Università degli Studi di Padova

Università degli Studi di Padova
Department of Mathematics "Tullio Levi-Civita"

PhD Course in *Mathematical Sciences* Curriculum: *Mathematics*
XXXVIII Cycle

Advances in symplectic billiards

Coordinator
Prof. Giovanni Colombo

Supervisors
Prof. Luca Baracco
Prof.ssa Olga Bernardi

PhD Candidate:
Alessandra Nardi
M. 2082873

30th September 2025

E la luna è una palla,
ed il cielo è un biliardo.

Abstract

In this PhD thesis, we present recent results on symplectic billiards. This class of mathematical billiards was introduced by P. Albers and S. Tabachnikov as a billiard dynamics having an area, rather than a length, as generating function. The thesis addresses three different problems related to this dynamical system.
The first concerns integrability: we establish a Bialy–Mironov type result for symplectic billiards.
The second question is related to the inverse problem, and asks whether the spectral properties of symplectic billiard dynamics allow us to reconstruct the shape of the domain. In this direction, we provide two results: one regarding the expansion of Mather's β-function, and the other concerning the rigidity property of the so-called area spectrum.
Finally, in the last chapter, we turn to the setting of conformal symplectic dynamics and propose a model of dissipative symplectic billiards. This system has a global attractor whose topological and dynamical complexity varies both in terms of the geometry of the billiard table and of the strength of the dissipation. We focus on the study of an invariant subset of the attractor, the so-called Birkhoff attractor.
The results of this thesis are contained in [17], [16], [18], and [15].

Sommario

In questa tesi di dottorato presentiamo alcuni risultati nell'ambito dei biliardi simplettici. Questa classe di biliardi matematici è stata recentemente introdotta da P. Albers e S. Tabachnikov con lo scopo di definire una dinamica di biliardo in cui la funzione generatrice fosse data da un'area, e non da una lunghezza, come invece accade nel caso del biliardo di Birkhoff. La tesi affronta tre diversi problemi legati a questo sistema dinamico.
La prima parte si concentra sullo studio dell'integrabilità, presentando un risultato di tipo Bialy–Mironov per i biliardi simplettici.
La seconda questione affrontata è legata al problema inverso, ed indaga quanto le proprietà spettrali della dinamica dei biliardi simplettici, permettano di ricostruire la forma del tavolo. In questa direzione forniamo due risultati: il primo relativo allo sviluppo asintotico della funzione β di Mather, il secondo riguardante le proprietà di rigidità del cosiddetto spettro delle aree.
Infine, l'ultimo capitolo rientra nell'ambito della dinamica conformemente simplettica e propone un modello di biliardi simplettici con dissipazione. Per questo sistema, viene studiato l'attrattore di Birkhoff al variare sia del parametro dissipativo che della forma del tavolo da biliardo.
I risultati di questa tesi sono contenuti in [17], [16], [18] e [15].

Contents

Aknowledgments **9**

Introduction **11**

1 Twist maps and billiards dynamics **17**
1.1 Monotone twist maps . . . 17
1.1.1 Mather's β-function . . . 21
1.2 Birkhoff billiards dynamics . . . 22
1.2.1 Integrable billiards and Birkhoff conjecture . . . 24
1.2.2 Periodic orbits, length spectrum and Inverse spectral problem . . . 26
1.3 Outer billiards . . . 29
1.4 Symplectic billiards . . . 30
1.4.1 Caustics and integrability for symplectic billiards . . . 33
1.4.2 Periodic orbits and area spectrum . . . 35

2 Bialy–Mironov type rigidity for centrally symmetric symplectic billiards **37**
2.1 Introduction . . . 37
2.2 Invariant curve of 4-periodic orbits . . . 40
2.3 Three technical lemmas . . . 41
2.4 Proof of Theorem 2.1.1 . . . 44

3 Higher order terms of Mather's β-function for symplectic and outer billiards **47**
3.1 Introduction and main result . . . 47
3.2 Preliminaries of affine differential geometry . . . 48
3.3 Asymptotic expansion for $\delta(\Omega, \mathcal{P}_n^i)$ . . . 49
3.4 Asymptotic expansion for $\delta(\Omega, \mathcal{P}_n^c)$ . . . 53
3.5 Proof of Theorem 3.1.1 . . . 55
3.6 Ellipses and circles . . . 58
3.6.1 Symplectic billiards . . . 58
3.6.2 Outer billiards . . . 59

4 Area spectral rigidity for axially symmetric and Radon domains **61**
4.1 Introduction . . . 61
4.2 Preliminaries . . . 63
4.3 Statement of results . . . 65
4.4 Normalizations and infinitesimal deformation function . . . 67
4.5 Preliminary results in affine differential geometry . . . 70
4.6 Proof of Theorems 4.4.5 and 4.4.6 . . . 72
4.6.1 Proof of Theorem 4.4.5 . . . 75
4.6.2 Proof of Theorem 4.4.6 . . . 79

5 Birkhoff attractors for dissipative symplectic billiards **83**
5.1 Introduction 83
5.2 Preliminaries 87
5.2.1 (Birkhoff) attractors for dissipative maps 87
5.2.2 Symplectic billiards 90
5.2.3 Dissipative symplectic billiards 93
5.3 Normally contracted Birkhoff attractors 97
5.4 Centrally symmetric dissipative billiards 100
5.4.1 An open and dense property in the centrally symmetric case 102
5.4.2 Fragility of invariant curves of rotation number 1/4 109
5.5 Topologically and dynamically complex Birkhoff attractors 113
5.6 Numerical simulations 117

Bibliography **118**

Aknowledgments

Before introducing this thesis, which collects the academic work of the past three years, it is fundamental for me to properly thank the people who were part of this journey and made it possible.

First of all, I'm deeply grateful to my supervisors, Professors Luca Baracco and Olga Bernardi. Their constant presence over the past three years has allowed me to never feel lost, even in moments in which new problems arose. I learned a lot from both of them, not only from a mathematical point of view, but also from the way they approach research: Olga's enthusiasm and Luca's thoughtfulness were both essential during this project. Finally, I am grateful for the foresight they had in proposing the topic for this thesis, demonstrating an understanding of how, and to what extent, I might be able to contribute.

A great acknowledgment also goes to Dr. Anna Florio. The months I spent in Paris, collaborating with her, were an essential part of this journey. I am grateful for the generosity with which she shared her time and her mathematical knowledge with me, and for the warmth she showed me during the past months, being a fundamental support through the uncertainty of this career. Great thanks also go to the professors who encouraged me, especially in the last year, when future prospects seemed uncertain. In particular, I would like to thank Martin Leguil and José Pedro Gaivão for the interesting discussions and for their support. Thanks to the people I have met at conferences around the world, particularly those who have become friends, whom I am always glad to meet again in different places.

Thanks to the members of the Department of Mathematics of the University of Padova, the place where I have grown throughout my entire academic journey. Thanks to those who gave me the chance to discover the pleasure of sharing mathematics with people around you, even with children. Torre Archimede is the place where I started thinking that mathematics has a deep social nature. Standing together in front of the blackboard, listening to each other's ideas, expanding the common knowledge of a problem by sharing our different visions of it: the beauty of this collective and creative process is something I had the opportunity to understand in this department. And that is something I am grateful for.

Thanks to the people I shared this journey with, to the office in corridor $7BC$. As you may now know, you have been an important support in all the academic and personal challenges of the past years. In particular, thanks, Marco, for bringing laughter and lightness to these years. Thanks, Liu, for your sympathy and understanding. Thanks, Gaia, for the natural empathy we shared. And thanks, Ishan, for being the most surprising friendship I could have asked for in the past three years. The presence of all of you allowed me to grow throughout these years; you all taught me something that I will keep in my heart, wherever life may bring us.

Aknowledgments

Questo dottorato ha rappresentato per me un percorso più profondo di quanto possa ammettere. Qualcuno, camminando, mi ha detto che non possiamo controllare il vento, ma possiamo aggiustare le vele. Ed io ora sono profondamente grata alla mia famiglia per aver retto queste vele con me. Grazie ai miei genitori, Lorena e Ludovico, e grazie a mio fratello, Filippo. In questi tre anni vi ho sentiti davvero accanto, con profondo affetto e una grande fiducia che spero di aver saputo meritare. Grazie alla mia nonna Anna, per quell'amore incondizionato che riesco a sentire anche ora che non c'è più.

Grazie di cuore a chi mi è stato amico in questi anni così densi e pieni di vita nuova. Grazie ad Erica e ad Emma, due donne incredibili, la cui amicizia sincera per me è stata un pilastro in questo percorso. Grazie a Giacomo, la cui amicizia ho sentito costantemente vicina, al di là di ogni distanza geografica. E poi grazie a Francesco, a Silvia, a Giacomo, a Giulia, a Vincenzo, a Lorenzo, a tutte le persone che mi ha fatto conoscere la matematica, ma con cui poi è subentrato un affetto sincero. Grazie a Rita, grazie a chi ho incontrato in viaggio, a Marta, a chi ha camminato al mio fianco con la luce o con il buio. Grazie alle persone con cui ho condiviso l'indirizzo, partendo da un terrazzo. Grazie a chi ha condiviso con me la musica, fino al diploma e oltre.

Grazie, di una gratitudine profonda e consapevole che questo viaggio, forse più umano che accademico, ha avuto dell'incredibile, proprio per merito di tutti coloro che ne hanno fatto parte. Con profondo affetto, dalle montagne del Paese che a breve mi ospiterà,

Alessandra

Introduction

A mathematical billiard is a dynamical system that describes the motion of a mass point (the billiard ball) inside a planar region (the billiard table) with, in general, a piecewise smooth boundary. The ball moves with constant speed and without friction, following a rectilinear path. The straightforwardness and versatility of this model have made mathematical billiards an object of interest in many different contexts. In fact, depending on the shape of the billiard table, they show a wide range of dynamical behaviors. Citing A. B. Katok [59], that of billiards is not a single mathematical theory, but rather a mathematician's "playground", where various methods and approaches, which can later be adapted to other dynamical systems, are tested and honed.

The standard mathematical billiards, known as *Birkhoff billiards*, were first introduced by G. D. Birkhoff in [30], as a mathematical tool to prove some dynamical applications of Poincaré's last geometric theorem and its generalizations. This dynamical system, which gives a prototype model of a Hamiltonian system, describes the trajectory of a massless particle moving with unit velocity inside a strictly convex planar cavity bounded by perfectly reflecting walls. The point mass therefore moves according to the standard *reflection law*, that is, the angle of incidence is equal to the angle of reflection, see Figure 1. The Birkhoff billiard map turns out to be a monotone twist map having the length of the segment between two consecutive bounces as generating function. In particular, it can be studied using Aubry–Mather's theory.

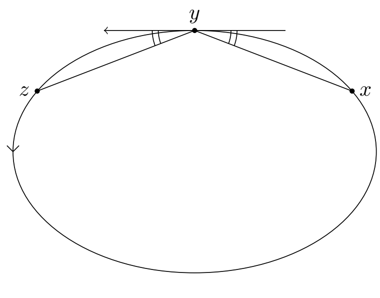


Figure 1: The Birkhoff billiard dynamics.

Later, in the Eighties, J. Moser in [77] popularized another type of billiard dynamics as a toy model for planetary motion. He named this model *outer (or dual) billiard*; in fact given a strictly convex planar domain Ω, the outer billiard map is defined on the exterior of $\partial\Omega$ as follows. A point A is mapped to $F(A)$ if and only if the segment joining A and $F(A)$ is tangent to $\partial\Omega$ exactly at its middle point and has positive orientation at the tangent point, see Figure 2.

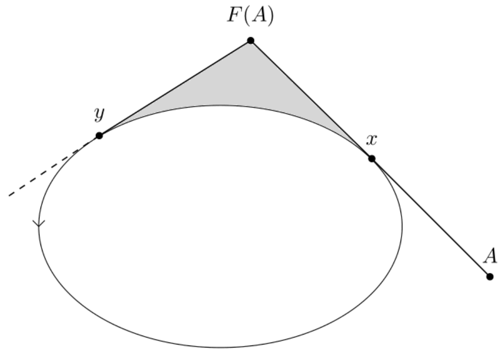


Figure 2: The outer billiard dynamics.

Again, the outer billiard dynamics gives a monotone twist map in which the generating function is given by the area of the curvilinear triangle shadowed in Figure 2.

In 2018, P. Albers and S. Tabachnikov introduced a new class of billiards called *symplectic billiards*, see [1]. As in the Birkhoff case, the billiard table is a strictly convex planar region with smooth boundary, and the dynamics is described as follows. Three points x, y, z on the boundary are three consecutive bounces of a symplectic billiard trajectory if and only if the tangent at the second point, y, is parallel to the line connecting x and z, see Figure 3. For what follows, it is relevant to stress that, while the Birkhoff billiard dynamics is invariant up to isometries, the symplectic billiard map commutes with affine transformations of the plane. For this reason, the results related to this class of billiards often involve affine or equi-affine –that is, unitary affine– transformations.

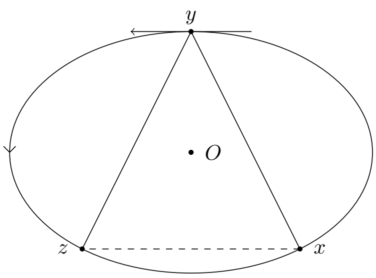


Figure 3: The symplectic billiard dynamics.

Also the symplectic billiard map is a monotone twist map, preserving an area form, and its generating function is the area of the parallelogram given by two consecutive bounces, i.e.,

$$\omega(x, y) = \det(x - O, y - O).$$

Recently, in [2], P. Albers and S. Tabachnikov introduced the so-called *fourth* or *outer length billiards*. These billiards, lately studied by L. Baracco, O. Bernardi and C. Fierobe in [14], are the "outer" counterpart to Birkhoff's billiards: the generating function is the length of the trajectory –like in the Birkhoff case– which lies outside the convex domain –like in the case of outer billiards. This class of planar billiards completes the scheme of billiards dynamics lying inside or outside a strictly convex domain, and having the area or the length as generating function.

Since Birkhoff's work [30], billiards have garnered considerable attention in various contexts, becoming a highly popular subject.

A crucial question in billiard dynamics is *integrability*, namely the existence of a regular foliation of the phase-space by invariant curves. There are different ways to define integrability, including local and/or perturbative notions. We refer to [55] for an exhaustive discussion in the case of Birkhoff billiards. A particularly strong notion of integrability, which is the first one that usually is investigated, is the *total integrability.* It requires the above foliation of the whole phase-space to consist only of not null-homotopic invariant curves. In 1993, M. Bialy, in his groundbreaking work [21], translated the total integrability condition of a twist map into a necessary integral inequality, which involves the construction of a discrete Jacobi field. The application of this framework to the setting of Birkhoff billiards forces the converse of the usual isoperimetric inequality and, consequently, leads to the celebrated result that

Totally integrable Birkhoff billiard tables are circles.

In the outer billiard case, the same question has been solved in 2023 by M. Bialy [24], who established the analogous rigidity for this class of billiards:

Totally integrable outer billiard tables are ellipses.

Here, one of the main differences from the Birkhoff billiards case, is the non-compactness of the phase-space. To overcome such a problem, the choice of a suitable generating function and the use of weights are crucial.
The use of the necessary integral condition led to the solution of the total integrability problem also for symplectic billiards by L. Baracco and O. Bernardi, in [13], who proved that:

Totally integrable symplectic billiard tables are ellipses.

In this case, the phase-space is compact; however, care has to be taken in finding a suitable choice of variables and an appropriate unitary affine transformation. Finally, for outer length billiards, it is known that elliptic tables are totally integrable, see [14][Proposition 4]. However, it is still unknown whether these are the unique cases; in the solution to this fundamental problem, the non-compactness of the phase-space may produce nontrivial issues.

It is then natural to wonder whether it is possible to relax total integrability by requiring a smaller part of the phase-space to admit the desired foliation. In 2022, M. Bialy and A. Mironov [27] obtained a result in this direction, inside the class of centrally symmetric domains. In what follows, strongly convex means with never vanishing curvature.

Given Ω a centrally symmetric C^2 strongly convex domain, if the billiard map of $\partial\Omega$ admits a continuous invariant curve of rotation number $1/4$, consisting only of 4-periodic orbits, and the region between this curve and the boundary of the phase-space cylinder corresponding to the identity is fully foliated by invariant closed curves, then $\partial\Omega$ is an ellipse.

The proof relies on the non-standard generating function from [26], the analysis of 4-periodic dynamics, and Hopf's method combined with Wirtinger's inequality. A perturbative version was later obtained by V. Kaloshin, C. E. Koudjinan, and K. Zhang in [54].
While for outer billiards the problem is still open, a Bialy–Mironov type result can be obtained in the symplectic billiards setting. In this case, the theorem follows more easily than in the Birkhoff one, since by means of some technical lemmas, we are allowed to trace back the problem to the totally integrable case.

The detailed discussion of this result can be found in Chapter 2, which presents the paper "Bialy–Mironov type rigidity for centrally symmetric symplectic billiards", joint work with L. Baracco and O. Bernardi, [16].

Another notion of integrability is linked to the existence of a first integral, which is well-known for Birkhoff billiards both in the case of circles and ellipses (see Section 1.2.1). The existence of first integrals for these types of billiard tables led to the so-called Birkhoff or Birkhoff–Poritsky conjecture, first stated in [80]. The conjecture can be stated as:

Are circles and ellipses the unique integrable Birkhoff billiards?

As of writing, this question remains unanswered. However, there are many interesting and technical results –global, local, and perturbative– in this direction, (see e.g. [42], [10], [46], [57], [52]). For other billiard dynamics, there are still no conjectures of this type, because it is an open question which domains play the same role as the ellipses for Birkhoff billiards.

Another intriguing question is to what extent knowledge of the dynamics allows one to reconstruct the shape of the billiard domain. While it is clear that the shape of the table completely determines dynamics, one can ask if it is possible to recover some geometric information by means of *spectral properties*, related to the action of periodic orbits. As mentioned above, every billiard map is a monotone twist map, preserving an area form, with generating function $L(x, y)$. Consequently, to every closed trajectory $\{x_j\}_{j=0}^{q}$ (i.e. $x_0 = x_q$), corresponds the action $\sum_{j=0}^{q-1} L(x_j, x_{j+1})$. In particular, in the setting of Birkhoff billiards, the action is the length of the corresponding periodic orbit.
It is now natural to define:

1. the *length spectrum* as the set of positive real numbers

$$\mathcal{L}(\Omega) = \mathbb{N}\{\text{action of all closed trajectories}\} \cup \mathbb{N}\{l(\partial\Omega)\},$$

   where $l(\partial\Omega)$ is the perimeter of Ω;

2. the *marked length spectrum* as the map

$$\mathcal{ML}(\Omega) : \mathbb{Q} \cap (0, \frac{1}{2}\Big] \to \mathbb{R}$$

   that associates to any p/q in lowest terms the maximal length of closed trajectories having rotation number p/q;

3. the Mather's β-function, as the map associating to each rotation number $\rho \in (0, 1/2)$,

$$\beta(\rho) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} L(x_i, x_{i+1}),$$

   where $\{x_i\}_{i\in\mathbb{Z}}$ is any minimizing billiard orbit with rotation number ρ. In particular, for rotation numbers of type $1/q$,

$$\beta(1/q) := -\frac{1}{q}\mathcal{ML}(\Omega)(1/q).$$

For other types of billiards, the analogous quantities are defined by means of the corresponding generating functions. In particular, for symplectic and outer billiards, we speak about *area spectrum* and *marked area spectrum*.

The problem of reconstructing the geometry of the table from the knowledge of the above spectral objects is linked to a well-known question in the literature, [53], that is:

Can we hear the shape of a drum?

In this direction, one can first ask if it is possible to reconstruct the shape of the billiard table from knowledge of its Mather's β-function. S. Marvizi and R. Melrose's theory of interpolating Hamiltonians [72] applied to the billiards map, gives as an outcome that the expansion at 0 of Mather's β function is:

$$\beta(1/q) \sim \frac{\beta_1}{q} + \frac{\beta_3}{q^3} + \frac{\beta_5}{q^5} + \frac{\beta_7}{q^7} + \dots$$

as $q \to \infty$. Many terms of such expansion for Birkhoff billiards have been computed by A. Sorrentino in [84]. For symplectic and outer billiards, the first two non-trivial coefficients, β_3 and β_5, we obtained in [66], while β_7 was computed in 2023 in a joint work with L. Baracco and O. Bernardi, see [17]. As a straightforward consequence, we point out that, while for symplectic billiards β_1 and β_3 allow to recognize an ellipse among all strictly-convex planar domains, for outer billiards the same rigidity result can be achieved by means of β_5 and β_7.

In Chapter 3, we give a precise description of these results and present the paper "Higher order terms of Mather's β-function for symplectic and outer billiards", joint work with L. Baracco and O. Bernardi, [17].

The analog result for the coefficients of Mather's β-function in the outer length billiards setting was obtained in [14]. In each case, the difficult part is the choice of suitable coordinates and a convenient parametrization for the boundary of the billiard table.

An alternative direction of study for the inverse spectral problem is the existence of *spectrally rigid* classes of domains. In the Birkhoff setting, this means that any family of domains with the same length spectrum in one of these classes is necessarily isometric. The *length spectral rigidity* has been investigated by J. De Simoi, V. Kaloshin, and Q. Wei in [41]. In particular, they proved that the class of finitely smooth, strictly convex, axially symmetric domains, sufficiently close to a circle, is length spectrally rigid, that is, any length-isospectral (or dynamically spectrally rigid) family of domains in this class is necessarily isometric.
Analogously, for symplectic billiards, one can investigate the *area spectral rigidity*, and ask if any family of domains in a suitable class, having the same area spectrum, is necessarily equi-affine.

Chapter 4 is devoted to construct two different classes of domains for which the area spectral rigidity holds, presenting the paper "Area spectral rigidity for axially symmetric and Radon domains", joint work with L. Baracco and O. Bernardi, [18]. The distinctive feature of this result is that it holds not only for the class of axially symmetric domains, similar to the work of J. De Simoi, V. Kaloshin, and Q. Wei, but also for a class of centrally symmetric domains.

As mentioned above, each of the billiards dynamics presented gives a conservative system, which means that the corresponding billiard maps preserve an area form. Recently, there has been growing interest in the study of *conformally symplectic dynamics*. This refers to the study of maps f for which the area is no longer preserved. In particular, when an area form σ is contracted, there exists a constant $a \in (0, 1)$, called *conformal ratio*, such that

$$f^*\sigma = a\,\sigma.$$

An interesting question is how to define a reasonable notion of billiard dynamics with dissipation. In this direction, O. Bernardi, A. Florio and M. Leguil in [20] recently introduced a dissipative version of Birkhoff billiards. In this model, the usual reflection law is changed so that the reflected angle bends toward the inner normal at the incidence point. See Figure 4. Moreover, different notions of billiards with some form of dissipation have previously been considered by other authors, see for example [40], [86], [67].

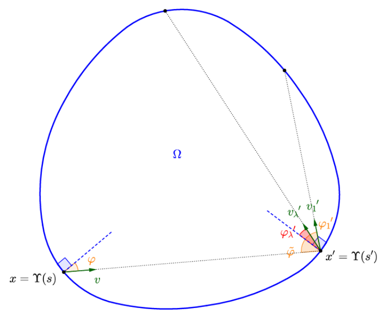


Figure 4: The dissipative Birkhoff billiard map compared with the conservative one.

The aim of Chapter 5 is to introduce a notion of *dissipative symplectic billiards*, via a conformal symplectic map. The dynamics is defined as follows. Fix $\lambda \in (0, 1]$. If $\gamma(t_1)$ and $\gamma(t_2)$ are two successive bounces, then the dissipative symplectic dynamics gives $\gamma(t_3)$ as next bounce if and only if the vector $\gamma(t_3) - \lambda\gamma(t_1)$ is parallel to $\gamma'(t_2)$. In other words, the segment passing through $\gamma(t_3)$ and parallel to $\gamma'(t_2)$ is approaching the origin, when λ approaches 0. See Figure 5. We will see that the dissipative billiard map T_λ is still a twist map, but it no longer preserves an area form. Clearly, $T_1 = T$ is the usual symplectic billiard map.

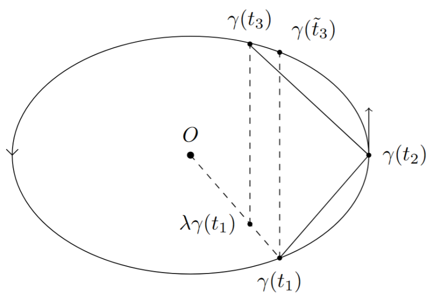


Figure 5: The dissipative symplectic billiard map compared with the conservative one.

The dissipative nature of these maps ensures the existence of a global attractor, named "*à la Conley*", and of the *Birkhoff attractor*, that is the minimal set, w.r.t. inclusion, which is invariant, compact, connected, and separating the phase-space. The paper "Birkhoff attractors for dissipative symplectic billiards", joint work with L. Baracco, O. Bernardi and A. Florio [15], presented in Chapter 5, aims to study the Birkhoff attractor for the dissipative symplectic billiard map, according to the dissipative parameter and to the geometry of the billiard table.

Beyond the applications to Birkhoff attractors, the paper presents and proves many properties for both dissipative symplectic billiard map and conservative symplectic billiards, including: the formula for the differential of both the dissipative and the symplectic billiard map, the discussion of the quality and quantity of 4-periodic orbits for the symplectic billiard dynamics in a centrally symmetric domain, and a result on the fragility of invariant curves of rotation number 1/4. These results may represent the starting point for further studies on symplectic billiard dynamics.

Chapter 1

Twist maps and billiards dynamics

ABSTRACT. The opening section of this chapter introduces monotone twist maps and summarizes key results concerning this class of functions. The presentation follows K. F. Siburg's exposition [83], with emphasis on the dynamical aspects of these maps. A more general and rigorous definition of twist maps, involving the use of lifts, will be provided in Chapter 5. The subsequent sections survey various billiard dynamics, with particular attention to symplectic billiards, and present two open problems in the context of mathematical billiards.

1.1 Monotone twist maps

Consider $\mathbb{S} \times (a,b)$ the annulus, with $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ and we allow $a = -\infty$ and / or $b = +\infty$. Given a diffeomorphism

$$\Phi : \mathbb{S} \times (a,b) \to \mathbb{S} \times (a,b),$$

we denote by

$$\phi : \mathbb{R} \times (a,b) \to \mathbb{R} \times (a,b), \qquad (x_0, y_0) \mapsto (x_1, y_1)$$

a lift of Φ to the universal cover. Then ϕ is a diffeomorphism and $\phi(x+2\pi, y) = \phi(x,y) + (2\pi, 0)$. We assume –if a (resp. b) is finite– that ϕ extends continuously to $\mathbb{R} \times \{a\}$ (resp. $\mathbb{R} \times \{b\}$) by a rotation of fixed angle:

$$\phi(x_0, a) = (x_0 + \rho_a, a) \qquad (\text{resp. } \ \phi(x_0, b) = (x_0 + \rho_b, b)). \tag{1.1.1}$$

Once the lift is fixed, the numbers ρ_a, ρ_b are unique. The choice of ρ_a (resp. ρ_b) if $a = -\infty$ (resp. $b = +\infty$) depends on the dynamics at infinity. The next definition of monotone twist map is well consolidated in literature; we refer, e.g., to [83][Page 2].

Definition 1.1.1. *A monotone twist map*

$$\begin{aligned} \phi : \mathbb{R} \times (a,b) &\to \mathbb{R} \times (a,b) \\ (x_0, y_0) &\mapsto (x_1, y_1) \end{aligned}$$

is a C^1-diffeomorphism satisfying the following conditions:

1. *$\phi(x_0 + 2\pi, y_0) = \phi(x_0, y_0) + (2\pi, 0)$;*

2. *ϕ preserves orientation and the boundaries of $\mathbb{R} \times (a,b)$;*

3. *ϕ extends to the boundaries by rotation, as in (1.1.1);*

4. *ϕ satisfies a monotone twist condition, that is*

$$\frac{\partial x_1}{\partial y_0} > 0; \tag{1.1.2}$$

5. *ϕ is exact symplectic, this means that there exists a generating function $L \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ for ϕ such that*

$$y_1 dx_1 - y_0 dx_0 = dL(x_0, x_1). \tag{1.1.3}$$

The interval $(\rho_a, \rho_b) \subset \mathbb{R}$ is called the twist interval of ϕ.

As a consequence of the exactness of ϕ, the area form $\sigma := dy \wedge dx$ is preserved by the monotone twist map.

Remark 1.1.2. In general, a monotone twist map can be defined on a domain of type

$$\{(x_0, y_0) \in \mathbb{S} \times \mathbb{R} : \varphi_-(x_0) \leq y_0 \leq \varphi_+(x_0)\} \tag{1.1.4}$$

for some $\varphi_-, \varphi_+ : \mathbb{S} \to \mathbb{R}$ continuous functions with $\varphi_- < \varphi_+$.

Clearly, $L(x_0 + 2\pi, x_1 + 2\pi) = L(x_0, x_1)$ and, due to the twist condition, the domain of L is the strip $\mathcal{P} = \{(x_0, x_1) : x_0 \in \mathbb{S},\ \rho_a + x_0 < x_1 < x_0 + \rho_b\}$. Moreover, the equality (1.1.3) reads

$$\begin{cases} y_0 = -L_1(x_0, x_1) \\ y_1 = L_2(x_0, x_1) \end{cases} \tag{1.1.5}$$

where $L_i(x_0, x_1) = \partial_i L(x_0, x_1)$, and the twist condition (1.1.2) becomes $L_{12} < 0$. As a consequence of the monotone twist condition and (1.1.5), $\{(x_i, y_i)\}_{i \in \mathbb{Z}}$ is an orbit of ϕ if and only if it satisfies the relations

$$L_2(x_{i-1}, x_i) = y_i = -L_1(x_i, x_{i+1}), \qquad \forall\, i \in \mathbb{Z}.$$

Equivalently, the corresponding bi-infinite sequence $x := \{x_i\}_{i \in \mathbb{Z}}$ is a so-called critical configuration of the discrete *action functional* on $\mathbb{R}^{\mathbb{Z}}$:

$$(\xi_i)_{i \in \mathbb{Z}} \mapsto \sum_{i \in \mathbb{Z}} L(\xi_i, \xi_{i+1}).$$

Example 1.1.3.

1. The simplest example is an *integrable* twist map that preserves the radial coordinate. The property of being area-preserving implies that the map has the following form:

$$\phi(x_0, y_0) = (x_0 + f(y_0), y_0)$$

with $f' > 0$, and the generating function (up to additive constants) is given by $h = h(x_1 - x_0)$ with $h' = f^{-1}$.

2. An example of a *non-integrable* monotone twist map is the standard map:

$$\phi : (x, y) \mapsto \left(x + y + \frac{k}{2\pi} \sin(2\pi x), y + \frac{k}{2\pi} \sin(2\pi x) \right)$$

with $k \geq 0$.

3. An interesting class of monotone twist maps arises from planar convex billiards, which will be presented in Section 1.2.

Definition 1.1.4. *Given L, a generating function of a monotone twist map ϕ, we say that a critical configuration x of ϕ is minimal if every finite segment of x minimizes the action functional with fixed end points, i.e.*

$$\sum_{i=k}^{l-1} L(x_i, x_{i+1}) \leq \sum_{i=k}^{l-1} L(\xi_i, \xi_{i+1})$$

for every finite segment $(\xi_k, \dots, \xi_l) \in \mathbb{R}^{l-k+1}$ with $\xi_k = x_k$ and $\xi_l = x_l$.

Given an orbit $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$, the map Φ induces a circle map on the first coordinate x_i. This leads to the definition of the rotation number of $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$.

Definition 1.1.5. *The rotation number of an orbit $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ of ϕ is*

$$\rho := \lim_{i\to\pm\infty} \frac{x_i}{i}$$

if such a limit exists.

The orbits for which we can easily compute the rotation number are the periodic orbits $((x_i, y_i))_{i\in\mathbb{Z}}$ with

$$x_{i+q} = x_i + p$$

for all $i \in \mathbb{Z}$ and for some $q, p \in \mathbb{N}$ with $q > 0$. Assuming q and p to be coprime, in order to have q as minimal period, the rotation number will be given by

$$\rho = \frac{p}{q}.$$

The core of the celebrated Aubry-Mather theory is to establish the existence of orbits for a monotone twist map of any given rotation number in the twist interval (ρ_a, ρ_b). The classical result for rational rotation number was given by G.D. Birkhoff in [30].

Theorem 1.1.6. *Let ϕ be a monotone twist map with twist interval (ρ_a, ρ_b), and $p/q \in (\rho_a, \rho_b)$ with $p, q \in \mathbb{N}$ coprime. Then ϕ possesses at least two periodic orbits of rotation number p/q.*

The proof of this result consists of using variational methods to construct specific orbits for monotone twist maps.

Remark 1.1.7. Birkhoff's theorem is sharp since –in general– one cannot expect more than two orbits of a given rotation number. This is the case of 2-periodic orbits of Birkhoff billiards in elliptic tables.

Of particular importance for the dynamics of a monotone twist map $\phi : \mathbb{R}\times(a, b) \to \mathbb{R}\times(a, b)$ are closed invariant curves. If we look at the projection $\Phi : \mathbb{S} \times (a, b) \to \mathbb{S} \times (a, b)$, invariant curves are divided into two classes: they can either be contractible or homotopically non-trivial. Lifted to the strip $\mathbb{R} \times (a, b)$, this means that we consider ϕ-invariant curves which are either closed or homotopic to $\mathbb{R}$.

Definition 1.1.8. *An invariant circle of a monotone twist map ϕ is an embedded, homotopically non-trivial, ϕ-invariant curve in $\mathbb{R} \times (a, b)$, respectively in its projection $\mathbb{S} \times (a, b)$.*

Another classical result by G.D. Birkhoff states that invariant circles must project injectively onto the base. More precisely, the following theorem holds.

Theorem 1.1.9. *Any invariant circle of a monotone twist map is the graph of a Lipschitz function.*

A proof of this theorem can be found, for example, in [82].

S. Aubry and J. N. Mather proved the extension of the above results to irrational rotation numbers. For more details, we refer to [12].

Theorem 1.1.10. *A monotone twist map possesses minimal orbits for every rotation number in its twist interval; for rational rotation numbers, there are always at least two periodic minimal orbits. Every minimal orbit lies on a Lipschitz graph over the x-axis. Moreover, if there exists an invariant circle, then every orbit on that circle is minimal.*

The existence of an invariant curve guarantees the following inequality involving the second derivatives of the generating function. The proof of the following proposition can be found in [74][Section 6].

Proposition 1.1.11. *Let $\Phi : \mathbb{S} \times [a, b]$ be a negative twist map with generating function L, that is $L_{12} > 0$, and let $(x_1, y_1) \in \mathbb{S} \times (a, b)$. Suppose that Φ preserves an area form on $\mathbb{S} \times [a, b]$. If there exists an invariant curve passing through (x_1, y_1), then*

$$L_{11}(x_1, x_2) + L_{22}(x_0, x_1) < 0,$$

where $\Phi(x_1, y_1) = (x_2, y_2)$ and $\Phi^{-1}(x_1, y_1) = (x_0, y_0)$.

Let us now introduce the notion of conjugate points for a monotone twist map. Consider a point $(x_0, y_0) \in \mathbb{S} \times (a, b)$ and the corresponding trajectory $(x_n, y_n) = \Phi(x_0, y_0)$, $n \in \mathbb{Z}$. The configuration $\{x_n\}_{n\in\mathbb{Z}}$ satisfies

$$L_1(x_n, x_{n+1}) + L_2(x_{n-1}, x_n) = 0 \quad \forall\, n \in \mathbb{Z}.$$

Definition 1.1.12. *A sequence $\{\xi_n\}_{n\in\mathbb{Z}} \subset T_{x_n}\mathbb{S}$ is called a Jacobi field along the configuration $\{x_n\}_{n\in\mathbb{Z}}$ if the following equation is satisfied for every $n \in \mathbb{Z}$*

$$L_{12}(x_{n-1}, x_n)\xi_{n-1} + (L_{22}(x_{n-1}, x_n) + L_{11}(x_n, x_{n+1}))\xi_n + +L_{12}(x_n, x_{n+1})\xi_{n+1} = 0. \tag{1.1.6}$$

Notice that since $T_x\mathbb{S} = \mathbb{R}$, each vector ξ is just a real number. It is also clear that a Jacobi field is fully determined by its two successive values.

Definition 1.1.13. *Two distinct points $x_M, x_N \in \{x_n\}_{n\in\mathbb{Z}}$ of the configuration are said to be conjugate if there exists a non-zero Jacobi field vanishing at both x_M and x_N.*

Suppose now that Φ is a positive twist map preserving an area form σ, that is $\Phi^*\sigma = \sigma$, with $\sigma = dy \wedge dx$. Moreover, we suppose that all the second derivatives of the generating function L are bounded, that is,

$$|L_{11}|, |L_{22}|, |L_{12}| \leq K \tag{1.1.7}$$

for some constant $K > 0$.
Consider a configuration $\{x_n\}_{n\in\mathbb{Z}}$ for the given monotone twist map, corresponding to the trajectory $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$. If $\{x_n\}_{n\in\mathbb{Z}}$ does not admit conjugate points, it is possible to construct a never vanishing Jacobi field $\{\nu_n\}_{n\in\mathbb{Z}}$ along $\{x_n\}_{n\in\mathbb{Z}}$. Then the function

$$F(x_n, y_n) := -L_{11}(x_n, x_{n+1}) - L_{12}(x_n, x_{n+1})\frac{\nu_{n+1}}{\nu_n} \tag{1.1.8}$$

is well-defined along the trajectory $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$.

Proposition 1.1.14. *If in the above hypotheses the monotone twist map has no conjugate points, then the following inequality holds:*

$$L_{11}(x_n, x_{n+1}) + 2L_{12}(x_n, x_{n+1}) + L_{22}(x_n, x_{n+1}) \leq F(x_{n+1}, y_{n+1}) - F(x_n, y_n). \tag{1.1.9}$$

Proof. We only sketch the proof following [13][Proposition 3.2], and referring to [21][Section 3] for a detailed proof in the Birkhoff billiards setting. First, by using the monotone twist condition and the absence of conjugate points assumption, it is possible to construct a proportional, strictly positive Jacobi field $\{\nu_n\}_{n\in\mathbb{Z}}$ and a corresponding well-defined function on $\mathbb{S} \times (a, b)$:

$$F(x_n, y_n) := -L_{11}(x_n, x_{n+1}) - L_{12}(x_n, x_{n+1})\frac{\nu_{n+1}}{\nu_n}.$$

As a consequence of estimates (1.1.7), $F \in L_1(\mathcal{P}, \sigma)$. Then, by the positive twist condition again, it can be easily proved –exactly as in [21][Lemma 4]– that the function $F : \mathbb{S} \times (a, b) \to \mathbb{R}$ satisfies the following inequality:

$$F(\Phi(x_1, y_1)) - F(x_1, y_1) \geq L_{11}(x_1, x_2) + 2L_{12}(x_1, x_2) + L_{22}(x_1, x_2).$$

□

1.1.1 Mather's β-function

The existence of orbits of any given rotation number allows us to build the so-called Mather's β-function (or minimal action).

Definition 1.1.15. *The Mather's β-function of ϕ is $\beta : (\rho_a, \rho_b) \to \mathbb{R}$ with*

$$\beta(\rho) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} L(x_i, x_{i+1})$$

where $\{x_i\}_{i\in\mathbb{Z}}$ is any minimal configuration of ϕ with rotation number ρ.

This function satisfies the following properties.

Property 1.1.16. *Let ϕ be a monotone twist map, and β its minimal action. The following hold:*

1. *β is strictly convex and, in particular, it is continuous;*
2. *β is differentiable at all irrational numbers;*
3. *if $\rho = p/q$ is rational, β is differentiable at ρ if and only if there is a ϕ-invariant circle of rotation number p/q consisting entirely of periodic minimal orbits;*
4. *if Γ_ρ is a ϕ-invariant circle of rotation number ρ, then β is differentiable at ρ with $\beta'(\rho) = \int_{\Gamma_\rho} y dx$.*

These are well-known results that can be found in [75].

The Mather's β-function is relevant because it contains information about the dynamical behaviour of the twist map. In the setting of mathematical billiards, it plays a crucial role in the comprehension of different rigidity phenomena.

1.2 Birkhoff billiards dynamics

A particularly interesting class of monotone twist maps is given by planar convex billiards.

A mathematical billiard consists of a planar domain (billiard table) and a point-mass (billiard ball) that moves freely inside the domain. In other words the point moves along a straight line with constant speed until it reaches the boundary.
Usual mathematical billiards are the so-called *Birkhoff billiards* in which the reflection off the boundary is subject to the reflection law. In other words, at each bounce at the boundary, the angle of incidence is equal to the angle of reflection. After that the point-mass continues its free motion with the new unitary velocity until the next bounce. G. D. Birkhoff first proposed this model in [31] to prove some dynamical applications of Poincaré's last geometric theorem, and as a mathematical playground for Hamiltonian dynamics.

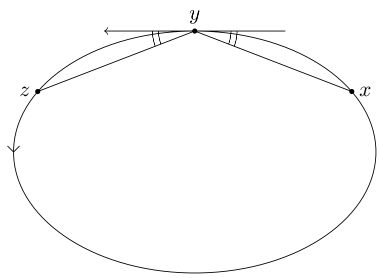


Figure 1.1: The Birkhoff billiard dynamics.

Let us now give a more precise description of the Birkhoff billiard dynamics and understand why it is a monotone twist map.

Consider $\Omega \subset \mathbb{R}^2$ a strictly convex bounded domain, with C^r-smooth boundary $\partial\Omega$, where $r \geq 3$. These assumptions are necessary to get a continuous billiard map and a complete flow, [49]. In this section, we will always work under such hypotheses. Let M be the space of unit tangent vectors (x, v) whose foot points $x \in \partial\Omega$ and v are inward directions. The Birkhoff billiard map is

$$\begin{array}{rccc} f: & M & \to & M \\ & (x, v) & \mapsto & (x', v'), \end{array}$$

where x' represents the point where the trajectory starting at x with velocity v hits the boundary, and v' the reflected velocity.

Let us suppose $\partial\Omega$ to have length $l(\partial\Omega) = 2\pi$. Parametrizing $\partial\Omega$ by arclength t, $(\gamma(t) \in \partial\Omega)$, and considering φ the angle between v and the positive tangent line of $\partial\Omega$, it is possible to read the Birkhoff billiard map as a map of the annulus $\mathbb{A} := \mathbb{S} \times (0, \pi)$ into itself. In fact, the billiard map f can be described as

$$\begin{array}{rccc} f: & \mathbb{A} & \to & \mathbb{A} \\ & (t, \varphi) & \mapsto & (t', \varphi'), \end{array}$$

and it can be extended to the closure $\bar{\mathbb{A}} = \mathbb{S} \times [0, \pi]$ by fixing $f(t, 0) = f(t, \pi) = Id$, for all $t \in \mathbb{S}$. In particular, by introducing the lift $\tilde{f} : \mathbb{R} \times [0, \pi] \to \mathbb{R} \times [0, \pi]$ of the above map, we have

$$\tilde{f}(t, 0) = (t, 0) \quad \text{and} \quad \tilde{f}(t, \pi) = (t, \pi) + (2\pi, 0) \qquad \forall t \in \mathbb{R}.$$

Denoting by

$$L(t,t') := -\|\gamma(t) - \gamma(t')\| \tag{1.2.1}$$

the opposite of the Euclidean distance between two consecutive bounces, the following proposition holds.

Proposition 1.2.1. *Let $\gamma(t)$ and $\gamma(t')$ be two consecutive bounces of the Birkhoff billiards map, and φ, φ' the angles between the corresponding tangents and the segment connecting $\gamma(t)$ and $\gamma(t')$. Then the following equalities hold*

$$\begin{cases} \frac{\partial L(t,t')}{\partial t} = \cos\varphi \\ \frac{\partial L(t,t')}{\partial t} = -\cos\varphi'. \end{cases} \tag{1.2.2}$$

Proof. Consider $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\gamma(t') = (\gamma_1(t'), \gamma_2(t'))$. Then

$$L(t,t') = -\|\gamma(t) - \gamma(t')\| = -\sqrt{(\gamma_1(t) - \gamma_1(t'))^2 + (\gamma_2(t) - \gamma_2(t'))^2}$$

and by differentiating, we obtain

$$\begin{aligned} \partial_1 L(t,t') =& \frac{\partial L(t,t')}{\partial t} = \frac{\gamma(t') - \gamma(t)}{|\gamma(t') - \gamma(t)|} \cdot \gamma'(t) = \cos\varphi, \\ \partial_2 L(t,t') =& \frac{\partial L(t,t')}{\partial t'} = -\frac{\gamma(t') - \gamma(t)}{|\gamma(t') - \gamma(t)|} \cdot \gamma'(t') = -\cos\varphi'. \end{aligned}$$

In particular, we get

$$-\cos\varphi' dt' + \cos\varphi dt = dL(t,t') \tag{1.2.3}$$

$\square$

Taking the differential of (1.2.3), we obtain

$$0 = d^2 L(t,t') = -\sin\varphi d\varphi \wedge dt + \sin\varphi' d\varphi' \wedge dt',$$

and as a consequence, the area form

$$\sigma := \sin\varphi d\varphi \wedge dt = d(-\cos\varphi dt)$$

is preserved along the dynamics, and moreover it is an exact symplectic form on $\mathbb{A}$.
Consider now the lift $\tilde{f}$ and let's introduce the coordinates $(t,r) := (t, -\cos\varphi) \in \mathbb{R} \times [-1,1]$.

Proposition 1.2.2. *In the coordinates (t,r), the Birkhoff billiard map*

$$\tilde{f} : \mathbb{R} \times (-1,1) \to \mathbb{R} \times (-1,1)$$

is a monotone twist map with generating function $L(t,t')$, preserving the standard area form $dt \wedge dr$.

Proof. Let $\tilde{f}(t,r) = (t', r')$. The only thing we still need to check is the monotone twist condition

$$\partial_{12} L(t,t') < 0. \tag{1.2.4}$$

Let φ and φ' be the angles such that $r = -\cos\varphi$ and $r' = -\cos\varphi'$, and notice that, by the definition of the phase-space, $\sin\varphi > 0$ and $\sin\varphi' > 0$. By computation we get

$$\begin{aligned}
\partial_{12}L(t,t') =& \partial_1\left(-\frac{\gamma(t')-\gamma(t)}{|\gamma(t')-\gamma(t)|}\cdot\gamma'(t')\right) = \\
=& -\gamma'(t)\frac{|\gamma(t')-\gamma(t)|}{|\gamma(t')-\gamma(t)|^2}\cdot\gamma'(t') + \left(\frac{\gamma(t')-\gamma(t)}{|\gamma(t')-\gamma(t)|^3}\cdot\gamma'(t)\right)(\gamma(t)-\gamma(t'))\cdot\gamma'(t') = \\
=& \frac{\gamma'(t)\cdot\gamma'(t')}{|\gamma(t')-\gamma(t)|} - \left(\frac{\gamma(t')-\gamma(t)}{|\gamma(t')-\gamma(t)|}\cdot\gamma'(t)\right)\left(\frac{\gamma(t')-\gamma(t)}{|\gamma(t')-\gamma(t)|}\cdot\gamma'(t')\right)\frac{1}{|\gamma(t')-\gamma(t)|} = \\
=& \frac{1}{|\gamma(t')-\gamma(t)|}\left[\cos(\varphi+\varphi') - \cos\varphi\cos\varphi'\right] = \\
=& -\frac{\sin\varphi\sin\varphi'}{|\gamma(t')-\gamma(t)|} < 0.
\end{aligned}$$

□

Remark 1.2.3. The Birkhoff billiard map is invariant up to isometries of the plane, since this class of transformations preserves angles.

1.2.1 Integrable billiards and Birkhoff conjecture

A physical phenomenon that is connected to mathematical billiards, are the so-called *whispering galleries*. Imagine to be inside a circular domed roof, trying to communicate with someone on the other side of the dome. An effective way to be heard clearly on the other side is by getting close to the circular wall and whispering along it. The sound waves get reflected and travel along the wall, always remaining close to it. In the context of billiards, a whispering gallery is called a *caustic*.

Definition 1.2.4. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^2$. A convex caustic is a closed C^1-curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection; see Figure 1.2.*

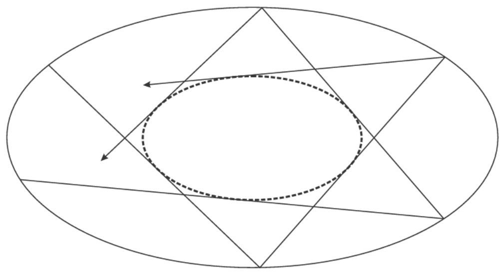


Figure 1.2: Example of convex caustic, picture from [82].

More generally, a caustic is defined as a continuous curve inside Ω with the above tangency property, which does not need to be differentiable, nor to bound a convex domain.

Remark 1.2.5. There exists a relation between convex caustics and invariant circles for the billiard map. In fact, to a convex caustic corresponds an invariant circle for the billiard map f; however, the converse is not entirely true. Birkhoff's Theorem 1.1.9 ensures that invariant circles of twist maps are graphs and therefore give rise to caustics; but such caustics need neither to be convex nor differentiable (see e.g. [61]).

Let us now present two examples of Birkhoff billiard dynamics.

Example 1.2.6. The simplest example is the one of *circular* billiard tables. Let Ω be the disk of unitary perimeter. On the annulus $\mathbb{A} = \mathbb{S} \times [0, \pi]$, the billiard map is given by

$$(t', \varphi') = (t + \varphi/\pi, \varphi).$$

Since the angle is preserved along the motion, this turns out to be an integrable twist map in the coordinates $(x, y) = (t, -\cos\varphi)$. The phase-space is fully foliated by invariant circles. Moreover, speaking about convex caustics, the disk Ω is foliated by concentric circular caustics.

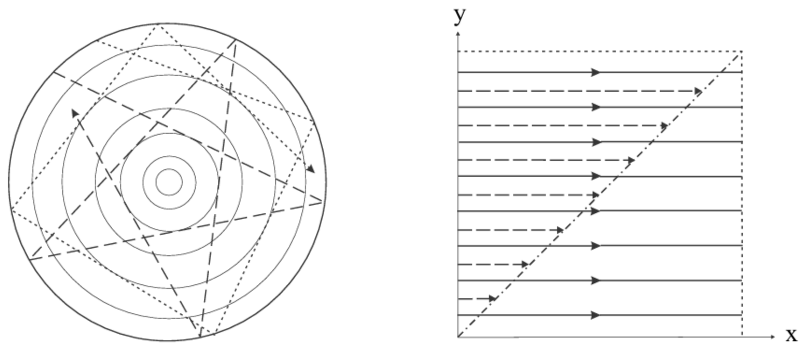


Figure 1.3: Caustics and phase portrait of Birkhoff billiard dynamics in circular tables, pictures from [82].

