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# Periodic and hypercomplex potentials. Properties and applications

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# Abstract

This Dissertation is devoted to the study of boundary value problems and concerns two research areas. *The first one* is related to the perturbation analysis of boundary value problems in perforated domains and its application to the investigation of effective properties of composite materials. We investigate the dependence of the solutions of transmission boundary value problems upon some parameters and their behavior when the parameter corresponding to the size of the inclusions tends to zero, and the other parameters tend to some fixed values. Then we apply our results to study the effective conductivity of periodic composites. We also investigate the behavior of the solution of the Dirichlet problem for the Poisson equation in the domain in $\mathbb{R}^3$ which consists of a periodic array of cylinders upon perturbation of the shape of the cross-section of the cylinders and the periodic structure. Moreover, we apply our results to study the behavior of the longitudinal permeability of a periodic array of cylinders upon such perturbation. *The second part of the Dissertation* is related to the development of tools for solving boundary value problems for functions taking values in commutative Banach algebras. In particular, we investigate the properties of logarithmic residues of monogenic (continuous and differentiable in the sense of Gateau) functions and the behavior of the certain Cauchy type integral on the boundary of its definition.

The Dissertation consists of two parts and is organized as follows.

Part I consists of three chapters. In Chapter 1 we investigate the asymptotic behavior of the solutions of singularly perturbed (ideal and nonideal nonlinear) transmission problems in a periodically perforated domain. In Chapter 2 we apply the results of Chapter 1 to study the asymptotic behavior of the effective thermal conductivity of a periodic two-phase dilute composite. Chapter 3 is devoted to the study of the behavior of the longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders and of the periodic structure.

Part II consists of two chapters. In Chapter 4 we introduce a three-dimensional commutative algebra over $\mathbb{C}$ with a one-dimensional radical and study the logarithmic residues of monogenic functions in this algebra. Chapter 5 is devoted to the investigation of a certain analog of Cauchy type integral taking values in the mentioned algebra and its limiting values on the boundary of the domain of definition. At the end of the Dissertation, we have enclosed three appendices with some results which we have exploited in the Dissertation.

# Contents

# Introduction

The Dissertation is devoted to the study of boundary value problems in finite-dimensional Euclidean spaces and in commutative Banach algebras.

In Euclidean spaces we are interested in the singular and regular perturbation analysis of boundary value problems (transmission problems for the Laplace equation, a Dirichlet problem for the Poisson equation) in perforated domains and its application to the investigation of effective properties of composite materials. We use techniques which allow us to look at the boundary value problems from the different point of view in comparison with the classical ones. Our techniques are mainly based on the *Functional Analytic Approach*, which has been proposed by Lanza de Cristoforis [76, 79] for the analysis of singular perturbation problems in perforated domains. Such an approach aims at representing the solution or related functionals in terms of real analytic functions of the singular perturbation parameter and of explicitly known functions of such a parameter (see, e.g., Dalla Riva and Musolino [35], Lanza de Cristoforis and Musolino [84]). The main advantage of this approach is the possibility to expand the investigated functionals into convergent power series of the mentioned parameter when it is small. Moreover, the coefficients of such power series can be explicitly determined by computing the solutions of recursive systems of integral equations (see, e.g., Dalla Riva, Musolino, and Rogosin [39]).

In commutative Banach algebras, we are interested in developing a theory similar to the theory of analytic functions of one complex variable within the framework of creating tools for solving boundary value problems in such algebras. We note that one of the first applications of analytic functions with values in commutative Banach algebras for describing spatial potential fields has been done by Ketchum [68], Ward [162], Wagner [161], and Kunz [73]. As a generalization of the ideas presented in these works, Mel'nichenko [100, 101] has proposed the *Algebraic-Analytic Approach* to elliptic equations of mathematical physics, which aims in finding commutative Banach algebras such that functions, which are differentiable in the sense of Gateaux and have values in these algebras, have components satisfying the given partial differential equation. In the Dissertation we consider one of those algebras, namely, the three-dimensional commutative Banach algebra over $\mathbb{C}$ with one-dimensional radical, and we study properties of monogenic (continuous and differentiable in the sense of Gateaux) functions and a certain analog of the Cauchy type integral. Based on the obtained results and using commutative Banach algebras, we plan to develop hypercomplex tools and methods for solving boundary value problems. Some advantages of using two-dimensional commutative algebras for solving boundary value problems can be found in Gryshchuk and Plaksa [57, 58]. In particular, using a hypercomplex analog of the Cauchy type integral, Gryshchuk and Plaksa reduced a Schwartz-type boundary value problem for biharmonic monogenic functions to a system of Fredholm integral equations on the real axes.

The Dissertation is organized as follows. The first part is devoted to the perturbation analysis of boundary value problems in perforated domains and its application to the investigation of effective properties of composite materials. The second part is devoted to the development of tools for solving boundary value problems for functions taking values in commutative Banach algebras.

Below, we describe in more details the structure of the Dissertation.

# PART I: Perturbation problems and applications

Part I consists of three chapters. In Chapter 1 we investigate the asymptotic behavior of the solutions of singularly perturbed (ideal and nonideal nonlinear) transmission problems in a periodically perforated domain. In Chapter 2 we apply the obtained results to study the asymptotic behavior of the effective thermal conductivity of a periodic two-phase dilute composite. Chapter 3 is devoted to the investigation of the behavior of the longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders and of the periodic structure.

## Asymptotic behavior of the solutions of transmission problems in a periodic domain

The asymptotic behavior of the solutions of transmission problems in periodic domains has been studied by many authors and by different approaches. The most common approach is the one of *Asymptotic Analysis*, aiming at computing asymptotic expansions of the solutions or of related functionals with respect to some parameter which tends to zero. We mention works of Ammari and Kang [7], Ammari, Kang, and Kim [8], and Ammari, Kang, and Lim [9], where the authors derive asymptotic expansions for some effective parameters of periodic dilute two-phase composites. For the application of asymptotic analysis to dilute and densely packed composites, we refer to Movchan, Movchan, and Poulton [115], and to Nieves [120] for transmission problems in solids with many inclusions. Concerning asymptotic methods for general elliptic problems we mention, e.g., Maz'ya, Nazarov, and Plamenewskij [98, 99] and Maz'ya, Movchan, and Nieves [97]. The technique of Asymptotic Analysis allows to produce asymptotic expansions and has revealed to be extremely versatile for a wide range of problems. On the other hand, one should note that this method usually does not allow to show that the power series associated with an asymptotic expansion is convergent.

Moreover, transmission problems in domains with periodic circular inclusions have been extensively analyzed with the method of *Functional Equations*, which aims at obtaining representation formulas in terms of power series of the radius of inclusions. As an example, we mention the works by Castro and Pesetskaya [26] and Drygas and Mityushev [44] for a transmission problem with nonideal (or imperfect) contact, the works by Mityushev [108], Kapanadze, Mishuris, and Pesetskaya [65, 66], and Rogosin, Dubatovskaya, and Pesetskaya [141] for a transmission problem with ideal (or perfect) contact, and, finally, the works of Mityushev, Obnosov, Pesetskaya, and Rogosin [111] and Mityushev and Rogosin [112, Ch. 5], where the two problems have been considered. One should note such a method applies to specific geometries as, for example, the cases of circular and elliptic inclusions and only in the two-dimensional case.

Here, instead, we use the alternative *Functional Analytic Approach*, which has been proposed by Lanza de Cristoforis for the analysis of singular perturbation problems in perforated domains. There are many papers in which such an approach was developed and adapted to analyze a variety of problems. Among them, we mention works of Lanza de Cristoforis [76, 77, 78, 79, 80], Dalla Riva and Lanza de Cristoforis [31], Lanza de Cristoforis and Musolino [83, 85], and Musolino and Mishuris [118]. Moreover, this approach has been applied to a mixed problem for the Poisson equation and to the Dirichlet problem for the Laplace equation in a domain with two moderately close holes by Dalla Riva and Musolino [36, 37], to the Dirichlet problem for the Laplace equation in a domain with a hole that approaches the outer boundary of the domain at a certain rate by Bonnaillie-Noël, Dalla Riva, Dambrine, and Musolino [25], and to the Dirichlet problem for the Laplace operator where holes are shrinking towards a point of the boundary that is the vertex of a plane sector by Costabel, Dalla Riva, Dauge, and Musolino [30]. Concerning transmission problems, by this approach, Dalla Riva and Musolino [35] have investigated a nonideal singularly perturbed linear transmission problem, and Lanza de Cristoforis and Musolino [84] have studied a quasi-linear heat transmission problem (see also Dalla Riva, Lanza de Cristoforis, and Musolino [32]). For other contributions for the analysis of nonlinear transmission problems, we refer to Dalla Riva and Mishuris [34] and Lanza de Cristoforis [80].

In the Dissertation we adapt the Functional Analytic Approach in order to study the behavior of solutions of singularly perturbed (ideal and nonideal nonlinear) transmission problems in a periodically perforated domain and, then, we apply obtained results to investigate the effective properties of composite materials.

In order to introduce the geometry of the problem, we fix once for all $n \in \mathbb{N}\setminus\{0,1\}$ and $(q_{11},\dots,q_{nn}) \in ]0,+\infty[^n$. Then, we define a periodicity cell $Q := \Pi_{i=1}^{n}]0,q_{ii}[$ and a diagonal matrix $q \equiv (\delta_{i,j}q_{jj})_{i,j\in\{1,\dots,n\}}$, where $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ if $i\neq j$ for all $i,j\in\{1,\dots,n\}$.

Then we fix $\alpha \in ]0,1[$ and take a bounded open connected subset of $\Omega$ of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\mathbb{R}^n\setminus \mathrm{cl}\Omega$ is connected, and $0\in\Omega$. Obviously, for a fixed point $p\in Q$, there exists $\epsilon_0 \in ]0,+\infty[$ such that

$$p + \epsilon\, \mathrm{cl}\Omega \subseteq Q$$

for all $\epsilon \in ]-\epsilon_0,\epsilon_0[$ and, to shorten our notation, we set $\Omega_{p,\epsilon} := p+\epsilon\Omega$ for all $\epsilon\in\mathbb{R}$. The set $\Omega$ represents the shape of each inclusion and if $\epsilon \in ]-\epsilon_0,\epsilon_0[$ then $\Omega_{p,\epsilon}$ plays the role of the inclusion in the fundamental cell $Q$ (see Figure 1). Then we introduce the periodic domains

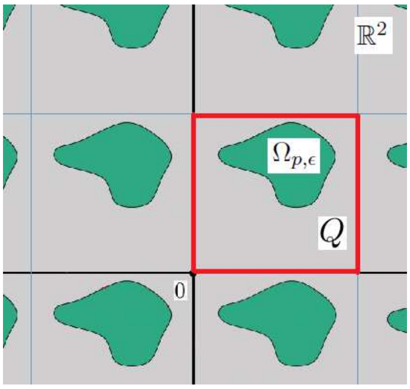


Figure 1: *The model of the composite in* $\mathbb{R}^2$

$$\mathbb{S}[\Omega_{p,\epsilon}] := \bigcup_{z\in\mathbb{Z}^n}(qz+\Omega_{p,\epsilon}), \quad \mathbb{S}[\Omega_{p,\epsilon}]^- := \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$$

for all $\epsilon \in ]-\epsilon_0,\epsilon_0[$. The set $\mathbb{S}[\Omega_{p,\epsilon}]$ corresponds to the region occupied by the periodic set of holes or inclusions; the matrix, instead, is represented by its complementary set $\mathbb{S}[\Omega_{p,\epsilon}]^-$.

Next, we take two positive constants $\lambda^+$, $\lambda^-$, a constant $c\in\mathbb{R}$, functions $f\in C^{0,\alpha}(\partial\Omega)_0$ and $g\in C^{1,\alpha}(\partial\Omega)$, a function $\rho : ]0,\epsilon_0[\to \mathbb{R}\setminus\{0\}$, and we consider the following *linear transmission problem* for a pair of functions $(u^+,u^-)\in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])\times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta u^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ u^+(x+qe_h) = u^+(x) & \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,, \quad \forall h\in\{1,\dots,n\}, \\ u^-(x+qe_h) = u^-(x) & \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,, \quad \forall h\in\{1,\dots,n\}, \\ \lambda^-\frac{\partial u^-}{\partial\nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+\frac{\partial u^+}{\partial\nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x\in\partial\Omega_{p,\epsilon}, \\ \frac{1}{\rho(\epsilon)}(u^+(x)-u^-(x)) = g((x-p)/\epsilon) & \forall x\in\partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} u^+ d\sigma = c \end{cases} \tag{1}$$

for all $\epsilon\in]0,\epsilon_0[$.

The analysis of problem (1) will allow us to study a (more general) transmission problem, which we introduce below (see problem (2)). So, let a matrix $B\in M_{1\times n}(\mathbb{R})$ and $F$ be a real analytic map from $\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $\mathbb{R}$. For $(\epsilon,f,g,c)\in]0,\epsilon_0[\times C^{0,\alpha}(\partial\Omega)_0\times C^{1,\alpha}(\partial\Omega)\times\mathbb{R}$, we consider the following *nonideal nonlinear transmission problem* for a pair of functions $(\mathbf{u}^+, \mathbf{u}^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

$$\begin{cases} \Delta \mathbf{u}^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta \mathbf{u}^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ \mathbf{u}^+(x+qe_h) = \mathbf{u}^+(x) + Be_h & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,\ \forall h \in \{1,\dots,n\}, \\ \mathbf{u}^-(x+qe_h) = \mathbf{u}^-(x) + Be_h & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,\ \forall h \in \{1,\dots,n\}, \\ \lambda^- \frac{\partial \mathbf{u}^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial \mathbf{u}^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ F\left[\frac{\partial \mathbf{u}^-}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot), \frac{\partial \mathbf{u}^+}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)\right] & \\ \qquad + \frac{1}{\rho(\epsilon)}(\mathbf{u}^+(x) - \mathbf{u}^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} \mathbf{u}^+ d\sigma = c \end{cases} \tag{2}$$

for all $\epsilon \in ]0, \epsilon_0[$.

Problems of this type usually arise in fluid dynamics and thermodynamics. If we assume that the set of inclusions and the matrix are filled with two different homogeneous and isotropic heat conductor materials, then the parameters $\lambda^+$ and $\lambda^-$ play the role of thermal conductivity of the materials which fill the inclusions and the matrix, respectively. Therefore, the solutions of problems (1) and (2) represent the temperature distribution on the set of inclusions $\mathbb{S}[\Omega_{p,\epsilon}]$ and in the matrix $\mathbb{S}[\Omega_{p,\epsilon}]^-$. Then the third and fourth conditions in problems (1) and (2) mean periodicity and periodicity up to a given linear function, respectively, of the temperature distribution. The fifth condition of such problems says that the normal component of the heat flux presents a jump which equals a given function. The sixth condition of the two problems is different: the one in (1) says that the temperature distribution presents a jump on the interface equal to a given function, while the one in (2) says that the jump on the interface of the temperature distribution plus a given function is proportional to a quantity which depends on the heat flux. In case $g = 0$, such a condition means that the jump of the temperature distribution is proportional to a quantity which depends nonlinearly on the heat flux and can be seen as a nonlinear counterpart of the linear nonideal transmission problem considered by Dalla Riva and Musolino [35], where the parameter $\rho(\epsilon)$ plays the role of the boundary thermal resistivity. Finally, the last condition of problems (1) and (2) is just a normalization condition, which we need in order to “fix” the solution.

Due to the presence of the factor $1/\rho(\epsilon)$, the boundary conditions may display a singularity as $\epsilon$ tends to 0. We consider the case in which $\lim_{\epsilon\to 0^+} \rho(\epsilon)/\epsilon$ exists in $\mathbb{R}$. This assumption will allow us to analyze problems (1) and (2) around the degenerate value $\epsilon = 0$, and if it holds then we set $r_* := \lim_{\epsilon\to 0^+} \rho(\epsilon)/\epsilon$. We also note that such an assumption is alternative to that considered by Dalla Riva and Musolino [35], where they assumed that $\lim_{\epsilon\to 0^+} \epsilon/\rho(\epsilon)$ exists in $\mathbb{R}$. Clearly, both assumptions are satisfied in case $\rho(\epsilon) = \epsilon$.

As we shall see, problems (1) and (2) have unique pairs of solutions which we denote by $(u^+[\epsilon, f, g, c], u^-[\epsilon, f, g, c])$ and by $(\mathbf{u}^+[\epsilon, f, g, c], \mathbf{u}^-[\epsilon, f, g, c])$, respectively. **Our aim is to investigate the behavior of** $(u^+, u^-)$ **and** $(\mathbf{u}^+, \mathbf{u}^-)$ when $\epsilon$ is close to the degenerate value $\epsilon = 0$, in correspondence of which the inclusions collapse to points. From the physical point of view, this situation corresponds to the case of a dilute composite.

We prove *our main result*, stating that the solutions of problems (1) and (2) can be represented in terms of real analytic maps of $\epsilon$ and of some additional functions. Our results imply that such solutions can be expanded into absolutely convergent power series what is used in the sequel of Dissertation in order to study the effective conductivity of composites.

Finally, we briefly outline *our strategy.* First, we convert problem (1) into a system of integral equations by exploiting layer potential representations. Taking the assumption on $\rho(\epsilon)$ into account, this system can be analyzed when $(\epsilon, f, g, c)$ is close to the degenerate quadruple $(0, f_0, g_0, c_0)$. We do so by means of the Implicit Function Theorem and we represent the unknowns of the system of integral equations in terms of analytic functions of $\epsilon$, $\rho(\epsilon)/\epsilon$, $f$ and $g$. Next we exploit the integral representations of the solutions in terms of the unknowns of the system of integral equations, and we deduce the representation of $u^+[\epsilon, f, g, c]$ and $u^-[\epsilon, f, g, c]$ in terms of real analytic maps of $\epsilon$, $\rho(\epsilon)/\epsilon$, $f$, $g$, and $c$. Finally, we convert problem (2) into a linear periodic transmission problem and we apply the previously discussed results for the derivation of representation for $(\mathbf{u}^+[\epsilon, f, g, c], \mathbf{u}^-[\epsilon, f, g, c])$.

## Effective conductivity of a periodic dilute composite

There is a vast literature devoted to the study of the effective properties of composites. We first mention *the Homogenization method* which is a common method to study the effective properties. It aims at giving a proper description of materials that are composed of several constituents, intimately mixed together (see Bensoussan, Lions, and Papanicolaou [18], Bakhvalov and Panasenko [12], Jikov, Kozlov, and Oleinik [63], Markov [96], and Milton [105]). We also note that due to the fact that the microstructure of the composite is close to periodic, one can apply *the periodic homogenization*, which allows reducing the investigation of the global problem for a composite to the investigation of the local problem for a cell of periodicity. The application of the periodic homogenization to study effective properties of heat conductors with interfacial contact resistance can be found in Monsurró [113, 114], Donato and Monsurró [42], and Faella, Monsurró, and Perugia [45]

Concerning the *Asymptotic Analysis*, we note that Ammari, Kang, and Touibi in [10] computed an asymptotic expansion of the effective electrical conductivity of a periodic dilute composite with the ideal contact condition, and Ammari, Kang, and Kim [8] have presented the asymptotic expansions of the effective electrical conductivity of periodic dilute composites. Also, Ammari, Garnier, Giovangigli, Jing and Seo [5] considered transmission problems in order to study the effective admittivity of cell suspensions, and Ammari, Giovangigli, Kwon, Seo, and Wintz [6] studied these problems in order to determine microscopic properties of cell cultures from spectral measurements of the effective conductivity. One should note that despite this approach is extremely versatile for a wide range of problems, it usually does not provide constructive formulas for all the coefficients of the power series associated with an asymptotic expansion.

Concerning *Functional Equation Method*, which is very useful to express the effective conductivity in terms of a convergent power series of the diameter of inclusions in two-dimensional case, we note the works of Castro, Pesetskaya, and Rogosin [27], Pesetskaya [122], Kapanadze, Miszuris, and Pesetskaya [67], and Drygaś and Mityushev [44]. We also note that this method can be applied to random composite materials as it was shown by Berlyand and Mityushev [20, 21].

We also mention the paper by Gryshchuk and Rogosin [56], where, using a method based on expansions in the Taylor's and Laurent's power series, the authors studied the effective conductivity of a two-dimensional circular composite material.

Concerning numerical results for the effective conductivity, we refer to Zuzovsky and Brenner [163] for studying composites with spherical inclusions, to Godin [54], and Alali and Milton [4], who obtained the series expansion of the effective electric conductivity tensor with exactly determined coefficients for composites with thin interphase regions. Moreover, Sciacca, Jou, and Mongiovì [149] have studied the effective thermal conductivity of narrow channels filled with helium II, which was used by Saluto, Jou, and Mongiovì [144] to analyze the effective thermal resistance. We also mention that Zheng, Yuan, Hu, and Luo [90] analyzed the effective thermal conductivity of the silicone/phosphor composite and its size effect.

Here, instead, we continue to adapt the *Functional Analytic Approach* and we investigate the properties of the composite in the dilute case, i.e., when the singular perturbation parameter $\epsilon$, which controls the size of the inclusions, tends to 0. By this approach, we can investigate functionals in terms of real analytic maps of the singular perturbation parameter $\epsilon$ and, in turn, such a result allows to justify representation formulas in terms of convergent power series of $\epsilon$. A preliminary step in the explicit computation of the series expansions has been performed in Dalla Riva, Musolino, and Rogosin [39], regarding the solution of a Dirichlet problem for the Laplace equation in a bounded domain with a small hole. Here, **we present the first extension of such a computation to periodic domains and to different transmission boundary conditions.**

Before we define the effective conductivity, we formulate three transmission boundary value problems: the first two in $\mathbb{R}^n$ for any $n \in \mathbb{N} \setminus \{0,1\}$, and the third one only for $n = 2$ but the results can be easily extended to the higher dimensions.

To begin with, we take two positive constants $\lambda^+$, $\lambda^-$. Then for each $j \in \{1, \dots, n\}$ and all $\epsilon \in ]0, \epsilon_0[$, we consider the so-called *ideal transmission problem*, which is a problem for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ satisfying

$$\begin{cases} \Delta u_j^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta u_j^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ u_j^+(x + q_{hh}e_h) = u_j^+(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \quad \forall h \in \{1, \dots, n\}, \\ u_j^-(x + q_{hh}e_h) = u_j^-(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \quad \forall h \in \{1, \dots, n\}, \\ \int_{\partial\Omega_{p,\epsilon}} u_j^+ d\sigma = 0 \end{cases} \tag{3}$$

with the following boundary conditions

$$\begin{cases} \lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = 0 & \forall x \in \partial\Omega_{p,\epsilon}, \\ u_j^+(x) - u_j^-(x) = 0 & \forall x \in \partial\Omega_{p,\epsilon}. \end{cases} \tag{4}$$

The analysis of the ideal transmission problem can be deduced by the analysis of a more general transmission problem which, in turn, can be considered as a particular case of problem (2). Such a more general transmission problem is a problem for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$, which satisfy system (3) and the following boundary conditions

$$\begin{cases} \lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \frac{1}{\epsilon^m}(u_j^+(x) - u_j^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon} \end{cases} \tag{5}$$

for a given number $m \in \{1, \dots, n-1\}$, and given functions $f \in C^{0,\alpha}(\partial\Omega)_0$ and $g \in C^{1,\alpha}(\partial\Omega)$. Clearly, if $(f, g, c) = (0, 0, 0)$ in (5) then the two problems coincide.

In order to formulate the third problem, which we consider in $\mathbb{R}^2$, we additionally introduce a function $\rho$ from $]0, \epsilon_0[$ to $]0, +\infty[$ equals $1/r_\#$ or $\epsilon/r_\#$, where $r_\#$ is a positive real number. Then, we consider the so-called *nonideal transmission problem* for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ satisfying system (3) with $n = 2$ and the following boundary conditions

$$\begin{cases} \lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) + \frac{1}{\rho(\epsilon)} \left( u_j^+(x) - u_j^-(x) \right) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}. \end{cases} \tag{6}$$

Finally, we introduce *the effective conductivity* matrix $\lambda^{\mathrm{eff}}[\epsilon]$ with $(k, j)$-entry $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ defined by means of the following.

$$\begin{aligned} \lambda^{\mathrm{eff}}_{kj}[\epsilon] := \frac{1}{|Q|_n} \Bigg( \lambda^+ \int_{\Omega_{p,\epsilon}} \frac{\partial u_j^+[\epsilon](x)}{\partial x_k} \, dx + \lambda^- \int_{Q \setminus \mathrm{cl}\Omega_{p,\epsilon}} \frac{\partial u_j^-[\epsilon](x)}{\partial x_k} \, dx \\ + \int_{\partial\Omega_{p,\epsilon}} f((x-p)/\epsilon) x_k \, d\sigma_x \Bigg) \end{aligned} \tag{7}$$

for all $\epsilon \in ]0, \epsilon_0[$ and all $k, j \in \{1, 2, \ldots, n\}$, where $(u_j^+[\epsilon], u_j^-[\epsilon])$ is the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of system (3) with boundary conditions (4) with $f \equiv 0$, or (5), or (6). We note that the integral of the function $f$ is not presented in the right hand side of expression (7) when we consider the transmission problem with the boundary condition (4). Moreover, we observe that expression (7) extends that of Benveniste and Miloh [19] to the case of nonhomogeneous boundary conditions and coincides with the classical expression when $f$ and $g$ are identically 0 in systems (5) and (6).

**Our aim is to investigate the asymptotic behavior of the effective conductivity** of a periodic two-phase composite. As *our main result*, for $\epsilon$ small we prove that the effective conductivity can be represented as a convergent power series in $\epsilon$ and we determine *all* the coefficients in terms of the solutions of explicit systems of integral equations. Moreover, we compute some coefficients in the series expansion of the effective conductivity in case when the inclusions are in the form of a disk. One should note that for the nonideal transmission problem, the real analytic dependence of the effective conductivity on the parameter $\epsilon$ around the degenerate value $\epsilon = 0$ has been proved by Dalla Riva and Musolino [35].

The different variations of formula (7) for the effective conductivity can be found in the papers by Benveniste and Miloh [19, 104], Lipton and Vernescu [92], Ammari, Kang, and Touibi [10], Drygaś and Mityushev [44], Castro, Pesetskaya, and Rogosin [27], and Dalla Riva and Musolino [35], where the authors considered different contact conditions in materials and the case when there are more then one inclusion in the unite cell $Q$. We also note that Benveniste and Miloh [19] introduced their expression for defining the effective conductivity of a composite with imperfect contact conditions by generalizing the dual theory of the effective behavior of composites with the ideal contact. Moreover, Dalla Riva and Musolino [35] introduced the effective conductivity as an extension of the classical definition to the case of nonhomogeneous boundary conditions.

We now briefly outline *our strategy.* We first consider problem (3) with boundary conditions (4) and (5). We begin by computing the power series expansions for two auxiliary functions and then we prove our main results: we describe the effective conductivity $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ in terms of real analytic functions and obtain the power series expansion for it with explicitly determined coefficients. After that, we compute some coefficients in the series expansion of $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ in case inclusions are in the form of a ball. Finally, we apply the similar strategy in order to consider problem (3) with boundary conditions (6).

## Shape analysis of the effective longitudinal permeability of a periodic array of cylinders

In this part of the Dissertation, we study the behavior of longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders and the periodic structure, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. The shape of the cross section of the cylinders is determined by the image of a base domain through a diffeomorphism $\phi$ and the periodicity cell is a rectangle of sides of length $l$ and $1/l$, where $l$ is a positive parameter. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, the velocity field has only one nonzero component which, by the Stokes equations, satisfies a Poisson equation. Then, by integrating the longitudinal component of the velocity field, for each pair $(l, \phi)$, one defines the longitudinal permeability $K_{II}[l, \phi]$. Here, we are interested in studying the behavior of $K_{II}[l, \phi]$ upon the pair $(l, \phi)$.

The longitudinal permeability of arrays of cylinders has been studied by several authors by exploiting different methods. For example, Hasimoto [61] has investigated the viscous flow past a cubic array of spheres and he has applied his results to the two-dimensional flow past a square array of circular cylinders. His techniques are based on the construction of a spatially periodic fundamental solution for the Stokes' system and apply to specific shapes (circular/spherical obstacles and square/cubic arrays). Schmid [148] has investigated the longitudinal laminar flow in an infinite square array of circular cylinders. Sangani and Yao [146, 147] have studied the permeability of random arrays of infinitely long cylinders. Mityushev and Adler [109, 110] have considered the longitudinal permeability of periodic rectangular arrays of circular cylinders. By means of complex variable techniques, they have transformed the boundary value problem defining the permeability into a functional equation and then they have derived a formula for the longitudinal permeability as a logarithmic term plus a power series in the radius of the cylinder. Finally, in Musolino and Mityushev [119] the asymptotic behavior of the longitudinal permeability of thin cylinders of arbitrary shape has been considered. They have proved that the longitudinal permeability can be written as the sum of a logarithmic term and a power series in a parameter which is proportional to a diameter of the cylinders.

Here, instead, we are interested in the dependence of the longitudinal permeability upon the sides of the rectangular array and the shape of the cross section of the cylinders (see Figure 2). In particular, in contrast with other approaches, we do not need to restrict ourselves to particular shapes, as circles or ellipses. In order to introduce the mathematical problem, for $l \in ]0,+\infty[$, we define

$$Q_l := ]0,l[\times]0,1/l[, \qquad q_l := \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}.$$

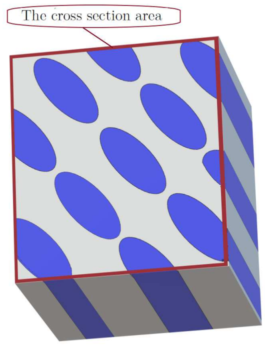


Figure 2: *The periodic array of cylinders in* $\mathbb{R}^3$

Clearly, $q_l\mathbb{Z}^2 \equiv \{q_l z : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the fundamental periodicity cell $Q_l$. Moreover, we find convenient to set $\widetilde{Q} := Q_1$. Then we take $\alpha \in ]0,1[$, fix a bounded open connected subset $\Omega$ of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\Omega$ is connected, and we consider a class of diffeomorphisms $\tilde{\mathcal{A}}_{\partial\Omega}$ from $\partial\Omega$ into $\widetilde{Q}$. If $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and the unbounded open connected components of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, respectively.

Then we consider the periodic domains $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] := \bigcup_{z\in\mathbb{Z}^2}(q_l z + q_l\mathbb{I}[\phi])$ and $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- := \mathbb{R}^2 \setminus \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]$. If $l \in ]0,+\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, the set $\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ is the region where a Newtonian fluid of viscosity $\mu$ is flowing at low Reynolds number. Then we assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (see, e.g., Adler [1, Ch. 4], Sangani and Yao [147], and Mityushev and Adler [109, 110]), one reduces the Stokes system to a Poisson equation for the nonzero component of the velocity field. Since we are working with dimensionless quantities, we may assume that the viscosity of the fluid and the pressure gradient are both set equal to one. For a more complete discussion on spatially periodic structures, we refer to Adler [1, Ch. 4]. Accordingly, if $l \in ]0,+\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \forall z \in \mathbb{Z}^2, \\ u(x) = 0 & \forall x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-. \end{cases} \tag{8}$$

If $l \in ]0,+\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, then the solution of problem (8) in the space $C^{1,\alpha}_{q_l}(\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$ of $q_l$-periodic functions in $\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ of class $C^{1,\alpha}$ is unique and we denote it by $u[l,\phi]$. From the physical point of view, the function $u[l,\phi]$ represents the nonzero component of the velocity field (see Mityushev and Adler [109, Sec. 2]). By means of the function $u[l,\phi]$, we can introduce the effective permeability $K_{II}[l,\phi]$ which we define as the integral of the opposite of the flow velocity over the unit cell (see Adler [1], Mityushev and Adler [109, Sec. 3]), i.e.,

$$K_{II}[l,\phi] := -\int_{Q_l \setminus q_l \mathbb{I}[\phi]} u[l,\phi](x)\,dx \qquad \forall l \in ]0,+\infty[\,, \quad \phi \in \tilde{\mathcal{A}}_{\partial\Omega}\,,$$

and **our aim is to investigate the regularity of the map** $(l,\phi) \mapsto K_{II}[l,\phi]$**.**

Shape analysis of functionals related to partial differential equations or quantities of physical relevance has been carried out by several authors and it is impossible to provide a complete list of contributions. Here we mention, for example the monographs by Henrot and Pierre [62], by Novotny and Sokołowski [121], and by Sokołowski and Zolésio [155]. Most of the works deals with differentiability properties. Here, instead, we are interested in proving higher regularity and we show that $K_{II}[l,\phi]$ depends analytically on $(l,\phi)$.

*Our main result* is the following: we have proved the fact that the map

$$(l,\phi) \mapsto K_{II}[l,\phi] \tag{9}$$

is real analytic. Such a result implies, in particular, that if we have a one-parameter analytic family of pairs $(l_\delta,\phi_\delta)_{\delta\in]-\delta_0,\delta_0[}$, with some $\delta_0$, then we can deduce the possibility to expand the permeability as a power series, i.e., $K_{II}[l_\delta,\phi_\delta] = \sum_{j=0}^{\infty} c_j \delta^j$ for $\delta$ close to zero. Moreover, by the analyticity of the map in (9), the coefficients in the series expansion of $K_{II}$ can be constructively determined by computing the differentials of $K_{II}[l_\delta,\phi_\delta]$ (cf. [136], Dalla Riva, Musolino, and the author [38] for the effective conductivity). Furthermore, another important consequence of our high regularity result is that it allows to apply differential calculus in order to find critical pairs $(l,\phi)$ *rectangle-shape* as a first step to find optimal configurations.

Finally, we briefly outline *our strategy* which is also in the framework of Functional Analytic Approach. As a first step, we transform the Dirichlet problem for the Poisson equation (8) in a Dirichlet problem for the Laplace equation which, it turn, we convert into an integral equation defined on $\partial\Omega$ by exploiting layer potential representations. Then we analyze the dependence of the solution of the integral equation upon $(l,\phi)$ by exploiting the Implicit Function Theorem and we prove that it depends real analytically on $(l,\phi)$. Finally, we exploit the obtained results and the integral representation of the solution of the boundary value problem to analyze $K_{II}[\mathrm{l},\phi]$.

## PART II: Properties of monogenic functions in a three-dimensional commutative algebra with one-dimensional radical

Part II consists of two chapters. In Chapter 4 we introduce a three-dimensional commutative algebra over $\mathbb{C}$ with one-dimensional radical and study the logarithmic residues of monogenic functions in this algebra. Chapter 5 is devoted to the investigation of the certain analog of Cauchy type integral taking values in the mentioned algebra and its limiting values on the boundary of definition.

There are many works devoted to developing hypercomplex methods and their application for solving problems of mathematical physics. Among them, we first note those which are related with noncommutative hypercomplex numbers: Sudbery [156], Gürlebeck and Sprössig [59, 60], Kravchenko and Shapiro [72], Kisil [69], Kisil and Ramírez de Arellano [70], Fokas and Pinotsis [46], Colombo, Sabadini, and Struppa [29], Pinotsis [123], and Shpakivskyi and Kuz'menko [154], *etc.* The analysis in noncommutative algebras is well developed and have many applications. Here, instead, we are interested in considering commutative algebras and developing techniques for their application to solving boundary value problems. One should note that such an analysis is at a preliminary stage and some results in this research area can be found in works of Kovalev and Mel'nichenko [71], Mel'nichenko [100, 101], Mel'nichenko and Plaksa [102, 103], Plaksa and Shpakivsky [126], and Gryshchuk and Plaksa [57, 58].

Below we describe the content of the chapters.

## Logarithmic residues of monogenic functions

The logarithmic residue in a Banach algebra means the contour integral of the logarithmic derivative of a hypercomplex function. It was considered by many authors in many algebras, for instance, in an algebra of all bounded linear operators on a complex Banach space (Mittenthal [107] and Bart, Ehrhardt, and Silbermann [14, 16]), matrix algebras (Bart, Ehrhardt, and Silbermann [17]), a biharmonic algebra (Grishchuk and Plaksa [55]), a three-dimensional algebra with two-dimensional radical (Plaksa and Shpakivskyi [129]). We note that Bart [13] considered the logarithmic residue for functions acting from the field of complex numbers $\mathbb{C}$ to a commutative Banach algebra. One of the main issues considered by Bart [13] (see also Bart, Ehrhardt, and Silbermann [14, 16]) is whether the vanishing of a logarithmic residue implies that a function takes only invertible values inside an integration contour, where contours are considered on the complex plane. We note that the answer is negative in general.

We also mention that Bart, Ehrhardt, and Silbermann [15] considered the logarithmic residues of locally analytic and meromorphic functions $f$ given in bounded Cauchy domains in the complex plane and taking values in a Banach algebra $\mathbb{A}$ with a unit element over $\mathbb{C}$. For definitions of locally analytic functions, meromorphic functions, and Cauchy domains, we refer, e.g., to Taylor and Lay [158, Sec. V.1.]. For instance, it was proven that if $f : \mathbb{C} \to \mathbb{A}$ is an analytic or meromorphic function in a bounded Cauchy domain in $\mathbb{C}$ then the logarithmic residue of $f$ is equal to a linear combination of idempotents of $\mathbb{A}$ with integer coefficients (see Bart, Ehrhardt, and Silbermann [15, Thm. 6.1, Thm. 7.1]). For the residues in multidimensional complex analysis and their applications, we refer to Aizenberg and Yuzhakov [3], Aizenberg, Tsikh, and Yuzhakov [2], and Tsikh [160].

The logarithmic residues of monogenic (continuous and differentiable in the sense of Gateaux) functions were considered in Grishchuk and Plaksa [55], and Plaksa and Shpakivskyi [129]. For instance, they calculated the logarithmic residue of monogenic function and it was shown that it is always an integer number. In the general case, it can be a hypercomplex number.

Here, instead, we consider the logarithmic residues of monogenic functions taking values in a three-dimensional commutative associative Banach algebra $\mathbb{A}_2$ over $\mathbb{C}$ with one-dimensional radical.

In order to define the logarithmic residue, we introduce some notation. Let $\mathbb{A}_2$ be a three-dimensional commutative associative Banach algebra over $\mathbb{C}$ with one-dimensional radical (see, e.g., Plaksa and the author [125], Mel'nichenko and Plaksa [103]). This algebra has a basis $\{I_1, I_2, \rho\}$ with the following multiplication rules for its elements

$$I_1{}^2 = I_1, \quad I_2{}^2 = I_2, \quad I_2\rho = \rho, \quad I_1 I_2 = \rho^2 = I_1\rho = 0.$$

The unit of $\mathbb{A}_2$ is represented as $1 = I_1 + I_2$.

Let $c = c_1 I_1 + c_2 I_2 + c_3\rho$, where $c_1, c_2, c_3 \in \mathbb{C}$. The element $c$ is *invertible* if and only if $c_1 \neq 0$ and $c_2 \neq 0$, moreover, the inverse element $c^{-1}$ is represented as $c^{-1} = 1/c_1 I_1 + 1/c_2 I_2 - c_3/c_2^2\rho$.

Then we take three vectors $e_1$, $e_2$, and $e_3$ in $\mathbb{A}_2$ that are linear independent over $\mathbb{R}$. We denote by

$$E_3 := \{xe_1 + ye_2 + ze_3 : \ x, y, z \in \mathbb{R}\}$$

the linear span over $\mathbb{R}$ in the algebra $\mathbb{A}_2$ generated by the vectors of basis $\{e_1 = 1, e_2, e_3\}$.

Let $\Omega$ be a domain in $E_3$ and the function $\Phi : \Omega \to \mathbb{A}_2$ be continuous in $\Omega$. We say that $\Phi$ is *monogenic* in $\Omega$ if $\Phi$ is differentiable in the sense of Gateaux at every point of $\Omega$, i.e., if for every $\zeta \in \Omega$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_2$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \qquad \forall\, h \in E_3.$$

Then we consider the linear continuous functionals $f_1,\ f_2 : \mathbb{A}_2 \to \mathbb{C}$ satisfying the equalities

$$f_1(I_1) = f_2(I_2) = 1, \quad f_1(I_2) = f_1(\rho) = f_2(I_1) = f_2(\rho) = 0,$$

and, for any nonnegative real number $R$ and a point $\zeta_0 \in E_3$, we set $\mathcal{K}_{0,R}(\zeta_0) := \{\zeta \in E_3 : 0 < |f_1(\zeta) - f_1(\zeta_0)| < R,\ 0 < |f_2(\zeta) - f_2(\zeta_0)| < R\}$

We define the logarithmic residue of monogenic functions in the following way. We take $\zeta_0 \in E_3$ and $R \in ]0, +\infty]$. If the functions $\Phi : \mathcal{K}_{0,R}(\zeta_0) \to \mathbb{A}_2$ and $\Phi'\Phi^{-1}$ are monogenic in the domain $\mathcal{K}_{0,R}(\zeta_0)$ then we say that the integral

$$\frac{1}{2\pi i} \int\limits_{\Gamma_r(\zeta_0)} \Phi'(\zeta)\big(\Phi(\zeta)\big)^{-1} d\zeta, \tag{10}$$

where $r \in ]0, R[$ and $\Gamma_r(\zeta_0)$ is a curve of a certain type in $\mathcal{K}_{0,R}(\zeta_0)$, is *the logarithmic residue of the monogenic function* $\Phi$ at the point $\zeta_0$.

**Our aim is to calculate the logarithmic residue of monogenic functions**, i.e., to calculate integral (10) over the curve $\Gamma_r(\zeta_0)$.

Now, we briefly outline *our strategy.* We start by proving some properties of Laurent series of monogenic functions in $\mathbb{A}_2$. Then we exploit the Laurent series to calculate the logarithmic residue of monogenic function. Using this result, we establish *our main results* for a curvilinear integral of the logarithmic derivative of a monogenic function along a family of curves. We show that the logarithmic residue depends on zeros, singular points, and, also, on those points at which the function takes values in the ideals of $\mathbb{A}_2$. Moreover, we show that the logarithmic residue is a hypercomplex number.

## Limiting values of a Cauchy type integral

Cauchy-type integrals are widely used for solving singular integral equations in boundary value problems for analytic functions of a complex variable. Among many works we mention those by Plemelj [130], Privalov [133], Muskhelishvili [116], Zygmund [164], Magnaradze [95], Salaev [143], and Babaev and Salaev [11], and Blaya, Reyes, and Kats [23].

One should mention the monographs by Gakhov [48] and by Muskhelishvili [116] where it is proven the existence of limiting values of Cauchy type integral

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \psi(t)(t-\xi)^{-1} dt, \qquad \xi \in \mathbb{C} \setminus \Gamma, \tag{11}$$

under classical conditions, namely, when $\Gamma$ is a smooth curve in the complex plane and a function $\psi : \Gamma \to \mathbb{C}$ is Hölder continuous. In case $\Gamma$ is a closed Jordan rectifiable curve Davydov [40] has obtained sufficient conditions for the existence of limiting values of integral (11) on $\Gamma$ from the interior and exterior domains bounded by $\Gamma$. Moreover, if

$$\sup_{t_0 \in \Gamma} \mu\{t \in \Gamma :\ |t - t_0| \le \varepsilon\} = O(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0, \tag{12}$$

where $\mu$ denotes the Lebesgue measure on $\Gamma$, and the modulus of continuity

$$\omega_{\psi,\Gamma}(\varepsilon) := \sup_{t_1,t_2 \in \Gamma,\ |t_1 - t_2| \le \varepsilon} |\psi(t_1) - \psi(t_2)|$$

of a function $\psi : \Gamma \to \mathbb{C}$ satisfies the Dini condition $\int_0^1 \omega_{\psi,\Gamma}(\eta)/\eta d\eta < \infty$, then integral (11) has limiting values in every point of $\Gamma$ from the mentioned above domains. This result was proven by Gerus [49], where he used results of Davydov [40].

Some analogs of integral (11) in commutative Banach algebras have been the subject of investigations since the last decades. In particular, Blaya, Peña, and Reyes [22, 24] investigated boundary properties of a Cauchy type integral over regular curves in the finite-dimensional Douglis algebras (see, e.g., Douglis [43]) and applied it to solve Riemann boundary value problem for some functions which act from $\mathbb{R}^2$ to such algebras (see also Gilbert and Zeng [53], and Gilbert and Buchanan [52]).

Unlike authors mentioned in the last paragraph, Plaksa and Shpakivskyi [127, 128] considered the integral of type (11) in a three-dimensional commutative algebra with two-dimensional radical, where $\Gamma$ is a plane curve in some three-dimensional real subspace of the algebra satisfying condition (12), and $\psi$ satisfies the Dini condition. Taking the structure of zero-divisors into account, it has been proven by Plaksa and Shpakivskyi [129] that such an integral is defined in two unbounded domains with the common cylindrical boundary for which the curve $\Gamma$ is the generatrix and that it has limiting values $\Gamma$. Also, under additional assumptions on the density $\psi$, the existence of limiting values on the whole cylindrical boundary from both domains has been proven in Plaksa and Shpakivskyi [127].

Here, instead, we consider a certain Cauchy type integral in the three-dimensional commutative algebra $\mathbb{A}_2$. One should note that the structure of zero-divisors in this algebra leads to an increase in the number of domains of definition for such an integral and to the complication of their geometry.

In order to formulate the problem we recall some notation. The algebra $\mathbb{A}_2$ has the the basis $\{I_1, I_2, \rho\}$ and $1 = I_1 + I_2$. Here we take three specific vectors $e_1 = 1$, $e_2 = iI_1 + \rho$, and $e_3 = iI_2$, which are linearly independent over $\mathbb{C}$, and we denote by $E_3 \subseteq \mathbb{A}_2$ the linear span of $\{e_1, e_2, e_3\}$ over $\mathbb{R}$.

Let $\Phi : E_3 \setminus \Sigma \to \mathbb{A}_2$ be a function defined as follows

$$\Phi(\zeta) := \frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1} d\tau \qquad \forall \zeta \in E_3 \setminus \Sigma, \tag{13}$$

where $\Gamma$ is a plain closed Jordan rectifiable curve, a function $\varphi : \Gamma \to \mathbb{R}$ is Dini-continuous, and $\Sigma$ is a set of singularithies of the function $\Phi$ in $E_3$. **Our aim is to investigate the behavior of $\Phi(\zeta)$ when $\zeta$ tends to a point of $\Sigma$.** One should note that the Cauchy type integral presented in (13) is a monogenic function in six domains obtained by removing the set $\Sigma$ from the linear space $E_3$, but it does not exist on $\Sigma$. *Our main result* is that we have established sufficient conditions for the existence of limiting values of (13) on $\Sigma$ and have shown the validity of analogue of Sokhotskii-Plemelj formulas. We also note that if $\Gamma$ is a straight line then integral (13) was considered in the paper of the author and Plaksa [140], and it was shown that the set of definition of $\Phi$ consists of four domains and integral (13) has different limiting values on $\Gamma$ when $\zeta$ tends to $\Gamma$ from each of such the domains.

Now, we briefly outline *our strategy*. Taking the representation of the unit element in $\mathbb{A}_2$ into account, we split the function $\Phi$ into three parts. We denote such parts by $\Phi_1$, $\Phi_2$, and $\Phi_3$, and study them separately. We first prove the existence of limiting values of $\Phi_1$ and $\Phi_2$ on the boundary of domains of definition. Then we analyze the behavior of $\Phi_3$ on the curve of integration $\Gamma$ and on the boundary of domains of definition. Then, using the obtained results, we prove the existence of limiting values of integral (13) on the boundary of domains of definition and establish the validity of analog of the Sokhotskii-Plemelj formulas.

**Note:** Some of the results presented in the Dissertation have been published in the following papers:

- Pukhtaievych, R. Asymptotic behavior of the solution of singularly perturbed transmission problems in a periodic domain. *Math. Methods Appl. Sci. 41*, 9 (2018), 3392–3413.

- PUKHTAIEVYCH, R. Effective conductivity of a periodic dilute composite with perfect contact and its series expansion. *Z. Angew. Math. Phys. 69*, 3 (2018), 69–83.
- DALLA RIVA, M., MUSOLINO, P., AND PUKHTAIEVYCH, R. Series expansion for the effective conductivity of a periodic dilute composite with thermal resistance at the two-phase interface. *Asymptot. Anal.* (2018 ACCEPTED).
- LUZZINI, P., MUSOLINO, P., AND PUKHTAIEVYCH, R. Shape analysis of the effective longitudinal permeability of a periodic array of cylinders. *In preparation* (2018).
- PUKHTAIEVYCH, R., AND PLAKSA, S. On logarithmic residue of monogenic functions in a three-dimensional commutative algebra with one-dimensional radical. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 25*, 3 (2017), 167–182.
- PUKHTAIEVYCH, R., AND PLAKSA, S. Some properties of a Cauchy type integral in a three-dimensional commutative algebra with one-dimensional radical. *Submitted* (2018).

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# Notation

Throughout the dissertation, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively, the symbol $\mathbb{N}$ denotes the set of natural numbers including 0.

Let $n \in \mathbb{N}$. We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We also denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$.

Let a set $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. We also set $\mathbb{D}^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean modulus of $x$ in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. For a number $k \in \{1, 2\}$ the symbol $M_{k\times n}(\mathbb{R})$ denotes the space of $k \times n$ matrices with real entries.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $r \in \mathbb{N} \setminus \{0\}$. Let $f \in C^r(\Omega)$. Then $Df$ denotes the vector $\left(\frac{\partial f}{\partial x_l}\right)_{l\in\{1,\ldots,n\}}$. For a multi-index $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$ we set $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^r(\Omega)$ of those functions $f$ whose derivatives $D^\eta f$ of order $|\eta| \le r$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted by $C^r(\mathrm{cl}\Omega)$. The subspace of $C^r(\mathrm{cl}\Omega)$ whose functions have $r$-th order derivatives which are uniformly Hölder continuous with exponent $\alpha \in ]0, 1[$ is denoted by $C^{r,\alpha}(\mathrm{cl}\Omega)$. The subspace of $C^r(\mathrm{cl}\Omega)$ of those functions $f$ such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))} \in C^{r,\alpha}(\mathrm{cl}(\Omega \cap \mathbb{B}_n(0, R)))$ for all $R \in ]0, +\infty[$ is denoted $C^{r,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$.

Now let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Then $C^r(\mathrm{cl}\Omega)$ and $C^{r,\alpha}(\mathrm{cl}\Omega)$ are endowed with their usual norm and are well known to be Banach spaces. We say that a bounded open subset $\Omega$ of $\mathbb{R}^n$ is of class $C^r$ or of class $C^{r,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^r$ or $C^{r,\alpha}$, respectively. We define the spaces $C^{k,\alpha}(\partial\Omega)$ for $k \in \{0, \ldots, r\}$ by exploiting the local parametrizations (cf., *e.g.*, Gilbarg and Trudinger [51, Sec. 6.2]). The trace operator from $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [51] (see also Lanza de Cristoforis [75, Sec. 2, Lem. 3.1, 4.26, Thm. 4.28], Lanza de Cristoforis and Rossi [87, Sec. 2]). We denote by $\nu_\Omega$ the outward unit normal to $\partial\Omega$ and by $d\sigma$ the area element on $\partial\Omega$. We retain the standard notation for the Lebesgue space $L^1(\partial\Omega)$ of integrable functions. By $|\partial\Omega|_{n-1}$, we denote the $n-1$-dimensional measure of $\partial\Omega$. To shorten our notation, we denote by $\fint_{\partial\Omega} \phi \, d\sigma$ the integral mean $\frac{1}{|\partial\Omega|_{n-1}} \int_{\partial\Omega} \phi \, d\sigma$ for all $\phi \in L^1(\partial\Omega)$. Also, if $\mathcal{X}$ is a vector subspace of $L^1(\partial\Omega)$ then we set $\mathcal{X}_0 \equiv \left\{f \in \mathcal{X} : \int_{\partial\Omega} f \, d\sigma = 0\right\}$ . For the definition and properties of real analytic operators, we refer, e.g., to Deimling [41, p. 150].

If $\Omega$ is an arbitrary open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, $\beta \in ]0, 1[$, we set

$$C^k_b(\mathrm{cl}\Omega) := \{u \in C^k(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \le k\},$$

and we endow $C^k_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^k_b(\mathrm{cl}\Omega)} := \sum_{|\gamma|\le k} \sup_{x\in\mathrm{cl}\Omega} |D^\gamma u(x)| \qquad \forall u \in C^k_b(\mathrm{cl}\Omega).$$

Then we set

$$C_b^{k,\beta}(\mathrm{cl}\Omega) := \{u \in C^{k,\beta}(\mathrm{cl}\Omega) :\ D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \le k\},$$

and we endow $C_b^{k,\beta}(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C_b^{k,\beta}(\mathrm{cl}\Omega)} := \sum_{|\gamma|\le k} \sup_{x\in \mathrm{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \mathrm{cl}\Omega|_\beta \qquad \forall u \in C_b^{k,\beta}(\mathrm{cl}\Omega),$$

where $|D^\gamma u : \mathrm{cl}\Omega|_\beta$ denotes the $\beta$-Hölder constant of $D^\gamma u$.

Next we turn to periodic domains. If $\Omega_Q$ is an arbitrary subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_Q \subseteq Q$, then we set

$$\mathbb{S}[\Omega_Q] := \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_Q) = q\mathbb{Z}^n + \Omega_Q, \qquad \mathbb{S}[\Omega_Q]^- := \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_Q].$$

Then a function $u$ from $\mathrm{cl}\mathbb{S}[\Omega_Q]$ or from $\mathrm{cl}\mathbb{S}[\Omega_Q]^-$ to $\mathbb{R}$ is $q$-periodic if $u(x + q_{hh}e_h) = u(x)$ for all $x$ in the domain of definition of $u$ and for all $h \in \{1, \dots, n\}$. If $\Omega_Q$ is an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_Q \subseteq Q$ and if $k \in \mathbb{N}$ and $\beta \in ]0, 1[$, then we denote by $C_q^k(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C_q^k(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and $C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ the subsets of the $q$-periodic functions belonging to $C_b^k(\mathrm{cl}\mathbb{S}[\Omega_Q])$, to $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q])$, to $C_b^k(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and to $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, respectively. We regard the sets $C_q^k(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C_q^k(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, $C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ as Banach subspaces of $C_b^k(\mathrm{cl}\mathbb{S}[\Omega_Q])$, of $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q])$, of $C_b^k(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, of $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, respectively.

Then, we introduce the Roumieu classes. For all bounded open subsets $\Omega'$ of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega') := \left\{ u \in C^\infty(\mathrm{cl}\Omega') :\ \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega')} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega')} := \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega')} \qquad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega'),$$

for all $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega'), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega')}\right)$ is a Banach space.

# Part I

# Perturbation problems and applications

# CHAPTER 1

# Asymptotic behavior of the solutions of transmission problems in a periodic domain

This chapter is mainly devoted to the study of the asymptotic behavior of the solutions of singularly perturbed transmission problems in a periodically perforated domain. The domain is obtained by making in $\mathbb{R}^n$ a periodic set of holes, each of them of a size proportional to a positive parameter $\epsilon$. We first consider an ideal transmission problem and investigate the behavior of the solution as $\epsilon$ tends to 0. In particular, we deduce a representation formula in terms of real analytic maps of $\epsilon$ and of some additional parameters. Then we apply such a result to a nonideal nonlinear transmission problem.

Our analysis is based on the *Functional Analytic Approach* proposed by Lanza de Cristoforis for the analysis of singular perturbation problems in perforated domains (see Lanza de Cristoforis [76, 79], Dalla Riva and Lanza de Cristoforis [31]). Such an approach aims at representing the solution or related functionals in terms of real analytic functions of the singular perturbation parameter and of explicitly known functions of such a parameter. By this approach, Dalla Riva and Musolino [35] have investigated a nonideal singularly perturbed linear transmission problem in a periodic domain, and Lanza de Cristoforis and Musolino [84] have investigated a quasi-linear heat transmission problem (see also Dalla Riva, Lanza de Cristoforis, and Musolino [32]). For other contributions for the analysis of nonlinear transmission problems, we refer to Dalla Riva and Mishuris [34] and Lanza de Cristoforis [80].

The chapter is organized as follows. In Section 1.1 we introduce some notation and pose the linear and nonlinear transmission problems. Sections 1.2 and 1.3 contain preliminary results. In Section 1.4 we formulate the linear transmission boundary value problem in terms of a system of integral equations and we study the dependence of the unknown of the system upon $\epsilon$ and some additional parameters. In Section 1.5 we show that the results of Section 1.4 can be exploited to prove our main Theorems 1.5.1 and 1.5.2. Finally, in Section 1.6 we apply the results of Section 1.5 to the nonideal nonlinear transmission problem.

Some of the results presented in this chapter have been published in the paper [135] by the author.

## 1.1 Preliminaries and notation

In this section we consider singularly perturbed transmission problems in a periodically perforated domain and introduce some notation.

In order to define the geometry of the problems, we fix once for all $n \in \mathbb{N} \setminus \{0, 1\}$ and $(q_{11}, \ldots, q_{nn}) \in ]0, +\infty[^n$. We introduce the periodicity cell $Q$ and the diagonal matrix $q$ by setting

$$Q := \Pi_{i=1}^{n}]0, q_{ii}[, \qquad q := \begin{pmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_{nn} \end{pmatrix}.$$

Clearly, $q\mathbb{Z}^n = \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell $Q$.

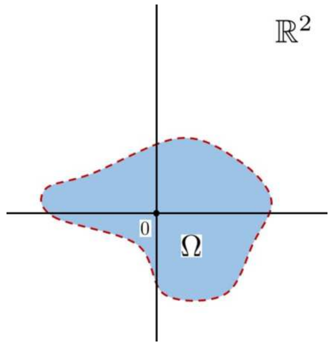


Figure 1.1: *The domain* $\Omega$

Then we take $\alpha \in ]0, 1[$. We fix once and for all in Chapter 1 and Chapter 2 a subset $\Omega$ of $\mathbb{R}^n$ satisfying the following assumption:

$$\begin{aligned} &\Omega \text{ is a bounded open connected subset of } \mathbb{R}^n \\ &\text{of class } C^{1,\alpha},\ \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ is connected, and } 0 \in \Omega. \end{aligned} \tag{1.1}$$

Let $p \in Q$ be fixed. Then there exists $\epsilon_0 \in \mathbb{R}$ such that

$$\epsilon_0 \in ]0, +\infty[ \quad \text{such that} \quad p + \epsilon\, \mathrm{cl}\Omega \subseteq Q \quad \forall \epsilon \in ] - \epsilon_0, \epsilon_0[, \tag{1.2}$$

and we also fix it. The set $\Omega$ represents the shape of each inclusion (see Figure 1.1). Then we set

$$\Omega_{p,\epsilon} := p + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R}.$$

For all $\epsilon \in ] - \epsilon_0, \epsilon_0[$, the set $\Omega_{p,\epsilon}$ plays the role of the inclusion in the fundamental cell $Q$ (see Figure 1.2).

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] := \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega_{p,\epsilon}), \quad \mathbb{S}[\Omega_{p,\epsilon}]^- := \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$$

for all $\epsilon \in ] - \epsilon_0, \epsilon_0[$. The set $\mathbb{S}[\Omega_{p,\epsilon}]$ corresponds to the region occupied by the periodic set of holes or inclusions; the matrix, instead, is represented by its complementary set $\mathbb{S}[\Omega_{p,\epsilon}]^-$ (see Figure 1.3).

Next, we take two positive constants $\lambda^+$, $\lambda^-$, a constant $c \in \mathbb{R}$, functions $f \in C^{0,\alpha}(\partial\Omega)_0$ and $g \in C^{1,\alpha}(\partial\Omega)$, a function $\rho : ]0, \epsilon_0[ \to \mathbb{R}\setminus\{0\}$, and we consider the following *linear transmission problem* for a pair of functions $(u^+, u^-) \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

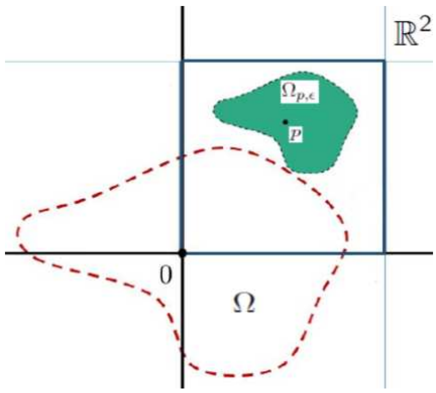


Figure 1.2: *The domain* $\Omega_{p,\epsilon}$

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta u^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ u^+(x + qe_h) = u^+(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,, \quad \forall h \in \{1, \dots, n\}, \\ u^-(x + qe_h) = u^-(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,, \quad \forall h \in \{1, \dots, n\}, \\ \lambda^- \frac{\partial u^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \frac{1}{\rho(\epsilon)}(u^+(x) - u^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} u^+ d\sigma = c \end{cases} \tag{1.3}$$

for all $\epsilon \in ]0, \epsilon_0[$.

The analysis of problem (1.3) will allow us to study a (more general) transmission problem, which we introduce below (see problem (1.4)). So let a matrix $B \in M_{1\times n}(\mathbb{R})$ and $F$ be a real analytic map from $\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $\mathbb{R}$. For $(\epsilon, f, g, c) \in ]0, \epsilon_0[ \times C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega) \times \mathbb{R}$,

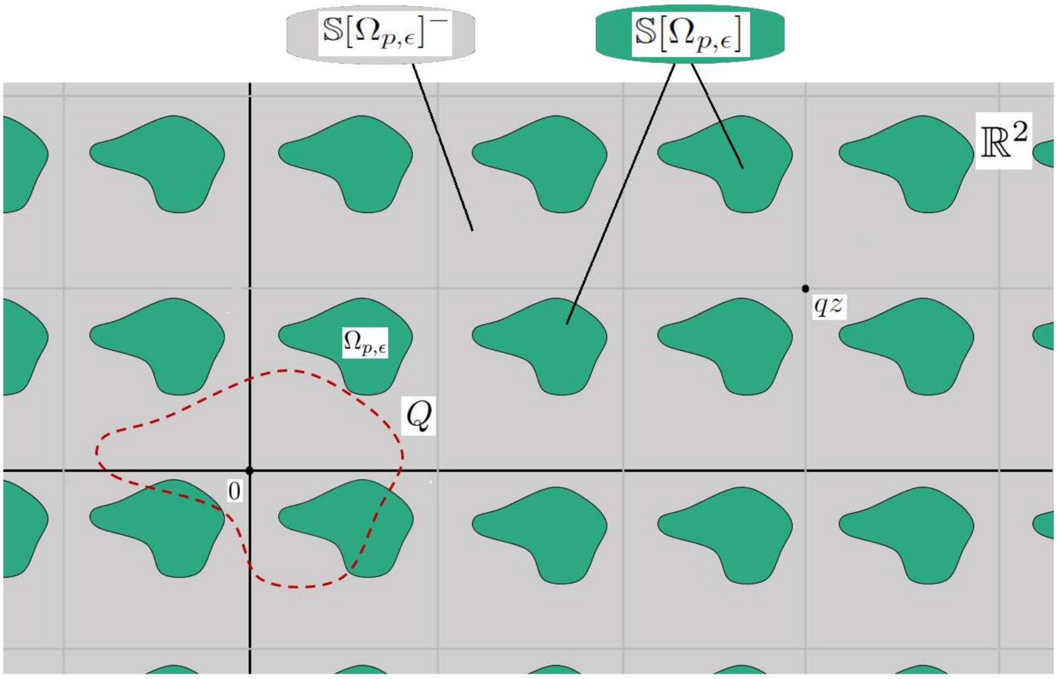


Figure 1.3: *The periodic domains* $\mathbb{S}[\Omega_{p,\epsilon}]$ *and* $\mathbb{S}[\Omega_{p,\epsilon}]^-$

we consider the following *nonideal nonlinear transmission problem* for a pair of functions $(\mathbf{u}^+, \mathbf{u}^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

$$\begin{cases} \Delta \mathbf{u}^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta \mathbf{u}^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ \mathbf{u}^+(x+qe_h) = \mathbf{u}^+(x) + Be_h & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,\ \forall h \in \{1,\dots,n\}, \\ \mathbf{u}^-(x+qe_h) = \mathbf{u}^-(x) + Be_h & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,\ \forall h \in \{1,\dots,n\}, \\ \lambda^- \frac{\partial \mathbf{u}^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial \mathbf{u}^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ F\left[\frac{\partial \mathbf{u}^-}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot), \frac{\partial \mathbf{u}^+}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)\right] & \\ \qquad +\frac{1}{\rho(\epsilon)}(\mathbf{u}^+(x) - \mathbf{u}^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} \mathbf{u}^+ d\sigma = c & \end{cases} \tag{1.4}$$

for all $\epsilon \in ]0, \epsilon_0[$.

If we assume that the set of inclusions and the matrix are filled with two different homogeneous and isotropic heat conductor materials, then the parameters $\lambda^+$ and $\lambda^-$ play the role of thermal conductivity of the materials which fill the inclusions and the matrix, respectively. Therefore, the solutions of problems (1.3) and (1.4) represent the temperature distribution on the set of inclusions $\mathbb{S}[\Omega_{p,\epsilon}]$ and in the matrix $\mathbb{S}[\Omega_{p,\epsilon}]^-$, under different conditions. More precisely, the third and fourth conditions in problems (1.3) and (1.4) mean periodicity and periodicity up to a given linear function, respectively, of the temperature distribution. The fifth condition of problems (1.3) and (1.4) says that the normal component of the heat flux presents a jump which equals a given function. The sixth condition of the two problems is different: the one in (1.3) says that the temperature distribution presents a jump on the interface equal to a given function, while the one in (1.4) says that the jump on the interface of the temperature distribution plus a given function is proportional to a quantity which depends on the heat flux. In case $g = 0$, such a condition means that the jump of the temperature distribution is proportional to a quantity which depends nonlinearly on the heat flux and can be seen as a nonlinear counterpart of the linear nonideal transmission problem considered by Dalla Riva and Musolino [35], where the parameter $\rho(\epsilon)$ plays the role of the boundary thermal resistivity. Finally, the last condition of problems (1.3) and (1.4) is just a normalization condition, which we need in order to "fix" the solution.

As we shall see, problems (1.3) and (1.4) have unique pairs of solutions which we denote by $(u^+[\epsilon, f, g, c], u^-[\epsilon, f, g, c])$ and by $(\mathbf{u}^+[\epsilon, f, g, c], \mathbf{u}^-[\epsilon, f, g, c])$, respectively. Our aim is to understand the behavior of those pairs when $\epsilon$ is close to the degenerate value $\epsilon = 0$, in correspondence of which the inclusions collapse to points. From the physical point of view, this situation corresponds to the case of a dilute composite.

Due to the presence of the factor $1/\rho(\epsilon)$, the boundary conditions may display a singularity as $\epsilon$ tends to 0. We consider the case in which

$$\lim_{\epsilon \to 0^+} \frac{\rho(\epsilon)}{\epsilon} \quad \text{exists finite in } \mathbb{R}. \tag{1.5}$$

Assumption (1.5) will allow us to analyze problems (1.3) and (1.4) around the degenerate value $\epsilon = 0$. We also note that we make no regularity assumption on the function $\rho$. If assumption (1.5) holds, then we set

$$r_* := \lim_{\epsilon \to 0^+} \frac{\rho(\epsilon)}{\epsilon}. \tag{1.6}$$

Incidentally, we observe that assumption (1.5) implies that

$$\lim_{\epsilon \to 0^+} \rho(\epsilon) = 0\,.$$

Assumption (1.5) is alternative to that considered by Dalla Riva and Musolino [35], where they assumed that $\lim_{\epsilon \to 0^+} \epsilon/\rho(\epsilon)$ exists finite in $\mathbb{R}$. Clearly, both assumptions are satisfied in case $\rho(\epsilon) = \epsilon$.

We also note that some results on the study of transmission problems can be found, for example, in Ammari, Kang, and Touibi [10], Ammari, Kang, and Kim [8], Ammari, Garnier, Giovangigli, Jing and Seo [5], Castro, Pesetskaya, and Rogosin [27], Pesetskaya [122], Kapanadze, Miszuris, and Pesetskaya [67], and Drygaś and Mityushev [44], Lanza de Cristoforis and Musolino [84], and Dalla Riva, Lanza de Cristoforis, and Musolino [32].

We complete this section recalling some properties of layer potentials which we use in order to convert the analysis of our transmission problems to that of systems of integral equations. To do so, we first recall that there exists a $q$-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z \in \mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n},$$

where $\delta_{qz}$ denotes the Dirac distribution with mass in $qz$ (see Theorem B.0.1 in Appendix B). The distribution $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) := -\sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{4\pi^2 |q^{-1}z|^2 |Q|_n} e^{2\pi i (q^{-1}z)\cdot x},$$

where the series converges in the sense of distributions on $\mathbb{R}^n$. Then $S_{q,n}$ is real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$ and is locally integrable in $\mathbb{R}^n$ (see Theorem B.0.1 in Appendix B). One should note that Ammari and Kang [7, p. 53] and Lanza de Cristoforis and Musolino [81] used the Poisson summation formula to obtain the construction for $S_{q,n}$ presented above, but there also exist other ways to construct and compute such a periodic fundamental solution. For example, Hasimoto [61] used Ewald's techniques in order to approximate $S_{q,n}$. Cichocki and Felderhof [28] obtained expressions for $S_{q,n}$ in the form of rapidly convergent series (see also Sangani, Zhang, and Prosperetti [145] and Poulton, Botten, McPhedran, and Movchan [131]). Finally, Mityushev and Adler [109] used elliptic functions to construct a formula for $S_{q,2}$. We also mention Berdichevskii [74] and Shcherbina [150] who described algorithms to construct the periodic fundamental solutions for certain equations.

