UNIVERSITÀ DEGLI STUDI DI PADOVA

Cohomological Field Theories, Quantum Hierarchies, and Quasimodular Forms

Author:
Ishan Jaztar SINGH

Supervisor:
Prof. Paolo ROSSI
Prof. Sergey SHADRIN

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Arithmetic and Complex Algebraic Geometry Group
Dipartimento di Matematica Tullio Levi-Civita

November 29, 2025

The research leading to these results has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska–Curie grant agreement No. 101034319, and from the European Union – NextGenerationEU.

UNIVERSITÀ DEGLI STUDI DI PADOVA

Abstract

Dipartimento di Matematica Tullio Levi-Civita

Doctor of Philosophy

Cohomological Field Theories, Quantum Hierarchies, and Quasimodular Forms

by Ishan Jaztar SINGH

This thesis studies the Quantum Double Ramification (qDR) hierarchy of Buryak–Rossi in the Gromov–Witten theory of elliptic curves, focusing on its explicit quasimodular structure and its relation to the Dubrovin–Zhang (DZ) hierarchy. Defined via Kontsevich's deformation quantization of the Poisson structure, the qDR hierarchy extends the classical Double Ramification hierarchy to a fully quantum setting. Beyond the rich quasimodular behaviour of the elliptic case, a further motivation is to understand the role of odd cohomology in non-semisimple theories, where Frobenius manifolds and their associated hierarchies acquire natural super-structures. These appear nontrivially in the recursive relations of the qDR hierarchy for elliptic curves.

A contribution of this work is a refinement of results by Oberdieck and Pixton on quasimodular forms and holomorphic anomaly equations in cohomological field theories (CohFTs). We derive explicit cyclic expressions for the Gromov–Witten classes of elliptic curves paired with Hodge classes, leading to new tautological relations and providing an alternative proof of Faber's socle intersection formula. This framework extends naturally to quadratic Hodge integrals, for which we conjecture new tautological relations derived from the holomorphic anomaly equation.

A central achievement of the thesis is the construction of a closed, dispersive, and modular expression for the quantum hierarchy associated with the Gromov–Witten theory of elliptic curves. This constitutes the first explicit nontrivial example of a quantum integrable hierarchy arising from a non-semisimple CohFT that incorporates fermionic fields corresponding to the odd cohomology of the elliptic curve.

Finally, we develop quasimodular and bi-cyclic formulations for the pairings of Gromov–Witten classes with stationary insertions and Hodge classes. These results yield an effective algorithm for evaluating descendant integrals relevant to the Gromov–Witten theory of elliptic curves and the reconstruction of the DZ hierarchy.

Acknowledgements

I owe my deepest gratitude to my supervisors, Paolo Rossi and Sergey Shadrin, without whose guidance and support this thesis would not have been possible. I am also grateful to the Departments of Mathematics at the Universities of Padova and Amsterdam for hosting me during this work and providing a stimulating and friendly environment to conduct my research.

To Paolo, I am grateful for his patience and for constantly encouraging me to explore with creative freedom. His ability to visualize objects geometrically and to see the bigger picture helped me clarify ideas, and I always left our discussions with new understandings. I also appreciated our conversations about career goals and plans, which were always supportive, constructive, and reassuring.

My thanks also go to Sergey for his openness and for introducing me to his way of thinking about mathematics. I learned greatly from his approach to problems, and the intensity of our meetings often gave the project a strong momentum. I also appreciated his efforts to lighten the mood with jokes, which made the process of doing research much more enjoyable.

I remain especially grateful to Guido Carlet, who first introduced me to this field of mathematics at a time when I was uncertain about which direction to pursue. Since then, I have enjoyed working in this area. Moreover, special thanks to Renzo Cavalieri, whose doctoral course at the beginning of my PhD gave me the intuition and confidence to continue, and to Alexandre Buryak, whose course and writings on integrable hierarchies were very useful for this project. My sincere thanks also go to Xavier Blot for always being open to discussion and patient with my many questions, and to David Klompenhouwer for the many informal seminars and our joint workshop.

The support of my friends has been invaluable and made this experience colourful. Without any particular order: Xavier for wine bars and long conversations, Weoy Howe for motivation and Berlin nights, Liu for laughter and football art, Marco for beers and late-night walks, Ale for openness and white-bar nights, David for the adventures and explorations, Nowras for kindness and forró nights, Elena for fun and debates, Giacomo for the talks that matter, Enrico for laughing and cooking, Momo for jokes and billiards, Chemy for improv nights and late-night philosophy, and Runlei for his passion. I am also thankful to my fellow office mates, Gaia, Martina, Giacomo, Pietro, Francesco, Tommaso, and Beatrice, for all the coffee breaks, dinners, and for making our office a truly memorable place.

I cherish my old friends, Nhimal, Jolene, Wei Xuan, Jui Wen, King Yun, Giovanny and Gabriel, for staying in touch during the years we were apart. Though we don't see each other often, I value the bond we maintained.

My deepest gratitude also goes to my siblings, Dimple, Joohi, Avtaar, Roshan, Zhi De and to my parents Jaztar and Ambika, for always being the biggest supporters in every endeavour I pursue. My gratitude to them is beyond words.

Above all, I am most thankful to my partner, Joey, who has been with me from the very beginning of my educational journey. Through all the ups and downs over the years, her presence has been a constant source of strength and support. Through the many experiences and memories we have shared, this journey has become all the more meaningful with her by my side.

Contents

Chapter 1

Moduli spaces, integrability and quasimodularity

1.1 Introduction

Moduli spaces parameterize families of algebro–geometric objects up to isomorphism and naturally arise in classification problems across algebraic geometry and mathematical physics. Among them, the moduli space of curves plays a central role, it appears in string theory, gauge theory, mirror symmetry, and in many contexts where geometry interacts with physics. This thesis focuses on the moduli space $\mathcal{M}_{g,n}$ of genus-g smooth curves with n marked points and its Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, whose points correspond to stable nodal curves [DM69]. Intersection theory on $\overline{\mathcal{M}}_{g,n}$ provides a rich enumerative framework governed by the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$, generated by ψ-, κ-, and boundary classes.

A long-standing problem in the field was to compute ψ-class intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n},$$

which encode subtle geometric and physical information. A major breakthrough came with Witten's conjecture [Wit91] and Kontsevich's proof [Kon92], showing that the generating function of ψ-class intersection numbers is a τ-function of the KdV hierarchy. Known as the Witten–Kontsevich theorem, this result revealed a deep correspondence between intersection theory on $\overline{\mathcal{M}}_{g,n}$ and integrable systems, and it established the conceptual foundations for a broad program relating Gromov–Witten theory and integrability.

Gromov–Witten theory extends this intersection-theoretic framework to maps from stable curves into a target variety X. Introduced by Kontsevich and Manin [KM94], it studies the moduli space $\overline{\mathcal{M}}_{g,n}(X,d)$ of stable maps of degree $d \in H_2(X,\mathbb{Z})$, producing invariants

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{k=1}^{n} \mathrm{ev}_k^*(\gamma_k)\, \psi_k^{i_k}.$$

These invariants generalize classical enumerative geometry, such as the count of rational curves in projective spaces. The formalism of *cohomological field theories* (CohFTs) [KM94] provides an abstract framework for these intersection numbers, a CohFT is a collection of classes

$$c_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n}$$

satisfying natural symmetry, unit, and gluing axioms. At genus zero, the classes $c_{0,n}$ endow the corresponding state space V with the structure of a Frobenius manifold [Dub96], whose geometry often governs an associated integrable hierarchy.