Example 1.2.7. Elliptic tables give another well-known example of billiard dynamics. Let $\Omega \subset \mathbb{R}^2$ be the ellipse of equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$$

with foci F_1 and F_2. Let M be the phase-space of the Birkhoff billiard map inside Ω, that is, the set of unit vectors (x, v) with foot point on the ellipse and v having inward direction. The function

$$\frac{x_1 v_1}{a_1^2} + \frac{x_2 v_2}{a_2^2}$$

is an integral of the billiard ball map. For a complete proof of this result, we refer, e.g., to [87][Theorem 4.4].
Geometrically, the presence of such an integral of the motion has the following interpretation. Each trajectory of the billiard map inside Ω either:

1. always intersects the open segment between the two foci, or

2. always passes through the two foci alternately, or

3. never intersects the closed segment between the foci.

Speaking about caustics, each trajectory that never intersects a focal point is always tangent to precisely one confocal conic section: in case 1., the caustic is given by the two branches of a confocal hyperbola; in case 3. the convex caustic is a confocal ellipse.

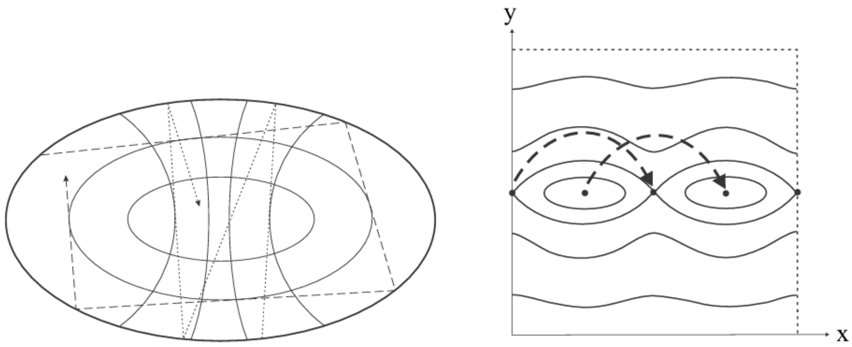


Figure 1.4: Caustics and phase portrait of Birkhoff billiard dynamics in elliptic tables, pictures from [82].

Notice that, although the phase portrait looks like the pendulum, the dynamics are quite different. The points $(0,0)$ and $(\pi,0)$ and their translates do not represent equilibria, but belong to two periodic orbits corresponding respectively to the larger and the smaller axis of the ellipse. The separatrices represent the orbits of case 2., passing through the foci. The invariant circles outside the island formed by the separatrices correspond to the convex caustics, while the homotopically trivial invariant curves correspond to the hyperbola.

Both these examples present the most deterministic behaviour for billiard ball maps, that is *integrability.* We refer to Definition 1.1.8 for the notion of invariant circles.

Definition 1.2.8. *A billiard table is said to be integrable if a subset of full measure of the phase-space admits a continuous foliation into invariant circles. We say that the billiard table is totally integrable if the whole phase-space is foliated by continuous invariant circles.*

As described in Example 1.2.6, circles are totally integrable Birkhoff billiard tables. In 1993, M. Bialy [21] proved the following result concerning global integrability.

Theorem 1.2.9. *If the phase-space of the billiard ball map is globally foliated by continuous invariant curves that are not null-homotopic, then it is a circular billiard.*

The proof of this result is based on the existence of a non-vanishing Jacobi field, which allows us to use inequality (1.1.9), and integrating it on the phase-space to go back to the opposite of the isoperimetric inequality, obtaining then a circle.

Although circular billiards are the only examples of totally integrable Birkhoff billiard tables, integrability remains, to this day, an open question. It is known, as shown in Example 1.2.7, that elliptic billiard tables are integrable. It is widely believed that these are the only examples.

Conjecture 1.2.10 (Birkhoff)**.** Circular and elliptic billiards are the only examples of integrable Birkhoff billiards.

The first indications of this question can be found in [31]. However, this problem is also known as the Birkhoff–Poritsky conjecture, since its first appearance as a conjecture was in a work by Poritsky [80]. Despite its long history, many interesting formulations of this problem, including perturbative and local ones, remain unanswered. For a detailed survey on the story and the state of the art of the Birkhoff conjecture, we refer to [56].

1.2.2 Periodic orbits, length spectrum and Inverse spectral problem

In order to distinguish topologically different billiard closed trajectories, we associate to each orbit its rotation number as defined for general twist maps in Definition 1.1.5. A more geometric definition can be given as follows, according to [83][Definition 3.1.2].

Definition 1.2.11. *The rotation number of a periodic billiard trajectory is the rational number*

$$\frac{m}{n} = \frac{\textit{winding number}}{\textit{number of reflections}} \in (0, \frac{1}{2}\Big],$$

where the winding number $m \geq 1$ is defined as follows. Fix the positive orientation of the boundary $\partial\Omega$ and pick any reflection point of of the trajectory on $\partial\Omega$; then follow the trajectory and measure how many times it goes around $\partial\Omega$ in the positive direction until it comes back to the starting point.

Notice that we restrict ourselves to rotation numbers in $(0, 1/2]$ since a closed trajectory of rotation number $\rho \in (0, 1/2]$ can be seen as one of rotation number $1 - \rho$, traversed in the backward direction.

Applied to convex billiards, Birkhoff's Theorem 1.1.6 shows that for every $m/n \in (0, 1/2]$ in lowest terms, there are at least two closed trajectories of rotation number m/n. One of them is an inscribed n-gon with winding number m, that maximizes the perimeter (i.e. minimizes the action functional) amongst all such n-gons; the other one is a saddle point of the action functional. As n grows, the corresponding trajectory of rotation number m/n shadows the boundary, giving a rather accurate approximation of $\partial\Omega$. It is then natural to ask if the knowledge of periodic orbits leads to the reconstruction of the geometry of the table.

Definition 1.2.12. (*i*) *The length spectrum of Ω is defined as the set*

$$\mathcal{L}(\Omega) := \mathbb{N}\{\textit{lengths of all the closed trajectories in } \Omega\} \cup \mathbb{N}\, l(\partial\Omega),$$

where $l(\partial\Omega)$ is the perimeter of the billiard table.

(*ii*) *The marked length spectrum of a strictly convex domain Ω is the map*

$$\mathcal{ML}(\Omega) : \mathbb{Q} \cap (0, \frac{1}{2}\Big] \to \mathbb{R}$$

that associates to any m/n in lowest terms the maximal length of closed trajectories having rotation number m/n.

Bikhoff's Theorem ensures that the marked length spectrum is a well-defined map, labeling the corresponding closed trajectory of maximal length by its rotation number. On the other hand, the length spectrum contains information on all closed trajectories in an "unformatted" form. By means of the marked length spectrum, it is also possible to express the Mather's β-function for Birkhoff billiards for any rational rotation number. In fact, starting from definition 1.1.15, one can see that for each $0 < m/n \leq 1/2$,

$$\beta(m/n) = -\frac{1}{n}\mathcal{ML}(\Omega)(m/n). \tag{1.2.5}$$

These objects lead to some natural questions.

What information on the geometry of the billiard domain do closed orbits carry? Do knowledge of the length spectrum or marked length spectrum allow one to reconstruct the billiard domain?

More precisely, V. Guillemin and R. Melrose in [48] asked the following question.

Question I. Let Ω_1 and Ω_2 be two strictly convex planar domains with smooth boundaries and assume that they have the same marked length spectrum, i.e., $\mathcal{ML}(\Omega_1) = \mathcal{ML}(\Omega_2)$. Is it true that Ω_1 and Ω_2 are isometric?

This question can be rephrased by means of Mather's β-function and of relation (1.2.5), as done by A. Sorrentino in [84].

Question I (bis). Let Ω_1 and Ω_2 be two strictly convex planar domains with smooth boundaries and assume that they have the same Mather's β-function, i.e., $\beta_{\Omega_1} = \beta_{\Omega_2}$. Is it true that Ω_1 and Ω_2 are isometric?

The problem of reconstructing the billiard shape from the length spectrum can be rephrased as:

Can one hear the shape of a billiard?

Moreover, this question turns out to be tightly related to the classical spectral problem *can one hear the shape of a drum?*, as formulated by Kac [53]. In fact, there exists a remarkable relation between the length spectrum of a Birkhoff billiard in a strictly convex compact domain Ω, and the spectrum of the Laplace operator in Ω with Dirichlet boundary condition:

$$\begin{cases} \Delta f + \lambda^2 f = 0 & \text{in } \Omega \\ f|_{\partial\Omega} = 0. \end{cases}$$

Let $\mathrm{Spec}_\Delta(\Omega) = \{0 < \lambda_1 \leq \lambda_2 \leq \dots\}$ be the Laplace spectrum of eigenvalues solving the problem. The famous question of M. Kac, in its original version, asks whether it is possible to recover the domain from the Laplace spectrum. Denoting by

$$w(t) := \mathrm{Re}\Bigg(\sum_{\lambda_n \in \mathrm{Spec}_\Delta(\Omega)} e^{i\lambda_n t} \Bigg)$$

the wave trace of the Laplace operator, Anderson and Melrose in [4] proved that

$$\text{sing.supp.}(w(t)) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}, \tag{1.2.6}$$

and generically, equality holds.
Given a class $\mathcal{M}$ of domains and a domain $\Omega \in \mathcal{M}$, we say that is *spectrally determined* in $\mathcal{M}$ if it is the unique element (modulo isometries) of $\mathcal{M}$ with its Laplace Spectrum: if $\Omega, \Omega' \in \mathcal{M}$ are isospectral, i.e., $\mathrm{Spec}_\Delta(\Omega) = \mathrm{Spec}_\Delta(\Omega')$, then Ω' is the image of Ω by an isometry (i.e., a composition of translations and rotations). The question of Kac can be thus formulated as follows, assuming that we have fixed a class of domains $\mathcal{M}$:

Inverse spectral problem. Is every $\Omega \in \mathcal{M}$ spectrally determined?

From relation (1.2.6), it is consequently natural to pose the same questions as above in this dynamical setting. We say that a domain Ω is *dynamically spectrally determined* in the class $\mathcal{M}$, if it is the unique element (modulo isometries) of $\mathcal{M}$ with its length spectrum.

Inverse dynamical spectral problem. Is every $\Omega \in \mathcal{M}$ dynamically spectrally determined?

The length spectral rigidity problem has been recently studied by J. De Simoi, V. Kaloshin, and Q. Wei in [41]. We refer to [58] for a detailed discussion of the inverse spectral problem.

1.3 Outer billiards

Another type of billiard dynamics was introduced by J. Moser in [77] as a toy model for planetary motion. He named this model *outer* (or dual) billiard. Given a strictly convex planar domain Ω, the outer billiard map is defined on the exterior of $\partial\Omega$ as follows. A point A is mapped to $F(A)$ iff the segment joining A and $F(A)$ is tangent to $\partial\Omega$ exactly at its middle point and has positive orientation at the tangent point. We refer to Figure 1.5.

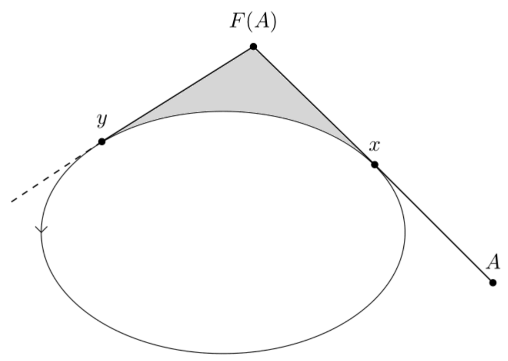


Figure 1.5: The outer billiard map reflection.

The natural phase-space for the outer billiard map is the cylinder $\mathbb{S} \times (0, +\infty)$. We notice that F is continuous and can be continuously extended to $\mathbb{S} \times [0, +\infty)$ by fixing $\mathbb{S} \times \{0\}$. We set $\rho = 1/2$ at $+\infty$ so that the twist interval is $(0, 1/2)$. Moreover –we refer e.g. to [28]– also such an F satisfies the twist condition, with the area of the curvilinear triangle of vertices $x, F(A)$ and y (see Figure 1.5 again) as a generating function. In view of all these facts, the marked area spectrum for the outer billiard map is defined as follows.

Definition 1.3.1. *The marked area spectrum for the outer billiard is the map*

$$\mathcal{MA}_o(\Omega) : \mathbb{Q} \cap (0, \frac{1}{2}) \to \mathbb{R}$$

that associates to any m/n in lowest terms the minimal area of the periodic trajectories having rotation number m/n.

We notice that periodic outer billiard minimal trajectories (with winding number = 1) correspond to convex polygons realizing the minimal (circumscribed) area, the so-called best approximating circumscribed polygons. Analogously to the previous section, we denote by $\mathcal{P}_n^c$ the set of circumscribed convex polygons with at most n vertices and

$$\delta(\Omega, \mathcal{P}_n^c) := inf\{\delta(\Omega, P_n) : P_n \in \mathcal{P}_n^c\} \tag{1.3.1}$$

where $\delta(\Omega, P_n) := Area(\Omega \bigtriangleup P_n)$ is the area of the symmetric difference of Ω and P_n, as in (1.4.5). Consequently:

$$\beta\left(\frac{1}{n}\right) = \frac{1}{n}\left(\mathcal{MA}_o(\Omega)\left(\frac{1}{n}\right) - Area(\Omega)\right) = \frac{1}{n}\delta(\Omega, \mathcal{P}_n^c). \tag{1.3.2}$$

For a detailed description of the outer billiard dynamics, see for example [88].
Regarding integrability, M. Bialy has recently established the following remarkable global result for this class of billiards [24].

Theorem 1.3.2. *If the outer billiard of $\partial\Omega$ is totally integrable, i.e., the phase-space is foliated by continuous rotational (i.e., non-contractable in the phase-space) invariant curves, then $\partial\Omega$ is an ellipse.*

1.4 Symplectic billiards

Symplectic billiards were introduced by P. Albers and S. Tabachnikov in 2018, in [1]. In the previous sections, we introduced two well-studied classes of mathematical billiards: Birkhoff billiards and outer billiards. Looking at their variational formulations, we can see that in the first case, periodic orbits correspond to inscribed polygons of extremal perimeter length; in the second case, periodic trajectories are realised by circumscribed polygons of extremal area.
It was then natural for the authors to consider two other planar billiards: inner with area and outer with length. Inner planar billiards with area as generating function are the so-called *symplectic billiards*, and their variational formulation is the following: if the points $x \in \partial\Omega$ and $z \in \partial\Omega$ are fixed, the position of the point $y \in \partial\Omega$ on the billiard curve is determined by the condition that the area of the triangle xyz is extremal. In this section, we present in detail the definition and the main properties of this interesting class of mathematical billiards, following the work of Albers and Tabachnikov [1].

Let Ω be a strictly-convex planar domain with smooth boundary $\partial\Omega$ and fixed orientation, and assume the origin of the plane to be inside Ω. Moreover, throughout the thesis, we suppose $\partial\Omega$ to have everywhere positive curvature, $k > 0$. Since Ω is strictly-convex, for every point $x \in \partial\Omega$ there exists a unique point x^* such that

$$T_x\partial\Omega = T_{x^*}\partial\Omega,$$

and we name x^* the *opposite* of x.

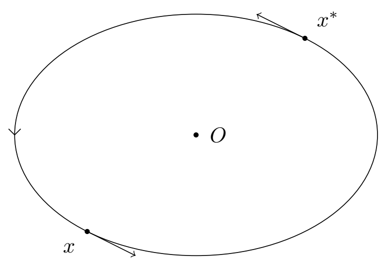


Figure 1.6: Opposite points.

We refer to

$$\hat{\mathcal{P}} := \{(x, y) \in \partial\Omega \times \partial\Omega : \; x < y < x^*\}$$

as the (open, positive) phase-space, where by $x < y < x^*$ we mean that, following the orientation of the boundary and starting from x, y is before x^*.

Consider ω to be the standard area form on $\mathbb{R}^2$, so given two vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ attached at a point $x \in \mathbb{R}^2$, $\omega(x; v, w)$ is the oriented area of the parallelogram spanned by v and w,

$$\omega(x; v, w) = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = v_1 w_2 - v_2 w_1.$$

In what follows, if the point x is the origin of $\mathbb{R}^2$, we denote the area form by $\omega(v, w)$. Notice also that, by direct computations, the following equality holds for every $\tau, v, w \in \mathbb{R}^2$,

$$\omega(\tau; v, w) = \omega(v, w) + \omega(v - w, \tau). \tag{1.4.1}$$

By mean of the area form, denoting by ν the outer normal of $\partial\Omega$, the phase-space $\hat{\mathcal{P}}$ can be read also as

$$\hat{\mathcal{P}} = \{(x,y) \in \partial\Omega \times \partial\Omega : \omega(\nu_x, \nu_y) > 0\}.$$

Lemma 1.4.1. *Given $(x,y) \in \hat{\mathcal{P}}$ there exists a unique point $z \in \partial\Omega$ such that*

$$z - x \in T_y\partial\Omega.$$

Furthermore, the new pair (y,z) lies again in the phase-space $\hat{\mathcal{P}}$.

Proof. Since $(x,y) \in \hat{\mathcal{P}}$, it is clear that $T_x\partial\Omega$ intersects $T_y\partial\Omega$ transversely, and by the strict convexity assumption we get

$$(x + T_y\partial\Omega) \cap \partial\Omega = \{x, z\}.$$

Moreover, $z \neq x$ since otherwise $T_y\partial\Omega = T_z\partial\Omega = T_x\partial\Omega$ that contradicts the hypothesis $x < y < x^*$. In other words, the line through x parallel to $T_y\partial\Omega$ intersects in a new point z.
To show that $(y,z) \in \hat{\mathcal{P}}$, we observe that if y is close to x, then so is z, and therefore $\omega(\nu_x, \nu_y) > 0$ implies $\omega(\nu_y, \nu_z) > 0$. Assume now that $x < y < x^*$ and $\omega(\nu_y, \nu_z) \leq 0$. By continuity and moving y close to x, we can arrange $\omega(\nu_y, \nu_z) = 0$. But this immediately implies that $T_y\partial\Omega = T_z\partial\Omega$, which implies $y = x$, and contradicts the hypothesis. □

The symplectic billiard map is defined as follows (see [1][Page 5]):

$$\hat{T} : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad (x,y) \mapsto (y,z)$$

where z is the unique point satisfying

$$z - x \in T_y\partial\Omega,$$

and Lemma 1.4.1 ensures that the map is well-defined.

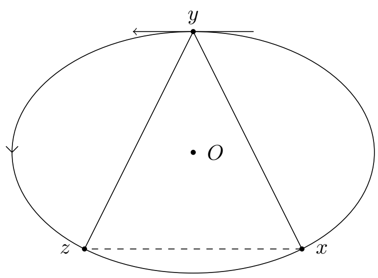


Figure 1.7: The symplectic billiard dynamics.

Remark 1.4.2. The same dynamics can be defined on the negative part of the phase-space $\{(x,y) \in \partial\Omega \times \partial\Omega : x^* < y < x\}$ simply by reversing the orientation.

The symplectic billiard map can be extended to the clousure of $\hat{\mathcal{P}}$ by continuity imposing

$$\hat{T}(x,x) = (x,x) \qquad \text{and} \qquad \hat{T}(x,x^*) = (x^*,x)$$

for every $x \in \partial\Omega$.

Lemma 1.4.3. *The continuous extension $\hat{T}(x,x^*) = (x^*,x)$ is characterized by the 2-periodicity, in other words $\hat{T}(x,y) = (y,x)$ is equivalent to $y \in \{x, x^*\}$.*

Proof. The continuous extension precisely gives one direction, so we just need to check the other one. Assuming $\hat{T}(x,y) = (y,x)$, if $(x,y) \in \hat{\mathcal{P}}$, then by Lemma 1.4.1,

$$(x + T_y\partial\Omega) \cap \partial\Omega = \{x, y\}$$

with $x \neq y$, but this contradicts the fact that $T_y\partial\Omega \cap \partial\Omega = \{y\}$. Therefore, either $(x,y) = (x,x)$ or $(x,y) = (x,x^*)$. □

Let us now check that the generating function for the symplectic billiard map is actually an area.

Lemma 1.4.4. *The function $L : \hat{\mathcal{P}} \to \mathbb{R}$, $(x,y) \mapsto L(x,y) := \omega(x,y)$ is a generating function for $\hat{T}$, that is*

$$\hat{T}(x,y) = (y,z) \iff \frac{d}{dy}\Big[L(x,y) + L(y,z)\Big] = 0.$$

Proof.

$$\begin{aligned}\frac{d}{dy}\Big[L(x,y) + L(y,z)\Big] &= \omega(x,v) + \omega(v,z) \qquad \forall v \in T_y\partial\Omega \\ &= \omega(x-z,v) = 0 \iff x - z \in T_y\partial\Omega.\end{aligned}$$

□

Remark 1.4.5. The area of the triangle xyz in Figure 1.7 equals

$$\frac{1}{2}[L(x,y) + L(y,z) + L(x,z)]$$

which differs from $L(x,y) + L(y,z)$ by a function of x and z, having no effect on the partial derivative with respect to y.

Let us now verify that, up to a good choice of the coordinates, the symplectic billiard map is monotone twist.

Let us suppose $\partial\Omega$ to have length $l(\partial\Omega) = 2\pi$, and let $\gamma(t)$ be a parametrization of the curve $\partial\Omega$, with $0 \leq t \leq 2\pi$. We denote by t^* the parameter giving $\gamma(t^*)$ the opposite of $\gamma(t)$. Through such parametrization and with an abuse of notation, we indicate phase-space

$$\hat{\mathcal{P}} = \{(t_1,t_2) \in \mathbb{S} \times \mathbb{S} : \gamma(t_1) < \gamma(t_2) < \gamma(t_1^*)\}$$

and the symplectic billiard map writes

$$\hat{T} : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad (t_1,t_2) \mapsto (t_2,t_3) \iff \gamma(t_3) - \gamma(t_1) \,||\, \gamma'(t_2).$$

Defining $L(t_1,t_2) := \omega(\gamma(t_1),\gamma(t_2))$ we get $\hat{T}(t_1,t_2) = (t_2,t_3)$ if and only if

$$L_2(t_1,t_2) + L_1(t_2,t_3) = 0. \tag{1.4.2}$$

Let us now introduce the new variables

$$s_1 := -L_1(t_1,t_2), \qquad s_2 := L_2(t_1,t_2) \tag{1.4.3}$$

and the 2-form on $\mathbb{S} \times \mathbb{S}$

$$\sigma := L_{12}(t_1,t_2)dt_1 \wedge dt_2. \tag{1.4.4}$$

We name

$$\mathcal{P} := \{(t_1,s_1) : \, s_1 = -L_1(t_1,t_2), (t_1,t_2) \in \hat{\mathcal{P}}\},$$

and notice that on $\mathcal{P}$, $\sigma = dt_1 \wedge ds_1$.

Lemma 1.4.6. *The variables (t,s) are coordinates on the phase-space $\hat{\mathcal{P}}$, and σ is an area form therein. The symplectic billiard map on $\mathcal{P}$, $T:\mathcal{P}\to\mathcal{P}$ is a monotone twist map, and the area form is T-invariant: $T^*\sigma=\sigma$.*

Proof. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we get

$$-\frac{\partial s_1}{\partial t_2}(t_1,t_2)=L_{12}(t_1,t_2)=\omega(\gamma(t_1),\gamma(t_2))=\omega(-i\gamma'(t_1),-i\gamma'(t_2))>0,$$

since $-i\gamma'(t)$ is the outward normal vector to $\partial\Omega$ at point $\gamma(t)$ and, since $(t_1,t_2)\in\hat{\mathcal{P}}$,

$$\omega(\nu_{\gamma(t_1)},\nu_{\gamma(t_2)})>0.$$

It follows that σ is an area form on $\hat{\mathcal{P}}$. It also follows that the Jacobian of the map

$$(t_1,t_2)\mapsto(t_1,s_1)$$

does not vanish, therefore (t_1,s_1) are coordinates. The twist condition $\partial t_2/\partial s_1<0$ follows as well: indeed, $\partial s_1/\partial t_2=-L_{12}(t_1,t_2)<0$. Finally, by taking the exterior derivative of (1.4.2) and wedge multiply by dt_2, we get that the 2-form $L_{12}(t_1,t_2)dt_1\wedge dt_2$ is invariant. □

The symplectic billiard map is therefore a monotone twist map. Thus, a variety of results about monotone twist maps apply to symplectic billiard. In particular, Theorem 1.1.6 gives that for every period $n\geq 3$ and any winding number $1\leq k\leq[n/2]$, the symplectic billiard has at least two distinct n-periodic orbits with winding number k. The twist interval is $(0,1/2)$.

Remark 1.4.7. We finally stress that analogously to what happens for isometries applied to the Birkhoff billiard map (Remark 1.2.3), the symplectic billiard map is invariant up to *affine transformations* of the plane. In fact, this class of transformations does not affect the tangent directions nor the parallelism condition.

1.4.1 Caustics and integrability for symplectic billiards

Let us now present an example of symplectic billiard dynamics.

Example 1.4.8. As in the Birkhoff billiards case, the easiest example of symplectic billiard dynamics is given by circles. Let Ω be a circle and let $\hat{T}:\hat{\mathcal{P}}\to\hat{\mathcal{P}}$ be symplectic billiard map inside Ω,

$$\hat{T}(x,y)=(y,z)\quad\Longleftrightarrow\quad x-z\in T_y\partial\Omega.$$

By a simple proof of synthetic geometry, one can easily notice that this occurs if and only if the angle of incidence is equal to the angle of reflection, see Figure 1.8.

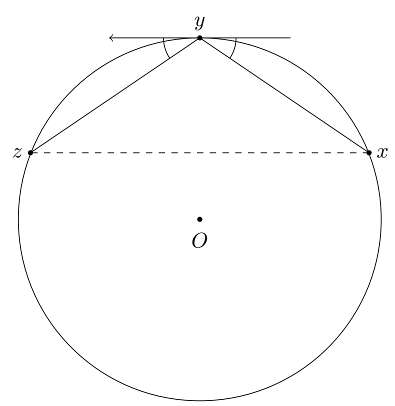


Figure 1.8: The symplectic billiard map in the circles.

This means that in circles

$$\text{Birkhoff billiard dynamics} = \text{Symplectic billiard dynamics}$$

and, in particular, allows us to conclude that *circular* symplectic billiard tables are *totally integrable.*

Moreover, since it is always possible to map a circle into an ellipse through an affine transformation, the invariance of the symplectic billiard map up to affinities (see Remark 1.4.7), directly implies that also *elliptic* symplectic billiard tables are *totally integrable.*

The previous example leads to the following natural question.

Question II. Are ellipses the only totally integrable symplectic billiard tables?

In 2024, L. Baracco and O. Bernardi [13] gave a positive answer to this question and proved the following result concerning global integrability for symplectic billiards.

Theorem 1.4.9. *The only totally integrable symplectic billiards are ellipses*

The proof, as Bialy did in the Birkhoff billiards case, is based on the integral inequality 1.1.9. After adjusting the phase-space, making use of a non-standard generating function, and through a unitary affine transformation the integral inequality leads to the opposite of the standard isoperimetric. In particular, the initial domain is forced to be an ellipse.

Unlike the Birkhoff billiards case, we still don't have any example of *integrable* but not totally integrable symplectic billiards dynamics. In particular, the following question remains open.

Question II (bis). Are there examples of integrable symplectic billiard tables which are not totally integrable? What could be the analog of the Birkhoff conjecture for symplectic billiards?

Analogously to what was done in Section 1.2.1 for Birkhoff billiards, *caustics* are defined for symplectic billiards as the continuous curves inside Ω with the property that each trajectory that is tangent to such a curve stays tangent to it after each iteration of the symplectic billiard map. As pointed out for Birkhoff billiards, there exists a relation between caustics and invariant curves. In particular, the absence of caustics implies the non-integrability of the symplectic billiard map inside Ω.

Theorem 1.4.10. *Let $\partial\Omega$ be a smooth closed convex curve whose curvature vanishes at some point. Then the symplectic billiard in Ω has no caustics.*

Proof. By Mather's analytic condition 1.1.11, a necessary condition for the existence of a caustic is $L_{22}(t_1,t_2) + L_{11}(t_2,t_3) < 0$, since $L_{12} > 0$. Let t be the arclength parameter and $\gamma(t)$ the arclength parametrization of $\partial\Omega$. Since $L(t_1,t_2) = \omega(\gamma(t_1),\gamma(t_2))$, we have

$$L_{22}(t_1,t_2) = \omega(\gamma(t_1),\gamma''(t_2)) = k(t_2)\omega(\gamma(t_1), i\gamma'(t_2)),$$

and likewise for $L_{11}(t_2,t_3)$, where k is the curvature of γ. If $k(t_2) = 0$, then

$$L_{22}(t_1,t_2) + L_{11}(t_2,t_3) = 0,$$

violating Mather's criterion. $\square$

This result generalizes to symplectic billiards the analog theorem for Birkhoff billiards by J. N. Mather, see [74]. In particular, when discussing integrability, we will then focus on *strongly* convex domains, i.e., those with non-vanishing curvature.

Restricting to strongly convex domains with infinitely smooth boundaries, the existence of caustics is provided by KAM theory. For usual Birkhoff billiards, this result is given by Lazutkin's Theorem, see [63].

Theorem 1.4.11. *Let Ω be a strongly convex domain and let γ be an infinitely smooth parametrization of $\partial\Omega$. Arbitrarily close to the curve γ, there exist smooth caustics for the symplectic billiard map; the union of these caustics has positive measure.*

For the proof of the result we refer to [1][Section 2.3.2].

1.4.2 Periodic orbits and area spectrum

As in Definition 1.2.12 for Birkhoff billiards, it is possible to construct analogous objects for the symplectic billiard map. Since the generating function is an area, we will speak about *area spectrum* and *marked area spectrum.*

Let $\{x_j\}_{j=0}^q$ be a periodic trajectory for the symplectic billiard map, that is,

$$\hat{T}(x_{j-1}, x_j) = (x_j, x_{j+1}) \quad \forall j = 1, \dots q-1, \qquad x_0 = x_q.$$

Its *action* is defined as

$$\sum_{j=0}^{q-1} \omega(x_j, x_{j+1})$$

and in particular, if the periodic trajectory winds once around $\partial\Omega$, $\frac{1}{2}\sum_{j=0}^{q-1}\omega(x_j, x_{j+1})$ is the area of the convex polygon inscribed in $\partial\Omega$ with vertices $\{x_j\}_{j=0}^q$.

Definition 1.4.12. *(i) The area spectrum for the symplectic billiard in Ω is the set of positive real numbers*

$$\mathcal{A}(\Omega) = \mathbb{N}\{\textit{action of all closed trajectories of } \hat{T}\,\} \cup \mathbb{N}\{A_\Omega\},$$

where A_Ω is the area of Ω.

(ii) The marked area spectrum for the symplectic billiard is the map

$$\mathcal{MA}_s(\Omega) : \mathbb{Q} \cap (0, \frac{1}{2}) \to \mathbb{R}$$

that associates to any m/n in lowest terms the maximal area of the periodic trajectories having rotation number m/n.

Clearly, periodic symplectic billiard maximal trajectories (with winding number $= 1$) correspond to convex polygons realizing the maximal (inscribed) area. We call them best approximating polygons inscribed in Ω. More precisely, let $\mathcal{P}_n^i$ the set of all convex polygons with at most n vertices that are inscribed in Ω. We define

$$\delta(\Omega, \mathcal{P}_n^i) := inf\{\delta(\Omega, P_n) : P_n \in \mathcal{P}_n^i\} \tag{1.4.5}$$

where $\delta(\Omega, P_n) := Area(\Omega \triangle P_n)$ is the area of the symmetric difference of sets Ω and P_n. Then:

$$\beta\left(\frac{1}{n}\right) = -\frac{2}{n}\mathcal{MA}_s(\Omega)\left(\frac{1}{n}\right) = -\frac{2}{n}\left(Area(\Omega) - \delta(\Omega, \mathcal{P}_n^i)\right). \tag{1.4.6}$$

We underline that the sign minus in the above equality comes from the use of the generating function $-\omega(x, y)$; in fact –according to Definition 1.1.15– the Mather's β-function is defined by using minimal, instead of maximal, trajectories.
The theory of interpolating Hamiltonians, [72], applied to symplectic billiards, provides an asymptotic expansion for $\mathcal{MA}_s(\Omega)(1/n)$ in negative even powers of n:

$$\mathcal{MA}_s(\Omega)(1/n) \sim a_0 + \frac{a_1}{n^2} + \frac{a_2}{n^4} + \frac{a_6}{n^6} + \ldots \qquad \text{as } n \to \infty.$$

The coefficient a_0 in this case is the area of the table Ω and $a_1 = -\lambda^3/12$, where λ indicates the total affine length of the curve $\partial\Omega$ (see Section 3.2 for details on affine parameterization). In affine geometry the isoperimetric equality for strictly convex closed curves is

$$\lambda^3 \leq 8\pi^2 A_\Omega$$

with equality if and only if $\partial\Omega$ is an ellipse. This immediately leads to the following consequence.

Theorem 1.4.13. *The first two coefficients, a_0 and a_1, make it possible to recognize an ellipse: one always has the inequality*

$$-3a_1 \leq 2\pi^2 a_0$$

with equality if and only if $\partial\Omega$ is an ellipse.

Remark 1.4.14. The statement of Theorem 1.4.13 can be rephrased as *one can hear the shape of an ellipse.* This leads to an interesting open question: can one interpret the sequence $a_0, a_1, a_2, \ldots$ as a true spectrum? Is there a differential operator whose spectrum is this sequence? In the Birkhoff billiards case, the relation with the Laplace operator is well-known.

We conclude this section by recalling that in [76] D.E. McClure and R.A. Vitale study circumscribed (resp. inscribed) polygons for which $D(\Omega, P_n)$ is minimal, where $D(\Omega, P_n)$ denotes different "distances" between Ω and P_n, including –besides the area of the symmetric difference– the Hausdorff distance and the difference of the perimeters of Ω and P_n. In such a paper, see Theorems 1 and 5, they prove that $inf\{D(\Omega, P_n) : P_n \in \mathcal{P}_n^c\} = O(1/n^2)$ (resp. $inf\{D(\Omega, P_n) : P_n \in \mathcal{P}_n^i\} = O(1/n^2)$) when $n \to +\infty$ for all considered measures D. Moreover, see [76][Theorems 2,3,6 and 7] sequences of polygons are explicitly constructed giving asymptotically efficient approximations of Ω.

Chapter 2

Bialy–Mironov type rigidity for centrally symmetric symplectic billiards

ABSTRACT. The aim of this chapter is to present the paper "Bialy–Mironov type rigidity for centrally symmetric symplectic billiards", joint work with L. Baracco and O. Bernardi, [16], which establishes a Bialy–Mironov type rigidity for centrally symmetric symplectic billiards. For a centrally symmetric C^2 strongly-convex domain Ω with boundary $\partial\Omega$, assume that the symplectic billiard map has a (simple) continuous invariant curve $\delta \subset \mathcal{P}$ of rotation number $1/4$ (winding once around $\partial\Omega$) and consisting only of 4-periodic orbits. If one of the parts between δ and each boundary of the phase-space is entirely foliated by continuous invariant closed (not null-homotopic) curves, then $\partial\Omega$ is an ellipse. The differences with Birkhoff billiards are essentially two: it is possible to assume the existence of the foliation in one of the parts of the phase-space detected by the curve δ, and the result is obtained by tracing back the problem directly to the totally integrable case.

2.1 Introduction

Crucial questions for any billiard dynamics comprehend integrability, which means the existence of a regular (i.e. at least C^0) foliation of the phase-space consisting of invariant, not null-homotopic curves. There are different ways to define integrability, including local and/or perturbative notions. We refer to [55] for an exhaustive discussion in the case of Birkhoff billiards. In particular, a billiard is called *totally –or full globally– integrable* if the foliation fill the whole phase-space. The first celebrated result that *totally integrable Birkhoff billiards are circles* was proved by M. Bialy [21] in 1993. It is based on a generalization of Hopf's argument, constructing a (discrete) Jacobi field without conjugated points, and uses the usual planar isoperimetric inequality. We refer to [92] for an alternative subsequent proof of this theorem based on genuine dynamical arguments. Finally, we notice that an additional quicker proof can also be performed by using different recent coordinates introduced in [26, Section 3.1], see also [22, Formula (14)]. In 2023, M. Bialy [24] establishes –with the same integral-geometric approach– the corresponding rigidity phenomenon for outer billiards: *if an outer billiard is totally integrable, then the boundary of the billiard table is an ellipse.* It is worth noting that, in such a case, a specific generating function as well as the use of suitable weights –in order to overcome the non-compactness of the phase-space– play a fundamental role. Finally, the result is reduced to the Blaschke-Santalo inequality. In the same year, the first two authors of the paper we aim to present, proved [13] that *the only totally integrable symplectic billiards are ellipses* by using –beyond the well-consolidated integral approach coming from Hopf's method– the affine equivariance of the symplectic billiard map in order to take back the problem to the isoperimetric inequality. We finally refer to [17] for other recent advances on symplectic (and outer) billiards.

It is consequently quite natural trying to apply the previous successful framework to search for other rigidity results, by relaxing the totally integrability assumption. In such a direction, a fundamental contribution is due to M. Bialy and A.E. Mironov [27] in 2022, proving the so-called Birkhoff-Poritsky conjecture for centrally symmetric C^2 strongly-convex (i.e. with positive curvature) domains Ω. Their main theorem can be stated as follows. *Assume that the Birkhoff billiard map of $\partial\Omega$ has a (simple) continuous invariant curve of rotation number* $1/4$ *(winding once around the phase-space cylinder) and consisting only of* 4*-periodic orbits. Moreover, suppose that the domain between this invariant curve and the boundary of the phase-space cylinder corresponding to the identity is entirely foliated by continuous invariant closed (not null-homotopic) curves. Then $\partial\Omega$ is an ellipse.* As clearly explained by the authors, the main ingredients of the proof are the use of the non-standard generating function for convex Birkhoff billiards introduced in [26], the accurate study of the properties of the above 4-periodic orbits and –in order to conclude– the Hopf's approach combined with Wirtinger's inequality. We also recall the contemporary paper [29] by M. Bialy and D. Tsodikovich where billiard tables with rotational symmetries of order greater than 3 (therefore strengthening the centrally symmetric hypothesis) are considered. In this article, the geometric assumptions allow the authors to weaken the total integrability to a region of the phase-space (in particular, to a neighborhood of a specific invariant curve formed by periodic points whose rotation number is linked to the order of symmetry of the billiard table). The results, essentially based on the remarkable structure of the invariant curves as well as the billiard tables, apply to Birkhoff, outer, symplectic and Minkowski billiards.

The aim of this chapter is to establish a Bialy-Mironov type rigidity for centrally symmetric symplectic billiards, as stated in the next theorem. In the sequel, we indicate $T : \mathcal{P} \to \mathcal{P}$ the symplectic billiard map and we refer to the beginning of Section 1.4 for details on the precise definition of the phase-space $\mathcal{P}$.

Theorem 2.1.1. *Let Ω be a centrally symmetric C^2 strongly-convex domain with boundary $\partial\Omega$. Assume that the symplectic billiard map $T : \mathcal{P} \to \mathcal{P}$ of $\partial\Omega$ has a (simple) continuous invariant curve $\delta \subset \mathcal{P}$ of rotation number* $1/4$ *(winding once around $\partial\Omega$) and consisting only of* 4*-periodic orbits. If one of the parts between δ and each boundary of the phase-space $\mathcal{P}$ is entirely foliated by continuous invariant closed (not null-homotopic) curves, then $\partial\Omega$ is an ellipse.*

We stress that, differently from Birkhoff case, the symmetric properties of the generating function for symplectic billiards in centrally symmetric tables allow to assume the existence of the foliation of invariant curves in one of the parts of the phase-space detected by the curve δ. Moreover, another difference from the "classical case" is that the proof of the above theorem is based on some technical lemmas (similar to Lemmas of [13, Section 4]) which allow us to trace back the problem to the totally integrable case studied in [13]. On the contrary, the Birkhoff case needs ad hoc arguments as explained right above. As a consequence, the techniques used to prove Theorem 2.1.1 don't fit if we consider smaller regions, e.g. by replacing δ with an invariant curve of rotation number $1/2^n$ with $n \geq 3$ or if we investigate on other billiard dynamics. We refer for example to the outer billiard case, which is particular both for the non-compactness of the phase-space and for the expression of the generating function.

Straightforward consequences of Theorem 2.1.1 are the following corollaries (corresponding respectively to Corollaries 1.3 and 1.2 in [27]).

Corollary 2.1.2. *If the symplectic billiard map $T : \mathcal{P} \to \mathcal{P}$ has a C^1 first integral with non vanishing gradient on one of the closed parts between δ and each boundary of the phase-space $\mathcal{P}$, then $\partial\Omega$ is an ellipse.*

In fact, a C^1 first integral with non vanishing gradient in a region of the phase-space induces a foliation by continuous invariant closed (not null-homotopic) curves. Since a boundary of this region is, by hypothesis, the curve δ, Theorem 2.1.1 immediately applies.

Corollary 2.1.3. *Suppose that one of the next two hypotheses holds.*

1. *A neighborhood of $\partial\Omega$ is C^1-foliated by convex caustics of rotation numbers* $(0, 1/4]$*;*

2. *A neighborhood of the center of symmetry of Ω is C^1-foliated by convex caustics of rotation numbers* $[1/4, 1/2)$.

Then $\partial\Omega$ is an ellipse.

The above corollary follows from these facts. By a standard argument, a foliation of differentiable convex caustics of given rotation numbers corresponds to a foliation of continuous invariant closed (not null-homotopic) curves with the same rotation numbers (see e.g. [82, Pages 44-45]). In order to conclude the existence of the curve δ, consisting only of periodic orbits, we need the foliation to be C^1. In fact, in such a case, the area form preserved by T induces an absolute continuous invariant measure on each leaf of the foliation. The invariance of such a measure assures that every invariant curve of rational rotation number is made up only of periodic orbits (non periodic orbits of rational rotation number should be asymptotic to periodic ones, [60, Propositions 11.1.4 and 11.2.2]). In particular, hypothesis 1. or 2. of Corollary 2.1.3 implies the ones of Theorem 2.1.1.
Remarkably, Theorem 2.1.1 can also be interpreted as a rigidity result in terms of the Mather β-function. In the context of rigidity, an important question is the following. Consider two strictly convex planar domains $\Omega_1, \Omega_2 \subset \mathbb{R}^2$, and assume their Mather β-functions coincide, i. e. $\beta_1 = \beta_2$. What can we say about the domains? Are they equal, up to unitary affine transformations of the plane? If Ω_1 is an ellipse, then Theorem 2.1.1 provides the following positive answer.

Theorem 2.1.4. *Let Ω_1 and Ω_2 be two strictly convex centrally symmetric planar domains, with C^2-smooth boundaries, and assume Ω_1 to be an ellipse. Suppose that the Mather β-functions satisfy*

$$\beta_1(\rho) = \beta_2(\rho), \quad \forall \rho \in \Big(0, \frac{1}{4}\Big].$$

Then Ω_2 is an ellipse, corresponding to Ω_1 up to a unitary affine transformation.

The analogue for Birkhoff billiards was proven by M. Bialy in [23, Theorem 2.3].
This result is important in view of its application to area-spectral rigidity, which will be investigated in Chapter 4, and in Chapter 3 through the study of the infinitesimal behaviour of the Mather β-function at 0.

The sequel is organised as follows. In Section 2.2, we discuss the consequences on the geometry of $\partial\Omega$ given by the existence of the curve δ and we recall the integral-geometric inequality to the base of the proof of Theorem 2.1.1. Section 2.3 –corresponding to [13, Section 4]– is devoted to technical facts on integrals involving the area form. Finally, Section 2.4 concludes the proof of Theorem 2.1.1.

2.2 Invariant curve of 4-periodic orbits

From now on, Ω is a centrally symmetric C^2 strongly-convex (i.e. with curvature > 0) domain and we assume the center of symmetry of Ω to be the origin. In such a case, beyond general properties recalled in Section 1.4, we also have

$$\omega(\gamma(t_1+\pi),\gamma(t_2)) = -\omega(\gamma(t_1),\gamma(t_2)) \quad \text{and} \quad \omega(\gamma(t_1),\gamma(t_2+\pi)) = -\omega(\gamma(t_1),\gamma(t_2)).$$

The fact that –in the centrally symmetric case– the generating function changes sign by adding π (half period) to one of the variables is a special property of symplectic billiards. In fact, in other billiards (e.g. Birkhoff, outer, fourth billiards, see [2] for details), the corresponding generating function depends also on the shape of the boundary between two points.

The next proposition is essentially contained in [1, Section 2.4.2.].

Proposition 2.2.1. *Let Ω be a centrally symmetric billiard table. Assume that the symplectic billiard map $T : \mathcal{P} \to \mathcal{P}$ of $\partial\Omega$ has a (simple) continuous invariant curve $\delta \subset \mathcal{P}$ of rotation number $1/4$ (winding once around $\partial\Omega$) and consisting only of 4-periodic orbits. Then each quadrilateral corresponding to the invariant curve δ is a parallelogram. In particular, $\partial\Omega$ is a Radon curve.*

Proof. Let consider a 4-periodic orbit in δ, $\{\gamma(t_i)\}_{i=1}^4$. As a consequence of the symplectic billiard dynamics, we have

$$\omega(\gamma'(t_1),\gamma'(t_3)) = 0 \quad \text{and} \quad \omega(\gamma'(t_2),\gamma'(t_4)) = 0.$$

By the centrally symmetric hypothesis, it follows

$$\gamma(t_3) = \gamma(t_1+\pi) \quad \text{and} \quad \gamma(t_4) = \gamma(t_2+\pi).$$

This means that the (inscribed) quadrilateral with vertices in $\{\gamma(t_i)\}_{i=1}^4$ is a parallelogram whose diagonals intersect in the origin (center of symmetry). Notice that the area of this parallelogram is half the action along $\{\gamma(t_i)\}_{i=1}^4$, which is constant for the invariant curve δ. As a consequence, every 4-periodic orbit for the symplectic billiard dynamics corresponds to an inscribed parallelogram having maximal area passing through a given vertex. In particular, $\partial\Omega$ is a Radon curve (according to [70, Beginning of Section 3]). $\square$

The simplest example of Radon curve is the ellipse. However, there are recent constructions of Radon curves based on methods both from Plane Minkowski Geometry [71] and from Convex Geometry [11]. We observe that the relation between the existence of an invariant curve of 4-periodic orbits and the Radon property was first explicitly mentioned for outer billiards in [25]. Moreover, the same paper –see [25, Example 4.11]– contains an analytic construction of an infinite family of Radon curves.

Let $\phi : \mathbb{S} \to \mathbb{S}$ be the function induced by δ and Φ the lift of ϕ to $\mathbb{R}$. Then, as a consequence of Proposition 2.2.1:

$$\Phi^2(t) = t+\pi, \quad \Phi^3(t) = \Phi(t)+\pi \quad \text{and} \quad \Phi^4(t) = t+2\pi. \tag{2.2.1}$$

In particular, $\Phi(t+\pi) = \Phi^3(t) = \Phi(t)+\pi$. Moreover, let $\Omega_{\gamma\delta}$ be the region in $[0,2\pi]^2$ corresponding to the part of the phase-space between γ and the curve δ:

$$\Omega_{\gamma\delta} := \{(t,s) : \ t \in [0,2\pi] \text{ and } t \le s \le \Phi(t)\}.$$

Similarly,

$$\Omega_{\delta\gamma^*} := \{(t,s) : \ t \in [0,2\pi] \text{ and } \Phi(t) \le s \le t+\pi\}.$$

From now on, $L(t_1, t_2) := \omega(\gamma(t_1), \gamma(t_2))$ denotes the generating function for Φ and L_{ij} (for $i, j = 1, 2$) are the usual partial derivatives:

$$L_{11}(t_1, t_2) = \omega(\gamma''(t_1), \gamma(t_2)), \quad L_{22}(t_1, t_2) = \omega(\gamma(t_1), \gamma''(t_2)) \quad \text{and} \quad L_{12}(t_1, t_2) = \omega(\gamma'(t_1), \gamma'(t_2)).$$

Therefore,

$$|L_{11}|, |L_{22}|, L_{12} \leq K$$

where K is a constant depending on the C^2-norm of γ.
The core of the next proposition is a nowadays well-consolidated argument by Bialy, based on the construction of a (discrete) Jacobi field without conjugated points. In the inequality's proof, the twist condition as well as the invariance of Ω with respect to T play a crucial role. We refer to [21, Section 3] for a detailed proof, also sketched in [13, Section 3].

Proposition 2.2.2. *Let Ω be a C^2 strongly-convex billiard table. Suppose that the symplectic billiard map $T : \mathcal{P} \to \mathcal{P}$ of $\partial\Omega$ has two (simple) continuous invariant closed (not null-homotopic) curves α and β, $\alpha < \beta$. Let $\Omega_{\alpha\beta}$ be the region in $[0, 2\pi]^2$ corresponding to the part of the phase-space between α and β, which we assume to be foliated by continuous invariant closed (not null-homotopic) curves. Then the following inequality holds:*

$$\iint_{\Omega_{\alpha\beta}} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)]\, d\sigma \leq 0. \tag{2.2.2}$$

The next section is devoted to study (2.2.2), in the special case where Ω is a centrally symmetric billiard table and $\Omega_{\alpha\beta}$ is $\Omega_{\gamma\delta}$ or $\Omega_{\delta\gamma^*}$, that is (by (1.4.4)):

$$\int_{\Omega_{\gamma\delta}} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)] L_{12}(t_1, t_2)\, dt_1 dt_2 \leq 0 \tag{2.2.3}$$

and

$$\int_{\Omega_{\delta\gamma^*}} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)] L_{12}(t_1, t_2)\, dt_1 dt_2 \leq 0. \tag{2.2.4}$$

2.3 Three technical lemmas

In this section, we prove three properties of integrals (2.2.3) and (2.2.4), which will be useful in the sequel.

