Università di Padova – Dipartimento di Matematica "Tullio Levi-Civita"

Scuole di Dottorato in Matematica Pura e Computazionale

Seminario Dottorato 2017/18

Preface

This document offers a large overview of the eight months' schedule of Seminario Dottorato 2017/18. Our "Seminario Dottorato" (Graduate Seminar) is a double-aimed activity. At one hand, the speakers (usually Ph.D. students or post-docs, but sometimes also senior researchers) are invited to think how to communicate their researches to a public of mathematically well-educated but not specialist people, by preserving both understandability and the flavour of a research report. At the same time, people in the audience enjoy a rare opportunity to get an accessible but also precise idea of what's going on in some mathematical research area that they might not know very well.
Let us take this opportunity to warmly thank the speakers once again, in particular for their nice agreement to write down these notes to leave a concrete footstep of their participation. We are also grateful to the collegues who helped us, through their advices and suggestions, in building an interesting and culturally complete program.

Padova, June 30th, 2018

Corrado Marastoni, Tiziano Vargiolu

Abstracts (from Seminario Dottorato's web page)

Wednesday 4 October 2017

Fine properties of functions of bounded variation in Carnot-Carathéodory spaces

Sebastiano DON (Padova, Dip. Mat.)

Functions of bounded variation (BV functions) can be seen as a generalization of Sobolev maps and arise naturally in many problems in Calculus of Variations. Carnot-Carathéodory spaces are particular metric spaces that arise from the study of hypoelliptic differential operators. In this seminar we will introduce the notions of approximate continuity point, approximate jump point and approximate differentiability point for a generic L^1 function and we will show the so-called fine properties of BV functions, first in the case of Euclidean spaces, and then in the case of Carnot-Carathéodory spaces. In particular, the rectifiability of the approximate jump set, the approximate differentiability almost everywhere and the decomposition formula for the measure derivative of a BV function will be shown.
The results are obtained in collaboration with Davide Vittone.

Wednesday 22 November 2017

Approximation and convergence in finite state Mean Field Games

Alekos CECCHIN (Padova, Dip. Mat.)

Mean Field Games represent limit models for symmetric non-zero sum non-cooperative dynamic games, when the number N of players tends to infinity. We focus on finite time horizon problems where the position of each agent belongs to a finite state space. Relying on a probabilistic representation of the dynamics in terms of Poisson random measures, we first show that any solution of the Mean Field Game provides an approximate symmetric Nash equilibrium for the N-player game. Then, under stronger assumptions for which uniqueness holds, we prove that the sequence on Nash equilibria converges to the Mean Field Game solution. We exploit the so-called Master Equation, which in this framework is a first order quasilinear PDE stated in the symplex of probability measures.

Wednesday 6 December 2017

Representation finite algebras and generalizations

Simone GIOVANNINI (Padova, Dip. Mat.)

An algebra is called "representation finite" if it has a finite number of indecomposable modules. Finite dimensional hereditary representation finite algebras are classified by Gabriel's Theorem: they are the path algebras of Dynkin quivers of type ADE. Recently, with the development of higher dimensional Auslander-Reiten theory, some interest has been raised by a generalization in dimension n of these algebras, which are called n-representation finite algebras. In this seminar we will recall some basic definitions and results about representation theory of finite dimensional algebras. Then we will give a naive idea of how some classical notions can be generalized to higher dimension and, finally, we will show some examples of 2-representation finite algebras.

Wednesday 20 December 2017

Stochastic models for energy forward markets

Marco PICCIRILLI (Padova, Dip. Mat.)

I will present a probabilistic modeling framework for forward prices, specifically designed for energy markets. Most of the presentation will be kept at an intuitive level, as far as this is possible and sensible. I will start by explaining the general framework of the talk and then move to our contribution, of course describing the underlying mathematical theory as well. This talk is based on joint work with Fred Espen Benth, Luca Latini and Tiziano Vargiolu.

Wednesday 17 January 2018

Norm attaining mappings

Marija SOLOVIOVA (Padova, Dip. Mat.)

This talk is about approximation by norm attaining mappings. We start with some basic notions of Functional Analysis. In the first part we recall the classical results in this field, like Bishop-Phelps-Bollobas' theorem, James' weak compactness theorem. In the second part we present our joint with Vladimir Kadets and Miguel Martin results of the paper "Norm-attaining Lipschitz functionals" (2016), where we introduce a concept of norm attainment for Lipschitz functionals. The seminar will be of introductory type.

Wednesday 14 February 2018

Effective conductivity of a composite material and its series expansion

Roman PUKHTAIEVYCH (Padova, Dip. Mat.)

In this talk we discuss the asymptotic behavior of the effective conductivity in a dilute two-phase composite with non-ideal contact conditions at the interface. The composite is obtained by introducing into an infinite homogeneous planar matrix a periodic set of inclusions of a different material and the diameter of each inclusion is proportional to a real positive parameter. After a brief introduction on composit materials, transmission boundary conditions, and the effective conductivity, we will discuss the way how to obtain the series expansion for the effective conductivity, and a fully constructive method to compute explicitly the coefficients of such series by solving recursive systems of integral equations. Also, we will solve some of them in case the inclusions are in the form of a disk. The talk will be of an introductory type and is intended for a general audience.

Wednesday 28 February 2018

Optimal cross-border electricity trading

Maria FLORA (Padova, Dip. Mat.)

Using econometric tools, we show statistical evidence of cross-effects on the price of electricity in neighbouring countries, due to electricity flows across interconnected locations. We build on this result to set up an optimal trading strategy, based not only on the price spreads observed over time among the selected interconnected countries, but also on the market impacts caused by the flows of electricity among them. Using the previous econometric analysis findings, we model the joint dynamics of electricity prices as including both temporary and permanent impacts of electricity trades, as well as driven by a common co-integration factor. We then pose an optimal control problem, and solve the resulting dynamic programming equation up to a system of Riccati equations, which we solve numerically to evaluate the performance of the strategy. We show that including cross-border effects in the trading strategy specification significantly improves the performance.

Wednesday 14 March 20178

Tilting approach to the theorem of Fontaine-Wintenberger

Yangyu FAN (Univ. Milano, Dip. Mat.)

In this basic notion talk, I will explain an approach to the Fontaine-Winterberger isomorphism between Galois groups in characteristic 0 and characteristic p using the tilting equivalence developed by Scholze and Kedlaya-Liu.

Wednesday 28 March 2018

Kernel-based methods: a general overview

Emma PERRACCHIONE (Padova, Dip. Mat.)

In this talk we present the general theory of Radial Basis Function (RBF) interpolation. In doing so, we follow the exposition line of the two books by G.E. Fasshauer (2007) and H. Wendland (2005). Such works provide a recent and extensive treatment about the theory of RBF-based meshless approximation methods. Thus, following their guidelines, we review the main theoretical features concerning positive definite functions and RBFs. Error bounds and error estimates for kernel-based interpolants will be presented as well. Moreover, all the results will be supported by basic numerical experiments carried out during the seminar with Matlab. To conclude, we also briefly review some recent research topics on RBF interpolation.

Wednesday 2 May 2018

Monodromy and Invariant Cycles

Pietro GATTI (Padova, Dip. Mat. and KU Leuven)

We will define the monodromy operator for a 1-parameter family of curves. Focusing on a concrete example, a family of tori, we will compute explicitly the monodromy and show that its invariant part has a geometrical interpretation. This is an instance of the classical Invariant Cycles Theorem. After giving an intuitive introduction to log-geometry, we will briefly explain how this setup let us reprove the Invariant Cycles Theorem for a Semistable Family of Curves.

Wednesday 16 May 2018

Abelian Model Structures

Marco TARANTINO (Padova, Dip. Mat.)

Model categories were introduced by Quillen in 1967 as an axiomatized setting in which it is possible to "do homotopy theory", by inverting a class of morphisms called weak equivalences. The construction involves the use of two more classes of morphisms, which, together with the weak equivalences, form what is called a model structure. In the case of abelian categories there are particular model structures, called abelian model structures, that can be constructed by means of objects rather than morphisms, using complete cotorsion pairs. We will present the theory of abelian model structures, showing how they can be applied to the particular case of R-modules to recover the derived category of the ring.

Wednesday 30 May 2018

Brascamp-Lieb inequalities and heat-flow monotonicity

Roberto BRAMATI (Padova, Dip. Mat.)

Many well-known multilinear integral inequalities in Euclidean analysis, such as Hölder's and Young's inequalities, share a common feature: they are instances of the so called Brascamp-Lieb inequalities. In this introductory talk we will describe these inequalities and show how they can be derived as a consequence of a monotonicity property associated to the heat flow. We will also discuss to what extent some of the ideas and techniques can be adapted to non-Euclidean settings. In particular we will present a family of inequalities on real spheres involving functions that possess some kind of symmetry.

Wednesday 27 June 2018

On a Particular Class of Singular Optimal Control Problems

Torben KOCH (Univ. Bielefeld, Center for Mathematical Economics)

Optimal control problems have been attracted by many researchers in recent years. Its applications can be found in different fields of sciences such as physics or economics. In this talk, we study a particular class of singular optimal control problems, that is, the control variable is represented by an increasing and right-continuous process. Therefore the control measure is allowed to be singular with respect to the Lebesgue measure. The optimal control and value of the problem are typically characterized by a partial differential equation, the so-called Hamilton-Jacobi-Bellman (HJB) equation. In a first step we study a deterministic setting in which the controlled system is governed by an ordinary differential equation. We derive the associated HJB equation by employing standard techniques such as Taylor's theorem. In a next step, we extend the model and consider a specific stochastic singular control problem. The model we have in mind is that of a firm which aims to maximize its profits from selling energy in the market. Here, the control variable represents the firm's installation strategy of solar panels in order to produce energy. We assume that the energy price follows an Ornstein-Uhlenbeck process and is affected by the installation strategy of the firm. We find that the optimal installation strategy is triggered by a threshold, the so-called free boundary which separates the waiting region, in which it is not optimal to install additional panels, and the installation region where it is optimal to install additional panels. Finally, our study is complemented by an analysis of the dependency of the optimal installation strategy on the model's parameters.

Fine properties of functions of bounded variation in Carnot-Carathéodory spaces

Sebastiano Don (*)

1 Introduction

Functions of bounded variations can be seen as the first generalization of Sobolev maps and they arise naturally from a lot of variational problems like free discontinuity problems in fracture mechanics (see [2] for an introduction to the topic). For this reason, an investigation of the precise structure of a measure derivative of BV functions in Euclidean spaces has been deeply developed in the literature.
A function of bounded variation in $\Omega \subseteq \mathbb{R}^n$ is a function $u \in BV(\Omega)$ which has a distributional derivative $Du = (D_1 u, \ldots, D_n u)$ that is represented by a (vector-valued) Radon measure with finite total variation $|Du|(\Omega) < +\infty$. For fine properties of BV functions we mean all the properties that have a role in the knowledge of the structure of the measure derivative like the so-called Federer-Vol'pert Theorem about the rectifiability of the jump set, or the Calderón-Zygmund Theorem about the approximate differentiability almost everywhere. The results in the classical framework can be in the end resumed by the decomposition

$$Du = \nabla u \mathscr{L}^n + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner \mathcal{J}_u + Du \llcorner \mathcal{C}, \tag{1.1}$$

where ∇u is the approximate differential, $\mathcal{J}_u$ is the approximate-jump set and

$$\mathcal{C} = \left\{ x \in \Omega : \liminf_{r \to 0} \frac{|Du|(B(x,r))}{r^{n-1}} = \lim_{r \to 0} \frac{r^n}{|Du|(B(x,r))} = 0 \right\}$$

is the set in which the Cantor part $Du \llcorner \mathcal{C}$ is concentrated.
More recently, both Sobolev maps and BV functions in the higher general context of metric measure spaces have also received a lot of interest in the literature (see for example [1], [9], [10], [14] and references therein). In this case, only few information are known about

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fine properties of BV functions.
The goal of this work is to provide results for BV functions that are analogous to the Euclidean case in a family of metric spaces consisting in a class of Carnot-Carathéodory spaces (CC spaces for short) satisfying some (natural) geometric hypotheses.
The first part will be then devoted to the introduction of the notions of points of approximate continuity, approximate jump, and approximate differentiability and consequently we will list the important classical results that we aim to generalize in a Carnot-Carathéodory setting. The second part will then contain a brief introduction to the definition of Carnot-Carathéodory space and BV function in CC spaces. The notions of approximate jump and approximate differentiability for a generic locally integrable function will be generalized in this framework. In the final part the main results will be shown, together with some rough ideas of the proofs.

2 Approximate calculus for locally integrable functions

Given an open subset $\Omega \subseteq \mathbb{R}^n$ and $u \in L^1(\Omega)$ we say that u has bounded variation in Ω (i.e. $u \in BV(\Omega)$) if its distributional derivative $Du = (D_1 u, \dots, D_n u)$ is given by a vector of Radon measures with finite total variation on Ω, i.e., for every $i = 1, \dots, n$ and for every $\varphi \in C^1_c(\Omega)$ we have

$$\int_\Omega u \partial_i \varphi d\mathscr{L}^n = -\int_\Omega \varphi d(D_i u),$$

and $|Du|(\Omega) < +\infty$. Notice that if $u \in W^{1,1}(\Omega)$, then $Du = \nabla u \mathscr{L}^n$ and so $u \in BV(\Omega)$.
A set E is said to be of finite perimeter in Ω if and only if $\chi_E \in BV(\Omega)$.
We recall that given a metric space (M, d) and given $\alpha > 0$, the Hausdorff measure of dimension α is defined as

$$\mathscr{H}^\alpha(E) := C_\alpha \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=0}^{\infty} (\mathrm{diam}(E_i))^\alpha : E \subseteq \bigcup_{i=0}^{\infty} E_i, \mathrm{diam}(E_i) < \delta \right\} \right),$$

where the constant C_α in $\mathbb{R}^n$ is chosen in order to guarantee $\mathscr{H}^n = \mathscr{L}^n$.
We also define the Hausdorff dimension of a set E as

$$\dim_{\mathscr{H}}(E) = \sup\{\alpha > 0 : \mathscr{H}^\alpha(E) = +\infty\} = \inf\{\alpha > 0 : \mathscr{H}^\alpha(E) = 0\}.$$

The following theorem is a generalization of the well-known Lebesgue differentiation theorem (see [11]) that was proved by Federer in [6].

Theorem 2.1 (Lebesgue-Federer) *Let (X, d, μ) be a separable and locally doubling metric measure space and let $u \in L^1_{loc}(\Omega, \mathbb{R}^k; \mu)$. Then, for μ-almost every $p \in \Omega$, there exists $\widetilde{u}(p) \in \mathbb{R}^k$ such that*

$$\lim_{r \to 0} \frac{1}{\mu(B(p, r))} \int_{B(p,r)} |u(y) - \widetilde{u}(p)| d\mu(y) = 0.$$

The value $\widetilde{u}(p)$ is called approximate limit of u in p. Moreover, if we denote by $\mathcal{S}_u$ the set of points for which u does not admit an approximate limit, then $\mathcal{S}_u$ is a Borel μ-negligible *set and the map* $\widetilde{u} : \Omega \setminus \mathcal{S}_u \to \mathbb{R}^k$ *is a Borel map that coincides* μ*-almost everywhere with* u*.*

In what follows, if u has an approximate limit at a point p, we will say that u is approximately continuous at p.

Definition 2.2 Let $u \in L^1_{loc}(\Omega;\mathbb{R}^k)$. The triple $(a, b, \nu) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{S}^{n-1}$ with $a \neq b$ is said to be an approximate jump triple for u at $p \in \Omega$ if

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B^+_\nu(p,r)} |u(y) - a| d\mathscr{L}^n(y) = \lim_{r\to 0} \frac{1}{r^n} \int_{B^-_\nu(p,r)} |u(y) - b| d\mathscr{L}^n(y) = 0,$$

where $B^+_\nu(p, r) = \{y \in B(p, r) : \langle \nu, y - p\rangle \geq 0\}$, $B^-_\nu(p, r) = \{y \in B(p, r) : \langle \nu, y - p\rangle \leq 0\}$. The set of approximate jumps for u is denoted by $\mathcal{J}_u \subseteq \mathcal{S}_u$.

Notice that an approximate jump triple is uniquely determined up to a change of sign of ν and a permutation of a and b. An approximate jump triple for u at p is therefore denoted by $(u^+(p), u^-(p), \nu_u(p))$.

Definition 2.3 Let $u \in L^1_{loc}(\Omega;\mathbb{R}^k)$ and let $p \in \Omega \setminus S_u$. We say that u is approximately differentiable at p if there exists a $(k \times n)$-matrix L such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(p,r)} \frac{|u(y) - \widetilde{u}(p) - L(y - p)|}{r} d\mathscr{L}^n(y) = 0.$$

If u is approximately differentiable at p the matrix L is uniquely determined and it is called approximate differential of u at p and denoted by $\nabla^{ap}u(p)$.

3 Fine properties in Euclidean case

We now list some of the well-known fine properties for BV functions in Euclidean spaces. For the reader's convenience we also recall the definition of reduced boundary and rectifiable set. Complete proofs of the facts in these section can be found in [2].

Definition 3.1 Let $E \subseteq \mathbb{R}^n$ be a set of (locally) finite perimeter. We say that a point $x \in \mathbb{R}^n$ is in the reduced boundary $\mathcal{F}E$ of E if the limit

$$\lim_{r\to 0} \frac{D\chi_E(B(x, r))}{|D\chi_E|(B(x, r))} =: \nu_E(x)$$

exists with $|\nu_E(x)| = 1$.

Definition 3.2 A set $E \subseteq \mathbb{R}^n$ is said to be countably $\mathscr{H}^{n-1}$-rectifiable if there exists a family (Γ_h) of C^1-hypersurfaces such that

$$\mathscr{H}^{n-1}\left(E \setminus \bigcup_{h=0}^{\infty} \Gamma_h\right) = 0.$$

Rectifiable sets play a crucial role in geometric measure theory. One of the reason of this importance is stated by the celebrated Rectifiability Theorem of De Giorgi [5] that says that any set of (locally) finite perimeter in $\mathbb{R}^n$ has reduced boundary that is (locally) $\mathscr{H}^{n-1}$-rectifiable. As we will see, the geometry of sets of finite perimeter has also consequence on the properties of functions of bounded variation.

A first problem that one could try to face is about the structure of the "bad" part of a BV function. The following theorem tells us that a BV function cannot have a too bad singular set, i.e., the singular set is rectifiable and it is almost completely formed by jump points. Moreover, the measure derivative restricted to the jump set has the most natural form: the Hausdorff measure in the direction of the normal to the jump and amplitude given by the amplitude of the jump. The proof requires two important tools: Coarea Formula for BV functions and Rectifiability Theorem.

Theorem 3.3 (Federer-Vol'pert) *For every $u \in BV(\Omega;\mathbb{R}^k)$ the set $\mathcal{S}_u$ is countably $\mathscr{H}^{n-1}$-rectifiable and $\mathscr{H}^{n-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$. Moreover $Du \llcorner \mathcal{J}_u = (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner \mathcal{J}_u$.*

The universal approximate differentiability Theorem 3.5 can be proved as a consequence of the following inequality and some classical measure theory argument. Inequality (3.2) can be proved by integrating on lines.

Lemma 3.4 *Let $x \in \mathbb{R}^n$, $r > 0$ and $u \in BV(B(x,r);\mathbb{R}^k)$. Suppose u is approximately continuous at x. Then*

$$\int_{B(x,r)} \frac{|u(y) - \widetilde{u}(x)|}{|y - x|} d\mathscr{L}^n(y) \leq \int_0^1 \frac{|Du|(x,tr)}{t^n} dt. \tag{3.2}$$

Theorem 3.5 (Calderòn-Zygmund) *Any function $u \in BV(\Omega;\mathbb{R}^k)$ is approximately differentiable at $\mathscr{L}^n$-almost every point of Ω. Moreover, the approximate differential $\nabla^{ap}u$ coincides almost everywhere with the density of the absolutely continuous part of Du with respect to $\mathscr{L}^n$.*

Finally we can state the theorem about the decomposition formula of the measure derivative. Notice that the Cantor part of the measure derivative is concentrated on a set with, roughly speaking, "measure-theoretic" dimension between $n-1$ and n.

Theorem 3.6 *For any $u \in BV(\Omega;\mathbb{R}^k)$ we have*

$$Du = \nabla^{ap}u\mathscr{L}^n + (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1} \llcorner \mathcal{J}_u + Du \llcorner \mathcal{C},$$

where

$$\mathcal{C} := \left\{ x \in \Omega : \lim_{r \to 0} \frac{r^n}{|Du|(B(x,r))} = \liminf_{r \to 0} \frac{|Du|(B(x,r))}{r^{n-1}} = 0 \right\}.$$

4 Carnot-Carathéodory spaces

A Carnot-Carathéodory space can be thought as a Riemannian manifold in which moving is allowed only in some directions. This constraint modifies the concept of distance between points and therefore the shape of the balls. Carnot-Carathéodory spaces appear naturally when studying visual cortex models, optical illusion models and image completion theory.

Let $X = (X_1, \dots, X_m)$ be a $m-$tuple of smooth vector fields in $\mathbb{R}^n$. We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ is X-subunit if there exists $h \in L^\infty([0, T]; \mathbb{R}^m)$ with $\|h\| \leq 1$ such that

$$\dot{\gamma}(t) := \sum_{i=1}^{m} h_i(t) X_i(\gamma(t)),$$

for almost every t in $[0, T]$.
We define the map $d_{cc} : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ letting

$$d_{cc}(x, y) := \inf \left\{ T > 0 : \exists \gamma \in AC([0, T], \mathbb{R}^n) \ X\text{-subunit}, \gamma(0) = x, \gamma(T) = y \right\}$$

Notice that d_{cc} does not define a distance in general. The set of X-subunit curves connecting two points can be empty (Take for example $\mathbb{R}^3$ with the vector fields $X_1 = \partial_x$ and $X_2 = \partial_y$).
The following Theorem gives us an important sufficient condition on the vector fields (known as Chow-Hörmander condition) to guarantee that d_{cc} is a metric. Its proof can be found in [4].

Theorem 4.1 (Chow, 1939) *Suppose that X is such that*

$$\dim \mathrm{Lie}(X_1, \dots, X_m)(p) = n \tag{4.3}$$

for every $p \in \mathbb{R}^n$. Then $d_{cc} < +\infty$ and therefore is a metric. In such a case the couple $(\mathbb{R}^n, d_{cc})$ (or equivalently $(\mathbb{R}^n, X)$) is said to be a Carnot-Carathéodory space of rank m. The balls with respect to the metric d_{cc} will be denoted by $B(x, r)$.

Notice that there are examples for which d_{cc} is a metric but (4.3) does not hold.
For every $p \in \mathbb{R}^n$ we define $\mathscr{L}^i(p)$ the linear span of all commutators of the vector fields $(X_1, \dots, X_m)$ up to order i computed at p.
A CC space $(\mathbb{R}^n, X)$ is said to be equiregular of step $s \in \mathbb{N}$ if there exist $n_0 = 0, \dots, n_s = n \in \mathbb{N}$ such that $\dim \mathscr{L}^i(p) = n_i$, for every $p \in \mathbb{R}^n$.

Example 4.2 The Grushin plane $\mathbb{R}^2$ with $X = \partial_x$ and $Y = x^2 \partial_y$ is a CC space that is not equiregular.

The following Theorem encloses basic important properties of equiregular CC spaces.

Theorem 4.3 *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of step s. Then the following facts hold.*

(i) *For every compact set $K \subseteq \mathbb{R}^n$, there exists $M > 0$ such that*

$$\forall x, y \in K : \quad \frac{1}{M}|x - y| \leq d_{cc}(x, y) \leq M|x - y|^{\frac{1}{s}}.$$

(ii) *The Hausdorff dimension of the metric space* $(\mathbb{R}^n, d_{cc})$ *is* $Q := \sum_{i=1}^{s} i(n_i - n_{i-1})$.

(iii) *For every compact set* $K \subseteq \mathbb{R}^n$*, there exists* $C > 0$ *and* $R > 0$ *such that*

$$\forall x \in K, \forall r \in]0, R[: \quad \frac{1}{C} r^Q \leq \mathscr{L}^n(B(x, r)) \leq C r^Q.$$

Point (i) tells us that d_{cc} induces the Euclidean topology but it is not metrically equivalent to the Euclidean metric. Property (iii) is known as *local* Q*-Ahlfors regularity* and it is a fundamental tool in the estimation of volumes of the balls built with the CC distance. Notice that the metric dimension Q as stated in point (ii) is always greater (except for the trivial Euclidean case) than the topological dimension n. Proofs of (i), (iii) can be found in [12]. A proof of (ii) can be found in [15].

Example 4.4 The Heisenberg Group $\mathbb{H}^1$ given by $\mathbb{R}^3$ endowed with the CC distance associated with $X = \partial_x - \frac{y}{2}\partial_z$ and $Y = \partial_y + \frac{x}{2}\partial_z$ is an equiregular Carnot-Carathéodory space of step 2 (since $[X, Y] = \partial_z$) and homogeneous dimension

$$Q = 2 + 2(1) = 4.$$

5 Approximate calculus in CC spaces

In this section we are going to introduce the notion of BV map in a CC space. In order to generalize the so-called fine properties of BV functions in analogy to the Euclidean setting, we need a notion of points of approximate jump and approximate differentiability in a CC space for a generic locally integrable function.

Definition 5.1 Let $(\mathbb{R}^n, X)$ be an CC space of rank m and let $u \in L^1(\Omega)$. Then u is a function of bounded X-variation in Ω ($u \in BV_X(\Omega)$) if the derivative in the sense of distributions $D_X u = (X_1 u, \ldots, X_m u)$ is given by an $m-$tuple of Radon measures such that for every $i = 1, \ldots, m$ and for every $\varphi \in C_c^1(\Omega)$ one has

$$\int_\Omega u X_i^* \varphi d\mathscr{L}^n = -\int_\Omega \varphi d(X_i u),$$

and $|D_X u|(\Omega) < +\infty$. Recall that if $Y f(p) = \sum_{i=1}^n a_i(p) \partial_i f(p)$ then the formal adjoint operator associated with the vector field Y is given by $Y^* f(p) = \sum_{i=1}^n \partial_i (a_i f)(p)$.

As in the Euclidean case, we say that a set has finite X-perimeter in Ω if $\chi_E \in BV_X(\Omega)$. We say also that a function $f \in C(\Omega; \mathbb{R}^k)$ is of class $C_X^1(\Omega)$ if the derivatives in the sense of distributions $X_1 f, \ldots, X_m f$ are represented by continuous functions.

In order to define the points of approximate jump we cannot split a ball using a hyperplane since no linear structure is present in our context. What indeed can be done is to identify the (horizontal) direction of the jump with an equivalence class of C_X^1 functions sharing the same differential and the same value at a fixed point. More precisely we give the following

Definition 5.2 Let $(\mathbb{R}^n, X)$ be an equiregular CC space, let $u \in L^1_{loc}(\Omega)$ and let $p \in \Omega$. A triple $(a, b, \nu) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{m-1}$ is said to be an approximate X-jump triple for u at p if there exist $R > 0$ and $f \in C^1_X(B(p,R))$ such that $Xf(p) = \nu$, $f(p) = 0$ and

$$\lim_{r\to 0} \frac{1}{r^Q} \int_{B_f^+(p,r)} |u - a| d\mathscr{L}^n = \lim_{r\to 0} \frac{1}{r^Q} \int_{B_f^-(p,r)} |u - b| d\mathscr{L}^n = 0,$$

where $B_f^{\pm}(p,r) := \{\xi \in B(p,r) : \pm f(\xi) > 0\}$.

Proposition 5.3 *The notion of approximate X-jump triple of u at p is independent on the choice of f and the triple (a, b, ν) is unique up to the equivalence relation*

$$\begin{aligned}(a_1, b_1, \nu_1) \cong (a_2, b_2, \nu_2) \Leftrightarrow &(a_1, b_1, \nu_1) = (a_2, b_2, \nu_2) \text{ or} \\ &(a_1, b_1, \nu_1) = (b_2, a_2, -\nu_2)\end{aligned}$$

and it is denoted by $(u^+(p), u^-(p), \nu_u(p))$.

The set of approximate X-jump points for u is denoted by $\mathcal{J}_u \subseteq \mathcal{S}_u$.

Definition 5.4 Let $(\mathbb{R}^n, X)$ be a CC space, let $u \in L^1_{loc}(\Omega)$ and let $p \in \Omega \setminus \mathcal{S}_u$. We say that u is approximate X-differentiable at p if there exist $R > 0$ and $f \in C^1_X(B(p,R))$ such that $f(p) = 0$ and

$$\lim_{r\to 0} \frac{1}{r^{Q+1}} \int_{B(p,r)} |u(y) - \widetilde{u}(p) - f(y)| d\mathscr{L}^n(y) = 0. \tag{5.4}$$

The idea of the approximate X-differentiability follows closely the one used for the approximate jumps. It can be proved that the definition of approximate X-differentiability does not depend on the choice of f, i.e., if f_1 and f_2 satisfy (5.4) then $Xf_1(p) = Xf_2(p)$, the common value is denoted by $D_X^{ap}u(p)$ and it is called approximate X-gradient for u at p.

6 Fine properties of BV functions on CC space

We now list some of the results about fine properties of BV functions in a class of CC spaces. We indeed need a further assumption on the space.

Definition 6.1 Let $(\mathbb{R}^n, X)$ be a CC space and let $S \subseteq \mathbb{R}^n$. We say that S is a C^1_X-hypersurface if, for every $p \in S$ there exists $R > 0$ and a map $f \in C^1_X(B(p,R))$ and for every $x \in S \cap B(p,R)$ one has $f(x) = 0$ and $|Xf|(x) \geq 1$.

Definition 6.2 Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q. A set E is said to be countably X-rectifiable if there exists a family of C^1_X-hypersurfaces

(Γ_h) such that
$$\mathscr{H}^{Q-1}\left(E \setminus \bigcup_{h=0}^{\infty} \Gamma_h\right) = 0.$$

Definition 6.3 We say that a CC space $(\mathbb{R}^n, X)$ satisfies property $\mathcal{R}$ (where $\mathcal{R}$ stands for *rectifiability*) if every set of locally finite X-perimeter has reduced boundary (defined as in the Euclidean case) that is locally X-rectifiable.

Outside Euclidean setting only a little is known about the validity of property $\mathcal{R}$. Franchi, Serapioni and Serra Cassano proved that the property holds true in Heisenberg groups ([8]) and more in general in any step 2 Carnot groups ([7]). In 2011 Marchi ([13]) proved that also type ★ Carnot groups satisfy property $\mathcal{R}$. Carnot groups are a particular case of equiregular CC spaces in which there is a group operation and a family of dilations that are compatible with the metric: for a precise approach and introduction to this topic see for example [16].
In a general setting the validity of property $\mathcal{R}$ represents a big open problem in Geometric Measure Theory.
This structure hypothesis about sets of finite perimeter has a direct consequence on the structure of the jump set for BV_X functions, as stated in the following.

Theorem 6.4 (D., Vittone, 2018) *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying property $\mathcal{R}$, and let $u \in BV_X(\Omega)$. Then the set $\mathcal{S}_u$ is countably X-rectifiable and $\mathscr{H}^{Q-1}(\mathcal{S}_u \setminus \mathcal{J}_u) = 0$.*

Theorem 6.5 (D., Vittone, 2018) *Let $(\mathbb{R}^n, X)$ be an equiregular CC space satisfying property $\mathcal{R}$ and let $u \in BV_X(\Omega)$. Then u is approximately X-differentiable at $\mathscr{L}^n$-almost every $p \in \Omega$ and $D_X^{ap}u$ coincides almost everywhere with the density $\nabla_X^a u$ of the absolutely continuous part of $D_X u$ with respect to $\mathscr{L}^n$.*

Notice that in a CC space an inequality of the form (3.2) is probably false. On the other hand a weaker form of the inequality can be proved in order to get Theorem 6.5. The required Lemma reads as follows.

Lemma 6.6 *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then for every $x \in \Omega$, there exists $C > 0$ such that for every $u \in BV_X(\Omega)$ with u approximately continuous at x and for every $0 < r < \min\{1, d(x, \mathbb{R}^n \setminus \Omega)\}$ one has*
$$\int_{B(x,r)} \frac{|u(y) - \widetilde{u}(x)|}{d_{cc}(x,y)} d\mathscr{L}^n(y) \leq C \left(\int_0^1 \frac{|D_X u|(B(x,tr))}{t^Q} dt + |D_X u|(B(x,r)) \right).$$

Finally, as in the Euclidean case, a precise decomposition formula holds. Notice that here the Cantor part is hidden in a set with "measure-theoretic" dimension between $Q-1$ and Q.

Theorem 6.7 (D., Vittone, 2018) *Let $(\mathbb{R}^n, X)$ be an equiregular CC space of homogeneous dimension Q satisfying property $\mathcal{R}$ and let $u \in BV_X(\Omega)$. Then the following formula holds*

$$D_X u = \nabla_X^a u \mathscr{L}^n + \vartheta(u^+ - u^-) \otimes \nu_u \mathscr{H}^{Q-1} \llcorner \mathcal{J}_u + D_X u \llcorner \mathcal{C},$$

where

$$\mathcal{C} := \left\{ x \in \Omega : \liminf_{r\to 0} \frac{|D_X u|(B(x,r))}{r^{Q-1}} = \lim_{r\to 0} \frac{r^Q}{|D_X u|(B(x,r))} = 0 \right\}$$

and ϑ is uniformly bounded away from zero on compact sets.

The proofs of the Theorems of this section mostly rely on some classical arguments (see for example [2]) and on an important compactness lemma, stated below in a quite general framework.

Theorem 6.8 (D., Vittone, 2018) *Let (X, d) be a locally compact and separable metric space, (d_j) be a sequence of metrics on X, λ, μ_j be Radon measures on X and let (u_j) be a sequence in $L^q_{loc}(X; \lambda)$. Suppose that*

(i) *d_j converges to d in $L^\infty_{loc}(X \times X)$;*

(ii) *(X, d, λ) is locally doubling;*

(iii) *for any compact set $K \subseteq X$ there exists $M_K > 0$ such that for all $j \in \mathbb{N}$ we have*

$$\|u_j\|_{L^q(K,\lambda)} + \mu_j(K) \le M_K;$$

(iv) *there exist $\delta > 0$, $C_p > 0$ and $\alpha > 0$ such that for every $j \in \mathbb{N}$ one has*

$$\|u_j - u_j(B^j)\|_{L^q(B^j;\lambda)} \le C_p r^\delta(B^j)\mu_j(\alpha B^j),$$

for any ball B^j with respect to the metric d_j.

Then there exist a subsequence (u_{j_h}) and $u \in L^q_{loc}(X; \lambda)$ such that (u_{j_h}) converges to u in $L^q_{loc}(X; \lambda)$ as $h \to +\infty$.

The previous lemma is applied to sequence of functions (u_j) with equibounded BV-norm with respect to some moving structure X^j that is converging in a sufficiently smooth way to the so-called nilpotent approximation $\widehat{X}$ of X. This is a blow-up procedure that enables us to understand what is the behaviour of a BV function around a point. In any metric setting the blow-up produces a tangent metric space in the Gromov-Hausdorff sense. The tangent structure $(\mathbb{R}^n, \widehat{X})$ of a CC space $(\mathbb{R}^n, X)$ in a point is a quotient of a Carnot group and a Carnot group whenever $(\mathbb{R}^n, X)$ is equiregular. For more information on the nilpotent approximation see for example [17].

References

[1] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces.* Advances in Mathematics 159/1 (2001), 51–67.

[2] L. Ambrosio, N. Fusco, D. Pallara, "Functions of bounded variation and free discontinuity problems". Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] G. Anzellotti, M. Giaquinta, *Funzioni BV e tracce.* Rendiconti del Seminario Matematico della Università di Padova, 60 (1978), 1–21.

[4] W. L Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.* Math. Ann. 117 (1939) 98–105.

[5] E. De Giorgi, *Nuovi teoremi relativi alle misure* $(r-1)-$*dimensionali in uno spazio a* r *dimensioni.* Ricerche Mat. 4 (1955), 95–113.

[6] H. Federer, "Geometric Measure Theory". Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[7] B. Franchi, R. Serapioni, F. Serra Cassano., *On the structure of sets of finite perimeter in step 2 Carnot groups.* . Geom. Anal. 13/3 (2003), 421–466.

[8] B. Franchi, R. Serapioni, F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group.* Math. Ann. 321/3 (2001), 479–531.

[9] P. Haiłasz, P. Koskela, "Sobolev met Poincaré". American Mathematical Soc., Vol. 688, 2000.

[10] J. Heinonen, "Lectures on analysis on metric spaces". Springer Science & Business Media, 2012.

[11] H. Lebesgue, "Leçons sur l'Intégration et la recherche des fonctions primitives professées au Collège de France". Reprint of the 1904 original. Cambridge Library Collection. Cambridge University Press, Cambridge, 2009.

[12] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields I: Basic properties.* Acta Math. 155 (1985), 103-147.

[13] M. Marchi, *Regularity of sets with constant intrinsic normal in a class of Carnot groups.* Ann. Inst. Fourier (Grenoble) 64/2 (2014), 429–455.

[14] M. Miranda Jr, *Functions of bounded variation on "good" metric spaces.* Journal de Mathématiques Pures et Appliquées 82/8 (2003), 975–1004.

[15] J. Mitchell, *On Carnot-Carathéodory metrics.* Journal of Differential Geometry 21/1 (1985), 35–45.

[16] R. Monti, *Distances, boundaries and surface measures in Carnot-Carathéodory spaces.* PhD Thesis in Mathematics, University of Trento, 2001. Available at `http://cvgmt.sns.it/paper/3706/cvgmt.sns.it`.

[17] R. Monti, A. Pigati, D. Vittone, *On tangent cones to length minimizers in Carnot-Carathéodory spaces. arXiv preprint arXiv*:1707.07990 (2017).

Approximation and convergence in finite state Mean Field Games

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Abstract. Mean Field Games represent limit models for symmetric non-cooperative non-zero sum dynamic games, when the number N of players tend to infinity. We focus on finite time horizon problems where the position of each agent belongs to a finite state space. Relying on a probabilistic representation of the dynamics in terms of Poisson random measures, we first show that a solution of the Mean Field Game provides an approximate symmetric Nash equilibrium for the N-player game. Then, under stronger assumptions for which uniqueness holds, we prove that the sequence on Nash equilibria converges to a Mean Field Game solution. We exploit the so-called Master Equation, which in this framework is a first order quasilinear PDE stated in the simplex of probability measures.

Mean field games were introduced about ten years ago by [7] and [6] as limit models for symmetric non-zero sum dynamic games, when the number N of players tends to infinity. They fit very well to model problems in economics.

We consider here games in continuous time where the position of each agent belongs to a finite state space $\Sigma = \{1, \dots, d\}$. Players are indistinguishable and control their transition rate from state to state in order to minimize a reward. In order to pass to the limit, we assume that players are symmetric and the interaction is of *mean field* type, that is, each agent chooses its action knowing only its position and the number of other players in any of the d states.

The notion of optimality at the prelimit level is (approximate) Nash equilibria. The problem of finding the equilibria for the N-player game is unfeasible to calculate practically, when N is large. For this reason *Mean Field Games* have been introduced; in fact, the problem becomes more tractable when considering in the limit an infinite number of players.

The connection between the N-player game and the limit can be understood in two opposite directions:

- Approximation: Mean Field Game solutions provide approximate Nash equilibria for the N-player game;
- Convergence: do Nash equilibria converge to Mean Field game solutions?

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As opposed to previous works on finite state Mean Field Games [5], we use a probabilistic representation of the system dynamics in terms of stochastic differential equations driven by Poisson random measures. This allows to perform a coupling between the limiting optimal trajectories and their prelimit counterparts, giving both the approximate Nash equilibria and a rigorous proof of the convergence. This latter requires the analysis of the so-called Master Equation, a PDE in the simplex of probability measures.

We first recall some preliminaries about stochastic calculus in our finite state setting and the related optimization problem. Then we introduce more in details the N-player game and the Mean Field Game. So we show how to build approximate Nash equilibria from limiting solutions. Finally, we heuristically introduce the Master Equation and state the main convergence theorems.

Preliminaries. Given a random variable $X : (\Omega, \mathcal{F}, P) \to \Sigma$ its *law* is the probability measure on Σ

$$Law(X) = P \circ X^{-1},$$

i.e. $Law(X).A = P(X \in A)$. So $Law(X) \in \mathcal{P}(\Sigma) \subset \mathbb{R}^d$ and $E[f(X)] = \int_\Sigma f dLaw(X)$. We say that X and Y are independent if $Law(X,Y) = Law(X) \otimes Law(Y)$, i.e. $P(X \in A, Y \in B) = P(X \in A)P(X \in B)$. A stochastic process $(X(t))_{t\in[0,T]}$ is viewed as a random variable in the space of trajectories $D([0,T], \Sigma)$. So the notions of independent and identically distributed processes can be given.

A process $(X(t))_{t\in[0,T]}$ with values in Σ is a continuous time *Markov chain* if

$$P\left[X(t+h) = y | X(t) = x\right] = \alpha_y(t,x) \cdot h + o(h)$$

for any $x \neq y \in \Sigma$. $\alpha \geq 0$ is called the transition rate of the (time-inhomogeneous) Markov chain. The law of the process $\mu(t) = Law(X(t))$ solves the *Kolmogorov-Fokker-Planck* ODE

$$\text{(KFP)} \qquad \begin{cases} \frac{d}{dt}\mu_x(t) = \sum_y \mu_y(t)\alpha_x(t,y), \\ \\ \mu(0) = Law(X(0)). \end{cases}$$

Let us now introduce the 1-player control problem. The player wants to choose his transition rate α in order to minimize the cost

$$J(\alpha) = E\left[\int_0^T L(X(t), \alpha(t, X(t)))dt + G(X(T))\right]$$

in the set of admissible feedback controls $\mathcal{A} = \left\{\alpha : [0,T] \times \Sigma \to \mathbb{R}^d \text{ measurable}\right\}$. Define the *value function*

$$u(t,x) = \inf_{\alpha\in\mathcal{A}} E\left[\int_t^T L(X^{t,x}(s), \alpha(s, X^{t,x}(s)))ds + G(X^{t,x}(T))\right]$$

where $X^{t,x}$ denotes the process which starts in t with $X(t) = x$.

The optimum can be fund through the following standard Verification Theorem. Let u be the solution to the *Hamilton-Jacobi-Bellman* ODE

$$\text{(HJB)} \qquad \begin{cases} -\frac{d}{dt}u(t,x) + H(x, \Delta u(t,x)) = 0 \\ u(T,x) = G(x), \end{cases}$$

where $\Delta u(x) = (u(1) - u(x), \dots, u(d) - u(x)) \in \mathbb{R}^d$ and $H(x,p) = \sup_a \{-a \cdot p - L(x,a)\}$ is the Hamiltonian. Moreover, let $\alpha^*(t,x) \in \operatorname{argmax}_a \{-a \cdot \Delta u(t,x) - L(x,a)\}$ Then u is the value function, and α^* is an optimal control. If L is strictly convex then the argmax is unique: denote by $a^*(x,p)$. The optimal control is also unique: $\alpha^*(t,x) = a^*(t, \Delta u(t,x))$.

As mentioned above, it is better to describe the dynamics as a stochastic differential equation. Let $\Xi \subset \mathbb{R}^d$ compact and ν be a finite positive measure on Ξ. Then a random variable $\mathcal{N} : \Omega \to \mathcal{I}([0,T] \times \Xi)$ is a Poisson random measure with intensity measure ν if, denote $\mathcal{N}_t(E) = \mathcal{N}([0,t] \times E)$,

- for any $E \subset \Xi$, $(\mathcal{N}_t(E))_{t \in [0,T]}$ is a Poisson process of parameter $\nu(E)$, i.e. a Markov chain with rate $\alpha_{x+1}(x) = \nu(E)$ and 0 otherwise;
- $(\mathcal{N}_t(E_1))_{t \in [0,T]}$ is independent of $(\mathcal{N}_t(E_2))_{t \in [0,T]}$ for E_1 and E_2 disjoint.

It is not hard to convince yourself that a Markov chain can be written as an SDE

$$X(t) = Z + \int_0^t \int_\Xi f(X(s^-), \alpha(s, X(s^-)), \xi) \mathcal{N}(ds, d\xi)$$

with a proper choice of f and ν.