Dubrovin and Zhang [DZ98; DZ01] constructed the integrable hierarchy associated with any semisimple CohFT, now known as the Dubrovin–Zhang (DZ) hierarchy. Their approach, later unified by Givental's quantization formalism [Giv01a; Giv04] and Teleman's classification theorem [Tel12], provides a correspondence between semisimple CohFTs and integrable hierarchies. In principle, every semisimple Gromov–Witten theory admits an associated hierarchy controlling its descendant invariants; in practice, explicitly constructing these hierarchies and identifying them with the corresponding geometric theories remains highly nontrivial.

The Double Ramification (DR) hierarchy, introduced by Buryak [Bur15], extends this integrable framework beyond the semisimple setting. It is defined through intersection theory on $\overline{\mathcal{M}}_{g,n}$ involving the double ramification cycle $\mathrm{DR}_g(A)$, top Hodge classes λ_g, and ψ-classes. The fundamental objects recursively determining the hierarchy are integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}\left(\bigotimes_{k=1}^{n} e_{\alpha_k}\right)\mathrm{DR}_g(A)\lambda_g.$$

The DR hierarchy and the DZ hierarchy have since been shown to coincide under broad assumptions [BGR19; BLS24a], providing a unified perspective on integrable hierarchies of topological type.

A further refinement was introduced by Buryak and Rossi [BR16a]: the *quantum* Double Ramification (qDR) hierarchy. This construction replaces λ_g with the full Chern polynomial of the Hodge bundle $\Lambda(\epsilon) = \sum_{i=0}^{g} \epsilon^i \lambda_i$, producing invariants

$$\int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}\left(\bigotimes_{k=1}^{n} e_{\alpha_k}\right)\mathrm{DR}_g(A)\Lambda(\epsilon).$$

The qDR hierarchy quantizes the DR hierarchy in the sense of Kontsevich's deformation of Poisson structures and incorporates higher-genus corrections. Its geometric and physical meaning is only beginning to be understood.

While the DZ and DR constructions are well-developed for semisimple and even-cohomology targets, much less is known for targets with nontrivial odd cohomology. Such targets require a *superalgebraic* extension of CohFTs and lead naturally to integrable systems with both bosonic and fermionic fields. The supergeometric versions of these constructions are often alluded to in the literature but rarely worked out explicitly, largely due to the scarcity of fully computable examples (see however [Dot+24]).

The elliptic curve is the simplest projective variety with nontrivial odd cohomology and therefore the natural testing ground for superextensions of DR and qDR hierarchies. Although its Gromov–Witten theory is understood thanks to the work of Okounkov and Pandharipande [OP06c; OP06b; OP06a], the corresponding classical integrable hierarchy is surprisingly simple. Buryak [Bur23] showed that, up to a change of variables, it reduces to a purely dispersionless system of evolutionary PDEs. This phenomenon extends to any target variety with nonpositive first Chern class [Bur+18], yielding classical hierarchies with very limited dynamics.

In contrast, the *quantum* DR hierarchy of the elliptic curve turns out to be unexpectedly rich. In this thesis, we compute the full qDR hierarchy for the Gromov–Witten theory of the elliptic curve. Dimension counting shows that only two potential $\hbar$-corrections to the primary Hamiltonians can appear; a term linear in $\hbar$, governed by intersection numbers computed by Oberdieck and Pixton [OP23], and a term quadratic in $\hbar$, which we show must vanish on geometric grounds. The vanishing follows from the behaviour of principal divisors under pushforward along branched covers between Riemann surfaces. If the source curve has positive genus, the image of the support of such a divisor cannot satisfy the necessary Jacobian constraints to contribute nontrivially.

The resulting qDR hierarchy exhibits a striking quasimodular structure in the degree parameter, reflecting the quasimodularity of elliptic Gromov–Witten classes [OP18; OP23] and related holomorphic anomaly equations [MRS15]. In particular, the primary Hamiltonians describe an interacting system of two bosonic and two fermionic fields, coupled through an integral transform with a kernel depending modularly on one of the bosonic fields. This is markedly different from traditional elliptic quantum integrable systems, where the modular parameter is a fixed background datum rather than a dynamical variable.

Several natural limits of the hierarchy further illuminate its structure. Besides the classical limit of [Bur23], one may perform a cusp degeneration of the modular parameter, yielding a trigonometric kernel, or apply a double scaling limit to obtain a new classical dispersive integrable hierarchy. The dispersionless limit of the full quantum theory has a primary Hamiltonian whose modular dependence is expressed entirely through the τ-derivative of the Eisenstein series of quasimodular weight 2.

Beyond the hierarchy itself, our computations produce new results on the geometry of Gromov–Witten classes of elliptic curves. We construct new tautological relations that imply Faber's socle intersection formula, and we propose a conjectural algorithm for quadratic Hodge integrals. Inspired by Buryak's construction of the Dubrovin–Zhang hierarchy for the elliptic curve [Bur23], we also obtain explicit cyclic-graph expressions for stationary Gromov–Witten classes and conjecture that these relations can be used to recursively determine the remaining components of Buryak's hierarchy.

Several questions remain open. Among them is the issue of nontriviality, is the quantum DR hierarchy Miura-trivial, like its classical elliptic limit, or does it define a genuinely new quantum integrable system? If it is nontrivial, can it be identified with an existing integrable model, or do its modularity dependent interactions represent a new class of dynamical structures? These questions lie at the intersection of geometry, representation theory, and integrability, and they provide a promising direction for future research.

Organization of the thesis.

Chapter 1 provides the necessary background on the relationship between cohomological field theories and integrable hierarchies of topological type. We review the geometry of the moduli space of stable curves, the Witten–Kontsevich theorem, and the structure of Frobenius manifolds, leading to Givental–Teleman's reconstruction formalism and the Dubrovin–Zhang hierarchy. The chapter concludes with a discussion of quantum integrability and quasimodularity, setting the motivation for the constructions developed later.

Chapter 2 focuses on the geometry and integrable structures of the double ramification cycle. We review its definition via the moduli space of relative and rubber maps, Hain's and Pixton's formulas, and the Buryak–Shadrin–Spitz–Zvonkine splitting formula. We also discuss relations arising from the DR/DZ equivalence and outline the construction of the Double Ramification hierarchy, including its Hamiltonian structure and recursion relations.

Chapter 3 presents a new proof of Faber's socle intersection number formula using the Gromov–Witten theory of elliptic curves. We derive new tautological relations from the study of quasimodular properties of Gromov–Witten classes, providing explicit intersection computations that link double ramification cycles with elliptic invariants. This chapter is a joint work with Xavier Blot and Sergey Shadrin [BSS25].

Chapter 4 develops the quantum double ramification (qDR) hierarchy for the Gromov–Witten theory of elliptic curves. We introduce the (super) quantum DR framework and conclude the chapter with a proof of a vanishing intersection result, which allows us to derive a closed formula for the quantum double ramification potential that initiates the recursion. This chapter is based on joint work with Paolo Rossi and Sergey Shadrin.

Chapter 5 investigates the quasimodular structure of the Gromov–Witten classes of elliptic curves. We study the interplay between quasimodularity, Eisenstein series, and Rankin–Cohen brackets, and construct explicit cyclic graph representations of elliptic Gromov–Witten classes paired with Hodge classes. Two complementary approaches are presented for evaluating descendant integrals relevant to the Gromov–Witten theory of elliptic curves and the Dubrovin–Zhang hierarchy. This chapter is also based on joint work with Paolo Rossi and Sergey Shadrin.

