the associated Hamilton-Jacobi equation, we derive an Hopf-Lax type representation formula of the solutions and we show how to recover the pair of entropic solution-flux of the conservation law from the gradient of the solution of the Hamilton-Jacobi equation. Finally, we consider the problem of a junction with a buffer (to store processed products) and with incoming and outgoing belts modeled by the class of conservation laws analyzed beforehand. Existence and uniqueness of the solution to the junction problem is established. The last chapter describes a mathematical model for a robotic root to be used in rescue technology. The root movement is described by two different Partial Integro-Differential Equations, one for the body and one for the tip. When the root encounters an obstacle in a "good" configuration, it moves around it by bending through a determined angular velocity minimizing the elastic deformation, the cost of moving sand and digging where the soil is dense. A restarting procedure is instead introduced to handle the crossing of obstacles in "bad" configuration. Some numeric simulation are also produced.

Italian Abstract

In questa tesi si studiano proprietà teoriche e controllistiche per tre differenti classi di PDE: Leggi di Conservazione Scalari con flusso discontinuo rispettivamente nello spazio e nella quantità conservata, ed Equazioni Integro-Differenziali alle derivate parziali. Nel primo capitolo si analizza l'insieme dei profili raggiungibili, ad un tempo fissato (con dato iniziale visto come controllo), da soluzioni di leggi di conservazione con flusso avente una singola discontinuità nello spazio e andamento strettamente convesso o strettamente concavo al di fuori di essa. Questa analisi mostra proprietà di compattezza di tale insieme che è essenziale al fine di studiare molti problemi variazionali che coinvolgono i profili delle soluzioni. Nel secondo capitolo si esamina una classe di leggi di conservazione con flusso discontinuo nella quantità conservata che emerge nei modelli per cinghie di distribuzione industriale e supply chain. Si introduce una appropriata nozione di coppia entropia- flusso entropico e si dimostra l'esistenza si una soluzione data dalla coppia densità-flusso mediante front-tracking, inoltre si prova un risultato di stabilità per queste soluzioni alla Kruzkhov. Successivamente viene analizzata l'equazione di Hamilton-Jacobi associata e per la quale si deriva una rappresentazione esplicita della soluzione del tipo Hopf-Lax. Il gradiente di questa soluzione fornisce la coppia densità-flusso. Conclusivamente si considera il problema al giunto con un buffer (in cui depositare i prodotti processati) con catene entranti e uscenti sulle quali il flusso è descritto dalle leggi di conservazione analizzate in precedenza. Si dimostra esistenza ed unicità della soluzione per il problema al giunto. L'ultimo capitolo descrive un modello matematico per una radice robotica da impiegare nelle rescue technology. Il movimento della radice è descritto da due equazioni integro-differenziali alle derivate parziali, una per il corpo e una per la punta. Quando la radice incontra un ostacolo in una "buona" configurazione, lo evita mediante una velocità angolare che minimizza la deformazione elastica, il costo per spostare sabbia e per scavare dove il suolo è denso. Quando invece la configurazione è "cattiva", si introduce una procedura di restarting per ovviare ad essa. Si forniscono per finire alcune simulazioni numeriche.

Introduction

In this thesis we study theoretical, structural and control type properties of three different classes of PDE: Conservation Laws with flux discontinuous in the space variable, Conservation Laws with flux discontinuous in the conserved quantity and Partial Integro-Differential Equations.
Conservation laws with discontinuous flux in space have been and still are a topic of great interest due to their large number of applications, ranging from road traffic with changing surface conditions, to sedimentation problems, to oil recovery problems and endovascular blood flow on vessels of different sizes. Important applications of the theory developed for such class of PDE arise also in the analysis of: network models, inverse problems for conservation laws with continuous flux, triangular systems of conservation laws.
The various studies performed so far concern the well-posedness of the Cauchy problem and the approximation of solutions by numerical schemes. However, until now no general control analysis has been conducted on this type of equations. As a first step in this direction, after an overview of the main entropy conditions introduced in the literature to achieve uniqueness of solutions for the Cauchy problem, we provide in Chapter 1 a characterization of a set of *attainable profiles* at a fixed time, for conservation laws that admit a single discontinuity in the space and are strictly convex (or strictly concave) outside it. This is the set of profiles that can be attained at a fixed time by solutions of this class of PDE with bounded initial data. Here, we regard the initial datum as a control affecting the behaviour of the solution. This analysis yields compactness properties of the attainable set which is instrumental to study many variational problems iintroduced in the past and involving the profiles of the solutions. At the end of the Chapter some applications to optimization problems for road traffic and oil extraction are analyzed.

Conservation laws with discontinuous flux in the conserved quantity is a much less studied topic in the literature, which presents entirely different feature from the previous one. These type of equations emerge in most modern supply chain models as well as in some empirical models for road traffic. Here, the discontinuity in the conserved variable yields solutions with multivalued flux. Such solutions can be approximated by solutions of conservation laws with regularized fluxes which exhibit shocks with arbitrarily large slope. This implies that, for these conservation laws, the fundamental property of finite wave propagation is lost. Therefore, for such a class of equations the very concept of solution needs to be better understood and the classical theory needs to be suitably adapted. In Chapter 2, after a brief summary of the known results on these equations, we analyze a specific class of conservation laws with flux discontinuous in the conserved quantity that emerges in studying industrial conveyor belt and in supply chain. We first introduce an appropriate notion of pair of entropic solution-flux, we provide existence of entropic solution-fluxes by front-tracking and we show a Kruzkhov's type stability of such solutions. Next, we analyze the associated Hamilton-Jacobi equation, we derive an Hopf-Lax type representation formula of the solutions and we show how to recover the pair of entropic solution-flux of the conservation law from the gradient of the solution of the Hamilton-

Jacobi equation. Finally, we consider the problem of a junction with a buffer (to store processed products) and with incoming and outgoing belts modelled by conservation laws with discontinuous flux in density (for the evolution of parts on the single chains) and by an ordinary differential equation (for the processing of parts inside the buffer). Here, the Hamilton Jacobi approach introduced for the single conveyor belt is crucial in order to prove existence and uniqueness of the solution to the junction problem. In fact, such a solution is obtained by a fixed type argument for a contractive map which is defined in terms of the Hopf-Lax type representation formula of the solutions along the single chains.

The last chapter concerns the control of a class of Partial Integro-Differential Equations describing the evolution of a robotic root. The idea of this robot stems from the need to expand those technologies known as "Rescue technologies" which are exploited in saving human lives. We speak of a robotic root because we want to imitate the behaviours and capacities of plants in order to optimize performance, especially in unstable environments like collapsed buildings. The goal of the root is to dig into the ground among the rubble where the radars pick up heat sources or pulsations that can be traced back to living beings. The root movement is described by two different partial differential equations, one for the body and one for the tip. In fact, while the body moves only in response to obstacles or high-density terrain, the tip also pursues a control that directs towards the target. Every time the robot encounters an obstacle in a configuration for which it is not possible to circumvent it, it is shortened by a suitable quantity described by special algorithms and grows away from the previous trajectory so as not to explore areas already known, i.e. it builds a map of the regions explored. When, on the other hand, the root encounters an obstacle in a good configuration, it moves around it by bending through a determined angular velocity minimizing the elastic deformation, the cost of moving sand and digging where the soil is dense. Finally we show some simulations of the behaviour of a robotic root in the presence of an obstacle.

We want to underline that the goal of this thesis is to understand and analyze theoretical aspects of some real models with all the necessary tools coming from Mathematical Analysis. In particular, among these, the main ones turn out to be the Theory of: Hyperbolic Conservation Laws, Hamilton-Jacobi Equations, Partial Integro-Differential Equations and the Mathematica Theory of Control.

Chapter 1

Scalar conservation laws with space discontinuous flux

Consider the following Cauchy problem for scalar conservation law in one space dimension

$$u_t + f(x,u)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{0.1}$$

$$u|_{t=0} = u_0 \qquad x \in \mathbb{R} \tag{0.2}$$

where the flux $f(x,u)$ is a discontinuous function given by

$$f(x,u) = f_l(u)\mathbb{1}_{x<0} + f_r(u)\mathbb{1}_{x>0} \tag{0.3}$$

Equations of the above type are topic of intense current research since they arise in a number of recent applications, for example in modeling two phase flow in a porous media ([43]), in sedimentation problems ([64]) and in traffic flow ([37])(with roads whose amplitude varies for work in progress or change of number of lanes), in Saint Venant model of blood flow ([10]).
This kind of problems appear also if we consider conservation laws on network (whose recent great interest is motivated by the application in data networks, supply chains, air traffic management, gas pipelines, irrigation channels) when the network is composed only by two arcs and a junction point. Regardless of how much f and u_0 are smooth, it is well known that classical solutions for this problem may not exist globally since discontinuities can develop in finite time, hence we need to interpret (0.1),(0.2) in a weak (distributional) sense. However, weak solutions are, in general, not unique, thus additional admissibility criteria are necessary to single out a unique solution; these criteria are called entropy condition. In the case of conservation laws with discontinuous flux, the classical entropy conditions are still not enough to ensure the uniqueness of weak solutions, therefore a wide theory has been developed on the right conditions that must be imposed along the discontinuity in order to obtain uniqueness. In particular our choice of entropy conditions will be dictated by the interest for any future applications to network problems.
Here we analyze the problem from the point of view of control theory, in particular the initial data u_0 will be considered as a control. Given a set $\mathcal{U} \subset L^\infty(\mathbb{R})$ of admissible controls, we characterize the set attainable profiles at a fixed time T

$$\mathcal{A}(T,\mathcal{U}) = \{u(\cdot,T) : u \text{ is a solution of } (0.1),(0.2) \text{ with } u_0 \in \mathcal{U}\}\,.$$

A brief survey on conservation laws with discontinuous flux

The interest captured by the innumerable applications of scalar conservation laws with discontinuous flux in space can be measured observing how the literature currently available on these equations is vast. The challenges regard their well-posedness, numerical analysis, regularity and controllability. The applications of conservation laws with discontinuous flux can be gathered in three different groups:

- the first group is represented by those models in which the discontinuity of the flux comes out from a spatial heterogeneous physical reality and is explicitly modeled. This is the case of traffic flow models with heterogeneous surface conditions and continuous sedimentation([64], [34]);
- a second group comes from the study of inverse problems of a standard scalar conservation law having a continuous flux ([14]);
- the last one arises in the reformulation of balance laws and triangular systems of conservation laws in term of conservation laws with discontinuous flux ([55], [54]).

For a deep analysis of the numerous applications see [22].

These equations admit many different L^1- contractive semigroup, but the physics of each model singles out the opportune solution. A large variety of admissibility criteria have been formulated in recent years and a unified perspective of these is offered by the work of Andreianov, Karlsen, Risebro [5] in which they show that the whole admissibility issue can be reduced to the selection of a family of "elementary solutions" which are piecewise constant weak solutions of the form

$$c(x) = c^l \mathbb{1}_{\{x<0\}} + c^r \mathbb{1}_{\{x>0\}}.$$

They refer such a family as "germ". This approach was suggested by a work of Garavello, Natalini, Piccoli, Terracina [38] in which the authors associate to a particular admissibility criteria a Riemann solver in $x = 0$.

In the wide range of available entropy conditions, we choose the one introduced in [2] that the authors refer as AB- interface entropy condition. This particular condition produces a unique solution of the conservation law satisfying the Kruzkov entropy condition out of $\{x = 0\}$ and an opportune inequality involving the left and right traces of this solution in $x = 0$. Existence of strong traces for solutions of scalar conservation laws defined on subsets of $\mathbb{R} \times \mathbb{R}^+$ reached by L^1 convergence, is proved in [58], [66] and [6]. In the next we will explain in detail the notion of AB- interface entropy condition and the associated AB- entropy solution, moreover we will recall some regularity results for this solutions.

Our contribution to the theory of conservation laws with discontinuous flux will consists in describing how a solution's profile is done at fixed time. We remark that what we state is a necessary and sufficient condition for a given function to be a solution . This means that for a function satisfying some conditions that we will state later, we are able to construct an initial datum such that this function is the unique solution for the Cauchy problem.

In the final analysis we study the compactness properties of the set of attainable profiles. It is easy to understand that compactness is essential for studying functionals that involve the profile of the solution. The simplest we can imagine is the functional distance from a target profile, but there are a lot of other possible choices.

The chapter is organized in the following way: in Section 1 we give an overview of conservation laws with flux discontinuous in the space and we highlight a distinctive feature of this type of equation, that is, the problem of selecting a single solution by means of appropriate entropy conditions along the discontinuity interface. To this end we will briefly describe the main entropy conditions introduced in past years and how to interpret them in terms of dissipative germs. In Section 2 we state the original results of the chapter, the full description of the set of attainable profiles for fixed time for scalar conservation laws with flux as (0.3). Before prooving the main theorem in Section 3 we introduce some preliminary lemmas which point out peculiarities of the entropy solution associated to the particular interface entropy condition chosen. In Section 4 and 5 we prove respectively the main theorem, that is the necessary and sufficient conditions for a function to belong to the set of attainable profiles and the topological characterization of this set. The analysis done for convex-convex fluxes can be adapted to the case of concave-concave fluxes and this is exactly what we do in Section 6 where we also compare how the meaning of the interface entropy condition changes by showing how to solve the different Riemann problems. Finally in Section 7 we apply the topological characterizazion of the attainable profiles to some variational problems emerging on porus media and traffic models.

1 Well posedness of the problem and entropy conditions

The basic difficulty that occurs in the well-posedness for initial value problem of (0.1) is due to the fact that given a value for the trace of the solution on one side of $\{x = 0\}$, the correspondent trace on the other side is not uniquely determined only by asking the correspondent fluxes to be equal. Therefore in the last decades several methods have been introduced in order to single out a unique solution of (0.1). They are mainly inspired by the classic criteria of admissibility for scalar conservation laws with continuous flux, such as vanishing viscosity and entropy conditions. After recalling the most important examples of there criteria, we show how we can interpret them in term of L^1-dissipative germ.

1.1 Diehl's Γ condition

In [35] Diehl introduced one of first entropy conditions for equation (0.1) with flux as (0.3), known as Γ condition. His analysis is limited to piecewise smooth solutions, i.e. bounded and $\mathcal{C}^1$ except along a finite number of $\mathcal{C}^1$ curves where both left and right limit exist. If u is such a function, we use the following notation:

$$u_\pm(t) = \lim_{\varepsilon\to 0} u(\pm\,\varepsilon, t),$$
$$u^\pm(t) = \lim_{\varepsilon\to 0} u_\pm(t+\varepsilon).$$

Moreover it is assumed that $u^\pm(t)$ are piecewise monotone. In the definition below, for every Riemann problem along $\{x = 0\}$ we introduce the functions $\check{f}_l(\mathbb{R}, u_-)$ and $\hat{f}_r(\mathbb{R}, u_+)$ consisting respectively in the minimal convex decreasing and increasing functions passing trough $f_l(u_-)$ and $f_r(u_+)$ when both f_l and f_r are convex. To these functions we can associate also the sets $N(f_l, u_-)$ and $P(f_r, u_+)$ where $\check{f}_l(\mathbb{R}, u_-)$ and $\hat{f}_r(\mathbb{R}, u_+)$ are strictly monotone. These elements are crucial in describing the Γ condition.

Definition 1.1. Given u_- and $u_+ \in \mathbb{R}$ and the flux functions f_l and f_r define (see Fig 1.1)

$$\check{f}_l(u, u_-) = \begin{cases} \max_{v\in[u,u_-]} f_l(v), & u \le u_- \\ \min_{v\in[u_-,u]} f_l(v), & u \ge u_- \end{cases}$$

$$\begin{aligned} N(f_l, u_-) = \{u_-\} &\cup \{u \ : \ u < u_l; \check{f}_l(u+\varepsilon, u_-) < \check{f}_l(u, u_-), \quad \text{for all } \varepsilon > 0\} \\ &\cup \{u \ : \ u > u_-; \check{f}_l(u-\varepsilon, u_-) > \check{f}_l(u, u_-), \quad \text{for all } \varepsilon > 0\} \end{aligned}$$

$$\hat{f}_r(u,u_+) = \begin{cases} \min_{v\in[u,u_+]} f_r(v), & u \le u_+ \\ \max_{v\in[u_+,u]} f_r(v), & u \ge u_+ \end{cases}$$

$$\begin{aligned} P(f_r,u_+) = \{u_+\} \cup \{u \ : \ u < u_+; \hat{f}_r(u+\varepsilon,u_+) > \hat{f}_r(u,u_+), \quad \text{for all } \varepsilon > 0\} \\ \cup \{u \ : \ u < u_+; \hat{f}_r(u-\varepsilon,u_+) < \hat{f}_r(u,u_+), \quad \text{for all } \varepsilon > 0\} \end{aligned}$$

Observe that $\check{f}_l(\cdot,u_-)$ is a decreasing function whose graph consists of strictly decreasing parts separated by plateaus where the function is constant. Analogously $\hat{f}_r(\cdot,u_+)$ is an increasing function with constant pieces.

Monotonicity of $\check{f}_l(\cdot,u_-)$ and $\hat{f}_r(\cdot,u_+)$ implies that their intersection is given by an interval, thus we introduce the following notation

$$\bar{U} = \bar{U}(u_-,u_+) = \{u \in \mathbb{R} \ : \ \check{f}_l(u,u_-) = \hat{f}_r(u,u_+)\},$$
$$\gamma = \check{f}_l(\bar{U},u_-),$$

In the case of convex fluxes if $f_l(u_l) \neq f_r(u_r)$ then $\bar{U}$ is just a point that we can denote by $\bar{u}$.

Condition Γ: For t fixed and given $u_-(t),\ u_+(t) \in \mathbb{R}$, $(u^-(t),u^+(t)) \in \Gamma(u_-(t),u_+(t)) = \{(\alpha,\beta) \in \mathbb{R}^2 : f_l(\alpha) = f_r(\beta) = \gamma\}$.

The following result ensures that if u is a solution up to the time t then the Riemann problem with left and right state given by $u_-(t),\ u_+(t)$ admits a solution satisfying the Γ condition.

Proposition 1.1. *Let $u_-,\ u_+ \in \mathbb{R}$ be given, if $\check{f}_l(\mathbb{R},u_-) \cap \hat{f}_r(\mathbb{R},u_+) \neq \emptyset$ then the set $(N(f_l,u_-) \times P(f_r,u_+)) \cap \Gamma(u_l,u_+)$ consists exactly in one pair representing left and right traces for the solution along the discontinuity $\{x=0\}$.*

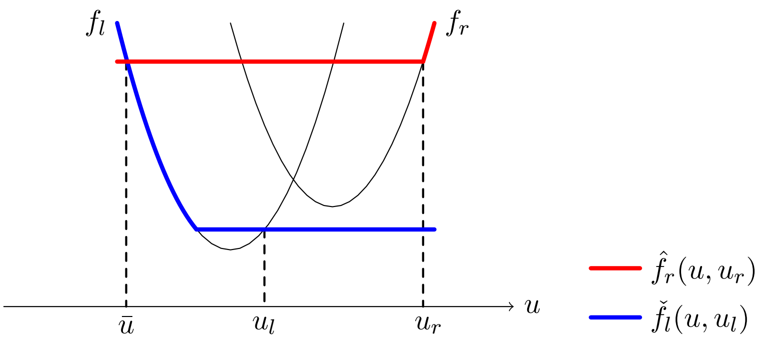


Figure 1.1: Example of how to construct $\check{f}_l(u,u_l)$, $\hat{f}_r(u,u_r)$ starting from the Riemann problem with states u_l and u_r

.

The Γ condition has been justified first in [34] by Godunov's method applied to (0.1), later in [35] it was proved the equivalence with the so called **viscous profile condition** obtained by studying stationary solutions of $u_t + f^\delta(u)_x = \varepsilon u_{xx}$ where f^δ is a smooth approximation of the discontinuous flux (0.3) .

1.2 Audusse-Perthame adapted entropies

In [10] Audusse and Perthame propose to adapt the definition of Kruzkov entropies to the discontinuous case by introducing partially adapted Kruzkov entropies

$$E_\alpha(x,u) = |u - k_\alpha(x)| \tag{1.1}$$

where $k_\alpha(x)$ satisfies

$$f(x, k_\alpha(x)) = \alpha \tag{1.2}$$

Hence the new adapted entropy condition is given by

$$\partial_t |u - k_\alpha(x)| + \partial_x [(f(x,u) - f(x, k_\alpha(x)))\,\mathrm{sgn}(u - k_\alpha(x))] \leq 0 \tag{1.3}$$

In this way the interface does not need a special treatment and no conditions along $\{x = 0\}$ are required. Uniqueness follows from an argument very similar to the classical one of Kruzkov and the main difficulty stands in working with the families of function $k_\alpha(x)$.

Definition 1.2. Let u and $v \in L^\infty([0,T],\mathbb{R}) \cap C^0([0,T], L^1_{loc}(\mathbb{R}))$ be respectively an entropy sub- and supersolution of (0.1) if an only if for all $\alpha \in \mathbb{R}^+$

$$\partial_t (u - k_\alpha(x)) + \partial_x [(f(x,u) - f(x, k_\alpha(x)))\,\mathrm{sgn}_+(u - k_\alpha(x))] \leq 0 \tag{1.4}$$

resp

$$\partial_t (v - k_\alpha(x)) + \partial_x [(f(x,v) - f(x, k_\alpha(x)))\,\mathrm{sgn}_-(v - k_\alpha(x))] \geq 0 \tag{1.5}$$

Theorem 1.2. *Let u and $v \in L^\infty([0,T],\mathbb{R}) \cap C^0([0,T], L^1_{loc}(\mathbb{R}))$ be respectively and entropy sub- and supersolution to the initial value problem (0.1) with initial data in $u_0, v_0 \in L^\infty([0,T],\mathbb{R})$. Assume that The flux f is of the form (0.3) with $f_{l,r}$ continuous, strictly convex and coercive, then for a.e. $t \in [0,T]$*

$$\int_a^b (u(x,t) - v(x,t))_+ dx \leq \int_{a-Mt}^{b+Mt} (u_0(x) - v_0(x))_+ dx. \tag{1.6}$$

Hence the previous results immediately implies uniqueness of the solution when $u_0 = v_0$. Theory of adapted entropies was developed for more general discontinuous fluxes but unlike the solutions obtained by means of the Γ condition, those selected by the adapted entropies cannot be derived by vanishing viscosity as shown in [48].

1.3 The Karlsen-Risebro-Towers entropy condition

In [56] the Karlsen, Risebro and Towers study entropy condition and uniqueness of solution for the following Cauchy problem

$$\begin{cases} u_t + f(\gamma(x), u)_x = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.7}$$

where $u_0 \in L^\infty(\mathbb{R})$ and the flux function $f : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ has a spatial dependence through the vector-valued parameter

$$\gamma(x) = \Big(\gamma_1(x), ..., \gamma_p(x)\Big). \tag{1.8}$$

The spatial varying coefficient γ is assumed to be piecewise $\mathcal{C}^1$ with finitely many jumps located at $\eta_1 < \eta_2 < ... < \eta_M$. They assume also that each γ_ν (ν=1,...p) is locally Lipschitz while the flux f is Lipschitz continuous in each variable. In this setting the following definition of entropy solution is suggested:

Definition 1.3. An entropy solution of the initial value problem (1.7) is a function $u \in L^\infty([0,T]\times\mathbb{R})$ that satisfies (1.7) in the distributional sense and the following Kruzkov-type entropy inequality for all $\phi \geq 0$ test function:

$$\begin{aligned}
&\int\int\Big(|u-c|\phi_t + \operatorname{sgn}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\phi_x\Big)dt\ dx \\
&\quad - \int\int_{([0,T]\times\mathbb{R})\setminus\{\eta_m\}_{m=1}^M} \operatorname{sgn}(u-c) f(\gamma(x),c)_x\ \phi dt\ dx \\
&\quad + \int_0^T \sum_{m=1}^M |f(\gamma(\eta_m+),c) - f(\gamma(\eta_m-),c)|\phi(\eta_m,t)dt \\
&\quad + \int_{\mathbb{R}} |u_0(x)-c|\phi(x,0)dx \geq 0 \qquad \text{for all } c\in\mathbb{R}
\end{aligned}$$

where

$$f(\gamma(x),c)_x = \gamma'(x)\cdot f_\gamma(\gamma(x),c) = \sum_{\nu=1}^p \gamma'_\nu(x) f_{\gamma_\nu}(\gamma(x),c).$$

Crossing condition For any jump in γ with associated left and right limits $(\gamma-,\gamma+)$, for any states u and v, the following crossing condition must hold:

$$f(\gamma+,u) - f(\gamma-,u) < 0 < f(\gamma+,v) - f(\gamma-,v) \quad \Longrightarrow \quad u < v. \tag{1.9}$$

From the geometric point of view this condition requires that either the graph of $f(\gamma-,\cdot)$ and $f(\gamma+,\cdot)$ do not cross, or if they intersect, the graph of $f(\gamma-,\cdot)$ lies above the graph of $f(\gamma+,\cdot)$ to the left of any crossing point.

Existence of traces Let $u = (x,t)$ be an entropy solution to the initial value problem (1.7). For $m = 1,...,M$ they assume that $u(\cdot,t)$ admits right and left traces at $x = \eta_m$ denoted by $u(\eta_m\pm,t)$.

Theorem 1.3. *Let u and v be two entropy solutions to the initial value problem (1.7) with initial data u_0,$v_0 \in L^\infty(\mathbb{R})$, respectively. If f satisfies the* ***crossing condition*** *and there exist right and left trace of $u(\cdot,t)$, $v(\cdot,t)$ at the jump point of γ, then for a.e. $t\in(0,T)$*

$$\int_{-r}^r |v(x,t)-u(x,t)|dx \leq C\int_{-r-Lt}^{r+Lt} |v_0(x) - u_0(x,t)|dx \tag{1.10}$$

with L Lipschitz constant of f with respect to the second variable and $C > 0$ finite constant. If $\gamma'(x) = 0$ for a.e. $x\in\mathbb{R}$, then $C = 1$.

When the crossing condition holds, also solutions selected by Karlsen-Risebro-Towers entropy condition can be derived by vanishing viscosity as proved in Section 5 of [5].

1.4 Connections

In [2] Mishra, Adimurthi and Verappa introduced the most commonly used entropy condition for scalar conservation laws with discontinuous flux, the so called AB-interface entropy condition associated to the connection (A,B). We briefly discuss how it works and some regularity results. We assume now that f_l and f_r in (0.3) are strictly convex fluxes in $\mathcal{C}^2(\mathbb{R};\mathbb{R})$ with θ_l and θ_r points of minima.

Definition 1.4. (Connection) Let $(A,B)\in\mathbb{R}^2$. Then (A,B) is called a connection (Fig. 1) if it satisfies

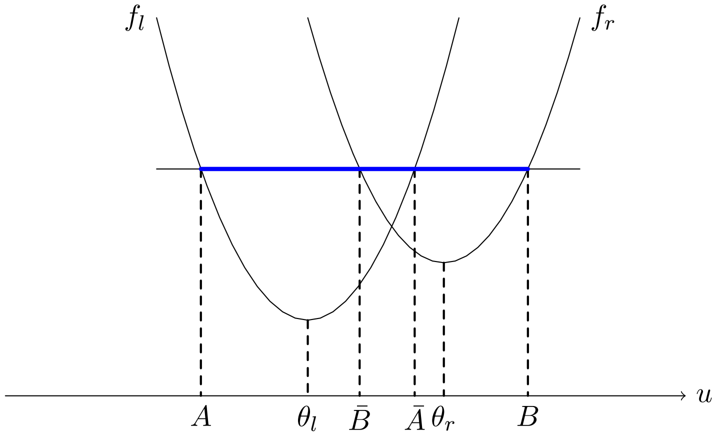


Figure 1.2: Example of AB connection with f_l, f_r strictly convex fluxes

(i) $f_l(A) = f_r(B)$,

(ii) $f_r'(B) \geq 0$ and $f_l'(A) \leq 0$.

Definition 1.5. Let $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that $u^\pm = u(0\pm, t)$ exists a.e $t > 0$. Then we define $I_{AB}(t)$, the interface entropy functional, by

$$\begin{aligned} I_{AB}(t) =& (f_l(u^-(t)) - f_l(A)) sgn(u^-(t) - A) \\ &- (f_r(u^+(t)) - f_r(B)) sgn(u^+(t) - B). \end{aligned}$$

The function u is said to satisfy the interface entropy condition relative to the connection (AB) if for a.e. $t > 0$

$$I_{AB}(t) \geq 0 \tag{1.11}$$

The meaning of condition (1.11) on the interface entropy functional is that the flux of the solution on the discontinuity interface must be greater or equal to the value of the flux on the AB connection. Indeed, by the Rankine-Hugoniot condition (2), inequality (1.11) is equivalent to

$$(f_l(u^-(t)) - f_l(A)) \Big[sgn(u^-(t) - A) - sgn(u^+(t) - B) \Big] \geq 0. \tag{1.12}$$

Even if (4.9) is satisfied when the two terms have the same sign, convexity of the fluxes implies that only the case with the two term positive can occur. Following [5], (1.11) can be rewritten in a more explicit way:

$$f_l(u^-(t)) = f_r(u^+(t)) \geq f_l(A) = f_r(B), \quad sgn^-(u^-(t) - A) sgn^+(u^+(t) - B) \leq 0. \tag{1.13}$$

where $sgn^- = sgn \vee 0$ and $sgn^+ = sgn \wedge 0$.

Definition 1.6. Let (AB) be a connection and $f(x, u) = f_l(u)\mathbb{1}_{x<0} + f_r(u)\mathbb{1}_{x>0}$. Let $u_0 \in L^\infty(\mathbb{R})$. Then $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is said to be an AB-entropy solution if:

(i) u is a weak solution of

$$\begin{aligned} u_t + f(x, u)_x = 0, \qquad & x \in \mathbb{R}, \quad t > 0 \\ u|_{t=0} = u_0 \qquad & x \in \mathbb{R}. \end{aligned}$$

(ii) u satisfies Kruzkov entropy condition away from the interface $\{x = 0\}$.

(iii) At the interface $\{x = 0\}$, u satisfies the AB interface entropy condition (1.1)

In [1] Adimurthi and al. proved that for each choice of a connection (A, B) with, an AB-entropy solution of (0.1) with bounded initial data, exists and is unique. Existence is proved by using convergence of some Godunov schemes based on solution of Riemann problem, for uniqueness they showed that each of these classes of entropy solutions form a contractive semigroup in L^1. Another important result due to Adimurthi, Dutta, Goshal and Gowda regards boundedness of total variation of the solution. In [40] they proved that for all the connections such that both $A \neq \theta_l$ and $B \neq \theta_r$, the entropy solutions for (0.1) with bounded initial data are in $BV_{loc}(\mathbb{R})$ for all positive times. But also when the connection passes through one of the two minimum of the fluxes the solution is BV_{loc} for all the times away from $(0, t)$ which are the only points where the total variation of the solution can explode (see for an explicit example [1]). However we will prove that right and left limit exists also in $(0, t)$ for all $t > 0$. This is very important for the analysis that we are going to do.

Also AB- entropy solutions can be obtained by vanishing viscosity as proved in [5]. Indeed fixed a connection (AB), it is possible to choose an artificial "adapted viscosity" of the form

$$\varepsilon(a(x, u))_{xx}$$

such that the stationary solution $c(x) = A\mathbb{1}_{x<0} + B\mathbb{1}_{x>0}$ of the limit equation is also solution of the viscous problem

$$u_t + f(x, u)_x = \varepsilon(a(x, u))_{xx}$$
$$f(x, u) = \begin{cases} f_l(u) & x < 0 \\ f_r(u) & x > 0 \end{cases} \quad a(x, u) = \begin{cases} a_l(u) & x < 0 \\ a_r(u) & x > 0 \end{cases}.$$

1.5 L^1 dissipative germs

The theory of L^1 dissipative germs that we are going to describe below, has the merit of synthesizing all the notions of entropy solutions for scalar conservation laws with discontinuous flux introduced so far by treating them in a systematic way. The starting assumption is that away from the point of discontinuity for the flux, the appropriate notion of entropy solution is still the one given by Kruzhkov [57].

Kruzhkov entropy condition Let φ_1, φ_2 be convex functions. Let $\psi_1'(s) = f_r'(s)\varphi_1'(s)$ and $\psi_2'(s) = f_l'(s)\varphi_2'(s)$. Then (φ_i, ψ_i) with $i = 1, 2$ are called entropy pairs associated to (0.1).
A weak solution $u \in L^\infty(\mathbb{R})$ of (0.1),(0.2) is said to satisfy *Kruzkov entropy condition* if for every entropy pairs (φ_i, ψ_i) and for every $\rho \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$ with $\rho \geq 0$

$$\int_0^\infty \int_0^\infty \left(\varphi_1(u)\frac{\partial \rho}{\partial t} + \psi_1(u)\frac{\partial \rho}{\partial x} \right) dxdt \geq -\int_0^\infty \psi_1(u(0+, t))\rho(0, t)dt,$$
$$\int_{-\infty}^0 \int_0^\infty \left(\varphi_2(u)\frac{\partial \rho}{\partial t} + \psi_2(u)\frac{\partial \rho}{\partial x} \right) dxdt \geq \int_0^\infty \psi_2(u(0-, t))\rho(0, t)dt.$$

Therefore, understanding the problem (0.1) is equivalent to study the coupling condition of two scalar conservation laws across the interface $\Sigma := \{x = 0\}$. The theory of strong boundary traces for conservation laws [66] allows to encode the coupling across Σ by the set $\mathcal{G}^*$ of the admissible couples of the left and right-sided traces of u at a.e. point of the interface.

$$q_l(u_l,\hat{u}_l) := \mathrm{sgn}(u_l-\hat{u}_l)(f_l(u_l)-f_l(\hat{u}_l))$$
$$\geq \mathrm{sgn}(u_r-\hat{u}_r)(f_r(u_r)-f_r(\hat{u}_r)) =: q_r(u_r-\hat{u}_r). \tag{1.14}$$

Definition 1.7. Any set $\mathcal{G}$ of pairs $(u_l,u_r)\in U\times U$ satisfying the Rankine-Hugoniot condition is called an admissibility germ. If (1.14) holds for all $(u_l,u_r),(\hat{u}_l,\hat{u}_r)\in\mathcal{G}$, then the germ $\mathcal{G}$ is called L^1-dissipative admissibility germ.

Definition 1.8. A function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ is a $\mathcal{G}$-entropy solution of (0.1) if it is a Kruzhkov entropy solution in the domains $\{\pm x>0\}$, it is a weak solution in the whole domain (i.e. the Rankine-Hugoniot condition holds), and the adapted entropy inequalities

$$|u-c(x)|_t + (\mathfrak{q}(x;u,c(x)))_x \leq 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}), \tag{1.15}$$

(where $\mathfrak{q}(x;u,c(x)) = q_l(\cdot,c_l)\mathbb{1}_{x<0}+q_r(\cdot,c_r)\mathbb{1}_{x>0}$ is the adapted entropy flux) hold for every $c(x)=c_l\mathbb{1}_{x<0}+c_r\mathbb{1}_{x>0}$ with $(c_l,c_r)\in\mathcal{G}$.

Definition 1.9. Let $\mathcal{G}$ be a germ. The dual germ of $\mathcal{G}$, denoted by $\mathcal{G}^*$, is the set of pairs $(\hat{u}_l,\hat{u}_r)\in U\times U$ such that (1.14) holds for all $(u_l,u_r)\in\mathcal{G}$, and the Rankine-Hugoniot condition $f_l(\hat{u}_l)=f_r(\hat{u}_r)$ is satisfied.

If $\mathcal{G}$ has not trivial extension, then it is called maximal. When $\mathcal{G}$ is L^1D it is contained in its dual, if it is maximal it also coincides with the dual.

It is immediate to observe that smaller is $\mathcal{G}$, easier is to check the constraint (1.15). In most of the cases considered by now, $\mathcal{G}$ is a singleton given by the couple (A,B) which is exactly the connection mentioned previously. Indeed this implies that only one adapted entropy inequality has to be checked.

Proposition 1.4. *(i) If $(u_l,u_r)\in\mathcal{G}$, then the function $u:=u_l\mathbb{1}_{x<0}+u_r\mathbb{1}_{x>0}$ id a $\mathcal{G}$-entropy solution of* (0.1).
(ii) A function $u(x)$ of the above form is a $\mathcal{G}$-entropy solution of (0.1) *if and only if $(u_l,u_r)\in\mathcal{G}^*$*

Theorem 1.5. *(Andreianov, Karlsen and Risebro [5])*

Let $\mathcal{G}$ be a given L^1D germ with the maximal extention $\mathcal{G}^*$ which is complete. Assume that the functions $f_{l,r}$ are locally Lipshitz continuous on $\mathbb{R}$. Then for any initial datum $u_0\in L^\infty(\mathbb{R})$ there exist a unique $\mathcal{G}$-entropy solution of the Cauchy problem (0.1)-(6.1).

There exist infinitely many complete maximal L^1D germs. Each germ contains an unique pair (A,B) extreme of a connection.

Analysis of previous admissibility criteria by germs Here we show how to interpret the conditions along the discontinuity interface $\{x=0\}$ described before in term of the germ theory.

- ***Vanishing viscosity germ associated to Diehl' s entropy condition***

  The germ associated to Diehl's Γ condition and denoted by $\mathcal{G}_{VV}$ is defined as follows

$$
\begin{aligned}
&(u_l, u_r) \in \mathcal{G}_D \iff \\
&f_l(u_l) = f_r(u_r) = s, \text{ and} \\
&\begin{cases}
\text{either} \quad u_l = u_r; & \\
\text{or } u_l < u_r \text{ and} & \text{there exists } u_o \in [u_l, u_r] \\
& \text{such that} f_l(z) \geq s \text{ for all } z \in [u_l, u_o] \\
& \text{and } f_r(z) \geq s \text{ for all } z \in [u_o, u_r]; \\
\text{or } u_l > u_r \text{ and} & \text{there exists } u_o \in [u_r, u_l] \\
& \text{such that} f_r(z) \leq s \text{ for all } z \in [u_r, u_o] \\
& \text{and } f_r(z) \leq s \text{ for all } z \in [u_o, u_l];
\end{cases}
\end{aligned}
$$

This germe is complete and maximal, hence by theorem 1.5, for any given initial data $u_0 \in L^\infty$ there exists a unique $\mathcal{G}_D$ entropy solution of (0.1).

- ***Audusse-Perthame germ***

  Consider the case of two strictly convex fluxes $f_l,\ f_r$ with minima given by θ_l, θ_r, in this context the Audusse-Perthame adapted entropies are entropies constructed from the step function $c(x) = c_l \mathbb{1}_{x<0} + c_r \mathbb{1}_{x>0}$ with (c_l, c_r) pair in the germ $\mathcal{G}_{AP}$. Here $\mathcal{G}_{AP}$ stands for "Audusse-Perthame" germ and is given by

$$
(u_l, u_r) \in \mathcal{G}_{AP} \iff f_l(u_l) = f_r(u_r), \qquad \operatorname{sgn}(u_l - \theta_l) = \operatorname{sgn}(u_r - \theta_r) \tag{1.16}
$$

  This germ is L^1D maximal and this confirms uniqueness already shown in [10].

- ***Karlsen-Risebro-Towers germ***

  The Karlsen-Risebro-Towers entropy condition corresponds to a particular germ that we call $\mathcal{G}_{KRT}$ and defined below

$$
\begin{aligned}
&(u_l, u_r) \in \mathcal{G}_{KRT} \text{ if and only if} f_l(u_l) = f_r(u_r) = s \\
&\text{and} \begin{cases}
\text{either} \quad u_l = u_r; & \\
\text{or } u_l < u_r & \text{and for all } z \in [u_l, u_r], \max\{f_l(z), f_r(z) \geq s\}; \\
\text{or } u_l < u_r & \text{and for all } z \in [u_r, u_l], \min\{f_l(z), f_r(z) \leq s\}.
\end{cases}
\end{aligned}
$$

  If the crossing condition (1.9) is satisfied, then $\mathcal{G}_{KRT}$ is an L^1D germ and is maximal.

- ***Connection germ*** The last germ is the one associated to the (A, B)-connection. It is given by $\mathcal{G}_{(A,B)} = \{(A, B)\}$ and admits a unique maximal extension $\left(\mathcal{G}_{(A,B)}\right)^*$ defined as follows

$$
\begin{aligned}
&(u_l, u_r) \in \left(\mathcal{G}_{(A,B)}\right)^* \\
&\iff f_l(u_l) = f_r(u_r) \geq f_l(A) = f_r(B), \qquad \operatorname{sgn}^-(u_l - A)\operatorname{sgn}^+(u_r - B) \leq 0.
\end{aligned}
$$

  Completeness of $\left(\mathcal{G}_{(A,B)}\right)^*$ implies uniqueness of solutions.

2 Attainable profiles

In this section we start describing the original results of this chapter, i.e. the full characterization of the attainable profiles. For this study we make large use of the theory of generalized characteristics developed by Dafermos in [27].
This kind of analysis is inspired to a previous work of the first author with A. Marson [4] in which they give a characterization of the set of attainable profiles for a initial value problem with boundary control for a scalar nonlinear conservation law.

Let us give some notations and definitions that will be used later.
Consider the scalar conservation law (0.1) with flux given by (0.3) and initial data (0.2). For our study we assume that

1 $f_{l,r} : [0, 1] \to \mathbb{R}$ are strictly convex and C^2;

2 $f_l(0) = f_r(0)$ and $f_l(1) = f_r(1)$;

3 $u_0 \in L^\infty(\mathbb{R})$.

By strictly convexity of f_l and f_r they admits a unique minimum which we call respectively θ_l and θ_r.
We consider solutions of (0.1),(0.2) in a weak sense, that is $u \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^+)$ such that

$$\int_{-\infty}^{\infty}\int_0^{\infty} u\frac{\partial\varphi}{\partial t} + f(x,u)\frac{\partial\varphi}{\partial x}dxdt + \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)dx = 0.$$

This condition is satisfied if and only if u is a weak solution of

$$u_t + f_l(u)_x = 0, \qquad x < 0, \quad t > 0, \tag{2.1}$$
$$u_t + f_r(u)_x = 0, \qquad x > 0, \quad t > 0. \tag{2.2}$$

and satisfies the Rankine-Hugoniot conditions

$$f_l(u^-(t)) = f_r(u^+(t)).$$

where $u^+(t) = \lim_{t\to 0+} u(x,t)$ and $u^-(t) = \lim_{t\to 0-} u(x,t)$.
But also when $f_l = f_r$, weak solutions are not necessarily unique, this leads us to the necessity of adding entropy conditions both in the interior part of the two domains $\mathbb{R}^- \times \mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^+$ and on the discontinuity interface $\{x = 0\}$. For the interior part it is natural to consider the Kruzkov entropy condition introduced in [57]. For the discontinuity interface $\{x = 0\}$ we use the notion of AB- interfece antropy condition described in detail in the previous section since it turns out to be realistic and useful for future applications on network problems.

Here we state the results concerning the characterization of the set of attainable profiles $\mathcal{A}(T)$, with some technical propositions and lemmas relevant for the proof of the main theorems. But before we need to recall the definition L^1-contractive semigroup associated to the conservation law (0.1).

Definition 2.1. $(\mathcal{S}_t)_{t>0} :\ L^\infty(R) \to L^\infty(R)$ *is the* L^1*-contractive semigroup associated to a conservation law (0.1) with AB-entropy condition if*

- $\mathcal{S}_t u_0(x) = u(x,t)$ *provides the unique AB-entropy solution with initial data* u_0*;*

- $\mathcal{S}_0 u = u$*;*

- $\mathcal{S}_{t+s}u = \mathcal{S}_t \circ \mathcal{S}_s u$ *for all* t, s;

- $\|\mathcal{S}_t u - \mathcal{S}_s v\|_{L^1} \leq \|u - v\|_{L^1} + |t - s|$.