Lemma 2.3.1. *For $L(t_1, t_2) = \omega(\gamma(t_1), \gamma(t_2))$ the next equalities hold:*

$$\begin{aligned}
&\iint_{\Omega_{\gamma\delta}} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)] L_{12}(t_1, t_2)\, dt_1 dt_2 \\
&= \iint_{\Omega_{\delta\gamma^*}} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)] L_{12}(t_1, t_2)\, dt_1 dt_2 \\
&= \iint_{[0,\pi]^2} [L_{11}(t_1, t_2) + 2L_{12}(t_1, t_2) + L_{22}(t_1, t_2)] L_{12}(t_1, t_2)\, dt_1 dt_2.
\end{aligned}$$

Proof. From (2.2.1), we conclude that (i) the graph of Φ in $[\Phi(0), \pi]$ is precisely the inverse of the graph of Φ in $[0, \Phi(0)]$, composed with the π-translation in the (positive) vertical direction; (ii) the graph of Φ in $[\pi, 2\pi]$ is the π-translation in the (positive) vertical direction of the graph of Φ in $[0, \pi]$. Now, as in the proof of [13, Lemma 4.1], let

$$\overline{\Omega}_{\gamma\delta} = \{(t_2, t_1) \in [0, 2\pi]^2 : (t_1, t_2) \in \Omega_{\gamma\delta}\}$$

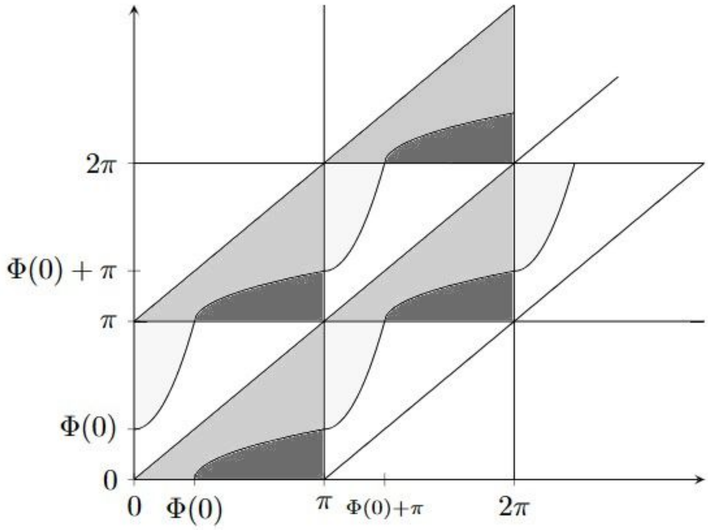


Figure 2.1: The same colored regions have equal area.

so that –referring to Figure 2.1– the same colored regions have equal area. Let

$$F(t_1,t_2) := [L_{11}(t_1,t_2) + 2L_{12}(t_1,t_2) + L_{22}(t_1,t_2)]L_{12}(t_1,t_2).$$

Since $L_{11}(t_1,t_2) = -L_{22}(t_2,t_1)$ and $L_{12}(t_1,t_2) = -L_{12}(t_2,t_1)$, we have that

$$\iint_{\Omega_{\gamma\delta}} F(t_1,t_2)\,dt_1dt_2 = \iint_{\Omega_{\gamma\delta}} F(t_2,t_1)\,dt_1dt_2 = \iint_{\overline{\Omega}_{\gamma\delta}} F(t_1,t_2)\,dt_1dt_2.$$

As a consequence, and by Property 5. recalled in Section 2.2, we get

$$2\iint_{\Omega_{\gamma\delta}} F(t_1,t_2)\,dt_1dt_2 = \iint_{\Omega_{\gamma\delta}} F(t_1,t_2)\,dt_1dt_2 + \iint_{\overline{\Omega}_{\gamma\delta}} F(t_1,t_2)\,dt_1dt_2 = 2\iint_{[0,\pi]^2} F(t_1,t_2)\,dt_1dt_2$$

and the statement immediately follows. □

Lemma 2.3.2. *For $L(t_1,t_2) = \omega(\gamma(t_1),\gamma(t_2))$ the next equality holds:*

$$\iint_{[0,\pi]^2} [L_{11}(t_1,t_2) + L_{22}(t_1,t_2)]L_{12}(t_1,t_2)dt_1dt_2 = -A(\Omega)\int_0^{\pi} \omega(\gamma''(t),\gamma'(t))dt,$$

where $A(\Omega)$ denotes the area of the billiard table Ω.

Proof. The proof is similar to the one of [13, Lemma 4.2]; however, in order to integrate in $[0,\pi]^2$ instead of in $[0,2\pi]^2$, the centrally symmetric hypothesis plays a crucial role.
Integrating by parts, we have

$$\iint_{[0,\pi]^2} L_{11}(t_1,t_2)L_{12}(t_1,t_2)\,dt_1dt_2 = \iint_{[0,\pi]^2} \omega(\gamma''(t_1),\gamma(t_2))\omega(\gamma'(t_1),\gamma'(t_2))\,dt_1dt_2$$
$$= \iint_{[0,\pi]^2} \left(\gamma_1''(t_1)\gamma_2(t_2) - \gamma_2''(t_1)\gamma_1(t_2)\right)\left(\gamma_1'(t_1)\gamma_2'(t_2) - \gamma_2'(t_1)\gamma_1'(t_2)\right)\,dt_1dt_2,$$

where γ_1 and γ_2 are the components of the curve γ. The terms

$$\iint_{[0,\pi]^2} \gamma_1''(t_1)\gamma_2(t_2)\gamma_2'(t_1)\gamma_1'(t_2)\,dt_1dt_2 = \iint_{[0,\pi]^2} \gamma_2''(t_1)\gamma_1(t_2)\gamma_1'(t_1)\gamma_2'(t_2)\,dt_1dt_2 = 0$$

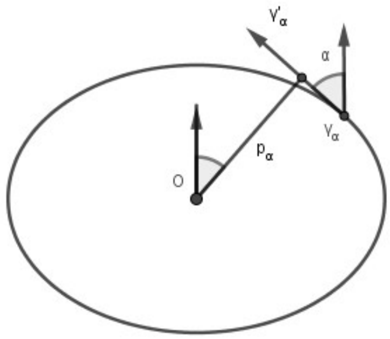


Figure 2.2: The angle α and the support function $p(\alpha)$.

since, by the centrally symmetric hypothesis, $\gamma'(0) = -\gamma'(\pi)$. As a consequence, we get

$$\iint_{[0,\pi]^2} L_{11}(t_1,t_2)L_{12}(t_1,t_2)\,dt_1dt_2 = \iint_{[0,\pi]^2} \gamma_1''(t_1)\gamma_2'(t_1)\gamma_2'(t_2)\gamma_1(t_2) - \gamma_2''(t_1)\gamma_1'(t_1)\gamma_2'(t_2)\gamma_1(t_2)\,dt_1dt_2$$

$$= \int_0^\pi \omega(\gamma''(t),\gamma'(t))\,dt \int_0^\pi \gamma_1(t)\gamma_2'(t)\,dt = -\frac{A(\Omega)}{2}\int_0^\pi \omega(\gamma''(t),\gamma'(t))\,dt.$$

Since the same argument holds for $\iint_{[0,\pi]^2} L_{22}(t_1,t_2)L_{12}(t_1,t_2)\,dt_1dt_2$, we conclude the proof. $\square$

We now parametrize the boundary by the direction of its tangent line and denote by $e_\alpha = (-\sin\alpha, \cos\alpha)$ the unitary vector forming an angle $\alpha \in [0,2\pi]$ with respect to the vertical direction $(0,1)$. Notice that for every α there exists a unique point $\gamma(\alpha)$ such that $\gamma'(\alpha) = \|\gamma'(\alpha)\|e_\alpha$. Let $p : [0,2\pi] \to \mathbb{R}^+$ be the corresponding support function, that is the distance from the origin of the tangent line to γ at $\gamma(\alpha)$. We refer to Figure 2.2. It is easy to see that

$$\gamma(\alpha) = p'(\alpha)e_\alpha - p(\alpha)Je_\alpha$$

where J is the rotation of $\frac{\pi}{2}$ in the positive verse. Consequently,

$$\gamma'(\alpha) = (p''(\alpha) + p(\alpha))e_\alpha \qquad \text{and} \qquad \gamma''(\alpha) = (p'''(\alpha) + p'(\alpha))e_\alpha + (p''(\alpha) + p(\alpha))Je_\alpha.$$

As already stressed in [13, Section 4], as a consequence of the strongly-convexity assumption of $\partial\Omega$, it holds that

$$(p''(\alpha) + p(\alpha)) > 0.$$

Therefore, the twist condition is also satisfied for $L(\alpha_1,\alpha_2) = \omega(\gamma(\alpha_1),\gamma(\alpha_2))$:

$$L_{12}(\alpha_1,\alpha_2) = (p''(\alpha_1) + p(\alpha_1))(p''(\alpha_2) + p(\alpha_2))\sin(\alpha_2 - \alpha_1) > 0$$

on $\mathcal{P} = \{(\alpha_1,\alpha_2) : \ 0 < \alpha_2 - \alpha_1 < \pi\}$.
From the centrally symmetric assumption, it immediately follows that the functions p, p' and p'' are π-periodic. Moreover, with such a parametrization, the result of Lemma 2.3.2 reads:

$$\iint_{[0,\pi]^2} [L_{11}(\alpha_1,\alpha_2) + L_{22}(\alpha_1,\alpha_2)]L_{12}(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2 = -A(\Omega)\int_0^\pi (p''(\alpha) + p(\alpha))^2\,d\alpha. \quad (2.3.1)$$

Lemma 2.3.3. *For $L(\alpha_1,\alpha_2)=\omega(\gamma(\alpha_1),\gamma(\alpha_2))$ the next equality holds:*

$$\begin{aligned}&2\iint_{[0,\pi]^2} L_{12}^2(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2\\ &=\left(\int_0^\pi (p''(\alpha)+p(\alpha))^2\,d\alpha\right)^2-\left(\int_0^\pi (p''(\alpha)+p(\alpha))^2\cos(2\alpha)\,d\alpha\right)^2-\left(\int_0^\pi (p''(\alpha)+p(\alpha))^2\sin(2\alpha)\,d\alpha\right)^2.\end{aligned} \tag{2.3.2}$$

Proof. Same proof of [13, Lemma 4.3]. We first observe that

$$L_{12}^2(\alpha_1,\alpha_2)=(p''(\alpha_1)+p(\alpha_1))^2(p''(\alpha_2)+p(\alpha_2))^2\sin^2(\alpha_1-\alpha_2).$$

Since

$$\sin^2(\alpha_1-\alpha_2)=\frac{1}{2}\left(1-\cos(2\alpha_1)\cos(2\alpha_2)-\sin(2\alpha_1)\sin(2\alpha_2)\right),$$

we take the double integral and conclude. □

2.4 Proof of Theorem 2.1.1

Under the hypotheses of Theorem 2.1.1, and by Proposition 2.2.2, one of the next integral inequalities necessarily holds:

$$\iint_{\Omega_{\gamma\delta}} [L_{11}(\alpha_1,\alpha_2)+2L_{12}(\alpha_1,\alpha_2)+L_{22}(\alpha_1,\alpha_2)]L_{12}(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2\le 0$$

or

$$\iint_{\Omega_{\delta\gamma^*}} [L_{11}(\alpha_1,\alpha_2)+2L_{12}(\alpha_1,\alpha_2)+L_{22}(\alpha_1,\alpha_2)]L_{12}(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2\le 0.$$

From Lemma 2.3.1, equalities (2.3.1) and (2.3.2) and the π-periodicity of p and p'', the above inequalities read

$$\begin{aligned}&4\iint_{\Omega_{\gamma\delta}} [L_{11}(\alpha_1,\alpha_2)+2L_{12}(\alpha_1,\alpha_2)+L_{22}(\alpha_1,\alpha_2)]L_{12}(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2\\ &=4\iint_{\Omega_{\delta\gamma^*}} [L_{11}(\alpha_1,\alpha_2)+2L_{12}(\alpha_1,\alpha_2)+L_{22}(\alpha_1,\alpha_2)]L_{12}(\alpha_1,\alpha_2)\,d\alpha_1 d\alpha_2\\ &=-2A(\Omega)\int_0^{2\pi}(p''(\alpha)+p(\alpha))^2\,d\alpha+\left(\int_0^{2\pi}(p''(\alpha)+p(\alpha))^2\,d\alpha\right)^2\\ &-\left(\int_0^{2\pi}(p''(\alpha)+p(\alpha))^2\cos(2\alpha)\,d\alpha\right)^2-\left(\int_0^{2\pi}(p''(\alpha)+p(\alpha))^2\sin(2\alpha)\,d\alpha\right)^2\le 0.\end{aligned}$$

This is the same inequality of [13, Beginning of Section 5]. We sketch the final argument for reader's convenience, see [13, Section 5] for all details.
First, by the affine equivariance of the symplectic billiard map, we can apply an affine transformation $\varphi_{a,\sigma}$ defined as the composition of the unitary, diagonal, linear transformation

$$\varphi_a(x,y)=\left(ax,\frac{y}{a}\right)$$

with $a>0$, after the rotation of angle $\sigma\in[0,2\pi]$. Let $\Omega_{a,\sigma}:=\varphi_{a,\sigma}(\Omega)$ and denote by $p_{a,\sigma}(\psi)$ its corresponding support function. By [13, Proposition 5.1], there exists a couple $(a,\sigma)\in\mathbb{R}_+\times[0,\frac{\pi}{2}]$ such that

$$\int_0^{2\pi} p_{a,\sigma}(\psi)\cos(2\psi)\,d\psi=\int_0^{2\pi} p_{a,\sigma}(\psi)\sin(2\psi)\,d\psi=0.$$

Secondly, we use the usual arc length as parameter to describe $\Omega_{a,\sigma}$, see [13, Remark 5.3]. With these specific choices of (i) affine transformation and (ii) parametrization, the above necessary integral inequality simply reads:

$$\left(\int_0^{2\pi} (p''_{a,\sigma} + p_{a,\sigma}) d\alpha\right)^2 \leq 4\pi A(\Omega_{a,\sigma}) \Longleftrightarrow (l(\partial\Omega_{a,\sigma}))^2 \leq 4\pi A(\Omega_{a,\sigma}),$$

where $l(\partial\Omega_{a,\sigma})$ is the perimeter length of $\partial\Omega_{a,\sigma}$. By the planar isoperimetric inequality, this means that the domain $\Omega_{a,\sigma}$ is necessarily a circle and –from the very construction of $\varphi_{a,\sigma}$– that the initial domain Ω is an ellipse. $\square$

Chapter 3

Higher order terms of Mather's β-function for symplectic and outer billiards

ABSTRACT. This chapter aims to present the article "Higher order terms of Mather's β-function for symplectic and outer billiards", joint work with L. Baracco and O. Bernardi, [17]. In that paper, we compute explicitly the formal Taylor expansion of Mather's β-function up to seventh-order terms for symplectic and outer billiards in a strictly-convex planar domain Ω. In particular, we specify the third terms of the asymptotic expansions of the distance (in the sense of the symmetric difference metric) between Ω and its best approximating inscribed or circumscribed polygons with at most n vertices. We use tools from affine differential geometry.

3.1 Introduction and main result

S. Marvizi and R. Melrose's theory, first stated and proved for Birkhoff billiards [72][Theorem 3.2], applies both to symplectic and outer billiards (see [1][Section 2.5] and [88][Section 8] respectively). It assures that the corresponding dynamics equals to the time-one flow of a Hamiltonian vector field composed with a smooth map fixing pointwise the boundary of the phase-space at all orders. We refer to [47][Section 2.1] for a detailed proof in the general case of (strongly) billiard-like maps. As an outcome, this result gives the following expansion at $\rho = 0$ of the corresponding minimal average function

$$\beta(\rho) \sim \beta_1\rho + \beta_3\rho^3 + \beta_5\rho^5 + \beta_7\rho^7 + \dots$$

in terms of odd powers of ρ. It is well-known –see e.g. [72][Section 7] again– that for usual billiards the sequence $\{\beta_k\}$ can be interpreted as a spectrum of a differential operator, see also Remark 2.11 in [1]. The question is open for symplectic and outer billiards. Coefficients $\beta_1, \dots, \beta_5$ are known. We refer e.g to [66] for detailed computations: Theorem 1 for the symplectic case and Theorem 2 for the outer one. Moreover, we suggest [2] for a recent discussion on this topic. The main result of this chapter, stated in the next theorem, is to provide coefficients β_7 both in the symplectic and outer case.

Theorem 3.1.1. *Let Ω be a strictly-convex planar domain with smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ has everywhere positive curvature. Denote by $k(s)$ the affine curvature of $\partial\Omega$ with affine parameter s. Let λ be the affine length of the boundary.*

(a) *The formal Taylor expansion at $\rho = 0$ of Mather's β-function for the symplectic billiard map has coefficients:*

$$\begin{aligned}
\beta_{2k} &= 0 \text{ for all } k\\
\beta_1 &= -2Area(\Omega)\\
\beta_3 &= \frac{\lambda^3}{6}\\
\beta_5 &= -\frac{\lambda^4}{5!}\int_0^\lambda k(s)ds\\
\beta_7 &= -\frac{9\lambda^6}{5\cdot 7!}\int_0^\lambda k^2(s)ds + \frac{\lambda^5}{15\cdot 5!}\left(\int_0^\lambda k(s)ds\right)^2.
\end{aligned}$$

(b) *The formal Taylor expansion at $\rho = 0$ of Mather's β-function for the outer billiard map has coefficients:*

$$\begin{aligned}
\beta_{2k} &= 0 \text{ for all } k\\
\beta_1 &= 0\\
\beta_3 &= \frac{\lambda^3}{24}\\
\beta_5 &= \frac{\lambda^4}{2\cdot 5!}\int_0^\lambda k(s)ds\\
\beta_7 &= \frac{421\lambda^6}{5\cdot 8!}\int_0^\lambda k^2(s)ds - \frac{\lambda^5}{5\cdot 5!}\left(\int_0^\lambda k(s)ds\right)^2.
\end{aligned}$$

As a straightforward consequence, in Corollary 3.5.2 we point out that the two coefficients β_5 and β_7 allow to recognize an ellipse among all strictly-convex planar domains, both for symplectic and outer billiards.

There are several –highly non trivial– questions about the structure of the formal expansion of coefficients β_{2h+1} as $h \to +\infty$ (we use only here the index h in order to avoid confusions with the affine curvature k). In the case of Birkhoff billiards, these problems have been discussed in different papers; we refer to [72][Theorem 5.19], [68][Conjecture 1] and [90][Conjecture 1.2]. Moreover, we observe that –according to the fact that the generating function is the length– from the explicit coefficients given in [84][Theorem 1.3], it arises that the derivative's degree of the standard curvature involved in the expression of the β_{2h+1}'s increases as $h \to +\infty$. In the symplectic and outer cases, our computations suggest that for $h \geq 4$ the terms β_{2h+1} may depend on the affine curvature through $\int_0^\lambda k^{h-1}(s)ds$ and $\left(\int_0^\lambda k(s)ds\right)^{h-1}$ even if the technique to obtain the β_7's should be refined for higher orders.

3.2 Preliminaries of affine differential geometry

Since the dynamics of symplectic and outer billiards –defined in Sections 1.4 and 1.3 respectively– are invariant with respect to affine transformations, we use the affine arc length to parametrize Ω.

We recall that the affine arc length and the affine length are given respectively by

$$s(t) = \int_0^t \kappa^{\frac{1}{3}}(\tau)d\tau \qquad 0 \leq t \leq l$$

and

$$\lambda = \int_0^l \kappa^{\frac{1}{3}}(\tau)d\tau$$

where t and l are the ordinary arc length parameter and ordinary length respectively and $\kappa(t)$ is the curvature of Ω. In the following, let $x(s)$ be an affine arc length parametrization of $\partial\Omega$. We denote by $x^{(i)}$ the i-th derivative of x and we omit the dependence on s. Then we have (see [85][Section 3]):

$$\omega(x^{(1)}, x^{(2)}) = 1, \qquad \omega(x^{(1)}, x^{(3)}) = 0 \tag{3.2.1}$$

and

$$k(s) := \omega(x^{(2)}, x^{(3)}) \tag{3.2.2}$$

is called the affine curvature of Ω.

The next lemma, which is a straightforward consequence of formulae (3.2.1) and (3.2.2), will be very useful through the chapter.

Lemma 3.2.1. *The following relations hold:*

$$\omega(x^{(1)}, x^{(4)}) = -k, \quad \omega(x^{(2)}, x^{(4)}) = k', \quad \omega(x^{(1)}, x^{(5)}) = -2k' \tag{3.2.3}$$

and

$$\omega(x^{(3)}, x^{(4)}) = k^2, \quad \omega(x^{(2)}, x^{(5)}) = k'' - k^2 \quad \omega(x^{(1)}, x^{(6)}) = -3k'' + k^2. \tag{3.2.4}$$

Proof. By deriving the second equality in (3.2.1) and taking into account (3.2.2) we have that $\omega(x^{(1)}, x^{(4)}) = -k$. Similarly, from (3.2.2) we get $\omega(x^{(2)}, x^{(4)}) = k'$. Moreover, by deriving the first identity of (3.2.3) we obtain the third one. In order to obtain the relations in (3.2.4) it is enough to recall that $x^{(3)} = -kx^{(1)}$ and to derive the identities in (3.2.3). □

3.3 Asymptotic expansion for $\delta(\Omega, \mathcal{P}_n^i)$

We gather in this section all the technical results in order to prove point (a) of Theorem 3.1.1.

We begin with a refinement of Lemma 1 in [66].

Proposition 3.3.1. *For $0 \le r \le s \le \lambda$, let $F(r,s)$ be the area of the region between the arc $\{x(t) : \ r \le t \le s\}$, and the line segment with end points $x(r)$ and $x(s)$. Then*

$$F(r,s) = \frac{1}{2}\left(\frac{(s-r)^3}{3!} - \frac{(s-r)^5}{5!}k(r) - \frac{3(s-r)^6}{6!}k'(r) - \frac{(s-r)^7}{7\cdot 5!}k''(r) + \frac{(s-r)^7}{7!}k^2(r) + o((s-r)^7)\right)$$

uniformly for all $0 \le r \le s \le \lambda$ as $(s-r) \to 0$.

Proof. Without loss of generality, we assume $r = 0$. The area of the region of Ω bounded by the segment $[x(0), x(s)]$ is given by

$$F(s) := F(0,s) = \frac{1}{2}\int_0^s \omega(x(t) - x(0), x^{(1)}(t))dt. \tag{3.3.1}$$

We consider the Taylor expansion of the function inside the integral and we have

$$\omega(x(t) - x(0), x^{(1)}(t)) = \sum_{l=2}^{m}\left(\sum_{1\le j<\frac{l+1}{2}} \frac{(h-j)\omega(x^{(j)}(0), x^{(h)}(0))}{j!h!}\right)t^l + o(t^m). \tag{3.3.2}$$

By Lemma 3.2.1 and formula (3.3.2) we get

$$\omega\left(x(t) - x(0), x'(t)\right) = \frac{t^2}{2} - \frac{k(0)}{4!}t^4 - \frac{3k'(0)}{5!}t^5 + \left(\frac{k^2(0)}{6!} - \frac{k''(0)}{5!}\right)t^6 + o(t^6)$$

and the statement follows after integration on $0 \le t \le s$. □

The remaining of this section is devoted to characterize the affine lengths of the n arcs in which $\partial\Omega$ is divided by the vertices of a best approximating polygon P_n inscribed in Ω.

Proposition 3.3.2. *For $n \geq 3$ let $P_n \in \mathcal{P}_n^i$ be a best approximating polygon inscribed in Ω. Let $x(s_{n,i})$, $i = 1, \dots, n$, be the ordered vertices of P_n and*

$$\lambda_{n,i} := s_{n,i} - s_{n,i-1}.$$

Then

$$\lambda_{n,i+1} = \lambda_{n,i} + \frac{1}{30} k'(s_{n,i}) \lambda_{n,i}^4 + \epsilon_1(s_{n,i}, \lambda_{n,i}) \tag{3.3.3}$$

where $\lim_{n\to\infty} \frac{\epsilon_1(s_{n,i},\lambda_{n,i})}{\lambda_{n,i}^5} = 0$ uniformly in i.

Proof. Since P_n is a best approximating polygon, its vertices satisfy:

$$\omega\left(x(s_{n,i-1}) - x(s_{n,i+1}), x^{(1)}(s_{n,i})\right) = 0 \quad \text{for } i = 1, ..., n \tag{3.3.4}$$

To study (3.3.4), we assume $s_{n,i} = 0$ and consider the equivalent equation:

$$x(s) - x(t) + hx^{(1)}(0) = 0 \text{ for some } h \in \mathbb{R}.$$

Excluding the trivial solution $t \equiv s$, we obtain (up to rename h):

$$\frac{x(s) - x(t)}{s - t} + hx^{(1)}(0) = 0. \tag{3.3.5}$$

Let us define

$$G(s,t) := \frac{x(s) - x(t)}{s - t}$$

and extend it, smoothly, to $s = t$ by setting $G(s, s) = x^{(1)}(s)$. Thus we have

$$G(s,t) = \frac{x(s) - x(t)}{s - t} = \sum_{k=1}^{m} \frac{x^{(k)}(0)}{k!} \left(\frac{s^k - t^k}{s - t}\right) + o((s-t)^{m-1})$$

and $s = t = 0$, $h = -1$ is a solution for (3.3.5). In order to apply the Implicit Function Theorem to solve (3.3.5) for h and t in terms of s, we compute the Jacobian matrix of $G(s,t) + hx^{(1)}(0)$ in $s = t = 0$, $h = -1$:

$$J_{h,t}\big(G(s,t) + hx^{(1)}(0)\big)_{|s=t=0,h=-1} = \left(x^{(1)}(0), \frac{x^{(2)}(0)}{2}\right).$$

Since $det J_{h,t} = \frac{1}{2}$, it is possible to solve (3.3.5) and find $h(s)$ and $t(s)$. Deriving we get

$$\frac{d}{ds}\left(G(s,t(s)) + h(s)x^{(1)}(0)\right) = \partial_s G(s,t(s)) + \partial_t G(s,t(s))t'(s) + h'(s)x^{(1)}(0) = 0. \tag{3.3.6}$$

Since for $s = 0$, $t(0) = 0$, we have

$$\frac{x^{(2)}(0)}{2} + \frac{x^{(2)}(0)}{2} t'(0) + h'(0)x^{(1)}(0) = 0,$$

and therefore we get $t'(0) = -1$.
In order to obtain the higher order derivatives of t in 0 it is enough to argue in the same way that is deriving (3.3.6) again and taking into account Lemma 3.2.1. In such a way we get

$$t(s) = -s + \frac{1}{30} k'(0)s^4 + o(s^5).$$

The previous expansion written for general $s_{n,i} \neq 0$ gives immediately formula (3.3.3).

□

The next proposition is a refinement of Lemma 2 in [66].

Proposition 3.3.3. *Under the same assumption of the previous proposition, it holds*

$$\lambda_{n,i} = \frac{\lambda}{n} - \frac{\lambda^2}{30n^3}\int_0^\lambda k(s)ds + \frac{\lambda^3}{30n^3}k\left(\frac{i\lambda}{n}\right) + o\left(\frac{1}{n^3}\right) \tag{3.3.7}$$

uniformly in i *as* $n \to \infty$.

Proof. Through the whole proof, we assume that $s_{n,0} = 0$. By a standard comparison argument of difference equations, we first prove that

$$\lambda_{n,i} = \frac{\lambda}{n} + O\left(\frac{1}{n^3}\right) \qquad \text{uniformly in } i \text{ as } n \to \infty \tag{3.3.8}$$

Let D be a constant such that $D > \frac{1}{30}\max|k'|$ and consider the two difference equations

$$v_{n,i+1} = v_{n,i}\left(1 - Dv_{n,i}^3\right), \quad V_{n,i+1} = V_{n,i}\left(1 + DV_{n,i}^3\right). \tag{3.3.9}$$

In view of Lemma 2 in [66], $\lambda_{n,1} = \frac{c_n}{n}$ for some uniform constant c_n (that is $|c_n|$ uniformly bounded in n). By taking the first terms of (3.3.9) both equal to $\frac{c_n}{n}$, it clearly holds

$$v_{n,i} \leq \lambda_{n,i} \leq V_{n,i}, \qquad i = 1, ..., n.$$

Moreover, the sequence $V_{n,i}$ is increasing and $V_{n,n}$ is uniformly bounded. In fact, let $k \geq 1$ be the largest integer such that $V_{n,k} \leq \frac{2c_n}{n}$. If $k < n$, then we should have $V_{n,k+1} > \frac{2c_n}{n}$ and therefore

$$\frac{2c_n}{n} < \frac{c_n}{n}\left(1 + D\left(\frac{2c_n}{n}\right)^3\right)^k < \frac{c_n}{n}\left(1 + D\left(\frac{2c_n}{n}\right)^3\right)^n \Rightarrow 2 < \left(1 + D\left(\frac{2c_n}{n}\right)^3\right)^n$$

Since the last term tends to 1, this contradicts the fact that $k < n$.
With an analogous argument, it follows that the sequence $v_{n,i}$ is decreasing and $v_{n,n}$ is uniformly bounded.
Therefore, up to rename the constant D, we have

$$\lambda_{n,1}\left(1 - \frac{D}{n^3}\right)^{i-1} \leq \lambda_{n,i} \leq \lambda_{n,1}\left(1 + \frac{D}{n^3}\right)^{i-1}.$$

Summing for $i = 1, \ldots, n$, we get

$$\lambda_{n,1}\frac{1 - \left(1 - \frac{D}{n^3}\right)^n}{\frac{D}{n^3}} \leq \lambda \leq \lambda_{n,1}\frac{1 - \left(1 + \frac{D}{n^3}\right)^n}{-\frac{D}{n^3}} \Rightarrow \lambda_{n,1}\frac{n}{1 + \frac{D}{n^2}} \leq \lambda \leq \lambda_{n,1}\frac{n}{1 - \frac{D}{n^2}}$$

so that $\lambda_{n,1} = \frac{\lambda}{n} + O\left(\frac{1}{n^3}\right)$, which corresponds to formula (3.3.8). Such a formula immediately gives $s_{n,i} = \frac{i\lambda}{n} + O\left(\frac{1}{n^2}\right)$ and therefore –up to renaming the function ϵ_1– expansion (3.3.3) of Proposition 3.3.2 can be equivalently written as

$$u_{n,i+1} = u_{n,i} + \frac{1}{30}k'\left(\frac{i\lambda}{n}\right)\frac{u_{n,i}^4}{n^3} + \epsilon_1\left(\frac{i\lambda}{n}, \frac{u_{n,i}}{n}\right) \tag{3.3.10}$$

where $u_{n,i} = n\,\lambda_{n,i}$. Let u_n be the solution of the Cauchy problem (corresponding to (3.3.10)):

$$\begin{cases} u_n'(t) = \frac{1}{30\lambda}k'(t)\frac{u_n^4(t)}{n^2} \\ u_n\left(\frac{\lambda}{n}\right) = u_{n,1} \end{cases} \tag{3.3.11}$$

and set $u_n(i) := u_n\left(\frac{i\lambda}{n}\right)$.

The second part of the proof is devoted to establish the next estimate:

$$u_n(i) - u_{n,i} = o\left(\frac{1}{n^2}\right) \tag{3.3.12}$$

uniformly in $i = 1, \dots, n$ as $n \to \infty$.
By integrating (3.3.11) between $\frac{i\lambda}{n}$ and $\frac{(i+1)\lambda}{n}$ via separation of variables, we get

$$-\frac{1}{3}\left(\frac{1}{u_n(i+1)^3} - \frac{1}{u_n(i)^3}\right) = \frac{1}{30\lambda n^2}\left[k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{i\lambda}{n}\right)\right],$$

that is

$$(u_n(i+1))^{-3} = \frac{1}{10\lambda n^2}\left[k\left(\frac{i\lambda}{n}\right) - k\left(\frac{(i+1)\lambda}{n}\right)\right] + u_n(i)^{-3},$$

implying

$$\begin{aligned} u_n(i+1) &= u_n(i)\left(1 + \frac{1}{10\lambda n^2}\left[k\left(\frac{i\lambda}{n}\right) - k\left(\frac{(i+1)\lambda}{n}\right)\right] u_n(i)^3\right)^{-1/3} \\ &= u_n(i)\left(1 + \frac{1}{30\lambda n^2}\left[k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{i\lambda}{n}\right)\right] u_n(i)^3\right) + O\left(\frac{1}{n^4}\right) \\ &= u_n(i) + \frac{1}{30 n^3}k'\left(\frac{i\lambda}{n}\right) u_n(i)^4 + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Taking into account the previous formula and the difference equation (3.3.10), we immediately have

$$u_{n,i+1} - u_n(i+1) = u_{n,i} - u_n(i) + \frac{1}{30n^3}k'\left(\frac{i\lambda}{n}\right)(u_{n,i}^4 - u_n(i)^4) + o\left(\frac{1}{n^3}\right).$$

Equivalently, $z_{n,i} := u_{n,i} - u_n(i)$ solves

$$z_{n,i+1} = z_{n,i} + \frac{C_{n,i}}{30n^3}k'\left(\frac{i\lambda}{n}\right) z_{n,i} + o\left(\frac{1}{n^3}\right) \tag{3.3.13}$$

where $C_{n,i} > 0$ are constants uniformly bounded in i for $n \to \infty$. As in the first part of the proof, let E be a constant such that $E > \frac{C_{n,i}}{30}\max|k'|$ (for every i and n) and consider the difference equation

$$Z_{n,i+1} = Z_{n,i}\left(1 + \frac{E}{n^3}\right) + \frac{c_n}{n^3},$$

with $c_n = o(1)$ and $Z_{n,1} = 0$. Comparing the $z_{n,i+1}$'s in (3.3.13) with the terms $Z_{n,i+1}$ of the previous difference equation, for every $i = 1, \dots, n$, we get

$$z_{n,i+1} \le Z_{n,i+1} = \frac{c_n}{E}\left(\left(1 + \frac{E}{n^3}\right)^i - 1\right) \le o(1)\left(\frac{1}{1 - \frac{iE}{n^3}} - 1\right) = o(1)\frac{i}{n^3} = o\left(\frac{1}{n^2}\right),$$

which corresponds to (3.3.12).

We finally make explicit the term of order 3 in formula (3.3.7). By integrating (3.3.11) between $\frac{\lambda}{n}$ and $\frac{(i+1)\lambda}{n}$, we have

$$\begin{aligned} u_n(i+1) &= u_n(1)\left(1 + \frac{1}{10\lambda n^2}\left[k\left(\frac{\lambda}{n}\right) - k\left(\frac{(i+1)\lambda}{n}\right)\right] u_n(1)^3\right)^{-1/3} \\ &= u_n(1)\left(1 + \frac{1}{30\lambda n^2}\left[k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{\lambda}{n}\right)\right] u_n(1)^3\right) + O\left(\frac{1}{n^4}\right). \end{aligned}$$

Moreover, by formula (3.3.12), $u_{n,i+1} = u_n(i+1) + o\left(\frac{1}{n^2}\right)$ or equivalently (since $\lambda_{n,i} = \frac{u_{n,i}}{n}$)

$$\lambda_{n,i+1} = \frac{u_n(i+1)}{n} + o\left(\frac{1}{n^3}\right).$$

Plugging the previous expression of $u_n(i+1)$ (in terms of $u_n(1) = n\,\lambda_{n,1}$) into formula above, we obtain

$$\lambda_{n,i+1} = \lambda_{n,1}\left[1 + \frac{1}{30\lambda n^2}\left(k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{\lambda}{n}\right)\right)n^3\lambda_{n,1}^3\right] + o\left(\frac{1}{n^3}\right).$$

In more detail, since $\lambda_{n,1} = \frac{\lambda}{n} + \frac{e_n}{n^3} + o\left(\frac{1}{n^3}\right)$ for some uniform constant (see formula (3.3.8)), we have

$$\lambda_{n,i+1} = \frac{\lambda}{n} + \frac{\lambda^3}{30n^3}\left(k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{\lambda}{n}\right)\right) + \frac{e_n}{n^3} + o\left(\frac{1}{n^3}\right). \tag{3.3.14}$$

Summing for $i = 0, \ldots, n-1$, we conclude that

$$\lambda = \sum_{i=0}^{n-1}\lambda_{n,i+1} = \lambda + \frac{\lambda^3}{30n^3}\sum_{i=0}^{n-1}\left(k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{\lambda}{n}\right)\right) + \frac{e_n}{n^2} + o\left(\frac{1}{n^2}\right),$$

that is

$$e_n = -\frac{\lambda^3}{30n}\sum_{i=0}^{n-1}\left(k\left(\frac{(i+1)\lambda}{n}\right) - k\left(\frac{\lambda}{n}\right)\right) + o(1)$$

and the limit for $n \to \infty$ leads to

$$e_n = -\frac{\lambda^2}{30}\int_0^\lambda k(s) - k\,(0)\,ds.$$

Plugging the expression for e_n into formula (3.3.14), we finally obtain (for $n \to \infty$):

$$\lambda_{n,i+1} = \frac{\lambda}{n} - \frac{\lambda^2}{30n^3}\int_0^\lambda k(s)ds + \frac{\lambda^3}{30n^3}k\left(\frac{(i+1)\lambda}{n}\right) + o\left(\frac{1}{n^3}\right),$$

which equals to (3.3.7). $\square$

3.4 Asymptotic expansion for $\delta(\Omega, \mathcal{P}_n^c)$

In this section, analogously to the previous one, we collect the technical results in order to prove point (b) of Theorem 3.1.1.

The next result is the analogous of Proposition 3.3.1 for circumscribed polygons.

Proposition 3.4.1. *For $0 \le r \le s \le \lambda$, let $H(r,s)$ be the area of the region between the tangents in $x(r)$ and $x(s)$ and the arc $\{x(t) : r \le t \le s\}$. Then*

$$H(r,s) = \frac{(s-r)^3}{24} + \frac{(s-r)^5}{2\cdot 5!}k(r) + \frac{3(s-r)^6}{6!}k'(r) + \frac{(s-r)^7}{7\cdot 5!}k''(r) + \frac{17(s-r)^7}{8!}k^2(r) + o((s-r)^7))$$

uniformly for all $0 \le r \le s \le \lambda$ as $(s-r) \to 0$.

Proof. Without loss of generality, we assume $r = 0$. Consider the vertex P of the polygon $P_n \in \mathcal{P}_n^c$ whose edges are tangent to Ω in $x(0)$ and $x(s)$:

$$P = x(s) + t_1 x'(s) = x(0) + t_2 x'(0) \Rightarrow \omega(x'(0), x(0) - x(s)) = t_1\omega(x'(0), x'(s))$$

so we get

$$P = x(s) + x'(s)\frac{\omega(x'(0), x(0) - x(s))}{\omega(x'(0), x'(s))}. \tag{3.4.1}$$

The area of the triangle of vertices $x(0)$, $x(s)$ and P is

$$-\frac{1}{2}\omega(P - x(s), x(0) - x(s)) = \frac{1}{2}\frac{\omega(x'(s), x(s) - x(0))\omega(x'(0), x(0) - x(s))}{\omega(x'(0), x'(s))}.$$

Taking $M, N, L, J \in \mathbb{N}$ large enough in order to approximate the above quantity at order 7, we obtain

$$\frac{1}{2}\omega\left(\sum_{h_1=1}^{M}\frac{x^{(h_1)}(0)s^{h_1-1}}{(h_1-1)!} + o(s^{M-1}), \sum_{h_2=1}^{N}\frac{x^{(h_2)}(0)s^{h_2}}{h_2!} + o(s^N)\right)\cdot$$

$$\cdot\left(-\sum_{h_3=2}^{L}\frac{\omega(x'(0), x^{(h_3)}(0))s^{h_3-1}}{h_3!} + o(s^{L-1})\right)\left(\sum_{h_4=2}^{J}\frac{\omega(x'(0), x^{(h_4)}(0))s^{h_4-2}}{(h_4-1)!} + o(s^{J-2})\right)^{-1} =$$

$$= \frac{1}{2}\left(\sum_{m=1}^{M}\sum_{j+h-1=m}\frac{\omega(x^{(h)}(0), x^{(j)}(0))s^m}{j!(h-1)!} + o(s^M)\right)\left(-\sum_{h_3=2}^{L}\frac{\omega(x'(0), x^{(h_3)}(0))s^{h_3-1}}{h_3!} + o(s^{L-1})\right)\cdot$$

$$\cdot\left(\sum_{h_4=2}^{J}\frac{\omega(x'(0), x^{(h_4)}(0))s^{h_4-2}}{(h_4-1)!} + o(s^{J-2})\right)^{-1} =$$

$$= \frac{1}{2}\left(\frac{1}{2}s^2 - \frac{k(0)}{4!}s^4 - \frac{3k'(0)}{5!}s^5 - \frac{k''(0)}{5!}s^6 + \frac{k^2(0)}{6!}s^6 + o(s^6)\right)\left(\frac{1}{2}s - \frac{k(0)}{4!}s^3 - \frac{2k'(0)}{5!}s^4 + \right.$$

$$\left. - \frac{3k''(0)}{6!}s^5 + \frac{k^2(0)}{6!}s^5 + o(s^5)\right)\left(1 - \frac{k(0)}{3!}s^2 - \frac{2k'(0)}{4!}s^3 - \frac{3k''(0)}{5!}s^4 + \frac{k^2(0)}{5!}s^4 + o(s^4)\right)^{-1} =$$

$$= \frac{1}{2}\left(\frac{1}{2}s^2 - \frac{k(0)}{4!}s^4 - \frac{3k'(0)}{5!}s^5 - \frac{k''(0)}{5!}s^6 + \frac{k^2(0)}{6!}s^6 + o(s^6)\right)\left(\frac{1}{2}s - \frac{k(0)}{4!}s^3 - \frac{2k'(0)}{5!}s^4 + \right.$$

$$\left. - \frac{3k''(0)}{6!}s^5 + \frac{k^2(0)}{6!}s^5 + o(s^5)\right)\left(1 + \frac{k(0)}{3!}s^2 + \frac{2k'(0)}{4!}s^3 + \frac{3k''(0)}{5!}s^4 - \frac{k^2(0)}{5!}s^4 + \frac{k^2(0)}{36}s^4 + o(s^4)\right) =$$

$$= \frac{1}{2}\left(\frac{1}{4}s^3 + \frac{3k^2(0)}{4\cdot 6!}s^7 + o(s^7)\right).$$

By difference of areas and taking into account the result of Proposition 3.3.1 for $F(0, s)$, we get

$$H(0, s) = \frac{1}{24}s^3 + \frac{k(0)}{2\cdot 5!}s^5 + \frac{k'(0)}{4\cdot 5!}s^6 + \frac{k''(0)}{14\cdot 5!}s^7 + \frac{17k^2(0)}{8!}s^7 + o(s^7).$$

□

The next two results correspond to Propositions 3.3.2 and 3.3.3 in the case of circumscribed best approximating polygons.

Proposition 3.4.2. *For $n \geq 3$ let $P_n \in \mathcal{P}_n^c$ be a best approximating polygon circumscribed to Ω. Let $x(s_{n,i})$, $i = 1, \ldots, n$, be the ordered tangency points of the edges of P_n to Ω and*

$$\lambda_{n,i} := s_{n,i} - s_{n,i-1}.$$

Then

$$\lambda_{n,i+1} = \lambda_{n,i} - \frac{8}{5!}k'(s_{n,i})\lambda_{n,i}^4 + \epsilon_2(s_{n,i}, \lambda_{n,i}) \tag{3.4.2}$$

where $\lim_{n\to\infty}\frac{\epsilon_2(s_{n,i},\lambda_{n,i})}{\lambda_{n,i}^5} = 0$ *uniformly in i.*

Proof. If $x(s_{n,1})$, $x(s_{n,2})$ and $x(s_{n,3})$, $0 \le s_{n,i} \le \lambda$, are three successive tangent points of P_n to Ω, the corresponding vertices of P_n are

$$P = x(s_{n,2}) + x'(s_{n,2})\frac{\omega(x'(s_{n,1}), x(s_{n,1}) - x(s_{n,2}))}{\omega(x'(s_{n,1}), x'(s_{n,2}))}, \qquad Q = x(s_{n,2}) + x'(s_{n,2})\frac{\omega(x'(s_{n,3}), x(s_{n,3}) - x(s_{n,2}))}{\omega(x'(s_{n,3}), x'(s_{n,2}))}.$$

Since P_n is a best approximating circumscribed polygon, we have

$$\frac{\omega(x'(s_{n,1}), x(s_{n,1}) - x(s_{n,2}))}{\omega(x'(s_{n,1}), x'(s_{n,2}))} = -\frac{\omega(x'(s_{n,3}), x(s_{n,3}) - x(s_{n,2}))}{\omega(x'(s_{n,3}), x'(s_{n,2}))}.$$

Setting $s_{n,2} = 0$, we define the function

$$G(s) = \frac{\omega(x'(s), x(s) - x(0))}{\omega(x'(s), x'(0))}.$$

The proof can be concluded as the one of Proposition 3.3.2, by solving $G(s) = -G(t)$ and applying the Implicit Function Theorem. $\square$

Finally, an argument analogous to the proof of Proposition 3.3.3 gives the next precise characterization of the $\lambda_{n,i}$'s.

Proposition 3.4.3. *Under the same assumption of the previous proposition, it holds*

$$\lambda_{n,i} = \frac{\lambda}{n} - \frac{8}{5!}\frac{\lambda^2}{n^3}\int_0^\lambda k(s)ds + \frac{8}{5!}\frac{\lambda^3}{n^3}k\left(\frac{i\lambda}{n}\right) + o\left(\frac{1}{n^3}\right)$$

uniformly in i as $n \to \infty$.

3.5 Proof of Theorem 3.1.1

We finally state and prove the higher order terms of $\delta(\Omega, \mathcal{P}_n^i)$ and $\delta(\Omega, \mathcal{P}_n^c)$ defined in Sections 1.4 and 1.3 respectively. This theorem is a refinement of Theorem 1 and Theorem 2 in [66]. In view of equalities (1.4.6) and (1.3.2), the proof of Theorem 3.1.1 is a straightforward application of this result.

Theorem 3.5.1. *Let Ω be a strictly-convex planar domain with smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ has everywhere positive curvature. Denote by $k(s)$ the affine curvature of $\partial\Omega$ with affine parameter s. Let λ be the affine length of the boundary.*

(*a*) *The formal expansion of $\delta(\Omega, \mathcal{P}_n^i)$ at $n \to \infty$ is given by*

$$\delta(\Omega, \mathcal{P}_n^i) = a_2\frac{1}{n^2} + a_4\frac{1}{n^4}\int_0^\lambda k(s)ds + \frac{1}{n^6}\left[a_6\int_0^\lambda k^2(s)ds + b_6\left(\int_0^\lambda k(s)ds\right)^2\right] + o\left(\frac{1}{n^6}\right)$$

with coefficients

$$a_2 = \frac{1}{12}\lambda^3, \qquad a_4 = -\frac{1}{2\cdot 5!}\lambda^4, \qquad a_6 = -\frac{9}{10\cdot 7!}\lambda^6, \qquad b_6 = \frac{1}{30\cdot 5!}\lambda^5.$$

(*b*) *The formal expansion of $\delta(\Omega, \mathcal{P}_n^c)$ at $n \to \infty$ is given by*

$$\delta(\Omega, \mathcal{P}_n^c) = a_2\frac{1}{n^2} + a_4\frac{1}{n^4}\int_0^\lambda k(s)ds + \frac{1}{n^6}\left[a_6\int_0^\lambda k^2(s)ds + b_6\left(\int_0^\lambda k(s)ds\right)^2\right] + o\left(\frac{1}{n^6}\right)$$

with coefficients

$$a_2 = \frac{1}{24}\lambda^3, \qquad a_4 = \frac{1}{2\cdot 5!}\lambda^4, \qquad a_6 = \frac{421}{5\cdot 8!}\lambda^6, \qquad b_6 = -\frac{1}{5\cdot 5!}\lambda^5.$$

Proof. Theorem 1 in [66] establishes that

$$\delta(\Omega, \mathcal{P}_n^i) - \frac{1}{12}\frac{\lambda^3}{n^2} + \frac{1}{2\cdot 5!}\frac{\lambda^4}{n^4}\int_0^\lambda k(s)ds = o\left(\frac{1}{n^4}\right)$$

as $n \to \infty$. We determine

$$\lim_{n\to\infty} n^6 \left(\delta(\Omega, \mathcal{P}_n^i) - \frac{1}{12}\frac{\lambda^3}{n^2} + \frac{1}{2\cdot 5!}\frac{\lambda^4}{n^4}\int_0^\lambda k(s)ds\right). \tag{3.5.1}$$

By means of Proposition 3.3.1, we get

$$\delta(\Omega, \mathcal{P}_n^i) = \frac{1}{2}\sum_{i=1}^n \left[\frac{\lambda_{n,i}^3}{3!} - \frac{\lambda_{n,i}^5}{5!}k(s_{n,i}) - \frac{3\lambda_{n,i}^6}{6!}k'(s_{n,i}) - \frac{\lambda_{n,i}^7}{7\cdot 5!}k''(s_{n,i}) + \frac{\lambda_{n,i}^7}{7!}k^2(s_{n,i}) + o(\lambda_{n,i}^7)\right]$$

$$= \frac{1}{2}\sum_{i=1}^n \left[\frac{1}{3!}\left(\frac{\lambda}{n} + \left(\lambda_{n,i} - \frac{\lambda}{n}\right)\right)^3 - \frac{\lambda_{n,i}^5}{5!}k(s_{n,i}) - \frac{3}{6!}\frac{\lambda^6}{n^6}k'(s_{n,i}) + \right.$$

$$\left. - \frac{1}{7\cdot 5!}\frac{\lambda^7}{n^7}k''(s_{n,i}) + \frac{1}{7!}\frac{\lambda^7}{n^7}k^2(s_{n,i}) + o\left(\frac{1}{n^7}\right)\right].$$

We compute (3.5.1) by dividing the limit in four terms. We stress that $\sum_{i=1}^n \lambda_{n,i} = \lambda$, so that $\sum_{i=1}^n(\lambda_{n,i} - \lambda/n) = 0$.