N-player game. We consider N identical players $X_1, \dots, X_N$, with $X_i(t) \in \Sigma$, and denote $\boldsymbol{x} = (x_1, \dots, x_N), \boldsymbol{X}_t = (X_1(t), \dots, X_N(t)) \in \Sigma^N$. Player i controls his transition rate α^i, that is

$$P\left[X_i(t+h) = y | \boldsymbol{X}(t) = \boldsymbol{x}\right] = \alpha_y^i(t, \boldsymbol{x})h + o(h).$$

Writing the dynamics as an SDE w.r.t. N i.i.d Poisson random measures, we have

$$X_i(t) = Z_i + \int_0^t \int_\Xi f(X_i(s^-), \alpha^i(s, \boldsymbol{X}_{s^-}), \xi) \mathcal{N}_i(ds, d\xi),$$

where $\boldsymbol{Z} = (Z_1, \dots, Z_N)$ is the vector of i.i.d initial states.

A *strategy vector* is $\boldsymbol{\alpha}(t, \boldsymbol{x}) = (\alpha^1(t, \boldsymbol{x}), \dots, \alpha^N(t, \boldsymbol{x}))$, the vector of transition rates chosen by the players. Let $\boldsymbol{\alpha}$ be a strategy vector and $\boldsymbol{X} = (X_1, \dots, X_N)$ be the corresponding solution. For $i = 1, \dots, N$, set

$$J^i(\boldsymbol{\alpha}) = E\left[\int_0^T c(X_i(t), \alpha^i(t, \boldsymbol{X}_t), m^{N,i}(t))dt + G(X_i(T), m^{N,i}(T))\right].$$

Given $\boldsymbol{x} = (x_1, \dots, x_N) \in \Sigma^N$ denote the *empirical measure* $m^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j} \qquad m^{N,i}_{\boldsymbol{x}} = \frac{1}{N-1}\sum_{j \neq i} \delta_{x_j}$. Then $m^N(t) = m^N_{\boldsymbol{X}(t)} \in \mathcal{P}(\Sigma)$ and $N \cdot [m^N(t)]_y$ is the number of players in state y, at time t. Hence any player aims at minimizing his cost knowing just his position and the fraction of players in each state.

Let us state the standard notion of (approximate) Nash equilibrium. For a strategy vector $\boldsymbol{\alpha}$ denote by $[\boldsymbol{\alpha}^{N,-i};\beta]$ the strategy vector such that $[\boldsymbol{\alpha}^{N,-i};\beta]_j = \begin{cases} \alpha^j & j \neq i \\ \beta & j = i. \end{cases}$

Definition 1 Let $\epsilon > 0$. A strategy vector $\boldsymbol{\alpha}$ is said to be an ϵ-*Nash equilibrium* if for each $i = 1, \ldots, N$

$$J_i^N(\boldsymbol{\alpha}) \leq J_i^N([\boldsymbol{\alpha}^{N,-i};\beta]) + \epsilon$$

for every β such that $[\boldsymbol{\alpha}^{N,-i};\beta]$ is a strategy vector. It is called a Nash equilibrium if $\epsilon = 0$.

Mean field game. We consider the limit as $N \to \infty$. The main idea is to replace the empirical measure by a deterministic flow of measures $m : [0,T] \to \mathcal{P}(\Sigma)$ and to suppose that players are asymptotically i.i.d (as propagation of chaos in statistical mechanics); so for N large, the law of any player is close to the empirical measure: $Law(X_i(t)) \approx m^N(t) \approx m(t)$.

In order to pass to the limit, we make the mean field assumption, i.e. $\alpha^i(t,\boldsymbol{x}) \approx \bar{\alpha}(t, x_i, m_{\boldsymbol{x}}^{N,i})$, hence in the limit $\alpha^i(t,\boldsymbol{X}(t)) = \bar{\alpha}(t, X_i(t), m(t)) = \bar{\alpha}(t, X_i(t))$, which says that any player chooses his action only knowing his position, meaning that agents are asymptotically independent. Therefore we consider ONE reference player.

Now, let us give the proper notion of the limiting optimization problem. The *mean field* limiting system consists of a single player whose state evolves according to the dynamics

$$X(t) = Z + \int_0^t \int_\Xi f(X(s^-), \alpha(s, X(s^-)), \xi)\mathcal{N}(ds, d\xi).$$

where $Law(Z) = Law(Z_i) = m_0$ and f and ν are as above. Denote by X^α the solution, which depends on the control $\alpha = \alpha(t,x) \in \mathcal{A}$.

Let $\mu(t) = Law(X(t))$, so $\mu : [0,T] \to \mathcal{P}(\Sigma)$ is called the flow of the process X. Any μ belongs to $\mathcal{L} \subset \mathcal{C}([0,T], \mathcal{P}(\Sigma))$, which is shown to be

$$\mathcal{L} = \{m : [0,T] \longrightarrow \mathcal{P}(\Sigma) : |m(t) - m(s)| \leq K|t-s|, \quad m(0) = m_0\}$$

where the constant is $K = 2\nu(U)\sqrt{d}$. For any $\alpha \in \mathcal{A}$ and $m \in \mathcal{L}$, let

$$J(\alpha, m) = E\left[\int_0^T c(s, X^\alpha(s), \alpha(s, X^\alpha(s)), m(s))ds + G(X^\alpha(T), m(T))\right].$$

Definition 2 A *solution of the mean field game* is a triple (α, m, X) such that

(a) $\alpha \in \mathcal{A}$, $m \in \mathcal{L}$ and $X = X^\alpha$;

(b) *Optimality*: $J(\alpha, m) \leq J(\beta, m)$ for every $\beta \in \mathcal{A}$;

(c) *Mean Field Condition*: $Law(X(t)) = m(t)$ for every $t \in [0,T]$, i.e. $Flow(X) = m$.

We observe that the solution can be viewed as a fixed point: given a generic flow of measures m find an optimal control α_m^* and then impose $Flow(X^{\alpha_m^*}) = m$.

This fixed point interpretation is the basis for the analytic point of view on Mean Field Games. Assume $c = L(x,a) + F(x,m)$. Given m, find the value function u_m and the optimal control α_m^* via the HJB equation. Then put the transition rate α_m^* into the KFP equation for $\mu(t) = Law(X^{\alpha_m^*}(t))$ and impose $\mu = m$. A solution of the *Mean Field Game system* is a couple (u,m) solving the system of ODEs

$$\begin{cases} -\frac{d}{dt}u(t,x) + H(x,\Delta^x u(t,x)) = F(x,m(t)) \\ \frac{d}{dt}m_x(t) = \sum_y m_y(t) a_x^*(y,\Delta^y u(t,y)), \\ u(T,x) = G(x,m(T)) \\ m(0) = m_0. \end{cases}$$

The *existence* of Mean Field Game solutions is equivalent to the existence of a fixed point of the map $\Phi : \mathcal{L} \to \mathcal{L}$, $\Phi(m) = Flow(X^{\alpha_m^*})$. We need continuity and $\mathcal{L} \subset \mathcal{C}([0,T],\mathcal{P}(\Sigma))$ to be compact and convex. Note that $\Phi : \mathcal{L} \to 2^{\mathcal{L}}$ if the optimal control is not unique. Thus existence holds assuming the continuity of the costs, via a topological fixed point theorem.

Uniqueness holds either for small T or under *monotonicity* assumption on F and G:

$$\sum_x [F(x,m) - F(x,\tilde{m})](m_x - \tilde{m}_x) \geq 0$$
$$\sum_x [G(x,m) - G(x,\tilde{m})](m_x - \tilde{m}_x) \geq 0.$$

For example, consider $F(x,m) = m_x$ or $F(x,m) = xE(m) = x\sum_x x m_x$. The monotonicity property means that players prefer to spread, instead of aggregate.

Approximation. We show here how to build approximate Nash equilibria from limiting solutions. Let (α, m, X) be a solution of the mean field game and consider the symmetric and decentralized strategy vector $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^N)$ made by N copies of α:

$$\alpha^i(t,\boldsymbol{x}) = \alpha(t,x_i).$$

Theorem 1 (C.-Fischer '17) *The strategy vector $\boldsymbol{\alpha}$ is an ϵ_N-Nash equilibrium for the N-player game, with $\epsilon_N \leq \frac{C}{\sqrt{N}}$.*

Only Lipschitz assumptions on the data are required. The proof relies on the probabilistic representation introduced above and a coupling argument. The processes

$$X_i(t) = Z_i + \int_0^t \int_\Xi f(X_i(s^-), \alpha(s, X_i(s^-)), \xi)\mathcal{N}_i(ds,d\xi)$$

related to the strategy vector $\boldsymbol{\alpha}$ are i.i.d. and $Law(X_i(t)) = m(t)$. Thanks to [4]

$$\sup_{t\in[0,T]} E|m_{\boldsymbol{X}}^N(t) - m(t)| \leq \frac{C}{\sqrt{N}}.$$

Then the claim follows comparing with the processes $\widetilde{\boldsymbol{X}}$ related to the perturbed strategy vector $[\boldsymbol{\alpha}^{N,-1};\beta]$

$$\begin{aligned}\widetilde{X}_1(t) &= Z_1 + \int_0^t \int_\Xi f(\widetilde{X}_1(s^-), \beta(s, \widetilde{\boldsymbol{X}}_{s^-}), \xi), \mathcal{N}_i(ds, d\xi),\\ \widetilde{X}_i(t) &= Z_i + \int_0^t \int_\Xi f(\widetilde{X}_i(s^-), \alpha(s, \widetilde{X}_i(s^-)), \xi)\mathcal{N}_i(ds, d\xi) \qquad i = 2, \dots, N.\end{aligned}$$

Convergence. For the opposite direction, we focus on the analytic interpretation and require stronger assumptions, that are the monotonicity and regularity of F and G and the strict convexity of H. Under these assumptions, the Nash equilibrium for the N-player game (exists and) is unique, provided by the solution of the *Nash system*

$$-\frac{dv^{N,i}}{dt} - \sum_{j=1,\ j\neq i}^{N} a^*(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} + H(x_i, \Delta^i v^{N,i}) = F(x_i, m_{\boldsymbol{x}}^{N,i}),$$

with the final condition $v^{N,i}(T, \boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i})$. It is a system of Nd^N ODEs for the value functions $v^{N,i}(t, \boldsymbol{x})$. The Nash equilibrium is then given by the strategy vector

$$\alpha^i(t, \boldsymbol{x}) = a^*(x_i, \Delta^i v^{N,i}(t, \boldsymbol{x})).$$

The convergence problem consists in finding a limit for the $v^{N,i}$. We make the mean field assumption $v^{N,i}(t, \boldsymbol{x}) = V^N(t, x_i, m_{\boldsymbol{x}}^{N,i})$. Let us proceed heuristically to give an intuition for the equation satisfied by the limit of the functions V^N, which should be $U(t, x, m)$, $x \in \Sigma$, $m \in \mathcal{P}(\Sigma)$.

$$\begin{aligned}\Delta^i v^{N,i}(t, \boldsymbol{x}) &= \left(V^N\left(t, y, m_{\boldsymbol{x}}^{N,i}\right) - V^N\left(t, x_i, m_{\boldsymbol{x}}^{N,i}\right)\right)_{y=1,\dots,d}\\ &\to \left(U(t, y, m) - U(t, x_i, m)\right)_{y=1,\dots,d} = \Delta^x U(t, x_i, m).\end{aligned}$$

For $j \neq i$ we should instead get, for $y = 1, \dots, d$,

$$\begin{aligned}&[\Delta^j v^{N,i}(t, \boldsymbol{x})]_y\\ &= V^N\left(t, x_i, m_{\boldsymbol{x}}^{N,i} + \frac{1}{N-1}(\delta_y - \delta_{x_j})\right) - V^N\left(t, x_i, m_{\boldsymbol{x}}^{N,i}\right)\\ &\sim \frac{1}{N-1}[D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)]_y,\end{aligned}$$

where $[D^m U(m,x)]_y = \frac{\partial U(m)}{\partial(\delta_y - \delta_x)}$ for $U : \mathcal{P}(\Sigma) \to \mathbb{R}$.

Note that $\int_\Sigma f dm_{\boldsymbol{x}}^{N,i} = \frac{1}{N-1}\sum_{j\neq i} f(x_j)$. Hence the term

$$\begin{aligned}
&\sum_{j=1, j\neq i}^{N} \alpha^{j,*}(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} \\
&\qquad \sim \frac{1}{N-1} \sum_{j=1,j\neq i}^{N} \alpha^*(x_j, \Delta^x U(t, x_j, m_{\boldsymbol{x}}^{N,i})) \cdot D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j) \\
&\qquad = \int_\Sigma \alpha^*(y, \Delta^y U(t, y, m_{\boldsymbol{x}}^{N,i})) \cdot D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, y) dm_{\boldsymbol{x}}^{N,i}(y) \\
&\qquad \to \int_\Sigma D^m U(t, x, m, y) \cdot \alpha^*(y, \Delta^y U(t, y, m)) dm(y).
\end{aligned}$$

Clearly $H(x_i, \Delta^i v^{N,i}) \to H(x, \Delta^x U)$.

We are then able to introduce the *Master Equation* for $U : [0,T] \times \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R}$:

$$\text{(M)} \quad \begin{cases} -\frac{\partial U}{\partial t} + H(x, \Delta^x U) - \int_\Sigma D^m U(t, x, m, y) \cdot \alpha^*(y, \Delta^y U(t, y, m)) dm(y) = F(x, m), \\ U(T, x, m) = G(x, m), \quad (x, m) \in \Sigma \times \mathcal{P}(\Sigma), \quad t \in [0, T]. \end{cases}$$

It is a first order quasilinear PDE stated in $\mathcal{P}(\Sigma)$. The Mean Field Game system is the system of characteristics of (M): if U solves (M) and m solves (KFP) then $u(t) = U(t, m(t))$ solves (HJB). The main result is the following

Theorem 2 (C.-Pelino '17) *There exists a unique classical solution U to (M). For any $t \in [0,T]$, $\boldsymbol{x} \in \Sigma^N$ and $N \geq 1$*

$$\frac{1}{N} \sum_{i=1}^{N} |v^{N,i}(t, \boldsymbol{x}) - U(t, x_i, m_{\boldsymbol{x}}^N)| \leq \frac{C}{N}.$$

The convergence proof follows the idea developed in [1], which uses a coupling argument between the limiting optimal trajectories and their prelimit counterparts. This coupling was the main motivation for introducing our probabilistic representation of the dynamics.

The key ingredient for the convergence is the existence of a classical solution U to the Master Equation. It is defined via the method of characteristics: let $u(t, x; t_0, m_0)$ and $m(t, x; t_0, m_0)$ be the solutions to the MFG system starting from t_0, with $m(t_0) = m_0$, and set

$$U(t_0, x, m_0) = u(t_0, x; t_0, m_0).$$

The convergence can be also studied in terms of the optimal trajectories. Fix $N \geq 1$ and let $\boldsymbol{Y} = (Y_1, \ldots, Y_N)$ be the optimal processes, i.e. with rates $\alpha^{i,\boldsymbol{Y}}(t, \boldsymbol{x}) = a^*(x_i, \Delta^i v^{N,i}(t, \boldsymbol{x}))$. Consider the processes $\boldsymbol{X}$ with rates induced by the Master Equation, $\alpha^{i,\boldsymbol{X}}(t, \boldsymbol{x}) = a^*(x_i, \Delta^i U(t, x_i, m_{\boldsymbol{x}}^{N,i}))$ and compare with the processes $\widetilde{\boldsymbol{Y}}$, i.i.d., in which the empirical measure is replaced by m deterministic: $\alpha^{i,\widetilde{\boldsymbol{Y}}}(t, \boldsymbol{x}) = a^*(x_i, \Delta^i U(t, x_i, m(t)))$. We remark that $Law(\widetilde{Y}_i(t)) = m(t)$.

Theorem 3 (C.-Pelino '17) *For any N and i:*

$$E\left[\sup_t |Y_i(t) - X_i(t)|\right] \le \frac{C}{N},$$
$$E\left[\sup_t |Y_i(t) - \widetilde{Y}_i(t)|\right] \le \frac{C}{N^{1/9}},$$
$$E\left[\sup_t |m^N_{\mathbf{Y}}(t) - m(t)|\right] \le \frac{C}{N^{1/9}}.$$

Conclusions. Let us finally summarize the main results and some perspectives. The existence of Mean Field Game solutions holds under mild assumptions; this is true also considering different type of controls: open loop, relaxed, feedback relaxed. These still provide $\frac{C}{\sqrt{N}}$-Nash equilibria for the N-player game, using the chattering lemma. Strong assumptions are needed to prove the convergence results, in particular for the existence of classical solutions to the Master Equation: monotonicity, uniqueness for MFG system. A Central Limit Theorem and a Large Deviation Principle are also proved for the asymptotic behavior of the empirical measures associated with the optimal trajectories for the N-player game.

References

[1] P. Cardaliaguet, F. Delarue, J. M. Lasry, P. L. Lions, *The master equation and the convergence problem in mean field games.* Preprint 2015, arXiv:1509.02505.

[2] A. Cecchin, M. Fischer, *Probabilistic approach to finite state Mean Field Games.* Preprint 2017, arXiv 1704.00984.

[3] A. Cecchin, G. Pelino, *Convergence, fluctuations and large deviations for finite state Mean Field Games via the Master Equation.* Preprint 2017, arXiv 1707.01819.

[4] N. Fournier, A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure.* Probab. Theory Relat. Fields 162 (2015), 707–738.

[5] D. Gomes, J. Mohr, R. R. Souza, *Continuous time finite state mean field games.* Appl. Math. Optim. 68/1 (2013), 99–143.

[6] M. Huang, R. P. Mallamé, P. E. Caines, *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle.* Commun. Inf. Syst. 6/3 (2006), 221–252.

[7] J.-M. Lasry and P.-L. Lions, *Mean field games.* Japan. J. Math. 2/1 (2007), 229–260.

Representation finite algebras and generalizations

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Abstract. An algebra is called representation finite if it has a finite number of indecomposable modules. Finite dimensional hereditary representation finite algebras are classified by Gabriel's Theorem: they are the path algebras of Dynkin quivers of type ADE. Recently, with the development of higher dimensional Auslander-Reiten theory, some interest has been raised by a generalization in dimension n of these algebras, which are called n-representation finite algebras. In this note we will recall some basic definitions and results about representation theory of finite dimensional algebras. Then we will give a naive idea of how some classical notions can be generalized to higher dimension and, finally, we will show some examples of 2-representation finite algebras.

1 Basic definitions and preliminaries

In this section we recall some basic definitions about representation theory of associative algebras.

Throughout this note we will work over the field complex numbers: actually many results are true in more generality, but we will use $\mathbb{C}$ already from the beginning in order to keep the exposition clear.

Definition 1.1 A $\mathbb{C}$-*algebra* is a $\mathbb{C}$-vector space equipped with a bilinear product

$$A \times A \to A, \qquad (a, b) \mapsto ab.$$

Since we only consider algebras over the complex numbers, we will drop the reference to the field and write "algebras" instead of "$\mathbb{C}$-algebras". From now on we will always assume that algebras are associative and have a unit element 1.

Example 1.2 Examples of algebras are the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices and the algebra $T_n(\mathbb{C})$ of $n \times n$ upper triangular matrices.

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1.1 Quivers

A large source of examples of algebras is given by path algebras of quivers, whose construction we now recall.

Definition 1.3 A *quiver* $Q = (Q_0, Q_1, s, t)$ is the data of two sets Q_0 and Q_1, and two maps $s, t\colon Q_1 \to Q_0$.

So a quiver is simply an oriented graph: Q_0 and Q_1 are, respectively, the sets of vertices and arrows, and for each arrow $\alpha \in Q_1$ the elements $s(\alpha)$ and $t(\alpha)$ are, respectively, the source and the target vertices of α.

We will always assume that quivers are finite, i.e. both Q_0 and Q_1 are finite sets.

Definition 1.4 A *path* in Q is a sequence of arrows $p = \alpha_1 \cdots \alpha_l$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, l-1$. We define $s(p) := s(\alpha_1)$ and $t(p) := t(\alpha_l)$ to be the source and target of p respectively.

If we have two paths $p = \alpha_1 \cdots \alpha_l$ and $q = \beta_1 \cdots \beta_m$ such that $t(p) = s(q)$, then we can define the concatenation of p and q as the path $pq := \alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m$. If $t(p) \neq s(q)$ we put $pq := 0$.

If $i \in Q_0$ is a vertex in Q, we also define e_i to be the trivial path relative to i. These paths have the property that $e_{s(p)}p = p = pe_{t(p)}$ for all paths p in Q.

Definition 1.5 The *path algebra of* Q, which we denote by $\mathbb{C}Q$, is the free vector space with basis all paths in Q, equipped with a product defined as follows. If p and q are paths in Q, we define their product as their concatenation pq introduced above; then we extend this by linearity on all $\mathbb{C}Q$.

We now give some examples of path algebras.

Example 1.6 $Q = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

The path algebra $\mathbb{C}Q$ has basis $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$. It is easy to see that we have an isomorphism $\mathbb{C}Q \cong T_3(\mathbb{C})$ between this path algebra the algebra of upper triangular 3×3 matrices. It is given by

$$\begin{gathered}
e_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha\beta \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{gathered}$$

Example 1.7 $Q = \ 1 \circlearrowleft \alpha$

The path algebra $\mathbb{C}Q$ has basis $\{e_1, \alpha, \alpha^2, \dots\}$.There is an isomorphism between $\mathbb{C}Q$ and the algebra $\mathbb{C}[x]$ of polynomials in one variable, given by mapping α to x.

It is convenient to look at quotients of path algebras by ideals generated by special elements called relations.

Definition 1.8 A *relation* on a quiver Q is an element of $\mathbb{C}Q$ of the form

$$\rho = \sum_{i=1}^{n} \lambda_i p_i,$$

where $\lambda_i \in \mathbb{C}$ and the p_i's are paths of length ≥ 2 with the same source and target.

Let $\rho_1, \dots, \rho_n$ be relations on Q and call $I = \langle \rho_1, \dots, \rho_n \rangle$ the two-sided ideal generated by them. We will call the quotient $\mathbb{C}Q/I$ the *path algebra of Q bounded by the relations* $\rho_1, \dots, \rho_n$.

Example 1.9

- $Q = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \;, \qquad I = \langle \alpha\beta, \beta\gamma \rangle$

  A basis of $\mathbb{C}Q/I$ is given by the equivalence classes of $e_1, e_2, e_3, \alpha, \beta, \gamma$. The relations are "zero relations", i.e. we have some paths in Q which become zero in the quotient.

- $$Q = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 4 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd} , \qquad I = \langle \alpha\beta - \gamma\delta \rangle$$

  A basis of $\mathbb{C}Q/I$ is given by the equivalence classes of $e_1, e_2, e_3, \alpha, \beta, \gamma, \delta, \alpha\beta$. Here the relations are "commutativity relations", i.e. we have two paths in Q which are identified in the quotient.

1.2 Modules

In representation theory one is interested in studying modules over algebras. Let us recall their definition.

Definition 1.10 Let A be an algebra. A left *A-module* is a $\mathbb{C}$-vector space M together with a left action of A, i.e. a bilinear map

$$A \times M \to M, \quad (a, m) \mapsto am$$

satisfying $1m = m$ and $(ab)m = a(bm)$.

A *homomorphism of left A-modules* is a linear map $f\colon M \to N$ between two modules M and N such that $f(am) = af(m)$ for all $a \in A$ and $m \in M$.

Right modules are defined similarly. In this note, unless stated otherwise, we will always use left modules, so we will drop the adjective and call them simply "modules".

2 Finite dimensional algebras

From now on we will concentrate on finite dimensional algebras, i.e. algebras which are finite dimensional as vector spaces. One of the aims of a branch of representation theory is to describe all finite dimensional modules (i.e. modules which are finite dimensional as vector spaces) over a finite dimensional algebra and the homomorphisms between them. For convenience we give the following definition.

Definition 2.1 Let A be a finite dimensional algebra. We define the category $\operatorname{mod} A$ of finite dimensional A-modules to be the category with:

- objects: finite dimensional A-modules,
- morphisms: homomorphisms of A-modules.

If A and B are two algebras, it could happen, even if they are not isomorphic, that $\operatorname{mod} A$ and $\operatorname{mod} B$ are equivalent as categories. For the reader who is not acquainted with category theory, this essentially means that we can give a full description of modules and homomorphisms over A if and only if we can do the same over B. In this case we say that A and B are *Morita equivalent*.

Example 2.2 If A is an algebra, then the algebra $M_n(A)$ of $n \times n$ matrices with coefficients in A is Morita equivalent to A.

Since we are interested in the category $\operatorname{mod} A$, the following theorem says that it is enough to study path algebras of quivers modulo relations.

Theorem 2.3 *Every finite dimensional algebra is Morita equivalent to the path algebra of a quiver bounded by relations.*

This theorem, as well as the approach to representation theory through quivers, goes back to Gabriel [Gab72]. Some references for an advanced treatment of this theory are the standard books [ARS95] and [ASS06].

We remark that there is a nice way to see modules over path algebras of quivers as "representations of quivers". We will not describe this construction in this note: the interested reader may look at the books mentioned above for further reference.

2.1 Auslander-Reiten theory

During the 70's Auslander and Reiten developed a theory in order to provide tools for studying the category $\operatorname{mod} A$. Very briefly, the idea is the following. Given a finite dimensional algebra A, we only need find some distinguished modules and morphisms, which are called "indecomposable modules" and "irreducible morphisms". Then we can construct a quiver, called the *Auslander Reiten quiver*, whose vertices are the indecomposable modules and whose arrows are the irreducible morphisms. From these data we can describe the whole category $\operatorname{mod} A$.

The details of this construction are beyond the scope of this note. The interested reader may look at [ASS06] or [ARS95] for further reference. Here we only recall the definition of indecomposable module, since it will be needed in the following.

Definition 2.4 An A-module M is *indecomposable* if, whenever $M \cong N_1 \oplus N_2$, then one of the N_i's is isomorphic to M and the other is 0.

The reason why all modules can be built from indecomposable ones is given by the following classical theorem.

Theorem 2.5 (Krull-Schmidt) *Let A be a finite dimensional algebra. Every finite dimensional A-module M has a decomposition*

$$M \cong \bigoplus_{i=0}^{n} M_i,$$

where each M_i is indecomposable. Moreover the indecomposable summands in this decomposition are uniquely determined up to isomorphism and up to reordering.

We have the following classification of finite dimensional algebras according to how many indecomposables they have.

Definition 2.6 An algebra A is:

- *representation finite* if there are only finitely many indecomposable A-modules up to isomorphism;
- *representation infinite* otherwise.

The situation is particularly nice when we work over hereditary algebras.

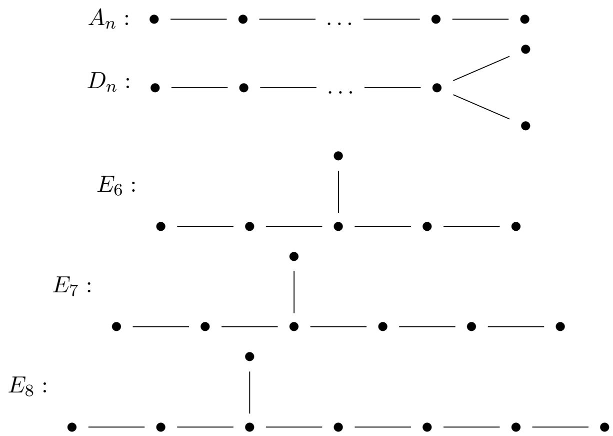
Definition 2.7 A finite dimensional algebra A is *hereditary* if one the following equivalent conditions holds.

- Every submodule of a projective A-module is again projective.
- $\operatorname{gl.dim} A \leq 1$.
- A is Morita equivalent to the path algebra of an acyclic quiver.

Recall that an A-module is projective if it is a direct summand of a free module. The global dimension of A, denoted by $\operatorname{gl.dim} A$, is the supremum of all the lengths of projective resolutions of A-modules. The non-expert reader may ignore the first two conditions in the previous definition (or eventually refer either to [ASS06], [ARS95] or any standard homological algebra book) and can just consider $\mathrm{gl.dim}\, A$ as a positive integer which can be associated to an algebra A.

The finite dimensional hereditary representation finite algebras are classified by the following theorem.

Theorem 2.8 ([Gab72]) *Let Q be an acyclic quiver. Then $\mathbb{C}Q$ is representation finite if and only if the unoriented graph underlying Q is a disjoint union of one of the following:*



The above graphs are called simply laced Dynkin diagrams. They were already known to mathematicians since they play a central role in the classification of semisimple Lie algebras.

3 Higher dimensional Auslander-Reiten theory

At beginning of the 2000's Iyama observed that Auslander-Reiten theory could be regarded in some sense as a "1-dimensional" theory: he then proposed a generalization of it to "dimension n". We will not give full details about this new theory. Some references are Iyama's original papers [Iya07a], [Iya07b] or the surveys [Iya08], [JK16].

Just to say a few words, the object of study in higher Auslander-Reiten theory is not the whole category $\mathrm{mod}\, A$ any more, but some special subcategories called *n-cluster tilting subcategories*. Moreover, in Auslander-Reiten theory the number 1 recurs in many places: usually it is enough to replace these 1's with n's to obtain an n-dimensional generalization of many definitions and statements of the classical theory. We will now see an example of this phenomenon by giving the n-dimensional analogue of the definition of a representation finite hereditary algebra.

Definition 3.1 ([IO13]) An algebra A is *n-representation finite* (n-RF for short) if

- $\mathrm{gl.dim}\, A \leq n$;
- a technical condition.

The technical condition above is related to the existence of an n-cluster tilting subcategory of $\mathrm{mod}\, A$. Taking $n = 1$ in the definition, this condition becomes equivalent to saying that $\operatorname{mod} A$ has a finite number of indecomposables. Moreover, $\operatorname{gl.dim} A \leq 1$ means that A is hereditary: hence we can see that the above definition is indeed a generalization of the one of a representation finite hereditary algebra.

3.1 Higher preprojective algebras

We saw before that 1-RF algebras are classified by Gabriel's Theorem. Unfortunately, if $n \geq 2$, no classification is known for n-RF algebras. However we can characterize n-RF algebras in terms of properties of their "higher preprojective algebras". This is useful in practice when one wants to find examples.

Higher preprojective algebras are the generalization of classical preprojective algebras. The latter were first introduced by Gel'fand and Ponomarev in 1979 and are defined as follows.

Definition 3.2 Let Q be an acyclic quiver. We define the double quiver $\overline{Q}$ by adding to Q, for each arrow $\alpha\colon i \to j$, an"inverse" arrow $\alpha^*\colon j \to i$. The *preprojective algebra* of Q is defined as

$$\Pi(Q) := \frac{\mathbb{C}\overline{Q}}{\left\langle \sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha \right\rangle}.$$

Example 3.3 Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Then $\overline{Q}$ is

$$1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3$$

and $\Pi(Q) =$ is the algebra $\mathbb{C}\overline{Q}$ modulo the relations

$$\alpha\alpha^* = 0, \quad \beta^*\beta = 0, \quad \alpha^*\alpha = \beta\beta^*.$$

Preprojective algebras have the following homological characterization: let Q be an acyclic quiver, $A = \mathbb{C}Q$. Then

$$\Pi(Q) \cong \bigoplus_{i \geq 0} \operatorname{Ext}^1_A(DA, A)^{\otimes_A i}.$$

Note that Q acyclic means $\operatorname{gl.dim} A \leq 1$; also the formula involves an Ext^1. The definition of preprojective algebras can be naturally generalized by replacing these 1's with n's:

Definition 3.4 ([IO13]) Let A be an algebra with $\operatorname{gl.dim} A \leq n$. We define the $(n+1)$-*preprojective algebra* of A as the tensor algebra

$$\Pi_{n+1}(A) = \bigoplus_{i \geq 0} \operatorname{Ext}^n_A(DA, A)^{\otimes_A i}.$$

Example 3.5 Let $A = \mathbb{C}Q/I$, where Q is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $I = \langle \alpha\beta \rangle$. Then $\operatorname{gl.dim} A = 2$ and $\Pi_3(A)$ is the path algebra of the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

modulo the relations

$$\alpha\beta = 0, \quad \beta\gamma = 0, \quad \gamma\alpha = 0.$$

Theorem 3.6 ([IO13]) *Let A be an algebra with* $\operatorname{gl.dim} A \leq n$. *Then A is n-representation finite if and only if*

- $\Pi_{n+1}(A)$ *is finite dimensional and selfinjective (i.e. it is injective as a module over itself) and*
- $\Pi_{n+1}(A)$ *satisfies a technical condition.*

4 2-representation finite algebras and quivers with potential

In this section we will see what happens when $n = 2$. In this case the technical condition in Theorem 3.6 is always satisfied. So an algebra A with $\operatorname{gl.dim} A \leq n$ is 2-RF if and only if $\Pi_3(A)$ is finite dimensional and selfinjective.

In the following we will see that 3-preprojective algebras have a nice description as quiver algebras with relations. We will be a bit sloppy with some definitions: one may refer to [HI11] for rigorous ones.

Definition 4.1 Let Q be a quiver. A *potential* W is a linear combination of cycles in Q up to cyclic permutation (i.e. if $\alpha_1 \cdots \alpha_m$ is a cycle, we identify all the cycles of the form $\alpha_i \cdots \alpha_m \alpha_1 \cdots \alpha_{i-1}$, $1 \leq i \leq m$). We will refer to the pair (Q, W) as a *quiver with potential*, or QP for short.

We define a set of relations from W in the following way. Let σ be the map defined on cycles by

$$\sigma(\alpha_1 \cdots \alpha_m) = \sum_i \alpha_i \cdots \alpha_m \alpha_1 \cdots \alpha_{i-1}.$$

Define the partial derivative $\partial_\alpha \colon \mathbb{C}Q \to \mathbb{C}Q$ with respect to an arrow α by setting

$$\partial_\alpha p := \begin{cases} q & \text{if } p = \alpha q, \\ 0 & \text{otherwise,} \end{cases}$$

if p is a path, and extending linearly to $\mathbb{C}Q$.

Definition 4.2 The *Jacobian algebra* of (Q, W) is defined as

$$\mathcal{P}(Q, W) := \mathbb{C}Q/\mathcal{J}(Q, W),$$

where $\mathcal{J}(Q, W)$ is the ideal generated by the relations $\partial_\alpha(\sigma(W))$ for $\alpha \in Q_1$.

Example 4.3 Let

$$Q = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

with potential $W = \alpha\beta\gamma$. Then

$$\sigma(W) = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta,$$

$$\partial_\alpha(\sigma(W)) = \beta\gamma, \quad \partial_\beta(\sigma(W)) = \gamma\alpha, \quad \partial_\gamma(\sigma(W)) = \alpha\beta,$$

hence

$$\mathcal{J}(Q, W) = \langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle.$$

We may note that the Jacobian algebra $\mathcal{P}(Q, W)$ of Example 4.3 is isomorphic to the 3-preprojective algebra $\Pi_3(A)$ of Example 3.5. As we can see in the following theorem, this is no coincidence.

Theorem 4.4 ([Kel11]) *Let* $\operatorname{gl.dim} A \leq 2$. *Then there exists a quiver with potential* (Q_A, W_A) *such that* $\Pi_3(A) \cong \mathcal{P}(Q_A, W_A)$.

It follows from Theorems 4.4 and 3.6 (recall that the technical condition in the latter comes from free in our case) that if A is 2-RF, then $\Pi_3(A)$ is finite dimensional selfinjective and $\Pi_3(A) \cong \mathcal{P}(Q_A, W_A)$ for a QP (Q_A, W_A). Is the converse true? More precisely, given a QP (Q, W) such that $\mathcal{P}(Q, W)$ is finite dimensional selfinjective (for short, a selfinjective QP), then is $\mathcal{P}(Q, W)$ isomorphic to $\Pi_3(A)$ for some 2-RF algebra?

Before answering this question let us observe a couple of things. Recall that $\Pi_3(A) = \bigoplus_{i \geq 0} \operatorname{Ext}^2_A(DA, A)^{\otimes i}$. We can consider it as a positively graded algebra, where the part of degree i is $\operatorname{Ext}^2_A(DA, A)^{\otimes i}$. Then we have that:

- the degree 0 part of $\Pi_3(A)$ is isomorphic to A;
- if $\Pi_3(A) \cong \mathcal{P}(Q_A, W_A)$, then this grading is induced by a grading on $\mathbb{C}Q$ such that W_A is homogeneous of degree 1.

It turns out that the existence of a grading satisfying these two properties is enough to get a positive answer to the previous question:

Theorem 4.5 ([HI11]) *Let (Q, W) be a selfinjective QP and put a positive grading on $\mathbb{C}Q$ such that W is homogeneous of degree 1. Then this induces a grading on $\mathcal{P} := \mathcal{P}(Q, W)$ such that its degree 0 part $\mathcal{P}_0$ is 2-RF and $\mathcal{P} \cong \Pi_3(\mathcal{P}_0)$.*

There is a nice way to define gradings such as the ones in the previous theorem.

Definition 4.6 Let (Q, W) be a QP. A *cut* is a subset C of the arrows of Q such that each cycle in W contains exactly one arrow of C.

Given a cut C, we can define a grading g_C on Q by

$$g_C(\alpha) = \begin{cases} 1 & \alpha \in C, \\ 0 & \alpha \notin C. \end{cases}$$

This induces a grading on the Jacobian algebra $\mathcal{P}(Q, W)$ since the relations on the quiver become automatically homogeneous.

Example 4.7 Let

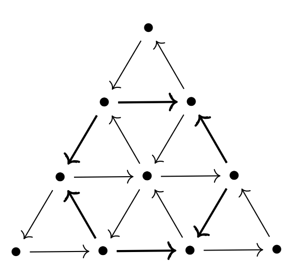
$$Q = \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

with $W = \alpha\beta\gamma$. Then $C = \{\gamma\}$ is a cut. The degree 0 part of $\mathcal{P}(Q, W)$ is

$$\mathbb{C}\left(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \right) \Big/ \langle \alpha\beta \rangle$$

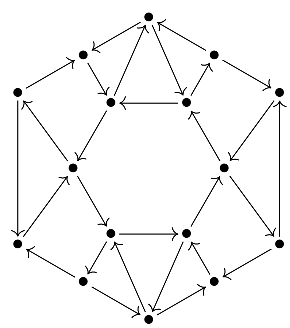
To sum up, we have seen that 2-RF algebras are essentially selfinjective QP's with cuts. A complete classification of them is not known yet. However many examples have been found and more are coming up. In the following we will show some of them.

Example 4.8 ([IO11]) A class of examples is given by QP's of the following form:



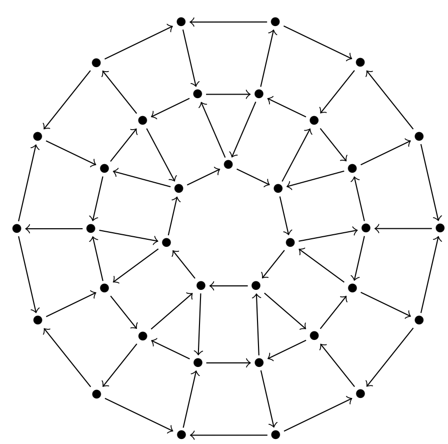
where the potential is $W = \sum \triangle - \sum \triangledown$ and the thick arrows represent an example of cut. By changing the size of the triangle we obtain an infinite family of QP's. The corresponding 2-RF algebras are called "of type A", since they are the 2-dimensional analogues of the representation finite algebras relative to Dynkin quivers of type A.

Example 4.9 A wide class of selfinjective QP's, which includes the ones in the previous example, was found by Herschend and Iyama [HI11]. They are called "planar", because they can be embedded in the plane. One example is this:

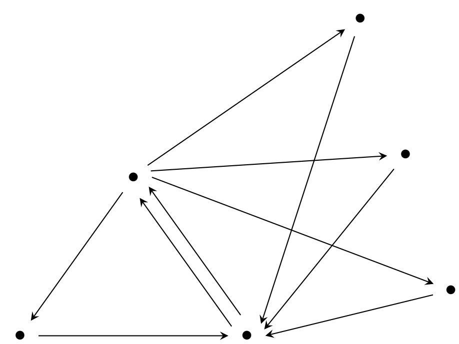


where the potential is given by the sum of all the faces, with the clockwise ones taken with coefficient 1 and the others with coefficient -1.

Even more examples of planar selfinjective QP's have been found by Pasquali [Pas17]. They are constructed from combinatorial objects called "Postnikov diagrams". Here is an example:



Example 4.10 There are also examples of selfinjective QP's which are not planar: they are obtained from planar ones by applying a construction which involves the action of a finite group on the QP. This is part of a work in progress by Pasquali and the author. An example (obtained with an action of $\mathbb{Z}/3\mathbb{Z}$ by rotations on the quiver of Example 4.8 is the following:



References

[ARS95] M. Auslander, I. Reiten, and S. Smalø, "Representation Theory of Artin Algebras". Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, New York, 1995.

[ASS06] I. Assem, D. Simson, A. Skowronski, "Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory". Vol. 65. Cambridge University Press, 2006.

[Gab72] P. Gabriel, *Unzerlegbare Darstellungen I.* Manuscripta Math. 6 (1972), 71–103.

[HI11] M. Herschend, O. Iyama, *Selfinjective quivers with potential and 2-representation-finite algebras.* Compos. Math. 147/6 (2011), 1885–1920.

[Iya07a] O. Iyama, *Auslander correspondence.* Adv. Math. 210/1 (2007), 51–82.

[Iya07b] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories.* Adv. Math. 210/1 (2007), 22–50.

[Iya08] O. Iyama, *Auslander-Reiten theory revisited.* In "Trends in representation theory of algebras and related topics", EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2008), 349–397.

[IO11] O. Iyama, S. Oppermann, *n-representation finite algebras and n-APR tilting.* Trans. Amer. Math. Soc. 363/12 (2011), 6575–6614.

[IO13] O. Iyama, S. Oppermann, *Stable categories of higher preprojective algebras.* Adv. Math. 244 (2013), 23–68.

[JK16] G. Jasso, S. Kvamme, *An introduction to higher Auslander-Reiten theory.* arXiv:1610.05458, 2016.

[Kel11] B. Keller, *Deformed Calabi-Yau completions* (with appendix by Michel Van den Bergh. J. Reine Angew. Math. 654 (2011), 125–180.

[Pas17] A. Pasquali, *Self-injective Jacobian algebras from Postnikov diagrams.* arXiv:1706.08756, 2017.

Stochastic models for energy forward markets

Marco Piccirilli (*)

Abstract. I will present a probabilistic modeling framework for forward prices, specifically designed for energy markets. Most of the presentation will be kept at an intuitive level, as far as this is possible and sensible. I will start by explaining the general framework of the talk and then move to our contribution, of course describing the underlying mathematical theory as well. These notes are based on joint work [3, 5] with Fred Espen Benth, Luca Latini and Tiziano Vargiolu.

1 Introduction

Energy markets are associated to the trade and supply of energy, such as electricity, gas, coal, oil, CO_2 emission certificates (EU ETS). Since their recent deregulations (1990s in Europe) energy markets are **rapidly evolving** sectors. From the modeling perspective, they are bringing to the attention of practitioners and researchers a bunch of interesting problems. Energy-related commodities share **peculiar features**, which make them different from classical financial markets:

- seasonality (weather, temperature and consumption),
- storability,
- technology (storage),
- renewable sources (**negative prices**).

Our focus is on **power** and **natural gas** markets (cf. [1]). In Europe there are more than twenty different energy exchanges. The two most liquid ones are the European Energy Exchange (EEX) in Leipzig and the Nord Pool Spot in Oslo. Trade is mainly organized in the **spot** market, where short-term trading takes place, and the **forward** market. The most active segment is often the forward market and, as such, the most liquid derivative products are **forward (or futures) contracts**.

Forward contracts consist of agreements between two parties: the seller (S) and the buyer (B). Today (at time t_0) (S) agrees to deliver a fixed amount of, say, power to (B) over a certain delivery period in the future (the time interval $[T_1, T_2]$) for a predetermined

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price (the strike price $F(t_0, T_1, T_2)$). The financial value of this contract, from stipulation to delivery, is called **forward price**

$$[t_0, T_1] \ni t \mapsto F(t, T_1, T_2)$$

and is what we aim to describe in a **dynamic probabilistic model**. This is determined by the market according to the "today" price (spot price) of power, and the market expectations on its future value. The **forward curve** or **term structure** is the plot of today forward prices as a function of the delivery period (see Figure 1).