Theorem 2.1. *In connection with problem (0.1),(0.2), for any fixed* $T > 0$,

$$\mathcal{A}(T) = \{S_T u_0 : u_0 \in L^\infty(\mathbb{R})\}$$

is given by the union of the following two sets:

$\mathcal{A}_1(T)$ *is the set of all the functions* $\omega \in BV_{loc}(\mathbb{R} \setminus \{0\})$ *for which there exists* $R > 0$ *such that*

$$f_r'(\omega(x)) \geq \frac{x}{T} \quad \text{for all } x \in (0, R), \tag{2.3}$$
$$f_r'(\omega(x)) < \frac{x}{T} \quad \text{for all } x \in (R, +\infty), \tag{2.4}$$
$$f_l'(\omega(x)) > \frac{x}{T} \quad \text{for all } x \in (-\infty, 0), \tag{2.5}$$
$$\varphi_1 := \begin{cases} x - f_l'(\omega(x))T & \text{if } x < 0 \\ -f_l'(f_l^{-1} f_r(\omega(x)))\left(T - \frac{x}{f_r'(\omega(x))}\right) & \text{if } 0 < x < R \\ x - f_r'(\omega(x))T & \text{if } x > R \end{cases} \tag{2.6}$$

is a not decreasing function

$\mathcal{A}_2(T)$ *is the set of all the functions* $\omega \in BV_{loc}(\mathbb{R} \setminus \{0\})$ *for which there exists* $L \in (-\infty, 0]$ *such that*

$$f_l'(\omega(x)) \leq \frac{x}{T} \quad \text{for all } x \in (L, 0), \tag{2.7}$$
$$f_l'(\omega(x)) > \frac{x}{T} \quad \text{for all } x \in (-\infty, L), \tag{2.8}$$
$$f_r'(\omega(x)) < \frac{x}{T} \quad \text{for all } x \in (0, +\infty), \tag{2.9}$$
$$\varphi_2(x) := \begin{cases} x - f_l'(\omega(x))T & \text{if } x < L, \\ -f_r'(f_r^{-1} f_l(\omega(x))\left(T - \frac{x}{f_l'(\omega(x))}\right) & \text{if } L < x < 0, \\ x - f_r'(\omega(x))T & \text{if } x > 0 \end{cases} \tag{2.10}$$

is a not decreasing function

$\mathcal{A}_3(T)$ *is the set of all the functions* $\omega \in BV_{loc}(\mathbb{R})$, *for which there exist* $-\infty < L \leq 0$ *and* $0 \leq R < \infty$ *such that*

$$f_l'(\omega(x)) = f_l'(A) \quad \text{for all } x \in (L, 0), \tag{2.11}$$
$$f_r'(\omega(x)) = f_r'(B) \quad \text{for all } x \in (0, R), \tag{2.12}$$
$$f_r'(\omega(x)) \leq \frac{x}{T} \quad \text{for all } x \in (R, +\infty), \tag{2.13}$$
$$f_l'(\omega(x)) \geq \frac{x}{T} \quad \text{for all } x \in (-\infty, L), \tag{2.14}$$
$$\varphi_3(x) := \begin{cases} x - f_l'(\omega(x))T & \text{if } x < L, \\ x - f_r'(\omega(x))T & \text{if } x > R \end{cases} \tag{2.15}$$

is a not decreasing function

The meaning of all the conditions on the tree sets is that the backward generalized characteristics cannot intersect in $\mathbb{R}\times(0,T)$ (see Figure 3.2). As mentioned in the previous section, the key point for using the theory of generalized characteristics is the existence of right and left limits in each point for a fixed time. This fact is well known for functions in $BV_{loc}(\mathbb{R})$. But when we consider entropy solution for critical connection, the total variation can explode in a neighborhood of the origin, therefore the next lemma states that right and left limits exist also here.

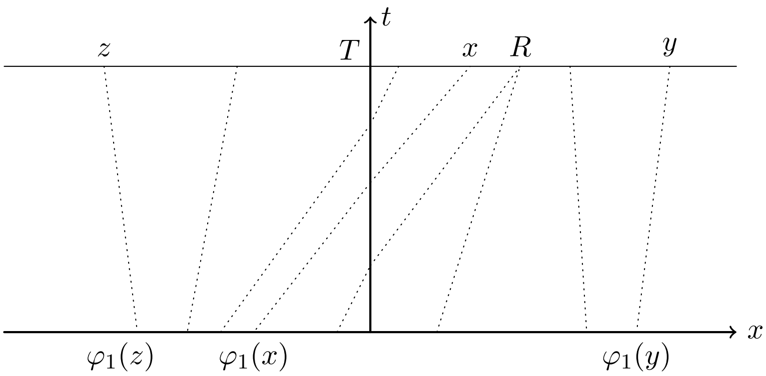


Figure 1.3: Characteristics's behavior for profiles in $\mathcal{A}_1(T)$

Lemma 2.2. *Let u be the entropy solution of (0.1),(0.2) with $u_0 \in L^\infty(\mathbb{R})$ associated to the connection passing through $max\{f_l(\theta_l), f_r(\theta_r)\}$. Then $\lim_{x\to 0^-} u(x,t)$ and $\lim_{x\to 0^+} u(x,t)$ exist for all $t > 0$.*

An interesting fact is that, for conservation laws whose flux verify our hypothesis, generalized characteristics do not intersect also on the discontinuity interface $\{x = 0\}$. This phenomena is explained in the following result.

Proposition 2.3. *For every choice of a connection (AB), the solution of (0.1),(0.2) cannot develop rarefactions on the discontinuity interface $\{x = 0\}$ for positive times.*

The complete characterization of the attainable sets for conservation laws with discontinuous flux can be used for studying some variational problems or questions regarding optimization, therefore this leads us to the necessity of doing a topological analysis of $\mathcal{A}(T,\mathcal{U})$. In order to achieve the closure of the attainable sets for (0.1),(0.2) we have to restrict the class of admissible initial data by using a multifunction G.

Theorem 2.4. *Let $G : \mathbb{R} \to \mathbb{R}$ a measurable uniformly bounded multifunction with convex and closed values. Denote*

$$\mathcal{U} = \{\tilde{u} \in L^\infty(\mathbb{R}) : \tilde{u}(t) \in G(t) \textit{ for a.e. } \quad t \in \mathbb{R}\}. \tag{2.16}$$

Then $\mathcal{A}(\mathcal{U},T)$, $T > 0$ is a compact subset of ${L^1}_{loc}(\mathbb{R})$

3 Proof of preliminary lemmas

We first prove 2.2 and 2.3 as they are indispensable for the proof of 2.1. Observe that although these are technical lemmas, they reveal some peculiarities of the behavior of the solutions of (0.1)-(0.2)

Proof. (Lemma 2.2) If $u(x,t)$ is in $BV_{loc}(R)$ right and left limits in the statement are naturally well defined. Therefore assume that the total variation of the solution is not finite in all the compact sets containing 0 for a fixed time $t > 0$. Without loss of generality we can assume that the total variation is exploding in the left neighborhoods of 0 (the proof is the same if we consider the right one). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence, with $x_n < 0$ for all $n \in \mathbb{N}$ and $x_n \to 0$, then there exist n_0 such that for all $n \geq n_0$ $f_r'(u(x_n,t) < \frac{x_n}{t}$. This fact holds since otherwise $u(x,t)$ restricted to $x \leq 0$ would not be affected by the discontinuity on the interface and therefore the total variation would be finite also in the neighborhoods of 0 contradicting our assumption.
From boundedness of the solution in L^∞ norm we get that also the sequence $u(x_n,t)$ is bounded therefore it admits a convergent subsequences. We need only to prove that all the subsequences converge to the same limit, thus we procede by contradiction and assume that there exist two subsequences, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that $u(y_n,t) \to v$ and $u(z_n,t) \to w$ with $v < w$. Set $d = w - v$, there exists $\bar{n} \in \mathbb{N}$ such that $|u(y_n,t) - v| \leq \frac{d}{3}$ and $|u(z_m,t) - w| \leq \frac{d}{3}$ for all $n, m \geq \bar{n}$, that is $u(y_n,t) \leq v + \frac{d}{3} < w - \frac{d}{3} \leq u(z_m,t)$ for all $n, m \geq \bar{n}$. In particular choosing $n, m \geq \bar{n}$ such that $z_m < y_n$, we can show that the backward generalized characteristics through (y_n,t) and (z_m,t) intersect in the set $(-\infty,0)\times(0,\infty)$. This means that if t_{z_m} and t_{y_n} are respectively the times of intersection of the backward generalized characteristics through (y_n,t) and (z_m,t) with the discontinuity interface, then $t_{z_m} > t_{y_n}$. The last inequality can be easily verified, indeed $u(y_n,t) < u(z_m,t)$ implies that $f_l'(u(y_n,t)) < f_l'(u(z_m,t))$ (by strict convexity of the flux), therefore we have

$$t_{z_m} = t - \frac{z_m}{f_l'(u(z_m,t))} > t - \frac{y_n}{f_l'(u(y_n,t))} = t_{y_n}.$$

By theory of generalized characteristics this is not possible, thus all the subsequences of $u(x_n,t)$ converge to the same limit. □

Thanks to this result we are sure that backward generalized characteristics are well defined on points which belong to the discontinuity interface. As regards Proposition 2.3, its deep meaning is that also on the discontinuity $\{x = 0\}$ it continues to be valid uniqueness of the forward characteristic for strict positive times as in the classical theory for continuous fluxes. Now we see the proof of this fact.

Proof. (Proposition 2.3) We proceed by contradiction and suppose that there exits $t_0 > 0$ such that a rarefaction arises on the discontinuity interface $\{x = 0\}$ and it opens onto $\mathbb{R}^+ \times \mathbb{R}^+$.
Set $\lim_{t\to t_0^+} u^+(t) = a$ and $\lim_{t\to t_0^-} u^+(t) = b$. Since the rarefaction opens in the quadrant I, it follows that $a < b$. Furthermore, by an easy substitution in (1.11), we get that $a \geq B$ in order to satisfy the interface entropy condition at time t_0.

We analyze separately the cases $a > B$ and $a = B$.

case 1: Assume that $a > B$ then, for preserving the Rankine-Hugoniot condition and (1.11) at time t_0, we must have

$$\lim_{t\to t_0^+} u^-(t) = \bar{a} \text{ and } \lim_{t\to t_0^-} u^-(t) = \bar{b}$$

with $f_l(\bar{a}) = f_r(a)$, $f_l(\bar{b}) = f_r(b)$, $0 < f_l'(\bar{a}) < f_l'(\bar{b})$ and $\bar{A} < \bar{a} < \bar{b}$.

Consider the sequences $\{t_n^+\}_{n\in\mathbb{N}}$, $\{t_n^-\}_{n\in\mathbb{N}}$ with $t_n^+ > t_0$ and $t_n^- < t_0$ for all $n \in \mathbb{N}$ such that $t_n^+, t_n^- \to t_0$ for $n \to \infty$. Then we have that $u^-(t_n^+) \to \bar{a}$ and $u^-(t_n^-) \to \bar{b}$.

Take $\varepsilon = \frac{(\bar{b}-\bar{a})}{3}$, there exist $n_0, n_1 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $m \geq n_1$ it holds

$$u^-(t_n^+) \leq \bar{a} + \varepsilon < \bar{b} - \varepsilon \leq u^-(t_m^-).$$

This implies, by strict convexity of f_l, that for all $n \geq n_0$ and $m \geq n_1$, $f_l'(u^-(t_n^+)) < f_l'(u^-(t_m^-))$. It is easy to verify that we can take $n \geq n_0$ and $m \geq n_1$ such that

$$\frac{t_n^+}{t_m^-} < \frac{f_l'(u^-(t_m^-))}{f_l'(u^-(t_n^+))}. \tag{3.1}$$

Consider now the backward generalized characteristics passing through $(0, t_n^+)$ and $(0, t_m^-)$, that is

$$\theta_0^n : t \mapsto f_l'(u^-(t_n^+))(t - t_n^+),$$
$$\theta_0^m : t \mapsto f_l'(u^-(t_m^-))(t - t_m^-)$$

defined respectively for $t \leq t_n^+$ and $t \leq t_m^-$. Inequality (3.1) implies that θ_0^n and θ_0^m intersect in a point belonging to $\mathbb{R}^- \times (0, t_m^-)$ (Figure 3, case 1) which is an absurd because the backward generalized characteristics cannot intersect in the open sets $\mathbb{R}^- \times \mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^+$.

case 2: Now assume that $a = B$. The only Riemann problem which can generate this rarefaction satisfying also the RH condtions is given by the couple $(\hat{b}, b)$ with $\hat{b} < A$ and $f_l(\hat{b}) = f_r(b)$, therefore we get $\lim_{x \to 0^-} u(x, t_0) = \hat{b}$. At the same time we have also that $\lim_{t \to t_0^-} u^-(t) = \bar{b}$ with $\bar{b} > \bar{A}$ and $f_l(\bar{b}) = f_r(b)$ for preserving both the interface entropy condition and the RH condition.
Consider the sequences $\{x_n\}_{n \in \mathbb{N}}$ and and $\{t_n\}_{n \in \mathbb{N}}$ such that $x_n < 0$, $x_n \to 0$, and $t_n < t_0$, $t_n \to t_0$, then $\lim_{n \to \infty} u(x_n, t_0) = \hat{b}$ and $\lim_{n \to \infty} u^-(t_n) = \bar{b}$. Let $\varepsilon < \min\{\theta_l - \hat{b}, \bar{b} - \theta_l\}$, then there exist n_0 and n_1 such that

$$u(x_n, t_0) \leq \hat{b} + \varepsilon \quad \text{for all } n > n_0, \tag{3.2}$$
$$u^-(t_m) \geq \bar{b} - \varepsilon \quad \text{for all } m > n_1. \tag{3.3}$$

The choice of ε implies that

$$f_l'(u(x_n, t_0)) < 0 \quad \text{for all } n > n_0, \tag{3.4}$$
$$f_l'(u^-(t_n)) > 0 \quad \text{for all } m > n_1. \tag{3.5}$$

In particular we can take n, m for which the following inequality is satisfied

$$x_n + f_l'(u^-(t_m))t_m - f_l'(u(x_n, t_0))t_0 > 0. \tag{3.6}$$

Consider now the backward generalized characteristics passing through (x_n, t_0) and $(0, t_m)$, that is

$$\theta_{x_n} : t \to x_n + f_l'(u(x_n, t_0))(t - t_0) \tag{3.7}$$
$$\theta_{t_m} : t \to f_l'(u^-(t_m))(t - t_m) \tag{3.8}$$

defined respectively for $t \leq t_0$ and $t \leq t_m$. The assumption (3.6) implies that θ_{x_n} and θ_{t_m} intersect in $\mathbb{R}^- \times (0, t_m)$ (see Figure 3, case 2), and this is again an absurd.

□

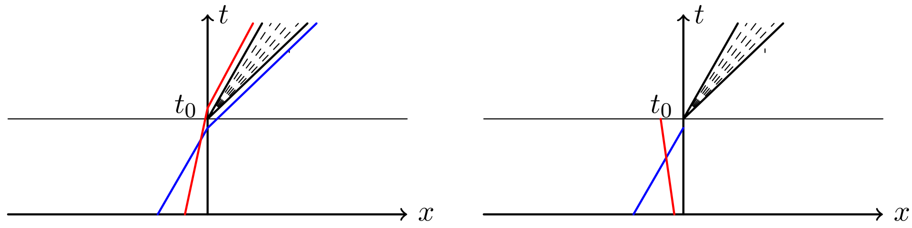


Figure 1.4: On the left case 1, on the right case 2

Now we are ready to prove (2.1).

4 Proof of Theorem 2.1

We proceed by dividing the proof into two steps: first we show that any solution of the problem (0.1),(0.2) satisfies all the conditions of one of the tree sets described in the statement, then we prove that for any function in $\mathcal{A}_1(T)$, $\mathcal{A}_2(T)$ and $\mathcal{A}_3(T)$, there exist $\tilde{u} \in L^\infty(\mathbb{R})$ such that $S_T(\tilde{u}) = \omega$.

4.1 Step 1.

The next technical result will highlight an important meaning of the conditions (2.6),(2.10) and (2.15) not evident at first glance. Throughout the following,

$$D^-\omega(x) = \liminf_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h}, \qquad D^+\omega(x) = \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} \tag{4.1}$$

will denote, respectively, the lower and upper Dini derivatives of a function ω at x, while f_r^{-1} will be the inverse of $f_{r|(-\infty,\theta_r]}$ and f_l^{-1} the inverse of $f_{l|[\theta_l,+\infty)}$.

Lemma 4.1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a bounded right continuous function having right and left limit in any point. If ω satisfies (2.3) − (2.5) then*

$$\varphi_1 := \begin{cases} x - f_l'(\omega(x))T & \text{if } x<0 \\ -f_l'(f_l^{-1}f_r(\omega(x)))\left(T - \frac{x}{f_r'(\omega(x))}\right) & \text{if } 0<x<R \\ x - f_r'(\omega(x))T & \text{if } x>R \end{cases}$$
$$\text{is a not decreasing function} \tag{4.2}$$

if and only if

$$D^+\omega(x) \le \begin{cases} \frac{1}{f_l''(\omega(x))T} & \text{if } x<0 \\ \frac{1}{x^2}\frac{f_r'(\omega(x))\left[f_l'(f_l^{-1}f_r(\omega(x))\right]^2}{f_l''(f_l^{-1}f_r(\omega(x))(f_r'(\omega(x))T-x)+\left[f_l'(f_l^{-1}f_r(\omega(x))\right]^2 f_r''(\omega(x))} & \text{if } 0<x<R \\ \frac{1}{f_r''(\omega(x))T} & \text{if } x>R \end{cases} \tag{4.3}$$

Analogously if ω satisfies (2.7) – (2.9), φ_2 is a not decreasing function iff

$$D^+\omega(x) \leq \begin{cases} \frac{1}{f_l''(\omega(x))T} & \text{if } x < L \\ \frac{1}{x^2}\frac{f_l'(\omega(x))\left[f_r'(f_r^{-1}f_l(\omega(x))\right]^2}{f_r''(f_r^{-1}f_l(\omega(x))\left(f_l'(\omega(x))T-x\right)+\left[f_r'(f_r^{-1}f_l(\omega(x))\right]^2 f_l''(\omega(x))} & \text{if } L < x < 0 \\ \frac{1}{f_r''(\omega(x))T} & \text{if } x > 0 \end{cases}$$

and if ω satisfies (2.11) – (2.14), φ_3 is a not decreasing function iff

$$D^+\omega(x) \leq \begin{cases} \frac{1}{f_l''(\omega(x))T} & \text{if } x < L \\ 0 & \text{if } L < x < R \\ \frac{1}{f_r''(\omega(x))T} & \text{if } x > R \end{cases}$$

Proof. We prove the equivalence just for the first case, in the others the procedure is the same. First observe that nondecreasing monotonicity of φ_1 is equivalent to

$$D^+\varphi_1 \geq 0 \quad \text{for all } x \in \mathbb{R}. \tag{4.4}$$

Suppose that x is a point of continuity for ω, then if $x < 0$ we get

$$\begin{aligned} &\limsup_{h\to 0} \frac{\varphi_1(x+h)-\varphi_1(x)}{h} = \limsup_{h\to 0} \frac{x+h-f_l'(\omega(x+h))T - x + f_l'(\omega(x))T}{h} \\ &= 1 - T\limsup_{h\to 0} \frac{f_l'(\omega(x+h)) - f_l'(\omega(x))}{\omega(x+h)-\omega(x)} \cdot \frac{\omega(x+h)-\omega(x)}{h} \\ &= 1 - f_l''(\omega(x))T \ \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} \geq 0, \end{aligned} \tag{4.5}$$

hence

$$D^+\omega(x) \leq \frac{1}{f_l''(\omega(x))T}. \tag{4.6}$$

Similarly for $x > R$ we get

$$D^+\omega(x) \leq \frac{1}{f_r''(\omega(x))T}. \tag{4.7}$$

Now if we consider $0 < x < R$, then

$$\begin{aligned} &\limsup_{h\to 0} \frac{\varphi_1(x+h)-\varphi_1(x)}{h} \\ &= \limsup_{h\to 0} \frac{1}{h} \cdot \left[f_l'(f_l^{-1}f_r(\omega(x+h)) \left(T - \frac{x+h}{f_r'(\omega(x+h))}\right) + f_l'(f_l^{-1}f_r(\omega(x)) \left(T - \frac{x}{f_r'(\omega(x))}\right)\right] \\ &= \frac{f_l''(f_l^{-1}f_r(\omega(x))f_r'(\omega(x))}{f_l'(f_l^{-1}f_r(\omega(x))} \left(T - \frac{x}{f_r'(\omega(x))}\right) \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} \\ &- f_l'(f_l^{-1}f_r(\omega(x))f_r''(\omega(x)) \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} + \frac{f_l'(f_l^{-1}f_r(\omega(x))f_r''(\omega(x))}{x^2} \geq 0 \end{aligned} \tag{4.8}$$

hence

$$D^+\omega(x) \le \frac{1}{x^2} \frac{f_r'(\omega(x))\left[f_l'(f_l^{-1}f_r(\omega(x))\right]^2}{f_l''(f_l^{-1}f_r(\omega(x))\,(f_r'(\omega(x))T-x)+\left[f_l'(f_l^{-1}f_r(\omega(x))\right]^2 f_r''(\omega(x))} \tag{4.9}$$

Observe that both terms of the sum in the denominator of (4.9) are positive, therefore erasing the first one it follows

$$D^+\omega(x) \le \frac{f_r'(\omega(x))}{x^2 f_r''(\omega(x))} \tag{4.10}$$

which is exactly the same bound obtained in [4].
In case of ω not continuous in x, assume that (4.4) holds, then $\omega(x-) > \omega(x)$. Indeed if we assume that it is false, convexity of f_l and f_r implies that $f_{r,l}'(\omega(x-)) < f_{r,l}'(\omega(x))$ and $f_l'(f_l^{-1}f_r(\omega(x-)) < f_l'(f_l^{-1}f_r(\omega(x))$, hence there exist $y < x$ such that $\varphi_1(y) > \varphi_1(x)$ and this contradicts (4.2).

There follows that

$$D^+\omega(x) = \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h}, \tag{4.11}$$

thus taking lim sup as $h \to 0^+$ in (4.5) and (4.8) we obtain (4.3).

Since ω and hence φ_1 are right continuous it follows that $\varphi_1(x-) > \varphi_1(x)$, due to the monotonicity of f_l and f_r, therefore it is sufficient to prove (4.4) for $h \to 0^+$. But this follows immediately from (4.5) and (4.8) with the same argument we used before.

□

Let $S_T(\tilde{u})$ be the solution at time T associated to (0.1) with initial datum $\tilde{u} \in L^\infty$. Since the solution is BV_{loc} in $\mathbb{R} \setminus \{0\}$, right and left limit of $S_T(\tilde{u})$ are well defined for all $x \in \mathbb{R}$. Assume that there exists $x > 0$ such that $f_r'(\omega(x)) \ge \frac{x}{T}$. We show that for all $0 < y < x$ it holds $f_r'(\omega(y)) \ge \frac{y}{T}$. Consider $\xi_x(t) = x + f_r'(\omega(x))(t-T)$ and $\xi_y(t) = y + f_r'(\omega(y))(t-T)$ the maximal backward generalized characteristics through (x,T) and (y,T), for $t \le T$. By Theorem 3.2 in [27], $\xi_x(t)$ and $\xi_y(t)$ are genuine, therefore they do not intersect in $\mathbb{R}^+ \times \mathbb{R}^+$. Let t_x and t_y the intersections of $\xi_x(t)$ and $\xi_y(t)$ with $\{x = 0\}$, we can observe by the previous consideration that $t_x \le t_y$. Since $f_r'(\omega(x)) \ge \frac{x}{T}$ it is easy to verify that $t_x \ge 0$. Therefore we get

$$t_y = T - \frac{y}{f_r'(\omega(y))} \ge 0$$

from which $f_r'(\omega(y)) \ge \frac{y}{T}$.

Remark 1. If $t_x > 0$, Proposition 2.3 implies that $t_x < t_y$ because we can not have rarefactions arising from $\{x = 0\}$ for positive time, thus we get $f_r'(\omega(y)) > \frac{y}{T}$ for all $0 < y < x$.

Next, suppose that $\{x > 0 : f_r'(\omega(x)) \ge \frac{x}{T}\}$ is not empty, denote with R the sup of this set. By definition, for all $x > R$ we have $f_r'(\omega(x)) < \frac{x}{T}$. Let $R < x_1 < x_2$ be given and trace that maximal backward characteristics $\xi_{x_1}(\cdot)$ and $\xi_{x_2}(\cdot)$ through (x_1,T) and (x_2,T), respectively. Their form is the following

$$\xi_{x_i}(\cdot) = x_i + f_r'(\omega(x_i))(t-T), \quad i = 1,2$$

as long as they exist. Since $x_1 < x_2$ and $\xi_{x_1}(\cdot)$ $\xi_{x_2}(\cdot)$ are genuine, $\xi_{x_1}(\cdot) \le \xi_{x_2}(\cdot)$ for $t \in [0,T]$ and in particular they can intersect only in $t = 0$. Thus we get

$$x_1 - f_r'(\omega(x_1))T \le x_2 - f_r'(\omega(x_2))T$$

that is the map which associates to all $x \in [R, +\infty)$ the value $x - f_r'(\omega(x))T$ is not decreasing. If the set $\{x > 0 : f_r'(\omega(x)) \geq \frac{x}{T}\}$ is empty, the previous map results to be not decreasing for all $x \in [0, +\infty)$.

Similarly to what was done before, assume that there exists $x < 0$ such that $f_l'(\omega(x)) \leq \frac{x}{T}$, we can show that for all $x < y < 0$ it holds $f_l'(\omega(y)) \leq \frac{y}{T}$. If $\{x > 0 : f_r'(\omega(x)) \leq \frac{x}{T}\}$ is not empty, denote with L the inf of this set. The map which associates to all $x \in (-\infty, L]$ the value $x - f_r'(\omega(x))T$ is not decreasing. If the set $\{x > 0 : f_l'(\omega(x)) \geq \frac{x}{T}\}$ is empty, the previous map results to be not decreasing for all $x \in (-\infty, 0]$.

Now we just have to analyze the following three situations: $R > 0$ and $L = 0$, $R = 0$ and $L < 0$, $R > 0$ and $L < 0$.

1. $R > 0$ and $L = 0$.
   For all $x \in [0, R)$, the map which associate to x the value $T - \frac{x}{f_r'(\omega(x))}$ (i.e. the abscissa in the intersection of the maximal backward characteristic through (x, T) with $\{x = 0\}$) is a not increasing map. Crossing the discontinuity in $(0, T - \frac{x}{f_r'(\omega(x))})$ the value of the solution along the characteristic which passes through (x, T) changes by preserving the Rankine-Hugoniot condition and the entropy interface condition and it is given by $f_l^{-1} f_r(\omega(x))$. Therefore the intersection of the backward characteristic through $(0, T - \frac{x}{f_r'(\omega(x))})$ in $\mathbb{R}^- \times \mathbb{R}^+$ with $t = 0$, it is given by $-f_l'(f_l^{-1} f_r(\omega(x))) \left(\mathrm{T} - \frac{x}{f_r'(\omega(x))}\right)$. The map which associate to $(0, T - \frac{x}{f_r'(\omega(x))})$ the value $-f_l'(f_l^{-1} f_r(\omega(x))) \left(\mathrm{T} - \frac{x}{f_r'(\omega(x))}\right)$ is not increasing, therefore it is not decreasing with respect to x for $x \in [0, R)$ because composition of two not increasing functions.

2. $R = 0$ and $L < 0$
   This case is identical to the previous one, with the only difference that we cross the discontinuity by using the inverse of the function $f_{r|[\theta_r, +\infty)}$.

3. $R > 0$ and $L < 0$
   Let t_R and t_L be the times in which the minimal backward characteristic through (R, T) and the maximal backward characteristic through (L, T) intersect $\{t = 0\}$. Denote with $t_0 = max\{t_R, t_L\}$. For all $t \in [t_0, T]$ it holds that $f_l'(u(0-, t)) < 0$ and $f_r'(u(0+, t)) > 0$, therefore the solution is undercompressive on the discontinuity interface for a set of times of positive measure and, by the interface entropy condition, this can happen only if $u(0-, t) = A$ and $u(0+, t) = B$ for all $t \in [t_0, T]$. Without loss of generality we can assume $t_0 = t_R$, therefore we have $t_R \geq t_L \geq 0$. Now we show that all the previous inequalities are actually equality. Indeed, if $t_R > t_L$, then for all $t \in [t_L, t_R]$ we should have $f_r'(u(0+, t)) > 0$ otherwise a backward characteristics through $(0, t)$ would intersect at least one of the backward generalized characteristics between the minimal and maximal through (R, T). This means that we have again a set of times of positive measure for in which the solution is undercompressive on the discontinuity interface, hence $u(0-, t) = A$ and $u(0+, t) = B$ for all $t \in (t_L, t_R)$, that is $f_r'(u(0+ .t)) = f_r'(B)$ in (t_L, t_R), but this is not possible since also in this situation, characteristics arising from $(0, t)$ would intersect at least one of the backward generalized characteristics between the minimal and maximal through (R, T), therefore $t_L = t_R$. The same argument tells us that t_L must be equal to 0.

4.2 Step 2.

Now we take a function $\mathcal{A}(T)$ and show how to build an initial data $\tilde{u}(x)$ such that the solution at time T of 0.1 is exactly the function considered. For this purpose (similarly to what is done in [4]) we adopt the following procedure:

1. For every x we trace the lines θ_x^-,θ_x^+ through (T,x) with slope $f_l(\omega(x-))$ and $f_l(\omega(x+))$ if $x<0$, $f_r(\omega(x-))$ and $f_r(\omega(x+))$ if $x>0$, which change in the opportune way intersecting $\{x=0\}$. These will be the minimal and maximal backward characteristics through (x,T). The assumptions on each set $\mathcal{A}_1(T)$,$\mathcal{A}_2(T)$,$\mathcal{A}_3(T)$ guarantee that $\{\theta_x^{\mp}: x<0\}$ do not intersect in the interior of $\mathbb{R}^-\times\mathbb{R}^+$ and $\{\theta_x^{\mp}: x>0\}$ in the interior of $\mathbb{R}^+\times\mathbb{R}^+$.

2. Since the solution is constant along minimal and maximal backward characteristics, for every $y\in\mathbb{R}$ for which there exists an $x\in\mathbb{R}$ such that $\theta_x^{\pm}(0)=y$, we define $\tilde{u}(y)=\omega(x)$ if $y,x<0$ or $x,y>0$, that is the backward generalized characteristic does not intersect the discontinuity interface,$\tilde{u}(y)=f_l^{-1}(f_r(\omega(x)))$ if $y<0$ and $x>0$, $\tilde{u}(y)=f_r^{-1}(f_l(\omega(x)))$ if $y>0$ and $x<0$. The set of remaining y is a disjoint union of open intervals in which $\tilde{u}$ is defined so as to produce a compression wave which generate a discontinuity at time T.

3. By using the fact that a solution is constant along genuine characteristics, we define a function $u:\mathbb{R}\times[0,T]\to\mathbb{R}$ which is candidate to be $S_{(\cdot)}(\tilde{u})$ and we prove that it is weak entropy solution for (0.1), (0.2) in $\mathbb{R}^-\times\mathbb{R}^+$ and $\mathbb{R}^+\times\mathbb{R}^+$.

4. We prove that u satisfies the interface entropy condition associated to the connection AB and that $u(T-,\cdot)=\omega$.

1. For this first point we take in consideration only $\omega\in\mathcal{A}_1(T)$, for the function in the other two sets it is the same. For each $x\in\mathbb{R}$ consider the lines

$$\theta_x^-(t) := \begin{cases} x+f_l'(\omega(x-))(t-T) & \text{if } x<0, t\in[0,T] \\ x+f_r'(\omega(x-))(t-T) & \text{if } x\in(0,R], t\in\left[T-\frac{x}{f_r'(\omega(x-))},T\right] \\ & \text{or } x>R, t\in[0,T] \\ f_l'(f_l^{-1}f_r(\omega(x-)))\left(t-T+\frac{x}{f_r'(\omega(x-))}\right) & \text{if } x\in(0,R], t\in\left[0,T-\frac{x}{f_r'(\omega(x-))}\right] \end{cases} \tag{4.12}$$

$$\theta_x^+(t) := \begin{cases} x+f_l'(\omega(x+))(t-T) & \text{if } x<0, t\in[0,T] \\ x+f_r'(\omega(x+))(t-T) & \text{if } x\in(0,R), t\in\left[T-\frac{x}{f_r'(\omega(x+))},T\right] \\ & \text{or } x\geq R, t\in[0,T] \\ f_l'(f_l^{-1}f_r(\omega(x+)))\left(t-T+\frac{x}{f_r'(\omega(x+))}\right) & \text{if } x\in(0,R), t\in\left[0,T-\frac{x}{f_r'(\omega(x+))}\right] \end{cases} \tag{4.13}$$

By the convexity of f_l,f_r we have that $\theta_x^-(t)\leq\theta_x^+(t)$ for all t. Now we show that for any $x<y$ the lines $\theta_x^{\pm}$ and $\theta_y^{\pm}$ do not intersect in $\mathbb{R}\times(0,T)$. The previous observation tells us we need only to prove that $\theta_x^+<\theta_y^-$ in $\mathbb{R}\times(0,T)$. If $x<0$ and $y>R$ it is immediate to verify that $\theta_x^+(t)<0\leq\theta_y^-(t)$. For $x<y<0$, by (2.6) we have $\varphi_1(x)\leq\varphi_1(y)$, therefore writing $\theta_x(t)$ and $\theta_y(t)$ as convex combination of x,$\varphi_1(x)$ and y,$\varphi_1(y)$ respectively, for all $0<t<T$ we get

$$\begin{aligned}\theta_x(t) = \lambda x+(1-\lambda)\left(x-f_l'(\omega(x)T\right) = \lambda x+(1-\lambda)\varphi_1(x) < \\ \lambda y+(1-\lambda)\varphi_1(y) = \lambda y+(1-\lambda)\left(y-f_l'(\omega(y)T\right) = \theta_y(t)\end{aligned} \tag{4.14}$$

where $\lambda = \frac{t}{T}$. The case $R \leq x < y$ can be checked in the same way. Consider now $0 < x < y < R$, if $f_r'(\omega(x)) \geq f_r'(\omega(y))$, we get also $f_l'(f_l^{-1} f_r(\omega(x))) \geq f_l'(f_l^{-1} f_r(\omega(y)))$, that is the inequality between the speeds of generalized characteristic is preserved crossing the discontinuity interface which implies easily that $\theta_x(t) < \theta_y(t)$ for $0 < t < T$. Therefore assume $f_r'(\omega(x)) < f_l'(\omega(y))$ and suppose that there exist a $\tau \in (0, T)$ such that $\theta_x(\tau) = \theta_y(\tau) = \xi$. If $\xi < 0$, then immediately follows that $\varphi_1(y) < \varphi_1(x)$ which is an absurd because φ_1 is a non increasing map. If $\xi \geq 0$, then $\theta_x(t)$ and $\theta_y(t)$ intersect $\{x = 0\}$ in t_x and t_y with $t_y \geq t_x$, therefore since inequality between the speeds of the characteristic is preserved crossing the discontinuity interface, we have again $\varphi_1(y) < \varphi_1(x)$. The remaining cases $x < 0 < y \leq R$ and $0 < x \leq R < y$ follow combining the previous situations.

2. Let ω be in $\mathcal{A}(T)$. In order to define the initial data $\tilde{u}$ which produces a solution of (0.1)-(0.2) that attains ω, consider the following partition of $\mathbb{R}$ (see Fig.1.5):

$$\begin{aligned}
\mathcal{I}_1 &\doteq \{x \in \mathbb{R} : \exists! \quad y : \theta_y^-(0) = x \text{ or } \theta_y^+(0) = x\}, \\
\mathcal{I}_2 &\doteq \{x \in \mathbb{R} : \exists y < z : \theta_y^+(0) = \theta_z^-(0) = x\}, \\
\mathcal{I}_3 &\doteq \{x \in \mathbb{R} : \nexists \quad y \in \mathbb{R} : \theta_y^-(0) = x \text{ or } \theta_y^+(0) = x, \\
&\qquad \exists\, x' < x < x'' \text{ and } y : \theta_y^-(0) = x', \theta_y^+(0) = x''\}.
\end{aligned}$$

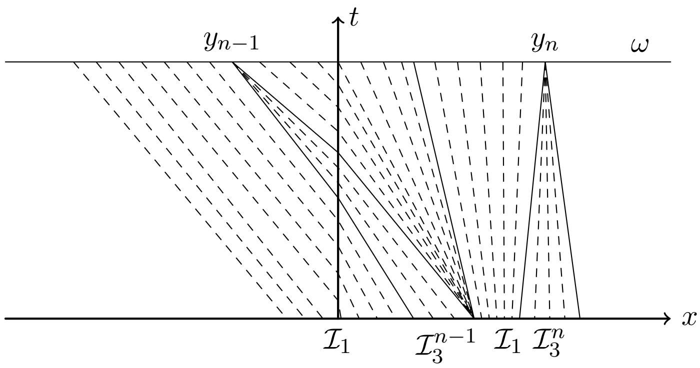


Figure 1.5: An example of partition of $\mathbb{R}$ associated to the profile ω; observe that $\sup \mathcal{I}_3^{n-1} \in \mathcal{I}_2$.

Some considerations about the previous partition are necessary for the next.
The set $\mathcal{I}_2$ contains at most countably many points, indeed since the backward generalized characteristic do not intersect in $\mathbb{R} \times \mathbb{R}^+$, for each $x \in \mathcal{I}_2$ the set

$$\mathcal{J}_x = \{y \in \mathbb{R} : \theta_y^-(0) = x \text{ or } \theta_y^+(0) = x\} \tag{4.15}$$

is an interval and $\mathcal{J}_x \cap \mathcal{J}_{x'} = \emptyset$ for any $x, x' \in \mathcal{I}_2$ with $x \neq x'$.

$\mathcal{I}_3$ is a disjoint union of at most countably many open intervals $\mathcal{I}_3^n$ of the form

$$\mathcal{I}_3^n = (x_n^1, x_n^2), \quad \theta_{y_n}^-(0) = x_n^1, \theta_{y_n}^+(0) = x_n^2 \tag{4.16}$$

with y_n point of discontinuity of ω. First we show that $\mathcal{I}_3$ is open in $\mathbb{R}$. Consider $x \in \mathcal{I}_3$ and assume by contradiction that $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \mathcal{I}_1 \cup \mathcal{I}_2$ is a sequence converging to x. Then there exist a sequence of $\{y_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}$ such that $\theta_{y_\nu}^{\pm}(0) = x_\nu$, in particular we can assume that $\theta_{y_\nu}^+(0) = x_\nu$. Since x_ν is convergent, it is also bounded, therefore by (4.13) also y_ν is bounded, hence it admits a converging subsequence which is still denoted with y_ν.

Set $\bar{y}$ its limit, we can assume up to a subsequence that $f'(\omega(y_\nu)) \to f'(\omega(\bar{y}))$. Then $x = \lim_{\nu\to\infty} x_\nu = \lim_{\nu\to\infty} \theta^+_{y_\nu}(0) = \theta^+_{\bar{y}}(0)$ which gives a contradiction.
It follows naturally that $\mathcal{I}_3$ is a disjoint union of at most countably many open intervals $\mathcal{I}_3^n = (x_n^1, x_n^2)$ with $x_n^1, x_n^2 \in \mathcal{I}_1 \cup \mathcal{I}_2$. Set

$$y_n^1 = \sup\{y \in \mathbb{R} : \theta_y^-(0) = x_n^1, \theta_y^+(0) = x_n^1\},$$
$$y_n^2 = \inf\{y \in \mathbb{R} : \theta_y^-(0) = x_n^2, \theta_y^+(0) = x_n^2\}.$$

Then $y_n^1 = y_n^2 = y_n$. Indeed $y_n^1 \le y_n^2$ because the characteristics do not interserct in $\mathbb{R}\times\mathbb{R}^+$, and if $y_n^1 < y_n^2$ then we can choose $y \in (y_n^1, y_n^2)$ for which there is a $x \in (x_n^1, x_n^2)$ such that $x = \theta_y^\pm(0)$, that is a contradiction. Finally it is possible to define the initial data (the control) which produces the given profile:

$$\tilde{u}(x) = \begin{cases} f_r^{-1} f_l(\omega(y\pm)) & \text{if } x \in \mathcal{I}_1, \quad x > 0 \quad \theta_y^\pm(0) = x, \\ f_l^{-1} f_r(\omega(y\pm)) & \text{if } x \in \mathcal{I}_1, \quad x < 0 \quad \theta_y^\pm(0) = x, \\ f_r^{-1} f_l(\omega(\sup \mathcal{J}_x -)) & \text{if } x \in \mathcal{I}_2, \quad x > 0, \\ f_l^{-1} f_r(\omega(\sup \mathcal{J}_x -)) & \text{if } x \in \mathcal{I}_2, \quad x < 0, \\ f_r{}^{-1} f_l f_l'^{-1}\left(\frac{y_n}{T-s}\right) & \text{if } x \in \mathcal{I}_3^n, \quad x > 0, \\ f_l{}^{-1} f_r f_r'^{-1}\left(\frac{y_n}{T-s}\right) & \text{if } x \in \mathcal{I}_3^n, \quad x < 0. \end{cases} \tag{4.17}$$

The parameter s in (4.17) is the unique value for which are satisfied

$$f_l f_l'^{-1}\left(\frac{y_n}{T-s}\right) = f_r f_r'^{-1}\left(-\frac{x}{s}\right) \text{ if } x > 0 \text{ and } y_n < 0,$$

$$f_r f_r'^{-1}\left(\frac{y_n}{T-s}\right) = f_l f_l'^{-1}\left(-\frac{x}{s}\right) \text{ if } x < 0 \text{ and } y_n > 0,$$

s is zero if both $(y_n, T), (x, 0)$ belong to $\mathbb{R}^- \times \mathbb{R}^+$ or $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover the compositions $f_l{}^{-1} f_r$ and $f_r^{-1} f_l$ are the identity map when the couples $(x, 0), (y, T)$ if $x \in \mathcal{I}_1$, $(x, 0), (\sup \mathcal{J}_x, T)$ if $x \in \mathcal{I}_2$, $(x, 0), (y_n, T)$ if $x \in \mathcal{I}_3^n$ belong to $\mathbb{R}^- \times \mathbb{R}^+$ or $\mathbb{R}^+ \times \mathbb{R}^+$ together. Observe that (4.17) is a pointwise definition given for $x \neq 0$. For $x = 0$ is not possible to do the same because of the doscontinuity of the flux, however right and left limits of $\tilde{u}(x)$ are well defined in 0. This can be proved using arguments similar to those used in (2.2).