We first consider

$$\frac{1}{2}\frac{1}{3!}\sum_{i=1}^n\left[\frac{\lambda}{n} - \left(\lambda_{n,i} - \frac{\lambda}{n}\right)\right]^3 - \frac{1}{12}\frac{\lambda^3}{n^2} =$$

$$= \frac{1}{12}\sum_{i=1}^n\left[\frac{\lambda^3}{n^3} - 3\frac{\lambda^2}{n^2}\left(\lambda_{n,i} - \frac{\lambda}{n}\right) + 3\frac{\lambda}{n}\left(\lambda_{n,i} - \frac{\lambda}{n}\right)^2 - \left(\lambda_{n,i} - \frac{\lambda}{n}\right)^3\right] - \frac{1}{12}\frac{\lambda^3}{n^2}.$$

Applying Proposition 3.3.3, this term is equal to

$$\frac{1}{4}\sum_{i=1}^n\left[\frac{1}{900}\frac{\lambda^5}{n^7}\left(\int_0^\lambda k(s)ds\right)^2 + \frac{1}{900}\frac{\lambda^6}{n^7}k^2\left(\frac{i\lambda}{n}\right) - \frac{2}{900}\frac{\lambda^6}{n^7}k\left(\frac{i\lambda}{n}\right)\int_0^\lambda k(s)ds + o\left(\frac{1}{n^7}\right)\right].$$

And the corresponding limit gives:

$$\lim_{n\to\infty}\frac{n^6}{4}\sum_{i=1}^n\left[\frac{1}{900}\frac{\lambda^5}{n^7}\left(\int_0^\lambda k(s)ds\right)^2 + \frac{1}{900}\frac{\lambda^6}{n^7}k^2\left(\frac{i\lambda}{n}\right) - \frac{2}{900}\frac{\lambda^6}{n^7}k\left(\frac{i\lambda}{n}\right)\int_0^\lambda k(s)ds + o\left(\frac{1}{n^7}\right)\right]$$

$$= \frac{1}{4\cdot 900}\lambda^6\int_0^\lambda k^2(s)ds - \frac{1}{4\cdot 900}\lambda^5\left(\int_0^\lambda k(s)ds\right)^2. \tag{3.5.2}$$

Second, we compute the limit:

$$\lim_{n\to\infty}\frac{n^6}{2}\sum_{i=1}^n\left[-\frac{\lambda_{n,i}^5}{5!}k(s_{n,i})\right] + \frac{1}{2\cdot 5!}\lambda^4 n^2\int_0^\lambda k(s)ds$$

$$= \lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^n\left(-k(s_{n,i})\lambda_{n,i} + \int_{s_{n,i}}^{s_{n,i+1}} k(s)ds\right)\right] + \frac{n^6}{2\cdot 5!}\sum_{i=1}^n -k(s_{n,i})\lambda_{n,i}\left(\lambda_{n,i}^4 - \frac{\lambda^4}{n^4}\right).$$

The limit of the first summand is zero, in fact:

$$\lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^{n}\left(-k(s_{n,i})\lambda_{n,i}+\int_{s_{n,i}}^{s_{n,i+1}}k(s)ds\right)\right]=\lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^{n}\int_{s_{n,i}}^{s_{n,i+1}}\left(k(s)-k(s_{n,i})\right)ds\right]$$

$$=\lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^{n}\left(\int_{s_{n,i}}^{s_{n,i+1}}(k'(s_{n,i})(s-s_{n,i})+k''(s_{n,i})\frac{(s-s_{n,i})^2}{2}+o((s-s_{n,i})^2)\right)ds\right]$$

$$=\lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^{n}k'(s_{n,i})\frac{\lambda_{n,i}^2}{2}+k''(s_{n,i})\frac{\lambda_{n,i}^3}{3!}\right]$$

where, in the last equality, we dropped $o((s-s_{n,i})^2)$ because after integration and summation it gives a negligible term. Now –again by Proposition 3.3.3– the above limit equals to

$$=\lim_{n\to\infty}\frac{1}{2\cdot 5!}\lambda^4 n^2\left[\sum_{i=1}^{n}k'(s_{n,i})\frac{\lambda_{n,i}}{2}\frac{\lambda}{n}+k''(s_{n,i})\frac{\lambda_{n,i}}{3!}\frac{\lambda^2}{n^2}\right].$$

The second term clearly converges to a constant times $\int_0^\lambda k''(s)ds$ which is 0. Also the first term converges to 0, and to see this it is sufficient to rewrite the summation as

$$\lim_{n\to\infty}\frac{1}{4\cdot 5!}\lambda^5 n\left[\sum_{i=1}^{n}\int_{s_{n,i}}^{s_{n,i+1}}(k'(s_{n,i})-k'(s))ds\right]\tag{3.5.3}$$

and then use the Taylor expansion of $k'(s)$ around $s_{n,i}$. Therefore, it remains to compute

$$\lim_{n\to\infty}\frac{n^6}{2\cdot 5!}\sum_{i=1}^{n}-k(s_{n,i})\lambda_{n,i}\left(\lambda_{n,i}^4-\frac{\lambda^4}{n^4}\right)=-\frac{1}{15\cdot 5!}\lambda^6\int_0^\lambda k^2(s)ds+\frac{1}{15\cdot 5!}\lambda^5\left(\int_0^\lambda k(s)ds\right)^2.\tag{3.5.4}$$

where, the last equality, is a straightforward application of Proposition 3.3.3. again, we have

By arguing as in (3.5.3), we obtain that the third term (of order 6) in the expansion of (3.5.1) does not give contribution:

$$\lim_{n\to\infty}\frac{n^6}{2}\sum_{i=1}^{n}-\frac{3}{6!}k'(s_{n,i})\lambda_{n,i}^6=0.$$

Finally, we compute the limit of the last term:

$$\lim_{n\to\infty}\frac{n^6}{2}\sum_{i=1}^{n}\left[-\frac{1}{7\cdot 5!}\frac{\lambda^7}{n^7}k''(s_{n,i})+\frac{1}{7!}\frac{\lambda^7}{n^7}k^2(s_{n,i})\right]=\lim_{n\to\infty}\frac{\lambda^6}{2}\sum_{i=1}^{n}\left[-\frac{1}{7\cdot 5!}\lambda_{n,i}k''(s_{n,i})+\frac{1}{7!}\lambda_{n,i}k^2(s_{n,i})\right]$$

$$=\frac{1}{2\cdot 7!}\lambda^6\int_0^\lambda k^2(s)ds.\tag{3.5.5}$$

The coefficients of point (a) of the theorem are obtained by summing up (3.5.2), (3.5.4) and (3.5.5). From (3.5.1), we immediately obtain that the term of order 5 has zero coefficient. This fact follows also from S. Marvizi and R. Melrose's theory, as recalled in Section 3.1.

Point (b) can be proved in the same way. □

As pointed out in [1][Theorem 6], in the symplectic billiard case, the first two coefficients β_1 and β_3 make it possible to distinguish an ellipse. In fact, the inequality

$$3\beta_3\leq -2\pi^2\beta_1$$

equals to the affine isoperimetric inequality

$$\lambda^3 \leq 8\pi^2 Area(\Omega),$$

which always holds for every strictly-convex closed curve $\partial\Omega$ and it is an equality only for ellipses. This is not the case of outer billiards, since $\beta_1 = 0$. However, in the next corollary we underline that coefficients β_5 and β_7 –even if not geometrically significant– allow to fix an ellipse, also in the outer billiard case.

Corollary 3.5.2. *Same assumptions of Theorem 3.1.1. The coefficients β_5 and β_7 recognize an ellipse. In particular:*

(*a*) *For symplectic billiards, one always has the inequality*

$$42\lambda^3\beta_7 \leq 5!(\beta_5)^2, \tag{3.5.6}$$

with equality if and only if $\partial\Omega$ is an ellipse.

(*b*) *For outer billiards, one always has the inequality*

$$7\lambda^3\beta_7 \geq 170(\beta_5)^2, \tag{3.5.7}$$

with equality if and only if $\partial\Omega$ is an ellipse.

Proof. By Cauchy-Schwartz inequality, it holds

$$\left(\int_0^\lambda k(s)ds\right)^2 \leq \lambda \int_0^\lambda k^2(s)ds,$$

with equality if and only if $k(s)$ is constant (that is, $\partial\Omega$ is an ellipse). As a consequence, in the symplectic billiard case, we get

$$\beta_7 \leq \left(-\frac{9}{5\cdot 7!} + \frac{1}{15\cdot 5!}\right)\lambda^5\left(\int_0^\lambda k(s)ds\right)^2 = \frac{\lambda^5}{7!}\left(\int_0^\lambda k(s)ds\right)^2 = \frac{5!}{42\lambda^3}(\beta_5)^2$$

which gives the result of point (*a*). Point (*b*) is obtained analogously. □

3.6 Ellipses and circles

In the case of circles and ellipses, all coefficients of the Mather's β-function –both for symplectic and outer billiards– can be easily obtained directly. In particular, by the affine equivariance of both maps, it is sufficient to consider the case of circular tables. In this final section, we compute these coefficients for circles (and therefore for ellipses) and check their consistency with the β_i's of Theorem 3.1.1.

3.6.1 Symplectic billiards

For a disc centered in 0 with radius R, the generating function is twice the area of the triangle $x0y$:

$$\omega(x,y) = 2Area(x0y)$$

and periodic orbits of rotation number $\frac{1}{n}$ ($n \geq 3$) correspond to inscribed regular polygons with n edges.

The generating function for an ellipse $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$ is obtained by applying the affine transformation $\begin{pmatrix} a/R & 0 \\ 0 & b/R \end{pmatrix}$, so that

$$\omega(x,y) = ab\sin\left(\frac{2\pi}{n}\right).$$

As a consequence, from Definition 1.1.15 of Mather's β-function, it follows

$$\beta\left(\frac{1}{n}\right) = ab\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\left(\frac{2\pi}{n}\right)^{2k+1}. \tag{3.6.1}$$

Since the affine length and curvature of the ellipse (and circle, for $a=b=R$) are respectively $\lambda = 2\pi(ab)^{1/3}$ and $k=(ab)^{-2/3}$, the above coefficients are consistent with the ones of point (a) of Theorem 3.1.1.

3.6.2 Outer billiards

For the circular table of boundary $x_1^2+x_2^2=R^2$, periodic orbits of rotation number $\frac{1}{n}$ ($n\geq 3$) correspond to circumscribed regular polygons with n edges and the generating function is the area:

$$R^2\left[\tan\left(\frac{\pi}{n}\right)-\frac{\pi}{n}\right]$$

of the grey region in Figure 3.1. Therefore, the generating function for an ellipse $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$ results:

$$\omega(x,y) = ab\left[\tan\left(\frac{\pi}{n}\right)-\frac{\pi}{n}\right]$$

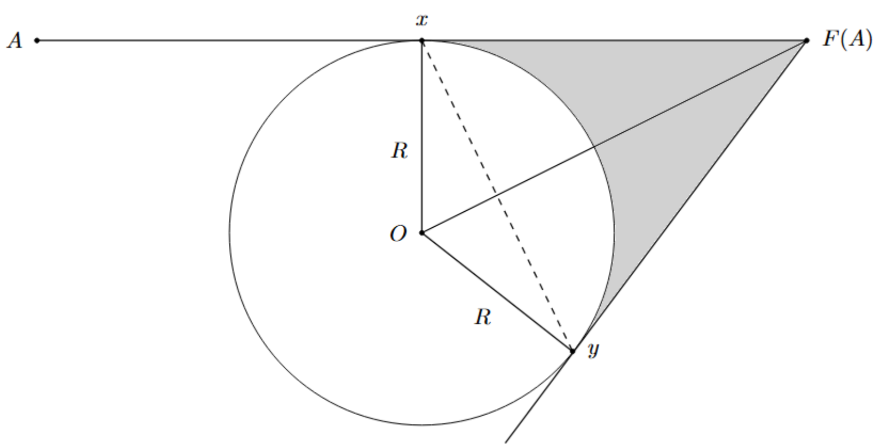


Figure 3.1: The outer billiard in circles.

and the corresponding Mather's β-function is

$$ab\sum_{k=2}^{\infty}\frac{(-1)^{k-1}4^k(4^k-1)B_{2k}}{(2k)!}\left(\frac{\pi}{n}\right)^{2k-1}$$

(the B_{2k}'s are the Bernoulli numbers), whose coefficients are consistent with the ones of point (b) of Theorem 3.1.1.

Chapter 4

Area spectral rigidity for axially symmetric and Radon domains

ABSTRACT. This chapter is devoted to presenting the preprint “Area spectral rigidity for axially symmetric and Radon domains”, joint work with L. Baracco and O. Bernardi, [18]. In this work, we prove that (a) any finitely smooth axially symmetric strictly convex domain, with everywhere positive curvature and sufficiently close to an ellipse and (b) any finitely smooth centrally symmetric strictly convex domain, with everywhere positive curvature, admitting locally maximizing centrally symmetric orbits of even period passing through a prescribed sequence of points and sufficiently close to an ellipse is area spectrally rigid. This means that any area-isospectral family of domains in these classes is necessarily equi-affine. We use techniques –adapted to symplectic billiards– inspired by the paper by J. De Simoi, V. Kaloshin and Q. Wei [41]. The novelty is that the result holds for axially symmetric domains, as in [41], as well as for a subset of centrally symmetric ones.

4.1 Introduction

Another direction of study is the existence of area spectrally rigid classes of domains; this means that any family of domains in one of this class with the same area spectrum is necessarily equi-affine. For Birkhoff billiards, the corresponding length spectral rigidity has been investigated by J. De Simoi, V. Kaloshin, and Q. Wei in [41]. In particular, they proved that the class of finitely smooth strictly convex axially symmetric domains, sufficiently close to a circle is length spectrally rigid, that is any length-isospectral (or dynamically spectrally rigid) family of domains in this class is necessarily isometric. Their technique can be resumed in the next steps. First of all, they reduce any family of axially symmetric domains to a normalized one by using uniquely isometries. Secondly, for every $q \geq 2$, they prove the existence of marked locally maximizing axially symmetric q-periodic orbits of rotation number $1/q$. Then, if the given family is length-isospectral, to each orbit as above, they associate a linear operator and rephrase the main result to the claim that this operator is injective. Finally, to prove the injectivity, they use precise estimates for a modification of the Lazutkin coordinates.

The first intent of the present chapter is to rephrase the result by De Simoi, Kaloshin, and Wei for symplectic billiards. As explained above, in this setting length spectral rigidity is replaced by area spectral rigidity; moreover –since the symplectic billiard map commutes with affine transformations– isometries are replaced by equi-affinities. Precise notions are contained in the next definition.

DEFINITION. For $r \geq 2$, let $\mathcal{D}^r$ be the set of strictly convex domains with C^{r+1} boundary and everywhere positive curvature. A C^1 parametric family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ is said to be

- area-isospectral if $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_{\tau'})$ for every $\tau, \tau' \in [-1, 1]$;
- equi-affine if there exists a family $(\mathcal{B}_\tau)_{|\tau|\leq 1}$ of affinities

$$\mathcal{B}_\tau : \mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto B_\tau x + b_\tau$$

  with $B_\tau \in SL(2, \mathbb{R})$ and $b_\tau \in \mathbb{R}^2$ such that

$$\Omega_\tau = \mathcal{B}_\tau \Omega_0 \qquad \forall \tau \in [-1, 1].$$

The main result of the present paper is then given by the next theorem.

THEOREM 1. Let $\mathcal{M}$ be the set of strictly convex domains with sufficiently (finitely) smooth boundary, everywhere positive curvature, axial symmetry, and sufficiently close to an ellipse. Let $(\Omega_\tau)_{|\tau|\leq 1}$ be a C^1 parameter family of domains in $\mathcal{M}$. If $(\Omega_\tau)_{|\tau|\leq 1}$ is area-isospectral then $(\Omega_\tau)_{|\tau|\leq 1}$ is equi-affine.

As explained for the seminal result in [41], the proof of this theorem passes through the construction for every $\Omega_\tau \in \mathcal{M}$ of marked axially symmetric periodic orbits of rotation number $1/q$ ($q \geq 3$). Studying the variation of their areas in terms of τ results sufficient to obtain information on the family $(\Omega_\tau)_{|\tau|\leq 1}$ itself and then to prove the theorem when Ω_0 is sufficiently close to an ellipse. Clearly –in general– it is not always possible to construct a selected family of periodic orbits providing information on the variation of the spectrum, and this is what happens when we replace axial symmetry with central symmetry, as explained here below.

The second intent of the present paper is, in fact, to construct another class of domains $\mathcal{M}'$ for which the area spectral rigidity holds. With this aim, we have substituted the family of axially symmetric domains with the family of centrally symmetric domains and we have followed the main lines of the proof of Theorem 1. It is worth noting that, in such a case, the result corresponding to Theorem 1 above is valid only if we restrict to the subset of centrally symmetric tables defined here below.

DEFINITION. A centrally symmetric domain $\Omega \in \mathcal{R}$ if and only if there exists a point $\gamma(0) \in \partial\Omega$ such that for every even $q \geq 4$ and for every $k \geq 2$:

(a) there exists a centrally symmetric periodic orbit of rotation number $1/2k$ which passes through $\gamma(0)$ and of maximal area among all centrally symmetric $2k$-periodic orbits;

(b) there exists a centrally symmetric periodic orbit of rotation number $1/2k$ which passes through $\gamma(1/4q)$ and of maximal area among all centrally symmetric $2k$-periodic orbits.

We notice that the additional (dynamical) hypothesis given by the previous definition –assuring the existence of locally maximizing centrally symmetric orbits of even period passing through a prescribed sequence of points– essentially fixes the lack of existence of marked centrally symmetric periodic orbits. In this framework, the second result of the present paper is given by the next theorem.

THEOREM 2. Let $\mathcal{M}'$ be the set of strictly convex domains with sufficiently (finitely) smooth boundary, everywhere positive curvature, belonging to $\mathcal{R}$, and sufficiently close to an ellipse.

Let $(\Omega_\tau)_{|\tau|\leq 1}$ be a C^1 parameter family of domains in $\mathcal{M}'$. If $(\Omega_\tau)_{|\tau|\leq 1}$ is area-isospectral then $(\Omega_\tau)_{|\tau|\leq 1}$ is equi-affine.

In light of the previous theorems, these further directions of investigation naturally arises. Can the hypothesis on the existence of locally maximizing centrally symmetric orbits of even period passing through prescribed points be removed in the centrally symmetric case? And in general, does the marked area spectrum uniquely determine –up to affinities– a domain? Or equivalently, does the Mather's β-function uniquely determine –up to affinities– a domain?

We underline that the recent paper [43] by C. Fierobe, A. Sorrentino, and A. Vig presents closely related results, with an approach similar to the original one of De Simoi, Kaloshin and Wei. In both papers, spectral rigidity is addressed by showing that isospectrality implies the Fourier coefficients of the deformation function lie in the kernel of a certain operator. For domains close to a circle (in the Birkhoff case) and to an ellipse (in the symplectic case), this operator is injective, providing trivial deformations. Our approach is instead more direct, since we derive precise inequalities for the Fourier coefficients of the deformation function and conclude that, for domains sufficiently close to an ellipse, these inequalities imply zero Fourier coefficients. In particular, from the detailed proofs of the theorems, it is clear both the necessity of the geometrical hypotheses on the family of domains and the role of the normalizations, which essentially compensate for the lack of periodic orbits of period smaller than 3.

The chapter is organized as follows. In Section 4.2 we define the area spectrum and area-isospectral and equi-affine families of domains. In Section 4.3 we introduce the two classes of domains for which the area spectral rigidity holds. In Section 4.4 we first introduce the normalization for each class of domains; clearly, by the affine equivariance of the symplectic billiard map, we reduce any family to a normalized one by using uniquely affinities. The normalization for axially symmetric domains is essentially the same as the one proposed in [41][Section 3]. Additionally, we introduce a normalization for area-isospectral domains in $\mathcal{R}$. Finally, we define the infinitesimal deformation function for the normalized families of domains and we rephrase the area isospectrality in terms of a linear system involving the deformation function. In Section 4.5 we recall some preliminaries in affine differential geometry and we prove a technical lemma –refinement of Proposition 3.3 in [17]– which will be fundamental in the proof of the two theorems. In fact, our proofs are performed by using the affine parametrization. In particular, if $\{s_j\}_{j=0}^{q}$ are the ordered affine parameters corresponding to a simple periodic trajectory for the symplectic billiard, we use –as in [17]– the Taylor expansions of both s_j and of $\lambda_j := s_j - s_{j-1}$. Section 4.6 is devoted to the detailed proofs of the main theorems. It is in the proof of the theorem for centrally symmetric domains that the hypothesis of the existence of marked locally maximizing centrally symmetric orbits of even period passing through prescribed points emerges as necessary to conclude.

4.2 Preliminaries

Let Ω be a strictly convex planar domain with smooth boundary $\partial\Omega$ and fixed positive counter-clockwise orientation. We assume that Ω contains the origin. By the strict convexity, for every point $x \in \partial\Omega$ there exists a unique (opposite) point x^* such that $T_x\partial\Omega = T_{x^*}\partial\Omega$. We refer to

$$\mathcal{P} := \{(x, y) \in \partial\Omega \times \partial\Omega : x < y < x^*\}$$

as the (open, positive) phase-space and we define the symplectic billiard map as follows (see [1][Page 5]):

$$\Phi : \mathcal{P} \to \mathcal{P}, \quad (x, y) \mapsto (y, z)$$

where z is the unique point satisfying

$$z - x \in T_y \partial\Omega.$$

We notice that Φ is continuous and can be continuously extended to $\bar{\mathcal{P}}$, so that $\Phi(x, x) = (x, x)$ and $\Phi(x, x^*) = (x^*, x)$ (these are the unique 2-periodic orbits of Φ). Moreover, see [1][Section 2] for exhaustive details, the symplectic billiard map turns out to be a twist map, preserving an area form, with generating function

$$\omega(x, y) = \det(x, y)$$

so that

$$\Phi(x, y) = (y, z) \Leftrightarrow \frac{d}{dy}\left[\omega(x, y) + \omega(y, z)\right] = 0.$$

We refer also to [13], [17], and [16] for recent advances in symplectic billiards.
To every closed trajectory $\{x_j\}_{j=0}^{q}$ of Φ in Ω ($x_0 = x_q$), it corresponds the action $\sum_{j=0}^{q-1} \omega(x_j, x_{j+1})$. In particular, if the periodic trajectory winds once around $\partial\Omega$, then $\frac{1}{2}\sum_{j=0}^{q-1} \omega(x_j, x_{j+1})$ is the area of the convex polygon inscribed in $\partial\Omega$ with vertices $\{x_j\}_{j=0}^{q}$.

Definition 4.2.1. *The area spectrum for the symplectic billiard in Ω is the set of positive real numbers*

$$\mathcal{A}(\Omega) = \mathbb{N}\{\textit{action of all closed trajectories of } \Phi\} \cup \mathbb{N}\{A_\Omega\},$$

where A_Ω is the area of Ω.

We underline the difference between the area spectrum and the so-called marked area spectrum, which is defined as the map that associates to any $1/q$, $q \geq 3$, the maximal area of (simple) convex polygons with q vertices inscribed in $\partial\Omega$. See e.g. [1][Beginning of Section 2.5.2] and [17][Definition 1.4]. We stress that the substantial difference between the two spectra is that the marked area spectrum preserves the information on the rotation number.

For $r \geq 2$, let $\mathcal{D}^r$ be the set of strictly convex domains with C^{r+1} boundary, *with everywhere positive curvature*. The proof of this property of the area spectrum is the same as the one in [41][Lemma 4.1].

Lemma 4.2.2. *Let $r \geq 2$. For any domain $\Omega \in \mathcal{D}^r$, $\mathcal{A}(\Omega)$ has zero Lebesgue measure.*

Given $\Omega \in \mathcal{D}^r$, we assume that the affine perimeter of $\partial\Omega$ is normalized to 1 and we indicate by

$$\mathbb{T} \ni s \mapsto \gamma_\Omega(s) \in \partial\Omega$$

the C^r affine arc-length parametrization of $\partial\Omega$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1]/\sim$ identifying $0 \sim 1$. In particular, for $\Omega_0, \Omega_1 \in \mathcal{D}^r$, let

$$d(\Omega_0, \Omega_1) := \sum_{k=0}^{r} \sup_{s \in [0,1]} \|\gamma_{\Omega_1}^{(k)}(s) - \gamma_{\Omega_0}^{(k)}(s)\|.$$

The next definition corresponds to Definition 2.9 in [41].

Definition 4.2.3. *Given $\Omega \in \mathcal{D}^r$ with unitary affine perimeter, let D be the disk of affine perimeter 1 tangent to Ω at the point $\gamma_\Omega(0)$. For $\delta > 0$, Ω is said to be δ-close to the circle if*

$$d(\Omega, D) \leq \delta. \tag{4.2.1}$$

A domain $\Omega \in \mathcal{D}^r$ of arbitrary affine perimeter is said to be δ-close to the circle if its rescaling of unitary affine perimeter satisfies (4.2.1).

Assumption 4.2.4. From now on, every domain $\Omega \in \mathcal{D}^r$ is assumed to be:

(a) parametrized by the affine parameter;

(b) normalized to unitary affine perimeter.

Given a C^1 parametric family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$, we use the notation

$$\gamma : [-1,1] \times \mathbb{T} \to \mathbb{R}^2, \quad (\tau, s) \mapsto \gamma(\tau, s) := \gamma_{\Omega_\tau}(s).$$

By hypothesis, $\gamma(\cdot, s)$ is C^1 for every $s \in \mathbb{T}$ and $\gamma(\tau, \cdot)$ is C^r for every $\tau \in [-1,1]$.
Notice that γ and $\tilde{\gamma}$ parametrize the same family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ if and only if there exists a C^1 family of C^r circle diffeomorphisms $\tilde{s} : [-1,1] \times \mathbb{T} \to \mathbb{T}$ such that

$$\gamma(\tau, s) = \tilde{\gamma}(\tau, \tilde{s}(\tau, s) = s + s_0(\tau)) \tag{4.2.2}$$

(or equivalently, $\tilde{\gamma}(\tau, \tilde{s}) = \gamma(\tau, s(\tau, \tilde{s}))$, where s denotes the inverse of $\tilde{s}$). Two parametrizations corresponding to the same family of domains are said to be equivalent.
We finally introduce the two fundamental notions of the present paper: area-isospectral and equi-affine families of domains.

Definition 4.2.5. *A C^1 parametric family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ is said to be*

(a) **area-isospectral** *if $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_{\tau'})$ for every $\tau, \tau' \in [-1,1]$;*

(b) **equi-affine** *if there exists a family $(\mathcal{B}_\tau)_{|\tau|\leq 1}$ of affinities*

$$\mathcal{B}_\tau : \mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto B_\tau x + b_\tau$$

with $B_\tau \in SL(2, \mathbb{R})$ and $b_\tau \in \mathbb{R}^2$ such that

$$\Omega_\tau = \mathcal{B}_\tau \Omega_0 \qquad \forall \tau \in [-1,1].$$

(c) **constant** *if $\Omega_\tau \equiv \Omega_0$.*

Finally, let fix a subset $\mathcal{M} \subset \mathcal{D}^r$.

Definition 4.2.6. *A domain $\Omega \in \mathcal{M}$ is called area spectrally rigid in $\mathcal{M}$ if any area-isospectral family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{M}$ with $\Omega_0 = \Omega$ is necessarily equi-affine.*

Remark 4.2.7. The aim of the present paper is to construct two different classes of domains for which the area spectral rigidity holds. As a consequence, we underline that Assumption 4.2.4 is not restrictive because if $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ is area-isospectral then the affine perimeter of $\partial\Omega_\tau$ is constant. In fact, under the hypothesis of area-isospectrality, the marked area spectrum for the symplectic billiard in Ω_τ is independent on τ. Equivalently, the corresponding formal Taylor expansion at 0 of the Mather's β-function is independent on τ. To conclude, it is then sufficient to remind that the term of order 3 is exactly $\lambda^3/6$ (see e.g. [66][Theorem 1]), where λ is the affine perimeter of the symplectic billiard table.

4.3 Statement of results

We start by recalling that a domain $\Omega \in \mathcal{D}^r$ is said to be *axially symmetric* if there exists a line $\Delta \subset \mathbb{R}^2$ such that Ω is invariant under the reflection along Δ. By convexity, $\Delta \cap \partial\Omega$ is given by two points. Chosen (arbitrarily) one of these points, we refer to it as the marked point of $\partial\Omega$ and to the other as the auxiliary point of $\partial\Omega$.
The next lemma –which is a consequence of the axial symmetry of Ω– corresponds to Lemma 4.3 in [41].

Lemma 4.3.1. *Let $\Omega \in \mathcal{D}^r$ be an axially symmetric domain. For every $q \geq 3$ there exists an axially symmetric periodic orbit of rotation number $1/q$ passing through the marked point of $\partial\Omega$ and of maximal area among all axially symmetric q-periodic orbits passing through the marked point.*

Proof. We recall that Ω is parametrized in affine arc-length by $\gamma(s)$; moreover, since $\lambda = 1$, the marked and the auxiliary points can be conventionally fixed to be $\gamma(0)$ and $\gamma(1/2)$ respectively. We distinguish even and odd period and we use the same arguments as in the proof of Lemma 4.3 in [41].

For $k \geq 2$, let $q = 2k$ even. In such a case, the desired q-periodic orbit passes through the marked and the auxiliary points. In fact, once fixed $s_0 = 0$ and $s_k = 1/2$, we look for $(s_1, \dots, s_{k-1})$ in the compact set $0 = s_0 \leq s_1 \leq \dots \leq s_{k-1} \leq s_k = 1/2$, maximizing

$$\sum_{j=0}^{k-1} \omega(\gamma(s_j), \gamma(s_{j+1})). \tag{4.3.1}$$

The maximum $(0, \bar{s}_1, \dots, \bar{s}_{k-1}, 1/2)$ must satisfy $0 < \bar{s}_1 < \dots < \bar{s}_{k-1} < 1/2$. Moreover,

$$0 = \partial_1\omega(\gamma(\bar{s}_j), \gamma(\bar{s}_{j+1})) + \partial_2\omega(\gamma(\bar{s}_{j-1}), \gamma(\bar{s}_j)) \qquad \forall j = 1, \dots, k-1.$$

Defining $\bar{s}_{2k-j} = -\bar{s}_j$ for every $j = 1, \dots k-1$, we obtain the desired $2k$-periodic orbit. See (a) in Figure 4.1.

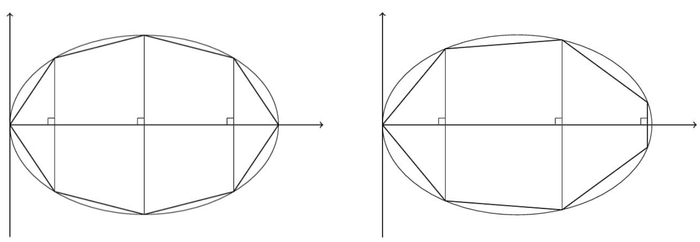
For $k \geq 1$, let $q = 2k+1$ odd. In such a case, differently from above, we fix only $s_0 = 0$ and we look for $(s_1, \dots, s_k)$ in the compact set $0 = s_0 \leq s_1 \leq \dots \leq s_k \leq 1/2$, maximizing

$$\sum_{j=0}^{k-1} \omega(\gamma(s_j), \gamma(s_{j+1})) + \frac{1}{2}\omega(\gamma(s_k), \gamma(-s_k)). \tag{4.3.2}$$

Again, the maximum $(0, \bar{s}_1, \dots, \bar{s}_k)$ must satisfy $0 < \bar{s}_1 < \dots < \bar{s}_{k-1} < \bar{s}_k < 1/2$. Moreover

$$\begin{aligned} 0 &= \partial_1\omega(\gamma(\bar{s}_j), \gamma(\bar{s}_{j+1})) + \partial_2\omega(\gamma(\bar{s}_{j-1}), \gamma(\bar{s}_j)) \qquad \forall j = 1, \dots, k-1, \\ 0 &= \partial_2\omega(\gamma(\bar{s}_{k-1}), \gamma(\bar{s}_k)) + \partial_1\omega(\gamma(\bar{s}_k), \gamma(-\bar{s}_k)). \end{aligned}$$

The $(2k+1)$-periodic orbit of the statement is then obtained by defining $\bar{s}_{2k+1-j} = -\bar{s}_j$ for every $j = 1, \dots k-1$. See (b) in Figure 4.1. $\square$



(a) (b)

Figure 4.1: Axially symmetric periodic orbits of even period (a) and odd period (b).

We continue by recalling that a domain $\Omega \in \mathcal{D}^r$ is said to be *centrally symmetric* if there exists a point $0 \in int(\Omega)$ such that Ω is invariant under the reflection with center 0.

In the sequel, we denote by $\mathcal{AS}_\delta^r$ the set of axially symmetric domains in $\mathcal{D}^r$ which are δ-close to the circle, in accord with Definition 4.2.3. Analogously, $\mathcal{CS}_\delta^r$ is the set of centrally symmetric domains in $\mathcal{D}^r$ which are δ-close to the circle.

The aim of the present paper is proving the next two rigidity results.

Theorem 4.3.2. *Let $r = 6$. There exists $\delta > 0$ such that any domain $\Omega \in \mathcal{AS}_\delta^r$ is area spectrally rigid in $\mathcal{AS}_\delta^r$. In other words, any area-isospectral family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{AS}_\delta^r$ with $\Omega_0 = \Omega$ is necessarily equi-affine.*

It is worth noting that the corresponding result in $\mathcal{CS}_\delta^r$ is valid only if we restrict to the subset $\mathcal{R}_\delta^r$ of $\mathcal{CS}_\delta^r$ defined here below.

Definition 4.3.3. *Given $\Omega \in \mathcal{CS}_\delta^r$, the domain $\Omega \in \mathcal{R}_\delta^r$ if and only if there exists a point $\gamma(0) \in \partial\Omega$ such that for every even $q \geq 4$ and for every $k \geq 2$:*

- (a) *there exists a centrally symmetric periodic orbit of rotation number $1/2k$ which passes through $\gamma(0)$ and of maximal area among all centrally symmetric $2k$-periodic orbits;*
- (b) *there exists a centrally symmetric periodic orbit of rotation number $1/2k$ which passes through $\gamma(1/4q)$ and of maximal area among all centrally symmetric $2k$-periodic orbits.*

Remark 4.3.4. We make some considerations on the previous definition. Hypothesis (a) imposes essentially the existence of a special point in $\partial\Omega$ playing the role of the marked point in the axially symmetric case. For this reason, we still indicate the point $\gamma(0)$ as the marked point of $\partial\Omega$. The sequence of points in hypothesis (b) (rather technical) will be necessary to conclude the proof in the centrally symmetric case, we refer to the detailed proof of Theorem 4.3.5.

Theorem 4.3.5. *Let $r = 6$. There exists $\delta > 0$ such that any domain $\Omega \in \mathcal{R}_\delta^r$ is area spectrally rigid in $\mathcal{R}_\delta^r$. In other words, any area-isospectral family of domains $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{R}_\delta^r$ with $\Omega_0 = \Omega$ is necessarily equi-affine.*

4.4 Normalizations and infinitesimal deformation function

We now introduce the normalization both in the axially and in the centrally symmetric case. Given a family $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ of axially symmetric domains, parametrized by γ, an associated normalized family $(\tilde{\Omega}_\tau)_{|\tau|\leq 1}$ is then constructed as follows:

- (a) By a translation, impose the marked point to be $\gamma(\tau, 0) = (0, 0)$ for every $\tau \in [-1, 1]$;
- (b) by a rotation, impose the axis of symmetry of every Ω_τ to be on $x > 0$;
- (c) finally, by a unitary diagonal transformation, impose the auxiliary point of every $\partial\Omega_\tau$ to be $\big(2(2\pi)^{-3/2}, 0\big)$.

Similarly, given a family $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ of centrally symmetric domains satisfying hypotheses of Definition 4.3.3, parametrized by γ, the associated normalized family $(\tilde{\Omega}_\tau)_{|\tau|\leq 1}$ is constructed as follows:

- (a) By a translation, impose the center of symmetry to be the origin $(0, 0)$ for every $\tau \in [-1, 1]$;
- (b) by a rotation, impose the marked point of every Ω_τ to be on $x > 0$;
- (c) finally, by a unitary affine transformation, impose the marked point of every $\partial\Omega_\tau$ to satisfy: $\gamma(\tau, 0) = \big((2\pi)^{-3/2}, 0\big)$ and $\partial_s\gamma(\tau, 0) \,||\, (0, 1)$.

We underline that all transformations used in points (a)–(c) of both normalizations do not change the (unitary) affine perimeter, and that circles and ellipses coincide with these normalizations. Moreover, given a family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{D}^r$ of axially symmetric or centrally symmetric domains satisfying hypotheses of Definion 4.3.3, we have used affinities to construct $(\tilde\Omega_\tau)_{|\tau|\le 1}$. As a consequence, $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\tilde\Omega_\tau)$ for every $\tau \in [-1,1]$ and therefore $(\Omega_\tau)_{|\tau|\le 1}$ is area isospectral if and only if $(\tilde\Omega_\tau)_{|\tau|\le 1}$ is area isospectral. Consequently, Theorems 4.3.2 and 4.3.5 can be respectively rephrased as follows. From now on –when it is clear from the context– we omit the tilde to indicate a normalized family.

Theorem 4.4.1. *Let $r = 6$. There exists $\delta > 0$ such that any domain $\Omega \in \mathcal{AS}^r_\delta$ is area spectrally rigid in $\mathcal{AS}^r_\delta$. In other words, any normalized area-isospectral family of domains $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}^r_\delta$ with $\Omega_0 = \Omega$ is necessarily constant.*

Theorem 4.4.2. *Let $r = 6$. There exists $\delta > 0$ such that any domain $\Omega \in \mathcal{R}^r_\delta$ is area spectrally rigid in $\mathcal{R}^r_\delta$. In other words, any normalized area-isospectral family of domains $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{R}^r_\delta$ with $\Omega_0 = \Omega$ is necessarily constant.*

Given a normalized family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{D}^r$ of domains in $\mathcal{AS}^r_\delta$ or $\mathcal{R}^r_\delta$, parametrized by γ, we define the infinitesimal deformation function $n_\gamma(\tau, s)$ as

$$n_\gamma(\tau, s) := \omega(\partial_\tau \gamma(\tau, s), T_\gamma(\tau, s)), \tag{4.4.1}$$

where $T_\gamma(\tau, s)$ is the unit tangent vector to $\partial\Omega_\tau$ at the point $\gamma(\tau, s)$, $T_\gamma(\tau, s) := \frac{\partial_s \gamma(\tau,s)}{\|\partial_s \gamma(\tau,s)\|}$. We refer to [41][Page 245] for the formula of the infinitesimal deformation function for (normalized) axially symmetric Birkhoff billiard tables. By the regularity properties of γ, n_γ is C^0 in τ and C^{r-1} in s.

Remark 4.4.3. We finally notice that, as a consequence of the normalizations, the infinitesimal deformation function $n_\gamma(\tau, s)$ satisfies the properties explained here below.

(a) For a normalized family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}^r_\delta$, $n_\gamma(\tau, 0) = n_\gamma(\tau, 1/2) = 0$ for every $|\tau| \le 1$.

(b) For a normalized family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{R}^r_\delta$, $n_\gamma(\tau, 0) = 0$ and

$$\partial_s n_\gamma(\tau, 0) = \omega(\partial^2_{s\tau}\gamma(\tau, 0), T_\gamma(\tau, 0)) + \omega(\partial_\tau \gamma(\tau, 0), \partial_s T_\gamma(\tau, 0)) = 0 \tag{4.4.2}$$

for every $|\tau| \le 1$. By central symmetry, also $n_\gamma(\tau, 1/2) = 0$ and

$$\partial_s n_\gamma(\tau, 1/2) = \omega(\partial^2_{s\tau}\gamma(\tau, 1/2), T_\gamma(\tau, 1/2)) + \omega(\partial_\tau \gamma(\tau, 1/2), \partial_s T_\gamma(\tau, 1/2)) = 0 \tag{4.4.3}$$

for every $|\tau| \le 1$. In fact, since the normalization imposes that for every $|\tau| \le 1$, $\gamma(\tau, 0) = \big((2\pi)^{-3/2}, 0\big)$ and $T_\gamma(\tau, 0) \,||\, (0, 1)$, it follows that $\partial_s T_\gamma(\tau, 0) \equiv 0$, and moreover $T_\gamma(\tau, 0) \,||\, \partial^2_{s\tau}\gamma(\tau, 0)$. The central symmetry assumption, provides the same properties for $\partial_s T_\gamma(\tau, 1/2)$ and $\partial^2_{s\tau}\gamma(\tau, 1/2)$.

We stress that the above properties will be fundamental in order to conclude the proof of Theorems 4.3.2 and 4.3.5 respectively; as it will become clear later, these normalizations will compensate for the lack of periodic orbits of period smaller than 3.

The statement and the proof of the next lemma is the same of Lemma 3.3 in [41].

Lemma 4.4.4. *Let $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ be a normalized family of domains in $\mathcal{AS}^r_\delta$ or $\mathcal{R}^r_\delta$, parametrized by γ. Then*

(a) for any other parametrization $\tilde\gamma$ as in (4.2.2), if $n_\gamma(\tau,\cdot)\equiv 0$ for some $\tau\in[-1,1]$ then $n_{\tilde\gamma}(\tau,\cdot)\equiv 0$ for the same $\tau\in[-1,1]$;

(b) $n_\gamma(\tau,s)=0$ for all $(\tau,s)\in[-1,1]\times\mathbb{T}$ if and only if $(\Omega_\tau)_{|\tau|\leq 1}$ is a constant family.

By using the lemma above, we can rephrase Theorems 4.3.2 and 4.3.5 (and their equivalent versions: Theorems 4.4.1 and 4.4.2) in terms of the infinitesimal deformation function.

Theorem 4.4.5. *Let $r=6$. There exists $\delta>0$ such that if $(\Omega_\tau)_{|\tau|\leq 1}$ is a normalized area-isospectral family of domains in $\mathcal{AS}^r_\delta$, then $n_\gamma\equiv 0$ for every parametrization γ.*

Theorem 4.4.6. *Let $r=6$. There exists $\delta>0$ such that if $(\Omega_\tau)_{|\tau|\leq 1}$ is a normalized area-isospectral family of domains in $\mathcal{R}^r_\delta$, then $n_\gamma\equiv 0$ for every parametrization γ.*

Consider a family of domains $(\Omega_\tau)_{|\tau|\leq 1}\subset\mathcal{D}^r$. For every $q\geq 3$, if $\mathbf{s}(\tau)=\{s_i(\tau)\}_{i=0}^q$ are the affine parameters for a q-periodic sequence corresponding to points in $\partial\Omega_\tau$, we indicate

$$\omega_q(\tau;\mathbf{s}(\tau)) := \sum_{j=0}^{q-1}\omega(\gamma(\tau,s_j(\tau)),\gamma(\tau,s_{j+1}(\tau))).$$

Clearly, if $\mathbf{s}(\tau)$ gives a q-periodic orbit, then $\omega_q(\tau;\mathbf{s}(\tau))$ is its action.
In the sequel, for the sake of simplicity, we omit the dependence on τ of $\mathbf{s}$ and s_i. We need to distinguish normalized family of domains $(\Omega_\tau)_{|\tau|\leq 1}\subset\mathcal{AS}^r_\delta$ and $(\Omega_\tau)_{|\tau|\leq 1}\subset\mathcal{R}^r_\delta$.

In the first case, for every $q\geq 3$, let $\mathbf{s}=\{s_j\}_{j=0}^q$ be the affine parameters corresponding to the ordered vertices of an axially symmetric q-periodic orbit for the symplectic billiard in Ω_τ, passing through the marked point of $\partial\Omega_\tau$. Moreover, let

$$[-1,1]\ni\tau\mapsto\Delta_q(\tau):=\max_{\mathbf{s}}\,\omega_q(\tau;\mathbf{s})\in\mathbb{R} \tag{4.4.4}$$

be the function associating to each domain Ω_τ twice the area of the q-periodic orbit constructed in Lemma 4.3.1. More precisely, $\Delta_q(\tau)$ is twice the maximal area among all axially symmetric q-periodic orbits for the symplectic billiard in Ω_τ, passing through the marked point of $\partial\Omega_\tau$.

In the second case, for every even $q\geq 4$, let $\mathbf{s}=\{s_j\}_{j=0}^q$ be the affine parameters corresponding to the ordered vertices of a centrally symmetric q-periodic orbit for the symplectic billiard in Ω_τ. In this latter case

$$[-1,1]\ni\tau\mapsto\tilde\Delta_q(\tau):=\max_{\mathbf{s}}\,\omega_q(\tau;\mathbf{s})\in\mathbb{R} \tag{4.4.5}$$

is the function associating to each domain Ω_τ twice the maximal area among all centrally symmetric q-periodic orbits for the symplectic billiard in Ω_τ.

Proposition 4.4.7. (*a*) *Let $(\Omega_\tau)_{|\tau|\leq 1}\subset\mathcal{AS}^r_\delta$ be a normalized area-isospectral family of domains parametrized by γ. If $\bar{\mathbf{s}}=(\bar s_0,\dots,\bar s_q)$ realizes the maximum (4.4.4), that is $\omega_q(\tau;\bar{\mathbf{s}})=\Delta_q(\tau)$, then*

$$\partial_\tau\omega_q(\tau,\bar{\mathbf{s}})=\sum_{j=0}^{q-1}\|\gamma(\tau,\bar s_{j+1})-\gamma(\tau,\bar s_{j-1})\|n_\gamma(\tau,\bar s_j)=0. \tag{4.4.6}$$

(*b*) *Let $(\Omega_\tau)_{|\tau|\leq 1}\subset\mathcal{R}^r_\delta$ be a normalized area-isospectral family of domains parametrized by γ. If $\bar{\mathbf{s}}=(\bar s_0,\dots,\bar s_q)$ realizes the maximum (4.4.5), that is $\omega_q(\tau;\bar{\mathbf{s}})=\tilde\Delta_q(\tau)$, then*

$$\partial_\tau\omega_q(\tau,\bar{\mathbf{s}})=\sum_{j=0}^{q-1}\|\gamma(\tau,\bar s_{j+1})-\gamma(\tau,\bar s_{j-1})\|n_\gamma(\tau,\bar s_j)=0. \tag{4.4.7}$$

Proof. Since the proof of (a) and (b) follows the same lines, we give all the details in axially symmetric case. We first prove that $\partial_\tau \omega_q(\tau, \bar{\mathbf{s}}) = 0$. By definition, $\Delta_q(\tau) \in \mathcal{A}(\Omega_\tau)$. Moreover, since the family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}^r_\delta$ is assumed to be C^1 parametric, Δ_q is a continuous function. By these facts and Lemma 4.2.2, it follows that if $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}^r_\delta$ is area-isospectral then Δ_q is necessarily constant. We recall that $\Delta_q(\tau)$ is –by definition– twice the maximal area among all axially symmetric q-periodic orbits for the symplectic billiard in Ω_τ, passing through the marked point of $\partial\Omega_\tau$. Consequently, if $\bar{\mathbf{s}}$ realizes the maximum at τ then

$$\omega_q(\tau', \bar{\mathbf{s}}) \le \Delta_q(\tau') = \Delta_q(\tau) = \omega_q(\tau; \bar{\mathbf{s}}) \qquad \forall \tau' \in [-1, 1],$$

where –in the first equality– we have used the fact that (in the area-isospectral case) Δ_q is constant. Since the family $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}^r_\delta$ is assumed to be C^1 parametric, $\tau \mapsto \omega_q(\tau; \bar{\mathbf{s}})$ is C^1 and the statement immediately follows. Finally, we verify the explicit expression for $\partial_\tau \omega_q(\tau, \bar{\mathbf{s}})$ by a direct computation:

$$\begin{aligned}
\partial_\tau \omega_q(\tau, \bar{\mathbf{s}}) &= \sum_{j=0}^{q-1} \partial_\tau \omega(\gamma(\tau, \bar{s}_j), \gamma(\tau, \bar{s}_{j+1})) \\
&= \sum_{j=0}^{q-1} [\omega(\partial_\tau \gamma(\tau, \bar{s}_j), \gamma(\tau, \bar{s}_{j+1})) + \omega(\gamma(\tau, \bar{s}_j), \partial_\tau \gamma(\tau, \bar{s}_{j+1}))] \\
&= \sum_{j=0}^{q-1} \omega(\partial_\tau \gamma(\tau, \bar{s}_j), \gamma(\tau, \bar{s}_{j+1}) - \gamma(\tau, \bar{s}_{j-1})) \\
&= \sum_{j=0}^{q-1} \|\gamma(\tau, \bar{s}_{j+1}) - \gamma(\tau, \bar{s}_{j-1})\| \omega(\partial_\tau \gamma(\tau, \bar{s}_j), T_\gamma(\tau, \bar{s}_j)) \\
&= \sum_{j=0}^{q-1} \|\gamma(\tau, \bar{s}_{j+1}) - \gamma(\tau, \bar{s}_{j-1})\| n_\gamma(\tau, \bar{s}_j).
\end{aligned}$$

where, in the fourth equality, we have used the symplectic billiard dynamics:

$$\gamma(\tau, \bar{s}_{j+1}) - \gamma(\tau, \bar{s}_{j-1}) = \|\gamma(\tau, \bar{s}_{j+1}) - \gamma(\tau, \bar{s}_{j-1})\| \, T_\gamma(\tau, \bar{s}_j) \qquad \forall j = 1, \dots, q-1.$$

□

4.5 Preliminary results in affine differential geometry

This section is devoted to collecting some notions and results in affine differential geometry which will be useful in the next sections. We recall that the affine arc length and the affine perimeter are given respectively by

$$s(t) = \int_0^t \kappa^{1/3}(r) dr \qquad 0 \le t \le l$$

and

$$\lambda = \int_0^l \kappa^{1/3}(r) dr$$

where t is the arc length and $\kappa(t)$ the (ordinary) curvature of $\partial\Omega$. As in the previous sections, let $\gamma(s)$ be the affine arc length parametrization of $\partial\Omega$. Then –see [85][Section 3]– γ is characterized by:

$$\omega(\gamma', \gamma'') = 1, \qquad \omega(\gamma', \gamma''') = 0$$

and

$$k := \omega(\gamma'', \gamma''')$$

is the affine curvature of $\partial\Omega$ (remind that the affine curvature of the circle of affine unitary perimeter is $(2\pi)^2$). Moreover, differentiating $\omega(\gamma', \gamma''') = 0$ and using the definition of k, we obtain

$$\gamma''' = -k\gamma'.$$

We underline that $\|\gamma'(s)\| = \rho(t(s))$, where ρ and t are the (usual) ray of curvature and arc length, respectively. From now on, we suppose $\lambda = 1$ (see Assumption 4.2.4).
The next technical lemma is a refinement of Proposition 3.3 in [17]. These coordinates and their expansions are the analogue, for symplectic billiards, of those computed by V.F. Lazutkin for the usual Birkhoff billiards [63].

Lemma 4.5.1. *For $q \geq 3$, let $\{\gamma(s_j)\}_{j=0}^q$ be the ordered vertices of a periodic trajectory for the symplectic billiard with rotation number $1/q$ and $s_0 = 0$ and $s_q = 1$. If*

$$\lambda_j := s_j - s_{j-1} \qquad \forall j = 1, \dots, q$$

then

$$\lambda_j = \frac{1}{q} - \frac{1}{30q^3}\int_0^1 k(s)ds + \frac{1}{30q^3}k\left(\frac{j}{q}\right) - \frac{1}{60q^4}k'\left(\frac{j}{q}\right) + O\left(\frac{1}{q^5}\right), \tag{4.5.1}$$

where, for some constant $C > 0$ independent on $q \geq 3$:

$$\left\|O\left(\frac{1}{q^5}\right)\right\| \leq C\,\frac{\|k'\|_{C^2}}{q^5}.$$

Moreover

$$s_j = \frac{j}{q} + \frac{1}{30q^2}\int_0^{\frac{j}{q}} k(s)ds - \frac{j}{30q^3}\int_0^1 k(s)ds + O\left(\frac{1}{q^4}\right). \tag{4.5.2}$$

Proof. We first prove formula (4.5.1). We start recalling that, by Propositions 3.2 and 3.3 in [17], it holds that

$$\begin{aligned} \lambda_{j+1} &= \lambda_j + \frac{1}{30}k'(s_j)\lambda_j^4 + O(\lambda_j^6) \qquad \text{and} \\ \lambda_j &= \frac{1}{q} - \frac{1}{30q^3}\int_0^1 k(s)ds + \frac{1}{30q^3}k\left(\frac{j}{q}\right) + \frac{\sigma(j,q)}{q^4} \end{aligned} \tag{4.5.3}$$

with $\sigma(j,q) = O(1)$.

Remark 4.5.2. We stress that, since we are applying the Implicit function Theorem, in order to obtain the first formula in (4.5.3), we need to ask the affine parametrization to be at least C^6, that is $r = 6$.

