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Four Essays on Dynamic Games Theory and Economic Applications

Coordinatore

Ch.mo Prof. Giovanni Colombo

Supervisore

Ch.mo Prof. Luca Grosset

Dottoranda

Chiara Brambilla

Ad Aakon,
e a me stessa.
Al futuro.

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Abstract

This thesis focuses on the study of differential games and their applications. In the first part, we describe the classical solution concepts of Nash and Stackelberg equilibria. To obtain results, we need tools coming from optimal control (OC), such as the Pontryagin Maximum Principle (PMP) and the Dynamic Programming (DP) approach. The aim is to study situations related to Stackelberg games. Whereas the theory of fixed final time hierarchical games is well defined, the variable time situation misses a proper formulation in the literature. We formalise and describe the solution procedure for this class of games by looking for an open-loop Stackelberg equilibrium (OLSE), where the final time becomes a decision variable for the leader. As typical for open-loop strategies, we face the time consistency issue by providing a couple of counterexamples. Then, we propose an applied model for infinite time horizon hierarchical games. Assume that a firm wants to invest in standard advertising and sponsored content on a media outlet highly concerned with its credibility. Since sponsored advertising may damage the media credibility, we allow the media to control its flow. We formalise the problem as a Stackelberg differential game where the media acts as the leader. We characterise a time-consistent OLSE despite the leader's upper bound on the follower's sponsored content control function. The introduction of this bound represents an innovation for this class of problems. The second part of the thesis is devoted to Markov chain-based models. We begin by proving that for the class of binary state continuous-time problems, the classical Hamilton–Jacobi–Bellman (HJB) equation can be substituted by an ordinary differential equation (ODE). This expedient allows us to obtain solutions for complex analytical situations. Later, we propose an application with infinitely many symmetric players based on offshoring and reshoring phenomena. We present a two-country model where the South country has lower production costs and the North country provides a huge tax discount to induce reshoring. We study the problem by considering a representative player. By employing the HJB to ODE transformation, we can compute a feedback solution for the problem and understand how national subsidies affect the process. Eventually, we modify the offshoring-reshoring model by making the incentives proportional to the amount of offshored businesses. As a result, the problem evolution turns out to be defined by coupled equations. The new formulation leads to a mean-field game (MFG) with no easy analytical solution. We obtain insights by numerical simulations.

Riassunto

Questa tesi tratta i giochi differenziali e le loro applicazioni.Nella prima parte della tesi, vengono definiti i concetti classici di equilibrio di Nash ed equilibrio di Stackelberg, per studiare i quali è necessario introdurre elementi della teoria del controllo ottimo (OC), tra cui il Principio del massimo di Pontryagin (PMP) e la programmazione dinamica (DP). Mentre, in generale, la teoria relativa a giochi di tipo gerarchico per problemi a tempo fisso è ben delineata, la letteratura è carente circa i problemi gerarchici a tempo variabile. A questo proposito, descriviamo la procedura per ottenere un equilibrio di Stackelberg open-loop (OLSE) in cui il tempo finale corrisponde ad una variabile decisionale per il leader del gioco. Per fare ciò, dobbiamo confrontarci con il problema dell'inconsistenza temporale delle soluzioni di tipo open-loop; problematica che viene enfatizzata attraverso alcuni controesempi. Successivamente, proponiamo un modello applicativo ad orizzonte infinito nel contesto pubblicitario. Si consideri un'azienda che vuole investire in due tipologie pubblicitarie, quella tradizionale e quella sponsorizzata, su una piattaforma mediatica. Poiché gli annunci sponsorizzati possono provocare danni alla credibilità percepita della piattaforma, assumiamo che il media abbia la facoltà di controllare la quantità di questa tipologia di advertising. Formalizziamo il problema attraverso un gioco alla Stackelberg ad orizzonte infinito, di cui il media è il leader. Nonostante la presenza di un vincolo imposto dal leader sulla variabile di flusso della pubblicità sponsorizzata del follower, siamo in grado di caratterizzare un equilibrio open-loop consistente nel tempo. Tale vincolo rappresenta un'innovazione in questa classe di giochi. Nella seconda parte della tesi, ci concentriamo su giochi basati su catene di Markov. Dopo aver dimostrato che, per problemi a tempo continuo e stato binario, la classica equazione di Hamilton–Jacobi–Bellman (HJB) può essere sostituita da un'equazione differenziale ordinaria (ODE), proponiamo un gioco con un numero infinito di giocatori nel contesto della rilocalizzazione della produzione aziendale. In un modello North-South, assumiamo che lo stato a sud sia caratterizzato da costi di produzione inferiori, mentre lo stato a nord mette in campo politiche di sconto fiscale con l'obiettivo di riportare all'interno la produzione. Considerando un giocatore rappresentativo e utilizzando quanto sopra, calcoliamo una soluzione di tipo feedback che ci permette di comprendere l'effetto dei sussidi nazionali nel processo di rilocalizzazione. Infine, modifichiamo tale modello rendendo gli incentivi proporzionali alla quantità di aziende che delocalizzano. Così facendo, il problema diventa un mean-field game (MFG) per il quale non è possibile ottenere una soluzione analitica. Analizziamo il problema tramite simulazioni numeriche.

Contents

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Listing of acronyms

DP .... Dynamic programming

FOC .... First order conditions

HJB .... Hamilton–Jacobi–Bellman

MFG .... Mean-Field game

MNE .... Markovian Nash equilibrium

NE .... Nash equilibrium

OC .... Optimal control

OCP .... Optimal control problem

ODE .... Ordinary differential equation

OLNE .... Open-loop Nash equilibrium

OLSE .... Open-loop Stackelberg equilibrium

PMP .... Pontryagin maximum principle

SE .... Stackelberg equilibrium

Notation

$\Gamma_i(\cdot, \cdot)$	Subgame
λ_i	Co-state function of the i-th player
$\mathcal{R}_i(\cdot)$	Set of the best response functions of the i-th player
$\mathcal{S}_i$	Set of strategies of the i-th player
Ψ_i	Terminal cost of the i-th player
ρ	Discount factor
f	Dynamics
g_i	Running cost of the i-th player
H_i^C	Current value Hamiltonian function of the i-th player
H_i	Hamiltonian function of the i-th player
J_i	Payoff functional of the i-th player
$T > 0$	Final time
$t \in [0, T]$	Time variable
u_i	Control function of the i-th player
$U_i \subset \mathbb{R}^p$	Control set of the i-th player
V_i^C	Current value function of the i-th player
V_i	Value function of the i-th player
$x \in \mathbb{R}^n$	State variable
$x_0 \in \mathbb{R}^n$	Initial state

1
Introduction

Optimisation theory deals primarily with maximising a real function by choosing the input value from a set of feasible values. In other words, we may also think of optimisation as a problem where an individual makes a decision in order to achieve a payoff, i.e., maximise a proper function [1]. Game theory is an extension of optimisation theory where more individuals, also called players, make choices to maximise their payoff. The main difference is that, in game theory, individuals are interdependent; namely, each player's gain depends not only on her/his choice but also on the behaviour of all other players. Since all players affect all payoffs, it is difficult to think of a combination of all player choices that would let all individuals achieve their goal of maximising their payoff. In fact, in general, such a solution does not exist. For this reason, the literature introduces new and different concepts of solutions based on the features of the game.

To give an idea of all the possibilities offered by this rich theory, we have different concepts of solutions for games concerning the fact that players cooperate or not, decide simultaneously or subsequently, have a complete or incomplete knowledge of the game and its evolution, etc. Another important distinction within games concerns how the choice happens. If players make only one instantaneous decision that completely determines the payoffs, then we are in a "one-shot" game, also known as a *static game*. Otherwise, there exist games where players make decisions over a whole interval of time. The decision variable becomes a function of time, also called *strategy*. This kind of game is also called *dynamic game*.

More specifically, my thesis focuses on *differential games*, which are special cases of dynamic games based on optimal control theory (individual dynamic continuous-time optimisation problems [2]) where a differential equation describes the game evolution. The beauty of this theory lies in the variety of concepts and possibilities of applications allowed. For instance, in differential games, the type of the selected strategy determines different concepts of solutions depending on which kind of information the individuals apply to make it, e.g. players consider only time or the evolved state of the game too, etc.

From an application point of view, dynamic games have become a standard tool for economic analysis [3]. Moreover, the interdependence of individuals typical of games is well-known in many areas of economics and management science [4]. Among the main applications, we find advertising models [5] and supply chain problems [6]. In both applications, hierarchical situations become increasingly relevant [7], making the corresponding concept of Stackelberg equilibrium a decisive matter.
The first part of the thesis concerns this topic.

1.1 Part I: Stackelberg differential games

A differential game played à la Stackelberg allows one to study situations where players have asymmetrical roles. For this reason, they are called leader and follower, and the game occurs as follows [4, Ch.5, p.113]: the leader first declares her/his strategy, and then the follower chooses her/his best response to the leader's announcement. At this point, knowing the follower's response, the leader picks her/his best strategy choice.

Chapter 3: Free final-time Stackelberg differential games In this chapter, we introduce the concept of free final-time Stackelberg differential games. This class of games is useful for describing economic problems in which time is a decision variable. We will assume time to be free and a decision variable for the leader. We will define a free final-time Stackelberg differential game and describe the analytical procedure to characterise such an equilibrium. Moreover, we propose a numerical example to directly show the characterisation of an open-loop equilibrium. Finally, we propose a couple of time-consistency counterexamples.

Chapter 4: A differential game model for sponsored content We propose an application model of Stackelberg differential games with an infinite time horizon. Let us consider a communication platform distinguished for its high-quality content, where advertising can take two different forms: traditional and sponsored (also known as native advertising in the marketing literature). Native advertising is a widely used marketing tool that aims to mimic the regular topics of the platform on which it is placed. Due to this striking resemblance, native advertising may be very effective, but at the same time, it may negatively influence the perceived credibility of the media outlet. In our model, a firm allocates investments to both traditional and native advertising on such a platform. Meanwhile, the media outlet must grapple with the trade-off between the profit accrued from publishing native advertising and the ensuing decline in credibility. We formalise this problem as a hierarchical infinite-time horizon linear state differential game, played à la Stackelberg, where the media outlet acts as the leader while the firm is the follower. Finally, we characterise a time-consistent open-loop equilibrium and obtain the conditions that make it optimal for the media outlet to accept native advertising. An innovative feature of our model is the presence of a leader control as a constraint to a follower control function. This may result in a time-inconsistent equilibrium. However, we gain in proving that, under suitable assumptions, we still have a time-consistent open-loop equilibrium for the model.

1.2 Part II: Markov based models with binary state

Interesting is also the application of Markov chain-based models. Considering the specific case of continuous-time Markov chains with binary state, we are able to analyse problems with infinitely many players.

Chapter 5: Continuous-Time Markov Decision Problems with Binary State We analyse a binary state continuous-time Markov decision problem. The standard Hamilton–Jacobi–Bellman equation is introduced and, with suitable assumptions on the probability rate and on the cost function, it can be replaced by a simpler backward differential equation. Through a numerical example, we show how to find an optimal feedback control using the results presented in this chapter.

Chapter 6: A Binary-State Continuous-Time Markov Chain Model for Offshoring and Reshoring We present a two-country model (North and South) that describes the phenomenon of offshoring and reshoring. The model is a continuous time-controlled Markov chain with binary states. The main trade-off involves production costs and transaction costs between one country and another. In the first part of this paper, we identify the key parameters of the model: the difference in unit production costs between the two countries considered, the marginal cost of transitioning between countries, and the incentive paid by the North country to all companies that have not relocated at the end of the planning interval. We aim to understand how national tax incentives can influence this process.

Chapter 7: From the Offshoring-Reshoring model to Mean-Field games As a development for the model presented in Chapter 6, we modify the structure of the problem by making the final incentive proportional to the amount of offshored businesses. Consequently, the problem turns into a mean-field game with no easy analytical solution. Once we have introduced the new formulation, we will base our analysis on numerical simulations.

1.3 Structure of the thesis

This thesis consists of seven chapters. The first part (Chapters 2 to 4) is devoted to introducing Stackelberg differential games and then to providing applications. The second part (Chapters 5 to 7) deals with the theory and application of differential games based on Markov processes. More precisely, games with a binary state structure. Since most of my thesis is based on four papers published during my doctorate, here are the main references.

- Chapter 3 is based on [8].
- Chapter 4 is based on [9].
- Chapter 5 is based on [10].
- Chapter 6 is based on [11].

Part I

Stackelberg differential games

2
Differential games

In this chapter, we introduce all the definitions and results that we will need in the thesis. Let us start by introducing the concept of differential games and their applications in the economic field. The theory illustrated below refers mainly to [1], [4], and [12] for games, and to [2], [13] and [14] for control theory.

2.1 Introduction

To simplify the comprehension of the problem, let us introduce the main theory referring to a two-player differential game. In a second part of the thesis (see Chapters 6 and 7), we are also going to introduce the concept of multi-player games with, in fact, infinitely many players.

We are considering continuous-time problems over a programming interval $[0, T]$, where T may be finite or infinite (in this last case the programming interval is $[0, +\infty)$, to avoid heavy notations the interval of time will be denoted as in the finite case when the infinite time horizon is not specified), fixed or variable depending on the game. At first, focus on fixed finite final-time problems. Let $x \in \mathbb{R}^n$, with $n \in \mathbb{N}$ be the *state variable* of the system, which evolves according to an ordinary differential equation (ODE), also called *dynamics* of the game

$$\dot{x}(t) = f(x, u_1, u_2, t) \tag{2.1}$$

with $t \in [0, T]$ and with initial condition $x(0) = x_0 \in \mathbb{R}^n$. The context is very similar to that given by the optimal control (OC) theory; however, the dynamics of the game (2.1) is affected by the actions taken by each player $u_1(\cdot)$ and $u_2(\cdot)$, which are also called *control functions* of the players. Given $U_1 \subset \mathbb{R}^p$ and $U_2 \subset \mathbb{R}^q$ (known as *control sets*), with $p,\ q \in \mathbb{N}$, let us assume the pointwise constraints $u_1(t) \in U_1$, $u_2(t) \in U_2$ for all $t \in [0, T]$.
The objective of each player is to maximise her/his *payoff function*

$$J_i(u_1, u_2) := \int_0^T g_i(x(s), u_1(s), u_2(s), s)\, ds + \Psi_i(x(T)) \tag{2.2}$$

with $i = 1, 2$. Moreover, function $g_i : \mathbb{R}^n \times U_1 \times U_2 \times [0, T] \to \mathbb{R}$ is also called *running cost*, and function $\Psi_i : \mathbb{R}^n \to \mathbb{R}$ is known as *terminal payoff*.

Game theory is based on fundamental axioms, namely that players are rational and think strategically. Players' rationality is associated with the presence of a main goal, represented by the maximisation of a payoff function. Furthermore, since each payoff is affected by the choice of all players, the interdependence will influence each player's choice. In a differential game, it is fundamental to specify the information available to the individuals because their strategies depend on the information available at each time t.
In this thesis, we consider only games with perfect information, which means that, at every instant of time, each player is completely aware of

- the dynamics f of the system,
- the control sets U_1 and U_2 of the two players,
- the payoffs J_1, J_2.
- time $t \in [0, T]$, and
- the initial state x_0.

However, when setting up a differential game, one also needs to specify upon which information the choice of action at time t is based. More precisely, we are going to present different cases related to the fact that players know about

- the current state of the system $x(t)$, and
- the control $u(\cdot)$ implemented by the other player.

Depending on what is above, we may have different concepts of strategies, where a strategy is a function that tells the player how to select one of his feasible actions for all possible events that may have occurred so far. If players ignore the state $x(t)$ of the system, they have to select an *open-loop strategy*; otherwise, they can pick a *Markovian/feedback strategy*. In addition, if one of the payers is informed of the opponent's strategy, the game is said to be *hierarchical*. Let us give a precise definition of these different situations.

Before the formal definitions, let me summarise the differential game as follows.

$$\text{Player I:} \quad \max_{u_1} J_1(u_1, u_2) \qquad\qquad \text{Player II:} \quad \max_{u_2} J_2(u_1, u_2)$$

$$J_i(u_1, u_2) = \int_0^T g_i(x(s), u_1(s), u_2(s), s)\, ds + \Psi_i(x(T))\,, \quad i = 1, 2 \tag{2.3}$$

$$\dot{x}(t) = f(x, u_1, u_2, t)\,, \quad x(0) = x_0$$

Definition 2.1.1 (Open-loop strategy). An open-loop strategy for the i-th player is any measurable function $u_i : [0, T] \to U_i$. The set of all available strategies for the i-th player $\mathcal{S}_i$ is the set of all possible measurable functions $t \to u_i(t) \in U_i$.

Definition 2.1.2 (Markovian strategy / Feedback strategy). A Markovian/feedback strategy for the i-th player is any measurable functions $u_i : [0, T] \times \mathbb{R}^n \to U_i$. The set of all available strategies for the i-th player $\mathcal{S}_i$ is the set of all possible measurable functions $(t, x) \to u_i(t, x) \in U_i$.

Whereas open-loop strategies can depend only on time, Markovian ones depend also on the current state $x(t)$ of the system.

Now, we can properly define the solution concept of *Nash* and *Stackelberg equilibrium* for differential games. This class of solutions is typical of the non-cooperative games in which we are interested.

2.2 Nash Equilibria

2.2.1 Open-loop Nash Equilibrium

Intuitively, a *Nash Equilibrium* (NE) for the game (2.3) is a couple of strategies $(u_1^*, u_2^*) \in U_1 \times U_2$ such that

$$J_1(u_1, u_2^*) \leq J_1(u_1^*, u_2^*) \qquad \forall u_1 \in U_1$$
$$J_2(u_1^*, u_2) \leq J_2(u_1^*, u_2^*) \qquad \forall u_2 \in U_2\,;$$

therefore, given the opponent best strategy, each player's Nash Equilibrium strategy guarantees the greatest payoff possible.

Fixed finite final time games

Consider now the game in (2.3) with fixed final time $T < +\infty$.

Definition 2.2.1 (Open-loop Nash Equilibrium). Function $t \to (u_1^*(t), u_2^*(t))$ is an *open-loop Nash equilibrium (OLNE)* for the game (2.3) if $u_1^*(\cdot)$ is a solution for the optimal control problem of Player I

$$\max_{u_1} J_1(u_1, u_2^*) = \int_0^T g_1(x(s), u_1(s), u_2^*(s), s)\, ds + \Psi_1(x(T))$$
$$\text{over all possible strategies } u_1 : [0, T] \to U_1 \text{ with dynamics}$$
$$\dot{x}(t) = f(x, u_1, u_2^*, t)\,, \quad x(0) = x_0$$

and $u_2^*(\cdot)$ is a solution for the optimal control problem of Player II

$$\max_{u_2} J_2(u_1^*, u_2) = \int_0^T g_2(x(s), u_1^*(s), u_2(s), s)\, ds + \Psi_2(x(T))$$
$$\text{over all possible strategies } u_2 : [0, T] \to U_2 \text{ with dynamics}$$
$$\dot{x}(t) = f(x, u_1^*, u_2, t)\,, \quad x(0) = x_0\,.$$

Searching for a Nash Equilibrium, we need to solve two optimal control problems simultaneously. The best way to proceed is to apply optimal control tools such as the *Pontryagin Maximum Principle (PMP)*.

Theorem 2.2.2. *Consider the game (2.3) with f, g_i and Ψ_i in C^1. Let (u_1^*, u_2^*) be a Nash equilibrium in the class of open–loop strategies with associated trajectory x^*. Then, there exist continuous co-state function $\lambda_i^* : [0, T] \to \mathbb{R}^n$, with $i = 1, 2$, such that*

1. $\lambda_i \neq 0$ *(non-triviality of multiplier)*

2. $$u_1^* \in \arg\max_{u \in U_1} H_1(x^*, u, u_2^*, \lambda_1^*, t)$$
   $$u_2^* \in \arg\max_{u \in U_2} H_2(x^*, u_1^*, u, \lambda_2^*, t) \quad \textit{for all } t \in [0, T] \qquad \textit{(maximum principle)}$$

3. $$\dot{\lambda}_i^*(t) = -\frac{\partial H_i(x^*, u_1^*, u_2^*, \lambda_i^*, t)}{\partial x} \quad \textit{a.e. in } [0, T] \qquad \textit{(adjoint equation)}$$

4. $\lambda_i^*(T) = \nabla_x \Psi_i(x^*(T))$ *(transversality condition).*

where $H_i(x, u_1, u_2, \lambda_i, t)$ is the Hamiltonian function of the $i - th$ player, with $i = 1, 2$,

$$H_i(x, u_1, u_2, \lambda_i, t) := g_i(x, u_1, u_2, t) + \lambda_i(t) \cdot f(x, u_1, u_2, t) \,. \tag{2.4}$$

Proof. The proof follows a double application of the Pontryagin Maximum Principle over both players' optimal control problems [2, Theorem 2, pag. 85]. □

Remark 1. Theorem 2.2.2 provides only the necessary conditions for Nash equilibria. However, assuming also that the control sets U_i are convex and functions $(x, u) \to H_1(x, u, u_2^*, \lambda_1^*, t)$ and $(x, u) \to H_2(x, u_1^*, u, \lambda_2^*, t)$ are concave for every $t \in [0, T]$, the above theorem also provides sufficient conditions [2, Theorem 4, p. 105].

Infinite time horizon games

In order to analyse the infinite final time case, it is useful to reformulate the game. More precisely, we will consider an autonomous problem.

$$\text{Player I:} \quad \max_{u_1} J_1(u_1, u_2) \qquad\qquad \text{Player II:} \quad \max_{u_2} J_2(u_1, u_2)$$
$$J_i(u_1, u_2) = \int_0^{+\infty} e^{-\rho s} g_i(x(s), u_1(s), u_2(s))\, ds \,, \quad i = 1, 2 \tag{2.5}$$
$$\dot{x}(t) = f(x(t), u_1(t), u_2(t)) \,, \quad x(0) = x_0$$

where $\rho > 0$ is called *discount factor*.
For infinite horizon games it is useful to introduce the *current value Hamiltonian function* for the i-th player:

$$H_i^C(x, u_1, u_2, \lambda_{0,i}, \lambda_i) := \lambda_{0,i}(t) \cdot g_i(x(t), u_1(t), u_2(t)) + \lambda_i(t) \cdot f(x(t), u_1(t), u_2(t))\,; \quad (2.6)$$

furthermore, the definition of OLSE is as described in (2.2.1) with small changes to conform it to the game in (2.5).

Theorem 2.2.3. *Consider the game (2.3) with f and g_i in C^1. Let (u_1^*, u_2^*) be a Nash equilibrium in the class of open–loop strategies with associated trajectory x^*. Then, there exist continuous co-state function $\lambda_i^* : [0, +\infty) \to \mathbb{R}^n$ and a constant $\lambda_{0,i} \geq 0$, with $i = 1, 2$, such that*

1. $(\lambda_{0,i}, \lambda_i) \neq (0, 0)$ *(non-triviality of multipliers)*

2. $u_1^* \in \arg\max_{u \in U_1} H_1^C(x^*, u, u_2^*, \lambda_{0,1}, \lambda_1^*, t)$
   $u_2^* \in \arg\max_{u \in U_2} H_2^C(x^*, u_1^*, u, \lambda_{0,2}, \lambda_2^*, t)$ *for all $t \in [0, +\infty)$* *(maximum principle)*

3. $\dot{\lambda}_i^*(t) = \rho\lambda_i(t) - \dfrac{\partial H_i^C(x^*, u_1^*, u_2^*, \lambda_{0,i}, \lambda_i^*, t)}{\partial x}$ *a.e. in $[0, +\infty)$* *(adjoint equation).*

Proof. See [13, Theorem 3.67, pag. 156]. □

Remark 2. The above result does not have transversality conditions. In fact, with the additional assumption of the concavity of the current Hamiltonian function H_i with respect to x, λ_i and t, and the normality of the control u_i (that is, $\lambda_{0,i} = 1$), we can obtain sufficient conditions with, in addition, the following:

$$\lim_{t \to +\infty} \lambda(t) \cdot (x(t) - x^*(t)) \geq 0$$

for all admissible trajectories $x(\cdot)$ [13, Theorem 3.70, pag. 159].

2.2.2 Markovian Nash Equilibrium

Definition 2.2.4 (Markovian Nash Equilibrium)**.** Function $(t, x) \to (u_1^*(t, x), u_2^*(t, x))$ is an *Markovian Nash equilibrium (MNE)* for the game (2.3) if $u_1^*(\cdot, \cdot)$ is a solution for the optimal control problem of Player I

$$\max_{u_1} J_1(u_1, u_2^*) = \int_0^T g_1(x(s), u_1(s, x(s)), u_2^*(s, x(s)), s)\, ds + \Psi_1(x(T))$$

over all possible strategies $u_1 : [0, T] \times \mathbb{R}^n \to U_1$ with dynamics

$$\dot{x}(t) = f(x, u_1, u_2^*, t), \quad x(0) = x_0$$

and $u_2^*(\cdot, \cdot)$ is a solution for the optimal control problem of Player II

$$\max_{u_2} J_2(u_1^*, u_2) = \int_0^T g_2(x(s), u_1^*(s, x(s)), u_2(s, x(s)), s)\, ds + \Psi_2(x(T))$$

over all possible strategies $u_2 : [0, T] \times \mathbb{R}^n \to U_2$ with dynamics

$$\dot{x}(t) = f(x, u_1^*, u_2, t), \quad x(0) = x_0 .$$

In order to determine a MNE, another useful tool coming from optima control is the *dynamic programming* (DP) approach.

Definition 2.2.5. Let (u_1^*, u_2^*) be a Markovian Nash equilibrium. We define the *value function* $V_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ for the players by

$$V_1(\tau, y) := \int_\tau^T g_1(x^*(s), u_1^*(s), u_2^*(s), s)\, ds + \Psi_1(x^*(T))$$
$$V_2(\tau, y) := \int_\tau^T g_2(x^*(s), u_1^*(s), u_2^*(s), s)\, ds + \Psi_2(x^*(T))$$

where $x^* = x^*(\cdot; \tau, y)$ is the unique solution of

$$\begin{cases} \dot{x}(t) = f(u_1^*, u_2^*, t) & \text{a.e. in } [\tau, T] \\ x(\tau) = y . \end{cases}$$

Fixed finite final time games

Theorem 2.2.6. *Consider the game (2.3), with f, g_i and Ψ_i in C^1. Let (u_1^*, u_2^*) be a couple of Markovian strategies with associated trajectory x^*. If there exist two continuously differentiable* *functions $V_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, with $i = 1, 2$, such that, for every $(t,x) \in [0,T] \times \mathbb{R}^n$, it holds*

$$\begin{aligned}
-\frac{\partial V_1(t,x)}{\partial t} &= \max_{u \in U_1} [g_1(x,u,u_2^*,t) + \nabla_x V_1(t,x) \cdot f(x,u,u_2^*,t)] \\
&= g_1(x,u_1^*,u_2^*,t) + \nabla_x V_1(t,x) \cdot f(x,u_1^*,u_2^*,t) \\
V_1(T,x) &= \Psi_1(x(T))
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial V_2(t,x)}{\partial t} &= \max_{u \in U_2} [g_2(x,u_1^*,u,t) + \nabla_x V_2(t,x) \cdot f(x,u_1^*,u,t)] \\
&= g_2(x,u_1^*,u_2^*,t) + \nabla_x V_2(t,x) \cdot f(x,u_1^*,u_2^*,t) \\
V_2(T,x) &= \Psi_2(x(T))
\end{aligned}$$

Then, (u_1^*, u_2^*) is a MNE and x^* is the associated optimal trajectory. Finally, V_i, for $i = 1, 2$, are the actual value functions for the problem (2.3) in the considered Nash equilibrium.

Proof. See [12, Theorem 2.1, pag. 9]. □

Infinite time horizon games

Let us consider the game defined in (2.5) and the current value Hamiltonian function defined in (2.6). Let us introduce the *current value function* V_i^C for the i-th player as

$$V_i(t,x) = e^{-\rho t} V_i^C(x) \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R}^n .$$

Theorem 2.2.7. *Consider the game (2.5), with f and g_i in C^1. Let (u_1^*, u_2^*) be a couple of Markovian strategies with associated trajectory x^*. If there exist two bounded continuously differentiable functions $V_i^C : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, 2$, such that, for every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$, it holds*

$$\begin{aligned}
\rho V_1^C(x) &= \max_{u \in U_1} \left[g_1(x,u,u_2^*,t) + \nabla_x V_1^C(t,x) \cdot f(x,u,u_2^*,t)\right] \\
&= g_1(x,u_1^*,u_2^*,t) + \nabla_x V_1^C(t,x) \cdot f(x,u_1^*,u_2^*,t)
\end{aligned}$$

$$\begin{aligned}
\rho V_2^C(x) &= \max_{u \in U_2} \left[g_2(x,u_1^*,u,t) + \nabla_x V_2^C(t,x) \cdot f(x,u_1^*,u,t)\right] \\
&= g_2(x,u_1^*,u_2^*,t) + \nabla_x V_2^C(t,x) \cdot f(x,u_1^*,u_2^*,t)
\end{aligned}$$

Then, (u_1^*, u_2^*) is a Markovian Nash equilibrium and x^* is the associated optimal trajectory. Finally, V_i^C, for $i = 1, 2$, are the actual current value functions for the problem (2.5) in the consid-

ered Nash equilibrium.

Proof. See [4, Theorem 4.1, pag. 92] □

2.3 Stackelberg Equilibria

In Section 2.2, we assume that the players decide simultaneously. However, hierarchical situations are very important for applications. For this reason, let us introduce the concept of hierarchical games. Assume that players make their choices subsequently; in this situation, the individual who decides next is aware of the opponent's choice, and then she/he has an advantage. For this reason, in hierarchical play, the player who lastly chooses is also called the *leader*, while the opponent is known as the *follower*.

To make it easier to understand how to deal with this kind of game, let me briefly summarise step-by-step its evolution. To simplify notation, let us denote by u_L the leader's strategy and u_F the follower's, and, similarly, all other game components.

1. First, the leader chooses and declares her/his strategy $t \to u_L(t)$;

2. Then, the follower assumes as given the leader's strategy and solves her/his optimal control problem to determine its optimal choice u_F. Since $u_F = u_F(u_L)$ depends on the declared leader's strategy, it is not yet the optimal follower choice, but the *follower's best response function* to the leader's declared strategy u_L.