Let $S_n$ be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) := \begin{cases} \frac{1}{2\pi} \log |x| & \forall x \in \mathbb{R}^2 \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2, \end{cases}$$

where $s_n$ denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n(0,1)$. $S_n$ is well known to be the fundamental solution of the Laplace operator (see, e.g., Folland [47, Thm. 2.17])

Then $S_{q,n} - S_n$ can be extended to an analytic function in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ and we find convenient to set

$$R_{q,n} := S_{q,n} - S_n \qquad \text{in } (\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$$

(see Theorem B.0.1 in Appendix B). Moreover, if $\epsilon \in \mathbb{R}$ and $x \in \mathbb{R}^n$ are such that $\epsilon x \in \mathbb{R}^n \setminus q\mathbb{Z}^n$ then the following trivial equality holds

$$S_{q,n}(\epsilon x) = \epsilon^{2-n} S_n(x) + \frac{\log |\epsilon|}{2\pi} \delta_{2,n} + R_{q,n}(\epsilon x). \tag{1.7}$$

We now recall the definition and some properties of the classical single layer potential. Let $\alpha \in ]0,1[$ and $\mathbb{I}$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then for all $\theta \in C^{0,\alpha}(\partial\mathbb{I})$, the single layer potential $v[\partial\mathbb{I}, \theta]$ is represented as follows

$$v[\partial\mathbb{I}, \theta](t) \equiv \int_{\partial\mathbb{I}} S_n(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n .$$

For the properties of single layer potential we refer to Theorem A.0.2 in Appendix A. Here, we just mention that $v[\partial\mathbb{I}, \theta]$ is continuous on $\mathbb{R}^n$ and harmonic in $\mathbb{R}^n \setminus \partial\mathbb{I}$, the function $v^+[\partial\mathbb{I}, \theta] := v[\partial\mathbb{I}, \theta]_{|\mathrm{cl}\mathbb{I}}$ belongs to $C^{1,\alpha}(\mathrm{cl}\mathbb{I})$, and the function $v^-[\partial\mathbb{I}, \theta] := v[\partial\mathbb{I}, \theta]_{|\mathbb{R}^n \setminus \mathbb{I}}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{I})$. Also, we set

$$w_*[\partial\mathbb{I}, \theta](t) := \int_{\partial\mathbb{I}} DS_n(t-s)\nu_{\mathbb{I}}(t)\theta(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

and recall that the function $w_*[\partial\mathbb{I}, \theta]$ belongs to $C^{0,\alpha}(\partial\mathbb{I})$, and we have

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} v^{\pm}[\partial\mathbb{I}, \theta] = \mp\frac{1}{2}\theta + w_*[\partial\mathbb{I}, \theta] \qquad \text{on } \partial\mathbb{I}$$

(see Theorem A.0.2 in Appendix A).

We now recall the definition and some properties of the periodic single layer potential. We fix once and for all a subset $\Omega_Q$ of $\mathbb{R}^n$ satisfying the following assumption:

$$\begin{gathered}\Omega_Q \text{ is a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{1,\alpha} \text{ such that} \\ \mathrm{cl}\Omega_Q \subseteq Q \text{ and } \mathbb{R}^n \setminus \mathrm{cl}\Omega_Q \text{ is connected.}\end{gathered} \tag{1.8}$$

Then for all $\theta \in C^{0,\alpha}(\partial\Omega_Q)$, the periodic single layer potential $v_q[\partial\Omega_Q, \theta]$ is represented as follows

$$v_q[\partial\Omega_Q, \theta](x) \equiv \int_{\partial\Omega_Q} S_{q,n}(x-y)\theta(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n .$$

For the properties of the periodic single layer potential we refer to Theorem B.0.2 in Appendix B. Here, we just mention that $v_q[\partial\Omega_Q, \theta]$ is continuous in $\mathbb{R}^n$, the function $v_q^+[\partial\Omega_Q, \theta] :=$

$v_q[\partial\Omega_Q,\theta]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]}$ belongs to $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, and $v^-_q[\partial\Omega_Q,\theta] := v_q[\partial\Omega_Q,\theta]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]^-}$ belongs to $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$. Then, we introduce the function $w_{q,*}[\partial\Omega_Q,\theta]$ by setting

$$w_{q,*}[\partial\Omega_Q,\theta](x) := \int_{\partial\Omega_Q} DS_{q,n}(x-y)\nu_{\Omega_Q}(x)\theta(y)\,d\sigma_y \qquad \forall x\in\partial\Omega_Q,\ \forall\theta\in C^{0,\alpha}(\partial\Omega_Q),$$

which belongs to $C^{0,\alpha}(\partial\Omega_Q)$. Then we have the following jump formulas:

$$\frac{\partial}{\partial\nu_{\Omega_Q}}v^{\pm}_q[\partial\Omega_Q,\theta] = \mp\frac{1}{2}\theta + w_{q,*}[\partial\Omega_Q,\theta] \qquad \text{on } \partial\Omega_Q$$

(see Theorem B.0.2 in Appendix B).

Finally, we briefly outline our strategy. First, we convert problem (1.3) into a system of integral equations by exploiting layer potential representations. Taking assumption (1.5) into account, this system can be analyzed when $(\epsilon,f,g,c)$ is close to the degenerate quadruple $(0,f_0,g_0,c_0)$. We do so by means of the Implicit Function Theorem and we represent the unknowns of the system of integral equations in terms of analytic functions of $\epsilon$, $\rho(\epsilon)/\epsilon$, $f$ and $g$. Next we exploit the integral representations of the solutions in terms of the unknowns of the system of integral equations, and we deduce the representation of $u^+[\epsilon,f,g,c]$ and $u^-[\epsilon,f,g,c]$ in terms of real analytic maps of $\epsilon$, $\rho(\epsilon)/\epsilon$, $f$, $g$, and $c$. Finally, we convert problem (1.4) into a linear periodic transmission problem and we apply the previously discussed results for the derivation of the representation for $(\mathbf{u}^+[\epsilon,f,g,c],\mathbf{u}^-[\epsilon,f,g,c])$.

## 1.2 A linear transmission problem for periodic functions

In this section we collect some results which we use in order to reformulate problem (1.3) in terms of a system of integral equations. We first have the following uniqueness result for a periodic transmission problem, whose proof is based on a standard energy argument for periodic harmonic functions.

**Proposition 1.2.1.** *Let $\lambda^+,\lambda^-\in]0,+\infty[$. Let $(v^+,v^-)\in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])\times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ be such that*

$$\begin{cases} \Delta v^+ = 0 & \text{in } \mathbb{S}[\Omega_Q],\\ \Delta v^- = 0 & \text{in } \mathbb{S}[\Omega_Q]^-,\\ v^+(x+qe_h) = v^+(x) & \forall x\in\mathrm{cl}\mathbb{S}[\Omega_Q]\,,\ \forall h\in\{1,\dots,n\},\\ v^-(x+qe_h) = v^-(x) & \forall x\in\mathrm{cl}\mathbb{S}[\Omega_Q]^-\,,\ \forall h\in\{1,\dots,n\},\\ \lambda^-\frac{\partial v^-}{\partial\nu_{\Omega_Q}}(x) - \lambda^+\frac{\partial v^+}{\partial\nu_{\Omega_Q}}(x) = 0 & \forall x\in\partial\Omega_Q,\\ v^+(x)-v^-(x) = 0 & \forall x\in\partial\Omega_Q,\\ \int_{\partial\Omega_Q} v^+\,d\sigma = 0\,. \end{cases} \tag{1.9}$$

*Then $v^+=0$ on $\mathrm{cl}\mathbb{S}[\Omega_Q]$ and $v^-=0$ on $\mathrm{cl}\mathbb{S}[\Omega_Q]^-$.*

*Proof.* By the Divergence Theorem and by the periodicity of $v^-$, we have

$$\begin{aligned} 0 \le \int_{\Omega_Q} |\nabla v^+(t)|^2 dt &= \int_{\partial\Omega_Q} v^+(t)\frac{\partial}{\partial\nu_{\Omega_Q}}v^+(t)d\sigma_t \\ &= \frac{\lambda^-}{\lambda^+}\int_{\partial\Omega_Q} v^-(t)\frac{\partial}{\partial\nu_{\Omega_Q}}v^-(t)d\sigma_t = -\frac{\lambda^-}{\lambda^+}\int_{Q\setminus\mathrm{cl}\Omega_Q}|\nabla v^-(t)|^2dt \le 0. \end{aligned}$$

Thus

$$\int_{\Omega_Q}|\nabla v^+(t)|^2dt = \int_{Q\setminus\mathrm{cl}\Omega_Q}|\nabla v^-(t)|^2dt = 0,$$

and so, by the periodicity of $v^+$ and $v^-$,

$$\begin{aligned} v^+(x) &= c^+ \quad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_Q], \\ v^-(x) &= c^- \quad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_Q]^-, \end{aligned}$$

for some $c^+$, $c^-$ in $\mathbb{R}$. By virtue of the sixth and the seventh equalities of (1.9), we conclude that $c^+ = 0 = c^-$ and the proposition is proved. □

In the following proposition we consider an operator which we will use in the sequel.

**Proposition 1.2.2.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. The operator $\Lambda$ from $C^{0,\alpha}(\partial\Omega_Q)_0$ to $C^{0,\alpha}(\partial\Omega_Q)_0$ defined by*

$$\Lambda[\mu] := \frac{1}{2}\mu + \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} w_{q,*}[\partial\Omega_Q, \mu] \qquad \forall \mu \in C^{0,\alpha}(\partial\Omega_Q)_0$$

*is a linear homeomorphism.*

*Proof.* First we note that the map $\mu \mapsto w_{q,*}[\partial\Omega_Q, \mu]$ from $C^{0,\alpha}(\partial\Omega_Q)_0$ to $C^{0,\alpha}(\partial\Omega_Q)_0$ is compact (see Theorem B.0.2($vii$) in Appendix B) and thus $\Lambda$ is a Fredholm operator of index 0. Thus, by Fredholm Theory and by the Open Mapping Theorem, in order to prove that $\Lambda$ is a linear homeomorphism, it suffices to prove that $\Lambda$ is injective. So let $\mu \in C^{0,\alpha}(\partial\Omega_Q)_0$ be such that

$$\frac{1}{2}\mu + \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} w_{q,*}[\partial\Omega_Q, \mu] = 0 \quad \text{on} \quad \partial\Omega_Q. \tag{1.10}$$

Then we note that equality (1.10) can be rewritten in the following way

$$\lambda^- \frac{\partial}{\partial \nu_{\Omega_Q}} v_q^-[\partial\Omega_Q, \mu] - \lambda^+ \frac{\partial}{\partial \nu_{\Omega_Q}} v_q^+[\partial\Omega_Q, \mu] = 0 \quad \text{on} \quad \partial\Omega_Q.$$

We observe that the pair of functions $(u^i, u^o) \equiv (v_q^+[\partial\Omega_Q, \mu], v_q^-[\partial\Omega_Q, \mu])$ satisfies the first six conditions of problem (1.9). Then the proof of Proposition 1.2.1 implies that there exists $c \in \mathbb{R}$ such that

$$v_q^+[\partial\Omega_Q, \mu] = c \qquad \text{in } \mathrm{cl}\mathbb{S}[\Omega_Q]\,, \qquad v_q^-[\partial\Omega_Q, \mu] = c \qquad \text{in } \mathrm{cl}\mathbb{S}[\Omega_Q]^-\,.$$

Then the jump formulae for the normal derivative of the periodic single layer potential on $\partial\Omega_Q$ imply that $\mu = 0$. Hence, $\Lambda$ is injective and thus a linear homeomorphism. □

We now study an integral operator which we need in order to solve a periodic transmission problem by means of periodic single layer potentials.

**Proposition 1.2.3.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $J := (J_1, J_2)$ be the operator from $(C^{0,\alpha}(\partial\Omega_Q)_0)^2$ to $C^{0,\alpha}(\partial\Omega_Q)_0 \times C^{1,\alpha}(\partial\Omega_Q)_0$ defined by*

$$\begin{aligned} J_1[\mu^i, \mu^o] &:= \lambda^- \left(\frac{1}{2}\mu^o + w_{q,*}[\partial\Omega_Q, \mu^o]\right) - \lambda^+ \left(-\frac{1}{2}\mu^i + w_{q,*}[\partial\Omega_Q, \mu^i]\right), \\ J_2[\mu^i, \mu^o] &:= v_q^+[\partial\Omega_Q, \mu^i]_{|\partial\Omega_Q} - \fint_{\partial\Omega_Q} v_q^+[\partial\Omega_Q, \mu^i]\, d\sigma - v_q^-[\partial\Omega_Q, \mu^o]_{|\partial\Omega_Q} \\ &\quad + \fint_{\partial\Omega_Q} v_q^-[\partial\Omega_Q, \mu^o]\, d\sigma\,, \end{aligned}$$

*for all $(\mu^i, \mu^o) \in (C^{0,\alpha}(\partial\Omega_Q)_0)^2$. Then $J$ is a linear homeomorphism.*

*Proof.* By continuity of $v_q^{\pm}[\partial\Omega_Q,\cdot]$ and of $w_{q,*}[\partial\Omega_Q,\cdot]$ (see Theorem B.0.2 in Appendix B), one verifies that $J$ is a continuous linear operator from $(C^{0,\alpha}(\partial\Omega_Q)_0)^2$ to $C^{0,\alpha}(\partial\Omega_Q)_0 \times C^{1,\alpha}(\partial\Omega_Q)_0$. As a consequence, by the Open Mapping Theorem, in order to show that $J$ is a homeomorphism, it suffices to prove that $J$ is bijection. Let $(\Psi,\Phi) \in C^{0,\alpha}(\partial\Omega_Q)_0 \times C^{1,\alpha}(\partial\Omega_Q)_0$. We first show that there exists at most one pair $(\mu^i,\mu^o)$ such that

$$J[\mu^i,\mu^o] = (\Psi,\Phi)\,. \tag{1.11}$$

So let us assume that there exists a pair $(\mu^i,\mu^o) \in (C^{0,\alpha}(\partial\Omega_Q)_0)^2$ such that (1.11) holds. We first note that by Lemma B.0.4 there exists a unique pair $(\mu^\sharp, c^\sharp) \in C^{0,\alpha}(\partial\Omega_Q)_0 \times \mathbb{R}$ such that

$$\Phi = v_q[\partial\Omega_Q,\mu^\sharp]_{|\partial\Omega_Q} + c^\sharp.$$

Accordingly, $J_2[\mu^i,\mu^o] = \Phi$ on $\partial\Omega_Q$ can be rewritten as

$$v_q[\partial\Omega_Q,\mu^i-\mu^o] - \fint_{\partial\Omega_Q} v_q[\partial\Omega_Q,\mu^i-\mu^o]\,d\sigma = \Phi \quad \text{on} \quad \partial\Omega_Q,$$

and we deduce that

$$\begin{aligned} \mu^\sharp &= \mu^i - \mu^o \quad \text{on} \quad \partial\Omega_Q, \\ c^\sharp &= -\fint_{\partial\Omega_Q} v_q[\partial\Omega_Q,\mu^i-\mu^o]\,d\sigma. \end{aligned} \tag{1.12}$$

By virtue of (1.12) we can rewrite equality $J_1[\mu^i,\mu^o] = \Psi$ as

$$\frac{1}{2}\mu^o + \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} w_{q,*}\left[\partial\Omega_Q,\mu^o\right] = \frac{1}{\lambda^- + \lambda^+}\Psi + \frac{\lambda^+}{\lambda^- + \lambda^+}\left(-\frac{1}{2}\mu^\sharp + w_{q,*}[\partial\Omega_Q,\mu^\sharp]\right). \tag{1.13}$$

Since the right hand side belongs to $C^{0,\alpha}(\partial\Omega_Q)_0$, Proposition 1.2.2 implies that there exists unique $\mu^o \in C^{0,\alpha}(\partial\Omega_Q)_0$ such that (1.13) holds and, accordingly, $\mu^o$ is uniquely determined. Then equation (1.12) uniquely determines $\mu^i$ and thus uniqueness follows. On the other hand, by reading backward the argument above, one deduces the existence of a pair $(\mu^i,\mu^o) \in (C^{0,\alpha}(\partial\Omega_Q)_0)^2$ such that (1.11) holds. □

By the jump formulae for the normal derivative of the periodic single layer potential, we can now deduce the validity of the following theorem.

**Theorem 1.2.4.** *Let $\lambda^+,\lambda^- \in ]0,+\infty[$. Let $(\Psi,\Phi,c) \in C^{0,\alpha}(\partial\Omega_Q)_0 \times C^{1,\alpha}(\partial\Omega_Q) \times \mathbb{R}$. Let $J$ be as in Proposition 1.2.3. Then the following statements hold.*

*(i) A pair $(\mu^i,\mu^o) \in (C^{0,\alpha}(\partial\Omega_Q)_0)^2$ satisfies the equality*

$$J[\mu^i,\mu^o] = \left(\Psi, \Phi - \fint_{\partial\Omega_Q} \Phi\,d\sigma\right) \tag{1.14}$$

*if and only if the pair $(v^+,v^-) \in C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]) \times C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ defined by*

$$v^+ \equiv v_q^+[\partial\Omega_Q,\mu^i] - \fint_{\partial\Omega_Q} v_q^+[\partial\Omega_Q,\mu^i]\,d\sigma + \frac{c}{|\partial\Omega_Q|_{n-1}}, \tag{1.15}$$

$$v^- \equiv v_q^-\left[\partial\Omega_Q,\mu^o\right] - \fint_{\partial\Omega_Q} v_q^-\left[\partial\Omega_Q,\mu^o\right]\,d\sigma - \fint_{\partial\Omega_Q} \Phi\,d\sigma + \frac{c}{|\partial\Omega_Q|_{n-1}}\,, \tag{1.16}$$

*solves problem*

$$\begin{cases} \Delta v^+ = 0 & \text{in } \mathbb{S}[\Omega_Q], \\ \Delta v^- = 0 & \text{in } \mathbb{S}[\Omega_Q]^-, \\ v^+(x+qe_h) = v^+(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_Q]\,,\ \forall h \in \{1,\dots,n\}, \\ v^-(x+qe_h) = v^-(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_Q]^-\,,\ \forall h \in \{1,\dots,n\}, \\ \lambda^- \frac{\partial v^-}{\partial \nu_{\Omega_Q}}(x) - \lambda^+ \frac{\partial v^+}{\partial \nu_{\Omega_Q}}(x) = \Psi(x) & \forall x \in \partial\Omega_Q, \\ v^+(x) - v^-(x) = \Phi(x) & \forall x \in \partial\Omega_Q, \\ \int_{\partial\Omega_Q} v^+ \, d\sigma = c\,. \end{cases} \tag{1.17}$$

(*ii*) *Problem* (1.17) *has a unique solution and it is delivered by the pair of functions defined in* (1.15), (1.16)*, where* $(\mu^i, \mu^o)$ *is the unique solution in* $(C^{0,\alpha}(\partial\Omega_Q)_0)^2$ *of equation* (1.14)*.*

*Proof.* Statement $(i)$ is a direct verification based on standard properties of periodic layer potentials (see Theorem B.0.2 in Appendix B). To prove statement $(ii)$ we first note that Proposition 1.2.1 implies that problem (1.17) has at most one solution. Then by Proposition 1.2.3, there exists a unique solution $(\mu^i, \mu^o) \in (C^{0,\alpha}(\partial\Omega_Q)_0)^2$ of (1.14), and, as a consequence, the validity of $(ii)$ follows. □

## 1.3 A non-periodic linear transmission problem

We now turn to non-periodic problems and we prove some results which we use in the sequel to analyze problem (1.3) around the degenerate case in which $\epsilon = 0$. We first have the following uniqueness result.

**Proposition 1.3.1.** *Let* $\lambda^+, \lambda^- \in ]0, +\infty[$*. Let* $(v^+, v^-) \in C^{1,\alpha}(\mathrm{cl}\Omega) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ *be such that*

$$\begin{cases} \Delta v^+ = 0 & \text{in } \Omega, \\ \Delta v^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega, \\ \lambda^- \frac{\partial v^-}{\partial \nu_\Omega}(t) - \lambda^+ \frac{\partial v^+}{\partial \nu_\Omega}(t) = 0 & \forall t \in \partial\Omega, \\ v^+(t) - v^-(t) = 0 & \forall t \in \partial\Omega, \\ \int_{\partial\Omega} v^+ \, d\sigma = 0\,, \\ \lim_{t\to\infty} v^-(t) \in \mathbb{R}. \end{cases} \tag{1.18}$$

*Then* $v^+ = 0$ *on* $\mathrm{cl}\Omega$ *and* $v^- = 0$ *on* $\mathbb{R}^n \setminus \Omega$*.*

*Proof.* By the Divergence Theorem and by the third and the fourth equalities of (1.18), we have

$$0 \le \int_\Omega |\nabla v^+(t)|^2 dt = \int_{\partial\Omega} v^+(t) \frac{\partial}{\partial \nu_\Omega} v^+(t) d\sigma_t = \frac{\lambda^-}{\lambda^+} \int_{\partial\Omega} v^-(t) \frac{\partial}{\partial \nu_\Omega} v^-(t) d\sigma_t.$$

By virtue of the sixth condition of (1.18), we set $\mathbb{R} \ni \xi \equiv \lim_{t\to\infty} v^-(t)$. Thus, $v^-(t) - \xi$ is harmonic in $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ and at infinity. Then by the Divergence Theorem and by the decay properties of $v^-(t) - \xi$ and of its radial derivative (see Folland [47, Prop. 2.75]), we deduce that

$$\begin{aligned} \frac{\lambda^-}{\lambda^+} \int_{\partial\Omega} v^-(t) \frac{\partial}{\partial \nu_\Omega} v^-(t) d\sigma_t =& \frac{\lambda^-}{\lambda^+} \int_{\partial\Omega} (v^-(t) - \xi) \frac{\partial}{\partial \nu_\Omega} (v^-(t) - \xi) d\sigma_t \\ &+ \frac{\xi\lambda^-}{\lambda^+} \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega} (v^-(t) - \xi) d\sigma_t = -\frac{\lambda^-}{\lambda^+} \int_{\mathbb{R}^n \setminus \mathrm{cl}\Omega} |\nabla v^-(t)|^2 dt \le 0. \end{aligned}$$

Thus

$$\int_{\Omega} |\nabla v^+(t)|^2 dt = \int_{\mathbb{R}^n \setminus \mathrm{cl}\Omega} |\nabla v^-(t)|^2 dt = 0,$$

and so

$$\begin{aligned} v^+(t) &= c^+ \quad \forall t \in \mathrm{cl}\Omega, \\ v^-(t) &= c^- \quad \forall t \in \mathbb{R}^n \setminus \mathrm{cl}\Omega, \end{aligned}$$

for some $c^+$, $c^- \in \mathbb{R}$. By virtue of the fifth and the fourth equalities of (1.18), we conclude that $c^+ = 0 = c^-$ and the proposition is proved. □

We can now study an integral operator which we need in order to solve (non-periodic) transmission problems in terms of classical single layer potentials.

**Proposition 1.3.2.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $K \equiv (K_1, K_2)$ be the operator from $(C^{0,\alpha}(\partial\Omega)_0)^2$ to $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$ defined by*

$$\begin{aligned} K_1[\theta^i, \theta^o] &:= \lambda^- \left( \frac{1}{2}\theta^o + w_*[\partial\Omega, \theta^o] \right) - \lambda^+ \left( -\frac{1}{2}\theta^i + w_*[\partial\Omega, \theta^i] \right) , \\ K_2[\theta^i, \theta^o] &:= v^+[\partial\Omega, \theta^i]_{|\partial\Omega} - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i]\, d\sigma - v^-[\partial\Omega, \theta^o]_{|\partial\Omega} + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o]\, d\sigma \end{aligned}$$

*for all $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$. Then $K$ is a linear homeomorphism.*

*Proof.* We first note that $K$ is linear and continuous. As a consequence, by the Open Mapping Theorem, in order to show that $K$ is a linear homeomorphism, it suffices to prove that $K$ is bijection. In other words, we need to prove that for each pair $(\Psi, \Phi) \in C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$ there exists a unique pair $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ such that

$$\begin{cases} \lambda^- \left( \frac{1}{2}\theta^o(t) + w_*[\partial\Omega, \theta^o](t) \right) \\ \qquad -\lambda^+ \left( -\frac{1}{2}\theta^i(t) + w_*[\partial\Omega, \theta^i](t) \right) = \Psi(t) \quad \forall t \in \partial\Omega\,, \\ v^+[\partial\Omega, \theta^i](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i]\, d\sigma \\ \qquad -v^-[\partial\Omega, \theta^o](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o]\, d\sigma = \Phi(t) \quad \forall t \in \partial\Omega. \end{cases} \tag{1.19}$$

We note that by Proposition A.2(*ii*) of Lanza de Cristoforis and Musolino [84], there exists and is unique a triple $(\theta^i, \theta^o, \xi) \in (C^{0,\alpha}(\partial\Omega)_0)^2 \times \mathbb{R}$ such that

$$\begin{cases} \frac{1}{2}\theta^o(t) + w_*[\partial\Omega, \theta^o](t) + \frac{1}{2}\frac{\lambda^+}{\lambda^-}\theta^i(t) - \frac{\lambda^+}{\lambda^-} w_*[\partial\Omega, \theta^i](t) = \frac{1}{\lambda^-}\Psi(t) \quad \forall t \in \partial\Omega\,, \\ v^+[\partial\Omega, \theta^i](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i]\, d\sigma - v^-[\partial\Omega, \theta^o](t) - \xi = \Phi(t) \quad \forall t \in \partial\Omega. \end{cases} \tag{1.20}$$

Since $\Phi \in C^{1,\alpha}(\partial\Omega)_0$, by integrating the second equality of (1.20) on $\partial\Omega$, we immediately have

$$\xi = -\fint_{\partial\Omega} v^-[\partial\Omega, \theta^o]\, d\sigma.$$

As a consequence, the existence and uniqueness of a pair $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ satisfying (1.19) follows and thus the proof is complete. □

By Propositions 1.3.1, 1.3.2, and by the jump formulae for the normal derivative of the classical single layer potential, we immediately deduce the validity of the following result concerning the solvability of a non-periodic transmission problem.

**Theorem 1.3.3.** *Let* $\lambda^+, \lambda^- \in ]0, +\infty[$. *Let* $K$ *be as in Proposition 1.3.2. Let* $(\Psi, \Phi) \in C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$. *Let* $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ *be such that*

$$K[\theta^i, \theta^o] = (\Psi, \Phi) \,. \tag{1.21}$$

*Let* $(v^+, v^-) \in C^{1,\alpha}(\mathrm{cl}\Omega) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ *be defined by*

$$v^+ \equiv v^+[\partial\Omega, \theta^i] - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i]\, d\sigma, \tag{1.22}$$

$$v^- \equiv v^-[\partial\Omega, \theta^o] - \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o]\, d\sigma \,. \tag{1.23}$$

*Then* $(v^+, v^-)$ *is the unique solution in* $C^{1,\alpha}(\mathrm{cl}\Omega) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ *of*

$$\begin{cases} \Delta v^+ = 0 & \text{in } \Omega, \\ \Delta v^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega, \\ \lambda^- \frac{\partial v^-}{\partial \nu_\Omega}(t) - \lambda^+ \frac{\partial v^+}{\partial \nu_\Omega}(t) = \Psi(t) & \forall t \in \partial\Omega, \\ v^+(t) - v^-(t) = \Phi(t) & \forall t \in \partial\Omega, \\ \int_{\partial\Omega} v^+ \, d\sigma = 0 \,, \\ \lim_{t\to\infty} v^-(t) \in \mathbb{R}. \end{cases} \tag{1.24}$$

*Proof.* We first note that Proposition 1.3.1 implies that problem (1.24) has at most one solution. By Proposition 1.3.2, there exists a unique solution $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ of equation (1.21). Then, by classical potential theory, we verify that the pair of functions defined by (1.22) and (1.23) solves problem (1.24). □

## 1.4 Formulation of the linear transmission problem in terms of integral equations

In Proposition 1.4.1 below, we formulate problem (1.3) in terms of integral equations on $\partial\Omega$. To do so, we exploit Theorem 1.2.4 and the rule of change of variables in integrals. By Theorem 1.2.4, one can reformulate such a problem in terms of a system of integral equations defined on the $\epsilon$-dependent domain $\partial\Omega_{p,\epsilon}$. Finally, by exploiting an appropriate change of variable, one can get rid of such a dependence and obtain an equivalent system of integral equations defined on the fixed domain $\partial\Omega$, as Proposition 1.4.1 below shows.

We now find convenient to introduce the following notation. We first introduce the maps $\Lambda$ and $\Lambda_\nu$ from $]-\epsilon_0, \epsilon_0[ \times C^{0,\alpha}(\partial\Omega)_0$ to $C^{1,\alpha}(\partial\Omega)$ and to $C^{0,\alpha}(\partial\Omega)$, respectively, by setting

$$\Lambda[\epsilon, \theta](t) := \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s) d\sigma_s \qquad \forall t \in \partial\Omega \,,$$

and

$$\Lambda_\nu[\epsilon, \theta](t) := \int_{\partial\Omega} DR_{q,n}(\epsilon(t-s))\nu_\Omega(t)\theta(s) d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \theta) \in ]-\epsilon_0, \epsilon_0[ \times C^{0,\alpha}(\partial\Omega)_0$.

Let $\lambda^+, \lambda^- \in ]0, +\infty[$, then we denote by $M := (M_1, M_2)$ the map from $]-\epsilon_0, \epsilon_0[ \times \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega) \times \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$ defined for all $t \in \partial\Omega$ by

$$\begin{aligned} M_1[\epsilon, \epsilon', f, g, \theta^i, \theta^o](t) := {} & \lambda^- \left(\frac{1}{2}\theta^o(t) + w_*[\partial\Omega, \theta^o](t) + \epsilon^{n-1}\Lambda_\nu[\epsilon, \theta^o](t)\right) \\ & - \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_*[\partial\Omega, \theta^i](t) + \epsilon^{n-1}\Lambda_\nu[\epsilon, \theta^i](t)\right) - f(t), \end{aligned} \tag{1.25}$$

$$
\begin{aligned}
M_2[\epsilon,\epsilon',f,g,\theta^i,\theta^o](t) := & \, v^+[\partial\Omega,\theta^i](t)+\epsilon^{n-2}\Lambda[\epsilon,\theta^i](t) \\
& - \fint_{\partial\Omega}\Big(v^+[\partial\Omega,\theta^i](s')+\epsilon^{n-2}\Lambda[\epsilon,\theta^i](s')\Big)\,d\sigma_{s'} \\
& - v^-[\partial\Omega,\theta^o](t)-\epsilon^{n-2}\Lambda[\epsilon,\theta^o](t) \\
& + \fint_{\partial\Omega}\Big(v^-[\partial\Omega,\theta^o](s')+\epsilon^{n-2}\Lambda[\epsilon,\theta^o](s')\Big)\,d\sigma_{s'} - \epsilon' g(t)+\epsilon'\fint_{\partial\Omega} g d\sigma
\end{aligned}
\tag{1.26}
$$

for all $(\epsilon,\epsilon',f,g,\theta^i,\theta^o)\in ]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0\times C^{1,\alpha}(\partial\Omega)\times\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$.

Then we have the following proposition.

**Proposition 1.4.1.** *Let $\lambda^+,\lambda^-\in]0,+\infty[$. Let $c\in\mathbb{R}$. Let $\rho$ be a function from $]0,\epsilon_0[$ to $\mathbb{R}\setminus\{0\}$. Let $\epsilon\in]0,\epsilon_0[$. Let $f\in C^{0,\alpha}(\partial\Omega)_0$. Let $g\in C^{1,\alpha}(\partial\Omega)$. Then the unique solution $(u^+[\epsilon,f,g,c],u^-[\epsilon,f,g,c])$ in $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])\times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of problem (1.3) is delivered by*

$$
\begin{aligned}
u^+[\epsilon,f,g,c](x) \equiv & \, v_q^+[\partial\Omega_{p,\epsilon},\hat\theta^i[\epsilon,f,g]((\cdot-p)/\epsilon)](x) \\
& - \fint_{\partial\Omega_{p,\epsilon}} v_q^+[\partial\Omega_{p,\epsilon},\hat\theta^i[\epsilon,f,g]((\cdot-p)/\epsilon)]\,d\sigma+\frac{c}{|\partial\Omega_{p,\epsilon}|_{n-1}} \qquad \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \\
u^-[\epsilon,f,g,c](x) \equiv & \, v_q^-[\partial\Omega_{p,\epsilon},\hat\theta^o[\epsilon,f,g]((\cdot-p)/\epsilon)](x) \\
& - \fint_{\partial\Omega_{p,\epsilon}} v_q^-[\partial\Omega_{p,\epsilon},\hat\theta^o[\epsilon,f,g]((\cdot-p)/\epsilon)]\,d\sigma \\
& - \rho(\epsilon)\fint_{\partial\Omega_{p,\epsilon}} g((y-p)/\epsilon)\,d\sigma_y+\frac{c}{|\partial\Omega_{p,\epsilon}|_{n-1}} \qquad \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,
\end{aligned}
$$

*where $(\hat\theta^i[\epsilon,f,g],\hat\theta^o[\epsilon,f,g])$ denotes the unique solution $(\theta^i,\theta^o)$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of*

$$
M\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g,\theta^i,\theta^o\right]=0. \tag{1.27}
$$

*Proof.* By a simple computation based on the rule of change of variables in integrals and on equality (1.7), we note that equation (1.27) can be written as

$$
\begin{aligned}
J[\theta^i((\cdot-p)/\epsilon), & \theta^o((\cdot-p)/\epsilon)] \\
& = \left(f((\cdot-p)/\epsilon),\rho(\epsilon)g((\cdot-p)/\epsilon)-\rho(\epsilon)\fint_{\partial\Omega_{p,\epsilon}} g((y-p)/\epsilon)\,d\sigma_y\right)
\end{aligned}
$$

on $\partial\Omega_{p,\epsilon}$, where the operator $J$ is defined as in Proposition 1.2.3 with $\Omega_Q\equiv\Omega_{p,\epsilon}$. Thus, there exists a unique pair $(\theta^i,\theta^o)$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that (1.27) holds. Finally, by Theorem 1.2.4, we deduce the validity of the proposition. □

Now, we are reduced to analyze system (1.27). We note that it does not make sense for $\epsilon=0$, but it makes perfectly sense if $\epsilon\to0$, which implies that $\rho(\epsilon)/\epsilon\to r_*$. So, we will analyze system (1.27) by replacing $(\epsilon,\rho(\epsilon)/\epsilon,f,g)$ by $(0,r^*,f_0,g_0)$ for some $f_0\in C^{0,\alpha}(\partial\Omega)_0$ and $g_0\in C^{1,\alpha}(\partial\Omega)$.

As a first step, we note that if $(\theta^i,\theta^o)\in\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ and if we let $\epsilon$ tend to 0, we obtain a system which we will call the "limiting system", and which has the following form

$$
\left\{
\begin{aligned}
& \lambda^-\left(\tfrac12\theta^o(t)+w_*[\partial\Omega,\theta^o](t)\right)-\lambda^+\left(-\tfrac12\theta^i(t)+w_*[\partial\Omega,\theta^i](t)\right)=f_0(t), \\
& v^+[\partial\Omega,\theta^i](t)-\fint_{\partial\Omega} v^+[\partial\Omega,\theta^i]d\sigma-v^-[\partial\Omega,\theta^o](t) \\
& \qquad\qquad +\fint_{\partial\Omega} v^-[\partial\Omega,\theta^o]d\sigma=r_*g_0(t)-r_*\fint_{\partial\Omega} g_0 d\sigma
\end{aligned}
\right.
\tag{1.28}
$$

for all $t\in\partial\Omega$.

Then we have the following theorem, which shows the unique solvability of system (1.28), and its link with a boundary value problem which we will call the "limiting boundary" value problem.

**Theorem 1.4.2.** *Let* $\lambda^+, \lambda^- \in ]0,+\infty[$. *Let* $\rho$ *be a function from* $]0,\epsilon_0[$ *to* $\mathbb{R} \setminus \{0\}$. *Let assumption* (1.5) *hold. Let* $r_*$ *be as in* (1.6). *Let* $f_0 \in C^{0,\alpha}(\partial\Omega)_0$. *Let* $g_0 \in C^{1,\alpha}(\partial\Omega)$. *Then the following statements hold.*

*(i) The "limiting system"* (1.28) *has one and only one solution* $(\tilde{\theta}^i, \tilde{\theta}^o) \in \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$.

*(ii) The "limiting boundary value problem"*

$$\begin{cases} \Delta u^+ = 0 & \text{in } \Omega, \\ \Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega, \\ \lambda^- \frac{\partial u^-}{\partial \nu_\Omega}(t) - \lambda^+ \frac{\partial u^+}{\partial \nu_\Omega}(t) = f_0(t) & \forall t \in \partial\Omega, \\ u^+(t) - u^-(t) = r_* g_0(t) - r_* \fint_{\partial\Omega} g_0 \, d\sigma & \forall t \in \partial\Omega, \\ \int_{\partial\Omega} u^+ \, d\sigma = 0\,, & \\ \lim_{t\to\infty} u^-(t) \in \mathbb{R} & \end{cases} \tag{1.29}$$

*has one and only one solution* $(\tilde{u}^+, \tilde{u}^-) \in C^{1,\alpha}(\mathrm{cl}\Omega) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$, *delivered by*

$$\tilde{u}^+ \equiv v^+[\partial\Omega, \tilde{\theta}^i] - \fint_{\partial\Omega} v^+[\partial\Omega, \tilde{\theta}^i] \, d\sigma, \tag{1.30}$$

$$\tilde{u}^- \equiv v^-[\partial\Omega, \tilde{\theta}^o] - \fint_{\partial\Omega} v^-[\partial\Omega, \tilde{\theta}^o] \, d\sigma\,. \tag{1.31}$$

*Proof.* We first note that the system of integral equations (1.28) can be rewritten as

$$K[\theta^i, \theta^o] = \left( f_0, \; r_* g_0 - r_* \fint_{\partial\Omega} g_0 \, d\sigma \right) \qquad \text{on } \partial\Omega,$$

where the operator $K$ is defined in Proposition 1.3.2. Then, by Proposition 1.3.2, there exists a unique pair $(\theta^i, \theta^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ such that (1.28) holds. Accordingly, the validity of statement $(i)$ follows. To prove $(ii)$, we observe that Theorem 1.3.3 and classical potential theory imply that the pair of functions delivered by (1.30)-(1.31) is the unique solution of problem (1.29). □

We are now ready to analyze system (1.27) for $(\epsilon, \epsilon', f, g)$ in a neighborhood of $(0, r_*, f_0, g_0)$ by means of the following.

**Theorem 1.4.3.** *Let* $\lambda^+, \lambda^- \in ]0,+\infty[$. *Let* $f_0 \in C^{0,\alpha}(\partial\Omega)_0$. *Let* $g_0 \in C^{1,\alpha}(\partial\Omega)$. *Let* $\rho$ *be a function from* $]0,\epsilon_0[$ *to* $\mathbb{R} \setminus \{0\}$. *Let assumption* (1.5) *hold. Let* $r_*$ *be as in* (1.6). *Let* $M$ *be as in* (1.25)-(1.26). *Let* $(\hat{\theta}^i[\cdot,\cdot,\cdot], \hat{\theta}^o[\cdot,\cdot,\cdot])$ *be as in Proposition 1.4.1. Let* $(\tilde{\theta}^i, \tilde{\theta}^o)$ *be as in Theorem 1.4.2. Then there exist* $\epsilon_1 \in ]0,\epsilon_0]$, *an open neighborhood* $\mathcal{U}_{r_*}$ *of* $r_*$ *in* $\mathbb{R}$, *an open neighborhood* $\mathcal{U}$ *of* $(f_0, g_0)$ *in* $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)$, *an open neighborhood* $\mathcal{V}$ *of* $(\tilde{\theta}^i, \tilde{\theta}^o)$ *in* $(C^{0,\alpha}(\partial\Omega)_0)^2$, *and a real analytic map* $(\Theta^i, \Theta^o)$ *from* $]-\epsilon_1, \epsilon_1[\times \mathcal{U}_{r_*} \times \mathcal{U}$ *to* $\mathcal{V}$ *such that*

$$\frac{\rho(\epsilon)}{\epsilon} \in \mathcal{U}_{r_*} \qquad \forall \epsilon \in ]0,\epsilon_1[,$$

*and such that the set of zeros of* $M$ *in* $]-\epsilon_1, \epsilon_1[\times \mathcal{U}_{r_*} \times \mathcal{U} \times \mathcal{V}$ *coincides with the graph of* $(\Theta^i, \Theta^o)$. *In particular,*

$$\left( \Theta^i \left[ \epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g \right], \Theta^o \left[ \epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g \right] \right) = \left( \hat{\theta}^i \left[ \epsilon, f, g \right], \hat{\theta}^o \left[ \epsilon, f, g \right] \right)$$

*for all* $(\epsilon, f, g) \in ]0,\epsilon_1[\times \mathcal{U}$, *and*

$$\left( \Theta^i[0, r_*, f_0, g_0], \Theta^o[0, r_*, f_0, g_0] \right) = (\tilde{\theta}^i, \tilde{\theta}^o)\,.$$

*Proof.* We plan to apply the Implicit Function Theorem to equation

$$M\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g, \theta^i, \theta^o\right] = 0$$

around the point $\left(0, r_*, f_0, g_0, \tilde{\theta}^i, \tilde{\theta}^o\right)$. By standard properties of integral operators with real analytic kernels and with no singularity, and by classical mapping properties of layer potentials (see Lanza de Cristoforis and Musolino [82, Sec. 4], Miranda [106], Lanza de Cristoforis and Rossi [87, Thm. 3.1]), we conclude that $M$ is a real analytic map from $]-\epsilon_0, \epsilon_0[\times \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega) \times \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$. By definition of $(\tilde{\theta}^i, \tilde{\theta}^o)$, we have $M[0, r_*, f_0, g_0, \tilde{\theta}^i, \tilde{\theta}^o] = 0$. By standard calculus in Banach spaces, the differential of $M$ at the point $(0, r_*, f_0, g_0, \tilde{\theta}^i, \tilde{\theta}^o)$ with respect to the variables $(\theta^i, \theta^o)$ is delivered by the formula

$$\left(\partial_{(\theta^i,\theta^o)} M[0, r_*, f_0, g_0, \tilde{\theta}^i, \tilde{\theta}^o]\right)(\bar{\theta}^i, \bar{\theta}^o) = K[\bar{\theta}^i, \bar{\theta}^o]$$

for all $(\bar{\theta}^i, \bar{\theta}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$, where the operator $K$ is defined in Proposition 1.3.2. Then, by Proposition 1.3.2, $\partial_{(\theta^i,\theta^o)} M[0, r_*, f_0, g_0, \tilde{\theta}^i, \tilde{\theta}^o]$ is a linear homeomorphism from $(C^{0,\alpha}(\partial\Omega)_0)^2$ onto $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$. Hence, the existence of $\epsilon_1$, $\mathcal{U}_{r_*}$, $\mathcal{U}$, $\mathcal{V}$, $\Theta^i$, $\Theta^o$ as in the statement follows by the Implicit Function Theorem for real analytic maps in Banach spaces (see, e.g., Deimling [41, Thm. 15.3]). □

## 1.5 A functional analytic representation theorem for the solutions of the linear transmission problem

In the following theorem we investigate the behavior of $u^+[\epsilon, f, g, c]$ when $\epsilon$ is small and positive and the triple $(f, g, c)$ belongs to $\mathcal{U} \times \mathbb{R}$.

**Theorem 1.5.1.** *Let the assumptions of Theorem 1.4.3 hold. Then there exists a real analytic map $U^+$ from $]-\epsilon_1, \epsilon_1[\times \mathcal{U}_{r_*} \times \mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$ such that*

$$u^+[\epsilon, f, g, c](p + \epsilon t) = \epsilon U^+\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](t) + \frac{\epsilon^{1-n} c}{|\partial\Omega|_{n-1}} \qquad \forall t \in \mathrm{cl}\Omega,$$

*for all $(\epsilon, f, g, c) \in ]0, \epsilon_1[\times \mathcal{U} \times \mathbb{R}$, where $u^+[\epsilon, f, g, c]$ is as in Proposition 1.4.1. Moreover,*

$$U^+[0, r_*, f_0, g_0](t) = \tilde{u}^+(t) \qquad \forall t \in \mathrm{cl}\Omega, \tag{1.32}$$

*where $\tilde{u}^+$ is defined as in Theorem 1.4.2.*

*Proof.* If $\epsilon \in ]0, \epsilon_1[$, then a simple computation based on the rule of change of variables in integrals shows that

$$\begin{aligned} u^+[\epsilon, f, g, c](p + \epsilon t) &= \epsilon v^+\left[\partial\Omega, \Theta^i\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right]\right](t) + \epsilon^{n-1}\Lambda\left[\epsilon, \Theta^i\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right]\right](t) \\ &- \epsilon \fint_{\partial\Omega}\left(v^+\left[\partial\Omega, \Theta^i\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right]\right](s') + \epsilon^{n-2}\Lambda\left[\epsilon, \Theta^i\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right]\right](s')\right) d\sigma_{s'} \\ &+ \frac{\epsilon^{1-n} c}{|\partial\Omega|_{n-1}} \end{aligned}$$

for all $t \in \mathrm{cl}\Omega$. (see also Proposition 1.4.1 and Theorem 1.4.3). Therefore, it is natural to set

$$\begin{aligned} U^+[\epsilon, \epsilon', f, g](t) := {} & v^+\left[\partial\Omega, \Theta^i\left[\epsilon, \epsilon', f, g\right]\right](t) + \epsilon^{n-2}\Lambda\left[\epsilon, \Theta^i\left[\epsilon, \epsilon', f, g\right]\right](t) \\ & - \fint_{\partial\Omega}\left(v^+\left[\partial\Omega, \Theta^i\left[\epsilon, \epsilon', f, g\right]\right](s') + \epsilon^{n-2}\Lambda\left[\epsilon, \Theta^i\left[\epsilon, \epsilon', f, g\right]\right](s')\right) d\sigma_{s'} \end{aligned}$$

for all $t \in \mathrm{cl}\Omega$ and for all $(\epsilon, \epsilon', f, g) \in ]-\epsilon_1, \epsilon_1[ \times \mathcal{U}_{r_*} \times \mathcal{U}$. By standard properties of integral operators with real analytic kernels and with no singularity, by classical mapping properties of layer potentials (see Lanza de Cristoforis and Musolino [82, Sec. 4], Miranda [106], Lanza de Cristoforis and Rossi [87, Thm. 3.1]) and by Theorem 1.4.3, we conclude that $U^+$ is real analytic. Moreover, Theorem 1.4.3 implies that $\Theta^i[0, r_*, f_0, g_0] = \tilde{\theta}^i$ and thus the validity of equality (1.32) follows (see also Theorem 1.4.2). □

Let $\tilde{\theta}^o$, $\tilde{u}^-$ be as in Theorem 1.4.2. Then by classical potential theory and by equality $\int_{\partial\Omega} \tilde{\theta}^o \, d\sigma = 0$, we deduce that

$$\tilde{l}^- := \lim_{t \to \infty} \tilde{u}^-(t) = -\fint_{\partial\Omega} v^-[\partial\Omega, \tilde{\theta}^o] \, d\sigma \,. \tag{1.33}$$

In the following theorem we investigate the behavior of $u^-[\epsilon, f, g, c]$ for $\epsilon$ small and positive.

**Theorem 1.5.2.** *Let the assumptions of Theorem 1.4.3 hold. Then there exists a real analytic map $C^-$ from $]-\epsilon_1, \epsilon_1[ \times \mathcal{U}_{r_*} \times \mathcal{U}$ to $\mathbb{R}$ such that*

$$C^-[0, r_*, f_0, g_0] = \tilde{l}^- - r_* \fint_{\partial\Omega} g_0 \, d\sigma, \tag{1.34}$$

*and such that the following statements hold.*

*(i) Let $\tilde{\Omega}$ be an open bounded subset of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\Omega} \cap (p + q\mathbb{Z}^n) = \emptyset$. Let $k \in \mathbb{N}$. Then there exist $\epsilon_{\tilde{\Omega}} \in ]0, \epsilon_1[$ and a real analytic map $U^-_{\tilde{\Omega}}$ from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{r_*} \times \mathcal{U}$ to $C^k(\mathrm{cl}\tilde{\Omega})$ such that $\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in ]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$, and such that*

$$u^-[\epsilon, f, g, c](x) = \epsilon C^- \left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon^n U^-_{\tilde{\Omega}} \left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](x) + \frac{\epsilon^{1-n} c}{|\partial\Omega|_{n-1}} \tag{1.35}$$

*for all $x \in \mathrm{cl}\tilde{\Omega}$ and for all $(\epsilon, f, g, c) \in ]0, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U} \times \mathbb{R}$, where $u^-[\epsilon, f, g, c]$ is as in Proposition 1.4.1. Moreover,*

$$\begin{aligned} U^-_{\tilde{\Omega}}&[0, r_*, f_0, g_0](x) \\ &= DS_{q,n}(x-p) \left( \int_{\partial\Omega} \nu_\Omega(s) \tilde{u}^-(s) \, d\sigma_s - \int_{\partial\Omega} s \frac{\partial}{\partial \nu_\Omega} \tilde{u}^-(s) \, d\sigma_s \right) \end{aligned} \tag{1.36}$$

*for all $x \in \mathrm{cl}\tilde{\Omega}$, where $\tilde{u}^-$ is defined as in Theorem 1.4.2.*

*(ii) Let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then there exist $\epsilon^\#_{\tilde{\Omega}} \in ]0, \epsilon_1[$ and a real analytic map $V^-_{\tilde{\Omega}}$ from $]-\epsilon^\#_{\tilde{\Omega}}, \epsilon^\#_{\tilde{\Omega}}[ \times \mathcal{U}_{r_*} \times \mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\tilde{\Omega})$ such that $p + \epsilon \mathrm{cl}\tilde{\Omega} \subseteq \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in ]-\epsilon^\#_{\tilde{\Omega}}, \epsilon^\#_{\tilde{\Omega}}[$, and*

$$u^-[\epsilon, f, g, c](p + \epsilon t) = \epsilon C^- \left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon V^-_{\tilde{\Omega}} \left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](t) + \frac{\epsilon^{1-n} c}{|\partial\Omega|_{n-1}}$$

*for all $t \in \mathrm{cl}\tilde{\Omega}$ and for all $(\epsilon, f, g, c) \in ]0, \epsilon^\#_{\tilde{\Omega}}[ \times \mathcal{U} \times \mathbb{R}$. Moreover,*

$$V^-_{\tilde{\Omega}}[0, r_*, f_0, g_0](t) = \tilde{u}^-(t) - \tilde{l}^- \qquad \forall t \in \mathrm{cl}\tilde{\Omega}, \tag{1.37}$$

*where $\tilde{u}^-$ is defined as in Theorem 1.4.2.*

*Proof.* We set

$$
\begin{aligned}
C^{-}[\epsilon,\epsilon',f,g] := -\fint_{\partial\Omega}\Big(& v^{-}[\partial\Omega,\Theta^{o}[\epsilon,\epsilon',f,g]](t) \\
&+\epsilon^{n-2}\int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\Theta^{o}[\epsilon,\epsilon',f,g](s)d\sigma_s\Big)d\sigma_t-\epsilon'\fint_{\partial\Omega} g\,d\sigma
\end{aligned}
\tag{1.38}
$$

for all $(\epsilon,\epsilon',f,g)\in]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$. By standard properties of integral operators with real analytic kernels and with no singularity, by classical mapping properties of layer potentials (see Lanza de Cristoforis and Musolino [82, Sec. 4], Miranda [106], Lanza de Cristoforis and Rossi [87, Thm. 3.1]) and by Theorem 1.4.3, we deduce that $C^{-}$ is real analytic. Then, by (1.33) and by equality $\Theta^{o}[0,r_*,f_0,g_0]=\tilde{\theta}^{o}$ one verifies the validity of (1.34).

We now consider the proof of statement $(i)$. By taking $\epsilon_{\tilde{\Omega}}$ small enough, we can assume that $\mathrm{cl}\tilde{\Omega}\subseteq\mathbb{S}[\Omega_{p,\epsilon}]^{-}$ for all $\epsilon\in[-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}]$. By Proposition 1.4.1 and Theorem 1.4.3, we have

$$
\begin{aligned}
u^{-}[\epsilon,f,g,c](x)=\ &\epsilon^{n-1}\int_{\partial\Omega}S_{q,n}(x-p-\epsilon s)\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right](s)\,d\sigma_s\\
&-\epsilon\fint_{\partial\Omega}\left(v^{-}\left[\partial\Omega,\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right]\right](t)+\epsilon^{n-2}\Lambda\left[\epsilon,\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right]\right](t)\right)d\sigma_t\\
&-\rho(\epsilon)\fint_{\partial\Omega}g\,d\sigma+\frac{\epsilon^{1-n}c}{|\partial\Omega|_{n-1}}\qquad\forall x\in\mathrm{cl}\tilde{\Omega},
\end{aligned}
$$

for all $\epsilon\in]0,\epsilon_{\tilde{\Omega}}[$. We note that if $\epsilon\in[-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}]$, then $\partial\mathbb{S}[\Omega_{p,\epsilon}]\cap\mathrm{cl}\tilde{\Omega}=\emptyset$. Also, we observe that if $x\in\mathrm{cl}\tilde{\Omega}$, then $x-p-\epsilon\beta s$ does not belong to $q\mathbb{Z}^n$ for any $s\in\partial\Omega$, $\epsilon\in]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[$, and $\beta\in[0,1]$. Accordingly, we can invoke the Taylor formula with integral remainder and write

$$
S_{q,n}(x-p-\epsilon s)-S_{q,n}(x-p)=-\epsilon\int_0^1 DS_{q,n}(x-p-\beta\epsilon s)s\,d\beta
$$

for all $(x,s)\in\mathrm{cl}\tilde{\Omega}\times\partial\Omega$ and $\epsilon\in]0,\epsilon_{\tilde{\Omega}}[$. Since $\int_{\partial\Omega}\Theta^{o}\left[\epsilon,\rho(\epsilon)/\epsilon,f,g\right]\,d\sigma=0$ for all $(\epsilon,f,g)\in]0,\epsilon_{\tilde{\Omega}}[\times\mathcal{U}$, we conclude that

$$
\begin{aligned}
u^{-}[\epsilon,f,g,c](x)=&-\epsilon^{n}\int_{\partial\Omega}\left(\int_0^1 DS_{q,n}(x-p-\beta\epsilon s)s\,d\beta\right)\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right](s)\,d\sigma_s\\
&-\epsilon\fint_{\partial\Omega}\left(v^{-}\left[\partial\Omega,\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right]\right](t)+\epsilon^{n-2}\Lambda\left[\epsilon,\Theta^{o}\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right]\right](t)\right)d\sigma_t\\
&-\rho(\epsilon)\fint_{\partial\Omega}g\,d\sigma+\frac{\epsilon^{1-n}c}{|\partial\Omega|_{n-1}}
\end{aligned}
\tag{1.39}
$$

for all $x\in\mathrm{cl}\tilde{\Omega}$ and for all $(\epsilon,f,g,c)\in]0,\epsilon_{\tilde{\Omega}}[\times\mathcal{U}\times\mathbb{R}$. Thus, it is natural to set

$$
U^{-}_{\tilde{\Omega}}[\epsilon,\epsilon',f,g](x):=-\int_{\partial\Omega}\left(\int_0^1 DS_{q,n}(x-p-\beta\epsilon s)s\,d\beta\right)\Theta^{o}[\epsilon,\epsilon',f,g](s)\,d\sigma_s
\tag{1.40}
$$

for all $x\in\mathrm{cl}\tilde{\Omega}$, and for all $(\epsilon,\epsilon',f,g)\in]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[\times\mathcal{U}_{r_*}\times\mathcal{U}$. Then the validity of (1.35) follows by definitions (1.38), (1.40) and equality (1.39). By standard properties of integral operators with real analytic kernels and with no singularity (see, e.g., Lanza de Cristoforis and Musolino [82, Sec. 3]) and by arguing exactly as in the proof of Theorem 5.1$(i)$ of Lanza de Cristoforis and Musolino [83], one verifies that $U^{-}_{\tilde{\Omega}}$ defines a real analytic map from $]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[\times\mathcal{U}_{r_*}\times\mathcal{U}$ to $C^{k}(\mathrm{cl}\tilde{\Omega})$. It remains to prove formula (1.36). Theorem 1.4.3 implies that $\Theta^{o}[0,r_*,f_0,g_0]=\tilde{\theta}^{o}$. Then we fix $k\in\{1,\dots,n\}$. By well known jump formulae for the normal derivative of the classical single layer potential, we have

$$
\int_{\partial\Omega}s_k\tilde{\theta}^{o}(s)\,d\sigma_s=\int_{\partial\Omega}s_k\frac{\partial}{\partial\nu_\Omega}v^{-}[\partial\Omega,\tilde{\theta}^{o}](s)\,d\sigma_s-\int_{\partial\Omega}s_k\frac{\partial}{\partial\nu_\Omega}v^{+}[\partial\Omega,\tilde{\theta}^{o}](s)\,d\sigma_s\,.
$$

Then by the Green Identity, we have

$$\int_{\partial\Omega} s_k \frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega,\tilde{\theta}^o](s)\,d\sigma_s = \int_{\partial\Omega} (\nu_\Omega(s))_k v^+[\partial\Omega,\tilde{\theta}^o](s)\,d\sigma_s\,.$$

Moreover, $\frac{\partial}{\partial\nu_\Omega}\tilde{u}^- = \frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\theta}^o]$ on $\partial\Omega$ and $\int_{\partial\Omega}(\nu_\Omega(s))_k\,d\sigma_s = 0$. Thus,

$$\int_{\partial\Omega}(\nu_\Omega(s))_k v^+[\partial\Omega,\tilde{\theta}^o](s)\,d\sigma_s = \int_{\partial\Omega}(\nu_\Omega(s))_k v^-[\partial\Omega,\tilde{\theta}^o](s)\,d\sigma_s = \int_{\partial\Omega}(\nu_\Omega(s))_k \tilde{u}^-(s)\,d\sigma_s\,.$$

As a consequence,

$$\int_{\partial\Omega} s_k\tilde{\theta}^o(s)\,d\sigma_s = \int_{\partial\Omega} s_k\frac{\partial}{\partial\nu_\Omega}\tilde{u}^-(s)\,d\sigma_s - \int_{\partial\Omega}(\nu_\Omega(s))_k\tilde{u}^-(s)\,d\sigma_s,$$

and accordingly (1.36) holds.

To prove statement $(ii)$ we use the same approach as in the proof of Theorem 5.1$(ii)$ of Lanza de Cristoforis and Musolino [83] (see, also, Dalla Riva and Musolino [35, Thm. 7.2]). Since there exists $R > 0$ such that $(\mathrm{cl}\tilde{\Omega}\cup\mathrm{cl}\Omega)\subseteq\mathbb{B}_n(0,R)$, it is convenient to consider a set $\Omega^*\equiv\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega$ instead of $\tilde{\Omega}$. The advantage of $\Omega^*$ with respect to $\tilde{\Omega}$ is that $\Omega^*$ is of class $C^1$ and that accordingly $C^2(\mathrm{cl}\Omega^*)$ is continuously embedded into $C^{1,\alpha}(\mathrm{cl}\Omega^*)$, a fact which we exploit below (see, e.g., Lanza de Cristoforis [75, Lem. 2.4$(ii)$]).

According to Lanza de Cristoforis and Musolino [83, Lem. A.5$(ii)$], there exists $\epsilon^{\#}_{\Omega^*}\in]0,\epsilon_1[$ such that $p+\epsilon\mathrm{cl}\Omega^*\subseteq Q$, and $p+\epsilon\Omega^*\subseteq\mathbb{S}[\Omega_{p,\epsilon}]^-$, for all $\epsilon\in[-\epsilon^{\#}_{\Omega^*},\epsilon^{\#}_{\Omega^*}]\setminus\{0\}$. Therefore, we find it convenient to set $\epsilon^{\#}_{\tilde{\Omega}}\equiv\epsilon^{\#}_{\Omega^*}$.

By equality $\int_{\partial\Omega}\Theta^o\left[\epsilon,\rho(\epsilon)/\epsilon,f,g\right](s)\,d\sigma_s = 0$ and by a simple computation based on the rule of change of variables in integrals and on equality (1.7), we have

$$\begin{aligned} u^-[\epsilon,f,g,c](p+\epsilon t) &= \epsilon^{n-1}\int_{\partial\Omega} S_{q,n}(\epsilon(t-s))\Theta^o\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right](s)\,d\sigma_s \\ &\quad + \epsilon C^-\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right] + \frac{\epsilon^{1-n}c}{|\partial\Omega|_{n-1}} = \epsilon\Bigg(\int_{\partial\Omega} S_n(t-s)\Theta^o\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right](s)\,d\sigma_s \\ &\quad + \epsilon^{n-2}\Lambda\left[\epsilon,\Theta^o\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right]\right](t)\Bigg) + \epsilon C^-\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right] + \frac{\epsilon^{1-n}c}{|\partial\Omega|_{n-1}} \end{aligned}$$

for all $t\in\mathrm{cl}\Omega^*$ and for all $(\epsilon,f,g,c)\in]0,\epsilon^{\#}_{\tilde{\Omega}}[\times\mathcal{U}\times\mathbb{R}$ (see also (1.38)). Thus it is natural to set

$$V^-_{\Omega^*}[\epsilon,\epsilon',f,g](t) := \int_{\partial\Omega} S_n(t-s)\Theta^o[\epsilon,\epsilon',f,g](s)\,d\sigma_s + \epsilon^{n-2}\Lambda\left[\epsilon,\Theta^o[\epsilon,\epsilon',f,g](t)\right] \qquad (1.41)$$

for all $t\in\mathrm{cl}\Omega^*$ and for all $(\epsilon,\epsilon',f,g)\in]-\epsilon^{\#}_{\tilde{\Omega}},\epsilon^{\#}_{\tilde{\Omega}}[\times\mathcal{U}_{r_*}\times\mathcal{U}$. Since $v^-[\partial\Omega,\cdot]_{|\mathrm{cl}\Omega^*}$ is linear and continuous from $C^{0,\alpha}(\partial\Omega)_0$ to $C^{1,\alpha}(\mathrm{cl}\Omega^*)$ and $\Theta^o$ is real analytic, the map from $]-\epsilon^{\#}_{\tilde{\Omega}},\epsilon^{\#}_{\tilde{\Omega}}[\times\mathcal{U}_{r_*}\times\mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\Omega^*)$ which takes $(\epsilon,\epsilon',f,g)$ to the function $\int_{\partial\Omega}S_n(t-s)\Theta^o[\epsilon,\epsilon',f,g](s)\,d\sigma_s$ of the variable $t\in\mathrm{cl}\Omega^*$ is real analytic (see, e.g., Miranda [106], Lanza de Cristoforis and Rossi [87, Thm. 3.1]). Clearly, we have $(p+\epsilon\mathrm{cl}\Omega^*)\cap(\partial\mathbb{S}[\Omega_{p,\epsilon}]\setminus Q)=\emptyset$ for all $\epsilon\in]-\epsilon^{\#}_{\tilde{\Omega}},\epsilon^{\#}_{\tilde{\Omega}}[$. As a consequence, standard properties of integral operators with real analytic kernels and with no singularity imply that the map from $]-\epsilon^{\#}_{\tilde{\Omega}},\epsilon^{\#}_{\tilde{\Omega}}[\times L^1(\partial\Omega)$ to $C^2(\mathrm{cl}\Omega^*)$ which takes $(\epsilon,\phi)$ to the function $\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\phi(s)\,d\sigma_s$ of the variable $t\in\mathrm{cl}\Omega^*$ is real analytic (see Lanza de Cristoforis and Musolino [82, Sec. 4]). Then by the analyticity of $\Theta^o$ and by the continuity of the embeddings of $C^{0,\alpha}(\partial\Omega)_0$ into $L^1(\partial\Omega)$ and of $C^2(\mathrm{cl}\Omega^*)$ into $C^{1,\alpha}(\mathrm{cl}\Omega^*)$, we conclude that the map from $]-\epsilon^{\#}_{\tilde{\Omega}},\epsilon^{\#}_{\tilde{\Omega}}[\times\mathcal{U}_{r_*}\times\mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\Omega^*)$ which takes $(\epsilon,\epsilon',f,g)$ to the second term in the right hand side of (1.41) is real analytic. So, by standard calculus in Banach space, we deduce that $V^-_{\Omega^*}$ is real analytic. Then we set $V^-_{\tilde{\Omega}}$ equal to the composition of $V^-_{\Omega^*}$ with the restriction operator from $C^{1,\alpha}(\mathrm{cl}\Omega^*)$ to $C^{1,\alpha}(\mathrm{cl}\tilde{\Omega})$. As a consequence, $V^-_{\tilde{\Omega}}$ is real analytic. Then, by Theorem 1.4.2, by equality $\Theta^o[0,r_*,f_0,g_0]=\tilde{\theta}^o$, and by (1.33), the validity of (1.37) follows. Thus, the proof is complete. □

## 1.6 A functional analytic representation theorem for the solutions of the nonideal nonlinear transmission problem

In this section we exploit the previous results in order to analyze the asymptotic behavior of the solution of problem (1.4). To do so, we want to find a way in which problem (1.4) can be converted into a periodic linear transmission one, and then obtain the desired results applying Proposition 1.4.1 to the new problem. We exploit such a strategy to prove the following proposition.

**Proposition 1.6.1.** *Let $\lambda^+,\lambda^-\in]0,+\infty[$. Let $c\in\mathbb{R}$. Let $B\in M_{1\times n}(\mathbb{R})$. Let $f\in C^{0,\alpha}(\partial\Omega)_0$. Let $g\in C^{1,\alpha}(\partial\Omega)$. Let $F$ be a real analytic map from $\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $\mathbb{R}$. Let $\rho$ be a function from $]0,\epsilon_0[$ to $\mathbb{R}\setminus\{0\}$. Let $\epsilon\in]0,\epsilon_0[$. Then problem (1.4) has a unique solution $(\mathbf{u}^+[\epsilon,f,g,c],\mathbf{u}^-[\epsilon,f,g,c])$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])\times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$. Moreover, such a pair is delivered by the formulas*

$$\mathbf{u}^+[\epsilon,f,g,c](x)\equiv u^+\left[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]\right](x)+Bq^{-1}x\quad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\tag{1.42}$$

$$\begin{aligned}\mathbf{u}^-[\epsilon,f,g,c](x)\equiv{}& u^-[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]](x)\\&+\rho(\epsilon)F\Bigg[\frac{\partial u^-[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]]}{\partial\nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)+Bq^{-1}\nu_\Omega(\cdot),\\&\frac{\partial u^+[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]]}{\partial\nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)+Bq^{-1}\nu_\Omega(\cdot)\Bigg]+Bq^{-1}x\quad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,\end{aligned}\tag{1.43}$$

*where the pair $(u^+[\cdot,\cdot,\cdot,\cdot],u^-[\cdot,\cdot,\cdot,\cdot])$ is defined as in Proposition 1.4.1, the function $\mathbf{f}[f](t)=f(t)+(\lambda^+-\lambda^-)Bq^{-1}\nu_\Omega(t)$ for all $t\in\partial\Omega$, and $\mathbf{c}[\epsilon,c]=-Bq^{-1}\int_{\partial\Omega_{p,\epsilon}}y\,d\sigma_y+c$.*

*Proof.* We first note that one can convert nonlinear problem (1.4) into a periodic linear one. Indeed, if the pair $(u^+[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]],u^-[\epsilon,\mathbf{f}[f],g,\mathbf{c}[\epsilon,c]])$ solves the problem

$$\begin{cases}\Delta u^+=0 & \text{in }\mathbb{S}[\Omega_{p,\epsilon}],\\ \Delta u^-=0 & \text{in }\mathbb{S}[\Omega_{p,\epsilon}]^-,\\ u^+(x+qe_h)=u^+(x) & \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\quad\forall h\in\{1,\dots,n\},\\ u^-(x+qe_h)=u^-(x) & \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,\quad\forall h\in\{1,\dots,n\},\\ \lambda^-\frac{\partial u^-}{\partial\nu_{\Omega_{p,\epsilon}}}(x)-\lambda^+\frac{\partial u^+}{\partial\nu_{\Omega_{p,\epsilon}}}(x)=\mathbf{f}[f]((x-p)/\epsilon) & \forall x\in\partial\Omega_{p,\epsilon},\\ \frac{1}{\rho(\epsilon)}(u^+(x)-u^-(x))=g((x-p)/\epsilon) & \forall x\in\partial\Omega_{p,\epsilon},\\ \int_{\partial\Omega_{p,\epsilon}}u^+\,d\sigma=\mathbf{c}[\epsilon,c],\end{cases}\tag{1.44}$$

then a straightforward computation implies that the pair of functions delivered by the right-hand side of (1.42)-(1.43) is a solution of problem (1.4). Conversely, if a pair $(\mathbf{u}^+,\mathbf{u}^-)$ solves problem (1.4), then one verifies that the pair of functions $(u^+_\sharp,u^-_\sharp)$ defined by

$$u^+_\sharp(x)\equiv\mathbf{u}^+(x)-Bq^{-1}x\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\tag{1.45}$$

$$u^-_\sharp(x)\equiv\mathbf{u}^-(x)-\rho(\epsilon)F\left[\frac{\partial\mathbf{u}^-}{\partial\nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot),\frac{\partial\mathbf{u}^+}{\partial\nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)\right]-Bq^{-1}x\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,\tag{1.46}$$

solves problem (1.44) and thus $(u^+_\sharp, u^-_\sharp) = (u^+[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]], u^-[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]])$. Furthermore, formulas (1.45)-(1.46) and the uniqueness of the solutions of problem (1.44) imply that if there are two pairs $(\mathbf{u}^+_1, \mathbf{u}^-_1)$ and $(\mathbf{u}^+_2, \mathbf{u}^-_2)$ which solve problem (1.4), then we have

$$\begin{aligned}
&\mathbf{u}^+_1(x) - Bq^{-1}x = \mathbf{u}^+_2(x) - Bq^{-1}x && \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\\
&\mathbf{u}^-_1(x) - \rho(\epsilon)F\left[\frac{\partial \mathbf{u}^-_1}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot), \frac{\partial \mathbf{u}^+_1}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)\right] - Bq^{-1}x &&\\
&\quad = \mathbf{u}^-_2(x) - \rho(\epsilon)F\left[\frac{\partial \mathbf{u}^-_2}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot), \frac{\partial \mathbf{u}^+_2}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot)\right] - Bq^{-1}x && \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,
\end{aligned}$$

from which one can deduce that

$$\begin{aligned}
&\mathbf{u}^+_1(x) = \mathbf{u}^+_2(x) && \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\\
&\mathbf{u}^-_1(x) = \mathbf{u}^-_2(x) && \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-.
\end{aligned}$$

Thus, we conclude that problem (1.4) has a unique solution and we denote such a solution by $(\mathbf{u}^+[\epsilon, f, g, c], \mathbf{u}^-[\epsilon, f, g, c])$ and, clearly, it is delivered by formulas (1.42)-(1.43). □

Our aim is to investigate the behavior of $\mathbf{u}^+[\epsilon, f, g, c]$ and $\mathbf{u}^-[\epsilon, f, g, c]$ when $(\epsilon, f, g, c)$ is close to the degenerate quadruple $(0, f_0, g_0, c_0)$. Proposition 1.6.1 tells us how to represent the solution $(\mathbf{u}^+[\epsilon, f, g, c], \mathbf{u}^-[\epsilon, f, g, c])$ of problem (1.4) in terms of the solution $(u^+[\cdot,\cdot,\cdot,\cdot], u^-[\cdot,\cdot,\cdot,\cdot])$ of the appropriate periodic linear problem. As a consequence, applying Theorems 1.5.1-1.5.2 for $(u^+[\cdot,\cdot,\cdot,\cdot], u^-[\cdot,\cdot,\cdot,\cdot])$, one can analyze the behavior of $\mathbf{u}^+[\epsilon, f, g, c]$ and $\mathbf{u}^-[\epsilon, f, g, c]$. Indeed, the following two theorems hold.