3. Given the initial data $\tilde{u}(x)$ we can define the solution $u(x,t)$ for $t \in (0,T)$. For each $x \in \mathcal{I}^n{}_3 \subseteq \mathcal{I}_3$ consider the line

$$\theta_x(t) = f'{}_r f_r{}^{-1} f_l(\tilde{u}(x)) \left(t + \frac{x}{f_l'(\tilde{u}(x))}\right) \text{ for } x < 0 \text{ and } 0 < t < T$$
$$\theta_x(t) = f'{}_l f_l{}^{-1} f_r(\tilde{u}(x)) \left(t + \frac{x}{f_r'(\tilde{u}(x))}\right) \text{ for } x > 0 \text{ and } 0 < t < T.$$

It is entirely contained in the open set $\{(\eta,\tau) : 0 < \tau < T, \theta^-_{y_n}(\tau) < \eta < \theta^+_{y_n}(\tau)\}$. Each θ_x cannot intersect one of the $\theta_y{}^\pm$ in $\mathbb{R}\times\mathbb{R}^+$. Our goal now is to show that for any $(\eta,\tau) \in \mathbb{R}\times(0,T)$ there exists a unique line through (η,τ) which belongs to $\Theta = \{\theta_y{}^\pm : y \in \mathbb{R}\} \cup \{\theta_x : x \in \mathcal{I}_3\}$. For the existence, observe that if $\eta \neq \theta_y{}^\pm(\tau)$ for any $y \in \mathbb{R}$ then there exists $x \in \mathcal{I}_3$ such that $\theta_x(\tau) = \eta$. Indeed the set

$$\mathcal{B}(\tau) := \{\eta \in \mathbb{R} : \nexists y \in \mathbb{R} : \theta_y{}^\pm(\tau) = \eta\}$$

is open. The proof of this fact is identical to what done for openess of $\mathcal{I}_3$. Now, consider the connected component (η_1, η_2) of $\mathcal{B}(\tau)$ which contains η. There exists $y \in \mathbb{R}$ such that ${\theta_y}^-(\tau) = \eta_1$ and ${\theta_y}^+(\tau) = \eta_2$. It is immediate that ${\theta_y}^-(0) = x_n^1$, ${\theta_y}^+(0) = x_n^2$ for a certain $n \in \mathbb{N}$, hence $y = y_n$. Moreover there exists $x \in (x_n^1, x_n^2)$ such that

$$\dot\theta_x(t) = \begin{cases} \frac{y_n-\eta}{T-\tau} & \text{if } \operatorname{sgn} y_n = \operatorname{sgn}\eta, \quad max\left\{0, T - \frac{y_n(T-\tau)}{y_n-\eta}\right\} < t < T \\ & \text{or } y_n = 0, \quad 0 < t < T, \\ {f_l}^{-1} f_r {f_r'}^{-1}\left(\frac{y_n-\eta}{T-\tau}\right) & \text{if } y_n > 0, \quad \eta \geq 0, \quad 0 < t < max\left\{0, T - \frac{y_n(T-\tau)}{y_n-\eta}\right\}, \\ {f_r}^{-1} f_l {f_l'}^{-1}\left(\frac{y_n-\eta}{T-\tau}\right) & \text{if } y_n < 0, \quad \eta \leq 0, \quad 0 < t < max\left\{0, T - \frac{y_n(T-\tau)}{y_n-\eta}\right\}, \\ \frac{\eta}{\tau-s} & \text{if } \operatorname{sgn} y_n \neq \operatorname{sgn}\eta, \quad 0 < t < s \\ \frac{y_n}{T-s} & \text{if } \operatorname{sgn} y_n \neq \operatorname{sgn}\eta, \quad s < t < T \end{cases}$$

where s is the unique value for which are satisfied

$$f_l {f_l'}^{-1}\left(\frac{\eta}{\tau-s}\right) = f_r {f_r'}^{-1}\left(\frac{y_n}{T-s}\right) \text{ if } y_n > 0 \text{ and } \eta < 0; \tag{4.18}$$

$$f_r {f_r'}^{-1}\left(\frac{\eta}{\tau-s}\right) = f_l {f_l'}^{-1}\left(\frac{y_n}{T-s}\right) \text{ if } y_n < 0 \text{ and } \eta > 0. \tag{4.19}$$

Consider now the function $u : \mathbb{R}\setminus\{0\} \times (0,T) \to \mathbb{R}$ given by

$$u(\eta,\tau) = \begin{cases} \omega(y\pm) & \text{if } \exists y \in \mathbb{R} : \theta_y^\pm(\tau) = \eta \text{ and } \operatorname{sgn} y = \operatorname{sgn}\eta, \\ {f_l}^{-1} f_r(\omega(y\pm)) & \text{if } \exists y \in \mathbb{R} : \theta_y^\pm(\tau) = \eta \text{ and } \eta < 0 < y, \\ {f_r}^{-1} f_l(\omega(y\pm)) & \text{if } \exists y \in \mathbb{R} : \theta_y^\pm(\tau) = \eta \text{ and } y < 0 < \eta, \\ \tilde u(x) & \text{if } \exists x \in \mathcal{I}_3 : \theta_x(\tau) = \eta \text{ and } \operatorname{sgn} y = \operatorname{sgn}\eta, \\ {f_l}^{-1} f_r(\tilde u(x)) & \text{if } \exists x \in \mathcal{I}_3 : \theta_x(\tau) = \eta \text{ and } \eta < 0 < x, \\ {f_r}^{-1} f_l(\tilde u(x)) & \text{if } \exists x \in \mathcal{I}_3 : \theta_x(\tau) = \eta \text{ and } x < 0 < \eta \end{cases}. \tag{4.20}$$

In the next we show that for a fixed time τ in $(0,T)$, (4.20) is continuous w.r.t the first variable in $\mathbb{R}^-$ and $\mathbb{R}^+$, on order to do this we prove the following three properties:
(i) if there exists $y \in \mathbb{R}$ such that $\theta_y^-(\tau) = \eta$ then $u(\cdot,\tau)$ is left continuous in η;
(ii) if there exists $y \in \mathbb{R}$ such that $\theta_y^+(\tau) = \eta$ then $u(\cdot,\tau)$ is right continuous in η;
(iii) if $\eta \in \mathcal{B}(\tau)$, $u(\cdot,\tau)$ is continuous in η.
Since the considerations necessary for proving these three points are similar, we analyze only the first one and in particular we show left continuity only in $\mathbb{R}^-$, for $\mathbb{R}^+$ it is analogous. Let $\eta < 0$ and y with $\theta_y^-(\tau) = \eta$, choose $\varepsilon > 0$, $\delta > 0$ such that $(y-\delta, y) \subset \mathbb{R}^+$ if $y > 0$ and

$$|\omega(z) - \omega(y-)| \leq \varepsilon \quad \text{for all } z \in (y-\delta, y). \tag{4.21}$$

First consider the case $y \leq 0$, by (4.20) $u(\eta,\tau) = \omega(y-)$. Let $\eta_\delta = \theta_{y-\delta}^+(\tau)$, by point 1 it holds $\eta_\delta < \eta$ therefore for all $\zeta \in (\eta_\delta, \eta)$,

$$|u(\zeta,\tau) - u(\eta,\tau)| \leq \varepsilon. \tag{4.22}$$

Indeed if $\zeta = \theta_z^\pm(\tau)$ for a $z \in (y-\delta, y)$ then

$$|u(\zeta,\tau) - u(\eta,\tau)| = |\omega(z\pm) - \omega(y-)| \leq \varepsilon; \tag{4.23}$$

if $\zeta \in \mathcal{B}(\tau)$ so that $\theta_{y_n}^-(\tau) < \zeta < \theta_{y_n}^+(\tau)$ for some $n \in \mathbb{N}$ and $\zeta = \theta_x(\tau)$ for some $x \in \mathcal{I}_3^n$, then

$$|u(\zeta,\tau)-u(\eta,\tau)|<|\omega(y_n-)-\omega(y-)|\leq\varepsilon \tag{4.24}$$

If $\eta<0<y$, in the previous estimate we need only to take in account the jump on the discontinuity interface by applying the change $f_l^{-1}f_r$, hence in (4.23) we have

$$\begin{aligned}&|u(\zeta,\tau)-u(\eta,\tau)|=|f_l^{-1}f_r(\omega(\zeta\pm))-f_l^{-1}f_r(\omega(y-))|\\&\leq\frac{1}{f_l'(\bar A)}|f_r(\omega(\zeta\pm))-f_r(\omega(y-))|\leq\frac{M}{f_l'(\bar A)}|\omega(\zeta\pm)-\omega(y-)|\\&\leq\frac{M}{f_l'(\bar A)}\varepsilon;\end{aligned}$$

where the former inequality can be justified observing that $f_l'(u)\geq f_l'(\bar A)$ for characteristics crossing the discontinuity with positive speed and $f_l{}^{-1}=\frac{1}{f_l'}$; the latter comes from the local lipschitzianity of strictly convex functions. Instead (4.24) becomes

$$\begin{aligned}&|u(\zeta,\tau)-u(\eta,\tau)|<|f_l^{-1}f_r(\omega(y_n-))-f_l^{-1}f_r(\omega(y-))|\\&\leq\frac{1}{f_l'(\bar A)}|f_r(\omega(y_n-))-f_r(\omega(y-))|\leq\frac{M}{f_l'(\bar A)}|\omega(y_n-)-\omega(y-)|\\&\leq\frac{M}{f_l'(\bar A)}\varepsilon.\end{aligned}$$

Combining (i),(ii),(iii) we are now able to derive the continuity of $u(\cdot,\tau)$ in $\mathbb{R}\setminus\{0\}$. In fact if $\eta=\theta_y^-(\tau)=\theta_y^+(\tau)$ for some $y\in\mathbb{R}$ or $\eta\in\mathcal{B}(\tau)$, continuity is obvious. Otherwise assume that $\eta=\theta_{y_n}^-(\tau)<\theta_{y_n}^+(\tau)$ for some $n\in\mathbb{N}$. Since $\zeta\in\mathcal{B}(\tau)$ for any $\zeta\in(\eta,\theta_{y_n}^+(\tau))$, if $\operatorname{sgn}y_n=\operatorname{sgn}\eta$ then

$$\lim_{\zeta\to\eta^+u(\zeta,\tau)}=\lim_{\zeta\to\eta^+}f'^{-1}_{l,r}\left(\frac{y_n-\zeta}{T-\tau}\right)=f'^{-1}_{l,r}(f'_{l,r}(\omega(y_n-)))=u(\eta,\tau);$$

if $\operatorname{sgn}y_n\neq\operatorname{sgn}\eta$ then

$$\begin{aligned}&\lim_{\zeta\to\eta^+}u(\zeta,\tau)=\lim_{\zeta\to\eta^+}f'^{-1}_{l,r}\left(\frac{\zeta}{\tau-s(\zeta)}\right)=f'^{-1}_{l,r}\left(\frac{\eta}{\tau-s(\eta)}\right)\\&=f_{l,r}^{-1}f_{r,l}^{-1}(\omega(y_n-))=u(\eta,\tau).\end{aligned}$$

In the same way it can be proved continuity in η when $\theta_{y_n}^-(\tau)<\theta_{y_n}^+(\tau)=\eta$ for some $n\in\mathbb{R}$.

Next step is to show that $u(\cdot,\tau)$ is differentiable almost everywhere and we do this proving that it u is locally Lipschitz continuous. We proceed finding a bound from above for the upper Dini's derivative and a bound from below for the lower Dini's derivative of u. This results are collected in the following lemma.

With the same argument of Step 1, the way in which we construct $u(\eta,\tau)$ starting from ω in $\mathcal{A}$, implies that u satisfies the assumption of one of the three attainable profiles. In particular assume that u is built starting from $\omega\in\mathcal{A}_1$. Then there exist $R'>0$ such that the map φ_1 defined as (2.6) replacing R with R', T with τ and ω with u is not decreasing. Therefore we compute the upper Dini derivative of $u(\eta,\tau)$ we get

$$D_\eta^+ u(\eta,\tau) \le \begin{cases} \frac{1}{f_l''(u(\eta,\tau)\tau} & \text{if } x<0 \\ \frac{1}{x^2}\frac{f_r'(u(\eta,\tau))\left[f_l'(f_l^{-1}f_r(u(\eta,\tau))\right]^2}{f_l''(f_l^{-1}f_r(u(\eta,\tau))(f_r'(u(\eta,\tau))\tau-\eta)+\left[f_l'(f_l^{-1}f_r(u(\eta,\tau))\right]^2 f_r''(u(\eta,\tau))} & \text{if } 0<x<R' \\ \frac{1}{f_r''(u(\eta,\tau))\tau} & \text{if } x>R \end{cases} \tag{4.25}$$

For the lower bound of the lower Dini's derivative the approach is different. If $D_\eta^- u(\eta,\tau) \ge 0$ we do not have to prove anything, otherwise assume that $\tau < T' < T$ and observe that by construction

$$u(\eta,\tau) = \begin{cases} u(\eta + f_{l,r}'(u(\eta,\tau))(t-\tau), t) \text{ or} \\ f_l^{-1} f_r \left(u \left(f_r' f_r^{-1} f_l(u(\eta,\tau)) \left(t-\tau+\frac{\eta}{f_r'(u(\eta,\tau))} \right), t \right) \right) \end{cases}$$

for all $t \in [\tau, T]$. The first or second expression is valid according to whether the characteristic intersects the discontinuity or not. Therefore, fixing τ, for all $\eta \in \mathbb{R}$ there exist a unique $z = z(\eta)$ such that

$$\eta = z + f_{l,r}'(u(z,T'))(\tau - T') \quad \text{and} \quad u(\eta,\tau) = u(z,T') \text{ or} \tag{4.26}$$

$$\eta = f_l' f_l^{-1} f_r(u(z,T')) \left(\tau - T' + \frac{z}{f_r'(u(z,t))} \right) \quad \text{and} \quad u(\eta,\tau) = f_l^{-1} f_r(u(z,T')). \tag{4.27}$$

If we are in case of (4.26) then

$$D_\eta^- u(\eta,\tau) = \liminf_{z\to z(\eta)} \frac{u(z,T') - u(z(\eta),T')}{(z - z(\eta)) + [f_{l,r}'(u(z,T')) - f_{l,r}'(u(z(\eta),T'))](\tau - T')}$$

$$\liminf_{z\to z(\eta)} \left(\frac{z - z(\eta)}{u(z,T') - u(z(\eta),T')} + \frac{[f_{l,r}'(u(z,T')) - f_{l,r}'(u(z(\eta),T'))]}{u(z,T') - u(z(\eta),T')}(\tau - T') \right)^{-1}.$$

Choose a sequence $\{z_\nu\}_{\nu\in\mathbb{N}}$ converging to $z(\eta)$ such that

$$D_\eta^- u(\eta,\tau) = \tag{4.28}$$

$$= \lim_{\nu\to\infty} \left(\frac{z_\nu - z(\eta)}{u(z_\nu,T') - u(z(\eta),T')} + \frac{[f_{l,r}'(u(z_\nu,T')) - f_{l,r}'(u(z(\eta),T'))]}{u(z_\nu,T') - u(z(\eta),T')}(\tau - T') \right)^{-1}, \tag{4.29}$$

by continuity of $u(\cdot,T')$

$$\lim_{\nu\to\infty} \frac{[f_{l,r}'(u(z_\nu,T')) - f_{l,r}'(u(z(\eta),T'))]}{u(z_\nu,T') - u(z(\eta),T')} = f_{l,r}''(u(z(\eta),T'))$$

and

$$\lim_{\nu\to\infty} \frac{z_\nu - z(\eta)}{u(z_\nu,T') - u(z(\eta),T')}$$

does exists. Call this limit λ, it should be negative otherwise for ν large enough, since $f_{l,r}'$ is increasing we get

$$\frac{u(\eta_\nu,\tau) - u(\eta,\tau)}{\eta_\nu - \eta} = \frac{u(z_\nu,T') - u(z(\eta),T')}{\eta_\nu - \eta} > 0 \tag{4.30}$$

where $\eta_\nu = z_\nu + f'_{l,r}(u(z_\nu, T'))(\tau - T')$ and this contradicts the previous assumption on $D^-_\eta u(\eta,\tau)$. By (4.28) we get

$$D^-_\eta u(\eta,\tau) \geq \frac{1}{f''_{l,r}(u(\eta,\tau))(\tau - T')} \tag{4.31}$$

If we are in case (4.27) we can take $\tau < T'' < T'$ such that

$$\eta = z + f'_l(u(z,T''))(\tau - T') \quad \text{and} \quad u(\eta,\tau) = u(z,T'') \tag{4.32}$$

and proceed as in the previous case.

Since $u(x,t)$ is locally Lipschitz continuous in $\mathbb{R} \setminus \{0\}$ it is also a.e. differentiable, and satisfies by construction $u_t + f'_l(u)u_x = 0$ for $x < 0$, $u_t + f'_r(u)u_x = 0$ for $x > 0$, and the interface entropy condition on $\{x = 0\}$.

5 Proof of Theorem 2.4

Let $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$. Since G is bounded,

$$\|S_t\tilde{u}_n(x)\|_\infty \leq \|\tilde{u}_n(x)\|_\infty \leq C \quad \text{for all } t \in [0,T], \text{for all } n \in \mathbb{N}. \tag{5.1}$$

Hence $S_T\tilde{u}_n$ is *weakly** relatively compact in $L^\infty(\mathbb{R})$ and we can assume

$$S_T\tilde{u}_n \overset{*}{\rightharpoonup} \omega \ \textit{in}\ L^\infty(\mathbb{R}) \tag{5.2}$$

$$S_{(\cdot)}\tilde{u}_n \overset{*}{\rightharpoonup} u \ \textit{in}\ L^\infty(\mathbb{R}\times\mathbb{R}^+) \tag{5.3}$$

with $\omega \in L^\infty(\mathbb{R})$ and $u \in L^\infty(\mathbb{R}\times\mathbb{R}^+)$. Our claim is that $\omega \in \mathcal{A}(T,\mathcal{U})$ and there exists a subsequence of $\{S_T\tilde{u}_n\}$ converging to ω in $L^1_{loc}(\mathbb{R})$. Indeed $S_T\tilde{u}_n$ is $BV_{loc}(\mathbb{R}\setminus\{0\})$ for all $n \in \mathbb{N}$ and we have an uniform bound on the sequence given by (5.1). Moreover boundedness of the upper derivative implies that the total variation of $S_T\tilde{u}_n$ is uniformly bounded in all the compact subsets of $\mathbb{R}\setminus\{0\}$ and $S_t\tilde{u}_n$ is locally Lipschitz in time respect the L^1-norm. By Helly's theorem there exists a subsequence $\{S_t\tilde{u}_{n_k}\}_{k\in\mathbb{N}}$ which converges to some function $\omega'(\cdot,t)$ in $L^1_{loc}(\mathbb{R}\setminus\{0\})$ for all $t \in [0,T]$. By (5.3) ω' must coincide with u and hence by (5.1) the original sequence $\{S_t\tilde{u}_n\}$ converge to $u(\cdot,t)$ in $L^1_{loc}(\mathbb{R}\setminus\{0\})$. in particular, by (5.2) $u(\cdot,T) = \omega$. Now we need only to check that ω is an AB-entropy solution of (0.1), corresponding to a datum $\tilde{u} \in \mathcal{U}$. By (5.1) and regularity of f_l, f_r it can be assumed that, $\{f_l(S_T\tilde{u}_n)\}_{n\in\mathbb{N}}, \{f_r(S_T\tilde{u}_n)\}_{n\in\mathbb{N}}$ converge respectively to $\{f_l(u)\}_{n\in\mathbb{N}}, \{f_r(u)\}_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}\setminus\{0\})$. Therefore for any non negative function $\rho \in \mathcal{C}^1$ with compact support in $\mathbb{R}\setminus\{0\}\times[0,T]$ and for any $k \in \mathbb{R}$ it holds:

$$\begin{aligned}&\int\int \{|u-k|\rho_t + (f_{l,r}(u) - f_{l,r}(k))\,\mathrm{sgn}\,(u-k)\rho_x\}dxdt = \\ &\lim_{n\to\infty}\int\int \{|S_t\tilde{u}_n - k|\rho_t + (f_{l,r}(S_t\tilde{u}_n) - f_{l,r}(k))\,\mathrm{sgn}\,(S_t\tilde{u}_n - k)\rho_x\}dxdt \\ &\geq 0\end{aligned}$$

which shows that u is weak kruzkhov entropic solution of (0.1) in $\mathbb{R}\setminus\{0\}\times[0,T]$. It is immediate to check that also the interface entropy condition is satisfied in the distributional sense, indeed for any non negative function $\rho \in \mathcal{C}^1$ with compact support in $\mathbb{R}\times[0,T]$

$$\begin{aligned}&\int\int \{|u-c(x)|\rho_t + (f(u) - f(c(x))\,\mathrm{sgn}\,(u-c(x))\rho_x\}dxdt = \\ &\lim_{n\to\infty}\int\int \{|S_t\tilde{u}_n - c(x)|\rho_t + (f(S_t\tilde{u}_n) - f(c(x))\,\mathrm{sgn}\,(S_t\tilde{u}_n - c(x))\rho_x\}dxdt \\ &\geq 0\end{aligned}$$

where $c(x) = A\mathbb{1}_{x<0} + B\mathbb{1}_{x>0}$ ad $f(x,u) = f_l(u)\mathbb{1}_{x<0} + f_r(u)\mathbb{1}_{x>0}$ as in (0.3).

Now observe that if we take the initial data in $\mathcal{U}' \subset \mathcal{U}$ which contains functions with compact support, then $\mathcal{A}(T, \mathcal{U}')$ is compact in $L^1(\mathbb{R})$ and this guarantees the existence of optimal controls for a class of minimization problems. Therefore we can state the following result.

Corollary 5.1. *Let $F : L^1(\mathbb{R}) \to \mathbb{R}$ be a lower semicontinuous functional and let $\mathcal{U}'$ the set of functions defined in 2.4 but with compact support. Then for every fixed $T > 0$ the optimal control problem*

$$\min_{\tilde{u} \in \mathcal{U}'} F(S_T \tilde{u}(\cdot)) \tag{5.4}$$

admits a solution.

6 The concave-concave case

The analysis on the attainable profiles that we did in the previous sections when the fluxes f_l and f_r are both convex can be authomatically extended to the case of fluxes both concave. Indeed if we consider the scalar conservation law

$$u_t + f(x,u)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{6.1}$$

$$u|_{t=0} = u_0 \qquad x \in \mathbb{R} \tag{6.2}$$

where $f(x,u) = f_l(u)\mathbb{1}_{x<0} + f_r(u)\mathbb{1}_{x>0}$ with f_l and f_r strictly concave functions on $\mathbb{R}$ and $u_0 \in L^\infty$, an AB-entropy solution is again a weak solution of (6.1) which satisfies Kruzkov entropy condition away from the interface $\{x = 0\}$ and the AB-interface entropy condition (1.1) on the discontinuity interface.

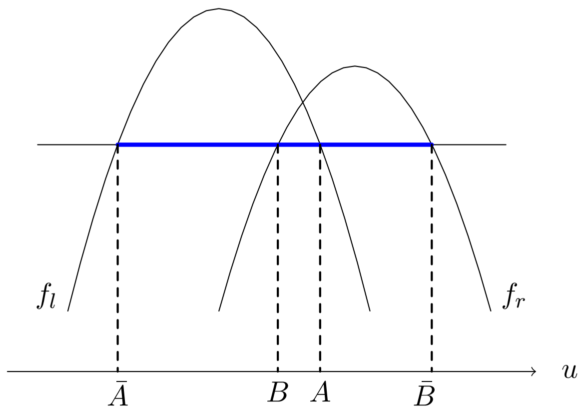


Figure 1.6: Example of AB connection with f_l, f_r strictly concave fluxes

However the meaning of the AB- interface entropy condition is different in the concave-concave flux case, since it implies that the flux of the solution along the discontinuity $\{x = 0\}$ must be lower or equal to the value of the flux on the connection. In Figure 5 we show how an AB-connection is in the case of concave-concave flux. By easy considerations we can show that the sets of attainable profiles for initial data $u_0 \in L^\infty$ and fixed time T is exactly the same that we described in Theorem 2.1. It is enough to observe that given a scalar conservation law $u_t + f(u)_x = 0$ with concave flux, it can be rewritten into another one with convex flux of the form $w_t + g(w)_x = 0$ where $w = -u$ and $g = -f \circ (-I)$ (I is the identity map).

Therefore in order to characterize the attainable profiles for the Cauchy problem (6.1)-(6.1) we can pass to the equivalent problem

$$w_t + g(x, w)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{6.3}$$

$$w|_{t=0} = -u_0 \qquad x \in \mathbb{R} \tag{6.4}$$

where $g(x, u) = g_l(u)\mathbb{1}_{x<0} + g_r(u)\mathbb{1}_{x>0}$ with $g_l = -f_l \circ (-I)$ and $g_r = -f_r \circ (-I)$ studying the unique entropy condition associated to the connection (-A)(-B). By Theorem 2.1 we are able to characterize the attainable profiles for the equation (6.3) for fixed time, hence we can switch to the equation (6.1) just changing sign.

In the next figures we compare the AB-entropy solutions of some Riemann problems in the case of convex-convex or concave-concave fluxes.

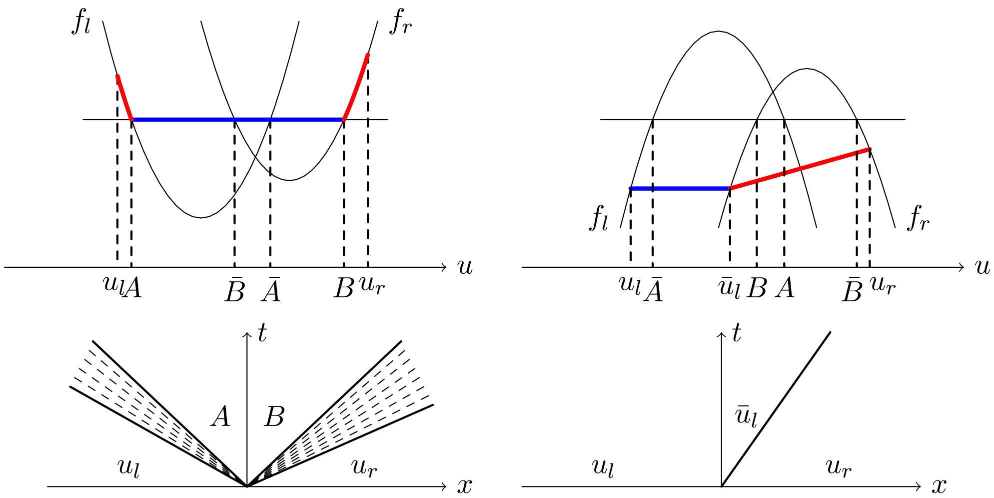


Figure 1.7: **Left**: $u_l < A < B < u_r$; **Right**: $u_l < \bar{A} < \bar{B} < u_r$

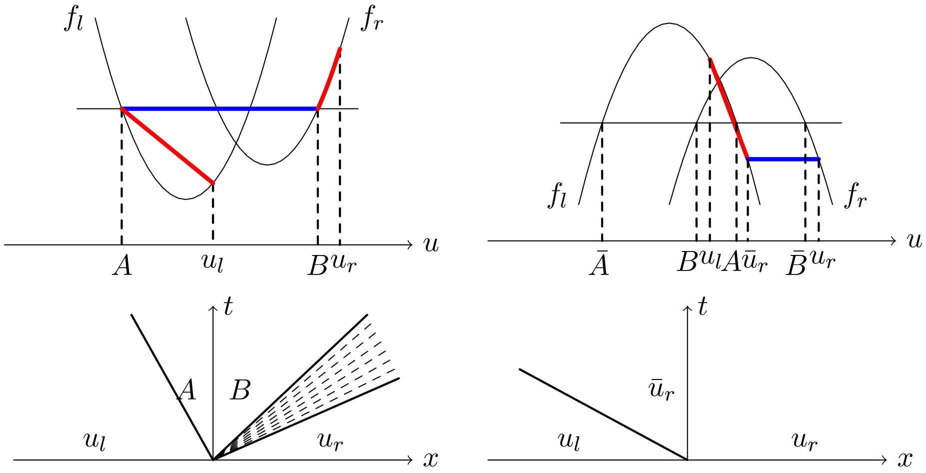


Figure 1.8: **Left**: $A < u_l < B < u_r$; **Right**: $\bar{A} < u_l < \bar{B} < u_r$

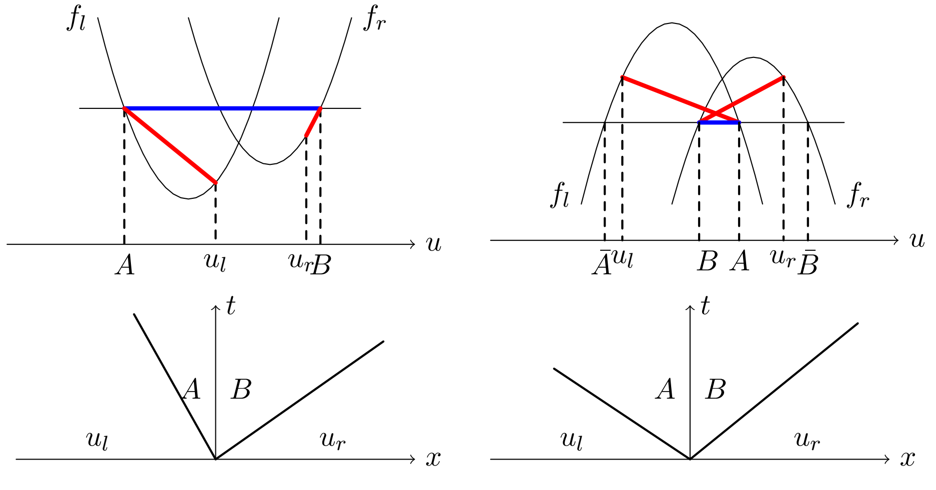


Figure 1.9: **Left**: $A < u_l < u_r < B$; **Right**: $\bar{A} < u_l < u_r < \bar{B}$

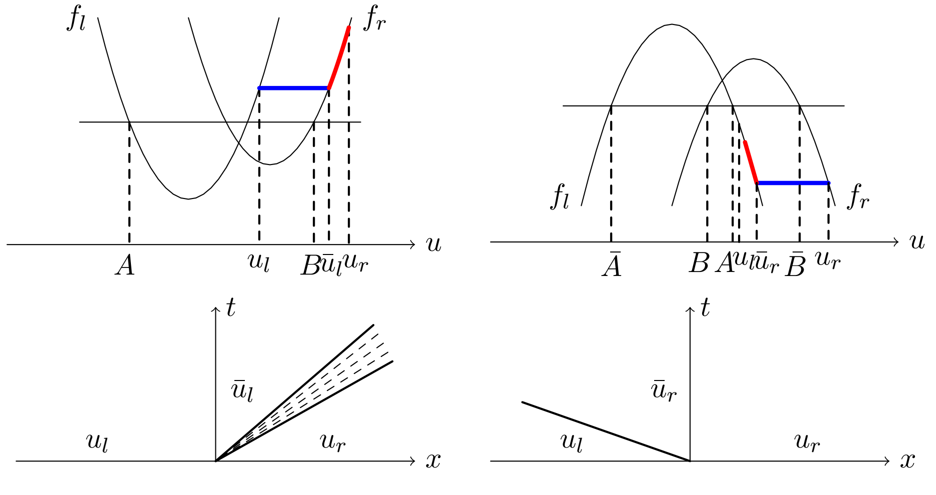


Figure 1.10: **Left**: $A < \bar{A} < u_l < B < u_r$; **Right**: $\bar{A} < A < u_l < \bar{B} < u_r$

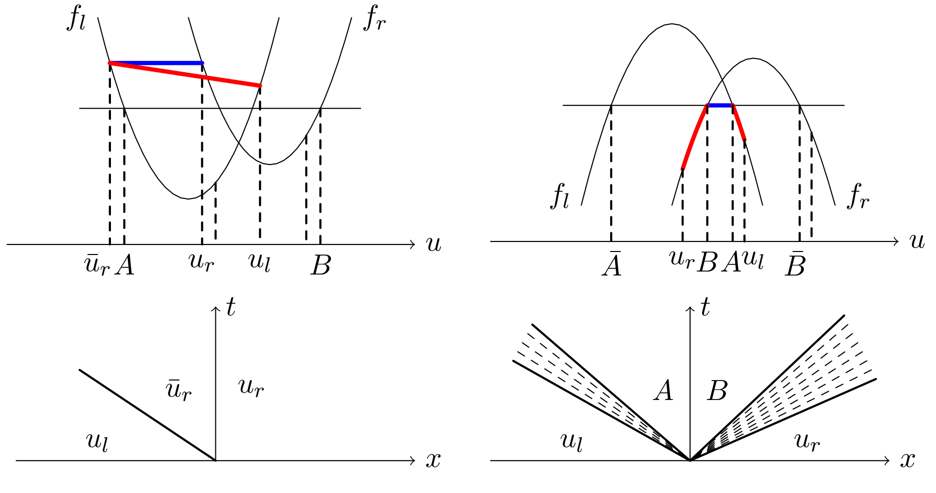


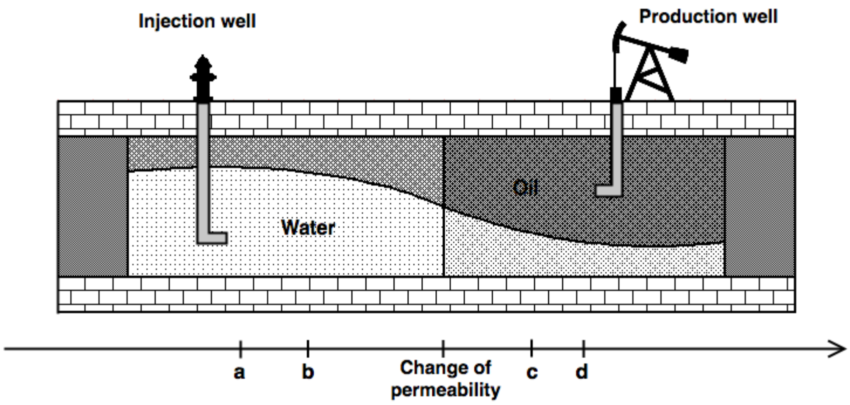
Figure 1.11: **Left**: $A < u_r < u_l < B$; **Right**: $\bar{A} < u_r < u_l < \bar{B}$

7 Some applications

After the theoretical results of the previous sections in the following we show how to apply them to some realistic problems. We briefly introduce the porus media model whose first simplification when the medium is heterogenous is a scalar conservation law with discontinuous flux in the convex-convex case and the LWR models for traffic flow which brings an example in the concave-concave case. We study some minimization problems involving profiles of the solutions in both the cases.

7.1 Porous Media Flows in Oil Field Development

In the oil industry, waterflooding or water injection is an inexpensive and simple engineering technique used in extraction of petroleum. It consists in drilling injection wells into an oil reservoir and introducing hot water. Water and oil inside the reservoir are immiscible, therefore, depending on the water rate injected, oil is displaced from the reservoir to the production wells.



Their flow can be described as the so-called two-phase flow in porous media, consisting of a saturation equation for the wetting phase and an equation for the total specific discharge. Assume for each phase the Darcy's Law

$$v = -\lambda(\nabla P - \rho G)$$

where v is the Darcy speed, λ is the mobility, P the phase pressure, ρ is the density and G the gravitational term. Combining Darcy's law with the source free equation of mass conservation for each phase

$$\rho_t + \nabla(v\rho)$$

we get

$$\alpha(\varphi\rho_w s_w)_t + (\alpha\rho_w F_w)_x = 0 \tag{7.1}$$

that is the one-dimensional saturation equation, ignoring the capillarity effects. In (7.1), α stands for the one dimensional cross-section area, φ is the rock porosity, ρ_w is the density of water and s_w is the saturation of the water at position x and time t. When α, φ, ρ_w are constant the saturation equation (7.1) can be simplified in a non-linear hyperbolic conservation law known as the Buckley - Leverett equation given by

$$(s_w)_t + \frac{1}{\varphi}(F_w)_x = 0. \tag{7.2}$$

The flux function F_w can depend both on position and saturation, $F_w = F_w(s_w, x)$. For a more detailed description of the physics of the problem we remand to [43].

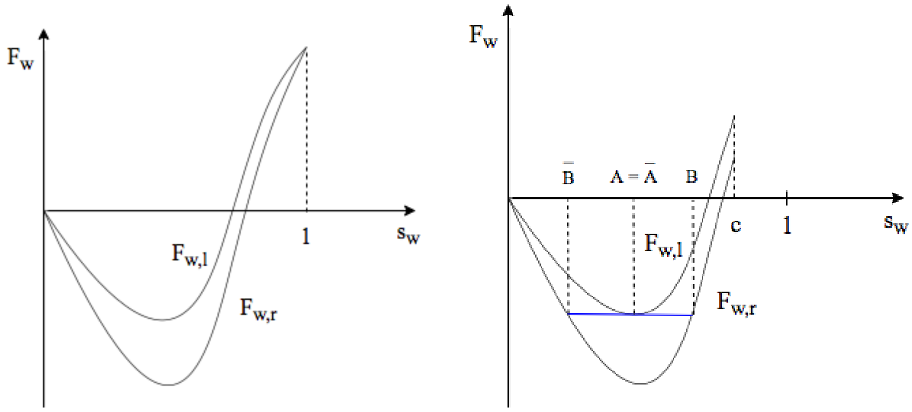


Figure 1.12: Both in continuous and discontinuous setting, F_w is in general not convex, but "bell-shaped", however it presents a region of convexity given by the interval $[0, c]$. If we restrict the initial data to this region, our analysis is still true. Observe that the connection, given by the blue segment on the right, passes through the maximum of the two minima.

In many applications, the porous medium is heterogenous hence the flow domain has to be divided into subdomains corresponding to different types of rock separated by lines or surfaces, along which the porosity and the absolute permeability of the rock type change, for example, the medium may consist of two rock types separated at the interface $\{x = 0\}$ and the flow is modeled by a scalar conservation law with a flux function discontinuous in the space variable as the following

$$F_w(s_w, x) = F_{w,l}(s_w)\mathbb{1}_{\{x<0\}} + F_{w,r}(s_w)\mathbb{1}_{\{x>0\}}. \tag{7.3}$$

In [53] the author derive an entropy inequality on the discontinuity interface by a regularisation procedure, where the physical capillary pressure term is added to the Buckley-Leverett equation. This entropy turns out to be our AB-interface entropy condition with AB critical connection, that is the connection passing through one of the two minima of the fluxes.

The first optimal control problem that we analyze is the classical minimization of the distance from a target function, which can represent the saturation regime (of water) necessary so that the gain received from the oil production is not lower than the cost for the injection of hot water into the reservoir. Hence if we consider a target function $l \in L^2([a, b])$ ([25]) with $[a, b]$ interval in which the porosity does not change, then the problem is

$$\min_{s_w \in \mathcal{U}} \int_a^b |S_T s_w(x) - l(x)|^2 dx. \tag{7.4}$$

where $\mathcal{U}$ is a set of controls as (2.16). Consider a minimizing sequence $\{s_{w,n}\}_{n\in\mathbb{N}}$, since $\mathcal{U}$ is uniformly bounded $\|S_T s_{w,n}\|_{L^\infty([a,b])} \leq C$ for all $n \in \mathbb{N}$, it follows that

$$\|S_T s_{w,n}\|_{L^1([a,b])} \leq C(b - a).$$

The estimate on the upper Dini derivative in Lemma 4.1 implies an uniform bound on the total variation of $S_T s_{w,n}$, therefore the sequence of solutions associated to the minimizing one is also uniformly bounded in the BV-norm. By the compact immersion of BV in L^2 there exists a subsequence that converges strongly in the L^2 norm. Hence by continuity of the L^2 norm the minimizing sequence reaches the minimum.

The second optimal control problems we consider is the maximization of net present value (NPV) of the waterflooding process ([47]). NPV is the difference between the present values of the expected cash inflows and outflows over the production period. Water injection and production costs are the two sources of cash outflow while oil production represents revenue generation. Assume that the injection cost R_i depends on the water saturation s_w in a region near the injection well ([a,b]), production cost R_p and oil production R_o depend on the water saturation near the production well ([c, d]) then we get the following variational problem

$$\max_{s_w \in \mathcal{U}} \int_0^T \left[\int_c^d \left(R_o(s_w(x,t)) - R_p(s_w(x,t)) \right) dx - \int_a^b R_i(s_w(x,t)) dx \right] dt \tag{7.5}$$

The evolution of water saturation is considered in the whole interval $[a, d]$ containing the change of porosity. We assume that R_i, R_p and R_o are polynomial functions of the water saturation. For initial data in $u \in \mathcal{U}$, the set of attainable profiles, at fixed time and in an interval not containing the change of porosity, is compact with respect to the L^1-topology. Proof of existence of a maximum is completely identical to the one we will prove in the next application, hence we omit it now.

7.2 LWR model for traffic flow

The LWR model, proposed by Lighthill and Whitham (1955) and by Richards (1956), describes the traffic flow at a macroscopic level, namely it considers speed, concentration and flows without taking into account the individual behaviour of the vehicles. In our application we deal with the simplest version, modelling the traffic flow on a one-dimensional highway.
Let $u(x, t)$ be the density of cars, its evolution is given by the scalar conservation law

$$u_t + [uV(u)]_x = 0 \tag{7.6}$$

where V represents the vehicle's velocity. In the classical model the velocity has a maximum representing the limit speed, and is inversely proportional to the density, since in presence of a large number of cars each driver goes slowly. This aspects are collected in the choice of a bounded decreasing speed which multiplied for the density produces a strictly concave flux.

In particular consider the LWR model on a junction made by one incoming and one outgoing road (this situation occur when the highway presents a changing surface condition), we get a scalar conservation law with discontinuous flux

$$F(x,u) = \begin{cases} F_l(u) & \text{x<0} \\ F_r(u) & \text{x>0} \end{cases}. \tag{7.7}$$

where right and left fluxes are of the form $F_{l,r}(u) = uV_{l,r}(u)$. As entropy on the discontinuity of the flux we can consider the AB-interface entropy condition with AB critical connection, that is the one passing trough the minimum of the two maxima of right and

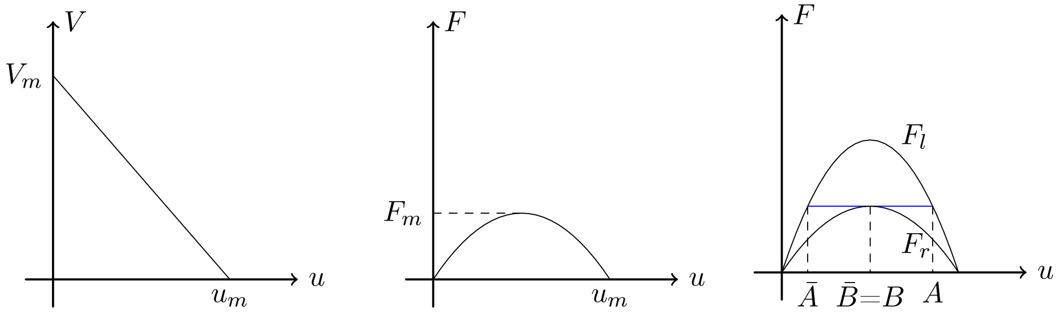


Figure 1.13: Velocity and flux in the LWR model, and a discontinuous flux with critical connection

left fluxes ([64]). We can consider again the problem of minimizing the distance from a target function $l \in L^2([a,b])$ with $[a,b]$ not containing the junction:

$$\min_{u\in\mathcal{U}} \int_a^b |S_T u(x) - l(x)|^2 dx. \tag{7.8}$$

Here $S_T u(x)$ is the unique AB- entropy solution at time T of (7.6) while l can be, for example, the optimal traffic density in a stretch of road near a school at time of exit. The approach to prove existence of a minimizer is exactly the same of (7.4)

In order to quantify the overall fuel consumption of all the vehicles in the LWR model, Ramdan and Sebold ([68]) introduce the average FC (Fuel Consumption) function, given by the following polynomial function of the speed

$$K(V) = 5.7\times10^{-12}V^6 - 3.6\times10^{-9}V^5 + 7.6\times10^{-7}V^4 - 6.1\times10^{-5}V^3 + 1.9\times10^{-3}V^2 + 1.6\times10^{-2}V + 0.99. \tag{7.9}$$

Multiplying K for the density $u(x,t)$ we get the Fuel Consumption rate of the whole road traffic. Now it is natural to consider the problem of minimizing the Fuel Consumption in space and time:

$$\min_{u\in\mathcal{U}} \int_0^T \int_a^b S_t u(x) K\left(V(S_t u(x))\right) dx dt. \tag{7.10}$$

For initial data in $u \in \mathcal{U}$ the set of attainable profiles at fixed time and in an interval not containing the origin is compact with respect to the L^1-topology, therefore we just need to show that the functional is continuous with respect to the L^1-topology in order to prove existence of minimizers for (7.10).

First observe that the map $\bar{u} \mapsto \int_a^b \bar{u}(x) K\left(V(\bar{u}(x))\right) dx$ is continuous as function from $\{\bar{u} \in \mathcal{A}(\mathcal{U},t)\}$ into $\mathbb{R}$ w.r.t. the L^1-norm and

$$\left| \int_a^b S_t u(x) K\left(V(S_t u(x))\right) dx \right| \leq C(b-a) \sup_{s\in[0,V_m]} K(s) \tag{7.11}$$

hence by the dominated convergence theorem also the map $\bar{u} \mapsto \int_0^T \int_a^b \bar{u}(x) K\left(V(\bar{u}(x))\right) dxdt$ is continuous w.r.t. the L^1-norm. Therefore given a minimizing sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, by Theorem 2.4 the associated sequence of solution $\{S_t\tilde{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{A}(\mathcal{U},t)$ is compact in L^1 topology which implies existence of a minimizer in (7.10).

Chapter 2

Conservation law models for supply chains

1 Conveyor belts and supply chain

Conveyor belts are component used in automata distribution and warehousing, whose origin dates back to 1892, by Thomas Robinson. They were introduced for carrying coal, ores and other products, but not too late they found wide use in other different sectors.
In fact nowadays conveyor systems have large application in industries for transportation of materials, goods and passengers, since they represent a quick and efficient technology which allows to move objects of different nature and has also some popular consumer application, as in supermarkets and airports (see Fig. 2.1 for an example of conveyor belt).
The conveyor also conceptually the load-bearing element of a supply chains, that is a system of organizations, people, activities, information, and resources involved in moving/processing a product or service from supplier to customer. In the last decade several mathematical models were developed in order to describe the flow of particles along a single conveyor belt. The main distinction is between the microscopic (discrete) models which track each part in the material flow and macroscopic (continuous) models relying on conservation laws which determines the motion of the part density ([30]).
The former models captures the most accurate dynamics but get computational extremely costly and produce inefficient simulation times, while the latter are inspired to continuous

Figure 2.1: Classic prototype conveyor belt

traffic flow models and captures phenomena such as queuing and congestion.
We will describe mathematically an infinite conveyor belt with finite capacity, modelled by a scalar conservation law whose flux is increasing, concave and multivalued at maximum density of particles representing the capacity of the chain. This choice for the flux was introduced for the first time in [8] to study shutdown of the production line due to a failure, and the time evolution of the recovery of the production line once the failure has been repaired. Therefore we have to deal with a scalar conservation law with discontinuous flux, and the discontinuity is in the conserved quantity. Even if there is now a vast literature for the case of spatial discontinuity as seen in the first chapter, the same cannot be said for the case that we want to analyze. However there is no way to avoid this peculiarity of the flux since conveyors used in industrial settings include tripping mechanisms which allow for workers to immediately shut down the conveyor when a problem arises ([31]). Hence to get a realistic description we need to consider this kind of discontinuity. The first to consider scalar conservation laws with flux discontinuous in the conserved quantity was Gimse in [42] in models for two phase flow in porous media.
Dias and al. in [33] analyze the limit case of a phase transition and study the problem by regularizing the flux function through some Friedrichs' mollifiers to fall back into classical theory (this is also the same approach used in [8]).
Carrillo proved existence and uniqueness of solutions in the case of a finite number of discontinuities of the flux in the density by passing though a continuous reformulation of the flux [24]. His results are extended to the case of fluxes that have at most countably number of monotone jumps by Buliceck and al. in [21].
A different point of view is given in [49] where the authors introduce an explicit transition phase approach enlarging the set of variable for the equation and considering non only the density but also the phase which can be free or congested.