3. At this point, the leader solves her/his optimal control problem by knowing the follower's best response function, obtaining so u_L^*.

4. Given the optimal leader choice, the follower can finally compute her/his optimal control u_F^*.

This process is independent of the kind of strategies the players select.

For the reader's convenience, I recall the game in (2.3) with the new notation.

$$\text{Leader:} \quad \max_{u_L} J_L(u_L, u_F) \qquad\qquad \text{Follower:} \quad \max_{u_F} J_F(u_L, u_F)$$

$$J_i(u_L, u_F) = \int_0^T g_i(x(s), u_L(s), u_F(s), s)\, ds + \Psi_i(x(T)) \,, \quad i = L, F \tag{2.7}$$

$$\dot{x}(t) = f(x, u_L, u_F, t) \,, \quad x(0) = x_0$$

where $u_L : [0, T] \to U_L$ and $u_F : [0, T] \to U_F$ are measurable functions ad $U_L,\ U_F$ are the control sets.

2.4 Open-loop Stackelberg equilibrium

Before we define Stackelberg equilibria, it is useful to formalise the concept of follower's best response function with respect to open-loop strategies.

Definition 2.4.1 (Follower's best response function)**.** Let us assume the leader to declare an admissible control $\hat{u}_L : [0, T] \to U_L$ for the game (2.7). The set of the best response function for the follower $\mathcal{R}_F(\hat{u}_L)$ is the set of all measurable functions $u_F : [0, T] \to U_F$ that solves the follower's optimal control problem

$$\max_{u_F} J_F(\hat{u}_L, u_F) = \int_0^T g_F(x(s), \hat{u}_L(s), u_F(s), s)\, ds + \Psi_F(x(T))$$
$$\dot{x}(t) = f(x, \hat{u}_L, u_F, t)\,, \quad x(0) = x_0\,.$$

Definition 2.4.2 (Open-loop Stackelberg equilibrium)**.** A pair of open-loop strategies (u_L^*, u_F^*) is a Stackelberg equilibrium (OLSE) for the game (2.7) if

1. $u_F^* \in \mathcal{R}_F(u_L^*)$;
2. given any admissible control u_L for the leader and any best response function $u_F \in \mathcal{R}_F(u_L)$ for the follower, it holds

$$J_L(u_L, u_F) \leq J_L(u_L^*, u_F^*)\,.$$

From the above definition, it may seem that the hierarchical play affects more the follower's problem than the leader's. In fact, the leader's optimal control problem is deeply impacted by the asymmetrical structure. In general, the follower cannot directly compute its best response function regardless of the leader's declared strategy. Therefore, when solving its optimal control problem, the leader has to take into account not only the opponent's best response but also its co-state functions if not explicitly computed.
More precisely, let us denote by $\lambda_F^* : [0, T] \to \mathbb{R}^n$ the follower's co-state function, that is, the

solution of

$$\begin{cases} \dot{\lambda}_F^*(t) = -\dfrac{\partial H_F(x^*, u_L^*, u_F^*, \lambda_F^*, t)}{\partial x} \\ \lambda_F^*(T) = \nabla_x \Psi_F(x^*(T)) \end{cases}$$

for almost every $t \in [0, T]$. Due to the dependence of the follower's control on the leader's strategy, the follower's best response function is a function $u_F = u_F^b(x, \hat{u}_L, \lambda_F, t)$. Therefore, the leader's OCP becomes

$$\max_{u_L} \left\{ \int_0^T g_L(x(s), u_L(s), u_F^b(x, u_L, \lambda_F, s), s)\, ds + \Psi_L(x(T)) \right\}$$

with dynamics

$$\begin{cases} \dot{\lambda}_F(t) = -\dfrac{\partial H_F(x, u_L, u_F^b(x, u_L, \lambda_F, t), \lambda_F, t)}{\partial x} & \lambda_F(T) = \nabla_x \Psi_F(x(T)) \\ \dot{x}(t) = f(x, u_L, u_F^b(x, u_L, \lambda_F, t), t) & x(0) = x_0 \,. \end{cases}$$

In order to consider the dependence on the follower's co-state function of the leader's optimal control problem, we introduce the following notation alteration:

$$\begin{aligned} g_L(x, u_L, \lambda_F, t) &:= g_L(x, u_L, u_F^b(x, u_L, \lambda_F, t), t) \\ f(x, u_L, \lambda_F, t) &:= f(x, u_L, u_F^b(x, u_L, \lambda_F, t), t) \,. \end{aligned} \tag{2.8}$$

Moreover, given the follower's Hamiltonian function

$$H_F(x, u_L, \lambda_F, t) := H_F(x, u_L, u_F^b(x, u_L, \lambda_F, t), \lambda_F, t)$$

the leader's Hamiltonian function become

$$\begin{aligned} H_L(x, u_L, \lambda_{L,0}, \lambda_{L,1}, \lambda_{L,2}, \lambda_F, t) &:= \lambda_{L,0} \cdot g_L(x, u_L, \lambda_F, t) + \\ &+ \lambda_{L,1}(t) \cdot f(x, u_L, \lambda_F, t) + \lambda_{L,2}(t) \cdot \left(-\frac{\partial H_F(x, u_L, \lambda_F, t)}{\partial x} \right) \,. \end{aligned} \tag{2.9}$$

Fixed finite final time games

Theorem 2.4.3 (Open-loop Stackelberg equilibrium). *Let (u_L^*, u_F^*) be an open-loop Stackelberg equilibrium for the game (2.7). Assume that, for every $(x, u_L, \lambda_F, t) \in \mathbb{R}^n \times U_L \times \mathbb{R}^n \times \mathbb{R}$, there* *exists a unique solution for*

$$u_F^b(x, u_L, \lambda_F, t) := \arg\max_{u \in U_F} g_F(x, u_L, u, t) + \lambda_F(t) \cdot f(x, u_L, u, t) \, .$$

Moreover, suppose that for every $t \in [0, T]$ *and* $u_L \in U_L$

$$\begin{aligned}
&(x, \lambda_F) \to g_L(x, u_L, \lambda_F, t) \\
&(x, \lambda_F) \to f(x, u_L, \lambda_F, t) \\
&(x, \lambda_F) \to H_L(x, u_L, \lambda_L, \lambda_F, t)
\end{aligned}$$

as defined in (2.8), and

$$x \to \nabla_x \Psi_L(x)$$

are continuously differentiable. Let x^* *and* λ_F^* *be the associated trajectory to the OLSE and the co-state function of the follower satisfying the follower's OCP necessary conditions. Then, there exist a constant* $\lambda_{L,0} \geq 0$ *and two absolute continuous functions* $\lambda_{L,i} : [0, T] \to \mathbb{R}^n$, $i = 1, 2$, *such that*

1. $(\lambda_{L,0}, \lambda_{L,1}, \lambda_{L,2}) \neq (0, 0, 0)$

2. $u_L^*(t) \in \arg\max_{u \in U_L} H_L(x^*, u_L, \lambda_{L,0}, \lambda_{L,1}, \lambda_{L,2}, \lambda_F^*, t)$ *for a. e.* $t \in [0, T]$

3. $$\begin{aligned}
\dot{\lambda}_{L,1}^*(t) &= -\frac{\partial H_L(x^*, u_L^*, \lambda_{L,0}^*, \lambda_{L,1}^*, \lambda_{L,2}^*, \lambda_F^*, t)}{\partial x} \\
\dot{\lambda}_{L,2}^*(t) &= -\frac{\partial H_L(x^*, u_L^*, \lambda_{L,0}^*, \lambda_{L,1}^*, \lambda_{L,2}^*, \lambda_F^*, t)}{\partial \lambda_F} \quad \textit{for a. e. } t \in [0, T]
\end{aligned}$$

4. $$\begin{aligned}
\lambda_{L,1}^*(T) &= \lambda_{L,0} \cdot \nabla_x \Psi_L(x^*(T)) - \lambda_{L,2} \cdot D_x^2 \Psi_F(x^*(T)) \\
\lambda_{L,2}^*(T) &= 0 \, ,
\end{aligned}$$

where $D_x^2 \Psi_F(x^*(T))$ *is the Hessian matrix of second derivatives of* Ψ_F*, at x.*

Proof. See [1, Theorem 4.4, pag. 386]. □

Infinite time horizon games

To avoid confusion, let me recall the infinite horizon problem under the leader-follower notation.

$$\text{Leader:} \quad \max_{u_L} J_L(u_L, u_F) \qquad\qquad \text{Follower:} \quad \max_{u_F} J_F(u_L, u_F)$$

$$J_i(u_L, u_F) = \int_0^{+\infty} e^{-\rho s} g_i(x(s), u_L(s), u_F(s))\, ds\,, \quad i = L, F \tag{2.10}$$

$$\dot{x}(t) = f(x(t), u_L(t), u_F(t))\,, \quad x(0) = x_0$$

Remark 3. In an infinite time horizon game, a Stackelberg equilibrium is subject to necessary conditions analogous to (2.4.3). To better understand the effects of $T = +\infty$, see also (2.2.3).

2.5 Time consistency

In finite and infinite time games, the co-dependency of the leader and follower's OC problems may result in a consistency over time issue for the equilibrium.

To better understand this concept, let me introduce some notation. Denote by $\Gamma(x_0, 0)$ the game defined in (2.3). I will focus on the finite time problem; the extension to the infinite horizon game is straightforward. For each $(\xi, \tau) \in \mathbb{R}^n \times [0, T]$, define the *subgame* $\Gamma(\xi, \tau)$ by modifying as follows the payoffs and the dynamics.

$$J_i(u_1, u_2) = \int_\tau^T g_i(x(s), u_1(s), u_2(s), s)\, ds + \Psi_i(x(T))\,, \quad i = 1, 2$$

$$\dot{x}(t) = f(x, u_1, u_2, t)\,, \quad x(\tau) = \xi\,,$$

that is, the game is defined in the programming interval $[\tau, T]$ with initial condition $x(\tau) = \xi$.

Definition 2.5.1 (Time consistency). Let (u_1^*, u_2^*) be an equilibrium for (2.3), with associated trajectory x^*. The equilibrium is said to be time-consistent if, for every $t \in [0, T]$, looking for the same type of equilibrium in the subgame $\Gamma(x(t), t)$, there exists a solution (v_1^*, v_2^*) such that $v_i^* = u_i^*|_{[t,T]}$ for $i = 1, 2$.

Definition 2.5.2 (Subgame perfectness). Let (u_1^*, u_2^*) be an equilibrium for (2.3), with associated trajectory x^*. The equilibrium is said to be subgame perfect if, for every $t \in [0, T]$ and

$\xi \in \mathbb{R}^n$, looking for the same type of equilibrium in the subgame $\Gamma(\xi, t)$, there exists a solution (v_1^*, v_2^*) such that $v_i^* = u_i^*|_{[t,T]}$ for $i = 1, 2$.

Since subgame perfectness does not require consistency only along the optimal trajectory, it is clear by definition that every equilibrium that is subgame perfect is also time consistent.

Remark 4. Every Markovian Nash equilibrium is time-consistent [4, Theorem 4.3, pag. 100]. The same does not hold for open-loop equilibria. This issue becomes even more significant for Stackelberg equilibria. A clear description of the time inconsistency issue for hierarchical play is described in [12]:

> In a Stackelberg game, the leader seeks to influence the follower's choice of strategy in a way that is favourable to the leader. For this purpose the leader announces - at time zero - the strategy that she will use for $t \in [0, T]$. However, given the option to re-optimise, the leader realises that although the follower has "kept her promises", she has herself no reason to keep hers. Indeed, it is in her own best interest to depart from the originally announced time path. The follower, however, sees through all this and does not believe that the leader will stick to her plan.

Nevertheless, under suitable assumptions, we are able to obtain time-consistent OLSE. For this aim, let me introduce another useful definition.

Definition 2.5.3 (Non-controllability of multipliers)**.** Consider a hierarchical game as (2.7). The follower's co-state variable $\lambda_F(\cdot)$ is said to be non-controllable by the leader if it is independent of the leader's control function $u_L(\cdot)$. Otherwise, it is said to be controllable.

In general, the above definition is not sufficient to guarantee an OLSE to be time-consistent. However, for the goal of this thesis, the non-controllability of the follower's co-state functions will be a sufficient condition for time consistency under special circumstances.

Theorem 2.5.4. *Every open-loop Stackelberg equilibrium for a linear-state game is subgame perfect.*

Proof. See [15, Theorem 1, pag 271]. □

The above discussion of differential games does not consider free final time problems, which are, instead, particularly suitable for in asymmetrical situations. In the next chapter, we introduce and provide examples of free final-time Stackelberg equilibria. As you will see, in the considered hierarchical situation, the final time becomes a decision variable of the leader of the game.

3

Free final time Stackelberg differential games

3.1 Introduction

This chapter is based on a joint work [8] with my Supervisor, Professor Luca Grosset. The goal is to analyse a new formulation of Stackelberg differential games in which the leader can control not only the dynamics of the game but also the length of the programming interval. This formulation of a free final time Stackelberg differential game was not explicitly considered in the literature and presents some interesting issues.

Free final time games are useful to describe economic problems with time as a decision variable; although, to the best of our knowledge, there are no papers analysing free final time Stackelberg differential games [4]. When the final time is free, it becomes a decision variable for the leader. In a free final time Stackelberg differential game, the sequential decision-making is similar to what is described in Chapter 2, except that, at first, the leader declares both his strategy and the final time, and finally, computes the final time as well. Our work contributed to the literature by defining a free final time Stackelberg differential game and by describing the analytical procedure to characterise such an equilibrium.

3.2 Free final time Stackelberg differential game

First of all, let me define the problem. Consider the following two-player differential game, where F denotes the follower and L the leader. The players' profits are

$$J_j = \int_0^T g_j(x(t), u_L(t), u_F(t), t)dt$$

where $j \in \{L, F\}$, and the motion equation is

$$\dot{x}(t) = f(x(t), u_L(t), u_F(t), t)$$

subject to the initial condition

$$x(0) = x_0 \in \mathbb{R}$$

and the final constraint

$$x(T) \geq \bar{x} \in \mathbb{R}\,.$$

The presence of a final condition for the state function $x(\cdot)$ is related to the fact that we are studying a free final time game and, hence, we need a condition to be able to talk about minimum time. Assume that functions g_F, g_L are continuously differentiable in all their variables and the controls $u_F(\cdot), u_L(\cdot)$ are in $L^1\left([0, +\infty), U_j\right)$, where $j \in \{L, F\}$ and $U_j \subset \mathbb{R}$, so that both objective functionals are well defined. Moreover, assume that f is continuously differentiable in all its variables and Lipschitz continuous in the state variable x uniformly with respect to the control variables u_L, u_F, so that the link between state and controls is well defined [2, Ch.2, p.73]. For the sake of simplicity, we deal with a one-dimensional instance of the problem; the multidimensional extension is straightforward.

Instead of a formal definition of a free final time Stackelberg differential game, it may be interesting to illustrate a procedure to characterise an open-loop equilibrium. This approach is useful because it directly refers to necessary conditions; hence, it is more practical and effective.

3.2.1 Follower's optimal control problem

First of all, in a free final time Stackelberg differential game the leader announces the control path $\hat{u}_L(\cdot)$ and the final time $\hat{T} \in [0, \overline{T}]$, where $\overline{T}$ is the maximum feasible final time. In the literature the final time is fixed, whereas here it is part of leader's strategy.

At this point, the follower has to find a best response function [4, Ch.2, p.17] to such a leader's strategy. To compute which, the follower solves the optimal control problem

$$\begin{aligned}
\max_{u_F(\cdot)} \quad & \int_0^{\hat{T}} g_F(x(t), \hat{u}_L(t), u_F(t), t)dt \\
\text{s.t.} \quad & \\
& \dot{x}(t) = f(x(t), \hat{u}_L(t), u_F(t), t) \\
& x(0) = x_0 .
\end{aligned}$$

Observe that the follower does not have to consider the final state constraint (otherwise the open-loop equilibrium would be time inconsistent, [15]). Follower's Hamiltonian function [2, Ch.2, p.85] is

$$H_F\left(x, \lambda_F, u_F, t \middle| \hat{u}_L(t), \hat{T}\right) = g_F(x, \hat{u}_L(t), u_F, t) + \lambda_F \cdot f(x, \hat{u}_L(t), u_F, t) . \tag{3.1}$$

Assume that the necessary conditions for follower's optimal control problem [2, Ch.2, p.85] are also sufficient. Hence, it is well defined the function

$$u_F^{\#}(x, \lambda_F, t| \hat{u}_L(t), \hat{T}) := \arg\max_{u_F} \left\{ H_F\left(x, \lambda_F, u_F, t \middle| \hat{u}_L(t), \hat{T}\right)\right\} . \tag{3.2}$$

Moreover, suppose that the two-point boundary value problem

$$\begin{cases}
\dot{x}(t) = f(x(t), \hat{u}_L(t), u_F^{\#}(x(t), \lambda_F(t), t| \hat{u}_L(t), \hat{T}), t) \\
x(0) = x_0 \\
\dot{\lambda}_F(t) = -\partial_x H_F\left(x(t), \lambda_F(t), u_F^{\#}(x(t), \lambda_F(t), t| \hat{u}_L(t), \hat{T}), t \middle| \hat{u}_L(t), \hat{T}\right) \\
\lambda_F(\hat{T}) = 0
\end{cases} \tag{3.3}$$

has a unique solution $(x^{\#}(t), \lambda_F^{\#}(t))$, for all $t \in [0, \hat{T}]$ and for all $\hat{u}_L(t) \in L^1([0, \hat{T}], U_L)$.

Finally, assume that U_F is a convex subset of $\mathbb{R}$ and the function

$$(x, u_F) \mapsto H_F\left(x, \lambda_F^{\#}(t), u_F, t \middle| \hat{u}_L(t), \hat{T}\right)$$

is concave for all $\hat{T} \in [0, \bar{T}]$, for all $t \in [0, \hat{T}]$ and for all $u_F \in U_F$. Under these hypotheses, the function

$$u_F^{\#}(x, \lambda_F, t| \hat{u}_L(t), \hat{T})$$

is the best response function of the follower to any leader's strategy [4, Ch.2, p.17].

3.2.2 Leader's optimal control problem

Now, focus on the optimal control problem of the leader.

$$\max_{T\in[0,\bar{T}],u_L(\cdot)} \int_0^T g_F(x(t),u_L(t),u_F^{\#}(x,\lambda_F,t|\,u_L(t),T),t)dt$$

$$\begin{aligned}
&\dot{x}(t) = f(x(t),u_L(t),u_F^{\#}(x(t),\lambda_F(t),t|\,u_L(t),T),t)\\
&x(0) = x_0\\
&x(T) \geq \bar{x}\\
&\dot{\lambda}_F(t) = -\partial_x H_F\left(x(t),\lambda_F(t),u_F^{\#}(x(t),\lambda_F(t),t|\,u_L(t),T),t\,\middle|\,u_L(t),T\right)\\
&\lambda_F(T) = 0\,.
\end{aligned}$$

This is a free final time optimal control problem; therefore, if we characterise the optimal time T^* and the optimal control $u_L^*(\cdot)$, we find an open-loop Stackelberg equilibrium for a free final time Stackelberg differential game.

This approach has two critical issues:

- The previous free final time optimal control problem is not standard since the differential equation for the adjoint function of the follower is backward, so the function $\lambda_F(\cdot)$ becomes a new state function for the leader, and, as a consequence, the standard necessary conditions for a free final time optimal control problem do not hold.

- Time consistency is crucial for an open-loop Nash equilibrium in a Stackelberg differential game; however, in this new framework, the standard condition about the controllability of the adjoint function of the leader is not straightforward because the leader can control the final time.

3.3 Numerical example

In this section, we recall a numerical example from [8] to show how to characterise an open-loop equilibrium for a free final time Stackelberg differential game. The objective functional of the leader is

$$J_L = \int_0^T e^{-t}\left(u_L(t) - \frac{1}{2}u_F(t)u_L^2(t)\right)dt$$

while the objective functional of the follower is

$$J_F = \int_0^T \left(u_F(t) - \frac{1}{2} u_F^2(t) \right) dt \,.$$

The motion equation is described by the Cauchy problem

$$\begin{cases} \dot{x}(t) = -u_L(t) - u_F(t) \\ x(0) = 4 \,. \end{cases} \tag{3.4}$$

Moreover, we assume that the leader has to satisfy the final constraint

$$x(T) \geq 0 \,.$$

Both controls must be positive, that is, $u_L(t), u_F(t) \geq 0$ for all $t \in [0, T]$, and the leader can choose the final time in the interval $[0, 2.5]$.

Let us start our analysis assuming that the leader proposes a strategy to the follower. We denote by $\hat{u}_L(\cdot)$ and $\hat{T}$ this strategy (with $\hat{T} > 0$). At this point, we find the best response function for the follower.

Follower's Hamiltonian function (3.1) is

$$H_F(x, u_F, \lambda_F, t) = u_F - \frac{1}{2} u_F^2 + \lambda_F \left(-\hat{u}_L(t) - u_F \right) \,.$$

We compute

$$\partial_{u_F} H_F(x, u_F, \lambda_F, t) = 1 - u_F - \lambda_F$$

and

$$\partial^2_{u_F u_F} H_F(x, u_F, \lambda_F, t) = -1 \,;$$

therefore, follower's best response function (3.2) is

$$u_F^{\#}(x, \lambda_F, t) = 1 - \lambda_F \,. \tag{3.5}$$

Moreover, from the adjoint equation and the transversality condition in (3.3), we have $\dot{\lambda}_F(t) = 0$ and $\lambda_F(\hat{T}) = 0$; therefore,

$$\lambda_F(t) = 0 \text{ for all } t \in [0, \hat{T}].$$

Thus, the best response function (3.5) becomes

$$u_F^{\#}(x, \lambda_F, t) = 1\,.$$

We notice that the Hamiltonian function of the follower is concave in the state and the control; hence, the sufficient conditions [2, Ch.2, p.105] are satisfied.
In this example:

- the adjoint equation is uncoupled from the motion equation, therefore we can explicitly solve it, and, as a consequence, the follower's adjoint equation does not become a motion equation for the leader's problem;
- because of the simplicity of the model, the follower's best response function does not depend on $\hat{T}$. This fact is crucial in order to obtain a time-consistent solution; indeed, the non-controllability of the follower's best response functions and, as a consequence, the follower's co-state functions by the leader's control functions guarantees the time-consistency of the solution for a linear-state game, as explained in Chapter 2.

Now, we study leader's optimal control problem. The Hamiltonian function is

$$H_L(x, u_L, \lambda_L, t) = \lambda_0 e^{-t}\left(u_L - \frac{1}{2}u_L^2\right) + \lambda_L(-u_L - 1)\,. \tag{3.6}$$

The necessary conditions [2, Ch.2, p.143] are

1. $(\lambda_0, \lambda_L(t)) \neq (0, 0)$ for all $t \in [0, T^*]$;
2. $u_L^*(t) \in \arg\max_w \{H_L(x^*(t), w, \lambda_L(t), t)\}$ for all $t \in [0, T^*]$;
3. $\lambda_0 \in \{0, 1\}$;
4. $\dot{\lambda}_L(t) = 0$ for a.e. $t \in [0, T^*]$;
5. $\lambda_L(T^*) \geq 0, \quad x^*(T^*) \geq 0, \quad \lambda_L(T^*)x^*(T^*) = 0$;
6. $H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = 0$.

Suppose, at first, that $\lambda_0 = 0$. Then $\lambda_L(t) = \bar{\lambda}$ for all $t \in [0, T^*]$ and $\bar{\lambda}$ must be strictly positive. However, by maximising the Hamiltonian function, we have $u_L^*(t) = 0$ for all $t \in$

$[0, T^*]$, which is not feasible because due to condition 5, indeed

$$\begin{aligned} H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) &= \lambda_L(T^*)(-u_L^*(T^*) - 1) \\ &= -\bar{\lambda} < 0 \,. \end{aligned}$$

Hence, we can assume that $\lambda_0 = 1$. We compute

$$\partial_{u_L} H_L(x, u_L, \lambda_L, t) = e^{-t}(1 - u_L) - \lambda_L$$

and

$$\partial^2_{u_L u_L} H_L(x, u_L, \lambda_L, t) = -e^{-t} \,;$$

thus, given $u_L(\cdot) \geq 0$,

$$u_L^{\#}(x, \lambda_L, t) = \left[1 - \bar{\lambda} e^t\right]^+ \,. \tag{3.7}$$

If $\bar{\lambda} = 0$, then (3.7) becomes $u_L^*(t) = 1$ for all $t \in [0, T^*]$; therefore, the solution is not feasible due to condition 6

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = \frac{e^{-T^*}}{2} > 0 \,.$$

Therefore, it must be $\bar{\lambda} > 0$. We notice that the map $t \mapsto 1 - \bar{\lambda} e^t$ is a strictly decreasing function, hence either $u_L^*(T^*) = 0$, or $u_L^*(T^*) > 0$, due to constraint $u_L(\cdot) \geq 0$. Suppose at first that $u_L^*(T^*) = 0$, then from 3.6, we have

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = -\bar{\lambda} < 0 \,,$$

hence this solution is not feasible. Therefore, for all $t \in [0, T^*]$

$$u_L^{\#}(x, \lambda_F, t) = 1 - \bar{\lambda} e^t \,. \tag{3.8}$$

From the motion equation (3.4), we obtain

$$x^*(t) = 4 - 2t + \bar{\lambda}(e^t - 1) \tag{3.9}$$

and, by the transversality condition, we get

$$x^*(T^*) = 4 - 2T^* + \bar{\lambda}(e^{T^*} - 1) = 0 \,,$$

which gives us

$$\lambda_L(t) = \bar{\lambda} = \frac{2(T^* - 2)}{e^{T^*} - 1}\,, \tag{3.10}$$

for all $t \in [0, T^*]$, that is feasible if and only if $T^* > 2$.

We can substitute (3.8), (3.9) and (3.10) into (3.6), obtaining so

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = e^{-T^*} u_L^*(T^*) \left(1 - \frac{1}{2}u_L^*(T^*) - \bar{\lambda}e^{T^*}\right) - \bar{\lambda}$$

$$= e^{-T^*}\left[1 - \frac{2(T^*-2)}{(e^{T^*}-1)}e^{T^*}\right]\left\{1 - \frac{1}{2}\left[1 - \frac{2(T^*-2)}{(e^{T^*}-1)}e^{T^*}\right] - \frac{2(T^*-2)}{(e^{T^*}-1)}e^{T^*}\right\} - \frac{2(T^*-2)}{(e^{T^*}-1)}$$

$$= \frac{e^{-T^*}}{2}\left[1 - \frac{2(T^*-2)}{(e^{T^*}-1)}e^{T^*}\right]^2 - \frac{2(T^*-2)}{(e^{T^*}-1)}\,.$$

Finally, by the free final time condition, we have

$$H_L(x^*(T^*), u_L^*(T^*), \lambda_L(T^*), T^*) = 0$$

$$\iff e^{-T^*}\left[(e^{T^*}-1) - 2(T^*-2)e^{T^*}\right]^2 - 4(T^*-2)(e^{T^*}-1) = 0$$

that we can rewrite as

$$a(T^*) = 4(T^*-2)^2 e^{T^*} - 8(T^*-2)\left(e^{T^*}-1\right) + e^{-T^*}(e^{T^*}-1)^2 = 0\,,$$

whose unique solution is $T^* \approx 2.1179$. Furthermore, note that $H_L(x^*(T), u_L^*(T), \lambda_L(T), T)$ is positive for $T < T^*$, while it becomes negative for $T > T^*$. Therefore, the sufficient conditions [2, Ch.2, p.145] are satisfied. Hence, we have completely characterised the open-loop equilibrium for the free final time Stackelberg differential game.

3.3.1 Time consistency

To better explain the time consistency issue for this kind of game, we display a couple of counterexamples.

Example 3.3.1. Consider the following one-dimensional free-final time linear-state Stackelberg differential game:

$$
\begin{aligned}
\max_{u_L} \quad & J_L[u_L(\cdot)] = \int_0^T \left[x(t) - \frac{1}{2}u_L^2(t) + u_L - \frac{3}{8} \right] dt \\
\max_{u_F} \quad & J_F[u_F(\cdot)] = \int_0^T \left[-x(t) - \frac{1}{2}u_F^2(t) \right] dt \\
& \dot{x}(t) = u_L(t)u_F(t) \\
& x(0) = 0
\end{aligned}
$$

where T is chosen by the leader.