**Theorem 1.6.2.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $c \in \mathbb{R}$. Let $B \in M_{1\times n}(\mathbb{R})$. Let $F$ be a real analytic map from $\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $\mathbb{R}$. Let $\rho$ be a function from $]0, \epsilon_0[$ to $\mathbb{R} \setminus \{0\}$. Let assumption* (1.5) *hold. Let $r_*$ be as in* (1.6). *Let $\epsilon_1$, $f_0$, $g_0$, $\mathcal{U}$, $\mathcal{U}_{r_*}$ be as in Theorem 1.4.3. Then the following statements hold.*

*(i) The "limiting system"*

$$\begin{cases}
\lambda^-\left(\frac{1}{2}\theta^o(t) + w_*[\partial\Omega, \theta^o](t)\right) - \lambda^+\left(-\frac{1}{2}\theta^i(t) + w_*[\partial\Omega, \theta^i](t)\right) & \\
\qquad\qquad = f_0(t) + (\lambda^+ - \lambda^-)Bq^{-1}\nu_\Omega(t) & \forall t \in \partial\Omega,\\
v^+[\partial\Omega, \theta^i](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i]d\sigma - v^-[\partial\Omega, \theta^o](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o]d\sigma & \\
\qquad\qquad = r_* g_0(t) - r_* \fint_{\partial\Omega} g_0 d\sigma & \forall t \in \partial\Omega
\end{cases}$$

*has one and only one solution $(\bar{\theta}^i, \bar{\theta}^o) \in \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$.*

*(ii) The "limiting boundary value problem"*

$$\begin{cases}
\Delta u^+ = 0 & \text{in } \Omega,\\
\Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega,\\
\lambda^- \frac{\partial u^-}{\partial \nu_\Omega}(t) - \lambda^+ \frac{\partial u^+}{\partial \nu_\Omega}(t) = f_0(t) + (\lambda^+ - \lambda^-)Bq^{-1}\nu_\Omega(t) & \forall t \in \partial\Omega,\\
u^+(t) - u^-(t) = r_* g_0(t) - r_* \fint_{\partial\Omega} g_0\, d\sigma & \forall t \in \partial\Omega,\\
\int_{\partial\Omega} u^+\, d\sigma = 0\,, &\\
\lim_{t\to\infty} u^-(t) \in \mathbb{R} &
\end{cases}$$

*has one and only one solution $(\bar{u}^+, \bar{u}^-) \in C^{1,\alpha}(\mathrm{cl}\Omega) \times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$, defined by*

$$\begin{aligned}
\bar{u}^+ &\equiv v^+[\partial\Omega, \bar{\theta}^i] - \fint_{\partial\Omega} v^+[\partial\Omega, \bar{\theta}^i]\, d\sigma,\\
\bar{u}^- &\equiv v^-[\partial\Omega, \bar{\theta}^o] - \fint_{\partial\Omega} v^-[\partial\Omega, \bar{\theta}^o]\, d\sigma\,.
\end{aligned}$$

*(iii) There exists a real analytic map* $\mathbf{U}^+$ *from* $]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$ *to* $C^{1,\alpha}(\mathrm{cl}\Omega)$ *such that*

$$\mathbf{u}^+[\epsilon,f,g,c](p+\epsilon t)=\epsilon\mathbf{U}^+\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},f,g\right](t)+\frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}}c\qquad\forall t\in\mathrm{cl}\Omega,$$

*for all* $(\epsilon,f,g,c)\in]0,\epsilon_1[\times\mathcal{U}\times\mathbb{R}$, *where* $(\mathbf{u}^+[\epsilon,f,g,c],\mathbf{u}^-[\epsilon,f,g,c])$ *is the unique solution of problem* (1.4). *Moreover,*

$$\mathbf{U}^+[0,r_*,f_0,g_0](t)=\bar{u}^+(t)+Bq^{-1}t-Bq^{-1}\fint_{\partial\Omega}s\,d\sigma_s\qquad\forall t\in\mathrm{cl}\Omega\,. \tag{1.47}$$

*Proof.* First we note that the validity of statements $(i)$ and $(ii)$ follows from Theorem 1.4.2.

Now, we turn to prove statement $(iii)$. By using Theorem 1.5.1, by representation (1.42), and by simple computations based on the rule of change of variables in integrals, we have

$$\mathbf{u}^+[\epsilon,f,g,c](p+\epsilon t)=\epsilon U^+\left[\epsilon,\frac{\rho(\epsilon)}{\epsilon},\mathbf{f}[f],g\right](t)+\epsilon Bq^{-1}t-\epsilon Bq^{-1}\fint_{\partial\Omega}s\,d\sigma_s+\frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}}c$$

for all $t\in\mathrm{cl}\Omega$, where $U^+$ is defined in Theorem 1.5.1 and $\mathbf{f}[f]$ is as in Proposition 1.6.1. So, it is natural to set

$$\mathbf{U}^+[\epsilon,\epsilon',f,g](t):=U^+[\epsilon,\epsilon',\mathbf{f}[f],g](t)+Bq^{-1}t-Bq^{-1}\fint_{\partial\Omega}s\,d\sigma_s\qquad\forall t\in\mathrm{cl}\Omega,$$

for all $(\epsilon,\epsilon',f,g)\in]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$. By the real analyticity of map $U$ in $]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$ (see Theorem 1.5.1), one can conclude that $\mathbf{U}^+$ is also a real analytic map from $]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$. Then, if $(\epsilon,\epsilon',f,g)$ tends to $(0,r_*,f_0,g_0)$, we have that

$$\mathbf{U}^+[0,r_*,f_0,g_0](t)=U^+[0,r_*,\mathbf{f}[f_0],g_0](t)+Bq^{-1}t-Bq^{-1}\fint_{\partial\Omega}s\,d\sigma_s\qquad\forall t\in\mathrm{cl}\Omega\,.$$

Finally, using the definition of $U^+$, Theorem 1.4.3, and also statements $(i)$ and $(ii)$ of this theorem, one can show that $U^+[0,r_*,\mathbf{f}[f_0],g_0](t)=\bar{u}^+(t)$ for all $t\in\mathrm{cl}\Omega$. Thus, the validity of (1.47) follows and the proof is complete. □

Before formulating the next theorem, we find it convenient to introduce the following notation. Let $\bar{\theta}^o$, $\bar{u}^-$ be as in Theorem 1.6.2. Then by classical potential theory and by equality $\int_{\partial\Omega}\bar{\theta}^o\,d\sigma=0$, we can set

$$\bar{l}^-:=\lim_{t\to\infty}\bar{u}^-(t)=-\fint_{\partial\Omega}v^-[\partial\Omega,\bar{\theta}^o]\,d\sigma\,.$$

**Theorem 1.6.3.** *Let* $\lambda^+,\lambda^-\in]0,+\infty[$. *Let* $c\in\mathbb{R}$. *Let* $B\in M_{1\times n}(\mathbb{R})$. *Let* $F$ *be a real analytic map from* $\left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ *to* $\mathbb{R}$. *Let* $\rho$ *be a function from* $]0,\epsilon_0[$ *to* $\mathbb{R}\setminus\{0\}$. *Let assumption* (1.5) *hold. Let* $r_*$ *be as in* (1.6). *Let* $\epsilon_1$, $f_0$, $g_0$, $\mathcal{U}$, $\mathcal{U}_{r_*}$ *be as in Theorem 1.4.3. Let* $\bar{u}^-$ *be as in Theorem 1.6.2(ii). Then there exists a real analytic map* $\mathbf{C}^-$ *from* $]-\epsilon_1,\epsilon_1[\times\mathcal{U}_{r_*}\times\mathcal{U}$ *to* $\mathbb{R}$ *such that*

$$\begin{aligned}\mathbf{C}^-[0,r_*,f_0,g_0]=\bar{l}^--r_*\fint_{\partial\Omega}g\,d\sigma-Bq^{-1}\fint_{\partial\Omega}s\,d\sigma_s&\\+r_*F\left[\frac{\partial\bar{u}^-}{\partial\nu_\Omega}+Bq^{-1}\nu_\Omega,\frac{\partial\bar{u}^+}{\partial\nu_\Omega}+Bq^{-1}\nu_\Omega\right]&\end{aligned} \tag{1.48}$$

*and such that the following statements hold.*

*(i) Let $\tilde{\Omega}$ be an open bounded subset of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\Omega} \cap (p + q\mathbb{Z}^n) = \emptyset$. Let $k \in \mathbb{N}$. Then there exist $\epsilon_{\tilde{\Omega}} \in ]0, \epsilon_1[$ and a real analytic map $\mathbf{U}^-_{\tilde{\Omega}}$ from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times \mathcal{U}_{r_*} \times \mathcal{U}$ to $C^k(\mathrm{cl}\tilde{\Omega})$ such that $\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in ]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[$, and such that*

$$\mathbf{u}^-[\epsilon, f, g, c](x) = Bq^{-1}x - Bq^{-1}p + \epsilon \mathbf{C}^-\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon^n \mathbf{U}^-_{\tilde{\Omega}}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](x) + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c$$

*for all $x \in \mathrm{cl}\tilde{\Omega}$ and for all $(\epsilon, f, g, c) \in ]0, \epsilon_{\tilde{\Omega}}[\times \mathcal{U} \times \mathbb{R}$. Moreover,*

$$\mathbf{U}^-_{\tilde{\Omega}}[0, r_*, f_0, g_0](x) = DS_{q,n}(x-p)\left(\int_{\partial\Omega} \nu_\Omega(s)\bar{u}^-(s)\, d\sigma_s - \int_{\partial\Omega} s \frac{\partial}{\partial \nu_\Omega}\bar{u}^-(s)\, d\sigma_s\right) \tag{1.49}$$

*for all $x \in \mathrm{cl}\tilde{\Omega}$.*

*(ii) Let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then there exist $\epsilon^{\#}_{\tilde{\Omega}} \in ]0, \epsilon_1[$ and a real analytic map $\mathbf{V}^-_{\tilde{\Omega}}$ from $]-\epsilon^{\#}_{\tilde{\Omega}}, \epsilon^{\#}_{\tilde{\Omega}}[\times \mathcal{U}_{r_*} \times \mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\tilde{\Omega})$ such that $p + \epsilon \mathrm{cl}\tilde{\Omega} \subseteq \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in ]-\epsilon^{\#}_{\tilde{\Omega}}, \epsilon^{\#}_{\tilde{\Omega}}[$, and*

$$\mathbf{u}^-[\epsilon, f, g, c](p + \epsilon t) = \epsilon \mathbf{C}^-\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon \mathbf{V}^-_{\tilde{\Omega}}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](t) + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c$$

*for all $t \in \mathrm{cl}\tilde{\Omega}$ and all $(\epsilon, f, g, c) \in ]0, \epsilon^{\#}_{\tilde{\Omega}}[\times \mathcal{U} \times \mathbb{R}$. Moreover,*

$$\mathbf{V}^-_{\tilde{\Omega}}[0, r_*, f_0, g_0](t) = \bar{u}^-(t) - \bar{l}^- + Bq^{-1}t \qquad \forall t \in \mathrm{cl}\tilde{\Omega}. \tag{1.50}$$

*Proof.* Let $\mathbf{f}[f]$, $\mathbf{c}[\epsilon, c]$ be as in Proposition 1.6.1. First, we consider the function

$$(\epsilon, f, g, c) \mapsto F\left[\frac{\partial u^-[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p + \epsilon \cdot) + Bq^{-1}\nu_\Omega(\cdot), \frac{\partial u^+[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p + \epsilon \cdot) + Bq^{-1}\nu_\Omega(\cdot)\right]$$

from $]0, \epsilon_1[\times \mathcal{U} \times \mathbb{R}$ to $\mathbb{R}$ (actually, we are going to show that it depends only on $(\epsilon, f, g)$). By Theorem 1.6.2(*iii*) and equality (1.42), we have

$$\frac{\partial u^+[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p + \epsilon t) + Bq^{-1}\nu_\Omega(t) = \frac{\partial}{\partial \nu_\Omega}\mathbf{U}^+\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](t) \tag{1.51}$$

for all $t \in \partial\Omega$. Moreover, by the fifth equality of (1.4), equalities (1.43) and (1.51), we have

$$\frac{\partial u^-[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p + \epsilon t) + Bq^{-1}\nu_\Omega(t) = \frac{\lambda^+}{\lambda^-}\frac{\partial}{\partial \nu_\Omega}\mathbf{U}^+\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right](t) + \frac{1}{\lambda^-}f(t) \tag{1.52}$$

for all $t \in \partial\Omega$. Furthermore, taking into account equalities (1.51)-(1.52) which do not depend on $c$, it is natural to define the map $\mathbf{F}$ from $]-\epsilon_1, \epsilon_1[\times \mathcal{U}_{r_*} \times \mathcal{U}$ to $\mathbb{R}$ by setting

$$\mathbf{F}[\epsilon, \epsilon', f, g] := F\left[\frac{\lambda^+}{\lambda^-}\frac{\partial}{\partial \nu_\Omega}\mathbf{U}^+[\epsilon, \epsilon', f, g](\cdot) + \frac{1}{\lambda^-}f(\cdot), \frac{\partial}{\partial \nu_\Omega}\mathbf{U}^+[\epsilon, \epsilon', f, g](\cdot)\right]$$

for all $(\epsilon, \epsilon', f, g) \in ]-\epsilon_1, \epsilon_1[ \times \mathcal{U}_{r_*} \times \mathcal{U}$. Then, by the real analyticity of the map $F$, we deduce that $\mathbf{F}$ is real analytic and that

$$F\Big[\frac{\partial u^-[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot) + Bq^{-1}\nu_\Omega(\cdot), \frac{\partial u^+[\epsilon, \mathbf{f}[f], g, \mathbf{c}[\epsilon, c]]}{\partial \nu_{\Omega_{p,\epsilon}}}(p+\epsilon\cdot) + Bq^{-1}\nu_\Omega(\cdot)\Big] = \mathbf{F}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right]$$

for all $(\epsilon, f, g, c) \in ]0, \epsilon_1[ \times \mathcal{U} \times \mathbb{R}$. We also note that

$$\begin{aligned} \mathbf{F}[0, r_*, f_0, g_0] &= F\left[\frac{\lambda^+}{\lambda^-}\Big(\frac{\partial \bar{u}^+}{\partial \nu_\Omega} + Bq^{-1}\nu_\Omega\Big) + \frac{1}{\lambda^-} f_0, \frac{\partial \bar{u}^+}{\partial \nu_\Omega} + Bq^{-1}\nu_\Omega\right] \\ &= F\left[\frac{\partial \bar{u}^-}{\partial \nu_\Omega} + Bq^{-1}\nu_\Omega, \frac{\partial \bar{u}^+}{\partial \nu_\Omega} + Bq^{-1}\nu_\Omega\right]. \end{aligned} \tag{1.53}$$

We now consider the proof of statement $(i)$. By Theorem 1.5.2, and by arguing as in the proof of Theorem 1.5.2$(i)$, and by equality (1.43), we have

$$\begin{aligned} \mathbf{u}^-[\epsilon, f, g, c](x) &= \epsilon C^-\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, \mathbf{f}[f], g\right] + \epsilon^n U^-_{\tilde{\Omega}}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, \mathbf{f}[f], g\right](x) \\ &+ \rho(\epsilon)\mathbf{F}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] - Bq^{-1}p - \epsilon Bq^{-1}\fint_{\partial\Omega} s\, d\sigma_s + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c + Bq^{-1}x \end{aligned}$$

for all $x \in \mathrm{cl}\tilde{\Omega}$ and all $(\epsilon, f, g, c) \in ]0, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U} \times \mathbb{R}$. It is natural to set

$$\begin{aligned} \mathbf{C}^-[\epsilon, \epsilon', f, g] &:= C^-[\epsilon, \epsilon', \mathbf{f}[f], g] + \epsilon'\mathbf{F}[\epsilon, \epsilon', f, g] - Bq^{-1}\fint_{\partial\Omega} s\, d\sigma_s, \\ \mathbf{U}^-_{\tilde{\Omega}}[\epsilon, \epsilon', f, g](x) &:= U^-_{\tilde{\Omega}}[\epsilon, \epsilon', \mathbf{f}[f], g](x) \qquad \forall x \in \mathrm{cl}\tilde{\Omega}, \end{aligned}$$

for all $(\epsilon, \epsilon', f, g) \in ]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{r_*} \times \mathcal{U}$. Clearly, by the real analyticity of $U^-_{\tilde{\Omega}}$, we deduce that $\mathbf{U}^-_{\tilde{\Omega}}$ is a real analytic map in $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[ \times \mathcal{U}_{r_*} \times \mathcal{U}$.

Now we note that Theorems 1.4.3 and 1.6.2$(i)$ imply that

$$\Big(\Theta^i[0, r_*, \mathbf{f}[f_0], g_0], \Theta^o[0, r_*, \mathbf{f}[f_0], g_0]\Big) = (\bar{\theta}^i, \bar{\theta}^o), \tag{1.54}$$

where the pair $\big(\Theta^i[\cdot,\cdot,\cdot,\cdot], \Theta^o[\cdot,\cdot,\cdot,\cdot]\big)$ is defined as in Theorem 1.4.3, and the pair $(\bar{\theta}^i, \bar{\theta}^o)$ as in Theorem 1.6.2$(i)$. Now, by virtue of equalities (1.38), (1.54), and Theorem 1.6.2$(ii)$, one can show that

$$C^-[0, r_*, \mathbf{f}[f_0], g_0] = \bar{l}^- - r_*\fint_{\partial\Omega} g\, d\sigma,$$

which together with equality (1.53) implies the validity of equality (1.48). Then, by exploiting equality (1.54) and Theorem 1.6.2$(ii)$, one verifies the validity of equality (1.49) in the same way as it is done for equality (1.36).

We now consider statement $(ii)$. By arguing as in the proof of Theorem 1.5.2$(ii)$ and by equality (1.43), we have

$$\begin{aligned} \mathbf{u}^-[\epsilon, f, g, c](p+\epsilon t) &= \epsilon C^-\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, \mathbf{f}[f], g\right] + \epsilon V^-_{\tilde{\Omega}}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, \mathbf{f}[f], g\right](t) \\ &- \epsilon Bq^{-1}\fint_{\partial\Omega} s\, d\sigma_s + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c + \rho(\epsilon)\mathbf{F}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon Bq^{-1}t \\ &= \epsilon\mathbf{C}^-\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, f, g\right] + \epsilon V^-_{\tilde{\Omega}}\left[\epsilon, \frac{\rho(\epsilon)}{\epsilon}, \mathbf{f}[f], g\right](t) + \epsilon Bq^{-1}t + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c \end{aligned}$$

for all $t \in \mathrm{cl}\tilde{\Omega}$ and for all $(\epsilon, f, g, c) \in ]0, \epsilon^{\#}_{\tilde{\Omega}}[\times \mathcal{U} \times \mathbb{R}$. Thus, it is natural to set

$$\mathbf{V}^{-}_{\tilde{\Omega}}[\epsilon, \epsilon', f, g](t) := V^{-}_{\tilde{\Omega}}[\epsilon, \epsilon', \mathbf{f}[f], g](t) + Bq^{-1}t \qquad \forall t \in \mathrm{cl}\tilde{\Omega}$$

and for all $(\epsilon, \epsilon', f, g) \in ] - \epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times \mathcal{U}_{r_*} \times \mathcal{U}$. By the real analyticity of $V^{-}_{\tilde{\Omega}}$, we have that $\mathbf{V}^{-}_{\tilde{\Omega}}$ is a real analytic map from $] - \epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times \mathcal{U}_{r_*} \times \mathcal{U}$. Then, by exploiting equality (1.54) and Theorem 1.6.2$(ii)$, one verifies equality (1.50) in the same way as it is done for equality (1.37). Thus, the proof is complete. □

As a conclusion, we note that our results imply that the solutions of (1.3) and (1.4) can be expanded into absolutely convergent power series. Moreover, one can compute explicitly such power series and adopt the technique presented in this chapter to compute expansions for the effective properties of periodic composite materials as you can see in the next chapter.

# CHAPTER 2

## Effective conductivity of a periodic dilute composite

In this chapter we study the asymptotic behavior of the effective thermal conductivity of a periodic two-phase dilute composite obtained by introducing into an infinite homogeneous matrix a periodic set of inclusions of a different material, each of them of a size proportional to a positive parameter $\epsilon$. We consider three mathematical models for composites of such a type assuming an ideal or nonideal contact at constituent interfaces. Each model is represented by a boundary value problem with specific transmission conditions. For instance, we will consider an ideal transmission condition when the normal component of the heat flux and the temperature field are continuous, and a nonideal transmission condition when the normal component of the heat flux is continuous but the temperature field displays a jump proportional to the normal heat flux. For $\epsilon$ small we prove that the effective conductivity can be represented as a convergent power series in $\epsilon$ and we determine all the coefficients in terms of the solutions of explicit systems of integral equations. We note that a preliminary step in the explicit computation of the series expansions has been performed in Dalla Riva, Musolino, and Rogosin [39], regarding the solution of a Dirichlet problem for the Laplace equation in a bounded domain with a small hole. Here, we present the first extension of such a computation to periodic domains and to different transmission boundary conditions.

The chapter is organized as follows. In Section 2.1 we introduce the effective conductivity of a periodic composite. Then, in Section 2.2 we investigate the asymptotic behavior the effective conductivity for a composite with ideal contact. We begin with some preliminaries collected in Subsection 2.2.1 and computing the power series expansions for two auxiliary functions in Subsection 2.2.2. Then, in Subsections 2.2.3 and 2.2.4, we prove our main results: we describe the effective conductivity in terms of real analytic functions (Theorem 2.2.9) and obtain its power series expansion with explicitly determined coefficients (Theorems 2.2.11 and 2.2.12). We complete the section by Subsection 2.2.5 devoted to the computation of the first few coefficients in the series expansion of the effective conductivity when the inclusions are in the form of a ball. Then, in Section 2.3 we apply the same strategy to study the asymptotic behavior the effective conductivity for a composite with nonideal contact.

Throughout this chapter, we retain the notation of Chapter 1.

Some of the results presented in this chapter have been published in the paper [136] by the author and in the paper [38] by Prof. Matteo Dalla Riva, Dr. Paolo Musolino, and the author.

### 2.1 Definition of the effective conductivity of a periodic composite

In this section we consider the effective properties of composite materials and derive a formula for defining the effective conductivity.

Mostly, when we study the mathematical model for composites, we need to determine the characteristics of the composite material as a single whole which are also called *the effective characteristics* of the composite material. Since the micro-structure of the composite is periodic, one can apply *the periodic homogenization* (see, e.g., Bensoussan, Lions, and Papanicolaou [18], Bakhvalov and Panasenko [12], Jikov, Kozlov, and Oleinik [63], Markov [96], Milton [105]). Due to the periodic homogenization, it suffices to study effective properties of a periodic composite only for one of its periodic cell, which is called fundamental or a cell of periodicity.

In this chapter, we consider three models of composite materials. To begin with, we recall that we fixed $\alpha \in ]0,1[$, a point $p \in Q$, a set $\Omega$ as in (1.1), and $\epsilon_0$ as in (1.2). Let us consider for each $j \in \{1,\dots,n\}$ the pair of functions $(u_j^+[\epsilon], u_j^-[\epsilon]) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ which are harmonic in $\mathbb{S}[\Omega_{p,\epsilon}]$ and $\mathbb{S}[\Omega_{p,\epsilon}]^-$, i.e.,

$$\begin{cases} \Delta u_j^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta u_j^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \end{cases}$$

are *quasi-periodic* in $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ and $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$, i.e.,

$$\begin{cases} u_j^+(x + q_{hh}e_h) = u_j^+(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \quad \forall h \in \{1,\dots,n\}, \\ u_j^-(x + q_{hh}e_h) = u_j^-(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \quad \forall h \in \{1,\dots,n\}, \end{cases}$$

and satisfy some transmission conditions on $\partial\Omega_{p,\epsilon}$ (see problems (2.8), (2.9), and (2.63) below), where the normal component of the heat flux has a jump on $\partial\Omega_{p,\epsilon}$ equals to some given function $f \in C^{0,\alpha}(\partial\Omega)_0$, i.e.,

$$\lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) \qquad \forall x \in \partial\Omega_{p,\epsilon}. \tag{2.1}$$

Then, by adapting the argument of Benveniste and Miloh [19], we introduce the effective conductivity.

Now, let us consider a periodic composite from a physical point of view and pick a piece of material which contains only the fundamental periodic cell $Q$ with a temperature field $T$ inside, where

$$T(x) \equiv \begin{cases} u_j^+(x) & \text{if} \quad x \in \Omega_{p,\epsilon}, \\ u_j^-(x) & \text{if} \quad x \in \mathrm{cl}Q \setminus \mathrm{cl}\Omega_{p,\epsilon}. \end{cases} \tag{2.2}$$

Then we are interesting in two quantities in $Q$. The first one is the heat flux, which we denote by $H$, and the second one is the temperature gradient $\nabla T$. These two quantities are related by the following equality

$$H = -\Lambda \nabla T \qquad \text{in} \quad Q, \tag{2.3}$$

where $\Lambda$ is the conductivity of the cell $Q$ and it is a matrix valued function on $Q$.

Some physical evidences lead to the conclusion that the composite behaves in average as a homogeneous material and our plan is to evaluate the average heat flux and the average temperature gradient in the cell. Thus we have to calculate the average heat flux and the average temperature gradient by means of quantities which can be measured on the boundary of $Q$. To do so, we suppose that $Q$ is filled with a homogeneous material with no sinks or sources of heat. Under this assumption $H \in C^1(\mathrm{cl}\, Q)$,

$$\mathrm{div} H = 0 \qquad \text{in} \quad Q,$$

and we calculate the average heat flux $< H >$ as seen from the boundary of $Q$ by means of the following

$$< H >= \frac{1}{|Q|_n} \int_Q H(x)dx = \frac{1}{|Q|_n} \int_{\partial Q} xH(x) \cdot \nu_Q(x) d\sigma_x. \tag{2.4}$$

Similarly, we obtain the formula for the average temperature gradient $\nabla T$. If the material is homogeneous and has no sinks or sources of heat then $T \in C^2(\mathrm{cl}\, Q)$, and $\Delta T = 0$, and we have

$$< \nabla T >= \frac{1}{|Q|_n} \int_Q \nabla T(x) dx = \frac{1}{|Q|_n} \int_{\partial Q} T(x) \nu_Q(x) d\sigma_x. \tag{2.5}$$

We observe that equalities (2.4) and (2.5) express the average heat flux and average temperature gradient by means of the normal component of the boundary heat flux and by means of the boundary temperature, respectively.

Then, denoting by $\lambda^{\mathrm{eff}}$ the effective conductivity matrix of the composite, one can deduce it by the following relation

$$< H >= -\lambda^{\mathrm{eff}} < \nabla T > \tag{2.6}$$

as an analog of equality (2.3). Our aim is to find the matrix $\lambda^{\mathrm{eff}}$.

Since we can see only the boundary of $Q$ and keeping identity (2.2) in mind, we observe that the boundary temperature is nothing else but $u_j^-$, and the $k$'s normal component of the boundary heat flux is nothing else but $\partial u_j^- / \partial x_k$, where $k \in \{1, 2, \dots, n\}$.

Also, taking the quasi-periodicity of $u_j^-$ in $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ and identity (2.2) into account, we observe that for all $x \in \partial Q$ such that $x + q_{hh} e_h \in \partial Q$

$$T(x + q_{hh} e_h) = T(x) + \delta_{h,j} q_{jj} \qquad \forall h \in \{1, \dots, n\}.$$

Then there exists a function $\mathcal{T} : \partial Q \to \mathbb{R}$ such that

$$T(x) = \mathcal{T}(x) + x_j \qquad \forall x \in \partial Q.$$

Hence, we have

$$\begin{aligned} < \nabla T >= & \frac{1}{|Q|_n} \int_{\partial Q} (\mathcal{T}(x) + x_j) \nu_Q(x) d\sigma_x \\ = & \frac{1}{|Q|_n} \int_{\partial Q} \mathcal{T}(x) \nu_Q(x) d\sigma_x + \frac{1}{|Q|_n} \int_{\partial Q} x_j \nu_Q(x) d\sigma_x. \end{aligned}$$

The first integral in the right-hand side is equal to zero due to periodicity of the function $\mathcal{T}$, and the second one is equal to $|Q|_n e_j$. Thus, we deduce that

$$< \nabla T >= e_j.$$

Now, we denote by $H_k$ the $k$'s normal component of the boundary heat flux. Then, using the Divergence theorem and equality (2.1), we calculate the average of $H_k$

$$\begin{aligned} < H_k >= & -\frac{1}{|Q|_n} \int_{\partial Q} x_k \lambda^- \frac{\partial u_j^-}{\partial x_k} d\sigma_x \\ = & -\frac{\lambda^-}{|Q|_n} \int_{\partial Q} x_k \frac{\partial u_j^-}{\partial x_k} d\sigma_x + \frac{1}{|Q|_n} \int_{\partial \Omega_{p,\epsilon}} x_k \lambda^- \frac{\partial u_j^-}{\partial x_k} d\sigma_x - \frac{1}{|Q|_n} \int_{\partial \Omega_{p,\epsilon}} x_k \lambda^- \frac{\partial u_j^-}{\partial x_k} d\sigma_x \\ = & -\frac{\lambda^-}{|Q|_n} \int_{Q \setminus \mathrm{cl}\Omega_{p,\epsilon}} \frac{\partial u_j^-}{\partial x_k} d\sigma_x - \frac{1}{|Q|_n} \int_{\partial \Omega_{p,\epsilon}} x_k \lambda^+ \frac{\partial u_j^+}{\partial x_k} d\sigma_x - \frac{1}{|Q|_n} \int_{\partial \Omega_{p,\epsilon}} x_k f((x-p)/\epsilon) d\sigma_x \\ = & -\frac{\lambda^-}{|Q|_n} \int_{Q \setminus \mathrm{cl}\Omega_{p,\epsilon}} \frac{\partial u_j^-}{\partial x_k} d\sigma_x - \frac{\lambda^+}{|Q|_n} \int_{\Omega_{p,\epsilon}} \frac{\partial u_j^+}{\partial x_k} d\sigma_x - \frac{1}{|Q|_n} \int_{\partial \Omega_{p,\epsilon}} x_k f((x-p)/\epsilon) d\sigma_x. \end{aligned}$$

The expressions for the average heat flux, for the average of the temperature gradient, and relation (2.6) lead to the following equality

$$\lambda^{\mathrm{eff}} e_j = - < H_k > \qquad \forall k, j \in \{1, 2, \dots, n\}.$$

Hence, we introduce the effective conductivity matrix $\lambda^{\mathrm{eff}}[\epsilon]$ with $(k, j)$-entry $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ defined by means of the following.

**Definition 2.1.1.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $(u_j^+[\epsilon], u_j^-[\epsilon])$ be the unique solution in the space $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of problem (2.8), or of problem (2.9) with $m \in \{1, \dots, n-1\}$, $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, $c \in \mathbb{R}$, or of problem (2.63) with the same $f$, $g$, and $c$. We set*

$$\lambda^{\mathrm{eff}}_{kj}[\epsilon] := \frac{1}{|Q|_n}\Bigg(\lambda^+ \int_{\Omega_{p,\epsilon}} \frac{\partial u_j^+[\epsilon](x)}{\partial x_k}\,dx + \lambda^- \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} \frac{\partial u_j^-[\epsilon](x)}{\partial x_k}\,dx + \int_{\partial\Omega_{p,\epsilon}} f((x-p)/\epsilon) x_k \, d\sigma_x\Bigg) \tag{2.7}$$

*for all $\epsilon \in ]0, \epsilon_0[$ and all $k, j \in \{1, 2, \dots, n\}$.*

We note that the integral of the function $f$ is not presented in the right-hand side of expression (7) when we consider the transmission problem with the boundary condition (4). Moreover, we observe that expression (7) extends that of Benveniste and Miloh [19] to the case of nonhomogeneous boundary conditions and coincides with the classical expression when $f$ and $g$ are identically 0 in systems (5) and (6).

Variations of formula (2.7) can be found in the papers of Benveniste and Miloh [19, 104], Lipton and Vernescu [92], Ammari, Kang, and Touibi [10], Drygaś and Mityushev [44], Castro, Pesetskaya, and Rogosin [27], and Dalla Riva and Musolino [35], where the authors considered different contact conditions in materials and the case when there are more than one inclusion in the unite cell $Q$. We also note that Benveniste and Miloh [19] introduced their expression for defining the effective conductivity of a composite with imperfect contact conditions by generalizing the dual theory of the effective behavior of composites with the ideal contact. Moreover, Dalla Riva and Musolino [35] introduced the effective conductivity as an extension of the classical definition to the case of nonhomogeneous boundary conditions.

## 2.2 Effective conductivity of a periodic dilute composite with ideal contact

In this section we consider the ideal transmission problem and study the effective conductivity of a periodic two-phase composite.

The section is organized as follows. In Subsection 2.2.1 we pose the problem, introduce some notation and prove some auxiliary statements. In Subsection 2.2.2 we compute the power series expansions for two auxiliary functions. In Subsections 2.2.3 and 2.2.4, we prove our main results: we describe the effective conductivity $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ in terms of real analytic functions (Theorem 2.2.9) and obtain the power series expansion for it with explicitly determined coefficients (Theorems 2.2.11 and 2.2.12). We complete the section by Subsection 2.2.5 devoted to the computation of some coefficients in the series expansion of $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ in case inclusions are in the form of a ball.

### 2.2.1 Preliminaries and notation

To pose the problem, we again take two positive constants $\lambda^+$, $\lambda^-$, and, for each $j \in \{1, \dots, n\}$ and a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$, we consider the following problem, which we call *the ideal transmission problem*:

$$
\begin{cases}
\Delta u_j^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\
\Delta u_j^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\
u_j^+(x + q_{hh}e_h) = u_j^+(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \quad \forall h \in \{1,\dots,n\}, \\
u_j^-(x + q_{hh}e_h) = u_j^-(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \quad \forall h \in \{1,\dots,n\}, \\
\lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = 0 & \forall x \in \partial\Omega_{p,\epsilon}, \\
u_j^+(x) - u_j^-(x) = 0 & \forall x \in \partial\Omega_{p,\epsilon}, \\
\int_{\partial\Omega_{p,\epsilon}} u_j^+ d\sigma = 0
\end{cases}
\tag{2.8}
$$

for all $\epsilon \in ]0, \epsilon_0[$.

The analysis of problem (2.8) can be deduced by the analysis of a more general transmission problem and which can be considered as a particular case of problem (1.4). In order to formulate this more general problem, we additionally take a number $m \in \{1, \dots, n-1\}$ and functions $f \in C^{0,\alpha}(\partial\Omega)_0$ and $g \in C^{1,\alpha}(\partial\Omega)$. Then, for each $j \in \{1, \dots, n\}$ we consider the following transmission problem for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

$$
\begin{cases}
\Delta u_j^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\
\Delta u_j^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\
u_j^+(x + q_{hh}e_h) = u_j^+(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \quad \forall h \in \{1,\dots,n\}, \\
u_j^-(x + q_{hh}e_h) = u_j^-(x) + \delta_{h,j}q_{jj} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \quad \forall h \in \{1,\dots,n\}, \\
\lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\
\frac{1}{\epsilon^m}(u_j^+(x) - u_j^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\
\int_{\partial\Omega_{p,\epsilon}} u_j^+ d\sigma = c
\end{cases}
\tag{2.9}
$$

for all $\epsilon \in ]0, \epsilon_0[$. Clearly, if $(f, g, c) = (0, 0, 0)$ in (2.9) then the two problems coincide.

The parameters $\lambda^+$ and $\lambda^-$ play the role of thermal conductivity of the materials which fill the inclusions and the matrix, respectively. Therefore, the solutions of problems (2.8) and (2.9) represent the temperature distribution on the set of inclusions $\mathbb{S}[\Omega_{p,\epsilon}]$ and in the matrix $\mathbb{S}[\Omega_{p,\epsilon}]^-$, under different conditions. More precisely, the third and the fourth conditions in problems (2.8) and (2.9) express a growth for the temperature distribution in the direction $e_h$ and periodicity in all other directions. The fifth and the sixth conditions of problem (2.8) say that the normal component of the heat flux and the temperature distribution are continuous on the interface. At the same time, the fifth condition in (2.9) says that the normal component of the heat flux presents a jump which equals a given function and the sixth condition in (2.9) says that the jump on the interface of the temperature distribution is proportional to a given function by means of the parameter $\epsilon^m$. Finally, the last condition of problems (2.8) and (2.9) is just a normalization condition, which we need in order to "fix" the solution.

Before studying the effective conductivity, we have to solve problem (2.9) what can be done by exploiting the results obtained in Chapter 1. To begin with, we recall some auxiliary properties of the periodic analog of the fundamental solution of the Laplace equation and then convert problem (2.9) into a system of integral equations.

We recall that if $S_n$ is the fundamental solution of the Laplace operator and $S_{q,n}$ is its periodic analog, then $R_{q,n} = S_{q,n} - S_n$ in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ is analytic in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$, and we have that

$$
\Delta R_{q,n} = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \delta_{qz} - \frac{1}{|Q|_n}
$$

in the sense of distributions (see Theorem B.0.1 in Appendix B). Since

$$
R_{q,n}(x_1, \dots, -x_l, \dots, x_n) = R_{q,n}(x_1, \dots, x_l, \dots, x_n)
$$

for all $l \in \{1, \ldots, n\}$, we have that $R_{q,n}$ is an even function and

$$(D^\gamma R_{q,n})(0) = 0 \qquad \forall \gamma \in \mathbb{N}^n \quad \text{with } |\gamma| \text{ odd}, \tag{2.10}$$

where $(D^\gamma R_{q,n})(0)$ means the value of the derivative $D^\gamma R_{q,n}$ at the point 0, and, moreover,

$$(\partial_i \partial_j R_{q,n})(0) = 0 \qquad \forall i, j \in \{1, \ldots, n\} \quad \text{such that } i \neq j. \tag{2.11}$$

Also, since $R_{q,n}$ is even and $(\Delta R_{q,n})(0) = -1/|Q|_n$, one verifies that if $Q$ is an $n$-dimensional cube, *i.e.*, $q_{11} = q_{22} = \cdots = q_{nn}$, then

$$(\partial_i^2 R_{q,n})(0) = -\frac{1}{n|Q|_n} \qquad \forall i \in \{1, \ldots, n\}. \tag{2.12}$$

As shown in Section 1.6, we can convert problem (2.9) into an equivalent system of integral equations by means of the periodic simple layer potential. Now, let $j \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, n-1\}$. To provide an integral equation formulation of problem (2.9), we also define the map $M_j := (M_{j,1}, M_{j,2})$ from $]-\epsilon_0, \epsilon_0[\times \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$ by setting

$$\begin{aligned}
M_{j,1}[\epsilon, \theta^i, \theta^o](t) :=& \lambda^- \left(\frac{1}{2}\theta^o(t) + w_*[\partial\Omega, \theta^o](t) + \epsilon^{n-1}\Lambda_\nu[\epsilon, \theta^o](t)\right) \\
&- \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_*[\partial\Omega, \theta^i](t) + \epsilon^{n-1}\Lambda_\nu[\epsilon, \theta^i](t)\right) \\
&- f(t) + (\lambda^- - \lambda^+)(\nu_\Omega(t))_j, \\
M_{j,2}[\epsilon, \theta^i, \theta^o](t) :=& v^+[\partial\Omega, \theta^i](t) + \epsilon^{n-2}\Lambda[\epsilon, \theta^i](t) - \fint_{\partial\Omega} \left(v^+[\partial\Omega, \theta^i] + \epsilon^{n-2}\Lambda[\epsilon, \theta^i]\right) d\sigma \\
&- v^-[\partial\Omega, \theta^o](t) - \epsilon^{n-2}\Lambda[\epsilon, \theta^o](t) + \fint_{\partial\Omega} \left(v^-[\partial\Omega, \theta^o] + \epsilon^{n-2}\Lambda[\epsilon, \theta^o]\right) d\sigma \\
&- \epsilon^{m-1} g(t) + \epsilon^{m-1} \fint_{\partial\Omega} g \, d\sigma
\end{aligned}$$

for all $t \in \partial\Omega$ and for all $(\epsilon, \theta^i, \theta^o) \in ]-\epsilon_0, \epsilon_0[\times \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$.

In the following proposition, we convert problem (2.9) into a system of integral equations by means of the operator $M_j$.

**Proposition 2.2.1.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $j \in \{1, \ldots, n\}$. Let $m \in \{1, \ldots, n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $\epsilon \in ]0, \epsilon_0[$. Then problem (2.9) has a unique solution $(u_j^+[\epsilon], u_j^-[\epsilon])$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$. Moreover, such a pair is delivered by the formulas*

$$\begin{aligned}
u_j^+[\epsilon](x) =& v_q^+[\partial\Omega_{p,\epsilon}, \hat\theta_j^i[\epsilon]((\cdot - p)/\epsilon)](x) - \fint_{\partial\Omega_{p,\epsilon}} v_q^+[\partial\Omega_{p,\epsilon}, \hat\theta_j^i[\epsilon]((\cdot - p)/\epsilon)] \, d\sigma \\
&+ x_j - \fint_{\partial\Omega_{p,\epsilon}} y_j \, d\sigma_y + \frac{c}{|\partial\Omega_{p,\epsilon}|_{n-1}} \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
u_j^-[\epsilon](x) =& v_q^-[\partial\Omega_{p,\epsilon}, \hat\theta_j^o[\epsilon]((\cdot - p)/\epsilon)](x) - \fint_{\partial\Omega_{p,\epsilon}} v_q^-[\partial\Omega_{p,\epsilon}, \hat\theta_j^o[\epsilon]((\cdot - p)/\epsilon)] \, d\sigma \\
&- \epsilon^m \fint_{\partial\Omega_{p,\epsilon}} g((y-p)/\epsilon) \, d\sigma_y \\
&+ x_j - \fint_{\partial\Omega_{p,\epsilon}} y_j \, d\sigma_y + \frac{c}{|\partial\Omega_{p,\epsilon}|_{n-1}} \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,
\end{aligned} \tag{2.14}$$

*where $(\hat\theta_j^i[\epsilon], \hat\theta_j^o[\epsilon])$ denotes the unique solution $(\theta^i, \theta^o)$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of*

$$M_j[\epsilon, \theta^i, \theta^o] = 0.$$

*Proof.* We first note that one can convert non-periodic problem (2.9) into a periodic one (see, also, the proof of Proposition 1.6.1). To begin with, we introduce the following notation: $\mathbf{f}_j[f](t) := f(t) + (\lambda^+ - \lambda^-)(\nu_\Omega(t))_j$ for all $t \in \partial\Omega$, and $\mathbf{c}_j[\epsilon, c] := -\int_{\partial\Omega_{p,\epsilon}} y_j \, d\sigma_y + c$. Then if the pair $(\bar{u}^+[\epsilon], \bar{u}^-[\epsilon])$ solves the problem

$$\begin{cases} \Delta \bar{u}^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}], \\ \Delta \bar{u}^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-, \\ \bar{u}^+(x + q_{hh}e_h) = \bar{u}^+(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \quad \forall h \in \{1, \dots, n\}, \\ \bar{u}^-(x + q_{hh}e_h) = \bar{u}^-(x) & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \quad \forall h \in \{1, \dots, n\}, \\ \lambda^- \frac{\partial \bar{u}^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial \bar{u}^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = \mathbf{f}_j[f]((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \frac{1}{\epsilon^m}(\bar{u}^+(x) - \bar{u}^-(x)) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon}, \\ \int_{\partial\Omega_{p,\epsilon}} \bar{u}^+ \, d\sigma = \mathbf{c}_j[\epsilon, c], \end{cases} \tag{2.15}$$

a straightforward computation implies that the pair of functions delivered by the right-hand side of (2.13)-(2.14) is a solution of problem (2.9). Conversely, if $(u^+[\epsilon], u^-[\epsilon])$ solves problem (2.9), then one verifies that the pair of functions $(u^+_\sharp, u^-_\sharp)$ defined by

$$\begin{aligned} u^+_\sharp(x) \equiv & u^+_j[\epsilon](x) - x_j \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}], \\ u^-_\sharp(x) \equiv & u^-_j[\epsilon](x) - x_j \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-, \end{aligned}$$

solves problem (2.15). Consequently, there is a bijective correspondence between the solutions of problem (2.9) and those of problem (2.15). Therefore, Proposition 2.2.1 implies that problem (2.9) has a unique solution which we denote by $(u^+_j[\epsilon], u^-_j[\epsilon])$ and which is delivered by formulas (2.13)-(2.14). □

In order to investigate the asymptotic behavior of the $(k,j)$-entry $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ of the effective conductivity tensor as $\epsilon \to 0^+$, we need to study the functions $u^+_j[\epsilon]$ and $u^-_j[\epsilon]$ for $\epsilon$ close to the degenerate value 0. On the other hand, Proposition 2.2.1 tells us how to represent $u^+_j[\epsilon]$ and $u^-_j[\epsilon]$ in terms of the densities $\hat{\theta}^i_j[\epsilon]$ and $\hat{\theta}^o_j[\epsilon]$. Therefore, the analysis of $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ for $\epsilon$ close to 0 can be deduced by the asymptotic behavior of $\hat{\theta}^i_j[\epsilon]$ and $\hat{\theta}^o_j[\epsilon]$. Accordingly, as a first step, in the following theorem we present a regularity result for $\hat{\theta}^i_j[\epsilon]$ and $\hat{\theta}^o_j[\epsilon]$ (see Theorem 1.4.3 and Theorem 1.6.2).

**Proposition 2.2.2.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $j \in \{1, \dots, n\}$. Let $m \in \{1, \dots, n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $\epsilon \in ]0, \epsilon_0[$. The following statements hold:*

*(i) $M_j$ is a real analytic map from $]-\epsilon_0, \epsilon_0[\times(C^{0,\alpha}(\partial\Omega)_0)^2$ to $C^{0,\alpha}(\partial\Omega)_0 \times C^{1,\alpha}(\partial\Omega)_0$.*

*(ii) There exists a unique pair $(\bar{\theta}^i_j, \bar{\theta}^o_j) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ satisfying the following "limiting system" of integral equations*

$$M_j\left[0, \bar{\theta}^i_j, \bar{\theta}^o_j\right] = 0.$$

*(iii) There exist $\epsilon_1 \in ]0, \epsilon_0[$ and a real analytic map $\epsilon \mapsto (\Theta^i_j[\epsilon], \Theta^o_j[\epsilon])$ from $]-\epsilon_1, \epsilon_1[$ to $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$M_j\left[\epsilon, \Theta^i_j[\epsilon], \Theta^o_j[\epsilon]\right] = 0 \qquad \forall \epsilon \in ]-\epsilon_1, \epsilon_1[\,. \tag{2.16}$$

*In particular,*

$$(\Theta^i_j[\epsilon], \Theta^o_j[\epsilon]) = (\hat{\theta}^i_j[\epsilon], \hat{\theta}^o_j[\epsilon]) \quad \forall \epsilon \in ]0, \epsilon_1[ \quad \text{and} \quad (\Theta^i_j[0], \Theta^o_j[0]) = (\bar{\theta}^i_j, \bar{\theta}^o_j),$$

*where the pair $(\hat{\theta}^i_j[\epsilon], \hat{\theta}^o_j[\epsilon])$ is defined in Proposition 2.2.1.*

*Proof.* By standard properties of integral operators with real analytic kernels and with no singularity, and by classical mapping properties of layer potentials (see Lanza de Cristoforis and Musolino [82, Sec. 4], Miranda [106], Lanza de Cristoforis and Rossi [87, Thm. 3.1]), one verifies that $M_j$ is a real analytic map from $]-\epsilon_0,\epsilon_0[\times \left(C^{0,\alpha}(\partial\Omega)_0\right)^2$ to $C^{0,\alpha}(\partial\Omega)_0\times C^{1,\alpha}(\partial\Omega)_0$, and statement $(i)$ is proven.

To verify $(ii)$, we note that the system of integral equations $M_j\left[0,\bar\theta^i_j,\bar\theta^o_j\right]=0$ can be written as

$$\begin{aligned}
\lambda^-\left(\frac{1}{2}\bar\theta^o(t)+w_*[\partial\Omega,\bar\theta^o](t)\right)-\lambda^+\left(-\frac{1}{2}\bar\theta^i(t)+w_*[\partial\Omega,\bar\theta^i](t)\right)&\\
=f(t)-(\lambda^--\lambda^+)(\nu_\Omega(t))_j&,\\
v^+[\partial\Omega,\bar\theta^i](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\bar\theta^i]d\sigma-v^-[\partial\Omega,\bar\theta^o](t)+\fint_{\partial\Omega}v^-[\partial\Omega,\bar\theta^o]d\sigma&\\
=\delta_{1,m}\left(g(t)-\fint_{\partial\Omega}g\,d\sigma\right)&
\end{aligned}$$

for all $t\in\partial\Omega$. Then, by Proposition 1.3.2, the above "limiting system" has a unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$.

Finally, the validity of $(iii)$ follows from Theorem 1.4.3. □

The real analyticity result of Proposition 2.2.2$(iii)$ implies that there exists $\tilde\epsilon\in]0,\epsilon_1[$ small enough such that $\Theta^i_j[\epsilon]$ and $\Theta^o_j[\epsilon]$ can be expanded into power series of $\epsilon$, *i.e.*,

$$\Theta^i_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^i_{j,h}}{h!}\epsilon^h,\qquad \Theta^o_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^o_{j,h}}{h!}\epsilon^h\,, \tag{2.17}$$

for some $\{\theta^i_{j,h}\}_{h\in\mathbb{N}}$, $\{\theta^o_{j,h}\}_{h\in\mathbb{N}}$ and for all $\epsilon\in]-\tilde\epsilon,\tilde\epsilon[$. Moreover,

$$\theta^i_{j,h}=\left(\partial^h_\epsilon\Theta^i_j[\epsilon]\right)_{|\epsilon=0}\,,\qquad \theta^o_{j,h}=\left(\partial^h_\epsilon\Theta^o_j[\epsilon]\right)_{|\epsilon=0}\,,$$

for all $h\in\mathbb{N}$. As a consequence,

$$\hat\theta^i_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^i_{j,h}}{h!}\epsilon^h,\qquad \hat\theta^o_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^o_{j,h}}{h!}\epsilon^h\,,$$

for all $\epsilon\in]0,\tilde\epsilon[$. Therefore, in order to obtain a power series expansion for $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ for $\epsilon$ close to 0, we want to exploit the expansion of $(\hat\theta^i_j[\epsilon],\hat\theta^o_j[\epsilon])$ (or equivalently of $(\Theta^i_j[\epsilon],\Theta^o_j[\epsilon])$). Since the coefficients of the expansions in (2.17) are given by the derivatives with respect to $\epsilon$ of $\Theta^i_j[\epsilon]$ and $\Theta^o_j[\epsilon]$, we would like to obtain some equations identifying $\left(\partial^h_\epsilon\Theta^i_j[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^h_\epsilon\Theta^o_j[\epsilon]\right)_{|\epsilon=0}$. The plan is to obtain such equations by differentiating equality (2.16) with respect to $\epsilon$, which then leads to

$$\partial^h_\epsilon\left(M_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]\right)=0\qquad\forall\epsilon\in]-\epsilon_1,\epsilon_1[\,,\qquad\forall h\in\mathbb{N}\,. \tag{2.18}$$

Then by taking $\epsilon=0$ in (2.18), we will obtain integral equations identifying $\left(\partial^h_\epsilon\Theta^i_j[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^h_\epsilon\Theta^o_j[\epsilon]\right)_{|\epsilon=0}$.

In order to compute the derivative in (2.18), we recall that

$$\partial^h_\epsilon\left(F[\epsilon](x)\cdot G[\epsilon](x)\right)=\sum_{l=0}^{h}\binom{h}{l}\partial^{h-l}_\epsilon\left(F[\epsilon](x)\right)\ \partial^l_\epsilon(G[\epsilon](x)) \tag{2.19}$$

and that

$$\partial_\epsilon^h(H(\epsilon x)) = \sum_{|\alpha|=h} \frac{h!}{\alpha!} x^\alpha (D^\alpha H)(\epsilon x) \tag{2.20}$$

for all $h \in \mathbb{N}$, $\epsilon \in \mathbb{R}$, $x \in \mathbb{R}^n$, and for smooth enough functions $F$, $G$, and $H$.

We begin with a preliminary lemma.

**Lemma 2.2.3.** *Let $\epsilon_0' \in ]0, \epsilon_0[$ . Let $\epsilon \mapsto \theta[\epsilon]$ be a real analytic map from $]-\epsilon_0', \epsilon_0'[$ to $C^{0,\alpha}(\partial\Omega)_0$. Possibly shrinking $\epsilon_0'$, assume that $\{\theta_h\}_{h\in\mathbb{N}}$ is a sequence in $C^{0,\alpha}(\partial\Omega)_0$ such that*

$$\theta[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta_h}{h!}\epsilon^h \qquad \forall \epsilon \in ]-\epsilon_0', \epsilon_0'[\,,$$

*where the series converges uniformly for $\epsilon \in ]-\epsilon_0', \epsilon_0'[$ in $C^{0,\alpha}(\partial\Omega)_0$. Then the following statements hold.*

*(i) The map $\epsilon \mapsto \Lambda[\epsilon, \theta[\epsilon]]$ from $]-\epsilon_0', \epsilon_0'[$ to $C^{1,\alpha}(\partial\Omega)$ is real analytic and we have*

$$\Lambda[0, \theta[0]] = 0, \qquad \left(\partial_\epsilon^h \left(\epsilon^{n-2}\Lambda[\epsilon, \theta[\epsilon]]\right)\right)_{|\epsilon=0} = 0 \qquad \forall h \in \{1, \dots, n-1\}, \tag{2.21}$$

*and*

$$\begin{aligned}&\left(\partial_\epsilon^h \left(\epsilon^{n-2}\Lambda[\epsilon, \theta[\epsilon]]\right)(t)\right)_{|\epsilon=0} \\ &= \sum_{l=2}^{h-n+2} \sum_{|\alpha|=l} \frac{h!}{\alpha!(h-n+2-l)!} (D^\alpha R_{q,n})(0) \int_{\partial\Omega} (t-s)^\alpha \theta_{h-n+2-l}(s)d\sigma_s\end{aligned} \tag{2.22}$$

*for all $t \in \partial\Omega$ and for all $h \in \mathbb{N} \setminus \{0, \dots, n-1\}$,*

*(ii) The map $\epsilon \mapsto \Lambda_\nu[\epsilon, \theta[\epsilon]]$ from $]-\epsilon_0', \epsilon_0'[$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic and we have*

$$\Lambda_\nu[0, \theta[0]] = 0, \qquad \left(\partial_\epsilon^h \left(\epsilon^{n-1}\Lambda_\nu[\epsilon, \theta[\epsilon]]\right)\right)_{|\epsilon=0} = 0 \qquad \forall h \in \{1, \dots, n-1\}, \tag{2.23}$$

*and*

$$\begin{aligned}&\left(\partial_\epsilon^h \left(\epsilon^{n-1}\Lambda_\nu[\epsilon, \theta[\epsilon]]\right)(t)\right)_{|\epsilon=0} \\ &= \sum_{l=1}^{h-n+1} \sum_{|\alpha|=l} \frac{h!}{\alpha!(h-n+1-l)!} (D^\alpha D R_{q,n})(0)\nu_\Omega(t) \int_{\partial\Omega} (t-s)^\alpha \theta_{h-n+1-l}(s)d\sigma_s\end{aligned} \tag{2.24}$$

*for all $t \in \partial\Omega$ and for all $h \in \mathbb{N} \setminus \{0, \dots, n-1\}$. Moreover,*

$$\int_{\partial\Omega} \partial_\epsilon^h \left(\epsilon \Lambda_\nu[\epsilon, \theta[\epsilon]]\right) d\sigma = 0 \quad \forall \epsilon \in ]-\epsilon_0', \epsilon_0'[, \quad \forall h \in \mathbb{N}\,. \tag{2.25}$$

*Proof.* We first consider statement (i). The analyticity of the map

$$\epsilon \mapsto \Lambda[\epsilon, \theta[\epsilon]]$$

from $]-\epsilon_0', \epsilon_0'[$ to $C^{1,\alpha}(\partial\Omega)$ follows by the analyticity of $\epsilon \mapsto \theta[\epsilon]$ and by standard properties of integral operators with real analytic kernels and with no singularity (see, e.g., Lanza de Cristoforis and Musolino [82, Sec. 4]). Clearly,

$$\Lambda[0, \theta[0]](t) = R_{q,n}(0) \int_{\partial\Omega} \theta[0](s)d\sigma_s = 0 \qquad \forall t \in \partial\Omega\,.$$

Using equalities (2.19)–(2.20), one verifies the validity of equality (2.22). By a straightforward computation, one verifies the second equality in (2.21) for all $h \leq n-1$. In particular, if $h = n-2$ it is verified by using the equality $\Lambda[0,\theta[0]] = 0$. If $h = n-1$, instead, by the fact that $\theta_1 \in C^{0,\alpha}(\partial\Omega)_0$ and by equality (2.10), one deduces that

$$\begin{aligned}\left(\partial_\epsilon^{n-1}\left(\epsilon^{n-2}\Lambda[\epsilon,\theta[\epsilon]]\right)(t)\right)_{|\epsilon=0} &= (n-1)!\,(\partial_\epsilon\Lambda[\epsilon,\theta[\epsilon]](t))_{|\epsilon=0}\\ &= (n-1)!\sum_{l=0}^{1}\sum_{|\alpha|=l}(D^\alpha R_{q,n})(0)\int_{\partial\Omega}(t-s)^\alpha\theta_{1-l}(s)d\sigma_s = 0 \qquad \forall t\in\partial\Omega\,,\end{aligned}$$

and, accordingly, equality (2.22) follows.

We now turn to prove (ii). Again, by the analyticity of $\epsilon \mapsto \theta[\epsilon]$ and by standard properties of integral operators with real analytic kernels and with no singularity (see, e.g., Lanza de Cristoforis and Musolino [82, Sec. 4]), we deduce the analyticity of the map

$$\epsilon \mapsto \Lambda_\nu[\epsilon,\theta[\epsilon]]$$

from $]-\epsilon_0',\epsilon_0'[$ to $C^{0,\alpha}(\partial\Omega)$. Moreover, by standard calculus in Banach spaces and by formulas (2.19)–(2.20), we deduce the validity of (2.24). Clearly,

$$\Lambda_\nu[0,\theta[0]](t) = DR_{q,2}(0)\nu_\Omega(t)\int_{\partial\Omega}\theta[0](s)d\sigma_s = 0 \qquad \forall t\in\partial\Omega,$$

which also implies

$$\left(\partial_\epsilon^{n-1}\left(\epsilon^{n-1}\Lambda_\nu[\epsilon,\theta[\epsilon]]\right)(t)\right)_{|\epsilon=0} = 0 \qquad \forall t\in\partial\Omega.$$

Also, by a straightforward computation, one can verify the validity of the second equality in (2.23) for all $h\in\{1,\dots,n-2\}$.

It remains to prove (2.25), we note that the map $\epsilon\mapsto H[\epsilon]$ from $]-\epsilon_0',\epsilon_0'[$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$ defined by the equality

$$H[\epsilon](t) \equiv \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta[\epsilon](s)d\sigma_s \qquad \forall t\in\mathrm{cl}\Omega,$$

is real analytic. Moreover,

$$\begin{aligned}\Delta_t\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta[\epsilon](s)d\sigma_s =& \epsilon^2\int_{\partial\Omega}(\Delta R_{q,n})(\epsilon(t-s))\theta[\epsilon](s)d\sigma_s\\ =& -\frac{\epsilon^2}{|Q|_n}\int_{\partial\Omega}\theta[\epsilon](s)d\sigma_s = 0 \qquad \forall t\in\mathrm{cl}\Omega,\end{aligned}$$

and thus $H[\epsilon]$ is harmonic in $\Omega$ for all $\epsilon\in]-\epsilon_0',\epsilon_0'[$. Therefore,

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}H[\epsilon]\,d\sigma = 0 \qquad \forall\epsilon\in]-\epsilon_0',\epsilon_0'[\,.$$

On the other hand, a straightforward computation shows that

$$\frac{\partial}{\partial\nu_\Omega}H[\epsilon] = \epsilon\Lambda_\nu[\epsilon,\theta[\epsilon]] \qquad \forall\epsilon\in]-\epsilon_0',\epsilon_0'[\,.$$

As a consequence,

$$\int_{\partial\Omega}\epsilon\Lambda_\nu[\epsilon,\theta[\epsilon]]\,d\sigma = 0 \qquad \forall\epsilon\in]-\epsilon_0',\epsilon_0'[. \tag{2.26}$$

By differentiating equality (2.26) with respect to $\epsilon$, we deduce that

$$0 = \partial_\epsilon^h\left(\int_{\partial\Omega}\epsilon\Lambda_\nu[\epsilon,\theta[\epsilon]]\,d\sigma\right) = \int_{\partial\Omega}\partial_\epsilon^h\left(\epsilon\Lambda_\nu[\epsilon,\theta[\epsilon]]\right)\,d\sigma \qquad \forall\epsilon\in]-\epsilon_0',\epsilon_0'[.$$

Thus the proof is complete. □

In view of Lemma 2.2.3, we find convenient to introduce the following notation:

$$\Lambda^h[\theta_0, \dots, \theta_{h-n}](t) := \sum_{l=2}^{h-n+2} \sum_{|\alpha|=l} \frac{h!}{\alpha!(h-n+2-l)!}(D^\alpha R_{q,n})(0) \int_{\partial\Omega} (t-s)^\alpha \theta_{h-n+2-l}(s) d\sigma_s$$

for all $(\theta_0, \dots, \theta_{h-n}) \in \left(C^{0,\alpha}(\partial\Omega)_0\right)^{h-n+1}$, all $t \in \partial\Omega$, and all $h \in \mathbb{N} \setminus \{0, \dots, n-1\}$, and

$$\begin{aligned}\Lambda^h_\nu&[\theta_0, \dots, \theta_{h-n}](t) \\ &:= \sum_{l=1}^{h-n+1} \sum_{|\alpha|=l} \frac{h!}{\alpha!(h-n+1-l)!}(D^\alpha D R_{q,n})(0)\nu_\Omega(t) \int_{\partial\Omega} (t-s)^\alpha \theta_{h-n+1-l}(s) d\sigma_s\end{aligned}$$

for all $(\theta_0, \dots, \theta_{h-n}) \in \left(C^{0,\alpha}(\partial\Omega)_0\right)^{h-n+1}$, and all $t \in \partial\Omega$, and all $h \in \mathbb{N} \setminus \{0, \dots, n-1\}$. We observe that by (2.10), $\Lambda^h[\theta_0, \dots, \theta_{h-n}]$ and $\Lambda^h_\nu[\theta_0, \dots, \theta_{h-n}]$ depend only on the $\theta_k$'s with $k$ odd if $h-n$ is odd, and only on the $\theta_k$'s with $k$ even if $h-n$ is even. As a consequence, we have the following.

**Lemma 2.2.4.** *Let $h \in \mathbb{N} \setminus \{0, \dots, n-1\}$ be such that $h-n$ is odd. If $\theta_0, \dots, \theta_{h-n} \in C^{0,\alpha}(\partial\Omega)_0$ are such that $\theta_j = 0$ for all odd $j \in \{1, \dots, h-n\}$ then $\Lambda^h[\theta_0, \dots, \theta_{h-n}](t) = 0$ and $\Lambda^h_\nu[\theta_0, \dots, \theta_{h-n}](t) = 0$ for all $t \in \partial\Omega$.*

### 2.2.2 Power series expansions of two auxiliary functions

In order to compute the asymptotic expansion of the effective conductivity, we start with the following proposition where we identify the coefficients of the power series expansions of $\Theta^i_j[\epsilon]$ and of $\Theta^o_j[\epsilon]$ in terms of the solutions of systems of integral equations.