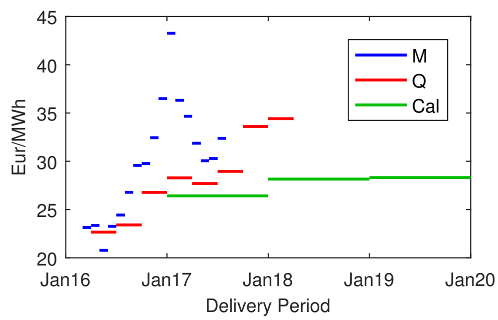

Figure 1. Futures prices at January 4, 2016 for delivery of electricity in the following months, quarters and years.

An important characteristics of energy forward markets is the peculiar trading mechanism (see Figure 2), which works as follows. For each given calendar year, as time passes forwards are unpacked first in quarters, then in the corresponding months. It may happen that the same delivery period is covered in the market by different contracts, e.g. one simultaneously finds quotes for Jan/17, Feb/17, Mar/17 and Q1/17.

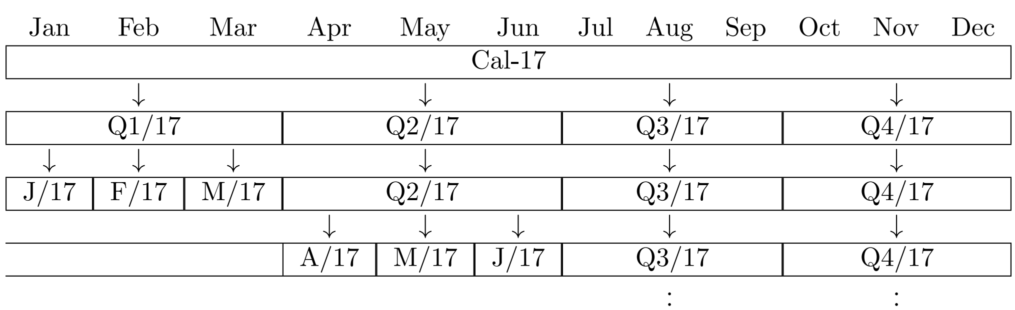

Figure 2. Cascade unpacking mechanism.

A fundamental empirical behavior of forward prices is the strong **seasonality** in both prices and volatility (see Figure 3). Moreover, price series exhibit **mean-reversion**: over time the process tends to drift towards its long-term equilibrium (see Figure 4).

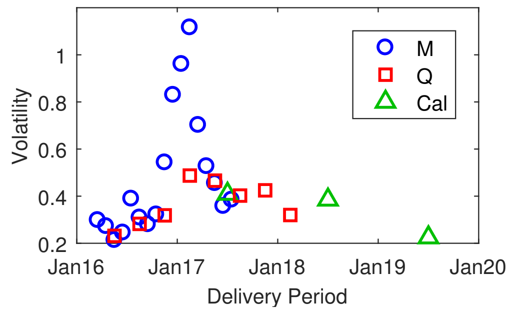

Figure 3. Realized volatilities of German power futures from January 4, 2016 to May 23, 2017. The **Samuelson effect** (=volatility increasing as time approaches delivery) is not sufficient to explain this term structure.

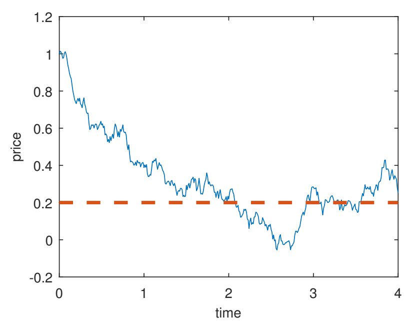

Figure 4. Simulation of a mean-reverting process (Ornstein-Uhlenbeck).

In Mathematical Finance a classical axiom is that pricing models must not admit arbitrage (cf. for instance [4]). An arbitrage opportunity is the **possibility of gaining** from an operation which does not need an initial capital and can not lead **in any case to a loss**. This concept can be translated in probabilistic terms as follows. If X_t is a random variable representing wealth at time t, then an arbitrage occurs if

$$X_0 = 0, \quad \mathbb{P}(X_T > 0) > 0, \quad \mathbb{P}(X_T \geq 0) = 1.$$

no initial capital, possibility of gain, no loss in any case.

The purpose of this work is to introduce a stochastic dynamics for energy forward curves which is:

- **Realistic**,
- **Tractable**,
- **Arbitrage-free**.

This will lead, of course, to a tradeoff in the modeling specification.

2 Mathematical framework

The theoretical framework is constructed as follows. Let $f(t,T)$ denote the forward price at time t for **instantaneous delivery** at time T. Analogously, for any $T_1 < T_2$, let $F(t,T_1,T_2)$ be the forward price at time t for **delivery period** $[T_1,T_2]$. The forward prices are **stochastic processes** solving the following parametric stochastic differential equations under the **real-world probability** measure $\mathbb{P}$ by

$$\begin{aligned} df(t,T) &= \big(-\lambda(t)f(t,T) + c(t,T)\big)\ dt + \theta(t,T)\ dW(t),\\ dF(t,T_1,T_2) &= \big(-\lambda(t)F(t,T_1,T_2) + C(t,T_1,T_2)\big)\ dt + \Sigma(t,T_1,T_2)\ dW(t). \end{aligned}$$

This is called the **Heath-Jarrow-Morton** methodology, especially developed for power markets by [2]. Under the physical measure $\mathbb{P}$ we have a time-dependent **mean-reversion** behavior, while the long term-mean and diffusion coefficients are both time and maturity dependent (**seasonality**). Since we have not performed a logarithmic transformation of the price, the dynamics are **additive**. In principle, this could generate negative prices, but this is reasonable for power and natural gas markets (given the presence of negative spot prices). Additivity is a very convenient property for the evaluation of energy derivative products (analytical formulas at hand). This is one reason why these models are becoming more and more **popular**.

In power and gas markets the instantantaneous forwards do not exist. However, they play the role of **building blocks** for the dynamics of traded forwards:

$$F(t,T_1,T_2) \overset{def}{=} \frac{1}{T_2-T_1}\int_{T_1}^{T_2} f(t,u)\ du,$$

for all $t \le T_1$ and $T_1 < T_2$. This constraint implies that overlapping contracts satisfy the following **no-arbitrage relations**:

$$\text{(NA)} \qquad F(t,T_1,T_n) = \frac{1}{T_n - T_1}\sum_{i=1}^{n-1}(T_{i+1}-T_i)F(t,T_i,T_{i+1}).$$

“Dynamic” arbitrage is prevented by constructing an **equivalent martingale measure** for the market, i.e. a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the processes $f(\cdot,T), F(\cdot,T_1,T_2)$ are $\mathbb{Q}$-martingales, or equivalently

$$\begin{aligned} df(t,T) &= \theta(t,T)\ dW^{\mathbb{Q}}(t),\\ dF(t,T_1,T_2) &= \Sigma(t,T_1,T_2)\ dW^{\mathbb{Q}}(t), \end{aligned}$$

for some $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$. The main **technical difficulty** consists in finding an equivalent martingale measure (mean-reversion). For this reason we introduce the **key assumption**:

$$\text{(1)} \qquad f(t,T) = \alpha(t,T)X(t) + \beta(t,T),$$

where α and β are deterministic and X is a universal source of randomness. This is not observable, but its coefficients do not appear explicitly in the resulting dynamics (no filtering needed). This assumption is equivalent to saying that we restrict our model to dynamics of type (1).

3 Generalized Lucia-Schwartz model

The class of models satisfying (1) is not void: the well-known **Lucia-Schwartz** model [6] turns out to be of this kind: this describes the spot price as the sum of two hidden state variables plus a deterministic seasonal component. Also, it is quite flexible in mean price levels, by incorporating the spot's seasonal component. The forward dynamics is

$$\begin{aligned} df(t,T) &= \lambda(t)\big(\phi(T) - f(t,T)\big)\ dt + e^{-\kappa(T-t)}\sigma_1\ dW_1(t) + \sigma_2\ dW_2(t), \\ dF(t,T_1,T_2) &= \lambda(t)\big(\Phi(T_1,T_2) - F(t,T_1,T_2)\big)\ dt + e^{\kappa t}\ \Gamma(T_1,T_2)\ dW_1(t) + \sigma_2\ dW_2(t). \end{aligned}$$

However, the volatility succeeds only in reproducing the Samuelson effect as an exponential decay (**no finer term structures**). Therefore, we propose a modification of the Lucia-Schwartz model such that both price level and volatility are allowed to have a **non-trivial term structure**:

$$\begin{aligned} df(t,T) &= \lambda(t)\big(\phi(T) - f(t,T)\big)\ dt + e^{-\kappa(T-t)}\sigma_1\ dW_1(t) + \psi(T)\ dW_2(t), \\ dF(t,T_1,T_2) &= \lambda(t)\big(\Phi(T_1,T_2) - F(t,T_1,T_2)\big)\ dt + e^{\kappa t}\ \Gamma(T_1,T_2)\ dW_1(t) + \Psi(T_1,T_2)\ dW_2(t), \end{aligned}$$

We still have a **Samuelson effect** in both the instanteneous (non-traded) forward prices $f(\cdot,T)$ and the traded forward prices $F(\cdot,T_1,T_2)$, but the second factor is able to describe the **volatility term structure** in a finer way. Since this model is derived from our framework, the dynamics is **mean-reverting** and **arbitrage-free**.

4 Estimation of model parameters

Estimation is performed **directly** on the market time series of traded forwards $F(\cdot,T_1,T_2)$ and no forward curve smoothing procedure is needed. For the **volatility** this task is not trivial, since the term structure is also maturity-dependent. We apply a method based on **quadratic variation/covariation** of price processes, which allows to estimate the diffusion coefficients. The estimation of the drift parameters is done by combining maximum likelihood (**ML**) with the technique of **Lagrange multipliers**, in order to take into account the **no-arbitrage constraints** (NA). For example, with the convention that $\Phi_{\mathrm{Q2/17}}$ denotes the parameter $\Phi(T_1,T_2)$ corresponding to the contract Q2/17, then

$$\Phi_{\mathrm{Q2/17}} = u_{\mathrm{Apr/17}}\Phi_{\mathrm{Apr/17}} + u_{\mathrm{May/17}}\Phi_{\mathrm{May/17}} + u_{\mathrm{Jun/17}}\Phi_{\mathrm{Jun/17}},$$

where the weights u_i are defined according to the number of days in the month/quarter (e.g. $u_{\mathrm{Apr/17}} = 30/91$).

5 Empirical results

We apply this estimation technique on the **Phelix Base Futures** market. We consider all the daily closing prices of each monthly, quarterly and calendar forward contract traded from January 4, 2016 to May 23, 2017.

We then do a simulation study and assess the **performance** of the model, in terms of reproducibility of both the data under investigation and the stylized features of energy futures markets. Firstly, we compare **simulated paths** of some exemplary futures contracts to the corresponding **observed trajectories**, so to discuss the qualitative behavior of model simulations. Secondly, we **compute fundamental statistics** of the model by averaging the results of a set of simulations and compare them to our data.

We report in Figures 5, 6, 7, 8 the simulated trajectories and the corresponding historical path of the forward price. For a more rigorous discussion of the fitting quality, we investigate the **statistical features** of the model and make a comparison with the historical data. A standard way to do it is by computing moments. In Table 1 we report the values of the first four moments, the minimum and the maximum of both **empirical and simulated returns**, i.e. the daily price increments, of all contracts. We run 1000 simulations and then average the results over all samples. The values are classified among different delivery periods in order to distinguish different behaviors among them.

	Mean	Std. Dev.	Skewness	Kurtosis	Min.	Max.
Phelix Base (M)	0.04	0.58	0.25	3.83	−1.32	1.62
Phelix Base (Q)	0.02	0.44	0.19	4.17	−1.36	1.44
Phelix Base (Cal)	0.03	0.41	−0.18	3.74	−1.35	1.26
Model (M)	0.00	0.67	0.01	4.71	−1.86	1.88
Model (Q)	0.00	0.43	0.06	4.87	−1.44	1.47
Model (Cal)	0.02	0.39	0.08	4.13	−1.25	1.32
Phelix Base	0.03	0.52	0.19	3.92	−1.33	1.53
Model	0.00	0.57	0.03	4.69	−1.68	1.70

Table 1. Empirical vs. simulated statistics (first four normalized moments, minimum and maximum) of the daily returns of all the contracts aggregated by delivery period. Model statistics are computed by averaging the results of the estimators over 1000 simulations, first, for each contract and, second, among contracts grouped by delivery period. The last two rows show the overall results for all the contracts.

6 Conclusions

We have designed a modeling approach for **energy forward curves** capable to hold together **mean-reversion** and **arbitrage theory**. This has been done through a suitable assumption (...and validated by proving certain results on **exponential martingales**...cf. [3]). Within this framework, we have specified a **generalized Lucia-Schwartz** (additive) model, which is sufficiently **tractable** for calibration purposes. Finally, we have introduced an *ad hoc* estimation procedure which takes into account the **peculiar trading mechanism** of these markets and applied it to German electricity futures.

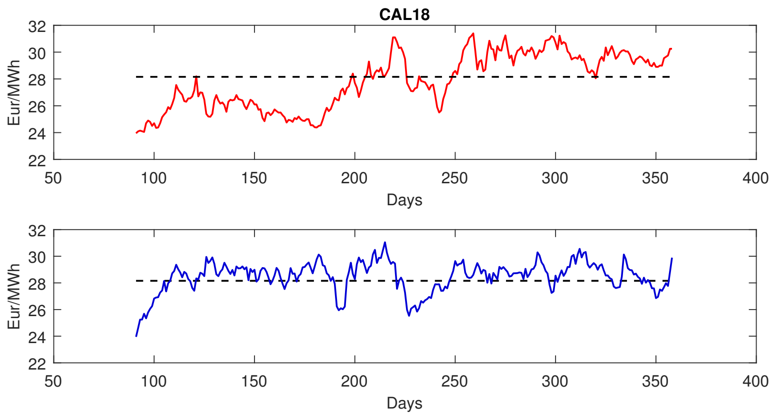

Figure 5. Historical (red) and simulated (blue) path of the contract Cal-18. The dotted line represents in both plots the estimated long-term mean of the contract.

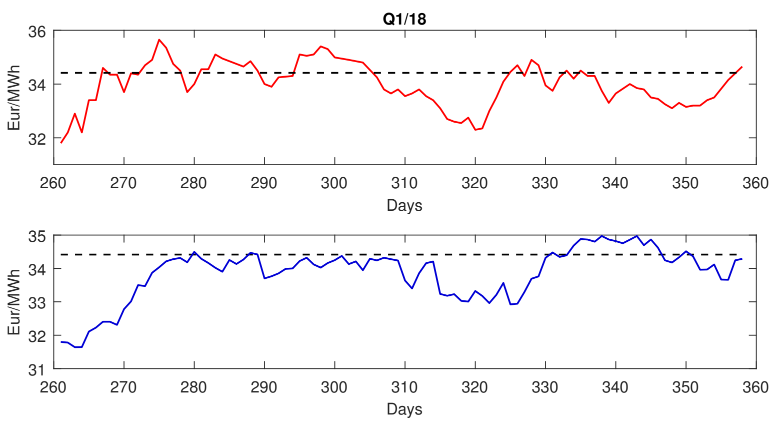

Figure 6. Historical (red) and simulated (blue) path of the contract Q1/18. The dotted line represents in both plots the estimated long-term mean of the futures price.

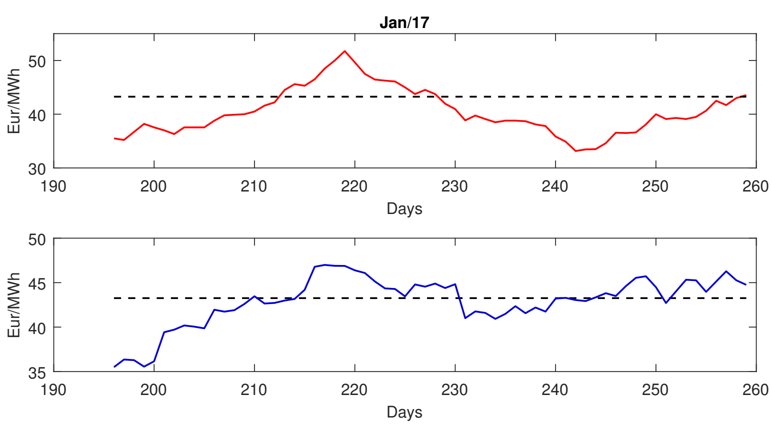

Figure 7. Historical (red) and simulated (blue) path of the contract Jan/17. The dotted line represents in both plots the estimated long-term mean of the futures price.

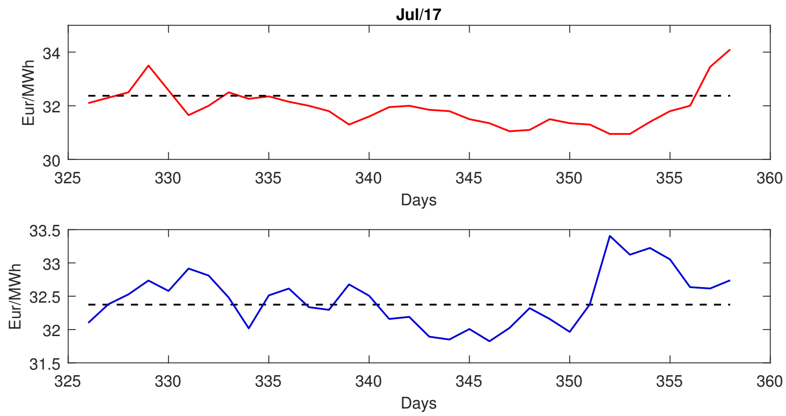

Figure 8. Historical (red) and simulated (blue) path of the contract Jul/17. The dotted line represents in both plots the estimated long-term mean of the futures price.

References

[1] F. E. Benth, J. S. Benth, S. Koekebakker, "Stochastic modelling of electricity and related markets". Advanced Series on Statistical Science and Applied Probability 11. World Scientific, 2008.

[2] F. E. Benth, S. Koekebakker, *Stochastic modeling of financial electricity contracts.* Energy Economics 30/3 (2008), 1116–1157.

[3] F. E. Benth, M. Piccirilli, T. Vargiolu, *Additive energy forward curves in a Heath-Jarrow-Morton framework.* Preprint (2018).

[4] T. Björk, "Arbitrage Theory in Continuous Time". Oxford University Press, 3 ed, 2009.

[5] L. Latini, M. Piccirilli, T. Vargiolu, *Mean-reverting no-arbitrage additive models for forward curves in energy markets.* Preprint (2018).

[6] J. J. Lucia and E. S. Schwartz, *Electricity Prices and Power Derivatives: Evidence from the Nordic Power Exchange.* Review of Derivatives Research 5/1 (2000), 5–50.

Norm attaining mappings

Marija Soloviova (*)

Abstract. The Bishop-Phelps-Bollobás theorem deals with approximation of linear continuous functionals and operators by norm attaining one. We introduce two concepts of norm attainment for Lipschitz functionals. We showed that the subset of strongly norm attaining Lipschitz functional is not dense in $\mathrm{Lip}_0(X)$. Then we introduce a weaker concept of directional norm attainment and demonstrate that for a uniformly convex X the set of directionally norm attaining Lipschitz functionals is strongly dense in $\mathrm{Lip}_0(X)$ and, moreover, an analogue of the Bishop-Phelps-Bollobás theorem is valid. This work was done jointly with Vladimir Kadets (Kharkiv V. N. Karazin National University) and Miguel Martín (Universidad de Granada).

1 Introduction and motivation

In this text, the letter X stands for a real Banach space. We denote, as usual, by S_X and B_X the unit sphere and the closed unit ball of X, respectively. A functional $x^* \in X^*$ *attains its norm*, if there is $x \in S_X$ with $x^*(x) = \|x^*\|$. If X is reflexive, then all $x^* \in X^*$ attain their norms and, according to the famous James theorem (see [9, Chapter 1, Theorem 3]), in every non-reflexive space there are functionals that do not attain their norm. Nevertheless, in every Banach space there are "many" norm attaining functional. Namely, the classical Bishop-Phelps theorem ([6], see also [9, Chapter 1]) states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. Moreover, for every closed bounded convex set $C \subset X$, the collection of functionals that attain their maximum on C is norm dense in X^*.

The fact that every functional can be approximated by norm attaining ones is quite useful, but sometimes one needs more. Namely, sometimes (in particular, when one works with numerical radius of operators) one needs to approximate a pair "element and functional" by a pair (x, x^*) such that x^* attains its norm in x. Such a modification of the Bishop-Phelps theorem was given by B. Bollobás [7]. Below we cite it in a slightly modified form with sharp estimates [8].

Theorem 1.1 (Bishop-Phelps-Bollobás theorem) *Let X be a Banach space. Suppose $x \in B_X$ and $x^* \in B_{X^*}$ satisfy $x^*(x) \geqslant 1-\delta$ ($\delta \in (0,2)$). Then there exists $(y, y^*) \in X \times X^*$*

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with $\|y\| = \|y^*\| = y^*(y) = 1$ *such that*

$$\max\{\|x-y\|, \|x^*-y^*\|\} \leqslant \sqrt{2\delta}. \tag{1.1}$$

In this project we are searching for possible extensions of Bishop-Phelps and Bishop-Phelps-Bollobás theorems for non-linear Lipschitz functionals $f : X \longrightarrow \mathbb{R}$.

Recall that the Banach space $\mathrm{Lip}_0(X)$ consists of functions $f : X \longrightarrow \mathbb{R}$ with $f(0) = 0$ which satisfy (globally) the Lipschitz condition. This space is equipped with the norm

$$\|f\| = \sup\left\{\frac{|f(x)-f(y)|}{\|x-y\|} : x, y \in X, x \neq y\right\}. \tag{1.2}$$

In other words, $\|f\|$ is the smallest Lipschitz constant of f. We refer the reader to the book [15] for background on Lipschitz spaces.

Remark that in this case, evidently, X^* is a closed subspace of $\mathrm{Lip}_0(X)$ with equality of norms.

As this work deals with possible extensions of the Bishop-Phelps and Bishop-Phelps-Bollobás theorems, we first have to say what we understand by a norm attaining Lipschitz functional. We have a couple of possible definitions for this.

First, the most natural definition of norm attainment for a functional $f \in \mathrm{Lip}_0(X)$ is the following.

Definition 1.2 A functional $f \in \mathrm{Lip}_0(X)$ *attains its norm in the strong sense* if there are $x, y \in X$, $x \neq y$ such that $\|f\| = \frac{|f(x)-f(y)|}{\|x-y\|}$. The subset of all functionals $f \in \mathrm{Lip}_0(X)$ that attain their norm in the strong sense is denoted $\mathrm{SA}(X)$.

Unfortunately, in the sense of the Bishop-Phelps theorem, this definition is too restrictive. Even in the one-dimensional case ($X = \mathbb{R}$), the subset $\mathrm{SA}(X)$ is not dense in $\mathrm{Lip}_0(X)$ (see Example 2.1).

It is then clear that a less restrictive way for a Lipschitz functional to attain its norm is needed to get density. We will use the following definition.

Definition 1.3 A functional $g \in \mathrm{Lip}_0(X)$ *attains its norm at the direction* $u \in S_X$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $X \times X$, with $x_n \neq y_n$, such that

$$\lim_{n\to\infty} \frac{x_n - y_n}{\|x_n - y_n\|} = u \quad \text{and} \quad \lim_{n\to\infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|.$$

In this case, we say that *g attains its norm directionally.* The set of all those $f \in \mathrm{Lip}_0(X)$ that attain their norm directionally is denoted by $\mathrm{DA}(X)$.

We start our consideration with two reasons of why the directional approach is natural in our framework.

(a) If X is finite-dimensional, then $\mathrm{DA}(X) = \mathrm{Lip}_0(X)$ by a compactness argument, so at least in this easiest case the directional Bishop-Phelps theorem does not fail.

(b) A linear functional attains its norm at direction u if and only if $f(u) = \|f\|$, so it attains its norm in the usual sense.

Our main result is a Bishop-Phelps-Bollobás type theorem for Lipschitz functional on uniformly convex spaces. Even though we only have one kind of examples, it make sense to introduce the following definition.

Definition 1.4 A Banach space X has the *directional Bishop-Phelps-Bollobás property for Lipschitz functionals* ($X \in \mathrm{LipBPB}$ for short), if for every $\varepsilon > 0$ there is such a $\delta > 0$, that for every $f \in \mathrm{Lip}_0(X)$ with $\|f\| = 1$ and for every $x, y \in X$ with $x \neq y$ satisfying $\frac{f(x)-f(y)}{\|x-y\|} > 1 - \delta$, there is $g \in \mathrm{Lip}_0(X)$ with $\|g\| = 1$ and there is $u \in S_X$ such that g attains its norm at the direction u, $\|g - f\| < \varepsilon$, and $\left\| \frac{x-y}{\|x-y\|} - u \right\| < \varepsilon$.

So, with this notation, our main result is to prove that uniformly convex spaces have the LipBPB (Theorem 4.3).

Remark that we are not able to answer some natural easy-looking questions related to our results. For example, we are not able to construct a Banach space which has no directional Bishop-Phelps-Bollobás property for Lipschitz functionals.

We finish this introduction recalling an important tool to construct Lipschitz functionals: the classical *McShane's extension theorem.* It says that if M is a subspace of a metric space E and $f : M \longrightarrow \mathbb{R}$ is a Lipschitz functional, then there is an extension to a Lipschitz functional $F : E \longrightarrow \mathbb{R}$ with the same Lipschitz constant; see [15, Theorem 1.5.6] or [5, p. 12/13].

2 Strongly attaining Lipschitz functionals

As announced in the introduction, there is no Bishop-Phelps type theorem for Lipschitz functionals in the strong sense of the attainment, even in the one-dimensional case.

Example 2.1 $\mathrm{SA}([0,1])$ *is not dense in* $\mathrm{Lip}_0([0,1])$.

In order to demonstrate this, we need the following easy lemma.

Lemma 2.2 *If $f \in \mathrm{Lip}_0(E)$ attains its norm on a pair $(x, y) \in E \times E$, $x \neq y$, and $z \in E \setminus \{x, y\}$ is such an element that $\rho(x, y) = \rho(x, z) + \rho(z, y)$, then f strongly attains its norm on the pairs (x, z) and (y, z), and*

$$f(z) = \frac{\rho(z, y)f(x) + \rho(x, z)f(y)}{\rho(x, y)}. \tag{2.1}$$

In particular, if E is a convex subset of a Banach space, then f is affine on the closed segment $\mathrm{conv}\{x, y\}$, *i.e.* $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$ *for every* $\theta \in [0, 1]$.

Proof. We may (and do) assume without loss of generality that $f(x) - f(y) \geqslant 0$ (otherwise we multiply f by -1). Since f attains its norm on the pair (x, y), we have

$$\begin{aligned} \|f\|\rho(x,y) &= f(x) - f(y) = f(x) - f(z) + f(z) - f(y) \\ &\leqslant \|f\|\rho(x,z) + \|f\|\rho(z,y) = \|f\|\rho(x,y). \end{aligned}$$

This means that the inequalities $f(x) - f(z) \leqslant \|f\|\rho(x,z)$ and $f(z) - f(y) \leqslant \|f\|\rho(z,y)$ which we used above are, in fact, equalities. So, $f(x) = f(z) + \|f\|\rho(x,z)$ and $f(y) = f(z) - \|f\|\rho(z,y)$. Substituting the last two formulas into the right-hand side of (2.1), we get the desired result. $\square$

Proof of Example 2.1. It is well known (for example, [15, Example 1.6.5] or [12, Propositions 6 and 2]) that $\mathrm{Lip}_0([0,1])$ is isometric to $L_\infty([0,1])$, and the corresponding bijective isometry $U : \mathrm{Lip}_0([0,1]) \longrightarrow L_\infty([0,1])$ is just the differentiation operator (the derivative of a Lipschitz function $f : [0,1] \longrightarrow \mathbb{R}$ exists almost everywhere). Under this isometry, every $f \in \mathrm{SA}([0,1])$ maps to a function, which is equal either to $\|f\|$ or to $-\|f\|$ on some non-void interval (here we use Lemma 2.2). Denote A a nowhere dense closed subset of $[0,1]$ of positive Lebesgue measure. and let $g \in \mathrm{Lip}_0([0,1])$ be the function, whose derivative equals $\mathbf{1}_A$ (the characteristic function of A) a.e. Then g cannot be approximated by functions from $\mathrm{SA}([0,1])$. Actually, we claim that

$$\|g - f\| = \|\mathbf{1}_A - f'\|_\infty \geqslant \frac{1}{2} \tag{2.2}$$

for every $f \in \mathrm{SA}(\mathbb{R})$.

In fact, $\|g\| = \|\mathbf{1}_A\|_\infty = 1$ so, if $\|f'\|_\infty \leqslant \frac{1}{2}$, then (2.2) follows from the triangle inequality. If $\|f'\|_\infty > \frac{1}{2}$, then, as we remarked before, $|f'(t)| > \frac{1}{2}$ on some open interval (a,b). But, since A is nowhere dense, there is a smaller interval $(c,d) \subset (a,b)$ such that $\mathbf{1}_A(t) = 0$ for $t \in (c,d)$. So $|(\mathbf{1}_A - U(f))(t)| > \frac{1}{2}$ on (c,d), which implies (2.2). Hence, $\mathrm{SA}([0,1])$ is not dense in $\mathrm{Lip}_0([0,1])$. $\square$

3 Two preliminary results

In this section we demonstrate two preliminary results on the way to Theorem 4.3. The first one is a weak version of the Bishop-Phelps-Bollobás theorem for Lipschitz functionals, valid for all Banach spaces, which can be of independent interest.

Lemma 3.1 (Preliminary LipBPB Theorem) *Let X be a Banach space, $f \in \mathrm{Lip}_0(X)$, $\|f\| = 1$, $\delta \in (0,2)$ and let $x, y \in X, x \neq y$ be such elements that*

$$\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta. \tag{3.1}$$

Then for every $h \in \mathrm{Lip}_0(X)$ with $\|h\| = 1$ and $\frac{h(x)-h(y)}{\|x-y\|} = 1$, there exists $g \in \mathrm{Lip}_0(X)$ with $\|g\| = 1$, $\|f - g\| < \sqrt{2\delta}$ and there exists a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$ for every n, such that

$$\frac{h(v_n) - h(w_n)}{\|v_n - w_n\|} > 1 - \sqrt{2\delta} \quad \textit{for all } n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1.$$

The proof of this result is based on the free Lipschitz space technique and can be found in [1].

Before stating the second result we need a couple of definitions.

Definition 3.2 A functional $g \in \mathrm{Lip}_0(X)$ *attains its norm in point* $v \in X$ *at the direction* $u \in S_X$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $X \times X$, with $x_n \neq y_n$, such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = v, \quad \lim_{n\to\infty} \frac{x_n - y_n}{\|x_n - y_n\|} = u \quad \text{and} \quad \lim_{n\to\infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|.$$

In this case, we say that g *attains its norm locally-directionally.* The set of all those $f \in \mathrm{Lip}_0(X)$ that attain their norm locally-directionally is denoted by $\mathrm{LDA}(X)$.

Definition 3.3 A Banach space X has the *local directional Bishop-Phelps-Bollobás property for Lipschitz functionals* ($X \in$ LLipBPB for short), if for every $\varepsilon > 0$ there is such a $\delta > 0$, that for every $f \in \mathrm{Lip}_0(X)$ with $\|f\| = 1$ and for every $x, y \in X$ with $x \neq y$ satisfying $\frac{f(x)-f(y)}{\|x-y\|} > 1-\delta$, there is $g \in \mathrm{Lip}_0(X)$ with $\|g\| = 1$ and there are $v \in X$, $u \in S_X$ such that g attains its norm in point v at the direction u, $\|g - f\| < \varepsilon$, $\left\| \frac{x-y}{\|x-y\|} - u \right\| < \varepsilon$, and $\mathrm{dist}(v, \mathrm{conv}\{x, y\}) < \varepsilon$.

The second preliminary result of this section, which also can be of independent interest, is a relaxation of the requirements for a Banach space to have the LLipBPB.

Lemma 3.4 *Let X be a Banach space. Suppose that for every $\varepsilon > 0$ there is such a $\delta > 0$ that for every $f \in \mathrm{Lip}_0(X)$ with $\|f\| = 1$ and for every pair $(x, y) \in X \times X$, $x \neq y$ with $\frac{f(x)-f(y)}{\|x-y\|} > 1 - \delta$ there is $g \in \mathrm{Lip}_0(X)$ with $\|g\| = 1$ and a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$ for every n, such that*

$$\lim_{n\to\infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1, \tag{3.2}$$

$\|g - f\| < \varepsilon$, $\left\| \frac{x-y}{\|x-y\|} - \frac{v_n-w_n}{\|v_n-w_n\|} \right\| < \varepsilon$, $\|v_n - w_n\| < \varepsilon$, and $\mathrm{dist}(v_n, \mathrm{conv}\{x, y\}) < \varepsilon$. Then $X \in$ LLipBPB.

The proof of Lemma 3.4 also can be found in [1].

Observe that the difference between the requirements of the lemma above and the local directional Bishop-Phelps-Bollobás property is that here the convergence of the sequences $\{v_n\}$, $\{w_n\}$ and $\left\{ \frac{v_n-w_n}{\|v_n-w_n\|} \right\}$ is not required.

4 Bishop-Phelps-Bollobás theorem for uniformly convex spaces

Let us recall the concept of uniform convexity.

Definition 4.1 A Banach space X is said to be *uniformly convex*, if for every $\varepsilon > 0$ there is such a $\delta > 0$, that for every pair $x, y \in B_X$ the condition $\|x - y\| \geqslant \varepsilon$ implies $\left\| \frac{x+y}{2} \right\| \leqslant 1 - \delta$. (Equivalently, $\|(x + y)/2\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$). The best possible value of δ is denoted $\delta_X(\varepsilon)$.

The unit ball of a uniformly convex space has many small slices. Recall, that if X is a Banach space, for given $x^* \in S_{X^*}$ and $\delta > 0$ the corresponding *slice* of the unit ball is defined as $S(B_X, x^*, \delta) := \{x \in B_X \, : \, x^*(x) > 1 - \delta\}$. The following easy result states a "uniform way" to find small slices on a uniformly convex space. A proof of it can be found in [3, Lemma 2.1].

Lemma 4.2 *Let X be a uniformly convex space and $\varepsilon > 0$. Then,* $\operatorname{diam} S\big(B_X, f, \delta_X(\varepsilon)\big) < \varepsilon$ *for every $f \in S_{X^*}$ and every $\varepsilon > 0$.*

We may now state and prove the main result.

Theorem 4.3 *Every uniformly convex Banach space X has the local directional Bishop-Phelps-Bollobás property for Lipschitz functionals.*

Proof. For a fixed $\varepsilon \in (0, 1/2)$ let us chose such a $\delta \in (0, \varepsilon^2/2)$ that $\sqrt{2\delta} < \frac{1}{2}\delta_X(\varepsilon)$. Let $f \in \mathrm{Lip}_0(X)$ with $\|f\| = 1$ and $x, y \in X$, $x \neq y$, such that $\frac{f(x)-f(y)}{\|x-y\|} > 1 - \delta$. Select $\tilde{x}, \tilde{y} \in \operatorname{conv}\{x, y\}$ in such a way that $\|\tilde{x} - \tilde{y}\| < \frac{1}{4} \min\{\varepsilon, \|\tilde{x}\|, \|\tilde{y}\|\}$, the vector $\tilde{x}\tilde{y}$ looks at the same direction that xy (i.e. $\frac{\tilde{x}-\tilde{y}}{\|\tilde{x}-\tilde{y}\|} = \frac{x-y}{\|x-y\|}$) and such that still $\frac{f(\tilde{x})-f(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} > 1 - \delta$.

Define $F \in \mathrm{Lip}_0(X)$ by the formula $F(z) = \max\{\|\tilde{x} - \tilde{y}\| - \|\tilde{x} - z\|, 0\}$. Then $\|F\| = 1$ and $\frac{F(\tilde{x})-F(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} = 1$. Let us denote $x^* \in S_{X^*}$ the supporting functional at the point $\frac{\tilde{x}-\tilde{y}}{\|\tilde{x}-\tilde{y}\|}$. Then, by linearity, $\frac{x^*(\tilde{x})-x^*(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} = 1$, so we can apply the preliminary LipBPB Theorem (Lemma 3.1) with f, (x, y) and $h = \frac{1}{2}(F + x^*) \in \mathrm{Lip}_0(X)$. According to it, there exist $g \in \mathrm{Lip}_0(X)$ with $\|g\| = 1$, $\|f - g\| < \sqrt{2\delta} < \varepsilon$ and a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$, such that

$$h\left(\frac{v_n - w_n}{\|v_n - w_n\|}\right) = \frac{1}{2}\left(\frac{F(v_n) - F(w_n)}{\|v_n - w_n\|} + \frac{x^*(v_n) - x^*(w_n)}{\|v_n - w_n\|}\right) > 1 - \sqrt{2\delta} > 1 - \frac{1}{2}\delta_X(\varepsilon) \tag{4.1}$$

for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1.$$

The inequality (4.1) and the fact that $\|x^*\| = \|F\| = 1$ imply that

$$\frac{F(v_n) - F(w_n)}{\|v_n - w_n\|} > 1 - \delta_X(\varepsilon) > 1 - \frac{\varepsilon}{4}, \tag{4.2}$$

and

$$\frac{x^*(v_n) - x^*(w_n)}{\|v_n - w_n\|} > 1 - \delta_X(\varepsilon).$$

The last condition means geometrically that $\frac{v_n - w_n}{\|v_n - w_n\|} \in S(B_X, x^*, \delta_X(\varepsilon))$. Since also $\frac{x-y}{\|x-y\|} = \frac{\tilde{x}-\tilde{y}}{\|\tilde{x}-\tilde{y}\|} \in S(B_X, x^*, \delta_X(\varepsilon))$, we get from Lemma 4.2 that

$$\left\| \frac{x - y}{\|x - y\|} - \frac{v_n - w_n}{\|v_n - w_n\|} \right\| < \varepsilon$$

for every $n \in \mathbb{N}$. The condition (4.2), in its turn, implies that $v_n \in \text{supp}F$ and $\|v_n - w_n\| < \varepsilon$. Since $v_n \in \text{supp}F$, we have that $\|v_n - \tilde{x}\| < \|\tilde{x} - \tilde{y}\| < \varepsilon$, so $\text{dist}(v_n, \text{conv}\{x, y\}) < \varepsilon$.

We have verified all the conditions of Lemma 3.4. The application of that Lemma shows that X has the local directional Bishop-Phelps-Bollobás property for Lipschitz functionals. $\square$

References

[1] V. Kadets, M. Martin, M. Soloviova, *Norm-attaining Lipschitz functionals.* Banach J. Math. Anal. 10/3 (2016), 621–637.

[2] M. D. Acosta, R. M. Aron, D. García and M. Maestre, *The Bishop-Phelps-Bollobás Theorem for operators.* J. Funct. Anal. 254 (2008), 2780–2799.

[3] M. Acosta, J. Becerra-Guerrero, D. García, and M. Maestre, *The Bishop-Phelps-Bollobás theorem for bilinear forms.* Trans. Amer. Math. Soc. 365 (2013), 5911–5932.

[4] R. F. Arens and J. Eells Jr., *On embedding uniform and topological spaces.* Pacific J. Math. 6 (1956), 397–403.

[5] Y. Benyamini and J. Lindenstrauss, "Geometric Nonlinear Functional Analysis, Vol. 1". Colloquium Publications no. 48. Amer. Math. Soc., 2000.

[6] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive.* Bull. Amer. Math. Soc 67 (1961), 97–98.

[7] B. Bollobás, *An extension to the theorem of Bishop and Phelps.* Bull. London Math. Soc. 2 (1970), 181–182.

[8] M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido, and F. Rambla-Barreno, *Bishop-Phelps-Bollobás moduli of a Banach space.* J. Math. Anal. Appl. 412 (2014), 697–719.

[9] J. Diestel, "Geometry of Banach spaces". Lecture Notes in Math. 485, Springer-Verlag, Berlin, 1975.

[10] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces.* Studia Math. 159 (2003), 121–141.

[11] Ye. Ivakhno, V. Kadets and D. Werner, *The Daugavet property for spaces of Lipschitz functions.* Math. Scand. 101 (2007), 261–279 and Corrigendum to: The Daugavet property for spaces of Lipschitz functions, Math. Scand. 104 (2009), 319.

[12] V. Kadets, *Lipschitz mappings of metric spaces.* Isv. Vyssh. Uchebn. Zaved. Mat. 1 (1985), 30–34. (Russian).

[13] M. Martín, J. Merí, and R. Payá, *On the intrinsic and the spatial numerical range.* J. Math. Anal. Appl. 318 (2006), 175–189.

[14] C. Stegall, *Optimization of functions on certain subsets of Banach spaces.* Math. Ann. 236/2 (1978), 171–176.

[15] N. Weaver, "Lipscitz Algebras". World Scientific Publishing Company, 1999.

On the effective conductivity of composite materials

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Abstract. The paper concerns the asymptotic behavior of the effective thermal conductivity of a periodic composite obtained by introducing into an infinite homogeneous matrix a periodic set of inclusions which are in the form of a disk. We assume that each component is of size proportional to a positive parameter ϵ. We show the idea how to represent the effective conductivity as a convergent power series in ϵ and we compute some coefficients of such series.

1 Introduction

In this paper, we study the asymptotic behavior of the effective conductivity of a two-phase composite. A composite material is a material which is made by combining two or more materials which usually have very different properties. Most composites are made of just two materials. One is the matrix which surrounds and binds together fragments of the other material, which is called the reinforcement. Such fragments can have different forms.

One can consider various mathematical problems for composites. Such problems have a different character and mathematical models of the composite material should be build with respect to a particular physical parameter (for example, such as thermal conductivity, electrical conductivity or elastic properties). The mathematical model has to include three main components: the description of the physical fields in each of the components (that is expressed by the equations of state), the characteristics an external medium (that can be done in the form of boundary conditions at the outer boundary of the material), and the indication of the method of the connection of components (that can be done in the form of conjugation conditions on the matrix-inclusion interface).

Mostly, when we study the mathematical model, we need to determine the characteristics of the composite material as a single whole which are also called the effective characteristics of the composite material.

A common method to study the properties of composite materials is the *Homogenization method* (see, e.g., Ammari and Kang [1], Milton [11] By this method, the character-

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istics of the investigated composite material are averaged in a certain way. Based on the averaged characteristics, it is possible to obtain a description of the effective properties of the material.

One should not that, in many cases, the micro-structure of the composite is close to periodic. And so, it is possible to use the homogenization method for periodic media to study the composite. The essence of such methods is to solve the local problem for a cell of periodicity and then, using such solution, one can solve the global problem with some averaged constants.

In order to construct the geometry of our composite, we introduce some notation. We denote the periodicity cell Q by setting

$$Q \equiv]0,1[\times]0,1[.$$

By $\mathbb{B}$ we denote a ball in $\mathbb{R}^2$ of radius 1 and with the center at the origin.

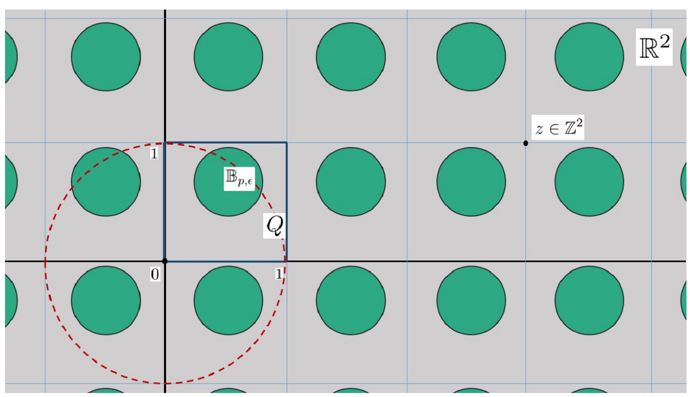

Figure 1. The model of the composite.