Conventions and notations

- The Einstein summation convention is applied for repeated upper and lower Greek indices.
- The symbol $*$ is used to represent any value within the appropriate range of a subscript or superscript.
- For a given topological space X, denote by $H_*(X)$ and $H^*(X)$ the homology and cohomology groups of X with coefficients in $\mathbb{C}$.
- The moduli space of stable curves is denoted by $\overline{M}_{g,n}$.
- The double ramification cycle associated with a vector $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ is denoted by

$$\mathrm{DR}_g(A) \in H^{2g}(\overline{M}_{g,n}).$$

- The tautological classes $\psi_i \in H^2(\overline{M}_{g,n})$ and $\lambda_j \in H^{2j}(\overline{M}_{g,n})$ denote the cotangent line classes at the i-th marking and the j-th Chern classes of the Hodge bundle, respectively.
- A cohomological field theory (CohFT) is given by multilinear maps

$$c_{g,n}\colon V^{\otimes n} \to H^*(\overline{M}_{g,n}),$$

  where V is a finite-dimensional $\mathbb{C}$–vector space equipped with a symmetric non-degenerate bilinear form η and unit e_0. Elements of V are denoted by $\{e_\alpha\}_{\alpha=1}^N$.
- The ring of differential polynomials is denoted by $\mathcal{A}$, and its space of local functionals by $\hat{\mathcal{A}}$.
- Formal variables u_j^α denote the jth x-derivative of the field variable u^α, i.e. $u_j^\alpha = \partial_x^j u^\alpha$.
- Fourier variables p_k^α are related to u^α by the expansion

$$u^\alpha = \sum_{k\in\mathbb{Z}} p_k^\alpha e^{\,kx}.$$

- The dispersion parameter is denoted by ε, and the quantization parameter is denoted by $\hbar$.
- The standard Poisson operator is denoted by $K = \eta\partial_x$, where η is the metric of the CohFT.
- Local Hamiltonians are denoted by $\overline{g}_{\alpha,d} = \int g_{\alpha,d}(x)\,dx$, where $g_{\alpha,d}$ are Hamiltonian densities.
- Bernoulli numbers are denoted by B_{2g}, with the convention $B_1 = -\frac{1}{2}$.
- The ring of integers modulo k is denoted by $\mathbb{Z}_k$.
- The symbol i is used in two contexts: in generating series of quantum hierarchies, it denotes the imaginary unit $\sqrt{-1}$; in other contexts, such as indices in combinatorial sums, it represents a positive integer.

1.2 Moduli space of stable curves

The moduli space M_g of algebraic curves of genus g classifies algebraic curves of genus g up to isomorphism. Its Deligne-Mumford compactification, $\overline{M}_g$, extends this classification to stable algebraic curves of genus g. To address more general enumerative geometry questions, one considers the moduli space $\overline{M}_{g,n}$ of stable algebraic curves of genus g with n marked points. This space plays a fundamental role in algebraic geometry, and its cohomology ring, $H^*(\overline{M}_{g,n})$, encodes the necessary information for formulating intersection theory on $\overline{M}_{g,n}$.

Since the full structure of the cohomology ring is often too intricate to analyze, attention is typically restricted to the tautological ring $RH^*(\overline{M}_{g,n})$. This is the smallest subring of $H^*(\overline{M}_{g,n})$ generated by natural geometric constructions on $\overline{M}_{g,n}$. A key feature of the tautological ring is the presence of ψ-classes, which can be expressed as first Chern classes of certain natural line bundles on $\overline{M}_{g,n}$. These ψ-classes, along with their associated strata, generate the tautological ring. However, determining the complete set of relations among these generators remains an open problem.

In this thesis, we provide an informal overview of $\overline{M}_{g,n}$, focusing on essential definitions and key results. For a more comprehensive treatment, we refer to [Zvo14; Sch20]. Additionally, an accessible and detailed review of the genus-zero theory of the moduli space of curves can be found in [KV07].

1.2.1 Stable curves and graphs

The algebraic curves parametrized by $\overline{\mathcal{M}}_{g,n}$ are connected, projective, complex curves of arithmetic genus g with at most nodal singularities, equipped with n pairwise distinct marked points lying in the smooth locus of the curve.

From an analytic perspective, one may view such a curve as a *Riemann surface with nodes*: this is a connected one–dimensional complex analytic space in which every point has a neighborhood biholomorphic either to a disc $\{|z| < 1\}$ (a smooth point) or to a neighborhood of the node $\{z_1 z_2 = 0\}$ in the bidisc $\{|z_1|, |z_2| < 1\}$. In this interpretation, $\overline{\mathcal{M}}_{g,n}$ parametrizes isomorphism classes of Riemann surfaces with nodes of arithmetic genus g together with n distinct marked points, all located at smooth points of the surface.

A pointed curve is written as

$$(C, p_1, \ldots, p_n),$$

where the p_i lie in the smooth locus of C and satisfy $p_i \neq p_j$ for $i \neq j$. Such a curve is called *stable* if its automorphism group $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite. Stability is equivalent to the numerical condition

$$2g - 2 + n > 0, \tag{1.1}$$

so the cases $(g, n) \in \{(0, 0), (0, 1), (0, 2), (1, 0)\}$ are unstable and do not appear in the moduli space. Each point of $\overline{\mathcal{M}}_{g,n}$ represents an isomorphism class

$$[(C, p_1, \ldots, p_n)].$$

To carry out intersection-theoretic constructions, it is essential to work with a compact parameter space. The moduli space $M_{g,n}$ of smooth pointed Riemann surfaces is not compact, families of smooth curves may degenerate by pinching a simple closed geodesic, producing nodal singularities in the limit. The Deligne–Mumford compactification

$$M_{g,n} \hookrightarrow \overline{M}_{g,n}$$

adds precisely these stable nodal curves as boundary points, yielding a compact moduli space. The boundary $\overline{M}_{g,n} \setminus M_{g,n}$ parametrizes curves that break into several components joined at nodes, while $M_{g,n}$ remains a dense open subset consisting of smooth curves. An intuitive depiction of such degenerations and the resulting boundary strata is shown in Figure 1.1.

In Figure 1.1(i), we present a topological perspective of *transversal* intersections in a nodal curve. This curve lies on the boundary of $\overline{M}_{4,8}$ as a result of the compactification. In Figure 1.1(ii), we provide a geometric perspective of a *reducible* nodal curve, where the transversal intersections are more clearly visible compared to (i). Additionally, we indicate the genus of each component on the side, indexed as $(0, 1, 2)$ in this case. In both Figures 1.1(i) and (ii), the curve is decomposed into three *irreducible* components, each belonging to a moduli space of lower genus with fewer marked points. Consequently, a curve $(C, p_1, \ldots, p_n)$ is considered stable if each of its irreducible components, $\tilde{C}$, satisfies the stability condition in Equation 1.1. More explicitly, each component $\tilde{C}$ must satisfy one of the following conditions:

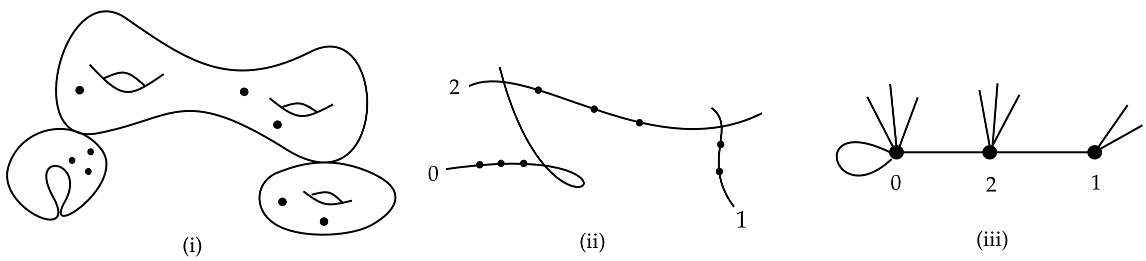


FIGURE 1.1: Equivalent descriptions of an isomorphism class in $\overline{M}_{4,8}$

- $\tilde{C}$ has genus zero and at least three marked points.
- $\tilde{C}$ has genus one and at least one marked point.
- $\tilde{C}$ has genus at least two.