Below we explain the original contributions of this chapter. We consider scalar conservation laws with flux flux concave and discontinuous at maximum density. First of all we prove existence an uniqueness of solution for bounded initial data having well defined limit at $+\infty$, then a stability result for solutions of the Kruzkhov type. What we are going to do is also an analysis of the Hamilton-Jacobi formulation associated to this conservation law. Although the HJ equation is also non-trivial to treat because of the discontinuity of the Hamiltonian in the gradient, it reveals the information required in the definition of solution for the conservation law in a sense explained later.
We will recall the definition of a viscosity solution in the case of discontinuous Hamiltonian, then we prove existence of solutions by approximating the Hamiltonian as done for the conservation law in [33] and [21] . We will show that it is also possible in our hypotheses to give an explicit formulation (Hopf Lax type) of the solution. As in the classical case, the explicit formulation can be computed through a variational problem and in this particular case through the combination of the optimal solution of two variational problems. After the analysis of the single chain, we proceed to insert it into a more complex structure, i.e. a supply chain on network. In particular we focus on the evolution of particles on a single junction point. In order to have a model as realistic as possible we require both incoming and outgoing chains to have finite capacity, moreover we assume that in the junction there is a buffer of finite capacity where to store products already processed waiting to enter in the outgoing chains. The evolution of the density of particles on the single chain is described by the scalar conservation law with flux satisfying the assumptions mentioned before, while the evolution of the buffer is described by an ODE. The Hamilton-Jacobi formulation introduced for the scalar conservation law turns out to be the key tool in order to achieve uniqueness of the solution for the junction problem.

The chapter is organized as follows: in Section 2.2 we recall the main models for the evolution of particle density on a single conveyor belt / chain introduced so far, in section 3 we briefly outline the available results for scalar conservation laws with flux discontinuous in the conserved quantity. Section 4 is devoted to the study of the Cauchy problem for a single conveyor belt, in particular we introduce an opportune notion of entropy condition and analyze first the single Riemann problem, then the interaction of several Riemann problems. Existence and Uniqueness of the solution are proved in Section 5 respectively by Front-Tracking and the stability result. In Section 6 we introduce the Hamilton-Jacobi reformulation of the Cauchy problem and we study an Hopf-Lax type formula for the solution which can be obtained by two combined optimization problems. In section 7 we clarify why the Hamilton-Jacobi approach is significant for our conservation law by introducing the Selection Principle. Finally in section 8 we move to the junction problem for which we prove both existence uniqueness of the solution of the Cauchy problem.

2 Models for supply chain

In this section we recall the main continuous models for conveyor belt and supply chain introduced in the last years which arise from such strictly applicative situations, these models offer important challenges also and above all from a theoretical point of view.

2.1 Armbruster- Degond-Ringhofer 2006

The first continuous model for conveyor belt was introduced in [7], inspired by traffic flow models for which a large body theory had already been developed. It is based on conservation laws of the form

$$\partial_t u + \partial_x \min\{N, u\} = 0 \tag{2.1}$$

where the variable $x \in [a, b]$ represents the position along the single chain, $u : [0, +\infty) \times [a, b] \to [0, +\infty)$, function of time and position, stands for the product density, and N is a bound on the rate of flux.

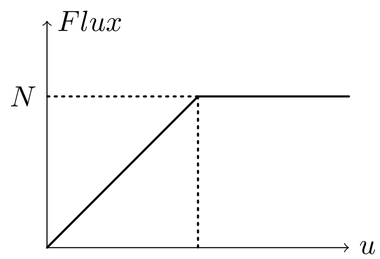


Figure 2.2: Flux in the conservation law 2.1

The number of parts processed is conserved, therefore a scalar conservation law is actually the most appropriate kind of equation to describe the physical behavior. Equation (2.1) is standard with a Lipschitsz continuous flux, hence it can be studied using the classical theory. However, although the flux is bounded, the density of parts can grow indefinitely which means that the chain has infinite capacity, this makes the model physically not very realistic. The defect was solved in the following model.

2.2 Armbruster - Gottlich - Herty 2011

In [8] the authors give a contribute to the body of continuous models by developing a model for supply chains or factories with finite work in progress. For evolution of parts they consider scalar conservation laws of the form

$$u_t + F(u)_x = 0 \tag{2.2}$$

with

$$F(u) = \begin{cases} a\,u & \text{if } u < M, \\ 0 & \text{if } u \geq M \end{cases}$$

where M is the maximum storage capacity in the processor.
The flux is discontinuous at maximum density and the most important consequence is the fact that the flux of the solution can not be uniquely determined through evaluation of u in M. Indeed assume that in a point a solution of (2.2) is exactly equal to M, then we are not able to say (in such a point) if the correspondent flux is 0 or $\lim_{s\to M^-} F(s)$. Since at

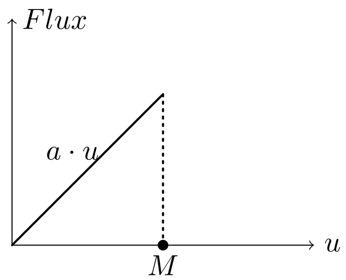


Figure 2.3: Flux in 2.2

the beginning it was not clear how to deal directly with this kind of equation, the problem was studied by considering continuous approximation of the flux. A more direct approach was introduced in [49].

2.3 Herty-Jorres-Piccoli 2013

In their work ([49]) equation (2.2) is studied by modifying the dynamic in the following way: the flux F is replaced with a flux G with argument the density $u \in [0, M]$ and a second argument $\mathcal{S}$ attaining value in the finite set $\{\mathcal{F}, \mathcal{C}\}$ and representing the status of the belt. Here $\mathcal{F}$ is the free phase and $\mathcal{C}$ the congested. More explicitly $G : [0, M] \times \{\mathcal{F}, \mathcal{C}\} \to \mathbb{R}$ is given by

$$G(u, \mathcal{S}) = \begin{cases} F(u) & \text{if } 0 \leq u < M, \quad \mathcal{S} = \mathcal{F} \\ \lim_{u\to M^-} F(u) & \text{if } u = M, \quad \mathcal{S} = \mathcal{F} \\ F(u) & \text{if } 0 \leq u < M, \quad \mathcal{S} = \mathcal{C} \\ 0 & \text{if } u = M, \quad \mathcal{S} = \mathcal{C} \end{cases} \tag{2.3}$$

where the third case never occurs and is added just to have a well defined function G on the full domain $[0, M] \times \{\mathcal{F}, \mathcal{C}\}$.
The evolution of $(u(t, x), \mathcal{S}(t, x))$ corresponding to (2.2) is given by a conservation law paired to a state constraint :

$$\begin{cases} u_t + G(u, \mathcal{S})_x = 0 \\ \mathcal{S}(t,x) = \mathcal{C}(t,x) \Longrightarrow u(t,x) = M \end{cases} . \tag{2.4}$$

The meaning of the state constraint is that the congested phase can appear only when $u(t, x) = M$, that is at maximal density.

In this setting they first prove existence of solutions to the Riemann problem and then existence of solution for the Cauchy problem with initial data given by the couple density-status, that is $(u_0, \mathcal{S}_0)$. In particular u_0 is an integrable function and $\mathcal{S}_0$ is constant in every non trivial interval where $u_0 = M$.

3 Preliminary result about fluxes discontinuous in the density

Here we present an oveview of the known results and approaches to scalar conservation laws with flux discontinuous in the conserved quantity that can be also interpreted as a transition phase.

3.1 Gimse

The first to be interested in scalar conservation laws with flux discontinuous in the conserved quantity was Gimse in [42]. In particular he considered the following equation

$$u_t + F(u)_x = 0$$

with flux function having a unique discontinuity at $u = \bar{u}$, so that

$$\lim_{u \to \bar{u}^-} F(u) \neq \lim_{u \to \bar{u}^+} F(u).$$

Hence, we can view F as a multivalued map where we set

$$F(\bar{u}) = \{ \lim_{u \to \bar{u}^-} F(u), \lim_{u \to \bar{u}^+} F(u)\}.$$

The study was motivated by several physical application, above all for two phase flow in porous media. Indeed in this model it is possible to have a discontinuous flux function when the flow properties change abruptly at some saturation.

The Riemann problem is studied and the solution turns out to be a couple given by the conserved quantity and the correspondent flux. The main contribution of his work concerns the full description of the interaction of several Riemann problems, moreover he introduces the notion of **zero-shock** which is a curve along which the conserved quantity is constant but the flux is discontinuous, i.e. the curve that identifies the phase transition.

For Gimse the initial data for the Cauchy problem is given by the couple conserved quantity-flux (u_0, f_0). In particular u_0 is a piecewise constant function and the flux satisfies $f_0 \in F(u_0)$.

At this point a notion of entropic solution is not yet available therefore the stability results are obtained only with restrictive assumption through which the analysis can be traced back to the classical case for continuous fluxes. More in detail, he asks the regions of congestion for the initial data to be concentrated inside a compact set, in this way there is no propagation of congestions from $\pm\infty$.

3.2 Dias, Figueira and Rodrigues

In the study of phase transitions in fluid dynamic or in elasticity it is common to find non convex flux associated to system of first order non linear conservation laws ([52], [67]). If we freeze one or more dependent variable we get a scalar conservation law for a continuous flux function $\bar{F}$ with the graph as in Fig 2.4 where the interval $[u_m, u_M]$ corresponds to the transition phase.

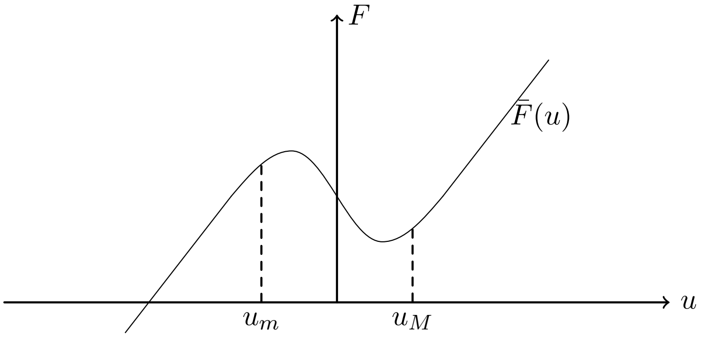


Figure 2.4: Flux $\bar{F}$

When the interval collapse to one point we get the discontinuous flux F in Fig 2.5 defined by

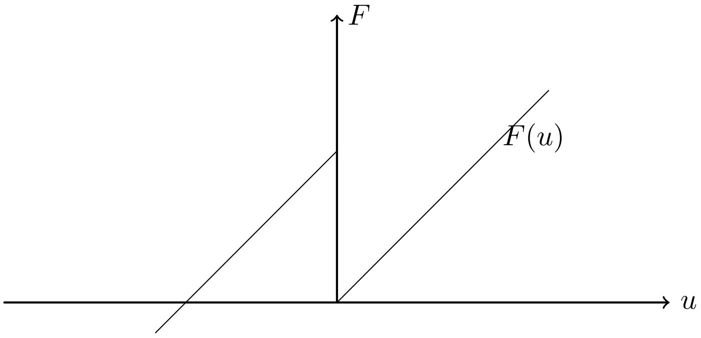


Figure 2.5: Flux $\bar{F}$

$$F(u) = u + 1 - \mathcal{H}(u)$$

where $\mathcal{H}$ is the Heaviside function. Motivated by this example Dias, Figueira and Rodrigues ([33]) started analyzing scalar conservation laws with flux discontinuous in the conserved quantity of the form

$$u_t + \tilde{F}(u)_x = 0 \tag{3.1}$$

where $\tilde{F}(u) = F(u)$ if $u \neq 0$ and $\tilde{F}(0) = [0, 1]$. They introduced the definition of weak solution below.

Definition 3.1. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is said a weak solution to the Cauchy problem for the equation (3.1), with initial data $u_0 \in L^\infty(\mathbb{R})$, if there exist $v \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ such that $v(t, x) \in \tilde{F}(u(t, x))$ a.e. and

$$\int_0^\infty \int_{\mathbb{R}} u\frac{\partial \varphi}{\partial t} + v\frac{\partial \varphi}{\partial x} dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x, 0)dx = 0$$

for each $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$.

Here the solution is given by the couple conserved quantity-flux and the initial data is just the conserved quantity. It is obtained by considering conservation laws associated to particular regularization of the multivalued flux. Such a procedure provides a preliminary selection of the flux in the initial data (at the discontinuity point). Existence of a solution for the Cauchy problem is proved for initial data of the form

$$u_0(x) = \sum_{i=1}^{p} u_i(x)\chi_{I_i}(x), \quad x \in \mathbb{R} \tag{3.2}$$

where for each i, χ_{I_i} is the characteristic function of the open interval I_i, $I_i \cap I_j = \emptyset$ if $i \neq j$ and u_i is given by a constant or a function in $C^2(\bar{I}_i) \cap L^\infty(I_i)$ with $u_i'(x) \neq 0$ for all $x \in I_i$. Their approach is based on continuous regularization of the flux and the study of the limits of the sequences of solutions obtained.

3.3 Czech-Polish Group

The last contribution to the phase transition theory for conservation laws we mention is given by the work of the Czech-Polish Group consisting of Bulicek, Gwiadza, Malek and Swierczewska-Gwiadza ([21]) and inspired by a previous one of Carrillo ([24]). The main idea is to appropriately redefine the conservation law in order to work with continuous flux.

Consider the scalar conservation law

$$u_t + F(u)_x = 0 \tag{3.3}$$

with initial data given only by the conserved quantity $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and the flux F given by a locally bounded regulated function (i.e. with left and right limits well defined in the points of discontinuity) of the form

$$F(s) = \lambda s + G(s) \text{ for } s \in [-M, M], \tag{3.4}$$

where

$$M = \|u_0\|_\infty, \quad \lambda \in \mathbb{R} \text{ and } G \text{ is strictly monotone so that } U = G^{-1} \in \mathcal{C}(G[-M, M]). \tag{3.5}$$

Definition 3.2. A couple (u, f) is a weak solution if

$$u \in L^\infty([0,T]; L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})), \quad f \in L^\infty([0,T]; L^\infty(\mathbb{R})), \tag{3.6}$$

$$f(x,t) \in F(u(x,t)) \text{ for a.a } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.7}$$

and the identity

$$-\int_0^\infty \int_{-\infty}^\infty u\phi_t dxdt - \int_0^\infty \int_{-\infty}^\infty f\phi_x dxdt = \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \tag{3.8}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^2)$.

As already mentioned, existence and uniqueness of solutions is proved by passing through a reparameterization of the equation with continuous flux.

If we now compare the definitions of solution for scalar conservation laws with discontinuous flux in the conserved quantity seen in the previous sections we can notice that for all of them the solution is defined as a pair (u, f).

On the other side, initial data is only the conserved quantity in works of Dias, Figueira and Rodrigues and the Czech-Polish group, while for Gimse it is given by the pair (u_0, f_0). So now the question is: how should we really state a Cauchy problem for phase transition? In particular how should the initial data be taken so that the problem is well posed? The answer to this question will be given in the next sections while we study scalar conservation law for conveyor belt.

4 The Cauchy problem for an infinite conveyor belt

In this section we introduce a Cauchy problem for the evolution of parts density on a conveyor belt of infinite length which will be the starting problem and whose resolution is our goal throughout this and the next sections.
Consider

$$\begin{cases} u_t + F(u)_x = 0 \\ \Big(u(0,x), f(0,x)\Big) = (u_0(x), f_0(x)) \end{cases} \tag{4.1}$$

where the flux F is a multifunction that satisfies the following assumptions:

$$\begin{aligned} &F : [0, M] \to [0, N], \text{ the restriction of F on } [0, M) \text{ is a single valued map such that} \\ &s \mapsto F(s) \text{ smooth }, \partial_s F > 0, \quad \partial_s^2 F \leq 0, \quad F(0) = 0 \\ &\text{and } F(M) = [0, N] \text{ with } N = \lim_{s \to M^-} F(s). \end{aligned} \tag{4.2}$$

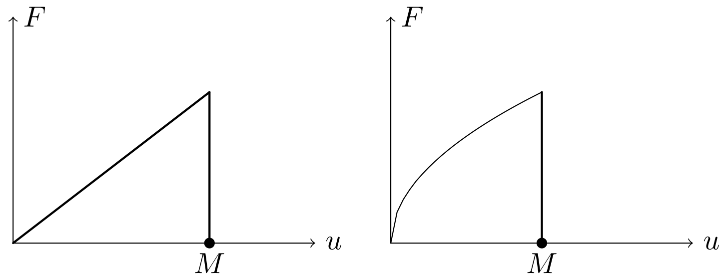


Figure 2.6: Two example of fluxes which satisfy (4.2)

and $u_0 \in L^\infty(\mathbb{R})$ and $f_0 = F(u_0)$ if $u_0 \leq M$ while $f_0 \in [0, N]$ if $u_0 = M$.

Definition 4.1. We say that a couple (u, f) is a weak solution of (4.1) if

$$u \in C([0,T]; L^\infty(\mathbb{R})), \quad f \in L^\infty([0,T]; L^\infty(\mathbb{R})), \tag{4.3}$$

$$f(x,t) \in F(u(x,t)) \text{ for a.a } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4.4}$$

and the identity

$$-\int_0^\infty \int_{-\infty}^\infty u\phi_t dxdt - \int_0^\infty \int_{-\infty}^\infty f\phi_x dxdt = \int_{-\infty}^\infty u_0(x)\phi(0,x)dx \tag{4.5}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^2)$.

We remember that for continuous flux, a function u is called entropy weak solution of (4.1) if it satisfies the equation in the sense of distribution and if

$$\eta(u)_t + q(u)_x \leq 0 \tag{4.6}$$

holds in the sense of distributions for all the entropy/entropy flux pairs (η, q) where η is a convex smooth function and q satisfies $q'(u) = \eta'(u)F'(u)$. The continuity of the flux plays an important role in proving existence of a solution for the Cauchy problem. However, in (4.1) the flux is a set-valued map, therefore the classical theory must be revised and adapted to the particular problem.

4.1 Entropy solution and Riemann solver

We now analyze entropy solutions for scalar conservation laws with flux discontinuous in the conserved quantity. In previous works entropy solutions were studied by passing through continuous reparamenterization of the flux and employing classical entropy inequality on such reparametrerized solutions.

After recalling the definition of entropy admissible solutions in the theory for smooth fluxes we introduce and justify a new version of Kruzkhov entropies, then we show how to solve a Riemann problem according to these.

Reformulation of Kruzkhov entropy conditions

Consider the scalar conservation law

$$u_t + F(u)_x = 0 \tag{4.7}$$

with F smooth flux. A weak solution u of (4.7) is said to be entropy admissible if it satisfies (4.6) in the distributional sense, for every pair (η, q), where η is a convex entropy and q the correspondent entropy flux. Condition (4.6) is sufficient to single out a unique solution for initial data in L^∞. In particular, for each constant k we can consider the functions

$$\eta(u) = |u - k| \qquad q(u) = \text{sgn}(u-k)(F(u) - F(k)) \tag{4.8}$$

(Kruzkov entropies) and say that a locally integrable function u is an entropy solution of (4.7) if

$$\int\int |u-k|\varphi_t + \text{sgn}(u-k)(F(u) - F(k))\varphi_x dxdt \geq 0 \tag{4.9}$$

for every positive $\mathcal{C}^1$ function φ with compact support in the half plane where $t > 0$. Continuity of the flux F implies that the entropy flux q is well defined also when $u = k$ and is equal to 0. Assume now that F satisfies assumption (4.2). Clearly it is no more possible to consider entropies like (4.8) since for $u = k = M$ the entropy flux is not well defined. Therefore we need introduce a modified version of the Kruzkov entropies, adapted to the case of the scalar conservation law with flux discontinuous in the conserved quantity. Since a solution of (4.1) is given by the couple representing the conserved quantity and the correspondent flux, the new entropies must depend on this couple. Also the constant k must be replaced with a couple (k, f_k) where $f_k = F(k)$ if $k < M$ and $f_k \in [0, N]$ if $k = M$. Hence we introduce the new pair of Kruzkhov entropies (η, q) which are maps defined on the graph of F. i.e

$$\mathcal{G}(F) = \{(u, f_u) \in \mathbb{R}^2;\ f_u \in F(u)\} \tag{4.10}$$

and given by

$$\eta_{(k,f_k)}(u, f_u) = |u - k| \tag{4.11}$$

$$q_{(k,f_k)}(u, f_u) = \begin{cases} \text{sgn}(u-k)(f_u - f_k) & \text{if } u \neq M \text{ or } u = M \text{ and } k \neq M \\ -|f_u - f_k| & \text{if } u = k = M \end{cases} \tag{4.12}$$

for all (u, f_u) and (k, f_k) in $\mathcal{G}(F)$. We will prove that also (4.11)-(4.12) single out a unique solution of (4.1).

Definition 4.2. A pair of functions (u, f_u) is an entropy solution of (4.7) if

$$\int\int \eta_{(k,f_k)}(u, f_u)\varphi_t + q_{(k,f_k)}(u, f_u)\varphi_x dxdt \geq 0 \tag{4.13}$$

for all the pairs $(k, f_k) \in \mathcal{G}(F)$ and every positive $\mathcal{C}^1$ function φ with compact support in the half plane where $t > 0$.

Remark 2. Entropy inequality (4.13) coincides with (4.9) when the flux of the solution is alway F_c.

In the next result we justify the new term in the entropy flux by considering Kruzkov entropies for scalar conservation laws with continuous fluxes converging to F in the sense of the graph. As in the classical theory of set-valued functions, we use the Hausdorff distance.

Proposition 4.1. *Let $F_n : [0, M] \to \mathbb{R}^+$ be a sequence of continuous fluxes converging to F in the sense of the graph. Given (u, f_u), (k, f_k) in $\mathcal{G}(F)$ with $u = k = M$, consider $(u_n, F_n(u_n))$, $(k_n, F_n(k_n)) \in \mathcal{G}(F_n)$ such that*

$$dist\Big((u_n, F_n(u_n)), (u, f_u)\Big) \longrightarrow 0$$
$$dist\Big((k_n, F_n(k_n)), (k, f_k)\Big) \longrightarrow 0$$

as $n \to \infty$. Then

$$q_{k,n}(u_n) \longrightarrow -|f_u - f_k| = q_{(k,f_k)}(u, f_u). \tag{4.14}$$

as $n \to \infty$.

Proof. Assume that $f_k > f_u$. For all $\varepsilon > 0$ such that $\varepsilon < \frac{f_k - f_u}{2}$

$$\exists\ n_k \text{ s.t. for all } n \geq n_k \quad \text{dist}\Big((k_n, F_n(k_n)), (k, f_k)\Big) \leq \varepsilon$$
$$\exists\ n_u \text{ s.t. for all } n \geq n_u \quad \text{dist}\Big((u_n, F_n(u_n)), (u, f_u)\Big) \leq \varepsilon$$

Therefore for all $n \geq \max\{n_k, n_u\}$ we get

$$|F_n(k_n) - f_k| \leq \text{dist}\Big((k_n, F_n(k_n)), (k, f_k)\Big) \leq \varepsilon \text{ and}$$
$$|F_n(u_n) - f_u| \leq \text{dist}\Big((u_n, F_n(u_n)), (u, f_u)\Big) \leq \varepsilon$$

which immediately implies

$$F_n(u_n) \leq f_u + \varepsilon < f_k - \varepsilon \leq F_n(k_n) \tag{4.15}$$

By definition (4.2), all the level sets of F consist into two points (which coincides when $F = N$). If $f_k < N$, let k' be the other point where $F = f_k$. Clearly we have that $k' < k$. For $\varepsilon < \min\{\frac{f_k - f_u}{2}, \frac{k-k'}{2}\}$ define $U = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ |\ \text{dist}((x, y), \text{Graph(F)}) < \varepsilon\}$. Then there exist $n_U \in \mathbb{N}$ such that $\text{Graph}(F_n) \subset U$ for all $n \geq n_U$. Hence by (4.15) and continuity of F_n it follows that $u_n > k_n$ for all $n \geq \max\{n_k, n_u, n_U\}$. Analogously we can prove that if $f_u > f_k$ for n large enough $F_n(k_n) < F_n(u_n)$ and $u_n < k_n$, thus (4.14) holds. ∎

Example. Assume that $F(u) = \frac{N}{M}u$ for $0 \leq u < M$ and $F(M) \in [0, N]$. It is immediate to check that it satisfies (4.2). Let F_n the sequence of continuous concave approximation converging to F in the sense of the graph described below:

$$F_n(u) = \begin{cases} \frac{nN}{nM-1}u & \text{if } 0 \leq u < M - \frac{1}{n} \\ -nN(u-M) & \text{if } M - \frac{1}{n} < u \leq M \end{cases}$$

Consider $f_u, f_k \in [0, N]$ and let u_n, k_n their preimage with respect to F_n restricted to $[M - \frac{1}{n}, M]$, i.e

$$u_n = F_n^{-1}(f_u) = M - \frac{f_u}{nM}, \qquad k_n = F_n^{-1}(f_k) = M - \frac{f_k}{nM}.$$

Both u_n, k_n converge to M as $n \to \infty$, and if $f_u < f_k$ then $u_n > k_n$ else if $f_u > f_k$ then $u_n < k_n$. This implies that

$$q_{k_n}(u_n) = \text{sgn}(u_n - k_n)(f_u - f_k) = -|f_u - f_k| = q_{(k,f_k)}(u, f_u)$$
$$\text{when } u = k = M$$

4.2 The single Riemann problem

We are going to show how to solve a single Riemann problem for scalar conservation laws with flux as in (4.2). The discontinuity in the conserved quantity brings a pathological behaviour in the interaction of several Riemann problems. However, once clear how to solve two coupled Riemann problems, it is immediate to derive an algorithm in order to find a solution for piecewise constant initial data. This kind of analysis was done for the first time by Gimse in [42] for scalar conservation laws with flux having a positive jump in the conserved quantity, here we present a more systematic treatment adapted to the case of negative jump which naturally appears modelling conveyor belts.

From now on we indicate with F_c the function given by

$$F_c = \begin{cases} F(u) & u \in [0, M) \\ \lim_{u \to M} F(u) & u = M \end{cases} \tag{4.16}$$

i.e. the continuous part of F.

Lemma 4.2. *Let F be a flux satisfying (4.2). Then the Riemann problem*

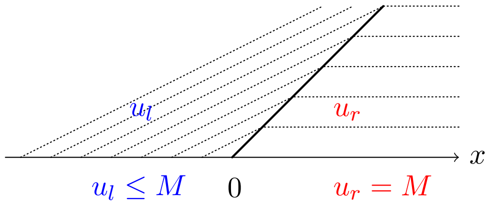
$$\begin{aligned} &u_t + F(u)_x = 0, \\ &(u_0, f_0)(x) = \begin{cases} (u_l, f_l) & x < 0, \\ (u_r, f_r) & x > 0, \end{cases} \end{aligned} \tag{4.17}$$

admits a unique selfsimilar entropy solution (u, f_u) which fulfils a maximum principle with respect to the conserved quantity u, i.e $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}$.

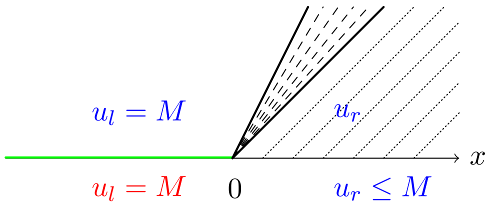
There may be the following cases:

i. If $u_l, u_r \leq M$ and $f_l = F_c(u_l)$, $f_r = F_c(u_r)$, it is the classical Riemann problem for a continuous monotone flux, hence by the Lax entropy condition there exists a unique solution (u, f_u) with u presenting a rarefaction if $u_l > u_r$ and a single shock if $u_l < u_r$, while f is simply given by $F_c(u)$.

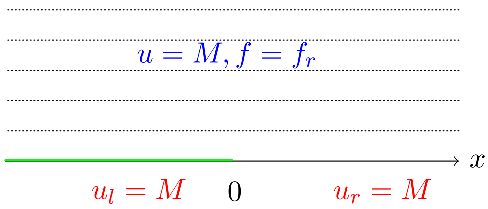
ii. *If $u_l < M$, $f_l = F_c(u_l)$ and $u_r = M$, $f_r \in [0, N)$, the unique solution is given by (u, f_u) where u presents a shock of speed $\frac{f_l - f_r}{u_l - u_r}$ connecting the left to the right state and f_u is just given by $F_c(u_l)$ for $x < \frac{f_l - f_r}{u_l - u_r} t$, by f_r for $x > \frac{f_l - f_r}{u_l - u_r} t$.*



iii. *If $u_l = M$, $f_l \in [0, N)$ and $u_r \leq M$, $f_r = F_c(u_r)$, the unique solution is given by (u, f) where u presents a rarefaction connecting the left state $u_l = M$ with associated flux $f_l = N$ to the right state $u_r \leq M$, $f_r = F_c(u_r)$. Hence $f_u = F_c(u)$.*



iv. *If $u_l, u_r = M$ and $f_l, f_r \in [0, N]$ the unique solution is given by (u, f_u) where $u = M$ and $f = f_r$. Hence, again the solution is constant.*



Proof. Solutions described in i, ii, iii, iv obviously satisfy the maximum principle, therefore it is only necessary to check that they are entropic solutions.
For the case i and iii, by Remark 2, the entropy condition coincides with the classical one for continuous flux which is clearly satisfied.
In the case ii the discontinuity generated by the shock wave $x = \lambda t$ with $\lambda = \frac{f_l - f_r}{u_l - u_r}$ in ii. is entropy admissible. Indeed condition (4.9) with entropies (4.12) implies that

$$\lambda\Big(\eta_{(k,f_k)}(u_r, f_r) - \eta_{(k,f_k)}(u_l, f_l)\Big) \geq q_{(k,f_k)}(u_l, f_l) - q_{(k,f_k)}(u_r, f_r). \tag{4.18}$$

which is satisfied by all the couples $(k, f_k) \in \mathcal{G}(F)$.
Finally the last case is just a check.

□

For the sake of clarity it is useful now to introduce a Riemann Solver that synthesizes what we saw in the previous lemma. Hence we define the following map

$$\mathcal{R} : \mathbb{R}^4 \to \mathbb{R}^4$$
$$\Big((u_l, f_l), (u_r, f_r)\Big) \mapsto \mathcal{R}\Big((u_l, f_l), (u_r, f_r)\Big) \tag{4.19}$$

where $\mathcal{R}$ is defined as

$$\mathcal{R}\Big((u_l, f_l), (u_r, f_r)\Big) := \begin{cases} \Big((u_l, f_l), (u_r, f_r)\Big) & \text{in case } i,\ ii \\ \Big((M, N), (u_r, f_r)\Big) & \text{in case } iii \\ \Big((M, f_r), (M, f_r)\Big) & \text{in case } iv \end{cases} \tag{4.20}$$

4.3 Shock interaction

The next step is to study how to construct a solution for piecewise constant initial data. The starting point is the interaction of two Riemann problems at time $t = 0$ when the solution of one of the two contains a wave with infinite speed, hence here we show how to solve it.

Consider the Cauchy problem (4.1) with initial data

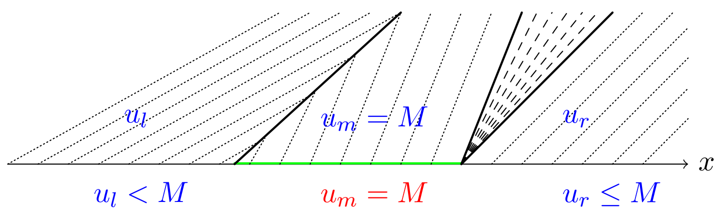
$$(u_0, f_0) = \begin{cases} (u_l, f_l) & x < a \\ (u_m, f_m) & a < x < b \\ (u_r, f_r) & x > b \end{cases} \tag{4.21}$$

where $a, b \in \mathbb{R}$

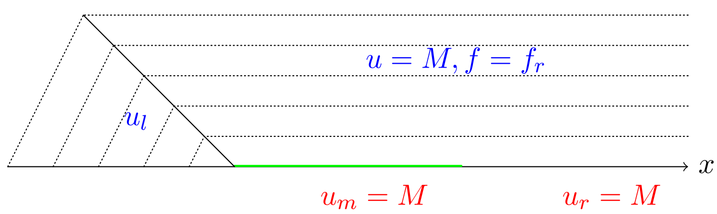
Case 1 If the two couples (u_m, f_m), (u_r, f_r) satisfy assumptions of point $i.$ or $ii.$ in 4.2, then the Riemann problem (u_l, f_l) (u_m, f_m) can be solved as in Lemma 4.2

Case 2 If (u_m, f_m), (u_r, f_r) satisfy assumptions of point $iii.$ in 4.2. Then the Riemann problem (u_l, f_l) (u_m, f_m) can be solved as prescribed in Lemma 4.2, hence the situation is identical to the previous one

Case 3 If (u_m, f_m), (u_r, f_r) satisfy assumptions of point $iv.$ in 4.2. then the Riemann problem (u_l, f_l) (u_m, f_m) is solved as (u_l, f_l) (M, N) following Lemma 4.2. This is due to the fact that the Riemann problem connecting the intermediate state with the left one produces a shock of speed $-\infty$ (Gimse's **zero shock**) which interacts at time $t = 0$ with the other Riemann problem.



Case 4 (u_m, f_m), (u_r, f_r) satisfy assumptions of point $v.$ in 4.2, the Riemann problem (u_l, f_l) (u_m, f_m) is solved as (u_l, f_l) (M, f_r) following Lemma 4.2.



Similarly to what was done in [42] we give an algorithmic procedure to determine the solution of (4.1) with piecewise initial data

i. Solve the Riemann problem starting from right to left, if a shock with speed $-\infty$ evolves, change the right state of the left next Riemann problem

ii. Once all the Riemann problems at time $t = 0$ are solved, let τ be the first positive time in which an interaction occur in space, let's solve again the Riemann problems according to i.

iii. when all the interactions at time $t = \tau$ are solved, proceed to the next interaction.

Through this procedure we can observe that for $t > 0$

$$(u, f)(t, x) = (M, \alpha) \text{ with } \alpha \in [0, N) \iff \lim_{x\to+\infty}(u_0, f_0) = (M, \alpha) \tag{4.22}$$

Remark 3. In general by the way we solve the Cauchy problem for piecewise initial data we can observe that if in a certain point (t, x) the solution is equal to (M, α) (as defined in (4.22)) then it is definitively equal to (M, α) on the line $\{t\} \times [x, +\infty)$. This means that if $\lim_{x\to+\infty}(u_0, f_0) = (M, \alpha)$ then there exists a curve which separates the region where the solution is definitively equal to (M, α) from the region where we have simply the solution of the scalar conservation law with flux given by F_c.

5 Existence and Uniqueness

We prove existence of a solution for (4.1) by the method of front tracking .
Given an initial data (u_0, f_0) with $0 \le u_0 \le M$ satisfying the following assumptions

$$\exists c \in [0, M] \text{ such that } (u_0 - c) \in BV(\mathbb{R}) \tag{5.1}$$

and

$$f_0 \in BV(\mathbb{R}), \quad f_0 = F(u_0) \quad \text{ if } u_0 < M \text{ otherwise } f_0 \in [0, N], \tag{5.2}$$

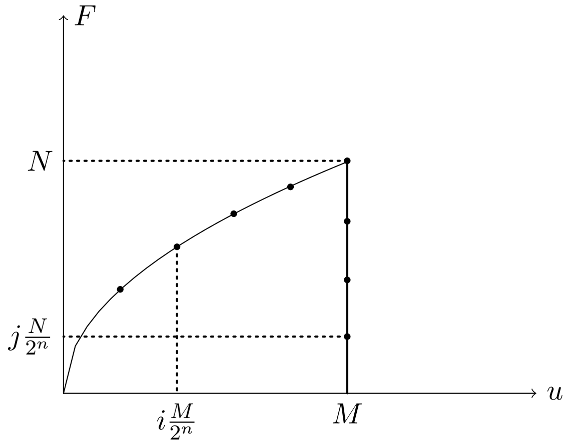
we construct a sequence $\{(u_n, f_n)\}_{n\in\mathbb{N}}$ of piecewise constant approximate solutions of (4.1) with $(u_n(0, \cdot), f_n(0, \cdot)) \to (u_0, f_0)$. The compactness argument for the approximate conserved quantities rests on Helly's theorem, but also on the fact that every solution has a very rigid structure as observed in the Remark 3. The convergence of the approximate flux is instead more delicate as we will see later.

5.1 Piecewise constant approximation

Fix $n \in \mathbb{N}$ and let F_n be the piecewise affine function which coincides with F in all the nodes $i\frac{M}{2^n}$ with $i = 0, ..., 2^n - 1$, and $F_n = [0, N]$ in M, i.e

$$\begin{aligned}
&F_n(s) = \frac{s - i\frac{M}{2^n}}{\frac{M}{2^n}} \cdot F\Big((i+1)\frac{M}{2^n}\Big) + \frac{(i+1)\frac{M}{2^n} - s}{\frac{M}{2^n}} \cdot F\Big(i\frac{M}{2^n}\Big) \\
&s \in \Big[i\frac{M}{2^n}, (i+1)\frac{M}{2^n}\Big] \\
&\text{and } F_n(M) = [0, N]\}
\end{aligned} \tag{5.3}$$

Let $\tilde{u}$ be a piecewise constant function with values in $\{i\frac{M}{2^n}, i = 0, ..., 2^n\}$, definitively constant outside a compact set. The associated approximated initial flux is given by $\tilde{f}(x) = F_c\Big(i\frac{M}{2^n}\Big)$ if $\tilde{u}(x) = i\frac{M}{2^n}$ for $i = 0, ..., 2^n - 1$ and $\tilde{f}(x) \in \{j\frac{N}{2^n}, j = 0, ..., 2^n\}$ if $\tilde{u}(x) = M$.



Now we show that the Cauchy problem

$$\begin{cases} u_t + F_n(u)_x = 0 \\ (u, f)(0, x) = (\tilde{u}, \tilde{f})(x) \end{cases} \tag{5.4}$$

admits a globally defined in time solution with values in $\{i\frac{M}{2^n}, i = 0, ..., 2^n\}$ for the conserved quantity and $\{F_c\Big(i\frac{M}{2^n}\Big), i = 0, ..., 2^n\} \cup \{j\frac{N}{2^n}, j = 0, ..., 2^n\}$ for the flux.

Consider the Riemann problem

$$(u, f)(0, x) = \begin{cases} (u^-, f^-) & \text{if } x < 0 \\ (u^+, f^+) & \text{if } x > 0 \end{cases} \tag{5.5}$$

where (u^-, f^-) and (u^+, f^+) admits values in the same sets of $(\tilde{u}, \tilde{f})$. We analyze separately the case $u^- < u^+$ and $u^- > u^+$.

1. Assume $u^- < u^+$. The largest convex function $F^\#$ such that

$$F^\#(s) \leq F_n(s) \quad \text{for all } s \in [u^-, u^+]$$

is the straight line with slope $\lambda = \frac{f^+ - f^-}{u^+ - u^-}$. This is clearly motivated by the concavity of F_n. By Lemma 4.2, the function

$$(v(x, t), f_v(x, t)), = \begin{cases} (u^-, f^-) & \text{if } x < \lambda t \\ (u^+, f^+) & \text{if } x > \lambda t \end{cases} \tag{5.6}$$

is a weak entropy solution for (5.4).

2. If $u^+ < u^-$, the smaller function $F_\#$ with convex subgraph such that

$$F_\#(s) \geq F_n(s) \quad \text{for all } s \in [u^+, u^-]$$

is given by the same F_n. Define $u^+ = v_0 < v_1 < ... < v_l = u^-$ the points of non differentiability for F_n and $f^+ = f_0, f_1, ..., f_l = f^-$ the correspondent values of the flux. If $u^- < M$ or $u^- = M$ and $f^- = N$, let λ_h be the sequence of speeds

$$\lambda_h = \frac{f_h - f_{h-1}}{v_h - v_{h-1}} \qquad h = 1, ..., l \tag{5.7}$$

while if $u^- = M$ and $f^- \in [0, N)$ set

$$\lambda_l = \frac{N - f_{l-1}}{M - v_{l-1}}, \tag{5.8}$$

λ_h as in (5.7) for $h = 1, ..., l-1$ and switch the value of the flux from f_- to N. We claim that the function

$$(v(x,t), f_v(x,t)), = \begin{cases} (u^-, N) & \text{if } x < \lambda_l t \\ (v_h, f_h) & \text{if } \lambda_h t < x < \lambda_{h-1} t, \\ (u^+, f^+) & \text{if } x > \lambda_1 t \end{cases} \tag{5.9}$$

is a weak entropy solution of (5.4).
Indeed let ϕ be a $\mathcal{C}^1$ non-negative function with compact support in the half plane where $t > 0$; fix the couple (k, f_k) with k constant function, $f_k = F_n(k)$ if $k < M$ and $f_k \in [0, N]$ if $k = M$. Define the characteristic function

$$\chi_{[v_{h-1}, v_h]}(k) := \begin{cases} 1 & \text{if } k \in [v_{h-1}, v_h] \\ 0 & \text{if } k \notin [v_{h-1}, v_h] \end{cases} \tag{5.10}$$

The following computation shows that (5.9) is an entropy solution :

$$\begin{aligned} &\int\int \{|v - k|\phi_t + q_{(k,f_k)}(v, f_v)\phi_x dx\ dt\} \\ &= \sum_{h=1}^{l} \int \left\{ \left(|v_h - k| - |v_{h-1} - k|\right)\lambda_h - \left[q_{(k,f_k)}(v_h, f_h) - q_{(k,f_k)}(v_{h-1}, f_{h-1})\right] \right\} \phi(t, \lambda_h t) dt \\ &= \sum_{h=1}^{l} \int \left[\left(v_h + v_{h-1} - 2k\right)\lambda_h + 2f_k - f_h - f_{h-1} \right] \chi_{[v_{h-1}, v_h]}(k) \phi(t, \lambda_h t) dt \\ &\geq 0 \end{aligned} \tag{5.11}$$

We can observe that the solutions obtained by solving the Riemann problem (5.4)−(5.5) have values only in the set $\{i\frac{M}{2^n}, i = 0, ..., 2^n\}$ for the conserved quantity and $\{F_c\left(i\frac{M}{2^n}\right), i = 0, ..., 2^n\} \cup \{j\frac{N}{2^n}, i = 0, ..., 2^n\}$ for the correspondent flux.

Consider now the Cauchy problem (5.4) with piecewise initial data $(\tilde{u}, \tilde{f})$ as described above and let $x_1, ..., x_\nu$ be the points where $\tilde{u}$ has the discontinuities. Solving the interacting Riemann problems according to the procedure introduced in Section 4.3, we obtain a local solution of (5.4) which can be prolonged up to a first time t_1 where some lines of discontinuity intersect. We can solve again the Riemann problems and extend the solution to the next time where a set of wave-front interactions take place and so on.

Now we show that the total number of interaction is finite, hence the solution can be prolonged for all $t \geq 0$. Indeed, let

$$\xi_1(t) < ... < \xi_m(t) \qquad (t < \tau) \tag{5.12}$$

the curves of finite speed where the solution has m discontinuities interacting at time $t = \tau$ and let $u_0, u_1, ..., u_m$ the constant values taken by u. If $\lim_{x\to\infty}(u_0(x), f_0(x)) = (M, \alpha)$ with $\alpha \in [0, N)$, it is possible to have one more curve of discontinuity, whose speed can be arbitrary large (with horizontal pieces). This could be a discontinuity in the density with arbitrary large negative slope, or in the flux with $-\infty$ slope. Denote with $\tilde{\xi}$ this curve, then it is always greater than ξ_m.

Consider the jumps

$$\begin{aligned}&u_i - u_{i-1} = u(t, \xi_i(t)^+) - u(t, \xi_{i-1}(t)^+) \text{ for } i = 1, ..., m \text{ and}\\ &M - u_m \text{ if } \tilde{\xi} \text{ occurs .}\end{aligned} \tag{5.13}$$

Observe that the last one can be not properly a jump in the classical sense, that is a jump in the conserved quantity, indeed if $u_m = M$ then the curve $\tilde{\xi}$ represents a phase transition, i.e. a jump in the flux of measure $\alpha - N$ and we treat it as normal shock.