Let $p \in Q$ be fixed. Then there exists $\epsilon_0 \in \mathbb{R}$ such that

$$\epsilon_0 \in]0,+\infty[\,, \qquad p + \epsilon\,\mathrm{cl}\mathbb{B} \subseteq Q \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\,. \tag{1}$$

To shorten our notation, we set

$$\mathbb{B}_{p,\epsilon} \equiv p + \epsilon\mathbb{B} \quad \forall \epsilon \in \mathbb{R}\,.$$

Then we introduce the periodic domains

$$\mathbb{S}_{p,\epsilon} \equiv \bigcup_{z\in\mathbb{Z}^2} (z + \mathbb{B}_{p,\epsilon})\,, \quad \mathbb{S}^-_{p,\epsilon} \equiv \mathbb{R}^2 \setminus \mathrm{cl}\mathbb{S}_{p,\epsilon}\,,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$.

We consider a thermal conductivity of such a composite and we assume that both the matrix and the set of inclusions are filled with two different homogeneous and isotropic heat conductor materials of conductivity λ^- and λ^+, respectively. In other words, each of tho materials is a material of uniform composition that cannot be mechanically separated into different materials, which also has identical values of a conductivity in all directions, and which can conduct heat.

Now, we describe some methods of the connection of components in the composit which can be expressed in the form boundary conditions on the matrix-inclusion boundary.

Let T^+ and T^- be the temperature distributions in $S_{p,\epsilon}$ and $S^-_{p,\epsilon}$. The most popular condition is the condition of the *ideal contact*:

$$\begin{aligned} & T^+ = T^- && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \\ & \lambda^- \frac{\partial T^-}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} = \lambda^+ \frac{\partial T^+}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \end{aligned}$$

for all $\epsilon \in]0,\epsilon_0[$, where $\nu_{\mathbb{B}_{p,\epsilon}}$ - the outward unit normal to $\partial\mathbb{B}_{p,\epsilon}$. These conditions mean the continuity of the temperature and of the heat flux on the boundary of $\partial\mathbb{B}_{p,\epsilon}$.

A natural type of boundary conditions is also the following one:

$$\begin{aligned} & T^+ - T^- = g && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \\ & \lambda^- \frac{\partial T^-}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} - \lambda^+ \frac{\partial T^+}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} = f && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \end{aligned}$$

where g and f are some given functions in some appropriate spaces. In this case, the temperature distribution and the normal component of the heat flux have jumps on the interface equal given functions g and f, respectively. Instead, we consider the composite in which we assume the *non-ideal contact* between the materials. It means that the following conditions hold:

$$\begin{aligned} & \lambda^- \frac{\partial T^-}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} - \lambda^+ \frac{\partial T^+}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} = 0 && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \\ & \lambda^+ \frac{\partial T^+}{\partial \nu_{\mathbb{B}_{p,\epsilon}}} + \frac{1}{\rho(\epsilon)}\big(T^+ - T^-\big) = 0 && \text{on} \quad \partial\mathbb{B}_{p,\epsilon}\,, \end{aligned}$$

for all $\epsilon \in]0,\epsilon_0[$, where $\rho :]0,\epsilon_0[\to]0,+\infty[$. The first one is an equality of heat fluxes. The second one means that the temperature field displays a jump proportional to the normal heat flux by means of a positive parameter $\rho(\epsilon)$. Such a discontinuity in the temperature field is a well-known phenomenon. A parameter $\rho(\epsilon)$ is called Kapitza resistance or a thermal boundary resistance.

We restrict ourselves to consider only one case, namely,

$$\rho(\epsilon) = \frac{1}{r_\#}, \qquad \text{where} \quad r_\# \in]0,+\infty[.$$

In order to define an effective conductivity, we have to consider a certain transmission problem. Let $\alpha \in]0,1[$. We take three positive constants λ^+, λ^-, $r_\sharp$, and for each $n \in \{1,2\}$ we consider the following transmission problem for a pair of functions $(u_n^+, u_n^-) \in$

$C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}_{p,\epsilon}) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}^-_{p,\epsilon})$:

$$
\begin{cases}
\Delta u^+_n = 0 & \text{in } \mathbb{S}_{p,\epsilon}\,, \\
\Delta u^-_n = 0 & \text{in } \mathbb{S}^-_{p,\epsilon}\,, \\
u^+_n(x+e_m) = u^+_n(x) + \delta_{m,n} & \forall x \in \mathrm{cl}\mathbb{S}_{p,\epsilon}\,, \forall m \in \{1,2\}\,, \\
u^-_n(x+e_m) = u^-_n(x) + \delta_{m,n} & \forall x \in \mathrm{cl}\mathbb{S}^-_{p,\epsilon}\,, \forall m \in \{1,2\}\,, \\
\lambda^- \frac{\partial u^-_n}{\partial \nu_{\mathbb{B}_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u^+_n}{\partial \nu_{\mathbb{B}_{p,\epsilon}}}(x) = 0 & \forall x \in \partial\mathbb{B}_{p,\epsilon}\,, \\
\lambda^+ \frac{\partial u^+_n}{\partial \nu_{\mathbb{B}_{p,\epsilon}}}(x) + r_\sharp\big(u^+_n(x) - u^-_n(x)\big) = 0 & \forall x \in \partial\mathbb{B}_{p,\epsilon}\,, \\
\int_{\partial\mathbb{B}_{p,\epsilon}} u^+_n(x)\, d\sigma_x = 0
\end{cases}
\tag{2}
$$

for all $\epsilon \in]0,\epsilon_0[$. Here $\{e_1,e_2\}$ denotes the canonical basis of $\mathbb{R}^2$.

In problem (2), the functions u^+_j and u^-_j play the role of the temperature field in the inclusions occupying the periodic set $\mathbb{S}_{p,\epsilon}$ and in the matrix occupying $\mathbb{S}^-_{p,\epsilon}$, respectively. The parameters λ^+ and λ^- represent the thermal conductivity of the materials which fill the inclusions and the matrix, respectively, while the parameter $1/r_\sharp$ is the interfacial thermal resistivity. The fifth and the sixth condition in (2) describe the jump of the normal heat flux and of the temperature field across the two-phase interface.

If $\epsilon \in]0,\epsilon_0[$, then the solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}_{p,\epsilon}) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}^-_{p,\epsilon})$ of problem (2) is unique and we denote it by $(u^+_n[\epsilon], u^-_n[\epsilon])$ (see Dalla Riva and Musolino [5]). We introduce the effective conductivity matrix $\lambda^{\mathrm{eff}}[\epsilon]$ with (m,n)-entry $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ defined by means of the following (see, *e.g.*, Benveniste and Miloh [3], Dalla Riva and Musolino [5]).

Definition 1.1 Let $\alpha \in]0,1[$. Let $p \in Q$. Let ϵ_0 be as in (1). Let $\lambda^+, \lambda^-, r_\sharp \in]0,+\infty[$. Let $(m,n) \in \{1,2\}^2$. We set

$$
\lambda^{\mathrm{eff}}_{mn}[\epsilon] \equiv \left(\lambda^+ \int_{\mathbb{B}_{p,\epsilon}} \frac{\partial u^+_n[\epsilon](x)}{\partial x_m}\, dx + \lambda^- \int_{Q \setminus \mathrm{cl}\mathbb{B}_{p,\epsilon}} \frac{\partial u^-_n[\epsilon](x)}{\partial x_m}\, dx \right) \qquad \forall \epsilon \in]0,\epsilon_0[\,,
$$

where $(u^+_n[\epsilon], u^-_n[\epsilon])$ is the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}_{p,\epsilon}) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}^-_{p,\epsilon})$ of problem (2).

Our aim is to investigate the behavior of $\lambda^{\mathrm{eff}}[\epsilon]$ as ϵ tends to 0.

Problems of this type have long been investigated with different approaches. One of the most common approaches to study the asymptotic behavior of functionals related to the solutions of singularly perturbed boundary value problems in domains with small holes and inclusions is that of *Asymptotic Analysis*, which allows to write out asymptotic expansions for $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ (see, *e.g.*, Ammari, Kang, and Touibi [2], Ammari and Kang [1]). We also mention *Functional Equation Method*, which is very useful in order to express the effective conductivity in terms of convergent power series of the diameter of the inclusion (cf., *e.g.*, Castro, Pesetskaya, and Rogosin [4], Drygaś and Mityushev [7]).

Our analysis in the paper is based on the *Functional Analytic Approach* proposed by Lanza de Cristoforis in [8] for the investigations of singular perturbation problems in perforated domains. The main aim of such approach is to represent the solution or related functionals in terms of real analytic maps of the singular perturbation parameter. A result of this type then allows to justify representation formulas in terms of convergent power series.

2 The simple layer potential and some preliminaries

As in Dalla Riva and Musolino [5], our approach is based on periodic potential theory, which allows us to convert problem (2) into a system of integral equations. To do so, we first need to introduce some notation.

We denote by S the function from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S(x) \equiv \frac{1}{2\pi} \log |x| \qquad \forall x \in \mathbb{R}^2 \setminus \{0\},$$

then S is a fundamental solution of the Laplace operator. Then we denote by S_q a periodic analog of S

$$S_q(x) \equiv - \sum_{z \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{4\pi^2 |z|^2} e^{2\pi i z \cdot x},$$

where the series converges in the sense of distributions on $\mathbb{R}^2$ (cf., *e.g.*, Ammari and Kang [1, p. 53], Lanza de Cristoforis and Musolino [9, Thm. 3.1, Thm. 3.5]). The difference $S_q - S$ admits an analytic extension to $(\mathbb{R}^2 \setminus \mathbb{Z}^2) \cup \{0\}$ and we denote such an extension by R_q, *i.e.*, we set

$$R_q \equiv S_q - S \qquad \text{in } (\mathbb{R}^2 \setminus \mathbb{Z}^2) \cup \{0\}.$$

We now introduce the classical simple layer potential: for all $\theta \in C^{0,\alpha}(\partial\mathbb{B})$ we set

$$v[\partial\mathbb{B}, \theta](t) \equiv \int_{\partial\mathbb{B}} S(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^2.$$

As is well known, $v[\partial\mathbb{B}, \theta]$ is continuous in $\mathbb{R}^2$, the function $v^+[\partial\mathbb{B}, \theta] \equiv v[\partial\mathbb{B}, \theta]_{|\mathrm{cl}\mathbb{B}}$ belongs to $C^{1,\alpha}(\mathrm{cl}\mathbb{B})$, and the function $v^-[\partial\mathbb{B}, \theta] \equiv v[\partial\mathbb{B}, \theta]_{|\mathbb{R}^2 \setminus \mathbb{B}}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathbb{B})$. Then we set

$$w_*[\partial\mathbb{B}, \theta](t) \equiv \int_{\partial\mathbb{B}} DS(t-s)\nu_{\mathbb{B}}(t)\theta(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{B},$$

and we recall that the function $w_*[\partial\mathbb{B}, \theta]$ belongs to $C^{0,\alpha}(\partial\mathbb{B})$ and we have

$$\frac{\partial}{\partial \nu_{\mathbb{B}}} v^{\pm}[\partial\mathbb{B}, \theta] = \mp \frac{1}{2}\theta + w_*[\partial\mathbb{B}, \theta] \qquad \text{on } \partial\mathbb{B}$$

(cf., *e.g.*, Miranda [12], Lanza de Cristoforis and Rossi [10, Thm. 3.1]).

We also introduce the maps Λ and Λ_ν from $]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\mathbb{B})_0$ to $C^{1,\alpha}(\partial\mathbb{B})$ and to $C^{0,\alpha}(\partial\mathbb{B})$, respectively, defined by

$$\Lambda[\epsilon, \theta](t) \equiv \int_{\partial\mathbb{B}} R_q(\epsilon(t-s))\theta(s) d\sigma_s \qquad \forall t \in \partial\mathbb{B},$$

and by

$$\Lambda_\nu[\epsilon, \theta](t) \equiv \int_{\partial\mathbb{B}} DR_q(\epsilon(t-s))\nu_{\mathbb{B}}(t)\theta(s) d\sigma_s \qquad \forall t \in \partial\mathbb{B},$$

for all $(\epsilon,\theta)\in]-\epsilon_0,\epsilon_0[\times C^{0,\alpha}(\partial\mathbb{B})_0$. Also, for $n\in\{1,2\}$, we define the map $M_n\equiv(M_{n,1},M_{n,2})$ from $]-\epsilon_0,\epsilon_0[\times(C^{0,\alpha}(\partial\mathbb{B})_0)^2$ to $(C^{0,\alpha}(\partial\mathbb{B})_0)^2$ by setting

$$\begin{aligned}M_{n,1}[\epsilon,\theta^i,\theta^o](t)\equiv\lambda^-\Big(\frac{1}{2}\theta^o(t)+\epsilon\Lambda_\nu[\epsilon,\theta^o](t)\Big)-\lambda^+\Big(-\frac{1}{2}\theta^i(t)+\epsilon\Lambda_\nu[\epsilon,\theta^i](t)\Big)\\+(\lambda^--\lambda^+)(\nu_{\mathbb{B}}(t))_n\qquad\forall t\in\partial\mathbb{B}\,,\end{aligned}$$

$$\begin{aligned}M_{n,2}[\epsilon,\theta^i,\theta^o](t)\equiv{}&\lambda^+\left(-\frac{1}{2}\theta^i(t)+\epsilon\Lambda_\nu[\epsilon,\theta^i](t)\right)+r_\#\epsilon\Bigg(v^+[\partial\mathbb{B},\theta^i](t)+\Lambda[\epsilon,\theta^i](t)\\&-\fint_{\partial\mathbb{B}}\Big(v^+[\partial\mathbb{B},\theta^i]+\Lambda[\epsilon,\theta^i]\Big)d\sigma-v^-[\partial\mathbb{B},\theta^o](t)-\Lambda[\epsilon,\theta^o](t)\\&+\fint_{\partial\mathbb{B}}\Big(v^-[\partial\mathbb{B},\theta^o]+\Lambda[\epsilon,\theta^o]\Big)d\sigma\Bigg)+\lambda^+(\nu_{\mathbb{B}}(t))_n\qquad\forall t\in\partial\mathbb{B}\,,\end{aligned}$$

for all $(\epsilon,\theta^i,\theta^o)\in]-\epsilon_0,\epsilon_0[\times(C^{0,\alpha}(\partial\mathbb{B})_0)^2$.

3 Series expansion of the effective conductivity

As shown in Dalla Riva and Musolino [5, Prop. 6.1], we can convert problem (2) into an equivalent system of integral equations by means of the operator M_n, as the following proposition shows.

Proposition 3.1 *Let $\epsilon\in]0,\epsilon_0[$. Let $r_\#\in]0,+\infty[$. Let $n\in\{1,2\}$. Then the unique solution $(u_n^+[\epsilon],u_n^-[\epsilon])$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}_{p,\epsilon})\times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}^-_{p,\epsilon})$ of problem (2) is delivered by*

$$\begin{aligned}u_n^+[\epsilon](x)\equiv{}&\epsilon^{n-1}\int_{\partial\mathbb{B}}S_q(x-p-\epsilon s)\hat\theta^i_n[\epsilon](s)\,d\sigma_s-\epsilon^{n-1}\fint_{\partial\mathbb{B}}\int_{\partial\mathbb{B}}S_q(\epsilon(t-s))\hat\theta^i_n[\epsilon](s)\,d\sigma_s d\sigma_t\\&+x_n-p_n-\epsilon\fint_{\partial\mathbb{B}}s_n\,d\sigma_s\qquad\forall x\in\mathrm{cl}\mathbb{S}_\epsilon\,,\\u_n^-[\epsilon](x)\equiv{}&\epsilon^{n-1}\int_{\partial\mathbb{B}}S_q(x-p-\epsilon s)\hat\theta^o_n[\epsilon](s)\,d\sigma_s-\epsilon^{n-1}\fint_{\partial\mathbb{B}}\int_{\partial\mathbb{B}}S_q(\epsilon(t-s))\hat\theta^o_n[\epsilon](s)\,d\sigma_s d\sigma_t\\&+x_n-p_n-\epsilon\fint_{\partial\mathbb{B}}s_n\,d\sigma_s\qquad\forall x\in\mathrm{cl}\mathbb{S}^-_\epsilon\,,\end{aligned}$$

where $(\hat\theta^i_n[\epsilon],\hat\theta^o_n[\epsilon])$ denotes the unique solution (θ^i,θ^o) in $(C^{0,\alpha}(\partial\mathbb{B})_0)^2$ of $M_n\Big[\epsilon,\theta^i,\theta^o\Big]=0$.

By Proposition 3.1, we can write the solution of (2) in terms of integral operators with some unknown densities which satisfy a system of integral equations.

In order to investigate the asymptotic behavior of the (m,n)-entry $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ of the effective conductivity tensor as $\epsilon\to0^+$, we need to study the functions $u_n^+[\epsilon]$ and $u_n^-[\epsilon]$ for ϵ close to the degenerate value 0. Moreover, by Proposition 3.1, we know how to represent $u_n^+[\epsilon]$ and $u_n^-[\epsilon]$ in terms of the densities $\hat\theta^i_n[\epsilon]$ and $\hat\theta^o_n[\epsilon]$ and, thus, the analysis of $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ for ϵ close to 0 can be deduced by the asymptotic behavior of $\hat\theta^i_n[\epsilon]$ and $\hat\theta^o_n[\epsilon]$.

We begin with the following theorem, where we present a regularity result for $\hat{\theta}^i_n[\epsilon]$ and $\hat{\theta}^o_n[\epsilon]$ for ϵ small and positive (cf. Dalla Riva and Musolino [5, Thm. 6.3]).

Proposition 3.2 *Let $n \in \{1,2\}$. There exist $\epsilon_1 \in]0,\epsilon_0[$ and a real analytic map $(\theta^i_n[\cdot],\theta^o_n[\cdot])$ from $]-\epsilon_1,\epsilon_1[$ to $(C^{0,\alpha}(\partial\mathbb{B})_0)^2$ such that*

$$M_n\left[\epsilon,\theta^i_n[\epsilon],\theta^o_n[\epsilon]\right]=0 \qquad \forall \epsilon \in]-\epsilon_1,\epsilon_1[\,. \tag{3}$$

In particular,

$$(\theta^i_n[\epsilon],\theta^o_n[\epsilon])=(\hat{\theta}^i_n[\epsilon],\hat{\theta}^o_n[\epsilon]) \quad \forall \epsilon \in]0,\epsilon_1[\quad \textit{and} \quad (\theta^i_n[0],\theta^o_n[0])=(\tilde{\theta}^i_n,\tilde{\theta}^o_n)\,,$$

where the pair $(\hat{\theta}^i_n[\epsilon],\hat{\theta}^o_n[\epsilon])$ is defined in Proposition 3.1.

Now we observe that the real analyticity result of Proposition 3.2 implies that there exists $\epsilon_2 \in]0,\epsilon_1[$ small enough such that we can expand $\theta^i_n[\epsilon]$ and $\theta^o_n[\epsilon]$ into power series of ϵ, *i.e.*,

$$\theta^i_n[\epsilon]=\sum_{k=0}^{+\infty}\frac{\theta^i_{n,k}}{k!}\epsilon^k\,, \qquad \theta^o_n[\epsilon]=\sum_{k=0}^{+\infty}\frac{\theta^o_{n,k}}{k!}\epsilon^k\,, \tag{4}$$

for some $\{\theta^i_{n,k}\}_{k\in\mathbb{N}}$, $\{\theta^o_{n,k}\}_{k\in\mathbb{N}}$ and for all $\epsilon \in]-\epsilon_2,\epsilon_2[$. Moreover,

$$\theta^i_{n,k}=\left(\partial^k_\epsilon\theta^i_n[\epsilon]\right)_{|\epsilon=0}\,, \qquad \theta^o_{n,k}=\left(\partial^k_\epsilon\theta^o_n[\epsilon]\right)_{|\epsilon=0}\,,$$

for all $k \in \mathbb{N}$. Therefore, in order to obtain a power series expansion for $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ for ϵ close to 0, we want to exploit the expansion of $(\hat{\theta}^i_n[\epsilon],\hat{\theta}^o_n[\epsilon])$. Since the coefficients of the expansions in (4) are given by the derivatives with respect to ϵ of $\theta^i_n[\epsilon]$ and $\theta^o_n[\epsilon]$, we would like to obtain some equations identifying $\left(\partial^k_\epsilon\theta^i_n[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^k_\epsilon\theta^o_n[\epsilon]\right)_{|\epsilon=0}$. One can obtain such equations by deriving with respect to ϵ equality (3), which then leads to

$$\partial^k_\epsilon\Big(M_n\left[\epsilon,\theta^i_n[\epsilon],\theta^o_n[\epsilon]\right]\Big)=0 \qquad \forall \epsilon \in]-\epsilon_1,\epsilon_1[\,, \qquad \forall k \in \mathbb{N}\,. \tag{5}$$

Then by taking $\epsilon=0$ in (5), we will obtain integral equations identifying $\left(\partial^k_\epsilon\theta^i_n[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^k_\epsilon\theta^o_n[\epsilon]\right)_{|\epsilon=0}$.

Using the described above procedure, one can write out such systems and, moreover, they can be easily solved. In the following proposition we write out such solutions and, thus, we identify the coefficients of the power series expansions of $\theta^i_n[\epsilon]$ and of $\theta^o_n[\epsilon]$.

Proposition 3.3 *Let $n \in \{1,2\}$. Let ϵ_1, $\theta^i_n[\cdot]$, and $\theta^o_n[\cdot]$ be as in Proposition 3.2. Then there exist $\epsilon_2 \in]0,\epsilon_1[$ and a sequence $\{(\theta^i_{n,k},\theta^o_{n,k})\}_{k\in\mathbb{N}}$ in $(C^{0,\alpha}(\partial\mathbb{B})_0)^2$ such that*

$$\theta^i_n[\epsilon]=\sum_{k=0}^{+\infty}\frac{\theta^i_{n,k}}{k!}\epsilon^k \qquad \textit{and} \qquad \theta^o_n[\epsilon]=\sum_{k=0}^{+\infty}\frac{\theta^o_{n,k}}{k!}\epsilon^k \qquad \forall \epsilon \in]-\epsilon_2,\epsilon_2[\,,$$

where the two series converge uniformly for $\epsilon \in]-\epsilon_2, \epsilon_2[$ in $(C^{0,\alpha}(\partial\mathbb{B})_0)^2$. Moreover, for all $t \in \partial\mathbb{B}$,

$$
\begin{aligned}
(\theta^i_{n,0}, \theta^o_{n,0}) =&(2t_n, -2t_n),\\
(\theta^i_{n,1}, \theta^o_{n,1}) =& \left(-\frac{4r_\#}{\lambda^+}t_n, \frac{4r_\#}{\lambda^+}t_n\right),\\
(\theta^i_{n,2}, \theta^o_{n,2}) =& \left(4\left(\frac{2(r_\#)^2}{\lambda^+}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)+\pi\right)t_n, -4\left(\frac{2(r_\#)^2}{\lambda^+}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)+\pi\right)t_n\right),\\
(\theta^i_{n,3}, \theta^o_{n,3}) =& \left(-2\frac{12(r_\#)^3}{\lambda^+}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)^2 t_n, 2\frac{12r_\#}{\lambda^-}\left((r_\#)^2\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)^2-2\pi\right)t_n\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\theta^i_{n,k}(t) =& 2\Lambda^k_\nu\left[\theta^i_{n,0},\dots,\theta^i_{n,k-2}\right](t) + \frac{2kr_\#}{\lambda^+}\Bigg(v^+[\partial\mathbb{B}, \theta^i_{n,k-1}](t)\\
&- \fint_{\partial\mathbb{B}} v^+[\partial\mathbb{B}, \theta^i_{n,k-1}]d\sigma - v^-[\partial\mathbb{B}, \theta^o_{n,k-1}](t) + \fint_{\partial\mathbb{B}} v^-[\partial\mathbb{B}, \theta^o_{n,k-1}]d\sigma\Bigg)\\
&+ \frac{2kr_\#}{\lambda^+}\Bigg(\Lambda^{k-1}[\theta^i_{n,0},\dots,\theta^i_{n,k-3}](t) - \fint_{\partial\mathbb{B}}\Lambda^{k-1}[\theta^i_{n,0},\dots,\theta^i_{n,k-3}]d\sigma\\
&- \Lambda^{k-1}[\theta^o_{n,0},\dots,\theta^o_{n,k-3}](t) + \fint_{\partial\mathbb{B}}\Lambda^{k-1}[\theta^o_{n,0},\dots,\theta^o_{n,k-3}]d\sigma\Bigg) \qquad \forall k \in \mathbb{N}\setminus\{0,1,2,3\},
\end{aligned}
$$

$$
\begin{aligned}
\theta^o_{n,k}(t) =& -2\Lambda^k_\nu\left[\theta^o_{n,0},\dots,\theta^o_{n,k-2}\right](t) - \frac{2kr_\#}{\lambda^-}\Bigg(v^+[\partial\mathbb{B}, \theta^i_{n,k-1}](t)\\
&- \fint_{\partial\mathbb{B}} v^+[\partial\mathbb{B}, \theta^i_{n,k-1}]d\sigma - v^-[\partial\mathbb{B}, \theta^o_{n,k-1}](t) + \fint_{\partial\mathbb{B}} v^-[\partial\mathbb{B}, \theta^o_{n,k-1}]d\sigma\Bigg)\\
&- \frac{2kr_\#}{\lambda^-}\Bigg(\Lambda^{k-1}[\theta^i_{n,0},\dots,\theta^i_{n,k-3}](t) - \fint_{\partial\mathbb{B}}\Lambda^{k-1}[\theta^i_{n,0},\dots,\theta^i_{n,k-3}]d\sigma\\
&- \Lambda^{k-1}[\theta^o_{n,0},\dots,\theta^o_{n,k-3}](t) + \fint_{\partial\mathbb{B}}\Lambda^{k-1}[\theta^o_{n,0},\dots,\theta^o_{n,k-3}]d\sigma\Bigg) \qquad \forall k \in \mathbb{N}\setminus\{0,1,2,3\}.
\end{aligned}
$$

where

$$
\Lambda^k[\theta_0,\dots,\theta_{k-2}](t) \equiv \sum_{j=2}^{k}\binom{k}{j}\sum_{h=0}^{j}\binom{j}{h}(\partial_1^h\partial_2^{j-h}R_q)(0)\int_{\partial\mathbb{B}}(t_1-s_1)^h(t_2-s_2)^{j-h}\theta_{k-j}(s)d\sigma_s
$$
$$
\forall t \in \partial\mathbb{B}, \quad \forall(\theta_0,\dots,\theta_{k-2}) \in (C^{0,\alpha}(\partial\mathbb{B})_0)^{k-1},
$$

for all $k \in \mathbb{N}\setminus\{0,1\}$, and

$$
\Lambda^k_\nu[\theta_0,\dots,\theta_{k-2}](t) \equiv k\sum_{j=1}^{k-1}\binom{k-1}{j}\sum_{h=0}^{j}\binom{j}{h}(\partial_1^h\partial_2^{j-h}DR_q)(0)\nu_{\mathbb{B}}(t)\int_{\partial\mathbb{B}}(t_1-s_1)^h(t_2-s_2)^{j-h}\theta_{k-1-j}(s)d\sigma_s
$$
$$
\forall t \in \partial\mathbb{B}, \quad \forall(\theta_0,\dots,\theta_{k-2}) \in (C^{0,\alpha}(\partial\mathbb{B})_0)^{k-1},
$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$.

We are now ready to formulate the main result, where we expand $\lambda^{\mathrm{eff}}_{mn}[\epsilon]$ as a power series and we provide an explicit and constructive expressions for the coefficients of the series.

Theorem 3.4 *Let* $m, n \in \{1, 2\}$. *Let* ϵ_2 *and* $\{(\theta^i_{n,k}, \theta^o_{n,k})\}_{k\in\mathbb{N}}$ *be as in Proposition 3.3. Then there exists* $\epsilon_5 \in]0, \epsilon_2[$ *such that*

$$\lambda^{\mathrm{eff}}_{mn}[\epsilon] = \lambda^- \delta_{m,n} + \epsilon^2 \sum_{k=0}^{+\infty} \frac{c_{(m,n),k}}{k!} \epsilon^k$$

for all $\epsilon \in]0, \epsilon_5[$, *where*

$$\begin{aligned}
c_{(m,n),0} =&\lambda^+ \int_{\partial\mathbb{B}} v^+[\partial\mathbb{B}, \theta^i_{n,0}](t) t_m \, d\sigma_t + (\lambda^+ - \lambda^-)\delta_{m,n} - \lambda^- \int_{\partial\mathbb{B}} v^-[\partial\mathbb{B}, \theta^o_{n,0}](t) t_m \, d\sigma_t,\\
c_{(m,n),1} =&\lambda^+ \int_{\partial\mathbb{B}} v^+[\partial\mathbb{B}, \theta^i_{n,1}](t) t_m \, d\sigma_t - \lambda^- \int_{\partial\mathbb{B}} v^-[\partial\mathbb{B}, \theta^o_{n,1}](t) t_m \, d\sigma_t \,,\\
c_{(m,n),k} =&\lambda^+ \int_{\partial\mathbb{B}} \Big(v^+[\partial\mathbb{B}, \theta^i_{n,k}](t) + \Lambda^k \left[\theta^i_{n,0}, \ldots, \theta^i_{n,k-2}\right](t) \Big) t_m \, d\sigma_t\\
&- \lambda^- \int_{\partial\mathbb{B}} \Big(v^-[\partial\mathbb{B}, \theta^o_{n,k}](t) + \Lambda^k \left[\theta^o_{n,0}, \ldots, \theta^o_{n,k-2}\right](t) \Big) t_m \, d\sigma_t \,,
\end{aligned}$$

for all $k \in \mathbb{N} \setminus \{0, 1\}$.

Then, by Theorem 3.4 and Proposition 3.3, if $m, n \in \{1, 2\}$, we can write out the series expansion of the effective conductivity and we have

$$\begin{aligned}
\lambda^{\mathrm{eff}}_{mn}[\epsilon] = \Bigg(\lambda^- - 2\pi\, \lambda^- \epsilon^2 + 4\pi\, r_\# \epsilon^3 - 4\pi \left((r_\#)^2 \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right) - \frac{1}{2}\pi\lambda^- \right) \epsilon^4 \\
+4\pi r_\# \left((r_\#)^2 \left(\frac{1}{\lambda^+} + \frac{1}{\lambda^-} \right)^2 - 2\pi \right) \epsilon^5 \Bigg) \delta_{m,n} + O(\epsilon^6)
\end{aligned}$$

as $\epsilon \to 0^+$, where the first five coefficient are written in terms of the simple function of $r_\#$, λ^+, λ^-.

We note that this approach can be exploited in order to investigate not only the effective conductivity but also other physical properties of composit materials.

The paper in based on the results obtained in [6].

References

[1] Ammari, H., Kang, H., "Polarization and moment tensors". Applied Mathematical Sciences 161, Springer, New York, 2007.

[2] Ammari, H., Kang, H., Touibi, K., *Boundary layer techniques for deriving the effective properties of composite materials.* Asymptot. Anal. 41 (2005), 119–140.

[3] Benveniste, Y., Miloh, T., *The effective conductivity of composites with imperfect thermal contact at constituent interfaces.* Internat. J. Engrg. Sci. 24 (1986), 1537–1552.

[4] Castro, L. P., Pesetskaya, E., Rogosin, S. V., *Effective conductivity of a composite material with non-ideal contact conditions.* Complex Var. Elliptic Equ. 54 (2009), 1085–1100.

[5] Dalla Riva, M., Musolino, P., *A singularly perturbed nonideal transmission problem and application to the effective conductivity of a periodic composite.* SIAM J. Appl. Math. 73 (2013), 24–46.

[6] Dalla Riva, M., Musolino, P., Pukhtaievych, R, *Series expansion for the effective conductivity of a periodic dilute composite with thermal resistance at the two-phase interface.* Preprint (2018).

[7] Drygaś, P., Mityushev, V., *Effective conductivity of unidirectional cylinders with interfacial resistance.* Quart. J. Mech. Appl. Math. 62 (2009), 235–262.

[8] Lanza de Cristoforis, M., *Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces.* Comput. Methods Funct. Theory 2 (2002), 1–27.

[9] Lanza de Cristoforis, M., Musolino, P., *A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients.* Far East J. Math. Sci. (FJMS) 52 (2011), 75–120.

[10] Lanza de Cristoforis, M., Rossi, L, *Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density.* J. Integral Equations Appl. 16 (2004), 137–174.

[11] Milton, G., "The theory of composites". Cambridge University Press 8, Cambridge, 2002.

[12] Miranda, C., *Sulle proprietà di regolarità di certe trasformazioni integrali.* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) 7 (1965), 303–336.

Optimal cross-border electricity trading

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Abstract. We show that electricity flows between interconnected locations have a direct and indirect effect on electricity prices in the different locations. The direct effect refers to how prices between two locations are affected when power is flowing between these two locations only. The indirect effect refers to how the flows between two locations affect the price of power in other locations that are part of the interconnected electricity network. Based on this result we propose a model of the joint dynamics of electricity prices where flows of electricity affect, directly and indirectly, prices in all locations, and model a common co-integration factor of prices. We solve the optimal control problem of an agent who uses the interconnector to take positions in a subset of locations that are part of the interconnected network. We reduce the Hamilton-Jacobi-Equation satisfied by the value function of the investor to a system of Riccati equations, which we solve analytically, and obtain the the optimal electricity trading strategy in closed-form. We show that including cross-border effects in the trading strategy specification significantly improves the performance of the strategy, that takes advantages of price differentials in interconnected locations. For example, for contracts with delivery at 1 p.m., we show that over a time horizon of one year, the optimal strategy delivers a profit that is approximately 250,000€ more than the profit of a naive strategy, which is based on the spread between locations (i.e., does not take into account cross-border effects).

Keywords: stochastic control, optimal control, electricity interconnector, co-integration, cross-border price impact.

1 Introduction

After it is produced by an electricity generator, electricity flows through transmission networks, which are run by Transmission System Operators (TSOs). Each country can have one or more TSOs; Italy and France, for example, have one (Terna and RTE, respectively), while Germany has four (50 Hertz, Amprion, TenneT and TransnetBW). Finally, DSOs (Distribution System Operators) distribute the electricity across the various households and businesses. In Italy, for example, ENEL is the biggest DSO.

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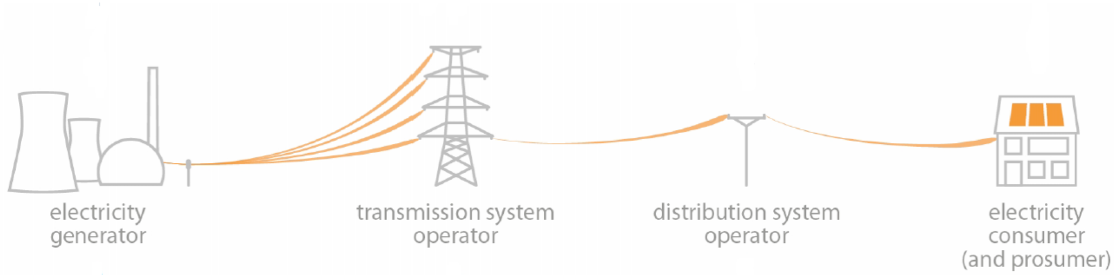

Source: *Understanding electricity markets in the EU.* European Parliament briefing (2016).

Transmission grids can be interconnected, meaning that electricity can flow across national borders into another country's grid. The physical structures enabling this connection between two different grids are called "interconnectors". The recent market coupling initiatives are aimed at integrating the European wholesale electricity markets, thus increasing security of supply while reducing price volatility across Europe, and interconnectors are at the core of the market coupling process.

Electricity is a commodity that can be traded in different markets, and the three main ones are the day-ahead market, the intra-day one and the forward one. The main difference among these three markets is that, in each one of them, one can trade contracts on electricity with a specific range of delivery times.

The forward and future market is mainly intended for hedging purposes, and it can be over-the-counter (OTC) or a centralized exchange market. In the latter case, it is a continuous trading market, where you can trade contracts where the delivery of electricity is sometime in the future: weeks, months, quarters, seasons, years.

The day-ahead market is a uniform auction system, where electricity contracts with delivery on the following day are traded. Each different hour of delivery is a different product, so that the price of electricity delivered at, for example, 4 p.m. on the following day, is different product where electricity is delivered at 5 p.m. of the following day. Moreover, electricity can also be traded in blocks, so that there are additional products, such as Block Baseload (covering hours 1 to 24), Block Peakload (covering hours 9 a.m. to 8 p.m.), Block Off-Peak (covering hours 9 p.m. to 8 a.m), among others. This market closes at midday of the day before delivery is scheduled, and demand and supply curves determine the price. In such a way, each electricity producer has incentive to bid the marginal cost of production, because if somebody else bids higher, and their aggregate supply is met by demand, that higher bid will be the final price they both will receive.

Finally, the intra-day market is a pay-as-bid system with continuous trading, where one can trade electricity contracts with delivery on the following or on the same day. In fact, each contract can be traded until 30 minutes before delivery begins, and, starting from 3 p.m. on a certain day, all hours of the following day can be traded.

These last two markets, the day-ahead and the intra-day ones, also differ in their level of interconnectedness: the day-ahead market is an integrated market, and prices are coupled, meaning that, provided that the interconnector capacity is enough, the price of electricity in the two interconnected countries will be the same. Moreover, imports and exports are implicit, in the sense that they are determined by an algorithm. The intra-day market, instead, is not as integrated, and agents can decide to go and buy electricity in another country, so that there are arbitrage opportunities.

The aim of this paper is to build on this opportunity and to develop an optimal trading strategy for an agent who uses the interconnectors to take simultaneous positions in electricity contracts in a set of locations in the intra-day market.

One of the biggest exchanges where electricity can be traded is EPEX Spot, the European power exchange for spot trading, covering Germany, France, the United Kingdom, the Netherlands, Belgium, Austria, Switzerland and Luxembourg. Thus, using data for all intraday contracts traded on EPEX, we first look for patterns in the dynamics of prices and flows of electricity across the different countries, and then choose a sub-set of locations where to base our trading strategy. Specifically, the locations we choose are Germany, Switzerland and France, because all of them are interconnected with each other, and thus the agent can buy or sell electricity in all directions in each of these three markets.

2 Cross-border effects

First, we look into how interconnecting different countries dynamically affects the flows and prices of electricity across the system, and to do so, we perform an econometric analysis using the data from EPEX. Specifically, we use tick data covering the period 01/01/2017-31/12/2017. We want to understand how the volumes of electricity traded can affect the electricity price over time, so to calibrate the agent's trading strategy accordingly.

We run the three stepwise robust OLS (ordinary least squares) regressions in (2.1), where the dependent variables are the price increments over 1 hour of trading, and the explanatory variables are the total volumes per trading hour, traded in all possible directions. The price increment is computed over 1 hour because the market is not a particularly liquid one, so that the effects of the different trades are evident over a somewhat appreciable amount of time.

$$\begin{aligned} \Delta \boldsymbol{P}_t = {} & \beta_1 \mathrm{Vol}^{FS}_{t-1} + \beta_2 \mathrm{Vol}^{SF}_{t-1} + \beta_3 \mathrm{Vol}^{GF}_{t-1} + \beta_4 \mathrm{Vol}^{FG}_{t-1} + \beta_5 \mathrm{Vol}^{GS}_{t-1} + \beta_6 \mathrm{Vol}^{SG}_{t-1} \\ & + \beta_7 \mathrm{Vol}^{OF}_{t-1} + \beta_8 \mathrm{Vol}^{FO}_{t-1} + \beta_9 \mathrm{Vol}^{OS}_{t-1} + \beta_{10} \mathrm{Vol}^{SO}_{t-1} + \beta_{11} \mathrm{Vol}^{OG}_{t-1} + \beta_{12} \mathrm{Vol}^{GO}_{t-1} + \varepsilon_t \,, \end{aligned} \tag{2.1}$$

with

$$\boldsymbol{P}_t = \begin{pmatrix} P^F_t \\ P^S_t \\ P^G_t \end{pmatrix} ,$$

where F stands for "France", S for "Switzerland", G for "Germany", and O for "other country", so that P^F_t is the French price of electricity at time t, P^S_t is the Swiss one, and P^G_t is the German one; Vol^{FS}_t represents the sum of all volumes of the transactions hour by hour, where the market area Buy is France, and the market area Sell is Switzerland, and so on. Finally, ε_t are normally distributed error terms.

In a stepwise regression, the choice of the predictive variables is carried out by an algorithm. In fact, the algorithm adds or removes terms of the multilinear model based on their statistical significance, so that the final choice of regressors has the maximum explanatory power.

We expect that, when the agent buys a certain amount of electricity in France to sell it in Switzerland, the French price will be negatively affected (in the sense that it will increase), while the Swiss one will proportionally decrease. When buying electricity, the agent is reducing the electricity supply, so that the price will increase. On the other hand, selling electricity will increase the supply, and prices will drop. The estimated coefficients of (2.1) confirm this intuition (see, for example, Table 1). We observe that when the dependent variable is ΔP_t^F, β_1 is positive, while β_2 is negative, β_3 is positive, while β_4 is negative, β_7 is negative, while β_8 is positive (when significant). If the dependent variable is instead ΔP_t^S, β_1 is negative, while β_2 is positive, β_5 is negative, while β_6 is positive, β_9 is negative, while β_{10} is positive (when significant).

Moreover, the coefficients referring to the same country pair are statistically different one from the other when taken in absolute value ($\beta_1 \neq -\beta_2$, for example).

	ΔP_t^F	ΔP_t^S	ΔP_t^G
Vol_t^{FS}	0.0032^{***}	-0.0007^{***}	0
Vol_t^{SF}	0	0	0
Vol_t^{GF}	0	0.0009^{***}	0
Vol_t^{FG}	0	-0.0007^{***}	-0.0060^{***}
Vol_t^{GS}	0	0	0.0064^{***}
Vol_t^{SG}	0	0	0
Vol_t^{OF}	0	0	-0.0013^{***}
Vol_t^{FO}	0.0022^{***}	0	-0.0038^{***}
Vol_t^{OS}	0	0	0
Vol_t^{SO}	0	0	0
Vol_t^{OG}	0	0	-0.0068^{***}
Vol_t^{GO}	0	0	0.0052^{***}

Table 1. OLS robust estimates for contracts with delivery at 3 p.m.. Dependent variables: ΔP_t^F, ΔP_t^S, ΔP_t^G. $^{***} = p < 0.01, ^{**} = p < 0.05, ^{*} = p < 0.1$.

We show that flows of electricity between two locations affect the prices of the two locations that receive/send electricity and also affect the prices of other locations which are not directly receiving electricity. That is, the price increment relative to a specific country is not only affected by the trades between that country and another, but is also affected by electricity trades happening somewhere else in the system. We label these effects "cross-border permanent impacts". When constructing our optimal trading strategy, we take into account the presence of these price impacts, because the agent's trades are going to permanently affect the price in a way proportional to the quantity traded.

The β coefficients in (2.1) give an indication of the magnitude and sign of the permanent price impacts that trading activity in each direction has on the price of electricity in each country of the interconnected network.

In our setup, we assume these impacts to be linear in the agent's speed of trading.

3 Electricity price modelling

Before defining the agent's optimal trading strategy, we analyze the dynamics of electricity prices over time.

From the mathematical point of view, electricity is a peculiar asset, which requires particular attention when one tries to model its behaviour. For example, in most European markets, electricity prices can be negative, and this may be a limitation when choosing the stochastic dynamics with which to model the electricity price: processes like the Geometric Brownian Motion are not an option in this case, as they imply positive values only. Prices can be negative because of the non-storability of electricity: sometimes it may be more convenient for a producer to pay in order to get rid of the electricity she is producing, rather than to temporarily shut down the plant and turn it on at a later time.

Moreover, when high demand brings on stream less efficient power generation sources, the electricity price exhibits spikes and jumps, and, in view of its non-storability, depending on different electricity usage throughout the year, it also displays a marked seasonal component.

Thus we start by de-seasonalizing prices. In this work, we decided to model the seasonal component $f(t)$ using sinusoidal functions with different periodicities (in this case, annual and semi-annual with different centers, see for example Lucia and Schwartz(2002), Seifert and Uhrig-Homburg(2007) and Pilipovic, D. (1998)). We employ the following seasonal function

$$f(t) = b_1 \sin(2\pi t) + b_2 \cos(2\pi t) + b_3 \sin(4\pi t) + b_4 \cos(4\pi t) + b_5 \, . \tag{3.1}$$

The seasonality parameters are calibrated using OLS (ordinary least squares), and Figure 1 reports the results of the calibration for intra-day contracts with delivery at 3 p.m..