For a detailed treatment of the explicit blow-up construction of the moduli space of stable genus-zero curves, the interested reader may consult [KV07]. In that case, $\overline{M}_{0,n}$ can be realized as a smooth projective variety obtained from

$$(\mathbb{P}^1)^{n-3}$$

by a sequence of blow-ups along diagonals; see [KV07] for a complete description. (In particular, $\overline{M}_{0,4} \cong \mathbb{P}^1$, but for $n \geq 5$ the moduli space is no longer a product of projective spaces.) In algebraic geometry, $\overline{M}_{g,n}$ is defined as a smooth Deligne–Mumford stack, while in the differential-geometric setting it is treated as a smooth complex orbifold. For a basic introduction to orbifolds, see [Zvo14], and for more comprehensive references, consult [ALR07].

By treating $\overline{M}_{g,n}$ as an orbifold with a group action given by the automorphism group of the curves, its cohomology and homology can be defined analogously to those of manifolds. However, there are subtleties due to the presence of the group action. Additionally, the (complex) dimension of $\overline{M}_{g,n}$ is well-defined and corresponds to the dimension of its associated complex orbifold:

$$\dim_{\mathbb{C}} \overline{M}_{g,n} = 3g - 3 + n.$$

A nontrivial one-dimensional example is the moduli space of elliptic curves, $\overline{M}_{1,1}$, which is constructed as $\mathbb{C}_+/\mathrm{PSL}(2,\mathbb{Z})$, where $\mathrm{PSL}(2,\mathbb{Z})$ is the modular group (see [Eyn18, p. 84]).

For practical purposes, points in $\overline{M}_{g,n}$ can be interpreted in terms of graphs that are dual to their corresponding curves. An example of such a graph is shown in Figure 1.1(iii). The vertices of the graph encode the geometric genus of the curve, while the legs (or external edges) attached to each vertex represent the marked points. The edges between vertices correspond to the gluing or intersection of different components of the curve.

Definition 1.1. A *stable graph* Γ is a tuple,

$$\Gamma = \Big(\mathrm{V}(\Gamma), \mathrm{H}(\Gamma), \mathrm{L}(\Gamma), g : \mathrm{V} \to \mathbb{Z}_{\geq 0} \,\Big|\, v : \mathrm{H} \to \mathrm{V},\ \iota : \mathrm{H} \to \mathrm{H},\ l : \mathrm{L} \to \mathrm{N}\Big)$$

where $\mathrm{N} = \{1, \ldots, n\}$ and the tuple satisfies the following:

i. $\mathrm{V}(\Gamma)$ is a finite set of vertices v, while g is a map $v \mapsto g(v)$ that provides the geometric genus of the vertex.

ii. $\mathrm{H}(\Gamma)$ is a finite set of half-edges h, while v is a map $h \mapsto v(h)$ that sends half-edges to the vertex its attached to.

iii. E(Γ) is a finite set of edges $e = (h, h')$, that contains pairs of half-edges, such that the involution $\iota : h \mapsto h'$ is an involution.

iv. $L(\Gamma) \subset H(\Gamma)$ is the set of half-edges that are fixed by ι, i.e. for any $h \in L(\Gamma)$, $\iota(h) = h$. The map l is a bijective map that sends half-edges to the index of the marked points $\{1, \ldots, n\}$.

v. The graph Γ is connected and the data surrounding each vertex $v \in V(\Gamma)$ satisfies stability condition:

$$2g(v) - 2 + n(v) > 0$$

Let $n(v)$ denote the number of half-edges around a vertex $v \in V(\Gamma)$, so that the number of marked points $n(\Gamma)$ of the graph is given by,

$$n(\Gamma) = \sum_{v \in V(\Gamma)} n(v) = |L(\Gamma)|$$

Since self-intersection of a curve is included in $\overline{M}_{g,n}$, we did not restrict loops in out definition, loops being an edge attached to the same vertex. Hence, the genus $g(\Gamma)$ of the graph is given by,

$$g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + 1 + |E(\Gamma)| - |V(\Gamma)|$$

Example 1. Consider the diagrams in 1.1, its stable graph is given by the indexed figure 1.2. We then have the following data:

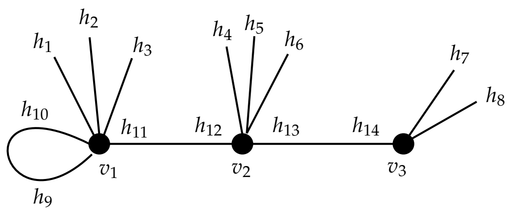


FIGURE 1.2: An example of a stable graph in $\overline{M}_{4,8}$

i. $V(\Gamma) = \{v_1, v_2, v_3\}$, $H(\Gamma) = \{h_1, \ldots, h_{14}\}$, $E(\Gamma) = \{(h_{11}, h_{12}), (h_{13}, h_{14})\}$ and $L(\Gamma) = \{h_1, \ldots, h_8\}$.

ii. The map g is defined by, $g(v_1) = 0, g(v_2) = 2$ and $g(v_3) = 1$. The map v is defined by, $v(\{h_1, h_2, h_3, h_9, h_{10}, h_{11}\}) = v_1$ and likewise for the other vertices. The map l is defined by $l(h_i) = i$ for $i = 1, \ldots, 8$.

iii. Hence, the number of marked points is $n(\Gamma) = 8$ and $g(\Gamma) = 4$.

Denote by $G_{g,n}$ the set of isomorphism classes of stable graphs with $n(\Gamma) = n$ and $g(\Gamma) = g$. For any $\Gamma \in G_{g,n}$, we define a moduli space M_Γ that classifies curves C whose associated stable graphs Γ_C are isomorphic to Γ. In fact, M_Γ is a closed subspace of $\overline{M}_{g,n}$, often referred to as a *stratum* of the moduli space of stable curves. Informally, it is plausible to observe that $\overline{M}_{g,n}$ is a disjoint union of all possible configurations of M_Γ:

$$\overline{M}_{g,n} = \coprod_{\Gamma \in G_{g,n}} M_\Gamma.$$

Additionally, the dimension of each stratum is given by:

$$\dim_{\mathbb{C}} M_\Gamma = \sum_{v \in V(\Gamma)} (3g(v) - 3 + n(v)) = \dim_{\mathbb{C}} \overline{M}_{g,n} - |E(\Gamma)|.$$

This formula can be easily verified in the genus-zero case (see [KV07, p. 35]).

Subspaces M_Γ of $\overline{M}_{g,n}$ with $|\mathrm{E}(\Gamma)| > 0$ are often referred to as *boundary cycles* of $\overline{M}_{g,n}$, while those with $|\mathrm{E}(\Gamma)| = 1$ are called *boundary divisors*. Moreover, it can be shown that the boundary of $\overline{M}_{g,n}$ is the disjoint union of all its boundary divisors, i.e.,

$$\partial\overline{M}_{g,n} = \overline{M}_{g,n} \backslash M_{g,n} = \coprod_{\Gamma \in \mathrm{G}_{g,n}, |\mathrm{E}(\Gamma)|=1} M_\Gamma.$$

For an illustration of these statements, see [Sch20, p. 28].

1.2.2 Gluing and forgetful morphisms

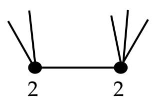
In the theory of the moduli space of stable curves, two maps arise *tautologically* (or naturally) from the construction of the *universal family* of $\overline{M}_{g,n}$. Proving these maps are well-defined requires additional algebro-geometric machinery, most of which will not be essential for the following chapter (see [Sch20]).