Searching for an open-loop Stackelberg equilibrium, we first consider the follower Hamiltonian function:

$$\mathrm{H}_F(x, u_L, u_F, \lambda) = -x - \frac{1}{2}u_F^2 + \lambda \cdot u_L u_F$$

where λ is the co-state function of the follower. Then from the Pontryagin necessary conditions:

$$
\begin{aligned}
u_F &= \arg\max_u \mathrm{H}_F(x, u_L, u, \lambda) \\
&= \arg\max_u \left[-\frac{1}{2}u_F^2 + \lambda \cdot u_L u_F \right] \\
\implies u_F(t) &= \lambda(t)u_L(t) \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{cases}
\dot{\lambda}(t) = 1 \\
\lambda(T) = 0
\end{cases}
$$

Consider now the leader's Hamiltonian function, where u_F is substituted by the expression in (3.11)

$$\mathrm{H}_L(x, \lambda, u_L, \mu, \nu) = x - \frac{1}{2}u_L^2 + u_L - \frac{3}{8} + \mu(\lambda u_L^2) + \nu(1)$$

where μ and ν are the leader's co-state functions. As before, we obtain

$$
\begin{aligned}
u_L &= \arg\max_u \mathrm{H}_L(x, \lambda, u, \mu, \nu) \\
&= \arg\max_u \left[-\frac{1}{2}u_L^2 + u_L + \mu\lambda u_L^2 \right]
\end{aligned}
$$

therefore,

$$\implies u_L(t) = \frac{1}{1 - 2\mu(t)\lambda(t)}$$

and

$$\begin{cases} \dot{\mu}(t) = -1 \\ \mu(T) = 0 \end{cases}$$

$$\begin{cases} \dot{\nu}(t) = -\mu(t)u_L^2(t) \\ \nu(0) = 0 \end{cases} \tag{3.12}$$

Moreover, to compute the final time, we have to consider the condition

$$\mathrm{H}_L(x, \lambda, u_L, \mu, \nu)|_{T^*} = 0\,.$$

By easy computation, we obtain

$$\lambda^*(t) = t - T^*\,,$$
$$\mu^*(t) = T^* - t\,,$$

which imply

$$u_L^*(t) = \frac{1}{1 + 2(T^* - t)^2}\,,$$
$$u_F^*(t) = \frac{t - T^*}{1 + 2(T^* - t)^2}\,,$$

and, finally,

$$\nu^*(t) = \frac{t(t - 2T^*)}{2(2T^{*2} + 1)(1 + 2(T^* - t)^2)}\,.$$

Thus, the state function $y(t; 0, 0, u_L^*, u_F^*) = x^*(t)$ is

$$x^*(t) = \frac{t(t - 2T^*)}{2(2T^{*2} + 1)(1 + 2(T^* - t)^2)}\,.$$

The functions above are defined for $t \in [0, T^*]$. To compute T^*, we observe that $\lambda^*(T^*) =$

$\mu^*(T^*) = 0$ and $u_L^*(T^*) = 1$. Moreover,

$$\nu^*(T^*) = x^*(T^*) = -\frac{T^{*2}}{2(2T^{*2}+1)}\,.$$

We can now observe that T^* is the unique positive solution of

$$\begin{aligned} \mathrm{H}_L(x,\lambda,u_L,\mu,\nu)|_{T^*} &= x^*(T^*) + \nu^*(T^*) + \frac{1}{8} \\ &= -\frac{T^{*2}}{2T^{*2}+1} + \frac{1}{8} = 0 \end{aligned} \tag{3.13}$$

$$\implies T^* = \frac{\sqrt{6}}{6}\,. \tag{3.14}$$

Given (u_L^*, u_F^*) the Stackelberg equilibrium of the game $\Gamma(0,0)$, to prove its time inconsistency, consider the subgame $\Gamma(\tau,\, y(\tau; 0, 0, u_L^*, u_F^*))$ in the interval $[\tau, T^*]$, for $\tau \in [0, T^*)$. The necessary conditions of the subgame are identical to those given for $\Gamma(0,0)$, except for the boundary conditions $x(\tau) = y(\tau; 0, 0, u_L^*, u_F^*)$ and $\nu(\tau) = 0$. Choose $\tau = 0.5$, then we have

$$\nu^*(t) = \frac{(2t-1)(2t-4T^*+1)}{4(4T^{*2}-4T^*+3)(1+2(T^*-t)^2)}$$

$$\implies \nu^*(T^*) = -\frac{(2T^*-1)^2}{4(4T^{*2}-4T^*+3)} \tag{3.15}$$

while the other equations remain unchanged. Even though these changes do not affect the functions $x^*(t)$, $\lambda^*(t)$, $\mu^*(t)$, the condition $\nu(\tau) = 0$ leads, for $\tau = 0.5$, to a different solution of (3.12). Moreover, the function ν^* is irrelevant in the discussion of almost all necessary conditions, except for (3.13). Substituting in (3.13) the new value of $\nu^*(T^*)$ obtained in (3.15), we have

$$-\frac{T^{*2}}{2(2T^{*2}+1)} - \frac{(2T^*-1)^2}{4(4T^{*2}-4T^*+3)} + \frac{1}{8} = 0\,.$$

By evaluating the above equation into the the optimal time (3.14) computed before, we get

$$\frac{3}{16} - \frac{10-4\sqrt{6}}{4(22-4\sqrt{6})} \neq 0\,.$$

Thus, the different initial condition $\nu(\tau) = 0$ implies a different final time T^*, which induces modifications for all functions $x^*(t),\ \lambda^*(t),\ \mu^*(t)$ and also for the controls $u_L^*(t)$ and $u_F^*(t)$. We can conclude that the game is time-inconsistent.

Example 3.3.2. Consider now the same game as before, with the modification that we are going to substitute the expression of $\lambda(t)$ in the leader's Hamiltonian function. We are allowed to do that because, starting from a linear-state game, $\lambda(t)$ is solvable before studying the leader's problem. The game is the same since we have to consider the leader's Hamiltonian function. Then,

$$\mathrm{H}_L(x,\lambda,u_L,\mu,t) = x - \frac{1}{2}u_L^2 + u_L - \frac{3}{8} + \mu u_L^2(t-T^*)$$

knowing $\lambda(t) = t - T^*$. Reasoning as before, we obtain the same expressions

$$\begin{aligned}
\mu^*(t) &= T^* - t \\
u_L^*(t) &= \frac{1}{1+2(T^*-t)^2}\,, \\
u_F^*(t) &= \frac{t-T^*}{1+2(T^*-t)^2}
\end{aligned}$$

and, as a consequence,

$$x^*(t) = \frac{t(t-2T^*)}{2(2T^{*2}+1)(1+2(T^*-t)^2)}\,.$$

To compute the final time, we put the leader's Hamiltonian evaluated at T^* equal to zero:

$$\begin{aligned}
\mathrm{H}_L(x,\lambda,u_L,\mu,t)|_{T^*} &= x^*(T^*) + \frac{1}{8} \\
&= -\frac{T^{*2}}{2(2T^{*2}+1)} + \frac{1}{8} \\
\implies T^* &= \frac{\sqrt{2}}{2}
\end{aligned}$$

However, this equation leads us to an absurd result. There is no finite final time that satisfies the equation.

3.4 Discussion

We introduced the definition of open-loop equilibrium for a free final time Stackelberg differential game and proposed a numerical example to prove that this equilibrium can be explicitly characterised. After our analysis, two issues remain open.

First of all, in the numerical example, the adjoint function of the follower can be explicitly solved, hence the leader has to solve a standard free final time optimal control problem. Moreover, the condition about the vanishing of the Hamiltonian function in the optimal final time is correct. However, this condition cannot be used in general because of the backward motion equation introduced by the adjoint function of the follower.

To define a free final time situation we need to specify some constraint, such as a final condition for the state variable. Moreover, time is set to be a decision variable for the leader of the game; this choice depends on the time inconsistency that may follow from letting the follower control the time interval. Indeed, time consistency is a key issue in Stackelberg differential games. In our numerical example, time consistency is trivially satisfied because the strategy of the follower is uniform with respect to the leader's. In general, this is not true, and time consistency becomes more relevant to preserve the credibility of an equilibrium. It seems interesting to characterise a class of problems with a simple structure such that time consistency is automatically satisfied.

In conclusion, even though this new definition seems to be a straightforward extension of the original Stackelberg differential game, its analysis appears rich in new stimulant situations. Even though we have not proposed an application for this model, it represents a future development. For instance, this kind of game may be involved in the analysis of situations where there is a leader who can decide the duration of a relationship with a follower, such as for rental agreements or life insurance policies.

4

A differential game model for sponsored content

4.1 Introduction

In this chapter, we describe an application of a Stackelberg differential game with an infinite time horizon. The application hereafter proposed allows us to introduce an innovative relationship between the leader and the follower by letting a leader's control to constraint a follower's control function. We will highlight the difficulties that may follow from this alteration of the typical Stackelberg structure. What follows refers to a publication on the *Journal of the Operational Research Society* [9] achieved in collaboration with Professor Alessandra Buratto and my Supervisor Professor Luca Grosset.

Let us introduce some context for the model application. Sponsored advertising, also known as native advertising, is a type of communication designed to resemble the regular content of the platform where it is embedded [16]. The adjective "native" refers to the coherence between advertising and the platform's content. Native advertising acts as advertorials, that is, sponsored messages in printed newspapers and magazines that look like an editorial or a news article. However, it is important to note that advertorial specifically pertains to printed publications, while native advertising extends its reach predominantly within digital media and includes advertising that resembles various formats such as news, search engine results, entertainment videos, and many other contexts [16]. As a matter of fact, the terms "native advertising" and "sponsored content" are not synonymous. A difference between them is that sponsored content is explicitly labelled as paid content, while native advertising aims to blend more seamlessly into the platform or medium where it appears. However, both forms are intended to promote products, services or brands, but their approach to disclosure and presentation varies among different countries where they are used. We are going to use these terms interchangeably. More precisely, by a Federal Trade Commission decision (2015), native advertising must be declared to allow consumers to identify it; some examples of labels that denote it are "sponsored by", "paid programme", etc.

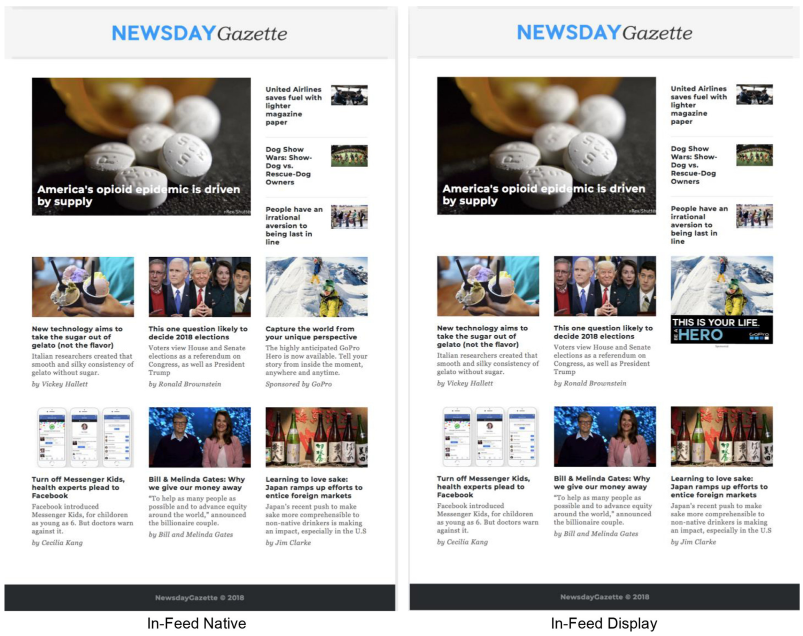


Figure 4.1: Reprinted from "Native Advertising in Online News: Trade-Offs Among Clicks, Brand Recognition, and Website Trustworthiness", by A. Aribarg and E. M. Schwartz, 2020, *Journal of Marketing Research*, **57**(1), page 23.

Sponsored advertising aims to create a fluid experience and embed advertising in the flow of the media platform; consequently, it is more difficult for the consumer to identify native content as sponsored [17, 18]. In this way, its communication tactic is opposite to the strategy of pop-up banners, whose purpose is to interrupt the user experience and make the advertisement recognisable. This peculiarity of native advertising is illustrated in Figure 4.1, which shows the same navigable news website featuring GoPro advertising (in a tile in the middle row in the far-right position) through sponsored advertising on the left page and standard advertising on the right page, respectively. In [17], they suitably designed the navigable news website illustrated in Figure 4.1 to understand how consumers respond to native versus display advertising. Since consumers do not think of content in terms of a promotional message, reception barriers become lower, and therefore advertising tolerance increases [19], providing an excellent opportunity to capture and retain consumer attention. The effectiveness of these sponsored messages is not without consequences. Native advertising has been criticised for violating journalistic norms regarding the distinction between advertising and proper content [20]. From the consumer's perspective, it can be disappointing to discover content as sponsored, and negative reactions might follow. If consumers feel deceived, they may lose faith in the credibility of the media outlet [21, 22, 23]. From an economic point of view, the media outlet has to consider the trade-off between the profit made by publishing sponsored advertising and the consequent loss of credibility. Our research aims to formalise this trade-off for a media platform and solve the problem using the approach taken from the theory of dynamic advertising models [5]. We describe the interactions between the media platform and the firm that wants to benefit from native advertising using the theory of differential games [4, 12]. First, we assume that the media outlet and the firm have asymmetrical roles; thus, we formalise the problem as a differential game played à la Stackelberg, where the media outlet acts as the leader. To obtain a solution, we search for an open-loop equilibrium.

We assume that native advertising has a detrimental effect on the evolution of a media outlet's credibility. Using our model, we aim to investigate the circumstances under which credibility can be sustained positively over the long term, even in the presence of native advertising.

Therefore, we pose the following research questions: (i) Under what conditions may the media outlet accept publishing native advertising? (ii) If native advertising is accepted, what is its optimal level at equilibrium? (iii) If the media outlet accepts publishing native advertising, does its credibility remain positive in the long term? Our model demonstrates that the acceptability and optimal level of native advertising depend on the extent to which it negatively impacts the credibility of the media outlet. Moreover, we demonstrate that admitting native advertising is not sufficient to guarantee positive credibility for the media outlet.

In a second part, we assume that the effectiveness of sponsored advertising is positively affected by the credibility of the media outlet: the higher the credibility of the media outlet, the greater the effectiveness of sponsored advertising. This improvement makes native advertising self-limiting; therefore, it is no longer necessary to introduce the upper bound control for the media outlet. Indeed, low levels of credibility imply low efficiencies for sponsored content and, consequently, small values of native advertising; hence, both players will seek optimal levels of credibility in the long run. Under this new assumption, it is possible to characterise a

Stackelberg equilibrium, but it is no longer time consistent. Therefore, we modify the initial model by assuming that players act simultaneously and identify the Markovian Nash equilibrium. The main research question investigated in this second part is the following: (iv) Under what conditions does a Markovian Nash equilibrium exist that allows for positive credibility in the long run? Under these new assumptions, the analysis becomes more complex; however, we can derive sufficient conditions to define an equilibrium. Within this scenario, the activation of native advertising depends on the long-term credibility of the media outlet.

4.2 The model

We assume that a firm wants to sustain its brand using an advertising campaign on a high-quality media platform. Two different types of advertising can be done: standard and native. Due to its striking resemblance to the main content of the platform on which it appears, native advertising is effective because consumers might not recognise it as a sponsored message. We modelled the firm's investment using the standard Nerlove–Arrow advertising model (see [24] for a recent and interesting analysis of this seminal model). If $G(t)$ represents the goodwill (brand value) of the firm at time t, the evolution of this state variable can be described by the motion equation

$$\dot{G}(t) = -\delta G(t) + g(a(t), n(t)), \tag{4.1}$$

where $\delta > 0$ is the natural decay parameter of the investment in goodwill. Furthermore, $a(t)$ is the control function that represents the standard advertising flow, while $n(t)$ represents the native advertising flow. The function $g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ summarises the effect in terms of goodwill of the investment in standard and sponsored advertising made by the firm. In the following, we assume that function g is linear, that is, $g(a, n) = \gamma_a a + \gamma_n n$, with γ_a, $\gamma_n > 0$, the standard/native advertising efficiencies, respectively. Under this assumption, the goodwill motion equation (4.1) becomes

$$\dot{G}(t) = -\delta G(t) + \gamma_a a(t) + \gamma_n n(t). \tag{4.2}$$

For all control functions $a(\cdot)$, $n(\cdot) \in L^1([0, +\infty); [0, +\infty))$ and for any initial condition $G(0) = G_0 > 0$, we obtain a unique positive solution to the state equation (4.2). We recall that this dynamics is in line with the original Nerlove–Arrow model and has been used by several dynamic advertising optimal control models, see [25], [26], and the references therein.

As in [27], we assume that the firm's profit is linear in goodwill value and quadratic in both advertising costs. Then, the discount profit of the firm is described by the following integral

$$J_F = \int_0^{+\infty} \left(\pi G(t) - c_a(a(t)) - c_n(n(t))\right) e^{-\rho t} \, \mathrm{d}t \,, \tag{4.3}$$

where the inside terms have the following meaning

- $\pi G(t)$ profit, net of advertising costs, $\pi > 0$;
- $c_a(a)$ standard advertising cost function;
- $c_n(n)$ native advertising cost function;
- $\rho > 0$ discount factor.

We let standard advertising and native advertising costs have the following form, respectively:

$$c_a(a) = \frac{\kappa_a}{2} a^2 + \theta_a a \,, \quad c_n(n) = \frac{\kappa_n}{2} n^2 + \theta_n n \,, \tag{4.4}$$

where κ_a, θ_a, κ_n, θ_n are positive constants.

Following Bachmann et al. (2019, p. 97), we consider advertising as a specific product offered by the publisher (the media outlet). In addition, some media companies hire employees to directly write native advertisements [20, p. 180]. This explains the presence of two different costs for traditional and native advertising in the firm's profit. The media outlet makes some profit from the advertising costs paid by the firm. In addition, subscriptions to the platform contribute to the profit made by the media outlet. The brand value of the firm is positively affected by sponsored advertising; however, from the point of view of the media outlet, consumers can feel deceived by this type of advertising and can react by losing their trust in the credibility of the media outlet itself [23].

To the best of our knowledge, there are no dynamic models that describe the credibility issue in an optimal control contest. We propose modelling the progression of credibility in a manner analogous to the treatment of goodwill. We assume that the credibility of the media outlet spontaneously decreases over time if its platform content is not updated recently [28]. Therefore, credibility is subject to natural decay (at a rate $\varepsilon > 0$) if the media outlet does not sustain it through high-quality content. So, we assume that the media outlet credibility is a state function that satisfies a linear motion equation of the form

$$\dot{C}(t) = -\varepsilon C(t) + w(t) - \alpha\, n(t) \tag{4.5}$$

and the initial condition $C(0) = C_0 > 0$. The media outlet can maintain its credibility by offering high-quality content, whose investment flow is described by the media outlet control function $w(\cdot) \in L^1([0, +\infty); [0, +\infty))$. However, native advertising may damage this credibility: whenever consumers recognise any subject as sponsored by a firm, they may lose trust in the media outlet and, consequently, the perceived media outlet credibility decreases. The parameter $\alpha > 0$ represents the negative effect (damage) of native advertising on the evolution of credibility. This assumption is supported by the results of marketing surveys, such as in [21], [22], and [18]. In our model, standard advertising does not influence the credibility of the media outlet; therefore, strategy $a(\cdot)$ does not enter the motion equation for credibility. The credibility of a media outlet is strictly related to sponsored content, not only for its intensity but also for the message conveyed. For example, paper [29] identifies common characteristics among companies that use greenwashing, showing that the publication of articles recognised as greenwashing can damage both the firm's goodwill and the credibility of the media. A different example comes from the Russia-Ukraine conflict. The war in Ukraine has dramatically affected the advertising industry, as numerous international media outlets have interrupted the advertising of Russian products fearing a loss of credibility. These aspects, which require a deep understanding of the content of advertising messages, would nonetheless be interesting to study in terms of the evolution of a media outlet's credibility concerning the alignment of advertising with editorial policy.

Another aspect to be considered is the measurement of media credibility. Paper [30] proposes a semi-quantitative approach focused on developing and validating a scale to gauge online-message credibility. Moreover, in [31], the authors examine how message credibility and media credibility impact trust and impulsive buying behaviour in digital marketing. The relationship between advertising credibility and sales remains an open area of investigation. To the best of our knowledge, our research is the first to employ differential games in this domain, thereby improving the quantitative dimension of the problem.

We assume that the profit of the media outlet increases with its credibility and includes both the costs to sustain high-quality content and the entries related to the costs the firm pays for the two types of advertising. Thus, the objective functional of the media outlet is described by the following integral

$$J_M = \int_0^{+\infty} \left(\eta\, C(t) - c_w(w(t)) + c_a(a(t)) + c_n(n(t))\right) e^{-\rho t}\, \mathrm{d}t\,, \tag{4.6}$$

where the inside terms have the following meaning

- $\eta\, C(t)$ profit related to credibility, $\eta > 0$;
- $c_w(w)$ cost to maintain the high-quality content offer positive;
- $c_a(a)$ revenue from standard advertising;
- $c_n(n)$ revenue from native advertising;
- $\rho > 0$ discount factor.

Following the previous approach, we define the cost function for high-quality content, denoted as $c_w(w)$, as strictly increasing and convex, and it is represented by the following equation

$$c_w(w) = \frac{\kappa_w}{2} w^2 + \theta_w w\,, \tag{4.7}$$

where κ_w and θ_w are positive parameters.

In a profit-oriented analysis, we observe that the negative effects of native advertising can reduce the credibility of the media outlet (4.5) and even induce recipients to abandon the platform if it fails to provide high-quality content.

Furthermore, damage to credibility could also affect media payoff (4.6). Indeed, the loss of profit due to the loss of credibility may not be balanced by the revenue from native advertising. Thus, the media outlet establishes an upper bound to limit sponsored advertising on its platform. Therefore, we introduce a nonnegative control $N(\cdot) \in L^1([0,+\infty);[0,+\infty))$ for the media outlet such that the following constraint for the native advertising strategy holds

$$n(t) \in [0, N(t)] \qquad \forall t \in [0,+\infty). \tag{4.8}$$

The control function $N(\cdot)$ may limit the native advertising flow $n(\cdot)$; thus, it may enter the media outlet payoff through the profit obtained by native advertising and the profit affected by the credibility. This condition contains an innovative aspect of this model, wherein a player's control function is constrained by an upper bound set by the opponent's strategy; thus, one of the problem constraints contains a decision variable. Indeed, in the related literature, an eventual upper bound for the advertising effort can usually be found as an exogenous parameter, see, e.g. [32].

The native advertising problem can be formalised as a differential game played à la Stackelberg, with payoffs (4.3), (4.6), and dynamics (4.2), (4.5).

4.3 Open-loop Stackelberg equilibrium

Assume that the players search for an open-loop Stackelberg equilibrium [4, Ch. 5, p. 113], where the media outlet acts as the leader and the firm as the follower. In the following results, we characterise the solution using the necessary and sufficient conditions in Seierstad & Sydsæter (1987, Ch. 3, p. 234, Th. 12, Th. 13). To evaluate the long-term sustainability of native advertising, we model an infinite-horizon game.

The game occurs as follows: First, the leader declares its strategy $(\overline{w}(\cdot), \overline{N}(\cdot))$; then the follower computes its best response function to the leader's declared strategies. To do that, the follower solves the optimal control problem defined starting from the expressions in (4.2), (4.3), (4.5), and (4.8):

$$\begin{aligned} \max_{a,n} \quad & J_F = \int_0^{+\infty} \left(\pi G(t) - c_a(a(t)) - c_n(n(t))\right) e^{-\rho t}\, \mathrm{d}t \\ & \dot{G}(t) = -\delta G(t) + \gamma_a a(t) + \gamma_n n(t), \qquad G(0) = G_0 \\ & \dot{C}(t) = -\varepsilon C(t) + \overline{w}(t) - \alpha\, n(t), \qquad C(0) = C_0 \\ & a(t) \geq 0, \qquad n(t) \in [0, \overline{N}(t)]\,. \end{aligned} \tag{4.9}$$

where the advertising costs are defined in (4.4). Analysing (4.9), we obtain the following result.

Proposition 4.3.1. *The follower's best response functions to the leader's controls $(\overline{w}(\cdot), \overline{N}(\cdot))$ are*

$$a^*(t) \equiv \left[\frac{\gamma_a \pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right]^+ \quad \text{and} \quad n^*(t;\overline{N}(t)) = \min\left\{\left[\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right]^+, \overline{N}(t)\right\} \tag{4.10}$$

for all $t \in [0, +\infty)$.

Proof. Let H_F^c denote the current Hamiltonian of the follower's problem (4.9)

$$\begin{aligned} H_F^c(G, C, a, n, \overline{w}(t), \overline{N}(t), \lambda_1, \lambda_2) &= \pi G - c_a(a) - c_n(n) \\ &\quad + \lambda_1(-\delta G + \gamma_a a + \gamma_n n) + \lambda_2(-\varepsilon C + \overline{w}(t) - \alpha n), \end{aligned}$$

where $c_a(a)$ and $c_n(n)$ are given in (4.4), and $\lambda_1(\cdot)$, $\lambda_2(\cdot)$ are the follower's co-state functions. From Pontryagin's Maximum Principle, we know that

$$\begin{aligned} (a, n) \in & \underset{a\geq 0,\, n\in[0,\overline{N}(t)]}{\arg\max} \; H_F^c(G, C, a, n, \overline{w}(t), \overline{N}(t), \lambda_1(t), \lambda_2(t)) \\ = & \underset{a\geq 0,\, n\in[0,\overline{N}(t)]}{\arg\max} \; \big[- c_a(a) - c_n(n) + \lambda_1(t)(\gamma_a a + \gamma_n n) - \lambda_2(t)\alpha\, n \big] \,. \end{aligned} \tag{4.11}$$

Recalling that $c_a(a) = \frac{\kappa_a}{2}a^2 + \theta_a a$ and $c_n(n) = \frac{\kappa_n}{2}n^2 + \theta_n n$, we notice that the expression within the brackets in (4.11) is a separable and concave function in a and n. Thus, we can rewrite (4.11) as

$$a \in \underset{a\geq 0}{\arg\max} \Big[- \frac{\kappa_a}{2}a^2 - \theta_a a + \lambda_1(t)\gamma_a a \Big] =: \varphi_a(a) \tag{4.12}$$

$$n \in \underset{n\in[0,\overline{N}(t)]}{\arg\max} \Big[- \frac{\kappa_n}{2}n^2 - \theta_n n + \lambda_1(t)\gamma_n n - \lambda_2(t)\alpha\, n \Big] =: \varphi_n(n) \,. \tag{4.13}$$

Moreover, the Hamiltonian function turns out to be concave in the state and control functions; hence, the Mangasarian sufficiency theorem holds. From (4.12), we get

$$\begin{aligned} \varphi_a'(a) &= -\kappa_a a - \theta_a + \lambda_1(t)\gamma_a \\ \varphi_a''(a) &= -\kappa_a < 0 \,; \end{aligned}$$

similarly, by (4.13),

$$\begin{aligned} \varphi_n'(n) &= -\kappa_n n - \theta_n + \lambda_1(t)\gamma_n - \alpha\,\lambda_2(t) \\ \varphi_n''(n) &= -\kappa_n < 0 \,; \end{aligned}$$

The FOC and the feasibility constraint for the follower's controls give

$$a(t) = \left[\frac{\gamma_a \lambda_1(t) - \theta_a}{\kappa_a} \right]^+ \tag{4.14}$$

and

$$n(t;\overline{N}(t)) = \min\left\{\left[\frac{\gamma_n\lambda_1(t) - \theta_n - \alpha\lambda_2(t)}{\kappa_n}\right]^+, \overline{N}(t)\right\}, \tag{4.15}$$

where the co-state functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ must satisfy

$$\begin{aligned}\dot{\lambda}_1(t) &= \rho\lambda_1(t) - \partial_G \mathrm{H}_F^c = (\rho+\delta)\lambda_1(t) - \pi \\ \dot{\lambda}_2(t) &= \rho\lambda_2(t) - \partial_C \mathrm{H}_F^c = (\rho+\varepsilon)\lambda_2(t)\end{aligned} \tag{4.16}$$

with transversality conditions

$$\lim_{t\to+\infty} e^{-\rho t}\lambda_1(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} e^{-\rho t}\lambda_2(t) = 0. \tag{4.17}$$

We observe that the co-state equations (4.16) and their transversality conditions (4.17) are independent of the leader controls $(\overline{w}(\cdot), \overline{N}(\cdot))$. As observed in Chapter 2, the non-controllability of the follower's control function by the leader's control is a key assumption to obtain a time-consistent solution. Otherwise, time consistency may not be guaranteed. For this reason, we can directly compute them

$$\lambda_1^*(t) \equiv \frac{\pi}{\rho+\delta} > 0\,, \qquad \lambda_2^*(t) \equiv 0. \tag{4.18}$$

Therefore, by substituting (4.18) into (4.14) and (4.15), we obtain the follower's best response functions

$$a^*(t) \equiv \left[\frac{\gamma_a\pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right]^+ \quad \text{and} \quad n^*(t;\overline{N}(t)) = \min\left\{\left[\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right]^+, \overline{N}(t)\right\}$$

for $t \in [0, +\infty)$.

□

We notice that both types of advertising are strictly positive only if their marginal costs at zero are less than a given threshold. For the standard advertising, we obtain that

$$a^*(t) > 0 \iff \frac{\theta_a}{\gamma_a} < \frac{\pi}{\rho+\delta}\,.$$

On the left-hand side of the second inequality, we have the ratio between the cost parameter and the effectiveness of standard advertising, while on the right-hand side, we find the discounted revenue parameter. A similar result can be obtained for native advertising:

$$n^*(t) > 0 \iff \frac{\theta_n}{\gamma_n} < \frac{\pi}{\rho + \delta} \quad \text{and} \quad \overline{N}(t) > 0 \,.$$

The primary distinction between the last two formulas lies in the fact that, in the latter, we must account for the upper bound represented by $\overline{N}(t)$. This upper bound is among the leader's controls, and it directly affects the optimal intensity of native advertising. The presence of the linear term in the cost functions can lead to the activation of only one of the two advertising channels. For example, if θ_n is too high, the native campaign will not be activated; conversely, without the linear terms, both types of advertising would always be active, which is not credible. Therefore, in this model, it is essential to consider not purely quadratic cost functions.
From (4.10), we observe that standard advertising flow $a^*(\cdot)$ is independent of both the leader's control functions, while native advertising effort $n^*(\cdot\,; \overline{N}(\cdot))$ depends on the leader's control $\overline{N}(\cdot)$.

Now, we formalise the leader's problem by considering the follower's best response functions $(a^*(\cdot), n^*(\cdot\,; N(\cdot)))$ given in (4.10). Since the follower's co-state functions (4.18) do not depend on the leader's controls, the time consistency of the equilibrium is guaranteed. Thus, omitting the follower's co-state equations, the leader's optimal control problem is defined starting from the expressions in (4.2), (4.5), and (4.6):

$$\begin{aligned} \max_{w,N} \quad & J_M = \int_0^{+\infty} \left(\eta\, C(t) + c_a(a^*(t)) + c_n(n^*(t; N(t))) - c_w(w(t))\right) e^{-\rho t}\, \mathrm{d}t \\ & \dot{G}(t) = -\delta G(t) + \gamma_a a^*(t) + \gamma_n n^*(t; N(t)), \quad G(0) = G_0 \\ & \dot{C}(t) = -\varepsilon C(t) + w(t) - \alpha\, n^*(t; N(t)), \qquad C(0) = C_0 \\ & w(t) \geq 0, \qquad N(t) \geq 0\,. \end{aligned} \tag{4.19}$$

where the advertising revenues are defined in (4.4), while the high-quality costs are defined in (4.7). We recall that the upper bound strategy $N(\cdot)$ enters the media outlet profit directly through the native advertising flow and indirectly through credibility. We say that native advertising is admissible for the media outlet when the imposed upper bound is strictly positive, that is, $N(t) > 0$ for all $t \in [0, +\infty)$. The following proposition characterises the optimal leader's strategies.