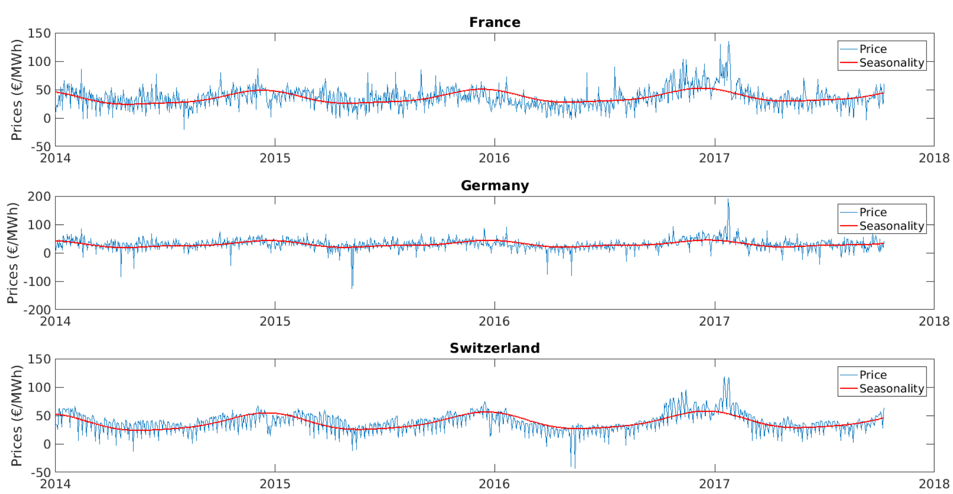

Figure 1. Electricity price for contracts with delivery at 3 p.m. for each country in the sample. The red solid line represents the calibrated seasonality function $f(t)$.

In our set up, because the three countries are interconnected and, as shown in the previous section, there is presence of cross-border effects, we model the prices as co-integrated. Co-integration was first defined in Engle and Granger (1987) as the property according to which a combination of two non-stationary processes can be stationary, and has since then found several applications, from macroeconomic analysis to fund management and portfolio selection. The core of the idea is to take advantage of the co-movement among the co-integrated stochastic variables in a dynamic specification framework.

In our model, the drift of the stochastic process we use to model electricity prices consists of an idiosyncratic component, which only affects the single country, and a systemic [?? i would say a common component] one, which is a proxy for all the common drivers of the electricity price in all countries, and which is what causes them to co-move. This common component is the co-integration factor. In this respect, our specification is similar to that of Cartea and Jaimungal (2016a), but differs from their one in the nature of the assets traded (electricity prices versus IT stocks), in the definition of the co-integration factor, in that we add a jump component to the dynamics, and most importantly, in that we consider the permanent impacts that cross-border trading has on the electricity price, as found in the previous section. Our electricity price dynamics are defined as:

$$\mathrm{d}P_t^k = \left(\theta_k + \sum_{i=1}^{n} \delta_{ki}\alpha_t^i\right)\mathrm{d}t + \sum_{i=1}^{n} \sigma_{ki}\mathrm{d}W_t^i + J(\psi_k, \xi_k)\mathrm{d}\Pi(\lambda_k)\,, \tag{3.2}$$

where $(P_t^k)_{t>0}$ is the de-seasonalized price of electricity in country k at time t (from now on, we will simply refer to the de-seasonalized electricity price as 'electricity price'), θ_k is the idiosyncratic component of the drift, δ_{ki} are country-specific constants, W_t^i are standard Brownian motions independent of each other, and σ_{ki} are the elements of the Cholesky decomposition of the instantaneous variance-covariance matrix of electricity prices. Jumps arrive as a Poisson process Π with intensity λ_k and have a normally distributed jump size with mean ψ_k and standard deviation ξ_k. Moreover,

$$\alpha_t^i = \sum_{j=1}^{n} a_{ij} P_t^j \tag{3.3}$$

is the co-integration factor for country i, where a_{ij} are constants. We can thus see that the price of electricity in each country also depends on the electricity price in the other ones.

A numerical maximization of the log-likelihood function returns the estimates for the parameters in (3.2). Figure 2 shows single simulated out-of-sample paths for the price process.

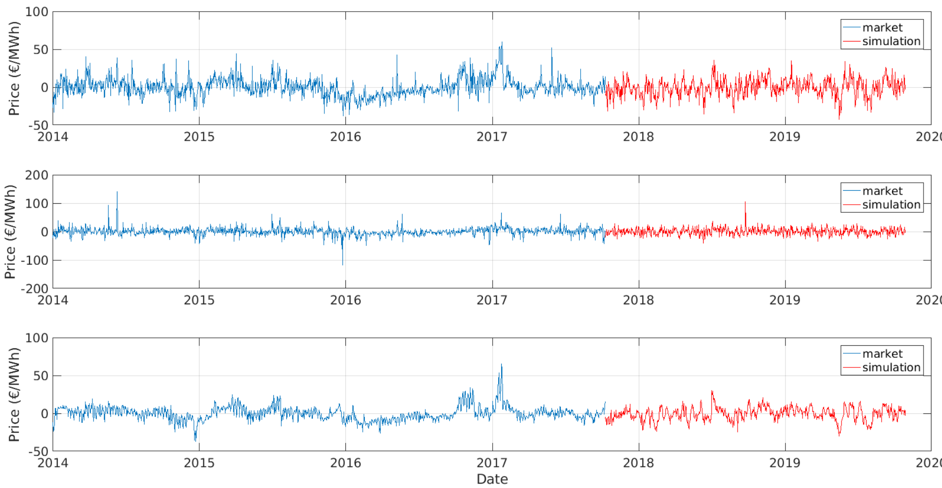

Figure 2. Historical and simulated electricity price paths for contracts with delivery at 3 p.m. for France (top panel), Germany (middle panel) and Switzerland (bottom panel). The blue solid line represents the historical price path, while the red one represents a single price simulation. Prices are expressed in €/MWh.

4 Optimal cross-border trading

Using the previous sections' results, we can finally define the agent's optimal trading strategy.

To do so, we set up a stochastic control problem, where the agent aims at maximizing her cash process $X(t, \mathbf{P}_t, \boldsymbol{\nu}_t)$ at each point in time over her trading horizon. Thus, her value function is

$$V(t, \mathbf{P}) = \sup_{\boldsymbol{\nu}\in\mathcal{A}} \mathbb{E}_{t,\mathbf{P}} \left[\int_t^T X(u, \mathbf{P}_u, \boldsymbol{\nu}_u)\,\mathrm{d}u \right], \tag{4.1}$$

where the set of admissible strategies $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{ \nu^{ij} \,:\, \mathbb{E}\left[\int_0^T (\nu_t^{ij})^2 \mathrm{d}u \right] < +\infty \right\} \qquad \forall\, i, j \neq i \in \mathbb{N}\,.$$

The cash process of course depends on the price differential between each two different countries in the sample, but also takes into account the fact that, the more we trade, the more we affect prices through the so called "cross-border effects", so that the agent's trading activity is penalized by these permanent price impacts. In our case, the vector of controls $\boldsymbol{\nu}$ is given by 6 elements, which correspond to the 6 optimal speeds of trading in all directions. Thus, we have

$$\boldsymbol{\nu}^{\intercal} = \begin{pmatrix} \nu_t^{SF} & \nu_t^{FS} & \nu_t^{GS} & \nu_t^{SG} & \nu_t^{GF} & \nu_t^{FG} \end{pmatrix}.$$

Since we are treating the three countries as distinct markets, we consider trading in opposite directions across each country couple as two separate positions. When the speed of trading ν^{ij} is positive, the agent, over a small time step Δt, is buying in country i a quantity $\nu^{ij}\Delta t$, and selling the same quantity in country j. On the contrary, when ν^{ij} is negative, the agent is buying contracts for $\nu^{ij}\Delta t$ MWh of electricity in country j to sell them in country i.

The dynamic programming principle suggests that (4.1) is the unique solution to the following Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t V(t,\mathbf{P}) + \sup_{\boldsymbol{\nu}\in\mathcal{A}} \left[\mathcal{L}^{\nu} V(t,\mathbf{P}) + X(t,\mathbf{P},\boldsymbol{\nu}_t)\right] = 0 \ , \tag{4.2}$$

where the infinitesimal generator $\mathcal{L}^{\nu}$ acts as follows

$$\begin{aligned}\mathcal{L}^{\nu} V(t,\boldsymbol{P}) &= (\boldsymbol{\theta} - \boldsymbol{\Phi}\,\mathbf{P}_t + \boldsymbol{H}\boldsymbol{\nu}_t)\, V_P(t,\boldsymbol{P}) + \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\Omega}^{\intercal}\mathcal{H}\,\boldsymbol{\Omega}\right] \\ &+ \sum_{k=1}^{n} \lambda_k \int_{-\infty}^{+\infty} \Delta_k(y) V(t,\mathbf{P}) \frac{1}{\sqrt{2\pi}\xi_k} \mathrm{e}^{\frac{-(y-\psi_k)^2}{2\xi_k^2}} \,\mathrm{d}y \ ,\end{aligned}$$

where $\mathrm{Tr}[\,\cdot\,]$ denotes the trace operator and $\mathcal{H}$ is the Hessian of V, namely a matrix with elements $\mathcal{H}_{i,j} = \frac{\partial^2 V}{\partial P_i \partial P_j}$. The operator $\Delta_k(y)V(t,\mathbf{P})$, due to the jump part, acts as follows (cf. Cartea et al. (2015) and Øksendal and Sulem(2007)):

$$\Delta_k(y)V(t,\mathbf{P}) = V(t,\mathbf{P} + y\mathbb{1}_k) - V(t,\mathbf{P}) \quad \forall\, k \in \{1,...,n\} \ ,$$

where the indicator function $\mathbb{1}_k$ is defined as

$$\mathbb{1}_1 = (1,0,\cdots,0)^{\intercal} \ , \qquad \mathbb{1}_2 = (0,1,\cdots,0)^{\intercal} \ , \ \cdots \ , \ \mathbb{1}_n = (0,0,\cdots,1)^{\intercal} \ .$$

Solving the supremum in (4.2) leads to the optimal controls in feedback form. Eq. (4.2) then becomes a partial integro-differential equation (PIDE), which admits a classical solution of the form

$$V(t,\mathbf{P}) = A(t) + \boldsymbol{D}^{\intercal}(t)\mathbf{P} + \mathbf{P}^{\intercal}\boldsymbol{E}(t)\mathbf{P} \, , \tag{4.3}$$

where $A(t)$, a scalar, and

$$D(t) = \begin{pmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \end{pmatrix} \ , \ E(t) = \begin{pmatrix} E_{11}(t) & E_{12}(t) & E_{13}(t) \\ E_{12}(t) & E_{22}(t) & E_{23}(t) \\ E_{13}(t) & E_{23}(t) & E_{33}(t) \end{pmatrix}$$

are the solution of the 10-ODEs system: specifically, E solves a matrix Riccati equation, D solves a linear differential equation and A solves an integrable equation. Thus, we get a closed-form solution to the PIDE, and an explicit formula for the optimal speed of trading in all directions. Finally, we can compute our cash process. Figures 3 and 4 showcase the results for the 6 optimal controls processes and for the cumulative cash process over a trading horizon of 1 year, respectively.

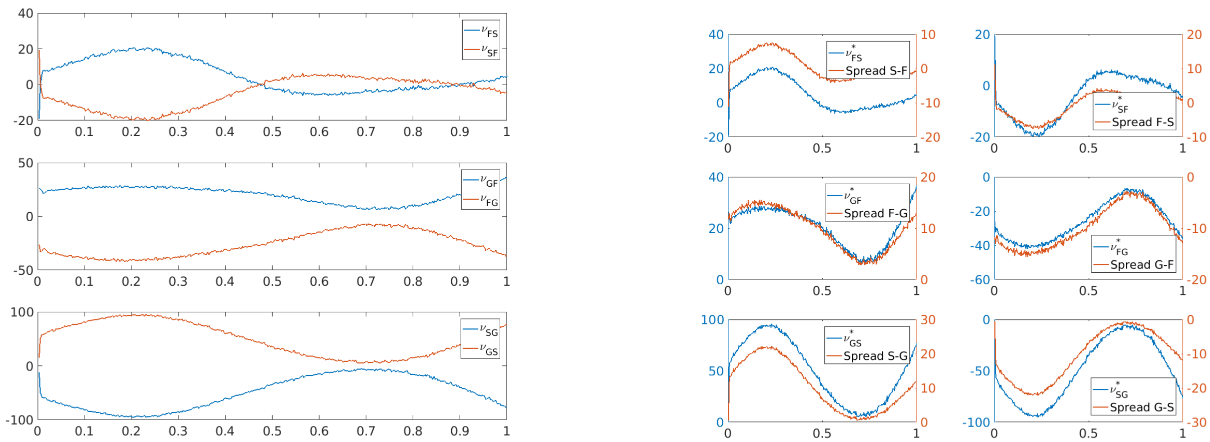

Figure 3. Optimal controls ν^* paired by country pair (left panel) and compared with the relative electricity price spread (right panel), trading in contracts with delivery at 3 p.m. over 1 year.

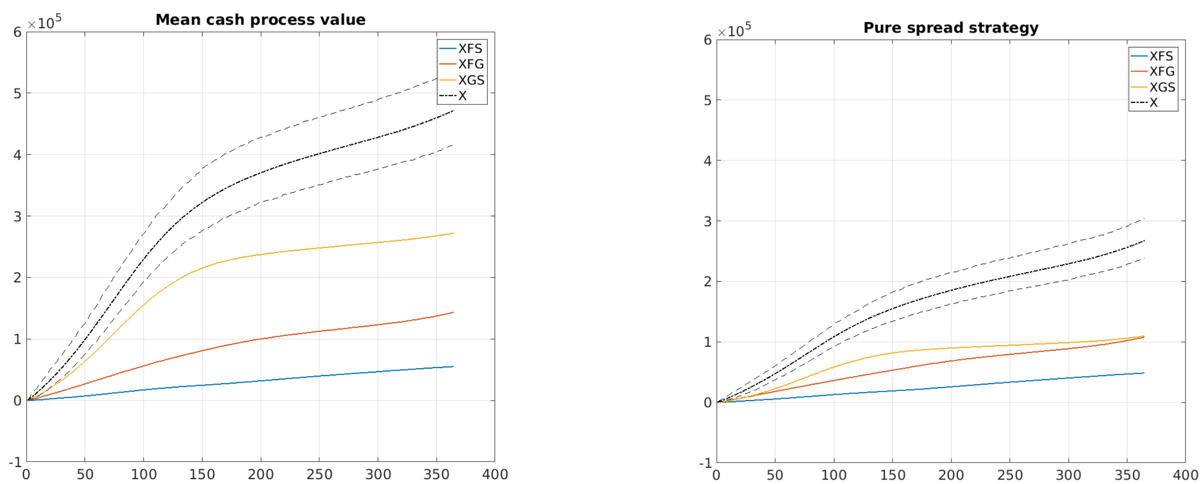

Figure 4. Comparison between the mean cash process value (expressed in €) obtained with the optimal trading strategy (left panel, black solid line) and that obtained using a strategy purely based on the electricity price spreads (right panel, black solid line), while trading in contracts with delivery at 3 p.m. The solid yellow, red and blue lines depict the profits resulting from trading in contracts between Germany-Switzerland, Germany-France and France-Switzerland, respectively. The dashed black outer bands show the confidence interval for the total mean cash process.

References

Cartea, Á., Jaimungal, S., Penalva, J., "Algorithmic and high frequency trading". (1st ed.) Cambridge: Cambridge University Press, 2015.

Cartea, Á., Jaimungal, S., *Algorithmic Trading of Co-Integrated Assets.* International Journal of Theoretical and Applied Finance, vol. 19, issue 6 (2016).

Engle, R., Granger, C. , *Co-integration and error correction: representation, estimation and testing.* Econometrica, 55 (1987), 251–276.

Lucia, J. J., Schwartz, E. S., *Electricity prices and power derivatives: Evidence from the Nordic power exchange.* Review of Derivatives Research, 5 (2002), 5-50.

Øksendal, B., Sulem, A., "Applied stochastic control of jump diffusions". 2nd ed., Springer, 2007.

Pilipovic, D., "Energy risk: valuing and managing energy derivatives". McGraw-Hill, New York, 1998.

Seifert, J., Uhrig-Homburg, M., *Modelling jumps in electricity prices: theory and empirical evidence.* Review of Derivatives Research, 10 (2007), 59–85.

Tilting approach to the theorem of Fontaine-Wintenberger

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1 Introduction

Let p be a rational prime, the Fontaine-Wintenberger theorem relates the Galois theory of certain infinite algebraic extension of $\mathbb{Q}_p$ and the Galois theory of certain characteristic p field. More precisely,

Theorem 1.1 (Fontaine-Wintenberger) *Let μ_{p^∞} be the group of all p-power roots of unity in an algebraic closure of $\mathbb{Q}_p$ and $F_p((t))$ be the field of Laurent series over F_p, then there exists a functorial degree-preserving one-one correspondence between finite extensions of $\mathbb{Q}_p(\mu_{p^\infty})$ and finite separable extensions of $F_p((t))$.*

In this note, we will explain a new approach to Thm. 1.1 using tilting and Witt vectors. First, we will show the following theorem using Galois theory of non-archimedean fields.

Theorem 1.2

(i) *Let $F_p((t))^{\mathrm{perf}} := \cup_{n\geq 1}\{x \in \overline{F_p((t))} : \ x^{p^n} \in F_p((t))\}$ be the perfection of $F_p((t))$ in some algebraic closure $\overline{F_p((t))}$ of $F_p((t))$, then composing with $F_p((t))^{\mathrm{perf}}$ induces a degree preserving one-one correspondence between finite (Galois) separable extensions of $F_p((t))$ and finite (Galois) extensions of $F_p((t))^{\mathrm{perf}}$.*

(ii) *Let $\mathbb{Q}_p^{\mathrm{cyc}}$ (resp. $F_p((t^{1/p^\infty}))$) be the p-adic (resp. t-adic) completion of $\mathbb{Q}_p(\mu_{p^\infty})$ (resp. $F_p((t))^{\mathrm{perf}}$), then p-adic (resp. t-adic) completion induces a degree preserving one-one correspondence between finite (Galois) extensions of $\mathbb{Q}_p(\mu_{p^\infty})$ (resp. $F_p((t))^{\mathrm{perf}}$) and finite (Galois) extensions of $\mathbb{Q}_p^{\mathrm{cyc}}$ (resp. $F_p((t^{1/p^\infty}))$).*

Then we will show the following result as a special case of tilting equivalence of perfectoid fields.

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Theorem 1.3 *There is a field isomorphism between $F_p((t^{1/p^\infty}))$ and the tilt $\mathbb{Q}_p^{\mathrm{cyc},\flat} := \varprojlim_{x\mapsto x^p} \mathbb{Q}_p^{\mathrm{cyc}}$ of $\mathbb{Q}_p^{\mathrm{cyc}}$. Moreover, tilting induces a degree preserving one-one correspondence between finite (Galois) extensions of $\mathbb{Q}_p^{\mathrm{cyc}}$ and finite (Galois) extensions of $F_p((t^{1/p^\infty}))$.*

Let us give a short description of contents in different sections.

In Section 2, we will introduce the notion of non-archimedean fields and briefly review their Galois theory. In particular, we will show Thm. 1.2.
In Section 3, we will introduce the notion of perfectoid fields and the basic property of tilting functor. Particularly, we will explain the isomorphism part of Thm. 1.3.
In Section 4, we will introduce an inverse functor of tilting using the theory of Witt vectors and then complete the proof of Thm. 1.3.

2 Galois theory of non-archimedean fields

Let us start with the definition of (ultra)-norms.

Definition 2.1 Let R be a ring, a function $|\,.\,|:\ R \to \mathbb{R}_{\geq 0}$ is a ***norm*** if

- For any $x \in R$ $|x| = 0$ if and only if $x = 0$;
- For any $x, y \in R$, $|xy| = |x||y|$.
- For any $x, y \in R$, $|x + y| \leq \max\{|x|, |y|\}$. (Strong triangle inequality)

If the image of $|.|$ is discrete (resp. $\{0, 1\}$), then $|.|$ is called ***discrete*** (resp. ***trivial***).

Example 3.2 Let p be a rational prime,

(i) The trivial norm on F_p which sends every non-zero element to 1.

(ii) The p-adic norm $|.|_p$ on $\mathbb{Q}$ which sends an element a/b with $a, b \in \mathbb{Z}$ to $p^{v_p(b)-v_p(a)}$. Here $v_p(a)$ is the non-negative number n such that $p^n \mid a$ and $p^{n+1} \nmid a$.

(iii) The t-adic norm $|.|_t$ on $F_p(t)$ which send a rational function f/g with $f, g \in F_p[t]$ to $p^{v_t(g)-v_t(f)}$. Here $v_t(f)$ is the non-negative number n such that $t^n \mid f$ and $t^{n+1} \nmid f$.

We have the following easy but fundamental result.

Proposition 3.3 *Let K be any field and $|.|:\ K \to \mathbb{R}_{\geq 0}$ be a norm, then the completion $\hat{K}$ of K with respect to $|.|$ is also a field and $|.|$ extends continuously to a norm, still denoted by $|.|$, on $\hat{K}$.*

Example 3.4

(i) The p-adic completion of $\mathbb{Q}$, denoted as $\mathbb{Q}_p$, is called ***the field of p-adic numbers*** and the subring $\{x \in \mathbb{Q}_p :\ |x|_p \leq 1\}$, denoted as $\mathbb{Z}_p$, is called ***the ring of p-adic integers***. Any element $a \in \mathbb{Q}_p$ has a unique expansion $a = \sum_{n\gg-\infty} a_n p^n$ such that $a_n \in \{0, 1..., p-1\}$ for all n. Moreover, $a \in \mathbb{Z}_p$ if and only if $a = \sum_{n\geq 0} a_n p^n$.

(ii) The t-adic completion of $k(t)$, denoted as $F_p((t))$, is called ***the field of Laurent series over*** F_p and the subring $\{x \in F_p((t)),\ |x|_t \leq 1\}$, denoted as $F_p[[t]]$, is called ***the ring of formal power series over*** F_p. Any element $f(t) \in F_p((t))$ has a unique expansion $f(t) = \sum_{n \gg -\infty} a_n t^n$ such that $a_n \in F_p$ for all n. Moreover, $|f(t)|_t \leq 1$ if and only if $f(t) = \sum_{n \geq 0} b_n t^n$.

The conception of non-archimedean fields arise as a natural generalization of $\mathbb{Q}_p$ and $F_p((t))$.

Definition 2.5 A non-Archimedean field is a pair $(K, |.|)$ where K is a field and $|.|$ is a non-trivial (ultra)-norm on K such that K is complete with respect to the topology induced by $|.|$. We will usually denote the sub-ring $\{x \in K :\ |x| \leq 1\}$ by $\mathcal{O}_K$.

In this note, what we mainly concern is the Galois theory of non-archimedean fields. Let us recall some notions in general Galois theory.

Definition 2.6

(i) Let K be any field and p be a rational prime, if $p = 0$ in K, then we say K is of characteristic p. If any rational prime is different from 0 in K, we say K is of characteristic 0.

(ii) Let K be any field and a field extension L/K, i.e. a field L containing K , is called finite if L is a finite dimensional K-vector space. If L/K is a finite field extension, then

- we denote $\dim_K L$ as $[L; K]$ and call it the degree of the extension L/K.
- we call L/K ***separable*** if for any $x \in L$, the minimal polynomial $f(X)$ of x over K does not have multiple roots. In this case, roots of $f(X)$ distinguished from x is called the Galois conjugates of x.
- we denote the set of all field homomorphisms from L to L whose restriction on K is identity by $\mathrm{Aut}_K(L)$. It is actually a group under composition.
- we call L/K Galois if $|\mathrm{Aut}_K(L)| = [L : K]$.

A field extension L/K is called ***algebraic*** if for any $x \in L$, there exists a finite sub-extension L'/K containing x. If L/K is algebraic, then we call L/K separable if any finite sub-extension L'/K is separable.

(iii) Let K be any field, we call K is ***algebraically closed*** (resp. ***separably closed***) if K do not have finite extension (resp. finite separable extension) other than K/K.

Example 2.7

(i) The field of rational numbers $\mathbb{Q}$ is of characteristic 0 and the finite field of order p F_P is of characteristic p.

(ii) The field of complex numbers $\mathbb{C}$ is a degree 2 Galois extension of the field of real numbers $\mathbb{R}$. Moreover, $\mathbb{C}$ is algebraically closed.

Fact 1 Let K be any field, then

- There exists a field extension L/K such that L is algebraically closed. The union of all finite (separable) sub-extensions of L/K, usually denoted as $\bar{K}$ (resp. K^{sep}) is algebraically closed (resp. separable closed). We will call $\bar{K}$ (resp. K^{sep}) the algebraic closure (resp. separable closure) of K (in L).
- If K is of characteristic 0, then any finite extension of K is separable.
- If K is of characteristic p, then any finite extension of K is separable if and only the Frobenius map $\phi: \ K \to K, \quad x \mapsto x^p$ is an isomorphism of field. In this case, we call the filed K ***perfect***.
- Galois extensions are always separable.

Now part (i) of Thm. 1.2 follows from the following general result of Galois theory. For details, see any textbook introducing Galois theory.

Proposition 2.8 *Let K be any field of characteristic p, and* $K^{\text{perf}} := \cap_{n\geq 1}\{x \in \bar{K}: \ x^{p^n} \in K\}$*, then sending an extension* L/K *to* $LK^{\text{perf}}/K^{\text{perf}}$ *induces a degree-preserving one-one correspondence between finite separable (Galois) extensions of K and finite (Galois) extensions of* K^{perf}*.*

Roughly speaking, the norm structure simplifies the Galois theory of non-archimedean fields. More precisely,

Proposition 2.9 *Let* $(K, |.|)$ *be any non-archimedean fields, then*

(i) *Let L be any finite extension of K, then there exists a unique norm on L whose restriction on K is* $|.|$*. By abuse of notations, this norm will also be denoted as* $|.|$*.*

(ii) *For any* $\alpha \in K^{\text{sep}}$ *with Galois conjugates* $\{\alpha_i\}_{i=2,\ldots,n}$*, if for* $\beta \in K^{\text{sep}}$*, we have*

$$|\beta - \alpha| < \min\{|a - \alpha_i| : \ i = 2..., n\},$$

then $K(\alpha) \subset K(\beta)$ [Krasner's Lemma]

(iii) *Let* $f(X) = \sum_{i=0}^{n} a_i X^i \in K[X]$ *be a separable polynomial of degree n, then for every* ϵ*, there exists a* $\delta > 0$ *such that for any* $g(X) = \sum_{i=0}^{n} b_i X^i \in K[X]$ *of degree n satisfying* $\max\{|b_i - a_i| : \ i = 1, ..., n\} < \delta$*, then for any root* α_i *of* $f(X)$*, there exists precisely one root* β_i *of* $g(X)$ *such that* $|\beta_i - \alpha_i| < \epsilon$*.*

Proof. (i) See [2, Thm.11]. (ii) can be found in p.69 op.cit. and (iii) can be found in p.70 op.cit. □

Particularly, part (ii) of Thm. 1.2 is a special case of the following result, which can be shown by Prop. 2.9.

Proposition 2.10 *Let* $(K, |.|)$ *be a non-Archimedean field, and L be any algebraic extension of K, then sending an extension* F/L *to its completion* $\hat{F}/\hat{L}$ *induces a degree* *preserving one-one correspondence between finite separable (Galois) extension of L and finite separable (Galois) extension of $\hat{L}$. In particular, if L is of characteristic 0 or L is perfect, then L is algebraically closed if and only if $\hat{L}$ is.*

3 Tilting

We start directly with definition of perfectoid fields and the tilting operator.

Definition 3.1 A non-archimedean field $(K, |.|)$ is called ***perfectoid*** if $|.|$ is not discretely valued, $|p| < 1$ and the Frobenius map

$$\mathcal{O}_K/p \to \mathcal{O}_K/p,\ x \mapsto x^p$$

is surjective.

Definition 3.2 For any ring R, we define the ***tilt*** $R^\flat$ of R to be the projective limit $\varprojlim_{x\mapsto x^p} R$. The multiplicative map from $R^\flat$ to R, projecting an element to its zeroth coordinate, is denoted by

$$\sharp:\ R^\flat \to R; \quad x \mapsto x^\sharp.$$

The tilt of non-archimedean field is actually a perfect non-archimedean field of characteristic p. The following proposition is straightforward.

Proposition 3.3 *Let $(K, |.|)$ be any non-archimedean field with $|p| < 1$, then*

(i) *The tilt $K^\flat$ is a field with the obvious multiplicative law and addition law*

$$x = (x_n)_{\geq 0}, y = (y_n)_{n\geq 0} \in K^\flat, x + y = (\lim_{m\to\infty} (x_{n+m} + y_{n+m})^{p^m})_{n\geq 0}.$$

(ii) *The pair $(K^\flat, |.|_\flat)$ is a non-archimedean field with*

$$|.|_\flat:\ K^\flat \xrightarrow{(x_n)_{\geq 0}\mapsto x_0} K \xrightarrow{|.|} \mathbb{R}_{\geq 0}.$$

(iii) *For any $\varpi \in \mathcal{O}_K$ such that $|p| \leq |\varpi| < 1$, the natural map*

$$\mathcal{O}_K^\flat \to \varprojlim_{x\mapsto x^p} \mathcal{O}_K/\varpi$$

is an isomorphism of rings. As a consequence, $\mathcal{O}_K^\flat$ is a complete perfect F_p-algebra.

In general, $K^\flat$ can be very small. For example, we have $\mathbb{Q}_p^\flat = F_p$. However, when K is perfectoid, then $K^\flat$ is big enough.

Proposition 3.4 *Let $(K, |.|)$ be a perfectoid field, then $|K^\times| = |K^{\flat,\times}|_\flat$.*

Now we can show the isomorphism part of Thm. 1.3.

Proposition 3.5 *The non-archimedean field $(\mathbb{Q}_p^{\mathrm{cyc}}, |.|_p)$ is perfectoid with tilting $F_p((t^{1/p^\infty}))$.*

Proof. Fix $\{\xi_n\}_{n\geq 1}$ a sequence of roots of unity such that $\xi_{n+1}^p = \xi_n, \xi_1^p = 1$, and $\xi_1 \neq 1$. Since the minimal polynomial of ξ_n is $\frac{X^{p^n}-1}{X^{p^{n-1}}-1}$, we have the norm of $\xi_n - 1$ is $|p|^{\frac{1}{(p-1)p^{n-1}}}$. So $\mathbb{Q}_p^{\mathrm{cyc}}$ is not discretely valued. Since $\xi_{n+1}^p = \xi_n$ for $n \geq 1$, and

$$\mathbb{Z}_p^{\mathrm{cyc}}/p = \mathbb{Z}_p[\xi_1, ..., \xi_n, ...]/p,$$

the Frobenius map is surjective. Moreover, we have

$$\begin{aligned} \mathbb{Z}_p^{\mathrm{cyc}}/(\xi_1 - 1) &= \mathbb{Z}_p[\xi_1 - 1, ..., \xi_n - 1, ...]/(\xi_1 - 1) \\ &\cong F_p[X_1, X_2, ..., X_n, ...]/(X_1, X_2^p - X_1, X_2 - X_3^p, X_3 - X_4^p, ...) \end{aligned}$$

by sending $\xi_n - 1$ to X_n. Identify $X_1 = t$, then we have

$$F_p[X_1, X_2, ..., X_n, ...]/(X_1, X_2^p - X_1, X_2 - X_3^p, X_3 - X_4^p, ...) = F_p[[t^{1/p^\infty}]]/t,$$

which implies that $\mathbb{Z}_p^{\mathrm{cyc},\flat} \cong F_p[[t^{1/p^\infty}]]$. □

The one-one correspondence part of Thm. 1.3 is a special case of the ***tilting equivalence*** of perfectoid fields.

Theorem 3.6 *Let $(K, |.|)$ be a perfectoid field with tilt $(K^\flat, |.|_\flat)$, then sending F to $F^\flat$ induces a degree-preserving one-one correspondence between finite (Galois) extensions of K and finite (Galois) extensions of $K^\flat$.*

4 Witt Vectors and an inverse functor of tilting

We start with the theory of Witt vectors, which allows us to construct characteristic 0 objects from characteristic p objects.

Fact 2

(i) The reduction modulo p map $\mathbb{Z}_p \to F_p$ has a multiplicative section $[.]$, usually called the Teichmüller lifting. More precisely $[0] = 0$ and for $a \in F_p^\times$, $[a]$ is the unique $(p-1)$-th root of unity which has residue a modulo p.

(ii) Any element $a \in \mathbb{Z}_p$ can be expressed uniquely as the convergent series $\sum_{n\geq 0}[a_n]p^n$ with $a_n \in F_p$.

Now we can equip $F_p^{\mathbb{N}}$ a new ring structure by pulling back the ring structure on $\mathbb{Z}_p$ via the bijection

$$F_p^{\mathbb{N}} \to \mathbb{Z}_p; (a_0, a_1, ..., a_n, ...) \mapsto \sum_{n\geq 0}[a_n^{p^{-n}}]p^n.$$

Denote this new sum and product by $+_W$ and $\times_W$.

Fact 3 For each $n \geq 0$, we have polynomials $S_n, P_n \in \mathbb{Z}_p[X_0, ..., X_n, Y_0, ..., Y_n]$ such that for any $a = (a_0, a_1, ...), b = (b_0, b_1, ...) \in F_p^{\mathbb{N}}$, we have

$$a +_W b = (S_n(a_0, a_1, ..., a_n, b_0, b_1, ..., b_n))_{n\geq 0}; \quad a \times_W b = (P_n(a_0, a_1, ..., a_n, b_0, b_1, ..., b_n))_{n\geq 0}.$$

Proposition 4.1 *Let R be any perfect F_p-algebra, then we can equip $R^{\mathbb{N}}$ with a ring structure whose unit element is $(1, 0, 0, 0, ...)$, zero element is $(0, 0, ...)$ and for any two elements $a = (a_0, a_1, ...), b = (b_0, b_1, ...) \in R^{\mathbb{N}}$, the summation (resp. production) of a and b is $(S_n(a_0, a_1, ..., a_n, b_0, b_1, ..., b_n))_{n\geq 0}$ (resp. $(P_n(a_0, a_1, ..., a_n, b_0, b_1, ..., b_n))_{n\geq 0}$). We will use $W(R)$ to denote $R^{\mathbb{N}}$ equipped with this ring structure. We moreover have that the map*

$$[.] : R \to W(R), a \mapsto (a, 0, 0.....)$$

is multiplicative and every element $a = (a_0, a_1, ..., a_n, ...) \in W(R)$ has a unique expansion $\sum_{n\geq 0}[a_n^{p^{-n}}]p^n$.

Now let $(K, |.|)$ be a perfectoid field with tilt $(K^\flat, |.|_\flat)$, recall $\mathcal{O}_K^\flat$ is a perfect F_p-algebra, we can recover $\mathcal{O}_K$ from $\mathcal{O}_K^\flat$ by

Theorem 4.2 (Fontaine) *The map*

$$\theta_K : \; W(\mathcal{O}_K^\flat) \to \mathcal{O}_K, \quad \sum_{n\geq 0}[a_n]p^n \mapsto \sum_{n\geq 0} a_n^\sharp p^n$$

is actually a surjective ring homomorphism. Moreover, the kernel of θ_K is principal with a generator of the form $[a] + p\alpha$ where $|a|_\flat < 1$ and α is an invertible element in $W(\mathcal{O}_K^\flat)$. If K is of characteristic 0, we can even assume $|a|_\flat = |p|_p$.

From now on, we will fix a perfectoid field $(K, |.|)$ of characteristic 0 with tilt $(K^\sharp, |.|_\sharp)$, and let $z = [a] + p\alpha$ be a generator of $\mathrm{Ker}(\theta_K)$ with $|\varpi| = |p|$ and $\alpha \in W(\mathcal{O}_K^\flat)^\times$.

Definition 4.3 For any perfect non-archimedean field $L/K^\flat$, we define $L^\sharp := W(\mathcal{O}_L)[1/p]/(z)$.

Theorem 4.4 *Let $(L, |.|_L)$ be any perfect non-archimedean field over $F^\flat$, then:*

(i) *The ring $L^\sharp$ is indeed a field over F.*

(ii) *We can define a norm $|.| : L^\sharp \to \mathbb{R}_{\geq 0}$ which makes $(L^\sharp, |.|)$ a perfectoid field over K. Moreover, under this norm, $\mathcal{O}_{L^\sharp} = W(\mathcal{O}_L)/(z)$.*

(iii) *The tilt of $L^\sharp$ is just L.*

Proof. See [1, Thm.1.4.13] □

A standard argument using Galois theory shows the following degree preserving result. For details, we refer to [1, Lem.1.5.3].

Proposition 4.5 *If L is a degree n (Galois) extension of $K^\flat$, then $L^\sharp$ is a degree n (Galois) extension of K.*

To show the functor $L \mapsto L^\sharp$ is one to one, we need the following lemma, which is [1, Lem.1.5.4].

Lemma 4.6 *If L is algebraically closed, then so is $L^\sharp$.*

Now we can complete the proof of Theorem 3.6.

Proof. Let M be the completion of any algebraic closure $\overline{K^\flat}$ of $K^\flat$, then $M^\sharp$ is an algebraically closed perfectoid field over F with tilt M. Note $\cup_L L^\sharp$ with L running over all finite sub-extension of $M/K^\flat$ is dense in $M^\sharp$, hence is also algebraically closed. Hence for any finite extension F/K, there must exist a finite extension $F^\flat/K^\flat$ such that $(F^\flat)^\sharp = F$. We are done. $\square$

References

[1] Kiran S. Kedlaya, *ew methods for (φ, Γ)-modules.* Research in the Mathematical Sciences 2/1 (2015), art. 20.

[2] Neal Koblitz, "p-adic Numbers, p-adic Analysis, and Zeta Functions". Volume 58. Springer Science & Business Media, 2012.

Kernel-based methods: a general overview

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Abstract. In these notes, we give a brief introduction to the general theory of Radial Basis Function (RBF) interpolation. The material is mainly taken from [2, 4], where we follow the exposition line of the books by M. Buhmann (2003), G.E. Fasshauer (2007) and H. Wendland (2005) [1, 3, 5]. Such works provide a recent and extensive treatment about the theory of RBF-based meshless approximation methods. Thus, following their guidelines, we review the main theoretical features concerning positive definite functions and we provide error bounds for the kernel-based interpolants.

1 Introduction

Given a set of data, i.e. measurements and locations at which these measurements were obtained, the aim is to find a function that matches the given measurements at their corresponding locations. Moreover, we focus on non-uniform data sites and this leads to the process of *scattered data interpolation*. More formally, the approximation problem we consider is the following.

Problem 1.1 *Given $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \dots, N\} \subseteq \Omega$ a set of distinct data points (or data sites or nodes), arbitrarily distributed on a domain $\Omega \subseteq \mathbb{R}^M$, with an associated set $\mathcal{F}_N = \{f_i = f(\boldsymbol{x}_i), i = 1, \dots, N\}$ of data values (or measurements or function values), which are obtained by sampling some (unknown) function $f : \Omega \longrightarrow \mathbb{R}$ at the nodes $\boldsymbol{x}_i$, the scattered data interpolation problem consists in finding a function $R : \Omega \longrightarrow \mathbb{R}$ such that*

$$R(\boldsymbol{x}_i) = f_i, \quad i = 1, \dots, N. \tag{1}$$

Usually, the interpolant R is expressed as a linear combination of some *basis functions* B_i,

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i.e.

$$R(\boldsymbol{x}) = \sum_{k=1}^{N} c_k B_k(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{2}$$

Since the interpolant is a linear combination of the basis functions, the scattered data interpolation problem always reduces to solve a linear system of the form $A\boldsymbol{c} = \boldsymbol{f}$, where the entries of A are given by $(A)_{ik} = B_k(\boldsymbol{x}_i)$, $i, k = 1, \ldots, N$, $\boldsymbol{c} = (c_1, \ldots, c_N)^T$ and $\boldsymbol{f} = (f_1, \ldots, f_N)^T$.

It is well-known that the system has a unique solution whenever the matrix A is non-singular. Thus, we present in a general framework the main theoretical results devoted to establish conditions under which Problem 1.1 is *well-posed*. Then, we introduce a particular class of basis functions, namely RBFs, and we provide error bounds.

The notes are organized as follows. In Section 2, we present the interpolation problem, which is solved via RBFs in Section 3. We then report error bounds for the RBF approximant in Section 4. The last section deals with conclusions.

2 The scattered data interpolation problem

In order to choose the basis functions for which Problem 1.1 is well-posed, i.e. a solution to such problem exists and is unique, we have to introduce the so-called *Haar systems*.

Definition 2.1 The finite-dimensional linear space $\mathcal{B} \subseteq C(\Omega)$, with basis $\{B_k\}_{k=1}^N$, is a Haar space on Ω if

$$\det A \neq 0,$$

for any set of distinct data points $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$. The set $\{B_k\}_{k=1}^N$ is called a Haar system.

For example, this is the case of the space of the univariate polynomials of degree $N-1$ which form a N-dimensional Haar space. However, in the multivariate case one can no longer ensure this result if one chooses the basis independent of the data sites. This is a consequence of the following theorem.

Theorem 2.1 ***(Haar-Mairhuber-Curtis)*** *Suppose that $\Omega \subseteq \mathbb{R}^M$, $M \geq 2$, contains an interior point. Then there exist no Haar spaces of continuous functions except for trivial ones, i.e. spaces spanned by a single function.*

From Theorem 2.1, we can understand that, if we want a well-posed multivariate scattered data interpolation problem, we should consider data-dependent approximation spaces. For this scope, we need to introduce positive definite matrices and functions.

Definition 2.2 A real symmetric matrix A is called positive semi-definite if the associated quadratic form is non-negative, i.e.

$$\sum_{i=1}^{N} \sum_{k=1}^{N} c_i c_k (A)_{ik} \geq 0, \tag{3}$$

for $\boldsymbol{c} = (c_1, \dots, c_N)^T \in \mathbb{R}^N$. If the quadratic form (3) is zero only for $\boldsymbol{c} \equiv 0$, then A is called positive definite.

In particular, we remark that if A is a positive definite matrix, then all its eigenvalues are positive and therefore A is non-singular. Thus, since we always require well-posed interpolation problems, we consider shifted basis functions. Specifically, we focus on functions B_k which are the shifts of a certain function Φ centred at $\boldsymbol{x}_k$, i.e. $B_k(\cdot) = \Phi(\cdot - \boldsymbol{x}_k)$. Indeed, we have the following theorem.

Theorem 2.2 *A real-valued continuous function Φ is positive definite on $\mathbb{R}^M$ if and only if it is even and*

$$\sum_{i=1}^{N} \sum_{k=1}^{N} c_i c_k \Phi(\boldsymbol{x}_i - \boldsymbol{x}_k) \geq 0, \tag{4}$$

for any N distinct data points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N \in \mathbb{R}^M$ and $\boldsymbol{c} = (c_1, \dots, c_N)^T \in \mathbb{R}^N$. The function Φ is called strictly positive definite on $\mathbb{R}^M$ if the quadratic form (4) is zero only for $\boldsymbol{c} \equiv \mathbf{0}$.

In what follows, we focus on RBFs as basis functions.

3 The scattered data interpolation problem via RBFs

Many interesting strictly positive definite functions belong to the class of radial functions. For this reason in (2) we focus on RBFs as basis functions.

Definition 3.1 A function $\Phi : \mathbb{R}^M \longrightarrow \mathbb{R}$ is called radial if there exists a univariate function $\phi : [0, \infty) \longrightarrow \mathbb{R}$ such that

$$\Phi(\boldsymbol{x}) = \phi(r), \quad \text{where} \quad r = \|\boldsymbol{x}\|,$$

and $\|\cdot\|$ is some norm on $\mathbb{R}^M$.

For several examples of radial basis functions and their regularities we refer the reader to Table 1 and Figure 1. Usually, since C^∞ functions might lead to instability, RBFs with finite regularities are strongly recommended in applications. Moreover, observe that the Buhmann's functions are independent of the shape parameter and, as the Wendland's ones, belong to the class of compactly supported RBFs. Indeed, the Buhmann's kernels listed below are defined for $0 \leq r \leq 1$.