The first map is the *gluing map*, denoted gl_Γ for a stable graph $\Gamma \in \mathrm{G}_{g,n}$. It is the morphism

$$\mathrm{gl}_\Gamma : \prod_{v \in \mathrm{V}(\Gamma)} \overline{M}_{g(v),n(v)} \longrightarrow \overline{M}_{g,n}.$$

It attaches components $(\Gamma_{\tilde{C}_1}, \Gamma_{\tilde{C}_2}, \ldots)$ to form an isomorphism class of the stable graph Γ, by gluing pairs of half-edges (h, h') from each component to create edges of a stable graph. The image of gl_Γ is the closure $\overline{M}_\Gamma$ of M_Γ in $\overline{M}_{g,n}$.

Example 2. To illustrate the gluing morphism, let $\Gamma \in \mathrm{G}_{4,5}$ be a stable graph.

$\Gamma =$ 

Then, there exist a gluing map,

$$\mathrm{gl}_\Gamma : \overline{M}_{2,3} \times \overline{M}_{2,4} \to \overline{M}_{4,5}$$

that is defined for points in $\overline{M}_{2,3} \times \overline{M}_{2,4}$, with a graphical interpretation given in Figure 1.3.

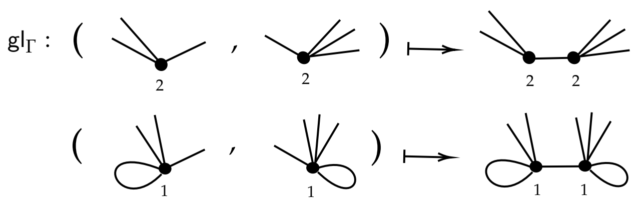


FIGURE 1.3: Example of the gluing morphism

The notion of gluing maps arises naturally from the construction of the universal family. The most important subclass consists of those whose images are boundary divisors; these correspond to sections of the universal family. Let us introduce two special kinds of gluing maps:

$$\rho : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \longrightarrow \overline{M}_{g,n},$$

where $g_1 + g_2 = g$ and $n_1 + n_2 = n$. This map glues the two additional marked points, forming an edge between the stable graphs.

We also have

$$\sigma : \overline{M}_{g-1,n+2} \longrightarrow \overline{M}_{g,n},$$

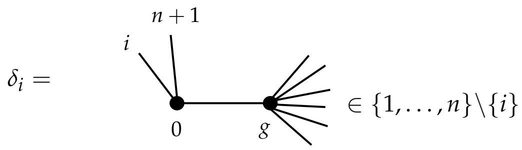


FIGURE 1.4: The graphical representation of the boundary divisor δ_i.

which glues two additional marked points to form a loop (self-intersection).

The universal sections $(\rho_1, \dots, \rho_n)$ of the universal family of $\overline{M}_{g,n}$ correspond to the gluing maps

$$\rho_i : \overline{M}_{0,3} \times \overline{M}_{g,n} \longrightarrow \overline{M}_{g,n+1}, \tag{1.2}$$

where the i-th and $(n+1)$-th points are marked on $\overline{M}_{0,3}$. Denote the image (a boundary divisor) of ρ_i by δ_i, as shown in Figure 1.4.

The second natural map is the *forgetful map*, denoted π, as it corresponds to the projection of the universal family, i.e.,

$$\pi \circ \rho_i = \mathrm{Id}.$$

It is the morphism

$$\pi : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}, \tag{1.3}$$

which sends a tuple $(C, p_1, \dots, p_{n+1})$ with $(n+1)$ marked points to a tuple $(C', p'_1, \dots, p'_n)$ with n marked points. When removing a marked point, two cases can occur:

i. The resulting curve is stable (all irreducible components remain stable). In this case, $C' = C$ and $p'_i = p_i$.

ii. The resulting curve is unstable. Then there exists a contraction map $\phi : C \to C'$ with $p'_i = \phi(p_i)$, contracting unstable irreducible components to a marked point.

These conditions ensure that π is well-defined.

1.2.3 Tautological ring and classes

A global understanding of the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ requires knowledge of its topological invariants, particularly its homology and cohomology rings. We follow the standard convention of working with singular homology $H_*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and singular cohomology $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ with rational coefficients.

Since $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack, its intersection theory is encoded in the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$, and there is a natural *cycle class map* (see [Ful98, Chap. 19]

$$\mathrm{cl}\colon A^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}),$$

which sends an algebraic cycle to its associated cohomology class. Whenever the choice of coefficients is clear from context, we omit $\mathbb{Q}$ and simply write $H^*(\overline{\mathcal{M}}_{g,n})$. Only basic intersection-theoretic notions will be needed here, as surveyed in [Nic11], with more detailed treatments in [HM91; Gat00].

Let X be a d-dimensional smooth connected complex projective variety. Denote by $\smile$ and $\frown$ the cup and cap products on $H^*(X)$, given by

$$\smile : H^k(X) \otimes H^l(X) \to H^{k+l}(X), \quad \alpha \otimes \beta \mapsto \alpha \smile \beta,$$
$$\frown : H_k(X) \otimes H^l(X) \to H_{k-l}(X), \quad \alpha \otimes \beta \mapsto \alpha \frown \beta,$$

satisfying the relation $\alpha \frown (\beta \smile \gamma) = (\alpha \frown \beta) \smile \gamma$. From the cap product and the identification $H_0(X) \simeq \mathbb{Q}$, we obtain the duality $H^k(X) \simeq H_k(X)^*$, where $H_k(X)^*$ is the dual space of $H_k(X)$.

The degree map is defined as

$$\deg : H^{2d}(X) \xrightarrow{\sim} \mathbb{Q}, \quad \deg(\alpha) = \int_X \alpha.$$

Since $\dim_{\mathbb{R}}(X) = 2d$, Poincaré duality provides the non-degenerate pairing

$$H^k(X) \otimes H^{2d-k}(X) \to \mathbb{Q}, \quad \alpha \otimes \beta \mapsto \deg(\alpha \smile \beta).$$

This implies the isomorphisms

$$H^k(X) \simeq H^{2d-k}(X)^* \simeq H_{2d-k}(X),$$

so that the map $H^k(X) \xrightarrow{\sim} H_{2d-k}(X)$ is given by $\alpha \mapsto [X] \frown \alpha$, where $[X]$ is the fundamental class of X (the Poincaré dual of X).

From a differential topology perspective, given a d-dimensional closed and oriented manifold X and a k-dimensional submanifold $S \subset X$, we associate S to its Poincaré dual $[S] \in H^{2(d-k)}(X)$ via

$$\int_S \iota^* \alpha = \int_X \alpha \smile [S],$$

where $\alpha \in H_c^{2k}(X)$ is a compactly supported k-form, and $\iota : S \hookrightarrow X$ is the inclusion map.

Given a morphism $f : X \to Y$, where $\dim X = d$ and $\dim Y = e$, the pushforward f_* is defined for the homology ring, while the pullback f^* is defined for the cohomology ring. Using Poincaré duality, where $H^l(X) \simeq H_{2d-l}(X)$ and $H_{2d-l}(Y) \simeq H^{2(e-d)+l}(Y)$, we obtain a well-defined pushforward in cohomology:

$$f_* : H^l(X) \to H^{2(e-d)+l}(Y).$$

The pushforward and pullback satisfy the following compatibility relations with respect to the cup and cap products:

$$f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta,$$
$$f_*(f^*\beta \smile \alpha) = \beta \smile f_*\alpha.$$

We now extend these notions to the cohomology ring $H^*(\overline{M}_{g,n})$ of the moduli space of stable curves. The *i-th cotangent line bundle* $\mathcal{L}_i$ is defined as

$$\mathcal{L}_i|_{(C,p_1,\dots,p_n)} = T^*_{p_i} C.$$

The *i-th ψ-class* is then given by the first Chern class of $\mathcal{L}_i$:

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{M}_{g,n}).$$

A nontrivial fact is that the ψ-classes are tautological for every $i = 1, \dots, n$, as they can be expressed as

$$\psi_i = -\pi_*([\delta_i] \smile [\delta_i]),$$

where $[\delta_i]$ is the Poincaré dual of the boundary divisor δ_i, illustrated in Figure 1.4.