Combining the two equalities, we obtain

$$\begin{aligned} \frac{\sigma(j+1,q)}{q^4} &= \frac{\sigma(j,q)}{q^4} + \frac{1}{30q^3}\left(k\left(\frac{j}{q}\right) - k\left(\frac{j+1}{q}\right) + \frac{1}{q}k'\left(\frac{j}{q}\right)\right) + O\left(\frac{1}{q^6}\right) \\ &= \frac{\sigma(j,q)}{q^4} - \frac{1}{60q^5}k''\left(\frac{j}{q}\right) + O\left(\frac{1}{q^6}\right), \end{aligned}$$

so that

$$\sigma(j,q) = \sigma(1,q) - \frac{1}{60q}\sum_{i=1}^{j-1} k''\left(\frac{i}{q}\right) + O\left(\frac{1}{q}\right) = \sigma(1,q) - \frac{1}{60}\left(k'\left(\frac{j}{q}\right) - k'\left(\frac{1}{q}\right)\right) + O\left(\frac{1}{q}\right), \tag{4.5.4}$$

where –in the last equality– we have used the fact that by Taylor expansion:

$$\frac{1}{q}\sum_{i=1}^{j-1} k''\left(\frac{i}{q}\right) - \int_{\frac{1}{q}}^{\frac{j}{q}} k''(s)ds = \frac{1}{q}\sum_{i=0}^{j-1} k''\left(\frac{i}{q}\right) - k'\left(\frac{j}{q}\right) + k'\left(\frac{1}{q}\right) = O\left(\frac{1}{q}\right).$$

Therefore:

$$\lambda_j = \frac{1}{q} - \frac{1}{30q^3}\int_0^1 k(s)ds + \frac{1}{30q^3}k\left(\frac{j}{q}\right) + \frac{\sigma(1,q) + \frac{1}{60}k'\left(\frac{1}{q}\right)}{q^4} - \frac{1}{60q^4}k'\left(\frac{j}{q}\right) + O\left(\frac{1}{q^5}\right).$$

We continue by summing up the last equality for $j = 1, \dots, q$. Recalling that $\sum_{j=1}^q \lambda_j = 1$, we obtain:

$$\begin{aligned}
0 =& \frac{1}{30q^3}\sum_{j=1}^{q} k\left(\frac{j}{q}\right) - \frac{1}{30q^2}\int_0^1 k(s)ds - \frac{1}{60q^4}\sum_{j=1}^{q} k'\left(\frac{j}{q}\right) + \frac{\sigma(1,q) + \frac{1}{60}k'\left(\frac{1}{q}\right)}{q^3} + O\left(\frac{1}{q^4}\right)\\
=& \frac{1}{30q^2}\sum_{j=1}^{q}\int_{\frac{j}{q}}^{\frac{j+1}{q}} k\left(\frac{j}{q}\right) - k(s)ds - \frac{1}{60q^4}\sum_{j=1}^{q} k'\left(\frac{j}{q}\right) + \frac{\sigma(1,q) + \frac{1}{60}k'\left(\frac{1}{q}\right)}{q^3} + O\left(\frac{1}{q^4}\right).
\end{aligned}$$

Since

$$\int_{\frac{j}{q}}^{\frac{j+1}{q}} k\left(\frac{j}{q}\right) - k(s)ds = -\frac{1}{2q^2}k'\left(\frac{j}{q}\right) + O\left(\frac{1}{q^3}\right) \qquad \text{and} \qquad \frac{1}{q}\sum_{j=1}^{q} k'\left(\frac{j}{q}\right) = O\left(\frac{1}{q}\right),$$

we finally obtain

$$\frac{\sigma(1,q) + \frac{1}{60}k'\left(\frac{1}{q}\right)}{q^3} = O\left(\frac{1}{q^4}\right)$$

By using (4.5.4), we immediately obtain formula (4.5.1).

Formula (4.5.2) comes by summing up expansion (4.5.1) for $i = 1, \dots, j$:

$$s_j = \sum_{i=1}^{j}\lambda_i = \frac{j}{q} - \frac{j}{30q^3}\int_0^1 k(s)ds + \frac{1}{30q^3}\sum_{i=1}^{j} k\left(\frac{i}{q}\right) - \frac{1}{60q^4}\sum_{i=1}^{j} k'\left(\frac{i}{q}\right) + O\left(\frac{1}{q^4}\right).$$

Since now, by Taylor's expansion

$$\frac{1}{30q^3}\sum_{i=1}^{j} k\left(\frac{i}{q}\right) = \frac{1}{30q^2}\int_0^{\frac{j}{q}} k(s)ds + \frac{1}{60q^4}\sum_{i=1}^{j} k'\left(\frac{i}{q}\right) + O\left(\frac{1}{q^4}\right),$$

formula (4.5.2) immediately follows. □

4.6 Proof of Theorems 4.4.5 and 4.4.6

For $\delta > 0$, consider $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{D}^r$ a family of normalized area-isospectral domains in $\mathcal{AS}_\delta^r$ and $\mathcal{R}_\delta^r$ respectively. The infinitesimal deformation function has the following properties.

(a) If $(\Omega_\tau)_{|\tau|\le 1} \subset \mathcal{AS}_\delta^r$ is a normalized area-isospectral family of domains, parametrized by γ, $n_\gamma(\tau,\cdot)$ is an even function. In particular, for every $\tau \in [-1,1]$ we can write the Fourier expansion of $n_\gamma(\tau,s)$ in the basis $\{e^{2\pi i k s}\}_{k\in\mathbb{Z}}$:

$$n_\gamma(\tau,s) = \sum_{|k|\ge 1}\hat{n}_k(\tau)e^{2\pi i k s}$$

where $\hat{n}_k(\tau)$ denotes the k-th Fourier coefficient of $n_\gamma(\tau,\cdot)$ and by symmetry $\hat{n}_k = \hat{n}_{-k}$. Since n_γ has zero-average, $\hat{n}_0 = 0$. We remind that, as a consequence of the normalization (see point (a) of Remark 4.4.3) $n_\gamma(\tau,0) = n_\gamma(\tau,1/2) = 0$ for every $|\tau| \leq 1$.

(b) If $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{R}^r_\delta$ is a normalized area-isospectral family of domains, parametrized by γ, $n_\gamma(\tau,\cdot)$ is a 1/2-periodic function. In particular, for every $\tau \in [-1,1]$ we can write the Fourier expansion of $n_\gamma(\tau,s)$ in the basis $\{e^{2\pi iks}\}_{k\in\mathbb{Z}}$:

$$n_\gamma(\tau,s) = \sum_{|k|\geq 1} \hat{n}_{2k}(\tau)e^{4\pi iks}.$$

In such a case, the normalization (see point (b) of Remark 4.4.3) implies:

$$n_\gamma(\tau,0) = n_\gamma(\tau,1/2) = 0 \qquad \text{and} \qquad \partial_s n_\gamma(\tau,0) = \partial_s n_\gamma(\tau,1/2) = 0 \qquad \forall \tau \in [-1,1].$$

Remark 4.6.1. Theorems 4.4.5 and 4.4.6 are especially easy to verify when Ω_0 is a circle. We give details for Theorem 4.4.5, the other theorem similarly follows. In this special case, for every $q \geq 3$ the axially symmetric periodic orbit of Lemma 4.3.1 is given by $\bar{\mathbf{s}} = \left\{\frac{j}{q}\right\}_{j=0}^q$. Consequently, from Proposition 4.4.7 we have that

$$\partial_\tau \omega_q(\tau,\bar{\mathbf{s}})\mid_{\tau=0} = \sum_{j=0}^{q-1} \left\|\gamma\left(0,\frac{j+1}{q}\right) - \gamma\left(0,\frac{j-1}{q}\right)\right\| n_\gamma\left(0,\frac{j}{q}\right) = 0.$$

Since $\left\|\gamma\left(0,\frac{j+1}{q}\right) - \gamma\left(0,\frac{j-1}{q}\right)\right\|$ is constant for each $j = 0,\dots,q-1$, we get

$$\sum_{j=0}^{q-1} n_\gamma\left(0,\frac{j}{q}\right) = \sum_{|k|\geq 1}\sum_{j=0}^{q-1} \hat{n}_k(0)e^{\frac{2\pi ikj}{q}} = 2q\sum_{k\geq 1}\hat{n}_{qk}(0) = 0,$$

implying that $\hat{n}_k(0) = 0$ for every $k \geq 3$. Moreover, from normalization, $n_\gamma(0,0) = 0 = n_\gamma(0,1/2)$ so that $\hat{n}_1(0) = \hat{n}_2(0) = 0$. This gives $n_\gamma(0,\cdot) \equiv 0$ and therefore the desired statement, $n_\gamma \equiv 0$.

Before entering into the details of the proof of Theorem 4.4.5, we premise a technical lemma.

Lemma 4.6.2. *Let $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{D}^r$ be a normalized family of domains in $\mathcal{AS}^r_\delta$ or $\mathcal{R}^r_\delta$ parametrized by γ. For $q \geq 3$, let $\mathbf{s} = \{s_j\}_{j=0}^q$ be the affine parameters corresponding to ordered vertices of a periodic trajectory for the symplectic billiard for the domain Ω_0, with rotation number $1/q$. Then*

$$\partial_\tau \omega_q(\tau,\mathbf{s})\mid_{\tau=0} = \sum_{j=0}^{q-1} n_\gamma(0,s_j)\,\rho(t(s_j))^{\frac{1}{3}}\left[\frac{2}{q} + \frac{1}{15q^3}\left(k\left(\frac{j}{q}\right) - \int_0^1 k(s)ds\right) - \frac{1}{3q^3}k(s_j) + O\left(\frac{1}{q^5}\right)\right], \tag{4.6.1}$$

where ρ denotes the (usual) ray of curvature of the domain Ω_0.

Proof. From Proposition 4.4.7, we know that

$$\partial_\tau \omega_q(\tau,\bar{\mathbf{s}})\mid_{\tau=0} = \sum_{j=0}^{q-1} \|\gamma(0,s_{j+1}) - \gamma(0,s_{j-1})\| n_\gamma(0,s_j),$$

so that we need to detect $\|\gamma(0,s_{j+1})-\gamma(0,s_{j-1})\|$. Naming $\gamma(0,s)=\gamma(s)$ and recalling that $\lambda_j = s_j - s_{j-1}$:

$$\gamma(s_{j+1})-\gamma(s_j)=\gamma'(s_j)\lambda_{j+1}+\frac{\gamma''(s_j)}{2}\lambda_{j+1}^2+\frac{\gamma'''(s_j)}{6}\lambda_{j+1}^3+\frac{\gamma''''(s_j)}{24}\lambda_{j+1}^4+O\left(\lambda_{j+1}^5\right),$$
$$\gamma(s_{j-1})-\gamma(s_j)=-\gamma'(s_j)\lambda_j+\frac{\gamma''(s_j)}{2}\lambda_j^2-\frac{\gamma'''(s_j)}{6}\lambda_j^3+\frac{\gamma''''(s_j)}{24}\lambda_j^4+O\left(\lambda_j^5\right).$$

Consequently, by expansion (4.5.1) and the fact that $\lambda_{j+1}-\lambda_j=O(1/q^4)$ (see first formula in (4.5.3)), the norm of their difference is

$$\|\gamma(s_{j+1})-\gamma(s_{j-1})\|=\left\|\gamma'(s_j)(\lambda_{j+1}+\lambda_j)+\frac{\gamma'''(s_j)}{6}(\lambda_{j+1}^3+\lambda_j^3)+O(1/q^5)\right\|.$$

Plugging formula (4.5.1) into the one above, we obtain that

$$\begin{aligned}\|\gamma(s_{j+1})-\gamma(s_{j-1})\| &= \|\gamma'(s_j)\|\left[\frac{2}{q}-\frac{1}{30q^3}\left(2\int_0^1 k(s)ds-k\left(\frac{j}{q}\right)-k\left(\frac{j+1}{q}\right)\right)+\right.\\ &\qquad\left.-\frac{1}{3q^3}\frac{\|\gamma'''(s_j)\|}{\|\gamma'(s_j)\|}-\frac{1}{60q^4}\left(k'\left(\frac{j}{q}\right)+k'\left(\frac{j+1}{q}\right)\right)+O\left(\frac{1}{q^5}\right)\right]=\\ &=\|\gamma'(s_j)\|\left[\frac{2}{q}-\frac{1}{30q^3}\left(2\int_0^1 k(s)ds-k\left(\frac{j}{q}\right)-k\left(\frac{j+1}{q}\right)\right)+\right.\\ &\qquad\left.-\frac{1}{3q^3}\frac{\|\gamma'''(s_j)\|}{\|\gamma'(s_j)\|}-\frac{1}{60q^4}\left(k'\left(\frac{j}{q}\right)+k'\left(\frac{j+1}{q}\right)\right)+O\left(\frac{1}{q^5}\right)\right]=\\ &=\rho(t(s_j))^{\frac{1}{3}}\left[\frac{2}{q}-\frac{1}{30q^3}\left(2\int_0^1 k(s)ds-k\left(\frac{j}{q}\right)-k\left(\frac{j+1}{q}\right)\right)+\right.\\ &\qquad\left.-\frac{1}{3q^3}k(s_j)-\frac{1}{60q^4}\left(k'\left(\frac{j}{q}\right)+k'\left(\frac{j+1}{q}\right)\right)+O\left(\frac{1}{q^5}\right)\right].\end{aligned}$$

By Taylor expansions, the order 4 term cancels out and we finally get

$$\|\gamma(s_{j+1})-\gamma(s_{j-1})\|=\rho(t(s_j))^{\frac{1}{3}}\left[\frac{2}{q}+\frac{1}{15q^3}\left(k\left(\frac{j}{q}\right)-\int_0^1 k(s)ds\right)-\frac{1}{3q^3}k(s_j)+O\left(\frac{1}{q^5}\right)\right],$$

which is the desired result. □

Corollary 4.6.3. *Let $(\Omega_\tau)_{|\tau|\le 1}\subset\mathcal{D}^r$ be a normalized family of domains in $\mathcal{AS}_\delta^r$ or $\mathcal{R}_\delta^r$, parametrized by γ and δ-close to the circle. For $q\ge 3$, let $\mathbf{s}=\{s_j\}_{j=0}^q$ be the affine parameters corresponding to ordered vertices of a periodic trajectory for the symplectic billiard for the domain Ω_0, with rotation number $1/q$. Then*

$$\partial_\tau\omega_q(\tau,\mathbf{s})\mid_{\tau=0}=\sum_{j=0}^{q-1}n_\gamma(0,s_j)\,\rho(t(s_j))^{\frac{1}{3}}\left[\frac{2}{q}-\frac{(2\pi)^2}{3q^3}+\frac{\beta(j/q)}{q^3}+O\left(\frac{\delta}{q^5}\right)\right],$$

where ρ denotes the (usual) ray of curvature of the domain Ω_0 and

$$\beta(j/q)=\frac{1}{15}\left[k\left(\frac{j}{q}\right)-\int_0^1 k(s)ds\right]+\frac{1}{3}\left((2\pi)^2-k\left(\frac{j}{q}\right)\right).$$

Proof. Notice that, under the hypothesis of δ-closeness to the circle, the reminder of expansion (4.6.1) is $O\left(\frac{\delta}{q^5}\right)$. Then it is sufficient to substitute expansion (coming from (4.5.2)):

$$k(s_j)=k\left(\frac{j}{q}\right)+O\left(\frac{\delta}{q^2}\right)$$

in (4.6.1) and sum and subtract $\frac{(2\pi)^2}{3q^3}$ (remind that $(2\pi)^2$ is the affine curvature of the circle of unitary affine perimeter). □

4.6.1 Proof of Theorem 4.4.5

Consider $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{AS}^r_\delta$ a family of normalized area-isospectral axially symmetric domains. For $q \geq 3$, let $\bar{\mathbf{s}} = \{s_j\}_{j=0}^q$ be the affine parameters corresponding to the ordered vertices of the maximizing axially symmetric q-periodic orbit for the symplectic billiard in Ω_0, as constructed in Lemma 4.3.1. By Proposition 4.4.7, $\partial_\tau \omega_q(\tau, \bar{\mathbf{s}}) \mid_{\tau=0}= 0$. This means that, as a consequence of Corollary 4.6.3,

$$\sum_{j=0}^{q-1} n_\gamma(0, s_j)\, \rho(t(s_j))^{\frac{1}{3}} \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = 0.$$

Since, by hypothesis, $\rho(t(s)) > 0$ for every $s \in \mathbb{T}$, we indicate

$$u(s) = n_\gamma(0, s)\, \rho(t(s))^{\frac{1}{3}}$$

and investigate on

$$\sum_{j=0}^{q-1} u(s_j) \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = 0.$$

Since $u(s)$ is even and –by isospectral hypothesis– with zero-average, we substitute its Fourier expansion in the basis $(\cos(2\pi k s))_{k\geq 1}$. We get

$$\sum_{j=0}^{q-1}\sum_{k\geq 1} \hat{u}_k \cos(2\pi k s_j) \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = 0. \tag{4.6.2}$$

Formula (4.5.2) in the closeness to the circle assumption reads

$$s_j = \frac{j}{q} + \frac{\alpha(j/q)}{q^2} + O\left(\frac{\delta}{q^4}\right), \qquad \alpha(j/q) := \frac{1}{30}\int_0^{\frac{j}{q}} k(s)ds - \frac{j}{30q}\int_0^1 k(s)ds.$$

We remark that, in the axially symmetric case, the function α is odd and of class C^r and the function β of Corollary 4.6.3 is even and of class C^{r-1}.
Substituting in (4.6.2) the above expansion of s_j, we obtain (up to rename β):

$$\frac{2}{q}\sum_{j=0}^{q-1}\sum_{k\geq 1} \hat{u}_k \left[\cos\left(2\pi k\left(\frac{j}{q} + \frac{\alpha(j/q)}{q^2} + O\left(\frac{\delta}{q^4}\right)\right)\right)\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta(j/q)}{q^2} + O\left(\frac{\delta}{q^4}\right)\right)\right] = 0.$$

We now use Taylor expansion for the term involving the cosine:

$$\cos\left(2\pi k\left(\frac{j}{q} + \frac{\alpha(j/q)}{q^2} + O\left(\frac{\delta}{q^4}\right)\right)\right) = \cos\left(\frac{2\pi k j}{q}\right) - 2\pi k \sin\left(\frac{2\pi j k}{q}\right)\frac{\alpha(j/q)}{q^2} + O\left(\frac{\delta k}{q^4}\right) + O\left(\frac{\delta^2 k^2}{q^4}\right),$$

and the Fourier expansions of α and β. We then obtain:

$$\begin{aligned}
&\sum_{k\geq 1}\hat{u}_k\left[q\cdot\delta_{q|k}\left(1 - \frac{2\pi^2}{3q^2}\right) + O\left(\frac{\delta}{q^3}\right) + O\left(\frac{\delta^2k^2}{q^3}\right) + O\left(\frac{\delta k}{q^3}\right)\right] + \\
&+ \frac{1}{q^2}\hat{u}_k\sum_{j=0}^{q-1}\sum_{p\in\mathbb{Z}} -2\pi i k\left(\exp\left(\frac{2\pi i j k}{q}\right) - \exp\left(-\frac{2\pi i j k}{q}\right)\right)\alpha_p \exp\left(\frac{2\pi i p j}{q}\right) + \\
&+ \left(\exp\left(\frac{2\pi i j k}{q}\right) + \exp\left(-\frac{2\pi i j k}{q}\right)\right)\beta_p \exp\left(\frac{2\pi i p j}{q}\right) = \\
&= \sum_{k\geq 1}\hat{u}_k\left[q\cdot\delta_{q|k}\left(1 - \frac{2\pi^2}{3q^2}\right) + O\left(\frac{\delta^2k^2}{q^3}\right) + O\left(\frac{\delta k}{q^3}\right)\right] + \hat{u}_k\frac{1}{q}\sum_{s\in\mathbb{Z}}\left[\beta_{sq-k} + \beta_{sq+k} + 2\pi i k(\alpha_{sq+k} - \alpha_{sq-k})\right] = 0.
\end{aligned}$$

Since $\alpha_p = -\alpha_{-p}$ and $\beta_p = \beta_{-p}$, we finally get:

$$\sum_{k\geq 1} \hat{u}_k \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \delta_{q|k} + \frac{2}{q^2} \sum_{\substack{s\in\mathbb{Z} \\ sq\neq k}} (\beta_{sq-k} - 2\pi i k \alpha_{sq-k}) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = 0,$$

or, equivalently (for all $q \geq 3$):

$$\sum_{k\geq 1} \hat{u}_k \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \delta_{q|k} + \frac{2}{q^2} \sum_{\substack{0\neq s\in\mathbb{Z} \\ sq\neq k}} (\beta_{sq-k} - 2\pi i k \alpha_{sq-k}) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = \tag{4.6.3}$$

$$= -\frac{2}{q^2} \sum_{k\geq 1} \hat{u}_k (\beta_k + 2\pi i k \alpha_k). \tag{4.6.4}$$

Since we prove that the right-hand side of equality (4.6.3) is 0 (we refer to the discussion at the end of the proof of the next theorem), Theorem 4.4.5 becomes a straightforward consequence of the next lemma, whose proof takes up most of the section.

Lemma 4.6.4. *If $u(s) = n_\gamma(0, s)\, \rho(t(s))^{\frac{1}{3}}$ solves*

$$\sum_{k\geq 1} \hat{u}_k \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \delta_{q|k} + \frac{2}{q^2} \sum_{\substack{0\neq s\in\mathbb{Z} \\ sq\neq k}} (\beta_{sq-k} - 2\pi i k \alpha_{sq-k}) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = 0 \tag{4.6.5}$$

for all $q \geq 3$, then $n_\gamma \equiv 0$.

Proof. For all $q \geq 3$, (4.6.5) equals to

$$-\hat{u}_q \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) = \sum_{k\geq 2} \hat{u}_{kq} \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) +$$
$$+ \sum_{k\geq 1} \hat{u}_k \left[\frac{2}{q^2} \sum_{\substack{0\neq s\in\mathbb{Z} \\ sq\neq k}} (\beta_{sq-k} - 2\pi i k \alpha_{sq-k}) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right].$$

Notice that

$$0.26 < \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) < 1 \qquad \forall q \geq 3$$

and take the absolute value in the last equality. We then obtain:

$$|\hat{u}_q| \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \leq$$
$$\leq \sum_{k\geq 2} |\hat{u}_{kq}| \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2}{q^2} \sum_{k\geq 1} |\hat{u}_k| \sum_{\substack{0\neq s\in\mathbb{Z} \\ sq\neq k}} |\beta_{sq-k} - 2\pi i k \alpha_{sq-k}| + \sum_{k\geq 1} |\hat{u}_k| \left| O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right|.$$

In the sequel, for $f \in C^{r-1}(\mathbb{T})$ with zero average, fixed an integer $1 \leq t \leq r-1$, we indicate with $\|f\|_t$ the norm

$$\|f\|_t := \max_{k\in\mathbb{Z}} |\hat{f}_k \cdot k^t| \Rightarrow |\hat{f}_k| \leq \frac{\|f\|_t}{|k|^t} \quad \forall k \neq 0, \tag{4.6.6}$$

where $\hat{f}_k$ denotes the k-th Fourier coefficient of f.
Closeness to the circle assumption implies

$$\|\alpha\|_t \leq A\delta, \qquad \|\beta\|_t \leq B\delta$$

for some uniform constants $A, B > 0$. Recalling that $u \in C^{r-1}$, and both functions α and β are at least C^{r-1}, for $1 \leq t \leq r-1$ we get

$$\begin{aligned}
&|\hat{u}_q| \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \leq \\
&\leq \sum_{k\geq 2} \frac{\|u\|_t}{(kq)^t} \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2}{q^2} \sum_{k\geq 1} \frac{\|u\|_t}{k^t} \sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq k}} |\beta_{sq-k} - 2\pi i k \alpha_{sq-k}| + \sum_{k\geq 1} \frac{\|u\|_t}{k^t} \left| O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right)\right| \\
&\leq \sum_{k\geq 2} \frac{\|u\|_t}{(kq)^t} \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2}{q^2} \sum_{k\geq 1} \frac{\|u\|_t}{k^t} \sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq k}} \left(\frac{B\delta}{|sq-k|^t} + \frac{A\delta k}{|sq-k|^t}\right) + \sum_{k\geq 1} \frac{\|u\|_t}{k^t} \left| O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right)\right| \\
&\leq \sum_{k\geq 2} \frac{\|u\|_t}{(kq)^t} \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{\bar{C}\|u\|_t}{q^2} \sum_{k\geq 1} \sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq k}} \frac{\delta k}{|sq-k|^t k^t} + \sum_{k\geq 1} \frac{\|u\|_t}{k^t} \left| O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right)\right|.
\end{aligned} \tag{4.6.7}$$

where $\bar{C} = 4\max\{A, B\} > 0$. Let us now look at each summand (the aim is showing that $\hat{u}_q = 0$ for every $q \geq 3$).

(a) The first term satisfies

$$\sum_{k\geq 2} \frac{\|u\|_t}{(kq)^t} \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \leq \frac{\|u\|_t}{q^t} (\zeta(t) - 1) \left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right).$$

Notice that to get the convergence of the series we need to ask $t \geq 2$.

(b) The third term satisfies

$$\sum_{k\geq 1} \frac{\|u\|_t}{k^t} \left| O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right)\right| \leq \frac{\|u\|_t}{q^4} (\delta^2 K_1 \zeta(t-2) + \delta K_2 \zeta(t-1))$$

for some constants $K_1, K_2 > 0$. For this term, in order to assure the convergence of the series, we need to ask $t \geq 4$.

(c) The second summand requires more work, and we use arguments analogous to the ones in

[41][Proof of Lemma 5.3]. Up to $\frac{\bar{C}\delta\|u\|_t}{q^2}$, the term to study is

$$
\sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq k}}\sum_{k\geq 1}\frac{1}{k^{t-1}|sq-k|^t}\leq 2\sum_{\substack{s\geq 1\\ sq\neq k}}\sum_{k\geq 1}\frac{1}{k^{t-1}|sq-k|^t}=
$$

$$
=2\sum_{s\geq 1}\left(\sum_{1\leq k<\frac{sq}{2}}\frac{1}{k^{t-1}(sq-k)^t}+\sum_{\frac{sq}{2}\leq k<sq}\frac{1}{k^{t-1}(sq-k)^t}+\sum_{k>sq}\frac{1}{k^{t-1}|sq-k|^t}\right)\leq
$$

$$
\leq 2\sum_{s\geq 1}\left(\sum_{1\leq k<\frac{sq}{2}}\frac{2^t}{k^{t-1}s^tq^t}+\sum_{\frac{sq}{2}\leq k<sq}\frac{2^{t-1}}{s^{t-1}q^{t-1}(sq-k)^t}+\sum_{k>sq}\frac{1}{s^{t-1}q^{t-1}|sq-k|^t}\right)\leq
$$

$$
\leq 2\left(\frac{2^t\zeta(t-1)\zeta(t)}{q^t}+\frac{2^{t-1}\zeta(t-1)\zeta(t)}{q^{t-1}}+\frac{\zeta(t-1)\zeta(t)}{q^{t-1}}\right)=\frac{2\zeta(t-1)\zeta(t)}{q^{t-1}}\left(\frac{2^t}{q}+2^{t-1}+1\right).
$$

Chosing $t=4$ and summing up the three estimates obtained in (a), (b) and (c), we obtain:

$$
|\hat{u}_q|\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)\leq\frac{\|u\|_4}{q^4}\left[(\zeta(4)-1)\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)+\right.
$$
$$
\left.+\frac{2\bar{C}\delta\zeta(3)\zeta(4)}{q}\left(\frac{16}{q}+9\right)+\delta^2K_1\zeta(2)+\delta K_2\zeta(3)\right].
$$

Since $\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)>0.26$ for every $q\geq 3$, we finally get

$$
|\hat{u}_q|\leq\frac{\|u\|_4}{q^4}\left((\zeta(4)-1)+\mathfrak{C}\delta\right)\qquad\forall q\geq 3
$$

where

$$
\mathfrak{C}:=\frac{10\bar{C}\zeta(3)\zeta(4)+K_1\zeta(2)+K_2\zeta(3)}{0.26}\tag{4.6.8}
$$

does not depend on $q\geq 3$ and $\delta>0$. Consequently, for $\delta>0$ sufficiently small, it holds

$$
|\hat{u}_q|<D\frac{\|u\|_4}{q^4}\qquad\forall q\geq 3,
$$

for a constant $D<1$. Note that –by the normalization conditions– we also have

$$
u(0)=\sum_{k=1}^{\infty}\hat{u}_k=0\qquad\text{and}\qquad u(0)+u\left(\frac{1}{2}\right)=2\sum_{k=1}^{\infty}\hat{u}_{2k}=0.
$$

From these it follows that also $|\hat{u}_2|<\frac{\|u\|_4}{16}$ and $|\hat{u}_1|<\|u\|_4$. At the end, we conclude that $\|u\|_4=0$ and therefore $n_\gamma\equiv 0$. □

We are now ready to conclude the proof of Theorem 4.4.5. From the explicit estimates contained in the proof of the previous lemma, it follows that all the terms involved in the linear system (4.6.5) are $O\left(\frac{1}{q^4}\right)$. Differently, the term in the right member of (4.6.3) is $O\left(\frac{1}{q^2}\right)$. Consequently, since the left members of (4.6.3) and (4.6.5) are exactly the same, if $u(s)=n_\gamma(0,s)\,\rho(t(s))^{\frac{1}{3}}$ solves (4.6.3), then necessarily $\sum_{k\geq 1}\hat{u}_k(\beta_k+2\pi ik\alpha_k)=0$ and therefore (as a consequence of Lemma 4.6.4) $u\equiv 0$. This means that $n_\gamma\equiv 0$, which is the desired result.

4.6.2 Proof of Theorem 4.4.6

We argue as in the proof of Theorem 4.4.5. Consider $(\Omega_\tau)_{|\tau|\leq 1} \subset \mathcal{R}^r_\delta$ a family of normalized area-isospectral domains.

Let $\bar{\mathbf{s}} = (\bar{s}_0, \dots, \bar{s}_q)$ be the affine parameters corresponding to an orbit realizing the maximum (4.4.5). Then, by Proposition 4.4.7, $\partial_\tau \omega_q(\tau, \bar{\mathbf{s}}) \mid_{\tau=0}= 0$. This means that, as a consequence of Corollary 4.6.3,

$$\sum_{j=0}^{q-1} n_\gamma\left(0, s_j\right) \rho(t(s_j))^{\frac{1}{3}} \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q+s_0)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = 0.$$

Since by strong convexity $\rho(t(s)) > 0$ for every $s \in \mathbb{T}$, as in the axially symmetric case we define $u(s) := n_\gamma\left(0, s\right) \rho(t(s))^{\frac{1}{3}}$ and prove that $u \equiv 0$. The function u has the same properties of n_γ so in particular it is $1/2$-periodic and its odd Fourier coefficients are all zero, which gives

$$\begin{aligned}&\sum_{j=0}^{q-1} u(s_j) \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q+s_0)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = \\ =&\sum_{j=0}^{q-1} \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_j} \left[\frac{2}{q} - \frac{(2\pi)^2}{3q^3} + \frac{\beta(j/q+s_0)}{q^3} + O\left(\frac{\delta}{q^5}\right)\right] = 0.\end{aligned}$$

The affine parameters of the q-periodic orbit, similarly to the axially symmetric case, have the following expansion:

$$s_j = s_0 + \frac{j}{q} - \frac{j}{30q^3}\int_0^1 k(s)ds + \frac{1}{30q^2}\int_{s_0}^{s_0+\frac{j}{q}} k(s)ds + O\left(\frac{1}{q^4}\right) = s_0 + \frac{j}{q} + \frac{\alpha(j/q, s_0)}{q^2} + O\left(\frac{\delta}{q^4}\right) \tag{4.6.9}$$

where

$$\alpha(j/q, s_0) := \frac{1}{30}\int_{s_0}^{s_0+\frac{j}{q}} k(s)ds - \frac{j}{30q}\int_0^1 k(s)ds.$$

We underline that both the functions $\alpha(\cdot, s_0)$ and $\beta(\cdot)$ are $1/2$-periodic so that only their even Fourier coefficients are non zero. By mean of (4.6.9) and by Taylor expansions, we obtain (up to rename β):

$$\begin{aligned}&\sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \sum_{j=0}^{q-1} \left[e^{\frac{4\pi i k j}{q}} + 4\pi i k e^{\frac{4\pi i k j}{q}} \frac{\alpha(j/q, s_0)}{q^2} + O\left(\frac{\delta k}{q^4}\right) + O\left(\frac{\delta^2 k^2}{q^4}\right)\right] \left[1 - \frac{2\pi^2}{3q^2} + \frac{\beta(j/q+s_0)}{q^2} + O\left(\frac{\delta}{q^4}\right)\right] \\ =& \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[q \cdot \delta_{q|2k}\left(1 - \frac{2\pi^2}{3q^2}\right) + O\left(\frac{\delta^2 k^2}{q^3}\right) + O\left(\frac{\delta k}{q^3}\right) + \frac{1}{q^2}\sum_{j=0}^{q-1} e^{\frac{4\pi i k j}{q}} \left(\beta(j/q+s_0) + 4\pi i k \alpha(j/q, s_0)\right)\right] \\ =& \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[q \cdot \delta_{q|2k}\left(1 - \frac{2\pi^2}{3q^2}\right) + O\left(\frac{\delta^2 k^2}{q^3}\right) + O\left(\frac{\delta k}{q^3}\right) + \frac{1}{q^2}\sum_{j=0}^{q-1}\sum_{p\in\mathbb{Z}} e^{\frac{4\pi i k j}{q}} e^{\frac{4\pi i p j}{q}} \left(\beta_{2p} e^{4\pi i p s_0} + 4\pi i k \alpha_{2p}\right)\right] \\ =&\ 0.\end{aligned}$$

We stress that, in the above formula, we have omitted the dependence on s_0 of the Fourier coefficients of α, and simply indicate $\alpha_p = \alpha_p(\cdot) = \alpha_p(\cdot, s_0)$. Therefore, similarly to the axially symmetric case, we have (for every even $q \geq 4$):

$$\sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \Bigg[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) \delta_{q|2k} + \frac{4\pi i k \alpha_0}{q^2} \delta_{q|2k} + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) + \\ + \frac{1}{q^2} \sum_{\substack{s\in\mathbb{Z}\\ sq\neq 2k}} \left(e^{4\pi i (sq-2k)s_0} \beta_{sq-2k} + 4\pi i k \alpha_{sq-2k}\right) \Bigg] = 0.$$

Equivalently, for every even $q \geq 4$:

$$\sum_{|k|\geq 1} \hat{u}_{kq} e^{2\pi i k q s_0} \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2\pi i k \alpha_0}{q} \right] + \\ + \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[\frac{1}{q^2} \sum_{\substack{s\in\mathbb{Z}\\ sq\neq 2k}} \left(e^{4\pi i (sq-2k)s_0} \beta_{sq-2k} + 4\pi i k \alpha_{sq-2k}\right) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = 0.$$

As in the axially symmetric case, we isolate in the right member the term with $s = 0$ so that the previous formula becomes:

$$\sum_{|k|\geq 1} \hat{u}_{kq} e^{2\pi i k q s_0} \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2\pi i k \alpha_0}{q} \right] + \\ + \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[\frac{1}{q^2} \sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq 2k}} \left(e^{4\pi i (sq-2k)s_0} \beta_{sq-2k} + 4\pi i k \alpha_{sq-2k}\right) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = \\ = -\frac{1}{q^2} \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[e^{-8\pi i k s_0} \beta_{-2k} + 4\pi i k \alpha_{-2k} \right]. \tag{4.6.10}$$

Now, arguing as at the end of the proof of Theorem 4.4.5, we see that all the terms involved in the left member of the linear system (4.6.10) are $O\left(\frac{1}{q^4}\right)$ while the term in the right member is $O\left(\frac{1}{q^2}\right)$. Consequently, if $u(s) = n_\gamma(0, s)\, \rho(t(s))^{\frac{1}{3}}$ solves (4.6.10), then necessarily, for every choice of the starting point s_0,

$$\sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[e^{-8\pi i k s_0} \beta_{-2k} + 4\pi i k \alpha_{-2k} \right] = 0. \tag{4.6.11}$$

In other words, solutions of (4.6.10) are solutions of

$$\sum_{|k|\geq 1} \hat{u}_{kq} e^{2\pi i k q s_0} \left[\left(1 - \frac{2\pi^2}{3q^2} + \frac{\beta_0}{q^2}\right) + \frac{2\pi i k \alpha_0}{q} \right] + \\ + \sum_{|k|\geq 1} \hat{u}_{2k} e^{4\pi i k s_0} \left[\frac{1}{q^2} \sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq 2k}} \left(e^{4\pi i (sq-2k)s_0} \beta_{sq-2k} + 4\pi i k \alpha_{sq-2k}\right) + O\left(\frac{\delta^2 k^2}{q^4}\right) + O\left(\frac{\delta k}{q^4}\right) \right] = 0$$

or equivalently of

$$\begin{aligned}
&-\left(\hat{u}_q e^{2\pi i q s_0}+\hat{u}_{-q}e^{-2\pi i q s_0}\right)\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)=\\
&=\sum_{|k|\geq 2}\hat{u}_{kq}e^{2\pi i k q s_0}\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)+\sum_{|k|\geq 1}\hat{u}_{kq}e^{2\pi i k q s_0}\frac{2\pi i k\alpha_0}{q}+\\
&+\sum_{|k|\geq 1}\hat{u}_{2k}e^{4\pi i k s_0}\left[\frac{1}{q^2}\sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq 2k}}\left(e^{4\pi i(sq-2k)s_0}\beta_{sq-2k}+4\pi i k\alpha_{sq-2k}\right)+O\left(\frac{\delta^2k^2}{q^4}\right)+O\left(\frac{\delta k}{q^4}\right)\right].
\end{aligned} \tag{4.6.12}$$

We first use hypothesis (a) in Definition 4.3.3, assuring the existence of a point $\gamma(\tau,0)\in\partial\Omega_\tau$ such that for every even $q\geq 4$ there exists a centrally symmetric periodic orbit of rotation number $1/q$ passing through $\gamma(\tau,0)$ and of maximal area among all centrally symmetric q-periodic orbits. For even $q\geq 4$, let $\bar{\mathbf{s}}=\{s_j\}_{j=0}^q$ be the affine parameters corresponding to such an orbit (hence with $s_0=0$). By the special choice of $\bar{\mathbf{s}}$, assured by hypothesis (a) of Definition 4.3.3, equalities (4.6.11) and (4.6.12) above hold in particular for $s_0=0$. Consequently, since $\hat{u}_{-q}=\overline{\hat{u}_q}$,

$$\hat{u}_q+\hat{u}_{-q}=2\mathfrak{Re}(\hat{u}_q)$$

and inequality (4.6.12) becomes:

$$\begin{aligned}
2|\mathfrak{Re}(\hat{u}_q)|\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)\leq&\sum_{|k|\geq 2}|\hat{u}_{kq}|\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)+\frac{2\pi|\alpha_0|}{q}\sum_{|k|\geq 1}|\hat{u}_{kq}||k|+\\
&+\sum_{|k|\geq 1}|\hat{u}_{2k}|\left[\frac{1}{q^2}\sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq 2k}}|\beta_{sq-2k}+4\pi k\alpha_{sq-2k}|+\left|O\left(\frac{\delta^2k^2}{q^4}\right)+O\left(\frac{\delta k}{q^4}\right)\right|\right]
\end{aligned} \tag{4.6.13}$$

We finally use hypothesis (b) in Definition 4.3.3, assuring the existence, for every even $q\geq 4$ and $k\geq 2$, of a centrally symmetric periodic orbit of rotation number $1/2k$ passing through $\gamma(\tau,1/4q)$ and of maximal area among all $2k$-periodic centrally symmetric orbits. Repeating the same argument as above, we obtain that equalities (4.6.11) and (4.6.12) hold for $s_0=1/4q$. Hence, since

$$\left(\hat{u}_q e^{\frac{\pi i}{2}}+\hat{u}_{-q}e^{-\frac{\pi i}{2}}\right)=-2\mathfrak{Im}(\hat{u}_q),$$

in such a case, (4.6.12) reads:

$$\begin{aligned}
2|\mathfrak{Im}(\hat{u}_q)|\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)\leq&\sum_{|k|\geq 2}|\hat{u}_{kq}|\left(1-\frac{2\pi^2}{3q^2}+\frac{\beta_0}{q^2}\right)+\frac{2\pi|\alpha_0|}{q}\sum_{|k|\geq 1}|\hat{u}_{kq}||k|+\\
&+\sum_{|k|\geq 1}|\hat{u}_{2k}|\left[\frac{1}{q^2}\sum_{\substack{0\neq s\in\mathbb{Z}\\ sq\neq 2k}}|\beta_{sq-2k}+4\pi k\alpha_{sq-2k}|+\left|O\left(\frac{\delta^2k^2}{q^4}\right)+O\left(\frac{\delta k}{q^4}\right)\right|\right].
\end{aligned}$$

In order to conclude, we notice now that the unique difference from the axially symmetric case is represented by the term $\frac{2\pi|\alpha_0|}{q}\sum_{|k|\geq 1}|\hat{u}_{kq}||k|$ which however is estimated as follows:

$$\frac{2\pi|\alpha_0|}{q}\sum_{|k|\geq 1}|\hat{u}_{kq}||k|\leq\frac{\|u\|_t}{q^{t+1}}(A'\delta\zeta(t-1)), \tag{4.6.14}$$

for a uniform constant $A' > 0$ (given by the closeness of the circle assumption) and $t \geq 3$. We now use estimates (a)–(c) already given in the proof of Theorem 3.5.1 as well as the last one (4.6.14). Choosing $t = 4$ and summing up all the contributions, we obtain

$$|\hat{u}_q| \leq (|\mathfrak{Re}(\hat{u}_q)| + |\mathfrak{Im}(\hat{u}_q)|) \leq \frac{\|u\|_4}{q^4}\left[2(\zeta(4)-1) + \delta(2\mathfrak{C} + A'\zeta(3))\right]$$

where $\mathfrak{C} > 0$ is the constant given in (4.6.8), which is independent on even $q \geq 4$ and $\delta > 0$. Consequently, for $\delta > 0$ sufficiently small, it holds

$$|\hat{u}_q| < E\frac{\|u\|_4}{q^4} \qquad \forall \text{ even } q \geq 4,$$

for a constant $E < 1$. In order to obtain the estimates for $|\mathfrak{Re}(\hat{u}_2)|$ and $|\mathfrak{Im}(\hat{u}_2)|$, which are necessary to conclude, we invoke the normalization conditions:

$$u(0) = \sum_{|k|\geq 1} \hat{u}_{2k} = \sum_{k\geq 1}(\hat{u}_{2k} + \overline{\hat{u}}_{2k}) = 2\sum_{k\geq 1}\mathfrak{Re}(u_{2k}) = 0. \tag{4.6.15}$$

Moreover, since

$$u'(s) = 4\pi i \sum_{|k|\geq 1} k\hat{u}_{2k}e^{4\pi iks} = 4\pi i \sum_{k\geq 1}\left[k\hat{u}_{2k}e^{4\pi iks} - k\overline{\hat{u}}_{2k}e^{-4\pi iks}\right],$$

we have that

$$u'(0) + u'\left(\frac{1}{2}\right) = 8\pi i \sum_{k\geq 1} k(\hat{u}_{2k} - \overline{\hat{u}}_{2k}) = -16\pi \sum_{k\geq 1} k\mathfrak{Im}(\hat{u}_{2k}) = 0. \tag{4.6.16}$$

From (4.6.15) and (4.6.16), we immediately get

$$\mathfrak{Re}(\hat{u}_2) = -\sum_{k>1}\mathfrak{Re}(\hat{u}_{2k}) \qquad \text{and} \qquad \mathfrak{Im}(\hat{u}_2) = -\sum_{k>1} k\mathfrak{Im}(\hat{u}_{2k}).$$

Finally, taking into account the definition of $\|u\|_4$ (see 4.6.6), the next estimate holds:

$$|\mathfrak{Re}(\hat{u}_2)| = \left|\sum_{k>1}\mathfrak{Re}(\hat{u}_{2k})\right| < \frac{\|u\|_4}{16}\sum_{k>1}\frac{1}{k^4} = \frac{\|u\|_4}{16}(\zeta(4)-1)$$

and similarly:

$$|\mathfrak{Im}(\hat{u}_2)| \leq \left|\sum_{k>1} k\mathfrak{Im}(\hat{u}_{2k})\right| < \frac{\|u\|_4}{16}(\zeta(3)-1).$$

It then follows that also $|\hat{u}_2| < \frac{\|u\|_4}{16}$ At the end, we conclude that $\|u\|_4 = 0$ so that $n_\gamma \equiv 0$ and the statement follows.

Chapter 5

Birkhoff attractors for dissipative symplectic billiards

ABSTRACT. This chapter presents the paper "Birkhoff attractors for dissipative symplectic billiards", joint work with L. Baracco, O. Bernardi and A. Florio [15]. The aim of this paper is to propose and study a dissipative variant of symplectic billiards within planar strictly convex domains. The associated billiard map is dissipative, thus it admits a compact invariant set, the so-called Birkhoff attractor. Its complexity depends on the rate of the dissipation as well as on the geometry of the billiard table. We prove that (a) for strong dissipation, the Birkhoff attractor is a normally contracted graph over the zero section; (b) for mild dissipation, the Birkhoff attractor within a centrally symmetric domain is an indecomposable continuum whose restricted dynamics has positive topological entropy. We compare these results with the case of dissipative Birkhoff billiards, studied in [20].

5.1 Introduction

Symplectic billiards were introduced by P. Albers and S. Tabachnikov in 2018 [1] and were subsequently studied by various authors who investigated integrability [13], [16], its Mather's β-function [17] and area spectral rigidity [18], [43], [89].

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^k boundary $\partial\Omega$, $k \geq 2$. Fix an origin $O \in \mathrm{int}(\Omega)$ and an orientation of $\partial\Omega$. Let $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z} \ni t \mapsto \gamma(t) \in \mathbb{R}^2$ be a C^k parametrization of $\partial\Omega$ such that $\gamma(\mathbb{S}) = \partial\Omega$. Given $t_1, t_2 \in \mathbb{S}$, if $\gamma(t_1)$ and $\gamma(t_2)$ are two successive bounces, then the next bounce under the symplectic dynamics occurs at $\gamma(\tilde{t}_3)$ if and only if the vector $\gamma(\tilde{t}_3) - \gamma(t_1)$ is parallel to $\gamma'(t_2)$.

The associated billiard map T is a twist map, which preserves an area form and whose generating function is

$$L(t_1, t_2) := \det(\gamma(t_1) - O, \gamma(t_2) - O) \,.$$

We refer to Section 5.2.2 for all details. In the present paper, we introduce a definition of *dissipative* symplectic billiard map, as explained right below.

Fix $\lambda \in (0, 1]$. If $\gamma(t_1)$ and $\gamma(t_2)$ are two successive bounces, then the dissipative symplectic dynamics gives $\gamma(t_3)$ as next bounce if and only if the vector $\gamma(t_3) - \lambda\gamma(t_1)$ is parallel to $\gamma'(t_2)$. In other words, the segment passing through $\gamma(t_3)$ and parallel to $\gamma'(t_2)$ is approaching the origin, when λ approaches O. See Figure 5.1. We will see that the dissipative billiard map T_λ is still a twist map, but it no longer preserves an area form. Clearly $T_1 = T$.

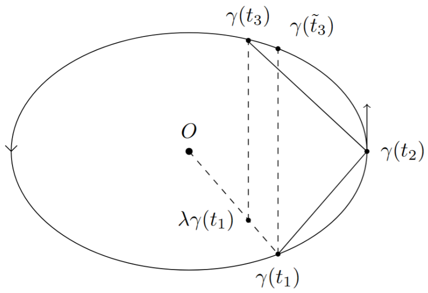


Figure 5.1: The dissipative symplectic billiard map compared with the conservative one.

A dissipative version of standard (Birkhoff) billiards has been recently introduced by Bernardi, Florio and Leguil in [20]. In such a case, the usual reflection law, which requires that the angle of incidence equals the angle of reflection, is changed in such a way that the reflected angle bends toward the inner normal at the incidence point. See Figure 5.2. Moreover, different notions of billiard with some form of dissipation have previously been considered by other authors, see for example [40], [86], [67], ...

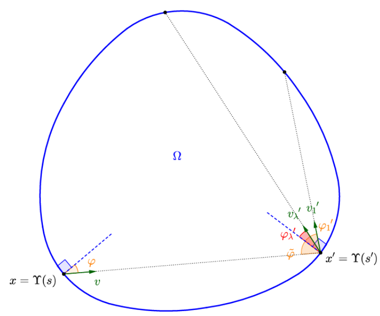


Figure 5.2: The dissipative Birkhoff billiard map compared with the conservative one.

Let us now move into some properties of dissipative symplectic billiards, see Section 5.2.3 for all details. We first remark that –up to an appropriate choice of coordinates– the phase space for the symplectic billiard map T is a bounded cylinder, denoted by $\mathcal{P}$. In the coordinates $(t, s) \in \mathcal{P}$, it results that

$$T_\lambda(t, s) = \mathcal{H}_\lambda \circ T(t, s)\,,$$

where $\mathcal{H}_\lambda$ is the λ-contraction map along the fiber, i.e., $\mathcal{H}_\lambda(t, s) = (t, \lambda s)$. Clearly, the map T_λ is no longer conservative; in fact, it is dissipative in the sense of Le Calvez [64], see Definition 5.2.1. In particular, T_λ contracts the standard area form, that is $T_\lambda^*(dt \wedge ds) = \lambda dt \wedge ds$. According to the vocabulary of symplectic dynamics, the map T_λ is conformally symplectic. Due to its dissipative character, the map T_λ admits a *global attractor* "à la Conley"

$$\Lambda_0 := \bigcap_{n \in \mathbb{N}} T_\lambda^n(\mathcal{P})\,,$$

which is not empty, compact, connected and T_λ-invariant. Moreover, it separates the phase space $\mathcal{P}$, that is its complement $\mathcal{P} \setminus \Lambda_0$ is a disjoint union of two connected open sets U_- and U_+. Following Birkhoff [32] and Le Calvez [65, Page 91]– it is possible to detect the smallest compact, connected, T_λ-invariant set that separates $\mathcal{P}$: the so-called *Birkhoff attractor*. Obtained from Λ_0 "by removing the whiskers", it is then defined as

$$\Lambda := \mathrm{cl}(U_-) \cap \mathrm{cl}(U_+)\,.$$

We need to underline that, differently from Λ_0 and despite its name, in general Λ is not an attractor in the usual sense, that is it is not the omega-limit set of one of its neighborhoods. We refer to Section 5.2.1 for all details on the Birkhoff attractor Λ.

The study of the properties and the structure of the Birkhoff attractor for dissipative maps of the annulus, begun by Birkhoff [32], pursued with Charpentier [35], Le Calvez [64, 65], Crovisier [37], ... Different criteria to obtain chaotic invariant continua for dissipative maps are proved in [22, 62, 78, 79, 34], ... Recently, the notion of Birkhoff attractor has been generalized in higher dimensions by Arnaud, Humilière and Viterbo, see [8, 91, 9], for conformally symplectic maps on cotangent bundles. This generalization relies on symplectic invariants of exact Lagrangian submanifolds.

More generally, in the last years, conformally symplectic dynamics has been an active research area, see [6, 7, 3], ... The dissipative systems associated to the discounted Hamilton-Jacobi equation have been studied in [39, 93, 36, 8], showing interesting properties of solutions of such equations through Weak KAM methods. In the dissipative Hamiltonian framework, invariant KAM-like tori for conformally symplectic flows are studied by Calleja, Celletti and de la Llave [33]; very recently, Gidea, de la Llave and Seara [45] have studied the geometry and the structure of normally hyperbolic manifolds and their scattering maps for conformally symplectic Hamiltonian flows.