Even if we characterize a (strictly) positive definite function in terms of a multivariate function Φ, when we deal with a radial function, i.e. $\Phi(\boldsymbol{x}) = \phi(\|\boldsymbol{x}\|)$, it is convenient to also refer to the univariate function ϕ as a positive definite radial function.

RBF	$\phi(r)$
Gaussian C^∞	$e^{-\varepsilon^2 r^2}$
Inverse Multiquadric C^∞	$(1+r^2/\varepsilon^2)^{-1/2}$
Generalized Multiquadric C^∞	$(1+r^2/\varepsilon^2)^{3/2}$
Matérn C^2	$e^{-\varepsilon r}(1+\varepsilon r),$
Matérn C^6	$e^{-\varepsilon r}(15+15\varepsilon r+6\,(\varepsilon r)^2+(\varepsilon r)^3)$
Wendland C^2	$(1-\varepsilon r)_+^4\,(4\varepsilon r+1)$
Wendland C^6	$(1-\varepsilon r)_+^8\,\big(32(\varepsilon r)^3+25(\varepsilon r)^2+8\varepsilon r+1\big)$
Buhmann C^2	$2r^4\log r-7/2r^4+16/3r^3-2r^2+1/6$
Buhmann C^3	$112/45r^{9/2}+16/3r^{7/2}-7r^4-14/15r^2+1/9$

Table 1. Examples of conditionally positive definite radial kernels depending on the shape parameter ε. The truncated power function is denoted by $(\cdot)_+$.

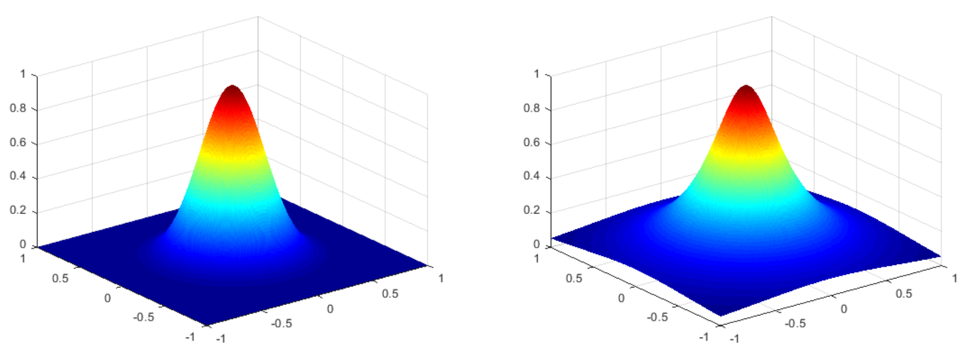

Figure 1. Left: Gaussian with $\varepsilon = 3$. Right: Inverse Multiquadric with $\varepsilon = 3$.

We now give a natural generalization of the definition of positive definite functions for RBFs.

Definition 3.2 A real-valued continuous even function Φ is called conditionally positive definite of order L on $\mathbb{R}^M$ if

$$\sum_{i=1}^{N}\sum_{k=1}^{N} c_i c_k \Phi\left(\boldsymbol{x}_i-\boldsymbol{x}_k\right) \geq 0, \tag{5}$$

for any N distinct data points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N \in \mathbb{R}^M$ and $\boldsymbol{c}=(c_1,\ldots,c_N)^T \in \mathbb{R}^N$ satisfying

$$\sum_{i=1}^{N} c_i p\left(\boldsymbol{x}_i\right) = 0,$$

for any real-valued polynomial p of degree at most $L-1$. The function Φ is called strictly conditionally positive definite of order L on $\mathbb{R}^M$ if the quadratic form (5) is zero only for $\boldsymbol{c} \equiv \boldsymbol{0}$.

Thus, in case of RBFs we need to modify (2) by adding a lower-degree M-variate polynomial term. Moreover, even if Definition 3.1 holds for a generic norm, in the sequel we consider the Euclidean norm $\|\cdot\|_2$ and thus a RBF interpolant is defined as follows.

Definition 3.3 Given $\mathcal{X}_N$ and $\mathcal{F}_N$, a RBF interpolant $R : \Omega \longrightarrow \mathbb{R}$ assumes the form

$$R(\boldsymbol{x}) = \sum_{k=1}^{N} c_k \phi\left(\|\boldsymbol{x} - \boldsymbol{x}_k\|_2\right) + \sum_{k'=1}^{l} c^{'}_{k'} p_{k'}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{6}$$

where $p_1, \dots, p_l$, are a basis for the l-dimensional linear space Π^M_{L-1} of polynomials of total degree less than or equal to $L-1$ in M variables, where

$$l = \binom{L-1+M}{L-1}.$$

Since the conditions (1) must be satisfied, solving the interpolation problem (6) leads to a linear system of the form

$$\underbrace{\begin{pmatrix} A & P \\ P^T & O \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^{'} \end{pmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix}}_{\boldsymbol{b}}, \tag{7}$$

where the entries of the interpolation matrix $\mathcal{A}$ are

$$(A)_{ik} = \phi\left(\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2\right), \quad i, k = 1, \dots, N,$$

$$(P)_{ik'} = p_{k'}(\boldsymbol{x}_i), \quad i = 1, \dots, N, \quad k^{'} = 1, \dots, l.$$

Moreover, $\boldsymbol{c} = (c_1, \dots, c_N)^T$, $\boldsymbol{c}^{'} = (c^{'}_1, \dots, c^{'}_l)^T$, $\boldsymbol{f} = (f_1, \dots, f_N)^T$, $\boldsymbol{0}$ is a zero vector of length l and O is a $l \times l$ zero matrix.

In order to establish conditions under which the interpolation problem is well-posed, we give the following definition.

Definition 3.4 A set $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \dots, N\} \subseteq \Omega$ of data points is called a $(L-1)$-unisolvent set if the only polynomial of total degree at most $L-1$ interpolating zero data on $\mathcal{X}_N$ is the zero polynomial. Essentially, if the data come from a polynomial of total degree less than or equal to $L-1$, then they are exactly fitted with (6).

The following theorem on polynomial reproduction assesses conditions under which the interpolation problem admits a unique solution [3].

Theorem 3.1 *If the function ϕ in (6) is strictly conditionally positive definite of order L on $\mathbb{R}^M$ and the set $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \dots, N\} \subseteq \Omega$ of data points forms a $(L-1)$-unisolvent set, then the system of linear equations (7) admits a unique solution.*

If $L = 0$ we have strictly conditionally positive definite functions of order zero, i.e. strictly positive definite functions. As a consequence, since the interpolation matrix is non-singular, the notation simplifies. Therefore, in this case, given $\mathcal{X}_N$ and $\mathcal{F}_N$, a RBF interpolant $R : \Omega \longrightarrow \mathbb{R}$ assumes the form

$$R(\boldsymbol{x}) = \sum_{k=1}^{N} c_k \phi\left(\|\boldsymbol{x} - \boldsymbol{x}_k\|_2\right), \quad \boldsymbol{x} \in \Omega. \tag{8}$$

Moreover, the entries of the interpolation matrix associated to the linear system

$$A\boldsymbol{c} = \boldsymbol{f}, \tag{9}$$

are given by

$$(A)_{ik} = \phi\left(\|\boldsymbol{x}_i - \boldsymbol{x}_k\|_2\right), \quad i,k = 1,\ldots,N.$$

Focusing on strictly positive definite functions, our aim is to give error bounds for the RBF interpolant.

4 Error bounds for RBF interpolants

We study here the more general situation where $\Phi : \mathbb{R}^M \times \mathbb{R}^M \longrightarrow \mathbb{R}$ is a strictly positive definite kernel, i.e. the entries of A are given by

$$(A)_{ik} = \Phi\left(\boldsymbol{x}_i, \boldsymbol{x}_k\right), \quad i,k = 1,\ldots,N. \tag{10}$$

The uniqueness result holds also in this general case.

For each positive definite and symmetric kernel Φ it is possible to define an associated real Hilbert space, the so-called *native space* $\mathcal{N}_\Phi(\Omega)$.

Definition 4.1 Let $\mathcal{H}$ be a real Hilbert space of functions $f : \Omega \longrightarrow \mathbb{R}$, with inner product $(\cdot,\cdot)_{\mathcal{H}}$. A function $\Phi : \Omega \times \Omega \longrightarrow \mathbb{R}$ is called a reproducing kernel for $\mathcal{H}$ if:

i. $\Phi(\cdot, \boldsymbol{x}) \in \mathcal{H}$, for all $\boldsymbol{x} \in \Omega$,

ii. $f(\boldsymbol{x}) = (f, \Phi(\cdot, \boldsymbol{x}))_{\mathcal{H}}$, for all $f \in \mathcal{H}$ and for all $\boldsymbol{x} \in \Omega$ (reproducing property).

Reproducing kernels are a classical concept in analysis. It is well-known that the reproducing kernel of a Hilbert space is unique and that existence is equivalent to the fact that the point evaluation functionals $\delta_{\boldsymbol{x}}$ are bounded linear functionals on Ω, i.e. there exists a positive constant $M_{\boldsymbol{x}}$ such that

$$|\delta_{\boldsymbol{x}} f| = |f(\boldsymbol{x})| \le M_{\boldsymbol{x}} ||f||_{\mathcal{H}},$$

for all $f \in \mathcal{H}$ and $\boldsymbol{x} \in \Omega$.

Additional properties are shown in the following theorem.

Theorem 4.1 *If $\mathcal{H}$ is a Hilbert space of functions $f : \Omega \longrightarrow \mathbb{R}$, with reproducing kernel Φ, then:*

i. $\Phi(\boldsymbol{x},\boldsymbol{y}) = (\Phi(\cdot,\boldsymbol{y}),\Phi(\cdot,\boldsymbol{x}))_{\mathcal{H}}$, *for* $\boldsymbol{x},\boldsymbol{y}\in\Omega$,

ii. $\Phi(\boldsymbol{x},\boldsymbol{y}) = \Phi(\boldsymbol{y},\boldsymbol{x})$, *for* $\boldsymbol{x},\boldsymbol{y}\in\Omega$,

iii. *convergence in Hilbert space norm implies pointwise convergence, i.e. if we have* $\|f-f_n\|_{\mathcal{H}} \longrightarrow 0$ *for* $n\longrightarrow\infty$ *then* $|f(\boldsymbol{x}) - f_n(\boldsymbol{x})| \longrightarrow 0$ *for all* $\boldsymbol{x}\in\Omega$.

After denoting with $\mathcal{H}^*$ the space of bounded linear functionals on $\mathcal{H}$, i.e. its *dual*, we state the following theorem.

Theorem 4.2 *Suppose* $\mathcal{H}$ *is a reproducing kernel Hilbert function space with reproducing kernel* $\Phi : \Omega\times\Omega \longrightarrow \mathbb{R}$. *Then* Φ *is positive definite. Moreover,* Φ *is strictly positive definite if and only if the point evaluation functionals* $\delta_{\boldsymbol{x}}$ *are linearly independent in* $\mathcal{H}^*$.

Theorem 4.2 provides one direction of the connection between strictly positive definite functions and reproducing kernels. However, we also want to know how to construct a reproducing kernel Hilbert space associated with strictly positive definite functions. To this aim, let us first note that from Definition 4.1 we have that $\mathcal{H}$ contains all functions of the form

$$f = \sum_{k=1}^{N} c_k \Phi(\cdot,\boldsymbol{x}_k),$$

with $\boldsymbol{x}_k\in\Omega$. As a consequence, we have that

$$||f||_{\mathcal{H}}^2 = \sum_{i=1}^{N}\sum_{k=1}^{N} c_i c_k \Phi(\boldsymbol{x}_i,\boldsymbol{x}_k).$$

Thus, we define the following space

$$H_{\Phi}(\Omega) = \mathrm{span}\{\Phi(\cdot,\boldsymbol{x}),\boldsymbol{x}\in\Omega\},$$

equipped with the bilinear form $(\cdot,\cdot)_{H_\Phi(\Omega)}$ defined as

$$\left(\sum_{i=1}^{m} c_i\Phi(\cdot,\boldsymbol{x}_i), \sum_{k=1}^{n} d_k\Phi(\cdot,\boldsymbol{x}_k)\right)_{H_\Phi(\Omega)} = \sum_{i=1}^{m}\sum_{k=1}^{n} c_i d_k \Phi(\boldsymbol{x}_i,\boldsymbol{x}_k).$$

By virtue of the above definition of the bilinear form, we can state the following theorem [3].

Theorem 4.3 *If* $\Phi : \Omega\times\Omega \longrightarrow \mathbb{R}$ *is a symmetric strictly positive definite kernel, then the bilinear form* $(\cdot,\cdot)_{H_\Phi(\Omega)}$ *defines an inner product on* $H_\Phi(\Omega)$. *Furthermore,* $H_\Phi(\Omega)$ *is a pre-Hilbert space with reproducing kernel* Φ.

Since Theorem 4.3 just shows that $H_\Phi(\Omega)$ is a pre-Hilbert space, i.e. need not be complete, we now define the native space $\mathcal{N}_\Phi(\Omega)$ of Φ to be the completion of $H_\Phi(\Omega)$ with respect to the norm $||\cdot||_{H_\Phi(\Omega)}$ so that $||f||_{H_\Phi(\Omega)} = ||f||_{\mathcal{N}_\Phi(\Omega)}$, for all $f\in H_\Phi(\Omega)$.

Now, focusing on strictly positive definite functions, our aim is to give error bounds for the RBF interpolant. At first, we need to express the interpolant in *Lagrange form*, i.e. using the so-called *cardinal basis functions*.

Theorem 4.4 *Suppose Φ is a strictly positive definite kernel. Then, for any set $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$ of distinct data points, there exist functions $u_k^* \in span\{\Phi(\cdot, \boldsymbol{x}_k), k = 1, \ldots, N\}$ such that $u_k^*(\boldsymbol{x}_i) = \delta_{ik}$.*

The resulting interpolant in cardinal form is given by

$$R(\boldsymbol{x}) = \sum_{k=1}^{N} f(\boldsymbol{x}_k) u_k^*(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

We now need to define the so-called *power function*. To this scope, for any strictly positive definite kernel $\Phi \in C(\Omega \times \Omega)$, any set of distinct points $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$, and any vector $\boldsymbol{u} \in \mathbb{R}^N$, we define the quadratic form

$$Q(\boldsymbol{u}) = \Phi(\boldsymbol{x}, \boldsymbol{x}) - 2\sum_{k=1}^{N} u_k \Phi(\boldsymbol{x}, \boldsymbol{x}_k) + \sum_{i=1}^{N}\sum_{k=1}^{N} u_i u_k \Phi(\boldsymbol{x}_i, \boldsymbol{x}_k).$$

Definition 4.2 Suppose $\Omega \subseteq \mathbb{R}^M$ and $\Phi \in C(\Omega \times \Omega)$ is strictly positive definite. For any distinct points of the set $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$ the power function is defined by

$$[P_{\Phi,\mathcal{X}_N}(\boldsymbol{x})]^2 = Q(\boldsymbol{u}^*(\boldsymbol{x})),$$

where $\boldsymbol{u}^*$ is the vector of cardinal functions from Theorem 4.4.

Taking into account the definition of the cardinal functions, computationally speaking, the power function can be calculated as

$$P_{\Phi,\mathcal{X}_N}(\boldsymbol{x}) = \sqrt{\Phi(\boldsymbol{x}, \boldsymbol{x}) - (b(\boldsymbol{x}))^T A^{-1} b(\boldsymbol{x})}, \tag{11}$$

where $b = (\Phi(\cdot, x_1), \ldots, \Phi(\cdot, x_N))^T$.

We are now able to give the following theorem.

Theorem 4.5 *Let $\Omega \subseteq \mathbb{R}^M$, $\Phi \in C(\Omega \times \Omega)$ be a strictly positive definite kernel and suppose that the points $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \ldots, N\} \subseteq \Omega$ are distinct. Then*

$$|f(\boldsymbol{x}) - R(\boldsymbol{x})| \le P_{\Phi,\mathcal{X}_N}(\boldsymbol{x}) ||f||_{\mathcal{N}_\Phi(\Omega)}, \quad \boldsymbol{x} \in \Omega,$$

where $f \in \mathcal{N}_\Phi(\Omega)$.

Theorem 4.5 allows to estimate the interpolation error by considering the smoothness of the data points measured in terms of the native space norm of f, which is independent of the data sites but dependent on Φ, and in terms of the power function, which is independent of the data values.

We can also express the error estimate in terms of the *mesh size*.

Definition 4.3 The fill distance, which is a measure of data distribution, is given by

$$h_{\mathcal{X}_N} = \sup_{\boldsymbol{x}\in\Omega}\left(\min_{\boldsymbol{x}_k\in\mathcal{X}_N} \|\boldsymbol{x}-\boldsymbol{x}_k\|_2\right). \tag{12}$$

The quantity (12) indicates how well the data fill out the domain Ω. A geometric interpretation of the fill distance is given by the radius of the largest possible empty ball that can be placed among the data locations inside Ω.

We now need some technical considerations.

Definition 4.4 $\Omega \subseteq \mathbb{R}^M$ satisfies an interior cone condition if there exist an angle $\theta \in (0, \pi/2)$ and a radius $\gamma > 0$ such that, for all $\boldsymbol{x} \in \Omega$, a unit vector $\boldsymbol{\xi}(\boldsymbol{x})$ exists such that the cone

$$C = \{\boldsymbol{x} + \lambda \boldsymbol{y} : \boldsymbol{y} \in \mathbb{R}^M, ||\boldsymbol{y}||_2 = 1, \boldsymbol{y}^T \boldsymbol{\xi}(\boldsymbol{x}) \geq cos(\theta), \lambda \in [0, \gamma]\},$$

is contained in Ω.

Theorem 4.6 *Suppose that $\Omega \subseteq \mathbb{R}^M$ is compact and satisfies an interior cone condition with angle $\theta = (0, \pi/2)$ and radius $\gamma > 0$. Suppose that there exist h_0, C_1, $C_2 > 0$ constants depending only on M, θ and γ, such that $h_{\mathcal{X}_N} \leq h_0$. Then, for all $\mathcal{X}_N = \{\boldsymbol{x}_i, i = 1, \dots, N\} \subseteq \Omega$ and every $\boldsymbol{x} \in \Omega$, there exist functions $v_k : \Omega \longrightarrow \mathbb{R}$, $k = 1, \dots, N$, such that:*

i. $\sum_{k=1}^N v_k(\boldsymbol{x}) p(\boldsymbol{x}_k) = p(\boldsymbol{x})$, *for all $p \in \Pi_{L-1}^M$, where Π_{L-1}^M be the set of polynomials of degree $L-1$,*

ii. $\sum_{k=1}^N |v_k(\boldsymbol{x})| \leq C_1$,

iii. $v_k(\boldsymbol{x}) = 0$ *provided that* $\|\boldsymbol{x} - \boldsymbol{x}_k\|_2 \geq C_2 h_{\mathcal{X}_N}$.

Then, the error bound in terms of fill distance is given by the following theorem.

Theorem 4.7 *Suppose $\Omega \subseteq \mathbb{R}^M$ is bounded and satisfies an interior cone condition. Suppose $\Phi \in C^{2k}(\Omega \times \Omega)$ is symmetric and strictly positive definite. Then, there exist positive constants h_0 and C, independent of $\boldsymbol{x}$, f and Φ, such that*

$$|f(\boldsymbol{x}) - R(\boldsymbol{x})| \leq C h_{\mathcal{X}_N}^k \sqrt{C_\Phi(\boldsymbol{x})} ||f||_{\mathcal{N}_\Phi(\Omega)},$$

provided $h_{\mathcal{X}_N} \leq h_0$ and $f \in \mathcal{N}_\Phi(\Omega)$, where

$$C_\Phi(\boldsymbol{x}) = \max_{|\boldsymbol{\beta}|=2k}\left(\max_{\boldsymbol{w},\boldsymbol{z}\in\Omega\cap B\left(\boldsymbol{x}, C_2 h_{\mathcal{X}_N}\right)} \left|D_2^{\boldsymbol{\beta}} \Phi(\boldsymbol{w}, \boldsymbol{z})\right|\right),$$

with $B(\boldsymbol{x}, C_2 h_{\mathcal{X}_N})$ denoting the ball of radius $C_2 h_{\mathcal{X}_N}$ centred at $\boldsymbol{x}$.

Theorem 4.7 states that the interpolation with a C^{2k} smooth kernel has approximation order k. Consequently, the approximation order k is arbitrarily high for infinitely smooth strictly positive definite functions, while for strictly positive definite functions with limited smoothness the approximation order is limited by the smoothness of the function. Thus, the choice of the RBF can affect the fitting process.

5 Concluding remarks

In this notes we gave the basic for knowledge RBF interpolation. Specifically, after establishing the general framework concerning the scattered data interpolation problem, we focused on its solution via RBFs and for the resulting interpolant we provided error bounds.

Of course, many topics and applications have been omitted. For more details the reader can refer to the books listed in the references. In particular, [3] also provides useful Matlab codes.

References

[1] M. D. Buhmann, "Radial Basis Functions: Theory and Implementation". Cambridge Monogr. Appl. Comput. Math., vol. 12, Cambridge Univ. Press, Cambridge, 2003.

[2] S. De Marchi, E. Perracchione, *Lectures on Radial Basis Functions.* Preprint (2018).

[3] G. E. Fasshauer, "Meshfree Approximations Methods with Matlab". World Scientific, Singapore, 2007.

[4] E. Perracchione, "RBF-Based Partition of Unity Method: Theory, Algorithms and Applications". Ph.D. Thesis, Torino, Italy, 2017.

[5] H. Wendland, "Scattered Data Approximation". Cambridge Monogr. Appl. Comput. Math., vol. 17, Cambridge Univ. Press, Cambridge, 2005.

Monodromy and Invariant Cycles

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1 A family of elliptic curves

This subsection is essentially an extract of [CMSP03, Ch. 1, §1.1], where the example of elliptic curves is treated more thoroughfully. We consider the equation

$$y^2 = x(x-1)(x-s)$$

over the complex numbers, where $s \in \mathbb{C} \setminus \{0,1\}$ is a fixed parameter and x and y denotes the variables. Its set of solutions

$$\{(x,y) \in \mathbb{A}^2_{\mathbb{C}} : y^2 = x(x-1)(x-s)\}$$

describes a subset of the affine plane $\mathbb{A}^2_{\mathbb{C}}$. This is an example of a plane algebraic curve over $\mathbb{C}$. One could consider the homogeneous equation associated

$$Y^2 Z = X(X-Z)(X-sZ),$$

the set of solutions now identifies with a subset of the projective plane $\mathbb{P}^2_{\mathbb{C}}$,

$$X_s := \{[X:Y:Z] \in \mathbb{P}^2_{\mathbb{C}} : Y^2 Z = X(X-Z)(X-sZ)\}.$$

This is an example of a projective algebraic plane curve. More precisely, this is an elliptic curve. We are interested in the topology of X_s.
It is not difficult to show that X_s is a double cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ ramified at the point 0, 1, s and ∞. Indeed for x in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,s,\infty\}$ there are two values of y such that (x,y) is a point in X_s. We will show how to see a double cover of the Riemann sphere ramified at 4 points as a torus. We cut the Riemann sphere from 0 to s and from 1 to ∞ as in Fig. 1.

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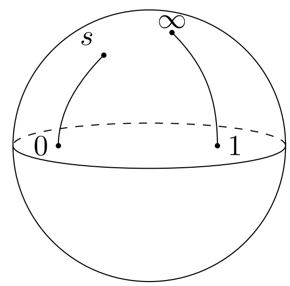

Figure 1

Then we "open" the cuts (Fig. 2). Now, after deforming continuously the shape, we glue two copies along the doubled cuts, see Fig. 3. The end result is a torus (Fig. 4).

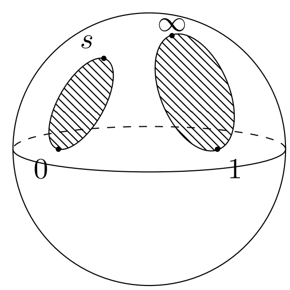

Figure 2

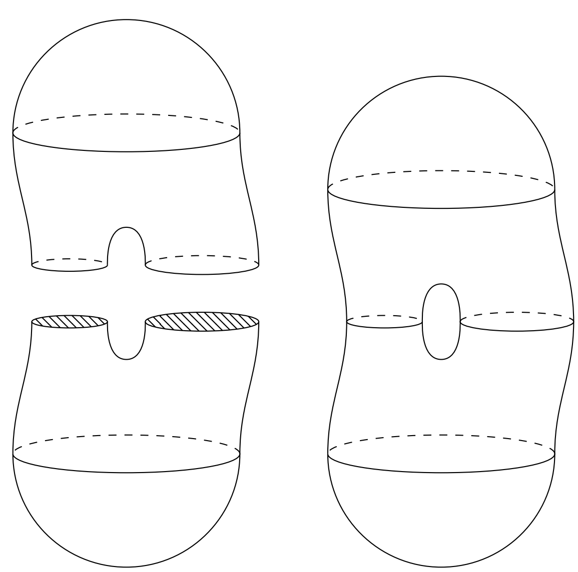

Figure 3

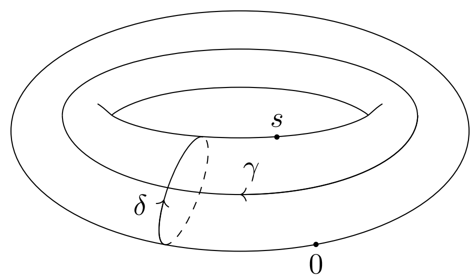

Figure 4

We study the variation of the topology of X_s as the parameter s moves. We will focus only on the case of s in S the complex unit disk. We look at the curves X_s as fibers of a morphism

$$f : X \longrightarrow S,$$

where $X = \{([U : V : W], s) \in \mathbb{P}^2_{\mathbb{C}}(\mathbb{C}) \times S : V^2W = U(U - W)(U - sW)\}$ and f is the projection to S, that is $X_s = f^{-1}(s)$. Moreover, we denote with S^* the punctured open disk $S \setminus \{0\}$ and with f^* the restriction of f to $X^* := X \setminus X_0$. One can show that the continuous map

$$f^* : X^* \longrightarrow S^*$$

is a smooth locally trivial fibration (see [Mil68, §1] for a proof of this claim). One implication of this fact it that for every $s \neq 0$, the fibers X_s are all smooth varieties and are one diffeomorphic one to eachother, for this reason we will call any of them "the" generic fiber. On the other hand, the curve X_0 may have some singularities, we will call it the special fiber.
Another consequence of the fact that f^* is a locally trivial fibration, is that we can lift homotopies. Meaning that for $s \in S^*$ fixed, we have a group homomorphism

$$\theta : \pi_1(S^*; s) \longrightarrow \mathrm{Aut}(X_s),$$

where the term on the right hand side is the group of homeomorphisms of X_s into itself. Identifying $\pi_1(S^*; s)$ with $\mathbb{Z}$, it makes sense to consider $\tau := \theta(1)$. This automorphism τ is the monodromy of X_s. We recall that, being a torus, the first homology of X_s is the vector space

$$\mathrm{H}_1(X_s; \mathbb{C}) = \langle \delta, \gamma \rangle,$$

where δ and γ are the loops in Fig. 4. We will consider the linear operator induced by τ on the homology

$$T : \mathrm{H}_1(X_s; \mathbb{C}) \longrightarrow \mathrm{H}_1(X_s; \mathbb{C})$$

and keep referring at it as the monodromy. Our goal is to compute the coinvariant part of this action and understand if it has a geometric meaning. More precisely, we want to find a cohomological description for $\mathrm{Coker}(T - \mathrm{Id})$. The first step is to compute the matrix for T in the basis $\{\delta, \gamma\}$.
We look at the cycles δ and γ on one half of the torus, observe that the dashed part of γ lives in the other part of the cover (Fig. 5).

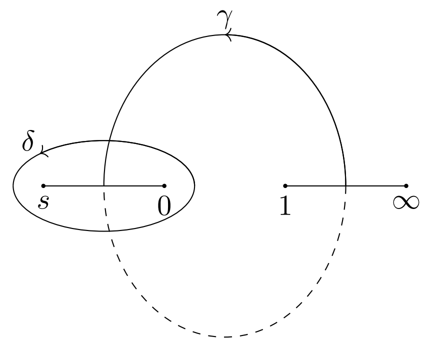

Figure 5

We first rotate s counterclockwise around zero by an angle of π like in Fig. 6, this operation corresponds to taking the square root of the monodromy: $T^{1/2}$. One easily sees that

$$T^{\frac{1}{2}}(\delta) = \delta.$$

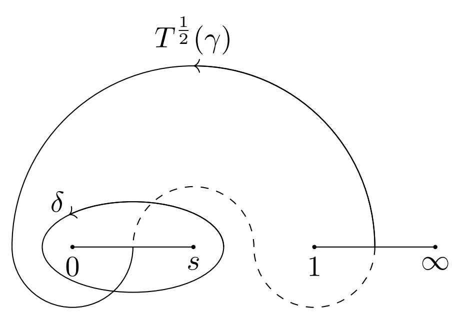

Figure 6

To compute $T^{1/2}(\gamma)$ we overlay Figures 5 and 6 and compute the intersection numbers on the picture with the convention $\delta \cdot \gamma = 1$. We see that $T^{1/2}(\gamma) \cdot \delta = -1$ and $T^{1/2}(\gamma) \cdot \gamma = 1$, so if $T^{1/2}(\gamma) = a\delta + b\gamma$ we obtain

$$\begin{aligned} -1 &= T^{\frac{1}{2}}(\gamma) \cdot \delta = (a\delta + b\gamma) \cdot \delta = -b, \\ 1 &= T^{\frac{1}{2}}(\gamma) \cdot \gamma = (a\delta + b\gamma) \cdot \gamma = a. \end{aligned}$$

So the matrix for $T^{1/2}$ in the basis $\{\delta, \gamma\}$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

hence the matrix for T in the same basis is

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

At this point is immediate to show that the coinvariant cycles are

$$\mathrm{Coker}(T - \mathrm{Id}) = \langle \gamma \rangle.$$

We want to show that there is a geometrical interpretation for the coinvariant part of this action. First we have to remark that the special fiber X_0 is a deformation retract of X, in other words, we have an isomorphism

$$\mathrm{H}_1(X;\mathbb{C}) \xrightarrow{\sim} \mathrm{H}_1(X_0;\mathbb{C}).$$

If we compose it with $\mathrm{H}_1(X_s;\mathbb{C}) \to \mathrm{H}_1(X;\mathbb{C})$, the morphism in homology induced by the inclusion of the generic fiber, we obtain a morphism

$$g : \mathrm{H}_1(X_s;\mathbb{C}) \longrightarrow \mathrm{H}_1(X_0;\mathbb{C}).$$

Since X_0 should not detect the action of the monodromy T, it is a natural guess that g is actually the projection to $\mathrm{Coker}(T - \mathrm{Id})$. Indeed Fig. 7 shows that

$$\mathrm{H}_1(X_0;\mathbb{C}) = \langle \gamma \rangle$$

because δ becomes contractible, hence g is the projection of $\langle \delta, \gamma \rangle$ onto $\langle \gamma \rangle$.

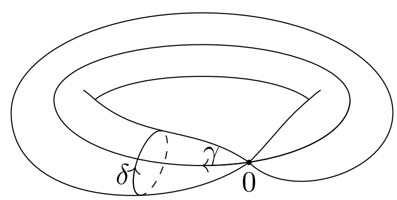

Figure 7

To summarize this result we say that

$$\mathrm{H}_1(X_s;\mathbb{C}) \xrightarrow{T-\mathrm{Id}} \mathrm{H}_1(X_s;\mathbb{C}) \xrightarrow{g} \mathrm{H}_1(X_0;\mathbb{C}) \longrightarrow 0$$

is an exact sequence. To consider analogous results in other geometrical settings, it would be useful to consider the dual of this sequence, hence we look at cohomology:

$$0 \longrightarrow \mathrm{H}^1(X_0;\mathbb{C}) \longrightarrow \mathrm{H}^1(X_s;\mathbb{C}) \xrightarrow{N} \mathrm{H}^1(X_s;\mathbb{C})$$

and we use the operator

$$N := -\frac{1}{2\pi i}\log(T_u)$$

where T_u is the unipotent part of T.

Remark 1 We did not change the meaning of the sequence because the dual of $\mathrm{Coker}(T-\mathrm{Id})$ is isomorphic to $\mathrm{Ker}(N)$.

We have just seen an instance of the (local) invariant cycles theorem. There are several statements of this result (see [Cle77, Ill94]), for this exposition we content ourselves with the following.

Theorem 1.1 (Clemens; Deligne; Katz; Schmid; etc.) *Let $f : X \longrightarrow S$ be a proper morphism of smooth complex analytic spaces (or smooth schemes of finite type over $\mathbb{C}$). Let us assume that S is the open unit disc (or an affine curve) and that f is smooth outside $s = 0$, then we have an exact sequence*

$$0 \longrightarrow \mathrm{H}^1(X_0; \mathbb{C}) \longrightarrow \mathrm{H}^1(X_s; \mathbb{C}) \xrightarrow{N} \mathrm{H}^1(X_s; \mathbb{C}).$$

2 p-adic invariant cycles

In this section we want to summarize some results in arithmetic geometry that draw on the analogies with the complex geoemtric setting of the previous section. To start we recall the definition of p-adic integers.

Definition 2.2 Let p be a prime number. The ring of p-adic integers is the following inverse limit

$$\mathbb{Z}_p := \varprojlim \frac{\mathbb{Z}}{p^n\mathbb{Z}}$$

where the arrow

$$\frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{p^n\mathbb{Z}}$$

is the quotient obtained from the inclusion $p^n\mathbb{Z} \subset p^{n+1}\mathbb{Z}$.

Remark 2 The ring $\mathbb{Z}_p$ can be identified with

$$\left\{ \sum_{i=0}^{\infty} a_i p^i \,:\, a_i \in \{0, 1, 2, \dots, p-1\} \right\}$$

where the ring operations are the ones that one expects.

Via the Zariski topology one can associate a topological space to any commutative ring with unity. We will recall its construction in general.
Let R be a commutative ring with unity. An ideal $\mathfrak{p} \subsetneq R$ is said to be a prime ideal if for any a and b in R

$$ab \in \mathfrak{p} \Longrightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

One defines

$$\mathrm{Spec}(R) := \{\mathfrak{p} \subset R \,:\, \mathfrak{p} \text{ is a prime ideal}\}.$$

Given an ideal $I \subset R$, non necessarily prime, its zero set is

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \,:\, I \subset \mathfrak{p}\}$$

The Zariski topology on $\mathrm{Spec}(R)$ is the coarsest topology for which the $V(I)$ are closed sets, for I varying among the ideals of R.
If we consider the case $R = \mathbb{Z}_p$ it is not difficult to show that $\mathrm{Spec}(\mathbb{Z}_p) = \{(0), (p)\}$, that the point (p) is closed, and $\eta := (0)$ is an open dense point. For this reason η is called the generic point.
Now we consider a proper curve

$$X \longrightarrow \mathrm{Spec}(\mathbb{Z}_p)$$

with semistable reduction. We refer to [HK94] for the precise definition. The (very) rough idea is that "locally" X is the zero set of the equation

$$xy = p$$

over $\mathbb{Z}_p$. This is an analogue of the family $f : X \longrightarrow S$ of the previous section with $\mathrm{Spec}(\mathbb{Z}_p)$ replacing the open unit disc S. Indeed the generic fiber X_η is a smooth algebraic curve over $\mathbb{Q}_p := \mathrm{Frac}(\mathbb{Z}_p)$ and the special fiber $X_{(p)}$ is an algebraic curve over $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$ not smooth. Continuing with the intuitive picture, if X is described "locally" by some equation over $\mathbb{Z}_p$, then X_η is described by the same equation considered over $\mathbb{Q}_p$ and $X_{(p)}$ is described by the equation modulo p.
In order to reconstruct a monodromy operator in the p-adic setting, we need to understand what should be the analogue of $\mathrm{H}^1(X_s; \mathbb{C})$. For this we recall the following.

Theorem 2.1 (de Rham's Theorem) *Let X be a complex manifold, then*

$$\mathrm{H}^1(X; \mathbb{C}) \simeq \mathrm{H}^1_{dR}(X).$$

In the previous statement, $\mathrm{H}^1_{dR}(X)$ denotes the hypercohomology of the de Rham complex of holomorphic differential forms on X. In particular, for our case

$$\mathrm{H}^1(X_s; \mathbb{C}) \simeq \mathrm{H}^1_{dR}(X_s).$$

If we consider the algebraic de Rham complex we can make sense of $\mathrm{H}^1_{dR}(X_\eta)$, this cohomology is the arithmetic counterpart of $\mathrm{H}^1(X_s; \mathbb{C})$. We conclude this section wtih some results towards a p-adic version of Theorem 1.1. We keep the notation and assumptions of this section.

Theorem 2.2 [HK94] *There is a monodromy operator*

$$N : \mathrm{H}^1_{dR}(X_\eta) \longrightarrow \mathrm{H}^1_{dR}(X_\eta)$$

Theorem 2.3 [Chi99] *We have an exact sequence*

$$0 \longrightarrow \mathrm{H}^1_{rig}(X_{(p)}) \longrightarrow \mathrm{H}^1_{dR}(X_\eta) \xrightarrow{N} \mathrm{H}^1_{dR}(X_\eta).$$

Here N is the operator of the previous theorem and $\mathrm{H}^1_{rig}(X_{(p)})$ is the rigid cohomology of the special fiber.

Remark 1 We want to point out that Theorem 2.3 shows some relation between the generic and the special fiber (as much as Theorem 1.1 does), so there is an interplay between characteristic 0 and p.

3 Our contribution

New contribution to the theory of p-adic invariant cycles came from [CI10], in this paper an explicit description of the monodromy N (cf. Theorem 2.2) is presented. This was fundamental for the work on invariant cycles with coefficients [CCDPI16]. Our project is to adapt the (in principle arithmetic) argument of [CI10] to the complex setting and to reobtain analogous results to the ones in [CCDPI16]. In this section we will briefly explain what was our strategy for the first part of the project.
We consider the following situation. Let $f : X \longrightarrow S$ be a proper separated morphism of finite type between smooth schemes over $\mathbb{C}$. We assume, moreover, that S is a smooth affine curve, f is of relative dimension 1 and smooth outside a fixed point $P \in S$. We denote with X_s the fiber above s and we assume X_P to be a simple normal crossing divisor. Our assumptions imply that X_P is the union of finitely many irreducible smooth curves that intersect eachother transversally

$$X_P = \bigcup_{v \in \mathcal{V}} X_v.$$

We construct the dual graph $\mathrm{Gr}(X_P) = (\mathcal{V}, \mathcal{E})$, where a vertex $v \in \mathcal{V}$ corresponds to a component X_v; an oriented edge $e \in \mathcal{E}$, with $e = [v, w]$, corresponds to a point X_e in the intersection $X_v \cap X_w$.
The inclusion

$$i_P : X_P \hookrightarrow X$$

induces a log-scheme structure on X (see [Kat99] for the definition). We will consider the scheme X_P as a log-scheme endowed with the pullback log-structure. Similarly, the inclusion of P in S makes P into a log-point. It will then make sense to talk about the relative logarithmic de Rham complex $\omega^\bullet_{X_P/P}$.

Theorem 3.1 [Ste76] *With the above notation,*

$$\mathrm{H}^1_{\log,dR}(X_P/P) := \mathbb{H}^1(X_P; \omega^\bullet_{X_P/P}) \simeq \mathrm{H}^1_{dR}(X_s)$$

for any $s \in S \setminus \{P\}$.

For any v in V we put

$$U_v := X_v \setminus \bigcup_{e \in \mathcal{E}} (X_e \cap X_v).$$

Then we can show that a cohomology class $[\omega]$ in $\mathrm{H}^1_{\log,dR}(X_P/P)$ can be represented by a hypercocycle

$$\left((\omega_v)_{v \in \mathcal{V}}, (f_e)_{e \in \mathcal{E}}\right)$$

where $\omega_v \in \Omega^1_{U_v}(U_v)$ and $f_e \in \mathbb{C}$, subject to the condition

$$\mathrm{Res}_{X_e}(\omega_v) + \mathrm{Res}_{X_e}(\omega_w) = 0$$

for any edge $e = [v, w]$. At this point we defined the monodromy operator

$$\tilde{N} : \mathrm{H}^1_{\log,dR}(X_P/P) \longrightarrow \mathrm{H}^1_{\log,dR}(X_P/P) \tag{1}$$

$$\left[\left((\omega_v)_{v\in\mathcal{V}}, (f_e)_{e\in\mathcal{E}}\right)\right] \longmapsto \left[\Big(0, \left(\mathrm{Res}_{X_e}(\omega_v)\right)_{e=[v,w]}\Big)\right] \tag{2}$$

This combinatorial monodromy is really computing the topological monodromy.

Theorem 3.2 *Under the isomorphism in Theorem 3.1, the operators $\tilde{N}$ and N coincide.*

This combinatorial monodromy comes with its invariant cycles sequence. We denote with $\mathrm{H}^1_{DB}(X_P)$ the hypercohomology of the Du Bois complex for X_P (see [DB81, §2.10]). From a result of Du Bois ([DB81, Théorème 4.5])

$$\mathrm{H}^1_{DB}(X_P) \simeq \mathrm{H}^1(X_P; \mathbb{C}).$$

Since the Du Bois cohomology is obtained from a variation of the de Rham complex, is not hard to show that we have a natural map

$$\mathrm{H}^1_{DB}(X_P) \longrightarrow \mathrm{H}^1_{\log,dR}(X_P/P).$$

This is really the inclusion of the kernel of $\tilde{N}$:

Theorem 3.3 *We have an exact sequence*

$$0 \longrightarrow \mathrm{H}^1_{DB}(X_P) \longrightarrow \mathrm{H}^1_{\log,dR}(X_P/P) \xrightarrow{\tilde{N}} \mathrm{H}^1_{\log,dR}(X_P/P).$$

References

[CCDPI16] Bruno Chiarellotto, Robert Coleman, Valentina Di Proietto, and Adrian Iovita, *On a p-adic invariant cycles theorem.* J. Reine Angew. Math.,711 (2016), 55–74.

[Chi99] Bruno Chiarellotto, *Rigid cohomology and invariant cycles for a semistable log scheme.* Duke Math. J., 97/1 (1999), 155–169.

[CI10] Robert Coleman and Adrian Iovita, *Hidden structures on semistable curves.* Astérisque 331 (2010), 179–254.

[Cle77] C. H. Clemens, *Degeneration of Kähler manifolds.* Duke Math. J. 44/2 (1977), 215–290.

[CMSP03] James Carlson, Stefan Müller-Stach, and Chris Peters, "Period mappings and period domains". Volume 85 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003.

[DB81] Philippe Du Bois, *Complexe de de Rham filtré d'une variété singulière.* Bull. Soc. Math. France, 109/1 (1981), 41–81.

[HK94] Osamu Hyodo and Kazuya Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles.* Astérisque 223 (1994), 221–268. Périodes p-adiques (Bures-sur-Yvette, 1988).

[Ill94] Luc Illusie, *Autour du théorème de monodromie locale.* Astérisque 223 (1994), 9–57. Périodes p-adiques (Bures-sur-Yvette, 1988).

[Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie.* In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pp. 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Mil68] John Milnor, "Singular points of complex hypersurfaces". Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[Ste76] Joseph Steenbrink, *Limits of Hodge structures.* Invent. Math. 31/3 (1975/76), 229–257.

Abelian Model Structures

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Abstract. Model categories were introduced by Quillen in 1967 as an axiomatized setting in which it is possible to "do homotopy theory", by inverting a class of morphisms called weak equivalences. The construction involves the use of two more classes of morphisms, which, together with the weak equivalences, form what is called a model structure. In the case of abelian categories there are particular model structures, called abelian model structures, that can be constructed by means of objects rather than morphisms, using complete cotorsion pairs. We will present the theory of abelian model structures, showing how they can be applied to the particular case of R-modules to recover the derived category of the ring.

1 Notions of category theory

Throughout this notes we will encounter many examples of categories, so we will start with a brief reminder.