If all the jumps have the same sign, the Riemann problem determined by the interaction is solved as in the classical theory by a single jump connecting u_0 to u_m or u_0 to M if $\tilde{\xi}$ occurs. Indeed by construction all the incoming fronts are entropy admissible, which means that

$$\begin{aligned}\dot{\xi}_i &= \frac{F_{c,n}(u_i) - F_{c,n}(u_{i-1})}{u_i - u_{i-1}} \text{ for } i = 1, ..., m,\\ \text{and } \dot{\tilde{\xi}} &= \frac{\alpha - F_{c,n}(u_m)}{M - u_m}\end{aligned} \tag{5.14}$$

where $F_{c,n}$ stands for the continuous part of the piecewise affine approximation of the flux. Since all the fronts are meeting in the same point, (5.12) implies that $\dot{\tilde{\xi}} < \dot{\xi}_m < ... < \dot{\xi}_1$.

By concavity of F_n we deduce that the single jump (u_0, u_m) with speed

$$\dot{\xi} = \frac{F_c(u_m) - F_c(u_0)}{u_m - u_0} \tag{5.15}$$

or the jump (u_0, M) with speed

$$\dot{\xi} = \frac{\alpha - F_c(u_0)}{M - u_0} \tag{5.16}$$

when $\tilde{\xi}$ occurs are entropy admissible. Moreover, the total variation of u is not changing and the number of discontinuity with finite possibly large slope is decreasing at least by 1.

If at least two of the jumps in (5.13) have opposite sign, the total number of wave front can increase, but the total strength of the outgoing fronts is given by the difference in modulus between the last state from the left and the last one from the right, and the total variation of the solution decreases at least by $\frac{M}{2^n}$. Since we are considering initial data with finite total variation, this last case can occur only a finite number of times.

Theorem 5.1. *Let F be a flux satisfying assumption (4.2) and let $(\tilde{u}, \tilde{f})$ an initial data where $\tilde{u}$ is a function as (5.1) and $\tilde{f}$ satisfies (5.2). Then the Cauchy Problem (4.1) admits an entropy weak solution $(u, f) = (u(t, x), f(t, x))$ defined for all $t > 0$ with*

$$T.V.\{u(t, x)\} \leq T.V.\{\tilde{u}(t, x)\}, \quad \|u(t, \cdot)\|_{L^\infty} \leq \|\tilde{u}\|_{L^\infty} \tag{5.17}$$

$$\text{and } f(t, x) = \begin{cases} F_c(u(t, x)) & \text{if } u < M \\ \alpha \in [0, N] & \text{if } u = M \end{cases}. \tag{5.18}$$

where $\alpha = \lim_{x \to +\infty} \tilde{f}$. In particular if $(u(\bar{t}, \bar{x}), f(\bar{t}, \bar{x})) = (M, \alpha)$ with $\alpha \in [0, N)$ then the solution is constantly equal to (M, α) for a.e. $x \geq \bar{x}$.

Proof. Set M:$= \|u(t, \cdot)\|_{L^\infty}$ and let $(\tilde{u}_n, \tilde{f}_n)$ a sequence of piecewise constant initial data such that

i. $\tilde{u}_n(x) \in \{i\frac{M}{2^n}, i = 0, ..., 2^n\}$ and $\tilde{f}_n(x) = F_c\Big(i\frac{M}{2^n}\Big)$ if $\tilde{u}_n(x) = i\frac{M}{2^n}$ for $i = 0, ..., 2^n - 1$, $\tilde{f}_n(x) \in \{j\frac{N}{2^n}, j = 0, ..., 2^n\}$ if $\tilde{u}_n(x) = M$ for all x,

ii. $\|\tilde{u}_n - \tilde{u}\|_{L^1} \to 0$ and $\|\tilde{f}_n - \tilde{f}\|_{L^1} \to 0$,

iii. $T.V.\{\tilde{u}_n\} \leq T.V.\{\tilde{u}\}$,

Observe that if the initial data is congested at $+\infty$ (i.e. $\lim_{x\to+\infty}(\tilde{u}, \tilde{f})(x) = (M, \alpha)$ with $\alpha \in [0, N)$) we can approximate it only with piecewise constant couples of data also congested ($\lim_{x\to+\infty}(\tilde{u}_n, \tilde{f}_n)(x) = (M, \alpha_n)$ with $\alpha_n \in [0, N)$ and $\alpha_n \to \alpha$). On the other side, if the initial data is not congested two different situations can occurr : if $\lim_{x\to+\infty} \tilde{u} < M$ then the couple $(\tilde{u}, \tilde{f})$ is approximable only with not congested piecewise constant couples, if $\lim_{x\to+\infty}(\tilde{u}, \tilde{f})(x) = (M, N)$ we can approximate it both with free and congested data.
Consider the sequence of piecewise constant entropy solutions (u_n, f_n) of the Cauchy problem (5.4), with initial data $(\tilde{u}_n, \tilde{f}_n)$, constructed by the front tracking procedure of the previous section. Since we solve the Riemann problems from right to left the unique congested state that can survive is the one at $+\infty$ and all the previous states $(M, \alpha(x))$ with $\alpha(x) \in [0, N)$ are switched in (M, N). Therefore if $(\tilde{u}_n, \tilde{f}_n)$ is a sequence of free initial data we have that for all $n \in \mathbb{N}$ and $t > 0$, $(u_n, f_n) = (u_n, F_{c,n}(u_n))$, hence all the approximate solutions are in free region, which means that they solve the scalar conservation law with the continuous part of the flux. Instead if $(\tilde{u}_n, \tilde{f}_n)$ are congested, all the solutions (u_n, f_n) have a rigid structure: for all $n \in \mathbb{N}$ there exists a connected piecewise affine curve γ_n separating the solution into two different regions, in the first one the solution is free, i.e. $(u_n, f_n) = (u_n, F_{c,n}(u_n))$, in the other it is congested, that is $(u_n, f_n) = (M, \alpha_n)$ with $\alpha_n = \lim_{x\to+\infty} \tilde{f}_n(x)$.

When the sequence (u_n, f_n) is free, we fall in the classical analysis. Indeed

$$T.V.\{u_n(t,\cdot)\} \leq T.V.\{\bar{u}\}, \quad |u_n(t,x)| \leq M \quad \text{for all } n, x, t, \tag{5.19}$$

and

$$\|u_n(t,\cdot) - u_n(t',\cdot)\|_{L^1} \leq L|t - t'|T.V.\{\bar{u}\} \quad \text{for all } t, t' > 0 \tag{5.20}$$

hence by Theorem 2.4 in [15] there exists a subsequence of solutions $\{u_k\}_{k\in\mathbb{N}}$ converging to a function u in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Moreover $F_{c,n}$ converges uniformly to F_c. Since $(u_n, F_{c,n}(u_n))$ is an entropy solution by construction it follows that

$$\begin{aligned}
&\int\int \{|u(t,x) - k|\phi_t + q_{(k,f_k)}(u(t,x), F_c(u(t,x)))\phi_x\}dx\ dt = \\
&\lim_{n\to\infty} \int\int \{|u_n(t,x) - k|\phi_t + q_{(k,f_{n,k})}(u(t,x), F_{c,n}(u(t,x)))\phi_x\}dx\ dt \geq 0
\end{aligned} \tag{5.21}$$

for every $\mathcal{C}^1$ non negative function ϕ with compact support in the half space $t > 0$. Here $f_{n,k}$ is equal to $F_{c,n}(k)$ if $k < M$ and $f_{n,k} \in [0, M]$ if $k = M$.

Analyze now the case of (u_n, f_n) congested. Condition (5.19) still holds by construction. Hence if we prove a condition analogous to (5.20), we can conclude the L^1_{loc} convergence of a subsequence of solutions $\{u_k\}_{k\in\mathbb{N}}$ to a function u.

Let $0 < t' < t$ and let α_h be a smooth approximation of the interval $[t', t]$ such that $\lim_{h\to\infty} \alpha_h = \chi_{[t',t]}$. Define

$$\varphi_h(\tau, y) = \alpha_h(\tau)\phi(y) \tag{5.22}$$

where ϕ is any smooth function with compact support. If we insert this into the weak formulation

$$\int\int (u_n \varphi_{h,t} + f_n \varphi_{h.x})\ dx\ dt + \int \varphi_h(0,x) u_n(0,x)\ dx = 0$$

and let $h \to \infty$, we get

$$\int \phi(y)\Big(u_n(t,y) - u_n(t',y)\Big)dy + \int_t^{t'} \int \phi_y f_n\ dy\ ds = 0.$$

It follow that

$$\begin{aligned}\|u_n(t,\cdot) - u_n(t',\cdot)\|_{L^1} &= \sup_{|\phi|\le 1} \int \phi(y)\Big(u_n(t,y) - u_n(t',y)\Big)dy \\ &= -\sup_{|\phi|\le 1} \int_t^{t'} \int \phi_y f_n\ dx \\ &\le \int_t^{t'} T.V.(f_n) ds.\end{aligned}$$

As mentioned before, if (u_n, f_n) is congested, then for all $n \in \mathbb{N}$ there exist a curve γ_n separating the free region were $f_n = F_{c,n}(u_n)$ from the congested region where $f_n = \lim_{x\to\infty} \bar{f}_n$, this implies that we can complete the previous estimate with

$$\|u_n(t,\cdot) - u_n(t',\cdot)\|_{L^1} \le \Big[TV\{\bar{u}\}L + N\Big]|t'-t|. \tag{5.23}$$

The last step is to show that also f_n converges in L^1_{loc} to a function f. If we prove that the sequence of curves γ_n converges to a curve which separates the congested region (set it Ω_u) from the free one ($(\mathbb{R}^+ \times \mathbb{R}) \setminus \Omega_u$), we get also the convergence of f_n to the function f given by

$$f(t,x) = \begin{cases} F_c(u(t,x)) & \text{if } (t,x) \in (\mathbb{R}^+ \times \mathbb{R}) \setminus \Omega_u \\ \lim_{x\to+\infty} \bar{f}(x) & \text{if } (t,x) \in \Omega_u \end{cases} \tag{5.24}$$

in L^1_{loc}. The proof of the convergence of the sequence γ_n will be done in a forthcoming work on the structural stability of entropy solution for the conservation laws with flux discontinuous in the conserved quantity. If we assume it true for now we can conclude again as in the non-congested case. □

5.2 Kruzkhov's type stability result

In this section we prove the classical theorem of Kruzkhov adapted to the case of scalar conservation law with flux discontinuous in the conserved quantity. This theorem provides an estimate on the L^1 distance of two bounded entropy admissible-solutions of (4.1) and in particular it implies uniqueness of the solution with in the class of L^∞ functions.

Theorem 5.2. *Let F be a a flux satisfying assumption (4.2).*

Let $(u, f_u), (v, f_v) \in L^\infty((0,+\infty), BV(\mathbb{R}))^2$ be entropy admissible solutions of (4.1) defined for $t \ge 0$ and let M, L such that

$$|u(t,x)| \le M, \quad |v(t,x)| \le M \qquad \textit{for all } t, x \tag{5.25}$$

$$|F_c(w) - F_c(w')| \le L|w - w'| \qquad \textit{for all } w, w' \in [0, M]. \tag{5.26}$$

Then, for every $R > 0$ and $\tau > \tau_0 \ge 0$ one has

$$\int_{|x|\le R} |u(\tau,x) - v(\tau,x)|dx \le \int_{-R+L(\tau_0-\tau)}^{R} |u(\tau_0,x) - v(\tau_0,x)|dx. \tag{5.27}$$

if $\{\tau\}\times(-R,R)$ is contained in the free region of both the solutions, and

$$\int_{|x|\le R}|u(\tau,x)-v(\tau,x)|dx \le \int_{-R+L(\tau_0-\tau)}^{R}|u(\tau_0,x)-v(\tau_0,x)|dx$$
$$\frac{2L+R\sup|F_c''|\,TV(u(\tau_0,\cdot))}{F_c'(M)^2}\left(\int_{-R-L(\tau-\tau_0)}^{R}|f_u(\tau_0,z)-f_{v,0}^{\infty}|dz\right.$$
$$\left.\int_{-R-L(\tau-\tau_0)}^{R}|f_{u,0}^{\infty}-f_v(\tau_0,z)|\,dz\right)+(\tau-\tau_0)|f_{u,0}^{\infty}-f_{v,0}^{\infty}| \quad (5.28)$$

with $f_{u,0}^{\infty}=\lim_{x\to+\infty}f_{0,u}(x)$ (flux of the initial data for (u,f_u)) and $f_{v,0}^{\infty}=\lim_{x\to+\infty}f_{0,v}(x)$ (flux of the initial data for (u,f_u)) if $\{\tau\}\times(-R,R)$ intersects at least one of the conjested region for the solutions.

Proof. Let $(u,f_u),(v,f_v)$ be entropy admissible solutions of (4.1).

If $f_u=F_c(u)$ and $f_v=F_c(v)$ a.e. on $\{\tau\}\times[-R,R]$, both the solutions are in free region which means that they solve the scalar conservation law with flux given by the continuous part F_c, hence the classical Kruzkhov theorem holds.
If $u,v=M$ and $f_u,f_v\in[0,N)$ a.e. on $\{\tau\}\times[-R,R]$, then both the solutions are in the congested region, therefore Kruzkhov theorem loses meaning.
The most complex case is that one in which the trapezoid intersects the congested regions of (u,f_u) and (v,f_v) with the side $\{\tau\}\times[-R,R]$ therefore in the following we show how to treat this case.

Given two couples (k,f_k) and $(k',f_{k'})$ such that $f_k\in F(k)$ and $f_{k'}\in F(k')$ and any smooth function $\varphi(s,x,t,y)\ge 0$ with compact support contained in the set where $s,t>0$, by assumption we have

$$\int\int\{|u(s,x)-k|\varphi_s(s,x,t,y)+q_{(k,f_k)}(u(s,x),f_u(s,x))\varphi_x(s,x,t,y)\}dx\,ds\ge 0, \quad (5.29)$$

$$\int\int\{|v(t,y)-k'|\varphi_t(s,x,t,y)+q_{(k',f_{k'})}(v(t,y),f_v(t,y))\varphi_y(s,x,t,y)\}dx\,ds\ge 0. \quad (5.30)$$

Set $k=v(t,y)$ in (5.29) and integrate w.r.t y,t, analogously set $k'=u(t,y)$ in (5.30) and integrate w.r.t s,x, then adding the two results we get

$$\int\int\int\int\{|u(s,x)-v(t,y)|(\varphi_s+\varphi_t)(s,x,t,y) \quad (5.31)$$
$$+q_{(v(t,y),f_v(t,y))}(u(s,x),f_u(s,x))(\varphi_x+\varphi_y)(s,x,t,y)\}dx\,dy\,ds\,dt\ge 0,. \quad (5.32)$$

Now consider $\rho:\mathbb{R}\to[0,1]$ a $\mathcal{C}^{\infty}$ such that

$$\int_{-\infty}^{\infty}\rho(z)dz=1,\quad \rho(z)=0 \text{ for all } z\notin[-1,1],$$

and define

$$\rho_n(z)=n\rho(nz),\quad \alpha_n(z)=\int_{-\infty}^{z}\rho_n(s)ds. \quad (5.33)$$

For any non-negative function $\psi=\psi(T,X)$ compactly supported in the half-space where $T>0$, define

$$\varphi(s,x,t,y)=\psi\left(\frac{s+t}{2},\frac{x+y}{2}\right)\rho_n\left(\frac{s-t}{2}\right)\rho_n\left(\frac{x-y}{2}\right), \quad (5.34)$$

by easy computation it follows that

$$(\varphi_s+\varphi_t)(s,x,t,y)=\psi_T\left(\frac{s+t}{2},\frac{x+y}{2}\right)\rho_n\left(\frac{s-t}{2}\right)\rho_n\left(\frac{x-y}{2}\right),$$
$$(\varphi_x+\varphi_y)(s,x,t,y)=\psi_X\left(\frac{s+t}{2},\frac{x+y}{2}\right)\rho_n\left(\frac{s-t}{2}\right)\rho_n\left(\frac{x-y}{2}\right).$$

For n large enough, the support of ψ is contained in the set where $s,t>0$. Replacing the right hand side of the last two identities in (5.32) we obtain

$$\begin{aligned}\int\int\int\int\rho_n\left(\frac{s-t}{2}\right)\rho_n\left(\frac{x-y}{2}\right)\Big\{|u(s,x)-v(t,y)|\psi_T\left(\frac{s+t}{2},\frac{x+y}{2}\right)\\+\operatorname{sgn}(u(s,x)-v(t,y))[f_u(s,x)-f_v(t,y)]\psi_X\left(\frac{s+t}{2},\frac{x+y}{2}\right)\Big\}dx\,dy\,ds\,dt\\ \geq 0.\end{aligned}\tag{5.35}$$

Now introduce the following change of variable in (5.35)

$$T=\frac{s+t}{2},\quad S=\frac{s-t}{2},\quad X=\frac{x+y}{2},\quad Y=\frac{x-y}{2}$$

and compute the limit as $n\to\infty$, renaming again the variables T,X we obtain

$$\int\int\Big\{|u(t,x)-v(t,x)|\psi_t\,(t,x)+q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\psi_x\,(t,x)\Big\}dx\,dt\geq 0.\tag{5.36}$$

for all the test function ψ with compact support in the half plane $t>0$.
Let $0<\tau_0<\tau$ and $R>0$ be given. We construct a smooth approximation ψ to the characteristic function of the in Fig (5.2) by setting

$$\psi(x,t)=\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]\cdot\big[\alpha_n(x+R-L(t-\tau))-\alpha_n(x-R)\big]\tag{5.37}$$

where α_n is the same defined in (5.33). Replacing the test function (5.37) in (5.36), it follows

$$\begin{aligned}\int\int|u(t,x)-v(t,x)|\big[\rho_n(t-\tau_0)-\rho_n(t-\tau)\big]\cdot\big[\alpha_n(x+R-L(t-\tau))\\-\alpha_n(x-R)\big]dx\,dt\\ \geq\int\int\Big\{|u(t,x)-v(t,x)|\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]L\rho_n(x+R-L(t-\tau))\\-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\\ \big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]\cdot\big[\rho_n(x+R-L(t-\tau))-\rho_n(x-R)\big]\Big\}dx\,dt\\ =\int\int\Big\{\big[L|u(t,x)-v(t,x)|-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\big]\rho_n(x+R-L(t-\tau))\\+q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\,dt\end{aligned}\tag{5.38}$$

Consider now the sets Ω_u and Ω_v where respectively (u,f_u) and (v,f_v) are congested. We can split the last integral and study singularly what happens on $\Omega_u\cap\Omega_v$, $\Omega_u\triangle\Omega_v$ and outside $\Omega_u\cup\Omega_v$.

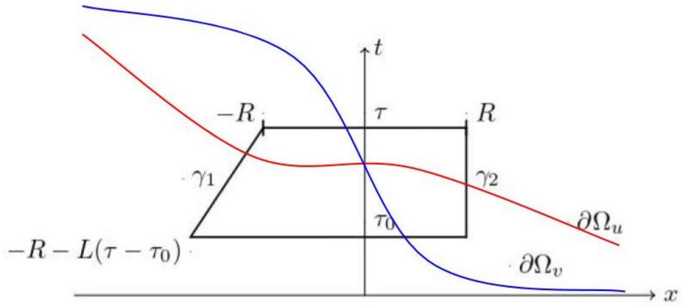


Figure 2.7: The blue and red curves separate the free from the congested region

Starting from the last one we get

$$\begin{aligned}
&\int\int_{\Omega_u\cap\Omega_v}\Big\{\big[L|u(t,x)-v(t,x)|-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\big]\rho_n(x+R-L(t-\tau))\\
&\qquad+q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\
&\quad=\int\int_{\Omega_u\cap\Omega_v}\Big\{\big[L|u(t,x)-v(t,x)|+|f_u(t,x)-f_v(t,x)|\big]\rho_n(x+R-L(t-\tau))\\
&\qquad\quad-|f_u(t,x)-f_v(t,x)|\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\
&\quad\geq-\int\int_{\Omega_u\cap\Omega_v}|f_u(t,x)-f_v(t,x)|\rho_n(x-R)\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt \qquad (5.39)
\end{aligned}$$

Consider now the set $(\mathbb{R}^+\times\mathbb{R})\setminus(\Omega_u\cup\Omega_v)$. Here both the solution are in free region. By monotony of F_c and (5.26) it follows that

$$\begin{aligned}
&\int\int_{(\mathbb{R}^+\times\mathbb{R})\setminus(\Omega_u\cup\Omega_v)}\Big\{\big[L|u(t,x)-v(t,x)|-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\big]\rho_n(x+R-L(t-\tau))\\
&\qquad+q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\
&=\int\int_{(\mathbb{R}^+\times\mathbb{R})\setminus(\Omega_u\cup\Omega_v)}\Big\{\big[L|u(t,x)-v(t,x)|-\mathrm{sgn}(u(t,x)-v(t,x))(f_u(t,x)-f_v(t,x))\big]\rho_n(x+R-L(t-\tau))\\
&\quad+\mathrm{sgn}(u(t,x)-v(t,x))(f_u(t,x)-f_v(t,x))\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\
&=\int\int_{(\mathbb{R}^+\times\mathbb{R})\setminus(\Omega_u\cup\Omega_v)}\Big\{\big[L|u(t,x)-v(t,x)|-|F_c(u(t,x))-F_c(v(t,x))|\big]\rho_n(x+R-L(t-\tau))\\
&\qquad+|F_c(u(t,x))-F_c(v(t,x))|\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\geq 0. \qquad (5.40)
\end{aligned}$$

Now the last case.

$$\begin{aligned}&\int\int_{\Omega_u\triangle\Omega_v}\Big\{\big[L|u(t,x)-v(t,x)|-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\big]\rho_n(x+R-L(t-\tau))\\&\qquad+q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\Big\}\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\&\qquad\geq\int\int_{\Omega_u\triangle\Omega_v}\Big[q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\\&\quad-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\end{aligned}\tag{5.41}$$

Assume that $(u,f_u)=(M,\delta)$ in $\Omega_u\setminus\Omega_v$ and $(v,f_v)=(M,\eta)$ in $\Omega_v\setminus\Omega_u$. Then we can rewrite (8.20) as follows

$$\begin{aligned}&\int\int_{\Omega_u\triangle\Omega_v}\Big[q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x-R)\\&-q_{(v(t,x),f_v(t,x))}(u(t,x),f_u(t,x))\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\&\qquad=\int\int_{\Omega_u\setminus\Omega_v}\Big[q_{(v(t,x),f_v(t,x))}(M,f^\infty_{u,0})\rho_n(x-R)\\&\quad-q_{(v(t,x),f_v(t,x))}(M,f^\infty_{u,0})\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\&\qquad+\int\int_{\Omega_v\setminus\Omega_u}\Big[q_{(M,f^\infty_{v,0})}(u(t,x),f_u(t,x))\rho_n(x-R)\\&\quad-q_{(M,f^\infty_{v,0})}(u(t,x),f_u(t,x))\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\&\geq-\int\int_{\Omega_u\setminus\Omega_v}|f^\infty_{u,0}-f_v(t,x)|\Big[\rho_n(x+R-L(t-\tau)+\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\\&-\int\int_{\Omega_v\setminus\Omega_u}|f_u(t,x)-f^\infty_{v,0}|\Big[\rho_n(x+R-L(t-\tau)+\rho_n(x+R-L(t-\tau))\Big]\big[\alpha_n(t-\tau_0)-\alpha_n(t-\tau)\big]dx\ dt\end{aligned}\tag{5.42}$$

Recalling that the maps $t\mapsto u(t,\cdot)$, $t\mapsto v(t,\cdot)$ are both continuous from $[0,+\infty)$ into the L^1_{loc}, if we compute the limit for $n\to+\infty$ in (5.38),(5.39),(8.31),(8.20),(5.42) we obtain

$$\begin{aligned}&\int_{|x|\leq R}|u(\tau,x)-v(\tau,x)|dx\\&\quad\leq\int_{-R+L(\tau_0-\tau)}^{R}|u(\tau_0,x)-v(\tau_0,x)|dx+\mathcal{H}^1(\mathrm{Graph}(\gamma_1)\cap(\Omega_u\cap\Omega_v))|f^\infty_{u,0}-f^\infty_{v,0}|\\&\qquad+\int_{I_1}|f_u(s,\gamma_1(s))-f^\infty_{v,0}|ds+\int_{I_2}|f_u(s,\gamma_1(s))-f^\infty_{v,0}|ds\\&\qquad\qquad+\int_{I_1'}|f^\infty_{u,0}-f_v(s,\gamma_1(s))|ds+\int_{I_2'}|f^\infty_{u,0}-f_v(s,\gamma_2(s))|ds\end{aligned}\tag{5.43}$$

where

$$\begin{aligned}I_1&=\{s\in[\tau_0,\tau]:(s,\gamma_1(s))\in(\Omega_v\setminus\Omega_u)\}\\I_2&=\{s\in[\tau_0,\tau]:(s,\gamma_2(s))\in(\Omega_v\setminus\Omega_u)\}\\I_1'&=\{s\in[\tau_0,\tau]:(s,\gamma_1(s))\in(\Omega_u\setminus\Omega_v)\}\\I_2'&=\{s\in[\tau_0,\tau]:(s,\gamma_2(s))\in(\Omega_u\setminus\Omega_v)\}\end{aligned}\tag{5.44}$$

Since $u \in L^\infty((0,\infty), BV(\mathbb{R}))$, there exist $\lim_{x\to\gamma_1(t)^+} u(t,x)$ and $\lim_{x\to\gamma_2(t)^-} u(t,x)$ for a.e. $t \in (\tau_0,\tau)$ therefore it is possible to define the following injective maps which associates to all the points in $\mathrm{Graph}(\gamma_1) \cap (\Omega_v \setminus \Omega_u)$ and $\mathrm{Graph}(\gamma_2) \cap (\Omega_v \setminus \Omega_u)$ the intersection respectively of the maximal and minimal backward generalized characteristic with the line $\{t=\tau\}$:

$$\Psi_1(s) = \gamma_1(s) + F_c'(u(s,\gamma_1(s)^+))(\tau_0 - s) \qquad \text{for a.e } s \in I_1 \tag{5.45}$$

$$\Psi_2(s) = \gamma_2(s) + F_c'(u(s,\gamma_2(s)^-))(\tau_0 - s) \qquad \text{for a.e } s \in I_2 \tag{5.46}$$

Since the backward generalized characteristic can not intersect each other, the map Ψ_1 is strictly increasing, while Ψ_2 is strictly decreasing and

$$\sup_{I_1} \Psi_1 < \inf_{I_2} \Psi_2 \tag{5.47}$$

We use these maps to change variable in the second-last integral of (5.43) and estimate the L^1-distance of the fluxes along γ_1 and γ_2 with the distance along $\{\tau_0\} \times [-R - L(\tau - \tau_0), R]$. Indeed

$$\int_{I_1} |F_c(u(s,\gamma_1(s))) - f_{v,0}^\infty| ds = \int_{\Psi_1(I_1)} |F_c(u(\tau_0,z)) - f_{v,0}^\infty| (\Psi_1^{-1})'(z) dz \tag{5.48}$$

where

$$\Psi_1^{-1}(z) = \frac{z + R - L\tau - F_c'(u(\tau_0,z))\tau_0}{L + F_c'(u(\tau_0,z))}. \tag{5.49}$$

It is possible to estimate from above the derivative of (3.11) as follows

$$\begin{aligned}(\Psi_1^{-1}(z))' &= \frac{L + F_c'(u(\tau_0,z)) - D(F_c'(u(\tau_0,z))[z + R + L(\tau+\tau_0)]}{\Big(L + F_c'(u(\tau_0,z))\Big)^2} \\ &\le \frac{L\Big(2 + \sup|F_c''|\, TV(u(\tau_0,\cdot))(\tau+\tau_0)\Big)}{F_c'(M)^2}\end{aligned} \tag{5.50}$$

Analogously we have

$$\begin{aligned}\int_{I_2} |f_u(s,\gamma_2(s)) - f_{v,0}^\infty| ds &= \int_{I_2} |F_c(u(s,R)) - f_{v,0}^\infty| ds \qquad (5.51) \\ &= \int_{\Psi_2(I_2)} |F_c(u(s,R)) - f_{v,0}^\infty|\ (\Psi_2^{-1})'(z) dz \qquad (5.52)\end{aligned}$$

with

$$\Psi_2^{-1}(z) = \frac{R - z}{F_c'(u(\tau_0,z))}. \tag{5.53}$$

Again we can estimate the derivative as follows

$$(\Psi_2^{-1}(z))' = \frac{F_c'(u(\tau_0,z)) + (R-z)D(F_c'(u(\tau_0,z))}{F_c'(u(\tau_0,z))^2}$$
$$\leq \frac{L + R \sup|F_c''|\, TV(u(\tau_0,\cdot))}{F_c'(M)^2} \tag{5.54}$$

By gathering (3.11) – (5.54) we get

$$\int_{I_1} |f_u(s,\gamma_1(s)) - f_{v,0}^\infty| ds + \int_{I_2} |f_u(s,\gamma_1(s)) - f_{v,0}^\infty| ds$$
$$\leq \frac{2L + \max\{L(\tau+\tau_0), R\} \sup|F_c''|\, TV(u(\tau_0,\cdot))}{F_c'(M)^2} \int_{-R-L(\tau-\tau_0)}^{R} |F_c(u(\tau_0,z)) - f_{v,0}^\infty|\, dz. \tag{5.55}$$

The same procedure allows to find the following inequality

$$\int_{I_1'} |f_{u,0}^\infty - f_v(s,\gamma_1(s))| ds + \int_{I_2'} |f_{u,0}^\infty - f_v(s,\gamma_1(s))| ds$$
$$\leq \frac{2L + \max\{L(\tau+\tau_0), R\} \sup|F_c''|\, TV(v(\tau_0,\cdot))}{F_c'(M)^2} \int_{-R-L(\tau-\tau_0)}^{R} |f_{u,0}^\infty - F_c(v(\tau_0,z))|\, dz. \tag{5.56}$$

If we assume that $R \geq L(\tau - \tau_0)$, we get (5.28).

□

This results immediately implies uniqueness of the solution for the Cauchy problem (4.1).

In the next section we study the Hamilton-Jacobi reformulation of problem (4.1). In fact, it provides a more concise way to study our conservation law as, as we shall see, the solution of the Hamilton-Jacobi contains all the information about the pair conserved quantity-flux.

6 The Hamilton-Jacobi reformulation

Classically, the HJ associated with a scalar conservation law is obtained by taking the Hamiltonian identically equal to the flux and in the Cauchy problem, the initial datum equal to the integral of the initial datum for the CL.

An immediate question that arises is the following: since the initial datum for a conservation law with flux discontinuous in the conserved quantity is constituted by a pair representing the density and the correspondent flux, how does the HJ inherit the information of the initial flux?

In the next we will show that in the case of a system like the following

$$\begin{cases} u_t + F(u)_x = 0 \\ (u,f)(0,x) = (u_0,f_0)(x) \end{cases} \tag{6.1}$$

with flux satisfying (4.2), $u_0 \in L^\infty(\mathbb{R})$ as in (5.1), f_0 as (5.2), the information about the initial flux f_0 is inherited by the Hamiltonian, which will be given by the flux suitably modified.

We are going to introduce a new Cauchy problem, our main goal in next sections is to study it and prove that it is exactly the Hamilton-Jacobi reformulation of (6.1).

Define $\tilde{F}$ as

$$\tilde{F}(s) = \begin{cases} F(s) & \text{if } s < M \\ \lim_{s\to M^-} F(s) & \text{if } \lim_{x\to+\infty} u_0(x) < M \\ & \text{and } s = M \\ & \\ \lim_{x\to+\infty} f_0(x) & \text{if } \lim_{x\to+\infty} u_0(x) = M \\ & \text{and } s = M \end{cases} \tag{6.2}$$

and consider the Cauchy problem

$$\begin{cases} \omega_t + \tilde{F}(\omega_x) = 0 \\ \omega(0,x) = \omega_0 \end{cases} \tag{6.3}$$

where $\omega_0 = \int_c^x u_0(s)ds$ which is clearly Lipschitz continuous.
Since the Legendre transform of $\tilde{F}$ is not well defined because of its discontinuity in M, it is not clear how to compute the solution of (6.3) using the classical Hopf-Lax formula for concave Hamiltonian, when a congestion occur at $+\infty$ in the initial data of CL. Therefore we show that we can obtain the solution as limit of a sequence of Lipschitz functions which are solutions of Cauchy problems with the same initial datum and continuous concave Hamiltonian $\tilde{F}_\varepsilon$ converging to $\tilde{F}$ in an opportune sense.

Before we recall the definition of viscosity sub- and supersolution for Hamilton-Jacobi equation with Hamiltonian defined in a dense subset of an open set of $\mathbb{R}$ (for proofs and more details see [41]).

In the following we denote with $\underline{h}$ and $\bar{h}$ the upper an lower semicontinuous envelope of a function h defined on a set L of a metric space X with values in $\mathbb{R} \cup \{\pm\infty\}$. If h_ε is a function with the same assumption as before, we define the *upper relaxed limit* as

$$\begin{aligned} \bar{h}(z) &= (\limsup_{\varepsilon\to 0}{}^* h_\varepsilon)(z) \\ &= (\limsup_{\epsilon\to 0})\{h_\delta(\eta); \eta \in L \cup B_\varepsilon(z), 0 < \eta < \varepsilon\}, \end{aligned} \tag{6.4}$$

and the *lower relaxed limit* as

$$\begin{aligned} \underline{h}(z) &= (\liminf_{\varepsilon\to 0}{}^* h_\varepsilon)(z) \\ &= (\liminf_{\epsilon\to 0})\{h_\delta(\eta); \eta \in L \cup B_\varepsilon(z), 0 < \eta < \varepsilon\}, \end{aligned} \tag{6.5}$$

Definition 6.1. Let Ω be an open set in $\mathbb{R}$, $T > 0$. Let H be a function defined on Ω with real values. Consider the Hamilton-Jacobi equation

$$\omega_t(x,t) + H(\omega_x(x,t)) = 0 \tag{6.6}$$

(i) Assume that H is upper semicontinuous in Ω with values in $\mathbb{R} \cup \{-\infty\}$. A supersolution of (6.6) is defined in the classical sense;

(ii) Assume that H is lower semicontinuous in Ω with values in $\mathbb{R} \cup \{\infty\}$. A subsolution of (6.6) is defined in the classical sense;

(iii) Assume that H is defined only on a dense subset of Ω an that $-\infty < \underline{H} \leq \bar{H} < \infty$.

- If ω is a subsolution of

$$\omega_t(x,t) + \underline{H}(\omega_x(x,t)) = 0 \tag{6.7}$$

in $\mathbb{R} \times (0,T)$, then ω is called subsolution of (6.6).

- If ω is a supersolution of

$$\omega_t(x,t) + \bar{H}(\omega_x(x,t)) = 0 \tag{6.8}$$

in $\mathbb{R} \times (0,T)$, then ω is called supersolution of (6.6).

The previous definition was introduced for the first time by H. Ishii and used after by Y.Giga for treating some HJ equations arising in mean curvature flow problems. In recent years different definitions of viscosity solution for HJ equations with discontinuous Hamiltonian were given (we refer to [23] for a comparison between the existing definitions), the last one is due to P.L. Lion and P.E.Souganidis ([59]), but the discontinuities were only in space. In our problem the situation is more delicate since we have a discontinuity in the gradient.

6.1 Existence by continuous approximation of the Hamiltonian

In order to construct a solution of (6.6) by approximating the equation, the classical locally uniform convergence of a sequence of continuous Hamiltonians cannot be used since we are dealing with a discontinuous Hamiltonian, therefore we replace it with the upper and lower relaxed limit (6.4),(6.5). The next is a stability result (due to Giga [41])) in its strong form.

Theorem 6.1. *Let Ω be an open set in $\mathbb{R}$ and $T > 0$. Let $\mathcal{O}$ be an open set in $\Omega \times (0,T)$. Assume that H_ε and H are lower (resp. upper) semicontinuous in $\bar{\Omega}$ with values in $\mathbb{R} \cup \{-\infty\}$ (resp. $\mathbb{R} \cup \{+\infty\}$) for $\varepsilon > 0$ satisfying the following inequality*

$$H \leq \liminf_{\varepsilon \to 0}{}^* H_\varepsilon \ \textit{in } \Omega \quad (\textit{resp. } H \geq \limsup_{\varepsilon \to 0}{}^* H_\varepsilon \ \textit{in } \Omega). \tag{6.9}$$

.

If ω_ε is a subsolution (resp. supersolution) of

$$\omega_t(x,t) + H_\varepsilon(\omega_x(x,t)) = 0 \tag{6.10}$$

in $\mathcal{O}$, then $\bar{u} = \limsup^*_{\varepsilon \to 0} \omega_\varepsilon$ (resp.$\liminf_{\varepsilon \to 0}{}^* \omega_\varepsilon$) is a subsolution of

$$\omega_t(x,t) + H(\omega_x(x,t)) = 0 \tag{6.11}$$

in $\mathcal{O}$ provided that $\bar{\omega}(z) < \infty$ (resp. $\underline{\omega}(z) > -\infty$) for each z in $\mathcal{O}$.

This is clearly the natural stability that has to be applied in our setting.

Some remarks on the graph convergence

We recall now some useful facts about the convergence in the sense of the graph in order to understand why it turns out to be naturally adequate for our purposes. This kind of convergence is the same used in [21] to approximate the flux and prove existence of solutions for the conservation laws. We will use the same notion for the continuous approximations of our Hamiltonian. For a systematic and detailed treatment of this topic we refer to [44].

Definition 6.2. A sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ graph converges to a function $f : \mathbb{R} \to \mathbb{R}$ if for each set $U \subset \mathbb{R} \times \mathbb{R}$ containing the graph $Gr(f)$ of the function f there is a positive integer k such that $Gr(f_n) \subset U$ for all $n \geq k$.

The key points to highlight are:

1. The graph convergence of f_n to f implies also the pointwise convergence to the same function and this is more in general true for sequence of functions defined on T_1 topological spaces.
2. Uniform convergence does not implies the graph confergence unless f_n and f are continuous.
3. The graph limit f of continuous functions f_n is almost continuous in the sense of Stallings, that is for all the open sets U containing the graph of f there is a continuous function g whose graph is contained in U.

Now we are ready to state the existence result.

Theorem 6.2. *Consider the Cauchy problem (6.3) with Hamiltonian $\tilde{F}$ satisfying assumption (4.2)-(6.2) and initial data $\omega_0(x) = \int_c^x u_0(z)dz$ with $u_0 \in L^\infty(\mathbb{R})$ as in (5.1). There exists a viscosity solution in the sense of definition (6.1) and it is Lipschitz continuous with Lipschitz constant given by $\|u_0\|_{L^\infty}$.*

Proof. Let $\{\tilde{F}_n\}_{n\in\mathbb{N}}$ be a sequence of continuous concave Hamiltonians $\tilde{F}_n : [0, M] \to \mathbb{R}$ converging to F in the sense of the graph. Thanks to the previous facts about the graph convergence it is immediate to check that the uniform limit of $\tilde{F}_n$ is given by the upper semicontinuous envelope of $\tilde{F}$ which coincides also with the upper relaxed limit as defined in (6.4), while the lower relaxed limit of $\tilde{F}_n$ coincides with the same $\tilde{F}$. For all $n \in \mathbb{N}$ the Cauchy problem

$$\begin{cases} \omega_t + \tilde{F}_n(\omega_x) = 0 \\ \omega(0, x) = \omega_0 \end{cases}$$

admits a unique viscosity solution ω_n which can be computed through the classical Hopf-Lax formula, and it is uniformly Lipschitz continuous with Lipschitz constant given by $\|u_0\|_{L^\infty}$.

The sequence of solutions $\{\omega_n\}$ is equi-Lipschits and uniformly bounded on compact sets, hence it converges locally uniformly to a Lipschitz continuous function ω.
By the stability results (6.1), ω is viscosity supersolution of the Cauchy problem with Hamiltonian given by the upper relaxed limit of $\tilde{F}_n$ (which coincides with the upper semi-continuous envelope of $\tilde{F}$), and a viscosity subsolution of the Cauchy problem with Hamiltonian given by the lower relaxed limit of $\tilde{F}_n$ (which coincides with the lower semicontinuous envelope of $\tilde{F}$) . It follows that ω is a viscosity solution of (6.3) in the sense of definition (6.1).

□

Remark 4. We remind here a stability result for the Legendre transfrom.

Theorem 6.3. *If f_n is an increasing sequence of upper semicontinuous concave functions, then the Legendre transform of the upper semicontinuous regularization of* $\lim f_n$ *is equal* $\lim f_n^*$ *(limit of the Legendre transform).*

It clarifies why it is not possible to use a classical Hopf-Lax formula for solving the problem (6.3). Indeed, the knot lies in the fact that if we compute the Legendre transform of a function satisfying assumptions (4.1) what we get that one of the upper semicontinuous envelope, hence the information of discontinuity is lost. For this result and in general for a systematic study of the Legendre transformation we refer to [50].

6.2 Hopf-Lax type formula

We first briefly recall the Hopf-Lax formula for HJ equation with concave continuous Hamiltonian and after we show how to generalize this one to the case of an Hamiltonian satisfying assumptions (4.2).
Consider the Cauchy problem for the Hamilton Jacobi equation

$$\begin{cases} v_t + H(v_x) = 0 \\ v(0,t) = \phi(x) \end{cases} \tag{6.12}$$

If the Hamiltonian is continuous, concave, depends only on the gradient and

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = -\infty, \tag{6.13}$$

and if the initial datum $\phi(x)$ is Lipschitz continuous, Hopf in 1965 established the following formula for a global Lipschitz solution of (6.12)

$$v(x,t) = \max_{y\in\mathbb{R}} \left\{ \phi(y) + tH^* \left(\frac{x-y}{t} \right) \right\} \tag{6.14}$$

with

$$H^*(q) = \inf_{z\in\mathbb{R}} \{qz - H(z)\}. \tag{6.15}$$

In the following result we state an Hopf-Lax type formula for the Cauchy problem (6.3), the solution is given for a concave discontinuous Hamiltonian.

Theorem 6.4. *Let $\tilde{F} : [0,M] \to [0,N]$ a function which satisfies (6.2), and consider the Cauchy problem (6.3) with initial datum $\omega_0(x) = \int_c^x u_0(s)ds$ and $u_0(x)$ as in (5.1). Then the function*

$$\omega(x,t) = max \left\{ \max_{y\in\mathbb{R}} \left\{ \omega_0(y) + t\tilde{F}^* \left(\frac{x-y}{t} \right) \right\}, \inf_{y\in\mathbb{R}} \{\omega_0(y) - M(y-x) - f_\infty t\} \right\} \tag{6.16}$$

with $\tilde{F}^*(q) = \inf_{z\in\mathbb{R}}\{qz - \tilde{F}(z)\}$ and $f_\infty = \lim_{x\to+\infty} f_0(x)$ (f_0 is the flux for the initial data of the conservation law), is a viscosity solution in the sense definition 6.1 and it is also Lipschitz continuous.

The inf inside the bracket represents the solution of a Cauchy problem in which the Hamiltonian is defined only in one point.

Remark 5. The way in which ω_0 is defined implies that $\omega_{0,x} \leq M$, thus the difference $\omega_0(y) - M \cdot y$ inside the inf is a decreasing function that reaches its infimum at $+\infty$. If this infimum is $-\infty$ then

$$(6.16) = \max_{y \in \mathbb{R}} \left\{ \omega_0(y) + t\tilde{F}^* \left(\frac{x-y}{t} \right) \right\} \tag{6.17}$$

otherwise

$$(6.16) = max \left\{ \sup_{y \in \mathbb{R}} \left\{ \omega_0(y) + t\tilde{F}^* \left(\frac{x-y}{t} \right) \right\}, Mx - f_\infty t + C_1 \right\} \tag{6.18}$$

where $C_1 = \lim_{y \to +\infty} \omega_0(y) - M \cdot y$.

Before proceeding with the proof of 6.4, we show that (6.16) satisfies the classical recursive property of the Hopf-Lax formula, this will play an important role proving that (6.16) is a viscosity solution of (6.3).