The aim of the present paper is to study the dynamical and topological complexity of the Birkhoff attractor in terms of the rate of the dissipation $\lambda \in (0,1)$ as well as in terms of the geometry of the billiard table.

In Section 5.2.3 we start by discussing some domains whose corresponding dissipative symplectic billiard dynamics T_λ has –independently on $\lambda \in (0,1)$– a particularly simple Birkhoff attractor, that is $\Lambda = \mathbb{S} \times \{0\}$. This is the case of any (centrally symmetric) Radon domain Ω, once fixed the origin O to be the center of symmetry, see Proposition 5.2.24 and Remark 5.2.25. It is worth noting that this is a first substantial difference compared to dissipative Birkhoff billiards, whose Birkhoff attractor is assured to be the zero section only in the case of the circle.

Section 5.3 is devoted to study Λ for dissipative symplectic billiards when the dissipation is strong, i.e., $0 < \lambda \ll 1$. In such a case, independently on the choice of the origin, T_λ exhibits a topologically simple Birkhoff attractor, that is Λ coincides with Λ_0 and it is a normally contracted graph over $\mathbb{S}$, as precised in the following statement.

Theorem. Let $\Omega \subset \mathbb{R}^2$ be a strongly convex (i.e., with never vanishing curvature) domain with C^k boundary, $k \geq 2$.

(a) There exists $\lambda(\Omega) \in (0,1)$ such that, for $\lambda \in (0, \lambda(\Omega))$, the Birkhoff attractor Λ coincides with the global attractor Λ_0 and it is a normally contracted C^1 graph over $\mathbb{S}$.

(b) There exists $\lambda'(\Omega) < \lambda(\Omega)$ such that, for $\lambda \in (0, \lambda'(\Omega))$, the Birkhoff attractor Λ is a C^{k-1} graph over $\mathbb{S}$ and it converges to $\mathbb{S} \times \{0\}$, as $\lambda \to 0$, in the C^1 topology.

We refer to Theorem 5.3.7 for all details. The proof is based on a cone-field criterion (see Proposition 5.3.1) and standard results in normally hyperbolic dynamics, and it follows the same lines of the proof of the corresponding result for dissipative Birkhoff billiards (see [20, Theorem 5.7]). However, in the Birkhoff dissipative dynamics, the analogous theorem does not hold in general, but only for a class of strictly convex domains satisfying a geometric *pinching* condition (see [20, Definition D]), while in the symplectic case, no hypothesis on the geometry of the table is needed.

In Section 5.4, we focus on dissipative symplectic billiard maps for a centrally symmetric domain Ω. In particular, we prove that, for a rate of dissipation small enough (i.e., when the dissipation is strong), not only the Birkhoff attractor is a graph, but we can describe quite explicitely the dynamics restricted to it. We refer to Propositions 5.4.2 and 5.4.7 for the detailed statements and we resume here the main result.

Proposition. Let Ω be a strongly convex centrally symmetric domain with C^k boundary, $k \geq 2$.

(*a*) There exists $\lambda(\Omega) \in (0,1)$ such that, for $\lambda \in (0, \lambda(\Omega))$, the Birkhoff attractor Λ intersects the zero section exactly in the 4-periodic points.

(*b*) There exists $\lambda'(\Omega) \in (0,1)$ such that, for $\lambda \in (0, \lambda'(\Omega))$, there exists an open and dense set of centrally symmetric domains, whose associated Birkhoff attractor has rotation number 1/4 and decomposes as

$$\Lambda = \bigcup_{i=1}^{l} \bigcup_{j=0}^{3} \overline{\mathcal{W}^u(T_\lambda^j(H_i); T_\lambda^4)}\,,$$

where $\{H_i\}_{i=1}^{l}$ is a finite family of 4-periodic points of saddle type, and $\mathcal{W}^u(H; T_\lambda^4)$ is the unstable manifold of a saddle point H, with respect to the dynamics of T_λ^4.

For dissipative Birkhoff billiards, the corresponding results are Corollary 3.4 and Theorem 5.14 in [20]. Even if the main ideas of the proofs in the two settings have common points, for dissipative symplectic billiards there is no need –as noticed above– of any geometric pinching condition of the table. Moreover, in the case of dissipative Birkhoff billiards, axial symmetry (instead of central symmetry) and 2-periodic orbits (instead of 4-periodic orbits) play a fundamental role.

Section 5.5 focuses on the Birkhoff attractor under conditions of weak dissipation, i.e., when the rate of dissipation is close to 1. In contrast with the previous cases, we obtain examples of complex Birkhoff attractors. A sufficient condition in order to observe topologically and dynamically intriguing phenomena is that $T_1 = T$ admits an instability region containing the zero section $\mathbb{S} \times \{0\}$. This follows by an adaptation of a result of Le Calvez (here Proposition 5.5.5): in such a case, the Birkhoff attractor Λ for the corresponding dissipative dynamics, if λ is close enough to 1, admits different upper and lower rotation numbers $\rho^- < \rho^+$, dynamical quantities defined in Section 5.2.1. This property has various consequences for Λ, all already observed in [20] for dissipative Birkhoff billiards. In particular:

(*i*) Λ is an *indecomposable continuum*, that is it cannot be written as the union of two compact, connected, non trivial sets (from a result by Charpentier, see [35]).

(*ii*) For every rational $\frac{p}{q} \in (\rho^-, \rho^+)$, there exists a periodic point in Λ of rotation number $\frac{p}{q}$ (which can be deduced from [22]).

(*iii*) If x is a saddle periodic point of rotation number $\frac{p}{q}$, for $\frac{p}{q} \in (\rho^-, \rho^+)$, then its unstable manifold is contained in Λ (see [64, Proposition 14.3]).

(*iv*) The map T_λ restricted to Λ has positive topological entropy, as a consequence of the existence of a rotational horseshoe (see [78, Theorem A]).

The original part of the section is therefore devoted to prove when hypotheses of Proposition 5.5.5 –hence guaranteeing "complicated" Birkhoff attractors– are satisfied, as resumed here below. We refer to Theorem 5.5.6 and Proposition 5.5.7 respectively.

Theorem. Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^k boundary, $k \geq 2$. The conservative symplectic billiard map T associated to Ω admits an instability region containing the zero section $\mathbb{S} \times \{0\}$ in the following two cases:

(a) If Ω belongs to an open and dense set of strongly convex, centrally symmetric billiard tables.

(b) If Ω has at least one point of zero curvature.

Case (a) is based on these two facts. Among strongly convex, centrally symmetric tables, an essential invariant curve for T passing through the zero section has necessarily rotation number 1/4, see Proposition 5.5.1; however, such curves are very easy to destroy with an arbitrary small perturbation of the table, see Proposition 5.4.8. We stress that, regarding the case (a), the hypothesis of central symmetry plays a fundamental role. This is another difference from dissipative Birkhoff billiards where, for weak dissipation, no properties on the geometry of the billiard table are required, see [20, Proposition 6.15]. Case (b), as for Birkhoff billiards, is a straightforward consequence of Mather's theorem on the non existence of caustics.

In Section 5.6, we present some numerical simulations to illustrate the above results. By using *Mathematica*, we compute the billiard map T_λ for specific domains, both centrally symmetric and not, and plot some orbits in the corresponding phase space.

Finally, we would like to emphasize that, beyond the applications to Birkhoff attractors, the paper presents and proves many properties for the map T_λ (with $\lambda \in (0,1]$, so also for the conservative case), including: the formula for the differential of T_λ (Lemma 5.2.18), the discussion of the quality and quantity of 4-periodic orbits for T in a centrally symmetric domain (Lemma 5.4.3 and Lemma 5.4.5), the result on the fragility of invariant curves of rotation number 1/4 for T (see Proposition 5.4.8), ... We consider that these results may be useful for further studies on symplectic billiard dynamics.

5.2 Preliminaries

This section is devoting to introduce two types of dynamics, namely, the dynamics of symplectic billiards, and the dynamics of dissipative maps on the 2-dimensional annulus. The aim of the present work is defining a combination of these two dynamics, i.e., *dissipative symplectic billiards*, and studying the main properties of the associated (Birkhoff) attractor. The model of dissipative symplectic billiard map is proposed in Subsection 5.2.3.

5.2.1 (Birkhoff) attractors for dissipative maps

In this subsection, we briefly recall the definition of attractor and Birkhoff attractor for a dissipative map of the annulus. We refer to [64] for further details.
Denote $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ and $\pi\colon \mathbb{R} \to \mathbb{S}$ a universal covering. Given $\varphi_\pm : \mathbb{S} \to \mathbb{R}$ two continuous functions such that $\varphi_- < \varphi_+$, let $\mathscr{C} \subset \mathbb{S} \times \mathbb{R}$ be the open, relatively compact subset

$$\mathscr{C} := \{(t,s) \in \mathbb{S} \times \mathbb{R} :\ \varphi_-(s) < t < \varphi_+(s)\}\,.$$

Consider the area form $dt \wedge ds$ on the annulus $\mathbb{S} \times \mathbb{R}$ and let m be the Lebesgue induced measure. Given a set A, we denote by $\mathrm{int}(A)$ the interior of the set, and by $\mathrm{cl}(A)$ the closure of the set.

Definition 5.2.1. *A map $f\colon \mathscr{C} \to \mathrm{int}(\mathscr{C})$ is a dissipative map if:*

1. *f is an homeomorphism of $\mathscr{C}$ into its image, homotopic to the identity;*

2. *f is a C^1 diffeomorphism of $\mathrm{int}(\mathscr{C})$ into its image;*

3. *there exists $\lambda \in (0,1)$ such that, for every $(t,s) \in \mathrm{int}(\mathscr{C})$, we have $0 < \det Df(t,s) \leq \lambda$.*

Example 5.2.2. Clearly, the easiest example of dissipative map one can think of is

$$f\colon \mathbb{S}\times[-1,1]\ni(t,s)\to(t,\lambda s)\in\times[-1,1]\,,$$

where $\lambda\in(0,1)$.

Given a dissipative map f, since $f(\mathscr{C})\subset\operatorname{int}(\mathscr{C})$, the following definition is well-posed.

Definition 5.2.3. *Let $f\colon\mathscr{C}\to\operatorname{int}(\mathscr{C})$ be a dissipative map. The* attractor *of f is the set*

$$\Lambda_0=\Lambda_0(f):=\bigcap_{n\in\mathbb{N}}f^n(\mathscr{C})\,.$$

Observe that Λ_0 is a compact, non-empty, connected, f-invariant set. Moreover, it *separates* the annulus $\mathscr{C}$, i.e., its complementary set $\mathscr{C}\setminus\Lambda_0$ is the union of two open, disjoint set $U_+\sqcup U_-$, such that the graph of $\varphi_\pm$ is contained in $\partial U_\pm$. The definition of the attractor depends on the initial domain $\mathscr{C}$. Moreover, the dynamics restricted to the attractor could be a priori further decomposed into smaller invariant pieces, i.e. the attractor could be not "minimal". This intuition –which will be clarified in Proposition 5.2.5– justifies the following definition, introduced by Birkhoff in [32].

Definition 5.2.4. *Let $f\colon\mathscr{C}\to\operatorname{int}(\mathscr{C})$ be a dissipative map and let Λ_0 be its attractor. In particular, we have $\mathscr{C}\setminus\Lambda_0=U_-\sqcup U_+$. The* Birkhoff attractor *of f is then*

$$\Lambda=\Lambda(f):=\operatorname{cl}(U_-)\cap\operatorname{cl}(U_+)\,.$$

Proposition 5.2.5. *Let $f\colon\mathscr{C}\to\operatorname{int}(\mathscr{C})$ be a dissipative map. A set $A\subset\mathscr{C}$ separates the annulus if $\mathscr{C}\setminus A$ is the disjoint union of two open sets, containing the upper and lower boundary of $\mathscr{C}$ respectively. Denote by $\mathcal{X}$ the set of compact, non-empty, f-invariant subsets of $\mathscr{C}$ that separate the annulus. Then, the Birkhoff attractor $\Lambda\in\mathcal{X}$ and it is the smallest one with respect to the inclusion.*

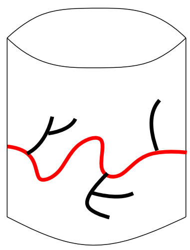


Figure 5.3: The red bold line corresponds to the Birkhoff attractor, while the black and red is the attractor.

Remark 5.2.6. Clearly, for the dissipative map of Example 5.2.2, the attractor coincides with the Birkhoff attractor, i.e. $\Lambda_0=\Lambda=\mathbb{S}\times\{0\}$. However, it is important to observe that the Birkhoff attractor is not a priori an attractor. There could be examples when the Birkhoff attractor is strictly contained in the attractor, as shown in Figure 5.3.

So far, the definition of (Birkhoff) attractor does not need any twist property: the intuition is that we obtain an interesting dynamical set, which a priori should have "lower" dimension (actually it has zero measure). Nevertheless, it could still be quite complicated, as we will see in Section 5.5. The detection of its possible topological complexity passes through the notion of upper and lower rotation number of the Birkhoff attractor. In order to have useful definitions, the map f needs to be a dissipative twist map, so we continue by recalling the definition of (exact) twist map.

Denote by $p_1\colon \mathbb{S}\times\mathbb{R}\to\mathbb{S}$ and $p_2\colon \mathbb{S}\times\mathbb{R}\to\mathbb{R}$ the projections over the first and second coordinates, respectively. Endow $\mathbb{S}\times\mathbb{R}$ with the 2-form $dt\wedge ds$: then, the 1-form $-s\,dt$ is a primitive of $dt\wedge ds$.

Definition 5.2.7. *Let $f\colon \mathbb{S}\times\mathbb{R}\to\mathbb{S}\times\mathbb{R}$ be a C^1 diffeomorphism, homotopic to the identity. We say that f is a* positive *(resp.* negative*)* twist map *if there exists $\epsilon>0$ such that, for every $(t,s)\in\mathbb{S}\times\mathbb{R}$, we have*

$$\frac{\partial p_1\circ f}{\partial s}(t,s)>\epsilon \qquad (\textit{resp. } <-\epsilon)\,.$$

Moreover, a twist map f is called conservative if $f^*(dt\wedge ds)=dt\wedge ds$ and f is exact if the 1-form $f^*(-s\,dt)+s\,dt$ is exact.

In the sequel, we will largely use the notion of *generating function* for a conservative twist map. We introduce it here through the following proposition (see [5, Proposition 1.8]).

Proposition 5.2.8. *Let $F\colon\mathbb{R}^2\to\mathbb{R}^2$ be a C^1 diffeomorphism. Then, F is a lift of a conservative twist map f if and only if there exists a C^2 function $S\colon\mathbb{R}^2\to\mathbb{R}$ such that*

1. *for every $T,\tilde T\in\mathbb{R}$, one has $S(T+2\pi,\tilde T+2\pi)=S(T,\tilde T)$;*
2. *there exists $\epsilon>0$ such that for all $T,\tilde T\in\mathbb{R}$ one has $\epsilon<-\frac{\partial^2 S}{\partial T\partial\tilde T}(T,\tilde T)$;*
3. *we have $F(T,s)=(\tilde T,\tilde s)\Leftrightarrow \tilde s=\frac{\partial S}{\partial\tilde T}(T,\tilde T)$ and $s=-\frac{\partial S}{\partial T}(T,\tilde T)$.*

We say that S is a generating function for F or f.

The notion of twist map and of generating function can be adapted also to the case of a map on a bounded annulus, as it will be the case for symplectic (conservative) billiards.

We return now to Birkhoff attractors. Since Λ separates the annulus, the set $\mathscr{C}\setminus\Lambda$ can be written as $U_-^\Lambda\sqcup U_+^\Lambda$, where $U_\pm^\Lambda$ is an open set containing the graph of $\varphi_\pm$ in its boundary. For every $(t,s)\in\mathscr{C}$, the upper and lower vertical lines are

$$V^+(t,s):=\{(t,y)\in\mathscr{C}:\ y\geq s\}\quad\text{and}\quad V^-(t,s):=\{(t,y)\in\mathscr{C}:\ y\leq s\}\,.$$

Define now

$$\Lambda^+:=\{(s,t)\in\Lambda:\ V^+(t,s)\setminus\{(t,s)\}\subset U_+^\Lambda\}$$

and

$$\Lambda^-:=\{(s,t)\in\Lambda:\ V^-(t,s)\setminus\{(t,s)\}\subset U_-^\Lambda\}\,.$$

Recalling that $\pi\colon\mathbb{R}\to\mathbb{S}$ is a universal covering of $\mathbb{S}$, we use $\Pi\colon\tilde{\mathscr{C}}\subset\mathbb{R}^2\to\mathscr{C}\subset\mathbb{S}\times\mathbb{R}$ for the induced universal covering on the considered annulus. With an abuse of notation, we denote by p_1,p_2 the projections on the first and second coordinate, respectively, on both $\mathbb{S}\times\mathbb{R}$ and $\mathbb{R}^2$. Let F be a continuous lift of f, where f is a dissipative, positive twist map.

Proposition 5.2.9. *Let $f\colon\mathscr{C}\to\operatorname{int}(\mathscr{C})$ be a dissipative, positive twist map. Let Λ be its Birkhoff attractor. The sequence*

$$\left(\frac{p_1\circ F-p_1}{n}\right)_{n\in\mathbb{N}}$$

converges uniformly on $\Pi^{-1}(\Lambda^+)$ (resp. on $\Pi^{-1}(\Lambda^-)$) to a constant ρ^+ (resp. ρ^-). This constant is called upper (resp. lower) rotation number.

The next result is due to Charpentier [35]: it provides a sufficient condition on the upper and lower rotation numbers for the existence of a "complicated" Birkhoff attractor.

Theorem 5.2.10. *Let $f\colon \mathscr{C} \to \mathrm{int}(\mathscr{C})$ be a dissipative map. If $\rho^+ - \rho^- > 0$, then the corresponding Birkhoff attractor is an indecomposable continuum, i.e., it cannot be written as the union of two compact, connected, non-trivial sets.*

5.2.2 Symplectic billiards

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex planar domain with C^k boundary $\partial\Omega$, $k \geq 2$. Assume that the perimeter of $\partial\Omega$ is normalized to 2π. Fix the origin $O \in \mathrm{int}(\Omega)$ and the positive counter-clockwise orientation on $\partial\Omega$. Let

$$\mathbb{S} \ni t \mapsto \gamma(t) \in \mathbb{R}^2$$

be a C^k parametrization of $\partial\Omega$, such that $\gamma(\mathbb{S}) = \partial\Omega$. Given two vectors $v_1, v_2 \in \mathbb{R}^2$, we denote by $\det(v_1, v_2)$ the determinant of the matrix whose columns are v_1, v_2: it corresponds to the signed area of the parallelogram determined by vectors v_1 and v_2. With an abuse of notation, given a point $x \in \mathbb{R}^2$, we will think of it also as a vector, considering $x - O$. Thus, given two points $x_1, x_2 \in \mathbb{R}^2$, the notation $\det(x_1, x_2)$ corresponds to $\det(x_1 - O, x_2 - O)$.

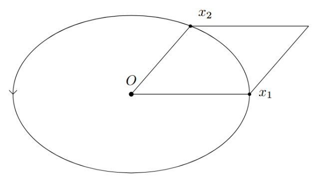


Figure 5.4: The determinant $\det(x_1 - O, x_2 - O)$ is the area of the parallelogram in figure.

Let us indicate then

$$L\colon (v, u) \in \mathbb{R}^2 \to L(v, u) := \det(v, u) \in \mathbb{R}\,.$$

In particular, for all $t_1, t_2 \in \mathbb{S}$, the notation

$$L(\gamma(t_1), \gamma(t_2))$$

denotes the signed area of the parallelogram of sides $\gamma(t_1) - O$ and $\gamma(t_2) - O$.
Given a point $\gamma(t) \in \partial\Omega$, we denote by $\gamma'(t)$ the tangent vector to $\partial\Omega$ at the point $\gamma(t)$, with respect to the fixed parametrization. Since Ω is strictly convex, for every point $\gamma(t) \in \partial\Omega$ there exists a unique (different) point $\gamma(t^*)$ such that

$$L(\gamma'(t), \gamma'(t^*)) = 0\,.$$

In other words, given $t \in \mathbb{S}$, there is a unique $t^* \in \mathbb{S}$ ($t^* \neq t$) such that the tangent vectors to $\partial\Omega$ at the points $\gamma(t)$ and $\gamma(t^*)$ are parallel.

Definition 5.2.11. *Given $t_1, t_2 \in \mathbb{S}$, we say that $(t_1, t_2) \in \mathbb{S} \times \mathbb{S}$ is* positive admissible *if for a lift T_1 of t_1 (i.e., $\pi(T_1) = t_1$) and the lift T_1^* of t_1^* such that $T_1 < T_1^* < T_1 + 2\pi$, there exists a lift T_2 of t_2 (i.e., $\pi(T_2) = t_2$) such that*

$$T_1 < T_2 < T_1^*.$$

We refer to

$$\hat{\mathcal{P}} = \{(t_1, t_2) \in \mathbb{S} \times \mathbb{S} :\ (t_1, t_2) \text{ is positive admissible}\} \tag{5.2.1}$$

as the (open, positive) phase space.

Definition 5.2.12. *The symplectic billiard map of the domain $\partial\Omega$, parametrized by $\gamma\colon \mathbb{S} \to \mathbb{R}^2$, is*

$$\hat{T} : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad (t_1, t_2) \mapsto (t_2, t_3)$$

where $\gamma(t_3) \in \partial\Omega$ is the unique point satisfying

$$L(\gamma'(t_2), \gamma(t_3) - \gamma(t_1)) = 0\,.$$

The next proposition summarizes the main properties of a symplectic billiard map. We refer to [1, Section 2] for further details and for the proofs.

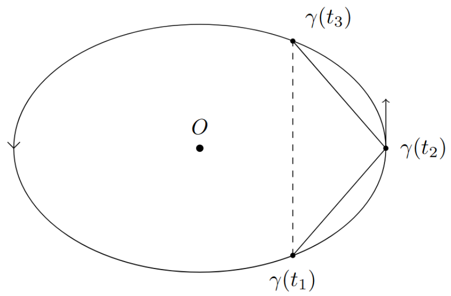


Figure 5.5: The symplectic billiard map reflection: after the points $\gamma(t_1)$ and $\gamma(t_2)$, the next bounce occurs at the point $\gamma(t_3)$.

Proposition 5.2.13. *Let $\Omega \subset \mathbb{R}^2$ be a C^k strictly convex smooth domain, $k \geq 2$. Let $\gamma\colon \mathbb{S} \to \mathbb{R}^2$ be a parametrization of $\partial\Omega$. Denote by $\hat{T}\colon \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ the associated symplectic billiard map. The following properties hold.*

1. *$\hat{T}$ is C^{k-1} and it extends continuously to the closure of $\hat{\mathcal{P}}$ so that*

$$\hat{T}(t,t) = (t,t) \qquad \text{and} \qquad \hat{T}(t,t^*) = (t^*,t)\,.$$

2. *For every $(t_1, t_2) \in \hat{\mathcal{P}}$ one has*

$$\hat{T}(t_1,t_2) = (t_2,t_3) \qquad \Longleftrightarrow \qquad L_2(t_1,t_2) + L_1(t_2,t_3) = 0 \tag{5.2.2}$$

   where we use the notation

$$L_2(t_1,t_2) := L(\gamma(t_1), \gamma'(t_2)) \qquad \text{and} \qquad L_1(t_2,t_3) := L(\gamma'(t_2), \gamma(t_3))\,.$$

3. *The map $\hat{T}$ does not depend on the choice of the origin O.*

4. *The map $\hat{T}$ commutes with any map obtained as affine transformation of the plane, since they preserve tangent directions.*

Up to a change of coordinates, it is possible to see the symplectic billiard map as a (negative) exact twist map (see Definition 5.2.7). In fact the function

$$\phi : \hat{\mathcal{P}} \to \mathbb{S} \times \mathbb{R}, \qquad (t_1,t_2) \mapsto (t_1, -L_1(t_1,t_2)) \tag{5.2.3}$$

is a diffeomorphism onto its image. The image $\phi(\hat{\mathcal{P}})$ is the open set

$$\mathcal{P} = \{(t,s) \in \mathbb{S} \times \mathbb{R} :\ s \in (\psi_1(t), \psi_2(t))\}\,,$$

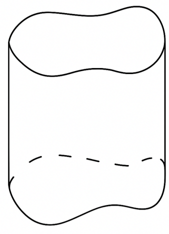


Figure 5.6: The phase space $\mathcal{P}$.

(see Figure 5.6), where

$$\psi_1 \colon t \in \mathbb{S} \to \psi_1(t) := -L_1(t, t^*) = -L(\gamma'(t), \gamma(t^*) - O) \in \mathbb{R}\,,$$

$$\psi_2 \colon t \in \mathbb{S} \to \psi_2(t) := -L_1(t, t) = -L(\gamma'(t), \gamma(t) - O) \in \mathbb{R}\,.$$

Observe that $\psi_1 < 0 < \psi_2$, so in particular the zero section $\mathbb{S} \times \{0\}$ is contained in $\mathcal{P}$. By the variational condition (1.1.3), denoting by $(t_0, t_1) \in \hat{\mathcal{P}}$ the point such that $\hat{T}(t_0, t_1) = (t_1, t_2)$, we have also that the second component of $\phi(t_1, t_2)$ equals $L_2(t_0, t_1)$. We can then consider the map

$$T := \phi \circ \hat{T} \circ \phi^{-1}|_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{P} \tag{5.2.4}$$

which preserves the area form $dt \wedge ds$. Moreover, the map T is a negative twist map, i.e. if $T(t_1, s_1) = (t_2, s_2)$ then

$$\frac{\partial t_2}{\partial s_1} < 0\,.$$

We refer to [1, Lemma 2.7] for the proof of this fact.

Remark 5.2.14. According to Proposition 5.2.8 and by (1.1.3), the function

$$S \colon (T_1, T_2) \in \mathbb{R}^2 \to S(T_1, T_2) := L(\gamma(\pi(T_1)), \gamma(\pi(T_2))) \in \mathbb{R}$$

is a generating function for the twist map T. In the sequel, with an abuse of notation, we will simply write $L(t_1, t_2)$ to refer to $L(\gamma(\pi(T_1)), \gamma(\pi(T_2)))$, where $\pi(T_i) = t_i$ for $i = 1, 2$; the notations L_i and L_{ij}, for $i, j \in \{1, 2\}$, will denote the partial derivatives of order one and two, respectively. In the sequel, in order to lighten the notation, we consider also the extension of the function γ to $\mathbb{R}$, and so drop π in the notation of the generating function.

Proposition 5.2.15. *Let $\Omega \subset \mathbb{R}^2$ be a C^k strictly convex domain, $k \geq 2$. Denote by $T \colon \mathcal{P} \to \mathcal{P}$ the associated symplectic billiard map. The map T^2 is a negative twist map.*

Proof. Following the same notation as before, we denote $T(t_1, s_1) = (t_2, s_2)$. Then $T^2(t_1, s_1) = (t_3, s_3)$ is a negative twist map if $\frac{\partial t_3}{\partial s_1} < 0$ or, equivalently,

$$\frac{\partial s_1}{\partial t_3} = -\frac{\partial L_1}{\partial t_3}(t_1, t_2(t_1, t_3)) = -L_{12}(t_1, t_2(t_1, t_3))\frac{\partial t_2}{\partial t_3}(t_1, t_3) < 0\,. \tag{5.2.5}$$

Fixed t_1, in the above formula, $t_2 := t_2(t_1, t_3)$ gives the unique –by strict convexity– point such that

$$L_2(t_1, t_2) + L_1(t_2, t_3) = 0\,. \tag{5.2.6}$$

Condition (5.2.5) is then easy to verify. In fact, $L_{12}(t_1, t_2) > 0$ since $L_{12}(t_1, t_2) = \det(\gamma'(t_1), \gamma'(t_2))$ and (t_1, t_2) is positive admissible, see Definition 5.2.11. Moreover, considering $t_2 = t_2(t_1, t_3)$ and differentiating (5.2.6) with respect to t_3, we obtain

$$\big(L_{22}(t_1, t_2) + L_{11}(t_2, t_3)\big)\frac{\partial t_2}{\partial t_3} + L_{12}(t_2, t_3) = 0\,.$$

Observe that $L_{22}(t_1, t_2) + L_{11}(t_2, t_3) = \det(\gamma''(t_2), \gamma(t_3) - \gamma(t_1))$, which is strictly negative since $\gamma''(t_2)$ points into the interior of the domain and $\gamma(t_3) - \gamma(t_1)$ is parallel and co-oriented with $\gamma'(t_2)$. Since also $L_{12}(t_2, t_3)$ is positive, we deduce that

$$\frac{\partial t_2}{\partial t_3} = -\frac{L_{12}(t_2, t_3)}{L_{11}(t_2, t_3) + L_{22}(t_1, t_2)} > 0\,.$$

Thus, (5.2.5) is true. This concludes the proof. □

5.2.3 Dissipative symplectic billiards

Let $T : \mathcal{P} \ni (t_1, s_1) \mapsto (t_2, s_2) \in \mathcal{P}$ be the symplectic billiard map on a strictly convex domain Ω with C^k boundary, $k \geq 2$. Let $\gamma\colon \mathbb{S} \to \mathbb{R}$ be a parametrization of the boundary $\partial\Omega$ and let $O \in \mathrm{int}(\Omega)$ be the fixed origin. Let $\lambda \in (0, 1)$: it will be our fixed dissipative parameter. Denote by

$$\mathcal{H}_\lambda : \mathcal{P} \to \mathcal{P}, \qquad \mathcal{H}_\lambda(t_1, s_1) = (t_1, \lambda s_1)$$

the λ-contraction map along the fiber. Observe that $\mathcal{H}_\lambda(\mathcal{P}) \subset \mathrm{int}(\mathcal{P})$.

Definition 5.2.16. *The dissipative symplectic billiard map on Ω is defined as*

$$T_\lambda := \mathcal{H}_\lambda \circ T : \mathcal{P} \to \mathcal{P}, \qquad T_\lambda(t_1, s_1) = (t_2, \lambda s_2)\,.$$

Since T preserves the standard area form $dt \wedge ds$ and T is a twist map, it can be shown that T_λ is a dissipative, twist map. In particular, T_λ dissipates the area of a constant factor λ:

$$T_\lambda^*(dt \wedge ds) = T^*(\mathcal{H}_\lambda^*(dt \wedge ds)) = \lambda dt \wedge ds\,.$$

According to the literature (see [8, 3, 6, 7, 70]), we can also say that the map T_λ is a conformally symplectic map: the symplectic form is contracted by the dynamics by some factor. In dimension 2, symplectic forms are area forms. We will be interested in studying the (Birkhoff) attractor of such dissipative, symplectic billiard maps. A straightforward consequence of the area dissipation is the next lemma.

Lemma 5.2.17. *Let $\lambda \in (0, 1)$. Consider the dissipative symplectic billiard map T_λ, introduced in Definition 5.2.16. Denoting by Λ its Birkhoff attractor, we have*

$$\Lambda \cap (\mathbb{S} \times \{0\}) \neq \emptyset\,.$$

Proof. In the proof, we indicate $\mathcal{P}^+ := \{(t, s) \in \mathcal{P} : s > 0\}$ and $\mathcal{P}^- := \{(t, s) \in \mathcal{P} : s < 0\}$. Suppose by contradiction that $\Lambda \cap (\mathbb{S} \times \{0\}) = \emptyset$. Then, without loss of generality and since $\mathbb{S} \times \{0\} \subset \mathcal{P}$, we can assume that

$$\Lambda \subset \mathcal{P}^+\,. \tag{5.2.7}$$

The Birkhoff attractor separates the annulus: denote then $\mathcal{P} \setminus \Lambda = U_-^\Lambda \sqcup U_+^\Lambda$. Since Λ is T_λ-invariant and by the definition of T_λ, we deduce that $T(\Lambda) = \mathcal{H}_{\frac{1}{\lambda}} \circ T_\lambda(\Lambda) = \frac{1}{\lambda}\Lambda \subset \mathcal{P}^+$. From this observation, by (5.2.7) and since T preserves boundaries, we deduce that

$$m(T(U_-^\Lambda)) = m(\mathcal{P}^-) + \frac{1}{\lambda} m(U_-^\Lambda \setminus \mathcal{P}^-) > m(U_\Lambda^-)\,,$$

which contradicts the conservative properties of T. □

According to (5.2.3), let

$$\phi^{-1}\colon \mathcal{P} \to \hat{\mathcal{P}}, \qquad \phi^{-1}(t_1, s_1) = \phi(t_1, -L_1(t_1, t_2)) = (t_1, t_2)\,. \tag{5.2.8}$$

Observe that it remains well-defined on the image of T_λ, since the map is dissipative. By $\mathcal{O}(t_1, s_1)$ and $\mathcal{O}(t_1, t_2)$ we denote the orbit of (t_1, s_1) under T_λ and of (t_1, t_2) under $\phi^{-1} \circ T_\lambda \circ \phi$, respectively.

The following lemma provides a formula for the differential of T_λ, which will be useful later on.

Lemma 5.2.18. *Let T_λ be a dissipative symplectic billiard map. Let $(t_1, s_1) \in \mathcal{P}$ and let $(t_1, t_2) = \phi^{-1}(t_1, s_1) \in \hat{\mathcal{P}}$. Then*

$$DT_\lambda(t_1, s_1) = -\frac{1}{L_{12}(t_1, t_2)} \begin{pmatrix} L_{11}(t_1, t_2) & 1 \\ -\lambda L_{12}^2(t_1, t_2) + \lambda L_{22}(t_1, t_2) \cdot L_{11}(t_1, t_2) & \lambda L_{22}(t_1, t_2) \end{pmatrix}\,; \tag{5.2.9}$$

in particular, we have that $\det DT_\lambda(t_1, s_1) = \lambda$.

Proof. From Definition 5.2.16, we have that $T_\lambda = \mathcal{H}_\lambda \circ T$; thus, for every $(t_1, s_1) \in \mathcal{P}$,

$$DT_\lambda(t_1, s_1) = D\mathcal{H}_\lambda(T(t_1, s_1))DT(t_1, s_1) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} DT(t_1, s_1)\,. \tag{5.2.10}$$

Since T is conservative, it follows immediately that $\det DT_\lambda(t_1, s_1) = \lambda$. Considering $(t_1, t_2) = \phi^{-1}(t_1, s_1)$, we have that

$$T \circ \phi(t_1, t_2) = (t_2, L_2(t_1, t_2)) = (t_2, L(\gamma(t_1), \gamma'(t_2))\,,$$

and so

$$D(T \circ \phi)(t_1, t_2) = \begin{pmatrix} 0 & 1 \\ \frac{\partial}{\partial t_1} L_2(t_1, t_2) & \frac{\partial}{\partial t_2} L_2(t_1, t_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ L_{12}(t_1, t_2) & L_{22}(t_1, t_2) \end{pmatrix}\,.$$

Since $DT(t_1, s_1) = D(T\circ\phi)(t_1, t_2)\,(D\phi(t_1, t_2))^{-1}$, and $(D\phi(t_1, t_2))^{-1} = \frac{1}{-L_{12}(t_1,t_2)} \begin{pmatrix} -L_{12}(t_1, t_2) & 0 \\ L_{11}(t_1, t_2) & 1 \end{pmatrix}$, we conclude that

$$DT(t_1, s_1) = \begin{pmatrix} 0 & 1 \\ L_{12}(t_1, t_2) & L_{22}(t_1, t_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{L_{11}(t_1,t_2)}{L_{12}(t_1,t_2)} & -\frac{1}{L_{12}(t_1,t_2)}\,. \end{pmatrix} \tag{5.2.11}$$

From (5.2.10) and (5.2.11), we conclude that

$$DT_\lambda(t_1, s_1) = -\frac{1}{L_{12}(t_1, t_2)} \begin{pmatrix} L_{11}(t_1, t_2) & 1 \\ -\lambda L_{12}^2(t_1, t_2) + \lambda L_{11}(t_1, t_2) L_{22}(t_1, t_2) & \lambda L_{22}(t_1, t_2) \end{pmatrix}\,,$$

as stated. ☐

In the next lemma, we proceed by giving a geometrical characterization of Definition 5.2.16. According to Definition 5.2.12 and to (5.2.4), we denote $\hat{T}_\lambda\colon \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ the map $\phi^{-1} \circ T_\lambda \circ \phi$.

Lemma 5.2.19. *Let $\gamma : \mathbb{S} \ni t \mapsto \gamma(t) \in \partial\Omega$ be a parametrization of $\partial\Omega$. Then $\hat{T}_\lambda(t_1, t_2) = (t_2, t_3)$ if and only if $\gamma(t_3) - \lambda\gamma(t_1) \in T_{\gamma(t_2)}\partial\Omega$.*

Proof. Fixed $(t_1, t_2) \in \mathbb{S} \times \mathbb{S}$, we indicate $\hat{T}(t_1, t_2) = (t_2, \tilde{t}_3)$ and $\hat{T}_\lambda(t_1, t_2) = (t_2, t_3)$. Equivalently, in the (t, s) coordinates:

$$T(t_1, s_1) = (t_2, s_2) \qquad \text{with } s_2 = -L_1(t_2, \tilde{t}_3)$$

and

$$T_\lambda(t_1, s_1) = (t_2, \lambda s_2) \qquad \text{with } \lambda s_2 = -L_1(t_2, t_3)\,.$$

Together with the variational condition (1.1.3) for $\hat{T}$, that is $L_2(t_1, t_2) + L_1(t_2, \tilde{t}_3) = 0$, we obtain that

$$\lambda L_2(t_1, t_2) + L_1(t_2, t_3) = 0 \Leftrightarrow L(\lambda\gamma(t_1), \gamma'(t_2)) + L(\gamma'(t_2), \gamma(t_3)) = L(\gamma'(t_2), -\lambda\gamma(t_1) + \gamma(t_3)) = 0\,,$$

and the statement follows. □

Remark 5.2.20. The definition of T_λ depends on the choice of the origin O, which is not the case for the (conservative) symplectic billiard map T.

In the sequel, we discuss the role of 4-periodic points in order to motivate the choice of the origin we will made in the next sections.

Lemma 5.2.21. *Let T be a symplectic billiard map. Then, there exists a choice of the origin O such that the set Π of 4-periodic points whose orbit is contained in the zero section is not empty. We will say that the origin O is* compatible.

Proof. Being T a conservative twist map, by a classical Birkhoff's theorem [30], T possesses at least two 4-periodic orbits. Let $\{t_i\}_{i=1}^4$ be the set of points in $\mathbb{S}$ corresponding to one of such orbits. In particular, by the definition of $\hat{T}$, we have

$$L(\gamma'(t_2), \gamma(t_3) - \gamma(t_1)) = L(\gamma'(t_4), \gamma(t_1) - \gamma(t_3)) = 0$$

and

$$L(\gamma'(t_3), \gamma(t_4) - \gamma(t_2)) = L(\gamma'(t_1), \gamma(t_2) - \gamma(t_4)) = 0\,;$$

this implies then

$$L(\gamma'(t_1), \gamma'(t_3)) = L(\gamma'(t_2), \gamma'(t_4)) = 0\,,$$

that is, the vectors $\gamma'(t_1)$ and $\gamma'(t_3)$ are parallel, as well as the vectors $\gamma'(t_2)$ and $\gamma'(t_4)$. Consequently, in the phase space $\hat{\mathcal{P}}$, a 4-periodic orbit for $\hat{T}$ is given by

$$\{(t_1, t_2), (t_2, t_1^*), (t_1^*, t_2^*), (t_2^*, t_1)\}\,.$$

Consider then the (inscribed) quadrilateral with vertices in $\{\gamma(t_i)\}_{i=1}^4$. Fix the origin $O \in \mathrm{int}(\Omega)$ as the intersection of the diagonals of the inscribed quadrilateral corresponding to (at least) one of the 4-periodic orbits for $\hat{T}$. As a consequence, by this choice, in the phase space $\mathcal{P}$, any such 4-periodic orbit is given by

$$\{(t_1, 0), (t_2, 0), (t_1^*, 0), (t_2^*, 0)\}\,, \tag{5.2.12}$$

i.e., its T-orbit is contained in the zero section, as required. This follows by the definition of the change of coordinates ϕ in (5.2.3), since $\phi(t_1, t_2) = (t_1, -L(\gamma'(t_1), \gamma(t_2) - O))$, and from the fact that $L(\gamma'(t_1), \gamma(t_2) - O) = 0$ since $\gamma(t_2) - O$ belongs to the line generated by $\gamma(t_2) - \gamma(t_4)$, by the choice of the origin, and since $\gamma(t_2) - \gamma(t_4)$ is parallel to $\gamma'(t_1)$. □

An immediate consequence is the following one.

Corollary 5.2.22. *Let Ω be a C^k strictly convex domain, $k \geq 2$. Choose a compatible origin O. Let T be the conservative symplectic billiard map and, for every $\lambda \in (0,1)$, let T_λ be the associated dissipative symplectic billiard map. For every $(t,0) \in \Pi$, one has*

$$T_\lambda(t,0) = T(t,0)\,. \tag{5.2.13}$$

We will see in the following remark that, for a centrally symmetric domain, the natural choice of the origin as the center of symmetry is compatible.

Remark 5.2.23. Let Ω be a (strictly convex) centrally symmetric domain, i.e., there is a point $O \in \mathbb{R}^2$ such that $\partial\Omega$ is invariant under the isometric involution $(x-O, y-O) \mapsto (-x-O, -y-O)$. The point O is called the center of symmetry. Fix it as the origin. Then, the phase space $\mathcal{P}$ becomes

$$\mathcal{P} = \{(t,s) \in \mathbb{S} \times \mathbb{R} : \ -\psi(t) < s < \psi(t)\}\,,$$

where $\psi(t) = -L_1(t,t) = -L(\gamma'(t), \gamma(t) - O)$. In such a case, for every $t \in \mathbb{S}$, the corresponding $t^* \in \mathbb{S}$ is just $t + \pi$. Thus, any 4-periodic orbit has the form

$$\{(t_1,t_2), (t_2, t_1+\pi), (t_1+\pi, t_2+\pi), (t_2+\pi, t_1)\}\,.$$

By this choice, it can be proved that the orbit of any 4-periodic point is contained in the zero section $\mathbb{S} \times \{0\}$, i.e., the set Π is the set of all 4-periodic points.

As a concluding example, we show now that the Birkhoff attractor for dissipative symplectic billiard maps on a (centrally symmetric) Radon domain is the zero section $\mathbb{S} \times \{0\}$. A Radon domain Ω is a centrally symmetric domain such that every point of the boudary $\partial\Omega$ is the vertex of an inscribed parallelogram of maximal area, see [69, Section 3].

Proposition 5.2.24. *Let Ω be a Radon domain and let the origin O be the center of symmetry. Then, for every $\lambda \in (0,1)$, the Birkhoff attractor of T_λ is $\Lambda = \mathbb{S} \times \{0\}$.*

Proof. By the property of centrally symmetric Radon domains recalled above, for the corresponding symplectic map T, the zero section $\mathbb{S} \times \{0\}$ is an invariant curve made up of 4-periodic points. By Corollary 5.2.22, since he chosen origin is compatible, for every $\lambda \in (0,1)$, the zero section is also T_λ-invariant. By minimality with respect to the inclusion (see Proposition 5.2.5), we conclude that $\mathbb{S} \times \{0\}$ is the Birkhoff attractor for T_λ. □

Remark 5.2.25. Let Ω be an elliptic domain. In particular, it is Radon: by Proposition 5.2.24, for every λ, the Birkhoff attractor is the zero section. Moreover, it always coincides with the attractor Λ_0, as we will argue now. As proved in [13], elliptic domains are the only completely integrable ones with respect to the symplectic reflection law. Up to an affine transformation, we can assume that Ω is the unitary circle. Considering as first coordinate $\alpha \in \mathbb{S}$ the counterclockwise oriented angle with the fixed direction $(1,0)$, the symplectic billiard map becomes $T(\alpha, s) = (\alpha + \arccos(s), s)$. Thus, the second coordinate is a first integral. For every $\lambda \in (0,1)$, the dissipative symplectic billiard map is $T_\lambda(\alpha, s) = (\alpha + \arccos s, \lambda s)$; in particular, the second coordinate is now a Lyapunov function. The corresponding neutral set, which is the attractor Λ_0, coincides with the zero section, i.e., with the Birkhoff attractor Λ. It could be interesting to construct a Radon domain (hence different to an ellipse) for which $\Lambda_\lambda \subsetneq \Lambda^0_\lambda$.

Remark 5.2.26. For dissipative Birkhoff billiards, elliptic domains play a special role: it is possible to completely described their Birkhoff attractor for any dissipative factor $\lambda \in (0,1)$ (see [20, Theorem 4.6]). Given an elliptic table with non-zero eccentricity, the 2-periodic orbits – corresponding to the trajectories along the major and minor axes– are hyperbolic orbits of saddle and sink type respectively, and they are not affected by dissipation. The Birkhoff attractor Λ

coincides with the global one Λ_0, and it is the closure of the unstable manifold for the 2-periodic hyperbolic orbit.

When the dissipation is strong enough and the eccentricity is small enough, the Birkhoff attractor turns out to be a normally contracted graph over $\mathbb{S} \times \{0\}$, see [20, Theorem 4.6, Proposition 5.16]. It is worth noting that the proof of [20, Theorem 4.6] relies on the existence of a first integral for the conservative elliptic Birkhoff billiard dynamics. In fact, such an integral becomes a Lyapunov function when the dissipation is turned on. Its neutral set is the set of 2-periodic points. The global attractor, which corresponds to the Birkhoff one, is then detected by using such a Lyapunov function.

For dissipative symplectic billiards, it is not known if there exists a class of tables which can replace the class of elliptic domains for dissipative Birkhoff billiards. The main reason is that it is not known if there exists a class of tables, for (conservative) symplectic billiards, whose dynamics is integrable, but not totally integrable.

5.3 Normally contracted Birkhoff attractors

In Proposition 5.2.24, we have seen that the Birkhoff attractor of any Radon billiard table is very simple, no matter the strength of the dissipation: it is always $\Lambda = \mathbb{S} \times \{0\}$. The aim of this section is showing that –when the dissipation is strong, i.e., $0 < \lambda \ll 1$ – the corresponding dissipative symplectic billiard map of a strongly convex C^k domain, $k \geq 2$, exhibits a topologically simple Birkhoff attractor, i.e., Λ is a normally contracted graph and it coincides with the attractor Λ_0. This section follows very closely [20, Section 5].

The next result is the main point to obtain a cone-field criterion and then invoke standard results in normally hyperbolic dynamics, recalled later on. In the next proposition, as in Theorem 5.3.7, the choice of the origin $O \in \operatorname{int}(\Omega)$ is arbitrary.

Proposition 5.3.1. *Let Ω be a C^k strongly convex domain, $k \geq 2$. Then, there exist $\lambda(\Omega) \in (0,1)$, $M > 0$, $\alpha > 0$ and a cone-field $\mathcal{C}_\alpha = (\mathcal{C}_\alpha(t,s))_{(t,s)\in\mathcal{P}}$ containing the horizontal direction*

$$\mathcal{C}_\alpha(t,s) = \{v \in T_{(t,s)}\mathcal{P} : v = (v_t, v_s), |v_s| \leq \alpha |v_t|\},$$

such that, for every $\lambda \in (0, \lambda(\Omega))$ and for every $(t,s) \in \mathbb{S} \times [-M \cdot \lambda(\Omega), M \cdot \lambda(\Omega)]$, one has

$$DT_\lambda(t,s)\mathcal{C}_\alpha(t,s) \subset \operatorname{int}(\mathcal{C}_\alpha(T_\lambda(t,s))) \cup \{0\}.$$

Proof. Consider the arc-length parametrization $\gamma \colon \mathbb{S} \to \partial\Omega$ of the billiard table Ω. Define the following positive constants

$$C_1 := \max_{t_1,t_2\in\mathbb{S}} |L_{11}(t_1,t_2) \cdot L_{22}(t_1,t_2) - L_{12}^2(t_1,t_2)|,$$

$$C_2 := \max_{t_1,t_2\in\mathbb{S}} |L_{22}(t_1,t_2)|$$

and

$$M(\Omega) := \max\left\{\max_{t\in\mathbb{S}} |L_1(t,t)|, \max_{t\in\mathbb{S}} |L_1(t,t^*)|\right\}.$$

Since γ is the arc-length parametrization, for every $t \in \mathbb{S}$, one has that $\gamma''(t) = k(t)\, i\gamma'(t)$, where $k(t) \in \mathbb{R}$ denotes the curvature of the table at the point $\gamma(t)$, and the multiplication by i is the rotation of angle $\frac{\pi}{2}$, so that $\gamma''(t)$ always points towards the inside of the domain. Recall that $L(\gamma(t_1), \gamma(t_2))$ refers to the determinant of the matrix whose columns are, respectively, $\gamma(t_1) - O$ and $\gamma(t_2) - O$.

By continuity of the involved function and compactness, we can choose $\lambda_1 \in (0,1)$ such that there exists $c_0 > 0$ so that for every $(t_1, s_1) \in \mathcal{P}$, if

$$|s_1| = |L_1(t_1,t_2)| = |\det(\gamma'(t_1), \gamma(t_2))| < M(\Omega)\cdot\lambda_1\,,$$

that is, if $\gamma'(t_1)$ is not so far from being parallel to $\gamma(t_2) - O$, then

$$|L_{11}(t_1,t_2)| = |L(\gamma''(t_1), \gamma(t_2)| = |k(t_1)|\,|L(i\gamma'(t_1), \gamma(t_2))| \geq c_0 > 0\,,$$

since $i\gamma'(t_1)$ is then not so far from being perpendicular to $\gamma(t_2) - O$. Fix some $0 < \alpha < c_0$. Let $\mathcal{C}_\alpha$ be the cone-field defined as

$$\mathcal{C}_\alpha(t,s) = \{v \in T_{(t,s)}\mathcal{P} : v = (v_t, v_s), |v_s| \leq \alpha|v_t|\}\,.$$

We are considering on each tangent space the coordinates inherited from $(t,s) \in \mathcal{P}$. By Lemma 5.2.18, for every $(t,s) \in \mathbb{S} \times [-M(\Omega)\cdot\lambda_1, M(\Omega)\cdot\lambda_1]$ and for every vector $v = (a,b) \in \mathcal{C}_\alpha(t,s)$, we get

$$\begin{aligned}
v' :=& DT_\lambda(t,s)v \\
=& -\frac{1}{L_{12}(t_1,t_2)}\begin{pmatrix} aL_{11}(t_1,t_2) + b \\ \lambda a\left[L_{11}(t_1,t_2)\cdot L_{22}(t_1,t_2) - L_{12}^2(t_1,t_2)\right] + \lambda b L_{22}(t_1,t_2)\end{pmatrix} \\
:=& -\frac{1}{L_{12}(t_1,t_2)}\begin{pmatrix} a' \\ b'\end{pmatrix}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|a'| &= |aL_{11}(t_1,t_2) + b| \geq |a||L_{11}(t_1,t_2)| - |b| \geq |a|(c_0 - \alpha), \\
|b'| &= |\lambda a\left[L_{11}(t_1,t_2)\cdot L_{22}(t_1,t_2) - L_{12}^2(t_1,t_2)\right] + \lambda b L_{22}(t_1,t_2)| \\
&\quad \leq \lambda|a|(C_1 + \alpha C_2)\,.
\end{aligned}$$

Given now $\mu_0 \in (0,1)$, it holds

$$DT_\lambda(t,s)\mathcal{C}_\alpha(t,s) \subset \mathcal{C}_{\mu_0\alpha}(T_\lambda(t,s)) \subset \operatorname{int}(\mathcal{C}_\alpha(T_\lambda(t,s))) \cup \{0\}\,,$$

where $\mathcal{C}_{\mu_0\alpha}(t,s) := \{v \in T_{(t,s)}\mathcal{P} :\ v = (v_t, v_s), |v_s| \leq \mu_0\alpha|v_t|\}$, provided that $\lambda \in (0, \lambda(\Omega))$, with

$$\lambda(\Omega) := \min\left\{\lambda_1, \frac{\mu_0\alpha(c_0-\alpha)}{K + \alpha C_0}\right\}\,.$$

□

The rest of the section follows [20, Section 5]: we are able to show that, when the dissipation is strong, the Birkhoff attractor of the corresponding dissipative symplectic map coincides with the attractor and it is a *normally contracted* graph. Even if the ideas are the same as in [20], for sake of clarity, we recall here the main definitions, coming from hyperbolic dynamics, we state the main results and give an idea of the proof.