**Proposition 2.2.5.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $j \in \{1, \dots, n\}$. Let $m \in \{1, \dots, n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $\epsilon_1$ and $\epsilon \mapsto (\Theta^i_j[\epsilon], \Theta^o_j[\epsilon])$ be as in Proposition 2.2.2(iii). Then there exist $\epsilon_2 \in ]0, \epsilon_1[$ and a sequence $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$\Theta^i_j[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^i_{j,h}}{h!}\epsilon^h \qquad \textit{and} \qquad \Theta^o_j[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^o_{j,h}}{h!}\epsilon^h \qquad \forall \epsilon \in ]-\epsilon_2, \epsilon_2[, \tag{2.27}$$

*where the two series converge uniformly for $\epsilon \in ]-\epsilon_2, \epsilon_2[$ in $C^{0,\alpha}(\partial\Omega)_0$. Moreover, the following statements hold.*

*(i) The pair of functions $(\theta^i_{j,0}, \theta^o_{j,0})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$\begin{aligned}\lambda^-\left(\frac{1}{2}\theta^o_{j,0}(t) + w_*[\partial\Omega, \theta^o_{j,0}](t)\right) - \lambda^+\left(-\frac{1}{2}\theta^i_{j,0}(t) + w_*[\partial\Omega, \theta^i_{j,0}](t)\right)& \\ = f(t) + (\lambda^+ - \lambda^-)(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega&,\end{aligned} \tag{2.28}$$

$$\begin{aligned}v^+[\partial\Omega, \theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}]d\sigma - v^-[\partial\Omega, \theta^o_{j,0}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}]d\sigma& \\ = \delta_{1,m}\left(g(t) - \fint_{\partial\Omega} g d\sigma\right) \qquad \forall t \in \partial\Omega&.\end{aligned} \tag{2.29}$$

*(ii) We have $(\theta^i_{j,h}, \theta^o_{j,h}) = (0,0)$ for all $h \in \{1, \dots, n-1\} \setminus \{m-1\}$. Moreover, if $m > 1$ then the pair of functions $(\theta^i_{j,m-1}, \theta^o_{j,m-1})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the*

*following system of integral equations*

$$\lambda^-\left(\frac{1}{2}\theta^o_{j,m-1}(t)+w_*[\partial\Omega,\theta^o_{j,m-1}](t)\right) \\ -\lambda^+\left(-\frac{1}{2}\theta^i_{j,m-1}(t)+w_*[\partial\Omega,\theta^i_{j,m-1}](t)\right)=0 \quad \forall t\in\partial\Omega,$$

$$v^+[\partial\Omega,\theta^i_{j,m-1}](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,m-1}]d\sigma-v^-[\partial\Omega,\theta^o_{j,m-1}](t) \\ +\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,m-1}]d\sigma=(m-1)!\left(g(t)-\fint_{\partial\Omega}g\,d\sigma\right) \quad \forall t\in\partial\Omega.$$

*(iii) For all $h\geq n$ the pair of functions $(\theta^i_{j,h},\theta^o_{j,h})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations which involves $\{(\theta^i_{j,l},\theta^o_{j,l})\}_{l=0}^{h-n}$*

$$\begin{aligned}\lambda^-\left(\frac{1}{2}\theta^o_{j,h}(t)+w_*[\partial\Omega,\theta^o_{j,h}](t)\right)-\lambda^+\left(-\frac{1}{2}\theta^i_{j,h}(t)+w_*[\partial\Omega,\theta^i_{j,h}](t)\right)\\ =\lambda^+\Lambda^h_\nu[\theta^i_{j,0},\ldots,\theta^i_{j,h-n}](t)-\lambda^-\Lambda^h_\nu[\theta^o_{j,0},\ldots,\theta^o_{j,h-n}](t) \quad \forall t\in\partial\Omega,\end{aligned} \tag{2.30}$$

$$\begin{aligned}&v^+[\partial\Omega,\theta^i_{j,h}](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,h}]d\sigma-v^-[\partial\Omega,\theta^o_{j,h}](t)+\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,h}]d\sigma\\ &=-\Lambda^h[\theta^i_{j,0},\ldots,\theta^i_{j,h-n}](t)+\fint_{\partial\Omega}\Lambda^h[\theta^i_{j,0},\ldots,\theta^i_{j,h-n}]d\sigma\\ &\quad+\Lambda^h[\theta^o_{j,0},\ldots,\theta^o_{j,h-n}](t)-\fint_{\partial\Omega}\Lambda^h[\theta^o_{j,0},\ldots,\theta^o_{j,h-n}]d\sigma \qquad \forall t\in\partial\Omega.\end{aligned} \tag{2.31}$$

*Proof.* We first note that Proposition 2.2.2(*iii*) implies the existence of $\epsilon_2\in]0,\epsilon_1[$ and a sequence $\{(\theta^i_{j,h},\theta^o_{j,h})\}_{h\in\mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that representation (2.27) holds. By standard properties of real analytic maps, one has

$$(\theta^i_{j,h},\theta^o_{j,h})=(\partial^h_\epsilon\Theta^i_j[0],\partial^h_\epsilon\Theta^o_j[0]) \qquad \forall h\in\mathbb{N}.$$

In order to determine $(\theta^i_{j,0},\theta^o_{j,0})$, we note that, by taking $\epsilon=0$ and keeping the first equality in (2.21) in mind, equality (2.16) can be immediately written as the system of integral equations (2.28)–(2.29). The existence and uniqueness of solution for this system are then ensured by Proposition 2.2.2(*ii*).

Now, we turn to prove statements $(ii)-(iii)$. Since equality (2.16) holds for all $\epsilon\in]-\epsilon_2,\epsilon_2[$ (see Proposition 2.2.2(*iii*)), the map $\epsilon\mapsto M_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]$ has derivatives which are equal to zero, *i.e.*,

$$\partial^h_\epsilon\left(M_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]\right)=0 \qquad \forall\epsilon\in]-\epsilon_2,\epsilon_2[, \quad \forall h\in\mathbb{N}\setminus\{0\}.$$

Then, a straightforward calculation shows that

$$\begin{aligned}&\partial^h_\epsilon\left(M_{j,1}[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]]\right)(t)\\ &=\lambda^-\left(\frac{1}{2}\partial^h_\epsilon\Theta^o_j[\epsilon](t)+w_*[\partial\Omega,\partial^h_\epsilon\Theta^o_j[\epsilon]](t)+\partial^h_\epsilon\left(\epsilon^{n-1}\Lambda_\nu[\epsilon,\Theta^o_j[\epsilon]](t)\right)\right)\\ &\quad-\lambda^+\left(-\frac{1}{2}\partial^h_\epsilon\Theta^i_j[\epsilon](t)+w_*[\partial\Omega,\partial^h_\epsilon\Theta^i_j[\epsilon]](t)+\partial^h_\epsilon\left(\epsilon^{n-1}\Lambda_\nu[\epsilon,\Theta^i_j[\epsilon]](t)\right)\right)=0,\end{aligned} \tag{2.32}$$

$$\begin{aligned}\partial_\epsilon^h \left(M_{j,2}[\epsilon, \Theta_j^i[\epsilon], \Theta_j^o[\epsilon]]\right)(t) =& v^+[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]](t) - \int_{\partial\Omega} v^+[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]]d\sigma \\ & - v^-[\partial\Omega, \partial_\epsilon^h \Theta_j^o[\epsilon]](t) + \int_{\partial\Omega} v^-[\partial\Omega, \partial_\epsilon^h \Theta_j^o[\epsilon]]d\sigma \\ & + \partial_\epsilon^h \left(\epsilon^{n-2}\left(\Lambda[\epsilon, \Theta_j^i[\epsilon]](t) - \int_{\partial\Omega} \Lambda[\epsilon, \Theta_j^i[\epsilon]]d\sigma\right)\right) \\ & - \partial_\epsilon^h \left(\epsilon^{n-2}\left(\Lambda[\epsilon, \Theta_j^o[\epsilon]](t) - \int_{\partial\Omega} \Lambda[\epsilon, \Theta_j^o[\epsilon]]d\sigma\right)\right) \\ & - \partial_\epsilon^h \left(\epsilon^{m-1}\left(g(t) - \int_{\partial\Omega} g d\sigma\right)\right) = 0\end{aligned} \tag{2.33}$$

for all $t \in \partial\Omega$ and for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$, and for all $h \in \mathbb{N} \setminus \{0\}$.

Now, taking equalities (2.21) and (2.23) into account, one verifies that for all natural $h \in \{1, \dots, n-1\}$, the system of integral equations (2.32)–(2.33) takes the following form

$$\lambda^- \left(\frac{1}{2}\theta_{j,h}^o(t) + w_*[\partial\Omega, \theta_{j,h}^o](t)\right) - \lambda^+ \left(-\frac{1}{2}\theta_{j,h}^i(t)(t) + w_*[\partial\Omega, \theta_{j,h}^i(t)](t)\right) = 0 \tag{2.34}$$

$$\begin{aligned}v^+[\partial\Omega, \theta_{j,h}^i](t) - \int_{\partial\Omega} v^+[\partial\Omega, \theta_{j,h}^i]d\sigma - v^-[\partial\Omega, \theta_{j,h}^o](t) + \int_{\partial\Omega} v^-[\partial\Omega, \theta_{j,h}^o]d\sigma \\ = \delta_{h,m-1}(m-1)!\left(g(t) - \int_{\partial\Omega} g d\sigma\right)\end{aligned} \tag{2.35}$$

for all $t \in \partial\Omega$. Proposition 1.3.2 implies that system (2.34)-(2.35) has only a unique solution for all $h \in \{1, \dots, n-1\}$. Moreover, $(\theta_{j,h}^i, \theta_{j,h}^o) = (0,0)$ for all $h \in \{1, \dots, n-1\} \setminus \{m-1\}$. Thus, the validity statement $(ii)$ follows.

Finally, we consider the case $h \geq n$. First we note that equations (2.32) and (2.33) take the form (2.30) and (2.31), respectively. Then, by observing that the right-hand sides of (2.30) and (2.31) belong to $C^{0,\alpha}(\partial\Omega)_0$ and $C^{1,\alpha}(\partial\Omega)_0$, respectively, Proposition 1.3.2 again ensures the existence and uniqueness of the solution $(\theta_{j,h}^i, \theta_{j,h}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ for the system (2.30)–(2.31) for all $h \geq n$. Thus, the proof is complete. □

We complete the section with two remarks which we will use in the sequel.

**Remark 2.2.6.** *We observe that the pair* $(\theta_{j,0}^i, \theta_{j,0}^o)$ *coincides with the pair* $(\bar{\theta}_j^i, \bar{\theta}_j^o)$ *(see Proposition 2.2.2(ii)), or, in others words, the pair of the first coefficients in the series expansions* (2.27) *is the unique solution of the "limiting system" of integral equations* $M_j\left[0, \theta^i, \theta^o\right] = 0$.

**Remark 2.2.7.** *By Lemma 2.2.4 and Proposition 2.2.5, we observe that if* $m$ *is odd and* $n$ *is even then* $(\theta_{j,h}^i, \theta_{j,h}^o) = (0,0)$ *for all odd* $h$ *and, as a consequence,*

$$\Theta_j^i[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta_{j,2h}^i}{(2h)!}\epsilon^{2h} \qquad \textit{and} \qquad \Theta_j^o[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta_{j,2h}^o}{(2h)!}\epsilon^{2h} \qquad \forall \epsilon \in ]-\epsilon_2, \epsilon_2[\,.$$

### 2.2.3 A functional analytic representation theorem for the effective conductivity

Definition 2.1.1 tells us how to represent $\lambda_{kj}^{\text{eff}}[\epsilon]$ in terms of the solution $(u_j^+[\epsilon], u_j^-[\epsilon])$ of problem (2.9). Thus, in order to study the asymptotic behavior of the effective conductivity, we find it convenient to study the asymptotic behavior of $u_j^+[\epsilon]$ and $u_j^-[\epsilon]$ first.

In the following theorem we show that $u_j^+[\epsilon]$ and $u_j^-[\epsilon]$ can be represented in terms of real analytic maps.

**Theorem 2.2.8.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $j \in \{1, \ldots, n\}$. Let $m \in \{1, \ldots, n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $(u_j^+[\epsilon], u_j^-[\epsilon])$ be the unique solution of problem (2.9) for all $\epsilon \in ]0, \epsilon_0[$. Let $\epsilon_1$ and $\epsilon \mapsto (\Theta_j^i[\epsilon], \Theta_j^o[\epsilon])$ be as in Proposition 2.2.2(iii). Then the following statements hold.*

*(i) Let $\mathbf{U}_j^+$ be the map from $]-\epsilon_1, \epsilon_1[$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\mathbf{U}_j^+[\epsilon](t) :=& v^+ \left[\partial\Omega, \Theta_j^i[\epsilon]\right](t) + \epsilon^{n-2}\Lambda[\epsilon, \Theta_j^i[\epsilon]](t) \\ &- \fint_{\partial\Omega} \Big( v^+ \left[\partial\Omega, \Theta_j^i[\epsilon]\right] + \epsilon^{n-2}\Lambda[\epsilon, \Theta_j^i[\epsilon]] \Big) d\sigma + t_j - \fint_{\partial\Omega} y_j d\sigma_y\end{aligned} \tag{2.36}$$

*for all $t \in \partial\Omega$ and for all $\epsilon \in ]-\epsilon_1, \epsilon_1[$. Then $\mathbf{U}_j^+$ is real analytic and*

$$u_j^+[\epsilon](p + \epsilon t) = \epsilon \mathbf{U}_j^+[\epsilon](t) + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c \qquad \forall t \in \partial\Omega, \tag{2.37}$$

*for all $\epsilon \in ]0, \epsilon_1[$.*

*(ii) Let $\mathbf{C}_j^-$ and $\mathbf{V}_j^-$ be the maps from $]-\epsilon_1, \epsilon_1[$ to $\mathbb{R}$ and to $C^{1,\alpha}(\partial\Omega)$, respectively, defined by*

$$\mathbf{C}_j^-[\epsilon] := -\fint_{\partial\Omega} \Big( v^- \left[\partial\Omega, \Theta_j^o[\epsilon]\right] + \epsilon^{n-2}\Lambda[\epsilon, \Theta_j^o[\epsilon]] \Big) d\sigma - \epsilon^{m-1} \fint_{\partial\Omega} g d\sigma - \fint_{\partial\Omega} y_j d\sigma_y$$

*and by*

$$\mathbf{V}_j^-[\epsilon](t) := v^-[\partial\Omega, \Theta_j^o[\epsilon]](t) + \epsilon^{n-2}\Lambda[\epsilon, \Theta_j^o[\epsilon]](t) + t_j \qquad \forall t \in \partial\Omega, \tag{2.38}$$

*for all $\epsilon \in ]-\epsilon_1, \epsilon_1[$. Then $\mathbf{C}_j^-$ and $\mathbf{V}_j^-$ are real analytic and*

$$u_j^-[\epsilon](p + \epsilon t) = \epsilon \mathbf{C}_j^-[\epsilon] + \epsilon \mathbf{V}_j^-[\epsilon](t) + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c \qquad \forall t \in \partial\Omega, \tag{2.39}$$

*for all $\epsilon \in ]0, \epsilon_1[$.*

*Proof.* By simple computations based on the rule of change of variables in integrals and by equality (2.13), we have

$$\begin{aligned}u_j^+[\epsilon](p + \epsilon t) =& \epsilon v^+ \left[\partial\Omega, \Theta_j^i[\epsilon]\right](t) + \epsilon^{n-1}\Lambda[\epsilon, \Theta_j^i[\epsilon]](t) \\ &- \fint_{\partial\Omega} \Big( \epsilon v^+ \left[\partial\Omega, \Theta_j^i[\epsilon]\right] + \epsilon^{n-1}\Lambda[\epsilon, \Theta_j^i[\epsilon]] \Big) d\sigma + t_j - \fint_{\partial\Omega} y_j d\sigma_y + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c\end{aligned}$$

for all $t \in \partial\Omega$ and for all $\epsilon \in ]0, \epsilon_1[$. Then by Theorem 1.5.1, one shows that $\mathbf{U}_j^+$ is real analytic on $]-\epsilon_1, \epsilon_1[$, and, by the definition of $\mathbf{U}_j^+$, the validity of statement $(i)$ follows.

Now we consider statement $(ii)$. As we have done above, by equality (2.14), we have

$$\begin{aligned}u_j^-[\epsilon](p + \epsilon t) =& \epsilon v^-[\partial\Omega, \Theta_j^o[\epsilon]](t) + \epsilon^{n-1}\Lambda[\epsilon, \Theta_j^o[\epsilon]](t) \\ &- \fint_{\partial\Omega} \Big( \epsilon v^- \left[\partial\Omega, \Theta_j^o[\epsilon]\right] + \epsilon^{n-1}\Lambda[\epsilon, \Theta_j^o[\epsilon]] \Big) d\sigma \\ &- \epsilon^m \fint_{\partial\Omega} g d\sigma + \epsilon t_j - \epsilon \fint_{\partial\Omega} y_j d\sigma_y + \frac{\epsilon^{1-n}}{|\partial\Omega|_{n-1}} c \qquad \forall t \in \partial\Omega\,.\end{aligned}$$

Then by arguing as in the proof of Theorem 1.5.2, one verifies that $\mathbf{C}_j^-$ and $\mathbf{V}_j^-$ are real analytic on $]-\epsilon_1, \epsilon_1[$ and the validity of equality (2.39). $\square$

Equalities (2.37) and (2.39) allow us to describe suitable restrictions of $u_j^+[\epsilon]$ and $u_j^-[\epsilon]$ in terms of real analytic maps. As a consequence, they can be exploited to obtain an analogous representation for $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$. We do so in the following theorem.

**Theorem 2.2.9.** *Let the assumptions of Theorem 2.2.8 hold. Let $k \in \{1,\dots,n\}$. Then there exists a real analytic function $\mathbf{\Lambda}_{kj}$ from $]-\epsilon_1,\epsilon_1[$ to $\mathbb{R}$ such that*

$$\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^-\delta_{k,j} + \epsilon^n \mathbf{\Lambda}_{kj}[\epsilon] \tag{2.40}$$

*for all $\epsilon \in ]0,\epsilon_1[$.*

*Proof.* We first note that if $\epsilon \in ]0,\epsilon_1[$ then, by the Divergence Theorem, we have

$$\int_{\Omega_{p,\epsilon}} \frac{\partial u_j^+[\epsilon](x)}{\partial x_k}\,dx = \int_{\partial\Omega_{p,\epsilon}} u_j^+[\epsilon](x)(\nu_{\Omega_{p,\epsilon}}(x))_k\,d\sigma_x = \epsilon^n \int_{\partial\Omega} \mathbf{U}_j^+[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t,$$

where $\mathbf{U}_j^+$ is as in (2.36). We find it convenient to introduce the function $\mathbf{\Lambda}^+_{kj}$ from $]-\epsilon_1,\epsilon_1[$ to $\mathbb{R}$ by setting

$$\mathbf{\Lambda}^+_{kj}[\epsilon] := \int_{\partial\Omega} \mathbf{U}_j^+[\epsilon](t)(\nu_\Omega(t))_k d\sigma_t \tag{2.41}$$

for all $\epsilon \in ]-\epsilon_1,\epsilon_1[$, which is real analytic on $]-\epsilon_1,\epsilon_1[$ due to Theorem 2.2.8($i$).

Next, let $\mathbf{V}_j^-$ be as in Theorem 2.2.8($ii$). Then, by the Divergence Theorem and by the periodicity of the function which takes $x$ to $u_j^-[\epsilon](x) - x_j$, we have

$$\begin{aligned}
\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} \frac{\partial u_j^-[\epsilon](x)}{\partial x_k}\,dx &= \int_{\partial(Q\setminus \mathrm{cl}\Omega_{p,\epsilon})} u_j^-[\epsilon](x)(\nu_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}}(x))_k\,d\sigma_x \\
&= \int_{\partial Q} u_j^-[\epsilon](x)(\nu_Q(x))_k\,d\sigma_x - \int_{\partial\Omega_{p,\epsilon}} u_j^-[\epsilon](x)(\nu_{\Omega_{p,\epsilon}}(x))_k\,d\sigma_x \\
&= \int_{\partial Q} \left(u_j^-[\epsilon](x) - x_j\right)(\nu_Q(x))_k\,d\sigma_x + \int_{\partial Q} x_j(\nu_Q(x))_k\,d\sigma_x \\
&\quad - \int_{\partial\Omega_{p,\epsilon}} u_j^-[\epsilon](x)(\nu_{\Omega_{p,\epsilon}}(x))_k\,d\sigma_x = -\int_{\partial\Omega_{p,\epsilon}} u_j^-[\epsilon](x)(\nu_{\Omega_{p,\epsilon}}(x))_k\,d\sigma_x + \delta_{k,j}|Q|_n \\
&= -\epsilon^n \int_{\partial\Omega} \mathbf{V}_j^-[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t + \delta_{k,j}|Q|_n \qquad \forall \epsilon \in ]0,\epsilon_1[.
\end{aligned}$$

We also find it convenient to introduce the function $\mathbf{\Lambda}^-_{kj}$ from $]-\epsilon_1,\epsilon_1[$ to $\mathbb{R}$ by setting

$$\mathbf{\Lambda}^-_{kj}[\epsilon] := \int_{\partial\Omega} \mathbf{V}_j^-[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t \tag{2.42}$$

for all $\epsilon \in ]-\epsilon_1,\epsilon_1[$, which is real analytic on $]-\epsilon_1,\epsilon_1[$ due to Theorem 2.2.8($ii$).

Finally, we can introduce the function $\mathbf{\Lambda}_{kj}$ from $]-\epsilon_1,\epsilon_1[$ to $\mathbb{R}$ by setting

$$\mathbf{\Lambda}_{kj}[\epsilon] := \frac{\lambda^+}{|Q|_n}\mathbf{\Lambda}^+_{kj}[\epsilon] - \frac{\lambda^-}{|Q|_n}\mathbf{\Lambda}^-_{kj}[\epsilon] + \frac{1}{|Q|_n}\int_{\partial\Omega} f(t)t_k\,d\sigma_t \qquad \forall \epsilon \in ]-\epsilon_1,\epsilon_1[. \tag{2.43}$$

The function $\mathbf{\Lambda}_{kj}$ is real analytic on $]-\epsilon_1,\epsilon_1[$. Then, keeping Definition 2.1.1 in mind, one verifies that equality (2.40) holds. □

### 2.2.4 Power series expansion of the effective conductivity

To compute the series expansion for the effective conductivity, as an intermediate step, we find it convenient first to compute the power series expansions for the functions $\mathbf{\Lambda}^+_{kj}$ and $\mathbf{\Lambda}^-_{kj}$, which are defined in the proof of Theorem 2.2.9. So, the following lemma holds.

**Lemma 2.2.10.** *Let $\lambda^+, \lambda^- \in ]0,+\infty[$. Let $k,j \in \{1,\dots,n\}$. Let $m \in \{1,\dots,n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $\epsilon_1$ be as in Proposition 2.2.2(iii) and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.2.5.*

*(i) Then there exists ${\epsilon_2}^+ \in ]0,\epsilon_1]$ such that the real analytic function $\mathbf{\Lambda}^+_{kj}$ defined in* (2.41) *can be represented as follows*

$$\begin{aligned}\mathbf{\Lambda}^+_{kj}[\epsilon] =& |\Omega|_n \delta_{k,j} + \int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,0}\right](t)(\nu_\Omega(t))_k \, d\sigma_t \\ &+ \frac{1-\delta_{1,m}}{(m-1)!}\left(\int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,m-1}\right](t)(\nu_\Omega(t))_k \, d\sigma_t\right)\epsilon^{m-1} \\ &+ \sum_{h=n}^{\infty}\frac{1}{h!}\biggl(\int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,h}\right](t)(\nu_\Omega(t))_k \, d\sigma_t \\ &+ \int_{\partial\Omega}\Lambda^h[\theta^i_{j,0},\dots,\theta^i_{j,h-n}](t)(\nu_\Omega(t))_k d\sigma_t\biggr)\epsilon^h\end{aligned}$$

*for all $\epsilon \in ]-{\epsilon_2}^+, {\epsilon_2}^+[$, where the series converges uniformly for all such $\epsilon$.*

*(ii) Then there exists ${\epsilon_2}^- \in ]0,\epsilon_1]$ such that the real analytic function $\mathbf{\Lambda}^-_{kj}$ defined in* (2.42) *can be represented as follows*

$$\begin{aligned}\mathbf{\Lambda}^-_{kj}[\epsilon] =& |\Omega|_n \delta_{k,j} + \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t \\ &+ \frac{1-\delta_{1,m}}{(m-1)!}\left(\int_{\partial\Omega} v^+\left[\partial\Omega, \theta^o_{j,m-1}\right](t)(\nu_\Omega(t))_k \, d\sigma_t\right)\epsilon^{m-1} \\ &+ \sum_{h=n}^{+\infty}\frac{1}{h!}\biggl(\int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,h}](t)(\nu_\Omega(t))_k \, d\sigma_t \\ &+ \int_{\partial\Omega}\Lambda^h[\theta^o_{j,0},\dots,\theta^o_{j,h-n}](t)(\nu_\Omega(t))_k d\sigma_t\biggr)\epsilon^h\end{aligned}$$

*for all $\epsilon \in ]-{\epsilon_2}^-, {\epsilon_2}^-[$, where the series converges uniformly for all such $\epsilon$.*

*Proof.* We first consider statement $(i)$. Let $\epsilon_1$ be as in Proposition 2.2.2$(iii)$. Using equalities (2.36), (2.41), and the Divergence Theorem, we can write $\mathbf{\Lambda}^+_{kj}[\epsilon]$ as follows

$$\begin{aligned}\mathbf{\Lambda}^+_{kj}[\epsilon] = |\Omega|_n \delta_{k,j} + &\int_{\partial\Omega} v^+\left[\partial\Omega, \Theta^i_j[\epsilon]\right](t)(\nu_\Omega(t))_k \, d\sigma_t \\ &+\epsilon^{n-2}\int_{\partial\Omega}\Lambda[\epsilon, \Theta^i_j[\epsilon]](t)(\nu_\Omega(t))_k d\sigma_t\end{aligned} \tag{2.44}$$

for all $\epsilon \in ]-\epsilon_1, \epsilon_1[$. Since $\mathbf{\Lambda}^+_{kj}$ is real analytic on $]-\epsilon_1, \epsilon_1[$ (see the proof of Theorem 2.2.9), there exist ${\epsilon_2}^+ \in ]0, \epsilon_1[$ and a sequence $\{a_h\}_{h\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\mathbf{\Lambda}^+_{kj}[\epsilon] \equiv \sum_{h=0}^{+\infty}\frac{a_h}{h!}\epsilon^h \qquad \forall \epsilon \in ]-{\epsilon_2}^+, {\epsilon_2}^+[,$$

where the series converges uniformly for all $\epsilon \in ]-{\epsilon_2}^+, {\epsilon_2}^+[$. Then, keeping equalities (2.27) in mind, by taking $\epsilon = 0$ in equality (2.44), and by using equality (2.21) if $n = 2$, one verifies that

$$a_0 = \mathbf{\Lambda}^+_{kj}[0] = |\Omega|_n \delta_{k,j} + \int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,0}\right](t)(\nu_\Omega(t))_k \, d\sigma_t.$$

Also, to compute the others coefficients, one can use the equality $a_h = (\partial_\epsilon^h(\mathbf{\Lambda}^+_{kj}[\epsilon]))_{|\epsilon=0}$ which holds for all $h \in \mathbb{N}\setminus\{0\}$. We note that the derivative of order $h \in \mathbb{N}\setminus\{0\}$ of $\mathbf{\Lambda}^+_{kj}[\epsilon]$ has the following form

$$\begin{aligned}\partial_\epsilon^h(\mathbf{\Lambda}^+_{kj}[\epsilon]) = &\int_{\partial\Omega} v^+\left[\partial\Omega, \partial_\epsilon^h \Theta^i_j[\epsilon]\right](t)(\nu_\Omega(t))_k\, d\sigma_t \\ &+ \int_{\partial\Omega} \partial_\epsilon^h\left(\epsilon^{n-2}\Lambda[\epsilon, \Theta^i_j[\epsilon]](t)\right)(\nu_\Omega(t))_k d\sigma_t.\end{aligned} \tag{2.45}$$

Then, taking $\epsilon = 0$ in (2.45), we obtain that

$$\begin{aligned}a_h = \left(\partial_\epsilon^h(\mathbf{\Lambda}^+_{kj}[\epsilon])\right)_{|\epsilon=0} = &\int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,h}\right](t)(\nu_\Omega(t))_k\, d\sigma_t \\ &+ \int_{\partial\Omega}\left(\partial_\epsilon^h\left(\epsilon^{n-2}\Lambda[\epsilon, \Theta^i_j[\epsilon]](t)\right)\right)_{|\epsilon=0}(\nu_\Omega(t))_k d\sigma_t\end{aligned}$$

for all $h \in \mathbb{N}\setminus\{0\}$. Then, equalities (2.21) and Proposition 2.2.5($ii$) imply that $a_h = 0$ for all $h \in \{1, \dots, n-1\}\setminus\{m-1\}$, and that

$$a_{m-1} = \int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,m-1}\right](t)(\nu_\Omega(t))_k\, d\sigma_t$$

in case $m > 1$. Then, since $a_h$ can be written for all $h \geq n$ as follows

$$a_h = \int_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,h}\right](t)(\nu_\Omega(t))_k\, d\sigma_t + \int_{\partial\Omega}\Lambda^h[\theta^i_{j,0}, \dots, \theta^i_{j,h-n}](t)(\nu_\Omega(t))_k d\sigma_t,$$

the validity of statement ($i$) follows.

Now, we consider statement ($ii$). Using equalities (2.38), (2.42), and the Divergence Theorem, we can write $\mathbf{\Lambda}^-_{kj}[\epsilon]$ as follows

$$\mathbf{\Lambda}^-_{kj}[\epsilon] = |\Omega|_n\delta_{k,j} + \int_{\partial\Omega} v^-[\partial\Omega, \Theta^o_j[\epsilon]](t)(\nu_\Omega(t))_k\, d\sigma_t + \epsilon^{n-2}\int_{\partial\Omega}\Lambda[\epsilon, \Theta^o_j[\epsilon]](t)(\nu_\Omega(t))_k d\sigma_t$$

for all $\epsilon \in ]-\epsilon_1, \epsilon_1[$. Then, by arguing as we have done for $\mathbf{\Lambda}^+_{kj}[\epsilon]$, one verifies the validity of statement ($ii$). □

In the following theorem we prove the main result of this section, where we expand $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ into a power series and we provide an explicit and constructive expression for the coefficients of such series.

**Theorem 2.2.11.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $k, j \in \{1, \dots, n\}$. Let $m \in \{1, \dots, n-1\}$. Let $f \in C^{0,\alpha}(\partial\Omega)_0$, $g \in C^{1,\alpha}(\partial\Omega)$, and $c \in \mathbb{R}$. Let $\epsilon_1$ be as in Proposition 2.2.2(iii) and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.2.5. Then there exists $\epsilon_3 \in ]0, \epsilon_1[$ such that*

$$\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^-\delta_{k,j} + \frac{1}{|Q|_n}\left(c_{(k,j),0} + (1-\delta_{1,m})\frac{c_{(k,j),m-1}}{(m-1)!}\epsilon^{m-1} + \sum_{h=n}^{+\infty}\frac{c_{(k,j),h}}{h!}\epsilon^h\right)\epsilon^n \tag{2.46}$$

*for all $\epsilon \in ]0, \epsilon_3[$, where*

$$
\begin{aligned}
c_{(k,j),0} =& \lambda^+ \int_{\partial\Omega} v^+ \left[\partial\Omega, \theta^i_{j,0}\right](t)(\nu_\Omega(t))_k \, d\sigma_t - \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t \\
&+ (\lambda^+ - \lambda^-)|\Omega|_n \delta_{k,j} + \int_{\partial\Omega} f(t) t_k \, d\sigma_t, \qquad (2.47)\\
c_{(k,j),m-1} =& \lambda^+ \int_{\partial\Omega} v^+ \left[\partial\Omega, \theta^i_{j,m-1}\right](t)(\nu_\Omega(t))_k \, d\sigma_t \\
&- \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,m-1}](t)(\nu_\Omega(t))_k \, d\sigma_t, \\
c_{(k,j),h} =& \lambda^+ \int_{\partial\Omega} \left(v^+ \left[\partial\Omega, \theta^i_{j,h}\right](t) + \Lambda^h[\theta^i_{j,0}, \dots, \theta^i_{j,h-n}](t)\right) (\nu_\Omega(t))_k d\sigma_t \\
&- \lambda^- \int_{\partial\Omega} \left(v^-[\partial\Omega, \theta^o_{j,h}](t) + \Lambda^h[\theta^o_{j,0}, \dots, \theta^o_{j,h-n}](t)\right) (\nu_\Omega(t))_k d\sigma_t \quad \forall h \geq n.
\end{aligned}
$$

*Moreover, if $m$ is odd and $n$ is even, then for all $\epsilon \in ]0, \epsilon_3[$*

$$
\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^- \delta_{k,j} + \frac{1}{|Q|_n} \left( c_{(k,j),0} + (1 - \delta_{1,m}) \frac{c_{(k,j),m-1}}{(m-1)!} \epsilon^{m-1} + \sum_{h=n/2}^{+\infty} \frac{c_{(k,j),2h}}{(2h)!} \epsilon^{2h} \right) \epsilon^n. \qquad (2.48)
$$

*Proof.* We first note that equality (2.46) follows by the equalities (2.40), (2.43), and Lemma 2.2.10. Moreover, if $m$ is odd and $n$ is even then Lemma 2.2.4 and Remark 2.2.7 imply that all odd coefficients $c_{(k,j),h}$ in (2.46) are equal to 0 and, thus, (2.46) takes the form (2.48). The proof of the theorem is now complete. □

Now, we restrict ourself to considering problem (2.8) instead of (2.9). As a consequence of Theorem 2.2.11, one immediately concludes the validity of the following proposition.

**Proposition 2.2.12.** *Let $\lambda^+, \lambda^- \in ]0, +\infty[$. Let $k, j \in \{1, \dots, n\}$. Let $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}} \subseteq (C^{0,\alpha}(\partial\Omega)_0)^2$ be the sequence defined as follows:*

- *the pair $(\theta^i_{j,0}, \theta^o_{j,0})$ is the unique solution of the following system of integral equations*

$$
\begin{aligned}
\lambda^- \left(\frac{1}{2}\theta^o_{j,0}(t) + w_*[\partial\Omega, \theta^o_{j,0}](t)\right) - \lambda^+ \left(-\frac{1}{2}\theta^i_{j,0}(t) + w_*[\partial\Omega, \theta^i_{j,0}](t)\right)& \\
= (\lambda^+ - \lambda^-)(\nu_\Omega(t))_j&, \\
v^+[\partial\Omega, \theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}] d\sigma - v^-[\partial\Omega, \theta^o_{j,0}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}] d\sigma = 0&
\end{aligned}
$$

  *for all $t \in \partial\Omega$,*

- *$(\theta^i_{j,h}, \theta^o_{j,h}) = (0, 0)$ for all $h \in \{1, \dots, n-1\}$,*

- *for all $h \geq n$ the pair of function $(\theta^i_{j,h}, \theta^o_{j,h})$ is the unique solution of system* (2.30)-(2.31).

*Then there exists $\epsilon_* \in ]0, \epsilon_0[$ such that*

$$
\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^- \delta_{k,j} + \frac{1}{|Q|_n} \left( c_{(k,j),0} + \sum_{h=n}^{+\infty} \frac{c_{(k,j),h}}{h!} \epsilon^h \right) \epsilon^n \qquad \forall \epsilon \in ]0, \epsilon_*[, \qquad (2.49)
$$

*where*

$$
\begin{aligned}
c_{(k,j),0} =&\lambda^+ \int_{\partial\Omega} v^+ \left[\partial\Omega, \theta^i_{j,0}\right](t)(\nu_\Omega(t))_k \, d\sigma_t \\
&- \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t + (\lambda^+ - \lambda^-)|\Omega|_n \delta_{k,j}, \\
c_{(k,j),h} =&\lambda^+ \int_{\partial\Omega} \left(v^+ \left[\partial\Omega, \theta^i_{j,h}\right](t) + \Lambda^h[\theta^i_{j,0}, \dots, \theta^i_{j,h-n}](t)\right)(\nu_\Omega(t))_k d\sigma_t \\
&- \lambda^- \int_{\partial\Omega} \left(v^-[\partial\Omega, \theta^o_{j,h}](t) + \Lambda^h[\theta^o_{j,0}, \dots, \theta^o_{j,h-n}](t)\right)(\nu_\Omega(t))_k d\sigma_t \quad \forall h \geq n.
\end{aligned}
$$

*Moreover, if $n = 2$, then for all $\epsilon \in ]0, \epsilon_*[$*

$$
\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^- \delta_{k,j} + \frac{1}{|Q|_2} \left( c_{(k,j),0} + \sum_{h=1}^{+\infty} \frac{c_{(k,j),2h}}{(2h)!} \epsilon^{2h} \right) \epsilon^2. \tag{2.50}
$$

### 2.2.5 Power series expression of the effective conductivity in a composite with spherical inclusions

In this section we introduce more restrictive assumptions on problem (2.8), which allow us to obtain simpler expressions for some coefficients $\{c_{(k,j),h}\}_{h\in\mathbb{N}}$ in the series expansion of the effective conductivity in terms of the given positive constants $\lambda^+$ and $\lambda^-$. So, we assume that

$$
(f, g, c) = (0, 0, 0), \quad Q \equiv ]0, 1[^n, \quad \text{and} \quad \Omega \equiv \mathbb{B}_n(0, 1).
$$

We begin by observing that if $(f, g, c) = (0, 0, 0)$ than the system of integral equations (2.28)-(2.29) takes the following form

$$
\begin{aligned}
&\lambda^- \left( \frac{1}{2} \theta^o_{j,0}(t) + w_*[\partial\Omega, \theta^o_{j,0}](t) \right) \\
&\qquad - \lambda^+ \left( -\frac{1}{2} \theta^i_{j,0}(t) + w_*[\partial\Omega, \theta^i_{j,0}](t) \right) = (\lambda^+ - \lambda^-)(\nu_\Omega(t))_j,
\end{aligned} \tag{2.51}
$$

$$
v^+[\partial\Omega, \theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}] d\sigma - v^-[\partial\Omega, \theta^o_{j,0}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}] d\sigma = 0 \tag{2.52}
$$

for all $t \in \partial\Omega$, which is equivalent to the system of the following equations

$$
\begin{aligned}
&\lambda^- \frac{\partial v^-[\partial\Omega, \theta^o_{j,0}]}{\partial\nu_\Omega}(t) - \lambda^+ \frac{\partial v^+[\partial\Omega, \theta^i_{j,0}]}{\partial\nu_\Omega}(t) = (\lambda^+ - \lambda^-)(\nu_\Omega(t))_j, \\
&v^+[\partial\Omega, \theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}] d\sigma - v^-[\partial\Omega, \theta^o_{j,0}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}] d\sigma = 0
\end{aligned}
$$

for all $t \in \partial\Omega$.

Then, one can verify that if the pair $(\theta^i_{j,0}, \theta^o_{j,0})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the system (2.51)-(2.52), then there exists a constant $c_0 \in \mathbb{R}$ such that

$$
\begin{aligned}
v^+[\partial\Omega, \theta^i_{j,0}](t) &= -\frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-(n-1)} t_j + c_0 \qquad \forall t \in \mathrm{cl}\Omega, \\
v^-[\partial\Omega, \theta^o_{j,0}](t) &= -\frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-(n-1)} \frac{t_j}{|t|^n} \qquad \forall t \in \mathbb{R}^n \setminus \Omega.
\end{aligned} \tag{2.53}
$$

Then, taking equalities (2.47) and (2.53) into account, by a simple computation, one verifies that

$$
c_{(k,j),0} = n\lambda^- \frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-(n-1)} |\Omega|_n \delta_{k,j},
$$

and, as a consequence of equality (2.49), we have that if $k, j \in \{1, \dots, n\}$ then

$$\lambda^{\text{eff}}_{kj}[\epsilon] = \lambda^- \left(1 + n\frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-(n-1)}|\Omega|_n \epsilon^n\right)\delta_{k,j} + O(\epsilon^{2n})$$

as $\epsilon \to 0^+$.

### Power series expression of the effective conductivity for the two-dimensional case

We want to find explicitly more coefficients in the series expression of $\lambda^{\text{eff}}_{kj}[\epsilon]$. To do so, we restrict ourself to consider the two-dimensional case. So, we assume that

$$n = 2, \quad Q \equiv ]0,1[\times]0,1[, \quad \text{and} \quad \Omega \equiv \mathbb{B}_2(0,1)$$

(see Figure 2.1).

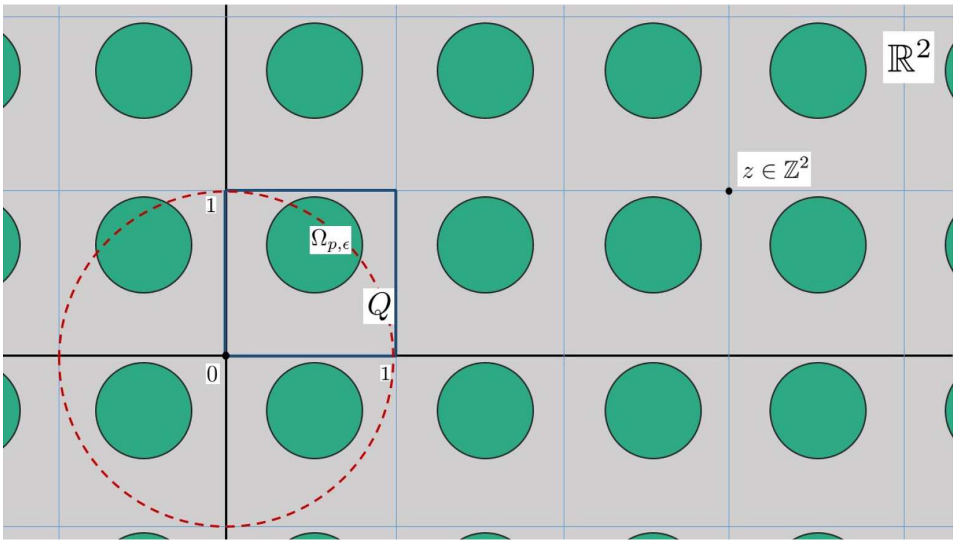


Figure 2.1: *The composite with the inclusions in the form of a disk*

Taking into account that $\Omega$ is the unit ball, we have that

$$\begin{aligned} w_*[\partial\Omega, \theta](t) &= \int_{\partial\Omega} D\left(S_{q,2}(t-s)\right)\nu_\Omega(t)\theta(s)\,d\sigma_s = \frac{1}{2\pi}\int_{\partial\Omega} D\left(\log|t-s|\right)\nu_\Omega(t)\theta(s)\,d\sigma_s \\ &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{t-s}{|t-s|^2}t\theta(s)\,d\sigma_s = \frac{1}{2\pi}\int_{\partial\Omega}\frac{(t_1-s_1, t_2-s_2)\cdot(t_1,t_2)}{|t-s|^2}\theta(s)\,d\sigma_s \\ &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{t_1^2 - t_1s_1 + t_2^2 - t_2s_2}{t_1^2 - 2t_1s_1 + s_1^2 + t_2^2 - 2t_2s_2 + s_2^2}\theta(s)\,d\sigma_s \\ &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{1 - t_1s_1 - t_2s_2}{2 - 2t_1s_1 - 2t_2s_2}\theta(s)\,d\sigma_s \\ &= \frac{1}{4\pi}\int_{\partial\Omega}\theta(s)\,d\sigma_s = 0 \end{aligned} \tag{2.54}$$

for all $\theta \in C^{0,\alpha}(\partial\Omega)_0$. Moreover, using the definitions of $\Lambda^2$ and $\Lambda^2_\nu$ (see Section 2.2.1), and equalities (2.10)-(2.12), by a straightforward computation, we have the following

$$
\begin{aligned}
\Lambda^2_\nu[\theta_0](t) =& 2\sum_{j=1}^{1}\binom{1}{j}\sum_{h=0}^{j}\binom{j}{h}(\partial_1^h\partial_2^{j-h}DR_q)(0)\nu_\Omega(t)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{j-h}\theta_0(s)d\sigma_s\\
=&2\sum_{h=0}^{1}\binom{1}{h}(\partial_1^h\partial_2^{1-h}DR_q)(0)\nu_\Omega(t)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{1-h}\theta_0(s)d\sigma_s\\
=&2\left(-\frac{1}{2}t_1\int_{\partial\Omega}(t_1-s_1)\theta_0(s)d\sigma_s-\frac{1}{2}t_2\int_{\partial\Omega}(t_2-s_2)\theta_0(s)d\sigma_s\right)\\
=&\int_{\partial\Omega}(t_1s_1+t_2s_2)\theta_0(s)d\sigma_s \qquad \forall t\in\partial\Omega,\quad \forall\theta_0\in C^{0,\alpha}(\partial\Omega)_0,
\end{aligned}
\tag{2.55}
$$

$$
\begin{aligned}
\Lambda^2[\theta_0](t) =& \sum_{j=1}^{2}\binom{2}{j}\sum_{h=0}^{j}\binom{j}{h}(\partial_1^h\partial_2^{j-h}R_q)(0)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{j-h}\theta_{2-j}(s)d\sigma_s\\
=&\sum_{h=0}^{2}\binom{2}{h}(\partial_1^h\partial_2^{2-h}R_q)(0)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{2-h}\theta_0(s)d\sigma_s\\
=&-\frac{1}{2}\int_{\partial\Omega}\left((t_1-s_1)^2+(t_2-s_2)^2\right)\theta_0(s)d\sigma_s\\
=&\int_{\partial\Omega}(t_1s_1+t_2s_2)\theta_0(s)d\sigma_s \qquad \forall t\in\partial\Omega,\quad \forall\theta_0\in C^{0,\alpha}(\partial\Omega)_0.
\end{aligned}
\tag{2.56}
$$

Then, keeping equalities (2.53) in mind, by using the jump formulae for the normal derivative of the simple layer potential (see Theorem A.0.2($v$) in Appendix A) and equality (2.54), one verifies that

$$
\theta^i_{j,0}(t)=2\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}t_j=\theta^o_{j,0}(t)\qquad \forall t\in\partial\Omega. \tag{2.57}
$$

Then, taking $h=2$ in the system of integral equations (2.30)-(2.31), one can rewrite it as follows

$$
\lambda^-\frac{\partial v^-[\partial\Omega,\theta^o_{j,2}]}{\partial\nu_\Omega}(t)-\lambda^+\frac{\partial v^+[\partial\Omega,\theta^i_{j,2}]}{\partial\nu_\Omega}(t)=\lambda^+\Lambda^2_\nu[\theta^i_{j,0}](t)-\lambda^-\Lambda^2_\nu[\theta^o_{j,0}](t), \tag{2.58}
$$

$$
v^+[\partial\Omega,\theta^i_{j,2}](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,2}]d\sigma-v^-[\partial\Omega,\theta^o_{j,2}](t)+\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,2}]d\sigma=0 \tag{2.59}
$$

for all $t\in\partial\Omega$. Also, using equalities (2.55), (2.56), and (2.57), one can show that

$$
\Lambda^2[\theta^i_{j,0}](t)=\Lambda^2[\theta^o_{j,0}](t)=\Lambda^2_\nu[\theta^i_{j,0}](t)=\Lambda^2_\nu[\theta^o_{j,0}](t)=2\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}\pi t_j \tag{2.60}
$$

for all $t\in\partial\Omega$. Again, if $(\theta^i_{j,2},\theta^o_{j,2})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the system (2.58)-(2.59), then, taking equalities (2.60) into account, one can verify that there exists a real constant $c_1$ such that

$$
\begin{aligned}
v^+[\partial\Omega,\theta^i_{j,2}](t)&=-2\pi\left(\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}\right)^2 t_j+c_1 \qquad \forall t\in\mathrm{cl}\Omega,\\
v^-[\partial\Omega,\theta^o_{j,0}](t)&=-2\pi\left(\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}\right)^2\frac{t_j}{|t|^2} \qquad \forall t\in\mathbb{R}^2\setminus\Omega.
\end{aligned}
\tag{2.61}
$$

Then by equalities (2.50), (2.53), (2.60), and (2.61), we deduce that if $k,j\in\{1,2\}$ then

$$
\lambda^{\mathrm{eff}}_{kj}[\epsilon]=\lambda^-\left(1+2\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}\pi\epsilon^2+2\left(\frac{\lambda^+-\lambda^-}{\lambda^++\lambda^-}\right)^2\pi^2\epsilon^4\right)\delta_{k,j}+O(\epsilon^6) \tag{2.62}
$$

as $\epsilon \to 0^+$. We note that series expansion (2.62) agrees with the series expansion of the effective conductivity obtained by Ammari, Kang, and Touibi in [10, Thm. 5.3], and Pesetskaya in [122, Sec. 5] for the case where the unit cell $Q$ has only one inclusion. Moreover, taking $\lambda^- = 1$, we observe that (2.62) agrees with the series expansion obtained by in Berlyand and Mityushev in [20, Sec. 4].

## 2.3 Effective conductivity of a periodic dilute composite with nonideal contact

In this section we consider a nonideal transmission problem in the two-dimensional case and study the effective conductivity of a periodic two-phase composite.

The section is organized as follows. In Subsection 2.3.1 we introduce the problem and collect some preliminaries on potential theory and on the integral equation formulation of problem (2.63). In Subsection 2.3.2 we compute the power series expansion under the assumption of present the constant interfacial thermal resistance. Finally, in Subsection 2.3.3 we analyze of the effective conductivity in the case the interfacial thermal resistance is proportional to a parameter related to the size of inclusions in a composite.

### 2.3.1 Preliminaries and notation

Within this section, we deal only with the two-dimensional case. Thus, we find it convenient to recall some notation of Section 1.1. We fix once for all $(q_{11}, q_{22}) \in ]0, +\infty[^2$. Then $Q = ]0, q_{11}[\times]0, q_{22}[$ is the periodicity cell and the diagonal matrix $q \equiv (\delta_{i,j} q_{jj})_{i,j\in\{1,2\}}$. Then $|Q|_2 = q_{11} q_{22}$ denotes the two-dimensional measure of the fundamental cell $Q$ and $q^{-1}$ denotes the inverse matrix of $q$. Clearly, $q\mathbb{Z}^2 = \{qz : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the fundamental cell $Q$.

We recall that the domain $\Omega$ is as in (1.1) and fixed, $\epsilon_0$ is as in (1.2) and fixed, and for a fixed point $p \in Q$ and all $\epsilon \in ]-\epsilon_0, \epsilon_0[$ the set $\Omega_{p,\epsilon} = p + \epsilon\Omega$ plays the role of the inclusion in the fundamental cell $Q$, where $\epsilon_0$ is as in (1.2). We also recall that $\mathbb{S}[\Omega_{p,\epsilon}] = \bigcup_{z\in\mathbb{Z}^2}(qz + \Omega_{p,\epsilon})$ and $\mathbb{S}[\Omega_{p,\epsilon}]^- = \mathbb{R}^2 \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ for all $\epsilon \in ]-\epsilon_0, \epsilon_0[$.

Next, we take two positive constants $\lambda^+$, $\lambda^-$, a function $f$ in $C^{0,\alpha}(\partial\Omega)_0$, a function $g$ in $C^{0,\alpha}(\partial\Omega)$, and a function $\rho$ from $]0, \epsilon_0[$ to $]0, +\infty[$, and for each $j \in \{1, 2\}$ we consider the following transmission problem for a pair of functions $(u_j^+, u_j^-) \in C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$:

$$
\begin{cases}
\Delta u_j^+ = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}],\\
\Delta u_j^- = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-,\\
u_j^+(x + q_{hh}e_h) = u_j^+(x) + \delta_{h,j} q_{hh} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],\ \forall h \in \{1,2\},\\
u_j^-(x + q_{hh}e_h) = u_j^-(x) + \delta_{h,j} q_{hh} & \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,\ \forall h \in \{1,2\},\\
\lambda^- \frac{\partial u_j^-}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) = f((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon},\\
\lambda^+ \frac{\partial u_j^+}{\partial \nu_{\Omega_{p,\epsilon}}}(x) + \frac{1}{\rho(\epsilon)}\Big(u_j^+(x) - u_j^-(x)\Big) = g((x-p)/\epsilon) & \forall x \in \partial\Omega_{p,\epsilon},\\
\int_{\partial\Omega_{p,\epsilon}} u_j^+(x)\, d\sigma_x = 0
\end{cases}
\tag{2.63}
$$

for all $\epsilon \in ]0, \epsilon_0[$.

In problem (2.63), the functions $u_j^+$ and $u_j^-$ play the role of the temperature field in the inclusions occupying the periodic set $\mathbb{S}[\Omega_{p,\epsilon}]$ and in the matrix occupying $\mathbb{S}[\Omega_{p,\epsilon}]^-$, respectively. The parameters $\lambda^+$ and $\lambda^-$ represent the thermal conductivity of the materials which fill the inclusions and the matrix, respectively, while the parameter $\rho(\epsilon)$ is the interfacial thermal resistance. The fifth and the sixth condition in (2.63) describe the jump of the normal heat flux and of the temperature field across the two-phase interface.

Such a discontinuity in the temperature field is a well-known phenomenon and has been largely investigated since 1941, when Kapitza carried out the first systematic study of thermal interface behavior in liquid helium (see, e.g., Swartz and Pohl [157], Lipton [91] and references therein).

Problem (2.63) has been investigated by Dalla Riva and Musolino [35] under the assumption that

$$\text{the limit } \lim_{\epsilon\to 0^+} \frac{\epsilon}{\rho(\epsilon)} \text{ exists finite in } \mathbb{R}. \tag{2.64}$$

As in the paper by Dalla Riva and Musolino [35], we consider the effective conductivity matrix $\lambda^{\mathrm{eff}}[\epsilon]$ with $(k,j)$-entry $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ defined by means of Definition 2.1.1, where $(u_j^+[\epsilon], u_j^-[\epsilon])$ denotes the unique solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of problem (2.63).

Dalla Riva and Musolino [35] have proved that under assumption (2.64) the effective conductivity can be continued real analytically in the parameter $\epsilon$ around the degenerate value $\epsilon = 0$.

In this section, we investigate two specific cases, namely, when $\rho(\epsilon) \equiv 1/r_\#$ or $\rho(\epsilon) \equiv \epsilon/r_\#$, where $r_\#$ is a positive real number, for which $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ can be expanded into a (convergent) power series for $\epsilon$. We observe that the first case corresponds to the situation where the thermal boundary resistance $\rho(\epsilon)$ is independent of $\epsilon$, whereas in the second case the resistance is proportional to the size of the contact interface $\partial\Omega_{p,\epsilon}$. This latter case has been considered also in the works of Castro, Pesetskaya, and Rogosin [27] and of Drygaś and Mityushev [44].

Now, let $j \in \{1,2\}$ and let $\rho(\cdot)$ be either $\epsilon \mapsto 1/r_\#$ or $\epsilon \mapsto \epsilon/r_\#$. To provide an integral equation formulation of problem (2.63), we define the map $N_j := (N_{j,1}, N_{j,2})$ from $]-\epsilon_0,\epsilon_0[\times (C^{0,\alpha}(\partial\Omega)_0)^2$ to $(C^{0,\alpha}(\partial\Omega)_0)^2$ by setting

$$\begin{aligned}
N_{j,1}[\epsilon,\theta^i,\theta^o](t) &:= \lambda^- \left(\frac{1}{2}\theta^o(t) + w_*[\partial\Omega,\theta^o](t) + \epsilon\Lambda_\nu[\epsilon,\theta^o](t)\right) \\
&\quad - \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_*[\partial\Omega,\theta^i](t) + \epsilon\Lambda_\nu[\epsilon,\theta^i](t)\right) - f(t) + (\lambda^- - \lambda^+)(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \\
N_{j,2}[\epsilon,\theta^i,\theta^o](t) &:= \lambda^+ \left(-\frac{1}{2}\theta^i(t) + w_*[\partial\Omega,\theta^i](t) + \epsilon\Lambda_\nu[\epsilon,\theta^i](t)\right) \\
&\quad + \frac{\epsilon}{\rho(\epsilon)} \left(v^+[\partial\Omega,\theta^i](t) + \Lambda[\epsilon,\theta^i](t) - \fint_{\partial\Omega} \left(v^+[\partial\Omega,\theta^i] + \Lambda[\epsilon,\theta^i]\right) d\sigma\right. \\
&\quad \left. - v^-[\partial\Omega,\theta^o](t) - \Lambda[\epsilon,\theta^o](t) + \fint_{\partial\Omega} \left(v^-[\partial\Omega,\theta^o] + \Lambda[\epsilon,\theta^o]\right) d\sigma\right) \\
&\quad - g(t) + \fint_{\partial\Omega} g \, d\sigma + \lambda^+(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega,
\end{aligned}$$

for all $(\epsilon,\theta^i,\theta^o) \in ]-\epsilon_0,\epsilon_0[\times (C^{0,\alpha}(\partial\Omega)_0)^2$.

By means of the operator $N_j$, we can convert problem (2.63) into a system of integral equations, as the following proposition shows (for a proof we refer to Dalla Riva and Musolino [35, Prop. 6.1]).

**Proposition 2.3.1.** *Let either $\rho(\epsilon) \equiv 1/r_\#$ for all $\epsilon \in ]-\epsilon_0,\epsilon_0[$ or $\rho(\epsilon) \equiv \epsilon/r_\#$ for all $\epsilon \in ]-\epsilon_0,\epsilon_0[$. Let $\epsilon \in ]0,\epsilon_0[$. Let $j \in \{1,2\}$. Then the unique solution $(u_j^+[\epsilon], u_j^-[\epsilon])$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of problem* (2.63) *is delivered by*

$$\begin{aligned}
u_j^+[\epsilon](x) = & v_q^+[\partial\Omega_{p,\epsilon}, \hat{\theta}_j^i[\epsilon]((\cdot - p)/\epsilon)](x) - \fint_{\partial\Omega_{p,\epsilon}} v_q^+[\partial\Omega_{p,\epsilon}, \hat{\theta}_j^i[\epsilon]((\cdot - p)/\epsilon)] \, d\sigma \\
& + x_j - \fint_{\partial\Omega_{p,\epsilon}} y_j \, d\sigma_y \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}],
\end{aligned}$$

$$
\begin{aligned}
u_j^-[\epsilon](x) =& v_q^-[\partial\Omega_{p,\epsilon}, \hat{\theta}_j^o[\epsilon]((\cdot - p)/\epsilon)](x) - \fint_{\partial\Omega_{p,\epsilon}} v_q^-[\partial\Omega_{p,\epsilon}, \hat{\theta}_j^o[\epsilon]((\cdot - p)/\epsilon)]\, d\sigma \\
&- \rho(\epsilon) \fint_{\partial\Omega_{p,\epsilon}} g((y-p)/\epsilon)\, d\sigma_y + x_j - \fint_{\partial\Omega_{p,\epsilon}} y_j \, d\sigma_y \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-,
\end{aligned}
$$

*where* $(\hat{\theta}_j^i[\epsilon], \hat{\theta}_j^o[\epsilon])$ *denotes the unique solution* $(\theta^i, \theta^o)$ *in* $(C^{0,\alpha}(\partial\Omega)_0)^2$ *of*

$$
N_j\left[\epsilon, \theta^i, \theta^o\right] = 0\,.
$$

Again, in order to investigate the asymptotic behavior of the $(k,j)$-entry $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ of the effective conductivity tensor as $\epsilon \to 0^+$, we need to study the functions $u_j^+[\epsilon]$ and $u_j^-[\epsilon]$ for $\epsilon$ close to the degenerate value 0. On the other hand, Proposition 2.3.1 tells us how to represent $u_j^+[\epsilon]$ and $u_j^-[\epsilon]$ in terms of the densities $\hat{\theta}_j^i[\epsilon]$ and $\hat{\theta}_j^o[\epsilon]$. Therefore, the analysis of $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ for $\epsilon$ close to 0 can be deduced by the asymptotic behavior of $\hat{\theta}_j^i[\epsilon]$ and $\hat{\theta}_j^o[\epsilon]$. Accordingly, as a first step, in the following theorem we present a regularity result for $\hat{\theta}_j^i[\epsilon]$ and $\hat{\theta}_j^o[\epsilon]$ for $\epsilon$ small and positive (see Dalla Riva and Musolino [35, Prop. 5.2, Thm. 6.2 and Thm. 6.3]).

**Proposition 2.3.2.** *Let either* $\rho(\epsilon) \equiv 1/r_{\#}$ *for all* $\epsilon \in ]-\epsilon_0, \epsilon_0[$ *or* $\rho(\epsilon) \equiv \epsilon/r_{\#}$ *for all* $\epsilon \in ]-\epsilon_0, \epsilon_0[$. *Let* $j \in \{1,2\}$. *The following statements hold.*

*(i)* $N_j$ *is a real analytic map from* $]-\epsilon_0, \epsilon_0[ \times (C^{0,\alpha}(\partial\Omega)_0)^2$ *to* $(C^{0,\alpha}(\partial\Omega)_0)^2$.

*(ii) There exists a unique pair* $(\tilde{\theta}_j^i, \tilde{\theta}_j^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ *such that* $N_j\left[0, \tilde{\theta}_j^i, \tilde{\theta}_j^o\right] = 0$.

*(iii) There exist* $\epsilon_1 \in ]0, \epsilon_0[$ *and a real analytic map* $\epsilon \mapsto (\Theta_j^i[\epsilon], \Theta_j^o[\epsilon])$ *from* $]-\epsilon_1, \epsilon_1[$ *to* $(C^{0,\alpha}(\partial\Omega)_0)^2$ *such that*

$$
N_j\left[\epsilon, \Theta_j^i[\epsilon], \Theta_j^o[\epsilon]\right] = 0 \qquad \forall \epsilon \in ]-\epsilon_1, \epsilon_1[. \tag{2.65}
$$

*In particular,*

$$
(\Theta_j^i[\epsilon], \Theta_j^o[\epsilon]) = (\hat{\theta}_j^i[\epsilon], \hat{\theta}_j^o[\epsilon]) \quad \forall \epsilon \in ]0, \epsilon_1[ \quad \text{and} \quad (\Theta_j^i[0], \Theta_j^o[0]) = (\tilde{\theta}_j^i, \tilde{\theta}_j^o),
$$

*where the pair* $(\hat{\theta}_j^i[\epsilon], \hat{\theta}_j^o[\epsilon])$ *is defined in Proposition 2.3.1.*

Now, we note again that the real analyticity result of Proposition 2.3.2 (iii) implies that there exists $\epsilon_2 \in ]0, \epsilon_1[$ small enough such that we can expand $\Theta_j^i[\epsilon]$ and $\Theta_j^o[\epsilon]$ into power series of $\epsilon$, i.e.,

$$
\Theta_j^i[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^i_{j,h}}{h!}\epsilon^h, \qquad \Theta_j^o[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^o_{j,h}}{h!}\epsilon^h, \tag{2.66}
$$

for some $\{\theta^i_{j,h}\}_{k\in\mathbb{N}}$, $\{\theta^o_{j,h}\}_{k\in\mathbb{N}}$ and for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$. Moreover,

$$
\theta^i_{j,h} = \left(\partial^h_\epsilon \theta^i_j[\epsilon]\right)_{|\epsilon=0}, \qquad \theta^o_{j,h} = \left(\partial^h_\epsilon \theta^o_j[\epsilon]\right)_{|\epsilon=0},
$$

for all $k \in \mathbb{N}$. As a consequence,

$$
\hat{\theta}_j^i[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^i_{j,h}}{h!}\epsilon^h, \qquad \hat{\theta}_j^o[\epsilon] = \sum_{h=0}^{+\infty} \frac{\theta^o_{j,h}}{h!}\epsilon^h,
$$

for all $\epsilon \in ]0, \epsilon_2[$. Therefore, in order to obtain a power series expansion for $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$ for $\epsilon$ close to 0, we want to exploit the expansion of $(\hat{\theta}_j^i[\epsilon], \hat{\theta}_j^o[\epsilon])$ (or equivalently of $(\Theta_j^i[\epsilon], \Theta_j^o[\epsilon])$). Since the coefficients of the expansions in (2.66) are given by the derivatives with respect to $\epsilon$ of $\Theta^i_j[\epsilon]$ and $\Theta^o_j[\epsilon]$, we would like to obtain some equations identifying $\left(\partial^h_\epsilon\theta^i_j[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^h_\epsilon\theta^o_j[\epsilon]\right)_{|\epsilon=0}$. The plan is to obtain such equations by differentiating with respect to $\epsilon$ equality (2.65), which then leads to

$$\partial^h_\epsilon\left(N_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]\right)=0 \qquad \forall\epsilon\in]-\epsilon_1,\epsilon_1[, \qquad \forall h\in\mathbb{N}. \tag{2.67}$$

Then by taking $\epsilon=0$ in (2.67), we will obtain integral equations identifying $\left(\partial^h_\epsilon\theta^i_j[\epsilon]\right)_{|\epsilon=0}$ and $\left(\partial^h_\epsilon\theta^o_j[\epsilon]\right)_{|\epsilon=0}$.

### 2.3.2 Power series expansion for the constant interfacial thermal resistance

Throughout this section, we consider the case where

$$\rho(\epsilon)\equiv 1/r_\# \qquad \forall\epsilon\in]-\epsilon_0,\epsilon_0[. \tag{2.68}$$

In order to compute the asymptotic expansion of the effective conductivity under assumption (2.68), we start with the following proposition where we identify the coefficients of the power series expansions of $\Theta^i_j[\epsilon]$ and of $\Theta^o_j[\epsilon]$ in terms of the solutions of systems of integral equations.

**Proposition 2.3.3.** *Let $j\in\{1,2\}$. Let $\epsilon_1$, $\epsilon\mapsto\Theta^i_j[\epsilon]$, and $\epsilon\mapsto\Theta^o_j[\epsilon]$ be as in Proposition 2.3.2. Then there exist $\epsilon_2\in]0,\epsilon_1[$ and a sequence $\{(\theta^i_{j,h},\theta^o_{j,h})\}_{h\in\mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$\Theta^i_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^i_{j,h}}{h!}\epsilon^h \qquad \text{and} \qquad \Theta^o_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^o_{j,h}}{h!}\epsilon^h \qquad \forall\epsilon\in]-\epsilon_2,\epsilon_2[, \tag{2.69}$$

*where the two series converge uniformly for $\epsilon\in]-\epsilon_2,\epsilon_2[$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$. Moreover, the following statements hold.*

*(i) The pair of functions $(\theta^i_{j,0},\theta^o_{j,0})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$-\frac{1}{2}\theta^i_{j,0}(t)+w_*[\partial\Omega,\theta^i_{j,0}](t)=\frac{1}{\lambda^+}\left(g(t)-\int_{\partial\Omega}g\,d\sigma\right)-(\nu_\Omega(t))_j, \tag{2.70}$$
$$\frac{1}{2}\theta^o_{j,0}(t)+w_*[\partial\Omega,\theta^o_{j,0}](t)=\frac{1}{\lambda^-}\left(g(t)-\int_{\partial\Omega}g\,d\sigma+f(t)\right)-(\nu_\Omega(t))_j \tag{2.71}$$

*for all $t\in\partial\Omega$.*

*(ii) The pair of functions $(\theta^i_{j,1},\theta^o_{j,1})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$\begin{aligned}-\frac{1}{2}\theta^i_{j,1}(t)+w_*[\partial\Omega,\theta^i_{j,1}](t)=&-\frac{r_\#}{\lambda^+}\Bigg(v^+[\partial\Omega,\theta^i_{j,0}](t)-\int_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,0}]d\sigma\\&-v^-[\partial\Omega,\theta^o_{j,0}](t)+\int_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,0}]d\sigma\Bigg),\end{aligned} \tag{2.72}$$

$$\begin{aligned}\frac{1}{2}\theta^o_{j,1}(t)+w_*[\partial\Omega,\theta^o_{j,1}](t)=&-\frac{r_\#}{\lambda^-}\Bigg(v^+[\partial\Omega,\theta^i_{j,0}](t)-\int_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,0}]d\sigma\\&-v^-[\partial\Omega,\theta^o_{j,0}](t)+\int_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,0}]d\sigma\Bigg)\end{aligned} \tag{2.73}$$

*for all $t\in\partial\Omega$.*

*(iii)* *The pair of functions* $(\theta^i_{j,2}, \theta^o_{j,2})$ *is the unique solution in* $(C^{0,\alpha}(\partial\Omega)_0)^2$ *of the following system of integral equations*

$$\begin{aligned} -\frac{1}{2}\theta^i_{j,2}(t) + w_*[\partial\Omega, \theta^i_{j,2}](t) = -\Lambda^2_\nu \left[\theta^i_{j,0}\right](t) - \frac{2r_\#}{\lambda^+}\Big( v^+[\partial\Omega, \theta^i_{j,1}](t) \\ - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}]d\sigma \Big), \end{aligned} \tag{2.74}$$

$$\begin{aligned} \frac{1}{2}\theta^o_{j,2}(t) + w_*[\partial\Omega, \theta^o_{j,2}](t) = -\Lambda^2_\nu[\theta^o_{j,0}](t) - \frac{2r_\#}{\lambda^-}\Big( v^+[\partial\Omega, \theta^i_{j,1}](t) \\ - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}]d\sigma \Big) \end{aligned} \tag{2.75}$$

*for all* $t \in \partial\Omega$.

*(iv)* *For all* $h \in \mathbb{N} \setminus \{0,1,2\}$ *the pair of functions* $(\theta^i_{j,h}, \theta^o_{j,h})$ *is the unique solution in* $(C^{0,\alpha}(\partial\Omega)_0)^2$ *of the following system of integral equations which involves* $\{(\theta^i_{j,k}, \theta^o_{j,k})\}_{k=0}^{h-1}$

$$\begin{aligned} -\frac{1}{2}\theta^i_{j,h}(t) + w_*[\partial\Omega, \theta^i_{j,h}](t) = -\Lambda^h_\nu \left[\theta^i_{j,0}, \ldots, \theta^i_{j,h-2}\right](t) - \frac{hr_\#}{\lambda^+}\Big( v^+[\partial\Omega, \theta^i_{j,h-1}](t) \\ - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,h-1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,h-1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,h-1}]d\sigma \Big) \\ - \frac{hr_\#}{\lambda^+}\Big( \Lambda^{h-1}[\theta^i_{j,0}, \ldots, \theta^i_{j,h-3}](t) - \fint_{\partial\Omega} \Lambda^{h-1}[\theta^i_{j,0}, \ldots, \theta^i_{j,h-3}]d\sigma \\ - \Lambda^{h-1}[\theta^o_{j,0}, \ldots, \theta^o_{j,h-3}](t) + \fint_{\partial\Omega} \Lambda^{h-1}[\theta^o_{j,0}, \ldots, \theta^o_{j,h-3}]d\sigma \Big), \end{aligned} \tag{2.76}$$

$$\begin{aligned} \frac{1}{2}\theta^o_{j,h}(t) + w_*[\partial\Omega, \theta^o_{j,h}](t) = -\Lambda^h_\nu \left[\theta^o_{j,0}, \ldots, \theta^o_{j,h-2}\right](t) - \frac{hr_\#}{\lambda^-}\Big( v^+[\partial\Omega, \theta^i_{j,h-1}](t) \\ - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,h-1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,h-1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,h-1}]d\sigma \Big) \\ - \frac{hr_\#}{\lambda^-}\Big( \Lambda^{h-1}[\theta^i_{j,0}, \ldots, \theta^i_{j,h-3}](t) - \fint_{\partial\Omega} \Lambda^{h-1}[\theta^i_{j,0}, \ldots, \theta^i_{j,h-3}]d\sigma \\ - \Lambda^{h-1}[\theta^o_{j,0}, \ldots, \theta^o_{j,h-3}](t) + \fint_{\partial\Omega} \Lambda^{h-1}[\theta^o_{j,0}, \ldots, \theta^o_{j,h-3}]d\sigma \Big) \end{aligned} \tag{2.77}$$

*for all* $t \in \partial\Omega$.