Definition 1 A *category* consists of:

- A class $\mathrm{ob}(\mathcal{C})$ of objects.
- For any pair of objects A and B in $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$
- For any three objects A, B, and C in $\mathcal{C}$ a binary operation

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

called *composition*. If $f : A \to B$ and $g : B \to C$, their composition is written $g \circ f$ or gf.

The composition satisfy the following axioms:

- (associativity) $h \circ (g \circ f) = (h \circ g) \circ f$, for a triple of composable morphisms;
- (identity) for every object X there is a morphism $1_X : X \to X$ such that $1_X \circ f = f$ and $g \circ 1_X = g$, whenever the compositions make sense.

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Homomorphisms are also called morphisms, maps, and arrows. Some examples of categories are:

- Sets: the category whose objects are sets and homomorphisms are set maps;
- Ab: the category of abelian groups with groups homomorphisms;
- Top: the category of topological spaces with continuous maps.

The transformations between categories are called functor and are defined as follows.

Definition 2 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, a *functor* is an assignment that associates:

- to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$
- to each map $f : X \to Y$ in $\mathcal{C}$ a map $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that:
  - $F(1_X) = 1_{F(X)}$
  - $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f and g in $\mathcal{C}$

The transformation between functors are called natural transformations and are defined as follows.

Definition 3 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors, a *natural transformation* $\eta : F \Rightarrow G$ is a family of morphisms $\eta_X : F(X) \to G(X)$ indexed by the objects X in $\mathcal{C}$ such that for any $f : X \to Y$ in $\mathcal{C}$ we have

$$\eta_Y \circ F(f) = G(f) \circ \eta_X.$$

The two maps $\eta_Y \circ F(f)$ and $G(f) \circ \eta_X$ can be represented as a diagram, i.e. a directed graph whose vertices are objects of a category and whose edges are morphisms (thus, the name arrow to indicate a morphism), as follows.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

The fact that the two compositions are the same is expressed by saying that the diagram is commutative. In general, a diagram is said to be commutative whenever the composition of morphisms does not depend on the choice of a path.

We can think of morphisms in a category as acting on the "elements" of the objects, although it was never required that the objects themselves are sets, so if we want to define properties of morphisms such as surjectivity, injectivity, etc. in a purely categorical way we need to formulate them in terms of objects and compositions of morphisms.

For example, the equivalent of "surjective" homomorphisms are called epimorphisms and are defined in the following way:

Definition 4 In a category $\mathcal{C}$ a morphism $f : X \to Y$ is called epimorphism if, for every pair of parallel morphisms $g_1, g_2 : Y \to Z$ such that the two compositions

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

$g_1 \circ f$ and $g_2 \circ f$ are equal, then $g_1 = g_2$. We say that f can be cancelled on the right.

Analogously, monomorphisms are defined as the morphisms that can be cancelled on the left and play the role of "injective" morphisms.

2 Abelian categories

Categories as we defined them above are too broad in scope for our setting, and in fact we need to require certain features from our categories in order to be able to work with them, namely we want homomorphisms that can be added and we want to talk about zero morphisms. Thus, we will restrict to abelian categories.

From our point of view, the notion of being "zero" will be antecedent to that of addition.

Definition 5 Let $\mathcal{C}$ be a category:

- an object $\emptyset$ is called *initial* if for every object X in $\mathcal{C}$ there exists exactly one morphism $\emptyset \to X$;
- dually, an object $*$ is called *terminal* if for every object X in $\mathcal{C}$ there exists exactly one morphism $X \to *$,
- if an object is both initial and terminal it is called a *zero object* and is usually indicated by 0.

For any two objects X and Y in a category $\mathcal{C}$ with a zero object, the unique morphism given by the composition $X \to 0 \to Y$ is called the zero morphism (between X and Y).

Another important notion is that of kernel.

Definition 6 Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism. A *kernel* (if it exists) of f is an object $\operatorname{Ker} f$ together with a map $\ker f : \operatorname{Ker} f \to X$ satisfying the following universal property:

- $f \circ \ker f = 0$, i.e. the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& \operatorname{Ker} f \arrow[ul, "\ker f"] \arrow[ur, "0"'] &
\end{tikzcd}$$

- For any object K and morphism $k : K \to X$ such that $f \circ k = 0$, there is a unique morphism $u : K \to \operatorname{Ker} f$ such that $k = \ker f \circ u$. I.e., if the solid part of the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& \operatorname{Ker} f \arrow[ul, "\ker f"'] \arrow[ur, "0"] & \\
& K \arrow[uul, "k"] \arrow[uur, "0"'] \arrow[u, dashed, "u"] &
\end{tikzcd}$$

  is commutative, there exists a unique dashed arrow u making the whole diagram commute.

The notion of cokernel is dual, i.e. it is the universal object C making the following diagram commute:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "c"'] & & Y \arrow[dl, "0"] \\
& C &
\end{tikzcd}$$

Another important concept is that of product (and its dual coproduct).

Definition 7 Let X, Y be two objects in a category $\mathcal{C}$, a product is an object $X \times Y$ together with maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any object Q such that the solid part of the following diagram is commutative

$$\begin{tikzcd}
Q \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& X \times Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

there exists a unique dashed arrow $Q \to X \times Y$ making the whole diagram commute.

A *biproduct* of X and Y is an object $X \oplus Y$ that is both a product and a coproduct. With these definitions we can define an abelian category as:

Definition 8 A category is *abelian* if

- it has a zero object;
- it has all binary biproducts;
- it has all kernels and cokernels;
- all monomorphisms and epimorphisms are normal, i.e. every monomorphism is a kernel and every epimorphism is a cokernel.

These axioms have some immediate consequences:

- The hom sets are abelian groups, usually indicated with the additive notation;
- The composition of morphisms is bilinear.

The categories we will work with, namely Mod-R and $\mathbf{C}(R)$ are both abelian. They are actually very special abelian categories, and in fact the results we will show do not hold in general for all abelian categories. They are, however, formulated just in terms of the abelian structure.

2.1 Exact sequences

If $\mathcal{C}$ is an abelian category, a *sequence* in $\mathcal{C}$ is a diagram

$$\cdots \longrightarrow X^{i-1} \xrightarrow{f^{i-1}} X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} \cdots$$

with $X^i \in \mathcal{C}$ and where any two successive morphisms compose to zero.

It we define the image of a morphism as $\mathrm{Im}\,(f) = \mathrm{Coker}\,\mathrm{Ker}\, f$, we can see that $f^i \circ f^{i-1}$ is zero if and only if $\mathrm{Im}\,(f^{i-1}) \subset \mathrm{Ker}\,(f^i)$.

The diagram can be finite or infinite in one or both directions.

A sequence is called *exact* if $\mathrm{Im}\,(f^{i-1}) = \mathrm{Ker}\,(f^i)$ for all $i \in I$.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

Example 1 Given a morphism $f : X \to Y$, its kernel $\mathrm{Ker}\,(f)$ fits into an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(f) \longrightarrow X \xrightarrow{f} Y$$

2.2 Projectivity and injectivity

Two important class of objects in any category are those of projectives and injectives.

Definition 9 An object P in a category $\mathcal{C}$ is called *projective* if for any epimorphism $f : X \to Y$ and any map $g : P \to Y$ there is a map $h : P \to X$ such that $g = f \circ h$, i.e. there exists a dashed arrow making the following diagram commutative

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The class of all projective objects is denoted Proj-$\mathcal{C}$.

Dually,

Definition 10 An object I in a category $\mathcal{C}$ is called *injective* if for any monomorphism $f : X \to Y$ and any map $g : X \to I$ there is a map $h : Y \to I$ such that $g = h \circ f$, i.e. there exists a dashed arrow making the following diagram commutative

$$\begin{tikzcd} I & \\ X \arrow[u, "g"] \arrow[r, "f"] & Y \arrow[ul, dashed, "h"'] \end{tikzcd}$$

The class of all injective objects is denoted Inj-$\mathcal{C}$.

3 Modules and cochain complexes

We will procede to define the categories of R-Modules and of cochain complexes of R-Modules.

Definition 11 Let $(R, +, \cdot, 0, 1)$ be a ring. An R-Modules M is an abelian group $(M, +, 0)$ with an operation:

$$\cdot : R \times M \longrightarrow M$$

$$(r, m) \longmapsto r \cdot m$$

satisfying:

(a) $r \cdot (m + n) = r \cdot m + r \cdot n$;

(b) $(r + s) \cdot m = r \cdot m + s \cdot m$;

(c) $(r \cdot s) \cdot m = r \cdot (s \cdot m)$;

(d) $1_R \cdot m = m$.

A homomorphism of R-Modules $f \in \mathrm{Hom}_R(M, N)$ is an R-linear homorphism of abelian groups.
The category of R-Modules is denoted Mod-R.

If k is a field, Mod-$k = \mathrm{Vect}_k$.

Definition 12 We call $\mathbf{C}$(Mod-R) (or shortly $\mathbf{C}(R)$) the category of unbounded complexes of R-Modules, defined as follows:

- The objects of $\mathbf{C}$(Mod-R) are the cochain complexes $X^\bullet$

  $$\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-i}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots$$

  where X^i belongs to Mod-R for all $i \in \mathbb{Z}$ and d^i are morphisms in Mod-R such that $d^i \circ d^{i-1} = 0$;

- A morphism $f^\bullet : X^\bullet \to Y^\bullet$ is a collection of morphisms $\{f^i\}_{i\in\mathbb{Z}}$ in Mod-R such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r, "d^{i-i}"] \arrow[d, "f^{i-1}"] & X^i \arrow[r, "d^i"] \arrow[d, "f^i"] & X^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d^{i-i}"] & Y^i \arrow[r, "d^i"] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

3.1 Cohomology

Consider a cochain complex

$$X^\bullet : \ \cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots$$

For all i there are short exact sequences

$$0 \longrightarrow \mathrm{Ker}\,(d^i) \longrightarrow X^i \xrightarrow{\tilde{d}^i} \mathrm{Im}\,(d^i) \longrightarrow 0.$$

Observe that $\mathrm{Im}\,(d^i) \subset \mathrm{Ker}\,(d^{i+1})$ since $X^\bullet$ is a complex.

Definition 13 If $X^\bullet \in \mathbf{C}(R)$, its n-th cohomology is defined as the module

$$H^n(X^\bullet) = \frac{\mathrm{Ker}\,(d^n)}{\mathrm{Im}\,(d^{n-1})}.$$

3.2 Exact complexes and projective complexes

Definition 14 A complex $X^\bullet$ is called *exact* if $\mathrm{Ker}\,(d^i) = \mathrm{Im}\,(d^{i-1})$ for all $i \in \mathbb{Z}$, or, equivalently, if $H^i(X^\bullet) = 0$.

We can give a complete description of all projective objects in the category $\mathbf{C}(R)$:

Lemma 1 *A complex $P^\bullet \in \mathbf{C}(R)$ is projective if and only if it is a split exact complex with projective terms, i.e.*

$$\begin{tikzcd}
\cdots \arrow[r] & P^{i-1} \arrow[rr, "d^{i-1}"] \arrow[dr, two heads] & & P^i \arrow[rr, "d^i"] \arrow[dr, two heads] & & P^{i+1} \arrow[r, "d^{i+1}"] & \cdots \\
& & \mathrm{Ker}\, d^i \arrow[ur, tail] & & \mathrm{Ker}\, d^{i+1} \arrow[ur, tail] & &
\end{tikzcd}$$

$P^i = \mathrm{Ker}\,(d^i) \oplus \mathrm{Ker}\,(d^{i+1})$ *for all i, iff* $\mathrm{Ker}\,(d^i) \in \mathrm{Proj}$-$R$ *for all i.*

4 The Ext^1 groups

For a pair of objects M and N in an abelian category $\mathcal{C}$, an extension ξ of M by N is a short exact sequence $0 \to N \to X \to M \to 0$.

We say that ξ and ξ' are equivalent and write it as $\xi \cong \xi'$ if there is a commutative diagram with exact rows

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
\xi': & 0 \arrow[r] & N \arrow[r] & X' \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Remark 1 For any commutative diagram with exact rows as above in an abelian category, the vertical arrow in the middle is actually an isomorphism.

An extension is called *split* if it is equivalent to

$$0 \to N \xrightarrow{\binom{1}{0}} N \oplus M \to M \to 0.$$

The set of all extensions of M by N modulo equivalence is denoted $\mathrm{Ext}^1_{\mathcal{C}}(M,N)$ and, moreover, there is a binary operation $+$ on extensions compatible with the equivalence making $\mathrm{Ext}^1_{\mathcal{C}}(M,N)$ into an abelian group. The 0 element is (the equivalence class of) the split extension.

5 Cotorsion pairs

Definition 15 Let $\mathcal{C}$ be an abelian category. A pair $(\mathcal{A},\mathcal{B})$ of subclasses of $\mathcal{C}$ is called a cotorsion pair if:

(a) $\mathcal{A} = {}^{\perp}\mathcal{B} = \{X \in \mathcal{C} | \mathrm{Ext}^1_{\mathcal{C}}(X,B) = 0 \ \forall B \in \mathcal{B}\}$,

(b) $\mathcal{B} = \mathcal{A}^{\perp} = \{X \in \mathcal{C} | \mathrm{Ext}^1_{\mathcal{C}}(A,X) = 0 \ \forall A \in \mathcal{A}\}$.

A cotorsion pair $(\mathcal{A},\mathcal{B})$ is called *complete* if for any object $X \in \mathcal{C}$ there are short exact sequences:

- $0 \to X \to B \to A \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$,
- $0 \to B' \to A' \to X \to 0$ with $B' \in \mathcal{B}$ and $A' \in \mathcal{A}$.

Example 2 Both $(\text{Proj-}R, \text{Mod-}R)$ and $(\text{Mod-}R, \text{Inj-}R)$ are cotorsion pairs. Moreover, they are complete.

5.1 Hereditariety

Definition 16 Let $\mathcal{X}$ be a full subcategory of an abelian category $\mathcal{C}$, then:

(a) $\mathcal{X}$ is *resolving* if whenever there is a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with X and X'' in $\mathcal{X}$, then $X' \in \mathcal{X}$,

(b) $\mathcal{X}$ is *coresolving* if whenever there is a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with X and X' in $\mathcal{X}$, then $X'' \in \mathcal{X}$,

Using the notions of resolving and coresolving classes, we can define hereditary cotorsion pairs as:

Definition 17 A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *hereditary* if $\mathcal{A}$ is resolving and $\mathcal{B}$ is coresolving.

Lemma 2 *Let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair in an abelian category $\mathcal{C}$. TFAE:*

(a) $\operatorname{Ext}^n_{\mathcal{C}}(A, B) = 0$ *for all* $A \in \mathcal{A}$, $B \in \mathcal{B}$, $n \geq 1$*;*

(b) $\operatorname{Ext}^2_{\mathcal{C}}(A, B) = 0$ *for all* $A \in \mathcal{A}$, $B \in \mathcal{B}$*;*

(c) $\mathcal{A}$ *is resolving;*

(d) $\mathcal{B}$ *is coresolving.*

Example 3 Both (Proj-R, Mod-R) and (Mod-R, Inj-R) are a complete hereditary cotorsion pairs.

6 Cotorsion pairs in $\mathbf{C}(R)$

Definition 18 Let $X^\bullet, Y^\bullet \in \mathbf{C}(R)$. A map $f : X^\bullet \to Y^\bullet$ is *null homotopic* if there are morphisms $s^i : X^i \to Y^{i-1}$ as in

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r, "d_X^{i-1}"] \arrow[d, "f^{i-1}"] & X^i \arrow[r, "d_X^i"] \arrow[d, "f^i"] \arrow[dl, "s^i"'] & X^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] \arrow[dl, "s^{i+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d_Y^{i-1}"'] & Y^i \arrow[r, "d_Y^i"'] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $f^i = s^{i+1} \circ d_X^i + d_Y^{i-1} \circ s^i$.

Let's define some classes of complexes starting from cotorsion pairs in the category of modules.

Definition 19 Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod-R. Define the following classes in $\mathbf{C}(R)$:

- $\widetilde{\mathcal{A}}$ is the class of exact complexes $A^\bullet$ such that $A \in \mathcal{A}$ and, moreover, $\operatorname{Ker} d_A^i \in \mathcal{A}$ for all $i \in \mathbb{Z}$. Same for $\widetilde{\mathcal{B}}$.
- $\operatorname{dg} \mathcal{A}$ is the class of complexes $A^\bullet$ such that $A^i \in \mathcal{A}$ and, moreover, any map $f : A^\bullet \to B^\bullet$ with $B^\bullet \in \widetilde{\mathcal{B}}$ is null homotopic. Similarly, $\operatorname{dg} \mathcal{B}$ is the class of complexes $B^\bullet$ with $B^i \in \mathcal{B}$ and such that any map $f : A^\bullet \to B^\bullet$ with $A \in \widetilde{\mathcal{A}}$ is null homotopic.

Lemma 3 *If $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in* Mod-R*, then $(dg\, \mathcal{A}, \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, dg\, \mathcal{B})$ are complete hereditary cotorsion pairs in $\mathbf{C}(R)$.*

Example 4 Let $\mathcal{E}$ be the class of all exact complexes in $\mathbf{C}(R)$, then the following are complete hereditary cotorsion pairs:

- $(\widetilde{\text{Proj-}R}, \mathbf{C}(R))$ and $(\mathbf{C}(R), \widetilde{\text{Inj-}R})$,
- $(\operatorname{dg} \text{Proj-}R, \mathcal{E})$ and $(\mathcal{E}, \operatorname{dg} \text{Inj-}R)$.

7 Model categories

We want to build the proper setting in which it is possible to invert a certain class of morphisms. As an example, consider the case of topological spaces, where we might want to invert moprhisms that are not really homeomorphisms, but just homotopies.

The proper setting in which to invert morphisms is that of model categories, i.e. categories with model structures. It requires some work to prove that a certain category has a model structure, however once done this immediately carries as a consequence the existence of a homotopy category in which the class of homotopies we chose is inverted.

7.1 Accessory definitions

Up to now we have always talked about classes of objects in a category. However, in order to define model structure we will need to talk about classes of morphisms. In the end, we will see that for abelian categories the morphisms of interest can be fully recovered from some cotorsion pairs, so we will switch back to talking about objects. In the meantime, we will give some definitions for classes of morphisms.

Definition 20 Let W be a class of morphisms in a category $\mathcal{C}$, then W is said to satisfy the *2-out-of-3* property if, given two composable morphisms f and g, whenver any two of f, g, and $g \circ f$ belong to W, so does the third.

Definition 21 Let W be a class of morphisms in a category $\mathcal{C}$, then W is said to be *closed under retracts* if, for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

such that $f \in W$ and the rows compose to the identity morphism, then $g \in W$ too.

Definition 22 Let f and g be two morphisms in $\mathcal{C}$. We say that f has the *left lifting property* for g and g has the *right lifting property* for f if for any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a diagonal morphism depitched by the dashed arrow such that both triangle commute.

7.2 Model structures

Definition 23 A *model structure* on a category $\mathcal{C}$ is a triple of classes of morphisms $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ called respectively cofibrations, weak equivalences and fibrations, satisfying:

(a) W satisfies the 2-out-of-3 property;

(b) Cof, W and Fib are closed under retracts;

(c) Maps in $\mathrm{Cof} \cap \mathrm{W}$ have the left lifting property with respect to maps in Fib, and similarly for Cof and $\mathrm{W} \cap \mathrm{Fib}$;

(d) Any map f in $\mathcal{C}$ can be factorized as $f = p \circ i$ with $p \in \mathrm{Fib}$ and $i \in \mathrm{Cof} \cap \mathrm{W}$. Moreover, f can be also factorized as $f = p' \circ i'$ with $p' \in \mathrm{W} \cap \mathrm{Fib}$ and $i' \in \mathrm{Cof}$.

Definition 24 A *model category* is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

7.3 Constructions in model categories

Maybe the most significative construction in model categories is that of fibrant and cofibrant replacements. We will see later that morphisms in the homotopy category can be represented by morphisms between cofibrant and fibrant objects.

Definition 25 Let X be an object in a model category $\mathcal{C}$.

- X is *cofibrant* if $0 \to X$ is a cofibration,
- X is *fibrant* if $X \to 0$ is a fibration.

Take $X \in \mathcal{C}$. From the factorization axiom the map $0 \to X$ can be factorized as

$$0 \xrightarrow{\in \mathrm{Cof}} CX \xrightarrow{\in \mathrm{W} \cap \mathrm{Fib}} X$$

CX is called a *cofibrant replacement*.

Dually $X \to 0$ can be factored as

$$X \xrightarrow{\in \mathrm{Cof} \cap \mathrm{W}} FX \xrightarrow{\in \mathrm{Fib}} 0$$

FX is called a *fibrant replacement*.

7.4 The homotopy category

Given a category $\mathcal{C}$ and a class W of weak equivalences in $\mathcal{C}$, form the free category $F(\mathcal{C}, \mathrm{W}^{-1})$ defined as follows:

- The objects of $F(\mathcal{C}, \mathrm{W}^{-1})$ are the objects of $\mathcal{C}$;
- A morphism is a finite string of composable arrows $(f_1, \ldots, f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of W.

We can picture a morphism between A and B in $F(\mathcal{C}, \mathrm{W}^{-1})$ as

$$A \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_3 \xleftarrow{w_3} X_4 \xrightarrow{f_4} \cdots \xleftarrow{w_n} B$$

Definition 26 With $\mathcal{C}$ and W as before, the homotopy category $\mathbf{Ho}(\mathcal{C})$ is the quotient category of $F(\mathcal{C}, \mathrm{W}^{-1})$ by the relations:

- $1_A = (1_A)$ for all objects A;
- $(f, g) = (g \circ f)$ for all composable arrows f and g in $\mathcal{C}$;
- $1_{\mathrm{dom}\, w} = (w, w^{-1})$ and $1_{\mathrm{codom}\, w} = (w^{-1}, w)$ for all w in W.

Remark 2 With this construction there is a functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ called quotient functor. Observe that under γ all maps in W map to isomorphisms in the category $\mathbf{Ho}(\mathcal{C})$. Moreover, $\mathbf{Ho}(\mathcal{C})$ satisfy a universal property (this time involving categories, so the arrows are functors):

(a) For any category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(w)$ is an isomorphism there is a unique functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $F = \mathbf{Ho}(F) \circ \gamma$, i.e. there is a unique dotted arrow such that the diagram

$$\begin{tikzcd} & \mathbf{Ho}(\mathcal{C}) \arrow[dd, dashed, "\mathbf{Ho}(F)"] \\ \mathcal{C} \arrow[ur, "\gamma"] \arrow[dr, "F"'] & \\ & \mathcal{D} \end{tikzcd}$$

is commutative.

(b) If $\mathcal{E}$ is a category and $\delta : \mathcal{C} \to \mathcal{E}$ satisfy the property (1), then there is a unique isomorphism of categories $I : \mathbf{Ho}(\mathcal{C}) \to \mathcal{E}$ such that $I \circ \gamma = \delta$.

Remark 3 Since the morphisms $\mathrm{Hom}_{\mathbf{Ho}(\mathcal{C})}(A, B)$ are defined as (equivalence classes) of diagrams, they do not necessarily form a set. Hence, the construction above is just formal, i.e. it does not necessarily yields a category. However, it does for model categories.

Theorem 1 *Let $\mathcal{C}$ be a model category, indicate with* $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ *the model structure. Denote by $\mathcal{C}_{cf}$ the full subcategory of objects that are both cofibrant and fibrant.*

(a) There is a category equivalence

$$(\mathcal{C}_{cf}/\sim) \to \boldsymbol{Ho}(\mathcal{C}).$$

In particular, $\boldsymbol{Ho}(\mathcal{C})$ is a legal category with small homomorphisms spaces.

(b) There are canonical isomorphisms

$$\frac{\mathrm{Hom}_{\mathcal{C}}(CX, FY)}{\sim} \xrightarrow{\cong} \mathrm{Hom}_{\boldsymbol{Ho}(\mathcal{C})}(X, Y)$$

for arbitrary X, $Y \in \mathcal{C}$.

8 Abelian Model Structures

The link between cotorsion pairs and model categories is given by a particular class of model structures, called abelian. These model strcutures are completely defined by the classes of of cofibrant and fibrant objects, and we will see that there is a bijection between certain couple of cotorsion pairs and abelian model structures.

Definition 27 Let $\mathcal{C}$ be an abelian category. An *abelian model structure* $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ on $\mathcal{C}$ is a model structure such that Cof coincide with the class of monomorphisms with cofibrant cokernel, and Fib with the class of epimorphisms with fibrant kernel.

We recall categorical properties of subclasses of objcts in a category.

Definition 28 Let $\mathcal{W}$ be class of objects in a category $\mathcal{C}$, we say that:

- $\mathcal{W}$ is *closed under retracts* if for any diagram

$$X \longrightarrow Y \longrightarrow X$$

  where the row composes to the identity, if Y belongs to $\mathcal{W}$, then so does X.

- $\mathcal{W}$ *satisfies the 2-out-of-3 property* if for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

  whenever two objects among X, X', X'' belong to $\mathcal{W}$, so does the third.

The bijection works as follows.

Theorem 2 *Let $\mathcal{C}$ be an abelian category. There is a bijective correspondence between abelian model structures on $\mathcal{C}$ and triples of classes $(\mathcal{A}, \mathcal{W}, \mathcal{B})$ satisfying the following conditions:*

(a) $\mathcal{W}$ is closed under retracts and satisfies the 2-out-of-3 property,

(b) $(\mathcal{A}, \mathcal{W} \cap \mathcal{B})$ and $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ are complete cotorsion pairs in $\mathcal{C}$.

The correspondence assigns to $(\mathcal{A}, \mathcal{W}, \mathcal{B})$ the model structure $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ such that:

(a) $i \in \mathrm{Cof}$ if and only if i is a monomorphism with cokernel in $\mathcal{A}$,

(b) $w \in \mathrm{W}$ if and only if $w = w_d \circ w_i$ where w_i is a monomorphism with cokernel in $\mathcal{W}$ and w_d is an epimorphism with kernel in $\mathcal{W}$,

(c) $p \in \mathrm{Fib}$ if and only if p is an epimorphism with kernel in $\mathcal{B}$.

The inverse assigns to $(\mathrm{Cof}, \mathrm{W}, \mathrm{Fib})$ the triple $(\mathcal{A}, \mathcal{W}, \mathcal{B})$ where $\mathcal{A}$, $\mathcal{W}$, and $\mathcal{B}$ are the classes of cofibrant, trivial and fibrant objects respectively.
We call the triples of objects $(\mathcal{A}, \mathcal{W}, \mathcal{B})$ satisfying the conditions above Hovey triples.

With this result we have shown that finding an abelian model structure amounts to finding two complete cotorsion pairs and a class of trivial objects $\mathcal{W}$. However, sometimes it is hard to give an explicit description of $\mathcal{W}$ and this can make the construction significantly harder. Fortunately, when the cotorsion pairs are hereditary in addition to being complete, we can find model structures using the following lemma.

Lemma 4 *Let $(\mathcal{A}, \mathcal{B}')$ and $(\mathcal{A}', \mathcal{B})$ be two complete hereditary cotorsion pairs in an abelian category $\mathcal{C}$ such that $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq \mathcal{B}$, and $\mathcal{A}' \cap \mathcal{B} = \mathcal{A} \cap \mathcal{B}'$. Then, there is a class $\mathcal{W} \subseteq \mathcal{C}$ such that $(\mathcal{A}, \mathcal{W}, \mathcal{B})$ is a Hovey triple.*

8.1 Examples

Our first example builds model structures in $\mathbf{C}(R)$ using the cotorsion pairs coming from the class of projective modules, from the class of injective modules, and finally from any complete hereditary cotorsion pair in Mod-R.

Example 5

(a) Recall that the projective cotorsion pair $(\text{Proj-}R, \text{Mod-}R)$ in Mod-R induces two cotorsion pairs in the category $\mathbf{C}(R)$, namely $(\text{dg}\,\text{Proj-}R, \mathcal{E})$ and $(\widetilde{\text{Proj-}R}, \mathbf{C}(R))$.

The inclusion $\mathcal{E} \in \mathbf{C}(R)$ is obvious and it is easy to check that $\widetilde{\text{Proj-}R} \subseteq \text{dg}\,\text{Proj-}R$. Actually, one can prove that $\text{dg}\,\text{Proj-}R \cap \mathcal{E} = \widetilde{\text{Proj-}R}$, so the conditions of the above lemma are satisfied and there is a corresponding Hovey triple $\mathcal{M} = (\text{dg}\,\text{Proj-}R, \mathcal{E}, \mathbf{C}(R))$. Observe that we have an explicit description of the class of trivial objects, which is exactly $\mathcal{E}$ in this case.

The homotopy category $\mathbf{Ho}(\mathcal{M}) = \mathbf{D}(R)$ is the derived category of the ring R.

(b) Similarly, from $(\text{Mod-}R, \text{Inj-}R)$ we get a Hovey triple $\mathcal{M}' = (\mathbf{C}(R), \mathcal{E}, \text{dg}\,\text{Inj-}R)$ such that $\mathbf{Ho}(\mathcal{M}') = \mathbf{D}(R)$.

(c) The same construction generalize to any complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R, giving the Hovey triple $\mathcal{M}'' = (\text{dg}\,\mathcal{A}, \mathcal{E}, \text{dg}\,\mathcal{B})$ such that $\mathbf{Ho}(\mathcal{M}'') = \mathbf{D}(R)$.

Remark 4 Observe that in the three examples above the homotopy category is always the same. This follows from the fact that the homotopy category is defined using only the weak equivalences, which in turn depend only on the class of trivial objects, which is the class of exact complexes $\mathcal{E}$ in all three cases.

We will now give another example in a category that is not abelian, however it has enough structure for all the results above to work anyway. Let Flat-R be the class of Flat R-Modules and define the cotorsion modules Cot-R as the class of modules such that $(\text{Flat-}R, \text{Cot-}R)$ is a complete cotorsion pair in Mod-R. It turns out that it is also an hereditary cotorsion pair.

Example 6 Consider the category $\mathbf{C}(\text{Cot-}R)$, i.e. the full subcategory of $\mathbf{C}(R)$ of complexes with terms in Cot-R. Moreover, take the exact structure to be the collection of degreewise split short exact sequences. This category, denoted by $\mathbf{C}(\text{Cot-}R)_{dw}$, is a Frobenius category whose class of projective-injective objects $\mathcal{W}$ is the class of contractible complexes with terms in Cot-R.

Then there are three abelian[(1)] model structures defined as follows:

- $\mathcal{M}_1 = (\mathbf{C}(\text{Cot-}R)_{dw}, \mathcal{W}, \mathbf{C}(\text{Cot-}R)_{dw})$,
- $\mathcal{M}_2 = (\mathbf{C}(\text{Cot-}R)_{dw}, [{}^{\perp}\widetilde{\text{Cot-}R} \cap \mathbf{C}(\text{Cot-}R)]_K, \widetilde{\text{Cot-}R})$,

[(1)] In this context they are actually referred to as *exact model structure*, due to the category being *exact* rather than abelian.

- $\mathcal{M}_3 = (\mathbf{C}(\text{Cot-}R)_{dw}, \widetilde{\text{Cot-}R}, [\text{dg Inj-}R]_K)$,

where $[\mathcal{X}]_K$ denotes the class of the complexes that are cochain homotopy equivalent to some complex in $\mathcal{X}$.

The homotopy categories are respectively:

- $\mathbf{Ho}(\mathcal{M}_1) = \mathbf{K}(\text{Cot-}R)$, i.e. the category $\mathbf{C}(\text{Cot-}R)$ modulo cochain homotopy,
- $\mathbf{Ho}(\mathcal{M}_2) = \dfrac{\widetilde{\text{Cot-}R}}{\sim}$, i.e. the subcategory of $\mathbf{K}(\text{Cot-}R)$ consisting of exact complexes with terms in Cot-R (this is a consequence of the fact that $\widetilde{\text{Cot-}R} = \mathbf{C}(\text{Cot-}R) \cap \mathcal{E}$ recently proved in [BCE17]),
- $\mathbf{Ho}(\mathcal{M}_3)$ is taken as the definition of $\mathbf{D}(\text{Cot-}R)$, the derived category of Cot-R.

Moreover, there is a recollement:

$$\begin{tikzcd} \dfrac{\widetilde{\text{Cot-}R}}{\sim} \arrow[r] & \mathbf{K}(\text{Cot-}R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathbf{D}(\text{Cot-}R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

References

[BCE17] Silvana Bazzoni, Manuel Cortés Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes.* eprint: arXiv : 1704.06672 (2017).

[BT17] Silvana Bazzoni and Marco Tarantino, *Recollements from Cotorsion Pairs.* eprint: arXiv:1712.04781 (2017).

Brascamp-Lieb inequalities and heat-flow monotonicity

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Abstract. Many well-known multilinear integral inequalities in Euclidean analysis, such as Hlder's and Young's inequalities, share a common feature: they are instances of the so called Brascamp-Lieb inequalities. In this note we will describe these inequalities and show how they can be derived as a consequence of a monotonicity property associated to the heat flow. We will also discuss to what extent some of the ideas and techniques can be adapted to non-Euclidean settings. In particular we will present a family of inequalities on real spheres involving functions that possess some kind of symmetry.

1 Heat equation on $\mathbb{R}^n$

Let H be a finite dimensional real vector space endowed with a scalar product $\langle \cdot, \cdot \rangle$. Without loss of generality we can set our problem in $\mathbb{R}^n$ with the standard scalar product, by fixing an orthonormal basis in H and using coordinates $(x_1, \dots, x_n)$. In this section we will recall some basic facts about the heat equation on $\mathbb{R}^n$. Let $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ be the standard Laplacian. We say that a function $u : \mathbb{R}_t^+ \times \mathbb{R}_x^n \to \mathbb{R}$ solves the Cauchy problem for the heat equation with initial datum $f : \mathbb{R}^n \to \mathbb{R}$ if

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \\ u(0,x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

This equation is used to model the heat diffusion in a medium. With this interpretation the function $u(t,x)$ represents the temperature field at time t in the point x. We will not bother about the smoothness assumptions on the initial datum and we will always assume that all functions have enough regularity to make the computations work.
It is known that for $t > 0$ one can obtain the solution to this problem by integrating the initial datum against shifted Gaussian functions:

$$u(t,x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy, \tag{1}$$

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where dy is the Lebesgue measure on $\mathbb{R}^n$. The function $H_n^t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ is called heat kernel. We now show some properties of a solution of the heat equation:

- Even if the expression (1) is not defined for $t = 0$, for f smooth enough one can see that $\lim_{t\to 0} u(t,x) = f(x)$ for all $x \in \mathbb{R}^n$, so that the initial datum is recovered in a limit sense.
- It is evident from expression (1) that if f is non-negative then so is $u(t,\cdot)$ for every $t > 0$.
- For every $t > 0$, $\int_{\mathbb{R}^n} H_n^t(x)dx = 1$.
- If the initial datum is integrable, the total mass of the solution is preserved at each time. This is a consequence of Fubini's theorem and invariance under translations of the Lebesgue measure:

$$\begin{aligned}\int_{\mathbb{R}^n} u(t,x)dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} H_n^t(x-y)f(y)dydx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} H_t^n(x-y)dx\right)f(y)dy = \int_{\mathbb{R}^n} f(y)dy.\end{aligned}$$

- If the initial datum is integrable, for fixed $x \in \mathbb{R}^n$ we see again from expression (1) that $\lim_{t\to\infty} u(t,x) = 0$. Intuitively this is quite natural, as the heat will tend to diffuse all over $\mathbb{R}^n$, and so the temperature cannot be concentrated at a specific point.

2 Loomis-Whitney inequality

In this section we will introduce Loomis-Whitney inequality, which will be our toy model to test the heat flow technique. Let $f_j : \mathbb{R}^{n-1} \to \mathbb{R}^+$. Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear map that forgets the j-th variable. For instance, the map π_1 is the linear map given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have the following

Theorem 2.1 (Loomis-Whitney inequality, [4]) *Under the above assumptions*

$$\int_{\mathbb{R}^n} f_1(\pi_1 x)\dots f_n(\pi_n x)dx \le \|f_1\|_{L^{n-1}(\mathbb{R}^{n-1})}\dots\|f_n\|_{L^{n-1}(\mathbb{R}^{n-1})}, \tag{2}$$

where $\|g\|_{L^{n-1}(\mathbb{R}^{n-1})} = \left(\int_{\mathbb{R}^{n-1}} |g(y)|^{n-1}dy\right)^{\frac{1}{n-1}}$.

We notice that on the right-hand side of (2) we have the same L^p norm for all functions, and that the integrations on the right-hand side are taken over $\mathbb{R}^{n-1}$, which is the space where the f_j are defined.
Loomis-Whitney inequality has some interesting applications: it can be used to obtain

the Sobolev-Gagliardo-Nirenberg inequality and it is related to a non-sharp version of the isoperimetric inequality. In fact, if E is a regular domain in $\mathbb{R}^n$, noticing that $\chi_E(x) \leq \prod_{j=1}^n \chi_{\pi_j(E)}(\pi_j x)$ and that $|\pi_j(E)| \leq |\partial E|$ for every j, an application of Loomis-Whitney inequality yelds:

$$\begin{aligned}|E| = \int_{\mathbb{R}^n} \chi_E(x)dx &\leq \int_{\mathbb{R}^n} \prod_{j=1}^n \chi_{\pi_j(E)}(\pi_j x)dx \\ &\leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \chi_{\pi_j(E)}(y)dy \right)^{\frac{1}{n-1}} = \prod_{j=1}^n |\pi_j(E)|^{\frac{1}{n-1}} \\ &\leq |\partial E|^{\frac{n}{n-1}}.\end{aligned}$$

In dimension $n = 2$, by Tonelli's theorem, the inequality is an equality:

$$\int_{\mathbb{R}^2} f_1(x_2)f_2(x_1)dx_1dx_2 = \int_{\mathbb{R}} f_1(x_2)dx_2 \int_{\mathbb{R}} f_2(x_1)dx_1.$$

In dimension $n = 3$ the inequality reads

$$\begin{aligned}&\int_{\mathbb{R}^3} f_1(x_2,x_3)f_2(x_1,x_3)f_3(x_1,x_2)dx_1dx_2dx_3 \\ &\leq \left(\int_{\mathbb{R}^2} f_1(x_2,x_3)^2dx_2dx_3 \right)^{1/2} \left(\int_{\mathbb{R}^2} f_2(x_1,x_3)^2dx_1dx_3 \right)^{1/2} \left(\int_{\mathbb{R}^2} f_3(x_1,x_2)^2dx_1dx_2 \right)^{1/2},\end{aligned}$$

or equivalently, by taking the square root of each function,

$$\int_{\mathbb{R}^3} f_1(x_2,x_3)^{1/2}f_2(x_1,x_3)^{1/2}f_3(x_1,x_2)^{1/2}dx_1dx_2dx_3 \leq \prod_{j=1}^3 \left(\int_{\mathbb{R}^2} f_j \right)^{1/2}.$$

From now on we will work with the Loomis-Whitney inequality in $\mathbb{R}^3$, in order to simplify the notation. There is a quick and direct way to prove this inequality which uses the following classical result in real analysis:

Theorem 2.2 (Multilinear Hölder's inequality) *Let $f_j : \mathbb{R}^n \to \mathbb{R}^+$ and let $1 \leq p_j \leq \infty$, $j = 1, \dots, m$ such that $\sum_{j=1}^m 1/p_j = 1$. Then*

$$\int_{\mathbb{R}^n} f_1(y) \dots f_m(y)dy \leq \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

We will need just the case $m = 2$, $p_1 = p_2 = 2$. In this case Hölder's inequality is also known as Cauchy-Schwarz inequality.
Starting from the left-hand side of (2), by applying Fubini's theorem we get

$$\begin{aligned}&\int_{\mathbb{R}^3} f_1(x_2,x_3)^{1/2}f_2(x_1,x_3)^{1/2}f_3(x_1,x_2)^{1/2}dx_1dx_2dx_3 \\ &= \int_{\mathbb{R}^2} f_1(x_2,x_3)^{1/2} \left(\int_{\mathbb{R}} f_2(x_1,x_3)^{1/2}f_3(x_1,x_2)^{1/2}dx_1 \right) dx_2dx_3.\end{aligned}$$

Next we apply Cauchy-Schwarz inequality twice, the first time $\mathbb{R}^2$ with measure dx_2dx_3 and the second time on $\mathbb{R}$ with measure dx_1:

$$\begin{aligned}
&\int_{\mathbb{R}^2} f_1(x_2,x_3)^{1/2}\left(\int_{\mathbb{R}} f_2(x_1,x_3)^{1/2} f_3(x_1,x_2)^{1/2}dx_1\right)dx_2dx_3\\
&\le \left(\int_{\mathbb{R}^2} f_1\right)^{1/2}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} f_2(x_1,x_3)^{1/2} f_3(x_1,x_2)^{1/2}dx_1\right)^2 dx_2dx_3\right)^{1/2}\\
&\le \left(\int_{\mathbb{R}^2} f_1\right)^{1/2}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} f_2(x_1,x_3)dx_1\int_{\mathbb{R}} f_3(x_1,x_2)dx_1\right) dx_2dx_3\right)^{1/2}\\
&= \left(\int_{\mathbb{R}^2} f_1\right)^{1/2}\left(\int_{\mathbb{R}^2} f_2\right)^{1/2}\left(\int_{\mathbb{R}^2} f_3\right)^{1/2},
\end{aligned}$$

where we concluded again by Fubini's theorem.
Before stating the main results of this section we observe that the pullbacks $f_j \circ \pi_j$ to $\mathbb{R}^3$ of the non-negative functions $f_j : \mathbb{R}^2 \to \mathbb{R}^+$, are functions that are constant on lines parallel to the j-th axis, i.e. they don't depend on the j-th variable. This symmetry can be encoded by the fact that, for all $x \in \mathbb{R}^3$

$$\partial_{x_j}\left(f_j \circ \pi_j\right)(x) = 0. \tag{3}$$

Actually, any function f on $\mathbb{R}^3$ with property (3) can be written as a function g on $\mathbb{R}^2$ pulled-back to $\mathbb{R}^3$ (g is simply the evaluation of the function f on any plane orthogonal to the j-th direction), so we obtain some kind of characterization of this type of symmetry.
One key feature of the heat equation is that this symmetries are preserved along the flow. To see this, let $f : \mathbb{R}^2 \to \mathbb{R}$ and consider its pullback to $\mathbb{R}^3$ by, say, the projection π_1, $f \circ \pi_1$, and the evolution of this function under the 3-dimensional heat equation $u(t,x)$. We have

$$\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^3} H_3^t(x-y) f(\pi_1 y)dy\\
&= (4\pi t)^{-3/2}\int_{\mathbb{R}^3} e^{-[(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2]/4t} f(y_2,y_3)dy_1dy_2dy_3\\
&= (4\pi t)^{-3/2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-[(x_1-y_1)^2]/4t}dy_1\right) e^{-|\pi_1 x-(y_2,y_3)|^2/4t} f(y_2,y_3)dy_2dy_3\\
&= (4\pi t)^{-1}\int_{\mathbb{R}^2} e^{-|\pi_1 x-\tilde{y}|^2/4t} f(\tilde{y})d\tilde{y}\\
&= \int_{\mathbb{R}^2} H_2^t(\pi_1 x - y) f(y)dy = \tilde{u}(t,\pi_1 x),
\end{aligned}$$

where we used Fubini's theorem. Notice that $\tilde{u}$ solves the 2-dimensional heat equation with initial datum f. So we showed that if the initial datum is a function which depends only on a projection of the point, so will be the solution at each time. Now we are ready to state the main result of this section. Consider $f_j : \mathbb{R}^2 \to \mathbb{R}^+$, for $j = 1,2,3$. We indicate with $u_j(t,x)$ the evolution of $f_j \circ \pi_j$ under the heat flow in $\mathbb{R}^3$.

Theorem 2.3 ([2]) *The functional*

$$\phi(t) = \int_{\mathbb{R}^3} u_1(t,x)^{1/2} u_2(t,x)^{1/2} u_3(t,x)^{1/2} dx$$

is non-decreasing for $t > 0$.

As a corollary we obtain:

Corollary 2.4 (Loomis-Whitney inequality)

$$\begin{aligned}&\int_{\mathbb{R}^3} f_1(\pi_1 x)^{1/2} f_2(\pi_2 x)^{1/2} f_3(\pi_3 x)^{1/2} dx \\ &\leq \left(\int_{\mathbb{R}^2} f_1\right)^{1/2} \left(\int_{\mathbb{R}^2} f_2\right)^{1/2} \left(\int_{\mathbb{R}^2} f_3\right)^{1/2}.\end{aligned}$$

First let's see how we obtain the corollary from the theorem.