Proposition 4.3.2. *Referring to the problem* (4.19)*, native advertising is admissible for the me-*

dia if and only if

$$\alpha < \bar{\alpha} = \frac{\rho+\varepsilon}{2\eta}\left(\frac{\gamma_n \pi}{\rho+\delta} + \theta_n\right). \tag{4.20}$$

Moreover, the leader's optimal control functions are

$$w^*(t) \equiv \left[\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right]^+ \quad \textit{and} \quad N^*(t) \in \begin{cases} [\tilde{n}, +\infty) & \alpha < \bar{\alpha} \\ \{0\} \cup [\tilde{n}, +\infty) & \alpha = \bar{\alpha} \\ \{0\} & \alpha > \bar{\alpha},\ \tilde{n} > 0 \\ [0, +\infty) & \alpha > \bar{\alpha},\ \tilde{n} = 0 \end{cases} \tag{4.21}$$

for all $t \in [0, +\infty)$*, where* $\tilde{n} = \left[\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right]^+$.

Proof. As already done for the follower's problem, H_L^c denotes the current Hamiltonian of the leader's problem (4.19)

$$\begin{aligned} \mathrm{H}_L^c(G, C, w, N, \mu_1, \mu_2, t) = \eta\, C - c_w(w) + c_a(a^*(t)) + c_n(n^*(t; N(t))) + \\ + \mu_1(-\delta G + \gamma_a a^*(t) + \gamma_n n^*(t; N)) + \mu_2(-\varepsilon C + w - \alpha\, n^*(t; N)), \end{aligned}$$

where $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are the co-state functions of the leader, which satisfy the following differential equations

$$\begin{aligned} \dot{\mu}_1(t) &= \rho\mu_1(t) - \partial_G \mathrm{H}_L^c = (\rho+\delta)\mu_1(t) \\ \dot{\mu}_2(t) &= \rho\mu_2(t) - \partial_C \mathrm{H}_L^c = (\rho+\varepsilon)\mu_2(t) - \eta \end{aligned} \tag{4.22}$$

and transversality conditions

$$\lim_{t\to+\infty} e^{-\rho t}\mu_1(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} e^{-\rho t}\mu_2(t) = 0. \tag{4.23}$$

From (4.22) and (4.23), we obtain

$$\mu_1^*(t) \equiv 0 \quad \text{and} \quad \mu_2^*(t) \equiv \frac{\eta}{\rho+\varepsilon}. \tag{4.24}$$

To maximise the current value Hamiltonian, we have to satisfy the necessary condition

$$(w, N) \in \underset{w \geq 0,\, N \geq 0}{\arg\max}\, \mathrm{H}_L^c(G, C, w, N, \mu_1^*(t), \mu_2^*(t), t)$$
$$= \underset{w \geq 0,\, N \geq 0}{\arg\max}\, \psi(w, N), \tag{4.25}$$

where, after using (4.24), we defined

$$\psi(w, N) := \; - c_w(w) + \frac{\eta}{\rho + \varepsilon} w + c_n(n^*(t; N)) - \frac{\eta}{\rho + \varepsilon} \alpha\, n^*(t; N). \tag{4.26}$$

Since the function $\psi(\cdot, \cdot)$ is separable in w and N for all $t \in [0, +\infty)$, let us first consider its maximisation with respect to w. Recall that $c_w(w) = \frac{\kappa_w}{2} w^2 + \theta_w w$. The first-order necessary condition is

$$\partial_w \psi(w, N) = -\kappa_w w - \theta_w + \frac{\eta}{\rho + \varepsilon} = 0,$$

while the second-order condition

$$\partial_w^2 \psi(w, N) = -\kappa_w < 0$$

guarantees the strict concavity of the function $\psi(\cdot, N)$, and together they lead to

$$w^*(t) \equiv \left[\frac{\eta}{\kappa_w(\rho + \varepsilon)} - \frac{\theta_w}{\kappa_w} \right]^+ . \tag{4.27}$$

Determining the optimal N from (4.26) is equivalent to maximising on $N \geq 0$ the following function

$$\tilde{\psi}(N) := c_n(n^*(t; N)) - \frac{\eta\alpha}{\rho + \varepsilon} n^*(t; N). \tag{4.28}$$

By replacing the expression for $(n^*(t; N))$, we obtain

$$\tilde{\psi}(N) = \frac{\kappa_n}{2} \left(\min \left\{ \left[\frac{\gamma_n \pi}{\kappa_n(\rho + \delta)} - \frac{\theta_n}{\kappa_n} \right]^+, N \right\} \right)^2 +$$
$$+ \left(\theta_n - \frac{\eta\alpha}{\rho + \varepsilon} \right) \min \left\{ \left[\frac{\gamma_n \pi}{\kappa_n(\rho + \delta)} - \frac{\theta_n}{\kappa_n} \right]^+, N \right\} . \tag{4.29}$$

Assume at first that the constant

$$\tilde{n} := \left[\frac{\gamma_n \pi}{\kappa_n(\rho + \delta)} - \frac{\theta_n}{\kappa_n} \right]^+ \tag{4.30}$$

is positive. Then, since all the parameters in the first argument of the minimum are strictly positive, it is clear that $\tilde{\psi}(\cdot)$ is a continuous function of N consisting of a first convex parabolic arc such that $\tilde{\psi}(0) = 0$ and a second constant path connected to the parabolic arc at the point $\tilde{n}$.

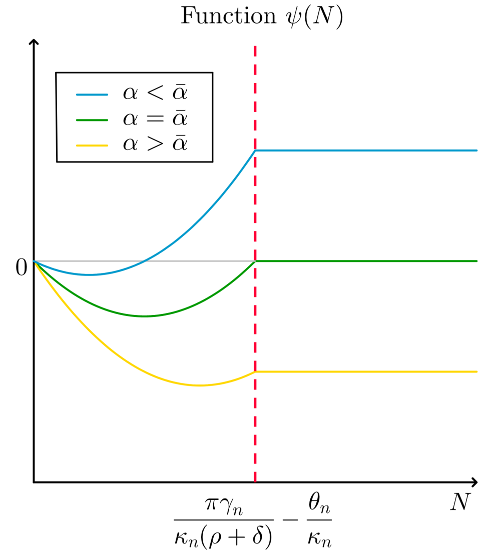


Figure 4.2: Representative graph of the behaviour of function $\tilde{\psi}$ for different values of α.

The function $\tilde{\psi}(\cdot)$ depends on α, and thus, recalling that

$$\bar{\alpha} = \frac{\rho + \varepsilon}{2\eta} \left(\frac{\gamma_n \pi}{\rho + \delta} + \theta_n \right),$$

three possible scenarios can occur according to the position of α with respect to $\bar{\alpha}$ (see Figure 4.2). Therefore, due to the structure of the function $\tilde{\psi}(\cdot)$, the optimal value of N depends on the sign of $\tilde{\psi}(\cdot)$ evaluated in $\tilde{n}$, that is,

$$\tilde{\psi}(\tilde{n}) = \left(\frac{\gamma_n \pi}{\kappa_n(\rho + \delta)} - \frac{\theta_n}{\kappa_n} \right) \left(\frac{\gamma_n \pi}{2(\rho + \delta)} + \frac{\theta_n}{2} - \frac{\eta \alpha}{\rho + \varepsilon} \right). \tag{4.31}$$

From Figure 4.2, we observe that if $\alpha < \bar{\alpha}$ then $\tilde{\psi}(\tilde{n}) > 0$, if $\alpha = \bar{\alpha}$ then $\tilde{\psi}(\tilde{n}) = 0$, and finally if $\alpha > \bar{\alpha}$ then $\tilde{\psi}(\tilde{n}) < 0$. So, the maximisation process gives us the following condition

$$N(t) \in \begin{cases} [\tilde{n}, +\infty) & \alpha < \bar{\alpha} \\ \{\, 0 \,\} \cup [\tilde{n}, +\infty) & \alpha = \bar{\alpha} \\ \{\, 0 \,\} & \alpha > \bar{\alpha} \end{cases} \tag{4.32}$$

for all $t \in [0, +\infty)$. Assume now that $\tilde{n} \leq 0$. In this case, $\tilde{\psi}(N) \equiv 0$; hence, $N(t) \in [0, +\infty)$. Summing up we have obtained

$$N^*(t) \in \begin{cases} [\tilde{n}, +\infty) & \alpha < \bar{\alpha} \\ \{\, 0 \,\} \cup [\tilde{n}, +\infty) & \alpha = \bar{\alpha} \\ \{\, 0 \,\} & \alpha > \bar{\alpha},\ \tilde{n} > 0 \\ [0, +\infty) & \alpha > \bar{\alpha},\ \tilde{n} = 0 \end{cases} \tag{4.33}$$

where $\tilde{n}$ is as in (4.30). This formulation emphasises the dependence of the leader's control $N(\cdot)$ on the effective damage α of native advertising on the credibility of the media outlet. We have obtained (4.21) with $\bar{\alpha}$ as in (4.20). □

We observe that high-quality investment to sustain credibility is positive if and only if

$$\theta_w < \frac{\eta}{\rho + \varepsilon}\,.$$

Summing up, if parameter α associated with the damage due to native advertising is low, then it is optimal for the media outlet to accept native advertising because it increases its profit; on the other hand, if α is high, then the media outlet must sustain its credibility by not accepting any native advertising.
Observe that the threshold $\bar{\alpha}$ decreases in η; therefore, the higher the marginal revenue with respect to credibility, the narrower the admissibility interval for native advertising.
Moreover, from Proposition 4.3.2, it follows that the leader's investment in credibility $w^*(\cdot)$ increases in η (weight of credibility in terms of revenue), while it decreases in κ_w (marginal cost of the investment in credibility).

After determining the follower's best response and the leader's optimal strategies, we can calculate the Open-Loop Stackelberg Equilibrium (OLSE) for the *native advertising problem*.

Proposition 4.3.3. *The open-loop Stackelberg equilibrium of the native advertising problem is* $((w^*, N^*), (a^*, n^*))$*, with*

$$w^*(t) \equiv \left[\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right]^+, \; N^*(t) \in \begin{cases} [\tilde{n}, +\infty) & \alpha < \bar{\alpha} \\ \{\,0\,\} \cup [\tilde{n}, +\infty) & \alpha = \bar{\alpha} \\ \{\,0\,\} & \alpha > \bar{\alpha},\ \tilde{n} > 0 \\ [0, +\infty) & \alpha > \bar{\alpha},\ \tilde{n} = 0 \end{cases} \tag{4.34}$$

$$a^*(t) \equiv \left[\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right]^+, \qquad n^*(t; N^*(t)) = \begin{cases} \tilde{n} & \alpha < \bar{\alpha} \\ \min\{\tilde{n}, N^*(t)\} & \alpha = \bar{\alpha} \\ 0 & \alpha > \bar{\alpha} \end{cases}$$

for all $t \in [0, +\infty)$*, where* $\bar{\alpha} = \frac{\rho+\varepsilon}{2\eta}\left(\frac{\gamma_n \pi}{\rho+\delta} + \theta_n\right)$ *and* $\tilde{n} = \left[\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right]^+$.

Proof. The proof follows directly from Propositions 4.3.1 and 4.3.2. □

In summary, if the damage parameter α is smaller than the threshold $\bar{\alpha}$, then native advertising is admissible and its flow is fixed at the optimal level $n^*(t; N^*(t)) \equiv \tilde{n}$. Once the leader accepts native advertising, it is not convenient to impose any upper bound on the follower's choice. Conversely, if α is greater than the threshold $\bar{\alpha}$, native advertising is not admissible. Moreover, in the specific situation when $\tilde{n}$ is zero, the optimal control $n^*(\cdot)$ is identically zero for all values of α. This happens because, in this case, the native advertising cost for the firm is too high.
The indeterminacy of $N^*(\cdot)$ in (4.34) does not pose a problem for our model because this control variable only governs the decision of whether to allow native advertising on the media platform or not. When native advertising is accepted by the media outlet, the upper bound $N^*(\cdot)$ is not active; while, when native advertising is not accepted, the value of $N^*(\cdot)$ is determined and equal to zero. Furthermore, when $\tilde{n}$ in (4.30) is zero, the upper bound is always an inactive constraint since it is the firm itself to cease its optimal sponsored advertising strategy due to high costs.
The three optimal controls $a^*(\cdot)$, $n^*(\cdot)$, and $w^*(\cdot)$ have the same structure, which is linked to linear profit and quadratic cost assumption: they are positive if and only if their marginal costs at zero are sufficiently small.

Remark 5. It is worth analysing the special case $\alpha = \bar{\alpha}$. In such a situation, accepting native advertising or rejecting it makes no difference to the leader. This means that there are no improvements for the leader by admitting native advertising or not, as its payoff is the same in both cases. To solve this indecision, the leader may select $N^*(\cdot)$ by considering other criteria that are not included in this model. For example, if the media outlet is highly concerned with its credibility, the severe loss of credibility could be a good reason not to admit native advertising.

Remark 6. The open-loop Stackelberg equilibrium obtained in Proposition 4.3.3 is consistent in time. We observe that the optimal co-state functions of the follower (4.18) are noncontrollable, as defined in Dockner et al. (2000, Def. 5.1). Furthermore, due to the state linearity of the model formulation, the controls of the leader and the follower do not depend on the initial state; therefore, the equilibrium is also subgame perfect; see Buratto, Grosset, & Viscolani (2012, Def. 3).

4.4 Analysis of the media outlet payoff and credibility

Since we have computed the open-loop Stackelberg equilibrium, we now study the positivity of the profit of the media outlet. Henceforth, to avoid trivial situations, we assume that

$$\frac{\gamma_a \pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a} > 0, \qquad \frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n} > 0, \qquad \frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w} > 0\,, \tag{4.35}$$

ensuring that $a^*(\cdot)$, $\tilde{n}$ and $w^*(\cdot)$ are positive. These hypotheses allow us to capture the trade-off between profit and credibility for the media outlet; moreover, it is certainly satisfied when the costs are purely quadratic (i.e., $\theta_a = \theta_n = \theta_w = 0$).

Proposition 4.4.1. *Under optimality conditions, the profit of the media outlet turns out to be positive for all choices of the parameters satisfying the assumptions in (4.35).*

Proof. Taking into account the optimal controls as in (4.34), we can distinguish the following three different scenarios for $\tilde{n} > 0$:

- If $\alpha < \bar{\alpha}$, then it is optimal to accept native advertising. Let us first compute the credibility of the media outlet. Substituting expressions in (4.34) into (4.5), we have

$$\begin{aligned}\dot{C}(t) &= -\varepsilon C(t) + w^*(t) - \alpha\, n^*(t)\\ &= -\varepsilon C(t) + \frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w} - \alpha\left(\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\end{aligned}$$

with initial condition $C(0) = C_0$. Hence,

$$C^*(t) = C_0 e^{-\varepsilon t} + \frac{1}{\varepsilon}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w} - \alpha\left(\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\right)\left(1 - e^{-\varepsilon t}\right), \quad (4.36)$$

and the leader's payoff (4.19) becomes

$$\begin{aligned}J_M^* &= \int_0^{+\infty} \left(\eta\, C^*(t) - c_w(w^*(t)) + c_a(a^*(t)) + c_n(n^*(t; N^*(t)))\right) e^{-\rho t}\, \mathrm{d}t\\ &= \int_0^{+\infty}\Bigg[\eta\left(C_0 e^{-\varepsilon t} + \frac{1}{\varepsilon}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w} - \alpha\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\right)\left(1-e^{-\varepsilon t}\right)\right)\\ &\quad - \frac{\kappa_w}{2}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)^2 - \theta_w\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right) + \frac{\kappa_a}{2}\left(\frac{\gamma_a\pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2\\ &\quad + \theta_a\left(\frac{\gamma_a\pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) + \frac{\kappa_n}{2}\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)^2 + \theta_n\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\Bigg] e^{-\rho t}\,\mathrm{d}t\\ &= \frac{\eta\, C_0}{\rho+\varepsilon} + \frac{\kappa_a}{2\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 + \frac{\theta_a}{\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) +\\ &\quad + \frac{1}{\rho}\left(\frac{\gamma_n\pi}{2(\rho+\delta)} + \frac{\theta_n}{2} - \alpha\frac{\eta}{(\rho+\varepsilon)}\right)\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right) +\\ &\quad + \frac{\kappa_w}{2\rho}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)^2. \end{aligned} \quad (4.37)$$

Under the initial assumptions and being $\alpha < \bar{\alpha}$, all the addends in (4.37) are positive.

- If $\alpha > \bar{\alpha}$, then it is not optimal to accept native advertising, and the leader's credibility becomes

$$C^*(t) = C_0 e^{-\varepsilon t} + \frac{1}{\varepsilon}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)\left(1 - e^{-\varepsilon t}\right). \quad (4.38)$$

As a consequence, the leader's payoff in (4.19) is

$$
\begin{aligned}
J_M^* &= \int_0^{+\infty} \left(\eta\, C^*(t) - c_w(w^*(t)) + c_a(a^*(t))\right) e^{-\rho t}\, \mathrm{d}t \\
&= \frac{\eta\, C_0}{\rho+\varepsilon} + \frac{\kappa_a}{2\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 + \frac{\theta_a}{\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) + \\
&\quad + \frac{\kappa_w}{2\rho}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)^2 .
\end{aligned} \tag{4.39}
$$

Under the initial assumptions, the optimal profit in (4.39) is trivially positive.

- If $\alpha = \bar{\alpha}$, then it is indifferent to accept native advertising or not because (4.37) coincides with (4.39), although (4.36) may be different from (4.38).

□

We recall that the leader's strategy $N(\cdot)$ has been introduced in (4.8) to limit the loss of credibility due to native advertising. Proposition 4.4.1 shows that such an upper bound guarantees the positivity of the media outlet's profit. Indeed, neglecting the control $N(\cdot)$ in the original problem, the optimal profit differs from that computed above only in the $\alpha > \bar{\alpha}$ scenario. In such a case, the optimal profit of the media outlet turns out to be as in (4.37), instead of (4.39), although with a negative fourth addend. Therefore, its positivity is not guaranteed.
It might seem counterintuitive that the introduction of a constraint leads to an improvement in the profit of an optimal control problem. However, in our model, the leader's problem gets a different formulation depending on whether we incorporate $N(\cdot)$ in the constraint (4.8) or not. Hence, it is convenient for the media outlet to impose its control $N(\cdot)$ on native advertising to guarantee the positivity of its profit. However, when it comes to the evolution of credibility, a different outcome emerges: accepting native advertising does not guarantee a positive outcome of credibility.

Remark 7. For completeness, we also compute the goodwill of the firm and its payoff in the different cases defined by $\bar{\alpha}$. Taking into account the optimal controls as in (4.34), we have:

- If $\alpha < \bar{\alpha}$, native advertising is accepted by the media outlet. We compute the firm's goodwill by substituting expressions in (4.34) into (4.2), we have

$$
\begin{aligned}
\dot{G}(t) &= -\delta G(t) + \gamma_a a^*(t) + \gamma_n n^*(t) \\
&= -\delta G(t) + \frac{\gamma_a^2 \pi}{\kappa_a(\rho+\delta)} - \frac{\gamma_a \theta_a}{\kappa_a} + \frac{\gamma_n^2 \pi}{\kappa_n(\rho+\delta)} - \frac{\gamma_n \theta_n}{\kappa_n}
\end{aligned}
$$

where $G(0) = G_0$. Therefore,

$$
G^*(t) = G_0 e^{-\delta t} + \frac{1}{\delta}\left(\frac{\gamma_a^2 \pi}{\kappa_a(\rho+\delta)} - \frac{\gamma_a \theta_a}{\kappa_a} + \frac{\gamma_n^2 \pi}{\kappa_n(\rho+\delta)} - \frac{\gamma_n \theta_n}{\kappa_n}\right)\left(1 - e^{-\delta t}\right), \quad (4.40)
$$

and the leader's payoff (4.9) becomes

$$
\begin{aligned}
J_F^* &= \int_0^{+\infty} \left(\pi G^*(t) - c_a(a^*(t)) - c_n(n^*(t))\right) e^{-\rho t}\, \mathrm{d}t \\
&= \int_0^{+\infty} \Bigg[\pi \left(G_0 e^{-\delta t} + \frac{1}{\delta}\left(\frac{\gamma_a^2 \pi}{\kappa_a(\rho+\delta)} - \frac{\gamma_a \theta_a}{\kappa_a} + \frac{\gamma_n^2 \pi}{\kappa_n(\rho+\delta)} - \frac{\gamma_n \theta_n}{\kappa_n}\right)\left(1 - e^{-\delta t}\right)\right) \\
&\quad - \frac{\kappa_a}{2}\left(\frac{\pi \gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 - \theta_a \left(\frac{\pi \gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) \\
&\quad - \frac{\kappa_n}{2}\left(\frac{\pi \gamma_n}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)^2 - \theta_n \left(\frac{\pi \gamma_n}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\Bigg] e^{-\rho t}\, \mathrm{d}t \\
&= \frac{\pi\, G_0}{\rho+\delta} + \frac{\kappa_a}{2\rho}\left(\frac{\pi \gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 + \frac{\kappa_n}{2\rho}\left(\frac{\pi \gamma_n}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)^2, \quad (4.41)
\end{aligned}
$$

which is trivially positive.

- If $\alpha > \bar{\alpha}$, then the media outlet does not allow native advertising, then

$$
G^*(t) = G_0 e^{-\delta t} + \frac{1}{\delta}\left(\frac{\gamma_a^2 \pi}{\kappa_a(\rho+\delta)} - \frac{\gamma_a \theta_a}{\kappa_a}\right)\left(1 - e^{-\delta t}\right). \quad (4.42)
$$

As a consequence, the leader's payoff in (4.19) is

$$
\begin{aligned}
J_F^* &= \int_0^{+\infty} \left(\pi G^*(t) - c_a(a^*(t))\right) e^{-\rho t}\, \mathrm{d}t \\
&= \frac{\pi\, G_0}{\rho+\delta} + \frac{\kappa_a}{2\rho}\left(\frac{\pi \gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2. \quad (4.43)
\end{aligned}
$$

The firm payoff remains positive, but it is lower than what was obtained in (4.41).

- If $\alpha = \bar{\alpha}$, then it makes no difference for the media outlet to admit native advertising or not from a profit point of view. However, both firm's payoff and goodwill will be influenced by the effective media choice.

Going back to the positivity of credibility, we can prove what was announced before.

Lemma 4.4.2. *Let us assume that the assumptions in (4.35) hold and that the media outlet accepts native advertising with* $\alpha < \bar{\alpha}$*, then the credibility of the media outlet remains positive for all* $t \in [0, +\infty)$ *if and only if*

$$\alpha < \frac{\kappa_n}{\kappa_w}\left(\frac{\eta}{\rho+\varepsilon} - \theta_w\right)\left(\frac{\gamma_n \pi}{\rho+\delta} - \theta_n\right)^{-1} =: \alpha_C\,. \tag{4.44}$$

Proof. Let us substitute the open-loop Stackelberg equilibrium (4.34) into the dynamics (4.5), then the optimal credibility trajectory with $\alpha < \bar{\alpha}$ is (4.36) for all $t \in [0, +\infty)$. This optimal trajectory remains positive for all $t \in [0, +\infty)$ if and only if (4.44) holds. □

Observe that condition $\alpha < \bar{\alpha}$ is not sufficient to ensure $\alpha < \alpha_C$ because the relative position of $\bar{\alpha}$ and α_C depends on parameters κ_n and κ_w. Indeed, the positivity of the credibility is guaranteed by the following sufficient condition.

Proposition 4.4.3. *Let us assume that the assumptions in (4.35) hold and that the media outlet accepts native advertising with* $\alpha < \bar{\alpha}$*, then the credibility of the media outlet remains positive for all* $t \in [0, +\infty)$ *only if*

$$\frac{\kappa_n}{\kappa_w} \geq \bar{\alpha}\left(\frac{\gamma_n \pi}{\rho+\delta} - \theta_n\right)\left(\frac{\eta}{\rho+\varepsilon} - \theta_w\right)^{-1}. \tag{4.45}$$

Proof. If (4.45) holds, the threshold α_C in Lemma 4.4.2 is greater than or equal to $\bar{\alpha}$; therefore, the credibility of the media outlet is greater than zero for any $t \in [0, +\infty)$. □

Remark 8. In contrast to credibility, firm goodwill is always positive in both its formulations (4.40) and (4.42).

Proposition 4.4.3 provides sufficient conditions to guarantee positive credibility. This can happen if κ_n and θ_n are high or if κ_w and θ_w are low. In other words, if the cost of native advertising is too large, its damage in terms of credibility is small. Similarly, when the cost of high-quality content is small, credibility can be easily sustained.

Suppose that the media outlet is highly concerned with its credibility. Then, it might consider preserving its credibility by always rejecting native advertising. Since for high values of the damage parameter α, the media outlet already forbids native advertising on its platform, in the following discussion we consider only the case $\alpha < \bar{\alpha}$. In this situation, we compare the optimal profit (4.37) with the profit that the media outlet would obtain under the assumption that no native advertising is admitted. We are interested in evaluating the cost of credibility, which refers to the reduction in profit per unit of credibility preserved when the media outlet chooses to protect its credibility instead of allowing native advertising. This definition is not straightforward; hence, we drive the reader in our analysis underlying the meaning of some formulas. For the reader's convenience, we recall here the optimal credibility (4.36) for $\alpha < \bar{\alpha}$

$$C^*(t) = C_0 e^{-\varepsilon t} + \frac{1}{\varepsilon}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w} - \alpha\left(\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\right)\left(1 - e^{-\varepsilon t}\right). \tag{4.46}$$

This equation describes the evolution of credibility when native advertising is optimally activated.

Assume that the media outlet refuses native advertising by fixing $N(t) \equiv 0$ for all $t \in [0, +\infty)$. This choice of the strategy $N(\cdot)$ implies that $n(t, N(t)) \equiv 0$ for all $t \in [0, +\infty)$, while the standard advertising strategy $a(\cdot)$ of the firm and the investment in high-quality content strategy $w(\cdot)$ of the media outlet are the same as in (4.34). Under this assumption, the credibility of the media outlet turns out to be

$$C^0(t) = C_0 e^{-\varepsilon t} + \frac{1}{\varepsilon}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)\left(1 - e^{-\varepsilon t}\right). \tag{4.47}$$

This equation describes the evolution of credibility when the media outlet decides *a priori* to avoid native advertising. Therefore, the instantaneous loss of credibility obtained by admitting native advertising can be defined as

$$\Delta C(t) = C^*(t) - C^0(t) = -\frac{\alpha}{\varepsilon}\left(\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right)\left(1 - \mathrm{e}^{-\varepsilon t}\right). \tag{4.48}$$

We note that the linearity of the motion equations allows us to eliminate the initial credibility C_0 in this difference. The limit when t tends to infinity allows us to obtain the constant long-term loss of credibility experienced by the media outlet by admitting native advertising

$$\Delta C = -\frac{\alpha}{\varepsilon}\left(\frac{\gamma_n \pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right) < 0\,. \tag{4.49}$$

The above loss of credibility corresponds to a gain in profit for the media outlet, which we can calculate explicitly. For the reader's convenience, we also recall the profit (4.37) at the Stackelberg equilibrium (4.34) for $\alpha < \bar{\alpha}$

$$\begin{aligned} J_M^* = \frac{\eta\, C_0}{\rho+\varepsilon} &+ \frac{\kappa_a}{2\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 + \frac{\theta_a}{\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) + \\ &+ \frac{1}{\rho}\left(\frac{\gamma_n\pi}{2(\rho+\delta)} + \frac{\theta_n}{2} - \alpha\frac{\eta}{\rho+\varepsilon}\right)\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right) + \\ &+ \frac{\kappa_w}{2\rho}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)^2, \end{aligned}$$

while the profit we get by refusing native advertising, with correspondent credibility (4.47), is

$$\begin{aligned} J_M^0 = \frac{\eta\, C_0}{\rho+\varepsilon} &+ \frac{\kappa_a}{2\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right)^2 + \frac{\theta_a}{\rho}\left(\frac{\pi\gamma_a}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right) + \\ &+ \frac{\kappa_w}{2\rho}\left(\frac{\eta}{\kappa_w(\rho+\varepsilon)} - \frac{\theta_w}{\kappa_w}\right)^2. \end{aligned} \tag{4.50}$$

So, the increase in profit obtained by accepting native advertising is

$$\Delta J_M = J_M^* - J_M^0 = \frac{1}{\rho}\left(\frac{\gamma_n\pi}{2(\rho+\delta)} + \frac{\theta_n}{2} - \alpha\frac{\eta}{\rho+\varepsilon}\right)\left(\frac{\gamma_n\pi}{\kappa_n(\rho+\delta)} - \frac{\theta_n}{\kappa_n}\right), \tag{4.51}$$

which is positive under assumptions $\alpha < \bar{\alpha}$ and (4.35).

Finally, we define as follows the *cost of credibility*

$$\left|\frac{\Delta J_M}{\Delta C}\right| = \frac{\varepsilon}{\alpha\rho}\left(\frac{\gamma_n\pi}{2(\rho+\delta)} + \frac{\theta_n}{2} - \alpha\frac{\eta}{\rho+\varepsilon}\right). \tag{4.52}$$

This formula describes the loss of profit incurred for every unit of saved credibility at the equilibrium.

The following Table shows the sensitivity analysis of the cost of credibility with respect to the parameters that characterise the leader's problem.

	ε	η	α
$\left\lvert\frac{\Delta J_M}{\Delta C}\right\rvert$	$+$	$-$	$-$

The strictly positive monotonicity in ε can be easily justified: the higher the credibility natural decay parameter, the higher the cost of credibility.
On the other hand, considering η: the higher the marginal profit with respect to credibility, the lower the cost of credibility. This happens because ΔC in (4.49) does not depend on η, while ΔJ_M in (4.51) exhibits a monotonically decreasing relationship with η. Let us observe that the negative effect in (4.51) corresponds exactly to the final term of the optimal profit J_M^* in (4.50), which is due to native advertising.
Finally, considering α: the higher the damage of native advertising to credibility, the lower the cost of credibility. This happens because $|\Delta C|$ increases in α; J_M^0 does not depend on α, while J_M^* decreases with respect to α. This behaviour can be justified by noting that, when α is high, the media outlet has to give up a large amount of credibility to make a small profit. To the limit, when $\alpha \to \bar{\alpha}$, the media outlet does not gain any profit from native advertising, so the cost of credibility vanishes.