*Proof.* We first note that Proposition 2.3.2 (iii) implies the existence of $\epsilon_2$ and of a family $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ such that (2.69) holds. By standard properties of real analytic maps, one has

$$(\theta^i_{j,h}, \theta^o_{j,h}) = (\partial^h_\epsilon \Theta^i_j[0], \partial^h_\epsilon \Theta^o_j[0]) \qquad \forall h \in \mathbb{N}.$$

By equality $\rho(\epsilon) \equiv 1/r_\#$ and by taking $\epsilon = 0$, equation (2.65) can be immediately written as the system of integral equations (2.70)–(2.71). The existence and uniqueness of solution for this system are then ensured by Proposition 2.3.2 (ii). Then observe that $N_j\left[\epsilon, \Theta^i_j[\epsilon], \Theta^o_j[\epsilon]\right] = 0$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$ (see Proposition 2.3.2 (iii)). Accordingly, the map $\epsilon \mapsto N_j\left[\epsilon, \Theta^i_j[\epsilon], \Theta^o_j[\epsilon]\right]$ has derivatives which are equal to zero, i.e., $\partial^h_\epsilon(N_j\left[\epsilon, \Theta^i_j[\epsilon], \Theta^o_j[\epsilon]\right]) = 0$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$ and all $h \in \mathbb{N} \setminus \{0\}$. Keeping equality $\rho(\epsilon) \equiv 1/r_\#$ in mind, a straightforward calculation shows that

$$\begin{aligned} &\partial^h_\epsilon\left(N_{j,1}\left[\epsilon, \Theta^i_j[\epsilon], \Theta^o_j[\epsilon]\right]\right)(t) \\ &= \lambda^-\left(\frac{1}{2}\partial^h_\epsilon\Theta^o_j[\epsilon](t) + w_*\left[\partial\Omega, \partial^h_\epsilon\Theta^o_j[\epsilon]\right](t) + \partial^h_\epsilon\left(\epsilon\Lambda_\nu\left[\epsilon, \Theta^o_j[\epsilon]\right]\right)(t)\right) \\ &\quad - \lambda^+\left(-\frac{1}{2}\partial^h_\epsilon\Theta^i_j[\epsilon](t) + w_*\left[\partial\Omega, \partial^h_\epsilon\Theta^i_j[\epsilon]\right](t) + \partial^h_\epsilon\left(\epsilon\Lambda_\nu\left[\epsilon, \Theta^i_j[\epsilon]\right]\right)(t)\right) = 0, \end{aligned} \tag{2.78}$$

$$
\begin{aligned}
&\partial_\epsilon^h \left(N_{j,2}\left[\epsilon, \Theta_j^i[\epsilon], \Theta_j^o[\epsilon]\right]\right)(t) = \lambda^+ \Big(-\frac{1}{2}\partial_\epsilon^h \Theta_j^i[\epsilon](t) + w_*\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right](t) \\
&+\partial_\epsilon^h \left(\epsilon\Lambda_\nu\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)(t)\Big) + \epsilon r_\# \Big(v^+\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right](t) + \partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)(t) \\
&- \fint_{\partial\Omega}\left(v^+\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right] + \partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)\right) d\sigma - v^-\left[\partial\Omega, \partial_\epsilon^h \Theta_j^o[\epsilon]\right](t) \\
&-\partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)(t) + \fint_{\partial\Omega}\left(v^-\left[\partial\Omega, \partial_\epsilon^h \Theta_j^o[\epsilon]\right] + \partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)\right) d\sigma\Big) \\
&+ hr_\# \Big(v^+\left[\partial\Omega, \partial_\epsilon^{h-1}\Theta_j^i[\epsilon]\right](t) + \partial_\epsilon^{h-1}\left(\Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)(t) - \fint_{\partial\Omega}\Big(v^+\left[\partial\Omega, \partial_\epsilon^{h-1}\Theta_j^i[\epsilon]\right] \\
&+\partial_\epsilon^{h-1}\left(\Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)\Big) d\sigma - v^-\left[\partial\Omega, \partial_\epsilon^{h-1}\Theta_j^o[\epsilon]\right](t) - \partial_\epsilon^{h-1}\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)(t) \\
&+ \fint_{\partial\Omega}\left(v^-\left[\partial\Omega, \partial_\epsilon^{h-1}\Theta_j^o[\epsilon]\right] + \partial_\epsilon^{h-1}\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)\right) d\sigma\Big) = 0
\end{aligned}
\tag{2.79}
$$

for all $t \in \partial\Omega$, all $\epsilon \in ]-\epsilon_2, \epsilon_2[$, and all $h \in \mathbb{N} \setminus \{0\}$. Then, by Lemma 2.2.4, one verifies that system (2.78)-(2.79) with $\epsilon = 0$ can be rewritten as (2.72)–(2.73) if $h = 1$, as (2.74)–(2.75) if $h = 2$, and as (2.76)–(2.77) for all $h \in \mathbb{N} \setminus \{0, 1, 2\}$.

One can easily verify that the integral on $\partial\Omega$ of the functions on right-hand sides of (2.72) and (2.73) vanishes. Then, by Lemma A.0.3 in Appendix A one proves that the solution $(\theta_{j,1}^i, \theta_{j,1}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ of (2.72)–(2.73) exists and is unique (we have already observed that the existence is granted by Proposition 2.3.2 (iii)). By Lemma 2.2.4 one also has that

$$
\int_{\partial\Omega} \Lambda_\nu^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right] d\sigma = 0 \qquad \text{and} \qquad \int_{\partial\Omega} \Lambda_\nu^h\left[\theta_{j,0}^o, \dots, \theta_{j,h-2}^o\right] d\sigma = 0
$$

for all $h \in \mathbb{N} \setminus \{0, 1\}$. Then a straightforward computation shows that the right hand sides of (2.74)–(2.77) belong to $C^{0,\alpha}(\partial\Omega)_0$ for all $h \in \mathbb{N} \setminus \{0, 1\}$. Hence, Lemma A.0.3in Appendix A ensures that the solution $(\theta_{j,h}^i, \theta_{j,h}^o) \in (C^{0,\alpha}(\partial\Omega)_0)^2$ of system (2.74)–(2.75) for $k = 2$ and of system (2.76)–(2.77) for all $h \in \mathbb{N} \setminus \{0, 1, 2\}$ exists and is unique (the existence also follows by Proposition 2.3.2 (iii)). The proof is now complete. □

Now, we proceed to compute the series expansion for the effective conductivity under assumption (2.68). To do so, we need the following two lemmas where we compute the power series expansions of two auxiliary maps.

**Lemma 2.3.4.** *Let $k, j \in \{1, 2\}$. Let $\epsilon \mapsto \Theta_j^i[\epsilon]$ and $\epsilon \mapsto \Theta_j^o[\epsilon]$ be as in Proposition 2.3.2. Let $\epsilon_2$ and $\{(\theta_{j,h}^i, \theta_{j,h}^o)\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.3. Let $U_j^+$ be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$
\begin{aligned}
U_j^+[\epsilon](t) :=& v^+\left[\partial\Omega, \Theta_j^i[\epsilon]\right](t) + \Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right](t) \\
&- \fint_{\partial\Omega}\left(v^+\left[\partial\Omega, \Theta_j^i[\epsilon]\right] + \Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right) d\sigma + t_j - \fint_{\partial\Omega} s_j \, d\sigma_s \qquad \forall t \in \partial\Omega.
\end{aligned}
$$

*Then $U_j^+$ is real analytic and there exists $\epsilon_3 \in ]0, \epsilon_2[$ such that*

$$
\begin{aligned}
&\int_{\partial\Omega} U_j^+[\epsilon](t)(\nu_\Omega(t))_k \, d\sigma_t = |\Omega|_2 \delta_{k,j} \\
&\quad + \int_{\partial\Omega} v^+[\partial\Omega, \theta_{j,0}^i](t)(\nu_\Omega(t))_k \, d\sigma_t + \epsilon \int_{\partial\Omega} v^+[\partial\Omega, \theta_{j,1}^i](t)(\nu_\Omega(t))_k \, d\sigma_t \\
&\quad + \sum_{h=2}^{+\infty} \frac{1}{h!}\left(\int_{\partial\Omega}\left(v^+[\partial\Omega, \theta_{j,h}^i](t) + \Lambda^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right](t)\right)(\nu_\Omega(t))_k \, d\sigma_t\right)\epsilon^h
\end{aligned}
\tag{2.80}
$$

*for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, where the series converges uniformly for $\epsilon \in ]-\epsilon_3, \epsilon_3[$.*

*Proof.* We first note that by Dalla Riva and Musolino [35, Thm. 7.1], $U_j^+$ is a real analytic map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$. Therefore, there exist $\epsilon_3 \in ]0, \epsilon_2[$ and a sequence $\{a_{j,h}\}_{h\in\mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$ such that

$$U_j^+[\epsilon](t) = \sum_{h=0}^{+\infty} \frac{a_{j,h}(t)}{h!}\epsilon^h \qquad \forall \epsilon \in ]-\epsilon_3, \epsilon_3[, \quad \forall t \in \partial\Omega,$$

where the series converges uniformly for $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\epsilon = 0$ and by Lemma 2.2.4, we verify that

$$a_{j,0}(t) \equiv v^+[\partial\Omega, \theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}]d\sigma + t_j - \fint_{\partial\Omega} s_j\, d\sigma_s \qquad \forall t \in \partial\Omega\,.$$

In order to compute the other coefficients, we take the derivative of order $h \in \mathbb{N}\setminus\{0\}$ of the map $\epsilon \mapsto U_j^+[\epsilon]$ and we obtain

$$\begin{aligned}\partial_\epsilon^h \left(U_j^+[\epsilon]\right)(t) =& v^+\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right](t) + \partial_\epsilon^h\left(\Lambda[\epsilon, \Theta_j^i[\epsilon]]\right)(t) \\ &- \fint_{\partial\Omega}\left(v^+\left[\partial\Omega, \partial_\epsilon^h\Theta_j^i[\epsilon]\right] + \partial_\epsilon^h\left(\Lambda[\epsilon,\Theta_j^i[\epsilon]]\right)\right)d\sigma \qquad \forall t \in \partial\Omega.\end{aligned}$$

Again, by taking $\epsilon = 0$ and by Lemma 2.2.4, we find

$$\begin{aligned}a_{j,1}(t) =& v^+[\partial\Omega, \theta^i_{j,1}](t) - \fint_{\partial\Omega} v^+\left[\partial\Omega, \theta^i_{j,1}\right]d\sigma \qquad \forall t\in\partial\Omega, \\ a_{j,h}(t) =& v^+[\partial\Omega, \theta^i_{j,h}](t) + \Lambda^h\left[\theta^i_{j,0}, \dots, \theta^i_{j,h-2}\right](t) \\ &- \fint_{\partial\Omega}\left(v^+[\partial\Omega, \theta^i_{j,h}] + \Lambda^h\left[\theta^i_{j,0},\dots,\theta^i_{j,h-2}\right]\right)d\sigma \quad \forall t \in \partial\Omega \quad \forall h \in \mathbb{N}\setminus\{0,1\}.\end{aligned}$$

As a consequence, possibly shrinking $\epsilon_3$, we have

$$\begin{aligned}\int_{\partial\Omega} U_j^+[\epsilon](t)(\nu_\Omega(t))_k\, d\sigma_t =& \int_{\partial\Omega}\sum_{h=0}^{+\infty}\frac{a_{j,h}(t)}{h!}\epsilon^h(\nu_\Omega(t))_k\, d\sigma_t \\ =& \sum_{h=0}^{+\infty}\frac{1}{h!}\left(\int_{\partial\Omega} a_{j,h}(t)(\nu_\Omega(t))_k\, d\sigma_t\right)\epsilon^h,\end{aligned}$$

where the series converges uniformly for $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Then we consider separately the cases $h = 0$, $h = 1$, and $h \in \mathbb{N}\setminus\{0,1\}$, and we have

$$\begin{aligned}&\int_{\partial\Omega} a_{j,0}(t)(\nu_\Omega(t))_k\, d\sigma_t \\ &\quad = \int_{\partial\Omega}\left(v^+[\partial\Omega,\theta^i_{j,0}](t) - \fint_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,0}]d\sigma + t_j - \fint_{\partial\Omega} s_j\, d\sigma_s\right)(\nu_\Omega(t))_k\, d\sigma_t \\ &\quad = \int_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,0}](t)(\nu_\Omega(t))_k\, d\sigma_t - \fint_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,0}]d\sigma\int_{\partial\Omega}(\nu_\Omega(t))_k\, d\sigma_t \\ &\qquad + \int_{\partial\Omega} t_j(\nu_\Omega(t))_k\, d\sigma_t - \fint_{\partial\Omega} s_j\, d\sigma_s\int_{\partial\Omega}(\nu_\Omega(t))_k\, d\sigma_t \\ &\quad = \int_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,0}](t)(\nu_\Omega(t))_k\, d\sigma_t + \int_{\partial\Omega} t_j(\nu_\Omega(t))_k\, d\sigma_t,\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Omega} a_{j,1}(t)(\nu_\Omega(t))_k\, d\sigma_t =& \int_{\partial\Omega}\left(v^+[\partial\Omega,\theta^i_{j,1}](t) - \fint_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,1}]d\sigma\right)(\nu_\Omega(t))_k\, d\sigma_t \\ =& \int_{\partial\Omega} v^+[\partial\Omega,\theta^i_{j,1}](t)(\nu_\Omega(t))_k\, d\sigma_t,\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Omega} a_{j,h}(t)(\nu_\Omega(t))_k\,d\sigma_t &= \int_{\partial\Omega}\Big(v^+[\partial\Omega,\theta^i_{j,h}](t)+\Lambda^h\left[\theta^i_{j,0},\dots,\theta^i_{j,h-2}\right](t)\\ &\quad -\fint_{\partial\Omega}\left(v^+[\partial\Omega,\theta^i_{j,h}]+\Lambda^h\left[\theta^i_{j,0},\dots,\theta^i_{j,h-2}\right]\right)d\sigma\Big)(\nu_\Omega(t))_k\,d\sigma_t\\ &= \int_{\partial\Omega}\left(v^+[\partial\Omega,\theta^i_{j,h}](t)+\Lambda^h\left[\theta^i_{j,0},\dots,\theta^i_{j,h-2}\right](t)\right)(\nu_\Omega(t))_k\,d\sigma_t.\end{aligned}$$

Moreover, by the Divergence Theorem one verifies that

$$\int_{\partial\Omega} t_j(\nu_\Omega(t))_k\,d\sigma_t = \int_\Omega \frac{\partial t_j}{\partial t_k}\,dt = |\Omega|_2\delta_{k,j}.$$

Accordingly, the validity of (2.80) follows. □

**Lemma 2.3.5.** *Let $k,j\in\{1,2\}$. Let $\epsilon\mapsto\Theta^i_j[\epsilon]$ and $\epsilon\mapsto\Theta^o_j[\epsilon]$ be as in Proposition 2.3.2. Let $\epsilon_2$ and $\{(\theta^i_{j,h},\theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.3. Let $V^-_j$ be the map from $]-\epsilon_2,\epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$V^-_j[\epsilon](t) := v^-\left[\partial\Omega,\Theta^o_j[\epsilon]\right](t)+\Lambda\left[\epsilon,\Theta^o_j[\epsilon]\right](t)+t_j \qquad \forall t\in\partial\Omega\,.$$

*Then $V^-_j$ is real analytic and there exists $\epsilon_4\in]0,\epsilon_2[$ such that*

$$\begin{aligned}\int_{\partial\Omega} V^-_j[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t &= |\Omega|_2\delta_{k,j}\\ &+\int_{\partial\Omega} v^-[\partial\Omega,\theta^o_{j,0}](t)(\nu_\Omega(t))_k\,d\sigma_t+\epsilon\int_{\partial\Omega} v^-[\partial\Omega,\theta^o_{j,1}](t)(\nu_\Omega(t))_k\,d\sigma_t\\ &+\sum_{h=2}^{+\infty}\frac{1}{h!}\left(\int_{\partial\Omega}\left(v^-[\partial\Omega,\theta^o_{j,h}](t)+\Lambda^h\left[\theta^o_{j,0},\dots,\theta^o_{j,h-2}\right](t)\right)(\nu_\Omega(t))_k\,d\sigma_t\right)\epsilon^h,\end{aligned}\tag{2.81}$$

*where the series converges uniformly for $\epsilon\in]-\epsilon_4,\epsilon_4[$.*

*Proof.* We first note that by Dalla Riva and Musolino [35, Thm. 7.2 (ii)] $V^-_j$ is a real analytic map from $]-\epsilon_2,\epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$. Therefore, there exist $\epsilon_4\in]0,\epsilon_2[$ and a sequence $\{b_{j,h}\}_{h\in\mathbb{N}}$ in $C^{1,\alpha}(\partial\Omega)$ such that

$$V^-_j[\epsilon](t)=\sum_{h=0}^{+\infty}\frac{b_{j,h}(t)}{h!}\epsilon^h \qquad \forall\epsilon\in]-\epsilon_4,\epsilon_4[\quad \forall t\in\partial\Omega,$$

where the series converges uniformly for $\epsilon\in]-\epsilon_4,\epsilon_4[$. By taking $\epsilon=0$ and by Lemma 2.2.4, we verify that

$$b_{j,0}(t)=v^-[\partial\Omega,\theta^o_{j,0}](t)+t_j \qquad \forall t\in\partial\Omega\,.$$

In order to compute the other coefficients, we take the derivative of order $h\in\mathbb{N}\setminus\{0\}$ of the map $\epsilon\mapsto V^-_j[\epsilon]$ and we obtain

$$\partial^h_\epsilon\left(V^-_j[\epsilon]\right)(t)=v^-[\partial\Omega,\partial^h_\epsilon\Theta^o_j[\epsilon]](t)+\partial^h_\epsilon\left(\Lambda[\epsilon,\Theta^o_j[\epsilon]]\right)(t) \qquad \forall t\in\partial\Omega\,.$$

Again, by taking $\epsilon=0$ and by Lemma 2.2.4, we find

$$\begin{aligned}&b_{j,1}(t)=v^-[\partial\Omega,\theta^o_{j,1}](t) \qquad \forall t\in\partial\Omega,\\ &b_{j,h}(t)=v^-[\partial\Omega,\theta^o_{j,h}](t)+\Lambda^h\left[\theta^o_{j,0},\dots,\theta^o_{j,h-2}\right](t) \qquad \forall t\in\partial\Omega\end{aligned}$$

and for all $h \in \mathbb{N} \setminus \{0,1\}$.

As a consequence, possibly shrinking $\epsilon_4$, we have

$$\int_{\partial\Omega} V_j^-[\epsilon](t)(\nu_\Omega(t))_k \, d\sigma_t = \int_{\partial\Omega} \sum_{h=0}^{+\infty} \frac{b_{j,h}(t)}{h!} \epsilon^h (\nu_\Omega(t))_k \, d\sigma_t = \sum_{h=0}^{+\infty} \frac{1}{h!} \left( \int_{\partial\Omega} b_{j,h}(t)(\nu_\Omega(t))_k \, d\sigma_t \right) \epsilon^h$$

where the series converges uniformly for $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Then we consider cases $h=0$, $h=1$, and $h \in \mathbb{N} \setminus \{0,1\}$ separately and we have

$$\begin{aligned}
\int_{\partial\Omega} b_{j,0}(t)(\nu_\Omega(t))_k \, d\sigma_t &= \int_{\partial\Omega} (v^-[\partial\Omega, \theta^o_{j,0}](t) + t_j)(\nu_\Omega(t))_k \, d\sigma_t \\
&= \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t + |\Omega|_2 \delta_{k,j}, \\
\int_{\partial\Omega} b_{j,1}(t)(\nu_\Omega(t))_k \, d\sigma_t &= \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}](t)(\nu_\Omega(t))_k \, d\sigma_t \\
\int_{\partial\Omega} b_{j,h}(t)(\nu_\Omega(t))_k \, d\sigma_t &= \int_{\partial\Omega} \left( v^-[\partial\Omega, \theta^o_{j,h}](t) + \Lambda^h \left[ \theta^o_{j,0}, \dots, \theta^o_{j,h-2} \right](t) \right) (\nu_\Omega(t))_k \, d\sigma_t.
\end{aligned}$$

Accordingly, the validity of (2.81) follows. □

We are now ready to prove the main result of this section, where we expand $\lambda^{\text{eff}}_{kj}[\epsilon]$ as a power series and we provide explicit and constructive expressions for the coefficients of the series.

**Theorem 2.3.6.** *Let $k, j \in \{1,2\}$.Let $\epsilon_2$ and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.3. Then there exists $\epsilon_5 \in ]0, \epsilon_2[$ such that*

$$\lambda^{\text{eff}}_{kj}[\epsilon] = \lambda^- \delta_{k,j} + \epsilon^2 \frac{1}{|Q|_2} \sum_{h=0}^{+\infty} \frac{c_{(h,j),h}}{h!} \epsilon^h$$

*for all $\epsilon \in ]0, \epsilon_5[$, where*

$$\begin{aligned}
c_{(k,j),0} =& \lambda^+ \int_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t + (\lambda^+ - \lambda^-)|\Omega|_2 \delta_{k,j} \\
& - \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k \, d\sigma_t + \int_{\partial\Omega} f(t) t_k \, d\sigma_t, \\
c_{(k,j),1} =& \lambda^+ \int_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,1}](t)(\nu_\Omega(t))_k \, d\sigma_t - \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}](t)(\nu_\Omega(t))_k \, d\sigma_t, \\
c_{(h,j),h} =& \lambda^+ \int_{\partial\Omega} \left( v^+[\partial\Omega, \theta^i_{j,h}](t) + \Lambda^h \left[ \theta^i_{j,0}, \dots, \theta^i_{j,h-2} \right](t) \right) (\nu_\Omega(t))_k \, d\sigma_t \\
& - \lambda^- \int_{\partial\Omega} \left( v^-[\partial\Omega, \theta^o_{j,h}](t) + \Lambda^h \left[ \theta^o_{j,0}, \dots, \theta^o_{j,h-2} \right](t) \right) (\nu_\Omega(t))_k \, d\sigma_t,
\end{aligned}$$

*for all $h \in \mathbb{N} \setminus \{0,1\}$.*

*Proof.* By Dalla Riva and Musolino [35, Thm. 8.1], if we set

$$\Lambda_{kj}[\epsilon] := \frac{\lambda^+}{|Q|_2} \int_{\partial\Omega} U_j^+[\epsilon](t)(\nu_\Omega(t))_k \, d\sigma_t - \frac{\lambda^-}{|Q|_2} \int_{\partial\Omega} V_j^-[\epsilon](t)(\nu_\Omega(t))_k \, d\sigma_t + \frac{1}{|Q|_2} \int_{\partial\Omega} f(t) t_k \, d\sigma_t$$

for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$, then we have

$$\lambda^{\text{eff}}_{kj}[\epsilon] = \lambda^- \delta_{k,j} + \epsilon^2 \Lambda_{kj}[\epsilon] \qquad \forall \epsilon \in ]0, \epsilon_2[.$$

Then the definition of $\{c_{(h,j),h}\}_{h\in\mathbb{N}}$ and Lemmas 2.3.4, 2.3.5 imply the validity of the statement. □

### Application to the effective conductivity in the composite with inclusions in the form of a disc

Introducing more restrictive assumptions, it is possible to obtain simpler expressions for the coefficients $c_{(h,j),h}$. For example, in this subsection we will assume that

$$Q \equiv ]0,1[\times]0,1[, \quad f \equiv 0, \quad g \text{ is a real constant, and } \Omega \equiv \mathbb{B}_2(0,1) \tag{2.82}$$

(see Figure 2.1 in Section 2.2), and will we write the first five coefficients as simple functions of $r_\#$, $\lambda^+$, and $\lambda^-$. We begin by observing that, with assumptions (2.82) the system of integral equations (2.70)–(2.71) takes the following form

$$-\frac{1}{2}\theta^i_{j,0}(t) + w_*[\partial\Omega, \theta^i_{j,0}](t) = -(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \tag{2.83}$$

$$\frac{1}{2}\theta^o_{j,0}(t) + w_*[\partial\Omega, \theta^o_{j,0}](t) = -(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \tag{2.84}$$

which is equivalent to the system of the following equations

$$\begin{aligned} \frac{\partial v^+[\partial\Omega, \theta^i_{j,0}]}{\partial \nu_\Omega}(t) &= -(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \\ \frac{\partial v^-[\partial\Omega, \theta^o_{j,0}]}{\partial \nu_\Omega}(t) &= -(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega\,. \end{aligned}$$

Hence, one can verify that if $(\theta^i_{j,0}, \theta^o_{j,0})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (2.83)-(2.84), then there exists a constant $c_0 \in \mathbb{R}$ such that

$$\begin{aligned} v^+[\partial\Omega, \theta^i_{j,0}](t) &= -t_j + c_0 \quad \forall t \in \mathrm{cl}\Omega, \\ v^-[\partial\Omega, \theta^o_{j,0}](t) &= \frac{t_j}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega. \end{aligned} \tag{2.85}$$

We now note that, since $\Omega$ is a unite ball, we have

$$w_*[\partial\Omega, \theta] = 0 \qquad \forall \theta \in C^{0,\alpha}(\partial\Omega)_0 \tag{2.86}$$

(see equality (2.54)). Then by (2.83)-(2.84) and (2.86) we have that

$$\theta^i_{j,0}(t) = 2t_j, \qquad \theta^o_{j,0}(t) = -2t_j \qquad \forall t \in \partial\Omega. \tag{2.87}$$

Next, taking equalities (2.85) into account and since $t_j = (\nu_\Omega(t))_j$ on $\partial\Omega$, the system of integral equations (2.72)–(2.73) takes the following form

$$-\frac{1}{2}\theta^i_{j,1}(t) + w_*[\partial\Omega, \theta^i_{j,1}](t) = \frac{2r_\#}{\lambda^+}(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \tag{2.88}$$

$$\frac{1}{2}\theta^o_{j,1}(t) + w_*[\partial\Omega, \theta^o_{j,1}](t) = \frac{2r_\#}{\lambda^-}(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \tag{2.89}$$

or, equivalently,

$$\begin{aligned} \frac{\partial v^+[\partial\Omega, \theta^i_{j,1}]}{\partial \nu_\Omega}(t) &= \frac{2r_\#}{\lambda^+}(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega, \\ \frac{\partial v^-[\partial\Omega, \theta^o_{j,1}]}{\partial \nu_\Omega}(t) &= \frac{2r_\#}{\lambda^-}(\nu_\Omega(t))_j \qquad \forall t \in \partial\Omega\,. \end{aligned}$$

If $(\theta^i_{j,1}, \theta^o_{j,1})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (2.88)-(2.89), then one can verify that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\begin{aligned} v^+[\partial\Omega, \theta^i_{j,1}](t) &= \frac{2r_\#}{\lambda^+}t_j + c_1 \quad \forall t \in \mathrm{cl}\Omega, \\ v^-[\partial\Omega, \theta^o_{j,1}](t) &= -\frac{2r_\#}{\lambda^-}\frac{t_j}{|t|^2} \quad \forall t \in \mathbb{R}^2 \setminus \Omega, \end{aligned} \tag{2.90}$$

and, moreover, by (2.88)-(2.89) and (2.86), one has

$$\theta^i_{j,1}(t) = -\frac{4r_\#}{\lambda^+}t_j, \qquad \theta^o_{j,1}(t) = \frac{4r_\#}{\lambda^-}t_j \qquad \forall t \in \partial\Omega. \tag{2.91}$$

Next, we rewrite the system of integral equations (2.74)–(2.75) as follows

$$\begin{aligned}\frac{\partial v^+[\partial\Omega, \theta^i_{j,2}]}{\partial\nu_\Omega}(t) &= -\Lambda^2_\nu\left[\theta^i_{j,0}\right](t) - \frac{2r_\#}{\lambda^+}\Big(v^+[\partial\Omega, \theta^i_{j,1}](t)\\ &-\fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}]d\sigma\Big) \quad \forall t \in \partial\Omega,\end{aligned} \tag{2.92}$$

$$\begin{aligned}\frac{\partial v^-[\partial\Omega, \theta^o_{j,2}]}{\partial\nu_\Omega}(t) &= -\Lambda^2_\nu\left[\theta^o_{j,0}\right](t) - \frac{2r_\#}{\lambda^-}\Big(v^+[\partial\Omega, \theta^i_{j,1}](t)\\ &-\fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,1}]d\sigma - v^-[\partial\Omega, \theta^o_{j,1}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,1}]d\sigma\Big) \qquad \forall t \in \partial\Omega,\end{aligned} \tag{2.93}$$

and, moreover, for $h = 3$ we rewrite the system of integral equations (2.76)–(2.77) as follows

$$\begin{aligned}\frac{\partial v^+[\partial\Omega, \theta^i_{j,3}]}{\partial\nu_\Omega}(t) &= -\Lambda^3_\nu\left[\theta^i_{j,0}, \theta^i_{j,1}\right](t) - \frac{3r_\#}{\lambda^+}\Big(v^+[\partial\Omega, \theta^i_{j,2}](t)\\ &-\fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,2}]d\sigma - v^-[\partial\Omega, \theta^o_{j,2}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,2}]d\sigma\Big)\\ &-\frac{3r_\#}{\lambda^+}\left(\Lambda^2[\theta^i_{j,0}](t) - \fint_{\partial\Omega}\Lambda^2[\theta^i_{j,0}]d\sigma - \Lambda^2[\theta^o_{j,0}](t) + \fint_{\partial\Omega}\Lambda^2[\theta^o_{j,0}]d\sigma\right),\end{aligned} \tag{2.94}$$

$$\begin{aligned}\frac{\partial v^-[\partial\Omega, \theta^o_{j,3}]}{\partial\nu_\Omega}(t) &= -\Lambda^3_\nu\left[\theta^o_{j,0}, \theta^o_{j,1}\right](t) - \frac{3r_\#}{\lambda^-}\Big(v^+[\partial\Omega, \theta^i_{j,2}](t)\\ &-\fint_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,2}]d\sigma - v^-[\partial\Omega, \theta^o_{j,2}](t) + \fint_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,2}]d\sigma\Big)\\ &-\frac{3r_\#}{\lambda^-}\left(\Lambda^2[\theta^i_{j,0}](t) - \fint_{\partial\Omega}\Lambda^2[\theta^i_{j,0}]d\sigma - \Lambda^2[\theta^o_{j,0}](t) + \fint_{\partial\Omega}\Lambda^2[\theta^o_{j,0}]d\sigma\right)\end{aligned} \tag{2.95}$$

for all $t \in \partial\Omega$.

We now exploit (2.92)-(2.93) and (2.94)-(2.95) to add other two explicit terms in our expansion. To do so, we first note that kipping in mind that $\Omega$ is a unit disk, using the definition of $\Lambda^3_\nu$ (see Section 2.2.1), and equalities (2.10)-(2.12), by a straightforward computation, we have the following

$$\begin{aligned}&\Lambda^3_\nu[\theta_1](t)\\ &=3\sum_{j=1}^{2}\binom{2}{j}\sum_{h=0}^{j}\binom{j}{h}(\partial_1^h\partial_2^{j-h}DR_q)(0)\nu_\Omega(t)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{j-h}\theta_{2-j}(s)d\sigma_s\\ &=3\binom{2}{1}\sum_{h=0}^{1}\binom{1}{h}(\partial_1^h\partial_2^{1-h}DR_q)(0)\nu_\Omega(t)\int_{\partial\Omega}(t_1-s_1)^h(t_2-s_2)^{1-h}\theta_1(s)d\sigma_s\\ &=6\left(-\frac{1}{2}t_1\int_{\partial\Omega}(t_1-s_1)\theta_1(s)d\sigma_s - \frac{1}{2}t_2\int_{\partial\Omega}(t_2-s_2)\theta_1(s)d\sigma_s\right)\\ &=3\int_{\partial\Omega}(t_1s_1+t_2s_2)\theta_1(s)d\sigma_s \qquad \forall t \in \partial\Omega, \quad \forall \theta_1 \in C^{0,\alpha}(\partial\Omega)_0.\end{aligned} \tag{2.96}$$

Then, using equalities (2.55), (2.56), (2.96), (2.87), and (2.91), one can show that

$$\begin{aligned}&\Lambda^2[\theta^i_{j,0}](t) = 2\pi t_j, &&\Lambda^2[\theta^o_{j,0}](t) = -2\pi t_j &&\forall t \in \partial\Omega,\\ &\Lambda^2_\nu[\theta^i_{j,0}](t) = 2\pi t_j, &&\Lambda^2_\nu[\theta^o_{j,0}](t) = -2\pi t_j &&\forall t \in \partial\Omega,\\ &\Lambda^3_\nu[\theta^i_{j,0}, \theta^i_{j,1}](t) = -\frac{12\pi r_\#}{\lambda^+}t_j, &&\Lambda^3_\nu[\theta^o_{j,0}, \theta^o_{j,1}](t) = \frac{12\pi r_\#}{\lambda^-}t_j &&\forall t \in \partial\Omega.\end{aligned} \tag{2.97}$$

Again, if $(\theta^i_{j,2},\theta^o_{j,2})$ and $(\theta^i_{j,3},\theta^o_{j,3})$ are the solutions in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the systems (2.92)-(2.93) and (2.94)-(2.95), respectively, then, taking equalities (2.90) and (2.97) into account, one can verify that there exists a real constant $c_2$ such that

$$\begin{aligned} v^+[\partial\Omega,\theta^i_{j,2}](t) =& -2\left(\frac{2(r_\#)^2}{\lambda^+}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)+\pi\right)t_j+c_2 \quad \forall t\in \mathrm{cl}\Omega,\\ v^-[\partial\Omega,\theta^o_{j,2}](t) =& 2\left(\frac{2(r_\#)^2}{\lambda^-}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)-\pi\right)\frac{t_j}{|t|^2} \quad \forall t\in\mathbb{R}^2\setminus\Omega\,. \end{aligned} \tag{2.98}$$

Similarly, by (2.98) one verifies that there exists a real constant $c_3$ such that

$$\begin{aligned} v^+[\partial\Omega,\theta^i_{j,3}](t) =& \frac{12(r_\#)^3}{\lambda^+}\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)^2 t_j+c_3 \quad \forall t\in \mathrm{cl}\Omega,\\ v^-[\partial\Omega,\theta^o_{j,3}](t) =& -\frac{12r_\#}{\lambda^-}\left((r_\#)^2\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)^2-2\pi\right)\frac{t_j}{|t|^2} \quad \forall t\in\mathbb{R}^2\setminus\Omega. \end{aligned} \tag{2.99}$$

Then, by Theorem 2.3.6 and equalities (2.85), (2.90), (2.98), and (2.99) we deduce that if $k,j\in\{1,2\}$ then

$$\begin{aligned} \lambda^{\mathrm{eff}}_{kj}[\epsilon] = \Bigg(&\lambda^- - 2\pi\,\lambda^-\epsilon^2 + 4\pi\, r_\#\epsilon^3 - 4\pi\left((r_\#)^2\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)-\frac{1}{2}\pi\lambda^-\right)\epsilon^4\\ &+4\pi r_\#\left((r_\#)^2\left(\frac{1}{\lambda^+}+\frac{1}{\lambda^-}\right)^2-2\pi\right)\epsilon^5\Bigg)\delta_{k,j}+O(\epsilon^6) \end{aligned}$$

as $\epsilon\to 0^+$.

### 2.3.3 Power series expansions for the variable interfacial thermal resistance

Throughout this section we consider the case where

$$\rho(\epsilon)\equiv \epsilon/r_\# \qquad \forall\epsilon\in]-\epsilon_0,\epsilon_0[. \tag{2.100}$$

As done in Section 2.3.2, in order to compute the asymptotic expansion of the effective conductivity under assumption (2.100), we start with the following proposition, where we identify the coefficients of the power series expansions of $\Theta^i_j[\epsilon]$ and of $\Theta^o_j[\epsilon]$ in terms of the solutions of systems of integral equations

**Proposition 2.3.7.** *Let $\epsilon_1$, $\epsilon\mapsto\Theta^i_j[\epsilon]$, and $\epsilon\mapsto\Theta^o_j[\epsilon]$ be as in Proposition 2.3.2. Then there exist $\epsilon_2\in]0,\epsilon_1[$ and a sequence $\{(\theta^i_{j,h},\theta^o_{j,h})\}_{h\in\mathbb{N}}$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$ such that*

$$\Theta^i_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^i_{j,h}}{h!}\epsilon^h \qquad \textit{and} \qquad \Theta^o_j[\epsilon]=\sum_{h=0}^{+\infty}\frac{\theta^o_{j,h}}{h!}\epsilon^h \qquad \forall\epsilon\in]-\epsilon_2,\epsilon_2[, \tag{2.101}$$

*where the two series converge uniformly for $\epsilon\in]-\epsilon_2,\epsilon_2[$ in $(C^{0,\alpha}(\partial\Omega)_0)^2$. Moreover, the following statements hold.*

*(i) The pair of functions $(\theta^i_{j,0},\theta^o_{j,0})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations*

$$\begin{aligned} \lambda^-\left(\frac{1}{2}\theta^o_{j,0}(t)+w_*[\partial\Omega,\theta^o_{j,0}](t)\right)-\lambda^+\left(-\frac{1}{2}\theta^i_{j,0}(t)+w_*[\partial\Omega,\theta^i_{j,0}](t)\right)&\\ -f(t)+(\lambda^- -\lambda^+)(\nu_\Omega(t))_j=0&, \end{aligned} \tag{2.102}$$

$$\begin{aligned}\lambda^+\Big(-\frac{1}{2}\theta^i_{j,0}(t)+w_*[\partial\Omega,\theta^i_{j,0}](t)\Big)+r_\#\Big(v^+[\partial\Omega,\theta^i_{j,0}](t)\\-\int_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,0}]d\sigma-v^-[\partial\Omega,\theta^o_{j,0}](t)+\int_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,0}]d\sigma\Big)\\-g(t)+\int_{\partial\Omega}g\,d\sigma+\lambda^+(\nu_\Omega(t))_j=0\end{aligned}\tag{2.103}$$

*for all $t\in\partial\Omega$.*

*(ii) For all $h\in\mathbb{N}$, we have $(\theta^i_{j,2h+1},\theta^o_{j,2h+1})=(0,0)$.*

*(iii) For all $h\in\mathbb{N}\setminus\{0\}$ the pair of functions $(\theta^i_{j,2h},\theta^o_{j,2h})$ is the unique solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of the following system of integral equations which involves $\{(\theta^i_{j,2h},\theta^o_{j,2h})\}_{h=0}^{k-1}$*

$$\begin{aligned}\lambda^-\Big(\frac{1}{2}\theta^o_{j,2h}(t)+w_*[\partial\Omega,\theta^o_{j,2h}](t)\Big)-\lambda^+\Big(-\frac{1}{2}\theta^i_{j,2h}(t)+w_*[\partial\Omega,\theta^i_{j,2h}](t)\Big)\\=\lambda^+\Lambda^{2h}_\nu[\theta^i_{j,0},\dots,\theta^i_{j,2h-2}](t)-\lambda^-\Lambda^{2h}_\nu[\theta^o_{j,0},\dots,\theta^o_{j,2h-2}](t),\end{aligned}\tag{2.104}$$

$$\begin{aligned}\lambda^+\Big(-\frac{1}{2}\theta^i_{j,2h}(t)+w_*[\partial\Omega,\theta^i_{j,2h}](t)\Big)+r_\#\Big(v^+[\partial\Omega,\theta^i_{j,2h}](t)\\-\int_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,2h}]d\sigma-v^-[\partial\Omega,\theta^o_{j,2h}](t)\\+\int_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,2h}]d\sigma\Big)=-\lambda^+\Lambda^{2h}_\nu[\theta^i_{j,0},\dots,\theta^i_{j,2h-2}](t)\\-r_\#\Big(\Lambda^{2h}[\theta^i_{j,0},\dots,\theta^i_{j,2h-2}](t)-\int_{\partial\Omega}\Lambda^{2h}[\theta^i_{j,0},\dots,\theta^i_{j,2h-2}]d\sigma\\-\Lambda^{2h}[\theta^o_{j,0},\dots,\theta^o_{j,2h-2}](t)+\int_{\partial\Omega}\Lambda^{2h}[\theta^o_{j,0},\dots,\theta^o_{j,2h-2}]d\sigma\Big)\end{aligned}\tag{2.105}$$

*for all $t\in\partial\Omega$, where we can take $(\theta^i_{j,2h+1},\theta^o_{j,2h+1})=(0,0)$ for all $h\in\mathbb{N}$.*

*Proof.* The existence of $\epsilon_2$ and $\{(\theta^i_{j,h},\theta^o_{j,h})\}_{h\in\mathbb{N}}$ for which (2.101) holds true is granted by Proposition 2.3.2 (iii). By equality $\rho(\epsilon)\equiv\epsilon/r_\#$ and by taking $\epsilon=0$, equation (2.65) can be written as the system of integral equations (2.102)–(2.103). The uniqueness of the solution for this system is then ensured by Proposition 2.3.2 (ii) (see also Dalla Riva and Musolino [35, Prop. 5.2]). Next, we observe that $N_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]=0$ for all $\epsilon\in]-\epsilon_2,\epsilon_2[$ (see Proposition 2.3.2 (iii)). Accordingly, the map $\epsilon\mapsto N_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]$ has all the derivatives equal to zero, i.e., $\partial^h_\epsilon(N_j\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right])=0$ for all $\epsilon\in]-\epsilon_2,\epsilon_2[$ and all $h\in\mathbb{N}\setminus\{0\}$. Keeping equality $\rho(\epsilon)\equiv\epsilon/r_\#$ in mind, a straightforward calculation shows that

$$\begin{aligned}&\partial^h_\epsilon\left(N_{j,1}\left[\epsilon,\Theta^i_j[\epsilon],\Theta^o_j[\epsilon]\right]\right)(t)\\&=\lambda^-\left(\frac{1}{2}\partial^h_\epsilon\Theta^o_j[\epsilon](t)+w_*\left[\partial\Omega,\partial^h_\epsilon\Theta^o_j[\epsilon]\right](t)+\partial^h_\epsilon\left(\epsilon\Lambda_\nu\left[\epsilon,\Theta^o_j[\epsilon]\right]\right)(t)\right)\\&-\lambda^+\left(-\frac{1}{2}\partial^h_\epsilon\Theta^i_j[\epsilon](t)+w_*\left[\partial\Omega,\partial^h_\epsilon\Theta^i_j[\epsilon]\right](t)+\partial^h_\epsilon\left(\epsilon\Lambda_\nu\left[\epsilon,\Theta^i_j[\epsilon]\right]\right)(t)\right)=0,\end{aligned}\tag{2.106}$$

$$
\begin{aligned}
&\partial_\epsilon^h \left(N_{j,2}\left[\epsilon, \Theta_j^i[\epsilon], \Theta_j^o[\epsilon]\right]\right)(t) \\
&\quad = \lambda^+ \left(-\frac{1}{2}\partial_\epsilon^h \Theta_j^i[\epsilon](t) + w_*\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right](t) + \partial_\epsilon^h\left(\epsilon\Lambda_\nu\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)(t)\right) \\
&\quad + r_\#\Bigg(v^+\left[\partial\Omega, \partial_\epsilon^h \Theta_j^i[\epsilon]\right](t) + \partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^i[\epsilon]\right]\right)(t) \\
&\quad - \int_{\partial\Omega}\left(v^+\left[\partial\Omega, \partial_\epsilon^h\Theta_j^i[\epsilon]\right] + \partial_\epsilon^h\left(\Lambda[\epsilon, \Theta_j^i[\epsilon]]\right)\right)d\sigma - v^-\left[\partial\Omega, \partial_\epsilon^h \Theta_j^o[\epsilon]\right](t) \\
&\quad -\partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)(t) + \int_{\partial\Omega}\left(v^-\left[\partial\Omega, \partial_\epsilon^h\Theta_j^o[\epsilon]\right] + \partial_\epsilon^h\left(\Lambda\left[\epsilon, \Theta_j^o[\epsilon]\right]\right)\right)d\sigma\Bigg) = 0
\end{aligned} \tag{2.107}
$$

for all $t \in \partial\Omega$, all $\epsilon \in ]-\epsilon_2, \epsilon_2[$, and all $h \in \mathbb{N}\setminus\{0\}$. By taking $\epsilon = 0$ in (2.106)–(2.107) and noting that $(\theta_{j,h}^i, \theta_{j,h}^o) = (\partial_\epsilon^h\Theta_j^i[0], \partial_\epsilon^h\Theta_j^o[0])$ for all $h \in \mathbb{N}$, we deduce that $(\theta_{j,1}^i, \theta_{j,1}^o)$ is a solution of the following system

$$
\lambda^-\left(\frac{1}{2}\theta_{j,1}^o(t) + w_*[\partial\Omega, \theta_{j,1}^o](t)\right) - \lambda^+\left(-\frac{1}{2}\theta_{j,1}^i(t) + w_*[\partial\Omega, \theta_{j,1}^i](t)\right) = 0, \tag{2.108}
$$

$$
\begin{aligned}
\lambda^+\left(-\frac{1}{2}\theta_{j,1}^i(t) + w_*[\partial\Omega, \theta_{j,1}^i](t)\right) &+ r_\#\Bigg(v^+[\partial\Omega, \theta_{j,1}^i](t) - \int_{\partial\Omega} v^+[\partial\Omega, \theta_{j,1}^i]d\sigma \\
&-v^-[\partial\Omega, \theta_{j,1}^o](t) + \int_{\partial\Omega} v^-[\partial\Omega, \theta_{j,1}^o]d\sigma\Bigg) = 0
\end{aligned} \tag{2.109}
$$

for all $t \in \partial\Omega$ (see also Lemma 2.2.4), and that, for $h \in \mathbb{N}\setminus\{0,1\}$, the pair $(\theta_{j,h}^i, \theta_{j,h}^o)$ is a solution of the following system

$$
\begin{aligned}
&\lambda^-\left(\frac{1}{2}\theta_{j,h}^o(t) + w_*[\partial\Omega, \theta_{j,h}^o](t)\right) - \lambda^+\left(-\frac{1}{2}\theta_{j,h}^i(t) + w_*[\partial\Omega, \theta_{j,h}^i](t)\right) \\
&\quad =\lambda^+\Lambda_\nu^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right](t) - \lambda^-\Lambda_\nu^h\left[\theta_{j,0}^o, \dots, \theta_{j,h-2}^o\right](t) \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{2.110}
$$

$$
\begin{aligned}
\lambda^+&\left(-\frac{1}{2}\theta_{j,h}^i(t) + w_*[\partial\Omega, \theta_{j,h}^i](t)\right) + r_\#\Bigg(v^+[\partial\Omega, \theta_{j,h}^i](t) - \int_{\partial\Omega} v^+[\partial\Omega, \theta_{j,h}^i]d\sigma \\
&-v^-[\partial\Omega, \theta_{j,h}^o](t) + \int_{\partial\Omega} v^-[\partial\Omega, \theta_{j,h}^o]d\sigma\Bigg) = -\lambda^+\Lambda_\nu^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right](t) \\
&- r_\#\Bigg(\Lambda^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right](t) - \int_{\partial\Omega}\Lambda^h\left[\theta_{j,0}^i, \dots, \theta_{j,h-2}^i\right]d\sigma \\
&-\Lambda^h\left[\theta_{j,0}^o, \dots, \theta_{j,h-2}^o\right](t) + \int_{\partial\Omega}\Lambda^h\left[\theta_{j,0}^o, \dots, \theta_{j,h-2}^o\right]d\sigma\Bigg) \qquad \forall t \in \partial\Omega.
\end{aligned} \tag{2.111}
$$

Since the right-hand sides of equalities (2.108)–(2.109) and (2.110)–(2.111) belong to the space $C^{0,\alpha}(\partial\Omega)_0$, the uniqueness of the solution to (2.108)–(2.109) and of the solution to (2.110)–(2.111) follows by Dalla Riva and Musolino [35, Prop. 5.2] (we have already observed that the existence is granted by Proposition 2.3.2 (iii)). Moreover, $(\theta_{j,1}^i, \theta_{j,1}^o) = (0,0)$. Also, by Lemma 2.2.4 and by the uniqueness of the solution to system (2.110)–(2.111), one can verify that $(\theta_{2h+1}^i, \theta_{2h+1}^o) = (0,0)$ for all $h \in \mathbb{N}\setminus\{0\}$. The validity of the proposition is now proved. □

By Proposition 2.3.7, we immediately deduce the validity of the following.

**Corollary 2.3.8.** *Let the assumptions of Proposition 2.3.7 hold. Then*

$$
\Theta_j^i[\epsilon] = \sum_{h=0}^{+\infty}\frac{\theta_{j,2h}^i}{(2h)!}\epsilon^{2h} \qquad \textit{and} \qquad \Theta_j^o[\epsilon] = \sum_{h=0}^{+\infty}\frac{\theta_{j,2h}^o}{(2h)!}\epsilon^{2h} \qquad \forall\epsilon \in ]-\epsilon_2, \epsilon_2[,
$$

*where* $\{(\theta_{j,2h}^i, \theta_{j,2h}^o)\}_{h\in\mathbb{N}}$ *is as in Proposition 2.3.7.*

Then, by exploiting Proposition 2.3.7, we can prove the following Lemmas 2.3.9 and 2.3.10. The proofs can be implemented by a straightforward modification of the proofs of the analogous Lemmas 2.3.4 and 2.3.5 and it is accordingly omitted.

**Lemma 2.3.9.** *Let $k,j \in \{1,2\}$. Let $\epsilon \mapsto \Theta^i_j[\epsilon]$ and $\epsilon \mapsto \Theta^o_j[\epsilon]$ be as in Proposition 2.3.2. Let $\epsilon_2$ and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.7. Let $U^+_j$ be the map from $]-\epsilon_2, \epsilon_2[$ to $C^{1,\alpha}(\partial\Omega)$ defined as in Lemma 2.3.4. Then $U^+_j$ is real analytic and there exists $\epsilon_3 \in ]0, \epsilon_2[$ such that*

$$\begin{aligned}\int_{\partial\Omega} U^+_j[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t &= \int_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}](t)(\nu_\Omega(t))_k\,d\sigma_t + |\Omega|_2\delta_{k,j}\\ &+ \sum_{h=1}^{+\infty} \frac{1}{(2h)!}\left(\int_{\partial\Omega}\left(v^+[\partial\Omega, \theta^i_{j,2h}](t) + \Lambda^{2h}[\theta^i_{j,0}, \dots, \theta^i_{j,2h-2}](t)\right)(\nu_\Omega(t))_k\,d\sigma_t\right)\epsilon^{2h}\end{aligned}$$

*for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, where the series converges uniformly for $\epsilon \in ]-\epsilon_3, \epsilon_3[$.*

**Lemma 2.3.10.** *Let $k,j \in \{1,2\}$. Let $\epsilon \mapsto \Theta^i_j[\epsilon]$ and $\epsilon \mapsto \Theta^o_j[\epsilon]$ be as in Proposition 2.3.2. Let $\epsilon_2$ and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.7. Let $V^-_j$ defined as in Lemma 2.3.5. Then $V^-_j$ is real analytic and there exists $\epsilon_4 \in ]0, \epsilon_2[$ such that*

$$\begin{aligned}\int_{\partial\Omega} V^-_j[\epsilon](t)(\nu_\Omega(t))_k\,d\sigma_t &= \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k\,d\sigma_t + |\Omega|_2\delta_{k,j}\\ &+ \sum_{h=1}^{+\infty} \frac{1}{(2h)!}\left(\int_{\partial\Omega}\left(v^+[\partial\Omega, \theta^o_{j,2h}](t) + \Lambda^{2h}[\theta^o_{j,0}, \dots, \theta^o_{j,2h-2}](t)\right)(\nu_\Omega(t))_k\,d\sigma_t\right)\epsilon^{2h}\end{aligned}$$

*for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$, where the series converges uniformly for $\epsilon \in ]-\epsilon_4, \epsilon_4[$.*

By Lemmas 2.3.9 and 2.3.10 and by arguing as in the proof of Theorem 2.3.6, one deduces the validity of the following result concerning the expansion of $\lambda^{\mathrm{eff}}_{kj}[\epsilon]$.

**Theorem 2.3.11.** *Let $k,j \in \{1,2\}$. Let $\epsilon_2$ and $\{(\theta^i_{j,h}, \theta^o_{j,h})\}_{h\in\mathbb{N}}$ be as in Proposition 2.3.7. Then there exists $\epsilon_5 \in ]0, \epsilon_2[$ such that*

$$\lambda^{\mathrm{eff}}_{kj}[\epsilon] = \lambda^-\delta_{k,j} + \epsilon^2 \frac{1}{|Q|_2} \sum_{h=0}^{+\infty} \frac{d_{(k,j),2h}}{(2h)!}\epsilon^{2h}$$

*for all $\epsilon \in ]0, \epsilon_5[$, where*

$$\begin{aligned}d_{(k,j),0} =&\lambda^+ \int_{\partial\Omega} v^+[\partial\Omega, \theta^i_{j,0}](t)(\nu_\Omega(t))_k\,d\sigma_t + (\lambda^+ - \lambda^-)|\Omega|_2\delta_{k,j}\\ &- \lambda^- \int_{\partial\Omega} v^-[\partial\Omega, \theta^o_{j,0}](t)(\nu_\Omega(t))_k\,d\sigma_t + \int_{\partial\Omega} f(t)t_k\,d\sigma_t,\\ d_{(k,j),2h} =&\lambda^+ \int_{\partial\Omega}\left(v^+[\partial\Omega, \theta^i_{j,2h}](t) + \Lambda^{2h}[\theta^i_{j,0}, \dots, \theta^i_{j,2h-2}](t)\right)(\nu_\Omega(t))_k\,d\sigma_t\\ &- \lambda^- \int_{\partial\Omega}\left(v^-[\partial\Omega, \theta^o_{j,2h}](t) + \Lambda^{2h}[\theta^o_{j,0}, \dots, \theta^o_{j,2h-2}](t)\right)(\nu_\Omega(t))_k\,d\sigma_t\end{aligned}$$

*for all $h \in \mathbb{N} \setminus \{0\}$.*

**Application to the effective conductivity in the composite with inclusions in the form of a disc**

As in Subsection 2.3.2 we consider assumption (2.82), but this time with $\rho(\epsilon) \equiv \epsilon/r_\#$. We will write the first 3 terms of the series expansion of $\lambda^{\mathrm{eff}}_{kj}$ in terms of simple functions of

$r_{\#}$, $\lambda^+$, and $\lambda^-$. We begin by noting that under assumption (2.82) the system of integral equations (2.102)–(2.103) takes the following form

$$\begin{aligned}\lambda^-\left(\frac{1}{2}\theta^o_{j,0}(t)+w_*[\partial\Omega,\theta^o_{j,0}](t)\right)-\lambda^+\left(-\frac{1}{2}\theta^i_{j,0}(t)+w_*[\partial\Omega,\theta^i_{j,0}](t)\right)&\\ +(\lambda^--\lambda^+)(\nu_\Omega(t))_j=0 \qquad \forall t\in\partial\Omega,&\end{aligned}$$

$$\begin{aligned}\lambda^+\left(-\frac{1}{2}\theta^i_{j,0}(t)+w_*[\partial\Omega,\theta^i_{j,0}](t)\right)+\lambda^+(\nu_\Omega(t))_j+r_{\#}\Bigg(v^+[\partial\Omega,\theta^i_{j,0}](t)&\\ -\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,0}]d\sigma-v^-[\partial\Omega,\theta^o_{j,0}](t)+\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,0}]d\sigma\Bigg)=0 \quad \forall t\in\partial\Omega,&\end{aligned}$$

which are equivalent to the following equations

$$\lambda^-\frac{\partial v^-[\partial\Omega,\theta^o_{j,0}]}{\partial\nu_\Omega}(t)-\lambda^+\frac{\partial v^+[\partial\Omega,\theta^i_{j,0}]}{\partial\nu_\Omega}(t)+(\lambda^--\lambda^+)(\nu_\Omega(t))_j=0 \quad \forall t\in\partial\Omega, \tag{2.112}$$

$$\begin{aligned}\lambda^+\frac{\partial v^+[\partial\Omega,\theta^i_{j,0}]}{\partial\nu_\Omega}(t)+\lambda^+(\nu_\Omega(t))_j+r_{\#}\Bigg(v^+[\partial\Omega,\theta^i_{j,0}](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,0}]d\sigma&\\ -v^-[\partial\Omega,\theta^o_{j,0}](t)+\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,0}]d\sigma\Bigg)=0 \qquad \forall t\in\partial\Omega\,.&\end{aligned} \tag{2.113}$$

If $(\theta^i_{j,0},\theta^o_{j,0})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (2.112)-(2.113), one can verify that there exists a constant $c_0\in\mathbb{R}$ such that

$$\begin{aligned}v^+[\partial\Omega,\theta^i_{j,0}](t)&=-\left(1-\frac{2\lambda^-r_{\#}}{\lambda^-\lambda^++\lambda^+r_{\#}+\lambda^-r_{\#}}\right)t_j+c_0 \qquad \forall t\in\mathrm{cl}\Omega,\\ v^-[\partial\Omega,\theta^o_{j,0}](t)&=\left(1-\frac{2\lambda^+r_{\#}}{\lambda^-\lambda^++\lambda^+r_{\#}+\lambda^-r_{\#}}\right)\frac{t_j}{|t|^2} \qquad \forall t\in\mathbb{R}^2\setminus\Omega\,.\end{aligned} \tag{2.114}$$

Then we recall that, by the jump formula for the normal derivative of the single layer potential (see Theorem A.0.2$(v)$ in Appendix A), if $\Omega=\mathbb{B}_2(0,1)$, then

$$\theta(t)=-2\frac{\partial v^+[\partial\Omega,\theta]}{\partial\nu_\Omega}(t)=2\frac{\partial v^-[\partial\Omega,\theta]}{\partial\nu_\Omega}(t) \qquad \forall t\in\partial\Omega, \tag{2.115}$$

for all $\theta\in C^{0,\alpha}(\partial\Omega)_0$. Therefore, by (2.114) and (2.115), one has

$$\begin{aligned}\theta^i_{j,0}(t)&=2\left(1-\frac{2\lambda^-r_{\#}}{\lambda^-\lambda^++\lambda^+r_{\#}+\lambda^-r_{\#}}\right)t_j \quad \forall t\in\partial\Omega,\\ \theta^o_{j,0}(t)&=-2\left(1-\frac{2\lambda^+r_{\#}}{\lambda^-\lambda^++\lambda^+r_{\#}+\lambda^-r_{\#}}\right)t_j \quad \forall t\in\partial\Omega.\end{aligned} \tag{2.116}$$

Now if $h=1$, the system of integral equations (2.104)–(2.105) takes the form

$$\lambda^-\frac{\partial v^-[\partial\Omega,\theta^o_{j,2}]}{\partial\nu_\Omega}(t)-\lambda^+\frac{\partial v^+[\partial\Omega,\theta^i_{j,2}]}{\partial\nu_\Omega}(t)=\lambda^+\Lambda^2_\nu[\theta^i_{j,0}](t)-\lambda^-\Lambda^2_\nu[\theta^o_{j,0}](t) \quad \forall t\in\partial\Omega, \tag{2.117}$$

$$\begin{aligned}\lambda^+\frac{\partial v^+[\partial\Omega,\theta^i_{j,2}]}{\partial\nu_\Omega}(t)+r_{\#}\Bigg(v^+[\partial\Omega,\theta^i_{j,2}](t)-\fint_{\partial\Omega}v^+[\partial\Omega,\theta^i_{j,2}]d\sigma-v^-[\partial\Omega,\theta^o_{j,2}](t)&\\ +\fint_{\partial\Omega}v^-[\partial\Omega,\theta^o_{j,2}]d\sigma\Bigg)=-\lambda^+\Lambda^2_\nu[\theta^i_{j,0}](t)-r_{\#}\Bigg(\Lambda^2[\theta^i_{j,0}](t)-\fint_{\partial\Omega}\Lambda^2[\theta^i_{j,0}]d\sigma&\\ -\Lambda^2[\theta^o_{j,0}](t)+\fint_{\partial\Omega}\Lambda^2[\theta^o_{j,0}]d\sigma\Bigg) \quad \forall t\in\partial\Omega,&\end{aligned} \tag{2.118}$$

since $\theta^i_{j,1} \equiv 0$ and $\theta^o_{j,1} \equiv 0$. Then, by equalities (2.55), (2.56), and (2.116), we obtain that

$$\begin{aligned} \Lambda^2[\theta^i_{j,0}](t) &= \Lambda^2_\nu[\theta^i_{j,0}](t) = 2\pi \left(1 - \frac{2\lambda^- r_\#}{\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#}\right) t_j \quad \forall t \in \partial\Omega, \\ \Lambda^2[\theta^o_{j,0}](t) &= \Lambda^2_\nu[\theta^o_{j,0}](t) = -2\pi \left(1 - \frac{2\lambda^+ r_\#}{\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#}\right) t_j \quad \forall t \in \partial\Omega. \end{aligned} \tag{2.119}$$

If $(\theta^i_{j,2}, \theta^o_{j,2})$ is the solution in $(C^{0,\alpha}(\partial\Omega)_0)^2$ of system (2.117)-(2.118), then, taking equalities (2.119) into account, one verifies that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\begin{aligned} v^+[\partial\Omega, \theta^i_{j,2}](t) &= -2\pi \left(1 - \frac{4\lambda^+\lambda^- r_\#^2}{(\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#)^2}\right) t_j + c_1 \qquad \forall t \in \mathrm{cl}\Omega, \\ v^-[\partial\Omega, \theta^o_{j,2}](t) &= -2\pi \left(1 - \frac{2\lambda^+ r_\#}{\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#}\right)^2 \frac{t_j}{|t|^2} \qquad \forall t \in \mathbb{R}^2 \setminus \Omega. \end{aligned} \tag{2.120}$$

Then by Theorem 2.3.11 and equations (2.114), (2.120), we have that if $k, j \in \{1, 2\}$, then

$$\begin{aligned} \lambda^{\mathrm{eff}}_{kj}[\epsilon] =& \lambda^- \Bigg(1 - 2\pi \left(1 - \frac{2\lambda^+ r_\#}{\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#}\right) \epsilon^2 \\ &+ 2\pi^2 \left(1 - \frac{2\lambda^+ r_\#}{\lambda^-\lambda^+ + \lambda^+ r_\# + \lambda^- r_\#}\right)^2 \epsilon^4 \Bigg) \delta_{k,j} + O(\epsilon^6) \end{aligned} \tag{2.121}$$

as $\epsilon \to 0^+$. Taking $\lambda^- = 1$, we observe that series expansion (2.121) agrees with the first terms in the series expansion of the effective conductivity obtained in Drygaś and Mityushev [44] for the case where the unit cell $Q$ contains only one inclusion.

# CHAPTER 3

## Shape analysis of the effective longitudinal permeability of a periodic array of cylinders

This chapter is devoted to the study of the behavior of the longitudinal permeability of a periodic array of cylinders upon perturbation of the shape of the cross section of the cylinders and the periodic structure when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. The shape of the cross section of the cylinders is determined by the image of a base domain through a diffeomorphism $\phi$ and the periodicity cell is a rectangle of sides of length $l$ and $1/l$, where $l$ is a positive parameter. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, the velocity field has only one non-zero component which, by the Stokes equations, satisfies a Poisson equation (see problem (3.2)). Then, by integrating the longitudinal component of the velocity field, for each pair $(l, \phi)$, one defines the longitudinal permeability $K_{II}[l, \phi]$. Here, we are interested in studying the behavior of $K_{II}[l, \phi]$ upon the pair $(l, \phi)$.

The chapter is organized as follows. In Section 3.1 we collect some preliminaries, introduce the problem and describe our strategy. Here we show that $K_{II}[l, \phi]$ can be represented as a sum of two integrals and we will study their dependence on $(l, \phi)$ separately. We show that such a dependence is analytical. In Section 3.2 we prove the analyticity of the first integral. Then, using some auxiliary results collected in Section 3.3, we prove the analyticity of the second integral in Section 3.4. Moreover, Section 3.4 contains our main result, namely, Theorem 3.4.7 on the analytical dependence of $K_{II}[l, \phi]$ upon the pair $(l, \phi)$.

We also note that throughout this chapter we retain the notation of Chapter 1 for the case $n = 2$ (see also the notation of Section 2.3)

Some of the results of this chapter are presented in the paper [94] by Paolo Luzzini, Paolo Musolino, and the author.

### 3.1 Preliminaries and notation

In order to introduce the mathematical problem, for $l \in ]0, +\infty[$, we introduce the periodicity cell $Q_l$ and the diagonal matrix $q_l$ by setting

$$Q_l := ]0, l[\times]0, 1/l[, \qquad q_l := \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix},$$

and for an arbitrary subset $\Omega_\sharp$ of $\mathbb{R}^2$ such $\mathrm{cl}\Omega_\sharp \subseteq Q_l$ we also introduce the following periodic domains (see Figure 3.1)

$$\mathbb{S}_{q_l}[\Omega_\sharp] := \bigcup_{z \in \mathbb{Z}^2} (q_l z + q_l \Omega_\sharp), \qquad \mathbb{S}_{q_l}[\Omega_\sharp]^- := \mathbb{R}^2 \setminus \mathrm{cl}\mathbb{S}_{q_l}[\Omega_\sharp].$$

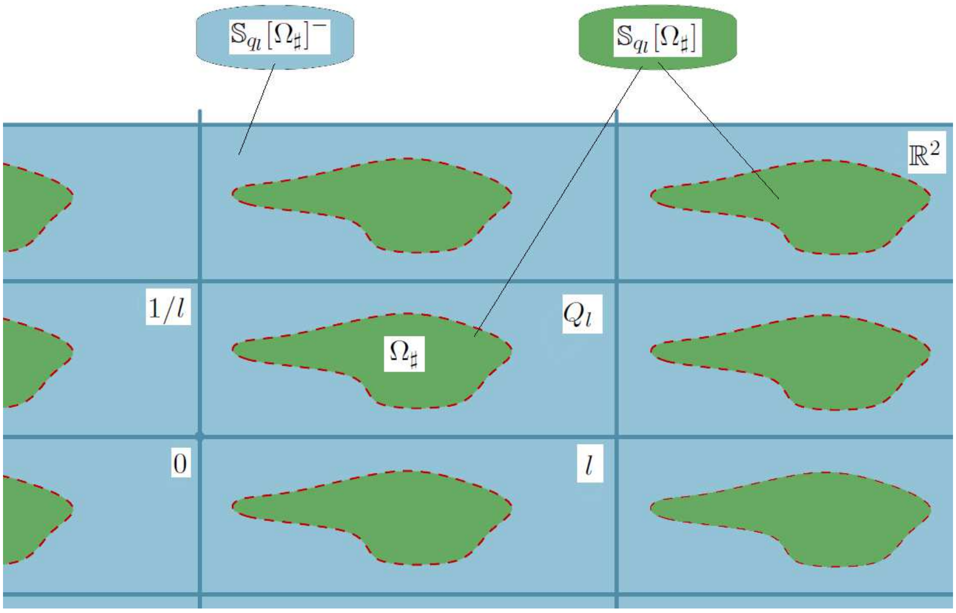


Figure 3.1: *The periodic domains* $\mathbb{S}_{q_l}[\Omega_\sharp]$ *and* $\mathbb{S}_{q_l}[\Omega_\sharp]^-$

Clearly, the area $|Q_l|_2$ of the cell $Q_l$ is equal to one and $q_l\mathbb{Z}^2 = \{q_l z : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the periodicity cell $Q_l$. Moreover, we find it convenient to set

$$\widetilde{Q} := Q_1 , \qquad \tilde{q} := q_1 .$$

Here we fix $\alpha \in ]0,1[$ and we fix once and for all throughout Chapter 3 a subset $\Omega$ of $\mathbb{R}^2$ satisfying the following assumption (cf. assumption (1.1)):

$$\begin{gathered}\Omega \text{ is a bounded open connected subset of } \mathbb{R}^2 \text{ of class } C^{1,\alpha} \text{ such that} \\ \mathbb{R}^2 \setminus \mathrm{cl}\Omega \text{ is connected.}\end{gathered} \tag{3.1}$$

In contrast to Chapters 1 and 2, we do not require here that the domain $\Omega$ contains 0.

In order to formulate the problem, we need to introduce some class of diffeomorphisms. Let $\Omega$ be as in (3.1) and let $\Omega'$ be a bounded open connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. We denote by $\mathcal{A}_{\partial\Omega}$ and by $\mathcal{A}_{\mathrm{cl}\Omega'}$ the set of functions of class $C^1(\partial\Omega, \mathbb{R}^2)$ and of class $C^1(\mathrm{cl}\Omega', \mathbb{R}^2)$ which are injective and whose differential is injective at all points $x \in \partial\Omega$ and at all points $x \in \mathrm{cl}\Omega'$, respectively. One can verify that $\mathcal{A}_{\partial\Omega}$ and $\mathcal{A}_{\mathrm{cl}\Omega'}$ are open in $C^1(\partial\Omega, \mathbb{R}^2)$ and $C^1(\mathrm{cl}\Omega', \mathbb{R}^2)$, respectively. Then we find convenient to set

$$\begin{aligned}\tilde{\mathcal{A}}_{\partial\Omega} &:= \{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq \widetilde{Q}\}, \\ \tilde{\mathcal{A}}_{\mathrm{cl}\Omega'} &:= \{\Phi \in \mathcal{A}_{\mathrm{cl}\Omega'} : \Phi(\mathrm{cl}\Omega') \subseteq \widetilde{Q}\}.\end{aligned}$$

If $\phi \in \mathcal{A}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and unbounded open connected components of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, respectively. Since $\phi(\partial\Omega) \subseteq \widetilde{Q}$, a simple topological argument shows that $\widetilde{Q} \setminus \mathrm{cl}\mathbb{I}[\phi]$ is also connected.