The *pullback relation* for the ψ-classes are given by

$$\pi^*\psi_i = \psi_i - [\delta_i].$$

This fundamental relation play a crucial role in computations.

Example 3. In genus zero, the i-th ψ-class admits an explicit expression:

$$\psi_i = [\delta_{i,jk}],$$

where $[\delta_{i,jk}]$ is the Poincaré dual of the boundary divisor $\delta_{i,jk}$. This divisor corresponds to a stable graph where the i-th marked point lies on one component, while the j-th and k-th marked points lie on the other, separated by an edge (see [Zvo14, p. 26]).

1.3 The Witten–Kontsevich Theorem

This section summarizes the key results of Witten's foundational work [Wit91], which connects ψ-class intersection numbers on $\overline{M}_{g,n}$ to the Korteweg–de Vries (KdV) integrable hierarchy. We begin by introducing the basic notation.

The *Witten–Kontsevich (WK) descendant correlators* are defined by

$$\langle \tau_{i_1} \cdots \tau_{i_n} \rangle_{g,n} = \int_{\overline{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n},$$

where $\psi_k \in H^2(\overline{M}_{g,n})$. Let $\mathbf{t} = (t_0, t_1, t_2, \ldots)$ be an infinite collection of formal variables. The *genus-g WK potential* is

$$F_g(\mathbf{t}) = \sum_{n \geq 0} \sum_{i_\bullet \geq 0} \langle \tau_{i_1} \cdots \tau_{i_n} \rangle_{g,n} \frac{t_{i_1} \cdots t_{i_n}}{n!}.$$

The full *WK descendant potential* is the formal series

$$\mathcal{F}^{\mathrm{WK}}(\mathbf{t}; \varepsilon) = \sum_{g=0}^{\infty} \varepsilon^{2g} F_g(\mathbf{t}),$$

where the dispersion parameter ε records the genus grading.

String and dilaton equations. Witten introduced two fundamental relations satisfied by the descendant potential. The *string equation* is

$$\frac{\partial \mathcal{F}^{\mathrm{WK}}}{\partial t_0} = \frac{t_0^2}{2} + \sum_{k \geq 0} t_{k+1} \frac{\partial \mathcal{F}^{\mathrm{WK}}}{\partial t_k},$$

and the *dilaton equation* is

$$\frac{\partial \mathcal{F}^{\mathrm{WK}}}{\partial t_1} = \frac{1}{24} + \frac{1}{3} \sum_{k \geq 0} (2k+1) t_k \frac{\partial \mathcal{F}^{\mathrm{WK}}}{\partial t_k}.$$

Taking the coefficient of ε^{2g} yields genus-by-genus versions of these equations.

Proposition 1.2 ([Wit91])**.** The string equation is equivalent to

$$\langle \tau_0 \, \tau_{i_1} \ldots \tau_{i_n} \rangle_{g,n+1} = \sum_{j=1}^{n} \langle \tau_{i_1} \cdots \tau_{i_j - 1} \cdots \tau_{i_n} \rangle_{g,n},$$

which holds for all stable pairs (g, n) satisfying $2g - 2 + n > 0$.

Corollary 1.3. Using the string equation inductively, one obtains the genus-zero formula

$$\langle \tau_{i_1} \cdots \tau_{i_n} \rangle_{0,n} = \frac{(n-3)!}{i_1! \cdots i_n!}.$$

Proposition 1.4 ([Wit91])**.** The dilaton equation is equivalent to

$$\langle \tau_1 \, \tau_{i_1} \cdots \tau_{i_n} \rangle_{g,n+1} = (2g - 2 + n) \, \langle \tau_{i_1} \cdots \tau_{i_n} \rangle_{g,n},$$

which again holds when $2g - 2 + n > 0$.

The KdV hierarchy. The key insight of Witten's conjecture is that the descendant potential governs a distinguished solution of the Korteweg–de Vries (KdV) hierarchy. To describe the hierarchy, let $u = u(\mathbf{t})$ be a smooth function and define

$$\dot{u} = \frac{\partial u}{\partial t_0}, \qquad \ddot{u} = \frac{\partial^2 u}{\partial t_0^2}, \quad \dots$$

Let $p_i(u, \dot{u}, \ddot{u}, \dots)$ be differential polynomials obtained recursively from

$$\dot{p}_{i+1} = \frac{1}{2i+1}\left(p_i \dot{u} + 2\dot{p}_i\, u + \frac{1}{4}\dddot{p}_i\right), \qquad p_1 = u.$$

The *KdV hierarchy* is the infinite system of PDEs

$$\frac{\partial u}{\partial t_i} = \frac{\partial p_{i+1}}{\partial t_0}, \qquad i \geq 0.$$

The first equation is the classical KdV equation:

$$\frac{\partial u}{\partial t_1} = u\frac{\partial u}{\partial t_0} + \frac{1}{12}\frac{\partial^3 u}{\partial t_0^3} = \frac{\partial}{\partial t_0}\left(\frac{u^2}{2} + \frac{1}{12}\frac{\partial^2 u}{\partial t_0^2}\right). \tag{1.4}$$

The parameter ε keeps track of genus in the expansion of $\mathcal{F}^{\mathrm{WK}}$, but it does not appear in the differential equations of the KdV hierarchy. Thus, to compare the WK potential with a solution of KdV, one specializes to $\varepsilon = 1$:

$$\overline{\mathcal{F}}^{\mathrm{WK}}(\mathbf{t}) := \mathcal{F}^{\mathrm{WK}}(\mathbf{t}; 1) = \sum_{g \geq 0} F_g(\mathbf{t}).$$

All derivatives in the KdV hierarchy are taken with respect to $\overline{\mathcal{F}}^{\mathrm{WK}}$.

Theorem 1.5 ([Wit91; Kon92])**.** *Let*

$$u = \frac{\partial^2 \overline{\mathcal{F}}^{\mathrm{WK}}}{\partial t_0^2}.$$

Then u satisfies the KdV equation (1.4)*. Moreover, together with the string and dilaton equations, $\overline{\mathcal{F}}^{\mathrm{WK}}$ satisfies the entire KdV hierarchy.*

This theorem provides a complete recursive method for computing all descendant correlators; see [Zvo14, p. 60] for explicit recursion formulas.

1.4 Gromov–Witten theory

A celebrated example in enumerative geometry is Kontsevich's formula for rational plane curves, which gives a recursive answer to the question: *How many plane rational curves of degree d pass through $3d-1$ points in general position?* This formula arises as a byproduct of Kontsevich and Manin's [KM94] development of Gromov–Witten theory. Beyond this example, Gromov–Witten theory provides the mathematical foundation for the well-known mirror conjecture. In this section, we review relevant aspects of Gromov–Witten theory. Standard references include [KM94; GP98; CK99] for algebraic Gromov–Witten classes, and [CI15; CI18] for Gromov–Witten potentials.