Theorem 6.5. *For each $x \in \mathbb{R}$ and $0 < s < t$ we have*

$$\omega(x,t) = max \left\{ \max_{y \in \mathbb{R}} \left\{ \omega(y,s) + (t-s)\tilde{F}^* \left(\frac{x-y}{t-s} \right) \right\}, \inf_{y \in \mathbb{R}} \{ \omega(y,s) + M(x-y) - f_\infty(t-s) \} \right\}. \tag{6.19}$$

Proof. 1. Assume that at time s

$$\omega(y,s) = \omega_0(z) + s\tilde{F}^* \left(\frac{z-y}{s} \right)$$

for $z \in \mathbb{R}$. Since $\tilde{F}^*$ is concave and $\frac{z-x}{t} = \left(1 - \frac{s}{t}\right) \frac{x-y}{t-s} + \frac{s}{t} \frac{y-z}{s}$ we have

$$\tilde{F}^* \left(\frac{z-x}{t} \right) \geq \left(1 - \frac{s}{t}\right) \tilde{F}^* \left(\frac{x-y}{t-s} \right) - \frac{s}{t} \tilde{F}^* \left(\frac{y-z}{s} \right).$$

Therefore

$$\begin{aligned} \omega(x,t) &\geq \omega_0(z) + t\tilde{F}^* \left(\frac{z-x}{t} \right) \\ &\geq \omega_0(z) + (t-s)\tilde{F}^* \left(\frac{x-y}{t-s} \right) + s\tilde{F}^* \left(\frac{y-z}{s} \right) \\ &= \omega(y,s) + (t-s)\tilde{F}^* \left(\frac{x-y}{t-s} \right). \end{aligned} \tag{6.20}$$

Assume now that at time s

$$\omega(y,s) = \omega_0(z) - M(z-y) - f_\infty s$$

from (6.16) we get

$$
\begin{aligned}
\omega(x,t) &\geq \inf_{h\in\mathbb{R}} \{\omega_0(h) - M(h-x) - f_\infty t\} \\
&= \inf_{h\in\mathbb{R}} \{\omega_0(h) - Mh + My - My + Mx + f_\infty s - f_\infty s - f_\infty t\} \\
&= \omega(y,s) - M(y-x) - f_\infty (t-s). \qquad (6.21)
\end{aligned}
$$

Combining (6.20) and (6.21) it follows that

$$
\omega(x,t) \geq \max\left\{ \sup_{y\in\mathbb{R}} \left\{ \omega(y,t) + (t-s)\tilde{F}^*\left(\frac{x-y}{t-s}\right)\right\}, \inf_{y\in\mathbb{R}} \{\omega(y,t) - M(y-x) - f_\infty(t-s)\}\right\}.
$$

2. Let $z \in \mathbb{R}$ such that

$$
\omega(x,t) = \omega_0(z) + t\tilde{F}^*\left(\frac{y-z}{s}\right)
$$

and set $y = \left(1 - \frac{s}{t}\right)x + \frac{s}{t}z$, then $\frac{y-x}{t-s} = \frac{z-x}{t} = \frac{z-x}{t}$.
Consequently

$$
\begin{aligned}
\omega(y,s) + (t-s)\tilde{F}^*\left(\frac{x-y}{t-s}\right) &\geq (t-s)\tilde{F}^*\left(\frac{x-y}{t-s}\right) + s\tilde{F}^*\left(\frac{y-z}{s}\right) + \omega_0(z) \\
&\geq \omega_0(z) + t\tilde{F}^*\left(\frac{x-z}{t}\right) = \omega(x,t). \qquad (6.22)
\end{aligned}
$$

If $z \in \mathbb{R}$ is such that

$$
\omega(x,t) = \omega_0(z) - M(z-x) - f_\infty t,
$$

it easily follow that

$$
\begin{aligned}
\omega(y,s) - M(y-x) - f_\infty(t-s) &\geq \inf_{h\in\mathbb{R}} \{\omega_0(h) - M(h-y) - f_\infty s\} - M(y-x) - f_\infty(t-s) \\
&= \inf_{h\in\mathbb{R}} \{\omega_0(h) - Mh + Mx - My + My - f_\infty s + f_\infty s - f_\infty t\} \\
&= \omega_0(z) - M(z-x) - f_\infty t = \omega(x,t). \qquad (6.23)
\end{aligned}
$$

From (6.22) and (6.23) it follows that

$$
\omega(x,t) \leq \max\left\{ \sup_{y\in\mathbb{R}} \left\{ \omega(y,t) + (t-s)\tilde{F}^*\left(\frac{x-y}{t-s}\right)\right\}, \inf_{y\in\mathbb{R}} \{\omega(y,t) - M(y-x) - f_\infty(t-s)\}\right\}
$$

which concludes the proof. ☐

Proof. Now we can prove 6.4. Lipschitz continuity is immediate, indeed

$$\omega_1(x,t) = \max_{y\in\mathbb{R}} \left\{ \omega_0(y) + t\tilde{F}^* \left(\frac{x-y}{t} \right) \right\} \tag{6.24}$$

is the unique viscosity solution of the Cauchy problem

$$\begin{cases} \omega_t + F_c(\omega_x) = 0 \\ \omega(0,t) = \omega_0(x) \end{cases}$$

with F_c is the constinuous part of the Hamiltonian. Therefore its Lipschitz continuity is well known by the classical theory, while the function

$$\omega_2(x,t) = \inf_{y\in\mathbb{R}} \{\omega(y,t) - M(y-x) - f_\infty t\} \tag{6.25}$$

is clearly affine, hence also Lipschitz. Since sup of two Lipschitz functions is still Lipschitz, we can conclude. In order to show that (6.16) is a viscosity solution, consider a test function $\varphi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^+)$ and assume that $\omega - \varphi$ has a minimum in (x_0, t_0). For (x,t) close enough to (x_0, t_0) with $t < t_0$ we have

$$(\omega - \varphi)(x_0, t_0) \le (\omega - \varphi)(x, t)$$

which implies that

$$\varphi(x_0,t_0) - \varphi(x,t) \ge \omega(x_0,t_0) - \omega(x,t).$$

Set $t_0 - t = h$ and $x = x_0 - hq$, by (6.5) it follows that

$$\varphi(x_0,t_0) - \varphi(x,t) \ge (t - t_0)\tilde{F}^* \left(\frac{x_0 - x}{t_0 - t} \right) = h\,(q)\,.$$

Thus dividing by h and doing the limit for $h \to 0$, we get

$$\varphi_t(x_0,t_0) + \varphi_x(x_0,t_0) \cdot q - (q) \ge 0$$

which holds for all $q \in \mathbb{R}$, in particular taking the inf for $q \in \mathbb{R}$ of the sum on the left side in the previous inequality we have

$$\varphi_t(x_0,t_0) + \bar{F}(\varphi_x(x_0,t_0)) \ge 0,$$

hence ω is a viscosity supersolution.

Assume now that $\omega - \varphi$ has a local maximum in (x_0, t_0), we need to distinguish two cases:

- $\omega(x_0,t_0) = \omega_1(x_0,t_0) \ge \omega_2(x_0,t_0)$
- $\omega(x_0,t_0) = \omega_2(x_0,t_0) > \omega_1(x_0,t_0)$.

Consider the first case and suppose by contradiction that

$$\varphi_t(x_0,t_0) + \underline{F}(\varphi_x(x_0,t_0)) \ge \theta > 0,$$

Since $\underline{F} \le \bar{F}$ it follows also that

$$\varphi_t(x_0,t_0)+\bar{F}(\varphi_x(x_0,t_0))\geq\theta>0. \tag{6.26}$$

By continuity of $\nabla\varphi$ and $\bar{F}$, inequality (6.26) holds also in a neighborhood of (x_0,t_0). Let $(x_1,t-h)$, with $h>0$ be a point in this neighborhood such that by (6.5)

$$\omega(x_0,t_0)=\omega(x_1,t-h)+h(\frac{x_0-x_1}{h})$$

.

Maximality of the point (x_0,t_0) implies that

$$\omega(x_0,t_0)-\omega(x_1,t-h)\geq\varphi(x_0,t_0)-\varphi(x_1,t-h).$$

Thanks to easy computations we get the following chain of inequalities

$$\begin{aligned}\varphi(x_0,t_0)-\varphi(x_1,t-h)=&h\int_0^1[\varphi_x(sx_0+(1-s)x_1,t_0+(s-1)h)\frac{x_0-x_1}{h}\\&+\varphi_t(sx_0+(1-s)x_1,t_0+(s-1)h)]ds\\&\geq h\int_0^1[\varphi_t(\cdots)+\bar{F}(\varphi_x(\cdots))+(\frac{x_0-x_1}{h})]ds\\&>h(\frac{x_0-x_1}{h})+\theta h\\&=\omega(x_0,t_0)-\omega(x_1,t-h)+\theta h.\end{aligned}$$

which gives an absurd since (x_0,t_0) is a maximum of $\omega-\varphi$.

The case $\omega(x_0,t_0)=\omega_2(x_0,t_0)>\omega_1(x_0,t_0)$ is immediate since by continuity of ω_1 and ω_2, $\omega_2-\omega_1$ must be positive also in a neighborhood of (x_0,t_0) where therefore $\omega(x,t)$ is an affine function of the form $Mx-f_\infty t+c$ with constant given by $\inf_{y\in\mathbb{R}}\{\omega_0(y)-y\}$. Hence $\nabla\varphi(x_0,t_0)=\nabla\omega(x_0,t_0)=(M,-f_\infty)$ which states that ω is also a subsolution. □

6.3 Variational formulation

As in the classical case of continuous and concave Hamiltonians, the Hopf-Lax type formula (6.16) that we stated before can be obtained by minimizing a cost functional. Therefore in the next we introduce a minimization problem that admits a unique minimizer given exactly by our Hopf-Lax type formula.

We shall first introduce a convenient way of writing the Hamiltonian $\tilde{F}$. Let F_c be the upper semicontinuous envelope of $\tilde{F}$, it is easy to check that it is given by F_c introduced in (4.16) and consider F_d ('d' stands for 'degenerate') the function given by

$$F_d=\begin{cases}+\infty & u\in[0,M)\\ f_\infty & u=M\end{cases}, \tag{6.27}$$

we observe immediately that

$$\tilde{F}=\min\{F_c,F_d\} \tag{6.28}$$

Hence a viscosity solution of (6.3) is also a viscosity supersolution of

$$\begin{cases}\omega_t+F_c(\omega_x)=0\\ \omega(0,x)=\omega_0\end{cases}\quad\text{(CPC)}\qquad\qquad\begin{cases}\omega_t+F_d(\omega_x)=0\\ \omega(0,x)=\omega_0\end{cases}\quad\text{(CPD)}$$

It follows that (6.16) can be derived through superposition of two variational problems of different nature, a maximization and minimization one.
The maximization problem is naturally associated to the Cauchy problem (CPC) and the minimization one to (CPD). Unfortunately, the competition between the concavity of F_c and the convexity of F_d does not allow a formulation with a single Lagrangian.

Optimization problem For $u_0 \in L^\infty(\mathbb{R})$ as (5.1) consider $\omega_0(x) = \int_c^x u_0(z)dz$, we want to analyze the following two variational problems

$$\text{maximize} \quad \mathcal{C}(x(\cdot)) \quad \text{and} \quad \text{minimize} \quad \mathcal{D}(\gamma(\cdot)) \tag{6.29}$$

where

$$\mathcal{C}(x(\cdot)) = \omega_0(x(0)) + \int_0^{\bar{t}} {F_c}^* \left(\dot{x}(t)\right) dt \tag{6.30}$$

and the maximum is sought among all the absolutely continuous curves

$$x : [0, \bar{t}] \to \mathbb{R} \quad \text{such that} \quad x(\bar{t}) = \bar{x}; \tag{6.31}$$

while

$$\mathcal{D}(\gamma(\cdot)) = \omega_0(x(a)) + \int_0^{\bar{t}} {F_d}^* \left(\frac{\dot{x}(s)}{\dot{t}(s)}\right) \dot{t}(s)ds. \tag{6.32}$$

and the minimum is studied among all the absolutely continuous curves

$$\gamma : [a, b] \to \mathbb{R} \cup \{\infty\} \times \mathbb{R}^+ \tag{6.33}$$
$$s \mapsto (x(s), t(s))$$

such that

$$\gamma(a) = (x(a), \bar{t}), \;\; \gamma(b) = (\bar{x}, \bar{t}) \text{ and } \dot{t} = 0 \text{ for a.e. } s \in [a, b]. \tag{6.34}$$

We attack the first variational problem using the direct method of calculus of variations. So let $\{x_n\}_{n\in\mathbb{N}}$ a maximizing sequence for (6.30), i.e

$$\lim_{n\to\infty} \left\{ \omega_0(x_n(0)) + \int_0^{\bar{t}} {F_c}^* \left(\dot{x}_n(t)\right) dt \right\} = A, \tag{6.35}$$

Applying the Jensen inequality to the concave function ${F_c}^*$ we obtain

$$\int_0^{\bar{t}} {F_c}^* \left(\dot{x}_n(t)\right) dt \le \bar{t} {F_c}^* \left(\frac{\bar{x} - x_n(0)}{\bar{t}} \right) \tag{6.36}$$

hence

$$\lim_{n\to\infty} \left\{ \omega_0(x_n(0)) + \bar{t} {F_c}^* \left(\frac{\bar{x} - x_n(0)}{\bar{t}} \right) \right\} = A, \tag{6.37}$$

The following argument shows that we can assume

$$\frac{\bar{x} - x_n(0)}{\bar{t}} \in [F_c'(M), F_c'(0)] \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.38}$$

indeed if we consider $x_n(0) \le x_n^- = \bar{x} - \bar{t} F_c'(0)$ then

$$\omega_0(x_n(0)) + \bar{t} {F_c}^* \left(\frac{\bar{x} - x_n(0)}{\bar{t}} \right) \le \omega_0(x_n^-) + \bar{t} {F_c}^* \left(\frac{\bar{x} - x_n^-}{\bar{t}} \right)$$

while if $x_n(0) \geq x_n^+ = \bar{x} - \bar{t}F_c'(M)$ we get

$$\omega_0(x_n(0)) + \bar{t}{F_c}^*\left(\frac{\bar{x} - x_n(0)}{\bar{t}}\right) \leq \omega_0(x_n^+) + M(x_n(0) - x_n^+) + \bar{t} \cdot M \cdot \frac{\bar{x} - x_n^+}{\bar{t}} - N \cdot \bar{t}$$
$$= \omega_0(x_n^+) + M(\bar{x} - x_n^+) - N \cdot \bar{t} = \omega_0(x_n^+) + \bar{t}{F_c}^*\left(\frac{\bar{x} - x_n^+}{\bar{t}}\right).$$

Since $\{x_n(0)\}_{n\in\mathbb{N}}$ is bounded, we can estract a subsequence converging to a point y. This implies

$$\omega_0(y) + \bar{t}{F_c}^*\left(\frac{\bar{x} - y}{\bar{t}}\right) = A \tag{6.39}$$

which means that the affine function

$$x(t) = y + t\frac{\bar{x} - y}{\bar{t}} \tag{6.40}$$

is an optimal control curve for the variational problem (6.30) and that the minimum is the first term in (6.16).

For the second variational problem it is enough to make some observations:

1. The Legendre Transform for ${F_d}^*$ is just given by the linear map $p \mapsto M \cdot p - f_\infty$ hence (6.32) can be rewritten simply as

$$\mathcal{D}(\gamma(\cdot)) = \omega_0(x(a)) + M(\bar{x} - x(a)) - f_\infty\bar{t}; \tag{6.41}$$

2. $\omega_0(x(a)) - M \cdot x(a)$ is a decreasing map, hence it reaches the infimum when $x(a) \to +\infty$ which means that a minimizing sequence of control curves converges to the semiline $[\bar{x}, +\infty)$. This implies that the the minimum is given by the second term inside the brackets in (6.16).

7 Selection principle

We have abundantly talked about the fact that a solution for a conservation law with flux discontinuous in the conserved quantity, as (4.1), is not given by a single function from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$, but by a pair representing the conserved quantity and the correspondent flux which can't be uniquely determined just by checking the value of F in u. However this definition is not constructive and the few cases of equation with this kind of discontinuity analyzed by now have been studied considering regularization of the flux.
In the next we show that passing through the Hamilton-Jacobi reformulation we can deal directly with the scalar conservation law without using continuous approximation of the flux. Indeed the gradient of the solution for the HJ contains both the informations given by (4.3)-(4.4).

Proposition 7.1. *Let ω be the solution of (6.3) given by the Hopf-Lax type formula (6.16). Then the couple $(\omega_x, -\omega_t)$ is the unique solution of (4.1) law in the sense of definition 3.2.*

Proof. We start observing that by Theorem 6.4 the solution ω of the Cauchy problem, given by the Hopf-Lax type formula, is Lipschitz continuous, hence by Radamacher's theorem, is differentiable for a.e. (t, x) and the couple $(\omega_x, -\omega_t)$ is well defined.
Chose any test function v smooth with compact support in $\mathbb{R}^+ \times \mathbb{R}$, multiply the HJ equation $\omega_t + F(\omega_x) = 0$ by v_x and integrate over $\mathbb{R}^+ \times \mathbb{R}$:

$$\int_0^\infty \int_{-\infty}^\infty [\omega_t + F(\omega_x)]\, v_x dxdt = 0. \tag{7.1}$$

Observe

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty \omega_t v_x dxdt &= -\int_0^\infty \int_{-\infty}^\infty \omega v_{tx}\; dxdt - \int_{-\infty}^\infty \omega v_x\; dx|_{t=0} \\ &= \int_0^\infty \int_{-\infty}^\infty \omega_x v_t\; dxdt - \int_{-\infty}^\infty \omega_x v\; dx|_{t=0}.\end{aligned}$$

Remember now that $\omega(0,x) = \omega_0(x) = \int_c^x u_0(z)dz$, thus $\omega_x(0,x) = u_0(x)$ for a.e. x. It follows that

$$\int_0^\infty \int_{-\infty}^\infty \omega_t v_x dxdt = \int_0^\infty \int_{-\infty}^\infty \omega_x v_t\; dxdt - \int_{-\infty}^\infty u_0 v|_{t=0}\; dx.$$

Finally if we substitute the last identity into (7.1) and use that $\omega_x = u$ and $F(u) = -\omega_t$ we get

$$\int_0^\infty \int_{-\infty}^\infty u v_t dxdt - \int_0^\infty \int_{-\infty}^\infty \omega_t v_x dxdt = \int_{-\infty}^\infty u_0 v|_{t=0} dx \tag{7.2}$$

We conclude that $(\omega_x, -\omega_t)$ satisfies the definition 4.1 and is exactly the unique entropy solution of (4.1). Indeed where the solution to the HJ is given by the first term in the Hopf-lax formula, then the solution of conservation law is in free region, while where the solution to the HJ is given by the second term, the solution for the conservation law is in the congested region. This tells us that the set of points where the two terms of the Hopf-Lax formula are equal corresponds exactly to the curve γ mentioned in the proof of existence by front-tracking. □

Remark 6. Essentially the previous result shows that $-\omega_t$ selects in a natural way the correct value of the flux of the solution for the conservation law (that is the second function in the couple), implementing what from now on we will call the **Selection Principle**.

7.1 Some examples

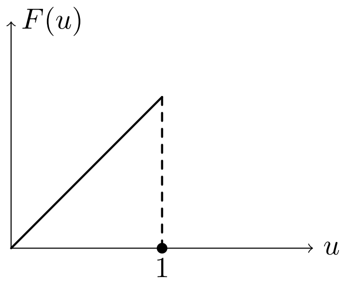
In this section we want to expose two significant examples that clearly illustrate what has been proved so far. We show in parallel the resolution of two Cauchy problems associated to the same conservation law whose flux satisfies assumption (4.2) and the correspondent Hamilton-Jacobi equation. For these two examples it is evident that the solution of the Hamilton-Jacobi provides the correct value of the flux for the solution of the conservation law, thus implementing the Selection Principle.

Example 1 Consider the Cauchy problem (4.1) with flux given by
and initial datum

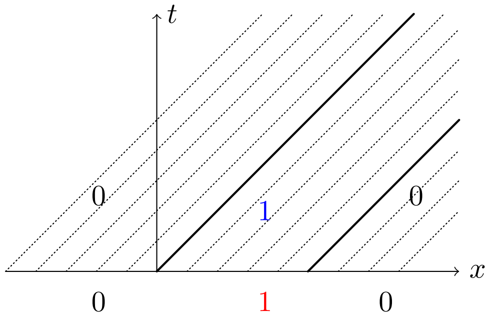
$$u_0^1(x) = \begin{cases} 0 & x \le 0 \\ 1 & 0 < x \le 1 \\ 0 & x > 1 \end{cases} \qquad f_0^1(x) = 0 \quad \text{for all } x \in \mathbb{R}$$

It is possible to compute the explicit solution using continuous approximation of the flux and considering the limit of the sequence of solutions. It turns out to be equal to (7.5)

The solution apparently has two parallel shocks, but there is an hidden shock of speed $-\infty$ in $(1, 0)$ which represents a phase transition (change of color between the two 1).



$$F(u) := \begin{cases} u & 0 \leq u < 1 \\ 0 & u = 1 \end{cases} \tag{7.3}$$

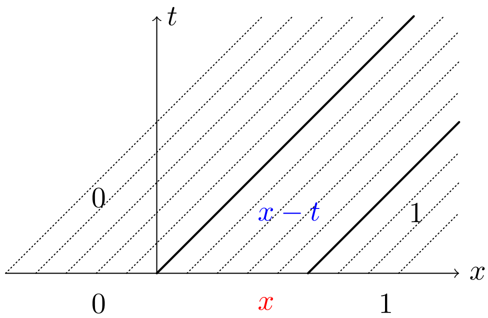


$$u^1 := \begin{cases} 0 & x < t \\ 1 & t < x < t+1 \\ 0 & t+1 < x \end{cases} \tag{7.4}$$

Consider now the Hamilton-Jacobi version of the Cauchy problem (4.1) with initial data given by

$$\omega_0^1(x) = \int_0^x u_0^1(s)ds = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

It admits a unique solution which can be computed though the Hopf Lax Type formula and given by



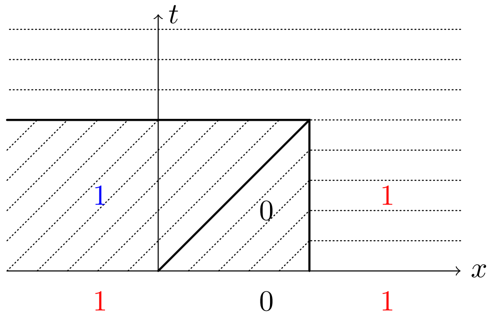
$$\omega^1 := \begin{cases} 0 & x < t \\ x - t & t < x < t+1 \\ 1 & t+1 < x \end{cases} \tag{7.5}$$

Example 2 The Cauchy problem (4.1) with flux (7.3) and initial data

$$u_0^2(x) = \begin{cases} 1 & x \leq 0 \\ 0 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

admits a unique solution given by (7.7).

The solution ω^2 presents apparently two shocks starting at time $t = 0$ which meet at time $t = 1$ and generate a new shock of speed $-\infty$. But again we have a third shock of speed $-\infty$ starting at time $t = 0$ in $x = 0$. Moreover:



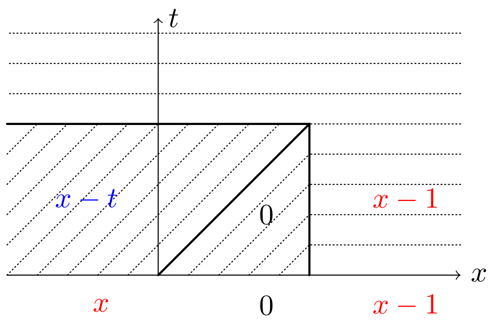
$$\omega^2 := \begin{cases} 1 & x < t \text{ and } t < 1 \\ 0 & t < x < 1 \\ 1 & 1 < x \text{ or } t > 1 \end{cases} \tag{7.6}$$

- we have two phase transition in different times denoted by the change of color of the value 1 for the solution (7.7);
- the characteristic with infinite speed are stopped only entering in shocks which connect with value of the solution different from the point of discontinuity of the flux;
- for the same value of the solution we have two values of the flux.

Passing to the Cauchy problem with the Hamilton-Jacobi equation with initial data

$$\omega_0^2(x) = \int_0^x u_0^2(s)ds = \begin{cases} x & x \leq 0 \\ 0 & 0 < x \leq 1 \\ x-1 & x > 1 \end{cases}$$

we get the solution (7.7)



$$\omega^2 := \begin{cases} x-t & x < t \text{ and } t < 1 \\ 0 & t < x < 1 \\ x-1 & 1 < x \text{ or } t > 1 \end{cases} \tag{7.7}$$

Conclusively if $\omega_x^{1,2} = 1$ then $-\omega_t^{1,2} = 1$ or 0, thus we can distinguish the value of the flux and where there is a phase transition.

7.2 Comparison with Herty-Jorres-Piccoli approach

We proved that the solution to the Hamilton-Jacobi reformulation of the Cauchy problem for the scalar conservaiton law with flux satisfying assumption(4.2), is Lipschitz continuous, hence differentiable a.e. and its gradient gives the solution for the conservation law in the sense of definition (4.1) (Selection principle). In this section we compare our analysis with the one introduced by Herty and al. in [49]. In this work the authors studied the Cauchy problem (4.1) by modifying the dynamic in the way we have already described in Section 2.3.

The following correspondences follow immediately

$$(u(x,t),\mathcal{F}) \text{ solution of } (2.4) \iff (\omega_x(t,x),-\omega_t(t,x)) \text{ solution of } (4.1)$$
$$\text{with } \omega(t,x)=\omega_c(t,x)$$

$$(u(x,t),\mathcal{C}) \text{ solution of } (2.4) \iff (\omega_x(t,x),-\omega_t(t,x)) \text{ solution of } (4.1)$$
$$\text{with } \omega(t,x)=\omega_d(t,x)$$

We remark that what we are saying is that we have congestion only when the solution of the HJ corresponds to the solution of (CPD), while we are in free phase if it is solution of (CPC)

In the next table we summarize what said by now about the correlation among the definition (4.1), the Selection Principle and the reformulation (2.4).

	Conservation laws	**Hamilton Jacobi**	**Phase transition**
1	$(u(t,x),f(t,x))$ solution, $0 \leq u(t,x) < M$ and $f(t,x) \in F(u(t,x))$	$(u(t,x),f(t,x))=$ $(\omega_x(t,x),-\omega_t(t,x))$ where $\omega(t,x)=\omega_c(t,x)$	$(u(t,x),\mathcal{F})$ solution
2	$(u(t,x),f(t,x))$ solution, $u(t,x) = M$ and $f(t,x) \in F(u(t,x))$	$(u(t,x),f(t,x))=$ $(\omega_x(t,x),-\omega_t(t,x))$ where $\omega(t,x)= \omega_c(t,x) \vee \omega_d(t,x)$	$(u(t,x),\mathcal{S})$ solution with $\mathcal{S}=\mathcal{F}$ or $\mathcal{C}$

So it is evident from the table that

- The definition of solution for **Conservation Laws** given in Section 4 helps to develop an independent theory for the case of fluxes discontinuous in in the conserved quantity, however it is not enough to use it if we want to get practical information about the state of the conveyor belt

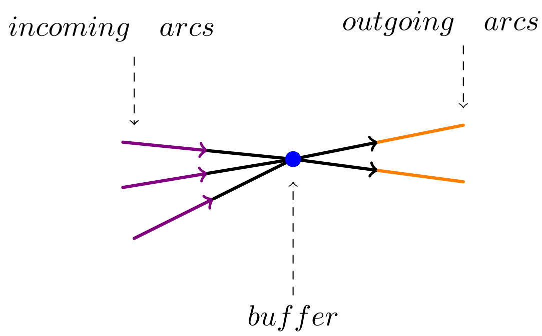
- The **Phase Transition** approach introduced by Herty et al. and based on the transformation of the dynamics is ingenious but forces us to check two variables instead of one

- The **Hamilton Jacobi** approach is clean and allows us to study the conservation law through the associated HJ without worrying about the flux value that can be determined thanks to the Selection Principle.

8 The Junction problem for Supply Chain with buffer

The analysis of conveyor belt models constitute the fundamental element through which we can study more complex structures such as the flow of goods in a supply chain. Below we introduce a model to describe supply chain on network, in particular we zoom on a single junction point, that is a node of the network in which we suppose to be present a buffer where products coming from incoming chains are stored waiting to be placed in the outgoing chains.

Consider a family of $n + m$ arcs joining at a node. We denote with indices $i \in \{1, ..., m\} = \mathcal{I}$ the in coming arcs, with $j \in \{1, ..., n\} = \mathcal{O}$ the outgoing arcs.



On the $k - th$ arc the density of parts is described by the scalar conservation law

$$u_t + F_k(u)_x = 0$$

with $t > 0$, $x \in [-\infty, 0]$ for incoming and $x \in [0, +\infty)$ for outgoing arcs. On the flux F_k we impose the following assumptions:

$$s \to F_k(s) \text{ smooth on } [0, M_k), \quad \partial_s^2 F_k \leq 0, \quad F(0) = 0 \text{ and } F_k(M_k) = [0, N_k]. \tag{8.1}$$

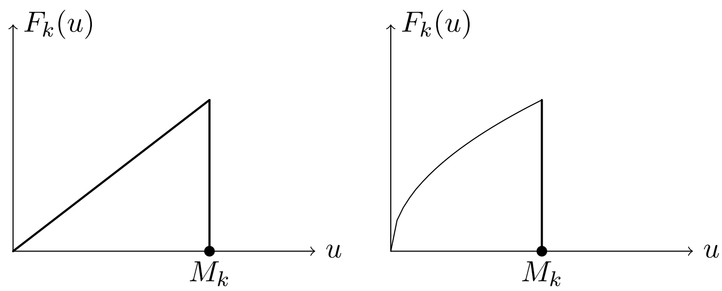


Figure 2.8: Two example of fluxes which satisfy (4.2)

The initial data on each arc is given by the couple

$$(u_k(0, x), f_k(0, x)) = (u_{k,0}(x), f_{k,0}(x)) \quad k = 1, ..., n + m \tag{8.2}$$

satisfying the following assumption

$$\text{for } i \in \mathcal{I} \quad u_{i,0} \in L^1(\mathbb{R}^-), \qquad f_{i,0}(x) = \begin{cases} F_{i,c}(u_{i,0}(x)) & \text{if } u_{i,0}(x) < M_i \\ \in [0, N_i] & \text{if } u_{i,0}(x) = M_i \end{cases} \tag{8.3}$$

and

$$\text{for } j \in \mathcal{O} \quad \exists c_j \in [0, M_j] \text{ such that } (u_{j,0} - c_j) \in BV(\mathbb{R}^+),$$

$$f_{j,0}(x) = \begin{cases} F_{j,c}(u_{j,0}(x)) & \text{if } u_{j,0}(x) < M_j \\ \in [0, N_j] & \text{if } u_{j,0}(x) = M_j \end{cases}, \quad f_{j,0} \in BV(\mathbb{R}^+) \tag{8.4}$$

where $F_{c,\cdot}$ stands again for the continuous part of the flux. We need to add a suitable set of boundary conditions in order to determine a unique solution. These boundary conditions provide additional constraints on the traces of the good densities

$$\bar{u}_k(t) = \begin{cases} \lim_{x \to 0^-} u_k(t, x) & k \in \mathcal{I} \\ \lim_{x \to 0^+} u_k(t, x) & k \in \mathcal{O} \end{cases} \tag{8.5}$$

and the correspondent fluxes

$$\bar{f}_k(t) = \begin{cases} \lim_{x \to 0^-} f_k(t, x) & k \in \mathcal{I} \\ \lim_{x \to 0^+} f_k(t, x) & k \in \mathcal{O} \end{cases} \tag{8.6}$$

near the junction.
Since we want to consider a realistic model, we assume that in the buffer in the junction point has limited capacity. The state of the buffer is represented by a function $q : [0, \infty) \to [0, M^b]$ which is the amount of goods from the incoming arcs waiting to enter in one of the outgoing arcs with equal probability.
Following [39], define

$$\alpha_i := \begin{cases} f_i & \text{if } u_i < M_i \\ N_i & \text{if } u_i = M_i \end{cases} \qquad i \in \mathcal{I} \tag{8.7}$$

the maximum possible flux at the end of an incoming chain, and

$$\alpha_j := \begin{cases} f_j & \text{if } u_j = M_j \\ N_j & \text{if } u_j < M_j \end{cases} \qquad j \in \mathcal{O} \tag{8.8}$$

We require that the incoming fluxes are given by

$$\bar{f}_i(t) = \min\left\{\alpha_i, \frac{M^b - q(t)}{|\mathcal{I}|}\right\} \quad i \in \mathcal{I}, \tag{8.9}$$

and the outgoing fluxes by

$$\bar{f}_j(t) = \begin{cases} \alpha_j & \text{if } q(t) > 0 \\ \min\left\{\alpha_j, \frac{\sum_{i \in \mathcal{I}} \bar{f}_i(t)}{|\mathcal{O}|}\right\} & \text{if } q(t) = 0 \end{cases} \tag{8.10}$$

Moreover we introduce the quantity

$$h(q(t)) = \min\left\{N_i, \frac{M - q(t)}{\mathcal{I}}\right\} \tag{8.11}$$

which is the maximum allowed incoming flux in the buffer at time t.

Conservation of the total number of parts implies that

$$\dot{q}(t) = \sum_{i \in \mathcal{I}} \bar{f}_i(t) - \sum_{j \in \mathcal{O}} \bar{f}_j(t). \tag{8.12}$$

Therefore the final model consists of a system of Conservation Laws coupled with an Ordinary differential equation

$$\begin{cases} \partial_t u(t,x) + \partial_x F_k(u(t,x)) = 0 \quad k \in \mathcal{I} \cup \mathcal{O} \\ \dot{q}(t) = \sum_{i\in\mathcal{I}} \bar{f}_i(t) - \sum_{j\in\mathcal{O}} \bar{f}_j(t) \\ (u_k(0,x), f_k(0,x)) = (u_{k,0}(x), f_{k,0}(x)) \\ q(0) = c \quad q_0 \geq 0 \end{cases} \tag{JP}$$

8.1 The Cauchy problem on a Junction

In this section we study the Cauchy problem (JP).

Incoming chain. For $i \in \mathcal{I}$ and $x \leq 0$, consider as initial data for the Hamilton Jacobi equation the function $\omega_{i,0}$ given by

$$\omega_{i,0} = \int_{-\infty}^{x} u_{i,0}(z)dz \tag{8.13}$$

Then the explicit solution is given by

$$\omega_i(t,x) = \max\Big\{ \max_{y\leq 0} \Big\{\omega_{i,0}(y) + tF^*_{i,c}\Big(\frac{x-y}{t}\Big)\Big\}, \\ Mx + \max_{0\leq t'\leq t,\ y\leq 0} \Big\{\omega_{i,0}(y) + t'F^*_{i,c}\Big(\frac{-y}{t'}\Big) - \int_{t'}^{t} h(q(s))ds\Big\}\Big\} \tag{8.14}$$

The function $\omega_i(t,x)$ can be interpreted as the amount of goods which at time t are in the region $(-\infty, x]$ if the incoming chain i. The difference

$$\omega_{i,0}(0) - \omega_i(t,0)$$

measure the amount of goods exited from the incoming arc in the time interval $[0,t]$.

Now we have to determine the total number of particles at time that have reached the buffer before the time t and that are waiting inside the buffer to enter the outgoing chains. Let $\eta_i(t)$ be defined as

$$\eta_i(t) = \max\Big\{k \in (-\infty, 0];\ \int_k^0 u_{i,0}(z)dz = \omega_{i,0}(0) - \omega_i(t,0)\Big\}. \tag{8.15}$$

It represents the position at time 0 of the particle that reaches the buffer at time t, then the number of particles waiting to enter in the outgoing chain j is given by

$$G_j(t) = \frac{q_0 + \sum_{i\in\mathcal{I}} \int_{\eta_i(t)}^0 u_{i,0}(z)dz}{|\mathcal{O}|}. \tag{8.16}$$

Outgoing chain For $j \in \mathcal{O}$ and $x > 0$ we consider as initial data for the Hamilton Jacobi equation the function

$$\omega_{j,0}(x) = \int_0^x u_{j,0}(z)dz. \tag{8.17}$$

Then the solution is given by

$$\omega_j(t,x) = \max\Big\{ \max_{y\geq 0} \Big\{\omega_{j,0}(y) + tF^*_{j,c}\Big(\frac{x-y}{t}\Big)\Big\},\ Mx + \lim_{y\to+\infty} \Big(\omega_{j,0}(y) - My\Big) - f^{\infty}_{j,0} \cdot t, \\ \max_{0\leq t'\leq t} \Big\{ - G_j(t) + (t-t')F^*_{j,c}\Big(\frac{x}{t-t'}\Big)\Big\}\Big\} \tag{8.18}$$

with G_j as in (8.16) and $f_{j,0}^{\infty} = \lim_{x\to+\infty} f_{j,0}$.

Here the quantity $\omega_{j,0}(x) - \omega_i(t,x)$ is the number of goods which crossed the buffer in the time interval $[0,t]$. So at time t, the length of the buffer is

$$q(t) = \sum_{i\in\mathcal{O}} G_j(t) + \sum_{j\in\mathcal{O}} \omega_j(t,0) \tag{8.19}$$

Definition 8.1. We say that the functions u_k (with $k \in \mathcal{I}\cup\mathcal{I}$) and q provide an admissible solution to the Cauchy problem (JP) with junction condition (8.9)-(8.10) if there exist Lipschitz continuous functions $\omega_k = \omega_k(t,x)$ such that

1. For $i \in \mathcal{I}$ ω_i satisfies (8.14);
2. For $j \in \mathcal{O}$ ω_j satisfies (8.18);
3. q satisfies (8.19);

Remark 7. The term Mx of the second maximization problem in (8.14) is a cost associated to an horizontal piece of optimal trajectory as proved in the analysis of conservation laws for the single conveyor belt and represents a congestion arriving from the boundary. We will find it also on the explicit formulation for the outgoing chain where it represents the congestion coming from $+\infty$.

8.2 Uniqueness of the solution

Here we show that the Cauchy problem (JP) associated with the model for processing goods near a buffer is well posed.

Theorem 8.1. *Consider the flux functions satisfying (4.2) and let the initial data as (8.3)-(8.4), then (JP) has a unique admissible solution in the sense of definition 8.1 globally defined for all* $t \geq 0$.

Proof. We show that on a small interval $[0,T]$ the solution of the Cauchy problem (JP) can be obtained as the unique fixed point of a contractive map. Let $t \mapsto q(t)$ be a Lipschitz continuous function whose Lipschitz constant is given by

$$L_q = \sum_{k\in\mathcal{I}\cup\mathcal{O}} N_k \tag{8.20}$$

and consider the sequence of maps

$$q \mapsto (\omega_i)_{i\in\mathcal{I}} \mapsto (G_j)_{j\in\mathcal{O}} \mapsto (\omega_j)_{j\in\mathcal{O}} \mapsto \Lambda(q). \tag{8.21}$$

where ω_i are the solutions for the Hamilton Jacobi equation on the incoming chain given in (8.14), the function F_j is the same in (8.16), ω_j are the solutions for the Hamilton Jacobi equation on the outgoing chain (8.18) and $\Lambda(q)$ is defined in (8.19).

In order to prove that (8.21) is a contractive map, consider two Lipschitz continuous functions q and $\bar{q}$ and define

$$\delta = \sup_{t\in[0,T]} \big|q(t) - \bar{q}(t)\big|. \tag{8.22}$$

For all $i \in \mathcal{I}$ the functions $h_i(q)$ are Lipschitz continuous with Lipschitz constant given by $\frac{L}{|\mathcal{I}|} = \frac{\max_{i\in\mathcal{I}} L_{c,i}}{|\mathcal{I}|}$, hence by (8.14) we can deduce the estimate

$$\sup_{i\in\mathcal{I},\ t\in[0,T],\ x\leq 0} \big|\omega_i(t,x) - \bar{\omega}_i(t,x)\big| \leq \frac{L}{|\mathcal{I}|} T\delta \tag{8.23}$$

which in particular implies

$$\sup_{i\in\mathcal{I},\ t\in[0,T]} \big|\omega_i(t,0)-\bar{\omega}_i(t,0)\big| \le \frac{L}{|\mathcal{I}|}T\delta. \tag{8.24}$$

Combining (8.24) with (8.16) we obtain

$$\big|G_j(t)-\bar{G}_j(t)\big| \le \frac{\sum_{i\in\mathcal{I}}\big|\omega_i(t,0)-\bar{\omega}_i(t,0)\big|}{|\mathcal{O}|} \le \frac{L}{|\mathcal{O}|}T\delta. \tag{8.25}$$

Now, if we pass on the outgoing chain, by (8.18) it follows

$$\sup_{j\in\mathcal{I},\ t\in[0,T]} \big|\omega_j(t,0)-\bar{\omega}_j(t,0)\big| \le \frac{L}{|\mathcal{O}|}T\delta. \tag{8.26}$$

In the last step of the composition we find

$$\begin{aligned}\big|\Lambda(q)(t)-\Lambda(\bar{q})(t)\big| &= \Big|\sum_{i\in\mathcal{O}}F_j(t)+\sum_{j\in\mathcal{O}}\omega_j(t,0)-\sum_{i\in\mathcal{O}}\bar{G}_j(t)-\sum_{j\in\mathcal{O}}\bar{\omega}_j(t,0)\Big| \\ &\le \sum_{j\in\mathcal{O}}\Big|G_j(t)-\bar{G}_j(t)\Big|+\sum_{j\in\mathcal{O}}\Big|\omega_j(t,0)-\bar{\omega}_j(t,0)\Big| \le 2LT\delta. \end{aligned} \tag{8.27}$$

Therefore, by taking $T=\frac{1}{4L}$ we can conclude that

$$\sup_{t\in[0,T]}\big|\Lambda(q)(t)-\Lambda(\bar{q})(t)\big| \le \frac{1}{2}\sup_{t\in[0,T]}\big|q(t)-\bar{q}(t)\big|. \tag{8.28}$$

which means that Λ is a strict contraction.

The next goal is to show that the map $t\mapsto\Lambda(q)(t)$ is Lipschitz continuous. Thus consider $i\in\mathcal{I}$, $x\le 0$ and $0<t_1\le t_2$. Assume that $\omega_i(t_1,x)=\omega_{i,0}(y)+t_1F^*_{i,c}\Big(\frac{x-y}{t_1}\Big)$ for some $y\le 0$. The concavity of $F_{i,c}$ implies that

$$\omega_i(t_2,x)\ge\omega_{i,0}(y)+t_2F^*_{i,c}\Big(\frac{x-y}{t_2}\Big)\ge\omega_{i,0}(y)+t_1F^*_{i,c}\Big(\frac{x-y}{t_1}\Big)\ge\omega_i(t_1,x)+(t_2-t_1)F^*_{i,c}(0).$$

It follow that

$$0\le\omega_i(t_1,x)-\omega_i(t_2,x)\le(t_2-t_1)N_i. \tag{8.29}$$

Analogously if

$$\omega_i(t_1,x)=Mx+\omega_{i,0}(y)+t'F^*_{i,c}\Big(\frac{-y}{t'}\Big)-\int_{t'}^{t_1}h_i(q(s))ds$$

for some $y\le 0$ and $0\le t'\le$, then

$$\omega_i(t_2,x)\ge Mx+\omega_{i,0}(y)+t'F^*_{i,c}\Big(\frac{-y}{t'}\Big)-\int_{t'}^{t_2}h(q(s))ds=\omega_i(t_1,x)-\int_{t_1}^{t_2}h_i(q(s))ds.$$

which implies again (8.29). Now letting $x\to 0$ and remembering that $h_i(q)\in[0,N_i]$ we conclude that $t\mapsto\omega_i(t,0)$ is Lipschitz continuous with constant N_i.

In the same way we can deduce that for all $j \in \mathcal{O}$, ω_j satisfies (8.29) with $0 \le t_1 < t_2$. Now by (8.15) and (8.16) we get

$$|G_j(t_1) - G_j(t_2)| \le \frac{1}{|\mathcal{O}|} \sum_{i \in \mathcal{I}} |\omega_i(t_1, 0) - \omega_i(t_2, 0)| \le \frac{(t_2 - t_1)}{|\mathcal{O}|} \sum_{i \in \mathcal{I}} N_i.$$

This implies that the function

$$t \mapsto \Lambda(q)(t) = \sum_{i \in \mathcal{O}} F_j(t) + \sum_{j \in \mathcal{O}} \omega_j(t, 0) \tag{8.30}$$

is Lipschitz continuous with Lipschitz constant given by $\sum_{k \in \mathcal{O} \cup \mathcal{I}} N_k = L_q$.

Consider the set $\mathcal{Z} \in C([0, T], \mathbb{R})$ of all the Lipschitz function q with Lipschitz constant L_q and such that $q(0) = q_0$. Given initial data (8.13),(8.17) and q_0, the map $q \mapsto \Lambda(q)(t)$ is a strict contraction of $\mathcal{Z}$ in itself, thus there exists a unique fixed point which provides the unique admissible solution to the Cauchy problem in the time interval $[0, T]$.