If $l \in ]0, +\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, the set $\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ is the region where a Newtonian fluid of viscosity $\mu$ is flowing at low Reynolds number. Then we assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (see, e.g., Adler [1, Ch. 4], Sangani and Yao [147], and

Mityushev and Adler [109, 110]), one reduces the Stokes system to a Poisson equation for the non-zero component of the velocity field. Since we are working with dimensionless quantities, we may assume that the viscosity of the fluid and the pressure gradient are both set equal to one. For a more complete discussion on spatially periodic structures, we refer to Adler [1, Ch. 4]. Accordingly, if $l \in ]0,+\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega}$, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x+q_l z) = u(x) & \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \,, \forall z \in \mathbb{Z}^2, \\ u(x) = 0 & \forall x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-. \end{cases} \tag{3.2}$$

which has the unique solution in the space $C^{1,\alpha}_{q_l}(\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$, and we denote by $u[l,\phi]$ this solution. From the physical point of view, the function $u[l,\phi]$ represents the non-zero component of the velocity field (see Mityushev and Adler [109, Sec. 2]). By means of the function $u[l,\phi]$, we can introduce *the effective permeability* $K_{II}[l,\phi]$ which we define as the integral of the opposite of the flow velocity over the unit cell (see Adler [1], Mityushev and Adler [109, Sec. 3]), i.e., we set

$$K_{II}[l,\phi] := -\int_{Q_l\setminus q_l\mathbb{I}[\phi]} u[l,\phi](x)\,dx \qquad \forall l \in ]0,+\infty[\,, \quad \forall \phi \in \tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2).$$

We are interested in studying the dependence of the longitudinal permeability upon the sides of the rectangular array and the shape of the cross section of the cylinders, and in proving higher regularity results. We also note that we do not need to restrict ourselves to particular shapes, like circles or ellipses. Our main result is Theorem 3.4.7, where we prove that the map

$$(l,\phi) \mapsto K_{II}[l,\phi] \tag{3.3}$$

form $]0,+\infty[\times\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ to $\mathbb{R}$ is real analytic.

Now, we outline our strategy. Clearly, in order to prove regularity properties of the map in (3.3), one can work locally. Thus, as the first step, we fix

$$\text{a function } \phi_0 \in \tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2), \tag{3.4}$$

and for $\phi_0$ as in (3.4) we find convenient to fix

$$\begin{aligned} &\text{an open connected Lipschitz subset } A_0 \text{ of } \mathbb{R}^2 \text{ such that} \\ &\mathbb{R}^2 \setminus \mathrm{cl}A_0 \text{ is connected and } \mathrm{cl}A_0 \subseteq \mathbb{I}[\phi_0]. \end{aligned} \tag{3.5}$$

Moreover, we also fix a point $p_0$ in $A_0$.

Since the norm in $\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ is stronger then the uniform norm, we have the following lemma.

**Lemma 3.1.1.** *There exist an open connected subset $B$ of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \mathrm{cl}B$ is connected and an open neighborhood $\mathcal{U}_0$ of $\phi_0$ in $\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ such that*

$$\mathrm{cl}A_0 \subseteq B \subseteq \mathrm{cl}B \subseteq \mathbb{I}[\phi] \qquad \forall \phi \in \mathcal{U}_0.$$

Then we want to transform the Dirichlet problem for the Poisson equation (3.2) in a Dirichlet problem for the Laplace equation. To do so, we need to have a function $B$ such that $\Delta B = 1$. We introduce such a function in the following lemma, which is an immediate consequence of Musolino [117, Thm. 2.1].

**Lemma 3.1.2.** *Let $l \in ]0,+\infty[$ and $\mathcal{U}_0$ be as in Lemma 3.1.1. Let $B_l$ be the function from $\mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2)$ to $\mathbb{R}$ defined by*

$$B_l(x) := -S_{q_l,2}(x - q_l p_0) \qquad \forall x \in \mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2).$$

*Then for all $\phi \in \mathcal{U}_0$ the following statements hold.*

*(i)* $B_{l|\mathrm{cl}\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-} \in C^{1,\alpha}_{q_l}(\mathrm{cl}\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-)$.

*(ii)* $\Delta B_l = 1$ *in* $\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-$.

Now, by means of Lemma 3.1.2, we can convert problem (3.2) for the Poisson equation into a nonhomogeneous Dirichlet problem for the Laplace equation. If $l \in ]0,+\infty[$ and $\phi \in \mathcal{U}_0$, we denote by $u_\#[l,\phi]$ the unique solution in $C^{1,\alpha}_{q_l}(\mathrm{cl}\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-)$ of the auxiliary boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-, \forall z \in \mathbb{Z}^2, \\ u(x) = -B_l(x) & \forall x \in \partial\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-. \end{cases} \tag{3.6}$$

Clearly, if $l \in ]0,+\infty[$, $\mathcal{U}_0$ is as in Lemma 3.1.1, and $\phi \in \mathcal{U}_0$, then

$$u[l,\phi] = B_l + u_\#[l,\phi] \qquad \text{in } \mathrm{cl}\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-,$$

and, accordingly,

$$K_{II}[l,\phi] = -\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_l(x)\,dx - \int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l,\phi](x)\,dx. \tag{3.7}$$

Then, we plan to investigate the dependence of two integrals in the right-hand side of (3.7) upon the pair $(l,\phi)$ separately. More precisely, we will investigate the map

$$(l,\phi) \mapsto \int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_l(x)\,dx \tag{3.8}$$

and the map

$$(l,\phi) \mapsto \int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l,\phi](x)\,dx, \tag{3.9}$$

which act from $]0,+\infty[ \times \tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ to $\mathbb{R}$. The results related to the map in (3.8) are presented in Section 3.2, and those related to the map in (3.9) are collected in Section 3.4.

## 3.2 Analyticity of the integral of the auxiliary function

In this section we investigate the map in (3.8). We briefly outline our strategy. First, we formulate two lemmas and introduce the exterior volume potential. Then we prove the analyticity of a certain map which we use in order to prove our main result, i.e., Proposition 3.2.5.

The first auxiliary result is proved by Lanza de Cristoforis and Rossi [88, Lem. 2.4, Prop. 2.5, Lem. 2.7] (see also Lanza de Cristoforis and Musolino [81, Lem. 4.1]).

**Lemma 3.2.1.** *Let $\beta \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ be such that $|\beta(x)| = 1$ and $\beta(x) \cdot \nu_\Omega(x) > 1/2$ for all $x \in \partial\Omega$. Then the following statements hold.*

*(i) There exists* $\delta_\Omega \in ]0,+\infty[$ *such that for all* $\delta \in ]0,\delta_\Omega[$ *the sets*

$$\begin{aligned}\Omega_{\beta,\delta} &:= \{x + t\beta(x) : x \in \partial\Omega,\ t \in ]-\delta,\delta[\},\\ \Omega^+_{\beta,\delta} &:= \{x + t\beta(x) : x \in \partial\Omega,\ t \in ]-\delta,0[\},\\ \Omega^-_{\beta,\delta} &:= \{x + t\beta(x) : x \in \partial\Omega,\ t \in ]0,\delta[\}\end{aligned}$$

*are connected and of class* $C^{1,\alpha}$*, and*

$$\begin{aligned}\partial\Omega_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\ t \in \{-\delta,\delta\}\},\\ \partial\Omega^+_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\ t \in \{-\delta,0\}\},\\ \partial\Omega^-_{\beta,\delta} &\equiv \{x + t\beta(x) : x \in \partial\Omega,\ t \in \{0,\delta\}\},\end{aligned}$$

*and*

$$\Omega^+_{\beta,\delta} \subseteq \Omega, \qquad \Omega^-_{\beta,\delta} \subseteq \mathbb{R}^2 \setminus \mathrm{cl}\Omega.$$

*(ii) Let* $\delta \in ]0,\delta_\Omega[$*. If* $\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\beta,\delta}}$ *then* $\Phi_{|\partial\Omega} \in \mathcal{A}_{\partial\Omega}$*.*

*(iii) If* $\delta \in ]0,\delta_\Omega[$*, then the set*

$$\mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta}} := \{\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\beta,\delta}} : \Phi(\Omega^+_{\beta,\delta}) \subseteq \mathbb{I}[\Phi_{|\partial\Omega}]\}$$

*is open in* $\mathcal{A}_{\mathrm{cl}\Omega_{\beta,\delta}}$ *and* $\Phi(\Omega^-_{\beta,\delta}) \subseteq \mathbb{E}[\Phi_{|\partial\Omega}]$ *for all* $\Phi \in \mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta}}$*.*

*(iv) If* $\delta \in ]0,\delta_\Omega[$ *and* $\Phi \in \mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta},\mathbb{R}^2)$*, then both* $\Phi(\Omega^+_{\beta,\delta})$ *and* $\Phi(\Omega^-_{\beta,\delta})$ *are open sets of class* $C^{1,\alpha}$*, and*

$$\partial\Phi(\Omega^+_{\beta,\delta}) = \Phi(\partial\Omega^+_{\beta,\delta}), \quad \partial\Phi(\Omega^-_{\beta,\delta}) = \Phi(\partial\Omega^-_{\beta,\delta}).$$

Keeping in mind that $\phi_0$ is in $\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$, which is a subspace of $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$, we have the following consequence of Lanza de Cristoforis and Rossi [88, Prop. 2.6] (see also Lanza de Cristoforis and Musolino [81, Lem. 4.1]).

**Lemma 3.2.2.** *Let* $\beta$*,* $\delta_\Omega$ *be as in Lemma 3.2.1. Then the following statements hold.*

*(i) There exist* $\delta_0 \in ]0,\delta_\Omega[$ *and* $\Phi_0 \in \mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta_0}} \cap C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0},\mathbb{R}^2)$ *such that* $\phi_0 = \Phi_{0|\partial\Omega}$*.*

*(ii) Let* $\delta_0$ *and* $\Phi_0$ *be as in (i). Then there exist an open neighborhood* $\mathcal{W}_0$ *of* $\phi_0$ *in* $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$*, and a real analytic extension operator* $\mathbf{E}_0$ *of* $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ *to* $C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0},\mathbb{R}^2)$ *which maps* $\mathcal{W}_0$ *to* $\mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta_0}} \cap C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0},\mathbb{R}^2)$ *and such that* $\mathbf{E}_0[\phi_0] = \Phi_0$ *and* $\mathbf{E}_0[\phi]_{|\partial\Omega} = \phi$ *for all* $\phi \in \mathcal{W}_0$*.*

Then, in the sequel of this section, we will exploit *the exterior periodic volume potential* $\mathcal{P}^-_{\tilde{q}}[\varphi]$, which we introduce in the following proposition.

**Proposition 3.2.3.** *The following statements are true.*

*(i) If* $\varphi \in L^\infty(\widetilde{Q} \setminus \mathrm{cl}A_0)$ *then the function* $\mathcal{P}^-_{\tilde{q}}[\varphi]$ *defined by*

$$\mathcal{P}^-_{\tilde{q}}[\varphi](x) := \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} S_{\tilde{q},2}(x-y)\varphi(y)\,dy \qquad \forall x \in \mathbb{R}^2. \tag{3.10}$$

*is continuous and* $\tilde{q}$*-periodic.*

*(ii) If* $\varphi \in C^{0,\alpha}(\widetilde{Q} \setminus \mathrm{cl}A_0)$ *then* $\mathcal{P}^-_{\tilde{q}}[\varphi]$ *is in the space* $C^{2,\alpha}_{\tilde{q}}(\widetilde{Q} \setminus \mathrm{cl}A_0)$*. Moreover,*

$$\Delta\mathcal{P}^-_{\tilde{q}}[\varphi](x) = \varphi(x) - \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} \varphi(y)\,dy \qquad \forall x \in \widetilde{Q} \setminus \mathrm{cl}A_0. \tag{3.11}$$

*(iii)* *Let $A$ be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}A_0 \subseteq A \subseteq \mathrm{cl}A \subseteq \widetilde{Q}$. Then there exists $\rho_0 \in ]0,+\infty[$ such that for all $\rho \in ]0,\rho_0[$ and $\varphi \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$, the restriction of $\mathcal{P}^-_{\tilde{q}}[\varphi_{|\widetilde{Q}\setminus\mathrm{cl}A_0}]$ to $\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-$ belongs to $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$. Moreover, the map $\varphi \mapsto \mathcal{P}^-_{\tilde{q}}[\varphi_{|\widetilde{Q}\setminus\mathrm{cl}A_0}]_{|\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-}$ from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$ is linear and continuous.*

*Proof.* First of all, by virtue of Section 4 of Dalla Riva, Lanza de Cristoforis, and Musolino [33], we observe that the kernel $S_{\tilde{q},2}$ in (3.10) belongs to a certain class of functions which belongs to those classes defined in Definitions 3.15 and 3.34 of [33]. Then we immediately deduce that Proposition 3.6 $(v)$ of [33] and Theorem 3.40 $(ii)$ of [33] can be applied to $\mathcal{P}^-_{\tilde{q}}[\varphi]$. Thus, statements $(i)$ and $(iii)$ are straightforward consequences of [33, Prop. 3.6 $(v)$] and of [33, Thm. 3.40 $(ii)$], respectively.

Then, by arguing as in the proof of Lanza de Cristoforis and Musolino [85, Prop. A.1], using Gilbarg and Trudinger [51, Lem. 4.2, Thm. 4.6], one proves statement $(ii)$.

□

We are now ready to formulate the following theorem, where we prove the analyticity of the auxiliary map which will be used in order to investigate the map in (3.8).

**Theorem 3.2.4.** *Let $\rho \in ]0,+\infty[$ and $\mathcal{U}_0$ be as in Lemma 3.1.1. Then the map*

$$(\phi, G) \mapsto \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} G(x)\, dx$$

*from $\mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ is real analytic.*

*Proof.* By virtue of equality (3.11), we can write the restriction $G_{|\widetilde{Q}\setminus\mathrm{cl}A_0}$ of the function $G$ to the set $\widetilde{Q}\setminus\mathrm{cl}A_0$ as follows

$$G(x) = \Delta\mathcal{P}^-_{\tilde{q}}[G_{|\widetilde{Q}\setminus\mathrm{cl}A_0}](x) + \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy \qquad \forall x \in \widetilde{Q}\setminus\mathrm{cl}A_0,$$

and, accordingly,

$$\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} G(x)\, dx = \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}[G_{|\widetilde{Q}\setminus\mathrm{cl}A_0}](x)\, dx + \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}\int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy\, dx. \tag{3.12}$$

Then we consider the two integrals in the right-hand side of (3.12) separately. We begin with the second one. By the Divergence Theorem, we have

$$\begin{aligned}\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}\int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy\, dx &= \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} dx \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy = \left(1 - \int_{\mathbb{I}[\phi]} dx\right)\int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy\\ &= \left(1 - \frac{1}{2}\int_{\phi(\partial\Omega)} x\cdot\nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x\right)\int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy.\end{aligned}$$

The map $G \mapsto \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} G(y)\, dy$ from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ is linear and continuous, and, thus, real analytic. It immediately follows from the linearity and continuity of the map $G \mapsto G_{|\widetilde{Q}\setminus\mathrm{cl}A_0}$ from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $L^1(\widetilde{Q}\setminus\mathrm{cl}A_0)$ and of the map $f \mapsto \int_{\widetilde{Q}\setminus\mathrm{cl}A_0} f(y)\, dy$ from $L^1(\widetilde{Q}\setminus\mathrm{cl}A_0)$ to $\mathbb{R}$.

Then, by Lemma C.0.4$(i)$ of Appendix C, we have that

$$\int_{\phi(\partial\Omega)} x\cdot\nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x = \int_{\partial\Omega} \phi(y)\cdot(\nu_{\mathbb{I}[\phi]}\circ\phi(y))\tilde{\sigma}[\phi](y)\, d\sigma_y.$$

Since the map $(f,g) \mapsto f\cdot g$ from $(C^{0,\alpha}(\partial\Omega,\mathbb{R}^2))^2$ to $C^{0,\alpha}(\partial\Omega)$ is bilinear and continuous, and the embedding of $C^{0,\alpha}(\partial\Omega)$ in $L^1(\partial\Omega)$ and the map $h \mapsto \int_{\partial\Omega} h\, d\sigma$ from $L^1(\partial\Omega)$ to $\mathbb{R}$

are linear and continuous operators, by using Lemma C.0.4 and we deduce that the map $\phi \mapsto \int_{\phi(\partial\Omega)} x \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x$ from $\mathcal{U}_0$ to $\mathbb{R}$ is real analytic. Accordingly, the map

$$(\phi, G) \mapsto \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \int_{\widetilde{Q}\setminus \mathrm{cl}A_0} G(y)\, dy\, dx$$

from $\mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ is real analytic.

Next, we consider the first integral in the right-hand side of equality (3.12) and our plan is to prove that the map

$$(\phi, G) \mapsto \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\, dx \tag{3.13}$$

from $\mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $\mathbb{R}$ is real analytic. By the Divergence Theorem and by the periodicity of the volume potential $\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]$, we have

$$\begin{aligned}
&\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\, dx \\
&\qquad = \int_{\partial\widetilde{Q}} D\left(\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\right)\cdot \nu_{\widetilde{Q}}(x)\, d\sigma_x - \int_{\phi(\partial\Omega)} D\left(\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\right)\cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x \\
&\qquad = -\int_{\phi(\partial\Omega)} D\left(\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\right)\cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x .
\end{aligned}$$

Let $\delta_0$, $\mathcal{W}_0$, and $\mathbf{E}_0$ be as in Lemma 3.2.2, $\mathcal{U} := \mathcal{U}_0 \cap \mathcal{W}_0$, and $B$ be as in Lemma 3.1.1. Clearly,

$$\mathrm{cl}A_0 \subseteq B \subseteq \mathrm{cl}B \subseteq \mathbb{I}[\phi] \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{U}.$$

By Proposition 3.2.3 (*iii*), there exists $\rho_0 \in ]0,\rho[$ such that the map $F \mapsto \mathcal{P}^-_{\tilde{q}}\left[F_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]_{|\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-}$ from $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-)$ is linear and continuous. Then taking the linearity and continuity of the embedding of $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ in $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ into account, we deduce that the map

$$G \mapsto \mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]_{|\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-}$$

from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-)$ is linear and continuous, and, thus, real analytic. Then possibly taking smaller $\rho_0 \in ]0,\rho[$, Proposition C.0.3 of Appendix C implies that $\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]_{|\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-}$ belongs to $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-)$ for each $j \in \{1,2\}$, and that the map

$$G \mapsto \frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]_{|\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-}$$

from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-)$ is linear and continuous. We also note that the restriction operator from $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[B]^-)$ to $C^0_{\omega,\rho_0}(\mathrm{cl}\widetilde{Q}\setminus B)$ is linear and continuous, and, thus, real analytic. Then, possibly shrinking $\delta_0$ and $\mathcal{U}$, we can assume that

$$\mathrm{cl}\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0}) \subseteq \widetilde{Q}\setminus \mathrm{cl}B \qquad \forall \phi \in \mathcal{U}.$$

and using Lemma 3.2.2 and by Theorem C.0.1, we obtain that for each $j \in \{1,2\}$ the map

$$(\phi, G) \mapsto \frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]\circ \mathbf{E}_0[\phi]$$

from $\mathcal{U} \times C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A_0]^-)$ to $C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0})$ is real analytic.

Then, keeping in mind that $\mathbf{E}_0[\phi]_{|\partial\Omega} = \phi$ for all $\phi \in \mathcal{U}$ (see Lemma 3.2.2 $(ii)$), we have

$$\begin{aligned}\int_{\phi(\partial\Omega)} & D\left(\mathcal{P}^-_{\tilde q}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\right)\cdot \nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x \\ &= \int_{\partial\Omega} D\mathcal{P}^-_{\tilde q}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]\circ \mathbf{E}_0[\phi](x)\cdot \nu_{\mathbb{I}[\phi]}\circ\phi(x)\tilde\sigma[\phi](x)\,d\sigma_x \\ &= \int_{\partial\Omega} D\mathcal{P}^-_{\tilde q}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right]\circ \phi(x)\cdot \nu_{\mathbb{I}[\phi]}\circ\phi(x)\tilde\sigma[\phi](x)\,d\sigma_x.\end{aligned}$$

By the linearity and continuity of the map $f \mapsto \int_{\partial\Omega} f\,d\sigma$ from $L^1(\partial\Omega)$ to $\mathbb{R}$, using Lemma C.0.4 and the linearity and continuity of the embedding of $C^{0,\alpha}(\partial\Omega)$ in $L^1(\partial\Omega)$, and of the trace operator from $C^{0,\alpha}(\mathrm{cl}\Omega_{\beta,\delta})$ to $C^{0,\alpha}(\partial\Omega)$, we conclude that the map

$$(\phi, G) \mapsto \int_{\phi(\partial\Omega)} D\left(\mathcal{P}^-_{\tilde q}\left[G_{|\widetilde{Q}\setminus \mathrm{cl}A_0}\right](x)\right)\cdot \nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x$$

from $\mathcal{U}\times C^0_{\tilde q,\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde q}[A_0]^-)$ to $\mathbb{R}$ is real analytic. Accordingly, the map in (3.13) is real analytic and, thus, the validity of the statement follows. □

We are now ready to analyze the dependence of the map in (3.8) on $(l,\phi)$.

**Proposition 3.2.5.** *Let $\mathcal{U}_0$ be as in Lemma 3.1.1. Then the map in* (3.8) *from* $]0,+\infty[\times\mathcal{U}_0$ *to* $\mathbb{R}$ *is real analytic.*

*Proof.* To begin with, we note that

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]} B_l(x)\,dx = \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} B_l(q_l x)\,dx = -\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} S_{q_l,2}(q_l(x-p_0))\,dx \tag{3.14}$$

for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. Our plan is to show first that the map $l\mapsto S_{q_l,2}(q_l(\cdot-p_0))$ from $]0,+\infty[$ to $C^0_{q_l,\omega,\rho}(\widetilde{Q}\setminus\mathbb{I}[\phi])$ is real analytic and then to apply Theorem 3.2.4 to the last integral in (3.14), that is sufficient to prove the validity of the proposition.

To achieve our goal, one can work locally. Thus, let $l_0$ be an arbitrary point in $]0,+\infty[$ and let $\mathcal{L}_0$ be an arbitrary neighborhood of $l_0$ such that $\mathcal{L}_0\subseteq]0,+\infty[$. Then we set

$$\mathcal{Q}_0 := \{q_l\in M_{2\times 2}(\mathbb{R}):\ l\in\mathcal{L}_0\}.$$

Clearly, $\mathcal{Q}_0$ is open and bounded in $M_{2\times2}(\mathbb{R})$ and $\mathrm{cl}\mathcal{Q}_0\subseteq M_{2\times2}(\mathbb{R})$.

Then we take an open bounded connected subset $W$ of $\mathbb{R}^2$ of class $C^\infty$ such that

$$\mathrm{cl}\widetilde{Q}\subseteq W \qquad\text{and}\qquad W\cap(z+\mathrm{cl}A_0)=\varnothing \qquad \forall z\in\mathbb{Z}^2\setminus\{0\}.$$

By Lanza de Cristoforis and Musolino [86, Thm. 8], there exists $\rho\in]0,+\infty[$ such that the map

$$q_l\mapsto S_{q_l,2}(q_l\,\cdot)_{|(\mathrm{cl}W\setminus A_0)-p_0}$$

from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}((\mathrm{cl}W\setminus A_0)-p_0)$ is real analytic. Since the translation operator from $C^0_{\omega,\rho}((\mathrm{cl}W\setminus A_0)-p_0)$ to $C^0_{\omega,\rho}(\mathrm{cl}W\setminus A_0)$ which takes $f$ to $f(\cdot-p_0)$ is linear and continuous, we have that the map

$$q_l\mapsto S_{q_l,2}(q_l(\cdot-p_0))_{|\mathrm{cl}W\setminus A_0}$$

from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}W\setminus A_0)$ is real analytic. Then, by virtue of the real analyticity of the map $l\mapsto q_l$ from $\mathcal{L}_0$ to $M_{2\times2}(\mathbb{R})$, we obtain that the map

$$l\mapsto S_{q_l,2}(q_l(\cdot-p_0))_{|\mathrm{cl}W\setminus A_0}$$

from $\mathcal{L}_0$ to $C^0_{\omega,\rho}(\mathrm{cl}W\setminus A_0)$ is real analytic.

Finally, taking Lemma C.0.2 of Appendix C into account, we deduce that the map

$$l\mapsto S_{q_l,2}(q_l(\cdot-p_0))$$

from $\mathcal{L}_0$ to $C^0_{\tilde q,\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde q}[A_0]^-)$ is real analytic. Using this result and applying Theorem 3.2.4 to the last integral in (3.14), we conclude the validity of the statement. □

## 3.3 Analyticity of the periodic double layer potential

This section consists of two lemmas which we will exploit in order to investigate the map in (3.9) upon the pair $(l, \phi)$. Namely, we study the dependence upon $l$ and $\phi$ of some integral operators related to the double layer potential.

We start with the following result.

**Lemma 3.3.1.** *Let $\beta$ and $\delta_\Omega$ be as in Lemma 3.2.1, and $\eta \in ]0,1[$. Let*

$$\tilde{\mathcal{A}}'_{\mathrm{cl}\Omega_{\beta,\delta}} := \mathcal{A}'_{\mathrm{cl}\Omega_{\beta,\delta}} \cap \tilde{\mathcal{A}}_{\mathrm{cl}\Omega_{\beta,\delta}} \qquad \forall \delta \in ]0, \delta_\Omega[.$$

*Then there exists $\delta_\eta \in ]0, \delta_\Omega[$ such that if $\delta \in ]0, \delta_\eta[$ then the map which takes*

$$(l, \Phi, \theta) \in ]0, +\infty[ \times \left(\tilde{\mathcal{A}}'_{\mathrm{cl}\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta}, \mathbb{R}^2)\right) \times C^{1,\alpha}(\partial\Omega)$$

*to the function $W^+[l, \Phi, \theta]$, which is defined as a continuous extension to $\mathrm{cl}\Omega^+_{\beta,\delta}$ of the function*

$$-\int_{q_l\Phi(\partial\Omega)} DS_{q_l,2}(q_l\Phi(x) - s) \cdot \nu_{q_l\mathbb{I}[\Phi]}(s)(\theta \circ \Phi^{(-1)} \circ q_l^{-1})(s)d\sigma_s \qquad \forall x \in \Omega^+_{\beta,\delta},$$

*is real analytic from $\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\Omega^+_{\beta,\delta})$, where*

$$\mathcal{O}(\eta) := \left\{ l \in ]0, +\infty[ : \ \max\{l^{-2}, l^2\} < \eta^{-1} \right\},$$

$$\mathcal{U}_{\eta,\delta} := \left\{ \Phi \in \tilde{\mathcal{A}}'_{\mathrm{cl}\Omega_{\beta,\delta}} \cap C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta}, \mathbb{R}^2) : \ |\det(D\Phi)| < \eta^{-1} \right\}.$$

*Proof.* To prove this lemma, we will follow the strategy of the proof of Corollary 5.7 in Lanza de Cristoforis and Musolino [81]. Thus, it is sufficient to show that maps

$$(l, \Phi, \theta) \mapsto W^+[l, \Phi, \theta], \qquad (l, \Phi, \theta) \mapsto \frac{\partial}{\partial x_k} W^+[l, \Phi, \theta] \quad \forall k \in \{1, 2\}, \tag{3.15}$$

are real analytic from $\mathcal{O}(\eta) \times \mathcal{U}_{\eta,\delta} \times C^{1,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\mathrm{cl}\Omega^+_{\beta,\delta})$. To do so, we first need to rewrite the operators $W^+$, $\frac{\partial}{\partial x_1} W^+$ and $\frac{\partial}{\partial x_2} W^+$ in terms of a single layer potential.

Let $\delta \in ]0, \delta_\Omega[$ and let $F$ be a linear and continuous operator from $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\mathbb{B}_2(0, R))$ with $R > \sup_{x \in \Omega \cup \Omega_{\beta,\delta}} |x|$, such that $F[\theta]_{|\partial\Omega} = \theta$ for all $\theta \in C^{1,\alpha}(\partial\Omega)$ (see, *e.g.*, Troianiello [159, Thm. 1.3 and Lem. 1.5]). Then, by using [81, Eq. (5.8) and (5.9)], we obtain that

$$W^+[l, \Phi, \theta] = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(V^+ \left[l, q_l \circ \Phi, \mathbf{n}_j[q_l \circ \Phi]\theta\right]\right) \left((D(q_l \circ \Phi))^{-1}\right)_{ij} \tag{3.16}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_k} \left(W^+[l, \Phi, \theta]\right) = &\sum_{r=1}^{2} \frac{\partial (q_l \circ \Phi)_r}{\partial x_k} \times \\ &\times \sum_{j,t=1}^{2} \frac{\partial}{\partial x_t} \left(V^+ \left[l, q_l \circ \Phi, M_{rj}[l, q_l \circ \Phi, \theta]\right]\right) \left((D(q_l \circ \Phi))^{-1}\right)_{ij} \\ &+ \sum_{r=1}^{2} \frac{\partial (q_l \circ \Phi)_r}{\partial x_k} \int_{\partial\Omega} \mathbf{n}_j[q_l \circ \Phi](y)\theta(y)\tilde{\sigma}[q_l \circ \Phi](s)d\sigma_s \end{aligned} \tag{3.17}$$

for all $k \in \{1,2\}$, where

$$\begin{aligned}M_{rj}[l, q_l\circ\Phi, \theta] :=&|(D(q_l\circ\Phi))^{-t}\cdot\nu_\Omega|^{-1}\times\\ &\times\Bigg[\left(\sum_{i=1}^{2}\left((D(q_l\circ\Phi))^{-1}\right)_{ir}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D(q_l\circ\Phi))^{-1}\right)_{ij}\right)\\ &-\left(\sum_{i=1}^{2}\left((D(q_l\circ\Phi))^{-1}\right)_{ij}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D(q_l\circ\Phi))^{-1}\right)_{ir}\right)\Bigg],\end{aligned}$$

and

$$\mathbf{n}_j[q_l\circ\Phi] := \left(\frac{(D(q_l\circ\Phi))^{-t}\cdot\nu_\Omega}{|(D(q_l\circ\Phi))^{-t}\cdot\nu_\Omega|}\right)_j,$$

and

$$V^+[l, q_l\circ\Phi, \mu](\cdot) := \int_{q_l\Phi(\partial\Omega)} S_{q_l,2}(q_l\Phi(\cdot)-s)\left(\mu\circ(q_l\circ\Phi)^{(-1)}\right)(s)d\sigma_s \qquad \forall\mu\in C^{1,\alpha}(\partial\Omega).$$

By the chain rule, we have

$$\begin{aligned}&(D(q_l\circ\Phi))_{ij} = (q_l)_{ii}(D\Phi)_{ij} && \forall i,j\in\{1,2\},\\ &((D(q_l\circ\Phi))^{-1})_{ij} = \frac{1}{(q_l)_{ii}}\left((D\Phi)^{-1}\right)_{ij} && \forall i,j\in\{1,2\},\\ &(D(q_l\circ\Phi))^{-t} = q_l^{-1}\cdot(D\Phi)^{-t}.\end{aligned} \tag{3.18}$$

Then we consider $V^+$, and we note that

$$\begin{aligned}V^+[l, q_l\circ\Phi, \mu](x) &= \int_{q_l\Phi(\partial\Omega)} S_{q_l,2}(q_l\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\circ q_l^{-1}\right)(s)d\sigma_s\\ &= \int_{\Phi(\partial\Omega)} S_{q_l,2}(q_l(\Phi(x)-s))\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s\end{aligned}$$

for all $\mu\in C^{1,\alpha}(\partial\Omega)$ and all $x\in\Omega^+_{\beta,\delta}$. Then we set

$$\tilde{S}_{\tilde{q},l,2}(x) := S_{q_l,2}(q_l x) \qquad \forall x\in\mathbb{R}^2\setminus\mathbb{Z}^2. \tag{3.19}$$

We note that the $\tilde{q}$-periodic function $\tilde{S}_{\tilde{q},l,2}(\cdot)$ is a $\tilde{q}$-periodic analog of the fundamental solution of the operator

$$\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2} + l^2\frac{\partial^2}{\partial x_2^2},$$

namely,

$$\left(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2} + l^2\frac{\partial^2}{\partial x_2^2}\right)\tilde{S}_{\tilde{q},l,2} = \sum_{z\in\mathbb{Z}^2}\delta_{\tilde{q}z} - 1,$$

in the sense of distributions (see Lanza de Cristoforis and Musolino [81, Sec. 1]). Thus, we can write

$$\begin{aligned}&\int_{\Phi(\partial\Omega)} S_{q_l,2}(q_l(\Phi(x)-s))\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s\\ &\qquad = \int_{\Phi(\partial\Omega)} \widetilde{S}_{\tilde{q},l,2}(\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s =: \widetilde{V}^+_{\tilde{q}}[l,\Phi,\mu](x) \qquad \forall x\in\Omega^+_{\beta,\delta},\end{aligned}$$

and for all $(l,\Phi,\mu)\in]0,+\infty[\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$.

Now, we can rewrite the operators $W^+$, $\frac{\partial}{\partial x_1}W^+$ and $\frac{\partial}{\partial x_2}W^+$ using the single layer potential $\widetilde{V}^+_{\tilde{q}}$. Thus, keeping equalities (3.16) and (3.17) in mind, using equalities (3.18), one has

$$W^+[l,\Phi,\theta] = -\sum_{m,i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\widetilde{V}^+_{\tilde{q}}\left[l,\Phi,\tilde{\mathbf{n}}_j[l,\Phi]\theta\right]\right)\frac{1}{(q_l)_{ii}}\left((D\Phi)^{-1}\right)_{im} \tag{3.20}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\left(W^+[l,\Phi,\theta]\right)\\
&=\sum_{r=1}^{2}\frac{\partial \Phi_r}{\partial x_k}(q_l)_{rr}\sum_{m,j,t=1}^{2}\frac{\partial}{\partial x_t}\left(\widetilde{V}^+_{\tilde{q}}\left[l,\Phi,\widetilde{M}_{rj}[l,\Phi,\theta]\right]\right)\frac{1}{(q_l)_{ii}}\left((D\Phi)^{-1}\right)_{im}\\
&\quad+\sum_{r=1}^{2}\frac{\partial \Phi_r}{\partial x_k}(q_l)_{rr}\int_{\partial\Omega}\tilde{\mathbf{n}}_j[l,\Phi](y)\theta(y)\tilde{\sigma}[q_l\circ\Phi](s)d\sigma_s
\end{aligned} \tag{3.21}$$

for all $k\in\{1,2\}$, where

$$\begin{aligned}
\widetilde{M}_{rj}[l,\Phi,\theta] &:= |q_l^{-1}\cdot(D\Phi)^{-t}\cdot\nu_\Omega|^{-1}\times\\
&\times\Bigg[\left(\sum_{i=1}^{2}\left((D\Phi)^{-1}\right)_{ir}\frac{1}{(q_l)_{ii}}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D\Phi)^{-1}\right)_{ij}\frac{1}{(q_l)_{ii}}\right)\\
&\quad-\left(\sum_{i=1}^{2}\left((D\Phi)^{-1}\right)_{ij}\frac{1}{(q_l)_{ii}}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D\Phi)^{-1}\right)_{ir}\frac{1}{(q_l)_{ii}}\right)\Bigg]
\end{aligned}$$

and

$$\tilde{\mathbf{n}}_j[l,\Phi] := \left(\frac{q_l^{-1}\cdot(D\Phi)^{-t}\cdot\nu_\Omega}{|q_l^{-1}\cdot(D\Phi)^{-t}\cdot\nu_\Omega|}\right)_j.$$

By Lanza de Cristoforis and Musolino [86, Thm. 7], the map $(l,x)\mapsto\tilde{S}_{\tilde{q},l,2}(x)$ from $]0,+\infty[\times\mathbb{R}^2\setminus\tilde{q}\mathbb{Z}^2$ to $\mathbb{R}$ is real analytic. Moreover, the map $l\mapsto(l^{-2},l^2)$ from $]0,+\infty[$ to $\mathbb{R}^2$ is real analytic. Accordingly, assumptions (1.8) of Lanza de Cristoforis and Musolino [81, pp. 78, 79] are satisfied and we can use the results of [81]. Then Proposition 5.1 of [81] implies that there exists $\delta_\eta\in]0,\delta_\Omega[$ such that, by Proposition 5.6 of [81], for all $\delta\in]0,\delta_\eta[$ the map

$$(l,\Phi,\theta)\mapsto\widetilde{V}^+_{\tilde{q}}[l,\Phi,\theta] \tag{3.22}$$

from $]0,+\infty[\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\Omega^+_{\beta,\delta})$ is real analytic.

If $\delta\in]0,\delta_\eta[$ then by the real analyticity of the pointwise product in Schauder spaces and by the real analyticity of the map which takes an invertible matrix with Schauder functions as entries to its inverse, and by the real analyticity of the linear and continuous map $F[\cdot]$ and of the trace operator, and by the real analyticity of the map in (3.22), and by identities (3.20) and (3.21), we conclude that maps (3.15) are real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\Omega^+_{\beta,\delta})$. Thus, the proof is complete. □

Then we have the following lemma where we prove the analyticity of the trace of the periodic double layer potential upon the periodicity parameter, the shape, and the density.

**Lemma 3.3.2.** *The map from $]0,+\infty[\times(\tilde{\mathcal{A}}_{\partial\Omega}\cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2))\times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ which takes a triple $(l,\phi,\theta)$ to the function*

$$W[l,\phi,\theta](x) := -\int_{q_l\phi(\partial\Omega)} DS_{q_l,2}(q_l\phi(x)-s)\cdot\nu_{q_l\mathbb{I}[\phi]}(q_l s)(\theta\circ\phi^{(-1)}\circ q_l^{-1})(s)d\sigma_s \qquad \forall x\in\partial\Omega$$

*is real analytic.*

*Proof.* Clearly, it is sufficient to show that if $(l_*, \phi_*, \theta_*)$ belongs to

$$]0, +\infty[\times(\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)) \times C^{1,\alpha}(\partial\Omega),$$

then the map

$$l, \phi, \theta \mapsto W[l, \phi, \theta]$$

is real analytic in a neighborhood of $(l_*, \phi_*, \theta_*)$. To do so, we will exploit Lemma 3.3.1.

Let $\beta$, $\delta_0$, $\mathbf{E}_0$, and $\mathcal{W}_0$ be as in Lemma 3.2.2, and $\delta \in ]0, \delta_0[$. Possibly shrinking $\mathcal{W}_0$, we can assume that there exists $\eta \in ]0, 1[$ such that

$$\sup_{\phi \in \mathcal{W}_0} \sup_{x \in \mathrm{cl}\Omega_{\beta,\delta}} |\det(D\mathbf{E}_0[\phi](x))| < \eta^{-1} \quad \text{and} \quad l_* \in \mathcal{O}[\eta],$$

where $\mathcal{O}[\eta]$ is as in Lemma 3.3.1. Moreover, possibly shrinking $\delta$ and $\mathcal{W}_0$, we can also assume that

$$\mathbf{E}_0[\phi](\mathrm{cl}\Omega_{\beta,\delta}) \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{W}_0.$$

Then using the jump formula for the double layer potential, we obtain that

$$W[l, \phi, \theta] = -\frac{1}{2}\theta + \Big(W^+ [l, \mathbf{E}_0[\phi], \theta]\Big)_{|\partial\Omega} \qquad \text{on} \qquad \partial\Omega, \tag{3.23}$$

for all $(l, \phi, \theta) \in \mathcal{O}[\eta] \times \mathcal{W}_0 \times C^{1,\alpha}(\partial\Omega)$, where $W^+$ is as in Lemma 3.3.1 for arbitrary $\delta \in ]0, \min\{\delta_0, \delta_\eta\}[$. Then, by Lemma 3.2.2 and Proposition 3.3.1, by the linearity and continuity of the trace operator from $C^{1,\alpha}(\mathrm{cl}\Omega^+_{\beta,\delta})$ to $C^{1,\alpha}(\partial\Omega)$ and by equality (3.23), one verifies the validity of the lemma. □

## 3.4 Analyticity of the effective longitudinal permeability

In order to investigate the map in (3.9), we would like to exploit some of the results of Musolino [117], where the behavior of a (singularly) perturbed Dirichlet problem for the Laplace equation is studied by means of periodic potential theory.

As we shall see, one can formulate a Dirichlet problem in terms of an integral equation. To show that, we first formulate the following auxiliary result on a boundary integral operators which is proved in Musolino [117, Prop. A.3].

**Lemma 3.4.1.** *Let $l \in ]0, +\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$. Then the map $\mu \mapsto M[\mu]$ from $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ to itself defined by*

$$M[\mu] := -\frac{1}{2}\mu + w_{q_l}[q_l\partial\mathbb{I}[\phi], \mu] \qquad \forall \mu \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$$

*is a linear homeomorphism from $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ to $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$.*

Then we have the following result where we establish a correspondence between the solution of a Dirichlet problem and the solution of an integral equation.

**Proposition 3.4.2.** *Let $l \in ]0, +\infty[$ and $\phi \in \tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^2)$. Let $\Gamma \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$. Then the following boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \forall z \in \mathbb{Z}^2, \\ u(x) = \Gamma(x) & \forall x \in q_l\partial\mathbb{I}[\phi] \end{cases} \tag{3.24}$$

*has a unique solution $u$ in $C^{1,\alpha}_{q_l}(\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-)$. Moreover*

$$u(x) = w^-_{q_l}[q_l\partial\mathbb{I}[\phi], \mu](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \tag{3.25}$$

*where $\mu$ is the unique solution in $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ of the following integral equation*

$$-\frac{1}{2}\mu(x)+w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x)=\Gamma(x) \qquad \forall x\in q_l\partial\mathbb{I}[\phi]. \tag{3.26}$$

*Proof.* By the maximum principle for periodic functions in $\mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$, problem (3.24) has at most one solution (see Musolino [117, Prop. A.1]). As a consequence, it suffices to prove that the function defined by (3.25) solves problem (3.24). By Lemma 3.4.1, there exists a unique solution $\mu\in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ of the integral equation (3.26). Then, by the properties of the periodic double layer potential (see Theorem B.0.3 in Appendix B), the function defined by (3.25) is the solution of problem (3.24) (see Musolino [117, Thm. 2.3]). □

Proposition 3.4.2 tells us that the unique solution $u_\sharp[l,\phi]$ of problem (3.6) is represented in form (3.25) with the unknown function $\mu$ which satisfies the following integral equation

$$-\frac{1}{2}\mu(x)+w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x)=S_{q_l,2}(x-q_lp_0) \qquad \forall x\in q_l\partial\mathbb{I}[\phi]. \tag{3.27}$$

Thus, in order to study the dependence of $u_\sharp[l,\phi]$ upon $(l,\phi)$, we first need to understand how $\mu$ depends on $(l,\phi)$. To do so, we find it convenient to transform equality (3.27) into the equality defined on the boundary of the domain $\Omega$. The following lemma holds.

**Lemma 3.4.3.** *Let $l\in]0,+\infty[$. Let $\mathcal{U}_0$ be as in Lemma 3.1.1 and $\phi\in\mathcal{U}_0$. Then the function $\theta\in C^{1,\alpha}(\partial\Omega)$ solves the equation*

$$\begin{aligned}-\frac{1}{2}\theta(t)-\int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t)-s))\cdot\nu_{q_l\mathbb{I}[\phi]}(q_ls)(\theta\circ\phi^{-1})(s)d\sigma_s&\\ -S_{q_l,2}(q_l(\phi(t)-p_0))=0 \qquad \forall t\in\partial\Omega,&\end{aligned} \tag{3.28}$$

*if and only if the function $\mu\in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$, with $\mu$ delivered by*

$$\mu(x)=(\theta\circ\phi^{-1}\circ q_l^{-1})(x) \qquad \forall x\in q_l\partial\mathbb{I}[\phi], \tag{3.29}$$

*solves the equation* (3.27)*. Moreover, equation* (3.28) *has a unique solution in $C^{1,\alpha}(\partial\Omega)$.*

*Proof.* The equivalence of equation (3.28) with the unknown $\theta$ and equation (3.27) with the unknown $\mu$ delivered by (3.29) is a straightforward consequence of the Theorem of change of variables in integrals.

Taking Lemma 3.1.2 $(i)$ into account, we can apply Lemma 3.4.1 to equation (3.27). Thus, we have the existence and uniqueness of a solution of equation (3.27) in $C^{1,\alpha}(\partial\Omega)$. Finally, due to the equivalence of equations (3.27) and (3.28), we deduce the existence and uniqueness of a solution of (3.28). Thus, the proof is complete. □

Now, our aim is to prove the analyticity of the function $\theta$ which solves equation (3.28) upon $(l,\phi)$ by exploiting the Implicit Function Theorem for real analytic maps. To shorten our notation, we find it convenient to introduce the map $\Lambda$ from $]0,+\infty[\times\mathcal{U}_0\times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ by setting

$$\begin{aligned}\Lambda[l,\phi,\theta](x):=-\frac{1}{2}\theta(t)-\int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t)-s))\cdot\nu_{q_l\mathbb{I}[\phi]}(q_ls)(\theta\circ\phi^{-1})(s)d\sigma_s&\\ -S_{q_l,2}(q_l(\phi(t)-p_0)) \qquad \forall t\in\partial\Omega,&\end{aligned}$$

for all $(l,\phi,\theta)\in]0,+\infty[\times\mathcal{U}_0\times C^{1,\alpha}(\partial\Omega)$, where $\mathcal{U}_0$ is defined in Lemma 3.1.1.

In order to apply the Implicit Function Theorem to the equation

$$\Lambda[l,\phi,\theta]=0\,,$$

we need to understand the regularity of $\Lambda$. As a first step, we show that the function $S_{q_l,2}(q_l(\phi(\cdot)-p_0))$ depends analytically on $(l,\phi)$.

**Lemma 3.4.4.** *Let $\mathcal{U}_0$ be as in Lemma 3.1.1. Then the map*

$$(l, \phi) \mapsto S_{q_l,2}(q_l(\phi(\cdot) - p_0))$$

*from* $]0, +\infty[\times\mathcal{U}_0$ *to* $C^{1,\alpha}(\partial\Omega)$ *is real analytic.*

*Proof.* We take a bounded open connected subset $W$ of $\mathbb{R}^2$ of class $C^\infty$ such that

$$\mathrm{cl}\widetilde{Q} \subseteq W \qquad \text{and} \qquad W \cap (z + \mathrm{cl}A_0) = \varnothing \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

By virtue of the proof of Proposition 3.2.5, we have that the map

$$l \mapsto S_{q_l,2}(q_l(\cdot - p_0))_{|\mathrm{cl}W\setminus A_0}$$

from $]0, +\infty[$ to $C^0_{\omega,\rho}(\mathrm{cl}W \setminus A_0)$ is real analytic.

Let $\delta_0$, $\mathcal{W}_0$, and $\mathbf{E}_0$ be as in Lemma 3.2.2, and $\mathcal{U} = \mathcal{U}_0 \cap \mathcal{W}_0$. Let $B$ be as in Lemma 3.1.1. Then, in particular, we have that

$$\mathrm{cl}A_0 \subseteq B \subseteq \mathrm{cl}B \subseteq \mathbb{I}[\phi] \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{U}.$$

Possibly shrinking $\delta_0$ and $\mathcal{U}$, we can assume that

$$\mathrm{cl}\mathbf{E}_0[\phi_0](\Omega_{\beta,\delta_0}) \subseteq \widetilde{Q} \setminus \mathrm{cl}B,$$

and

$$\mathrm{cl}\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0}) \subseteq \widetilde{Q} \setminus \mathrm{cl}B \qquad \forall \phi \in \mathcal{U}.$$

Applying Lemma 3.2.2 and Lemma C.0.1 of Appendix C to the map

$$(l, \phi) \mapsto S_{q_l,2}(q_l(\cdot - p_0)) \circ \mathbf{E}_0[\phi]$$

from $]0, +\infty[\times\mathcal{U}$ to $C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0})$, we immediately obtain that it is real analytic. Then, to complete the proof, it is sufficient to note that the trace operator from $C^{1,\alpha}(\mathrm{cl}\Omega_{\beta,\delta_0})$ to $C^{1,\alpha}(\partial\Omega)$ is linear and continuous. □

We are now ready to show that the solution of equation (3.28) depends analytically on $(l, \phi)$ by using the Implicit Function Theorem for real analytic maps in Banach spaces.

**Proposition 3.4.5.** *Let $\mathcal{U}_0$ be as in Lemma 3.1.1. Then the following statements hold.*

*(i) For each $(l, \phi) \in ]0, +\infty[\times\mathcal{U}_0$, there exists a unique $\theta$ in $C^{1,\alpha}(\partial\Omega)$ such that $\Lambda[l, \phi, \theta] = 0$, and we denote such a function by $\theta[l, \phi]$.*

*(ii) The map $(l, \phi) \mapsto \theta[l, \phi]$ from $]0, +\infty[\times\mathcal{U}_0$ to $C^{1,\alpha}(\partial\Omega)$ is real analytic.*

*Proof.* Statement $(i)$ is a straightforward consequence of Lemma 3.4.3.

Next, we consider statement (ii). Keeping in mind the definition of $\Lambda[l, \phi, \theta]$, we observe that Lemmas 3.3.2 and 3.4.4 imply that the map

$$(l, \phi, \theta) \mapsto \Lambda[l, \phi, \theta]$$

from $]0, +\infty[\times\mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ is real analytic.

To prove statement $(ii)$ we will work locally. Thus, fixing $(l_1, \phi_1)$ in $]0, +\infty[\times\mathcal{U}_0$, by standard calculus in normed spaces, the differential $\partial_\theta\Lambda[l_1, \phi_1, \theta[l_1, \phi_1]]$ of $\Lambda$ at $(l_1, \phi_1, \theta[l_1, \phi_1])$ with respect to the variable $\theta$ is delivered by the following formula:

$$\partial_\theta\Lambda[l_1, \phi_1, \theta[l_1, \phi_1]](\psi)(t) = -\frac{1}{2}\psi(t) - \int_{\phi(\partial\Omega)} DS_{q_{l_1},2}(q_{l_1}(\phi(t) - s)) \cdot \nu_{q_{l_1}\mathbb{I}[\phi]}(q_{l_1}s)(\psi \circ \phi^{-1})(s)d\sigma_s \qquad \forall t \in \partial\Omega,\ \forall \psi \in C^{1,\alpha}(\partial\Omega).$$

By Lemma 3.4.1 and by the proof of Lemma 3.4.3, we deduce that $\partial_\theta \Lambda[l_1, \phi_1, \theta[l_1, \phi_1]]$ is a linear homeomorphism of $C^{1,\alpha}(\partial\Omega)$ onto $C^{1,\alpha}(\partial\Omega)$. Then by the Implicit Function Theorem for real analytic maps in Banach spaces, we deduce that the map $(l, \phi) \mapsto \theta[l, \phi]$ is real analytic in some neighborhood of $(l_1, \phi_1)$. Since $(l_1, \phi_1)$ is an arbitrary pair in $]0, +\infty[\times \mathcal{U}_0$, the validity of the statement follows. □

Finally, in the following theorem we prove the analyticity of the map in (3.9) upon the pair $(l, \phi)$.

**Theorem 3.4.6.** *Let $\mathcal{U}_0$ be as in Lemma 3.1.1. Then the map in* (3.9) *from* $]0, +\infty[\times \mathcal{U}_0$ *to* $\mathbb{R}$ *is real analytic.*

*Proof.* By virtue of Proposition 3.4.5 $(i)$, using equalities (3.25) and (3.29), we can write the solution $u_\#[l, \phi]$ of problem (3.6) as follows

$$u_\#[l, \phi](x) = w^-_{q_l}[q_l\phi(\partial\Omega), \theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$$

and for all $(l, \phi) \in ]0, +\infty[\times \mathcal{U}_0$. Accordingly,

$$\int_{Q_l \setminus q_l\mathbb{I}[\phi]} u_\#[l, \phi]\, dx = \int_{Q_l \setminus q_l\mathbb{I}[\phi]} w^-_{q_l}[q_l\phi(\partial\Omega), \theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1}]\, dx$$

for all $(l, \phi) \in ]0, +\infty[\times \mathcal{U}_0$. By the definition of the periodic double layer potential, we have

$$\begin{aligned}
& w^-_{q_l}[q_l\phi(\partial\Omega), \theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1}](x) \\
&= \int_{q_l\phi(\partial\Omega)} \frac{\partial}{\partial\nu_{q_l\mathbb{I}[\phi]}(y)} S_{q_l,2}(x - y)(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})(y)\, d\sigma_y \\
&= -\int_{q_l\phi(\partial\Omega)} DS_{q_l,2}(x - y) \cdot \nu_{q_l\mathbb{I}[\phi]}(y)(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})(y)\, d\sigma_y \\
&= -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \int_{q_l\partial\phi(\Omega)} S_{q_l,2}(x - y)(\nu_{q_l\mathbb{I}[\phi]}(y))_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})(y)\, d\sigma_y \\
&= -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} v^-_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})](x) \qquad \forall x \in \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-
\end{aligned}$$

and for all $(l, \phi) \in ]0, +\infty[\times \mathcal{U}_0$. Therefore, we have

$$\begin{aligned}
& \int_{Q_l \setminus q_l\mathbb{I}[\phi]} w^-_{q_l}[q_l\phi(\partial\Omega), \theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1}](x)\, dx \\
& \qquad = -\sum_{j=1}^{2} \int_{Q_l \setminus q_l\mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v^-_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})](x)\, dx,
\end{aligned} \tag{3.30}$$

for all $(l, \phi) \in ]0, +\infty[\times \mathcal{U}_0$.

Now we fix $j \in \{1, 2\}$. Using consequently the Divergence Theorem, the periodicity and the continuity on $\partial\Omega$ of the periodic single layer potential, we obtain

$$\begin{aligned}
& \int_{Q_l \setminus q_l\mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v^-_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})](x)\, dx \\
&= \int_{\partial Q_l} v^-_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})](x)(\nu_{Q_l}(x))_j\, d\sigma_x \\
&- \int_{q_l\phi(\partial\Omega)} v^-_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l, \phi] \circ \phi^{-1} \circ q_l^{-1})](x)(\nu_{q_l\mathbb{I}[\phi]}(x))_j\, d\sigma_x
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{q_l\phi(\partial\Omega)} v_{q_l}^{-}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})](x)(\nu_{q_l\mathbb{I}[\phi]}(x))_j\,d\sigma_x \qquad (3.31)\\
&= -\int_{\partial\Omega} v_{q_l}^{-}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})](q_l\phi(x))((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)(x)\tilde{\sigma}[\phi](x)\,d\sigma_x\\
&= -\int_{\partial\Omega} v_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})](q_l\phi(x))((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)(x)\tilde{\sigma}[\phi](x)\,d\sigma_x,
\end{aligned}
$$

fo all $(l,\phi)\in]0,+\infty[\times\mathcal{U}_0$. Then for $\tilde{S}_{\tilde{q},l,2}$ defined as in (3.19), we have

$$
\begin{aligned}
&v_{q_l}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})](q_l\phi(x)) \qquad (3.32)\\
&\quad= \int_{q_l\phi(\partial\Omega)} S_{q_l,2}(q_l\phi(x)-y)(\nu_{q_l\mathbb{I}[\phi]}(y))_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})(y)\,d\sigma_y\\
&\quad= \int_{\phi(\partial\Omega)} S_{q_l,2}(q_l(\phi(x)-y))(\nu_{q_l\mathbb{I}[\phi]}(q_ly))_j(\theta[l,\phi]\circ\phi^{-1})(y)\,d\sigma_y\\
&\quad= \int_{\phi(\partial\Omega)} \tilde{S}_{\tilde{q},l,2}(\phi(x)-y)(\nu_{q_l\mathbb{I}[\phi]}(q_ly))_j(\theta[l,\phi]\circ\phi^{-1})(y)\,d\sigma_y\\
&\quad=: \tilde{v}_{l,\tilde{q}}[\phi(\partial\Omega), (((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])\circ\phi^{-1})](\phi(x)) \qquad \forall x\in\partial\Omega
\end{aligned}
$$

and for all $(l,\phi)\in]0,+\infty[\times\mathcal{U}$. Here $\tilde{v}_{l,\tilde{q}}[\phi(\partial\Omega),\cdot]$ is the $\tilde{q}$-periodic single layer potential associated to $\tilde{S}_{\tilde{q},l,2}$.

Now, we want to prove that the map

$$
(l,\phi)\mapsto \tilde{v}_{l,\tilde{q}}[\phi(\partial\Omega), (((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])\circ\phi^{-1})]\circ\phi \qquad (3.33)
$$

from $]0,+\infty[\times\mathcal{U}_0$ to $C^{1,\alpha}(\partial\Omega)$ is real analytic, by using Theorem 5.10 $(i)$ of Lanza de Cristoforis and Musolino [81]. To do so, we first note that due to the real analyticity of the map $(l,\phi)\mapsto q_l\phi$ from $]0,+\infty[\times\mathcal{U}_0$ to $\mathcal{A}_{\partial\Omega}\cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$, Lemma C.0.4 $(ii)$ implies that the map

$$
(l,\phi)\mapsto(\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi
$$

from $]0,+\infty[\times\mathcal{U}_0$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic. Then, using Proposition 3.4.5 $(ii)$, one verifies that the map

$$
(l,\phi)\mapsto((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi]
$$

from $]0,+\infty[\times\mathcal{U}_0$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic. Finally, taking also the real analyticity of the map $l\mapsto(l^{-2},l^2)$ from $]0,+\infty[$ to $\mathbb{R}^2$ into account, we can apply [81, Thm 5.10 $(i)$] to the map in (3.33), and, thus, the real analyticity of (3.33) follows.

Then using the real analyticity of the map in (3.33), identity (3.32), equality (3.31), Lemma C.0.4 $(i)$, and the linearity and continuity of the map $f\mapsto\int_{\partial\Omega}f\,d\sigma$ from $L^1(\partial\Omega)$ to $\mathbb{R}$, we conclude that the map

$$
(l,\phi)\mapsto\int_{Q_l\setminus q_l\mathbb{I}[\phi]}\frac{\partial}{\partial x_j}v_{q_l}^{-}[q_l\phi(\partial\Omega), (\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1})](x)\,d\sigma_x
$$

from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$ is real analytic. Accordingly, equality (3.30) implies that the map

$$
(l,\phi)\mapsto\int_{Q_l\setminus q_l\mathbb{I}[\phi]} w_{q_l}^{-}[q_l\phi(\partial\Omega), \theta[l,\phi]\circ\phi^{-1}\circ q_l^{-1}](x)\,dx,
$$

is real analytic from $]0,+\infty[\times\mathcal{U}_0$ to $\mathbb{R}$. Thus the validity of the statement follows. □

Finally, using representation formula (3.7) for the effective permeability $K_{II}[l,\phi]$, Proposition 3.2.5, and Theorem 3.4.6, we immediately have the following result on the analyticity of the map $(l,\phi)\mapsto K_{II}[l,\phi]$.

**Theorem 3.4.7.** *The map* $(l,\phi) \mapsto K_{II}[l,\phi]$ *from* $]0,+\infty[\times \left(\tilde{\mathcal{A}}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\right)$ *to* $\mathbb{R}$ *is real analytic.*

Theorem 3.4.7 implies, in particular, that if we have a one-parameter analytic family of pairs $(l_\delta,\phi_\delta)_{\delta\in]-\delta_0,\delta_0[}$, with some $\delta_0$, then we can deduce the possibility to expand the permeability as a power series, i.e., $K_{II}[l_\delta,\phi_\delta] = \sum_{j=0}^{\infty} c_j\delta^j$ for $\delta$ close to zero. Moreover, by the analyticity of the map $(l,\phi) \mapsto K_{II}[l,\phi]$, the coefficients in the series expansion of $K_{II}$ can be constructively determined by computing the differentials of $K_{II}[l_\delta,\phi_\delta]$ as it has been done for the effective conductivity in Chapter 2. Furthermore, another important consequence of our high regularity result is that it allows applying differential calculus in order to find critical pairs $(l,\phi)$ *rectangle-shape* as a first step to find optimal configurations.

# Part II

# Properties of monogenic functions in a three-dimensional commutative algebra with one-dimensional radical

# CHAPTER 4

## Logarithmic residues of monogenic functions

This chapter is devoted to the analysis of monogenic functions in the three-dimensional commutative algebra $\mathbb{A}_2$ over $\mathbb{C}$ with one-dimensional radical and some their properties. In particular, we calculate logarithmic residues of monogenic functions acting from a three-dimensional real subspace of $\mathbb{A}_2$ into $\mathbb{A}_2$. We show that the logarithmic residue of a monogenic function is a hypercomplex number (cf. Grishchuk and Plaksa [55], and Plaksa and Shpakivskyi [129]) and it depends not only on zeros and singular points of the function but also on points at which the function takes values in ideals of $\mathbb{A}_2$.

We note that here we deal with a certain specific three-dimensional algebra and the obtained results are ones of the first steps to develop tools for solving boundary value problems in any finite-dimensional Banach algebras.

The chapter is organized as follows. In Section 4.1 we introduce the algebra $\mathbb{A}_2$ mentioned above, monogenic functions taking values in this algebra, and some additional notation used in the sequel. In Section 4.2 we study the logarithmic residues of monogenic functions in $\mathbb{A}_2$.

Some of the results presented in this chapter have been published in the paper [137] by the author and Prof. Sergiy Plaksa.

### 4.1 Three-dimensional commutative algebra with one-dimensional radical and monogenic functions

In this section we introduce the three-dimensional commutative algebra $\mathbb{A}_2$ over $\mathbb{C}$ with one-dimensional radical and basic notation. To begin with, we note that there exist four three-dimensional commutative associative Banach algebras over $\mathbb{C}$ (see, e.g., Mel'nichenko and Plaksa [103]) and we consider one of them.

Let $\mathbb{A}_2$ be a three-dimensional commutative associative Banach algebra over $\mathbb{C}$ with one-dimensional radical (see, e.g., Plaksa and the author [125], and Mel'nichenko and Plaksa [103]). This algebra has a basis $\{I_1, I_2, \rho\}$ with the following multiplication rules for its elements

$$I_1{}^2 = I_1, \quad I_2{}^2 = I_2, \quad I_2\rho = \rho, \quad I_1 I_2 = \rho^2 = I_1\rho = 0.$$

The unit of $\mathbb{A}_2$ is represented as $1 = I_1 + I_2$.

Let $c = c_1 I_1 + c_2 I_2 + c_3\rho$, where $c_1, c_2, c_3 \in \mathbb{C}$. The element $c$ is *invertible* if and only if $c_1 \neq 0$ and $c_2 \neq 0$, moreover, the inverse element $c^{-1}$ is represented as

$$c^{-1} = \frac{1}{c_1} I_1 + \frac{1}{c_2} I_2 - \frac{c_3}{c_2^2}\rho.$$

There are two maximal ideals in $\mathbb{A}_2$:

$$\mathcal{I}_1 := \{t_1 I_2 + t_2\rho : t_1, t_2 \in \mathbb{C}\}, \qquad \mathcal{I}_2 := \{t_1 I_1 + t_2\rho : t_1, t_2 \in \mathbb{C}\}.$$

Both ideals together include all non-invertible elements of the algebra $\mathbb{A}_2$ and consist of such elements only. We denote by $\mathcal{R} := \mathcal{I}_1 \cap \mathcal{I}_2$ the radical of this algebra. $\mathcal{R}$ is a one-dimensional subspace of $\mathbb{A}_2$.

We consider the linear continuous functionals $f_1$, $f_2 : \mathbb{A}_2 \to \mathbb{C}$ satisfying the equalities

$$f_1(I_1) = f_2(I_2) = 1, \quad f_1(I_2) = f_1(\rho) = f_2(I_1) = f_2(\rho) = 0.$$

The maximal ideals $\mathcal{I}_1$, $\mathcal{I}_2$ are kernels of the functionals $f_1$, $f_2$, respectively. For an arbitrary set $X$ in $\mathbb{A}_2$, we find it convenient to set

$$D_1(X) := \{\xi \in \mathbb{C} : \ \xi = f_1(a) \quad \forall a \in X\},$$

$$D_2(X) := \{\xi \in \mathbb{C} : \ \xi = f_2(a) \quad \forall a \in X\}.$$

Obviously, the sets $D_1(X)$ and $D_2(X)$ are the images of the set $X$ in $\mathbb{C}$ under the mappings $f_1$ and $f_2$, respectively.

Now we want to construct the three-dimensional linear subspace over $\mathbb{R}$ in the algebra $\mathbb{A}_2$. To begin with, we take three vectors $e_1$, $e_2$, and $e_3$ of the following form

$$e_1 = 1, \quad e_2 = p_1 I_1 + p_2 I_2 + p_3 \rho, \quad e_3 = q_1 I_1 + q_2 I_2 + q_3 \rho,$$

where $p_k, q_k \in \mathbb{C}$ for all $k \in \{1, 2, 3\}$, and assume that they are linear independent over $\mathbb{R}$. Then we denote by

$$E_3 := \{x e_1 + y e_2 + z e_3 : \ x, y, z \in \mathbb{R}\}$$

the linear span over $\mathbb{R}$ in $\mathbb{A}_2$ generated by the vectors of basis $\{e_1 = 1, e_2, e_3\}$. We also assume that

$$D_k(E_3) \equiv \mathbb{C} \qquad \forall k \in \{1, 2\}. \tag{4.1}$$

Obviously, condition (4.1) holds if and only if at least one of the numbers $p_k$ or $q_k$ belongs to $\mathbb{C} \setminus \mathbb{R}$ for all $k \in \{1, 2\}$.

Since $e_1$, $e_2$, and $e_3$ are linearly independent over $\mathbb{R}$, we can define the *Euclidean norm* in $E_3$ in the following way

$$\|a\| := \sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2} \tag{4.2}$$

for all $a = a_1 e_1 + a_2 e_2 + a_3 e_3 \in E_3$, where $a_1, a_2, a_3 \in \mathbb{R}$. Moreover, if this triple is linearly independent over $\mathbb{C}$ then $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{A}_2$, and the norm in $\mathbb{A}_2$ is defined as in (4.2) for all $a = a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{A}_2$, where $a_1, a_2, a_3 \in \mathbb{C}$.

In what follows, let $x, y, z \in \mathbb{R}$ and $x$, $y$, and $z$ with arbitrary sub-indexes be also real. We find it convenient to set

$$\begin{aligned} \zeta &:= x e_1 + y e_2 + z e_3, & \zeta_0 &:= x_0 e_1 + y_0 e_2 + z_0 e_3, \\ \xi_1 &:= x + p_1 y + q_1 z, & \xi_{10} &:= x_0 + p_1 y_0 + q_1 z_0, \\ \xi_2 &:= x + p_2 y + q_2 z, & \xi_{20} &:= x_0 + p_2 y_0 + q_2 z_0, \end{aligned}$$

$$e_1^* := (\mathrm{Re}\, p_1 \mathrm{Im}\, q_1 - \mathrm{Im}\, p_1 \mathrm{Re}\, q_1) e_1 - \mathrm{Im}\, q_1 e_2 + \mathrm{Im}\, p_1 e_3,$$

$$e_2^* := (\mathrm{Re}\, p_2 \mathrm{Im}\, q_2 - \mathrm{Im}\, p_2 \mathrm{Re}\, q_2) e_1 - \mathrm{Im}\, q_2 e_2 + \mathrm{Im}\, p_2 e_3,$$

$$L_1(\zeta) := \{\zeta + t e_1^* : \ t \in \mathbb{R}\}, \quad L_2(\zeta) := \{\zeta + t e_2^* : \ t \in \mathbb{R}\} \qquad \forall \zeta \in E_3.$$

Here, $\xi_k$ and $\xi_{k0}$ are the images of the point $\zeta$ and $\zeta_0$, respectively, by the functional $f_k : \mathbb{A}_2 \to \mathbb{C}$, for all $k \in \{1, 2\}$. The union of the straight lines $L_1(\zeta)$ and $L_2(\zeta)$ represent the set of points in $E_3$ such that the element $\tau - \zeta$ is non-invertible for all $\tau \in L_1(\zeta) \cup L_2(\zeta)$. Furthermore, setting $\xi_{k\tau} := f_k(\tau)$ for all $k \in \{1, 2\}$ and $\tau \in E_3$, we have that for any $\zeta \in E_3$ and all $\tau \in E_3 \setminus (L_1(\zeta) \cup L_2(\zeta))$ the following equality is true:

$$(\tau - \zeta)^{-1} = \frac{1}{\xi_{1\tau} - \xi_1} I_1 + \frac{1}{\xi_{2\tau} - \xi_2} I_2 - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} \rho \tag{4.3}$$

(see, e.g., Mel'nichenko and Plaksa [103, Sec. 8]).

We will often consider domains which are convex in some directions.