1.4.1 Moduli space of stable maps

Let X be a smooth complex projective variety and let $d \in H^2(X, \mathbb{Z})$. A tuple $(C, p_1, \dots, p_n, f)$ is called a *stable map* if:

- $(C, p_1, \dots, p_n)$ is a nodal, genus-g, compact Riemann surface with n marked points.
- $f : C \to X$ is a morphism such that every irreducible component $\widetilde{C}$ of C satisfies one of the following:
    i. $\widetilde{C}$ has genus 0, at least three special points (marked or nodes), and is contracted (i.e., $f|_{\widetilde{C}}$ is constant).
    ii. $\widetilde{C}$ has genus 1, at least one special point, and is contracted.
    iii. $\widetilde{C}$ has genus at least 2, or it is not contracted.

A *stable map of class d* is a stable map $(C, p_1, \dots, p_n, f)$ such that $f_*([C]) = d$, where $[C] \in H_2(C, \mathbb{Z})$ is the fundamental class of C. If $d = [C']$ for a curve $C' \subset X$ and f is injective, then $f_*([C]) = d$ means that f is a parametrization of $C' = f(C)$.

In the degenerate case where X is zero-dimensional (i.e. a point), a stable map is the same as a stable curve. The *moduli space of stable maps* $\overline{M}_{g,n}(X, d)$ parametrizes stable maps $f : (C, p_1, \dots, p_n) \to X$ of degree $d \in H_2(X, \mathbb{Z})$, up to isomorphism. In particular,

$$X = \mathrm{pt},\ d = 0 \quad \Rightarrow \quad \overline{M}_{g,n}(\mathrm{pt}, 0) \simeq \overline{M}_{g,n},$$

and for general X, if $d = 0$ then all stable maps are constant, so

$$\overline{M}_{g,n}(X, 0) \simeq \overline{M}_{g,n} \times X.$$

In contrast with $\overline{M}_{g,n}$, the moduli space $\overline{M}_{g,n}(X, d)$ is in general *not* smooth and typically not an orbifold. Its deformation theory is governed by a perfect obstruction theory, and its "dimension" is captured by the *expected dimension*

$$\operatorname{edim} \overline{M}_{g,n}(X, d) = (1 - g)(\dim X - 3) - \int_d c_1(TX) + n,$$

which is always an integer. The actual geometric dimension of $\overline{M}_{g,n}(X, d)$ may differ from this expected value due to obstructions.

The obstruction theory determines the *virtual fundamental class*

$$[\overline{M}_{g,n}(X, d)]^{\mathrm{vir}} \in H_{2\cdot \mathrm{edim}}\big(\overline{M}_{g,n}(X, d)\big),$$

which plays the role of a fundamental class in Gromov–Witten theory. Unless otherwise stated, we work with rational cohomology.

Several important morphisms are associated with $\overline{M}_{g,n}(X, d)$:

- *Evaluation maps:* For each marked point $i = 1, \dots, n$, the evaluation map

$$\mathrm{ev}_i : \overline{M}_{g,n}(X, d) \longrightarrow X \tag{1.5}$$

  sends $(C, p_1, \dots, p_n, f) \mapsto f(p_i)$.

- *Forgetful morphisms:*
    - Forgetting the map:

$$\mu : \overline{M}_{g,n}(X, d) \longrightarrow \overline{M}_{g,n} \tag{1.6}$$

      discards f and stabilizes the resulting curve.
    - Forgetting a marked point:

$$\pi : \overline{M}_{g,n+1}(X, d) \longrightarrow \overline{M}_{g,n}(X, d) \tag{1.7}$$

      removes the $(n + 1)$-st marked point and stabilizes if necessary.

- *Boundary divisor maps:* These are special gluing maps corresponding to universal sections on $\overline{M}_{g,n+1}(X,d)$. For example,
$$\rho_i : \overline{M}_{g,n}(X,d) \longrightarrow \overline{M}_{g,n+1}(X,d) \tag{1.8}$$
glues a genus-0 component containing the i-th marked point, producing a boundary divisor (recall $\overline{M}_{0,3} \times \overline{M}_{g,n} \simeq \overline{M}_{g,n}$; see Figure 1.4).

1.4.2 Gromov–Witten classes

We now introduce the central object of interest, the *Gromov–Witten class*. For $g, n \geq 0$ and $d \in H_2(X, \mathbb{Z})$, it is defined as a $\mathbb{Q}$-multilinear map

$$\begin{aligned} I_{g,n,d} : (H^*(X))^{\otimes n} &\longrightarrow H^*(\overline{M}_{g,n}), \\ \gamma_1 \otimes \cdots \otimes \gamma_n &\longmapsto I_{g,n,d}(\gamma_1, \ldots, \gamma_n), \end{aligned} \tag{1.9}$$

which may be regarded, via the Künneth isomorphism, as an element of

$$I_{g,n,d} \in H^*(\overline{M}_{g,n}) \otimes \left((H^*(X))^*\right)^{\otimes n}.$$

The class is defined by the formula

$$I_{g,n,d}(\gamma_1, \ldots, \gamma_n) = \mu_* \left(\mathrm{ev}_1^*(\gamma_1) \cdots \mathrm{ev}_n^*(\gamma_n)[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}\right), \tag{1.10}$$

where $\mu : \overline{M}_{g,n}(X,d) \to \overline{M}_{g,n}$ is the forgetful map, $\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \to X$ are the evaluation morphisms, and μ_* denotes the pushforward in cohomology (using Poincaré duality).

The associated *primary Gromov–Witten invariants* are

$$\langle I_{g,n,d}(\gamma_1, \ldots, \gamma_n)\rangle = \int_{\overline{M}_{g,n}} I_{g,n,d}(\gamma_1, \ldots, \gamma_n). \tag{1.11}$$

Descendants and ψ-classes. For each marked point i, let $\mathcal{L}_i$ be the i-th *cotangent line bundle* over $\overline{M}_{g,n}(X,d)$, and let

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{M}_{g,n}(X,d))$$

be its first Chern class (the *i-th ψ-class*). The *descendant Gromov–Witten invariants* are defined by

$$\langle \tau_{i_1}(\gamma_1) \cdots \tau_{i_n}(\gamma_n)\rangle^X_{g,n,d} = \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\, \psi_1^{i_1} \cdots \mathrm{ev}_n^*(\gamma_n)\, \psi_n^{i_n}, \tag{1.12}$$

where $\gamma_k \in H^*(X)$ and $i_k \in \mathbb{Z}_{\geq 0}$ for $k = 1, \ldots, n$.

These classes satisfy a number of important properties, which can be found in [CK99, p. 191]. In the next section, we will see that cohomological field theories generalize Gromov–Witten classes, retaining several of their key properties.

Remark 1. In general, computing Gromov–Witten invariants for a given variety X is a highly non-trivial problem. A celebrated exception is Kontsevich's formula for enumerating degree-d rational curves in $\mathbb{P}^2$, passing through $3d - 1$ points in general position (see [CK99, p. 196]). A powerful approach to extracting further information about these invariants is to package them into *generating functions*, particularly those encoding descendant Gromov–Witten invariants. This perspective, inspired by Witten's conjecture [Wit91], reveals deep connections between Gromov–Witten theory and integrable hierarchies, providing recursive structures that can be exploited in explicit computations.