To complete the proof we just need to show that the solution can be extended on a sequence of time intervals $[T_n, T_{n+1}]$ with $T_{n+1} \to +\infty$ as n goes to $+\infty$. By gathering (8.11) with the equation for the evolution of the buffer, we obtain the following differential inequality

$$\frac{d}{dt}(M - q(t)) \ge -(M - q(t))$$

whose solution is given by

$$M - q(t) \ge e^{-t}(M - q_0). \tag{8.31}$$

This means that $M - q(t)$ is always greater then zero for finite time and the buffer is full only when $t \to +\infty$. Hence we can repeat the previous construction for all the sequence of time intervals such that $T_{n+1} - T_n = \frac{1}{4L}$.

□

8.3 Variational formulation of the Junction Problem

This section is devoted to the study of two optimization problems. In both the problems we show that the optimal solutions are piecewise affine functions, and the correspondent value functions are exactly the Hopf-Lax type formulas introduced in the previous section for the boundary value problem associated to the H-J equation on the incoming and outgoing chain.

Optimization problem on the incoming chain. For any $i \in \mathcal{I}$, given an initial data $\omega_{i,0}$ as in (8.13), and the length of the buffer q, consider the two variational problem below

$$\text{maximize: } J_i^1(x(\cdot)) := \omega_{i,0}(x(0)) + \int_0^{\bar{t}} F_{i,c}^*(\dot{x}(t))dt \tag{8.32}$$

$$\text{maximize: } J_i^2(x(\cdot)) := \omega_{i,0}(x(0)) + \int_0^{\bar{t}} L(x(t), \dot{x}(t))dt + M_i\bar{x} \tag{8.33}$$

where the payoff function L in (8.33) is defined as

$$L(x(t), \dot{x}(t)) = \begin{cases} F_{i,c}^*(\dot{x}(t)) & \text{if } x(t) < 0 \\ -h_i(q(t)) & \text{if } x(t) = 0 \end{cases}. \tag{8.34}$$

The maximum in (8.32) is sought among all the absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

$$x(\bar{t}) = \bar{x}, \qquad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}] \tag{8.35}$$

while in (8.33) among all the absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

$$x(\bar{t}) = 0, \qquad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]. \tag{8.36}$$

We focus for a moment on the second variational problem. The following lemma is the key tool in order to understand the structure of the optimal control curve.

Lemma 8.1. *Consider an absolutely continuous function $x : [0, \bar{t}] \mapsto (-\infty, 0]$ which satisfies assumptions (8.36). Define*

$$a = \min\{t \in [0, \bar{t}] : x(t) = 0\}, \qquad b = \max\{t \in [0, \bar{t}] : x(t) = 0\}$$

and the curve

$$\tilde{x}(t) = \begin{cases} 0 & \text{if } x \in [a, b], \\ x(t) & \text{if } x \neq [a, b]. \end{cases} \tag{8.37}$$

Then $\tilde{x}$ satisfies assumption (8.36) and achieves a larger payoff.

Proof. The proof ca be obtained by adapting the same of Lemma 1 in [17] in the case of $g_i = F^*_{i,c}$. ☐

Proposition 8.2. *Let a continuous function $t \mapsto q(t)$ be given, together with the initial data $\omega_{i,0}$. For $i \in \mathcal{I}$, define ω_i as in (8.14) and consider the variational problems (8.32),(8.33). Then the following holds.*

1. *For every time $\bar{t} > 0$ and $\bar{x} < 0$ there exist optimal solutions $x^*_1(\cdot)$ of (8.32) and $x^*_2(\cdot)$ of (8.33). The solutions are piecewise affine and satisfy $\dot{x}^*_{1,2}(t) \in [F'_{i,c}(M_i), F'_{i,c}(0)]$ for a.e. $t \in [0, \bar{t}]$ such that $x^*_{1,2}(t) < 0$.*

2. *The maximum attainable values of (8.32) and (8.33) are respectively the first and second terms inside the bracket of (8.14).*

3. *The maximum between the two value functions of (8.32) and (8.33) is a function ω_i such that $(\omega_{i,x}, -\omega_{i,t})$ is an entropy weak solution to the conservation law*

$$\partial_t u(t, x) + \partial_x F_i(u(t, x)) = 0 \tag{8.38}$$

with initial data (8.3) and boundary fluxes (8.9).

Proof. We know by the classical theory of Hamilton-Jacobi equation with concave Hamiltonian that the variational problem (8.32) admits a unique maximum given by

$$\max_{y \leq 0} \left\{ \omega_{i,0}(y) + t F^*_{i,c}\left(\frac{x - y}{t}\right) \right\}, \tag{8.39}$$

that is the first term inside the bracket in (8.14). We also know that the optimal trajectory $x^*_1(\cdot)$ is given by a straight line connecting the point $(\bar{t}, \bar{x})$ to a point $(0, y)$ with $y \leq 0$ and such that $\dot{x}^*_1(\cdot) \in [F'_{i,c}(M_i), F'_{i,c}(0)]$.
Analogously, by Proposition 1. in [17], we find that the functional J_2 admits a unique maximum given by

$$M_i \bar{x} + \max_{0 \leq t' \leq t,\ y \leq 0} \left\{ \omega_{i,0}(y) + t' F^*_{i,c}\left(\frac{-y}{t'}\right) - \int_{t'}^{t} h(q(s)) ds \right\} \tag{8.40}$$

which is the second term inside the bracket in (8.14). In this case the optimal trajectory has the form

$$x_2^*(t) = \begin{cases} \frac{t'-t}{t'}y & \text{if } t \in [0,t'] \\ 0 & \text{if } t \in [0,t'] \end{cases} \tag{8.41}$$

where $y \leq 0$ and $0 \leq t' \leq t$. The fact that the set of times where $x_2^* = 0$ in (8.41) is connected is a directed consequence of Lemma 8.1. Also in this case it is known that $\dot{x}_2^*(\cdot) \in [F'_{i,c}(M_i), F'_{i,c}(0)]$ when $x_2^*(\cdot) < 0$. Then point 1. and 2. are proved, it remains only to prove point 3.

Observe that $\omega_i(t,x) = \max\{8.39, 8.40\}$ is a Lipschitz continuous function, this comes from the Lipschitz continuity of $\omega_{i,0}$ and $h_i(q)$. Moreover it provides a a viscosity solution to the Hamilton-Jacobi equation

$$\omega_t + \tilde{F}_i(t, \omega_x) = 0 \tag{8.42}$$

where $\tilde{F}_i$ is defined as follows

$$\tilde{F}_i(t,s) = \begin{cases} F_i(s) & \text{if } s < M_i \\ \\ h(q(t)) & \text{if } s = M_j \end{cases}. \tag{8.43}$$

Therefore $(\omega_{i,x}, -\omega_{i,t})$ is an entropy solution of (8.38) on the domain $\mathbb{R}^+ \times \mathbb{R}^-$.

We need just to check that the boundary condition are satisfied. Consider the curve $\gamma_i(s) = (t_i(s), x_i(s))$ contained in $\mathbb{R}^+ \times \mathbb{R}^-$ which divides the open set into two regions:

- A_{id} the set of points where the solution to the Hamilton Jacobi equation is given by the value function of (8.32);

- A_b the set of points where the solution is given by the value function of (8.33).

Consider the following two cases.

CASE 1. If $\gamma_j(s) = (t_j(s), 0)$, then for each $x \leq 0$ there exists a point y_x such that $\omega_i(t,x) = \omega_{i,0}(y_x) + tF^*_{i,c}\left(\frac{x-y_x}{t}\right)$. The map $x \mapsto y_x$ is non decreasing hence there exist the limit $y_x \to y$ as $x \to 0^-$ where $y < 0$. By continuity we find that

$$\omega_i(t,0) = \omega_{i,0}(y) + tF^*_{i,c}\left(\frac{-y}{t}\right)$$

thus the optimal trajectory has speed $\frac{-y}{t} > 0$ and

$$\bar{u}_i(t) = \lim_{x\to 0^-} u_i(t,x) = \lim_{x \to 0^-} \omega_{i,x}(t,x) \tag{8.44}$$

is well defined. It is clear also that $\bar{u}_i(t) \leq M_i$. Therefore the maximum outgoing flux is $\bar{f}_i(t) = F_{i,c}(\bar{u}_i(t)) = \omega_{i,t}(t,0)$. To complete the proof, it remains to show that

$$\bar{f}_i(t) \leq h_i(q(t)) = \min\left\{N_i, \frac{M^b - q(t)}{|\mathcal{I}|}\right\}. \tag{8.45}$$

Assume that (8.45) fails, by the continuity of q there exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ small such that

$$\bar{f}_i(t) > h_i(q(\tau)) + \delta_0 \quad \text{for all } \tau \in [t - \varepsilon_0, t]. \tag{8.46}$$

In this case the trajectory

$$x^b(s) = \begin{cases} \left(1 - \frac{s}{t-\varepsilon}\right)y & \text{if} s < t - \varepsilon \\ 0 & \text{if} s \geq t - \varepsilon \end{cases} \tag{8.47}$$

achieves a larger payoff in (8.33) for $\varepsilon \in (0, \varepsilon_0]$ sufficiently small. The proof of this fact is identically similar to the CASE 1 of Proposition 1 in [17].

Hence condition (8.9) is satisfied.

CASE 2. If $\gamma_j(s) = (t_j(s), x(s))$ with $x(s) > 0$ then $(\bar{t}, \bar{x}) \in A_b$ implies that we can find a δ such that $(t, x) \in A_b$ for all $t \in [\bar{t} + \delta, \bar{t} - \delta]$ and $x \in [\bar{x}, 0)$ and

$$\omega_i(t,x) = M_i x + \omega_{i,0}(y) + t' F^*_{i,c}\Big(\frac{-y}{t'}\Big) - \int_{t'}^{t} h(q(s))ds \tag{8.48}$$

with the same $y \le 0$ and $t' \ge 0$. Therefore $\omega_{i,t}(t,x) = h(q(t))$ and letting $(t,x) \to (t,0)$ we find $\bar{f}_i(t) = h(q(t))$ which proves (8.9). □

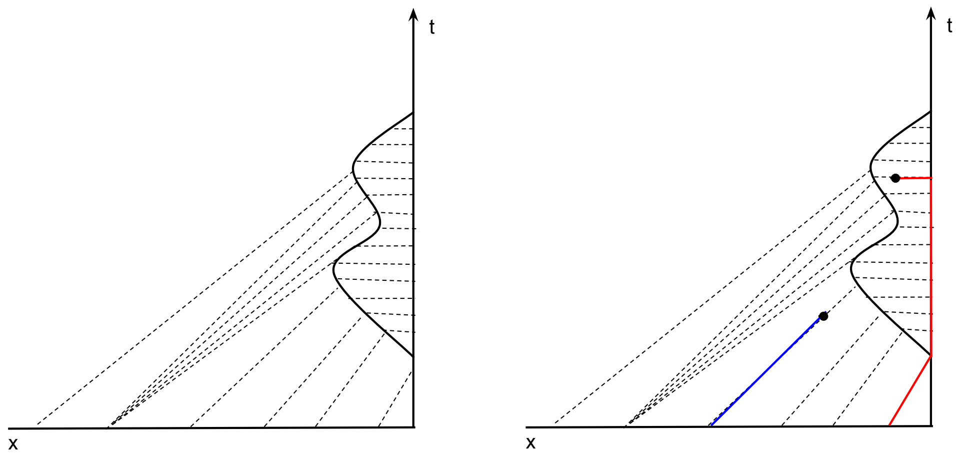


Figure 2.9: On the left we see the curve which separates the two families of optimal trajectory on the incoming chain. On the right we show an example of trajectory given by the straight line and one passing trough the boundary.

Optimization problem on the outgoing chain. For any $j \in \mathcal{O}$ and fixed point $(\bar{t}, \bar{x})$ with $t > 0$ and $x > 0$, given an initial data $\omega_{j,0}$ as in (8.17) and G_j in (8.16), consider the following two variational problem

$$\text{maximize: } J^1_j(x(\cdot)) := \omega_{i,0}(x(0)) + \int_0^{\bar{t}} F^*_{j,c}(\dot{x}(t))dt \tag{8.49}$$

$$\text{maximize: } J^2_j(x(\cdot)) := -G_j(\tau) + \int_\tau^{\bar{t}} F^*_{j,c}(\dot{x}(t))dt. \tag{8.50}$$

The maximum is sought among all the absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

$$x(\bar{t}) = \bar{x}, \qquad x(t) \ge 0 \quad \text{for all } t \in [0, \bar{t}]. \tag{8.51}$$

Proposition 8.3. *For $j \in \mathcal{O}$, let $t \mapsto G_j(t) \ge 0$ be given together with the initial data $\omega_{j,0}$ as in (8.17) and consider the variational problem (8.49),(8.50). Then the following holds.*

1. *For every time $\bar{t} > 0$ and $\bar{x} > 0$ there exist optimal solutions $x^*_1(\cdot)$ of (8.49) and $x^*_2(\cdot)$ of (8.50). The solutions are affine and satisfy $\dot{x}^*_{1,2}(t) \in [F'_{i,c}(M_i), F'_{i,c}(0)]$ for a.e. $t \in [0, \bar{t}]$.*

2. *The maximum attainable values of (8.49) and (8.50) are respectively the first and third terms inside the bracket of (8.18).*

3. *The maximum among the two value functions of (8.49), (8.50) and*

$$M\bar{x} + \lim_{y\to+\infty}\Big(\omega_{j,0}(y) - M_j y\Big) - f^{\infty}_{j,0}\cdot t \tag{8.52}$$

is a function ω_j such that $(\omega_{j,x}, -\omega_{j,t})$ is an entropy weak solution of the conservation law

$$\partial_t u(t,x) + \partial_x F_j(u(t,x)) = 0 \tag{8.53}$$

with initial (8.4) and boundary condition (8.10).

Proof. As in the proof of Proposition 8.2, we know that the optimal trajectory $x_1^*(\cdot)$ of (8.49) is a straight line $(\bar{t},\bar{x})$ to $(0,y)$ for some $y \geq 0$ and $\dot{x}_1^*(t) \in [F'_{i,c}(M_i), F'_{i,c}(0)]$, moreover the value function is the first term inside the bracket in (8.18).

By Proposition 3 of [17] we have also that (8.50) admits a unique maximum and the optimal trajectory is given by a is a straight line $x_2^*(\cdot)$ connecting $(\bar{t},\bar{x})$ to $(\tau,0)$ for some $\tau \geq 0$, $\dot{x}_2^*(t) \in [F'_{j,c}(M_i), F'_{j,c}(0)]$ and the value function is the third term inside the bracket in (8.18). Hence both point 1. and 2. are proved.

Denoted with ω_j the maximum among the value functions of (8.49), (8.50) and (8.52), it is a Lipschitz continuous function. Moreover it is immediate to check that ω_j provides a viscosity solution to the Hamilton-Jacobi equation

$$\omega_t + \tilde{F}_j(\omega_x) = 0 \tag{8.54}$$

where $\tilde{F}_j$ is defined as follows

$$\tilde{F}_j(s) = \begin{cases} F_j(s) & \text{if } s < M_j \\ \lim_{s\to M_j^-} F_j(s) & \text{if } \lim_{x\to+\infty} u_{j,0}(x) < M_j \\ & \text{and } s = M \\ \\ \lim_{x\to+\infty} f_{j,0}(x) & \text{if } \lim_{x\to+\infty} u_{j,0}(x) = M_j \\ & \text{and } s = M_j \end{cases} . \tag{8.55}$$

By the theory developed in Section 6, the couple $(\omega_{j,x}, -\omega_{j,t})$ exists a.e. and provides a weak entropy solution to the scalar conservation law (8.53) on the open domain $\mathbb{R}^+\times\mathbb{R}^+$. The initial data $\omega_j(0,x) = \omega_{j,0}(x)$ is also satisfied. So the last step is to show that also the boundary conditions are satisfied. Thus consider the curve $\gamma_j(s) = (t_j(s), x_j(s))$ contained in $\mathbb{R}^+\times\mathbb{R}^+$ which divides the open set into two regions:

- A_{id} the set of points where the solution to the Hamilton Jacobi equation is given by the value function of (8.49) or by (8.52);
- A_b the set of points where the solution is given by the value function of (8.50).

We need to analyze two cases:

CASE 1. If $\gamma_j(s) = (t_j(s), 0)$, then $\omega_j(0,t) = \lim_{y\to+\infty}\Big(\omega_{j,0}(y) - M_j y\Big) - f_{j,\infty}t$ (we omit the dependence of t on s). This follows by point 1, since the optimal trajectory for (8.49) has strictly positive speed. We have also that

$$(\omega_{j,x}(t,0), -\omega_{j,t}(t,0)) = (u_j(t,0), -f_j(t,0)) = (M_j, -f_{j,\infty}). \tag{8.56}$$

If $q(t) > 0$ then (8.10) is automatically satisfied. If $q(t) = 0$ then $\omega_j(t,0) = -F_j(t)$. Assume that at time t both $\omega_j(t,0)$ and $F_j(t)$ are differentiable, we have

$$0 = \lim_{h\to 0}\frac{\omega_j(t+h,0) - \omega_j(t,0)}{h} - \omega_{j,t}(t,0) \geq \lim_{h\to 0}\frac{-F_j(t+h) + F_j(t)}{h} + f_{j,\infty} = -F'_j(t) + f_{j,\infty}.$$

On the other hand, by (8.16), for a.e. $t > 0$,

$$G_j'(t) = \frac{1}{|\mathcal{O}|} \sum_{i \in \mathcal{I}} f_i(t,0)$$

$$f_j(t,0) = f_{j,\infty} \leq \frac{1}{|\mathcal{O}|} \sum_{i \in \mathcal{I}} f_i(t,0)$$

. therefore (8.10) is again satisfied.

CASE 2. If $\gamma_j(s) = (t_j(s), x(s))$ with $x(s) > 0$ then by definition in every point $(t,x) \in A_b$ the solution to the Hamilton-Jacobi equation (8.54) is given by the maximum of (8.50). If $(\bar{t}, \bar{x}) \in A_b$, it follows that $(\bar{t}, x) \in A_b$ for all $x \in (0, \bar{x})$ and there exists a time $\tau_x \in [0, \bar{t})$ such that

$$\omega_j(\bar{t}, x) = -G_j(\tau_x) + (\bar{t} - \tau_x) F^*_{j,c}\Big(\frac{x}{\bar{t} - \tau_x}\Big). \tag{8.57}$$

By the characterization of the attainable profiles on the outgoing chain for concave fluxes proved in Chapter 1, we know that for $x \in (0, \bar{x})$, the map $x \mapsto \tau_x$ is non increasing, which implies that $\lim_{x \to 0^+} \tau_x$ is well defined and coincides $\bar{t}$, otherwise we would have that there exists an optimal trajectory with speed less than $F'_{j,c}(M_j)$ and this contradicts point 1. We conclude that the following limits exist

$$\lim_{x \to 0} \frac{x}{\bar{t} - \tau_x} = \ell \quad \text{with} \quad \ell \in [F'_{j,c}(M), F'_{j,c}(0)] \tag{8.58}$$

and by (8.57), (8.58)

$$\omega_j(\bar{t}, 0) = \lim_{x \to 0^+} -G_j(\tau_x) + (\bar{t} - \tau_x) F^*_{j,c}\Big(\frac{x}{\bar{t} - \tau_x}\Big) = -G_j(\bar{t}). \tag{8.59}$$

Now if $q > 0$, this means that the particles which reached the buffer before the time $\bar{t}$ are not exited, hence there is at list one outgoing chain congested and the outgoing flux is maximum on the chains where there is not congestion, hence

$$\omega_{j,t}(\bar{t}, 0) = N_j. \tag{8.60}$$

If $q = 0$, then all the particles reaching the buffer at time $\bar{t}$ try to enter the outgoing chain with the maximum flux allowed by the capacity, hence

$$\omega_{j,t}(\bar{t}, 0) = \min\Big\{N_j, \frac{M - q(\bar{t})}{\mathcal{O}}\Big\}. \tag{8.61}$$

We conclude the boundary conditions are satisfied.

□

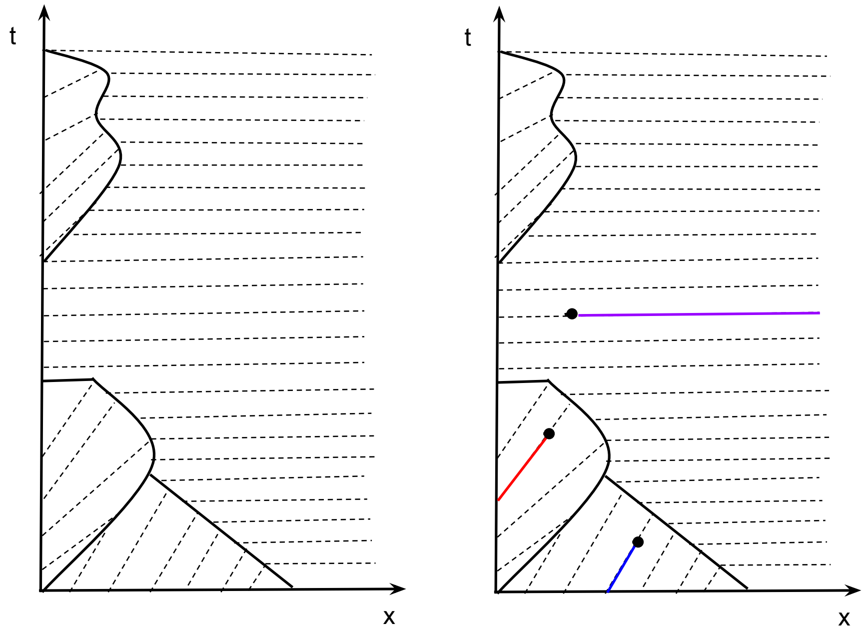


Figure 2.10: On the left we see again the curve which separates the tree families of optimal trajectory on the outgoing chain. On the right we show examples of trajectory arriving respectively from the boundary, from $t = 0$ and from $+\infty$.

Chapter 3

Soil Searching by an Artificial Root

1 Bio-inspired robotics

1.1 Soil exploration and plants

Soil exploration machanisms are widely studied thanks to their vast range of applications which vary from searching for resources in the soil, to planetary exploration and rescuing people trapped by cataclysms. These technologies can be divided into two main species: invasive including drilling devices and not invasive as radars. Drilling devices used today are able to create straight and vertical drilling holes but they are not ideal for small subterranean research because of the power and heavy equipment required. Moreover they are inappropriate in unstable lands, even more so in the presence of human lives to be saved. Robotic technology for soil exploration and monitoring has been scarcely developed compared with that one available for exploration and monitoring above ground and underwater (above all talking about autonomous systems). This motivates the growing interest in biologically inspired solutions [61]. Several penetration devices have been developed which mimic techniques used by animals such as wood wasps, locusts, and clam, but plants are the most efficient at soil exploration among living organisms, since they are able to grow and adapt inside the medium. Plant roots have the ability to find low-resistance pathways and exploit cracks in the soil, overcoming soil penetration resistance.

Figure 3.1: Roots growing in the rocks.

Their behaviour is incredible adaptive, above all interacting with the environment and most of these abilities are a consequence of apical growth, i.e. a growth process wherein new cells are added at the root apex by mitosis while mature cells of the root remain stationary and in contact with the soil.

1.2 Continuum robots inspired by plants

Continuum robots are biologically inspired robots with capability to bend anywhere along their length, this structural advantage allows them to reach spaces that are difficult by traditional rigid-link structured to negotiate. A sub-class is given by the thin-continuum robots designed to be long in order to reach region of high density in the soil, delicate region in human body and navigate in environments with solid obstacles ([74],[45]). Many of the currently designed robots are inspired by the stems and roots of plants, for the reasons mentioned in the previous section([71],[72],[46],[75]).

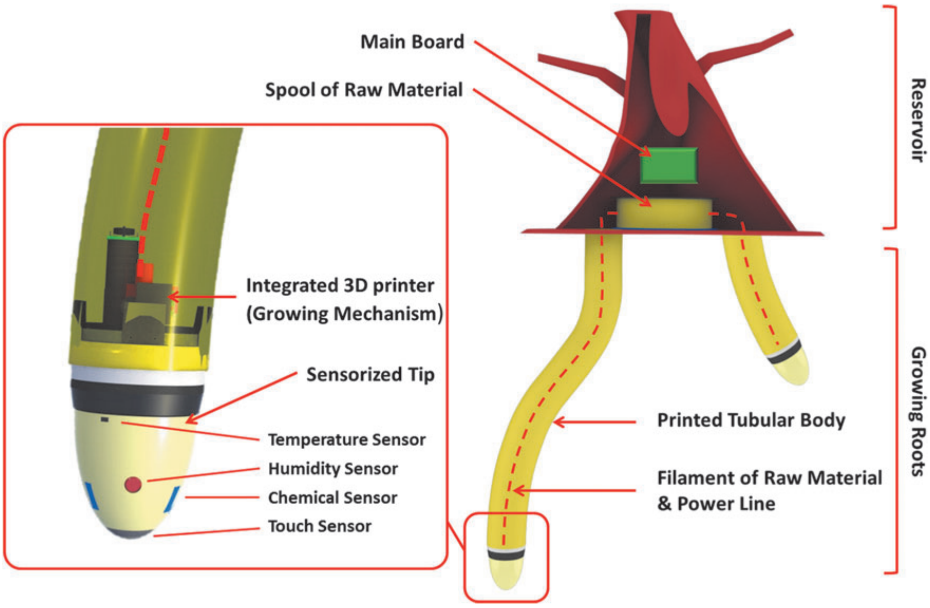


Figure 3.2: Robotic model inspired by plants root. The original picture is contained in [71].

All the many engineering models are not accompanied by as many mathematical models, therefore it is necessary to develop the latter appropriately, in order to better understand the operation and optimization of these new technologies. Our goal in this chapter is to introduce a mathematical model for a continuum robot, with apical growth which explores the soil in order to reach a certain target. It imitates root of plants, hence it is able to bend when the soil is too hard or when it finds an obstacle. The evolution will be described by a system of two partial differential equation, one for the growth of the body, and the other for the extension of the tip. The angular velocity of the root is obtained as solution of an optimization problem with state constraint, indeed it is physically meaningful that the robot bends minimizing the deformation energy and the cost to move sand or to drill in high-density regions.

1.3 Mathematical models in botany

Before starting to study robotic models inspired by plants, it is essential to understand the mathematics of plants themselves. The interest in this topic started some centuries ago, but it is recently attracting enormous attention. The current research focuses on the behaviour of individual components or on the overall structure of the plant ([13],[12]) and the interaction with the environment or with other plants. In the first group we underline above all the works on the growth and symmetry of the leaves and their organization ([77],[51],[76]), on the growth of the roots ([32]), on the behaviour of stems and phenomena as gravitropism, i.e. the slow reorientation of plant growth in response to gravity ([73],[11])

or more in general tropism phenomena due to time-varying stimuli ([62]) . In the second group we can mention the studies about optimal shape of the crowns and the network organization of the roots ([20]), the plant soil interaction ([70],[69]) and the competition among plants ([16]).
What is interesting to observe is the great variety of mathematical methods and theories involved in these studies:

- *Discrete Dynamical Systems* are used for studying phyllotaxis problems, i.e. the arrangement of leaves on a plant stem ([9]);
- *Optimal Ramified Transport* appears in studying the optimal shape of tree branches assuming that the primary goal of tree leaves (tree roots) is to gather sunlight (water and nutrients from the soil, respectively) ([20]);
- *Scalar Conservation Laws* are used to describe the variations of properties such as meristem density, which remains constant within the system in the absence of external sources or sinks ([36]);
- *Smoothed-Particle Hydrodynamics* framework is adopted to study growth in plant tissues at different scales ([63]);
- *Homogenization* approach is used in the analysis of water transport in plant tissues with periodic microstructures ([26]).
- *Control Theory* is the main tool for studying stabilization of growth, in the vertical direction for stems ([3]);
- *Game Theory* allows to model competition among plants ([16]);
- *Mean Field Approximation* appears in modelling heterogeneous population of plants in competition ([65]).

1.4 A model for Stem Growth

For the elastic model of robotic roots that we will introduce later there exists a correspondent model of growth for the stems that has been abundantly analyzed in [19],[18],[3] and that we recall briefly below.

Assume that new cells are generated at the tip of the stem and they grow in size, they induce at time $t \geq 0$ an elongation during the time interval $[s, s + ds]$ whose measure is $d\ell = ds$. Denote by $\mathbf{k}(t, s)$ the unit tangent vector to the stem at the point $P(t, s)$, so that

$$\mathbf{k}(t,s) = \frac{P_s(t,s)}{|P_s(t,s)|}, \qquad P_s(t,s) := \frac{\partial}{\partial s} P(t,s). \tag{1.1}$$

We impose also the curvature to vanish at the tip, which means that

$$\frac{\partial}{\partial s}\mathbf{k}(t,s)\Big|_{s=t} = 0. \tag{1.2}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$ with $\mathbf{e}_1$ oriented in the upward direction, at every point P(t,s), with $s \in [0, t]$ the change of position in response to gravity is described by

$$\frac{\partial}{\partial t} P(t,s) = \int_0^s \mu e^{-\beta(t-\sigma)} \big(\mathbf{k}(t,\sigma) \times \mathbf{e}_3\big) \times \Big(P(t,s) - P(t,\sigma)\Big) d\sigma \tag{1.3}$$

where $\mu > 0$ is a constant which measure the strength of the response, $e^{-\beta(t-s)}$ is the **stiffness factor** and $\mu e^{-\beta(t-\sigma)}\big(\mathbf{k}(t,\sigma) \times \mathbf{e}_3\big)$ is an angular velocity determined by the response to the gravity in the point $P(t,\sigma)$.
In [3] the authors analyzed equations (1.3),(1.2) and proved that the vertical direction is stable for certain values of the parameters μ, β.

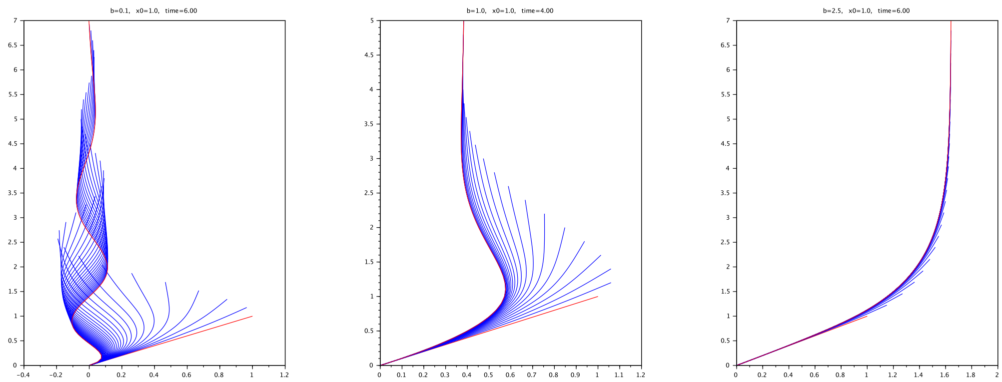


Figure 3.3: Numerical simulations of stem growth, at discrete time with $\mu = 1$ and stiffness constant $\beta = 0.1$ on the left, $\beta = 1.0$ in the center and $\beta = 2.5$ on the right.

In [19],[18] two more terms are added to the evolution equation (1.3), a first term describing the clinging phenomena of the stem to an obstacle Ω, a second one that interprets the reaction thanks to which the stem does not penetrate inside Ω. Therefore the final evolution equation is given by

$$\begin{aligned}\frac{\partial}{\partial t}P(t,s) = &\int_0^s \mu e^{-\beta(t-\sigma)}\big(\mathbf{k}(t,\sigma)\times\mathbf{e}_3\big)\times\Big(P(t,s)-P(t,\sigma)\Big)d\sigma \\ &+\int_0^s e^{-\beta(t-\sigma)}\big(\nabla\psi(P(t,\sigma))\times\mathbf{k}(t,\sigma)\big)\times\Big(P(t,s)-P(t,\sigma)\Big)d\sigma \\ &+\mathbf{v}(s) \end{aligned} \tag{1.4}$$

where the function ψ is a map describing the ability of the stem to feel the presence of an obstacle within a certain distance, the vector field $\mathbf{v}$ belongs to the cone of admissible reaction of the obstacle. Existence and uniqueness of solution to equation (1.4) are proved when the initial configuration is not a so called *breakdown configuration*, whose definition will be recalled later in the chapter. An important role in these works is played by the explicit formulation of the reaction $\mathbf{v}(s)$ that we lose in our study. Indeed for the stem it is assumed that the angular velocity produced by the obstacle reaction minimizes only the elastic defomation, while in our case we will ask to minimize a Lipschitz cost.

The chapter is organized as follows: in section 2 we introduce basic element of the evolution of our artificial root, in Section 2.1 and 2.2 we describe respectively a rigid and an elastic model for the robot, introducing a restarting procedure when it is no more able to grow because of an obstacle. But we will focus above all on the elastic model since it is more flexible and challenging from the mathematical point of view, this is done in Section 3. Section 4 is devoted to the construction of the solution for the model and finally in section and finally in Section 5 we show some numerical implementation. All the models proposed, except for some technical results in 4, are completely original and represents a first step in the development of a mathematical theory for bio-robotics.

2 Growing an artificial root

We seek to model the movement of an artificial root, which penetrates the ground searching for water, chemical compounds, cavities where earthquake survivors may be trapped, etc...

For this purpose, we need to assign

(I) An equation describing the scalar velocity at which the length of the root grows.

(II) An equation describing how the orientation $\mathbf{k}(t)$ of the tip of the root changes in time.

(III) A rule determining when the growth stops, the root shrinks to an earlier configuration, and then starts growing in a new direction.

2.1 A rigid root

In this first model we assume that the root is rigid. The portion of the root that has already grown does not move. In this case, apart from restarting times, the growth can be determined by a second order ODE.

1. At any time t, the root will be described by a $\mathcal{C}^{1,1}$ curve

$$s \mapsto \gamma(t,s), \qquad s \in [0,t], \tag{2.1}$$

parameterized by arc-length. For simplicity we assume that the root grows at unit speed, so that at time t its length is simply $\ell(t) = t$. We denote by

$$\mathbf{k}(t,s) \doteq \gamma_s(t,s) \tag{2.2}$$

the unit tangent vector to the curve.

2. For convenience we shall denote by

$$P(t) \doteq \gamma(t,t), \qquad \mathbf{k}(t) \doteq \mathbf{k}(t,t) = \gamma_s(t,t) \tag{2.3}$$

respectively the position and the orientation of the root tip.

Two types of control will be implemented. Namely:

(i) At every time t, the orientation of the tip is modified.

(ii) At a finite number of times $0 < t_1 < t_2 < \cdots$, if the root hits an obstacle, a restarting procedure is adopted and growth is restarted in a different direction.

3. At any time t, we assume that the direction $\mathbf{k}(t)$ of the root tip can change, subject to a minimum radius of curvature. Namely,

$$\dot{\mathbf{k}}(t) = \mathbf{u}(t) \times \mathbf{k}(t), \qquad |\mathbf{u}(t)| \leq \kappa_0 \tag{2.4}$$

Here $\mathbf{u}(\cdot)$ is a measurable control function, describing the angular velocity at which we bend the tip of the root.

4. We now introduce a feedback rule, assigning the control $\mathbf{u}$ in terms of the position of the tip. Toward this goal, we consider two scalar functions.

- A function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$, given at the beginning and never modified afterwards. Roughly speaking one may think of $\phi(x)$ as the expected distance from a (random) target.

- A function $\psi : \mathbb{R}^3 \mapsto \mathbb{R}$, measuring a kind of closeness to previously reached points. A bit more precisely, if $\Gamma(t)$ is the union of all points reached by the artificial root during previously failed attempts, we could define

$$\psi(x) \doteq \int_{y\in\Gamma(t)} e^{-|x-y|}\,dy, \tag{2.5}$$

The growth of the root can then be described by

$$\dot{P}(t) = \mathbf{k}(t), \qquad \dot{\mathbf{k}}(t) = \mathbf{u}(t)\times\mathbf{k}(t), \tag{2.6}$$

where the control vector $\mathbf{u}$ is determined by

$$\mathbf{u}(t) = \arg\max_{|\omega|\le\kappa_0} \Big\langle \omega\times\mathbf{k}(t)\,,\,\nabla\phi(P(t)) - \nabla\psi(P(t))\Big\rangle. \tag{2.7}$$

Note: in (2.5), instead of $e^{-|x-y|}$, one could use some other kernel $K(|x-y|)$, where $s \mapsto K(s)$ is a smooth, decreasing function.

In some cases, the control in (2.7) may not be unique. To achieve a well posed evolution equation, it is convenient to replace (2.7) with

$$\mathbf{u}(t) = \arg\max_{|\omega|\le\kappa_0} \Big\{\big\langle \omega\times\mathbf{k}(t)\,,\,\nabla\phi - \nabla\psi\big\rangle - \varepsilon|\omega|^2\Big\}. \tag{2.8}$$

The strict convexity of the right hand side yields a unique maximizer, depending Lipschitz continuously on x. Of course, other approximations are possible, with $\mathcal{C}^\infty$ dependence on $\mathbf{k}, \nabla\phi, \nabla\psi$.

Having done this, the local existence and uniqueness of solutions to the evolution problem is trivial.

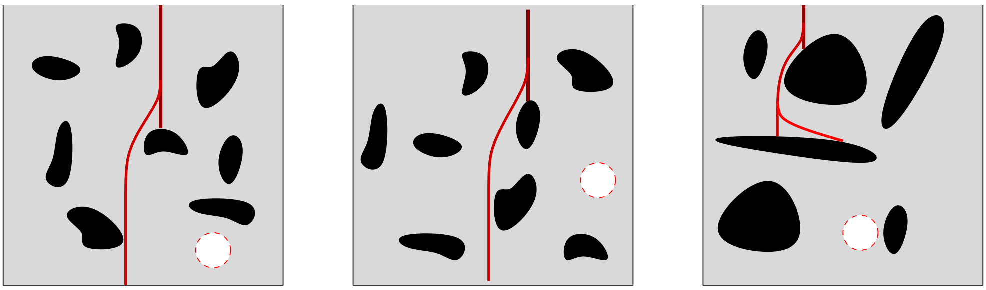


Figure 3.4: Rocks with different average sizes should determine different restarting strategies. The white circles denote the targets.

5. In addition, there must be a rule prescribing when the growth should stop and the root should go back to a previous configuration, and restart.

We denote by $\gamma_i(\cdot,\cdot)$ the curve constructed at the i-th trial run, and write $P_i(t) = \gamma_i(t,t)$ for the position of the tip of this curve at time t. The i-th curve starts with length t_i^- and grows up to a maximum length t_i^+.

Here $h : \mathbb{R}^3 \mapsto [0,1]$ is some scalar function, not known a priori, accounting for the "hardness" of the soil. For example, $h = 0$ when the soil offers no resistance (say, inside a crack), while $h >> 1$ along an impenetrable obstacle (say, a wall).

A reasonable stopping rule is then

$$t_i^+ \doteq \min\Big\{t > t_i^- \,;\quad h(P(t)) \leq h_0\Big\}. \tag{2.9}$$

Here $h_0 > 0$ is a threshold value, assigned a priori. The idea is that, if the tip of the root encounters very hard soil or a rock, it is better go back and restart in another direction.

The restarting procedure must also be carefully defined. Calling t_i^+ the length of the root when we decide to restart, we choose a new length $t_{i+1}^- < t_i^+$ and set

$$\gamma(s, t_{i+1}^-) = \gamma(s, t_i^+) \qquad\qquad s \in [0, t_{i+1}^-]. \tag{2.10}$$

In general, the new length will be determined by a restarting function:

$$t_{i+1}^- = R\Big(t_i^+, t_i^-, \psi(P(t_i^+))\Big). \tag{2.11}$$

By (2.11), we are saying that the amount by which the root shrinks depends on

- the length t_i^+ and the change in the length $t_i^+ - t_i^-$, achieved during the previous failed attempt,
- the density $\psi(P(t_i^+))$, measuring how much the region near $P(t_i^+)$ has already been explored.

A first basic example of restarting function that we can consider is

$$R_1\Big(t_i^+, t_i^-, \psi(P(t_i^+))\Big) = t_i^+ + c\Big(e^{-\psi(P(t_i^+))} - 1\Big)\Big(t_i^+ - t_i^-\Big) \tag{2.12}$$

with $c > 1$. We can interpret it as follows: if the value of $\psi(P(t_i^+))$ is large (which means that $P(t_i^+)$ is close to many trajectories already explored in previous attempts), then we need to go back a strictly greater quantity than the last change of length $t_i^+ - t_i^-$, if $\psi(P(t_i^+))$ is small, we do not need to restart so far from the last starting point $P(t_i^-)$. A second prototype restarting function could be the following

$$R_2\Big(t_i^+, t_i^-, \psi(P(t_i^+))\Big) = t_i^+ + c\Big(1 + 2\rho\chi_{[0,\rho]}(|P(t_i^+) - P(t_i^-)|)\Big)\Big(e^{-\psi(P(t_i^+))} - 1\Big)\Big(t_i^+ - t_i^-\Big) \tag{2.13}$$

which differs from the first by a single factor measuring the length of the last attempt. Indeed if the last elongation is small (in the interval $[0, \rho]$), then probably the root has not enough space to move, therefore it is reasonable to shorten a quantity much greater then $t_i^+ - t_i^-$.

Remark. In practice, the rate of growth of the root will likely depend on the hardness of the soil at the tip. In other words, at time t the total length will not be $\ell(t) = t$. Rather, it may increase at a rate

$$\dot\ell(t) = \frac{1}{1 + h\big(\gamma(\ell(t))\big)}, \tag{2.14}$$

so that

$$\dot P(t) = \frac{d}{dt}\gamma(t, \ell(t)) = \frac{\mathbf{k}(\ell(t))}{1 + h\big(\gamma(\ell(t))\big)}. \tag{2.15}$$

This more general case can be reduced to the previous setting by using a rescaled time $\tau(t) = \ell(t)$. In this way, as in [3, 18, 19], we can assume that at time τ the curve $\gamma(\tau, \cdot)$ has length τ. This length thus increases at unit rate.

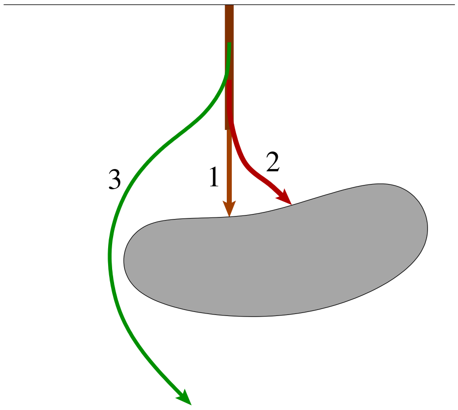


Figure 3.5: Three attempts at growing a root past a rock.

2.2 An elastic root

In the previous model the growing root was completely rigid. Since the previously constructed portions of the root do not move, the equation of growth reduces to a simple ODE.

We now consider an alternative model, where the root is allowed to change its shape, in response to obstacles. The instantaneous deformation of the root is determined as a minimizer of an elastic deformation energy, plus a cost for displacing the nearby soil.

Let $s \mapsto \gamma(t, s)$, $s \in [0, t]$, be an arc-length parameterization of the root at time t. Moreover, let $h = h(x) \geq 0$ be a function describing the hardness of the soil at the point x, and fix a constant $\alpha > 0$.

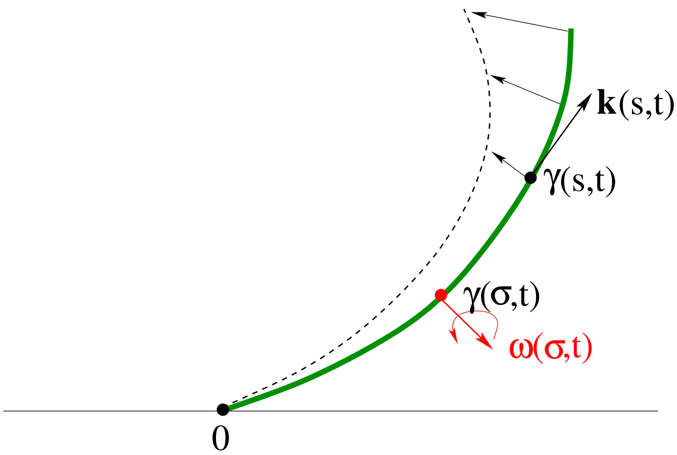


Figure 3.6: An infinitesimal rotation at the point $\gamma(\sigma, t)$ produces a displacement of all points $\gamma(t, s)$ with $s \in [\sigma, t]$.