**Definition 4.1.1.** *Let $\Omega$ be a domain in $E_3$ and $k \in \{1,2\}$. We say that $\Omega$ is* convex in the direction $L_k$ *if the set $\Omega \cap L_k(\zeta)$ is connected for all $\zeta \in \Omega$.*

Then, for two arbitrary vectors $a, b \in E_3$ defined as $a = a_1e_1 + a_2e_2 + a_3e_3$ and $b = b_1e_1 + b_2e_2 + b_3e_3$, we denote

$$a \times b := (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3.$$

It is an analog of the vector product in $\mathbb{R}^3$.

Let $\alpha$ be an angle in $]-\pi/2, 0[\cup]0, \pi/2[$. We find it convenient to set

$$\hat{e}_1 := \begin{cases} e_1^* \times e_1, & \text{if} \quad e_1^* = e_2^*, \\ e_1^* \times e_2^*, & \text{if} \quad e_1^* \neq e_2^*, \end{cases}$$

$$\hat{e}_2 := \begin{cases} e_1^* \times (e_1^* \times e_1), & \text{if} \quad e_1^* = e_2^*, \\ \frac{e_1^*}{\|e_1^*\|}\sin\alpha + \frac{e_2^*}{\|e_2^*\|}\cos\alpha, & \text{if} \quad e_1^* \neq e_2^*, \end{cases}$$

$$\Pi_{\alpha,\zeta} := \{\zeta + t_1\hat{e}_1 + t_2\hat{e}_2 : \ t_1, t_2 \in \mathbb{R}\} \qquad \forall \zeta \in E_3.$$

If $\alpha = \pi/4$ then we will simply write $\Pi_\zeta$ instead of $\Pi_{\alpha,\zeta}$ for all $\zeta \in E_3$. The set $\Pi_{\alpha,\zeta}$ is a specific plane $E_3$ which we need in order to introduce the logarithmic residue.

Now we define monogenic functions by means of the following.

**Definition 4.1.2.** *Let $\Omega$ be a domain in $E_3$ and the function $\Phi : \Omega \to \mathbb{A}_2$ be continuous in $\Omega$. We say that $\Phi$ is* monogenic *in $\Omega$ if $\Phi$ is differentiable in the sense of Gateaux at every point of $\Omega$, i.e., if for every $\zeta \in \Omega$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_2$ such that*

$$\lim_{\varepsilon \to 0+} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right)\varepsilon^{-1} = h\Phi'(\zeta) \qquad \forall\, h \in E_3.$$

$\Phi'(\zeta)$ is called the *Gateaux derivative* of the function $\Phi$ at the point $\zeta$.

We recall the following result proved by Shpakivskyi [152, Thm. 5.4].

**Theorem 4.1.3.** *Let a domain $\Omega \subseteq E_3$ be convex in the direction $L_k$ and $f_k(\Omega) = \mathbb{C}$ for all $k \in \{1,2\}$. Then every monogenic function $\Phi : \Omega \to \mathbb{A}_2$ can be expressed in the form*

$$\Phi(\zeta) = F_1(\xi_1)\, I_1 + F_2(\xi_2) I_2 + \left((p_3y + q_3z)F_2'(\xi_2) + F_0(\xi_2)\right)\rho \qquad \forall\, \zeta \in \Omega\,, \tag{4.4}$$

*where $F_1$ is a holomorphic function in the domain $D_1(\Omega)$, and $F_2$ and $F_0$ are holomorphic functions in the domain $D_2(\Omega)$.*

We also note that representation (4.4) was also proved by Plaksa and the author [125, Thm. 4] for the case where the basis $\{e_1, e_2, e_3\}$ satisfies the following conditions:

$$e_1^2 + e_2^2 + e_3^2 = 0, \qquad e_k^2 \neq 0 \qquad \forall k \in \{1,2,3\}.$$

If these conditions are satisfied then the basis $\{e_1, e_2, e_3\}$ is called a harmonic basis (see Mel'nichenko and Plaksa [103, pp. 17, 18]). Additionally, we note that if the basis $\{e_1, e_2, e_3\}$ is harmonic then any monogenic function is the solution of the Laplace equation in the domain of its definition.

To complete the section, we recall two results which we exploit in the sequel. The first one is an analog of the Cauchy integral theorem for monogenic functions in $\mathbb{A}_2$, and the second one is an analog of the classical Cauchy integral formula. The proofs can be found in Shpakivskyi [152, Thm. 4.2] and [152, Thm. 6.1], respectively.

**Theorem 4.1.4.** *Let $\Omega$ be a domain in $E_3$ convex in directions $L_1$ and $L_2$. Let $\Phi : \Omega \to \mathbb{A}_2$ be a monogenic function in $\Omega$. Let $\gamma \subseteq \Omega$ be a closed Jordan rectifiable curve which is homotopic to a point of $\Omega$. Then*

$$\int_\gamma \Phi(\zeta)d\zeta = 0.$$

**Theorem 4.1.5.** *Let $\Omega$ be a domain in $E_3$ convex in directions $L_1$ and $L_2$. Let $\Phi : \Omega \to \mathbb{A}_2$ be a monogenic function in $\Omega$. Then for every point $\zeta_0 \in \Omega$ the following equality is true:*

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int_\gamma \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta,$$

*where $\gamma$ is a closed Jordan rectifiable curve in $\Omega$, that surrounds once the set $L_1(\zeta_0) \cup L_2(\zeta_0)$.*

## 4.2 On the logarithmic residues of monogenic functions

In this section we consider the logarithmic residues of monogenic functions taking values in the algebra $\mathbb{A}_2$ and we calculate the logarithmic residues of monogenic functions which act from a three-dimensional real subspace $E_3$ into $\mathbb{A}_2$.

To begin with, we note that the logarithmic residue in a Banach algebra is a contour integral of the logarithmic derivative of a hypercomplex function. The logarithmic residues of monogenic functions were considered by Grishchuk and Plaksa in [55] and by Plaksa and Shpakivskyi in [129], where the logarithmic residue of monogenic functions was calculated and it was shown that it is always an integer number. In general case, it can be a hypercomplex number, as it is shown in this section

The section is organized as follows. In Subsection 4.2.1, we introduce some standard notation and the logarithmic residue. In Subsection 4.2.2 we consider some properties of Laurent series of monogenic functions in $\mathbb{A}_2$. Subsection 4.2.3 is devoted to the logarithmic residues. In this subsection, we exploit the Laurent series to calculate the logarithmic residue of monogenic function (see Lemma 4.2.11). Using this result, we establish the validity of Theorem 4.2.15 and Theorem 4.2.16 for a curvilinear integral of the logarithmic derivative of a monogenic function along a family of curves.

### 4.2.1 Preliminaries and notation

Let $c = c_1 I_1 + c_2 I_2 + c_3 \rho$, where $c_1, c_2, c_3 \in \mathbb{C}$. We define the logarithm in $\mathbb{A}_2$ in the same way as it is done by Lorch [93, p. 422], and it takes the following form in the basis $\{I_1, I_2, \rho\}$

$$\ln c := (\ln c_1)I_1 + (\ln c_2)I_2 + \frac{c_3}{c_2}\rho, \tag{4.5}$$

where $\ln c_1$, $\ln c_2$ are the principal branches of appropriate logarithmic functions.

Then, for any nonnegative real numbers $r$ and $R$ such that $r < R$, and a point $\zeta_0 \in E_3$, we find it convenient to set

$$\mathcal{K}_R(\zeta_0) := \{\zeta \in E_3 : |\xi_1 - \xi_{10}| < R, |\xi_2 - \xi_{20}| < R\},$$
$$\mathcal{K}_{r,R}(\zeta_0) := \{\zeta \in E_3 : 0 \le r < |\xi_1 - \xi_{10}| < R \le \infty,\ 0 \le r < |\xi_2 - \xi_{20}| < R \le \infty\},$$

and we define the logarithmic residue as follows.

**Definition 4.2.1.** *Let $\zeta_0 \in E_3$ and $R \in ]0, +\infty]$. Let $\Phi : \mathcal{K}_{0,R}(\zeta_0) \to \mathbb{A}_2$ be a monogenic function in the domain $\mathcal{K}_{0,R}(\zeta_0)$, and the function $\Phi'\Phi^{-1}$ be monogenic in $\mathcal{K}_{0,R}(\zeta_0)$. We say that the integral*

$$\frac{1}{2\pi i} \int\limits_{\Gamma_r(\zeta_0)} \Phi'(\zeta)\big(\Phi(\zeta)\big)^{-1} d\zeta, \tag{4.6}$$

*where* $r \in ]0, R[$ *and* $\Gamma_r(\zeta_0) := \{\zeta \in \Pi(\zeta_0) : \; |\xi_{10} - f_1(\zeta)| = r, |\xi_{20} - f_2(\zeta)| = r\}$, *is* the logarithmic residue *of the function* $\Phi$ *at the point* $\zeta_0$.

By virtue of Theorem 4.1.4, which is the analog of the Cauchy integral theorem, we conclude that the value of logarithmic residue is independent of $r$ for $0 < r < R$.

Finally, we note that the logarithmic residues of monogenic functions which take values in commutative Banach algebras different from $\mathbb{A}_2$ were considered in Grishchuk and Plaksa [55], and in Plaksa and Shpakivskyi [129]. They calculated the logarithmic residue of monogenic functions which is always an integer.

### 4.2.2 Properties of Laurent series of monogenic functions

In order to investigate the logarithmic residue, we will exploit the property of monogenic functions in $\mathbb{A}_2$ to be expanded into the convergent Laurent series.

To prove such a property, we will use the following auxiliary statement, which we need in order to apply Theorem 4.1.3 to monogenic functions in $\mathcal{K}_{r,R}(\zeta_0)$.

**Lemma 4.2.2.** *Let* $\zeta_0 \in E_3$, $R \in ]0, +\infty]$ *and* $r \in [0, R[$. *Then the domain* $\mathcal{K}_{r,R}(\zeta_0)$ *can be represented as a union of domains* $\widetilde{\mathcal{K}}_{r,R}(\zeta_0)$ *and* $\widehat{\mathcal{K}}_{r,R}(\zeta_0)$, *each of them is convex in both directions* $L_1$ *and* $L_2$, *and* $\widetilde{\mathcal{K}}_{r,R}(\zeta_0) \cap \widehat{\mathcal{K}}_{r,R}(\zeta_0)$ *is an open set.*

*Proof.* The proof is constructive. We first note that the case $L_1 = L_2$ is trivial. It is enough to set $\widetilde{\mathcal{K}}_{r,R}(\zeta_0) = \widehat{\mathcal{K}}_{r,R}(\zeta_0) = \mathcal{K}_{r,R}(\zeta_0)$.

Next, we suppose that $L_1$ and $L_2$ do not coincide. Let

$$\Pi^*_{\zeta_0} := \{\zeta_0 + t_1 e_1^* + t_2 e_2^* : \; t_1, t_2 \in \mathbb{R}\}$$

be a plane in $E_3$. If $r \neq 0$ then the planes $\Pi^*_{\zeta_0 + r\hat{e}_1}$ and $\Pi^*_{\zeta_0 - r\hat{e}_1}$ split $\mathcal{K}_{r,R}(\zeta_0)$ into six parts. By construction, there are four parts located between the planes and we denote them by $P_1$, $P_2$, $P_3$, $P_4$. One can uniquely indicate two pairs of sets from $\{P_1, P_2, P_3, P_4\}$ such that the union of the sets in the pair is convex in both directions $L_1$ and $L_2$. Without loss of generality, we assume that the indicated pairs are $\{P_1, P_2\}$ and $\{P_3, P_4\}$. Then, we set

$$\widetilde{\mathcal{K}}_{r,R}(\zeta_0) := \mathcal{K}_{r,R}(\zeta_0) \setminus \mathrm{cl}(P_1 \cup P_2), \qquad \widehat{\mathcal{K}}_{r,R}(\zeta_0) := \mathcal{K}_{r,R}(\zeta_0) \setminus \mathrm{cl}(P_3 \cup P_4).$$

By construction, one observes that both sets $\widetilde{\mathcal{K}}_{r,R}(\zeta_0)$ and $\widehat{\mathcal{K}}_{r,R}(\zeta_0)$ are open and convex in directions $L_1$, $L_2$. As a conclusion, $\widetilde{\mathcal{K}}_{r,R}(\zeta_0) \cap \widehat{\mathcal{K}}_{r,R}(\zeta_0)$ is an open set.

If $r = 0$ then we take one plane $\Pi^*_{\zeta_0}$ and split it into four parts by the straight lines $L_1(\zeta_0)$ and $L_2(\zeta_0)$. Then, by arguing as above, we deduce the validity of the lemma. □

We observe that by the construction of $\widetilde{\mathcal{K}}_{r,R}(\zeta_0)$ and $\widehat{\mathcal{K}}_{r,R}(\zeta_0)$ the following conditions are true:

$$D_1\left(\widetilde{\mathcal{K}}_{0,R}(\zeta_0)\right) \equiv D_1\left(\widehat{\mathcal{K}}_{0,R}(\zeta_0)\right) \equiv \{\xi \in \mathbb{C} : \; 0 < |\xi - \xi_{10}| < R\},$$
$$D_2\left(\widetilde{\mathcal{K}}_{0,R}(\zeta_0)\right) \equiv D_2\left(\widehat{\mathcal{K}}_{0,R}(\zeta_0)\right) \equiv \{\xi \in \mathbb{C} : \; 0 < |\xi - \xi_{20}| < R\}.$$

Then, we define the following properties for curves in $E_3$.

**Definition 4.2.3.** *Let* $\zeta_0$ *be a point in* $\Omega$ *and* $C(\zeta_0)$ *be a circle in* $\Omega$ *with the center at* $\zeta_0$. *Let* $k \in \{1, 2\}$. *We say that the circle* $C(\zeta_0)$ surrounds the set $L_k(\zeta_0)$ *if* $f_k(\zeta_0)$ *lies inside the domain in* $\mathbb{C}$ *bounded by* $D_k(C(\zeta_0))$.

**Definition 4.2.4.** *Let* $\zeta_0$ *be a point in* $\Omega$ *and* $\gamma$ *be a closed curve in* $\Omega$. *Let* $k \in \{1, 2\}$. *We say that* $\gamma$ surrounds once the set $L_k(\zeta_0)$ *if there exists a circle* $C(\zeta_0) \subseteq \Omega$ *which is homotopic to* $\gamma$ *in the domain* $\Omega \setminus L_k(\zeta_0)$ *and surrounds* $L_k(\zeta_0)$.

Now, using Lemma 4.2.2, we prove the following result on the Laurent series expansion for a monogenic function.

**Theorem 4.2.5.** *Let $\zeta_0 \in E_3$ and $R \in ]0, +\infty]$. Let $\Phi : \mathcal{K}_{0,R}(\zeta_0) \to \mathbb{A}_2$ be monogenic in $\mathcal{K}_{0,R}(\zeta_0)$. Let $a_n$, $b_n$, and $c_n$ be the coefficients of the Laurent series of the functions in decomposition* (4.4)*:*

$$\begin{aligned} F_1(\xi_1) &= \sum_{n=-\infty}^{\infty} a_n(\xi_1 - \xi_{10})^n, \qquad F_2(\xi_2) = \sum_{n=-\infty}^{\infty} b_n(\xi_2 - \xi_{20})^n, \\ F_0(\xi_2) &= \sum_{n=-\infty}^{\infty} c_n(\xi_2 - \xi_{20})^n . \end{aligned} \tag{4.7}$$

*Then $\Phi$ can be represented as a sum of absolutely convergent Laurent series*

$$\Phi(\zeta) \equiv \sum_{n=-\infty}^{\infty} d_n(\zeta - \zeta_0)^n , \tag{4.8}$$

*where $(\zeta - \zeta_0)^n := ((\zeta - \zeta_0)^{-1})^{-n}$ for all negative integer $n$, and*

$$d_n \equiv a_n I_1 + b_n I_2 + ((n+1)(p_3 y_0 + q_3 z_0) b_{n+1} + c_n)\rho. \tag{4.9}$$

*Proof.* By Lemma 4.2.2, the domain $\mathcal{K}_{0,R}(\zeta_0)$ can be represented as a union of domains $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$ and $\widehat{\mathcal{K}}_{0,R}(\zeta_0)$, which are convex in both directions $L_1$ and $L_2$, and $\widetilde{\mathcal{K}}_{0,R}(\zeta_0) \cap \widehat{\mathcal{K}}_{0,R}(\zeta_0)$ is an open set.

We first consider the domain $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$. Clearly, $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$ is convex in both directions $L_1$ and $L_2$, and the function $\Phi$ is monogenic in $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$. Then, by Theorem 4.1.3, equality (4.4) holds in $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$ and, moreover, the function $F_1$ is analytic in $D_1\left(\widetilde{\mathcal{K}}_{0,R}(\zeta_0)\right)$ and the functions $F_2$ and $F_0$ are analytic in $D_2\left(\widetilde{\mathcal{K}}_{0,R}(\zeta_0)\right)$ and, thus, the series in (4.7) are absolutely convergent. Using (4.4) and (4.7) we have:

$$\begin{aligned} \Phi(\zeta) = &\sum_{n=-\infty}^{\infty} a_n(\xi_1 - \xi_{10})^n I_1 + \sum_{n=-\infty}^{\infty} b_n(\xi_2 - \xi_{20})^n I_2 \\ &+ (p_3 y + q_3 z) \sum_{n=-\infty}^{\infty} n b_n(\xi_2 - \xi_{20})^{n-1}\rho + \sum_{n=-\infty}^{\infty} c_n(\xi_2 - \xi_{20})^n \rho \qquad \forall\, \zeta \in \widetilde{\mathcal{K}}_{0,R}(\zeta_0). \end{aligned}$$

Now, using the following equalities:

$$\begin{aligned} (\zeta - \zeta_0)^n I_1 &= (\xi_1 - \xi_{10})^n I_1, \\ (\zeta - \zeta_0)^n I_2 &= (\xi_2 - \xi_{20})^n I_2 + n(p_3(y - y_0) + q_3(z - z_0))(\xi_2 - \xi_{20})^{n-1}\rho, \\ (\zeta - \zeta_0)^n \rho &= (\xi_2 - \xi_{20})^n \rho, \end{aligned}$$

which hold for all integer $n$, we obtain representation (4.8) for the function $\Phi$, where the series is absolutely convergent in $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$.

Now, we consider the domain $\widehat{\mathcal{K}}_{0,R}(\zeta_0)$. In the same way as for $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$, we obtain the validity of (4.8) in $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$. Then, since the monogenic function $\Phi$ has representation (4.8) in both domains $\widetilde{\mathcal{K}}_{0,R}(\zeta_0)$ and $\widehat{\mathcal{K}}_{0,R}(\zeta_0)$, and $\widetilde{\mathcal{K}}_{0,R}(\zeta_0) \cap \widehat{\mathcal{K}}_{0,R}(\zeta_0)$ is an open set, using Theorem 2 of [139], we deduce the validity of the theorem. □

We also note that the coefficients $d_n$ can be represented (cf. Shpakivskyi [151, Thm. 4]) by the formula

$$d_n = \frac{1}{2\pi i} \int_{\gamma} \Phi(\tau)(\zeta - \zeta_0)^{-n-1} d\tau \qquad \forall n \in \mathbb{Z},$$

where $\gamma$ is an arbitrary closed Jordan rectifiable curve in $\mathcal{K}_{0,R}(\zeta_0)$ surrounding once the straight lines $L_1(\zeta_0)$, $L_2(\zeta_0)$.

Next, we define types of singular points of a monogenic function.

**Definition 4.2.6.** *Let $\Phi$ be a monogenic function in a domain $\Omega \subseteq E_3$ and let $\zeta_0 \in \Omega$. We say that the point $\zeta_0$ is*

- a removable singularity *of $\Phi$ if there exists a finite limit*
$$\lim_{\zeta\to\zeta_0,\ \zeta\notin L_1(\zeta_0)\cup L_2(\zeta_0)} \Phi(\zeta) = A;$$
- a pole *of $\Phi$ if there exists an infinite limit*
$$\lim_{\zeta\to\zeta_0,\ \zeta\notin L_1(\zeta_0)\cup L_2(\zeta_0)} \Phi(\zeta) = \infty;$$
- an essential singularity *of $\Phi$ if a limit of $\Phi(\zeta)$ does not exist as $\zeta \to \zeta_0$ and $\zeta \notin L_1(\zeta_0) \cup L_2(\zeta_0)$.*

It is known that the isolated singularity can be only removable. Otherwise, if $\Phi$ has a non-removable singularity at the point $\zeta_0 \in \Omega$ then all points of the set $\Omega \cap L_1(\zeta_0)$ or the set $\Omega \cap L_2(\zeta_0)$, or both these sets are singular for $\Phi$ (cf., [139, Sec. 3]).

### 4.2.3 Main result on logarithmic residues of monogenic functions

First of all, we note that one needs to consider logarithmic residues not only at zeros and singular points of the function $\Phi$ but also at those points where values of $\Phi$ belong to the ideals of $\mathbb{A}_2$.

Before formulating the main theorem, we prove some auxiliary results. The statement is a necessary and sufficient condition for the existence of integral (4.6).

**Lemma 4.2.7.** *Let $\zeta_0 \in E_3$ and $R \in ]0, +\infty]$. Let $\Phi : \mathcal{K}_{0,R}(\zeta_0) \to \mathbb{A}_2$ be monogenic in $\mathcal{K}_{0,R}(\zeta_0)$. The following statements are equivalent:*

*(i) There exists $R_1 < R$ such that integral* (4.6) *exists for all $r \in ]0, R_1[$.*

*(ii) There exists $R_2 < R$ such that $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R_2}(\zeta_0)$.*

*Proof.* First, we prove that $(i)$ implies $(ii)$. Assume, for the sake of contradiction, that the statement does not hold. In other words, for all arbitrarily small $R_2 > 0$ there exist points $\zeta \in \mathcal{K}_{0,R_2}(\zeta_0)$ for which $\Phi(\zeta) \in \mathcal{I}_k$ for some $k \in \{1, 2\}$. Then the inner point $\xi_{k0}$ of the domain $D_k(\mathcal{K}_{R_2}(\zeta_0))$ is a limit point of the set of zeros of the holomorphic function $F_k$ appearing in equality (4.4). Hence, according to the Uniqueness theorem for holomorphic functions of a complex variable (see, e.g., Rudin [142, p. 209]), $F_k \equiv 0$ and, in view of equality (4.4), we conclude that all values of the function $\Phi$ belong to the ideal $\mathcal{I}_k$. Therefore, integral (4.6) does not exist and we have a contradiction.

Now, we prove that $(ii)$ implies $(i)$. It is enough to note that the assumption $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R_2}(\zeta_0)$ implies that $\Phi'(\zeta)\big(\Phi(\zeta)\big)^{-1}$ exists for all $\zeta \in \mathcal{K}_{0,R_2}(\zeta_0)$ and is monogenic in $\mathcal{K}_{0,R_2}(\zeta_0)$. We set $R_1 = R_2$ and, thus, integral (4.6) exists for all $r \in ]0, R_1[$. □

**Lemma 4.2.8.** *Let $\Phi : \mathcal{K}_{0,R}(\zeta_0) \to \mathbb{A}_2$ be a monogenic function in the domain $\mathcal{K}_{0,R}(\zeta_0)$ and $d_n$ be defined in* (4.8)*. If $\Phi(\zeta) \notin \mathcal{I}_k$ for any $k \in \{1, 2\}$ and all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$, the set $\mathbb{Z}_k := \{n \in \mathbb{Z} :\ d_n \notin \mathcal{I}_k\}$ is nonempty.*

*Proof.* Since $\Phi$ is a monogenic function in the domain $\mathcal{K}_{0,R}(\zeta_0)$, it can be represented in form (4.8). Assume, for the sake of contradiction, that $\mathbb{Z}_k$ is empty. Then $d_n(\zeta - \zeta_0)^n \in \mathcal{I}_k$ for all $n \in \mathbb{Z}$, which implies that $\Phi(\zeta) \in \mathcal{I}_k$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$. We have a contradiction. □

By virtue of Lemma 4.2.8, the assumption $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$ implies that both $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are nonempty. Additionally, we assume that

$$\mathbb{Z}_1,\, \mathbb{Z}_2 \quad \text{are bounded from below} \tag{4.10}$$

and we set

$$n_1 := \min_{n\in\mathbb{Z}_1} n \qquad \text{and} \qquad n_2 := \min_{n\in\mathbb{Z}_2} n.$$

**Lemma 4.2.9.** *Let $\Phi$ be a monogenic function in $\mathcal{K}_{0,R}(\zeta_0)$. Let assumption* (4.10) *hold. Then there exist two monogenic functions $\phi_1$, $\phi_2$ in the domain $\mathcal{K}_R(\zeta_0)$ and a monogenic function $\psi$ in $\mathcal{K}_{0,R}(\zeta_0)$ such that*

$$\Phi(\zeta) = (\zeta-\zeta_0)^{n_1}\phi_1(\zeta)I_1 + (\zeta-\zeta_0)^{n_2}\phi_2(\zeta)I_2 + \psi(\zeta)\rho \qquad \forall \zeta \in \mathcal{K}_{0,R}(\zeta_0). \tag{4.11}$$

*Proof.* Using relations (4.8) and (4.9), definitions of $\mathbb{Z}_1$, $\mathbb{Z}_2$ and assumption (4.10), we have

$$\begin{aligned}
\Phi(\zeta) =& \sum_{n=-\infty}^{\infty} a_n(\zeta-\zeta_0)^n I_1 + \sum_{n=-\infty}^{\infty} b_n(\zeta-\zeta_0)^n I_2 \\
&+ \sum_{n=-\infty}^{\infty} \big((n+1)(p_3y_0+q_3z_0)b_{n+1}+c_n\big)(\zeta-\zeta_0)^n\rho \\
=&(\zeta-\zeta_0)^{n_1}\sum_{n\in\mathbb{Z}_1} a_n(\zeta-\zeta_0)^{n-n_1}I_1 + (\zeta-\zeta_0)^{n_2}\sum_{n\in\mathbb{Z}_2} b_n(\zeta-\zeta_0)^{n-n_2}I_2 \\
&+ \sum_{n=-\infty}^{\infty} \big((n+1)(p_3y_0+q_3z_0)b_{n+1}+c_n\big)(\zeta-\zeta_0)^n\rho,
\end{aligned}$$

where $a_n$, $b_n$, $c_n$ are defined in (4.9). To complete the proof it is natural to set

$$\begin{aligned}
\phi_1(\zeta) &:= \sum_{n\in\mathbb{Z}_1} a_n(\zeta-\zeta_0)^{n-n_1}, \qquad \phi_2(\zeta) := \sum_{n\in\mathbb{Z}_2} b_n(\zeta-\zeta_0)^{n-n_2}, \\
\psi(\zeta) &:= \sum_{n=-\infty}^{\infty} \big((n+1)(p_3y_0+q_3z_0)b_{n+1}+c_n\big)(\zeta-\zeta_0)^n.
\end{aligned}$$

□

In the following lemma, we consider some properties of functions $\phi_1$, $\phi_2$ presented in the statement of Lemma 4.2.9.

**Lemma 4.2.10.** *Let assumptions of Lemma 4.2.9 hold. Moreover, let $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$. Then $\phi_1(\zeta), \phi_2(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$.*

*Proof.* We first assume, for the sake of contradiction, that the statement does not hold for $\phi_1$. Let there exist $\zeta_1 \in \mathcal{K}_{0,R}(\zeta_0)$ such that $\phi_1(\zeta_1) \in \mathcal{I}_1$. Then $(\zeta_1-\zeta_0)^{n_1}\phi_1(\zeta_1)I_1 = 0$, and, by virtue of (4.11), $\Phi(\zeta_1) \in \mathcal{I}_1$. We have a contradiction.

Now, we note that $\phi_1$ can be rewritten as follows

$$\begin{aligned}
\phi_1(\zeta) = \sum_{n\in\mathbb{Z}_1} a_n(\xi_1-\xi_{10})^{n-n_1}I_1 + \sum_{n\in\mathbb{Z}_1} a_n(\xi_2-\xi_{20})^{n-n_1}I_2 \\
+ \sum_{n\in\mathbb{Z}_1} (n-n_1)a_n(\xi_2-\xi_{20})^{n-n_1-1}(p_3(y-y_0)+q_3(z-z_0))\rho.
\end{aligned}$$

and we assume that there exists $\zeta_2 \in \mathcal{K}_{0,R}(\zeta_0)$ such that $\phi_1(\zeta_2) \in \mathcal{I}_2$. This assumption implies that

$$\sum_{n\in\mathbb{Z}_1} a_n\delta^{n-n_1} = 0, \quad \text{where} \quad \delta = f_2(\zeta_2) - \xi_{20}.$$

Let us consider the set

$$\mathcal{K}(\zeta_0,\delta) := \{\zeta \in \mathcal{K}_{0,R}(\zeta_0): \ |\xi_1-\xi_{10}| = |\delta|, \ |\xi_2-\xi_{20}| = |\delta|\}.$$

Then, there exists $\tilde{\zeta}_2 \in \mathcal{K}(\zeta_0,\delta)$ such that $f_1(\tilde{\zeta}_2)-\xi_{10}=\delta$. Thus,

$$\sum_{n\in\mathbb{Z}_1} a_n(f_1(\tilde{\zeta}_2)-\xi_{10})^{n-n_1}=0,$$

which implies that $\phi_1(\tilde{\zeta}_2)\in\mathcal{I}_1$, and, by arguing as above, we have a contradiction. Considering $\phi_2$ in a similar way as $\phi_1$, we deduce the validity of the lemma. □

In the following lemma we find the logarithmic residue of $\Phi$ at the point $\zeta_0$.

**Lemma 4.2.11.** *Let $\Phi$ be a monogenic function in $\mathcal{K}_{0,R}(\zeta_0)$ and $\Phi(\zeta)\notin\mathcal{I}_1\cup\mathcal{I}_2$ for all $\zeta\in\mathcal{K}_{0,R}(\zeta_0)$. Moreover, let $\Phi$ have representation* (4.11), *where $\phi_1$, $\phi_2$ are monogenic functions in the domain $\mathcal{K}_R(\zeta_0)$ and $\psi$ is monogenic in $\mathcal{K}_{0,R}(\zeta_0)$. Then*

$$\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)} \Phi'(\zeta)\big(\Phi(\zeta)\big)^{-1}d\zeta = n_1I_1+n_2I_2 \tag{4.12}$$

*for an arbitrary $r\in\mathbb{R}$ such that $0<r<R$.*

*Proof.* By Lemma 4.2.7, the integral on the left hand side of equality (4.12) exists. Moreover, by Lemma 4.2.10, $\phi_1$ and $\phi_2$ do not take values in the ideals of $\mathbb{A}_2$ for all $\zeta\in\mathcal{K}_{0,R}(\zeta_0)$, that implies the existence of $(\phi_1(\zeta))^{-1}$ and $(\phi_2(\zeta))^{-1}$ for all $\zeta\in\mathcal{K}_{0,R}(\zeta_0)$.

By (4.11), we immediately have

$$\begin{aligned}\Phi'(\zeta) =& n_1(\zeta-\zeta_0)^{n_1-1}\phi_1(\zeta)I_1+(\zeta-\zeta_0)^{n_1}\phi_1'(\zeta)I_1+n_2(\zeta-\zeta_0)^{n_2-1}\phi_2(\zeta)I_2\\ &+(\zeta-\zeta_0)^{n_2}\phi_2'(\zeta)I_2+\psi'(\zeta)\rho \qquad \forall\zeta\in\mathcal{K}_{0,R}(\zeta_0),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\big(\Phi(\zeta)\big)^{-1} =& (\zeta-\zeta_0)^{-n_1}(\phi_1(\zeta))^{-1}I_1+(\zeta-\zeta_0)^{-n_2}(\phi_2(\zeta))^{-1}I_2\\ &-(\zeta-\zeta_0)^{-2n_2}(\phi_2(\zeta))^{-2}\psi(\zeta)\rho \qquad \forall\zeta\in\mathcal{K}_{0,R}(\zeta_0).\end{aligned} \tag{4.14}$$

Taking into account (4.13) and (4.14), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)} \Phi'(\zeta)\big(\Phi(\zeta)\big)^{-1}d\zeta =& I_1\frac{n_1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}(\zeta-\zeta_0)^{-1}d\zeta + I_1\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\phi_1'(\zeta)(\phi_1(\zeta))^{-1}d\zeta\\ &+I_2\frac{n_2}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}(\zeta-\zeta_0)^{-1}d\zeta+I_2\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\phi_2'(\zeta)(\phi_2(\zeta))^{-1}d\zeta\\ &-\rho\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\Big[(\zeta-\zeta_0)^{-3n_2}(\phi_2(\zeta))^{-3}\psi(\zeta)\Big]'d\zeta\\ =&(n_1I_1+n_2I_2)\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}(\zeta-\zeta_0)^{-1}d\zeta+I_1\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\phi_1'(\zeta)(\phi_1(\zeta))^{-1}d\zeta\\ &+I_2\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\phi_2'(\zeta)(\phi_2(\zeta))^{-1}d\zeta-\rho\frac{1}{2\pi i}\int\limits_{\Gamma_r(\zeta_0)}\Big[(\zeta-\zeta_0)^{-3n_2}(\phi_2(\zeta))^{-3}\psi(\zeta)\Big]'d\zeta\\ =:& (n_1I_1+n_2I_2)\Lambda_1+I_1\Lambda_2+I_2\Lambda_3+\rho\Lambda_4.\end{aligned}$$

By virtue of Theorem 4.1.5, we have $\Lambda_1=1$. Then, using Theorem 4.1.4, we obtain the equality $\Lambda_2=\Lambda_3=0$ because the functions $\phi_1'(\zeta)(\phi_1(\zeta))^{-1}$ and $\phi_2'(\zeta)(\phi_2(\zeta))^{-1}$ are monogenic in the domain $\mathcal{K}_R(\zeta_0)$. Finally, taking into account the continuity of the function $(\zeta-\zeta_0)^{-3n_2}(\phi_2(\zeta))^{-3}\psi(\zeta)$ on the curve $\Gamma_r(\zeta_0)$, we obtain the equality $\Lambda_4=0$. □

The following result follows from Lemma 4.2.11.

**Corollary 4.2.12.** *Let assumptions of Lemma 4.2.11 hold. If $n_1 = n_2$ then the logarithmic residue of the monogenic function $\Phi$ at the point $\zeta_0$ is an integer.*

Here, we find it convenient to introduce the set $L_\Phi(\zeta_0)$ by setting

$$L_\Phi(\zeta_0) := \begin{cases} L_1(\zeta_0) \cup L_2(\zeta_0) & \text{if either } \Phi(\zeta_0) \in \mathcal{R} \text{ or } \xi_{10} \text{ is} \\ & \text{a non-removable singularity for } F_1 \text{ and} \\ & \xi_{20} \text{ is a non-removable singularity for } F_2, \\ L_1(\zeta_0) & \text{if either } \Phi(\zeta_0) \in \mathcal{I}_1 \setminus \mathcal{R} \text{ or } \xi_{10} \text{ is a non-removable} \\ & \text{singularity for } F_1 \text{ and } \xi_{20} \text{ is not a non-removable} \\ & \text{singularity for } F_2, \\ L_2(\zeta_0) & \text{if either } \Phi(\zeta_0) \in \mathcal{I}_2 \setminus \mathcal{R} \text{ or } \xi_{10} \text{ is not a non-removable} \\ & \text{singularity for } F_1 \text{ and } \xi_{20} \text{ is a non-removable} \\ & \text{singularity for } F_2. \end{cases}$$

**Definition 4.2.13.** *Let the function $\Phi$ be monogenic in the domain $\mathcal{K}_{0,R}(\zeta_0)$ and $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$. Let either $\Phi(\zeta_0) \in \mathcal{I}_1 \cup \mathcal{I}_2$ or $\zeta_0$ is a non-removable singular point of $\Phi$. Then we say that $\zeta_0$ is* a singular point of the logarithmic derivative *of function $\Phi$*

Obviously, if $\zeta_0$ is a such a point then every point of $\mathcal{K}_R(\zeta_0) \cap L_\Phi(\zeta_0)$ is a singular point of the logarithmic derivative of $\Phi$.

**Definition 4.2.14.** *Let the function $\Phi$ be monogenic in the domain $\mathcal{K}_{0,R}(\zeta_0)$ and $\Phi(\zeta) \notin \mathcal{I}_1 \cup \mathcal{I}_2$ for all $\zeta \in \mathcal{K}_{0,R}(\zeta_0)$. Let either $\Phi(\zeta_0) \in \mathcal{I}_1 \cup \mathcal{I}_2$ or $\zeta_0$ is a non-removable singular point of $\Phi$. Let assumption* (4.10) *holds. Then we say that a hypercomplex number $n_1 I_1 + n_2 I_2$ is* a singularity index of the logarithmic derivative *of function $\Phi$ at the point $\zeta_0$.*

For an arbitrary set $\Omega \subseteq E_3$, we find it reasonable to set

$$\mathcal{S}_\Phi(\Omega) := \{\zeta \in \Omega : \ \zeta \text{ is a non-removable singularity of } \Phi\},$$
$$\mathcal{I}_\Phi(\Omega) := \{\zeta \in \Omega : \Phi(\zeta) \in \mathcal{I}_1 \cup \mathcal{I}_2\}.$$

Let $G \subseteq \Pi_\zeta$ be a domain in $\Pi_\zeta$. Then cl$G$ and $\partial G$ denote the closure and the boundary of $G$ in the induced topology of $\Pi_\zeta$, respectively.

Now, we can formulate our main result, namely, two theorems on the logarithmic residue for monogenic functions taking values in the algebra $\mathbb{A}_2$.

**Theorem 4.2.15.** *Let $\Omega$ be a domain in $E_3$ and $\Phi$ be a monogenic function in $\Omega \setminus \mathcal{S}_\Phi(\Omega)$. Let $\zeta_0$ be an arbitrary point in $\Omega$. Let $G \subseteq \Pi_{\zeta_0}$ be a domain in $\Pi_{\zeta_0}$ such that* cl$G \subseteq \Omega$ *and $\partial G$ be a closed Jordan rectifiable curve. Let $\partial G$ do not contain singular points of the logarithmic derivative of function $\Phi$, $\mathcal{S}_\Phi(G) \cup \mathcal{I}_\Phi(G) = \{\zeta_k\}_{k=1}^m$, where $m$ is finite, and there exist $R \in ]0, +\infty]$ such that assumption* (4.10) *holds in $\mathcal{K}_{0,R}(\zeta_k)$ for all $k \in \{1, 2, \dots, m\}$. Moreover, let $n_{1k} I_1 + n_{2k} I_2$ denoting the singularity index of the logarithmic derivative of function $\Phi$ at the point $\zeta_k$ be finite for all $k \in \{1, 2, \dots, m\}$. Then*

$$\frac{1}{2\pi i} \int_\Gamma \Phi'(\zeta) \left(\Phi(\zeta)\right)^{-1} d\zeta = \sum_{k=1}^m (n_{1k} I_1 + n_{2k} I_2)$$

*where $\Gamma$ is an arbitrary closed Jordan rectifiable curve in the domain $\Omega \setminus \left(\mathcal{S}_\Phi(\Omega) \cup \mathcal{I}_\Phi(\Omega)\right)$ which is homotopic to $\partial G$ in this domain.*

*Proof.* Let $R \in ]0, +\infty]$ be such that the sets $\mathcal{K}_{0,R}(\zeta_k) \subseteq \Omega$ are pairwise disjoint for all $k \in \{1, 2, \dots, m\}$. Since $\Gamma$ is homotopic to $\partial G$ in $\Omega \setminus (\mathcal{S}_\Phi(\Omega) \cup \mathcal{I}_\Phi(\Omega))$, by Theorem 4.1.4, we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta = \frac{1}{2\pi i}\int\limits_{\partial G} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta = \frac{1}{2\pi i}\sum_{k=1}^{m}\int\limits_{\Gamma_r(\zeta_k)} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta$$

for any $r \in ]0, R[$.

Now, to complete the proof one can apply Lemma 4.2.11. □

The following theorem is true.

**Theorem 4.2.16.** *Let assumptions of Theorem 4.2.15 hold. Then*

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta = (N_{F_1} - P_{F_1})I_1 + (N_{F_2} - P_{F_2})I_2, \tag{4.15}$$

*where $N_{F_k}$, $P_{F_k}$ are the numbers of zeros and poles, respectively, of the function $F_k$ in the domain $D_k(G)$ for $k = 1, 2$.*

*Proof.* Since $\Gamma$ is homotopic to $\partial G$ in $\Omega \setminus (\mathcal{S}_\Phi(\Omega) \cup \mathcal{I}_\Phi(\Omega))$, we conclude that $\Gamma$ does not contain singularities of the logarithmic derivative of function $\Phi$. Then the following equality is true:

$$\frac{1}{2\pi i}\int\limits_{\Gamma} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta = \frac{1}{2\pi i}\int\limits_{\partial G} \Phi'(\zeta)(\Phi(\zeta))^{-1} d\zeta = \frac{1}{2\pi i}\triangle_{\partial G} \ln \Phi(\zeta), \tag{4.16}$$

where $\triangle_{\partial G} \ln \Phi(\zeta)$ denotes the increment of function $\ln \Phi(\zeta)$ as $\zeta$ passes the curve $\partial G$. Equalities (4.5) and (4.4) yield the equality

$$\ln \Phi(\zeta) = \ln F_1(\xi_1)I_1 + \ln F_2(\xi_2)I_2 + \frac{(p_3 y + q_3 z)F_2'(\xi_2) + F_0(\xi_2)}{F_2(\xi_2)}\rho$$

for all $\zeta \in \Gamma$, where $\xi_1 \in D_1(\partial G)$, $\xi_2 \in D_2(\partial G)$.

Since $\Phi$ does not take values in the ideals on the curve $\partial G$, by virtue of (4.4), we conclude that the function $F_2$ is not equal to zero on the curve $D_2(\partial G)$ in the complex plane. Therefore, the function $((p_3 y + q_3 z)F_2'(\xi_2) + F_0(\xi_2))/F_2(\xi_2)$ is continuous on the curve $D_2(\partial G)$ and, hence, its increment in passing this curve is equal to zero.

Thus, $\triangle_{\partial G} \ln \Phi(\zeta) = \triangle_{D_1(\partial G)} F_1(\xi_1)I_1 + \triangle_{D_2(\partial G)} F_2(\xi_2)I_2$ and, in view of the principle of argument for analytic functions of a complex variable (see, e.g., Rudin [142, Ch. 10]), equality (4.16) is transformed into (4.15). □

Finally, we formulate the result which follows from Theorem 4.2.16.

**Corollary 4.2.17.** *Let the assumptions of Lemma 4.2.11 hold. Let functions $F_1$, $F_2$ be as in (4.4). Then the logarithmic residue of a monogenic function $\Phi$ at the point $\zeta_0$ is an integer if and only if the logarithmic residue of $F_1$ at the point $\xi_{10}$ and the logarithmic residue of $F_2$ at the point $\xi_{20}$ coincide. If so, the logarithmic residues of all these functions coincide.*

# CHAPTER 5

## Limiting values of a Cauchy type integral

In this chapter we study a certain analog of the Cauchy type integral taking values in the algebra $\mathbb{A}_2$. A certain analog in commutative Banach algebras has been considered by Plaksa and Shpakivskyi [127, 128]. They considered a three-dimensional commutative algebra with two-dimensional radical where such an integral is defined in two unbounded domains with the common cylindrical boundary. In Plaksa and Shpakivskyi [129] the authors proved the existence of limiting values of it on the whole cylindrical boundary from both domains.

Here, instead, we consider a certain Cauchy type integral in the algebra $\mathbb{A}_2$ of the following form

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1} d\tau,$$

where a curve $\Gamma$ and a function $\varphi$ satisfy some appropriate assumptions. One should note that the structure of zero-divisors in $\mathbb{A}_2$ leads to an increment in the number of domains of definition for such an integral and to the complication of their geometry. For example, such an integral is defined in six domains obtained by removing two specific cylindrical surfaces from the linear space $E_3$. We consider the limiting behavior of the function $\Phi$ on these surfaces and establish sufficient conditions for the existence of limiting values of $\Phi$ on these two surfaces and show the validity of analogs of Sokhotskii-Plemelj formulas.

Now, we briefly outline our strategy and the structure of the chapter. Taking the representation of the unit element in $\mathbb{A}_2$ into account, we split the function $\Phi$ into three parts denoted by $\Phi_1$, $\Phi_2$, and $\Phi_3$, and study them separately. First, in Section 5.1 we introduce some notation and collect some preliminary results. In Section 5.2 we prove the existence of limiting values of $\Phi_1$ and $\Phi_2$ on the boundary of domains of definition. Sections 5.3 and 5.4 are devoted to the behavior of $\Phi_3$ on the curve of integration and on the boundary of domains of definition, respectively. In Section 5.5, using Propositions 5.2.2, 5.3.3, and 5.4.1, we prove the existence of limiting values of $\Phi$ on the boundary of domains of definition and establish the validity of analogs of the Sokhotskii-Plemelj formulas.

Throughout this chapter, we retain the notation of Chapter 4.

Some of the results of this chapter are presented in the paper [138] by the author and Prof. Sergiy Plaksa.

### 5.1 Preliminaries and notation

In this section we introduce new notation and recall some of Section 4.1.

In contrast to Chapter 4, we consider here the vectors $e_1$, $e_2$, $e_3$ in $\mathbb{A}_2$ defined as follows

$$e_1 = I_1 + I_2, \qquad e_2 = iI_1 + \rho, \qquad e_3 = iI_2.$$

Since they are linearly independent over $\mathbb{C}$, $\{e_1, e_2, e_3\}$ is also a basis of $\mathbb{A}_2$, and for all $a = a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{A}_2$, where $a_1, a_2, a_3 \in \mathbb{C}$, we can define the *norm* in $\mathbb{A}_2$ as follows

$$\|a\| := \sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2}.$$

We note that the following inequalities holds:

$$\|ab\| \leq c_1 \|a\| \, \|b\| \qquad \forall a, b \in \mathbb{A}_2, \tag{5.1}$$

where $c_1$ is a positive real constant which does not depend on $a$ or $b$. One can take $c_1$ equals 14.

Now we define a curvilinear integral in $E_3$. Let $\Gamma \subseteq E_3$ be a rectifiable curve, $\Gamma_{\mathbb{R}} := \{(x, y, z) \in \mathbb{R}^3 : \; xe_1 + ye_2 + ze_3 \in \Gamma\}$ be the congruent curve in $\mathbb{R}^3$, and a continuous function $\psi : \Gamma \to \mathbb{A}_2$ be of the form

$$\psi(\zeta) = \sum_{k=1}^{2} (u_k(x, y, z) + iv_k(x, y, z))I_k + (u_3(x, y, z) + iv_3(x, y, z))\rho \qquad \forall \zeta \in \Gamma,$$

where $u_k : \Gamma_{\mathbb{R}} \to \mathbb{R}$ and $v_k : \Gamma_{\mathbb{R}} \to \mathbb{R}$ are continuous functions defined on $\Gamma_{\mathbb{R}}$ for all $k \in \{1, 2, 3\}$. We define an integral of $\psi$ along $\Gamma$ by the equality

$$\begin{aligned}
\int\limits_{\Gamma} \psi(\zeta)d\zeta := &\left[\int\limits_{\Gamma_{\mathbb{R}}} (u_1(x, y, z)dx - v_1(x, y, z)dy) + i \int\limits_{\Gamma_{\mathbb{R}}} (v_1(x, y, z)dx + u_1(x, y, z)dy)\right] I_1 \\
&+ \left[\int\limits_{\Gamma_{\mathbb{R}}} (u_2(x, y, z)dx - v_2(x, y, z)dz) + i \int\limits_{\Gamma_{\mathbb{R}}} (v_2(x, y, z)dx + u_2(x, y, z)dz)\right] I_2 \\
&+ \left[\int\limits_{\Gamma_{\mathbb{R}}} (u_3(x, y, z)dx + u_2(x, y, z)dy - v_3(x, y, z)dz)\right. \\
&\left. + i \int\limits_{\Gamma_{\mathbb{R}}} (v_3(x, y, z)dx + v_2(x, y, z)dy + u_3(x, y, z)dz)\right] \rho,
\end{aligned}$$

where $d\zeta = dxe_1 + dye_2 + dze_3$ and all integrals in the right-hand side are taken in the Lebesgue-Stieltjes sense (see, e.g., Kamke [64], Privalov [134, p. 26]). Moreover, if $u_1(x, y, z)$ and $v_1(x, y, z)$ do not depend on $z$ then

$$\begin{aligned}
&\int\limits_{\Gamma_{\mathbb{R}}} (u_1(x, y)dx - v_1(x, y)dy) + i \int\limits_{\Gamma_{\mathbb{R}}} (v_1(x, y)dx + u_1(x, y)dy) \\
&\qquad = \int\limits_{D_1(\Gamma)} (u_1(x, y) + iv_1(x, y))d\xi_1,
\end{aligned}$$

where $d\xi_1 = dx + idy$, and if $u_2(x, y, z)$ and $v_2(x, y, z)$ do not depend on $y$ then

$$\begin{aligned}
&\int\limits_{\Gamma_{\mathbb{R}}} (u_2(x, z)dx - v_2(x, z)dz) + i \int\limits_{\Gamma_{\mathbb{R}}} (v_2(x, z)dx + u_2(x, z)dz) \\
&\qquad = \int\limits_{D_1(\Gamma)} (u_2(x, z) + iv_2(x, z))d\xi_2,
\end{aligned}$$

where $d\xi_2 = dx + idz$. Thus, one can write the integral of $\psi$ along $\Gamma$ as follows

$$\int_{\Gamma} \psi(\zeta)d\zeta = \int_{D_1(\Gamma)} (u_1(x,y) + iv_1(x,y))d\xi_1 I_1 + \int_{D_2(\Gamma)} (u_2(x,z) + iv_2(x,z))d\xi_2 I_2$$
$$+ \Bigg[ \int_{\Gamma_{\mathbb{R}}} (u_3(x,y,z)dx + u_2(x,z)dy - v_3(x,y,z)dz) \tag{5.2}$$
$$+ i \int_{\Gamma_{\mathbb{R}}} (v_3(x,y,z)dx + v_2(x,z)dy + u_3(x,y,z)dz) \Bigg] \rho.$$

We also note that for any rectifiable curve $\Gamma$ and a continuous function $\psi : \Gamma \to \mathbb{A}_2$ the following estimate holds:

$$\left\| \int_{\Gamma} \psi(\tau)d\tau \right\| \le c_2 \int_{\Gamma} \|\psi(\tau)\| \, \|d\tau\|, \tag{5.3}$$

where $c_2$ is a positive real constant which does not depend neither on $\Gamma$ nor on $\psi$ (cf. Shpakivskyi [153, Lem. 5.1]). One can take $c_2$ equals $6\sqrt{2}$.

Let $\Gamma$ be a closed Jordan rectifiable curve, $\zeta \in \Gamma$, and $\mu$ denote the linear Lebesgue measure on $\Gamma$, then we set

$$\theta_{\zeta,\Gamma}(\varepsilon) := \mu\{\tau \in \Gamma : \ \|\tau - \zeta\| \le \varepsilon\}, \qquad \theta_{\Gamma}(\varepsilon) := \sup_{\zeta \in \Gamma} \theta_{\zeta,\Gamma}(\varepsilon) \qquad \forall \varepsilon \in [0, +\infty[.$$

These metric characteristics have been introduced by Salaev [143].

In what follows, we fix

- an angle $\alpha$ in $]-\pi/2, 0[ \cup ]0, \pi/2[$,
- a point $\upsilon := x_\upsilon e_1 + y_\upsilon e_2 + z_\upsilon e_3$ in $E_3$.

We observe that for any $\tau_1, \tau_2$ in the plane $\Pi_{\alpha,\upsilon}$ (see Section 4.1)

$$\|\tau_1 - \tau_2\| \, |\sin\alpha| \le |f_2(\tau_1) - f_2(\tau_2)| \le \|\tau_1 - \tau_2\|. \tag{5.4}$$

Next, we impose the following assumption on the curve $\Gamma$ :

$$\Gamma \text{ is a closed Jordan rectifiable curve, } \Gamma \subseteq \Pi_{\alpha,\upsilon} \text{ and } \theta_\Gamma(\varepsilon) = O(\varepsilon) \text{ as } \varepsilon \to 0. \tag{5.5}$$

Moreover, we denote the modulus of continuity of a function $\varphi : \Gamma \to \mathbb{R}$ on $\Gamma$ by

$$\omega_{\varphi,\Gamma}(\varepsilon) := \sup_{\tau_1,\tau_2 \in \Gamma, \ \|\tau_1 - \tau_2\| \le \varepsilon} |\varphi(\tau_1) - \varphi(\tau_2)| \qquad \forall \varepsilon \in [0, +\infty[,$$

and we say that $\varphi$ is *Dini-continuous* on $\Gamma$, if

$$\int_0^1 \frac{\omega_{\varphi,\Gamma}(\eta)}{\eta} d\eta < \infty. \tag{5.6}$$

One should note that every Dini-continuous function on $\Gamma$ is continuous on $\Gamma$.

Let $\Omega$ be a bounded domain in $\Pi_{\alpha,\upsilon}$. We denote the closure and the boundary of $\Omega$ in the induced topology of the plane $\Pi_{\alpha,\upsilon}$ by $\mathrm{cl}\Omega$ and $\partial\Omega$, respectively. In what follows, let a domain $\Omega \subseteq \Pi_{\alpha,\upsilon}$ be bounded by $\Gamma$, and we set

$$\begin{aligned} \Sigma_1 &:= \{\zeta \in E_3 : \ f_1(\zeta) \in D_1(\Gamma)\}, & \Sigma_2 &:= \{\zeta \in E_3 : \ f_2(\zeta) \in D_2(\Gamma)\}, \\ \Pi_1^+ &:= \{\zeta \in E_3 : \ f_1(\zeta) \in D_1(\Omega)\}, & \Pi_1^- &:= \{\zeta \in E_3 : \ f_1(\zeta) \notin D_1(\mathrm{cl}\Omega)\}, \\ \Pi_2^+ &:= \{\zeta \in E_3 : \ f_2(\zeta) \in D_2(\Omega)\}, & \Pi_2^- &:= \{\zeta \in E_3 : \ f_2(\zeta) \notin D_2(\mathrm{cl}\Omega)\}. \end{aligned}$$

Obviously, $E_3 = \Pi_1^+ \cup \Pi_1^- \cup \Sigma_1 = \Pi_2^+ \cup \Pi_2^- \cup \Sigma_2$.

Let $d_\Gamma$ and $k_\alpha$ be defined as follows

$$d_\Gamma := \max_{\tau_1,\tau_2\in\Gamma} \|\tau_1 - \tau_2\| \qquad \text{and} \qquad k_\alpha := \left\lfloor \frac{1}{|\sin\alpha|} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$ for all $x \in \mathbb{R}$. Obviously, $d_\Gamma$ is the diameter of the curve $\Gamma$. Then, for any $\varepsilon \in ]0, +\infty[$ and $\zeta_0 \in \Gamma$, we set

$$\Gamma_\varepsilon(\zeta_0) := \{\tau \in \Gamma : \|\tau - \zeta_0\| \le k_\alpha \varepsilon\}, \qquad \Omega_\varepsilon(\zeta_0) := \{\tau \in \Omega : \|\tau - \zeta_0\| < k_\alpha \varepsilon\}.$$

Obviously, if $k_\alpha \varepsilon > d_\Gamma$ then $\Gamma_\varepsilon(\zeta_0) = \partial\Omega_\varepsilon(\zeta_0) = \Gamma$ for any $\zeta_0 \in \Gamma$.

In the sequel, we need the next result which follows immediately from Plaksa [124, Prop. 1].

**Lemma 5.1.1.** *Let* $\Gamma$ *be as in* (5.5), $\epsilon \in [0, d_\Gamma]$, *and a function* $\varphi : \Gamma \to \mathbb{R}$ *satisfy* (5.6). *Then the following statements hold:*

*(i)* $\int\limits_{[0,\epsilon]} \frac{\omega_{\varphi,\Gamma}(\eta)}{\eta} d\theta_\Gamma(\eta) \le \int\limits_0^\epsilon \frac{\theta_\Gamma(2\eta)}{\eta^2} \omega_{\varphi,\Gamma}(2\eta) d\eta.$

*(ii)* $\int\limits_{[\epsilon,d_\Gamma]} \frac{\omega_{\varphi,\Gamma}(\eta)}{\eta^2} d\theta_\Gamma(\eta) \le \int\limits_{\epsilon/2}^{d_\Gamma} \frac{\theta_\Gamma(2\eta)}{\eta^3} \omega_{\varphi,\Gamma}(2\eta) d\eta.$

Our aim is to study a function $\Phi : E_3 \setminus (\Sigma_1 \cup \Sigma_2) \to \mathbb{A}_2$ defined as follows

$$\Phi(\zeta) := \frac{1}{2\pi i} \int\limits_\Gamma \varphi(\tau)(\tau - \zeta)^{-1} d\tau \qquad \forall \zeta \in E_3 \setminus (\Sigma_1 \cup \Sigma_2) \tag{5.7}$$

where the curve $\Gamma$ is as in (5.5), and the function $\varphi : \Gamma \to \mathbb{R}$ is Dini-continuous. One should note that Cauchy type integral (5.7) is a monogenic function in six domains obtained by removing the set $\Sigma_1 \cup \Sigma_2$ from the linear space $E_3$, but it does not exist on $\Sigma_1 \cup \Sigma_2$. And we are interested in the investigation of the behavior of $\Phi(\zeta)$ when $\zeta$ tends to a point on $\Sigma_1 \cup \Sigma_2$.

Taking equality (4.3) and $d\tau = d\xi_{1\tau} I_1 + d\xi_{2\tau} I_2 + dy_\tau \rho$ into account, the function $\Phi$ can be represented as a sum of three functions

$$\Phi_k(\zeta) := \frac{1}{2\pi i} \int\limits_\Gamma \frac{\varphi(\tau)}{\xi_{k\tau} - \xi_k} d\xi_{k\tau} I_k \qquad \forall \zeta \in E_3 \setminus \Sigma_k, \quad k \in \{1, 2\},$$

$$\Phi_3(\zeta) := \frac{1}{2\pi i} \int\limits_\Gamma \varphi(\tau) \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \rho \qquad \forall \zeta \in E_3 \setminus \Sigma_2.$$

Hence, the analysis of $\Phi$ on $\Sigma_1 \cup \Sigma_2$ can be deduced from the behavior of $\Phi_k$, $k \in \{1, 2, 3\}$, on this set and it is convenient to begin our investigation with these three functions.

## 5.2 On the existence of limiting values of $\Phi_k$ on the surface $\Sigma_k$, $k \in \{1, 2\}$

In this section we prove the existence of limiting values of $\Phi_1$ and $\Phi_2$ on the boundary of domains of definition.

First of all, we note that $\Phi_k$ is continuous in the domains $\Pi_k^+$ and $\Pi_k^-$ for all $k \in \{1, 2\}$. In order to investigate the limiting behavior of $\Phi_k$ on the surface $\Sigma_k$, one can exploit corresponding results on a Cauchy-type integral in $\mathbb{C}$. To do so, we introduce some additional notation and formulate auxiliary statements.

Let $\Gamma$ be a closed Jordan rectifiable curve in $\Pi_{\alpha,\upsilon}$. Clearly, for any $\tau \in \Pi_{\alpha,\upsilon}$

$$z_\tau - z_\upsilon = (y_\tau - y_\upsilon) \tan \alpha,$$

and, accordingly,

$$z_\tau = y_\tau \tan \alpha - y_\upsilon \tan \alpha + z_\upsilon, \tag{5.8}$$

$$y_\tau = z_\tau \cot \alpha - z_\upsilon \cot \alpha + y_\upsilon. \tag{5.9}$$

Keeping equalities (5.8), (5.9) in mind, we find it convenient to introduce the functions $\varphi_1 : D_1(\Gamma) \to \mathbb{R}$ and $\varphi_2 : D_2(\Gamma) \to \mathbb{R}$ defined as follows

$$\varphi_1(\xi_{1\tau}) = \varphi_1(x_\tau, y_\tau) := \varphi(x_\tau e_1 + y_\tau e_2 + (y_\tau \tan \alpha - y_\upsilon \tan \alpha + z_\upsilon) e_3), \tag{5.10}$$

$$\varphi_2(\xi_{2\tau}) = \varphi_2(x_\tau, z_\tau) := \varphi(x_\tau e_1 + (z_\tau \cot \alpha - z_\upsilon \cot \alpha + y_\upsilon) e_2 + z_\tau e_3) \tag{5.11}$$

for all $\tau = x_\tau e_1 + y_\tau e_2 + z_\tau e_3 \in \Gamma$. Obviously, $\varphi_1(\xi_{1\tau}) = \varphi(\tau)$ for all $\xi_{1\tau} \in D_1(\Gamma)$ and $\tau \in \Gamma$ such that $\xi_{1\tau} = f_1(\tau)$, and moreover, $\varphi_2(\xi_{2\tau}) = \varphi(\tau)$ for all $\xi_{2\tau} \in D_2(\Gamma)$ and $\tau \in \Gamma$ such that $\xi_{2\tau} = f_1(\tau)$

Using the monotonicity of the modulus of continuity $\omega_{\varphi,\Gamma}$ and of the function $\theta_{\zeta,\Gamma}$, one verifies the following lemma.

**Lemma 5.2.1.** *For $k \in \{1, 2\}$, the following statements hold:*

*(i) Let $\tilde{\zeta}_0$ be a point on $\Sigma_k$. If $\zeta \in E_3 \setminus \Sigma_k$ tends to $\tilde{\zeta}_0$ then $\xi_k$ tends to $f_k(\tilde{\zeta}_0)$ and $|\xi_k - f_k(\tilde{\zeta}_0)| \leq \|\zeta - \tilde{\zeta}_0\|$.*

*(ii) If $\Gamma$ is as in (5.5) then $D_k(\Gamma)$ is a closed Jordan rectifiable curve in $\mathbb{C}$, and $\theta_{D_k(\Gamma)}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.*

*(iii) If $\varphi : \Gamma \to \mathbb{R}$ is Dini-continuous on $\Gamma$ then $\varphi_k : D_k(\Gamma) \to \mathbb{R}$ is Dini-continuous on $D_k(\Gamma)$.*

In the following proposition we prove that $\Phi_k$ has limiting values on the surface $\Sigma_k$, $k \in \{1, 2\}$.

**Proposition 5.2.2.** *Let $\Gamma$ be as in (5.5) and $\varphi : \Gamma \to \mathbb{R}$ satisfy (5.6). Then for each $k \in \{1, 2\}$ and all $\tilde{\zeta}_0 \in \Sigma_k$, the function $\Phi_k(\zeta)$ has limiting values $\Phi_k^\pm(\tilde{\zeta}_0)$ when $\zeta$ approaches $\tilde{\zeta}_0$ from $\Pi_k^\pm$. Moreover,*

$$\Phi_k^+(\tilde{\zeta}_0) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau) - \varphi(\zeta_0)}{\xi_{k\tau} - \xi_{k0}} d\xi_{k\tau} I_k + \varphi(\zeta_0) I_k,$$

$$\Phi_k^-(\tilde{\zeta}_0) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau) - \varphi(\zeta_0)}{\xi_{k\tau} - \xi_{k0}} d\xi_{k\tau} I_k$$

*for all $\tilde{\zeta}_0 \in \Sigma_k$, where the point $\zeta_0 \in \Gamma$ is such that $f_k(\zeta_0) = f_k(\tilde{\zeta}_0)$.*

*Proof.* Using equalities (5.2), (5.10), and (5.11), we can represent $\Phi_k$ as follows

$$\Phi_k(\zeta) = F_k(\xi_k) I_k \qquad \forall \zeta \in E_3 \setminus \Sigma_k,$$

where $F_k(\xi_k) = \frac{1}{2\pi i} \int_{D_k(\Gamma)} \varphi_k(\xi_{k\tau}) / (\xi_{k\tau} - \xi_k) d\xi_{k\tau}$. Thus, the analysis of $\Phi_k$ nearby $\Gamma$ can be deduced from the behavior of $F_k$ nearby $D_k(\Gamma)$.

Assumptions on $\Gamma$ and Lemma 5.2.1(*iii*) imply that $D_k(\Gamma)$ is a closed Jordan rectifiable curve and $\theta_{D_k(\Gamma)}(\varepsilon) = O(\varepsilon)$ as $\varepsilon$ tends to 0. Then, the Jordan-Leray separation theorem ensures that $\mathbb{C} \setminus D_k(\Gamma)$ has exactly two open connected components and it is convenient to denote by $F_k^+$ and $F_k^-$ the limiting values of the function $F_k$ from interior $\Omega^+$ and exterior $\Omega^-$ domains bounded by $D_k(\Gamma)$. Obviously, the limiting values of $\Phi$ on $\Gamma$ depend on the limiting values of $F_k$ on $D_k(\Gamma)$ and, moreover, $\Phi_k^{\pm} = F_k^{\pm} I_k$.

Also, assumptions on $\varphi$ and Lemma 5.2.1(*ii*) imply that $\varphi_k : D_k(\Gamma) \to \mathbb{R}$ is Dini-continuous on $D_k(\Gamma)$. Thus, by Gerus [49], the function $F_k$ has limiting values on $D_k(\Gamma)$ from $\Omega^+$ and $\Omega^-$, and, furthermore, taking the equality $f_k(\tilde{\zeta}_0) = \xi_{k0}$ into account, we have

$$\begin{aligned} F_k^+(\xi_{k0}) &= \frac{1}{2\pi i} \int\limits_{D_k(\Gamma)} \frac{\varphi_k(\xi_{k\tau}) - \varphi_k(\xi_{k0})}{\xi_{k\tau} - \xi_{k0}} d\xi_{k\tau} + \varphi_k(\xi_{k0}), \\ F_k^-(\xi_{k0}) &= \frac{1}{2\pi i} \int\limits_{D_k(\Gamma)} \frac{\varphi_k(\xi_{k\tau}) - \varphi_k(\xi_{k0})}{\xi_{k\tau} - \xi_{k0}} d\xi_{k\tau} \end{aligned} \tag{5.12}$$

for all $\xi_{k0} \in D_k(\Gamma)$. Finally, using equalities (5.10)-(5.12), one verifies the validity of the statement. □

Proposition 5.2.2 implies that

$$\Phi_k^+(\tilde{\zeta}_0) - \Phi_k^-(\tilde{\zeta}_0) = \varphi(\zeta_0) I_k \qquad \forall \tilde{\zeta}_0 \in \Sigma_k, \ k \in \{1,2\},$$

and, thus, $\Phi_k$ is a function which is monogenic in $\Pi_k^+ \cup \Pi_k^-$ and has a jump equals $\varphi(\zeta_0)I_k$ at each point of the surface $\tilde{\zeta}_0 \in \Sigma_k$, where the point $\zeta_0 \in \Gamma$ is such that $f_k(\zeta_0) = f_k(\tilde{\zeta}_0)$.

## 5.3 On the existence of limiting values of $\Phi_3$ on the curve $\Gamma$

In this section we investigate the behavior of $\Phi_3(\zeta)$ when $\zeta$ approaches the curve $\Gamma$ along some curve $\gamma \subseteq \Pi_2^{\pm}$. To do so, we need to introduce some assumptions on $\gamma$, namely, we assume that $\gamma$ is a curve in $E_3 \setminus \Sigma_2$ such that

$$\exists\, m \in [0,1[ \quad \forall \zeta \in \gamma, \quad \forall \tau \in \Gamma : \quad |y - y_\tau| \leq m \|\zeta - \tau\|. \tag{5.13}$$

Assuming that $|y - y_\tau| \neq 0$, we observe that a triangle with the vertexes $\zeta = xe_1 + ye_2 + ze_3$, $\tau = x_\tau e_1 + y_\tau e_2 + z_\tau e_3$, and $xe_1 + y_\tau e_2 + ze_3$ is the right triangle with the right angle at the vertex $xe_1 + y_\tau e_2 + ze_3$, and, thus, condition (5.13) is equivalent to the following one

$$\exists\, n \in [0,\infty[ \quad \forall \zeta \in \gamma, \quad \forall \tau \in \Gamma : \quad |y - y_\tau| \leq n |\xi_2 - \xi_{2\tau}|. \tag{5.14}$$

We note that parameters $m$ and $n$ can be considered as the values of cosine and of cotangent, respectively, of the angle at the vertex $\tau$. Clearly, both equalities hold automatically, if $|y - y_\tau| = 0$. One should also note that in comparison with (5.14), assumption (5.13) is more natural and says that $\gamma$ can not be a tangent to the line $\{x_\tau e_1 + te_2 + z_\tau e_3 \in E_3, \ t \in \mathbb{R}\}$, while assumption (5.14) is more convenient to use.

To begin with, we need to prove some auxiliary statements that will be done in the two following lemmas.

**Lemma 5.3.1.** *Let* $\Gamma$ *be as in* (5.5) *and* $\varphi : \Gamma \to \mathbb{R}$ *satisfy* (5.6). *Let* $\zeta$ *tend to a point* $\zeta_0 \in \Gamma$ *along a curve* $\gamma \subseteq E_3 \setminus \Sigma_2$ *satisfying* (5.13) *and* $\varepsilon = \|\zeta - \zeta_0\|$. *Then*

$$\left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \rho \right\| \to 0 \tag{5.15}$$

*as* $\varepsilon \to 0$, *and*

$$\left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) \left( \frac{1}{\xi_{2\tau} - \xi_{20}} dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau} \right) \rho \right\| \to 0 \tag{5.16}$$

*as* $\varepsilon \to 0$.