Remark 9. In the above analysis, we have decided to consider ΔJ_M in (4.51) along the optimal trajectories instead of evaluating it at the steady state, to ensure that the variation in profit ΔJ_M is positive if and only if $\alpha < \bar{\alpha}$. Using steady-state trajectories would not guarantee a positive variation in profit for any $\alpha < \bar{\alpha}$.

Remark 10. In general, open-loop solutions are not credible in dynamic contexts because they do not allow the review of strategies over time as new information becomes available. However, in our case, the Stackelberg open-loop equilibrium identified coincides with the feedback solution. This fact depends on the linear state structure of the game and holds although the presence of a leader's constraint on the follower's native advertising flow.

4.5 Native advertising and credibility, a different perspective

An interesting development of the model described above can be achieved by incorporating the influence of the media outlet's credibility on the effectiveness of native advertising. In this section, we formulate a possible extension of the problem by assuming that the effectiveness of sponsored advertising is no longer constant but follows an increasing linear function of credibility. In other words, the greater the credibility of the media outlet, the more effective native advertising becomes. However, introducing this dependence on the media outlet's credibility leads to a not-time-consistent Stackelberg equilibrium. Following the above assumption, the goodwill motion equation (4.2) takes the following form

$$\dot{G}(t) = -\delta G(t) + \gamma_a a(t) + \gamma_n C(t)\, n(t)\,, \tag{4.53}$$

where the native advertising effectiveness is given by $\gamma_n\, C$, i.e. it linearly depends on the credibility of the media outlet. Let us now analyse this issue using two different formulations.

4.5.1 Stackelberg Equilibrium

Let us consider the optimal control problem of the follower (4.9), where the credibility motion equation is replaced by (4.53).

Proposition 4.5.1. *Any Stackelberg equilibrium for this new formulation of the problem is time-inconsistent.*

Proof. The firm's Hamiltonian function is

$$\begin{aligned} \mathrm{H}^c_F(G, C, a, n, \overline{w}(t), \overline{N}(t), \lambda_1, \lambda_2) &= \pi G - c_a(a) - c_n(n) \\ &\quad + \lambda_1(-\delta G + \gamma_a a + \gamma_n Cn) + \lambda_2(-\varepsilon C + \overline{w}(t) - \alpha\, n)\,. \end{aligned}$$

The necessary conditions derived from Pontryagin's Maximum Principle result in a standard advertising strategy similar to equation (4.14), but they yield a slightly different optimal strategy for native advertising:

$$n(t; \overline{N}(t)) = \min\left\{ \left[\frac{\gamma_n\, C\lambda_1(t) - \theta_n - \alpha\lambda_2(t)}{\kappa_n} \right]^+, \overline{N}(t) \right\}. \tag{4.54}$$

Moreover, the second adjoint equation of the follower becomes

$$\dot{\lambda}_2(t) = \rho\lambda_2(t) - \partial_C H^c_F = (\rho + \varepsilon)\lambda_2(t) - \gamma_n n(t; \overline{N}(t))\lambda_1(t), \tag{4.55}$$

from which we obtain a follower's adjoint equation that is directly controllable by the leader through the control $\overline{N}(t)$ and indirectly via state function $C(t)$. Consequently, any Stackelberg equilibrium for this new formulation of the problem turns out to be time-inconsistent; see [4, p.117]. □

Remark 11. Time inconsistency of the Stackelberg equilibrium is a direct consequence of the controllability of the follower's adjoint equation by the leader through the control $\overline{N}(t)$ and the state function $C(t)$, see [4, p.116].

To bypass the inconsistency issue and obtain a result describing the interaction between the effectiveness of sponsored advertising and the credibility of the media outlet, we decided to modify the type of equilibrium that we want to consider, shifting from the Stackelberg open-loop equilibrium to the Markovian Nash equilibrium. Before analysing this new problem, it is important to emphasise the consequences of a change in the type of solution we are looking for in the problem formulation. Indeed, the inclusion of credibility-dependent effectiveness for native advertising reduces the necessity for the upper bound control N. Clearly, in the original model, the presence of N was essential for the media outlet to safeguard its credibility, since the firm had no interest in preserving credibility. However, after the introduction of a linear dependence of sponsored advertising on media credibility, the intensity of the firm's native advertising depends directly on credibility. As a consequence, too much damage to credibility will result in low effectiveness of sponsored advertising. For this reason, we assume henceforth that the media outlet cannot control the maximum intensity of native advertising. This assumption is also consistent with the fact that we are no longer considering a Stackelberg game. The choice between a Nash equilibrium formulation and a Stackelberg equilibrium formulation depends on the specific characteristics of the strategic interaction being analysed and the assumptions made about the players' decision-making power and information. Furthermore, this hypothesis simplifies the problem, allowing us to obtain a closed-form solution.

4.5.2 Markovian Nash Equilibrium

Let us now formalise the "Nash" framework of the problem, where the firm's strategies are the two types of advertising a, n, while the media strategy is the high-quality content w. Let us recall that the control N can be omitted in this new version of the problem. The firm and the media outlet want to maximise their respective discounted profits, which are the same as in (4.3) and (4.6). The goodwill state equation is now the one in (4.53) and the evolution of credibility remains (4.5).

We recall that the main differences between the formulation of the game solved in Section 4.3 and the current one lie in the firm problem. This is because the evolution of the firm's brand value is now affected by the credibility of the media outlet, and in turn, this credibility can be negatively impacted by sponsored advertising.

Proposition 4.5.2. *The optimal high-quality content and standard advertising flows are the same as in (4.3.3), whereas the optimal value for native advertising is*

$$n^*(C(t), q_2(t)) = \left[\frac{\gamma_n \pi\, C(t)}{\kappa_n(\rho+\delta)} - \frac{\theta_n + \alpha\, q_2(t)}{\kappa_n}\right]^+ , \tag{4.56}$$

where $q_2(t)$ is the shadow value of credibility for the firm's problem. Furthermore, when searching for non-trivial solutions, the credibility remains positive in the long run and its value at equilibrium is

$$C^{eq} = \frac{w^*(\rho+\delta)[\kappa_n(\rho+\varepsilon) + \alpha\gamma_n] + \alpha\theta_n(\rho+\delta)(\rho+\varepsilon)}{\kappa_n\varepsilon(\rho+\varepsilon)(\rho+\delta) + \alpha\gamma_n(\varepsilon(\rho+\delta) + \pi(\rho+\varepsilon))} . \tag{4.57}$$

Finally, the following activation condition for native advertising holds

$$\pi\gamma_n\, C^{eq} > \theta_n(\rho+\delta) . \tag{4.58}$$

Proof. Looking for a Markovian Nash Equilibrium, we obtain the same optimal high-quality content flow w^* as in (4.21). The same does not hold for the firm's problem because the evolution of goodwill now depends on the credibility of the media outlet. Consider the following current-value Hamiltonian function of the firm:

$$\begin{aligned} H_F^c(G, C, a, n, q_1, q_2) = \pi G - c_a(a) - c_n(n) + q_1(-\delta G + \gamma_a a + \gamma_n \cdot C \cdot n) + \\ + q_2(-\varepsilon C + w^* - \alpha\, n) \end{aligned}$$

where q_1 and q_2 are the firm co-state functions of the firm for the Markovian Nash equilibrium. Then, the optimal value for the standard advertising is constant and equal to the previous one:

$$a^* = \left[\frac{\gamma_a \pi}{\kappa_a(\rho+\delta)} - \frac{\theta_a}{\kappa_a}\right]^+ . \tag{4.59}$$

In contrast, the optimal value for native advertising is

$$n^*(C(t), q_2(t)) = \left[\frac{\gamma_n \pi\, C(t)}{\kappa_n(\rho+\delta)} - \frac{\theta_n + \alpha\, q_2(t)}{\kappa_n}\right]^+ . \tag{4.60}$$

Once again, we are interested in nontrivial solutions, specifically $n^*(\cdot) > 0$. Let us assume that

$$\frac{\gamma_n \pi\, C(t)}{\kappa_n(\rho+\delta)} - \frac{\theta_n + \alpha\, q_2(t)}{\kappa_n} > 0 ;$$

then, a sufficient condition on the activation of native advertising is

$$\frac{\gamma_n \pi\, C(t)}{\alpha(\rho+\delta)} - \frac{\theta_n}{\alpha} > q_2(t). \tag{4.61}$$

The co-state variable $q_2(t)$ represents the shadow value of credibility. In other words, $q_2(t)$ is the highest (hypothetical) price that a rational decision-maker would be willing to pay for an additional unit of credibility at time t when the problem is solved optimally. If the shadow price is sufficiently small, native advertising is activated. Similarly, if the credibility of the media is too low, native advertising is not activated.

Our goal is to obtain a closed-form formula to compare these results with those of the previous model. To calculate the optimal native advertising strategy in (4.60), it is necessary to know both the credibility function and its associated co-state function $q_2(t)$. Hence, we aim to solve the following differential equations

$$\begin{cases} \dot{G}(t) = -\delta G(t) + \gamma_a a^*(t) + \gamma_n\, C(t)\, n^*(C(t), q_2(t)) \\ \dot{C}(t) = -\varepsilon C(t) + w^*(t) - \alpha\, n^*(C(t), q_2(t)) \\ \dot{q}_2(t) = (\rho+\varepsilon) q_2(t) - \gamma_n\, n^*(C(t), q_2(t)) \end{cases} \tag{4.62}$$

It is important to note that the optimal high-quality investment is the same as in (4.21). Since we are looking for equilibrium points, we also assume that in the long run $\dot{G} = 0$, $\dot{C} = 0$, and also $\dot{q}_2 = 0$. Hence, substituting the native advertising optimal control in the differential equations we obtain the steady state conditions

$$\begin{cases} 0 = -\delta G^{eq} + \gamma_a a^* + \gamma_n\, C^{eq} \left(\dfrac{\gamma_n \pi\, C^{eq}}{\kappa_n(\rho+\delta)} - \dfrac{\theta_n}{\kappa_n} - \dfrac{\alpha\, q_2^{eq}}{\kappa_n} \right) \\ 0 = -\varepsilon C^{eq} + w^* - \alpha \left(\dfrac{\gamma_n \pi\, C^{eq}}{\kappa_n(\rho+\delta)} - \dfrac{\theta_n}{\kappa_n} - \dfrac{\alpha\, q_2^{eq}}{\kappa_n} \right) \\ 0 = (\rho+\varepsilon) q_2^{eq} - \gamma_n \left(\dfrac{\gamma_n \pi\, C^{eq}}{\kappa_n(\rho+\delta)} - \dfrac{\theta_n}{\kappa_n} - \dfrac{\alpha\, q_2^{eq}}{\kappa_n} \right) \end{cases}$$

From the third one, we get

$$q_2^{eq} = \frac{\gamma_n^2 \pi\, C^{eq} - \gamma_n \theta_n (\rho+\delta)}{(\rho+\delta)[\kappa_n(\rho+\varepsilon) + \alpha\gamma_n]}\,. \tag{4.63}$$

Then, the steady-state equation for credibility gives

$$C^{eq} = \frac{w^*(\rho+\delta)[\kappa_n(\rho+\varepsilon)+\alpha\gamma_n] + \alpha\theta_n(\rho+\delta)(\rho+\varepsilon)}{\kappa_n\varepsilon(\rho+\varepsilon)(\rho+\delta)+\alpha\gamma_n(\varepsilon(\rho+\delta)+\pi(\rho+\varepsilon))}\,. \tag{4.64}$$

From the above equation, it is clear that the credibility remains positive in the long run. This is justified by the fact that, within this model, the firm is also vested in the credibility of the media outlet, given that credibility plays a role in the goodwill equation. Let us now concentrate on the native advertising activation condition. Substituting the equilibria (4.63) into (4.61), we get

$$\begin{aligned}
&\frac{\gamma_n\pi C^{eq}}{\alpha(\rho+\delta)} - \frac{\theta_n}{\alpha} > \frac{\gamma_n^2\pi C^{eq} - \gamma_n\theta_n(\rho+\delta)}{(\rho+\delta)[\kappa_n(\rho+\varepsilon)+\alpha\gamma_n]}\\
\implies &\frac{\gamma_n\pi}{(\rho+\delta)}\frac{C^{eq}}{\alpha} - \frac{\gamma_n^2\pi C^{eq}}{(\rho+\delta)[\kappa_n(\rho+\varepsilon)+\alpha\gamma_n]} > \frac{\theta_n}{\alpha} - \frac{\gamma_n\theta_n}{\kappa_n(\rho+\varepsilon)+\alpha\gamma_n}\\
\implies &\frac{\gamma_n\pi C^{eq}}{(\rho+\delta)}\left(\frac{1}{\alpha} - \frac{\gamma_n}{\kappa_n(\rho+\varepsilon)+\alpha\gamma_n}\right) > \theta_n\left(\frac{1}{\alpha} - \frac{\gamma_n}{\kappa_n(\rho+\varepsilon)+\alpha\gamma_n}\right).
\end{aligned}$$

So that, it can be proved that equation (4.61) holds true if and only if

$$\pi\gamma_n C^{eq} > \theta_n(\rho+\delta)\,, \tag{4.65}$$

given that

$$\frac{1}{\alpha} - \frac{\gamma_n}{\kappa_n(\rho+\varepsilon)+\alpha\gamma_n} > 0 \iff \kappa_n(\rho+\varepsilon) > 0.$$

Therefore, the activation condition for native advertising strongly depends on credibility.

□

We observe that the optimal native advertising effort increases as the credibility value increases and declines as the credibility shadow price increases: the higher the credibility of the media, the greater the optimal level of native advertising, whereas the more sensitive credibility is to changes in initial data, the lower the optimal level of native advertising. Moreover, it is optimal to activate native advertising in the long run if the efficiency of the sponsored advertising weighted by the marginal revenue given by goodwill is higher than the marginal cost at zero for native weighted by the discount factor and the decay parameter.

Remark 12. The above analysis defines a Markovian Nash equilibrium only if it is a stable solution.

4.6 Discussion

We addressed the issue of a media outlet facing the dilemma of balancing the profit gained from publishing native advertising against the ensuing loss of credibility. We formalised a hierarchical differential game where the media outlet acts as the leader, whereas the firm that invests in advertising on such a platform acts as the follower. We introduce a constraint on native advertising flow that has a dual meaning: it represents something innovative for this type of problem and emphasises why the media outlet is the leader of the game; for further extension, we could avoid the upper limit by introducing a high cost in loss of credibility. We characterised a time-consistent open-loop Stackelberg equilibrium for the game and obtained the conditions under which it is optimal for the media outlet to accept native advertising. Regarding research questions, we demonstrated that the admissibility of native advertising depends on its effective damage to the credibility of the media outlet. If the damage due to native advertising is low, the media outlet accepts native advertising because it increases the outlet's profit, while if the damage due to native advertising is high, the media outlet preserves its credibility by not accepting native advertising. Furthermore, in Proposition 4.3.3, the optimal level of native advertising and the optimal upper bound fixed by the media outlet to counteract the negative effect on its credibility were computed. Finally, we showed that the admissibility of native advertising is not sufficient to guarantee positive credibility for the media outlet; indeed, we introduced an additional condition on the problem parameters to ensure this. In contrast, admitting native advertising is sufficient to ensure a positive profit for the media outlet itself.

In the last part of the chapter, we generalised the model by assuming a linear dependence between the effectiveness of native advertising and the credibility of the media outlet. Under these assumptions, the Stackelberg equilibrium is not time-consistent. Hence, we reformulated the differential game and calculated a Markovian Nash equilibrium to describe the trade-off for the media outlet between loss of credibility and revenues generated by accepting native advertising. The primary outcome of this revised model is that, unlike standard advertising, the activation of sponsored advertising is not solely tied to its cost; it is also linked to the credibility level of the media outlet at equilibrium.

A possible extension of this model consists of studying the alignment between the content of an advertising message and the evolution of the credibility of a media outlet. Accepting standard or native advertisements that are not in line with the editorial policies of a media outlet can damage the outlet's credibility and consequently its profits. For example, the use of greenwashing is a widespread practice and, to our knowledge, there are no dynamic models that explore the trade-off between gain from misleading advertising and loss of credibility.

Part II

Markov based models with binary state

5

Continuous-Time Markov decision problems with binary state

5.1 Introduction

This part of my thesis focuses on the analysis of the Hamilton–Jacobi–Bellman equation for binary state continuous-time Markov decision problems. We aim to prove that the standard HJB equation can be replaced by a simpler backward differential equation under suitable assumptions on the probability rate and the cost function. The results displayed below refer to a paper published in collaboration with Professor Elena Sartori and my Supervisor, Professor Luca Grosset [10].

The theory of continuous-time Markov decision processes is a rapidly growing area of research with numerous practical applications; see, e.g., [33], and [34]. We focus on binary state processes and prioritise an analytical solution over a computational one. The description of the evolution of binary state processes draws inspiration from the Curie-Weiss model [35], but recent models have also incorporated human rationality by allowing decision-making agents to modify their states. Initially, agents were assumed to update their state according to a predetermined probabilistic transition rate based on their surrounding environment. However, this approach does not take into account the rationality of humans. Therefore, to change their states, agents face decision problems and try to minimise or maximise a suitable objective (such as cost, gain, or level of satisfaction). To model this situation, we introduce in the transition rate a control function, which is chosen by the agents once they have solved their optimisation problem. This leads to the definition of a controlled Markov chain. It is well known that the Hamilton–Jacobi–Bellman equation (HJB) associated with a controlled continuous-time finite-state Markov process can be difficult to treat. In our work, we use techniques developed in [36], [37] and [38], and references therein, to derive closed or semi-closed expressions for optimal control and state evolution.

5.2 The model

In this section, we describe the continuous-time Markov decision process that we are going to study, which is defined in the programming interval $[0, T]$ with $T > 0$. Let $S = \{-1, 1\}$ be the state space and let $[0, v]$, with $v > 0$, be the control set. For $t \in [0, T]$ denote by X_t the state process and by $U_t \in [0, v]$ the control process. We formally define the dynamics of the process by

$$\mathbb{P}\left(X_{t+h} = -x \,|\, X_t = x, U_t = u\right) = \ell(x, u)h + o(h), \tag{5.1}$$

where $\ell(x, \cdot) : [0, v] \to [0, +\infty)$ is a continuous function for all $x \in S$. Now, assume that the system is controlled using a feedback function:

$$U_t = u(t, X_{t^-}), \tag{5.2}$$

where

$$u : [0, T] \times \{-1, 1\} \to [0, v] \tag{5.3}$$

is a measurable function. We denote by $\mathcal{U}$ the set of all feasible feedback control functions. We recall that for all feasible feedback functions, there exists a continuous-time Markov process defined by (5.1).

For the reader's convenience, we sketch how control and state are connected. We observe that the following approach is a bit more general than what we would need because it can be applied not only for the feedback controls, but also for more general non-anticipating controls. Nevertheless, we prefer to keep this approach since it is the most popular in stochastic optimal control problems. Let us denote by $B(S)$ the space of bounded functions with real value in S equipped with the supremum norm.

For all $t \in [0, T]$ and for all measurable control functions u, consider the following operator:

$$\begin{aligned} \Lambda_t^u: \quad B(S) &\rightarrow B(S) \\ f &\mapsto \Lambda_t^u f, \end{aligned} \tag{5.4}$$

where

$$\Lambda_t^u f(x) := \ell(x, u(t,x)) \left(f(-x) - f(x) \right)$$

for $x \in S$. Let $\mathcal{D} = \mathcal{D}([0,T], S)$ be the space of right-continuous functions with finite left limit defined in $[0, T]$, taking values in the finite binary set $S = \{-1, 1\}$. We endow this space with the Skorokhod topology and denote by $\mathcal{S}$ the Borel σ-algebra on $\mathcal{D}$. In the measurable space $(\mathcal{D}, \mathcal{S})$, we denote by $(X_t)_{t \in [0,T]}$ the canonical process: $X_t(\omega) = \omega(t)$.

Let μ be a probability measure on S. A probability measure $\mathbb{P}^u_{s,\mu}$ on $(\mathcal{D}, \mathcal{S})$ is a solution to the martingale problem characterised by (5.4) if and only if the following two conditions hold:

1. $\mathbb{P}^u_{s,\mu}(X_s \in A) = \mu(A)$ for $s \in [0, T], A \subset S$;

2. for all functions $f \in B(S)$, the process $f(X_t) - \int_0^t \Lambda_r^u f(X_r) dr$ is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(X_r, r \leq t)$.

The process X_t characterised by the previous martingale problem is the unique continuous-time Markov chain with infinitesimal generator given by (5.4). Let us recall that the definition in (5.1) is equivalent to what follows (5.4). As explained in [39], we can obtain the same result through the transition matrix. In the following discussion, we rely on the generator structure for the mathematical computational benefits, whereas we prefer to use the probability rate for its economic interpretability and to obtain insights about the problem.

This discussion allows us to clarify the connection between a feedback control function and the associated state function. Therefore, we denote by $\mathbb{E}^u_{s,\mu}$ the expectation with respect to the probability measure $\mathbb{P}^u_{s,\mu}$. Furthermore, if μ is the measure that concentrates all probability in the state $x \in S$, we write $\mathbb{P}^u_{s,x}$ and $\mathbb{E}^u_{s,x}$.

5.3 Stochastic optimal control

In this section, we present the finite-time optimal control problem we are dealing with. We are looking for a feasible feedback control function u that minimises

$$J(u) := \mathbb{E}^{u}_{0,x}\left\{\int_0^T c(t, X_t, U_t)dt + g(X_T)\right\}, \tag{5.5}$$

where $x \in S$, $c(\cdot,\cdot,\cdot)$ is a continuous function and $g(\cdot)$ is given.
Let's introduce the optimal value function:

$$V(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}^{u}_{t,x}\left\{\int_t^T c(s, X_s, U_s)ds + g(X_T)\right\}.$$

This function is the key point to solving our optimal control problem because it solves a particular equation. More in detail, the HJB equation associated with the previous optimal control problem can be written as

$$\begin{cases} \partial_t V(t,x) + \min\limits_{w\in[0,v]} \{c(t,x,w) + \Lambda^w_t V(t,x)\} = 0 \\ V(T,x) = g(x), \end{cases} \tag{5.6}$$

where $x \in \{-1, 1\}$, and $t \in [0, T)$.

In the following Theorem, we explain the connection between the optimal value function, a solution of (5.6), an optimal feedback control function.

Theorem 5.3.1 (HJB Equation). *Let us assume that:*

- $S = \{-1, 1\}$*;*
- $\ell(x,\cdot) : [0, v] \to [0, +\infty)$ *is a continuous function for all* $x \in S$*;*
- $c : [0, T] \times S \times [0, v] \to \mathbb{R}$ *is a continuous function.*

Then there exists a unique function, bounded and continuously differentiable with respect to the first variable, which is a solution of the HJB equation. Moreover, an optimal control in the feedback form $u^* : [0, T] \times \{-1, 1\} \to [0, v]$ *is given by*

$$c(t,x,u^*(t,x)) + \Lambda^{u^*}_t V(t,x) = \min_{w\in[0,v]} \{c(t,x,w) + \Lambda^w_t V(t,x)\}. \tag{5.7}$$

Proof. See [40] Theorem 2.4. □

This standard result is very powerful when we want to solve an optimal control problem driven by a stochastic differential equation. However, when considering a controlled continuous-time Markov chain with binary state, the HJB equation does not seem to be so useful. In the next section, we present an approach that allows us to use all the information of the HJB equation to find an optimal control.

5.4 From HJB to a backward ODE

At this point, we show how the HJB equation can be replaced by an ordinary backward differential equation. The idea is to use the algebraic difference between the evaluation of the HJB equation in both states of the system to obtain useful information to construct an optimal feedback control. Thus, the HJB equation, which seemed hardly treatable, allows us to obtain an ordinary differential equation that must be solved backwards. Henceforth, we use the discrete gradient notation: for all $x \in S$ we define

$$\nabla_x f(x) := f(-x) - f(x).$$

Now we have all the necessary information to enunciate and prove the following result.

Theorem 5.4.1 (Backward ODE). *Assume that the optimal control problem presented in the previous sections has the following formulation:*

- $\ell(x,u) = \alpha(x) + \beta u$, *with* $\alpha(x) \geq 0$ *for all* $x \in S$ *and* $\beta > 0$*;*
- $c(t,x,u) = \gamma(t)x + \dfrac{\kappa(t)}{2}u^2$ *, with* $\gamma(\cdot), \kappa(\cdot) \in C^0([0,T],\mathbb{R})$*, and* $\kappa(t) > 0$.

Moreover, suppose that we can solve the backward ODE:

$$\begin{cases} \dot{z}(t) = 2\gamma(t) - \nabla_x \alpha(1) + \dfrac{\beta^2}{2\kappa(t)}\,|z(t)|\,z(t) \\ z(T) = \nabla_x g(1) \end{cases}$$

and, finally, assume that the solution $z(t)$ of this ODE satisfies the inequality:

$$[z(t)]^- \leq \frac{\upsilon\kappa(t)}{\beta}, \tag{5.8}$$

so that the control constraint is inactive. Therefore, the optimal feedback control is

$$u^*(t,x) = \frac{\beta}{\kappa(t)}[z(t)x]^-.$$

Proof. Under the assumptions, the HJB equation becomes:

$$\partial_t V(t,x) \ + \min_{w\in[0,v]} \left\{ \gamma(t)x + \frac{\kappa(t)w^2}{2} + \alpha(x) + \beta w \nabla_x V(t,x) \right\} = 0. \tag{5.9}$$

We are searching for

$$w \in \arg\min_{[0,v]} \left\{ \gamma(t)x + \frac{\kappa(t)w^2}{2} + \alpha(x) + \beta w \nabla_x V(t,x) \right\} =: \varphi(w) \, .$$

Since

$$\varphi'(w) = \kappa(t)w + \beta \nabla_x V(t,x)$$
$$\varphi''(w) = \kappa(t) > 0 \, ,$$

we get

$$w = \min \left\{ \frac{\beta}{\kappa(t)} [\nabla_x V(t,x)]^-, v \right\} . \tag{5.10}$$

If the constraint in (5.8) is active, the result (5.10) would be v, and we lose the internal solution of the optimal control problem we are interested in.
Since $\frac{\beta}{\kappa(t)}[\nabla_x V(t,x)]^- \ \leq \ v$ by the above assumption, we can rewrite the HJB equation by substituting (5.10) into (5.9)

$$\partial_t V(t,x) + \gamma(t)x + \frac{\beta^2}{\kappa(t)} \left\{ \frac{1}{2} ([\nabla_x V(t,x)]^-)^2 \right\} + \\ + \alpha(x) + \frac{\beta^2}{\kappa(t)} \left\{ [\nabla_x V(t,x)]^- \, \nabla_x V(t,x) \right\} = 0. \tag{5.11}$$

Let us introduce a new key variable

$$z(t) := \nabla_x V(t,1).$$

Observe that

$$z(t) = \nabla_x V(t,1) = V(t-1) - V(t,1) = -\nabla_x V(t,-1).$$

Now we rewrite the previous HJB equation (5.11) for $x = 1$ and for $x = -1$; we obtain the system:

$$\begin{cases} \partial_t V(t,1) + \gamma(t) + \alpha(1) + \dfrac{\beta^2}{\kappa(t)} \left\{ \dfrac{1}{2}([z(t)]^-)^2 + [z(t)]^- z(t) \right\} = 0 \\ \partial_t V(t,-1) - \gamma(t) + \alpha(-1) + \dfrac{\beta^2}{\kappa(t)} \left\{ \dfrac{1}{2}([-z(t)]^-)^2 - [-z(t)]^- z(t) \right\} = 0. \end{cases}$$

Recalling that, for all $z \in \mathbb{R}$, we have that $[-z]^- = [z]^+$, and, taking the difference between the two equations, we get

$$\dot{z}(t) - 2\gamma(t) + \nabla_x \alpha(1) + \frac{\beta^2}{\kappa(t)} \left\{ \frac{1}{2} \left(([z(t)]^+)^2 - ([z(t)]^-)^2 \right) \right\} + \\ + \frac{\beta^2}{\kappa(t)} \left\{ -z(t) \left([z(t)]^+ + [z(t)]^- \right) \right\} = 0.$$

Finally, for all $z \in \mathbb{R}$ it holds $[z]^+ + [z]^- = |z|$ and $[z]^+ - [z]^- = z$; hence we obtain:

$$\dot{z}(t) = 2\gamma(t) - \nabla_x \alpha(1) + \frac{\beta^2}{2\kappa(t)} |z(t)|\, z(t).$$

The final condition $V(T,x) = g(x)$ for all $x \in S$ gives us $z(T) = \nabla_x V(T,1) = \nabla_x g(1)$. Therefore, HJB becomes

$$\begin{cases} \dot{z}(t) = 2\gamma(t) - \nabla_x \alpha(1) + \dfrac{\beta^2}{2\kappa(t)} |z(t)|\, z(t) \\ z(T) = \nabla_x g(1), \end{cases}$$

which is exactly the backward ODE we are looking for. □

5.5 A numerical example

We formally define the dynamics of the process with $x \in S$ as follows:

$$\mathbb{P}(X_{t+h} = -x \,|\, X_t = x, U_t = u) = (1 + x + u)h + o(h). \tag{5.12}$$

Even though this is only an example introduced to display the process discussed so far, the choice of a transition probability that is linear in the state variable is without loss of generality. Since the state variable $x \in S$ can only assume two values, every map $S \to \mathbb{R}$ could be represented by a linear structure.