If the soil around the tip is very hard, or if obstacles are encountered, the root may be forced to bend. This can be achieved by means of a field of angular velocities $\omega(t, \cdot)$. As in [18, 19] the evolution of the curve γ is described by

$$\gamma_t(t, s) \;=\; \int_0^s \omega(t, \sigma) \times \big(\gamma(t, s) - \gamma(t, \sigma)\big)\, d\sigma, \tag{2.1}$$

while

$$\dot{P}(t) \;=\; \mathbf{k}(t) + \int_0^t \omega(t, \sigma) \times \big(\gamma(t, s) - \gamma(t, \sigma)\big)\, d\sigma, \tag{2.2}$$

As in (2.4), this equation must be supplemented with a boundary condition describing how the orientation of the tip changes in time:

$$\dot{\mathbf{k}}(t) \;=\; \left(\mathbf{u}(t) + \int_0^t \omega(t,\sigma)\,d\sigma\right) \times \mathbf{k}(t). \tag{2.3}$$

Here the feedback control $\mathbf{u}(t)$ can be chosen as in (2.8), in order to approach the target as fast as possible. Notice that the integral term in (2.3) accounts for the change in the orientation of the tip coming from the elastic deformation.

At each given time t, following [18] we assume that the field of angular velocities $\omega(\cdot)$ in (2.1)–(2.3) is determined by an instantaneous minimization problem, which we now describe.

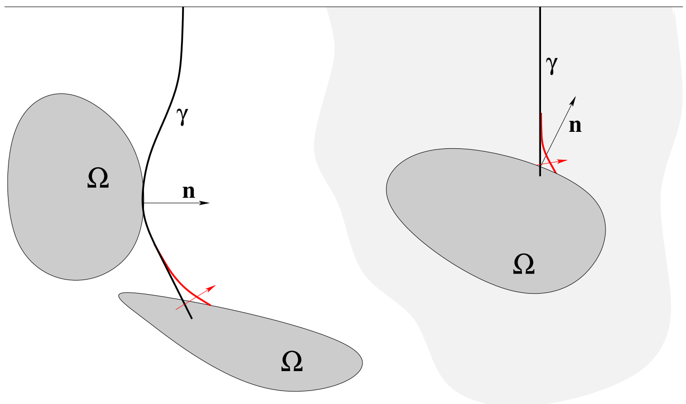


Figure 3.7: By an infinitesimal elastic deformation, the curve γ is kept outside the obstacle. Left: outside the obstacle there is empty space. Right: the obstacle is surrounded by softer soil.

Assume that the external environment consists of soil whose hardness is described by a scalar function $h = h(x) \geq 0$, together with impenetrable obstacles, whose union we denote by $\Omega \subset \mathbb{R}^3$. We assume that Ω is an open set with $\mathcal{C}^2$ boundary $\partial\Omega$. The unit outer normal vector at a point $x \in \partial\Omega$ will be denoted by $\mathbf{n}(x)$. At a contact point $\gamma(t,s) \in \partial\Omega$, this unit outer normal is denoted by

$$\mathbf{n}(t,s) \;\doteq\; \mathbf{n}(\gamma(t,s)).$$

In our model, the angular velocity $\omega(t,\cdot) \in \mathbf{L}^2([0,t])$ is determined as the solution to the following optimization problem.

(OP) *Find an angular velocity $\omega \in \mathbf{L}^2([0,t])$ which minimizes the functional*

$$\begin{aligned} J(\omega) \;&\doteq\; \int_0^t |\omega(s)|^2\,ds + \int_0^t h(\gamma(t,s)) \cdot \Big|\gamma_t(t,s) \times \mathbf{k}(t,s)\Big|\,ds + \alpha\, h(P(t)) \cdot \big\langle \dot{P}(t), \mathbf{k}(t)\big\rangle_+ \\ &=\; J_1(\omega) + J_2(\omega) + J_3(\omega)\,. \end{aligned} \tag{2.4}$$

Here γ_t and $\dot{P}$ are recovered from $\omega(\cdot)$ by the formulas (2.1)-(2.2), while

$$\langle \mathbf{v}, \mathbf{w}\rangle_+ \;\doteq\; \max\{\langle \mathbf{v}, \mathbf{w}\rangle\,,\ 0\}$$

denotes the positive part of an inner product. The global minimizer of the functional (2.4) is sought under the constraints

$$\Big\langle \mathbf{n}(t,s)\,,\, \gamma_t(t,s)\Big\rangle \;\geq\; 0 \qquad \textit{for all } s \textit{ such that } \gamma(t,s) \in \partial\Omega. \tag{2.5}$$

In the case where $P(t) \doteq \gamma(t,t) \in \partial\Omega$, *we impose the additional constraint*

$$\Big\langle \mathbf{n}(t,t)\,,\,\dot P(t)\Big\rangle \;\geq\; 0. \tag{2.6}$$

Remark 2.1. The three terms on the right hand side of (2.4) can be interpreted as

[elastic bending energy] + [soil hardness]×[swept area] + [soil hardness]×[tip penetration].

The coefficient $\alpha > 0$ allows to better calibrate the relative weight of these terms.

Remark 2.2. In [18, 19], the surface of the obstacle was modeled as a smooth frictionless surface. In alternative, one could impose that there is some friction between the obstacle and the tip of the root. For example, one can impose that the tip can move only if

$$\big\langle \mathbf{n}(t,t)\,,\,\mathbf{k}(t,t)\big\rangle \;\geq\; \kappa_0\,, \tag{2.7}$$

for some $\kappa_0 \in]-1, 0]$.

Remark 2.3. In the first model, considered in Section 2.1, the root is completely rigid, in the sense that the portion that is already grown does not change its position at any later time. On the other hand, in this second model the dynamics is more complex, because it takes into account the elasticity of the root.

The difference is only in the dynamics. The optimal control (orienting the tip in the direction where it is more likely to find the target), as well as the restarting strategy, can be chosen in the same way as in the model discussed in the first two sections.

3 Solutions to the elastic root model

From a theoretical point of view, the model with an elastic root is much more interesting. Toward a proof of well-posedness, two steps are required.

1. At any time $t \geq 0$, the instantaneous growth is determined by solving an minimization problem with constraints. We need to prove that this optimization problem admits a unique solution. Moreover, this minimizer depends continuously on the data, except when a “breakdown configuration" is reached, as in the model studied in [19].

2. Following [18], we should then introduce a suitable weighted distance among root configurations, and prove that the evolution problem is well posed.

Aim of this section is to prove a local existence theorem, valid as long as a “breakdown configuration" is not reached, as shown in Fig. 3.8.

Definition 3.1. *We say that a curve* $\gamma : [0, t_0] \mapsto \mathbb{R}^3 \backslash \Omega$ *is in a* **breakdown configuration** *w.r.t. the obstacle* Ω *if the following holds*

(B) *The tip* $\gamma(t_0)$ *touches the obstacle perpendicularly, namely*

$$\gamma(t_0) \;\in\; \partial\Omega\,, \qquad\qquad \gamma_s(t_0) \;=\; -\mathbf{n}(\gamma(t_0)). \tag{3.1}$$

Moreover,

$$\gamma_{ss}(s) \;=0 \qquad \text{for all } s \in\,]0, t_0[\ \text{ such that } \gamma(s) \notin \partial\Omega\,. \tag{3.2}$$

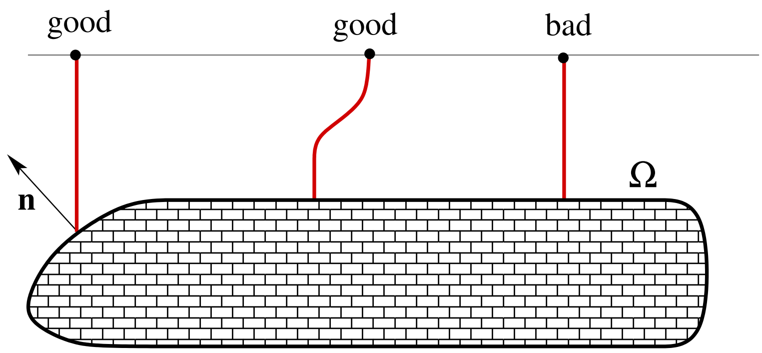


Figure 3.8: For the two initial configurations on the left and in the center, the constrained growth equations for the root admit a unique solution. The root configuration on the right satisfies the condition **(B)**. In such case, the Cauchy problem is ill posed.

3.1 Computing the instantaneous velocity of the root.

At each time $t \geq 0$, the velocity of points on the curve $\gamma(t, \cdot)$ and of the root tip $P(t)$ is determined by (2.1) and (2.2), respectively. In our model, the instantaneous bending rate $\omega(t, \cdot) \in \mathbf{L}^2([0, t]\,; \mathbb{R}^3)$ is determined as the unique global minimizer of the functional (2.4), subject to the constraints (2.5)-(2.6). We recall that $h : \mathbb{R}^3_- \mapsto \mathbb{R}$ is a smooth scalar function, describing the hardness of the soil.

Lemma 3.2. *If $\gamma(t, \cdot)$ is not a breakdown configuration, then the functional (2.4) has a unique minimizer, subject to (2.1)-(2.2) and the constraints (2.5)-(2.6).*

Proof.
1. Call $\mathcal{A} \subseteq \mathbf{L}^2([0, t])$ the set of angular velocities for which constraints (2.5)-(2.6) are satisfied. Observe that the maps

$$\omega(\cdot) \ \mapsto \ \gamma_t(t, \cdot), \qquad \omega(\cdot) \ \mapsto \ \dot{P}(t)$$

are linear and affine, respectively.

If $\gamma(t, \cdot)$ does not satisfy all conditions in **(B)**, then the same arguments used in Lemma 1 of [19] show that $\mathcal{A}$ is a nonempty, closed, affine subspace of $\mathbf{L}^2$.

Indeed assume that $\mathbf{v} \in H^2([0, t]; \mathbb{R}^3)$ is a velocity field produced by the obstacle reaction when $\gamma(t, \cdot)$ touches $\partial\Omega$ or a reaction due to hardness of the soil with

$$\mathbf{v}(0) = \mathbf{v}'(0) = 0, \qquad \langle \mathbf{v}'(s) \ , \ \mathbf{k}(t, s) \rangle = 0 \quad \text{for all } s \in [0, t]. \tag{3.3}$$

Then there exist a unique angular velocity field $\omega \in \mathbf{L}^2([0, t]; \mathbb{R}^3)$ such that

$$\|\omega\|_{\mathbf{L}^2} \leq C \|\mathbf{v}\|_{H^2} \tag{3.4}$$

$$\langle \omega(s), \mathbf{k}(t, s) \rangle = 0 \qquad \text{for all } s \in [0, t]. \tag{3.5}$$

$$\mathbf{v}(s) = \int_0^s \omega(\sigma) \times (\gamma(t, s) - \gamma(t, \sigma)) d\sigma \quad \text{for all } s \in [0, t]. \tag{3.6}$$

To see it, we first observe that by condition (3.3) and differentiating (3.6) it results that $\omega(\cdot)$ satisfies (3.6) if and only if

$$\mathbf{v}'(s) = \int_0^s \omega(\sigma) d\sigma \times \mathbf{k}(t, s) \quad \text{for all } s \in [0, t]. \tag{3.7}$$

Now , consider a family of orthonormal frames $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ such that $\mathbf{e}_1(s) = \mathbf{k}(t,s)$ for all $s \in [0,t]$. We want to find two scalar functions $\omega_2, \omega_3 : [0.t] \to \mathbb{R}$ ($\omega_1 = 0$ by condition (3.5)), components of the vector function

$$\omega(s) = \omega_2(s)\mathbf{e}_2(s) + \omega_3(s)\mathbf{e}_3(s) \tag{3.8}$$

satisfying (3.6).

By the orthogonality assumption in (3.6), we get two scalar functions z_2 and z_3 such that

$$\mathbf{v}'(s) = z_2(s)\mathbf{e}_2(s) + z_3(s)\mathbf{e}_3(s) = \int_0^s \big(\omega_2(\sigma)\mathbf{e}_2(\sigma) + \omega_3(\sigma)\mathbf{e}_3(\sigma)\big)d\sigma \times \mathbf{e}_1(s) \tag{3.9}$$

Projecting respectively along $\mathbf{e}_2(s)$ and $\mathbf{e}_3(s)$ we get

$$z_2(s) = \int_0^s \langle \mathbf{e}_2(\sigma) \times \mathbf{e}_1(s),\ \mathbf{e}_2(s)\rangle \omega_2(\sigma)d\sigma + \int_0^s \langle \mathbf{e}_3(\sigma) \times \mathbf{e}_1(s),\ \mathbf{e}_2(s)\rangle \omega_3(\sigma)d\sigma;$$
$$z_3(s) = \int_0^s \langle \mathbf{e}_2(\sigma) \times \mathbf{e}_1(s),\ \mathbf{e}_3(s)\rangle \omega_2(\sigma)d\sigma + \int_0^s \langle \mathbf{e}_3(\sigma) \times \mathbf{e}_1(s),\ \mathbf{e}_3(s)\rangle \omega_3(\sigma)d\sigma.$$

By a property of the triple product $\langle \cdot \times \cdot\ ,\ \cdot \rangle$ we can rewrite

$$z_2(s) = \int_0^s \Big[\langle \mathbf{e}_2(\sigma),\ \mathbf{e}_3(s)\rangle \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\ \mathbf{e}_3(s)\rangle \omega_3(\sigma)\Big]d\sigma$$
$$z_3(s) = -\int_0^s \Big[\langle \mathbf{e}_2(\sigma),\ \mathbf{e}_2(s)\rangle \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\ \mathbf{e}_2(s)\rangle \omega_3(\sigma)\Big]d\sigma$$

Since all the function involved in (3.9) are in H^1 we can differentiate again and obtain the following system

$$\begin{cases} \omega_3(s) = z_2'(s) - \int_0^s \Big[\langle \mathbf{e}_2(\sigma),\ \mathbf{e}_3'(s)\rangle \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\ \mathbf{e}_3'(s)\rangle \omega_3(\sigma)\Big]d\sigma \\ \omega_2(s) = -z_3'(s) - \int_0^s \Big[\langle \mathbf{e}_2(\sigma),\ \mathbf{e}_2'(s)\rangle \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\ \mathbf{e}_2'(s)\rangle \omega_3(\sigma)\Big]d\sigma \end{cases} \tag{3.10}$$

which is a linear system of of Volterra integral equation, whose unique solution can be obtained by a fix point argument. Indeed, introducing the vector notation

$$U = \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}, \qquad Z = \begin{pmatrix} -z_3 \\ z_2 \end{pmatrix}, \tag{3.11}$$

(3.10) can be rewritten as

$$U(s) = Z'(s) + \int_0^s B(s,\sigma)U(\sigma)d\sigma = \mathcal{P}[U](s) \tag{3.12}$$

where the matrix $B(s,\sigma)$ has norm

$$|B(s,\sigma)| \le 2|\mathbf{e}_2'(s)| + 2|\mathbf{e}_3'(s)| := b(s). \tag{3.13}$$

The operator $\mathcal{P}$ defined in (3.12) is a strict contraction on the space $\mathbf{L}^1([0,T];\mathbb{R}^2)$ with the equivalent norm

$$\|U\| := \int_0^t \exp\left\{-2\int_0^s b(\sigma)d\sigma|U(s)|\right\}|U(s)|ds \tag{3.14}$$

In fact, for any $U_1, U_2 \in L^1$, if we integrate by part we get

$$\begin{aligned}
\|\mathcal{P}[U_1]-\mathcal{P}[U_2]\| &\leq \int_0^t \exp\left\{-2\int_0^s b(\sigma)d\sigma\right\} b(s)\Big(\int_0^s |U_1(\sigma)-U_2(\sigma)|\Big)ds \\
&= \int_0^t \frac{1}{2}\exp\left\{-2\int_0^s b(\sigma)d\sigma\right\}|U_1(s)-U_2(s)|ds \\
&- \frac{1}{2}\exp\left\{-2\int_0^t b(\sigma)d\sigma\right\}\Big(\int_0^s |U_1(\sigma)-U_2(\sigma)|d\sigma\Big) \\
&\leq \int_0^t \frac{1}{2}\exp\left\{-2\int_0^s b(\sigma)d\sigma\right\}|U_1(s)-U_2(s)|ds \\
&= \frac{1}{2}\|U_1-U_2]\|
\end{aligned}$$

therefore by the contraction principle, equation (3.12) has a unique solution in $\mathbf{L}^1([0,T];\mathbb{R}^2)$, moreover we have

$$\|U\|_{\mathbf{L}^1} \leq C_1\|Z'\|_{\mathbf{L}^1} \leq C_2\|Z'\|_{\mathbf{L}^2} \tag{3.15}$$

with C_1, C_2 constants depending on t and $b(\cdot)$. Conclusively we obtain that

$$\begin{aligned}
\|U\|^2_{\mathbf{L}^2} &\leq 2\|Z'\|^2_{\mathbf{L}^2} + \int_0^t b^2(s)\Big(\int_0^s U(\sigma)\sigma\Big)^2 ds \\
&\leq 2\|Z'\|^2_{\mathbf{L}^2} + 2\|b\|^2_{\mathbf{L}^2}\|U\|^2_{\mathbf{L}^1} \\
&\leq (2+2\|b\|^2_{\mathbf{L}^2}C_2^2)\|Z'\|_{\mathbf{L}^2}
\end{aligned}$$

hence also ω_1 and ω_2 are in $\mathbf{L}^2$. To conclude we just need to observe that $\|Z'\|_{\mathbf{L}^2} = \mathcal{O}(1)\cdot\|\mathbf{v}\|_{H^2}$ and $\|b\|_{\mathbf{L}^2}\mathcal{O}(1)\|\gamma\|_{H^2}$. Condition (3.5) plays an essential role in uniques of the angular velocity ω, indeed without that we can produce infinitely many solution for the system (3.10).

2. We now observe that the functional J in (2.4) is non-negative, strictly convex, and coercive on $\mathbf{L}^2$.

Let $(\omega_n)_{n\geq 1}$ be a minimizing sequence in $\mathcal{A} \subseteq \mathbf{L}^2$. By coercivity, the norms $\|\omega_n\|_{\mathbf{L}^2}$ are uniformly bounded. By possibly extracting a subsequence, we can thus assume the weak convergence $\omega_n \rightharpoonup \omega^*$.

Since the functional J is convex, it is lower semicontinuous w.r.t. weak convergence. Therefore

$$J(\omega^*) \;\leq\; \liminf_{n\to\infty} J(\omega_n) \;=\; \inf_{\omega\in\mathcal{A}} J(\omega).$$

Recalling that $\mathcal{A}$ is closed, we conclude that $\omega^* \in \mathcal{A}$ is a global minimizer.

Finally, the strict convexity of J implies that this minimizer is unique. □

3.2 Solutions to the evolution problem.

Given a feedback control $\mathbf{u} = \mathbf{u}(x)$ accounting for the desired bending of the tip, we can now give a precise definition of "solution" the the evolution problem describing the growth of the artificial root.

At each time t, the position of the root is described by a map $\gamma(t,\cdot)$ from $[0,t]$ into $\mathbb{R}^3$. Of course, the domain of this map grows with time. Following [19], it is convenient to reformulate our model as an evolution problem on a functional space independent of t. We thus fix $T > t_0$ and consider the Hilbert-Sobolev space $H^2([0,T];\mathbb{R}^3)$. Any function $\gamma(t,\cdot) \in H^2([0,t];\mathbb{R}^3)$ will be canonically extended to $H^2([0,T];\mathbb{R}^3)$ by setting

$$\gamma(t,s) \doteq \gamma(t,t) + (s-t)\gamma_s(t,t) \qquad \text{for} \quad s \in [t,T]\,. \tag{3.16}$$

Notice that the above extension is well defined because $\gamma(t,\cdot)$ and $\gamma_s(t,\cdot)$ are continuous functions. In the following we shall study functions defined on a domain of the form

$$\mathcal{D}_T \doteq \big\{(t,s)\,;\quad 0 \le s \le t, \quad t \in [t_0,T]\big\}, \tag{3.17}$$

and extended to the rectangle $[t_0,T] \times [0,T]$ as in (3.16).

$$\gamma(t_0,s) = \overline{\gamma}(s), \qquad s \in [0,t_0]. \tag{3.18}$$

Definition 3.3. *We say that a function $\gamma = \gamma(t,s)$, defined for $(t,s) \in [t_0,T] \times [0,T]$ is a solution to the root growth problem if the following holds.*

(i) The map $t \mapsto P(t,\cdot)$ is Lipschitz continuous from $[t_0,T]$ into $H^2([0,T];\mathbb{R}^3)$.

(ii) For a.e. $t \in [t_0,T]$ the partial derivative $\partial_t\gamma(t,\cdot)$ is given by

$$\gamma_t(t,s) = \int_0^s \omega(t,\sigma) \times \big(\gamma(t,s) - \gamma(t,\sigma)\big)\, d\sigma, \tag{3.19}$$

for $s \in [0,t]$, while

$$\gamma_t(t,s) = \int_0^t \omega(t,\sigma) \times \big(\gamma(t,s) - \gamma(t,\sigma)\big)\, d\sigma + \mathbf{u}(t) \times \big(\gamma(t,s) - \gamma(t,t)\big) \tag{3.20}$$

if $s \in [t,T]$. Here the angular velocity $\omega(t,\cdot)$ is the unique solution to the optimization problem **(OP)**, *in Section 2.2.*

(iii) The initial conditions hold:

$$\gamma(t_0,s) = \begin{cases} \overline{\gamma}(s) & \text{if} \quad s \in [0,t_0], \\ \overline{\gamma}(t_0) + (s-t_0)\overline{\gamma}_s(t_0) & \text{if} \quad s \in [t_0,T]. \end{cases} \tag{3.21}$$

(iv) The pointwise constraints hold:

$$\gamma(t,s) \notin \Omega \qquad \text{for all } t \in [t_0,T], \quad s \in [0,t]. \tag{3.22}$$

4 Construction of solutions

4.1 Technical lemmas and known estimates

We now recall some notations, technical results and estimates contained in [19] that will be useful in order to prove the main theorem of this chapter.

Let $\bar{\gamma}$ be a curve in $H^2([0,t_0];\mathbb{R}^3)$, satisfying the following assumptions

$$\bar{\gamma}(0) = 0 \in \mathbb{R}^3, \qquad |\bar{\gamma}'(s)| = 1, \qquad \bar{\gamma}(s) \notin \Omega \quad \text{for all } s \in [0,t_0]. \tag{4.1}$$

and such that it is not a breakdown configuration as described in 3.1. Given $T > t_0$, it is possible to extend $\bar\gamma$ to a map $[0,T] \to \mathbb{R}^3$ as we showed in (3.16). For a fixed radius ρ and $T > t_0$ we can define the following tubolar neighborhood of $\bar\gamma$ in $H([0,T],\mathbb{R}^3)$

$$\mathcal{V}_\rho := \Big\{\gamma \in H([0,T],\mathbb{R}^3), \quad \gamma(0) = 0, \quad \gamma'(0) = \bar\gamma'(0), \\ |\gamma'(s)| = 1 \text{ for all } s \in [0,t], \quad \int_0^t |\gamma''(s) - \bar\gamma''(s)|^2 ds \le \rho\Big\} \tag{4.2}$$

The following result shows that given a curve satisfying (4.1) and which it is not a breakdown configuration, then there exists a tubolar neighborhood as described before such that each curve in this set can be pushed away from the obstacle by a small rotation.

Lemma 4.1. *Let $\bar\gamma : [0,t_0] \mapsto \mathbb{R}^3 \setminus \Omega$ an initial curve which satisfies (4.1) and such that it does not satisfies simultaneously all conditions in (3.1),(3.2). Then there exists $T > t_0$, ρ, $\delta > 0$, and a constant C_0 such that the following holds.*

Set $\phi : \mathbb{R}^3 \mapsto \mathbb{R}^+$ the function signed distance defined as

$$\phi(x) := \begin{cases} d(x,\partial\Omega) & \text{if } x \notin \Omega \\ -d(x,\partial\Omega) & \text{if } x \in \Omega \end{cases}. \tag{4.3}$$

Let $t \in [t_0,T]$ and consider any curve $\gamma \in \mathcal{V}_\rho$. Then there exists $\omega : [0,t] \mapsto \mathbb{R}^3$, with

$$\|\omega\|_{\mathbf{L}^2} \le C_0 \tag{4.4}$$

such that, for all $s \in [0,t]$ with $|\phi(\gamma(s))| \le \delta$ one has

$$\Big\langle \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma)), \nabla\phi(\gamma(s))\Big\rangle \ge 1 \tag{4.5}$$

The proof is based on the analysis of three different cases in which one of the condition (3.1),(3.2) is not satisfied.
We introduce now the following quantity

$$E(t,\gamma,\Omega) := \sup\Big\{d(\gamma(s),\partial\Omega); \quad s \in [0,t], \gamma(s) \in \Omega\Big\} \tag{4.6}$$

that measures the maximum depth at which the initial portion of a curve γ penetrates inside the obstacle Ω. In the same hypothesis of 4.1 we can prove that there exists a tubolar neighborhood $\mathcal{V}_\rho$ of $\bar\gamma$ such that for every curve γ inside there exist a unique angular velocity $\bar\omega$ minimizer of **OP** which push out the curve and his norm is bounded by

$$\|\bar\omega\|_{\mathbf{L}^2} \le 2C_0 \cdot E(t,\gamma,\Omega) \tag{4.7}$$

for some constant C_0 independent of $\gamma \in \mathcal{V}_\rho$.

We introduce now a nonlinear push out operator $\mathcal{P}$. Given an angular velocity $\omega \in \mathbb{R}^3$, let $R[\omega]$ be the 3×3 rotation matrix

$$R[\omega] := e^A := \sum_{k_0}^{\infty} \frac{A^k}{k!}, \qquad A := \begin{pmatrix} 0 & -\omega_3 & -\omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{4.8}$$

It is well known that for every $\bar v \in \mathbb{R}^3$, the vector $R[\omega]\bar{\mathbf{v}}$ is the solution at time $t = 1$ of the following system

$$\dot{\mathbf{v}}(t) = \omega \times \mathbf{v}(t), \qquad \mathbf{v}(0) = \bar{\mathbf{v}}. \tag{4.9}$$

If γ is a curve in $H^2([0,t],\mathbb{R}^3)$, and $\bar{\omega}$ is the unique minimizer of the associated **OP**, then we use the previous notation to define

$$\mathcal{P}[\gamma](s) := \int_0^s R\Big[\int_0^\sigma \bar{\omega}(\xi)d\xi\Big]\gamma'(\sigma)d\sigma \tag{4.10}$$

By (4.7), the depth at which $\mathcal{P}[\gamma]$ can penetrate inside Ω can be estimate as follows

$$E(\mathcal{P}[\gamma],\omega) \leq \mathcal{O}(1)\|\bar{\omega}\|^2_{\mathbf{L}^2} \leq C\cdot E(\gamma,\omega) \tag{4.11}$$

with C uniform constant.

4.2 The existence result for a single attempt

Here we state and prove the main result of the chapter, that is the existence of a single trajectory for the root. By now we said that the root stops growing only because of a breakdown configuration, but since it is a robot, the growth can be interrupted also because of the physical limits of the machine itself, like a bound on the extension.

Theorem 4.2. *Let a $\mathcal{C}^1$ feedback control function $u = u(x,\mathbf{k})$ be given. Assume that the initial configuration $\overline{\gamma} \in H^2([0,t_0];\mathbb{R}^3)$ is not a breakdown configuration. Then there exists $T > t_0$ and a solution to the root evolution problem, defined on an interval $[t_0,T]$.*

Proof. First observe that if we differentiate (3.19) and (3.20) with respect to s and then integrate with respect to t, we get the following equation

$$\mathbf{k}(t,s) = \mathbf{k}(t_0,s) + \int_{t_0}^t \Big(\int_0^{\tau\wedge s}\omega(\sigma)d\sigma\Big)\times\mathbf{k}(\tau,s)d\tau + \mathcal{H}(s-t_0)\int_{t_0}^{t\wedge s}u(\tau)\times\mathbf{k}(\tau,s)d\tau \tag{4.1}$$

with $s\in[0,T]$ and $\mathcal{H}$ the Heaviside function .

We proceed by constructing a sequence of approximate solutions, to this end we use a splitting operator scheme. Let $\overline{\gamma}\in H^2([0,t_0];\mathbb{R}^3)$ be a the initial configuration and assume that it is not a breakdown configuration.
Fix a time step ε and set $t_k = t_0 + k\varepsilon$. Assume that the approximate tangent vector to the curve $\mathbf{k} = \mathbf{k}$(t,s) has been constructed for all the times $t\in[0,t_k]$ and $s\in[0,t]$ starting from

$$\begin{cases}\mathbf{k}(t_0,s) & \text{if } s\in[0,t_0]\\ \mathbf{k}(t_0,t_0) & \text{if } s\in[t_0,T]\end{cases} \tag{4.2}$$

To extend the solution on $[t_k,t_{k+1}]$ we proceed as follows: we define

$$\mathbf{k}(t,s) \;=\; R\left[(t\wedge s - t_k)u(t_k)\mathcal{H}(s-t_k)\right]\mathbf{k}(t_k,s) \tag{4.3}$$

for all $t\in[t_k,t_{k+1}[$ and $s\in[0,T]$ and ω_k the angular velocity obtained as solution of (**OP**) at time $t=t_k$. Taking $t=t_{k+1}$, the previous construction produces a curve

$$s \;\mapsto\; \gamma(t_{k+1}-,s) = \int_0^s \mathbf{k}(t,\sigma)d\sigma. \tag{4.4}$$

Observe that the physical meaningful portion of the curve, that is the part with $s\in[0,t_{k+1}]$ may lie inside the obstacle therefore we need to use a push-out operator and replace $\gamma(t_{k+1}-,s)$ by a new curve, by setting

$$\gamma(t_{k+1},s) \;=\; \mathcal{P}\left[\gamma(t_{k+1}-,\cdot)\right](s) \;=\; \int_0^s R\left[\int_0^\sigma \overline{\omega}_{k+1}(\xi)d\xi\right]\gamma_s(t_{k+1}-,\sigma)d\sigma \tag{4.5}$$

with $\varpi_{k+1}(\cdot)$ solution to (**OP**) at time $t = t_{k+1}$. This is equivalent to say that

$$\mathbf{k}(t_{k+1}, s) \;=\; \int_0^s R\left[\int_0^\sigma \varpi_{k+1}(\xi)d\xi\right]\mathbf{k}(t_{k+1}-, s) \tag{4.6}$$

.

By (4.7) and (4.11), as long as the approximation remains inside the neighborhood $\mathcal{V}_\rho$, there exists a constant C_3 such that

$$\|\bar{\omega}_{k+1}\|_{\mathbf{L}([0,t_k+1])} \le C_3 E\big(t_{k+1}, \gamma(t_{k+1}-,\cdot), \Omega\big) \tag{4.7}$$

$$E\big(t_{k+1}, \gamma(t_{k+1},\cdot), \Omega\big) \le C_3 E^2\big(t_{k+1}, \gamma(t_{k+1}-,\cdot), \Omega\big) \tag{4.8}$$

This last inequality express the fact that every time we apply the non linear push-out (4.5), this entails a deformation of the curve of ε^2 if the depth of the curve inside the obstacle is of the order of ε. When we apply the rotation (4.3), the root penetration depth increases of a quantity of the order of ε, that is

$$E\big(t_{k+1}, \gamma(t_{k+1}-,\cdot), \Omega\big) \le E\big(t_k, \gamma(t_k,\cdot), \Omega\big) + C_4\varepsilon \tag{4.9}$$

for some constant C_4. Thus for $\varepsilon > 0$ small enough, the estimates (4.8)-(4.9) yelds the implication

$$E\big(t_k, \gamma(t_k,\cdot), \Omega\big) \le \varepsilon \quad\Longrightarrow\quad E\big(t_{k+1}, \gamma(t_{k+1},\cdot), \Omega\big) \le \varepsilon \tag{4.10}$$

Since by assumption the initial curve $\bar{\gamma}$ lies outside the obstacle, this means that $E\big(t_0, \bar{\gamma}(t_0,\cdot), \Omega\big) = 0$, this by induction, for all $k \ge 1$ it follows that

$$E\big(t_{k+1}, \gamma(t_{k+1},\cdot), \Omega\big) \le \varepsilon \tag{4.11}$$

which combined with (4.7)-(4.8) implies

$$\|\bar{\omega}_{k+1}\|_{\mathbf{L}^2[0,t_{k+1}]} \le C_5\varepsilon. \tag{4.12}$$

Thanks to the previous construction , for every time step $\varepsilon > 0$ small enough, we get a piecewise continuous approximate solution $\mathbf{k}_\varepsilon = \mathbf{k}_\varepsilon(t,s)$ defined for all $s \in [0,t]$ and $t \in [t_0, T_\varepsilon]$, where T_ε is given by

$$T_\varepsilon = \sup\Big\{\tau \in [t_0, T] :\ \gamma_\varepsilon(\tau, s) = \int_0^s \mathbf{k}_\varepsilon(\tau,\sigma)d\sigma \in \mathcal{V}_\rho\Big\}.$$

As long as the approximation $\gamma_\varepsilon(\tau,\cdot)$ remains inside $\mathcal{V}_\rho$ we have the following estimates

$$\|\mathbf{k}_\varepsilon(t,\cdot) - \mathbf{k}_\varepsilon(t',\cdot)\|_{H^1([0,t_k])} \le C_6|t-t'| \quad \text{for all } t,t' \in [t_k, t_{k+1}], \tag{4.13}$$

$$\|\mathbf{k}_\varepsilon(t_{k+1},\cdot) - \mathbf{k}_\varepsilon(t_{k+1}-,\cdot)\|_{H^1([0,t_{k+1}])} \le C_6\varepsilon, \tag{4.14}$$

for some constant C_6 independent of k, ε. The first inequality derives from the boundedness of $u(t,\mathbf{k})$, while the second one is a consequence of (4.12). Gathering (4.13) and (4.14) we obtain the next estimate

$$\|\mathbf{k}_\varepsilon(\tau,\cdot) - \mathbf{k}_\varepsilon(t,\cdot)\|_{H^1([0,t])} \le C_7(\varepsilon + \tau - t) \tag{4.15}$$

for all $t < \tau$ and some constant C_7. Since by construction

$$\gamma_\varepsilon(t_k, 0) = 0 \text{ and } \mathbf{k}_\varepsilon(t,0) = \bar{\mathbf{k}}(0) \quad \text{for all } t \ge 0 \tag{4.16}$$

then we can conclude that (4.15) implies

$$\gamma_\varepsilon(t,\cdot) \in \mathcal{V}_\rho \quad \text{for all } t \in [0,T], \tag{4.17}$$

for some $T > 0$ independent of ε. The estimates implies also that for ε small enough, all the approximations $\mathbf{k}_\varepsilon = \mathbf{k}_\varepsilon(t,s)$ are well defined.

In order to get the converge of a subsequence we need to observe that:
1. All the functions $t \mapsto \mathbf{k}_\varepsilon(t,\cdot)$ have uniformly bounded total variation as map from $[t_0,T]$ into $H^1([0,T])$.
2. Since $\gamma_\varepsilon(t,\cdot) \in \mathcal{V}_\rho$, for any fixed t we have that $\mathbf{k}_\varepsilon(t,\cdot)$ are uniformely bounded in $H^1([0,T])$.
Therefore by a weak version of Helly's selection principle for BV functions with value in metric spaces ([29],[60]), there exists a subsequence $\mathbf{k}_{\varepsilon_n}$ and a function $\mathbf{k} \in BV([t_0,T], H^1([0,T]), \mathbb{R}^3)$ such that $\mathbf{k}_{\varepsilon_n}(t,\cdot) \rightharpoonup \mathbf{k}(t,\cdot)$ in H^1 for every $t \in [t_0,T]$. By the compact embedding of $H^1([0,T])$ into $C([0,T])$ we get also the uniform convergence up to subsequences.

The next step is to prove that by taking the limit of $\mathbf{k}_\varepsilon$ as $\varepsilon \to 0$ we obtain exactly (4.1). We showed before that $\mathbf{k}_\varepsilon$ converges uniformly for $(t,s) \in [t_0,T] \times [0,T]$ to a function $\mathbf{k}$ and this implies that

$$\gamma_\varepsilon(t,s) = \int_0^s \mathbf{k}_\varepsilon(t.\sigma)d\sigma \rightrightarrows \int_0^s \mathbf{k}(t.\sigma)d\sigma \to \gamma(t,s). \tag{4.18}$$

Therefore if we consider the sequence of convex functionals

$$J_\varepsilon(\omega) = \int_0^t |\omega(s)|^2\, ds + \int_0^t h(\gamma_\varepsilon(t,s)) \cdot \Big|\gamma_{\varepsilon,t}(t,s) \times \mathbf{k}_\varepsilon(t,s)\Big|\, ds + \alpha\, h(P(t)) \cdot \big\langle \dot{P}_\varepsilon(t), \mathbf{k}_\varepsilon(t)\big\rangle_+ \tag{4.19}$$

with $\omega \in \mathbf{L}^2([0,t])$, it is equibounded in a neighborhood of each point and converges pointwise to the convex functional J in (2.4). By Theorem 5.12 in [28], the sequence $\{J_\varepsilon\}$ Γ-converges to J in $\mathbf{L}^2([0,t])$ as $\varepsilon \to 0$. Hence we have also the convergence of the sequence of minimizers ω_ε to the minimizer ω of J in $\mathbf{L}^2([0,t])$. The $\mathbf{L}^2$-convergence in a bounded domain implies the $\mathbf{L}^1$ convergence and hence the pointwise convergence a.e. in $[0,t]$.

Now thanks to the matrix estimate

$$\Big|\mathbf{v} + (\omega_1 + ... + \omega_n) \times \mathbf{v} - R[\omega_1] \circ \cdots \circ R[\omega_n]\mathbf{v}\Big| = \mathcal{O}(1)\Big(\sum_i |\omega_i|\Big)^2 |\mathbf{v}|$$

the uniform convergence of $\mathbf{k}_\varepsilon$, and pointwise convergence almost everywhere of ω_ε we conclude that

$$\mathbf{k}(t,s) - \mathbf{k}(t_0,s) = \lim_{\varepsilon\to 0} \mathcal{H}(s-t_0) \sum_{k=0}^{k_\varepsilon(t)} (t_{k+1} \wedge s - t_k) u(t_k) \times \mathbf{k}_\varepsilon(t_k^\varepsilon, s)$$
$$+ \lim_{\varepsilon\to 0} \sum_{k=0}^{k_\varepsilon(t)} \Big(\int_0^{t_k \wedge s} \omega_k^\varepsilon(\sigma) d\sigma\Big) \times \mathbf{k}(t_k^\varepsilon, s).$$

The final step is to show that the curve γ obtained is differentiable for a.e. $t \in [t_0,T]$. By the previous passages, the derivative $\mathbf{k}_t$ is well defined for a.e. $(\tau,\sigma) \in [t_0,T] \times [0,T]$ and satisfies a uniform bound $|\mathbf{k}_t| \leq C$. Hence there exists a negligible set of times in $[t_0,T]$ such that for all t not inside this set, the partial derivative $\mathbf{k}_t(t,s)$ exists for a.e.

$s \in [0, T]$. By the Lebesgue dominated convergence theorem and for a.e. $t \in [t_0, T]$ we obtain

$$\gamma_t(t,s) = \lim_{\varepsilon \to 0} \frac{\gamma(t+\varepsilon, s) - \gamma(t,s)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \int_0^s \frac{\mathbf{k}_t(\tau, \sigma)}{\varepsilon} d\sigma d\tau$$
$$= \int_0^s \frac{\mathbf{k}(t+\varepsilon, \sigma) - \mathbf{k}(t, \sigma)}{\varepsilon} d\sigma = \int_0^s \mathbf{k}_t(t, \sigma) d\sigma$$

This implies the Lipschitz continuity of γ as map from $[t_0, T]$ to $H^2([0,T], \mathbb{R}^3)$.

□

5 Numerical implementation

In this section we present some numerical simulations for the elastic model proposed in Section 2.2. We first show how to rewrite it in the planar case, and the simulations will be done in dimension two. For any vector $\mathbf{v} = (v_1, v_2)$ let $\mathbf{k}^\perp = (-v_2, v_1)$ be the perpendicular vector obtained by a counterclockwise rotation of $\pi/2$. Then the equation for the evolution of the tangential speed of the root is given by

$$\mathbf{k}_t(t,s) = \Big(\int_0^{t \wedge s} \omega(\sigma) d\sigma + \mathcal{H}(s-t) u(t) \Big) \mathbf{k}^\perp(t,s) \tag{5.1}$$

The root is discretized with uniform arc-length Δs and time step is taken such that $\Delta t = \Delta s$. Given an initial configuration, represented by an array of nodes, we construct a new node following the direction of the target. In our simulation the target will be the whole axis $y = 0$, which means that the first goal of the root is to grow in depth compatibly with the prescribed bound on the curvature, that is, the root cannot suddenly change direction. Simulations are carried out in Matlab.

Simulation 1 In the first simulation the obstacle is a disc with center $(a, b) = (1.4, 1)$ and radius $r = 0.3$. The root has origin in $(2, 2)$ and initial shape $y = x$ for $1.4 \leq x \leq 2$. We fix the bound on the curvature $\kappa_0 = 4$. As we can see in Fig 3.9, the root bends, avoid the obstacle and grows perpendicularly.

Simulation 2 Here we want to show how the restarting procedure works when the root is in breakdown configuration. The obstacle is now a disc with center $(a', b') = (1, 1.5)$ and radius $r = 0.3$. The root has origin in $(1, 2.5)$ and initial shape is $x = 1$ for $1.8 \leq y \leq 2.5$. The bound on the curvature is again $\kappa_0 = 4$. As we can see in Fig 3.10, the original root meets the obstacle perpendicularly with no curvature, hence by applying the restarting algorithm (2.12) the new attempt grows far from the previous one. In this way we can map the region already explored.

The main difference with respect to the simulations conducted in the case of the stems lies in the fact that the evolution of the curve is described by two different equations, one for the body and one for the tip. Therefore the way in which we add nodes is different, in fact in our case we have to take into account the control.
Furthermore the robotic root must be able to restart once a break configuration has been encountered. This involves having to take into account all previous attempts in order not to travel trajectories already investigated. We underline the fact that in our simulations

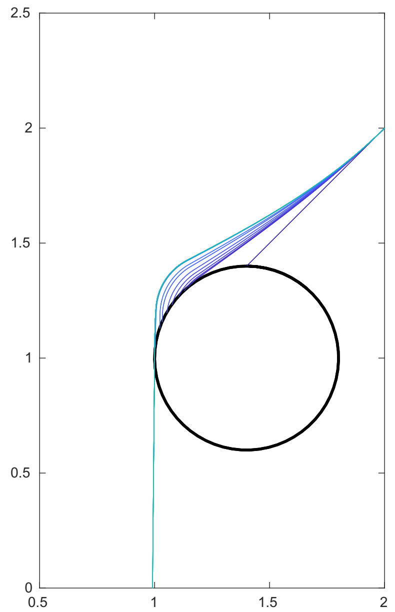


Figure 3.9: Artificial root not in breakdown configuration whose goal is to growth in depth

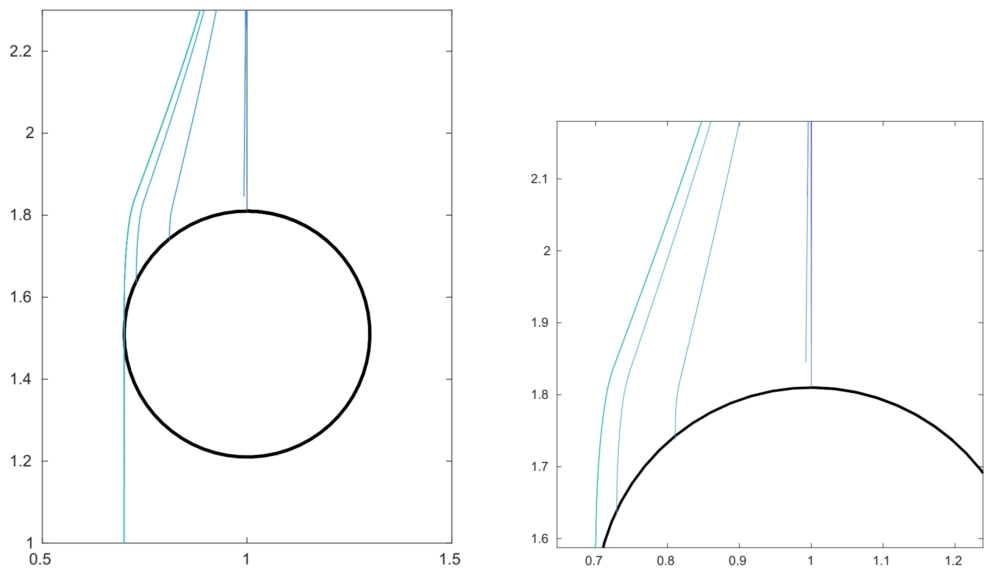


Figure 3.10: Artificial root in breakdown configuration which restarts

we are assuming zero soil density, i.e. the case in which for example the root descends into an empty cavity in which only stones are present.

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