Remark 5.3.2. As we will see in the main statements, the results of this section hold for **every** C^k strongly convex domain, with $k \geq 2$, while, in the Birkhoff dissipative dynamics, the analogous results hold for a class of strictly convex domains satisfying some geometric *pinching* condition (see [20, Definition D]).

Let us start by recalling some well-known definitions and results in normally hyperbolic dynamics. We refer to [19, 38, 51, 81].

Definition 5.3.3. *Let M be a compact Riemmannian manifold without boundary and $f\colon M \to M$ be a C^k diffeomorphism, with $k \geq 1$. A compact invariant set K has a dominated splitting if $T_K M = E \oplus F$, where the Df-invariant continuous subbundles E and F are such that there exists $C > 0$ and $\nu \in (0,1)$ so that, for every $x \in K$*

$$\|Df^n(x)|_E\| \cdot \|Df^{-n}(f^n(x))|_F\| \leq C\nu^n \qquad \forall n \geq 0\,.$$

Definition 5.3.4. *Let M be a compact Riemmannian manifold without boundary and $f\colon M \to M$ be a C^k diffeomorphism, with $k \geq 1$. Let N be a closed C^1 f-invariant manifold. Then N is normally contracted if N has a dominated splitting $T_N M = E^s \oplus TN$ such that E^s is uniformly contracted, i.e., there exists $n_0 \in \mathbb{N}$ and $\mu \in (0,1)$ such that for any $n \geq n_0$ one has $\|Df^n(x)|_{E^s}\| \leq \mu^n$ for every $x \in N$.*

We say that N is k-normally contracted if N is normally contracted and there exists $C > 0$ and $\nu \in (0,1)$ such that for every $x \in N$ and for every $0 \leq j \leq k$ one has $\|Df^n(x)|_{E^s}\| \cdot \|Df^{-n}(f^n(x))|_{TN}\|^j \leq C\nu^n$ for all $n \geq 0$.

Proposition 5.3.1 represents the main point to obtain a cone-field criterion and deduce the following corollary.

Corollary 5.3.5. *Let Ω be a strongly convex domain with C^k boundary, $k \geq 2$. Let $\lambda(\Omega) \in (0,1)$ be given by Proposition 5.3.1. Then, for every $\lambda \in (0, \lambda(\Omega))$, the attractor Λ_0 has a dominated splitting $E^s \oplus E^c$ with E^s uniformly contracted. Moreover, every $(t,s) \in \Lambda_0$ has a stable manifold $\mathcal{W}^s(t,s)$ that is transversal to the horizontal. Furthermore, there exists $0 < \lambda'(\Omega) < \lambda(\Omega)$ such that for some constant, $C > 0$ and $0 < \nu < 1$, for every $\lambda \in (0, \lambda'(\Omega))$, for every $x \in \Lambda_0$ and for every $1 \leq j \leq k-1$*

$$\|DT_\lambda^n(x)|_{E^s}\| \cdot \|DT_\lambda^{-n}(T_\lambda^n(x))|_{E^c}\|^j \leq C\nu^n, \quad \forall n \geq 0\,. \tag{5.3.1}$$

Proof. The proof follows from the application of the cone-field criterion (see [38, Theorem 2.6]), and it is *verbatim* the proof of [20, Proposition 5.5]. □

Remark 5.3.6. If it would be possible to say *a priori* that Λ_0 is a C^1 manifold, then the previous corollary would say that Λ_0 is l-normally contracted.

Theorem 5.3.7. *Let Ω be a strongly convex domain with C^k boundary, $k \geq 2$.*

(*a*) *Let $\lambda(\Omega) \in (0,1)$ given in Proposition 5.3.1. Then for $\lambda \in (0, \lambda(\Omega))$, the Birkhoff attractor Λ coincides with the attractor Λ_0 and it is a normally contracted C^1 graph on $\mathbb{S} \times \{0\}$.*

(*b*) *Let $\lambda'(\Omega) < \lambda(\Omega)$ given in Corollary 5.3.5. Then for $\lambda \in (0, \lambda'(\Omega))$, $\Lambda = \Lambda_0$ is a C^{k-1} graph and Λ converges to $\mathbb{S} \times \{0\}$, as $\lambda \to 0$, in the C^1 topology.*

We give here the main ideas of the proof and refer to the proof of [20, Theorem 5.7] for full details.

Idea of the proof. The proof is *verbatim* the proof of [20, Theorem 5.7]. Let us give here the main ideas. Consider $\lambda \in (0, \lambda(\Omega))$ and $(\mathcal{C}_\alpha(t,s))_{(t,s)\in\mathcal{P}}$ given by Proposition 5.3.1; define the set of graphs

$$\mathcal{F} := \{\gamma : \mathbb{S} \to \left[-\lambda, \lambda\right] \text{ s.t. } \gamma \in C^1(\mathbb{S}) \text{ and } (1, \gamma'(t)) \in \mathcal{C}_\alpha(t,s), \forall t \in \mathbb{S}\}.$$

Recall that p_1, p_2 are the projections on the first and the second coordinate, respectively. By mean of the billiard map T_λ, it is possible to construct the graph transform

$$\mathcal{G}_{T_\lambda} : \mathcal{F} \to \mathcal{F}, \qquad \gamma \mapsto (t, p_2 \circ T_\lambda(g_\lambda^{-1}(t), \gamma(g_\lambda^{-1}(t))))\,,$$

where

$$g_\lambda : \mathbb{S} \to \mathbb{S}, \qquad t \mapsto p_1 \circ T_\lambda(t, \gamma(t))\,.$$

In order to obtain $\Lambda = \Lambda_0$, it is sufficient to prove that $\mathcal{G}_{T_\lambda}$ is a contraction on $\mathcal{F}$ for the norm $\|\cdot\|_\infty$. Its only fixed point corresponds precisely to the global attractor and, since it is a graph, it coincides with the Birkhoff attractor. The regularity is a direct consequence of the previous corollary, and the convergence comes from the constructed cone-field. □

5.4 Centrally symmetric dissipative billiards

In this section, we consider dissipative symplectic billiard maps for a centrally symmetric domain Ω, i.e., a domain Ω such that there exists $O \in \operatorname{int}(\Omega)$ for which Ω is invariant by involution $(x - O, y - O) \mapsto (-x - O, -y - O)$. Under the assumption of strong dissipation, not only Theorem 5.3.7 holds, i.e., the Birkhoff attractor is a normally contracted graph, but we will prove the following results.

(a) for $\lambda \in (0, 1)$ small enough, the Birkhoff attractor intersects the zero section exactly in the 4-periodic points, that is $(t_1, 0) \in \Pi$ if and only if $(t_1, 0) \in \Lambda_\lambda$;

(b) for $\lambda \in (0, 1)$ small enough, there exists an open and dense set of centrally symmetric domains, whose associated Birkhoff attractor has rotation number $1/4$ and decomposes as

$$\Lambda = \bigcup_{i=1}^{l} \bigcup_{j=0}^{3} \overline{\mathcal{W}^u(T_\lambda^j(H_i); T_\lambda^4)}\,,$$

where $\{H_i\}_{i=1}^l$ is a finite family of 4-periodic points of saddle type, and $\mathcal{W}^u(H; T_\lambda^4)$ is the unstable manifold of a saddle point H, with respect to the dynamics of T_λ^4.

Recall that $\hat{T} \colon \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ denotes the symplectic billiard map in the coordinates $(t_1, t_2) \in \hat{\mathcal{P}}$. Let us introduce the maps

$$\hat{I} : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad \hat{I}(t_1, t_2) = (t_2^*, t_1)\,, \tag{5.4.1}$$

$$\hat{I}_2 : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad \hat{I}_2(t_1, t_2) = (t_1, t_0^*)\,. \tag{5.4.2}$$

Notice also that

$$\hat{I}^2(t_1, t_2) = \hat{I} \circ \hat{I}(t_1, t_2) = (t_1^*, t_2^*).$$

The corresponding maps I and I_2 in $\mathcal{P}$ can be described respectively as

$$I \circ \phi(t_1, t_2) = I(t_1, -L_1(t_1, t_2)) = (t_2^*, -L_1(t_2^*, t_1))$$

and

$$I_2 \circ \phi(t_1, t_2) = I_2(t_1, -L_1(t_1, t_2)) = (t_1, -L_1(t_1, t_0^*))\,.$$

Lemma 5.4.1. *Let Ω be a strictly convex C^k domain, $k \geq 2$. Let $O \in \operatorname{int}(\Omega)$ be compatible (see Lemma 5.2.21). The following properties hold.*

1. *Recall that $\mathcal{P} = \{(t, s) \in \mathbb{S} \times \mathbb{R} : \ \psi_1(t) \leq s \leq \psi_2(t)\}$. Then, $I_2(\operatorname{graph}(\psi_1)) = \operatorname{graph}(\psi_2)$ and $I_2(\operatorname{graph}(\psi_2)) = \operatorname{graph}(\psi_1)$. In particular, if Ω is centrally symmetric, one has*

$$I_2(graph(\psi)) = -graph(\psi)\,,$$

where $\psi(t) = -L_1(t, t)$.

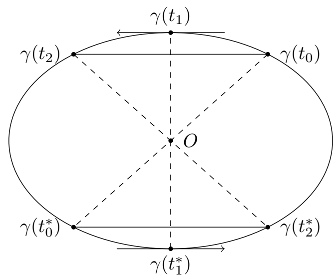


Figure 5.7: For the symplectic billiard map on centrally symmetric domains, $\hat{T}(t_0, t_1) = (t_1, t_2)$ if and only if $\hat{T}(t_0^*, t_1^*) = (t_1^*, t_2^*)$. In particular, $\gamma'(t_1)$ is parallel to $\gamma(t_2^*) - \gamma(t_0^*)$.

2. *$I(\Pi) = \Pi$, where Π is the set of 4-periodic points whose orbit is contained in the zero section.*

3. *If Ω is centrally symmetric, then $I^2(\Lambda) = \Lambda$ for every $\lambda \in (0,1)$, i.e., $(t,s) \in \Lambda$ if and only if $(t^*, s) \in \Lambda$.*

4. *If Ω is centrally symmetric, then $I(\Lambda) = \mathcal{H}_{-\frac{1}{\lambda}}(\Lambda)$ for every $\lambda \in (0,1)$, where $\mathcal{H}_{-\frac{1}{\lambda}}(t,s) = (t, -\frac{s}{\lambda})$.*

Proof. Point 1 follows from the definition of the function I_2, and since points of the graph of ψ_1 (resp. ψ_2) correspond to points of type (t,t) (resp. (t,t^*)) in $\hat{\mathcal{P}}$. For point 2, let $(t_1, 0) \in \Pi$: since the origin is compatible, its orbit is given by

$$\{(t_1, 0), (t_2, 0), (t_1^*, 0), (t_2^*, 0)\}$$

and in particular, we have $I(t_1, 0) = I(t_1, -L_1(t_1, t_2)) = (t_2^*, -L_1(t_2^*, t_1)) = (t_2^*, 0)$. Concerning point 3, since I is continuous and by the properties of the Birkhoff attractor, the set $I^2(\Lambda)$ is a compact, connected set which separates $\mathcal{P}$. Observe that, when Ω is centrally symmetric, we have that $I^2(t,s) = (t^*, s)$, since, in such a case, $L_1(t, \tilde{t}) = L_1(t^*, \tilde{t}^*)$. Moreover, the map I^4 is the identity. Therefore, one has

$$I^2 \circ T_\lambda(t_1, s_1) = I^2(t_2, \lambda s_2) = (t_2^*, \lambda s_2)$$

and

$$T_\lambda \circ I^2(t_1, s_1) = T_\lambda(t_1^*, s_1) = (t_2^*, \lambda s_2)\,,$$

that is, $I^2 \circ T_\lambda = T_\lambda \circ I^2$. Thus, by the T_λ-invariance of Λ, we obtain $I^2(\Lambda) = I^2 \circ T_\lambda(\Lambda) = T_\lambda \circ I^2(\Lambda)$, i.e., the set $I^2(\Lambda)$ is T_λ-invariant. By the minimality property of the Birkhoff attractor, we deduce that $\Lambda \subset I^2(\Lambda)$. By applying I^2, we conclude that $I^2(\Lambda) \subset \Lambda$, that is $\Lambda = I^2(\Lambda)$, as required.

Let us prove point 4. We first notice that $\hat{I}_2 = \hat{T} \circ \hat{I}$, since, by the centrally symmetric hypothesis on Ω, the vector $\gamma(t_2) - \gamma(t_0)$ is parallel to the vector $\gamma(t_0^*) - \gamma(t_2^*)$, see Figure 5.7. As a consequence,

$$I_2 = T \circ I\,. \tag{5.4.3}$$

In addition

$$I_2(t_1, s_1) = (t_1, -s_1)\,. \tag{5.4.4}$$

The above equality is a consequence of the geometry and the dynamics of symplectic billiards in centrally symmetric domains, since

$$(t_1, s_1) = (t_1, -L_1(t_1, t_2)) \xmapsto{I_2} (t_1, -L_1(t_1, t_0^*)) = (t_1, L_1(t_1, t_2)) = (t_1, -s_1)\,,$$

where in the last equality, we have used the fact that

$$L(\gamma'(t_1),\gamma(t_2)-\gamma(t_0)) = L(\gamma'(t_1),\gamma(t_0^*)-\gamma(t_2^*)) = 0 \quad \Longrightarrow \quad -L_1(t_1,t_0^*) = -L_1(t_1,t_2^*) = L_1(t_1,t_2)\,.$$

By equality (5.4.3), by point 3 and by the definition of T_λ, we get

$$I_2 \circ I(\Lambda) = T \circ I \circ I(\Lambda) = T(\Lambda) = \mathcal{H}_{\frac{1}{\lambda}} \circ T_\lambda(\Lambda) = \mathcal{H}_{\frac{1}{\lambda}}\Lambda\,.$$

Finally, applying I_2^{-1} on both sides of the previous equation and using (5.4.4), we conclude

$$I(\Lambda) = I_2^{-1}\left(\mathcal{H}_{\frac{1}{\lambda}}(\Lambda)\right) = \mathcal{H}_{-\frac{1}{\lambda}}(\Lambda)\,.$$

□

We are now ready to prove the statement of point (a).

Proposition 5.4.2. *Let Ω be a strongly convex, centrally symmetric C^k domain, $k \geq 2$. Let $\lambda(\Omega) \in (0,1)$ be given by Proposition 5.3.1. Then, for every $\lambda \in (0,\lambda(\Omega))$, one has $\Lambda \cap (\mathbb{S}\times\{0\}) = \Pi$.*

Proof. By Theorem 5.3.7, for $\lambda \in (0,\lambda(\Omega))$, the Birkhoff attractor is a normally contracted graph over $\mathbb{S}$. Since it is a graph and by point 4 of Lemma 5.4.1, we deduce that $\Lambda \cap I(\Lambda) = \Lambda \cap (\mathbb{S}\times\{0\})$. Thus, by point 3 of Lemma 5.4.1, if $(t_1,0) \in \Lambda \cap (\mathbb{S}\times\{0\}) = \Lambda \cap I(\Lambda)$, then $I(t_1,0) \in I(\Lambda) \cap \Lambda = \Lambda \cap (\mathbb{S}\times\{0\})$. By the definition of I, we obtain that

$$I(t_1,0) = I(t_1,-L_1(t_1,t_2)) = (t_2^*,-L_1(t_2^*,t_1)) = (t_2^*,0)\,;$$

in particular, the vector $\gamma'(t_1)$ is parallel to $\gamma(t_2)-O$ and $\gamma'(t_2)$ is parallel to $\gamma(t_1)-O$. We then deduce that the point $(t_1,0) \in \Pi$, proving then $\Lambda \cap (\mathbb{S}\times\{0\}) \subset \Pi$.

Moreover, again by Theorem 5.3.7, we also know that the Birkhoff attractor coincides with the global attractor, when $\lambda \in (0,\lambda(\Omega))$. Let $(t_1,0) \in \Pi$: then clearly $(t_1,0) \in \mathbb{S}\times\{0\}$ and, since $(t_1,0) \in \Lambda_0$ and $\Lambda_0 = \Lambda$, we easily conclude that $(t_1,0) \in \Lambda \cap (\mathbb{S}\times\{0\})$ as required.

□

5.4.1 An open and dense property in the centrally symmetric case

To prove point (b), we first need to discuss the quantity and quality of 4-periodic orbits for centrally symmetric domains and then to introduce the topology with respect to which we will state our result. Let Ω be a C^k centrally symmetric domain, $k \geq 2$, and let $\gamma\colon \mathbb{S} \to \mathbb{R}^2$ be a parametrization of $\partial\Omega$. Let (t_1,t_2) be a 4-periodic point. Consider then the quantity:

$$k_{1,2} = k_{1,2}(t_1,t_2) := \frac{L_{11}(t_1,t_2)\cdot L_{22}(t_1,t_2)}{L_{12}^2(t_1,t_2)}\,. \tag{5.4.5}$$

Lemma 5.4.3. *Let Ω be a C^k strictly convex centrally symmetric domain, $k \geq 2$. Let (t_1,t_2) be a 4-periodic point. For $\lambda \in (0,1)$, denote by $\mu_1 = \mu_1(\lambda)$ and $\mu_2 = \mu_2(\lambda)$ the eigenvalues of $DT_\lambda^4(t_1,s_1(t_1,t_2))$, with $|\mu_1| \leq |\mu_2|$. Then the next cases occur.*

- (a) *If $k_{1,2} > 1$, then $0 < \mu_1 < \lambda^4 < 1 < \mu_2$, and the 4-periodic point is a* ***saddle****.*
- (b) *If $k_{1,2} = 1$, then $\mu_1 = \lambda^4$, $\mu_2 = 1$, and the 4-periodic point is* ***parabolic****.*

(c) *If* $k_{1,2} \in (0,1)$, *then the* 4*-periodic point is a* ***sink****. In particular, let*

$$\lambda_- = \lambda_-(t_1,t_2) := \frac{1-\sqrt{1-k_{1,2}}}{1+\sqrt{1-k_{1,2}}} \in (0,1)\,.$$

Thus:

(i) *If* $\lambda \in (0,\lambda_-)$, *then* $\mu_1,\mu_2 \in \mathbb{R}$ *and* $\lambda^4 < \mu_1 < \mu_2 < 1$.

(ii) *If* $\lambda = \lambda_-$, *then* $\mu_1 = \mu_2 = \lambda^2$.

(iii) *If* $\lambda \in (\lambda_-,1)$ *and* $k_{1,2} \neq \frac{2\lambda}{(\lambda+1)^2}$, *then* $\overline{\mu_1} = \mu_2$ *and* $|\mu_1| = |\mu_2| = \lambda^2$.

(iv) *If* $\lambda \in (\lambda_-,1)$ *and* $k_{1,2} = \frac{2\lambda}{(\lambda+1)^2}$, *then* $\mu_1 = \mu_2 = -\lambda^2$.

Proof. We already know by Lemma 5.2.18 the explicit formula for the differential of the dissipative symplectic billiard map. We are studying 4-periodic orbits, hence, up to choose a good parametrization, all of the form:

$$\{(t_1,t_2),(t_2,t_1+\pi),(t_1+\pi,t_2+\pi),(t_2+\pi,t_1)\}\,, \tag{5.4.6}$$

see Remark 5.2.23. In such a case, from the central symmetric hypothesis, the next equalities hold:

$$L_{12}(t_2,t_1+\pi) = L_{12}(t_1,t_2), \quad L_{11}(t_2,t_1+\pi) = L_{22}(t_1,t_2), \quad L_{22}(t_2,t_1+\pi) = L_{11}(t_1,t_2)\,.$$

From now on, we omit the dependence of every L_{ij} on (t_1,t_2). By using the previous equalities, a direct computation gives:

$$\begin{aligned} A_\lambda &:= DT_\lambda^2(t_1,s_1(t_1,t_2)) = DT_\lambda(t_2,s_1(t_2,t_1+\pi))\cdot DT_\lambda(t_1,s_1(t_1,t_2)) \\ &= \frac{1}{L_{12}^2}\begin{pmatrix} -\lambda L_{12}^2 + (1+\lambda)L_{11}\cdot L_{22} & (1+\lambda)L_{22} \\ (1+\lambda)(-\lambda L_{12}^2 + \lambda L_{11}\cdot L_{22})\cdot L_{11} & -\lambda L_{12}^2 + (\lambda+\lambda^2)L_{11}\cdot L_{22} \end{pmatrix}, \end{aligned}$$

and, for the 4-periodic orbit (5.4.6), one has $DT_\lambda^4(t_1,s_1(t_1,t_2)) = A_\lambda^2$. Consequently, to understand the nature of the 4-periodic points of T_λ, we just need to study the eigenvalues of the matrix A_λ. The determinant and the trace of A_λ are respectively

$$\det A_\lambda = \lambda^2 \qquad \text{and} \qquad trA_\lambda = -2\lambda + (1+\lambda)^2\left(\frac{L_{11}\cdot L_{22}}{L_{12}^2}\right).$$

The characteristic polynomial is then

$$\chi_\lambda(x) = x^2 - \left[(1+\lambda)^2 k_{1,2} - 2\lambda\right]x + \lambda^2\,,$$

which is exactly as in the Birkhoff dissipative case with $\lambda_1 = \lambda_2 = \lambda$ and $k_{1,2} := \frac{L_{11}\cdot L_{22}}{L_{12}^2}$, see [20, Appendix A], where we refer to the notation of the Appendix A there. Repeating then *verbatim* the proof in [20], we conclude. □

Let us introduce some further notations that will be largely used in the sequel. Denote by

$$e_\theta = (-\sin\theta, \cos\theta)$$

the unit vector which forms an angle $\theta \in \mathbb{S}$ with the fixed vertical direction $(0,1)$. For every θ there exists a unique point $\gamma(t_\theta) := \gamma(\theta) \in \partial\Omega$ such that $\gamma'(\theta) = \|\gamma'(\theta)\|e_\theta$. Let $O \in \mathrm{int}(\Omega)$ be the center of symmetry of the table. Let $p\colon \mathbb{S} \to \mathbb{R}_+$ be the support function, defined as the

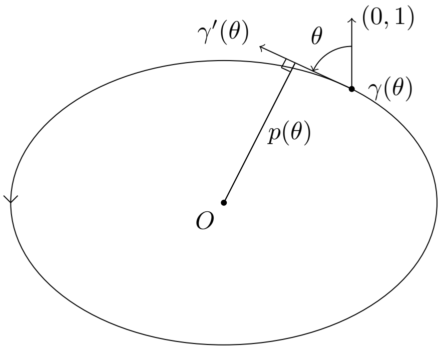


Figure 5.8: The support function $p(\theta)$ at the point $\theta \in \mathbb{S}$.

distance of the origin from the affine line $\gamma(\theta) + \mathbb{R}\gamma'(\theta)$. Let J be the rotation of angle $\frac{\pi}{2}$. Then we have

$$\begin{cases} \gamma(\theta) & = p'(\theta)e_\theta - p(\theta)Je_\theta\,; \\ \gamma'(\theta) & = (p(\theta) + p''(\theta))e_\theta\,; \\ \gamma''(\theta) & = (p'(\theta) + p'''(\theta))e_\theta + (p(\theta) + p''(\theta))Je_\theta\,; \end{cases} \tag{5.4.7}$$

see [16]. We refer to Figure 5.8. We remind that the radius of the osculating circle at $\gamma(\theta)$ is

$$\rho(\theta) = p(\theta) + p''(\theta)\,,$$

see for example [44, Formula 2.9]. We are going to use the angles θ as coordinates. In the centrally symmetric case (see Remark 4.6.11) Π is the set of all 4-periodic points for $\{T_\lambda\}_{\lambda\in[0,1]}$. A point (θ_1, θ_2) corresponding to a point in Π has then the following 4-periodic orbit:

$$\{(\theta_1,\theta_2),(\theta_2,\theta_1+\pi),(\theta_1+\pi,\theta_2+\pi),(\theta_2+\pi,\theta_1)\}\,. \tag{5.4.8}$$

With an abuse of notation, we will indicate the symplectic billiard map in the coordinates (θ_1, θ_2) also by T. For a 4-periodic point corresponding to the couple of angles (θ_1, θ_2), we denote the relative quantity $k_{1,2}$, defined in (5.4.5), also as $k_{1,2} = k_{1,2}(\theta_1, \theta_2)$.

Remark 5.4.4. We explicit now the quantity $\frac{L_{11}L_{22}}{L_{12}^2}$, that appears in Lemma 5.4.3, in terms of (θ_1, θ_2). We start by recalling that at the points of a 4-periodic orbit, the vectors $\gamma'(\theta_i)$ and $\gamma(\theta_{i+1})$ are parallel, so $\det(e_{\theta_i}, \gamma(\theta_{i+1})) = 0$, that is

$$\begin{aligned} \det(e_{\theta_1},\gamma(\theta_2)) &= p'(\theta_2)\sin(\theta_2-\theta_1) - p(\theta_2)\cos(\theta_2-\theta_1) = 0\,, \\ \det(e_{\theta_2},\gamma(\theta_1)) &= -p'(\theta_1)\sin(\theta_2-\theta_1) - p(\theta_1)\cos(\theta_2-\theta_1) = 0\,. \end{aligned} \tag{5.4.9}$$

As a consequence:

$$\begin{aligned} L_{12}(t_{\theta_1},t_{\theta_2}) &= \rho(\theta_1)\rho(\theta_2)\sin(\theta_2-\theta_1)\,, \\ L_{22}(t_{\theta_1},t_{\theta_2}) &= -p(\theta_1)\rho(\theta_2)\sin(\theta_2-\theta_1) + p'(\theta_1)\rho(\theta_2)\cos(\theta_2-\theta_1)\,, \\ L_{11}(t_{\theta_1},t_{\theta_2}) &= -p(\theta_2)\rho(\theta_1)\sin(\theta_2-\theta_1) - p'(\theta_2)\rho(\theta_1)\cos(\theta_2-\theta_1)\,. \end{aligned}$$

So that

$$L_{11}(t_{\theta_1},t_{\theta_2})\cdot L_{22}(t_{\theta_1},t_{\theta_2}) =$$
$$\rho(\theta_1)\rho(\theta_2)\left[p(\theta_1)p(\theta_2)\sin^2(\theta_2-\theta_1) - p'(\theta_1)p'(\theta_2)\cos^2(\theta_2-\theta_1) + \right.$$
$$\left.\big(p(\theta_1)p'(\theta_2) - p'(\theta_1)p(\theta_2)\big)\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)\right]\,.$$

Using now the equalities (5.4.9), previous formula simplifies to

$$L_{11}(t_{\theta_1}, t_{\theta_2}) \cdot L_{22}(t_{\theta_1}, t_{\theta_2}) = \rho(\theta_1)\rho(\theta_2)\left[p(\theta_1)p(\theta_2) - p'(\theta_1)p'(\theta_2)\right] = \frac{\rho(\theta_1)\rho(\theta_2)p(\theta_1)p(\theta_2)}{\sin^2(\theta_2-\theta_1)}\,. \tag{5.4.10}$$

Moreover:

$$L_{12}^2(t_{\theta_1}, t_{\theta_2}) = \rho^2(\theta_1)\rho^2(\theta_2)\sin^2(\theta_2-\theta_1)\,. \tag{5.4.11}$$

From (5.4.10) and (5.4.11), we have that

$$\frac{L_{11}L_{22}}{L_{12}^2}(\theta_1,\theta_2) = \frac{p(\theta_1)p(\theta_2)}{\rho(\theta_1)\rho(\theta_2)\sin^4(\theta_2-\theta_1)}\,.$$

By using again equalities (5.4.9), we obtain:

$$(p')^2(\theta_i)\sin^2(\theta_2-\theta_1) = p^2(\theta_i)\cos^2(\theta_2-\theta_1) = p^2(\theta_i)(1-\sin^2(\theta_2-\theta_1)) \Rightarrow \sin^2(\theta_2-\theta_1) = \frac{p^2(\theta_i)}{((p')^2+p^2)(\theta_i)}\,,$$

for $i=1,2$. We then conclude that

$$k_{1,2} := \frac{L_{11}L_{22}}{L_{12}^2} = \frac{(p'^2+p^2)(\theta_1)(p'^2+p^2)(\theta_2)}{\rho(\theta_1)\rho(\theta_2)p(\theta_1)p(\theta_2)}\,. \tag{5.4.12}$$

In the following lemma, we see how it is possible to perturb any strongly convex, centrally symmetric domain in order to assure that all the 4-periodic points of the dynamics of the perturbed table are non-degenerate (in particular, they are only a finite number). Recall that a 4-periodic point is non-degenerate for the map T, if the differential map DT^4 at the point does not admit the values ± 1 as eigenvalues.

Lemma 5.4.5. *Let Ω be a C^k strongly convex, centrally symmetric domain, $k \geq 2$, and let $p\colon \mathbb{S} \to \mathbb{R}$ be its support function of class C^k. Then, for every $\varepsilon > 0$, there exists a strongly convex, centrally symmetric smooth domain Ω_ε with support function $p_\varepsilon\colon \mathbb{S} \to \mathbb{R}$ such that*

$$\|p - p_\varepsilon\|_k < \varepsilon\,,$$

where $\|\cdot\|_k$ denotes the C^k-norm, and such that the symplectic billiard associated with the domain Ω_ε has a finite number of 4-periodic orbits, all of which are non-degenerate.

Proof. Let (θ_1, θ_2) correspond to a 4-periodic point in Π; then its orbit is of the form (5.4.8). In particular, the vector $\gamma'(\theta_2)$ is parallel to the vector $\gamma(\theta_1) - O$, and so, by (5.4.7), one has $\det(e_{\theta_2}, \gamma(\theta_1)) = 0$, that is

$$p(\theta_1)\cos(\theta_2-\theta_1) + p'(\theta_1)\sin(\theta_2-\theta_1) = 0 \quad\Leftrightarrow\quad \tan\left((\theta_2-\theta_1) - \frac{\pi}{2}\right) = \frac{p'}{p}(\theta_1)\,,$$

where we use the fact that the support function p is positive and that $\theta_2 \neq \theta_1$. There exist lifts of the angles θ_1, θ_2 –which, for the sake of simplicity, we continue to indicate by θ_1, θ_2– such that $\theta_2 - \theta_1 \in (0,\pi)$. Therefore, we have

$$\theta_2 = \arctan\left(\frac{p'}{p}(\theta_1)\right) + \frac{\pi}{2} + \theta_1\,. \tag{5.4.13}$$

Apply then the same argument to the subsequent points of the orbit: since $\theta_3 = \theta_1 + \pi$, we obtain that the angle θ_1 has to solve

$$G(\theta) := \frac{p'}{p}\left(\arctan\left(\frac{p'}{p}(\theta)\right) + \theta + \frac{\pi}{2}\right) + \frac{p'}{p}(\theta) = 0\,. \tag{5.4.14}$$

We proceed by observing that, up to perturb the support function (and so the billiard table), equation (5.4.14) admits a finite number of solutions. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of π-periodic trigonometric polynomials that approximates the support function p in the C^k-norm. Substitute then p_n in (5.4.14): one obtains an equation whose left-hand side is a real analytic function, which therefore has a finite number of solutions. This fact immediately gives that every strictly convex domain can be perturbed into a strictly convex domain with a finite number of 4-periodic orbits.

The last part of the proof is devoted to prove that it is possible to further perturb the domain in such a way that all 4-periodic orbits become non-degenerate. Let $n \in \mathbb{N}$ be large enough and fix the support function p_n. With an abuse of notation, we still denote by Ω the billiard table corresponding to the support function p_n and we designate p_n by p. Let $g(\theta)$ be a π-periodic function to be chosen later. We consider the perturbed domain whose support function is given by $p_\varepsilon(\theta) := e^{\varepsilon g(\theta)}p(\theta)$, and define the function

$$f(\theta,\varepsilon) := \frac{p'}{p}(\theta) + \varepsilon g'(\theta)\,.$$

Observe that $\frac{p'_\varepsilon}{p_\varepsilon}(\theta) = f(\theta,\varepsilon)$. The 4-periodic orbits of the billiard table associated to the support function p_ε are then determined by solving the following equation, analogous to(5.4.14):

$$G(\theta,\varepsilon) := f\left(\arctan(f(\theta,\varepsilon)) + \theta + \frac{\pi}{2}, \varepsilon\right) + f(\theta,\varepsilon) = 0\,, \tag{5.4.15}$$

that we can write also as

$$G(\theta,\varepsilon) = \frac{p'_\varepsilon}{p_\varepsilon}\left(\arctan\left(\frac{p'_\varepsilon}{p_\varepsilon}(\theta)\right) + \theta + \frac{\pi}{2}\right) + \frac{p'_\varepsilon}{p_\varepsilon}(\theta) = 0\,. \tag{5.4.16}$$

Let us show that a 4-periodic orbit for the table with support function p_ε is degenerate if and only if $\partial_\theta G(\theta,\varepsilon) = 0$. Let θ_1 correspond to a 4-periodic point for the table associated to the support function p_ε, i.e., $G(\theta_1,\varepsilon) = 0$. The point is degenerate if and only if, according to formula (5.4.12) and point (b) of Lemma 5.4.3, we have

$$k_{1,2} = \frac{(p'^2_\varepsilon + p^2_\varepsilon)(\theta_1)(p'^2_\varepsilon + p^2_\varepsilon)(\theta_2)}{\rho_\varepsilon(\theta_1)\rho_\varepsilon(\theta_2)p_\varepsilon(\theta_1)p_\varepsilon(\theta_2)} = 1\,, \tag{5.4.17}$$

where $\rho_\varepsilon = p_\varepsilon + p''_\varepsilon$ and, with an abuse of notation, we still denote by θ_2 the quantity $\arctan\left(\frac{p'_\varepsilon}{p_\varepsilon}(\theta_1)\right) + \theta_1 + \frac{\pi}{2}$. Consider then

$$\begin{aligned}
\partial_\theta G(\theta_1,\varepsilon) &= \frac{p''_\varepsilon p_\varepsilon - p'^2_\varepsilon}{p^2_\varepsilon}(\theta_2)\left(\frac{1}{1+\frac{p'^2_\varepsilon}{p^2_\varepsilon}(\theta_1)}\left(\frac{p''_\varepsilon p_\varepsilon - p'^2_\varepsilon}{p^2_\varepsilon}\right)(\theta_1) + 1\right) + \frac{p''_\varepsilon p_\varepsilon - p'^2_\varepsilon}{p^2_\varepsilon}(\theta_1)\\
&= \frac{p''_\varepsilon p_\varepsilon - p'^2_\varepsilon}{p^2_\varepsilon}(\theta_2)\left(\frac{(p''_\varepsilon + p_\varepsilon)p_\varepsilon}{p'^2_\varepsilon + p^2_\varepsilon}(\theta_1)\right) + \frac{p''_\varepsilon p_\varepsilon - p'^2_\varepsilon}{p^2_\varepsilon}(\theta_1)\\
&= \frac{(p''_\varepsilon + p_\varepsilon)p_\varepsilon - (p'^2_\varepsilon + p^2_\varepsilon)}{p^2_\varepsilon}(\theta_2)\left(\frac{(p''_\varepsilon + p_\varepsilon)p_\varepsilon}{p'^2_\varepsilon + p^2_\varepsilon}(\theta_1)\right) + \frac{(p''_\varepsilon + p_\varepsilon)p_\varepsilon - (p'^2_\varepsilon + p^2_\varepsilon)}{p^2_\varepsilon}(\theta_1)\\
&= \left(\frac{\rho_\varepsilon}{p_\varepsilon}(\theta_2) - \frac{p'^2_\varepsilon + p^2_\varepsilon}{p^2_\varepsilon}(\theta_2)\right)\left(\frac{\rho_\varepsilon p_\varepsilon}{p'^2_\varepsilon + p^2_\varepsilon}(\theta_1)\right) + \frac{\rho_\varepsilon}{p_\varepsilon}(\theta_1) - \frac{p'^2_\varepsilon + p^2_\varepsilon}{p^2_\varepsilon}(\theta_1)\,.
\end{aligned}$$

From (5.4.17), we have that $\frac{p_\varepsilon'^2+p_\varepsilon^2}{\rho_\varepsilon p_\varepsilon}(\theta_2) = \frac{\rho_\varepsilon p_\varepsilon}{p_\varepsilon'^2+p_\varepsilon^2}(\theta_1)$, and we obtain from the previous equalities:

$$\begin{aligned}
\partial_\theta G(\theta_1, \varepsilon) &= \frac{p_\varepsilon'^2 + p_\varepsilon^2}{p_\varepsilon^2}(\theta_2)\left[1 - \frac{\rho_\varepsilon p_\varepsilon}{p_\varepsilon'^2 + p_\varepsilon^2}(\theta_1)\right] - \frac{p_\varepsilon'^2 + p_\varepsilon^2}{p_\varepsilon^2}(\theta_1)\left[1 - \frac{\rho_\varepsilon p_\varepsilon}{p_\varepsilon'^2 + p_\varepsilon^2}(\theta_1)\right] \\
&= \left[1 - \frac{\rho_\varepsilon p_\varepsilon}{p_\varepsilon'^2 + p_\varepsilon^2}(\theta_1)\right]\left[\frac{p_\varepsilon'^2 + p_\varepsilon^2}{p_\varepsilon^2}(\theta_2) - \frac{p_\varepsilon'^2 + p_\varepsilon^2}{p_\varepsilon^2}(\theta_1)\right] \\
&= \left[1 - \frac{\rho_\varepsilon p_\varepsilon}{p_\varepsilon'^2 + p_\varepsilon^2}(\theta_1)\right]\left[\frac{p_\varepsilon'^2}{p_\varepsilon^2}(\theta_2) - \frac{p_\varepsilon'^2}{p_\varepsilon^2}(\theta_1)\right].
\end{aligned}$$

Since we are looking at a point satisfying (5.4.16), it holds $\frac{p'_\varepsilon}{p_\varepsilon}(\theta_2) = -\frac{p'_\varepsilon}{p_\varepsilon}(\theta_1)$, and we conclude that

$$\partial_\theta G(\theta_1, \varepsilon) = 0\,, \tag{5.4.18}$$

for every θ_1 that corresponds to a degenerate 4-periodic point. Note that both conditions, (5.4.16) and (5.4.18), are expressed by real-analytic equations.

If no solution of (5.4.15) also satisfies (5.4.18) for sufficiently small ε, we are done. Suppose instead, by contradiction, that this is not the case, i.e., there exists a sequence $\varepsilon_n \to 0$ and corresponding values $\theta_n \to \theta_0$ such that $G(\theta_n, \varepsilon_n) = \partial_\theta G(\theta_n, \varepsilon_n) = 0$ for every n. Clearly, θ_0 corresponds to a degenerate 4-periodic orbit of the unperturbed domain Ω. Note that

$$\partial_\varepsilon G(\theta, 0) = g'\left(\arctan(f(\theta,0)) + \theta + \frac{\pi}{2}\right) + g'(\theta)\left(\frac{\partial_\theta f\left(\arctan(f(\theta,0)) + \theta + \frac{\pi}{2}, 0\right)}{1 + f^2(\theta,0)} + 1\right).$$

Choose g so that $\partial_\varepsilon G(\theta_0, 0) \neq 0$, for every θ_0 corresponding to a 4-periodic orbit of the table Ω. This is possible because we only need to check a finite number of points.

Thanks to the assumptions on g, by the implicit function theorem we have that the solutions of (5.4.15) in a neighborhood of θ_0 are given by the graph of a real-analytic function $\varepsilon(\theta)$. Since Ω has only a finite number of 4-periodic orbits, the function $\varepsilon(\theta)$ is not identically zero. There exists then $N \in \mathbb{N}$ and a function c, which does not vanish in a neighborhood of any θ_0, corresponding to a 4-periodic orbit for Ω, such that

$$\varepsilon(\theta) = (\theta - \theta_0)^N c(\theta)\,. \tag{5.4.19}$$

We then have

$$\varepsilon_n = c(\theta_n)(\theta_n - \theta_0)^N\,,$$

and, by assumption,

$$\partial_\theta G(\theta_n, \varepsilon(\theta_n)) = 0\,.$$

It follows, by real-analyticity, that

$$\partial_\theta G(\theta, \varepsilon(\theta)) = 0 \quad \text{for all } \theta\,.$$

Then, again by the implicit function theorem, in a neighborhood of the graph of $\varepsilon(\cdot)$ we can write

$$\partial_\theta G(\theta, \varepsilon) = G(\theta, \varepsilon)\, h(\theta, \varepsilon)\,,$$

for some suitable function h. Differentiating with respect to θ, we get

$$\partial_\theta^2 G(\theta, \varepsilon) = G(\theta, \varepsilon)\left(h^2(\theta, \varepsilon) + \partial_\theta h(\theta, \varepsilon)\right)\,,$$

and proceeding inductively, we find

$$\partial_\theta^n G(\theta, \varepsilon(\theta)) = 0 \quad \text{for all } n\,.$$

As a consequence, by real-analyticity, we deduce that G is independent of θ in an open set by (5.4.19), and hence everywhere. This is the required contradiction and concludes the proof. □

We want now to make precise the topology used in order to be able to talk about open and dense sets in point (b). Denote by $C^\infty(\mathbb{S},\mathbb{R}^2)$ the set of smooth functions $\gamma\colon \mathbb{S}\to\mathbb{R}^2$ and endow it with the C^2-norm $\|\cdot\|_2$. Consider the open set

$$\mathcal{B}_2^0 := \{\gamma\in C^\infty(\mathbb{S},\mathbb{R}^2):\ \gamma\colon \mathbb{S}\to\mathbb{R}^2 \text{ is an embedding}\}\,;$$

we will be interested in the subset

$$\mathcal{B}_2 := \{\gamma\in \mathcal{B}_2^0:\ \gamma(\mathbb{S}) \text{ is strongly convex and centrally symmetric}\}$$

endowed with the restricted topology induced by the norm $\|\cdot\|_2$. The following result is then immediate from Lemma 5.4.5.

Corollary 5.4.6. *There exists an open and dense set $\mathcal{U}\subset\mathcal{B}_2$ such that, for every $\gamma\in\mathcal{U}$, the symplectic billiard map associated to γ has a finite number of 4-periodic orbits, all non-degenerate.*

We are now ready to prove the following statement, which corresponds to point (b).

Proposition 5.4.7. *There exists an open and dense set $\mathcal{U}\subset\mathcal{B}_2$ such that for any $\gamma\in\mathcal{U}$ the following property holds. Let $\lambda'(\Omega)\in(0,1)$ as given in Theorem 5.3.7; there exists $\lambda''(\Omega)\in(0,\lambda'(\Omega))$ such that for every $\lambda\in(0,\lambda''(\Omega))$, the Birkhoff attractor Λ has rotation number $1/4$ and*

$$\Lambda=\bigcup_{i=1}^{l}\bigcup_{j=0}^{3}\overline{\mathcal{W}^u(T_\lambda^j(H_i);T_\lambda^4)}$$

for some finite collection $\{H_i\}_{i=1}^l$ of 4-periodic points of saddle type, where $\mathcal{W}^u(H_i;T_\lambda^4)$ denotes the unstable manifold of a hyperbolic point H_i with respect to the dynamics T_λ^4.

Proof. The proof is an adaptation of the proof of Theorem 5.14 in [20]. Let $\lambda'(\Omega)\in(0,1)$ be given by Theorem 5.3.7. Then, for every $\lambda\in(0,\lambda'(\Omega))$, the (Birkhoff) attractor is a C^1 normally contracted graph of a function $\eta_\lambda:\mathbb{S}\to\mathbb{R}$. Let us define

$$g_\lambda:\mathbb{S}\to\mathbb{S},\qquad t_1\mapsto p_1\circ T_\lambda(t_1,\eta_\lambda(t_1))$$

the circle map induced by T_λ on the attractor, where p_1 is the projection on the first coordinate. As proved in Claim 5.15 in [20], $g_\lambda\to g_0$ in the C^1 topology when $\lambda\to 0$. We proceed to prove that g_0 is a circle diffeomorphism. By Lemma 5.2.18, we get

$$g_\lambda'(t_1)=-\frac{1}{L_{12}(t_1,t_2)}\left[L_{11}(t_1,t_2)+\eta_\lambda'(t_1)\right]\,.$$

In particular $g_0'(t_1)=-\frac{L_{11}(t_1,t_2)}{L_{12}(t_1,t_2)}$.

We want to show that g_0 is a circle diffeomorphism: in particular, it is sufficient to assure that $g_0'\neq 0$ at every t_1. Its expression is, more explicitely,

$$g_0'(t_1)=\frac{p}{\rho}(\theta_2)+\frac{p'}{\rho}(\theta_2)\cot(\theta_2-\theta_1)\,. \tag{5.4.20}$$

We recall that $\Pi\subset\mathbb{S}\times\{0\}$ does not depend on $\lambda\in[0,1]$. Moreover, by Lemma 5.4.5, for a C^k-open and dense subset of centrally symmetric billiard tables, Π consists of non-degenerate 4-periodic points, i.e., saddles or sinks, hence persisting under perturbations. In the sequel, in order to conclude, we prove that the circle dynamics of g_λ (which is a small perturbation of g_0 in the C^1 topology) essentially recovers the one of T_λ on Λ. Let $(t_1,0)\in\Pi$. In particular, we have

$\cot(\theta_2 - \theta_1) = \frac{p'}{p}(\theta_2)$ and $\cot(\theta_1 - \theta_2) = \frac{p'}{p}(\theta_1)$, thanks to the centrally symmetric hypothesis of the table. Then, according to (5.4.20):

$$(g_0^4)'(t_1) = (g_0'(t_1)g_0'(t_2))^2 = \left(\frac{p'^2 + p^2}{\rho p}(\theta_2)\,\frac{p'^2 + p^2}{\rho p}(\theta_1)\right)^2 = k_{1,2}^2 \neq 1$$

so that, for the circle diffeomorphism g_0, the 4-periodic point t_1 is repelling or attracting (respectively when $k_{1,2} > 1$ or $k_{1,2} < 1$). Since for $\lambda > 0$ small enough g_λ is C^1-close to g_0 and the points of Π are generically isolated, there exists $\lambda''(\Omega) \in (0, \lambda'(\Omega))$ such that for every $\lambda \in (0, \lambda''(\Omega))$ the 4-periodic point t_1 for g_0 admits a continuation for g_λ. Thus, the restriction of T_λ on Λ still has rotation number 1/4. As in the end of the proof of Theorem 5.14 in [20], by considering the α-limit and the ω-limit sets of the points in $\Lambda \setminus \Pi$, we conclude that

$$\Lambda_\lambda = \bigcup_{i=1}^{l} \bigcup_{j=0}^{3} \overline{\mathcal{W}^u(T_\lambda^j(H_i); T_\lambda^4)}$$

for some finite collection $\{H_i\}_{i=1}^l$ of 4-periodic points of saddle type. $\square$

5.4.2 Fragility of invariant curves of rotation number 1/4

In this subsection, we will see that, among centrally symmetric tables, invariant curves of rotation number $\frac{1}{4}$, for (conservative) symplectic billiard maps, are very easy to destroy. This result is used in Section 5.5 to obtain topologically complicated Birkhoff attractors in the dissipative framework.

Proposition 5.4.8. *There exists an open and dense set of strongly convex, centrally symmetric billiard tables $\mathcal{U} \subset \mathcal{B}_2$ such that, for every $\gamma \in \mathcal{U}$, the associated symplectic billiard map does not have an invariant curve of rotation number $\frac{1}{4}$.*

We split the proof of Proposition 5.4.8 in two parts, in order to show first the openness and then the density property.

Lemma 5.4.9. *The set of strongly convex, centrally symmetric billiard tables in $\mathcal{B}_2$ whose associated symplectic billiard map has an invariant curve of rotation number $\frac{1}{4}$ is closed.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of centrally symmetric tables and denote by T_n the symplectic billiard map associated to γ_n. Assume that every T_n exhibits an invariant curve Γ_n of rotation number $\frac{1}{4}$. Moreover, assume that the sequence of γ_n is converging to γ in the C^2 topology. The sequence of corresponding billiard maps $(T_n)_{n\in\mathbb{N}}$ is going to the billiard map T, associated to γ, in the C^1 topology. By Birkhoff's theorem [30], since each T_n is a twist map, each invariant curve Γ_n is a graph of a Lipschitz function over $\mathbb{S}$. Moreover, up to consider n large enough, since the Lipschitz constant depends on the twist condition and since the tables are converging in the C^2 topology, there exists a constant $C > 0$ such that all invariant curves Γ_n are graphs of C-Lipschitz functions. Therefore, up to subsequences, Γ_n converges to a curve Γ, which is the graph of a C-Lipschitz function and is T-invariant. Moreover, the curve Γ has still rotation number $\frac{1}{4}$. We deduce that the set of tables with invariant curves of rotation number $\frac{1}{4}$ is closed. $\square$

Proposition 5.4.10. *The set of strongly convex, centrally symmetric billiard tables whose associated symplectic billiard map does not have an invariant curve of rotation number $\frac{1}{4}$ is dense among $\mathcal{B}_2$.*

Proof. Let Ω be a centrally symmetric, strictly convex smooth table with an invariant curve of rotation number 1/4. By Corollary 5.4.6, we can perform a first C^2 perturbation to obtain a table Ω' which has a finite number of 4-periodic orbits, all non-degenerate. If the billiard map associated to Ω' has no invariant curve of rotation number 1/4, then we are done. Assume that this is not the case, i.e., the billiard map associated to Ω' has an invariant curve Γ of rotation number 1/4: in particular, there are a finite number of 4-periodic points on Γ. Since Γ is invariant, we deduce that all the 4-periodic points on the curve are hyperbolic ones. Indeed, since the points are non-degenerate, they are either hyperbolic (i.e., the corresponding differential map DT^4 has eigenvalues of modulus different from 1) or elliptic ones with irrational angle (i.e., the eigenvalues of DT^4 are $e^{2\pi i\rho}$ with $\rho \in \mathbb{R}\setminus\mathbb{Q}$). However, if by contradiction a point is elliptic, then, after some iterations, the vertical vector $(0,1)$ would be sent by the differential dynamics to a vector pointing downward: this would contradict the existence of the invariant curve and the orientation preserving hypothesis of the map. By Poincaré classification theorem for circle homeomorphisms, the curve Γ is composed by heteroclinic or homoclinic connections between 4-periodic hyperbolic points, see Figure 5.9.