We want to choose a feasible feedback control function u which takes values in $[0, 2]$, in order to minimise

$$J^u_{0,x} := \mathbb{E}^u_{0,x}\left\{ \int_0^1 \frac{-X_t}{2} + \frac{U_t^2}{4} dt + \frac{X_T}{4} \right\} ;$$

then, the backward ODE is

$$\begin{cases} \dot{z}(t) = 1 + |z(t)|\, z(t) \\ z(1) = -\dfrac{1}{2}. \end{cases}$$

In a left neighbourhood of 1, for example in $(1 - \varepsilon, 1]$ with $\varepsilon > 0$, the ODE becomes

$$\begin{cases} \dot{z}(t) = 1 - (z(t))^2 \\ z(1) = -\dfrac{1}{2}. \end{cases}$$

The solution to the previous Cauchy problem is

$$z(t) = 1 - \frac{6e^2}{e^{2t} + 3e^2} \,.$$

We observe that, for $t \in [0, 1]$, this function is always negative (i.e., $\varepsilon > 1$); moreover, since the function is strictly negative and strictly increasing

$$[z(t)]^- \leq \, - z(0) = \frac{1 - 3e^3}{1 + 3e^2} < 1 \,.$$

Hence, the constraint on the control $u \in [0, 2]$ is not active and the optimal feedback is feasible. Using the optimal feedback control

$$u^*(t,x) = \left[\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2} \cdot x\right]^- ,$$

we can find the evolution of the expected value of the process. Let us define $m_x^*(t) := \mathbb{E}_{0,x}^{u^*}(X_t)$; using the infinitesimal generator we obtain

$$\begin{aligned}
\dot{m}_x^*(t) &= -2\mathbb{E}_{0,x}^{u^*}\left\{X_t\left(1 + X_t + \left[\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2} \cdot X_t\right]^-\right)\right\} \\
&= -2\mathbb{E}_{0,x}^{u^*}\left\{X_t + 1 + X_t\left(\frac{X_t+1}{2}\left[\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2}\right]^- + \frac{X_t-1}{2}\left[\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2}\right]^+\right)\right\} ;
\end{aligned}$$

since z_t <0 in the programming interval, it holds

$$= -2\mathbb{E}_{0,x}^{u^*}\left\{X_t + 1 + \frac{1+X_t}{2}\left(\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2}\right)\right\}$$

which becomes

$$\dot{m}_x^*(t) = \left(\frac{e^{2t} - 3e^2}{e^{2t} + 3e^2} - 2\right)(m_x^*(t) + 1) \tag{5.13}$$

with initial condition:

$$m_x^*(0) = \mathbb{E}_{0,x}^{u^*}(X_0) = x\,.$$

Equation (5.13) is a linear ODE whose coefficients are continuous functions for all $t \in [0, 1]$; therefore, there exists a unique solution $m_x^*(t)$ to the previous Cauchy problem. For $x = -1$, the solution is constant: $m_{-1}^*(t) \equiv -1$. On the other hand, for $x = 1$, we can explicitly find the analytical form

$$m_1^*(t) = \frac{e^{-t}(1458 - e^t(t+3)^6)}{(t+3)^6}\,.$$

Since the above expression is long and inexpressive, we prefer to plot its graph in Figure 5.1. Moreover, using the evolution of the expected value of the optimal process starting from $x = 1$, we can find the evolution of the probability of each state:

$$p(t) := \mathbb{P}_{0,1}^{u^*}(X_t = 1) = \frac{1 + m_1^*(t)}{2}\,. \tag{5.14}$$

The probability is displayed in Figure 5.1.

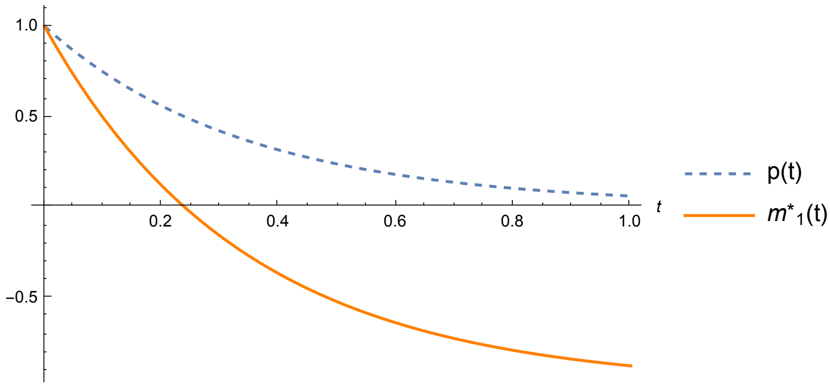


Figure 5.1: Probability rate $p(t)$ and expected value $m_1^*(t)$ graphs.

We notice that, using the optimal feedback control, the process moves towards the state -1. When the initial position is -1, the process remains in this state; otherwise, when the initial position is 1, the process changes its state with a strictly positive probability rate.

5.6 Discussion

In this chapter, we analyse the evolution of a continuous-time Markov decision process characterised by a binary state. We introduce the standard Hamilton–Jacobi–Bellman equation and prove that, under suitable analytical formulations of the rate of transition and of the cost function, we can replace the HJB equation with a backward ODE. Then, all information useful to characterise an optimal feedback control is now contained in the solution of the backward ODE. Using a numerical example, we show how to find an optimal feedback control.

This research can be improved in various directions. First, we can try to extend the family of functions that satisfies the hypotheses of Theorem 5.4.1. Subsequently, we can investigate whether what is proven in this chapter is valid for analogous problems with an infinite-time horizon, too. Finally, it may be interesting to study whether this approach is useful for analysing the interaction between multiple players.

6

A binary-state continuous-time Markov chain model for Offshoring and Reshoring

6.1 Introduction

As an application of the results of the previous chapter, we present a binary state continuous-time controlled Markov chain model representing a two-country model (North and South) that describes the phenomenon of offshoring and reshoring. The final goal of our exercise is to understand how national tax incentives can influence this process. What follows is based on a paper published in collaboration with Professor Elena Sartori and my Supervisor, Professor Luca Grosset [11].

First of all, let us introduce some context for the formulation of the problem. In the last decades, the practice of establishing a business, or part of it, in a different place than home-country has become more and more popular [41]. This phenomenon, known as offshoring, applies to several sectors, from manufacturing to services [42]. However, in recent years, the global socioeconomic situation has increased the reverse phenomenon: reshoring [43]. Reshoring is a voluntary corporate strategy regarding partial or total relocation of production (insourced or outsourced) in the home country to meet local, regional or global demands, as defined by [44]. The explanations for this change of direction are numerous and highly different. Among them, there are well-known economic reasons, such as the reduction in the labour cost gap between the home country and the host country or logistics costs, but also planning risks, such as global supply chain risks and impact of production/delivery time [43, 45]. Furthermore, in determining the spread of reshoring, there are problems related to the lack of skilled workers in the host country or the lack of knowledge about the foreign destination. Last but not least, today, increasing attention is paid to brand value in terms of social responsibility and quality [46]. Therefore, the need to increase customer satisfaction and the built-in effect play a key role in deciding whether to re-establish a business. An aspect that has particularly interested consumers in recent years is the focus on green policies, leading them to prefer products with a low environmental footprint [47].

We investigate the conditions under which offshoring and reshoring occur, as well as how a government can influence this process. The purpose of our model is to analyse the economic relationships that exist between essential quantitative variables in the offshoring and reshoring process. For simplicity, we model a situation in which the motivation for reshoring is only related to the presence of national relocation subsidies, which take place at the end of the programming interval. Only one foreign country is taken into account and offshoring is determined by the lower production costs in that country. A similar situation has been studied in [48] but with the focus only on the offshoring phenomenon. The final goal is to understand the trade-off between offshoring and reshoring phenomena as a function of the national taxation incentives. Moreover, we are also interested in representing the evolution of the proportion of offshored and home-placed businesses.

6.2 The model

6.2.1 The microscopic model

Let us consider a North-South model (see [49]) in which companies have the flexibility to select their production locations. In [49], the author builds a growth model that focuses on research and development (R&D) activities and examines the relationship between the North and South regions. Specifically, the model investigates how firms producing final goods in the North inherently influence the extent of international outsourcing of intermediate goods to the South. We do not focus on growth models, but rather analyse how the presence of incentives can drive companies' decisions to reshore their production. However, we continue to consider only two possible countries to understand which are the essential parameters that favour offshoring and reshoring. We present the N-player game, followed by its formal limit with infinitely many companies. Although our analysis focuses primarily on the limit model, the N-player game serves as its motivating force, giving a clearer understanding of the parameters involved and providing a sort of microfoundation of the model studied. Strictly speaking, our approach does not fit the conventional definition of a game, as it lacks direct interaction among players. However, conceptualising it as a game is convenient for two reasons. First, it allows us to apply the standard theory of continuous time finite state mean-field games [50, 51]. Second, this approach facilitates the presentation of a direct extension of the model in Section 6.5.

Let us assume that N companies sell their product on the domestic market (North country). We identify each company with the index $i = 1, \dots, N$. Let us assume that all of these companies have the same programming interval $[0, T]$. The instantaneous demand for the product produced by the i-th company in the programming interval is assumed to be constant and equal to $\delta_i > 0$. The selling price for one unit of goods is also assumed to be constant in the same programming interval and equal to $\pi_i > 0$. However, the production cost varies depending on whether the i-th company is located in the North or in the South country. There are no inventories, so everything that is produced is immediately sold.

The decision to produce in the North or in the South country represents the state variable for each company and is denoted by X_t^i, where $t \in [0, T]$. Therefore, X_t^i can take two possible values:

- we set $X_t^i = -1$ if, at time t, the company denoted by the index i is producing in the South country;

- on the other hand, $X_t^i = +1$ if, at time t, it is producing in the North country.

Therefore, the state of the i-th company is a binary state variable, that is, $X_t^i \in \{-1, +1\}$, depending on where the production is located. We assume that the unit production cost in the South country for the i-th company is $\rho_i > 0$, while the production cost in the North country is higher and equal to $\rho_i + \xi_i > 0$, where $\xi_i > 0$.

To treat this model using finite state continuous time Markov chains, it is necessary to assume that each company can control the probability rate of transition from one state to another. In our case, the i-th company can control the probability of changing its state, that is, transitioning from production in the North country to production in the South country, or vice versa. We assume that the i-th company can control the transition rate between one state and another. The higher the transition rate, the greater the probability that the company will change the position of the production plant by moving it from one country to another. In this way, our model falls into what is called a continuous time Markov decision problem with binary states [10]. In more detail, we assume that the dynamics of the problem can be described as follows.

$$\mathbb{P}\left(X^i_{t+h} = -X^i_t \,|\, X^1_s, \ldots X^N_s, s \leq t\right) = U^i_t \cdot h + o(h)\,, \tag{6.1}$$

where $U^i_t = \phi^i(t, X^1_t, \ldots X^N_t) \geq 0$ is the control function in feedback form of the i-th company. This function is right-continuous at t and depends on the entire state $(X^1_t, \ldots X^N_t)$ of the system at time t. Each company has complete information on the behaviour of all others and decides whether to activate its own control, which consists of increasing the transition rate from the current state to the opposite, meaning U^i_t. In other words, each company decides the probability rate with which it changes the production location from the current state to the opposite one (from North to South or from South to North). The activation of this control is costly for the i-th company, and we assume that the activation cost is quadratic, i.e., the instantaneous cost function for the activation of the transition rate is

$$c^i(U^i_t) = \frac{\kappa^i}{2}(U^i_t)^2 + \theta^i U^i_t\,. \tag{6.2}$$

The choice of a quadratic cost function is standard for this type of problem [12]; the model could be implemented with a generic convex function that satisfies the uniqueness of the solution.

The North country perfectly knows that the production cost in the South country is more convenient, and, therefore, in order to limit the phenomenon of offshoring, it proposes an incentive to all companies that, at the end of the programming interval, have their production localised in the North country.

The problem for the i-th company can be formulated as follows:

$$J^i[U^i_s] = \mathbb{E}\left\{\int_0^T \left[\delta^i\left(\pi^i - \rho^i - \frac{\xi^i}{2}(1 + X^i_t)\right) - \frac{\kappa^i}{2}(U^i_t)^2 - \theta^i U^i_t\right] dt + \frac{\gamma^i}{2}(X^i_T + 1)\right\}. \tag{6.3}$$

For the reader's convenience, we introduce a simple table (see Table 6.1) to describe the instantaneous profit of the i-th company assuming that $U^i_s = 0$, hence, when the i-th company does not activate its transition rate.

In this model, the objective function of each company is independent of the states of the other companies, making the formulation we just described redundant. However, if the parameters

Table 6.1: Profit and incentive for i-th company.

State at time t	Instantaneous profit	State at time T	Final incentive
-1	$\delta^i(\pi^i - \rho^i)$	-1	0
$+1$	$\delta^i(\pi^i - (\rho^i + \xi^i))$	$+1$	γ^i

we described depended on the states of the other companies, this approach would be exactly what is necessary to describe the problem. For example, if the final incentive depended on the number of companies that relocated their production, then the parameter γ^i would no longer be a constant but a function of the state of all other companies: $\gamma^i = \gamma^i(X_T^1, ...X_T^N)$.

Let us highlight the importance of the microscopic view by assuming that there are only two companies on the market and that the North country decides to invest a budget $\beta > 0$. Companies producing in the North country at the end of the programming interval receive the entire or a part of the incentive. More precisely, if both companies produce in the North country at time T, each receives an incentive equal to $\beta/2$. Otherwise, if only one company produces in the North country at the end of the programming interval, that company receives the full incentive β. Finally, if no company produces in the North country at time T, then the incentive is not spent. In this situation, the i-th player incentive $\gamma^i(X_1, X_2)$ becomes

$$\gamma^i(X_T^i, X_T^j) = \beta \frac{(X_T^i + 1)/2}{1 + (X_T^j + 1)/2} \, ,$$

with $i, j \in \{1, 2\}$ and $i \neq j$. Similarly, we can move on from 2 to N companies competing for the same incentive budget, and we compute

$$\gamma^i(X_T^1, ..., X_T^N) = \beta \frac{(X_T^i + 1)/2}{1 + \sum_{j=1, j\neq i}^{N} (X_T^j + 1)/2} \, .$$

In the formula above, the average behaviour of the other players is highly relevant. Let us define

$$m^{N-1}(T) := \sum_{j=1, j\neq i}^{N} \frac{X_T^j}{N-1} \, .$$

Therefore, we can rewrite the i-th player incentive as follows:

$$\gamma^i(X_T^1, \dots, X_T^N) = \beta \frac{(X_T^i + 1)/2}{1 + (N-1)(m^{N-1}(T) + 1)/2}\,. \tag{6.4}$$

This reasoning directly connects the original problem to a mean-field game, wherein the strategic interaction among players is present in the scrap value function. When N goes to infinity, the function $m^{N-1}(T)$ can be well approximated by the empirical mean. In mean-field games with Markov feedback Nash equilibria, this convergence process is a challenging task. Convergence can be achieved by identifying the potential limit points of sequences of N-player Nash equilibria, which serve as solutions, to some extent, of the limit model. This approach tackles the convergence issue in mean-field games (see [37] and the references therein for a clear example of the challenges associated with this type of problem). Although, from a mathematical point of view, this is certainly one of the most interesting situations, it could be complicated to understand the problem without clarifying the economic meaning of the parameters $\delta^i, \pi^i, \rho^i, \xi^i, \kappa^i, \theta^i$. For this reason, in this work, we limit ourselves to a constant function γ^i which corresponds to a fixed incentive that is assigned to all companies that produce in the North country at time T. This approach allows us to obtain closed-form solutions that would not be possible to achieve with direct interactions among different companies. Nevertheless, we introduce the problem in all its generality to allow for an immediate extension to mean-field game theory of the results described here. The approach of continuous time Markov chain in optimal decision process is widely used in applications; for a recent paper on this topic, we suggest [52] and the references therein.

6.2.2 The macroscopic model

The microscopic model described above can be solved using the techniques presented in [37]. Unfortunately, the solution to the problem with N companies is not very informative because it requires estimating a very large number of parameters and allows obtaining few quantitative indications that clarify the trade-offs between the parameters. At this point, we prefer to introduce the macroscopic model as the formal limit for N that tends to infinity of the microscopic one, assuming that all companies are exchangeable. This approach is standard, but it is not trivial: when the model involves mean-field games, the passage to the limit is not straightforward, and additional equilibria may arise (see, e.g., [37, 38]). We are not interested in this situation and instead we formalise the problem directly, corresponding to the formal limit for a *repre-*

sentative player. In this case, all parameters, which were indexed by "i" before, now become independent of i because companies are considered indistinguishable.

Table 6.2: Table of variables and parameters.

Variables	Description
X_t	production State of the representative player (state function)
U_t	transition rate of the representative player (control function)

Parameters	Description
T	fixed final time
δ	instantaneous demand for the product
π	selling price for one unit of goods
ρ	unit production cost in the South country
ξ	additional production cost in the North country
κ, θ	quadratic and linear activation costs of the transition rate
γ	final incentive

It is a very strong simplification, but it allows us to identify the explicit trade-off between the model parameters, a result that cannot be obtained in the microscopic formulation. We reiterate that we follow the structure of mean-field games even though there is no strategic interaction among players in this model. In Section 6.5 we propose an extension that justifies this approach to the problem. The limit as N goes to $+\infty$ can be obtained at a heuristic level. We expect that the average state obeys a *Law of Large Numbers*, so it converges to a deterministic limit. The representative agent, whose state is denoted by X_t, moves according to the following feedback-controlled dynamics:

$$\mathbb{P}\left(X_{t+h} = -X_t \,|\, X_s, s \leq t\right) = U_t \cdot h + o(h)\,, \tag{6.5}$$

where $U_t = \phi(t, X_t) \geq 0$ is the feedback control function of the *representative company*. Note that X_t evolves as a controlled binary state continuous time Markov chain with transition rate

U_t. We can rewrite the representative company's objective functional as follows:

$$J[U_s] = \mathbb{E}\left\{ \int_0^T \left[\delta\left(\pi - \rho - \frac{\xi}{2}(1 + X_t)\right) - \frac{\kappa}{2}U_t^2 - \theta U_t \right] dt + \frac{\gamma}{2}(1 + X_T) \right\}. \qquad (6.6)$$

To complete the macroscopic analysis, let us introduce the expected value of a representative company, m_t, as

$$m_t = \mathbb{E}\left[X_t\right] \qquad (6.7)$$

for $t \in [0, T]$. By definition, $m_t \in [-1, 1]$ throughout the programming interval. This quantity is very important for the policy planner of the North country, as it can be used to describe the proportion of companies that have offshored their production in the South country. Specifically, if $m_t = 1$, then at time t all companies have remained in the North country; if $m_t = 0$, then half of the companies have offshored their production in the South country; and finally, if $m_t = -1$, then all companies have moved their production to the South country.

The two main research questions of this work can now be explicitly formulated:

- What is the trade-off between offshoring costs (described by parameters κ and θ) and the different production costs in the two countries (described by parameters ρ and ξ)?
- How can the North country use the incentives described by the parameter γ to avoid the phenomenon of offshoring, and what is the impact of this type of incentives on the evolution over time of the quantity m_t that is related to the proportion of companies that have offshored their production?

6.3 The HJB equation and the evolution of m_t

The solution procedure for the stochastic optimal control problem characterised by the objective functional (6.6), by the motion equation (6.5), and by the initial condition X_0 is the same as described in [10]. As usual, X_0 is independent of the evolution (6.5) of the stochastic process $(X_t)_{t\in[0,T]}$.

Theorem 6.3.1. *The optimal control in feedback form for the representative company is*

$$U_t^* = \phi^*(t, X_t^*) = \frac{[z_t \cdot X_t^* - \theta]^+}{\kappa}. \qquad (6.8)$$

where the function z_t satisfies the following backward ODE:

$$\dot{z}_t = \begin{cases} -\dfrac{(z_t+\theta)^2}{2\kappa} - \delta\xi & z_t < -\theta \\ -\delta\xi & |z_t| \leq \theta \\ \dfrac{(z_t-\theta)^2}{2\kappa} - \delta\xi & z_t > \theta \end{cases} \tag{6.9}$$

with boundary condition

$$z_T = -\gamma\,. \tag{6.10}$$

Proof of Theorem 6.3.1. We face this stochastic control problem using dynamic programming. This approach is standard in applications of Economics and Finance (see, e.g., [53]). For the stochastic optimal control problem (6.5) and (6.6), the value function has the following form:

$$V(t,x) = \sup_{U_s\geq 0} \mathbb{E}\left\{ \int_t^T \left[\delta\left(\pi - \rho - \frac{\xi}{2}(1+X_s)\right) - \frac{\kappa}{2}U_s^2 - \theta U_s \right] ds + \frac{\gamma}{2}(1+X_T) \,\Big|\, X_t = x \right\}. \tag{6.11}$$

The HJB equation associated with this problem is

$$\begin{cases} \partial_t V(t,x) + \max\limits_{u\geq 0}\left\{ u\nabla_x V(t,x) + \delta\left(\pi - \rho - \dfrac{\xi}{2}(1+x)\right) - \dfrac{\kappa}{2}u^2 - \theta u \right\} = 0 \\ V(T,x) = \dfrac{\gamma}{2}\,(1+x)\,, \end{cases} \tag{6.12}$$

where, as usual, $\nabla_x V(t,x) := V(t,-x) - V(t,x)$ is the discrete gradient of the value function. Recalling $u \geq 0$, we observe that the function

$$u \mapsto u\nabla_x V(t,x) + \delta(\pi - \rho + \xi(1+x)) - \frac{\kappa u^2}{2} - \theta u$$

is strictly concave with respect to u. Hence, by first-order necessary condition, its global maximum point is the zero of its first derivative $u \mapsto \nabla_x V(t,x) - \kappa u - \theta$, when it lies in the interval $[0,+\infty)$; otherwise, it is equal to 0. This control in feedback form can be written as

$$u^* = \frac{[\nabla_x V(t,x) - \theta]^+}{\kappa}\,, \tag{6.13}$$

where we recall that $[a]^+ := \max\{0,a\}$ and $[a]^- := \max\{0,-a\}$ are the positive and the negative part, respectively. Following [10], we want to obtain a backward ODE from the HJB equation. For this aim, following the process explained in the previous chapter, we introduce the auxiliary variable

$$z_t := \nabla_x V(t,1) = V(t,-1) - V(t,1)\,. \tag{6.14}$$

We interpret $V(t,-1)$ as the optimal value obtained when starting at $x=-1$ at time t, analogously, $V(t,1)$. Therefore, $V(t,-1) - V(t,1)$ is the algebraic increment obtained by the transition from the state $x=1$ to $x=-1$ at time t. Since we are taking the maximum in the HJB equation (6.12), assuming that the initial state at t is $x=1$, the transition from state $x=1$ to $x=-1$ is convenient whenever $z_t = V(t,-1) - V(t,1) > 0$.
To obtain the backward ODE, we substitute (6.13) into the HJB equation (6.12) and evaluate the result in both states:

$$\begin{aligned}\partial_t V(t,1) = &-\frac{[\nabla_x V(t,1)-\theta]^+}{\kappa}\nabla_x V(t,1) - \delta\,(\pi-\rho-\xi) + \\ &+\frac{\kappa}{2}\left(\frac{[\nabla_x V(t,1)-\theta]^+}{\kappa}\right)^2 + \theta\frac{[\nabla_x V(t,1)-\theta]^+}{\kappa}\end{aligned}$$

$$\begin{aligned}\partial_t V(t,-1) = &-\frac{[\nabla_x V(t,-1)-\theta]^+}{\kappa}\nabla_x V(t,-1) - \delta\,(\pi-\rho) + \\ &+\frac{\kappa}{2}\left(\frac{[\nabla_x V(t,-1)-\theta]^+}{\kappa}\right)^2 + \theta\frac{[\nabla_x V(t,-1)-\theta]^+}{\kappa}\,.\end{aligned}$$

Let us observe that

$$\nabla_x V(t,-1) = V(t,1) - V(t,-1) = -\nabla_x V(t,1) = -z_t\,.$$

We can recognise the auxiliary variable in the above equations:

$$\begin{aligned}\partial_t V(t,1) &= -\frac{[z_t-\theta]^+}{\kappa}z_t - \delta\,(\pi-\rho-\xi) + \frac{\kappa}{2}\left(\frac{[z_t-\theta]^+}{\kappa}\right)^2 + \theta\frac{[z_t-\theta]^+}{\kappa}\\ \partial_t V(t,-1) &= \frac{[-z_t-\theta]^+}{\kappa}z_t - \delta\,(\pi-\rho) + \frac{\kappa}{2}\left(\frac{[-z_t-\theta]^+}{\kappa}\right)^2 + \theta\frac{[-z_t-\theta]^+}{\kappa}\,.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\dot{z}_t &= \partial_t V(t,-1) - \partial_t V(t,1) \\
&= \frac{z_t}{\kappa}\left[[-z_t-\theta]^+ + [z_t-\theta]^+\right] + \frac{1}{2\kappa}\left[\left([-z_t-\theta]^+\right)^2 - \left([z_t-\theta]^+\right)^2\right] \\
&\qquad\qquad\qquad\qquad + \frac{\theta}{\kappa}\left[[-z_t-\theta]^+ - [z_t-\theta]^+\right] - \delta\xi .
\end{aligned}
$$

Recalling classical relations such as $[-a]^+ = [a]^-$, $|a| = [a]^+ + [a]^-$ and $a = [a]^+ - [a]^-$, we obtain

$$
\dot{z}_t = \frac{(z_t+\theta)}{\kappa}[z_t+\theta]^- + \frac{(z_t-\theta)}{\kappa}[z_t-\theta]^+ + \frac{1}{2\kappa}\left[\left([z_t+\theta]^-\right)^2 - \left([z_t-\theta]^+\right)^2\right] - \delta\xi . \quad (6.15)
$$

Now, let's consider the three different cases separately.

- If $z_t < -\theta$, then $z_t+\theta < 0$ and $z_t-\theta < 0$, meaning that $[z_t+\theta]^- = -(z_t+\theta)$ and $[z_t-\theta]^+ = 0$. In this case (6.15) reads
$$
\dot{z}_t = -\frac{(z_t+\theta)^2}{2k} - \delta\xi.
$$
- If $|z_t| \leq \theta$, then $z_t+\theta > 0$ and $z_t-\theta < 0$, meaning that $[z_t+\theta]^- = 0$ and $[z_t-\theta]^+ = 0$. In this case (6.15) reads
$$
\dot{z}_t = -\delta\xi.
$$
- If $z_t > \theta$, then $z_t+\theta > 0$ and $z_t-\theta > 0$, meaning that $[z_t+\theta]^- = 0$ and $[z_t-\theta]^+ = (z_t-\theta)$. In this case (6.15) reads
$$
\dot{z}_t = \frac{(z_t-\theta)^2}{2k} - \delta\xi.
$$

Hence, we can formalise the evolution of the auxiliary variable as

$$
\dot{z}_t = \begin{cases} -\dfrac{(z_t+\theta)^2}{2\kappa} - \delta\xi & z_t < -\theta \\ -\delta\xi & |z_t| \leq \theta \\ \dfrac{(z_t-\theta)^2}{2\kappa} - \delta\xi & z_t > \theta \,. \end{cases} \quad (6.16)
$$

We observe that equation (6.16) is differentiable within the different intervals defined by the marginal cost θ. Moreover, we can write this ODE in the following compact way:

$$\dot{z}_t = \frac{\operatorname{sgn}(z_t)}{2\kappa}\left([\,|z_t| - \theta]^+\right)^2 - \delta\xi\,. \tag{6.17}$$

Using the boundary condition for the HJB equation we obtain

$$V(T,x) = \frac{\gamma}{2}(1+x)\ ,$$

and by the definition of the function z_t, it yields

$$z_T = V(T,-1) - V(T,1) = -\gamma\,. \tag{6.18}$$

The optimal control in feedback form is characterised by the following function:

$$\phi^*(t,x) = \frac{[\nabla_x V(t,x) - \theta]^+}{\kappa} = \frac{[z_t \cdot x - \theta]^+}{\kappa} \tag{6.19}$$

since $\nabla_x V(t,x) = V(t,-x) - V(t,x)$, corresponds to z_t if $x=1$, to $-z_t$ if $x=-1$. Now, putting together (6.16), (6.18), and (6.19) we can conclude. □

Theorem 6.3.1 identifies the optimal strategy for the representative company. However, the most relevant information for the policy maker is related to the macroscopic evolution of the offshoring phenomenon. This information is related to the temporal evolution of the aggregate variable m_t, and in the next result we characterise the ODE that describes this dynamics.

Theorem 6.3.2. *Let us assume that the representative company uses the optimal control $\phi^*(t,X_t^*)$ characterised in Theorem 6.3.1, hence the evolution of the deterministic variable $m_t = \mathbb{E}(X_t^*)$ is described by the following equation*

$$\dot{m}_t = -\frac{\operatorname{sgn}(z_t)}{\kappa}\cdot[|z_t| - \theta]^+ \cdot (1 + \operatorname{sgn}(z_t)\cdot m_t)\,, \tag{6.20}$$

where the function z_t satisfies (6.9)-(6.10) and $m_0 = \mathbb{E}(X_0)$.

Proof of Theorem 6.3.2. By definition of m_t, $\dot{m}_t = \frac{d}{dt}\mathbb{E}(X_t^*)$. Note that we are computing the expected value of a continuous-time finite-state Markov chain with transition rate $U_t^* =$

$\phi^*(t, X_t^*)$; then if we apply Dynkin's formula (see, e.g., [54]), we obtain

$$\dot{m}_t = \frac{d}{dt}\mathbb{E}(X_t^*) = \mathbb{E}(\nabla_x X_t^* \cdot \phi^*(t, X_t^*)) = -\frac{2}{\kappa} \cdot \mathbb{E}(X_t^* \cdot [z_t X_t^* - \theta]^+) , \tag{6.21}$$

where the last equality holds true because $\nabla_x X_t^* = -X_t^* - (X_t^*) = -2X_t^*$ and $\phi^*(t, X_t^*)$ has been characterised in Theorem 6.3.1. Now, using the fact that the state is binary, we can explicitly compute the previous expected value, yielding the following result.

$$\begin{aligned} \dot{m}_t &= -\frac{2}{\kappa}\, \mathbb{E} \left\{ [z_t - \theta]^+ \cdot \frac{1 + X_t^*}{2} - [-z_t - \theta]^+ \cdot \frac{1 - X_t^*}{2} \right\} \\ &= -\frac{1}{\kappa} \left\{ [z_t - \theta]^+ (1 + m_t) - [-z_t - \theta]^+ (1 - m_t) \right\} \\ &= -\frac{\operatorname{sgn}(z_t)}{\kappa} \cdot [|z_t| - \theta]^+ \cdot (1 + \operatorname{sgn}(z_t) \cdot m_t) . \end{aligned}$$

□

We note that the results described in this section hold for any value of the parameter γ. Therefore, we have simultaneously addressed both the case $\gamma = 0$, which means the absence of incentives, and the case $\gamma > 0$, indicating the presence of economic support from the North country to the companies that maintain domestic production.