Proof. (of Corollary) By the Theorem 2.3, we get that

$$\lim_{t\to 0} \phi(t) \leq \lim_{t\to\infty} \phi(t).$$

The limit as $t \to 0$ is readily seen to be the left-hand side. For the right-hand side we notice that

$$\begin{aligned}\phi(t) &= \int_{\mathbb{R}^3} \prod_{j=1}^{3} \left((4\pi t)^{-1} \int_{\mathbb{R}^2} e^{-|\pi_j x - y|^2/4t} f_j(y) dy\right)^{1/2} dx \\ &= (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \prod_{j=1}^{3} \left(\int_{\mathbb{R}^2} e^{-|\pi_j t^{-1/2} x - t^{-1/2} y|^2/4} f_j(y) dy\right)^{1/2} dx \\ &= (4\pi t)^{-3/2} t^{3/2} \int_{\mathbb{R}^3} \prod_{j=1}^{3} \left(\int_{\mathbb{R}^2} e^{-|\pi_j \tilde{x} - t^{-1/2} y|^2/4} f_j(y) dy\right)^{1/2} d\tilde{x},\end{aligned}$$

performing the change of variable $\tilde{x} = t^{-1/2} x$. Taking the limit as $t \to \infty$ we get

$$\lim_{t\to\infty} \phi(t) = (4\pi)^{-3/2} \prod_{j=1}^{3} \left(\int_{\mathbb{R}^2} f_j\right)^{1/2} \int_{\mathbb{R}^3} e^{-(|\pi_1 x|^2 + |\pi_2 x|^2 + |\pi_3 x|^2)/8} dx.$$

But $|\pi_j x|^2 = \langle \pi_j^* \pi_j x, x\rangle$. Notice that

$$\pi_1^* \pi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly for $j = 2, 3$. Hence

$$|\pi_1 x|^2 + |\pi_2 x|^2 + |\pi_3 x|^2 = \langle (\pi_1^*\pi_1 + \pi_2^*\pi_2 + \pi_3^*\pi_3) x, x\rangle = \langle 2\,\mathrm{Id}\, x, x\rangle = 2|x|^2.$$

We may then conclude that

$$\lim_{t\to\infty}\phi(t)=\prod_{j=1}^{3}\left(\int_{\mathbb{R}^2} f_j\right)^{1/2}(4\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|x|^2/4}dx=\prod_{j=1}^{3}\left(\int_{\mathbb{R}^2} f_j\right)^{1/2}.$$

□

In order to prove Theorem 2.3 we first state a Lemma. Let $\vec{a}(x)=(a_1(x),\ldots,a_n(x))$ be a vector field on $\mathbb{R}^n$. Recall that the divergence of $\vec{a}$ is the scalar field defined as

$$\operatorname{div}(\vec{a})(x)=\sum_{j=1}^{n}\partial_{x_j}a_j(x).$$

Let $u:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^+$ and $\vec{v}:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^n$.

Lemma 2.5 ([2]) *Let $u,\vec{v}$ as above be sufficiently smooth and such that $\vec{v}u$ is rapidly decreasing at spatial infinity. Suppose that u satisfies the transport inequality*

$$\partial_t u(t,x)+\operatorname{div}\left(\vec{v}(t,x)u(t,x)\right)\geq 0$$

for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$. Then the functional

$$\phi(t)=\int_{\mathbb{R}^n}u(t,x)dx$$

is non-decreasing for $t>0$.

Sketch of proof. By the divergence theorem

$$\int_{\mathbb{R}^n}\operatorname{div}\left(\vec{v}(t,x)u(t,x)\right)dx=0,$$

for all $t>0$, since $u\vec{v}$ is rapidly decrasing at spatial infinity. Let $t_1<t_2$. Formally, we have

$$\begin{aligned}
\phi(t_2)-\phi(t_1)&=\int_{t_1}^{t_2}\partial_t\phi(t)dt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\partial_t u(t,x)dxdt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\partial_t u(t,x)+\operatorname{div}\left(\vec{v}(t,x)u(t,x)\right)dxdt\geq 0.
\end{aligned}$$

So $\phi(t_2)\geq\phi(t_1)$. This naive argument can be made rigorous with some small modifications. □

Now we are ready to prove Theorem 2.3.

Proof. By Lemma 2.5 it sufficies to prove that

$$u(t,x) := \prod_{j=1}^{3} u_j(t,x)^{1/2}$$

satisfies the transport inequality

$$\partial_t u(t,x) + \operatorname{div}\left(\vec{v}(t,x)u(t,x)\right) \geq 0.$$

We take as $\vec{v} = -1/2 \sum_{j=1}^{3} \nabla(\log u_j)$. Observe that, since $u_j(t,x)$ depends only on $\pi_j x$, one has

$$\nabla(\log u_j) = \pi_j^* \pi_j \nabla(\log u_j).$$

We have

$$\begin{aligned}
&\partial_t \prod_{j=1}^{3} u_j(t,x)^{1/2} + \operatorname{div}\left(\vec{v}(t,x) \prod_{j=1}^{3} u_j(t,x)^{1/2}\right) \\
&= u(t,x) \left(\frac{1}{2} \sum_{j=1}^{3} \frac{\partial_t u_j}{u_j} + \operatorname{div}(\vec{v}) + \frac{1}{2} \left\langle \vec{v}, \sum_{j=1}^{3} \frac{\nabla u_j}{u_j} \right\rangle \right).
\end{aligned}$$

The first factor is positive and the second factor can be written as

$$\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{3} \frac{1}{u_j} \left(\partial_t u_j + \operatorname{div}\left(-\frac{\nabla u_j}{u_j} u_j \right) \right) - \frac{1}{2} \sum_{j=1}^{3} \frac{1}{u_j} \operatorname{div}\left(-\frac{\nabla u_j}{u_j} u_j \right) + \\
&+ \operatorname{div}(\vec{v}) + \frac{1}{2} \left\langle \vec{v}, \sum_{j=1}^{3} \frac{\nabla u_j}{u_j} \right\rangle.
\end{aligned}$$

Recalling that $\operatorname{div}(\nabla) = \Delta$, we see that the first summand is zero, since each of the functions u_j solve a heat equation. The remaining part can be arranged as

$$\frac{1}{2} \left(\sum_{j=1}^{3} \left\langle \pi_j^* \pi_j \left(\vec{v} + \nabla \log u_j \right), \left(\vec{v} + \nabla \log u_j \right) \right\rangle \right) \geq 0,$$

since $\pi_j^* \pi_j$ is positive semi-definite. This completes the proof. □

3 Brascamp-Lieb inequalities

The heat-flow method that we used to prove Loomis-Whitney inequality can be generalized to prove a larger family of inequalities, the so-called Brascamp-Lieb inequalities. A Brascamp-Lieb inequality is an inequality of the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(B_j x) dx \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{n_j})}, \tag{4}$$

with $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+$ non-negative functions, $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ linear maps, and $p_j \geq 1$. We already saw two examples of this kind of inequalities:

- Multilinear Hölder's inequality, where $m \in \mathbb{N}$, $B_j = \mathrm{Id}_{\mathbb{R}^n}$ for all j and $\sum p_j^{-1} = 1$.
- Loomis-Whitney inequality, where $m = n$, $B_j = \pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ are the projections defined above, and $p_j = (n-1)$ for all j.

For the sake of completeness we state a necessary and sufficient condition for the finiteness of the constant C in inequality (4).

Theorem 3.1 *([1], [2]) The constant C in inequality (4) is finite if and only if*

$$\sum n_j p_j^{-1} = n$$

and for all subspaces $V \subseteq \mathbb{R}^n$,

$$\dim(V) \leq \sum p_j^{-1} \dim(B_j V).$$

4 Heat equation on the sphere $\mathbb{S}^{n-1}$

In the rest of the note we will try to adapt the heat flow method to a non Euclidean context: the sphere. Consider the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, with normalized uniform measure $d\sigma$. On the sphere we can consider a laplacian Δ, which can be seen as the restriction of the laplacian on $\mathbb{R}^n$ in spherical coordinates, or as the Laplace-Beltrami operator of the sphere seen as a Riemannian manifold.
Again we can consider the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) & (t,x) \in \mathbb{R}^+ \times \mathbb{S}^{n-1} \\ u(0,x) = f(x) & x \in \mathbb{S}^{n-1} \end{cases}$$

We will write $u(t,x) = (e^{t\Delta} f)(x)$, using a notation coming from semigroup theory. We now show some properties for the solution of a heat equation on the sphere.

- If f is non-negative then so is $u(t,\cdot)$ for every $t > 0$. This is less evident than the $\mathbb{R}^n$ case, but it can be seen quite easily via Fourier series expansion.
- The total mass is preserved at each time:

$$\int_{\mathbb{S}^{n-1}} u(t,y) d\sigma(y) = \int_{\mathbb{S}^{n-1}} f d\sigma.$$

- For fixed $x \in \mathbb{S}^{n-1}$ we see that $\lim_{t\to\infty} u(t,x) = \int_{\mathbb{S}^{n-1}} f d\sigma$. This is a difference with the case of $\mathbb{R}^n$, but intuitively it states the fact that in the compact setting of the sphere, the heat cannot "escape", so it will stabilize around the mean of the initial temperature at every point (notice that the quantity on the right-hand side is indeed the mean of the initial temperature, being $d\sigma$ normalized).

5 Brascamp-Lieb type inequalities on spheres

As a first instance of inequality on spheres we introduce the Carlen-Lieb-Loss inequality. Let $\pi_j : \mathbb{S}^{n-1} \to [-1,1]$ be the projection on the j-th coordinate and let $f_j : [-1,1] \to \mathbb{R}^+$, for $j = 1,\dots,n$. Consider their pullbacks on the sphere $\tilde{f}_j := f_j \circ \pi_j : \mathbb{S}^{n-1} \to \mathbb{R}^+$. The function $f_j \circ \pi_j$ depends only on one variable x_j, i.e. it is constant on $n-2$ dimensional subspheres which are intersections of $\mathbb{S}^{n-1}$ with the hyperplanes $x_j = k$.

Theorem 5.1 ([3]) *Under the above hypothesis, the following inequality holds:*

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^{n} \tilde{f}_j d\sigma \leq \prod_{j=1}^{n} \|\tilde{f}_j\|_{L^2(\mathbb{S}^{n-1})}.$$

Moreover the inequality is sharp, in the sense that for each $p < 2$ there exist functions $f_1,\dots,f_n$ with $\|\tilde{f}_j\|_{L^p(\mathbb{S}^{n-1})} < \infty$ such that the left-hand side diverges.

Notice that, with the constraint of having the same exponent on the right hand siede, a direct application of Hölder's inequality would give

$$\int_{\mathbb{S}^{n-1}} f_1(x_1)\dots f_n(x_n) d\sigma \leq \|f_1\|_{L^n(\mathbb{S}^{n-1})} \dots \|f_n\|_{L^n(\mathbb{S}^{n-1})}.$$

The sum of the reciprocals of the exponents is 1. By continuous embeddings of $L^p(\mathbb{S}^{n-1})$ spaces this sum can be made smaller, by making each exponent bigger. In the Carlen-Lieb-Loss inequality the exponents are smaller than n and the sum of the reciprocals is much bigger than 1: it is $n/2$. The feature that helps improving on Hölder's inequality is the symmetry of the functions involved.
For $n \geq 3$ it makes sense to consider blue functions depending on k variables, with $1 \leq k \leq n-1$. These functions are constant on $n-k-1$ dimensional subspheres, which are intersections of $\mathbb{S}^{n-1}$ with $(n-k)$-planes. We will use the notation $f(x_{i_1},\dots,x_{i_k})$ to indicate a function defined on the k-dimensional unit ball B_k in $\mathbb{R}^k$, pulled back to the sphere by the projection $\pi_{i_1,\dots,i_k}$ on the k variables involved.
As noted in the $\mathbb{R}^n$ case with the condition (3), these symmetries can be encoded in terms of annihilation by certain differential operators.
For functions depending on k variables we have the following result.

Proposition 5.2 *Let $k \leq n-2$ and let f_j be nonnegative functions on the sphere $\mathbb{S}^{n-1}$, for $j = 1,\dots,\binom{n}{k} := C(n,k)$, each depending on a different set of k variables. Then the following inequality holds:*

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^{C(n,k)} f_j d\sigma \leq \prod_{j=1}^{C(n,k)} \|f_j\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})} \tag{5}$$

with $\tilde{p} = \binom{n}{k} - \binom{n-2}{k}$. The inequality is sharp in the sense of Carlen-Lieb-Loss.

Since $\tilde{p} < \binom{n}{k}$, inequality (5) doesn't follow directly from Hölder's inequality. Notice that the Carlen-Lieb-Loss result is recovered for $k = 1$.

Remark 5.3 For a function f depending only on k variables $x_{i_1}, \dots, x_{i_k}$ we have

$$\int_{\mathbb{S}^{n-1}} f(x_{i_1}, \dots, x_{i_k}) d\sigma \sim \int_{B_k} f(x_{i_1}, \dots, x_{i_k})(1 - x_{i_1}^2 - \dots - x_{i_k}^2)^{\frac{n-k-2}{2}} dx_{i_1} \dots dx_{i_k},$$

so that

$$\|f\|_{L^p(\mathbb{S}^{n-1})} \lesssim \|f\|_{L^p(B_k)}.$$

Indicating with π_j the projections on all possible k-tuples of variables, we can rewrite the above inequality as

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^{C(n,k)} f_j(\pi_j x) d\sigma \lesssim \prod_{j=1}^{C(n,k)} \|f_j\|_{L^{\tilde{p}}(B_k)}.$$

This looks like a Brascamp-Lieb inequality, since on the left-hand side we have an integration of a product of functions pulled back to the sphere via some projections, compared with the product of L^p norms of the functions on their space of definition.

6 Heat flow monotonicity

To prove Proposition 5.2 we rely on the following theorem. Take $C(n,k)$ functions f_j, each depending on a different set of k variables.

Theorem 6.1 *The functional*

$$\phi(t) = \int_{\mathbb{S}^{n-1}} \prod_{j=1}^{C(n,k)} \left(e^{t\Delta} f_j^p\right)^{1/p} d\sigma \tag{6}$$

is non-decreasing for $p \geq \tilde{p}$.

As a corollary, for $p \geq \tilde{p}$ we get Proposition 5.2, by comparing the limits as $t \to 0$ and as $t \to \infty$:

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^{C(n,k)} f_j d\sigma = \lim_{t\to 0} \phi(t) \leq \lim_{t\to\infty} \phi(t) = \prod_{j=1}^{C(n,k)} \|f_j\|_{L^p(\mathbb{S}^{n-1})}.$$

We will not go into details but we describe the key ingredients in the proof of Theorem 6.1.

- An expression for the laplacian involving certain vector fields (a basis the Lie algebra of the group $SO(n)$, adapted to coordinates $(x_1, \dots, x_n)$).
- A closed formula for the time derivative of the functional (6).
- Symmetry preservation under the heat flow of the symmetries of the initial data.
- Some kind of non-degeneracy condition between the different classes of symmetry of the functions involved.

To conclude we mention that the proof of Theorem 6.1 can be generalized to the case of functions with different types of symmetry. The critical (sharp) exponent $\tilde{p}$ obtained in the general case is related in a combinatorical way to the interplay between the different symmetries involved.

References

[1] Barthe, F., *On a reverse form of the Brascamp-Lieb inequality.* Invent. Math. 134 (1998), 335–361.

[2] Bennett, J.,Carbery, A., Christ, M., Tao, T., *The Brascamp–Lieb inequalities: finiteness, structure and extremals.* Geometric and Functional Analysis 5 (2008), 1343–1415.

[3] Carlen, E. A., Lieb, E. H., Loss, M., *A sharp analog of Young's inequality on SN and related entropy inequalities.* The Journal of Geometric Analysis 3 (2004), 487–520.

[4] Loomis, L. H.,Whitney, H., *An inequality related to the isoperimetric inequality.* Bull. Amer. Math. Soc. 55 (1949), 961–962.

On a Particular Class of Singular Optimal Control Problems

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Abstract. Optimal control problems have been attracted by many researchers in recent years. Its applications can be found in different fields of sciences such as physics or economics. In this talk, we study a particular class of singular optimal control problems, that is, the control variable is represented by an increasing and right-continuous process. Therefore the control measure is allowed to be singular with respect to the Lebesgue measure. The optimal control and value of the problem are typically characterized by a partial differential equation, the so-called *Hamilton-Jacobi-Bellman* (HJB) equation. In a first step we study a *deterministic* setting in which the controlled system is governed by an ordinary differential equation. We derive the associated HJB equation by employing standard techniques such as Taylor's theorem. In a next step, we extend the model and consider a specific stochastic singular control problem. The model we have in mind is that of a firm which aims to maximize its profits from selling energy in the market. Here, the control variable represents the firm's installation strategy of solar panels in order to produce energy. We assume that the energy price follows an *Ornstein-Uhlenbeck process* and is affected by the installation strategy of the firm. We find that the optimal installation strategy is triggered by a threshold, the so-called *free boundary* which separates the *waiting region*, in which it is not optimal to install additional panels, and the *installation region* where it is optimal to install additional panels. Finally, our study is complemented by an analysis of the dependency of the optimal installation strategy on the model's parameters.

1 Introduction

In an optimal control problem the agent controls a parameter in a mathematical model to produce an optimal output, using some optimization technique. More precisely, the agent seeks to optimize an objective function, which is a function depending on the state and the control variable. The control variable usually affects the evolution of the state variable. A popular solution technique is the so-called Hamilton-Jacobi-Bellman (HJB) equation, a partial differential equation which characterizes the solution of the problem. Optimal control theory finds a wide range of applications in different fields. The interested reader may refer to La Torre et al. [3]. Here, we consider a control variable which is an increasing

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and right-continuous process. We therefore allow the control measure to be singular with respect to the Lebesgue measure. These kind of problems are known as singular control problems. For instance, in a stochastic framework, Chiarolla and Haussmann [2] study an irreversible investment model with a finite time horizon by introducing a singular control on the capital expansion. Also, Al Motairi and Zervos [1] consider an irreversible capital expansion in which additional investment has a negative effect on the underlying economic indicator.

In Section 2 we derive the HJB-equation in a specific deterministic setting with an infinite time horizon. Then, in Section 3 we consider a stochastic version of the model introduced in Section 2, and apply it to an irreversible installation problem. We consider a firm which aims to maximize its profits from selling energy in the market. The firm's control variable represents the production of solar energy that can be increased by installing additional solar panels. The energy price is hereby assumed to follow an *Ornstein-Uhlenbeck process* and is affected by the firm's strategy.

2 A Deterministic Singular Control Problem: The HJB equation

We consider a control variable which is a nonnegative and increasing right-continuous function $Y : [0,\infty) \to [0,\infty)$ such that

$$Y(0-) = y \geq 0, \text{ and } Y(t) \leq \overline{y}, \text{ for all } t \geq 0,$$

where $Y(0-) = \lim_{\epsilon \downarrow 0} Y(0-\epsilon)$ and $\overline{y} > y$. The set of those functions is denoted by $\mathcal{Y}$.

Following a control strategy $Y \in \mathcal{Y}$, the state of the system X evolves as

$$\dot{X}^Y(t) = b(X^Y(t), Y(t)), \quad X(0) = x \in \mathbb{R}, \tag{2.1}$$

and the objective function is given by

$$\mathcal{J}(x,y,Y) = \int_0^\infty e^{-\rho t} f(X^Y(t), Y(t))dt - c\int_0^\infty e^{-\rho t} dY(t), \tag{2.2}$$

where $b, f : \mathbb{R} \times [0,\overline{y}] \mapsto \mathbb{R}$ to be specified and $\rho, c > 0$. The last term in (2.2) is intended as a standard Lebesgue-Stieltjes integral. It has the clear interpretation that the control Y can be increased at marginal cost $c > 0$. The functions b and f are such that the ordinary differential equation (2.1) admits a unique solution X^Y and the objective function $\mathcal{J}$ is finite for any $Y \in \mathcal{Y}$.

The associated value function is defined by

$$V(x,y) := \sup_{Y \in \mathcal{Y}} \mathcal{J}(x,y,Y). \tag{2.3}$$

We say that $Y^\star \in \mathcal{Y}$ is an optimal control if

$$V(x,y) = \mathcal{J}(x,y,Y^\star).$$

Notice that

$$V(x,\overline{y}) = \int_0^\infty e^{-\rho t} f(X^{\overline{Y}}(t),\overline{y})dt, \tag{2.4}$$

where $\overline{Y}(t) = \overline{y}$ for any $t \geq 0$ is the only admissible strategy if $y = \overline{y}$.

The goal is to find a partial differential equation, the so-called HJB equation, which characterizes the value function V and $Y^\star$. This is achieved in the following.

2.1 HJB equation

The basic idea is to split the problem over time. This approach is known as *the Dynamic Programming Principle* (DPP). The HJB equation is then the infinitesimal version of the DPP. Applied to this setting, we employ that there are only two possible actions at each time: we can either increase the control variable Y immediately, or we do not act at all. The latter action is associated to

$$V(x,y) \geq \int_0^{\Delta t} e^{-\rho t} f(X^Y(t),y)dt + e^{-\rho \Delta t} V(X^Y(\Delta t),y), \tag{2.5}$$

by waiting a short time period Δt at time zero. The inequality holds since this action might be suboptimal. By using Taylor's Theorem, we can find

$$\begin{aligned} &e^{-\rho \Delta t} V(X^Y(\Delta t),y) \\ &= V(x,y) - \rho V(x,y)\Delta t + \dot{X}^Y(0) V_x(x,y)\Delta t + o(\Delta t), \end{aligned} \tag{2.6}$$

where $o(\cdot)$ denotes the terms of higher order than one.
From (2.5) and (2.6) we have

$$\begin{aligned} 0 \geq &\int_0^{\Delta t} e^{-\rho t} f(X^Y(t),y)dt - \rho V(x,y)\Delta t \\ &+ \dot{X}^Y(0) V_x(x,y)\Delta t + o(\Delta t). \end{aligned} \tag{2.7}$$

Dividing by Δt and letting $\Delta t \downarrow 0$ we get

$$0 \geq b(x,y)V_x(x,y) - \rho V(x,y) + f(x,y).$$

Now, suppose we immediately increase Y by an amount $\epsilon > 0$, and then continue optimally. This implies

$$V(x,y) \geq V(x,y+\epsilon) - c\epsilon,$$

and again dividing by ϵ and letting $\epsilon \downarrow 0$ we have

$$0 \geq V_y(x,y) - c. \tag{2.8}$$

Since one of those actions should be optimal, the inequalities (2.7) and (2.8) suggests that V coincides with a function $w \in C^1$ satisfying

$$\max\{b(x,y)w_x(x,y) - \rho w(x,y) + f(x,y), w_y(x,y) - c\} = 0. \tag{2.9}$$

with *boundary condition* (cf. (2.4))

$$w(x,\overline{y}) = \int_0^\infty e^{-\rho t} f(X^{\overline{Y}}(t), \overline{y}) dt.$$

We now introduce the so-called inaction region:

$$\{(x,y) \in \mathbb{R} \times [0,\bar{y}] : \, b(x,y)w_x(x,y) - \rho w(x,y) + f(x,y) = 0\},$$

in which we expect that it is optimal not to increase the control variable Y, and the action region:

$$\{(x,y) \in \mathbb{R} \times [0,\bar{y}] : \, w_y(x,y) - c = 0\},$$

in which we expect that it is optimal to increase Y. A way to solve the HJB equation (2.9) is to make the conjecture that for any $y \in [0,\bar{y}]$ the inaction region and the action region are separated by a function $F(y)$ to be found.

In the next Section, we extend the previous setting by introducing stochastic dynamics on the state variable, and find a characterization of the function F. The problem is applied to an irreversible installation model.

3 Irreversible Installation of Solar Panels

Uncertainty is usually described by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ carrying a standard one-dimensional $(\mathcal{F}_t)_t$-Brownian motion W.

We consider a company installing solar panels, and selling the energy produced by the panels instantaneously in the spot market. We assume that investments in solar panels are irreversible in the sense that the firm cannot reduce the number of panels. In absence of the company's economic activities the fundamental energy price $(X_t^x)_t$ evolves stochastically according to an Ornstein-Uhlenbeck process

$$dX_t^x = \kappa\big(\mu - X_t^x\big)dt + \sigma dW_t, \quad X_0^x = x > 0, \tag{3.1}$$

for some constants $\mu \in \mathbb{R}$ (called the mean-reversion level) and $\kappa, \sigma > 0$ (called the mean-reversion speed, and volatility respectively).

The level of solar energy can be increased by installing additional panels at constant costs $c \geq 0$ per unit of energy. We identify the amount of solar energy produced by the number of installed panels at time t, by Y_t, as the company's control variable. The control $Y = (Y_t)_t$ is an $(\mathcal{F}_t)_t$-adapted nonnegative and increasing right-continuous process. Moreover, we assume that only a finite number of solar panels can be installed, i.e. we introduce the constraint $Y_t \leq \overline{y}\, a.s.$. The initial level of solar energy is given by $y \geq 0$, i.e. $Y_{0-} = y$. We denote the set of those strategies by $\mathcal{Y}$.

Following an installation strategy $Y \in \mathcal{Y}$, the controlled price dynamics are given by

$$dX_t^Y = \kappa(\mu - X_t^Y) - \beta Y_t dt + \sigma dW_t, \quad X^Y(0) = x \in \mathbb{R},\, \beta > 0. \tag{3.2}$$

The dynamics of X^Y implies that the company's actions have a price impact on the mean reversion level.

The total expected profits, net of the total expected costs, are then given by

$$\mathcal{J}(x,y,Y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} X_t^Y Y_t dt - c\int_0^\infty e^{-\rho t} dY_t\right], \tag{3.3}$$

where $\rho > 0$ is the discount factor of the company. It aims to find $Y^\star \in \mathcal{I}$ which maximizes (3.3), so to determine

$$V(x,y) := \sup_{Y \in \mathcal{Y}} \mathcal{J}(x,y,Y). \tag{3.4}$$

We now define the function

$$R(x,y) := \mathcal{J}(x,y,Y^0) = \frac{xy}{\rho+\kappa} + \frac{\mu\kappa y}{\rho(\rho+\kappa)} - \frac{\beta y^2}{\rho(\rho+\kappa)},$$

where $Y^0(t) = y$ for all $t \geq 0$. The function R gives the expected profits of the firm following a non-installation strategy Y^0. The next preliminary result helps us to solve the problem in the following steps.

Proposition 3.1 *There exist a constant $K > 0$ such that for all $(x,y) \in \mathbb{R} \times [0,\overline{y}]$ one has*

$$|V(x,y)| \leq K\big(1 + |x|\big). \tag{3.5}$$

Moreover, $V(x,\overline{y}) = R(x,\overline{y})$, and V is increasing in x.

Proceeding as in Section 2, we can find the associated HJB-equation. We hereby use a powerful tool from stochastic calculus, that is Itô's formula which is in some sense the counterpart of Taylor's Theorem in the deterministic setting. The only two options "waiting" and "installing" give rise to the HJB-equation

$$\max\big(\mathcal{L}^y w(x,y) - \rho w(x,y) + xy, w_y(x,y) - c\big) = 0, \tag{3.6}$$

with boundary condition $w(x,\overline{y}) = R(x,\overline{y})$. Here

$$\mathcal{L}^y w(x,y) := \frac{1}{2}\sigma^2 w_{xx}(x,y) + (\kappa(\mu - x) - \beta y) w_x(x,y). \tag{3.7}$$

The first term on the right-hand side of (3.7) is an additional component in the HJB equation compared to (2.9). It is a consequence from the uncertainty in our model.

With reference to the HJB equation (3.6) we introduce the waiting region

$$\mathbb{W} = \{(x, y) \in \mathbb{R} \times [0, \bar{y}] : \mathcal{L}^y w(x, y) - \rho w(x, y) + xy = 0\},$$

in which it is optimal not to install additional panels, and the installation region

$$\mathbb{I} = \{(x, y) \in \mathbb{R} \times [0, \overline{y}) \, : \, w_y(x, y) - c = 0\},$$

in which it is optimal to install additional panels. Informally described, an optimal strategy $Y^\star$ keeps the joint process $(X^{Y^\star}, Y^\star)$ in $\mathbb{W}$ with minimal effort, i.e. $Y^\star$ increases whenever $(X^{Y^\star}, Y^\star)$ enters $\mathbb{I}$.

The following theorem shows indeed that an appropriate solution to the HJB equation identifies with the value function, and the strategy $Y^\star$ discussed as above is optimal.

Theorem 3.2 (Verification Theorem) *Suppose there exists a function $w \in C^{2,1}$, which solves the HJB equation (3.6) with $w(x, \bar{y}) = R(x, \bar{y})$, and satisfies the growth condition*

$$|w(x, y)| \leq K\big(1 + |x|\big), \tag{3.8}$$

for a constant $K > 0$. Moreover, suppose that a process $Y^\star$ previously described exists. Then

$$V(x, y) = w(x, y), \quad (x, y) \in \mathbb{R} \times [0, \bar{y}],$$

and $Y^\star$ is optimal; that is, $V(x, y) = \mathcal{J}(x, y, Y^\star)$.

We now aim to find a function w which solves (3.6). For this purpose, we make a guess that there exists an injective function $F : [0, \overline{y}] \mapsto \mathbb{R}$ separating the waiting region $\mathbb{W}$ (price is "small") and the installation region $\mathbb{I}$ (price is sufficiently "high"), i.e. such that

$$\mathbb{W} = \{(x, y) \in \mathbb{R} \times [0, \overline{y}) \, : \, x < F(y)\} \cup (\mathbb{R} \times \{\overline{y}\}), \tag{3.9}$$

$$\mathbb{I} = \{(x, y) \in \mathbb{R} \times [0, \overline{y}) \, : \, x \geq F(y)\}. \tag{3.10}$$

In $\mathbb{W}$, our candidate value function w must satisfy

$$\mathcal{L}^y w(x, y) - \rho w(x, y) + xy = 0. \tag{3.11}$$

- A particular solution to (3.11) is given by $R(x, y)$.
- It is well known that the homogeneous part of (3.11) admits two strictly positive solutions $\phi_y(x)$ (strictly decreasing) and $\psi_y(x)$ (strictly increasing). The variable y enters here as a parameter.

For $(x, y) \in \mathbb{W}$ we thus consider a candidate value function of the form

$$w(x, y) = A(y)\psi_y(x) + B(y)\phi_y(x) + R(x, y),$$

for some functions $A(y)$ and $B(y)$ to be found. The fundamental solutions are represented by a *cylinder function*. Moreover, ψ_y increases exponentially as $x \uparrow \infty$ and ϕ_y increases exponentially as $x \downarrow -\infty$. Due to the linear growth condition of $V(\cdot, y)$, cf. Proposition 3.1, and in light of the structure of the waiting region, we make the ansatz

$$w(x,y) = A(y)\psi_y(x) + R(x,y), \quad (x,y) \in \mathbb{W}.$$

The goal is to find the functions A and F.

Remark 3.3 The function $\psi_y(\cdot)$ is given by

$$\psi_y(x) = e^{\frac{\kappa(x-(\mu-\frac{\beta}{\kappa}y))^2}{2\sigma^2}} D_{-\frac{\rho}{b}}\Big(- \frac{x-(\mu-\frac{\beta}{\kappa}y)}{\sigma}\sqrt{2\kappa}\Big), \tag{3.12}$$

where

$$D_\beta(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\beta)} \int_0^\infty t^{-\beta-1} e^{-\frac{t^2}{2}-xt} dt, \quad \beta < 0, \tag{3.13}$$

and $\Gamma(\cdot)$ is the Euler's Gamma function. Employing $\psi_y(x) = \psi_0\Big(x + \frac{\beta}{\kappa}y\Big)$, we find $\frac{\partial \psi_y(x)}{\partial y} = \frac{\beta}{\kappa}\psi_y'(x)$.

In the installation region, we must have

$$w_y(x,y) - c = 0, \text{ for all } (x,y) \in \mathbb{I},$$

implying

$$w_{yx}(x,y) = 0, \text{ for all } (x,y) \in \mathbb{I}.$$

Suppose $w \in C^{2,1}$ along the boundary $(F(y), y)$ between $\mathbb{W}$ and $\mathbb{I}$. We derive the following conditions

$$A'(y)\psi_y(x) + \frac{\beta}{\kappa}A(y)\psi_y'(x) + R_y(x,y)|_{x=F(y)} = c, \tag{3.14}$$

$$A'(y)\psi_y'(x) + \frac{\beta}{\kappa}A(y)\psi_y''(x) + R_{xy}(x,y)|_{x=F(y)} = 0. \tag{3.15}$$

Solving the previous system (3.14) and (3.15) for A, we find

$$A(y) = \frac{\kappa}{\beta(\rho+\kappa)} \times \frac{\psi_y'(F(y))\Big(\bar{c} + \frac{2\beta y}{\rho} - F(y)\Big) + \psi_y(F(y))}{\psi_y'(F(y))^2 - \psi_y''(F(y))\psi_y(F(y))}, \tag{3.16}$$

where $\bar{c} := c(\rho+\kappa) - \frac{\mu\kappa}{\rho}$.

Due to the boundary condition, cf. Proposition 3.1, we must have that

$$w(x,\overline{y}) = R(x,\overline{y}) \quad \Rightarrow \quad A(\overline{y}) = 0. \tag{3.17}$$

Hence, from (3.17), there should exist some $x_\infty = F(\overline{y}) \in \mathbb{R}$ such that $H(x_\infty) = 0$, where

$$H(x) := \psi_{\overline{y}}'(x)\Big(\bar{c} + \frac{2\beta\overline{y}}{\rho} - x\Big) + \psi_{\overline{y}}(x). \tag{3.18}$$

Lemma 3.4 *There exists a unique solution to the equation $H(x) = 0$.*

Now, by differentiating (3.16), and using, for example, (3.15), one obtains an ordinary differential equation

$$F'(y) = \mathcal{F}(y, F(y)), \tag{3.19}$$

with boundary condition $F(\overline{y}) = x_\infty$.

In particular, we can derive

$$\mathcal{F}(y, F(y)) = \frac{\frac{\beta}{\kappa}\left(2\psi_y''(F(y))\psi_y(F(y)) - \psi_y'(F(y))^2\right) A'(y) + \left(\frac{\beta}{\kappa}\right)^2 \psi_y'''(F(y))A(y) + \frac{(2\kappa+\rho)\beta}{\rho\kappa(\rho+\kappa)}\psi_y'(F(y))}{\psi_y(F(y))\left(-\psi_y''(F(y))A'(y) - \frac{\beta}{\kappa}\psi_y'''(F(y))A(y)\right)}.$$

In the following we define the function $N : \mathbb{R}_+ \mapsto \mathbb{R}$ such that $N(y) = -\psi_y''(F(y))A'(y) - \frac{\beta}{\kappa}\psi_y'''(F(y))A(y)$ which is a factor in the denominator of $\mathcal{F}$. For verification, the following Assumption is needed.

Assumption 3.5 The ordinary differential equation (3.19) with boundary condition $F(\overline{y}) = x_\infty$ admits a strictly increasing function F on $[0, \overline{y}]$ such that $N|_{[0,\overline{y}]}$ is strictly positive.

Due to Assumption 3.5 there exists a strictly increasing inverse function F^{-1} on the interval $[x_0, x_\infty]$, where $x_0 = F(0)$.

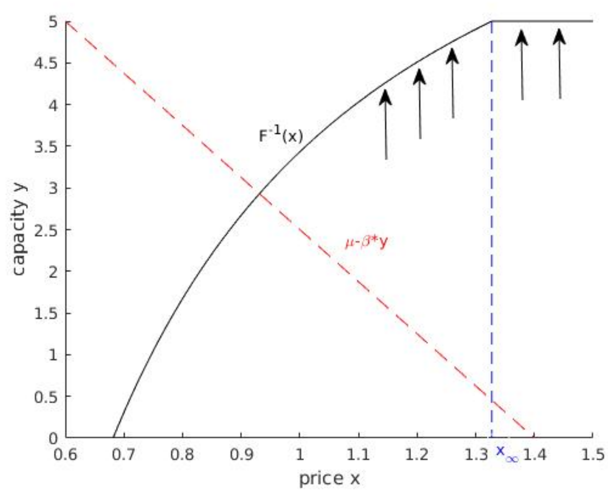


Figure 1. Boundary $F^{-1}(x)$ with $\mu = 1.4$, $\kappa = 1$, $\sigma = 0.8$, $\rho = 3/8$, $c = 0.3$, $\beta = 0.15$, $\overline{y} = 5$.

In Figure 1 we find a plot of the function F^{-1} and the mean reversion level of the controlled process X^Y depending on the current installation level (dashed line). The plot of F^{-1}

leads us to the following conjecture about the form of the candidate value function w: if the the current price x is sufficiently high, i.e. $x > F(y)$ and the number of solar panels that can be installed is sufficiently large, i.e. $x \in (x_0, x_\infty)$, then the company pushes the joint process (X^Y, Y) immediately to the boundary curve $\{(x, y) \in \mathbb{R} \times [0, \bar{y}] : x = F(y)\}$ in direction $(0, 1)$, so to produce additional $F^{-1}(x) - y$ units of solar energy. The associated payoff to this action is then the difference of the continuation value starting from the new state $(x, F^{-1}(x))$ and the costs of installation $c(F^{-1}(x) - y)$. If the maximum number of solar panels that can be installed is not sufficiently large, i.e. $x > x_\infty$, then the company immediately installs the maximum number of panels, that is $Y_0 = \bar{y}$, and the associated payoff to such a strategy is $R(x, \bar{y}) - c(\bar{y} - y)$. Moreover, the current mean-reversion level provides the price levels at which the energy price is likely to move around. Consequently the higher the production of solar energy, the higher the expected time to install additional panels.

In light of the previous discussion, we define the candidate value function as

$$w(x,y) = \begin{cases} A(y)\psi\big(x + \frac{\beta}{\kappa}y\big) + R(x,y), & \text{if } x \le F(y), \\ w\big(x, F^{-1}(x)\big) - c(F^{-1}(x) - y), & \text{if } x \in (x_0, x_\infty) \text{ and } y < F^{-1}(x), \\ R(x, \bar{y}) - c(\bar{y} - y), & \text{if } x > x_\infty. \end{cases} \tag{3.20}$$

Indeed we can prove that the candidate value function w identifies with V and the installation strategy $Y^\star$ discussed above is optimal. More formally:

Theorem 3.6 *Let $\Delta := (\bar{y} - y)\mathbb{1}_{\{x \ge x_\infty\}} + (F^{-1}(x) - y)\mathbb{1}_{\{x > x_0, y < F^{-1}(x)\}}$, $\tau := \inf\{t \ge 0 : K_t = \bar{y} - (y + \Delta)\}$, and (X, K) defined on $[0, \tau]$ such that*

$$\begin{aligned} X_t &\le F(y + \Delta + K_t), \\ dX_t &= \kappa\Big(\mu - X_t\Big)dt - \beta(y + \Delta + K_t)dt + \sigma dW_t, \\ dK_t &= \mathbb{1}_{\{X_t = F(y + \Delta + K_t)\}} dK_t, \end{aligned} \tag{3.21}$$

with increasing K, and starting point $(X_0, K_0) = (x, 0)$. Then the function w is a $C^{2,1}$ solution to the HJB equation (3.6), and identifies with the value function V from (3.4). The optimal installation strategy, denoted by $Y^\star$, is given by

$$\begin{cases} Y^\star_{0-} = y \\ Y^\star_t = \begin{cases} y + \Delta + K_t, & t \in [0, \tau), \\ y + \Delta + K_\tau, & t \ge \tau. \end{cases} \end{cases}$$

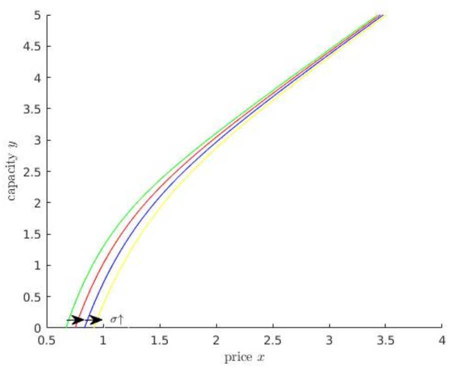
3.1 Comparative Statics

In this section, we study the sensitivity on the model parameters μ,σ and κ numerically.

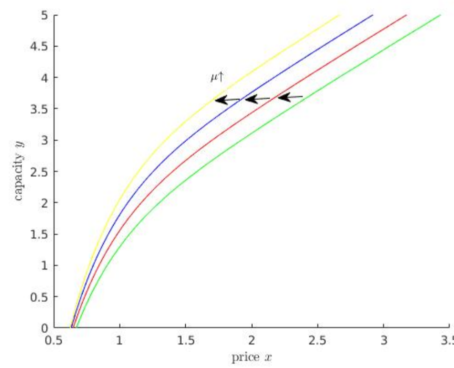
We first study the behavior of the free boundary with respect to the volatility displayed in Figure 2(a). Here the volatility parameter σ takes values from 0.8 to 1.1. As a consequence we can observe that the boundary F^{-1} is shifted to the right; that is the firm's willingness to install additional panels is undertaken at higher prices. The firm might be afraid of receiving negative future prices due to higher uncertainty.

Now let the mean-reversion level μ vary in $[0.4, 0.7]$. Figure 2(b) shows that the critical threshold F^{-1} moves to left. A lower value for the critical threshold F^{-1} leads the firm to undertake the installation at lower prices. This observation can be explained by the fact that the company is eager to act earlier, the higher the expected future profits.

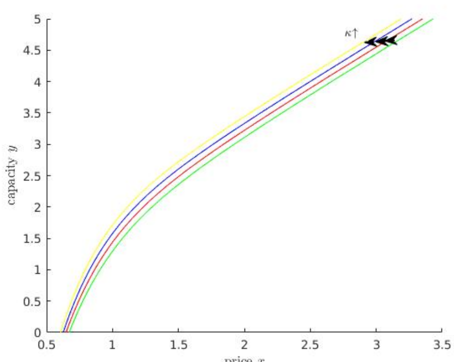
Finally, the dependency on κ is analysed in Figure 2(c), that is we let κ vary. We find that higher values for the mean reversion speed κ shift the boundary F^{-1} to the left, that is the company installs solar panels at lower prices. Since a higher mean reversion speed reduces its ratio with respect to σ, the uncertainty is decreased, and hence we observe a converse behavior with respect to Figure 2(a).



(a) Boundary $F^{-1}(x)$ with $\mu = 0.4$, $\kappa = 1$, $\rho = 3/8$, $c = 0.3$, $\beta = 0.15$, $\bar{y} = 5$ and $\sigma = 0.8$ (green), $\sigma = 0.9$ (red), $\sigma = 1$ (blue), $\sigma = 1.1$ (yellow).



(b) Boundary $F^{-1}(x)$ with $\sigma = 0.8$, $\kappa = 1$, $\rho = 3/8$, $c = 0.3$, $\beta = 0.15$, $\bar{y} = 5$ and $\mu = 0.4$ (green), $\mu = 0.5$ (red), $\mu = 0.6$ (blue), $\mu = 0.7$ (yellow).



(c) Boundary $F^{-1}(x)$ with $\sigma = 0.8$, $\mu = 0.4$, $\rho = 3/8$, $c = 0.3$, $\beta = 0.15$, $\bar{y} = 5$ and $\kappa = 1$ (green), $\kappa = 1.1$ (red), $\kappa = 1.2$ (blue), $\kappa = 1.3$ (yellow).

Figure 2. Sensitivity of the boundary $F^{-1}(x)$ with respect to μ, σ and κ.

References

[1] Al Motairi, H., Zervos, M., *Irreversible Capital Accumulation with Economic Impact.* Appl. Math. Optim. 75/3 (2017) 521–551.

[2] Chiarolla, M.B., Haussmann, U.B., *Explicit Solution of a Stochastic, Irreversible Investment Problem and its Moving Threshold.* Math. Oper. Res. 30 (2005), 91–108.

[3] Kunze, H., La Torre, D., Malik, T., Marsiglio, S., Ruiz-Galan, M., *Optimal Control: Theory and Application to Science, Engineering, and Social Sciences.* Abstr. Appl. Anal. 890527 (2015).