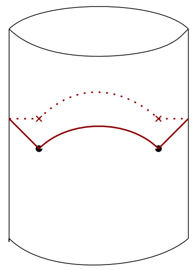


Figure 5.9: Invariant curve of rotation number $\frac{1}{4}$ with homoclinic (heteroclinic) connexions.

Denote by T the symplectic billiard map associated to the table. With an abuse of notation, denote by T also a lift of it. According to the notation used in [12], we can consider bi-infinite configurations $(t_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$. Orbits for the map T are then in correspondence, thanks to the twist condition, to configurations that are critical for the formal action functional $\mathcal{L}\colon \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$, $(t_i)_{i\in\mathbb{Z}} \mapsto \sum_{i\in\mathbb{Z}} L(\gamma(t_i),\gamma(t_{i+1}))$. That is, a bi-infinite sequence corresponds to an orbit if and only if

$$L_1(t_i,t_{i+1}) + L_2(t_{i-1},t_i) = 0 \qquad \forall i \in \mathbb{Z}\,. \tag{5.4.21}$$

Let $\gamma\colon \mathbb{S} \to \mathbb{R}^2$ be the smooth arc-length parametrization of Ω', $|\partial\Omega'| = 2\pi$. Fix $\varepsilon > 0$ and consider the perturbed domain corresponding to

$$\bar\gamma\colon t \in \mathbb{S} \to \bar\gamma(t) := \gamma(t) + \delta(t)\, n(t) \in \mathbb{R}^2\,,$$

where $n(t)$ is the unit normal vector at $\gamma(t)$, pointing inside the domain Ω'. Up to choose the smooth function δ small enough, we obtain a perturbation of the initial table which is ε-C^2 close to Ω'. Our aim is then proving that we can choose the function δ so that the map of the perturbed domain does not have an invariant curve or rotation number 1/4.

Consider $(t_i^-)_{i\in\mathbb{Z}}, (t_i^+)_{i\in\mathbb{Z}}$ two sequences corresponding to nearby 4-periodic hyperbolic points contained in the invariant curve Γ. The curve Γ is the graph of a Lipschitz function $\eta\colon \mathbb{S} \to \mathbb{R}$. Let $(t_i)_{i\in\mathbb{Z}}$ be a sequence corresponding to a point $P_0 = (t_0,\eta(t_0)) \in \Gamma$, which belongs to the unstable manifold of $(t_0^-,\eta(t_0^-)) = (t_0^-,0)$ and to the stable manifold of $(t_0^+,\eta(t_0^+)) = (t_0^+,0)$. In particular it holds $t_i^- < t_i < t_i^+ < t_i^- + \pi$ for every $i \in \mathbb{Z}$ and

$$\lim_{i\to+\infty} t_i^+ - t_i = \lim_{i\to-\infty} t_i - t_i^- = 0\,.$$

Let $U \subset \mathbb{R}$ be a neighborhood of t_0 such that $t_i \notin U$ for every $i \neq 0$. This is possible because, at $\pm\infty$, the t_i accumulate on 4-periodic points, which are finite. Assume that the function δ is a positive π-periodic function whose support is contained in U (i.e., its projection on $\mathbb{S}$) and such that $\delta(t_0) = \delta'(t_0) = 0$ and $\delta''(t_0) \neq 0$.

Observe in particular that all the 4-periodic points, which were maximizing the area of the inscribed quadrilateral, are still 4-periodic orbits of the perturbed symplectic billiard map and moreover they are still maximizing the area of the inscribed quadrilateral for the perturbed domain, i.e., they still belong to the Aubry-Mather set of rotation number 1/4.

Since the characterization of orbits (5.4.21) only involves γ and γ' along the bi-infinite sequence corresponding to the orbit, and since both $\bar{\gamma}$ and $\bar{\gamma}'$ do not change along the sequence $(t_i)_{i\in\mathbb{Z}}$ by hypothesis, we deduce that the sequence $(t_i)_{i\in\mathbb{Z}}$ corresponds to an orbit also for the perturbed symplectic billiard map. Moreover, it is still a heteroclinc or homoclinic orbit, intersection of the unstable manifold of $(t_0^-, 0)$ and the stable manifold of $(t_0^+, 0)$ (for the perturbed dynamics).

Therefore, we only have to show that this intersection is transverse. Observe that, for Ω', the tangent spaces to the stable and unstable manifolds at $(t_0, \eta(t_0))$ clearly coincide and are generated by the vector $(1, \eta'(t_0))$.

From now on, let $P_i := (t_i, \eta(t_i))$. Consider the perturbed dynamics and the point P_1: the tangent space to the stable manifold of $(t_1^+, \eta(t_1^+))$ at the point P_1 is still generated by the vector $(1, \eta'(t_1))$, since the dynamics remains unchanged on the future of the point.

Denote then by T_γ and $T_{\bar{\gamma}}$ the symplectic billiard maps associated to γ and $\bar{\gamma}$ respectively. It is then sufficient to show that

$$T_{P_1} W^s_{\bar{\gamma}}(t_1^+, 0) \pitchfork T_{P_1} W^u_{\bar{\gamma}}(t_1^-, 0)\,.$$

As explained right above:

$$\langle w \rangle := \langle (1, \eta'(t_1)) \rangle = T_{P_1} W^s_{\bar{\gamma}}(t_1^+, 0) = T_{P_1} W^s_{\gamma}(t_1^+, 0)\,.$$

Moreover, let denote

$$\langle u \rangle := T_{P_1} W^u_{\bar{\gamma}}(t_1^-, 0)\,.$$

Then we have:

$$w = DT_\gamma(t_0, s_0)\, DT_\gamma(t_{-1}, s_{-1}) \begin{pmatrix} 1 \\ \eta'(t_{-1}) \end{pmatrix} \in \langle w \rangle$$

and

$$u = DT_{\bar{\gamma}}(t_0, s_0)\, DT_{\bar{\gamma}}(t_{-1}, s_{-1}) \begin{pmatrix} 1 \\ \eta'(t_{-1}) \end{pmatrix} \in \langle u \rangle\,.$$

We need to prove that w and u are linearly independent. We start by recalling that for $i \in \mathbb{Z}$ (see formula (5.2.9) with $\lambda = 1$):

$$DT_\gamma(t_i, s_i) = -\frac{1}{L_{12}(t_i, t_{i+1})} \begin{pmatrix} L_{11}(t_i, t_{i+1}) & 1 \\ -L_{12}^2(t_i, t_{i+1}) + L_{22}(t_i, t_{i+1}) L_{11}(t_i, t_{i+1}) & L_{22}(t_i, t_{i+1}) \end{pmatrix}.$$

Consequently, by denoting

$$A_\gamma := L_{12}(t_{-1}, t_0) L_{12}(t_0, t_1) DT_\gamma(t_0, s_0)\, DT_\gamma(t_{-1}, s_{-1})\,,$$

we have that:

$$A_\gamma = \begin{pmatrix} L_{11}(t_{-1}, t_0) L_{11}(t_0, t_1) + F(t_{-1}, t_0) & L_{22}(t_{-1}, t_0) + L_{11}(t_0, t_1) \\ L_{11}(t_{-1}, t_0) F(t_0, t_1) + L_{22}(t_0, t_1) F(t_{-1}, t_0) & L_{22}(t_{-1}, t_0) L_{22}(t_0, t_1) + F(t_0, t_1) \end{pmatrix},$$

where, for $i \in \mathbb{Z}$:

$$F(t_i, t_{i+1}) := -L_{12}^2(t_i, t_{i+1}) + L_{22}(t_i, t_{i+1}) L_{11}(t_i, t_{i+1})\,.$$

Similarly, let denote

$$B_{\bar{\gamma}} := L_{12}(t_{-1}, t_0) L_{12}(t_0, t_1) DT_{\bar{\gamma}}(t_0, s_0)\, DT_{\bar{\gamma}}(t_{-1}, s_{-1})\,.$$

Observe that, since we have choosen the arc-length parametrization and $\delta(t_0) = \delta'(t_0) = 0$, we get

$$\bar{\gamma}''(t_0) = \gamma''(t_0) + \delta''(t_0) n(t_0)\,.$$

By straightforward computations, we obtain that

$$B_{\bar{\gamma}} = A_\gamma + \det(\delta''(t_0) n(t_0), \gamma(t_1) - \gamma(t_{-1})) C_\gamma \tag{5.4.22}$$

where

$$C_\gamma = \begin{pmatrix} L_{11}(t_{-1}, t_0) & 1 \\ L_{11}(t_{-1}, t_0) L_{22}(t_0, t_1) & L_{22}(t_0, t_1) \end{pmatrix}.$$

Moreover, we notice that also A_γ can be written in terms of C_γ:

$$A_\gamma = (L_{22}(t_{-1}, t_0) + L_{11}(t_0, t_1)) C_\gamma + D_\gamma\,, \tag{5.4.23}$$

where

$$D_\gamma = \begin{pmatrix} -L_{12}^2(t_{-1}, t_0) & 0 \\ -L_{12}^2(t_0, t_1) L_{11}(t_{-1}, t_0) - L_{12}^2(t_{-1}, t_0) L_{22}(t_0, t_1) & -L_{12}^2(t_0, t_1) \end{pmatrix}.$$

As a consequence of (5.4.23) and (5.4.22), $w \in \langle A_\gamma(1, \eta'(t_{-1})) \rangle$ and $u \in \langle B_{\bar{\gamma}}(1, \eta'(t_{-1})) \rangle$ are linearly independent if and only if $C_\gamma(1, \eta'(t_{-1}))$ and $D_\gamma(1, \eta'(t_{-1}))$ are linearly independent. In the sequence, we proceed by checking this condition. We observe that

$$C_\gamma \begin{pmatrix} 1 \\ \eta'(t_{-1}) \end{pmatrix} = (L_{11}(t_{-1}, t_0) + \eta'(t_{-1})) \begin{pmatrix} 1 \\ L_{22}(t_0, t_1) \end{pmatrix}.$$

Moreover:

$$D_\gamma \begin{pmatrix} 1 \\ \eta'(t_{-1}) \end{pmatrix} = -L_{12}^2(t_{-1}, t_0) \begin{pmatrix} 1 \\ L_{22}(t_0, t_1) \end{pmatrix} - L_{12}^2(t_0, t_1)(L_{11}(t_{-1}, t_0) + \eta'(t_{-1})) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By the twist condition, $L_{12} > 0$. Therefore, the two vectors above are lineraly independent if and only if there exists t_{-1} such that $L_{11}(t_{-1}, t_0) + \eta'(t_{-1}) \neq 0$, or equivalently:

$$L_{11}(t_{-1}, p_1 \circ T_\gamma(t_{-1}, \eta(t_{-1})) + \eta'(t_{-1}) \neq 0\,.$$

If this not the case, it means that the Lipschitz invariant graph η satisfies a.e. the equation:

$$L_{11}(t_{-1}, p_1 \circ T_\gamma(t_{-1}, \eta(t_{-1}))) + \eta'(t_{-1}) = 0$$

and consequently η is at least C^1. However, in such a case, the 4-periodic points $(t_i^{\pm}, 0)$ would be degenerate, since the stable and unstable directions should coincide, giving the desired contradiction.

We have then showed that for a perturbation δ such that $\delta(t_0) = \delta'(t_0) = 0$ and $\delta''(t_0) \neq 0$, there are 4-periodic hyperbolic points within the Aubry-Mather set of rotation number 1/4 with transverse heteroclinic or homoclinic intersections. In particular, the map $T_{\bar{\gamma}}$ associated to the perturbed domain does not have an invariant curve of rotation number 1/4. □

The proof of Proposition 5.4.8 is concluded: it is sufficient to put together Lemma 5.4.9 and Proposition 5.4.10.

5.5 Topologically and dynamically complex Birkhoff attractors

Birkhoff attractors described in Section 5.3 do not give an idea of their possible complexity, which in general occurs for mild dissipation (that is for λ close to 1). A sufficient condition in order to observe such a topologically and dynamically complex phenomena is that the conservative billiard map admits an instability region containing the zero section. In fact, by an adaptation of a result of Le Calvez (here Proposition 5.5.5), in such a case the Birkhoff attractor Λ for the corresponding dissipative dynamics, with λ close to 1, admits different upper and lower rotation numbers $\rho^- < \rho^+$ (defined in Section 5.2.1). This property has various consequences for Λ, all already observed in [20] for Birkhoff billiards. In particular:

(*i*) Λ is an *indecomposable continuum*, that is it cannot be written as the union of two connected non-trivial sets (from a result by Charpentier, see [35]).

(*ii*) For every rational $p/q \in (\rho^-, \rho^+)$, there exists a periodic point in Λ of the rotation number $\frac{p}{q}$ (which can be deduced from [22]).

(*iii*) If x is a saddle periodic point of rotation number $\frac{p}{q}$, for $\frac{p}{q} \in (\rho^-, \rho^+)$, then its unstable manifold is contained in Λ (see [64, Proposition 14.3]).

(*iv*) The map T_λ restricted to Λ has positive topological entropy, as a consequence of the existence of a rotational horseshoe (see [78, Theorem A]).

We remind that an essential curve in $\mathbb{S}\times\mathbb{R}$ is a topological embedding of $\mathbb{S}$ which is not homotopic to a point.

In the next proposition, we prove that, for a centrally symmetric table, any essential curve for the corresponding symplectic billiard dynamics passing through $\mathbb{S}\times\{0\}$ has necessarily rotation number $1/4$.

Proposition 5.5.1. *Let $\Omega \subset \mathbb{R}^2$ be a C^k strictly convex, centrally symmetric domain, $k \geq 2$. Denote by $T\colon \mathcal{P} \to \mathcal{P}$ the associated symplectic billiard map. Any essential invariant curve $\Gamma \subset \mathcal{P}$ passing through $\mathbb{S}\times\{0\}$ has rotation number $\rho(\Gamma) = 1/4$.*

Proof. Recall that $\hat{T} : \hat{\mathcal{P}} \ni (t_0, t_1) \mapsto (t_1, t_2) \in \hat{\mathcal{P}}$ denotes the symplectic billiard map in the coordinates (t_1, t_2). Let define

$$\hat{I}_2 : \hat{\mathcal{P}} \ni (t_1, t_2) \mapsto (t_1, t_0^*) \in \hat{\mathcal{P}}$$

and

$$* : \hat{\mathcal{P}} \ni (t_1, t_2) \mapsto (t_1^*, t_2^*) \in \hat{\mathcal{P}}$$

Notice that $*$ is an involution, i.e., $*\circ* = Id$. Moreover, from the centrally symmetric assumption –see Figure 5.7– we have that $\hat{T}$ and $*$ commute: this implies that also $\hat{I}_2$ and $*$ commute. Thus we obtain

$$\hat{T} \circ \hat{I}_2(t_1, t_2) = \hat{T}(t_1, t_0^*) = (t_0^*, t_{-1})$$

and

$$\hat{I}_2 \circ \hat{T}^{-1}(t_1, t_2) = \hat{I}_2(t_0, t_1) = (t_0, t_{-1}^*)\,,$$

so that, in particular:

$$\hat{T} \circ \hat{I}_2 = * \circ \hat{I}_2 \circ \hat{T}^{-1} = \hat{I}_2 \circ * \circ \hat{T}^{-1}\,. \tag{5.5.1}$$

In the sequel, we simply denote by $\rho(\cdot)$ the rotation number with respect to T and by $\rho(\cdot, T^2)$ the rotation number with respect to T^2. We divide the proof in two points.

(i) We start by proving that $\rho(\Gamma)$ is necessarily rational. Let $\hat{\Gamma} \subset \hat{\mathcal{P}}$ be the image through $\phi : \mathcal{P} \to \hat{\mathcal{P}}$ –see formula (5.2.8)– of such a curve Γ and suppose –by contradiction– that Γ has irrational rotation number. First, we deduce, as a consequence of the centrally symmetric hypothesis, that also $*(\hat{\Gamma})$ is invariant: indeed $\hat{T} \circ * = * \circ \hat{T}$ and, since $\hat{T}(\hat{\Gamma}) = \hat{\Gamma}$, the required invariance follows. Moreover, the two invariant curves $\hat{\Gamma}$ and $*(\hat{\Gamma})$ must intersect, as we are going to show. With an abuse of notation, we denote by $*$ also the map corresponding to $*$ on $\mathcal{P}$, i.e.,

$$*\colon (t,s) \in \mathcal{P} \to *(t,s) = (t^*, s) \in \mathcal{P} \,.$$

Up to choose a good parametrization of the domain, we can assume that $t^* = t + \pi$ for every $t \in \mathbb{S}$. The curves $\Gamma, *(\Gamma)$ are both invariant, thus they are graphs. To fix the idea, we have

$$\Gamma = \{(t, \eta(t)) :\ t \in \mathbb{S}\} \quad \text{and} \quad *(\Gamma) = \{(t, \eta(t+\pi)) :\ t \in \mathbb{S}\} \,.$$

Since $\int_{\mathbb{S}} \eta(t)\,dt = \int_{\mathbb{S}} \eta(t+\pi)\,dt$, we deduce that $\Gamma \cap *(\Gamma) \neq \emptyset$. Thus, the curve $*(\Gamma)$ has the same rotation number as Γ. If, by contradiction, this rotation number is irrational, then $*(\Gamma)$ and Γ must coincide (by e.g. [12][Section 4]). Then, by (5.5.1):

$$\hat{T}(\hat{I}_2(\hat{\Gamma})) = \hat{I}_2 \circ * \circ \hat{T}^{-1}(\hat{\Gamma}) = \hat{I}_2(\hat{\Gamma}) \,.$$

Observe that the map corresponding to $\hat{I}_2$ in $\mathcal{P}$ is the map $I_2\colon (t,s) \in \mathcal{P} \to (t,-s) \in \mathcal{P}$. The image corresponding to the curve $\hat{I}_2(\hat{\Gamma})$ is then $I_2(\Gamma)$. The previous equality means that also $I_2(\Gamma)$ is an invariant essential curve and therefore, since Γ is a graph,

$$\Gamma \cap (\mathbb{S} \times \{0\}) \neq \emptyset \Leftrightarrow \Gamma \cap I_2(\Gamma) \neq \emptyset \,.$$

Since Γ and $I_2(\Gamma)$ intersect, these two curves have the same rotation number, i.e., $\rho(\Gamma) = \rho(I_2(\Gamma))$. Again, since the rotation number is supposed irrational, we deduce that $\Gamma = I_2(\Gamma) = \mathbb{S} \times \{0\}$. This contradicts the fact that the zero section contains 4 periodic points and, thus, that its rotation number should be $1/4 \in \mathbb{Q}$.

(ii) We finally prove that $\rho(\Gamma)$ is necessarily $1/4$. We start by recalling that $\rho(\Gamma) = \rho(*(\Gamma))$, since they are both invariant curves and the intersection of Γ and $*(\Gamma)$ is not empty, as explained above. Denote by p/q the rotation number of Γ. If $\Gamma \cap *(\Gamma) \subset \mathbb{S} \times \{0\}$, then $\Gamma \cap *(\Gamma)$ in an invariant set necessarily given by 4-periodic points. Hence $\rho(\Gamma) = 1/4$. We then suppose –on the contrary– that $\Gamma \cap *(\Gamma) \not\subset \mathbb{S} \times \{0\}$. To conclude, we need to consider also the essential curve $I_2(\Gamma)$, which is T^2-invariant. In fact, applying twice (5.5.1) and since $\hat{T}$ and $*$ commute, we get:

$$\hat{T}^2 \circ \hat{I}_2(\hat{\Gamma}) = \hat{T} \circ * \circ \hat{I}_2(\hat{\Gamma}) = * \circ \hat{T} \circ \hat{I}_2(\hat{\Gamma}) = * \circ * \circ \hat{I}_2(\hat{\Gamma}) = \hat{I}_2(\hat{\Gamma}) \,.$$

From now on, with abuse of notation, we continue to indicate with T its lift to $\mathbb{R}^2$. By Proposition 5.2.15, T^2 is a twist map so we can consider the rotation number of the essential T^2-invariant curves Γ, $*(\Gamma)$ and $I_2(\Gamma)$:

$$\frac{2p}{q} = 2\rho(\Gamma) = \rho(\Gamma, T^2) = \rho(*(\Gamma), T^2) = \rho(I_2(\Gamma), T^2) \,.$$

In the last two equalities, we have respectively used the facts that the rotation numbers are the same because both $\Gamma \cap *(\Gamma)$ and $\Gamma \cap I_2(\Gamma)$ are nonempty.
By using classical Aubry-Mather theory (see [12] and also [5, Subsection 3.4]), in the sequel we will prove that $\Gamma \cap *(\Gamma) \cap I_2(\Gamma) \neq \emptyset$ since this intersection necessarily contains every $2p/q$-periodic point for T^2.
Recall that T^2 is still a twist map and that we are denoting by T^2 also a lift of T^2. Since Γ,

$*(\Gamma)$ and $I_2(\Gamma)$ are all T^2-invariant graphs, they consist of points of T^2-minimizing orbits (see [73, Proposition 2.8], [12] or [5, Subsection 5.2]). Equivalently, $\Gamma, *(\Gamma), I_2(\Gamma)$ are contained in

$$\mathcal{M}_{\frac{2p}{q}}(T^2) := \{(t,s) \in \mathcal{P} \text{ having a } T^2\text{-minimizing orbit of rotation number } 2p/q\}\,.$$

We recall that set $\mathcal{M}_{\frac{2p}{q}}(T^2)$ is the disjoint union of 3 invariant sets:

$$\mathcal{M}_{\frac{2p}{q}}(T^2) = \mathcal{M}^{per}_{\frac{2p}{q}}(T^2) \sqcup \mathcal{M}^{+}_{\frac{2p}{q}}(T^2) \sqcup \mathcal{M}^{-}_{\frac{2p}{q}}(T^2)\,,$$

where, denoting by p_1 the projection on the first coordinate,

$$\mathcal{M}^{per}_{\frac{2p}{q}}(T^2) = \{(t,s) \in \mathcal{M}_{\frac{2p}{q}}(T^2) |\ p_1 \circ T^{2q}(t,s) = p_1(t,s) + 2p\}\,,$$

$$\mathcal{M}^{+}_{\frac{2p}{q}}(T^2) = \{(t,s) \in \mathcal{M}_{\frac{2p}{q}}(T^2) |\ p_1 \circ T^{2q}(t,s) > p_1(t,s) + 2p\}\,,$$

and

$$\mathcal{M}^{-}_{\frac{2p}{q}}(T^2) = \{(t,s) \in \mathcal{M}_{\frac{2p}{q}}(T^2) |\ p_1 \circ T^{2q}(t,s) < p_1(t,s) + 2p\}\,.$$

Since both $\mathcal{M}^{per}_{\frac{2p}{q}}(T^2) \sqcup \mathcal{M}^{+}_{\frac{2p}{q}}(T^2)$ and $\mathcal{M}^{per}_{\frac{2p}{q}}(T^2) \sqcup \mathcal{M}^{-}_{\frac{2p}{q}}(T^2)$ are well-ordered sets, we have that the fiber of every $P \in \mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$ intersects the set $\mathcal{M}_{\frac{2p}{q}}(T^2)$ uniquely in P. This means that:

$$p_1^{-1}\left(p_1\left(\mathcal{M}^{per}_{\frac{2p}{q}}(T^2)\right)\right) \bigcap \mathcal{M}_{\frac{2p}{q}}(T^2) = \mathcal{M}^{per}_{\frac{2p}{q}}(T^2)\,, \tag{5.5.2}$$

see [12, Section 5]. We deduce that the set $\mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$ is necessarily contained in every essential T^2-invariant curve of rotation number $2p/q$. In fact, let Γ be an essential T^2-invariant curve of rotation number $\frac{2p}{q}$: then, it is contained in $\mathcal{M}_{\frac{2p}{q}}(T^2)$ and it is a graph $\{(t, \eta(t)) :\ t \in \mathbb{R}\}$. Let $P \in \mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$ and let $p_1(P)$ be its projection on the first coordinate. From (5.5.2), we deduce that the point $(p_1(P), \eta(p_1(P)) \in \Gamma$ must be the point P, i.e., $P \in \Gamma$. In particular:

$$\emptyset \neq \mathcal{M}^{per}_{\frac{2p}{q}}(T^2) \subseteq \Gamma \cap *(\Gamma) \cap I_2(\Gamma)\,.$$

Since $\Gamma \cap I_2(\Gamma) \subset \mathbb{S} \times \{0\}$, $\mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$ is given by periodic points for T^2 contained in $\mathbb{S}\times\{0\}$. Given $P \in \mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$, the T^2-orbit of $T(P)$ is clearly contained in Γ, since Γ is actually T-invariant. Consequently, since $T(P)$ is periodic for T^2, its orbit belongs to $\mathcal{M}^{per}_{\frac{2p}{q}}(T^2)$ and therefore it is contained in $\mathbb{S} \times \{0\}$. This means that the T-orbit of P is entirely contained in the zero section, and therefore $\rho(\Gamma) = 1/4$.

□

In the sequel, the union of all T-invariant essential curves in $\mathcal{P}$ will be indicated by $\mathcal{V}(T)$. The next definition can be formulated for a general twist map (see e.g. Definition 6.9 in [20]).

Definition 5.5.2. *An instability region for $T : \mathcal{P} \to \mathcal{P}$ is an open bounded connected component of $\mathcal{P} \setminus \mathcal{V}(T)$ that contains in its interior an essential curve.*

We conclude this preamble by recalling a result in [20] (see Proposition 6.10 and Corollary 6.12) whose proof adapts a former argument due to Le Calvez (see [64, Section 8]). To do that, we remind the notion of twist map with respect to $\beta \in (0, \frac{\pi}{2})$ (given for general twist maps in [50, Section 1.2], see also [65] or [20, Section 6.1]).

Definition 5.5.3. *Let $U \subset \mathcal{P}$ be open. We say that $T : U \to T(U)$ is a positive (resp. negative) twist map on U with respect to $\beta \in (0, \frac{\pi}{2})$ if it is a C^1 diffeomorphism onto its image and, for every $(t,s) \in U$, the angle formed by the unitary vertical vector $(0,1) \in T_{(t,s)}\mathcal{P}$ and $DT(t,s)(0,1)$ is in $(\beta - \pi, -\beta)$ (resp. $(\beta, \pi - \beta)$), where we have fixed at every tangent plane $T_{(t,s)}\mathcal{P}$ the counter-clockwise orientation.*

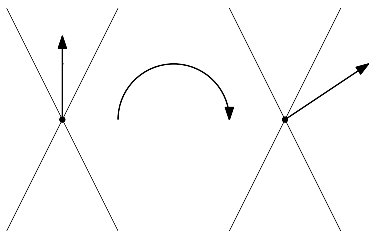


Figure 5.10: Definition of twist map.

We refer to Figure 5.10.

Remark 5.5.4. We notice that, whenever the boundary of the symplectic billiard table is C^2 strictly convex, T is a negative twist map according to the previous Definition 5.5.3. In fact, let $(t_1, s_1) \in \mathcal{P}$. The image of the vertical direction $(0,1)$ by the differential of T –see formula (5.2.9)– is

$$DT(t_1, s_1)(0,1) = -\frac{1}{L_{12}(t_1,t_2)} \begin{pmatrix} 1 \\ L_{22}(t_1,t_2) \end{pmatrix}$$

and, independently from the point $(t_1, s_1) \in \mathcal{P}$, we have that $L_{22}(t_1,t_2)$ is uniformly bounded.

Proposition 5.5.5. *Let Ω be a C^k strictly convex domain, $k \geq 2$. Suppose that there exists*

$$\mathcal{I} := \{(t,s) : \ \phi^-(t) < s < \phi^+(t)\} \subseteq \mathcal{P}, \qquad \phi^- < \phi^+$$

where $\phi^{\pm} : \mathbb{S} \to \mathbb{R}$ are continuous maps such that:

(*a*) *$T : \mathcal{I} \to \mathcal{I}$ is a positive twist map with respect to $\beta \in (0, \frac{\pi}{2})$.*

(*b*) *$\mathcal{I}$ is an instability region for T containing $\mathbb{S} \times \{0\}$.*

Then there exist $\lambda_0 \in (0,1)$ such that, for any $\lambda \in [\lambda_0, 1)$, the Birkhoff attractor Λ of the corresponding dissipative symplectic billiard map T_λ has $\rho^+ - \rho^- > 0$, with $\frac{1}{4} \in (\rho^-, \rho^+) \mod \mathbb{Z}$.

The rest of the section is devoted to proving that hypothesis of Proposition 5.5.5 are satisfied in the following two cases.

(*a*) For an open and dense set of strongly convex, centrally symmetric billiards tables. Indeed, using Proposition 5.4.8, we will see that there exists an open and dense set of tables exhibiting a region of instability containing the zero section, see Theorem 5.5.6.

(*b*) For symplectic billiard maps when the strictly convex billiard table has a point with zero curvature. This result –here Proposition 5.5.7– is a straightforward consequence of Mather's theorem on non-existence of caustics which holds also for symplectic billiards (as proved in [1, Theorem 2]).

We proceed with precise statements and proofs of (*a*) and (*b*), which are Theorem 5.5.6 and Proposition 5.5.7, respectively. In particular, Theorem 5.5.6 immediately follows from Proposition 5.5.1.

Theorem 5.5.6. *The set of strongly convex, centrally symmetric billiard tables whose associated symplectic billiard map does have an instability region containing $\mathbb{S} \times \{0\}$ is open and dense among $\mathcal{B}_2$.*

Proof. According to Proposition 5.5.1, any essential invariant curve $\Gamma \subset \mathcal{P}$ passing through $\mathbb{S}\times\{0\}$ has rotation number $\rho(\Gamma) = 1/4$. To conclude is then sufficient to apply Proposition 5.4.8, assuring that the set of strongly convex, centrally symmetric billiard tables whose associated symplectic billiard map does not have an invariant curve of rotation number $1/4$, is open and dense among $\mathcal{B}_2$. □

Proposition 5.5.7. *If the curvature of the boundary of a C^2 strictly convex billiard table vanishes at some point, then the whole associated $\mathcal{P}$ is an instability region.*

Proof. According to Theorem 2 in [1], when the curvature of the boundary of the billiard table vanishes at some point, then the associate symplectic billiard map T has no caustics. This means that the whole phase space $\mathcal{P}$ is an instability region. □

5.6 Numerical simulations

In this section, we present some numerical simulations to illustrate the results discussed in the previous sections. With the aid of *Mathematica*, we compute the billiard map T_λ for specific domains, both centrally symmetric and non-symmetric, and plot some orbits in the corresponding phase space.

To this end, we represent the domain using the angle θ, defined as the angle between the tangent vector and a fixed reference direction, along with the support function as defined in Section 4. We then choose to represent the phase space using the coordinates θ and $\psi = \theta_2 - \theta_1$, that is, the angular difference between two consecutive points of the orbit (up to consider suitable lifts of angles). As a result, the phase space reduces to $\mathbb{S} \times [0, \pi]$.

We then plot several orbits $T_\lambda^n(\theta_0, \psi_0)$, choosing random initial values for θ_0 and ψ_0. Each orbit is assigned a different color according to its initial value ψ_0, with cooler colors corresponding to values of ψ_0 close to 0, and progressively warmer colors assigned as ψ_0 increases.

The obtained simulations let appear the attractor (not necessarily the Birkhoff attractor). To gain some insight into the structure of the attractor, we display only the points of the orbits for $n > n_0$, where n_0 is chosen appropriately.

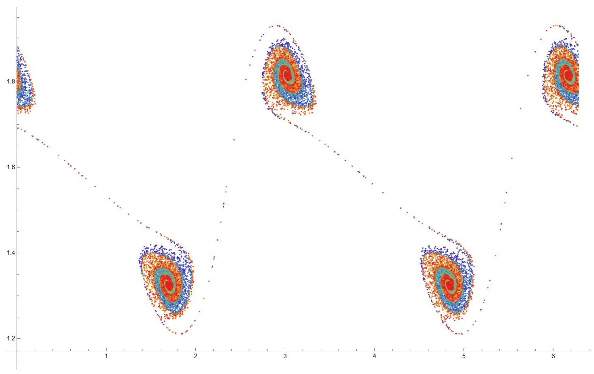


Figure 5.11: $p(\theta) = 1 + \frac{\sin 2\theta}{8}$, $\lambda = 0.9$, $n_0 = 30$.

In Figure 5.11, we consider a non-symmetric billiard table. From the simulation, it seems that the attractor and the Birkhoff attractor coincide, even if the latter is not a graph over $\mathbb{S}$. A clearly visible 4-periodic orbit can be identified, around which the attractor wraps.

Figure 5.12, again for a non-symmetric table, by contrast, seems to give an example in which the attractor strictly contains the Birkhoff attractor. There are extra components in the attractor, which, we guess, are due to the presence of a 3-periodic orbit.

In Figure 5.13, we consider the case of a billiard table that is centrally symmetric and has

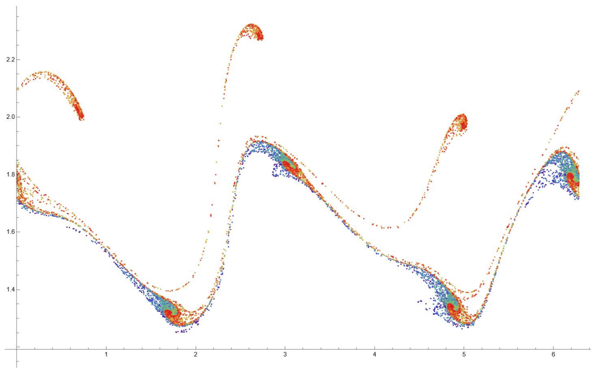


Figure 5.12: $p(\theta) = 1 + \frac{\sin 2\theta}{8} + \frac{\cos 3\theta}{27}$, $\lambda = 0.71$, $n_0 = 10$.

two points of zero curvature. In this example, we use polar coordinates to represent the table. It is well known from the previous section, that in this case, the entire phase space forms a region of instability, and that for dissipation values close to 1, the attractor becomes topologically complex. Notably, there are blue-colored points in the upper part and red-colored points in the lower part of the figure. This indicates that the attractor is highly intricate and entangled.

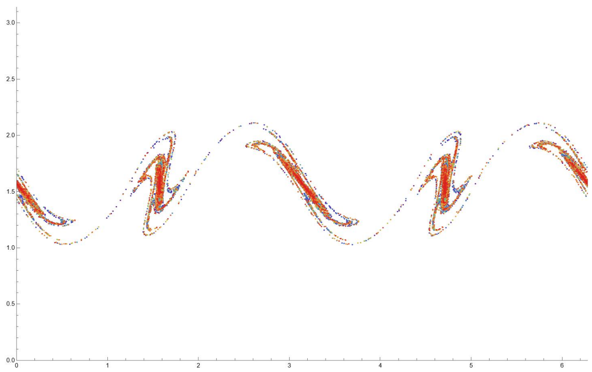


Figure 5.13: $r(\theta) = 1 - \frac{\cos 2\theta}{5}$, $\lambda = 0.71$, $n_0 = 10$.

Bibliography

[1] Peter Albers and Serge Tabachnikov. Introducing symplectic billiards. *Adv. Math.*, 333:822–867, 2018.

[2] Peter Albers and Serge Tabachnikov. Monotone twist maps and Dowker-type theorems. *Pacific J. Math.*, 330(1):1–24, 2024.

[3] S. Allais and M.-C. Arnaud. The dynamics of conformal Hamiltonian flows: dissipativity and conservativity. *Rev. Mat. Iberoam.*, 40(3):987–1021, 2024.

[4] K. G. Andersson and R. B. Melrose. The propagation of singularities along gliding rays. *Invent. Math.*, 41:197–232, 1977.

[5] M.-C. Arnaud. Hyperbolicity for conservative twist maps of the 2-dimensional annulus. *Publ. Mat. Urug.*, 16:1–39, 2016.

[6] M.-C. Arnaud and J. Fejoz. Invariant submanifolds of conformal symplectic dynamics. *J. Éc. polytech. Math.*, 11:159–185, 2024.

[7] M.-C. Arnaud, A. Florio, and V. Roos. Vanishing asymptotic Maslov index for conformally symplectic flows. *Ann. H. Lebesgue*, 7:307–355, 2024.

[8] M.-C. Arnaud, V. Humilière, and C. Viterbo. Higher Dimensional Birkhoff attractors. 2024. arXiv:2404.00804.

[9] T. Asano, S. Guillermou, Y. Ike, and C. Viterbo. Regular Lagrangians are smooth Lagrangians, 2024. arXiv:2407.00395, to appear in J. Math. Soc. Japan.

[10] Artur Avila, Jacopo De Simoi, and Vadim Kaloshin. An integrable deformation of an ellipse of small eccentricity is an ellipse. *Ann. Math. (2)*, 184(2):527–558, 2016.

[11] Vitor Balestro, Horst Martini, and Ralph Teixeira. A new construction of Radon curves and related topics. *Aequationes Math.*, 90(5):1013–1024, 2016.

[12] V. Bangert. Mather sets for twist maps and geodesics on tori. In *Dynamics reported, Vol. 1*, volume 1 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 1–56. Wiley, Chichester, 1988.

[13] Luca Baracco and Olga Bernardi. Totally integrable symplectic billiards are ellipses. *Adv. Math.*, 454:17, 2024. Id/No 109873.

[14] Luca Baracco, Olga Bernardi, and Corentin Fierobe. Starting the study of outer length billiards. Preprint, arXiv:2504.11915 [math.DS] (2025), 2025.

[15] Luca Baracco, Olga Bernardi, Anna Florio, and Alessandra Nardi. Birkhoff attractors for dissipative symplectic billiards. Preprint, arXiv:2509.13086 [math.DS] (2025), 2025.

[16] Luca Baracco, Olga Bernardi, and Alessandra Nardi. Bialy-mironov type rigidity for centrally symmetric symplectic billiards. *Nonlinearity*, 37(12):12, 2024. Id/No 125025.

[17] Luca Baracco, Olga Bernardi, and Alessandra Nardi. Higher order terms of Mather's β-function for symplectic and outer billiards. *J. Math. Anal. Appl.*, 537(2):Paper No. 128353, 20, 2024.

[18] Luca Baracco, Olga Bernardi, and Alessandra Nardi. Area spectral rigidity for axially symmetric and Radon domains. Preprint, arXiv:2410.12644 [math.DS] (2025), 2025.

[19] P. Berger and A. Bounemoura. A geometrical proof of the persistence of normally hyperbolic submanifolds. *Dyn. Syst.*, 28(4):567–581, 2013.

[20] O. Bernardi, A. Florio, and M. Leguil. Birkhoff attractors of dissipative billiards. *Ergodic Theory and Dynamical Systems*, 2024. https://doi.org/10.1017/etds.2024.68.

[21] Misha Bialy. Convex billiards and a theorem by E. Hopf. *Math. Z.*, 214(1):147–154, 1993.

[22] Misha Bialy. Gutkin billiard tables in higher dimensions and rigidity. *Nonlinearity*, 31(5):2281–2293, 2018.

[23] Misha Bialy. Mather β-function for ellipses and rigidity. *Entropy*, 24(11):1600, 2022.

[24] Misha Bialy. Integrable outer billiards and rigidity. *J. Mod. Dyn.*, 20:51–65, 2024.

[25] Misha Bialy, Gil Bor, and Serge Tabachnikov. Self-bäcklund curves in centroaffine geometry and Lamé's equation. *Commun. Am. Math. Soc.*, 2:232–282, 2022.

[26] Misha Bialy and Andrey E. Mironov. Angular billiard and algebraic Birkhoff conjecture. *Adv. Math.*, 313:102–126, 2017.

[27] Misha Bialy and Andrey E. Mironov. The Birkhoff-Poritsky conjecture for centrally-symmetric billiard tables. *Ann. Math. (2)*, 196(1):389–413, 2022.

[28] Misha Bialy, Andrey E. Mironov, and Lior Shalom. Outer billiards with the dynamics of a standard shift on a finite number of invariant curves. *Exp. Math.*, 30(4):469–474, 2021.

[29] Misha Bialy and Daniel Tsodikovich. Billiard tables with rotational symmetry. *Int. Math. Res. Not.*, 2023(5):3970–4003, 2023.

[30] G. D. Birkhoff. Surface transformations and their dynamical applications. *Acta Math.*, 43:1–119, 1920.

[31] G. D. Birkhoff. On the periodic motions of dynamical systems. *Acta Math.*, 50:359–379, 1927.

[32] G. D. Birkhoff. Sur quelques courbes fermées remarquables. *Bull. Soc. Math. France*, 60:1–26, 1932.

[33] R. Calleja, A. Celletti, and R. de la Llave. KAM theory for some dissipative systems. In *New frontiers of celestial mechanics—theory and applications*, volume 399 of *Springer Proc. Math. Stat.*, pages 81–122. Springer, Cham, [2022] ©2022.

[34] M. Capiǹski, M. Gröger, A. Passeggi, and F. A. Tal. Conditions Implying Annular Chaos: Quantitative results and Computer Assisted Proofs, 2025. arXiv:2506.06608.

[35] M. Charpentier. Sur quelques propriétés des courbes de m. birkhoff. *Bulletin de la Société Mathématique de France*, 62:193–224, 1934.

[36] Q. Chen, A. Fathi, M. Zavidovique, and J. Zhang. Convergence of the solutions of the nonlinear discounted Hamilton-Jacobi equation: the central role of Mather measures. *J. Math. Pures Appl. (9)*, 181:22–57, 2024.

[37] S. Crovisier. Langues d'Arnold généralisées des applications de l'anneau déviant la verticale. *C. R. Math. Acad. Sci. Paris*, 334(1):47–52, 2002.

[38] S. Crovisier and R. Potrie. Introduction to partially hyperbolic dynamics. ICTP, Trieste (2015).

[39] A. Davini and M. Zavidovique. Convergence of the solutions of discounted Hamilton-Jacobi systems. *Adv. Calc. Var.*, 14(2):193–206, 2021.

[40] M. M. Day. Polygons circumscribed about closed convex curves. *Trans. Amer. Math. Soc.*, 62:315–319, 1947.

[41] Jacopo De Simoi, Vadim Kaloshin, and Qiaoling Wei. Dynamical spectral rigidity among $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle. *Ann. Math. (2)*, 186(1):277–314, 2017.

[42] Amadeu Delshams and Rafael Ramírez-Ros. Poincaré-melnikov-arnold method for analytic planar maps. *Nonlinearity*, 9(1):1–26, 1996.

[43] Corentin Fierobe, Alfonso Sorrentino, and Amir Vig. Deformational spectral rigidity of axially symmetric symplectic billiards. *Nonlinearity*, 38(10):28, 2025. Id/No 105010.

[44] H. Flanders. A proof of minkowski's inequality for convex curves. *The American Mathematical Monthly*, 75(6):581–593, 1968.

[45] M. Gidea, R. de la Llave, and T. M-Seara. Geometric, topological and dynamical properties of conformally symplectic systems, normally hyperbolic invariant manifolds, and scattering maps, 2025. arXiv:2508.14794.

[46] Alexey Glutsyuk. On polynomially integrable Birkhoff billiards on surfaces of constant curvature. *J. Eur. Math. Soc. (JEMS)*, 23(3):995–1049, 2021.

[47] Alexey Glutsyuk. On infinitely many foliations by caustics in strictly convex open billiards. *Ergodic Theory Dynam. Systems*, 44(5):1418–1467, 2024.

[48] Victor Guillemin and Richard Melrose. A cohomological invariant of discrete dynamical systems. E. B. Christoffel: the influence of his work on mathematics and the physical sciences, int. Symp., Aachen 1979, 672-679 (1981)., 1981.

[49] Benjamin Halpern. Strange billiard tables. *Trans. Amer. Math. Soc.*, 232:297–305, 1977.

[50] M. Herman. *Sur les courbes invariantes par les difféomorphismes de l'anneau. Vol. 1*, volume 103-104 of *Astérisque*. Société Mathématique de France, Paris, 1983. With an appendix by Albert Fathi, With an English summary.

[51] M. W. Hirsch, C. C. Pugh, and M. Shub. Invariant manifolds. *Bull. Amer. Math. Soc.*, 76:1015–1019, 1970.

[52] Guan Huang, Vadim Kaloshin, and Alfonso Sorrentino. Nearly circular domains which are integrable close to the boundary are ellipses. *Geom. Funct. Anal.*, 28(2):334–392, 2018.

[53] Mark Kac. Can one hear the shape of a drum? *Am. Math. Mon.*, 73:1–23, 1966.

[54] V. Kaloshin, C. E. Koudjinan, and Ke Zhang. Birkhoff conjecture for nearly centrally symmetric domains. *Geom. Funct. Anal.*, 34(6):1973–2007, 2024.

[55] Vadim Kaloshin and Alfonso Sorrentino. On the integrability of Birkhoff billiards. *Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci.*, 376(2131):16, 2018. Id/No 20170419.

[56] Vadim Kaloshin and Alfonso Sorrentino. On the integrability of Birkhoff billiards. *Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci.*, 376(2131):16, 2018. Id/No 20170419.

[57] Vadim Kaloshin and Alfonso Sorrentino. On the local Birkhoff conjecture for convex billiards. *Ann. Math. (2)*, 188(1):315–380, 2018.

[58] Vadim Kaloshin and Alfonso Sorrentino. Inverse problems and rigidity questions in billiard dynamics. *Ergodic Theory Dyn. Syst.*, 42(3):1023–1056, 2022.

[59] A. B. Katok. Billiard table as a playground for a mathematician. Prasolov, Victor (ed.) et al., Surveys in modern mathematics. The Independent University of Moscow Seminars. (ISBN 0-521-54793-8/pbk). London Mathematical Society Lecture Note Series 321, 216-242 (2005)., 2005.

[60] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encycl. Math. Appl.* Cambridge: Cambridge Univ. Press, 1995.

[61] Oliver Knill. On nonconvex caustics of convex billiards. *Elem. Math.*, 53(3):89–106, 1998.

[62] A. Koropecki. Realizing rotation numbers on annular continua. *Math. Z.*, 285(1-2):549–564, 2017.

[63] V. F. Lazutkin. The existence of caustics for a billiard problem in a convex domain. *Math. USSR, Izv.*, 7:185–214, 1974.

[64] P. Le Calvez. Propriétés des attracteurs de Birkhoff. *Ergodic Theory Dynam. Systems*, 8(2):241–310, 1988.

[65] P. Le Calvez. Étude topologique des applications déviant la verticale. / patrice le calvez. *Ensaios Matemáticos*, 2:7–102, 1990.

[66] Monika Ludwig. Asymptotic approximation of smooth convex bodies by general polytopes. *Mathematika*, 46(1):103–125, 1999.

[67] R. Markarian, E. J. Pujals, and M. Sambarino. Pinball billiards with dominated splitting. *Ergodic Theory Dynam. Systems*, 30(6):1757–1786, 2010.

[68] P. Martín, R. Ramírez-Ros, and A. Tamarit-Sariol. Exponentially small asymptotic formulas for the length spectrum in some billiard tables. *Exp. Math.*, 25(4):416–440, 2016.

[69] H. Martini and K. J. Swanepoel. Equiframed curves—a generalization of Radon curves. *Monatsh. Math.*, 141(4):301–314, 2004.

[70] Horst Martini and Konrad J. Swanepoel. Equiframed curves – a generalization of Radon curves. *Monatsh. Math.*, 141(4):301–314, 2004.

[71] Horst Martini and Konrad J. Swanepoel. Antinorms and Radon curves. *Aequationes Math.*, 72(1-2):110–138, 2006.

[72] Shahla Marvizi and Richard Melrose. Spectral invariants of convex planar regions. *J. Differential Geom.*, 17(3):475–502, 1982.

[73] J. N. Mather. Variational construction of orbits of twist diffeomorphisms. *J. Amer. Math. Soc.*, 4(2):207–263, 1991.

[74] John N. Mather. Glancing billiards. *Ergodic Theory Dyn. Syst.*, 2:397–403, 1982.

[75] John N. Mather and Giovanni Forni. Action minimizing orbits in Hamiltonian systems. In *Transition to chaos in classical and quantum mechanics. Lectures given at the 3rd session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, July 6-13, 1991*, pages 92–186. Berlin: Springer-Verlag, 1994.

[76] Donald E. McClure and Richard A. Vitale. Polygonal approximation of plane convex bodies. *J. Math. Anal. Appl.*, 51:326–358, 1975.

[77] Jürgen Moser. Is the solar system stable? *Mitt. Dtsch. Math.-Ver.*, 4(4):17–28, 1996.

[78] A. Passeggi, R. Potrie, and M. Sambarino. Rotation intervals and entropy on attracting annular continua. *Geom. Topol.*, 22(4):2145–2186, 2018.

[79] A. Passeggi and F. A. Tal. Conditions implying annular chaos, 2023. arXiv:2305.02963.

[80] Hillel Poritsky. The billiard ball problem on a table with a convex boundary - An illustrative dynamical problem. I. The invariant integral. *Ann. Math. (2)*, 51:446–470, 1950.

[81] M. Sambarino. A (short) survey on dominated splittings. In *Mathematical Congress of the Americas*, volume 656 of *Contemp. Math.*, pages 149–183. Amer. Math. Soc., Providence, RI, 2016.

[82] Karl Friedrich Siburg. A dynamical systems approach to Birkhoff's theorem. *Enseign. Math. (2)*, 44(3-4):291–303, 1998.

[83] Karl Friedrich Siburg. *The principle of least action in geometry and dynamics*, volume 1844 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[84] Alfonso Sorrentino. Computing Mather's β-function for Birkhoff billiards. *Discrete Contin. Dyn. Syst.*, 35(10):5055–5082, 2015.

[85] Buchin Su. Affine differential geometry. Beijing, China: Science Press; New York: Gordon and Breach, Science Publishers, Inc. IV, 248 p. $ 63.75 (1983)., 1983.

[86] S. Tabachnikov. Outer billiards. *Uspekhi Mat. Nauk*, 48(6(294)):75–102, 1993.

[87] Serge Tabachnikov. *Geometrie und Billard*. Springer-Lehrb. Heidelberg: Springer Spektrum, 2013.

[88] Serge Tabachnikov and Filiz Dogru. Dual billiards. *Math. Intelligencer*, 27(4):18–25, 2005.

[89] D. Tsodikovich. Local rigidity for symplectic billiards. *J. Geom. Anal.*, 35(10):26, 2025. Id/No 306.

[90] Amir Vig. Compactness of Marked Length Isospectral Sets of Birkhoff Billiard Tables. Preprint, arXiv:2310.05426 [math.DS] (2023), 2023.

[91] C. Viterbo. On the supports in the Humilière completion and γ-coisotropic sets, 2022. arXiv:2204.04133.

[92] Maciej P. Wojtkowski. Two applications of Jacobi fields to the billiard ball problem. *J. Differ. Geom.*, 40(1):155–164, 1994.

[93] M. Zavidovique. Convergence of solutions for some degenerate discounted Hamilton-Jacobi equations. *Anal. PDE*, 15(5):1287–1311, 2022.