*Proof.* First, we consider (5.15). Let us take a point $\hat{\xi}_2$ on the curve $D_2(\Gamma_\varepsilon(\zeta_0))$ satisfying the following condition

$$|\hat{\xi}_2 - \xi_2| = \min_{\xi \in D_2(\Gamma_\varepsilon(\zeta_0))} |\xi - \xi_2|,$$

and let $\zeta_2 \in \Gamma_\varepsilon(\zeta_0)$ be such that $f_2(\zeta_2) = \hat{\xi}_2$. Using inequality (5.3), we obtain

$$\begin{aligned}
&\left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_2) + \varphi(\zeta_2) - \varphi(\zeta_0)) \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \rho \right\| \\
&\quad \leq c_2 \|\rho\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} |\varphi(\tau) - \varphi(\zeta_2)| \left| \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right| \\
&\quad + |\varphi(\zeta_2) - \varphi(\zeta_0)| \|\rho\| \left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \right\| \\
&\quad \leq c_2 \|\rho\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_2)|}{|\xi_{2\tau} - \xi_2|} \left( 1 + \frac{|y_\tau - y|}{|\xi_{2\tau} - \xi_2|} \right) \|d\tau\| \\
&\quad + |\varphi(\zeta_2) - \varphi(\zeta_0)| \|\rho\| \left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \right\| =: \Lambda_{21} + \Lambda_{22},
\end{aligned}$$

where $c_2$ is as in (5.3). Taking the equivalence of conditions (5.13) and (5.14) into account, we have

$$\Lambda_{21} \leq (1+n) c_2 \|\rho\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_2)|}{|\xi_{2\tau} - \xi_2|} \|d\tau\|,$$

where the constant $n$ is as in (5.14). By the triangle inequality, one shows that $|\xi_{2\tau} - \xi_2| \geq |\xi_{2\tau} - \hat{\xi}_2|/2$ , and then, using inequality (5.4), we can estimate $\Lambda_{21}$ in the following way:

$$\begin{aligned}
\Lambda_{21} &\leq \frac{2c_2(1+n)\|\rho\|}{|\sin\alpha|} \int\limits_{\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_2)|}{\|\tau - \zeta_2\|} \|d\tau\| \\
&\leq \frac{2c_2(1+n)\|\rho\|}{|\sin\alpha|} \int\limits_{[0, 2k_\alpha \varepsilon]} \frac{\omega_{\varphi,\Gamma}(\eta)}{\eta} d\theta_{\zeta_2,\Gamma}(\eta),
\end{aligned}$$

where the last integral is taken in the Lebesgue-Stieltjes sense. Then, by Lemma 5.1.1($i$) and assumption (5.5), one verifies that $\Lambda_{21}$ vanishes when $\varepsilon$ tends to 0.

Now, we consider $\Lambda_{22}$. Using the equality $\Gamma_\varepsilon(\zeta_0) = \partial\Omega_\varepsilon(\zeta_0) \setminus (\partial\Omega_\varepsilon(\zeta_0) \setminus \Gamma_\varepsilon(\zeta_0))$, and since

$$|\varphi(\zeta_2) - \varphi(\zeta_0)| \leq \sup_{\tau_1, \tau_2 \in \Gamma_\varepsilon(\zeta_0), \|\tau_1 - \tau_2\| \leq \|\zeta_2 - \zeta_0\|} |\varphi(\tau_1) - \varphi(\tau_2)| \leq \omega_{\varphi,\Gamma}(k_\alpha \varepsilon),$$

we obtain the following inequality:

$$\begin{aligned}
\Lambda_{22} \leq \|\rho\| \omega_{\varphi,\Gamma}(k_\alpha \varepsilon) &\left( \left\| \int\limits_{\partial\Omega_\varepsilon(\zeta_0)} \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \right\| \right. \\
&\left. + \left\| \int\limits_{\partial\Omega_\varepsilon(\zeta_0) \setminus \Gamma_\varepsilon(\zeta_0)} \left( \frac{1}{\xi_{2\tau} - \xi_2} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \right) \right\| \right).
\end{aligned}$$

We proceed to estimate $\Lambda_{22}$. Using (5.4), we have that for all $\tau \in \partial\Omega_\varepsilon(\zeta_0) \setminus \Gamma_\varepsilon(\zeta_0)$

$$\begin{aligned} |\xi_{2\tau} - \xi_2| =& |\xi_{2\tau} - \xi_{20} + \xi_{20} - \xi_2| \geq |\xi_{2\tau} - \xi_{20}| - |\xi_{20} - \xi_2| \\ &\geq |\sin\alpha| \|\tau - \zeta_0\| - \|\zeta - \zeta_0\| \geq (k_\alpha |\sin\alpha| - 1)\varepsilon > 0. \end{aligned} \tag{5.17}$$

Then, using inequalities (5.3), (5.14), (5.17), and the equality $d\tau = d\xi_{1\tau} I_1 + d\xi_{2\tau} I_2 + dy_\tau \rho$, we obtain the following:

$$\begin{aligned} \Lambda_{22} \leq& \|\rho\| \omega_{\varphi,\Gamma}(k_\alpha \varepsilon) \left( \left\| \int\limits_{\partial\Omega_\varepsilon(\zeta_0)} d\left(\frac{y_\tau - y}{\xi_{2\tau} - \xi_2}\right) \right\| + \frac{(1+n)c_2}{\varepsilon(k_\alpha |\sin\alpha| - 1)} \int\limits_{\partial\Omega_\varepsilon(\zeta_0) \setminus \Gamma_\varepsilon(\zeta_0)} \|d\tau\| \right) \\ \leq& \frac{(1+n)c_2 \|\rho\| \omega_{\varphi,\Gamma}(k_\alpha \varepsilon)}{\varepsilon(k_\alpha |\sin\alpha| - 1)} 2\pi k_\alpha \varepsilon \leq \frac{2\pi k_\alpha (1+n) c_2 \|\rho\|}{(k_\alpha |\sin\alpha| - 1)} \omega_{\varphi,\Gamma}(k_\alpha \varepsilon) \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Thus, (5.15) is proven.

Now we consider (5.16). By a straightforward estimation, using (5.3) and (5.14), one can have the following:

$$\begin{aligned} &\left\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) \left( \frac{1}{\xi_{2\tau} - \xi_{20}} dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau} \right) \rho \right\| \\ &\qquad\qquad \leq (1+n) c_2 \|\rho\| \int\limits_{\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_{20}|} \|d\tau\|. \end{aligned}$$

Further, we estimate the last integral in the same way as $\Lambda_{21}$. $\square$

**Lemma 5.3.2.** *Let assumptions of Lemma 5.3.1 hold. Then*

$$\begin{aligned} &\Bigg\| \int\limits_{\Gamma \setminus \Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) \Bigg( \frac{\xi_2 - \xi_{20}}{(\xi_{2\tau} - \xi_2)(\xi_{2\tau} - \xi_{20})} dy_\tau \\ &\qquad\qquad - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} + \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau} \Bigg) \rho \Bigg\| \to 0 \end{aligned}$$

*as* $\varepsilon \to 0$, *where* $\varepsilon = \|\zeta - \zeta_0\|$.

*Proof.* To begin with, we note that

$$\begin{aligned} &\left( \frac{\xi_2 - \xi_{20}}{(\xi_{2\tau} - \xi_2)(\xi_{2\tau} - \xi_{20})} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} + \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau} \right) \rho \\ &\qquad = (\zeta - \zeta_0)(\tau - \zeta)^{-1}(\tau - \zeta_0)^{-1} d\tau I_2 - \frac{\xi_2 - \xi_{20}}{(\xi_{2\tau} - \xi_2)(\xi_{2\tau} - \xi_{20})} d\xi_{2\tau} I_2 \end{aligned} \tag{5.18}$$

for all $\tau \in \Gamma \setminus \Gamma_\varepsilon(\zeta_0)$ and $\zeta \in \Pi_2^+ \cup \Pi_2^-$, what can be verified by using (4.3).

Then, without loss of generality, we assume that $\varepsilon$ is less than $d_\Gamma / k_\alpha$. Using equalities (4.3), (5.18), and $I_2 = I_2^2$, inequalities (5.1) and (5.3), we obtain the following relations:

$$\begin{aligned} \Lambda :=& \Bigg\| \int\limits_{\Gamma \setminus \Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0)) \Bigg( \frac{\xi_2 - \xi_{20}}{(\xi_{2\tau} - \xi_2)(\xi_{2\tau} - \xi_{20})} dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2} d\xi_{2\tau} \\ &+ \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau} \Bigg) \rho \Bigg\| \leq \left\| (\xi_2 - \xi_{20}) \int\limits_{\Gamma \setminus \Gamma_\varepsilon(\zeta_0)} \frac{\varphi(\tau) - \varphi(\zeta_0)}{(\xi_{2\tau} - \xi_2)(\xi_{2\tau} - \xi_{20})} d\xi_{2\tau} I_2 \right\| \\ &+ \left\| (\zeta - \zeta_0) \int\limits_{\Gamma \setminus \Gamma_\varepsilon(\zeta_0)} (\varphi(\tau) - \varphi(\zeta_0))(I_2(\tau - \zeta)^{-1})(I_2(\tau - \zeta_0)^{-1}) d\tau \right\| \end{aligned}$$

$$
\begin{aligned}
&\leq c_1^2 c_2 \|\zeta - \zeta_0\| \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} |\varphi(\tau) - \varphi(\zeta_0)| \times \\
&\quad \times \left\| \left( \frac{1}{\xi_{2\tau} - \xi_2} I_2 - \frac{y_\tau - y}{(\xi_{2\tau} - \xi_2)^2}\, \rho \right) \left( \frac{1}{\xi_{2\tau} - \xi_{20}} I_2 - \frac{y_\tau - y_0}{(\xi_{2\tau} - \xi_{20})^2}\, \rho \right) \right\| \|d\tau\| \\
&\quad + c_2 \|I_2\| |\xi_2 - \xi_{20}| \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} |d\xi_{2\tau}| \\
&\leq c_1^2 c_2 \|\zeta - \zeta_0\| \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \|I_2\| \|d\tau\| \\
&\quad + c_1^2 c_2 \|\zeta - \zeta_0\| \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \left( \frac{|y_\tau - y|}{|\xi_{2\tau} - \xi_2|} + \frac{|y_\tau - y_0|}{|\xi_{2\tau} - \xi_{20}|} \right) \|\rho\| \|d\tau\| \\
&\quad + c_2 \|I_2\| |\xi_2 - \xi_{20}| \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \|d\tau\|,
\end{aligned}
$$

where $c_1$ and $c_2$ are positive constants defined in inequalities (5.1) and (5.3), respectively. Since $\tau, \zeta_0 \in \Pi_{\alpha,\upsilon}$, it follows that

$$
\frac{|y_\tau - y_0|}{|\xi_{2\tau} - \xi_{20}|} \leq \frac{|\cot \alpha||z_\tau - z_0|}{|\xi_{2\tau} - \xi_{20}|} \leq |\cot \alpha|. \tag{5.19}
$$

Thus, taking condition (5.14), inequalities (5.19), $|\xi_2 - \xi_{20}| \leq \|\zeta - \zeta_0\| = \varepsilon$, and the equality $d\tau = d\xi_{1\tau} I_1 + d\xi_{2\tau} I_2 + dy_\tau \rho$ into account, we have

$$
\begin{aligned}
\Lambda &\leq c_1^2 c_2 \|I_2\| \varepsilon \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \|d\tau\| \\
&\quad + c_1^2 c_2 \|\rho\| \varepsilon \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} (n + |\cot \alpha|) \|d\tau\| \\
&\quad + c_2 \|I_2\| \varepsilon \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \|d\tau\| \leq C\varepsilon \int\limits_{\Gamma \backslash \Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_0)|}{|\xi_{2\tau} - \xi_2||\xi_{2\tau} - \xi_{20}|} \|d\tau\|
\end{aligned}
$$

where the constant $n$ is as in (5.14), and the constant $C := c_1 c_2 (c_1 (\|I_2\| + \|\rho\|(n + |\cot \alpha|)) + \|I_2\|)$ does not depend on $\varepsilon$. Since $\tau \in \Gamma \setminus \Gamma_\varepsilon(\zeta_0)$, we have $\|\tau - \zeta_0\| \geq k_\alpha \varepsilon$ that together with (5.4) imply the estimation $|\xi_{2\tau} - \xi_{20}| \geq k_\alpha |\sin \alpha| \varepsilon$, and, thus,

$$
\varepsilon \leq \frac{1}{k_\alpha |\sin \alpha|} |\xi_{2\tau} - \xi_{20}|. \tag{5.20}
$$

Then, using inequalities $|\xi_{2\tau} - \xi_2| \geq |\xi_{2\tau} - \xi_{20}| - |\xi_2 - \xi_{20}|$, $|\xi_2 - \xi_{20}| \leq \|\zeta - \zeta_0\| = \varepsilon$, and (5.20), we obtain the following inequality:

$$
|\xi_{2\tau} - \xi_2| \geq \left( 1 - \frac{1}{k_\alpha |\sin \alpha|} \right) |\xi_{2\tau} - \xi_{20}| = \frac{k_\alpha |\sin \alpha| - 1}{k_\alpha |\sin \alpha|} |\xi_{2\tau} - \xi_{20}|. \tag{5.21}
$$

Finally, by virtue of inequalities (5.21) and (5.4), we continue to estimate $\Lambda$:

$$\begin{aligned}\Lambda \leq & \frac{C\varepsilon k_\alpha |\sin\alpha|}{k_\alpha|\sin\alpha|-1} \int\limits_{\Gamma\setminus\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau)-\varphi(\zeta_0)|}{|\xi_{2\tau}-\xi_{20}|^2}\|d\tau\| \\ \leq & \frac{Ck_\alpha|\sin\alpha|\varepsilon}{\sin^2\alpha(k_\alpha|\sin\alpha|-1)} \int\limits_{\Gamma\setminus\Gamma_\varepsilon(\zeta_0)} \frac{|\varphi(\tau)-\varphi(\zeta_0)|}{\|\tau-\zeta_0\|^2}\|d\tau\| \\ \leq & \frac{Ck_\alpha\varepsilon}{|\sin\alpha|(k_\alpha|\sin\alpha|-1)} \int\limits_{[k_\alpha\varepsilon, d_\Gamma]} \frac{\omega_{\varphi,\Gamma}(\eta)}{\eta^2} d\theta_{\zeta_0,\Gamma}(\eta),\end{aligned}$$

and, using Lemma 5.1.1(*ii*) and assumption (5.5), we verify that $\Lambda$ vanishes as $\varepsilon$ tends to zero. □

In the following proposition we prove the existence of some limiting values of $\Phi_3$ on the curve $\Gamma$.

**Proposition 5.3.3.** *Let $\Gamma$ be as in* (5.5) *and $\varphi : \Gamma \to \mathbb{R}$ satisfy* (5.6)*. Then there exists the limiting value $\widehat{\Phi}_3(\zeta_0)$ of the function $\Phi_3(\zeta)$ when $\zeta$ tends to a point $\zeta_0 \in \Gamma$ along a curve $\gamma \subseteq E_3 \setminus \Sigma_2$ satisfying* (5.13)*. Moreover,*

$$\widehat{\Phi}_3(\zeta_0) = \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho \qquad \forall \zeta_0 \in \Gamma.$$

*Proof.* To begin with, we note that $\Phi_3$ can be represented as follows

$$\begin{aligned}\Phi_3(\zeta) = & \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho \\ & + \frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma \left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho\end{aligned} \tag{5.22}$$

for all $\zeta \in E_3 \setminus \Sigma_2$ and for any $\zeta_0 \in \Gamma$. By a straightforward integration, one can compute the second integral in (5.22). Accordingly, we have

$$\frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma \left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho = \frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma d\left(\frac{y_\tau - y}{\xi_{2\tau}-\xi_2}\right)\rho = 0.$$

Now we consider the first integral in (5.22). Our aim is to prove that $\Phi_3$ has the limiting values $\widehat{\Phi}_3(\zeta_0)$ at all points $\zeta_0 \in \Gamma$, when $\zeta$ tends to the point $\zeta_0$ along a curve $\gamma \subseteq E_3 \setminus \Sigma_2$ satisfying (5.13). And to do so, without loss of generality, we take $\varepsilon = \|\zeta - \zeta_0\|$ smaller than $d_\Gamma/k_\alpha$ and consider the following difference

$$\begin{aligned}&\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho \\ &- \int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho \\ &= \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho \\ &- \int\limits_{\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho \\ &+ \int\limits_{\Gamma\setminus\Gamma_\varepsilon(\zeta_0)} (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{\xi_2-\xi_{20}}{(\xi_{2\tau}-\xi_2)(\xi_{2\tau}-\xi_{20})}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau} + \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho\end{aligned}$$

Since the integrals in the right hand side have been estimated in Lemmas 5.3.1-5.3.2, we use those estimations and obtain the following result:

$$\lim_{\zeta\to\zeta_0,\,\zeta\in\gamma} \int_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho$$
$$= \int_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho = \widehat{\Phi}_3(\zeta_0)$$

for any $\gamma$ satisfying (5.13) and $\zeta_0 \in \Gamma$. $\square$

## 5.4 On the existence of limiting values of $\Phi_3$ on the surface $\Sigma_2$

The method applied in Section 5.3 to prove the existence of $\widehat{\Phi}_3$ on the curve $\Gamma$ does not work when we consider the whole surface $\Sigma_2$ instead of $\Gamma$. But the existence of limiting values of $\Phi_3$ on the surface $\Sigma_2$ can be proved under some additional assumptions.

The following proposition is true.

**Proposition 5.4.1.** *Let $\Gamma$ be as in* (5.5)*. Let $\varphi : \Gamma \to \mathbb{R}$ be absolutely continuous on $\Gamma$ and satisfy* (5.6)*, and $\varphi'$ satisfy* (5.6)*. Then for all $\tilde{\zeta}_0 := \tilde{x}_0 e_1 + \tilde{y}_0 e_2 + \tilde{z}_0 e_3 \in \Sigma_2$, the function $\Phi_3(\zeta)$ has limiting values $\Phi_3^\pm(\tilde{\zeta}_0)$ when $\zeta$ approaches $\tilde{\zeta}_0$ from $\Pi_2^\pm$. Moreover,*

$$\begin{aligned}
\Phi_3^+(\tilde{\zeta}_0) = {} & \frac{1}{2\pi i}\int_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho \\
& - \frac{y_0-\tilde{y}_0}{2\pi i}\int_\Gamma (\varphi'(\tau)-\varphi'(\zeta_0))(\tau-\zeta_0)^{-1}d\tau\rho - (y_0-\tilde{y}_0)\varphi'(\zeta_0)\rho \quad \forall \tilde{\zeta}_0 \in \Sigma_2, \\
\Phi_3^-(\tilde{\zeta}_0) = {} & \frac{1}{2\pi i}\int_\Gamma (\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau - \frac{y_\tau - y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho \\
& - \frac{y_0-\tilde{y}_0}{2\pi i}\int_\Gamma (\varphi'(\tau)-\varphi'(\zeta_0))(\tau-\zeta_0)^{-1}d\tau\rho \quad \forall \tilde{\zeta}_0 \in \Sigma_2,
\end{aligned} \tag{5.23}$$

*where the point $\zeta_0 \in \Gamma$ is such that $f_2(\zeta_0) = f_2(\tilde{\zeta}_0)$.*

*Proof.* Keeping the definition of $\Phi_3$ abd equalities (5.9) and (5.2) in mind, we can represent the function $\Phi_3$ as follows

$$\begin{aligned}
\Phi_3(\zeta) = {} & \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\left(\frac{1}{\xi_{2\tau}-\xi_2}dy_\tau - \frac{y_\tau-y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\right)\rho = \frac{\cot\alpha}{2\pi i}\int_{D_2(\Gamma)} \frac{\varphi_2(\xi_{2\tau})}{\xi_{2\tau}-\xi_2}dz_\tau\rho \\
& - \frac{1}{2\pi i}\int_{D_2(\Gamma)} \varphi_2(\xi_{2\tau})\frac{z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\rho
\end{aligned} \tag{5.24}$$

for all $\zeta \in E_3 \setminus \Sigma_2$. The absolutely continuity of $\varphi$ on $\Gamma$ implies that $\varphi_2$ is absolutely continuous on $D_2(\Gamma)$ due to the definition of $\varphi_2$. Also, taking the absolutely continuity of $(z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y)/(\xi_{2\tau}-\xi_2)^2$ on $D_2(\Gamma)$ into account, one can integrate by parts the last integral in (5.24) which is taken in the Lebesgue-Stieltjes sense (see, e.g., Kamke [64, Sec.

9.42]). Hence, we have the following:

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2(\xi_{2\tau})\frac{z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{(\xi_{2\tau}-\xi_2)^2}d\xi_{2\tau}\\
=&\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2(\xi_{2\tau})(z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y)d\left(\frac{1}{\xi_{2\tau}-\xi_2}\right)\\
=&-\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \frac{1}{\xi_{2\tau}-\xi_2}d\left(\varphi_2(\xi_{2\tau})(z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y)\right)\\
=&-\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{\xi_{2\tau}-\xi_2}d\xi_{2\tau} - \frac{\cot\alpha}{2\pi i}\int\limits_{D_2(\Gamma)}\frac{\varphi_2(\xi_{2\tau})}{\xi_{2\tau}-\xi_2}dz_\tau.
\end{aligned}
\tag{5.25}
$$

Then, equalities (5.24) and (5.25) imply that

$$
\Phi_3(\zeta) = -\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho
$$

for all $\zeta \in E_3 \setminus \Sigma_2$.

Now, taking equality (5.9) into account, we rewrite $\Phi_3$ as follows

$$
\begin{aligned}
\Phi_3(\zeta) =& -\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{(z_\tau - z_0 + z_0)\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho\\
=& -\frac{\cot\alpha}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau - z_0}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho\\
&-\frac{z_0\cot\alpha - z_\upsilon\cot\alpha + y_\upsilon - y}{2\pi i}\int\limits_{D_2(\Gamma)}\frac{\varphi_2'(\xi_{2\tau})}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho\\
=& -\frac{\cot\alpha}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau - z_0}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho - \frac{y_0 - y}{2\pi i}\int\limits_{D_2(\Gamma)}\frac{\varphi_2'(\xi_{2\tau})}{\xi_{2\tau}-\xi_2}d\xi_{2\tau}\rho\\
=&:-\cot\alpha\,\Lambda_1(\xi_2)\rho - (y_0 - y)\Lambda_2(\xi_2)\rho
\end{aligned}
\tag{5.26}
$$

for all $\zeta \in E_3 \setminus \Sigma_2$. Then we want to investigate the behavior of $\Lambda_1(\cdot)\rho$ and $\Lambda_2(\cdot)\rho$ on $\Sigma_2$. We begin with $\Lambda_1$. Setting $\phi(\xi_{2\tau}) := \varphi'(\xi_{2\tau})(z_\tau - z_0)$ for all $\xi_{2\tau} \in D_2(\Gamma)$, we observe that $\phi$ vanishes at $\xi_{20}$ and is Dini-continuous on $D_2(\Gamma)$ due to the Dini-continuity of $\varphi'$ on $\Gamma$ (see Lemma 5.2.1(*iii*)). Now, we apply results of Gerus [49] to $\Lambda_1$ and deduce that $\Lambda_1$ is continuous on $D_2(\Gamma)$. Accordingly, we have

$$
\Lambda_1(\xi_2)\rho \to \frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau - z_0}{\xi_{2\tau}-\xi_{20}}d\xi_{2\tau}\rho \quad \text{as} \quad \zeta \to \tilde{\zeta}_0. \tag{5.27}
$$

Moreover, the integral in (5.27) can be written as follows (see Plaksa and Shpakivskyi [127, p. 125]):

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \varphi_2'(\xi_{2\tau})\frac{z_\tau - z_0}{\xi_{2\tau}-\xi_{20}}d\xi_{2\tau}\rho = -\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)}\frac{\varphi_2(\xi_{2\tau})-\varphi_2(\xi_{20})}{\xi_{2\tau}-\xi_{20}}dz_\tau\rho\\
&\quad+\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)}\frac{(\varphi_2(\xi_{2\tau})-\varphi_2(\xi_{20}))(z_\tau - z_0)}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\rho\\
&=-\frac{1}{2\pi i}\int\limits_{D_2(\Gamma)}(\varphi_2(\xi_{2\tau})-\varphi_2(\xi_{20}))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dz_\tau - \frac{z_\tau - z_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho.
\end{aligned}
\tag{5.28}
$$

Now, we turn to the consideration of $\Lambda_2(\cdot)\rho$. The Dini-continuity of $\varphi'$ on $\Gamma$ and Lemma 5.2.1$(iii)$ imply the Dini-continuity of $\varphi_2'$ on $D_2(\Gamma)$. Thus, the integral $\Lambda_2(\xi_2)$ can be continuously extended onto $D_2(\Gamma)$ (see Gerus [49]) from the interior and exterior domains bounded by $D_2(\Gamma)$, and, consequently, $\Lambda_2(\xi_2)\rho$ has limiting values at all points $\tilde{\zeta}_0 \in \Sigma_2$. Denote by $\Lambda_2^{\pm}(\xi_{20})\rho$ the limiting values of $\Lambda_2(\xi_2)\rho$ when $\zeta \to \tilde{\zeta}_0$, $\zeta \in \Pi_2^{\pm}$. Since the condition $\zeta \to \tilde{\zeta}_0$ implies that $\xi_2 \to \xi_{20}$, one can easily see that

$$\begin{aligned}
\Lambda_2^{+}(\xi_{20})\rho =& \frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \frac{\varphi_2'(\xi_{2\tau}) - \varphi_2'(\xi_{20})}{\xi_{2\tau} - \xi_{20}} d\xi_{2\tau}\rho + \varphi_2'(\xi_{20})\rho,\\
\Lambda_2^{-}(\xi_{20})\rho =& \frac{1}{2\pi i}\int\limits_{D_2(\Gamma)} \frac{\varphi_2'(\xi_{2\tau}) - \varphi_2'(\xi_{20})}{\xi_{2\tau} - \xi_{20}} d\xi_{2\tau}\rho.
\end{aligned} \tag{5.29}$$

Finally, using conditions (5.26), (5.26), (5.28), and (5.29), we obtain

$$\begin{aligned}
\Phi_3^{+}(\tilde{\zeta}_0) =& \frac{\cot\alpha}{2\pi i}\int\limits_{D_2(\Gamma)} (\varphi_2(\xi_{2\tau}) - \varphi_2(\xi_{20}))\left(\frac{1}{\xi_{2\tau} - \xi_{20}} dz_\tau - \frac{z_\tau - z_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau}\right)\rho\\
&- \frac{y_0 - \tilde{y}_0}{2\pi i}\int\limits_{D_2(\Gamma)} \frac{\varphi_2'(\xi_{2\tau}) - \varphi_2'(\xi_{20})}{\xi_{2\tau} - \xi_{20}} d\xi_{2\tau}\rho - (y_0 - \tilde{y}_0)\varphi_2'(\xi_{20})\rho \qquad \forall \tilde{\zeta}_0 \in \Sigma_2,\\
\Phi_3^{-}(\tilde{\zeta}_0) =& \frac{\cot\alpha}{2\pi i}\int\limits_{D_2(\Gamma)} (\varphi_2(\xi_{2\tau}) - \varphi_2(\xi_{20}))\left(\frac{1}{\xi_{2\tau} - \xi_{20}} dz_\tau - \frac{z_\tau - z_0}{(\xi_{2\tau} - \xi_{20})^2} d\xi_{2\tau}\right)\rho\\
&- \frac{y_0 - \tilde{y}_0}{2\pi i}\int\limits_{D_2(\Gamma)} \frac{\varphi_2'(\xi_{2\tau}) - \varphi_2'(\xi_{20})}{\xi_{2\tau} - \xi_{20}} d\xi_{2\tau}\rho \qquad \forall \tilde{\zeta}_0 \in \Sigma_2,
\end{aligned}$$

where $\Phi_3^{\pm}(\tilde{\zeta}_0)$ are the limiting values of the function $\Phi_3(\zeta)$ when $\zeta$ tends to the point $\tilde{\zeta}_0 \in \Sigma_2$ from $\Pi_2^{\pm}$. Then, using equalities (5.9), (5.2), and (4.3), one can easily represent $\Phi_3^{\pm}(\tilde{\zeta}_0)$ as in (5.23). □

We observe that for all $\tilde{\zeta}_0 \in \Gamma$, the value of the functions $\Phi_3^{\pm}(\tilde{\zeta}_0)$ and $\widehat{\Phi}_3(\tilde{\zeta}_0)$ coincide.

## 5.5 On the existence of limiting values of $\Phi$ on the boundary of domains of its definition

In this section we present sufficient conditions for the existence of limiting values of the function $\Phi$ on the set $\Sigma_1 \cup \Sigma_2$. These results follow from Propositions 5.2.2, 5.3.3, 5.4.1, and the following equality

$$\Phi(\zeta) = \Phi_1(\zeta) + \Phi_2(\zeta) + \Phi_3(\zeta) \qquad \forall \zeta \in E_3 \setminus (\Sigma_1 \cup \Sigma_2). \tag{5.30}$$

We find it convenient to formulate the results in two theorems. In the first one, we prove the existence of limiting values of $\Phi$ on the set $(\Sigma_1 \setminus \Sigma_2) \cup \Gamma$.

**Theorem 5.5.1.** *Let* $\Gamma$ *be as in* (5.5). *Let* $\varphi : \Gamma \to \mathbb{R}$ *satisfy* (5.6). *Then the following statements hold:*

*(i) For all* $\tilde{\zeta}_0 \in \Sigma_1 \setminus \Sigma_2$*, the function* $\Phi(\zeta)$ *has limiting values* $\Phi^{+}(\tilde{\zeta}_0)$, $\Phi^{-}(\tilde{\zeta}_0)$, $\Phi^{\pm}(\tilde{\zeta}_0)$, *and* $\Phi^{\mp}(\tilde{\zeta}_0)$*, when* $\zeta$ *approaches* $\tilde{\zeta}_0$ *from the sets* $\Pi_1^{+} \cap \Pi_2^{+}$, $\Pi_1^{-} \cap \Pi_2^{-}$, $\Pi_1^{+} \cap \Pi_2^{-}$, *and* $\Pi_1^{-} \cap \Pi_2^{+}$,

*respectively. Moreover,*

$$
\begin{aligned}
\Phi^+(\tilde{\zeta}_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau+\varphi(\zeta_0),\\
\Phi^-(\tilde{\zeta}_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau,\\
\Phi^\pm(\tilde{\zeta}_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau+\varphi(\zeta_0)I_1,\\
\Phi^\mp(\tilde{\zeta}_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau+\varphi(\zeta_0)I_2,
\end{aligned}
\tag{5.31}
$$

*where the point $\zeta_0\in\Gamma$ is such that $f_1(\zeta_0)=f_1(\tilde{\zeta}_0)$.*

*(ii) For all $\zeta_0\in\Gamma$, the function $\Phi(\zeta)$ has limiting values $\widehat{\Phi}^+(\zeta_0)$, $\widehat{\Phi}^-(\zeta_0)$, $\widehat{\Phi}^\pm(\zeta_0)$, and $\widehat{\Phi}^\mp(\zeta_0)$, when $\zeta$ approaches $\zeta_0$ along a curve $\gamma\subseteq E_3\setminus(\Sigma_1\cup\Sigma_2)$ satisfying* (5.13) *from the sets $\Pi_1^+\cap\Pi_2^+$, $\Pi_1^-\cap\Pi_2^-$, $\Pi_1^+\cap\Pi_2^-$, and $\Pi_1^-\cap\Pi_2^+$, respectively. Moreover,*

$$
\begin{aligned}
\widehat{\Phi}^+(\zeta_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta_0)^{-1}d\tau+\varphi(\zeta_0),\\
\widehat{\Phi}^-(\zeta_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta_0)^{-1}d\tau,\\
\widehat{\Phi}^\pm(\zeta_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta_0)^{-1}d\tau+\varphi(\zeta_0)I_1,\\
\widehat{\Phi}^\mp(\zeta_0) &= \frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta_0)^{-1}d\tau+\varphi(\zeta_0)I_2,
\end{aligned}
$$

*and*

$$\widehat{\Phi}^+(\zeta_0)-\widehat{\Phi}^-(\zeta_0)=\varphi(\zeta_0)\qquad\forall\zeta_0\in\Gamma.$$

*Proof.* We begin with $(i)$. To prove it, we want to exploit equality (5.30). First, we note that the function $\Phi_1$ has limiting values on $\Sigma_1\setminus\Sigma_2$ due to Proposition 5.2.2.

Then, since $\Phi_2$ and $\Phi_3$ are continuous on $\Sigma_1\setminus\Sigma_2$, we can represent their sum in the following form:

$$
\begin{aligned}
&\Phi_2(\tilde{\zeta}_0)+\Phi_3(\tilde{\zeta}_0)\\
&=\frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\xi_{2\tau}-f_2(\tilde{\zeta}_0)}d\xi_{2\tau}I_2+\frac{1}{2\pi i}\int\limits_\Gamma \varphi(\tau)\left(\frac{1}{\xi_{2\tau}-f_2(\tilde{\zeta}_0)}dy_\tau-\frac{y_\tau-y}{(\xi_{2\tau}-f_2(\tilde{\zeta}_0))^2}d\xi_{2\tau}\right)\rho\\
&=\frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_2+\frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma \frac{1}{\xi_{2\tau}-f_2(\tilde{\zeta}_0)}d\xi_{2\tau}I_2\\
&\quad+\frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma d\left(\frac{y_\tau-y}{\xi_{2\tau}-f_2(\tilde{\zeta}_0)}\right)\rho
\end{aligned}
$$

$$
=\frac{1}{2\pi i}\int\limits_\Gamma (\varphi(\tau)-\varphi(\zeta_0))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_2+\frac{\varphi(\zeta_0)}{2\pi i}\int\limits_\Gamma \frac{1}{\xi_{2\tau}-f_2(\tilde{\zeta}_0)}d\xi_{2\tau}I_2 \tag{5.32}
$$

for all $\tilde{\zeta}_0 \in \Sigma_1 \setminus \Sigma_2$, where the point $\zeta_0 \in \Gamma$ is such that $f_1(\zeta_0) = f_1(\tilde{\zeta}_0)$. To get equality (5.32), we used the following relation

$$(\tau - \tilde{\zeta}_0)^{-1} d\tau I_2 = \frac{1}{\xi_{2\tau} - f_2(\tilde{\zeta}_0)} d\xi_{2\tau} I_2 + \frac{1}{\xi_{2\tau} - f_2(\tilde{\zeta}_0)} dy_\tau \rho - \frac{y_\tau - y}{(\xi_{2\tau} - f_2(\tilde{\zeta}_0))^2} d\xi_{2\tau} \rho, \tag{5.33}$$

that can be verified by using (4.3).

Moreover, the following equality holds:

$$\frac{\varphi(\zeta_0)}{2\pi i} \int_\Gamma \frac{1}{\xi_{2\tau} - f_2(\tilde{\zeta}_0)} d\xi_{2\tau} I_2 = \begin{cases} \varphi(\zeta_0) I_2, & \text{if} \quad \tilde{\zeta}_0 \in \Pi_2^+, \\ 0, & \text{if} \quad \tilde{\zeta}_0 \in \Pi_2^-. \end{cases} \tag{5.34}$$

Now, one can see that equality (5.30), Proposition 5.2.2, and equalities (5.32), (5.34) imply the validity of statement $(i)$.

We turn to the consideration of $(ii)$. To prove it, we just note that functions $\Phi_1$ and $\Phi_2$ have limiting values on $\Gamma$ due to Proposition 5.2.2, and $\Phi_3$ has the limiting values $\widehat{\Phi}_3$ on $\Gamma$ due to Proposition 5.3.3. Thus, the validity of statement $(ii)$ follows from equality (5.30) and two mentioned propositions. □

In the following theorem we impose stricter conditions on the function $\varphi$ what allows us to extend the existence of limiting values of the function $\Phi$ onto the whole surface $\Sigma_2$.

**Theorem 5.5.2.** *Let* $\Gamma$ *be as in* (5.5). *Let* $\varphi : \Gamma \to \mathbb{R}$ *be absolutely continuous on* $\Gamma$ *and satisfy* (5.6), *and* $\varphi'$ *satisfy* (5.6). *Then the following statements hold:*

*(i) For all* $\tilde{\zeta}_0 = \tilde{x}_0 e_1 + \tilde{y}_0 e_2 + \tilde{z}_0 e_3 \in \Sigma_2 \setminus \Sigma_1$, *the function* $\Phi(\zeta)$ *has limiting values* $\Phi^+(\tilde{\zeta}_0)$, $\Phi^-(\tilde{\zeta}_0)$, $\Phi^\pm(\tilde{\zeta}_0)$, *and* $\Phi^\mp(\tilde{\zeta}_0)$, *when* $\zeta$ *approaches* $\tilde{\zeta}_0$ *from the sets* $\Pi_1^+ \cap \Pi_2^+$, $\Pi_1^- \cap \Pi_2^-$, $\Pi_1^+ \cap \Pi_2^-$, *and* $\Pi_1^- \cap \Pi_2^+$, *respectively. Moreover,*

$$\begin{aligned}
\Phi^+(\tilde{\zeta}_0) &= \frac{1}{2\pi i} \int_\Gamma (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau + \widetilde{\varphi}(\zeta_0), \\
\Phi^-(\tilde{\zeta}_0) &= \frac{1}{2\pi i} \int_\Gamma (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau, \\
\Phi^\pm(\tilde{\zeta}_0) &= \frac{1}{2\pi i} \int_\Gamma (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau + \widetilde{\varphi}(\zeta_0) I_1, \\
\Phi^\mp(\tilde{\zeta}_0) &= \frac{1}{2\pi i} \int_\Gamma (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau + \widetilde{\varphi}(\zeta_0) I_2,
\end{aligned} \tag{5.35}$$

*where* $\widetilde{\varphi}(\cdot) := \varphi(\cdot) - (y_0 - \tilde{y}_0)\varphi'(\cdot)\rho$ *on* $\Gamma$, *and the point* $\zeta_0 \in \Gamma$ *is such that* $f_2(\zeta_0) = f_2(\tilde{\zeta}_0)$.

*(ii) For all* $\tilde{\zeta}_0 = \tilde{x}_0 e_1 + \tilde{y}_0 e_2 + \tilde{z}_0 e_3 \in \Sigma_2 \cap \Sigma_1$, *the function* $\Phi(\zeta)$ *has limiting values* $\Phi^+(\tilde{\zeta}_0)$, $\Phi^-(\tilde{\zeta}_0)$, $\Phi^\pm(\tilde{\zeta}_0)$, *and* $\Phi^\mp(\tilde{\zeta}_0)$, *when* $\zeta$ *approaches* $\tilde{\zeta}_0$ *from the sets* $\Pi_1^+ \cap \Pi_2^+$, $\Pi_1^- \cap \Pi_2^-$, $\Pi_1^+ \cap \Pi_2^-$, *and* $\Pi_1^- \cap \Pi_2^+$, *respectively. Moreover,*

$$\begin{aligned}
\Phi^{+}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0^*))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_1+\varphi(\zeta_0^*)I_1\\
&+\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2+\widetilde{\varphi}(\zeta_0)I_2,\\
\Phi^{-}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0^*))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_1\\
&+\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2,\\
\Phi^{\pm}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0^*))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_1\\
&+\varphi(\zeta_0^*)I_1+\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2,\\
\Phi^{\mp}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0^*))(\tau-\tilde{\zeta}_0)^{-1}d\tau I_1\\
&+\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2+\widetilde{\varphi}(\zeta_0)I_2,
\end{aligned}\tag{5.36}$$

*where* $\widetilde{\varphi}(\cdot)$ *is the same as in* (5.35)*, and the points* $\zeta_0^*, \zeta_0 \in \Gamma$ *are such that* $f_1(\zeta_0^*) = f_1(\tilde{\zeta}_0)$ *and* $f_2(\zeta_0) = f_2(\tilde{\zeta}_0)$*. Moreover,*

$$\Phi^{+}(\tilde{\zeta}_0)-\Phi^{-}(\tilde{\zeta}_0)=\widetilde{\varphi}(\zeta_0) \qquad \forall\tilde{\zeta}_0\in\Gamma.$$

*Proof.* We first consider the functions $\Phi_2$ and $\Phi_3$. By virtue of Propositions 5.2.2 and 5.4.1, $\Phi_2$ and $\Phi_3$ have limiting values at all points of $\Sigma_2$. Furthermore, using equality (5.33) with $\tilde{\zeta}_0 = \zeta_0$, we observe that

$$\begin{aligned}
\Phi_2^{+}(\tilde{\zeta}_0)+\Phi_3^{+}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi(\tau)-\varphi(\zeta_0)}{\xi_{2\tau}-\xi_{20}}d\xi_{2\tau}I_2+\varphi(\zeta_0)I_2\\
&+\frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0))\left(\frac{1}{\xi_{2\tau}-\xi_{20}}dy_\tau-\frac{y_\tau-y_0}{(\xi_{2\tau}-\xi_{20})^2}d\xi_{2\tau}\right)\rho\\
&-\frac{y_0-\tilde{y}_0}{2\pi i}\int_{\Gamma}(\varphi'(\tau)-\varphi'(\zeta_0))(\tau-\zeta_0)^{-1}d\tau\rho-(y_0-\tilde{y}_0)\varphi'(\zeta_0)\rho\\
=&\frac{1}{2\pi i}\int_{\Gamma}(\varphi(\tau)-\varphi(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2+\varphi(\zeta_0)I_2\\
&-\frac{y_0-\tilde{y}_0}{2\pi i}\int_{\Gamma}(\varphi'(\tau)-\varphi'(\zeta_0))(\tau-\zeta_0)^{-1}d\tau\rho\\
&-(y_0-\tilde{y}_0)\varphi'(\zeta_0)\rho=\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2+\widetilde{\varphi}(\zeta_0)I_2
\end{aligned}\tag{5.37}$$

for all $\tilde{\zeta}_0 \in \Sigma_2$. Similarly, we obtain

$$\Phi_2^{-}(\tilde{\zeta}_0)+\Phi_3^{-}(\tilde{\zeta}_0)=\frac{1}{2\pi i}\int_{\Gamma}(\widetilde{\varphi}(\tau)-\widetilde{\varphi}(\zeta_0))(\tau-\zeta_0)^{-1}d\tau I_2 \qquad \forall\tilde{\zeta}_0\in\Sigma_2.\tag{5.38}$$

Now, we consider the function $\Phi_1$. First, let $\tilde{\zeta}_0 \in \Sigma_2 \setminus \Sigma_1$. Obviously, $\Phi_1$ is continuous on $\Sigma_2 \setminus \Sigma_1$, and, thus,

$$\begin{aligned}\Phi_1(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 \\ =& \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau) - \varphi(\zeta_0)}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 + \frac{\varphi(\zeta_0)}{2\pi i}\int\limits_{\Gamma} \frac{1}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 \\ =& \frac{1}{2\pi i}\int\limits_{\Gamma} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau I_1 + \frac{\varphi(\zeta_0)}{2\pi i}\int\limits_{\Gamma} \frac{1}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1\end{aligned}$$

for all $\tilde{\zeta}_0 \in \Sigma_2 \setminus \Sigma_1$, where the point $\zeta_0 \in \Gamma$ is such that $f_2(\zeta_0) = f_2(\tilde{\zeta}_0)$. Also, since $\widetilde{\varphi} I_1 \equiv \varphi I_1$ on $\Gamma$, we can write $\Phi_1$ as follows

$$\Phi_1(\tilde{\zeta}_0) = \frac{1}{2\pi i}\int\limits_{\Gamma} (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0))(\tau - \tilde{\zeta}_0)^{-1} d\tau I_1 + \frac{\widetilde{\varphi}(\zeta_0)}{2\pi i}\int\limits_{\Gamma} \frac{1}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 \tag{5.39}$$

for all $\tilde{\zeta}_0 \in \Sigma_2 \setminus \Sigma_1$. Moreover,

$$\frac{\widetilde{\varphi}(\zeta_0)}{2\pi i}\int\limits_{\Gamma} \frac{1}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 = \begin{cases} \widetilde{\varphi}(\zeta_0) I_1 & \text{if } \tilde{\zeta}_0 \in \Pi_1^+, \\ 0 & \text{if } \tilde{\zeta}_0 \in \Pi_1^-. \end{cases} \tag{5.40}$$

Then, equalities (5.30), (5.37), (5.38), (5.39), and (5.40) imply the validity of statement $(i)$.

Now, we consider the case when $\tilde{\zeta}_0 \in \Sigma_2 \cap \Sigma_1$. Clearly, $\Phi_1$ has the limiting values $\Phi_1^{\pm}(\tilde{\zeta}_0)$ on $\Sigma_2 \cap \Sigma_1$ due to Proposition 5.2.2. Furthermore, using the identity $\widetilde{\varphi} I_1 \equiv \varphi I_1$ on $\Gamma$, $\Phi_1^{\pm}(\tilde{\zeta}_0)$ can be expressed by the following formulas:

$$\begin{aligned}\Phi_1^{+}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau) - \varphi(\zeta_0^*)}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 + \varphi(\zeta_0^*) I_1 \\ =& \frac{1}{2\pi i}\int\limits_{\Gamma} (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0^*))(\tau - \tilde{\zeta}_0)^{-1} d\tau I_1 + \widetilde{\varphi}(\zeta_0^*) I_1, \\ \Phi_1^{-}(\tilde{\zeta}_0) =& \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau) - \varphi(\zeta_0^*)}{\xi_{1\tau} - f_1(\tilde{\zeta}_0)} d\xi_{1\tau} I_1 = \frac{1}{2\pi i}\int\limits_{\Gamma} (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\zeta_0^*))(\tau - \tilde{\zeta}_0)^{-1} d\tau I_1\end{aligned} \tag{5.41}$$

for all $\tilde{\zeta}_0 \in \Sigma_1 \cap \Sigma_2$, where $\zeta_0^* \in \Gamma$ is such that $f_1(\zeta_0^*) = f_1(\tilde{\zeta}_0)$.

Then, the validity of statement $(ii)$ follows from equalities (5.30), (5.37), (5.38), and (5.41). □

Theorems 5.5.1, 5.5.2 tell us that, under the suitable assumption, the integral defined in the right-hand side of (5.7) can be continuously extended onto the boundary of each domain of definition. We also observe that its limiting values are represented by different formulas on different parts of the union $\Sigma_1 \cup \Sigma_2$.

**Remark 5.5.3.** *We note that one can prove the existence of limiting values of $\Phi$ by using assumptions on the curve $\Gamma$ and the function $\varphi$ different from those in Theorems 5.5.1, 5.5.2. In particular, one can use the assumption that the curves $D_1(\Gamma)$ and $D_2(\Gamma)$ are quasiconformal (see, e.g., Lehto and Virtanen [89]). Indeed, if $D_1(\Gamma)$ and $D_2(\Gamma)$ are quasiconformal, the function $\varphi : \Gamma \to \mathbb{R}$ is continuously differentiable on $\Gamma$, and $\varphi'$ satisfies* (5.6)*, then one can prove the validity of equalities* (5.31), (5.35) *and* (5.36)*. The proof is based on Lemma 4 of Gerus [50] and Theorems 5.5.1, 5.5.2, and it can be done similarly to the proof of Theorem 4.2 in Plaksa and Shpakivskyi [127].*

We also note that one can consider analogs of the Cauchy type integral in other commutative algebras. Comparing Theorems 5.5.1, 5.5.2 and the results of Plaksa and Shpakivskyi [127], one can note that an increase in dimensionality of the radical of a commutative algebra leads to an increase in number of domains of definition of such an integral, and to an increase in number of forms for its limiting values.

# APPENDIX A

## Results of classical potential theory on the layer potentials

In this appendix we collect some results of potential theory for the classical layer potentials.

Let $\alpha \in ]0,1[$ and $\mathbb{I}$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\nu_{\mathbb{I}}$ denote the outward unit normal to $\partial\mathbb{I}$. We define functions harmonic at infinity by means of the following (see, e.g., Folland [47, Prop. 2.74, p. 114]):

**Definition A.0.1.** *We say that a harmonic function $u$ on $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{I}$ is harmonic at infinity if it satisfies the following condition*

$$\sup_{|x|\geq R} |x|^{n-2}|u(x)| < \infty$$

*for some $R > 0$ such that $\mathrm{cl}\mathbb{I} \subseteq \mathbb{B}_n(0, R)$.*

We now introduce the classical single (or simple) layer potential $v[\partial\mathbb{I}, \theta]$ and the double layer potential $w[\partial\mathbb{I}, \theta]$ with moments $\theta$ for all $\theta \in L^2(\partial\mathbb{I})$ by setting

$$\begin{aligned} v[\partial\mathbb{I}, \theta](t) &:= \int_{\partial\mathbb{I}} S_n(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n, \\ w[\partial\mathbb{I}, \theta](t) &:= \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}} S_n(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n. \end{aligned}$$

We have the following result.

**Theorem A.0.2.** *Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R > 0$ be such that $\mathrm{cl}\mathbb{I} \subseteq \mathbb{B}_n(0, R)$. Then the following statements hold.*

*(i) Let $\theta \in C^0(\partial\mathbb{I})$. Then the function $v[\partial\mathbb{I}, \theta]$ is continuous on $\mathbb{R}^n$ and harmonic in $\mathbb{R}^n \setminus \partial\mathbb{I}$. If $n = 2$ then the function $v^-[\partial\mathbb{I}, \theta] \equiv v[\partial\mathbb{I}, \theta]_{|\mathbb{R}^n\setminus\mathbb{I}}$ is harmonic at infinity if and only if $\int_{\partial\mathbb{I}} \theta d\sigma = 0$. If so, then $\lim_{t\to\infty} v^-[\partial\mathbb{I}, \theta](t) = 0$. If $n \geq 3$, then the function $v^-[\partial\mathbb{I}, \theta]$ is harmonic at infinity.*

*(ii) If $\theta \in C^{0,\alpha}(\partial\mathbb{I})$, then $v^+[\partial\mathbb{I}, \theta] \equiv v[\partial\mathbb{I}, \theta]_{|\mathrm{cl}\mathbb{I}} \in C^{1,\alpha}(\mathrm{cl}\mathbb{I})$, and the map of $C^{0,\alpha}(\partial\mathbb{I})$ to $C^{1,\alpha}(\mathrm{cl}\mathbb{I})$ which takes $\theta$ to $v^+[\partial\mathbb{I}, \theta]$ is linear and continuous.*

*(iii) If $\theta \in C^{0,\alpha}(\partial\mathbb{I})$, then $v^-[\partial\mathbb{I}, \theta]_{|\mathbb{R}^n\setminus\mathbb{I}} \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{I})$, and the map of $C^{0,\alpha}(\partial\mathbb{I})$ to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \mathbb{I})$ which takes $\theta$ to $v^-[\partial\mathbb{I}, \theta]_{|\mathbb{R}^n\setminus\mathbb{I}}$ is linear and continuous.*

*(iv) Let $\theta \in C^{0,\alpha}(\partial\mathbb{I})$. If $n = 2$ and $\int_{\partial\mathbb{I}} \theta d\sigma = 0$, then the function $v^-[\partial\mathbb{I}, \theta] \in C^{1,\alpha}(\mathbb{R}^n \setminus \mathbb{I})$. If $n \geq 3$, then the function $v^-[\partial\mathbb{I}, \theta] \in C^{1,\alpha}(\mathbb{R}^n \setminus \mathbb{I})$.*

*(v) If $\theta \in C^{0,\alpha}(\partial\mathbb{I})$, then the function*

$$w_*[\partial\mathbb{I}, \theta](t) \equiv \int_{\partial\mathbb{I}} DS_n(t-s)\nu_{\mathbb{I}}(t)\theta(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I}\,,$$

*belongs to $C^{0,\alpha}(\partial\mathbb{I})$. Moreover, the following jump formulas hold:*

$$\begin{aligned}
\frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I}, \theta] &= -\frac{1}{2}\theta + w_*[\partial\mathbb{I}, \theta] \qquad \text{on} \quad \partial\mathbb{I}, \\
\frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I}, \theta] &= \frac{1}{2}\theta + w_*[\partial\mathbb{I}, \theta] \qquad \text{on} \quad \partial\mathbb{I}.
\end{aligned}$$

*Proof.* For the proof we refer to Folland [47]. The proof of regularity results can be found in Miranda [106] and Lanza de Cristoforis and Rossi [87, Thm. 3.1]. □

In the Dissertation we also use the following results.

**Lemma A.0.3.** *The maps $\theta \mapsto \frac{1}{2}\theta + w_*[\partial\mathbb{I}, \theta]$ and $\theta \mapsto -\frac{1}{2}\theta + w_*[\partial\mathbb{I}, \theta]$ are bounded linear isomorphisms from $C^{0,\alpha}(\partial\mathbb{I})_0$ to itself.*

*Proof.* Since $\theta \in C^{0,\alpha}(\partial\mathbb{I})_0$, we have that $\int_{\partial\mathbb{I}} \theta d\sigma = 0$ and, thus, the validity of the statement follows from Folland [47, Ch. 3.E]. □

# APPENDIX B

## Results on periodic layer potentials

In this appendix we collect some results on the periodic layer potentials and we begin with the following one for a periodic analog of the fundamental solution of the Laplace operator.

**Theorem B.0.1.** *The generalized series*

$$S_{q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{4\pi^2|q^{-1}z|^2|Q|_n} e^{2\pi i(q^{-1}z)\cdot x},$$

*defines a tempered distribution in $\mathbb{R}^n$ such that $S_{q,n}$ is q-periodic, that is,*

$$S_{q,n}(\cdot + q_{jj}e_j) = S_{q,n}(\cdot) \qquad \forall j \in \{1, 2, \dots, n\},$$

*and such that*

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{|Q|_n},$$

*where $\delta_{qz}$ denotes the Dirac measure with mass at $qz$, for all $z \in \mathbb{Z}^n$. Moreover, the following statements hold.*

*(i)* $S_{q,n}$ *is real analytic in* $\mathbb{R}^n \setminus q\mathbb{Z}^n$

*(ii)* $R_{q,n} = S_{q,n} - S_n$ *is real analytic in* $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$, *and we have*

$$\Delta R_{q,n} = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \delta_{qz} - \frac{1}{|Q|_n}.$$

*(iii)* $S_{q,n} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

*(iv)* $S_{q,n}(x) = S_{q,n}(-x)$ *for all* $x \in \mathbb{R}^n \setminus q\mathbb{Z}^n$.

*Proof.* For the proof we refer to Lanza de Cristoforis and Musolino [81, Sec. 3] (see also Musolino [117, Thm. 2.1]). □

Then let $\alpha \in ]0,1[$ and $\Omega_Q$ be as in (1.8). We now introduce the periodic single (or simple) layer potential $v_q[\partial\Omega_Q, \theta]$ and the double layer potential $w_q[\partial\Omega_Q, \theta]$ with moments $\theta$ for all $\theta \in L^2(\partial\mathbb{I})$ by replacing $S_n$ by $S_{q,n}$ in the definitions of classical layer potentials (see Appendix A). Thus, for all $\theta \in L^2(\partial\mathbb{I})$, we set

$$v_q[\partial\Omega_Q, \theta](x) := \int_{\partial\Omega_Q} S_{q,n}(x-y)\theta(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n,$$

$$w_q[\partial\Omega_Q, \theta](x) := \int_{\partial\Omega_Q} \frac{\partial}{\partial\nu_{\Omega_Q}(y)} S_{q,n}(x-y)\theta(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n.$$

We have the following results.

**Theorem B.0.2.** *Let $\alpha \in ]0,1[$. Let $\Omega_Q$ be as in (1.8) and $\theta \in C^{0,\alpha}(\partial\Omega_Q)$. Then the following statements hold.*

*(i) The function $v_q[\partial\Omega_Q,\theta]$ is continuous on $\mathbb{R}^n$ and $q$-periodic, i.e.,*

$$v_q[\partial\Omega_Q,\theta](t+q_{jj}e_j) = v_q[\partial\Omega_Q,\theta](t) \qquad \forall t \in \mathbb{R}^n, \quad \forall j \in \{1,2,\dots,n\}.$$

*Moreover,*

$$\Delta v_q[\partial\Omega_Q,\theta](t) = -\frac{1}{|Q|_n}\int_{\partial\Omega}\theta(s)d\sigma_s \qquad \forall t \in \mathbb{S}[\Omega_Q] \cup \mathbb{S}[\Omega_Q]^-.$$

*(ii) The functions*

$$v_q^+[\partial\Omega_Q,\theta] \equiv v_q[\partial\Omega_Q,\theta]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]}, \qquad v_q^-[\partial\Omega_Q,\theta] \equiv v_q[\partial\Omega_Q,\theta]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]^-}$$

*belong to $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q])$ and $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, respectively.*

*(iii) The map $\theta \mapsto v_q^+[\partial\Omega_Q,\theta]_{|\mathrm{cl}\Omega_Q}$ of $C^{0,\alpha}(\partial\Omega_Q)$ to $C^{1,\alpha}(\mathrm{cl}\Omega_Q)$ is linear and continuous. Let $V$ be a bounded open connected subset of $\mathbb{R}^n$ such that $\mathrm{cl}Q \subseteq V$ and*

$$\mathrm{cl}V \cap \mathrm{cl}(\Omega_Q + qz) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

*Set*

$$W := V \setminus \mathrm{cl}\Omega_Q.$$

*The map $\theta \mapsto v_q^-[\partial\Omega_Q,\theta]_{|\mathrm{cl}W}$ from $C^{0,\alpha}(\partial\Omega_Q)$ to $C^{1,\alpha}(\mathrm{cl}W)$ is linear and continuous.*

*(iv) The function*

$$w_{q,*}[\partial\Omega_Q,\theta](x) \equiv \int_{\partial\Omega_Q} DS_{q,n}(x-y)\nu_{\Omega_Q}(x)\theta(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_Q\,,$$

*belongs to $C^{0,\alpha}(\partial\Omega_Q)$ and the following jump relations hold:*

$$\frac{\partial}{\partial\nu_{\Omega_Q}}v_q^+[\partial\Omega_Q,\theta] = -\frac{1}{2}\theta + w_{q,*}[\partial\Omega_Q,\theta] \qquad \text{on} \quad \partial\Omega_Q,$$
$$\frac{\partial}{\partial\nu_{\Omega_Q}}v_q^-[\partial\Omega_Q,\theta] = \frac{1}{2}\theta + w_{q,*}[\partial\Omega_Q,\theta] \qquad \text{on} \quad \partial\Omega_Q.$$

*(v) The map $\theta \mapsto w_{q,*}[\partial\Omega_Q,\theta]$ from $C^{0,\alpha}(\partial\Omega_Q)$ to $C^{0,\alpha}(\partial\Omega_Q)$ is compact.*

*(vi) If $\theta \in C^{0,\alpha}(\partial\Omega_Q)_0$ then $w_{q,*}[\partial\Omega_Q,\theta] \in C^{0,\alpha}(\partial\Omega_Q)_0$.*

*(vii) The map $\theta \mapsto w_{q,*}[\partial\Omega_Q,\theta]$ from $C^{0,\alpha}(\partial\Omega_Q)_0$ to $C^{0,\alpha}(\partial\Omega_Q)_0$ is compact.*

*Proof.* For the proof of $(i)$-$(iv)$ we refer to Lanza de Cristoforis and Musolino [81, Thm. 3.7]. The proof of $(v)$-$(vii)$ can be found in Dalla Riva and Musolino [35, Lem. 4.2$(iii)$]. □

We also formulate some properties of the periodic double layer potential, which are proved in Musolino [117].

**Theorem B.0.3.** *Let $\alpha \in ]0,1[$. Let $\Omega_Q$ be as in (1.8) and $\theta \in C^{0,\alpha}(\partial\Omega_Q)$. Then the following statements hold.*

*(i) Let $\theta \in C^{0,\alpha}(\partial\Omega_Q)$. Then $w_q[\partial\Omega_Q,\theta]$ is $q$-periodic and*

$$\Delta(w_q[\partial\Omega_Q,\theta])(x) = 0 \qquad \forall x \in \mathbb{R}^n \setminus \partial\mathbb{S}[\Omega_Q].$$

*(ii)* *If* $\theta \in C^{1,\alpha}(\partial\Omega_Q)$ *then the restriction* $w_q[\partial\Omega_Q, \theta]_{|\mathbb{S}[\Omega_Q]}$ *can be extended uniquely to an element* $w_q^+[\partial\Omega_Q, \theta]$ *of* $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q])$*, and the restriction* $w_q[\partial\Omega_Q, \theta]_{|\mathbb{S}[\Omega_Q]^-}$ *can be extended uniquely to an element* $w_q^-[\partial\Omega_Q, \theta]$ *of* $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$*, and we have*

$$w_q^{\pm}[\partial\Omega_Q, \theta] = \pm\frac{1}{2}\theta + w_q[\partial\Omega_Q, \theta] \qquad \textit{on } \partial\Omega_Q.$$

*(iii)* *The map* $\theta \mapsto w_q^+[\partial\Omega_Q, \theta]$ *from* $C^{k,\alpha}(\partial\Omega_Q)$ *to* $C_q^{k,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q])$ *is continuous for all* $k \in \{1, 2\}$*. The map* $\theta \mapsto w_q^-[\partial\Omega_Q, \theta]$ *from* $C^{k,\alpha}(\partial\Omega_Q)$ *to* $C_q^{k,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ *is continuous for all* $k \in \{1, 2\}$*.*

We also introduce the following known results.

**Lemma B.0.4.** *Let* $\alpha \in ]0, 1[$*. Let* $\Omega_Q$ *be as in* (1.8)*. Then the following statements hold.*

*(i)* *The map from* $C^{0,\alpha}(\partial\Omega_Q)_0 \times \mathbb{R}$ *to the Banach subspace of* $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ *of those functions which are harmonic in* $\mathbb{S}[\Omega_Q]^-$ *which takes a pair* $(\theta, c)$ *to* $v_q^-[\partial\Omega_Q, \theta] + c$ *is a linear homeomorphism.*

*(ii)* *The map from* $C^{0,\alpha}(\partial\Omega_Q)_0 \times \mathbb{R}$ *to the Banach subspace of* $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_Q])$ *of those functions which are harmonic in* $\mathbb{S}[\Omega_Q]$ *which takes a pair* $(\theta, c)$ *to* $v_q^+[\partial\Omega_Q, \theta] + c$ *is a linear homeomorphism.*

*Proof.* The proof can be found in Lanza de Cristoforis and Musolino [85, Lem. A.5], [83, Lem. 3.2]. □

# APPENDIX C

## Some results on the Roumieu spaces and the composition of operators

In this appendix we collect some technical results that we have used in Chapter 3. We retain here all the notation of Chapter 3 and, briefly, we recall

$$\widetilde{Q} = ]0,1[\times]0,1[, \qquad \tilde{q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We first introduce the following slight variant of Preciso [132, Prop. 1.1, p. 101] on the real analyticity of a composition operator (see also Lanza de Cristoforis and Musolino [82, Prop. 5.2] and Lanza de Cristoforis [78, Prop. 9, p. 214])

**Theorem C.0.1.** *Let $\alpha \in ]0,1]$, $\rho \in ]0,+\infty[$. Let $\Omega_1$, $\Omega'$ be a bounded open subsets of $\mathbb{R}^2$. Let $\Omega'$ be of class $C^1$. Then the composition operator $T$ from $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{1,\alpha}(\mathrm{cl}\Omega', \Omega_1)$ to $C^{1,\alpha}(\mathrm{cl}\Omega')$ defined by*

$$T[u,v] := u \circ v, \qquad \forall (u,v) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{1,\alpha}(\mathrm{cl}\Omega', \Omega_1),$$

*is real analytic.*

Also, in the following lemma, we show that it is sufficient to work on a suitable neighbourhood of the periodicity cell when we deal with periodic functions in the Roumieu class.

**Lemma C.0.2.** *Let $\rho \in ]0,+\infty[$. Let $A$ be an open connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \mathrm{cl}A$ is connected and such that*

$$\mathrm{cl}A \subseteq \widetilde{Q}.$$

*Let $W$ be a bounded open connected subset of $\mathbb{R}^2$ such that*

$$\mathrm{cl}\widetilde{Q} \subseteq W \quad \textit{and} \quad \mathrm{cl}W \cap (z + \mathrm{cl}A) = \emptyset \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

*Then the restriction operator from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$ onto the subspace*

$$C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}W \setminus A) \equiv \Big\{ v \in C^0_{\omega,\rho}(\mathrm{cl}W \setminus A) : \\ \exists\, u \in \mathbb{C}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-) \text{ such that } u \text{ is } \tilde{q}\text{-periodic}, v = u_{|\mathrm{cl}W \setminus A} \Big\},$$

*of $C^0_{\omega,\rho}(\mathrm{cl}W \setminus A)$ induces a linear homeomorphism*

*Proof.* If $u \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$, then its restriction $u_{|\mathrm{cl}W\setminus A}$ belongs to $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}W \setminus A)$. Indeed by the $\tilde{q}$-periodicity of $u$ we have that

$$\sup_{\beta\in\mathbb{N}^2} \frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}W\setminus A)} = \sup_{\beta\in\mathbb{N}^2} \frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}\widetilde{Q}\setminus A)}$$

Conversely let $v \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}W \setminus A)$, then there exists a unique $\tilde{q}$-periodic function $u$ from $\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-$ to $\mathbb{C}$ such that $v = u_{|\mathrm{cl}W\setminus A}$ and clearly $u \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$. Then the restriction operator is a bijection from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[A]^-)$ to $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}W \setminus A)$. Since it is clearly linear and continuous, then the Open Mapping Theorem implies the validity of the statement. □

We also have the following result.

**Proposition C.0.3.** *Let $\rho \in ]0,+\infty[$ and $\rho_0 \in ]0,\rho[$. Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\tilde{\Omega} \subseteq \widetilde{Q}$. Let $\kappa \in \mathbb{N}^2$ with $|\kappa| = 1$. If $u \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)$, then $D^\kappa u \in C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)$. Moreover, the operator $u \mapsto D^\kappa u$ from $C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)$ to $C^0_{\tilde{q},\omega,\rho_0}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)$ is linear and continuous.*

*Proof.* For $u \in C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)$ we have

$$\begin{aligned}
\sup_{\beta\in\mathbb{N}^2} \frac{\rho_0^{|\beta|}}{|\beta|!}\|D^\beta D^\kappa u\|_{C^0_{\tilde{q}}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)} &= \sup_{\beta\in\mathbb{N}^2} \frac{\rho_0^{|\beta|}}{|\beta|!}\frac{(|\beta|+1)!}{\rho^{|\beta|+1}}\frac{\rho^{|\beta|+1}}{(|\beta|+1)!}\|D^{\beta+\kappa}u\|_{C^0_{\tilde{q}}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)} \\
&= \sup_{\beta\in\mathbb{N}^2} \frac{1}{\rho_0}\left(\frac{\rho_0}{\rho}\right)^{|\beta|+1}(|\beta|+1)\frac{\rho^{|\beta|+1}}{|\beta|+1}\|D^{\beta+\kappa}u\|_{C^0_{\tilde{q}}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)} \\
&\le \frac{1}{\rho_0}\sup_{\beta\in\mathbb{N}^2}\left\{\left(\frac{\rho_0}{\rho}\right)^{|\beta|+1}(|\beta|+1)\right\}\sup_{\beta\in\mathbb{N}^2}\left\{\frac{\rho^{|\beta|+1}}{|\beta|+1}\|D^{\beta+\kappa}u\|_{C^0_{\tilde{q}}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)}\right\} \\
&\le \frac{1}{\rho_0}\sup_{\beta\in\mathbb{N}^2}\left\{\left(\frac{\rho_0}{\rho}\right)^{|\beta|+1}(|\beta|+1)\right\}\|u\|_{C^0_{\tilde{q},\omega,\rho}(\mathrm{cl}\mathbb{S}_{\tilde{q}}[\tilde{\Omega}]^-)}
\end{aligned}$$

Since $\rho_0 < \rho$, we have that

$$\sup_{\beta\in\mathbb{N}^2}\left\{\left(\frac{\rho_0}{\rho}\right)^{|\beta|+1}(|\beta|+1)\right\} < +\infty$$

and, thus, the validity of the proposition follows. □

Finally, we formulate the following technical lemma (see Lanza de Cristoforis and Rossi [87, p. 166], and Lanza de Cristoforis [78, Prop. 1]).

**Lemma C.0.4.** *Let $\alpha \in ]0,1[$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathbb{R}^2 \setminus \mathrm{cl}\Omega$ is connected. Then the following statements hold.*

*(i) For each $\phi \in \mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ there exists a unique $\tilde{\sigma}[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\phi] > 0$ and*

$$\int_{\phi(\partial\Omega)} w(s)\,d\sigma_s = \int_{\partial\Omega} w\circ\phi(y)\tilde{\sigma}[\phi](y)\,d\sigma_y, \qquad \forall\omega \in L^1(\phi(\partial\Omega)).$$

*Moreover, the map $\phi \mapsto \tilde{\sigma}[\phi]$ from $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.*

*(ii) The map $\phi \mapsto \nu_{\mathbb{I}[\phi]}\circ\phi$ from $\mathcal{A}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ to $C^{0,\alpha}(\partial\Omega,\mathbb{R}^2)$ is real analytic.*

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