6.4 Offshoring without incentives

6.4.1 Technical analysis

Assume that there are no incentives that the North country implements to support domestic production. This assumption corresponds to setting the value $\gamma = 0$ in the results obtained in Section 6.3. The final condition (6.10) becomes $z_T = 0$. Let us solve the backward ODE equation (6.9). By continuity, there exists $\varepsilon > 0$ such that, for $t \in [T - \varepsilon, T]$, there is $|z_t| \leq \theta$. Hence, for every $t \in [T - \varepsilon, T]$,

$$\dot{z}_t = -\delta\xi$$

and we can solve the following Cauchy problem

$$\begin{cases} \dot{z}_t = -\delta\xi \\ z_T = 0 \end{cases} \tag{6.22}$$

as long as $|z_t| \le \theta$. Therefore,

$$z_t = \delta\xi(T - t)\,. \tag{6.23}$$

This solution holds until $|z_t| \le \theta$, hence let us define $\bar{t}$ using the equation $z_{\bar{t}} = \theta$. We obtain

$$\bar{t} = T - \frac{\theta}{\delta\xi}\,. \tag{6.24}$$

The variable z_t in the last part of the programming interval (that is, for $t \in [\max\{\bar{t}, 0\}, T]$), is described by (6.23). If $\bar{t} \le 0$, then this function fully describes the optimal feedback strategy. Otherwise, if $\bar{t} > 0$, then, in the first part of the programming interval $[0, \bar{t})$, we solve the following Cauchy problem

$$\begin{cases} \dot{z}_t = \dfrac{(z-\theta)^2}{2\kappa} - \delta\xi \\ z_{\bar{t}} = \theta\,. \end{cases}$$

Hence, we can obtain that, for all $t \in [0, \bar{t})$,

$$z_t = \theta + \sqrt{2\delta\xi\kappa}\tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{t} - t)\right)\,. \tag{6.25}$$

Using equations (6.23) and (6.25), we can compute the corresponding transition rate. By (6.8), if $\bar{t} \le 0$, then

$$\phi^*(t,x) = \frac{[z_t x - \theta]^+}{\kappa} = \frac{[\delta\xi(T-t)x - \theta]^+}{\kappa} \equiv 0\,, \tag{6.26}$$

for every $t \in [0, T]$ and for every $x \in \{-1, +1\}$ since $|z_t| \le \theta$. On the other hand, if $\bar{t} > 0$, then the transition rate is

$$\phi^*(t,x) = \frac{\left[\theta(x-1) + \sqrt{2\delta\xi\kappa}\tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{t}-t)\right)x\right]^+}{\kappa}$$

which can be rewritten as

$$\phi^*(t,x) = \begin{cases} \sqrt{\frac{2\delta\xi}{\kappa}}\tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{t}-t)\right) & x = 1 \\ 0 & x = -1 \end{cases} \tag{6.27}$$

for $t \in [0, \bar{t})$.

Using this optimal control, we can characterise the evolution of the expected value of the process m_t. If $\bar{t} \leq 0$, recalling (6.21), we have $\dot{m}_t \equiv 0$. So, denoted by m_0 the initial condition of the average distribution, it holds $m_t \equiv m_0$, for all $t \in [0, T]$. On the other hand, if $\bar{t} > 0$, then

$$\dot{m}_t = -\sqrt{\frac{2\delta\xi}{\kappa}} \tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{t} - t)\right)(1 + m_t) \tag{6.28}$$

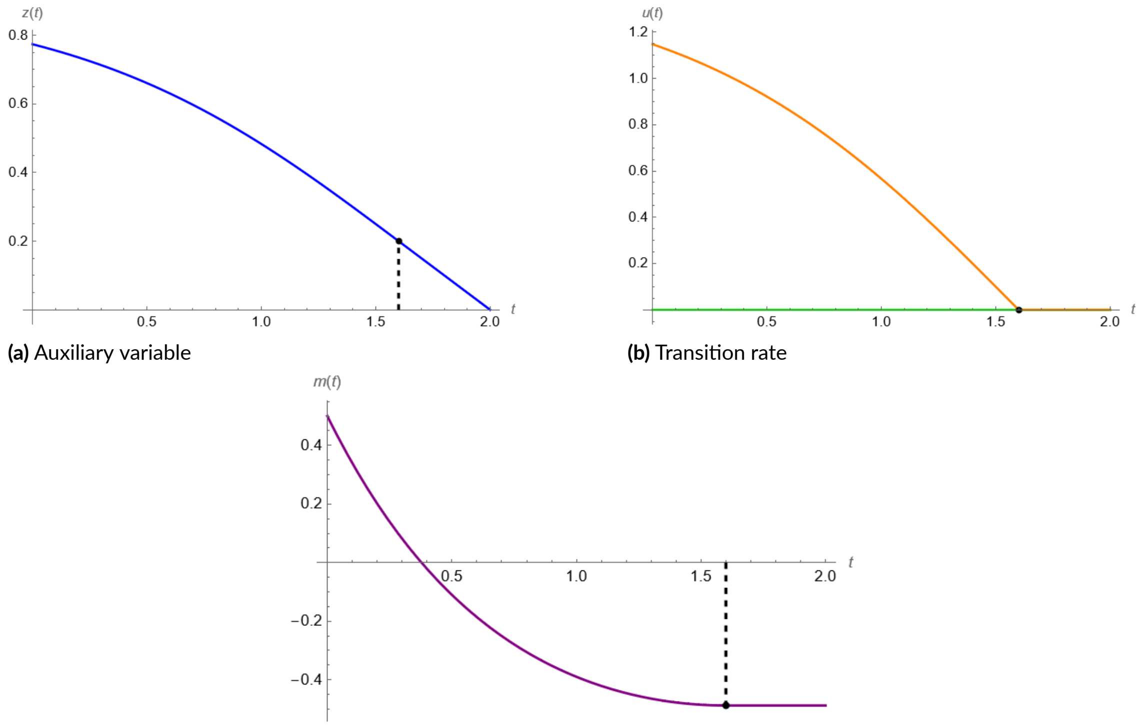
for $t \in [0, \bar{t})$, and $m_t \equiv m_{\bar{t}}$ for $t \in [\bar{t}, T]$. Since the expression in (6.28) holds for $t \in [0, \bar{t})$, it follows that $\dot{m}_t < 0$. Indeed, solving the differential equation (6.28) with initial condition $m(0) = m_0$, we obtain, for $t \in [0, \bar{t})$,

$$m_t = c_0 \cosh^2\left[\sqrt{\frac{\delta\xi}{2\kappa}}(t - \bar{t})\right] - 1 \tag{6.29}$$

where

$$c_0 = (m_0 + 1) \cosh^{-2}\left[\sqrt{\frac{\delta\xi}{2\kappa}}\bar{t}\right].$$

To make it easier to understand the meaning under the above results, we display here the principal functions' graph for specific values of parameters such that there exists $\bar{t} > 0$ as defined in (6.24). More precisely, Figure 6.1a displays the outline of the auxiliary variable z_t, which is always positive and always decreasing. These observations make us understand that it is always convenient for a business to move the production location from the North to the South country. Moreover, the earlier the delocalisation the greater the gain; indeed, moving the production location to the South country as soon as possible will result in enjoying lower production costs for more time. Figure 6.1b shows the transition rate u_t, where the green line refers to the transition rate for $x = -1$ (reshoring phenomenon) and the orange line to $x = +1$ (offshoring phenomenon). This graph agrees with what above. We observe only an offshoring phenomenon with a higher transition rate near the beginning of the programming interval. The instant of time where offshoring stops is (6.24) and corresponds to the dashed vertical line in Figure 6.1a, that is, when the instantaneous gain $z_{\bar{t}}$ becomes lower than the linear cost of delocalisation θ. After $\bar{t}$, the delocalisation costs are not balanced by the lower production costs due to the brief remaining interval. Finally, Figure 6.1c refers to the expected production location m_t. As expected, m_t is strictly decreasing for $t \in [0, \bar{t})$ and stabilises near the end of the programming interval.



(a) Auxiliary variable

(b) Transition rate

(c) Expected production location

Figure 6.1: Absence of incentives. Parameters value: $T = 2, m_0 = 0.5, \rho = 1, \pi = 1, \delta = 1, \xi = 0.5, \kappa = 0.5, \theta = 0.2, \gamma = 0.5$. Simulation with Wolfram Mathematica 12.3.

6.4.2 Interpretation of the results

First of all, it is important to note that from (6.27) the transition rate is active from state $+1$ to state -1, only. The unique process that we can describe in this situation is the process of offshoring. Moreover, since the model is introduced on a finite horizon, in the last part of the programming interval, no phenomenon occurs because the marginal cost of transition, described by the parameter θ, is too large and makes it not convenient to take any action, since one can benefit from this action for a too short period of time. What changes significantly in the optimal control of the representative company is the positivity of time $\bar{t}$, which effectively represents the presence or absence of the offshoring phenomenon. More specifically, we can say that if $\bar{t} \leq 0$, then there is no offshoring phenomenon, whereas if $\bar{t} > 0$, there is relocation to the South country. Therefore, we present a sensitivity analysis with respect to the time parameter $\bar{t}$.

Table 6.3: Sensitivity analysis for $\bar{t}$.

	T	θ	∂	ξ
$\bar{t}$	+	-	+	+

We recall that the greater the value of $\bar{t}$, the greater the phenomenon of offshoring. The parameters that increase the value of $\bar{t}$ increase the location of production in the South country. The parameters T and ∂ are related to the production and sales of the representative company. Parameters θ and ξ can be influenced by both the North country and the South country. In the following proposition, we clarify the economic phenomena associated with the modification of these two parameters.

Proposition 6.4.1. *The North country can contrast the offshoring process by decreasing the difference between two production costs which is described by the parameter ξ; a standard way to obtain this result consists in imposing a duty on the goods imported by the South country.*

The South country can promote the offshoring process by decreasing the marginal cost of the transition described by the parameter θ; a standard way to obtain this result consists of offering incentives to sustain costs.

Proof of Proposition 6.4.1. Let us consider (6.24). We notice that $\bar{t}$ is positive if and only if

$$T\partial\xi - \theta > 0$$

The quantity $T\partial$ represents the cumulative demand throughout the programming interval. Therefore, $T\partial\xi$ is the net profit that a representative company can obtain by moving production from a North country, where the production cost of a unit of good is $\rho + \xi$, to the South country, where this production cost is just ρ.

The North country can contrast the offshoring process by decreasing the value of ξ. The standard way to obtain this result is to impose a duty on importation so that the difference between the production cost in the North and in the South country decreases (see, e.g., [55, 56]).

However, the opposite situation concerns the South country, which aims to increase the offshoring phenomenon in order to boost the companies producing within its own territory. In this case, the best policy is to offer incentives that reduce the marginal transition cost described by the parameter θ (see, e.g., [57]). □

6.5 Offshoring and reshoring in presence of fixed relocalisation incentives

6.5.1 Technical analysis

Assume now the presence of North country's incentives for relocalisation. We will proceed similarly as before, but here we consider $\gamma > 0$. Since the final condition (6.10) is $z_T = -\gamma$, we cannot be sure if, near the end of the programming interval, $|z_t| \leq \theta$ or $z_t < -\theta$. Consequently, we solve the backward equation (6.9) by considering the two cases separately.

Low incentives: $\gamma \leq \theta$

Assume, first, that the incentive parameter γ is lower or equal to the marginal cost for delocalisation θ. Then, the analysis is similar to what was done in Section 6.4.1. By equation (6.9), there must exist $\varepsilon > 0$ such that, for $t \in [T - \varepsilon, T]$, so that $|z_t| \leq \theta$ holds. More precisely,

$$\begin{cases} \dot{z}_t = -\delta\xi \\ z_T = -\gamma \end{cases} \tag{6.30}$$

as long as $|z_t| \leq \theta$. The solution to the above Cauchy problem is

$$z_t = \delta\xi(T - t) - \gamma \tag{6.31}$$

and it holds until $z_{\bar{\bar{t}}} = \theta$, that is, for $t \in [\max(\bar{\bar{t}}, 0), T]$, with

$$\bar{\bar{t}} = T - \frac{\theta + \gamma}{\delta\xi}. \tag{6.32}$$

Equation (6.31) describes the variable z_t near the end of the programming interval. If $\bar{\bar{t}} \leq 0$, then the equation is true for all $t \in [0, T]$. Otherwise, if $\bar{\bar{t}} > 0$, equation (6.31) holds for $t \in [\bar{t}, T]$, while in the remaining interval $z(\cdot)$ is the same as in (6.25). So,

$$z_t = \theta + \sqrt{2\delta\xi\kappa} \tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{t} - t)\right) \tag{6.33}$$

for all $t \in [0, \bar{\bar{t}})$.

Recalling the transition rate (6.8), we obtain

$$\phi^*(t,x) \equiv 0 \tag{6.34}$$

for $\bar{\bar{t}} \leq 0$ and for every $x \in \{-1, +1\}$. On the other hand, if $\bar{\bar{t}} > 0$, the transition rate is

$$\phi^*(t,x) = \begin{cases} \sqrt{\frac{2\delta\xi}{\kappa}} \tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{\bar{t}} - t)\right) & x = 1 \\ 0 & x = -1 \end{cases} \tag{6.35}$$

for $t \in [0, \bar{\bar{t}})$, and $\phi^*(t,x) \equiv 0$ in the remaining interval.

By substituting (6.31) and (6.33) into Equation (6.20), we can also calculate the expected production location. As before, if $\bar{\bar{t}} \leq 0$, then $\dot{m}_t \equiv 0$; therefore, $m_t \equiv m_0$. In contrast, if $\bar{\bar{t}} > 0$, then a similar equation to (6.28) with $\bar{\bar{t}}$ instead of $\bar{t}$ is valid for $t \in [0, \bar{\bar{t}})$

$$\dot{m}_t = -\sqrt{\frac{2\delta\xi}{\kappa}} \tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{\bar{t}} - t)\right)(1 + m_t). \tag{6.36}$$

Solving the above differential equation with initial condition $m(0) = m_0$, we obtain, for $t \in [0, \bar{\bar{t}})$,

$$m_t = c_0 \cosh^2\left[\sqrt{\frac{\delta\xi}{2\kappa}}(t - \bar{\bar{t}})\right] - 1 \tag{6.37}$$

where

$$c_0 = (m_0 + 1)\cosh^{-2}\left[\sqrt{\frac{\delta\xi}{2\kappa}}\,\bar{\bar{t}}\right].$$

Finally, $m_t \equiv m_{\bar{\bar{t}}}$ while $t \in [\bar{\bar{t}}, T]$.

High incentives: $\gamma > \theta$

Let us now consider the incentive parameter γ to be higher than the marginal cost θ to move the production location. Under this assumption, the final condition (6.10) is such that there must exist $\varepsilon > 0$ such that for $t \in (T - \varepsilon, T]$ it holds $z_t < -\theta$. By equation (6.9), we are solving the Cauchy problem

$$\begin{cases} \dot{z}_t = -\dfrac{(z+\theta)^2}{2\kappa} - \delta\xi \\ z_T = -\gamma \end{cases} \tag{6.38}$$

as long as $z_t < -\theta$. Therefore,

$$z_t = -\theta + \sqrt{2\delta\xi\kappa}\tan\left[\sqrt{\frac{\delta\xi}{2\kappa}}(T-t) + \arctan\left(\frac{\theta-\gamma}{\sqrt{2\delta\xi\kappa}}\right)\right] \tag{6.39}$$

for $t \in (\max(\tilde{t}, 0), T]$, where $\tilde{t}$ is defined by $z_{\tilde{t}} = -\theta$. We compute the switching time

$$\tilde{t} = T - \sqrt{\frac{2\kappa}{\delta\xi}}\arctan\left(\frac{\gamma-\theta}{\sqrt{2\delta\xi\kappa}}\right). \tag{6.40}$$

As before, we can observe different situations depending on the sign of $\tilde{t}$. If $\tilde{t} \leq 0$, then z_t is defined as in (6.39) for all $t \in [0, T]$; otherwise, (6.39) holds only for $t \in [\tilde{t}, T]$, while a different equation has to be considered for $t < \tilde{t}$. In the latter case, we enter the region $|z_t| \leq \theta$ at $\tilde{t}$. In this interval, by (6.9), it holds

$$z_t = \delta\xi(\tilde{t} - t) - \theta \tag{6.41}$$

until $|z_t| \leq \theta$, that is, for $t \in [\max\{\bar{\bar{t}}, 0\}, \tilde{t}]$, with

$$\bar{\bar{t}} = \tilde{t} - \frac{2\theta}{\delta\xi}. \tag{6.42}$$

Once again, we can distinguish two different cases. If $\bar{\bar{t}} \leq 0$, then (6.41) holds for all $t \in [0, \tilde{t}]$, otherwise equation (6.41) holds for $t \in [\bar{\bar{t}}, \tilde{t}]$, while z_t is defined as in (6.33) in the remaining interval. Observe that the switching time $\bar{\bar{t}}$ is different between the current case and the low incentives case.

Unlike the previous cases, the analysis of the transition rate highlights the presence of reshoring. Providing incentives is not sufficient to induce reshoring, it is also necessary to cross a minimum threshold to make incentives attractive to businesses. However, it is necessarily to understand that higher incentives correspond to higher costs for the North country. By (6.8) and (6.39), we get

$$\phi^*(t,x) = \begin{cases} 0 & x = 1 \\ -\sqrt{\frac{2\delta\xi}{\kappa}} \tan\left[\sqrt{\frac{\delta\xi}{2\kappa}}(T-t) + \arctan\left(\frac{\theta-\gamma}{\sqrt{2\delta\xi\kappa}}\right)\right] & x = -1 \end{cases} \tag{6.43}$$

for $t \in (\max\{\tilde{t}, 0\}, T]$. Moreover, if $\tilde{t} > 0$,

$$\phi^*(t,x) \equiv 0\,, \tag{6.44}$$

for $t \in [\max\{\bar{\bar{t}}, 0\}, \tilde{t}]$ and, eventually, if $\bar{\bar{t}} > 0$

$$\phi^*(t,x) = \begin{cases} \sqrt{\frac{2\delta\xi}{\kappa}} \tanh\left[\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{\bar{t}} - t)\right] & x = 1 \\ 0 & x = -1 \end{cases} \tag{6.45}$$

for $t \in [0, \bar{\bar{t}})$.

The reshoring phenomenon described in (6.43) also affects the expected location of production (6.7). For $t \in (\max\{\tilde{t}, 0\}, T]$, we obtain

$$\dot{m}_t = -\sqrt{\frac{2\delta\xi}{\kappa}} \tan\left[\sqrt{\frac{\delta\xi}{2\kappa}}(T-t) + \arctan\left(\frac{\theta-\gamma}{\sqrt{2\delta\xi\kappa}}\right)\right](1-m_t)\,. \tag{6.46}$$

If $\tilde{t} > 0$, then $\dot{m}_t \equiv 0$ for $t \in [\max\{\bar{\bar{t}}, 0\}, \tilde{t}]$. Furthermore, if $\bar{\bar{t}} > 0$, then

$$\dot{m}_t = -\sqrt{\frac{2\delta\xi}{\kappa}} \tanh\left(\sqrt{\frac{\delta\xi}{2\kappa}}(\bar{\bar{t}} - t)\right)(1+m_t)\,. \tag{6.47}$$

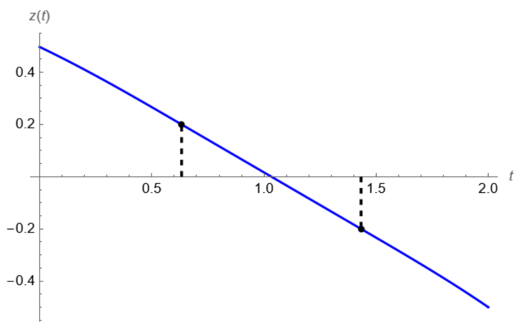
for $t \in [0, \bar{\bar{t}})$. Solving the differential equations (6.47) and (6.46) with initial condition $m(0) = m_0$, we obtain (6.37) for $t \in [0, \bar{\bar{t}})$, $m_t \equiv m_{\bar{\bar{t}}}$ for $t \in [\bar{\bar{t}}, \tilde{t}]$, and for $t \in (\tilde{t}, T]$

$$m_t = c_1 \cos^2\left[\sqrt{\frac{\delta\xi}{2\kappa}}(T-t) + \arctan\left(\frac{\theta-\gamma}{\sqrt{2\delta\xi\kappa}}\right)\right] + 1 \tag{6.48}$$

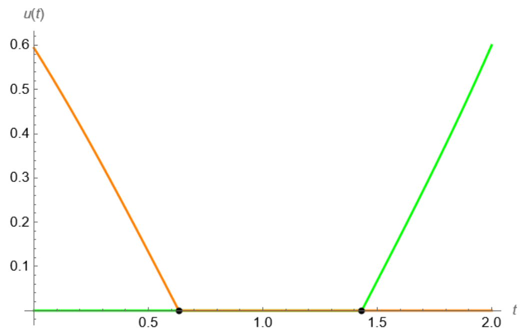
where $c_1 = c_0 - 2$, with c_0 as in (6.37).

Once again, we show the graphs of the principal functions for the same parameters as above.

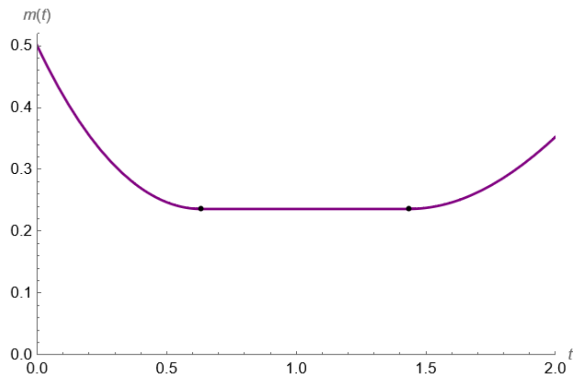
We consider only the most general case, that is, the one with high incentives and with both instant of time (6.40) and (6.42) positive. In Figure 6.2a, we observe the outline of the auxiliary variable z_t, which is always decreasing, but not only positive. These observations make us understand that it may also be convenient for a business to move the production location back to the North country, but only near the end of the interval. However, near $t = 0$, the situation is very similar to the one without incentives. In Figure 6.2b, the transition rate u_t follows the same colour scheme as before: the green line refers to $x = -1$ (reshoring phenomenon) while the orange line refers to $x = +1$ (offshoring phenomenon). An offshoring phenomenon is still taking place near the beginning of the programming interval; however, we also observe a positive transition rate for the reshoring process near the final time T. I recall that if we had considered the low-incentive case, no reshoring would have occurred. These offshoring / reshoring phenomena are also represented in Figure 6.2c by the expected production location m_t.



(a) Auxiliary variable



(b) Transition rate



(c) Expected production location

Figure 6.2: High incentives. Parameters value: $T = 2$, $m_0 = 0.5, \rho = 1, \pi = 1, \delta = 1, \xi = 0.5, \kappa = 0.5, \theta = 0.2$, $\gamma = 0.5$. Simulation with Wolfram Mathematica 12.3.

6.5.2 Interpretation of the results

In contrast to what is observed in Section 6.4.2, Equation (6.43) shows an active transition rate from state -1 to state $+1$. As a consequence, in the presence of incentives, a reshoring process can take place. As we shall see later, the fundamental difference compared to the solutions described in Section 6.4 concerns the presence of the parameter γ: if this quantity is sufficiently large, then our model also describes the phenomenon of reshoring. However, also in this situation, an offshoring phenomenon could occur if the programming interval is long enough or incentives are too low, as described by equations (6.45) and (6.35), respectively. Once again, both cases analysed illustrate the presence of an interval where no phenomenon occurs. This happens because of the marginal cost of transition θ, which may make it difficult to take any action for a too short period of time. In this second case, the switching times $\bar{\bar{t}}$ and $\tilde{t}$ also play a fundamental role in the optimal control of the representative company. Thus, we can summarise here by saying that if incentives are too low, then an offshoring phenomenon may occur when $\bar{\bar{t}} > 0$. Otherwise, an offshoring phenomenon takes place if $\bar{\bar{t}} > 0$, but relocalisation to the North country is always present near the end of the programming interval. We also observe that when $\gamma > \theta$, the starting time of the relocation to the North country $\tilde{t}$ depends decreasingly on the incentive parameter γ. In fact, the greater the incentives γ, the wider the interval to bring back production to the North country. Similarly, the end of the eventual offshoring interval $\bar{\bar{t}}$ is decreasing in γ; hence, the higher the incentives, the narrower the delocalisation to the South country. For $\bar{\bar{t}}$, the same observation holds for low incentives values, that is, when $\gamma \leq \theta$. Given the dependence of the change of production location on the switching times, we present a sensitivity analysis regarding time $\bar{\bar{t}}$ and $\tilde{t}$.

Table 6.4: Sensitivity analysis for $\bar{\bar{t}}$ and $\tilde{t}$.

	T	θ	δ	ξ	γ
$\bar{\bar{t}}$	+	-	+	+	-
$\tilde{t}$	+	+	+	+	-

We recall that the greater $\bar{\bar{t}}$, the long-lasting the offshoring phenomenon; the greater $\tilde{t}$, the shorter the relocalisation to the North country. All parameters that positively affect $\bar{\bar{t}}$, increase the location of production in the South country. Those parameters that positively affect $\tilde{t}$, increase the relocalisation to the North country. In particular, a higher cost to relocate the production location θ always corresponds to more limited offshoring and reshoring phenomena. In contrast, higher incentive values γ imply a reduction of the outsourcing interval and an increase in relocation phenomena. The other parameters have a negative effect on the production location by increasing the transfer to the South and limiting the relocation to the North country. We can summarise all this information in the following proposition.

Proposition 6.5.1. *The incentives paid by the North country allow to achieve a reshoring phenomenon from the South country to the North country if and only if these incentives (described by the parameter γ) are greater than the marginal transition cost (described by the parameter θ).*

Proof of Proposition 6.5.1. If $\gamma \leq \theta$, then the optimal control in feedback form is described by formula (6.35), hence only offshoring is allowed. On the other hand, if $\gamma > \theta$ then, at the end of the programming interval, the optimal control in feedback form is described by formula (6.43), hence the representative company moves its production from South to North in order to obtain the incentives. □

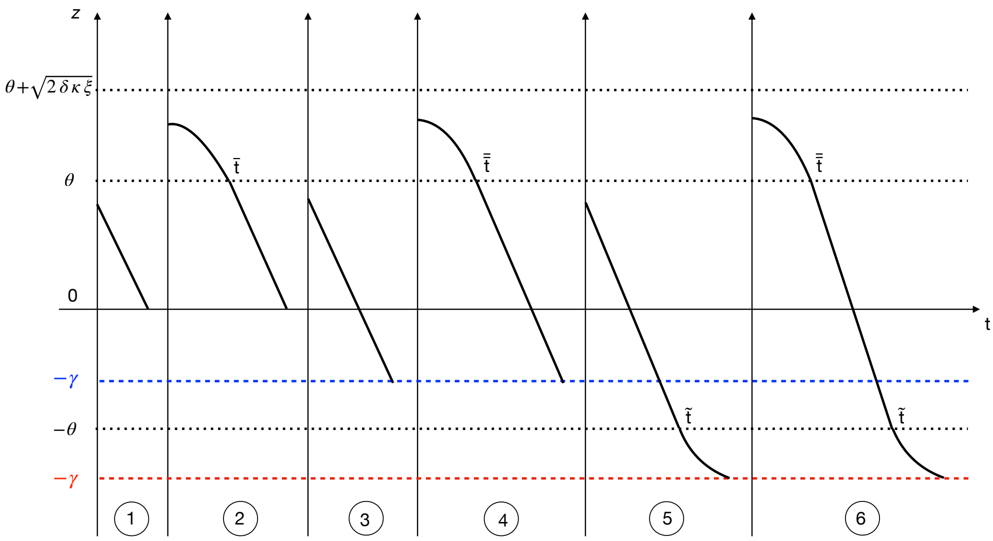


Figure 6.3: Summary of the graphs of the function z_t in all possible instances of the problem.

Figure 6.3 summarises all the results obtained in both Section 4 and Section 5. In the figure, we include two values of γ to describe their possible mutual positions with respect to the variable

θ. The higher dashed blue line corresponds to $\gamma \leq \theta$, while the lower red line corresponds to $\gamma > \theta$. We can describe in detail the various behaviours as a function of the value of the function z.

① In this case, there is no offshoring because the marginal cost is greater than the savings obtained from production costs; see equation (6.26).

② Initially, the offshoring phenomenon is present, but it stops in the second part of the programming interval because the savings in production no longer justify the transaction cost; see equation (6.27).

③ Even in the presence of incentives from the North country, there is neither offshoring nor reshoring because the transaction costs are too high compared to other parameters; see equation (6.34).

④ In this situation, incentives from the North country are too weak, and only the offshoring phenomenon occurs in the first part of the programming interval; see Equation (6.35).

⑤ The presence of incentives from the North country is sufficiently intense to allow the reshoring phenomenon in the last part of the programming interval (eventually, in the whole programming interval if $\tilde{t} \leq 0$). See equation (6.43).

⑥ This is the most informative situation: offshoring occurs in the first part of the programming interval when there is enough time to take advantage of the low production costs in the South country; reshoring, on the other hand, occurs at the end of the programming interval and is stimulated by sufficiently high incentives paid by the North country; see equations (6.43), (6.44), and (6.45).

Starting from the most interesting situation (point 6 of the previous list), we numerically solved the differential equation (6.20) describing the evolution of the mean of the stochastic process X_t^*. The expected production location m_t shown in Figure 6.2c allows us to visually understand the problem. In the image, the programming interval can be separated into three different parts. The left section, ending at $\bar{\bar{t}}$ (corresponding to the first black point), shows a decreasing expected production location, which means that an offshoring phenomenon is occurring. The central section is characterised instead by a flat value. This corresponds to the interval of time in which no actions occur. On the right, starting at $\tilde{t}$, we can observe an increasing trend implied by the reshoring phenomenon.

6.6 Discussion

In this chapter, we have presented a two-country model (North and South) that describes the phenomenon of offshoring and reshoring. From a mathematical point of view, the model is a continuous time-controlled Markov chain with binary state.

In the first part, we have identified the parameter values for which the offshoring phenomenon occurs. We have found results well known in the literature: the main trade-off involves production costs and transaction costs between one country and another [55].

In the second part, we hypothesised that the country with higher production costs can counteract the offshoring phenomenon by using incentives paid at the end of the planning interval to all companies that have not relocated their production. The most informative solution we obtain can be described as follows: a company initially relocates to take advantage of lower production costs, and only at the end of the planning interval does it use a reshoring policy to benefit from the incentives. The key parameters identified by our model are three: the difference in unit production costs between the two countries considered, the marginal cost of transitioning between countries, and the incentive paid by the North country to companies that have not relocated at the end of the planning interval.

The main purpose is to introduce a new model and identify the main trade-offs among the various parameters of the problem. This entire chapter is structured to make this extension very easy and to allow for the easy extension of the research presented in this work to even more detailed and informative situations.

7

From the Offshoring-Reshoring model to Mean-Field games

7.1 Introduction

This chapter presents a possible extension of the model described in Chapter 6. We assume that policymakers provide variable incentives, depending on the amount of offshored businesses at the final time. This alteration in the problem structure will result in a mean-field game with no easy analytical solution. The primary goal is to introduce a new model and identify the main trade-offs among the various parameters of the problem. The main references for this part of the thesis are [36], [37], [38] and [58].

7.2 Offshoring and reshoring in presence of variable relocalisation incentives

7.2.1 Technical analysis

Assume now that North country's incentives for relocalisation depend on the amount of offshored businesses. We rewrite the representative company's objective functional (6.6) as follows:

$$J[U_s] = \mathbb{E}\left\{ \int_0^T \left[\delta\left(\pi - \rho - \frac{\xi}{2}(1+X_t) \right) - \frac{\kappa}{2}U_t^2 - \theta U_t \right] dt + \frac{\gamma}{2}(1+X_T)(1-m_T) \right\}. \quad (7.1)$$

The only difference with (6.6) is in the final incentive: the higher the amount of offshored businesses ($m_T \approx -1$), the higher the incentive. On the other hand, if at time T, only a few businesses have offshored, then policymakers provide low incentives. We can explain this structure by the need for policymakers to limit the disbursement of incentives because the North country's costs may be too expensive. Another way to interpret this new formulation is that a fixed share of incentives is now divided among all firms that relocate.

The analysis is similar to what was done in Chapter 6 with $\gamma > 0$. However, the final condition (6.10) is now equal to

$$z_T = -\gamma(1 - m_T)\,. \quad (7.2)$$

Observe that $z_T \leq 0$, nevertheless we cannot be sure if, near the end of the programming interval, $|z_t| \leq \theta$ or $z_t < -\theta$. Consequently, we consider two separate cases for the backward equation (6.9).

Low incentives: $\gamma(1 - m_T) \leq \theta$

If the final incentive $\gamma(1 - m_T)$ is lower or equal to the marginal cost for delocalisation θ, the analysis is similar to what was done in Section 6.4.1. By equation (6.9), there must exist $\varepsilon > 0$ such that, for $t \in [T - \varepsilon, T]$, so that $|z_t| \leq \theta$ holds. More precisely,

$$\begin{cases} \dot{z}_t = -\delta\xi \\ z_T = -\gamma(1 - m_T) \end{cases} \quad (7.3)$$

as long as $|z_t| \leq \theta$. If the programming interval is long enough and if there are suitable values of the parameters, the auxiliary variable evolution may become

$$\begin{cases} \dot{z}_t = \dfrac{(z-\theta)^2}{2\kappa} - \delta\xi \\ z_{\bar{t}} = \theta\,. \end{cases} \quad (7.4)$$

in the remaining initial part of the programming interval and under the condition $z_t > \theta$. The evolution of the expected production location in Theorem 6.3.2 still holds; we recall it here for simplicity:

$$\dot{m}_t = -\frac{\operatorname{sgn}(z_t)}{\kappa} \cdot [|z_t| - \theta]^+ \cdot (1 + \operatorname{sgn}(z_t) \cdot m_t) \,. \tag{7.5}$$

By (7.3), we have $\dot{m}_t \equiv 0$ while (7.4) implies

$$\dot{m}_t = -\frac{z_t - \theta}{\kappa}(1 + m_t) \,.$$

The problem can be summarised into the following coupled equations

$$\begin{cases} \dot{z}_t = \dfrac{(z_t - \theta)^2}{2\kappa} - \delta\xi & \\ & z_t > \theta \\ \dot{m}_t = -\dfrac{(z_t - \theta)}{\kappa}(1 + m_t) \,, \qquad m_0 & \\ & \\ \dot{z}_t = -\delta\xi \,, \qquad z_T = -\gamma(1 - m_T) & \\ & |z_t| \leq \theta \\ \dot{m}_t = 0 \,. & \end{cases} \tag{7.6}$$

Recall that, after computing the solution for the above problem, we also need to verify that the initial hypothesis of low incentives $\gamma(1 - m_T) \leq \theta$ is satisfied. As in Chapter 6, from a solution point of view, the case of low incentives does not result in a reshoring phenomenon. For this reason, we focus directly on the following part.

High incentives: $\gamma(1 - m_T) > \theta$

Consider now the final incentive $\gamma(1 - m_T)$ to be higher than the marginal cost θ to move the production location. Under this assumption, the final condition (7.2) is such that there must exist $\varepsilon > 0$ such that for $t \in (T - \varepsilon, T]$ it holds $z_t < -\theta$. Reasoning as before, by equation (6.9), we get

$$\dot{z}_t = -\frac{(z_t + \theta)^2}{2\kappa} - \delta\xi$$

and

$$\dot{m}_t = -\frac{(z_t + \theta)}{\kappa}(1 - m_t)$$

for until $z_t < -\theta$. Possibly, if $z_t < -\theta$ does not hold throughout the whole programming interval, we get the same equations as in (7.6) in the remaining part. We can recap the most general case into the following MFG system

$$
\begin{cases}
\dot{z}_t = \dfrac{(z_t-\theta)^2}{2\kappa} - \delta\xi & \\
 & z_t > \theta \\
\dot{m}_t = -\dfrac{(z_t-\theta)}{\kappa}(1+m_t)\,, \qquad m_0 & \\
 & \\
\dot{z}_t = -\delta\xi & \\
 & |z_t| \leq \theta \\
\dot{m}_t = 0 & \\
 & \\
\dot{z}_t = -\dfrac{(z_t+\theta)^2}{2\kappa} - \delta\xi\,, \qquad z_T = -\gamma(1-m_T) & \\
 & z_t < -\theta \\
\dot{m}_t = -\dfrac{(z_t+\theta)}{\kappa}(1-m_t)\,. &
\end{cases}
\tag{7.7}
$$

Also in this case, after computing the solution for the above problem, we also need to verify that the initial hypothesis of high incentives $\gamma(1-m_T) > \theta$ is satisfied.

7.2.2 Flow graphs

Pure quadratic delocalisation cost: $\theta = 0$

Solving (7.7) is not straightforward; therefore, to better understand the problem, it may be useful to start with an easier formulation choosing the marginal cost to move the production location $\theta = 0$. Under this assumption the central region defined by $|z_t| \leq \theta$ vanishes and problem in (7.7) becomes

$$
\begin{cases}
\dot{z}_t = \dfrac{z_t^2}{2\kappa} - \delta\xi & \\
 & z_t > 0 \\
\dot{m}_t = -\dfrac{z_t}{\kappa}(1+m_t)\,, \qquad m_0 & \\
 & \\
\dot{z}_t = -\dfrac{z_t^2}{2\kappa} - \delta\xi\,, \qquad z_T = -\gamma(1-m_T) & \\
 & z_t < 0 \\
\dot{m}_t = -\dfrac{z_t}{\kappa}(1-m_t)\,, &
\end{cases}
\tag{7.8}
$$

which can be summarised as

$$\begin{cases} \dot{z}_t = \dfrac{1}{2\kappa} z_t |z_t| - \delta\xi \\ z_T = -\gamma(1 - m_T) \\ \dot{m}_t = -\dfrac{1}{\kappa} |z_t| (\operatorname{sgn} z_t + m_t) \\ m_0 \end{cases} \tag{7.9}$$

for all $t \in [0, T]$.

We try to obtain some insight of this problem by defining its flow graph. Let us search for an equilibrium by solving $\dot{z}_t = 0$ e $\dot{m}_t = 0$. From (7.9), we have $\dot{z}_t = 0 \iff z_t|z_t| = 2\delta\xi\kappa$, which has a solution only if $z_t > 0$ with

$$z_t = \sqrt{2\delta\xi\kappa} =: \bar{z}\,.$$

We observe that

- if $z_t > \bar{z}$, then $\dot{z}_t > 0$;
- if $z_t = \bar{z}$, then $\dot{z}_t = 0$;
- if $z_t < \bar{z}$, then $\dot{z}_t < 0$.

The final condition for the auxiliary variable implies $z_T \leq 0$, then the region defined by $z_t \geq \bar{z}$ is forbidden. As a result, z_t is always decreasing and, therefore, the final condition itself defines a lower limit that depends on the final expected production location. Indeed, at any instant of time, the value of z_t cannot be lower than its value at the final time T.

Similarly, we reason for the expected production location, solving $\dot{m}_t = 0$. Let us recall that $m_t \in [-1, +1]$ for all $t \in [0, T]$, then

- if $z_t > 0$, then $\dot{m}_t = 0$ if and only if $m_t = -1$, otherwise $\dot{m}_t < 0$;
- if $z_t = 0$, then $\dot{m}_t \equiv 0$;
- if $z_t < 0$, then $\dot{m}_t = 0$ if and only if $m_t = +1$, otherwise $\dot{m}_t > 0$;

This analysis is graphically displayed in Figure 7.1, obtained by a numerical simulation for the same parameter values as in Figure 6.1. In Figure 7.1, the black dashed line corresponds to the upper limit beyond which there is a forbidden region, while the red line displays the final condition $z_T = -\gamma(1 - m_T)$, so it also defines the lower limit with another forbidden region.

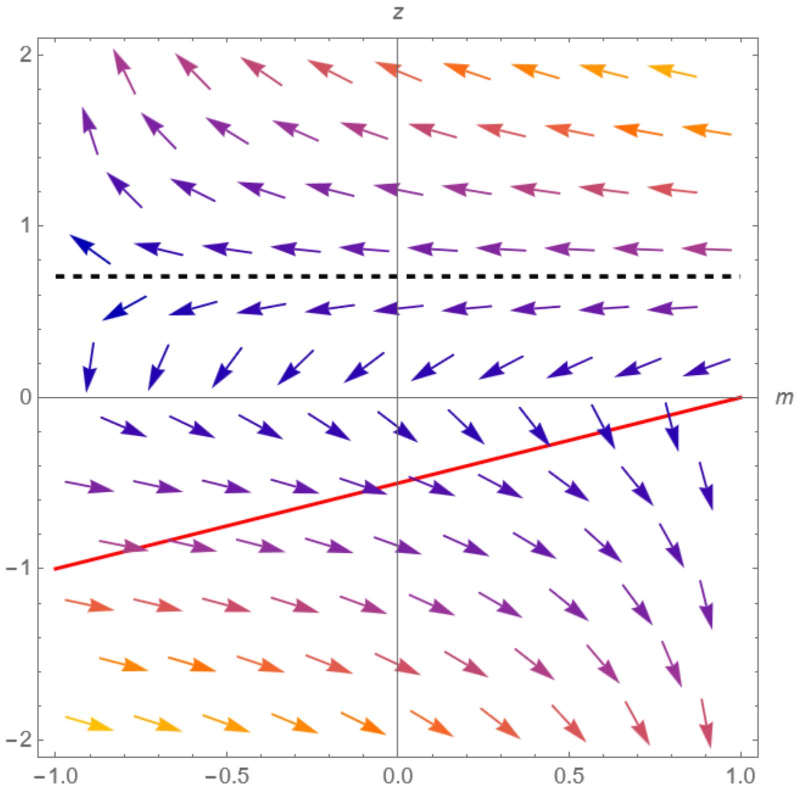


Figure 7.1: Mean-field game flow graph for $\theta = 0$.

The presence of these forbidden areas depends on the final condition $z_T = -\gamma(1 - m_T)$: since the flow has to end on the red line, by considering the flow direction denoted by the arrows in Figure 7.1, we observe that there is no possible initial condition above the dashed black line or below the red line. In both of these cases, indeed, the flow is not able to match the final condition. For this reason, I prefer to display the flow direction there to better explain why we are talking about non-reachable regions.
First of all, we observe that there is no solution for $m_T = 1$. Indeed, system (7.9) has a solution with $m_T = 1$ only if $m_0 = 1$ and $T = 0$. Similarly, we can also exclude the existence of a solution for $m_T = -1$. This final condition for the expected value implies $z_T = -2\gamma$, and therefore, near T, it holds $\dot{z}_t > 0$; as a consequence, $T = 0$ which is an absurd.
The graph also shows that if there is a positive instantaneous gain for the representative company from moving the production from the North country to the South one at $t = 0$, i.e.

$z(0) > 0$, an offshoring phenomenon always precedes re-localisation. In any case, the reshoring occurs near the end of the problem with an intensity highly related to γ and T. From a graphical point of view, γ influences the reshoring by changing the slope of the final condition represented by the red line, while the higher T the farther away the system can go backwards on flow lines. Figure 7.2 shows the representative flow for $\theta = 0$. In the image, the dashed grey lines highlight forbidden regions.

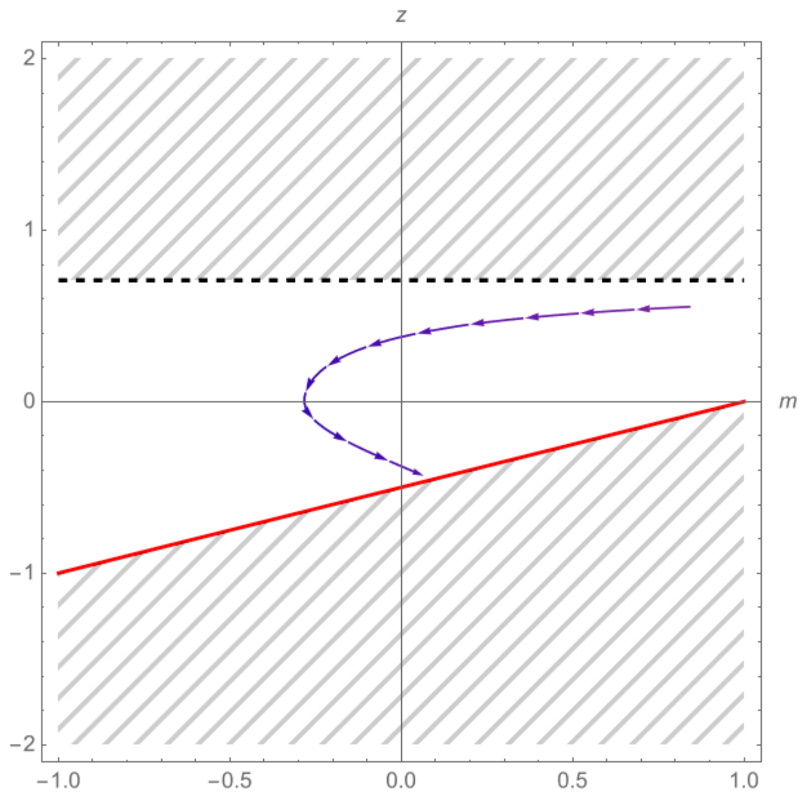
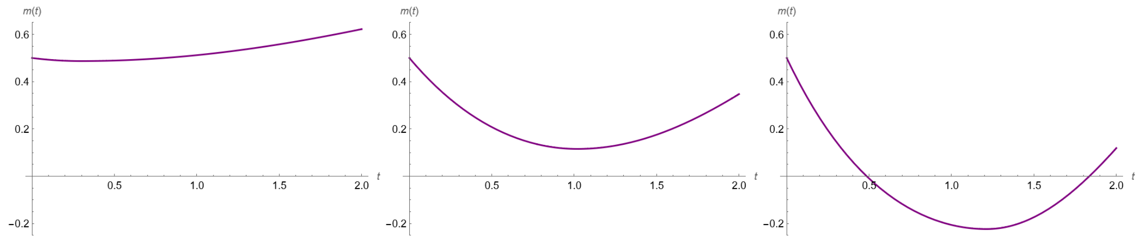


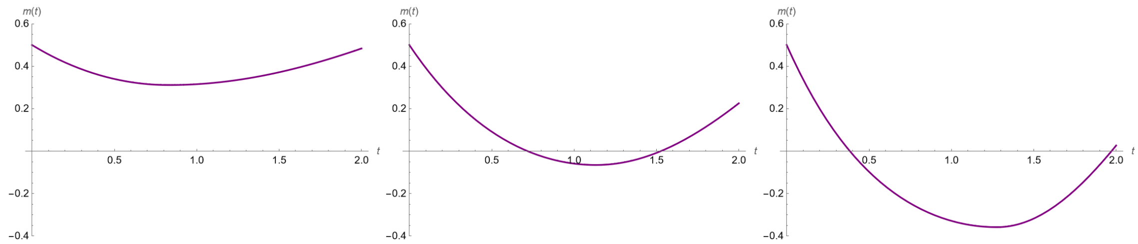
Figure 7.2: Representative mean-field game flow for $\theta = 0$.

To understand how the main parameters affect the expected production location, I display below the result of different simulations. If not specified, the parameter values are: $T = 2$, $m_0 = 0.5, \rho = 1$, $\pi = 1$, $\delta = 1$, $\xi = 0.5$, $\kappa = 0.5$, $\theta = 0$, $\gamma = 0.5$. All simulations obtained with Wolfram Mathematica 12.3.



(a) $\partial = 0.2$ **(b)** $\partial = 0.6$ **(c)** $\partial = 1$

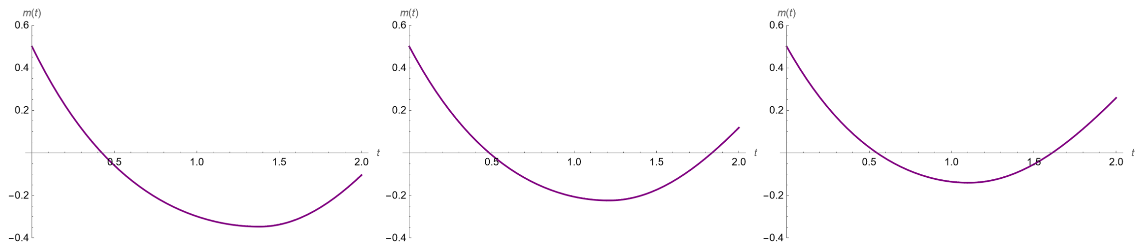
Figure 7.3: Impact of the demand ∂ on the expected production location m_t.



(a) $\xi = 0.2$ **(b)** $\xi = 0.4$ **(c)** $\xi = 0.6$

Figure 7.4: Impact of the additional production cost in the North country ξ on the expected production location m_t.

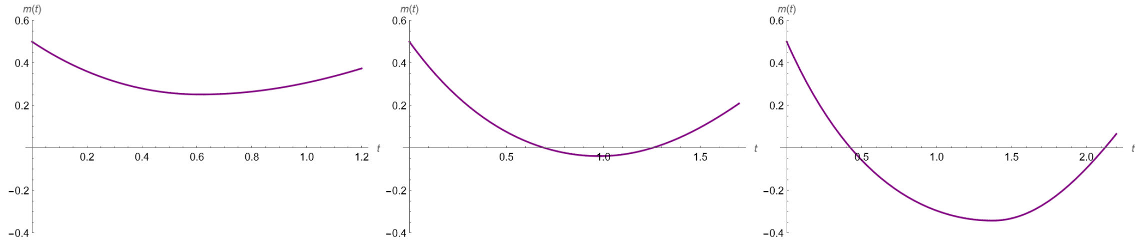
Looking at Figures 7.3 and 7.4, we understand that, once fixed an instant of time, the expected production location m_t decreases in the demand ∂ and the additional production cost for the North country ξ. Clearly, the higher the demand or the additional production cost, the more intense offshoring to enjoy as much as possible the lower production costs of the South country.



(a) $\gamma = 0.3$ **(b)** $\gamma = 0.5$ **(c)** $\gamma = 0.7$

Figure 7.5: Impact of the incentive parameter γ on the expected production location m_t.

In contrast with what observe above, m_t increases instantaneously in the incentive parameter γ (see Figure 7.5). Of course, the higher the incentives provide by the North country policymakers, the weaker offshoring.

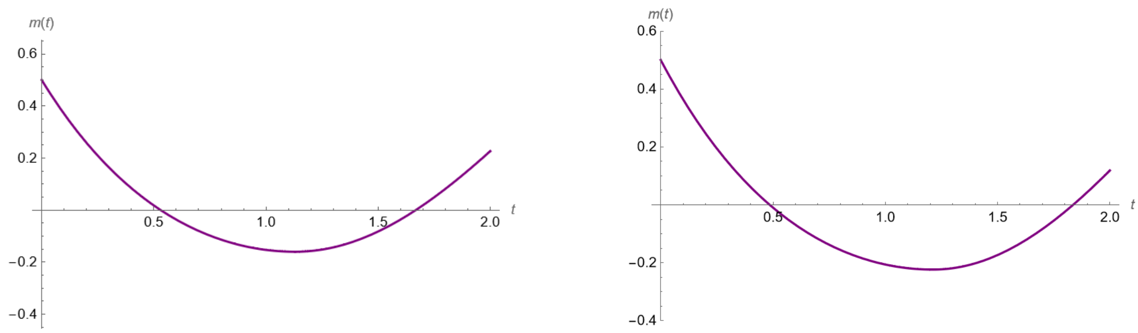


(a) $T = 1.2$ **(b)** $T = 1.7$ **(c)** $T = 2.2$

Figure 7.6: Impact of the programming interval length T on the expected production location m_t.

Eventually, Figure 7.6 displays the impact of the length of the programming interval on the expected production location. We observe that the greater the final time T the greater the offshoring interval and the delocalisation velocity. This behaviour is the effect of the fact that incentives are efficient only near T, and therefore, the smaller the programming interval the more efficient the incentives.

The parameter γ also reflects the goal of the policy makers of the North country. For this analysis, it is useful to compare the evolution of the expected production location where we have fixed incentives and variable incentives, see Figure 7.7. For the same parameter values defined above ($\theta = 0$), the presence of fixed incentives guarantees an instantaneous better result for the expected production location. However, at the end of the programming interval, what policymakers effectively have to pay to the representative company producing in the North country at T is exactly γ in the fixed incentive case, while it is $\gamma(1 - m_T)$ in the other case. Therefore, if the objective of the North country is to bring back a reasonable amount of the production with $m_T > 0$, but it is not a complete reshoring (which is not possible in the variable incentive case), providing variable incentives seems to be a good compromise between the final amount of reshored businesses and the final cost to sustain the reshoring campaign.



(a) Fixed incentive expected production location m_t **(b)** Variable incentive expected production location m_t

Figure 7.7: Comparison of the expected production location m_t in the fixed and variable incentive cases.

Linear and quadratic delocalisation cost: $\theta > 0$

We try now to repeat the above analysis for the more general problem (7.7). By solving $\dot{z}_t = 0$, we obtain the following.

- If $z_t > \theta$ and
  - if $z_t > \theta + \sqrt{2\delta\xi\kappa}$, then $\dot{z}_t > 0$;
  - else if $z_t = \theta + \sqrt{2\delta\xi\kappa}$, then $\dot{z}_t \equiv 0$;
  - else if $z_t < \theta + \sqrt{2\delta\xi\kappa}$, then $\dot{z}_t < 0$.
- Otherwise, if $z_t \leq \theta$, we can easily observe that $\dot{z}_t < 0$.

We proceed in a similar manner for the expected production location.

- if $z_t > \theta$ and
  - if $m_t = -1$, then $\dot{m}_t \equiv 0$;
  - else if $m_t > -1$, then $\dot{m}_t < 0$.
- Otherwise, if $|z_t| \leq \theta$, it holds $\dot{m}_t \equiv 0$.
- Lastly, if $z_t < -\theta$ and
  - if $m_t = +1$, then $\dot{m}_t \equiv 0$;
  - else if $m_t < 1$, then $\dot{m}_t > 0$.

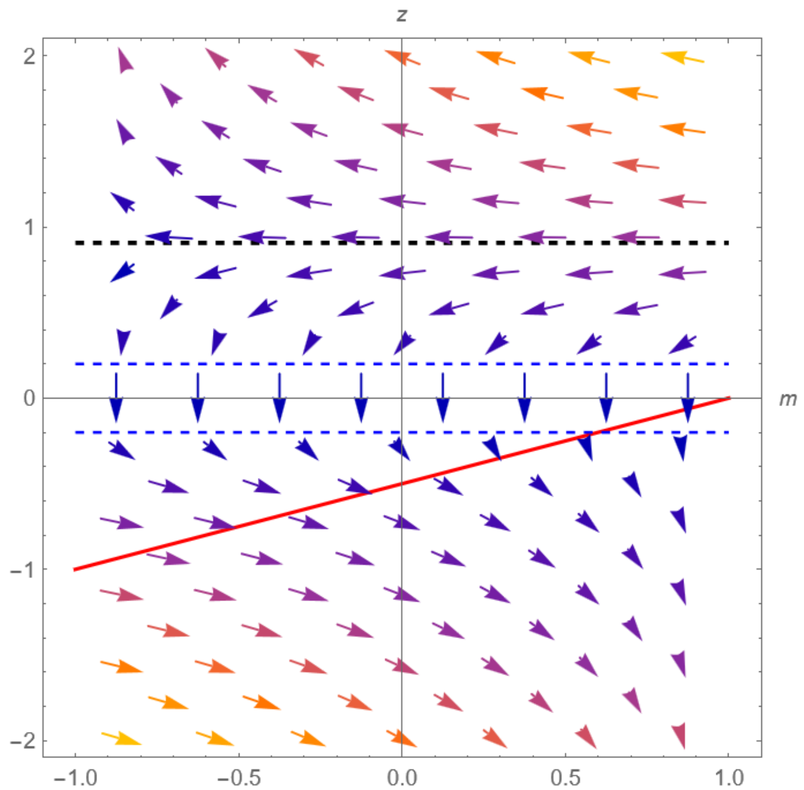


Figure 7.8: Mean-field game flow graph for $\theta = 0.2$.

Figure 7.8 shows a representative graph for the above analysis. As in Figure 7.1, the final condition on z determines a limit for the reachable areas of the graphs. Indeed, once again, since the flow must end on the red line, it is not possible to reach the area below the red line or above the dashed black line. The main difference from Figure 7.1 is the presence of a central region for $|z_t| \leq \theta$ outlined by the horizontal dashed blue lines, where the expected production location m_t does not change, which means that neither offshoring nor reshoring occurs. This region is determined by the fact that, given higher re-localisation costs, there exists a period where neither offshore nor reshore is convenient for businesses. It seems to be a similar situation to the one with fixed incentives shown in Figure 6.2c. Moreover, while the slope of the final condition is the same as in Figure 7.1, the flow arrows change and the upper limit increases by θ. This part of my thesis concerns a work-in paper; therefore, we are not yet able to provide graphs similar to those in the previous section. Specifically, we are interested in the impact of parameter θ on the expected production location. However, intuitively we understand that the higher the marginal linear cost for changing the production location θ, the greater the region where there is no delocalisation in Figure 7.8. Therefore, an increase in delocalisation cost results in a reduction in reshoring, while the upper limit of offshoring would increase by θ, so we cannot say the same. Figure 7.9 shows the representative flow for $\theta > 0$; the forbidden regions are highlighted by dashed grey lines.

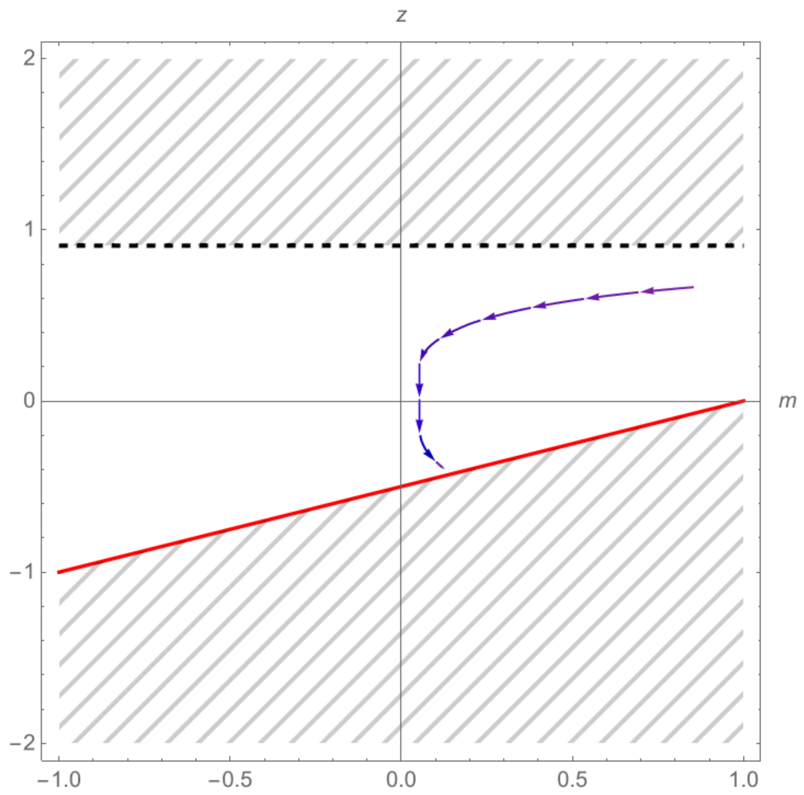


Figure 7.9: Representative mean-field game flow for $\theta > 0$.

7.3 Discussion

In this chapter, we make the offshoring-reshoring problem even more interesting from both an economic and a mathematical point of view by introducing the dependence of one parameter on the average behaviour of the companies. More specifically, the parameter γ (representing incentives) now depends on the number of companies that delocalise: the higher this average, the higher the incentives. This alteration of the problem turns it into a mean-field game that is difficult to treat but rich in information. Through numerical simulations, we obtained insight regarding how the parameters affect the problem.

This part of my thesis concerns a work-in paper; therefore, we cannot provide a complete analysis of the problem. Moreover, we are interested in studying further extensions of the game. For example, we are examining the case where there are no incentives, but the production cost gap varies depending on the amount of offshored businesses: the higher the number of offshored businesses, the lower the production cost gap. Furthermore, we are interested in reformulating the problem by setting an infinite-time horizon so that the existence of an equilibrium point seems possible. This will allow us to introduce a stability analysis to understand how changes in the parameters affect the stability of the equilibrium.

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