1.4.3 Gromov–Witten potentials

We follow the notation and conventions of [CI15; CI18]. Recall that for a smooth projective variety X, we have the moduli space $\overline{M}_{g,n}(X,d)$ of n-pointed, genus-g stable maps to X of degree $d \in H_2(X,\mathbb{Z})$. The *Gromov–Witten descendant invariants* are defined by

$$\langle \tau_{i_1}(\gamma_1)\cdots\tau_{i_n}(\gamma_n)\rangle^X_{g,n,d} = \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1)\psi_1^{i_1}\cdots \mathrm{ev}_n^*(\gamma_n)\psi_n^{i_n}, \tag{1.13}$$

where $\gamma_k \in H^*(X,\mathbb{Q})$, $\mathrm{ev}_k : \overline{M}_{g,n}(X,d) \to X$ are the evaluation maps, $\psi_k \in H^2(\overline{M}_{g,n}(X,d),\mathbb{Q})$ are the cotangent line classes, and $i_k \in \mathbb{Z}_{\geq 0}$.

Let H_X be the even part of $H^*(X,\mathbb{Q})$, a $\mathbb{Q}$-vector space of rank N. Choose dual bases $(\phi_i)_{i=0}^N$ and $(\phi^j)_{j=0}^N$ of H_X such that:

i. $\phi_0 = \mathbf{1} \in H_X$ is the identity;

ii. $(\phi_i)_{i=1}^r$ is a $\mathbb{Z}$-basis for $H^2(X,\mathbb{Z}) \subset H_X$;

iii. Each ϕ_i is homogeneous;

iv. They are dual with respect to the Poincaré pairing $\langle\cdot,\cdot\rangle : H_X \otimes H_X \to \mathbb{Q}$, i.e., $\langle \phi_i, \phi_j\rangle = \eta_{ij} = \int_X \phi_i\phi_j$ and $\phi^i = \sum_j \eta^{ij}\phi_j$, where (η^{ij}) is the inverse of (η_{ij}).

Novikov ring. Let $r = \mathrm{rk}\, H^2(X,\mathbb{Z})$ and define the *Novikov ring*:

$$\Lambda = \mathbb{Q}[\![q_1,\ldots,q_r]\!] = \left\{ \sum_{d\in H^2(X,\mathbb{Z})} a_d q^d \,\middle|\, a_d \in \mathbb{Q},\ \ |\{d : a_d \neq 0\}| < \infty \right\}, \tag{1.14}$$

where $q^d = q_1^{d_1}\cdots q_r^{d_r}$ if $d = \sum_{i=1}^r d_i\phi_i \in H^2(X,\mathbb{Z})$. Let $E(X) \subset H^2(X,\mathbb{Z})$ be the semigroup of effective 2-cycles in X; the variables q^d are taken in the semigroup ring of $E(X)$.

Genus-zero potential. For $t \in H_X$, write $t = \sum_i t^i\phi_i$. The *genus-zero Gromov–Witten potential* is the formal series

$$F_0^X = \sum_{d\in E(X)}\sum_{n\geq 0} \frac{q^d}{n!}\langle \tau_0(t)\cdots\tau_0(t)\rangle^X_{0,n,d} \in \Lambda[\![t^0,\ldots,t^N]\!]. \tag{1.15}$$

Expanding t in the chosen basis gives

$$\langle \tau_0(t)^n\rangle^X_{0,n,d} = \int_{[\overline{M}_{0,n}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(t)\cdots\mathrm{ev}_n^*(t) \tag{1.16}$$

$$= \sum_{i_1,\ldots,i_n} t^{i_1}\cdots t^{i_n}\int_{[\overline{M}_{0,n}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_{i_1})\cdots\mathrm{ev}_n^*(\phi_{i_n}). \tag{1.17}$$

Quantum product. Define the *(big) quantum product* on H_X by

$$\phi_i\cdot\phi_j = \sum_{k=0}^N \frac{\partial^3 F_0^X}{\partial t^i\,\partial t^j\,\partial t^k}\phi^k. \tag{1.18}$$

This product is bilinear over Λ and defines a formal family of algebras $H_X\otimes\Lambda$ parametrized by (t^i). The structure $(H_X, \eta, \cdot, \mathbf{1})$ is called the *big quantum cohomology*.

Proposition 1.6. We have $F_0^X \in [\![t^0, q_1e^{t^1},\ldots,q_re^{t^r}, t^{r+1},\ldots,t^N]\!]$.

Remark 2 (Convergence at genus zero)**.** In many cases, F_0^X converges to an analytic function

$$F_0^X \in \mathbb{Q}\{t^0, e^{t^1}, \dots, e^{t^r}, t^{r+1}, \dots, t^N\}$$

after setting $q_1 = \cdots = q_r = 1$. In this case, the quantum product defines a genuine analytic family of algebras H_X parametrized by (t^i).

Higher genus potentials. Let

$$\mathbf{t} = (t_0, t_1, \dots) \in H_X^{\oplus\infty} := \bigoplus_{i=0}^{\infty} H_X$$

denote a sequence of elements of H_X with *finite* support, where each

$$t_i = \sum_{\alpha=0}^{N} t_i^\alpha \, \phi_\alpha$$

is expanded in the fixed homogeneous basis $\{\phi_\alpha\}_{\alpha=0}^N$ of H_X. While individual correlators involve only finitely many insertions, generating functions such as the Gromov–Witten potentials naturally require allowing infinitely many nonzero t_i. In that setting, one replaces the direct sum by the completed product

$$H_X[\![z]\!] := \prod_{i=0}^{\infty} H_X \, z^i,$$

the space of formal H_X-valued power series in z. We adopt this completed framework when writing $\Lambda[\![\mathbf{t}]\!]$ below, meaning the ring of formal power series in the coordinates $\{t_i^\alpha\}$, completed with respect to the grading $v(t_i^\alpha) = i + 1$.

The *genus-g Gromov–Witten potential* is

$$F_g^X = \sum_{d \in E(X)} \sum_{n, i_* \geq 0} \frac{q^d}{n!} \langle \tau_{i_1}(t_{i_1}) \cdots \tau_{i_n}(t_{i_n}) \rangle_{g,n,d}^X \;\in\; \Lambda[\![\mathbf{t}]\!], \tag{1.19}$$

and the *total descendant potential* is

$$\mathcal{F}^X = \sum_{g=0}^{\infty} \varepsilon^{2g} \, F_g^X. \tag{1.20}$$

Proposition 1.7. We have $\mathcal{F}^X \in \Lambda[\![\mathbf{t}, \varepsilon]\!]$, where the valuation is given by $v(t_i^\alpha) = i + 1$.

Ancestor potentials. Let $\pi_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ forget the last m marked points, $\rho : \overline{M}_{g,n+m}(X,d) \to \overline{M}_{g,n+m}$ forget the map f, and define $\overline{\psi}_i = (\pi_m \circ \rho)^* \psi_i$. The *Gromov–Witten ancestor invariants* are

$$\langle \overline{\tau}_{i_1}(\gamma_1) \cdots \overline{\tau}_{i_n}(\gamma_n); \tau_0(\beta_1) \cdots \tau_0(\beta_m) \rangle_{g,n+m,d}^X \tag{1.21}$$

$$= \int_{[\overline{M}_{g,n+m}(X,d)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \overline{\psi}_1^{i_1} \cdots \mathrm{ev}_n^*(\gamma_n) \overline{\psi}_n^{i_n} \, \mathrm{ev}_{n+1}^*(\beta_1) \cdots \mathrm{ev}_{n+m}^*(\beta_m), \tag{1.22}$$

where $\gamma_k, \beta_j \in H_X$.

Let $\mathbf{s} = (s_0, s_1, \dots) \in H_X^{\oplus\infty}$ and $t \in H_X$. The *genus-g ancestor potential* is

$$\overline{F}_g^X = \sum_{d \in E(X)} \sum_{n,m,i_* \geq 0} \frac{q^d}{n!m!} \langle \overline{\tau}_{i_1}(s_{i_1}) \cdots \overline{\tau}_{i_n}(s_{i_n}); \tau_0(t)^m \rangle_{g,n+m,d}^X. \tag{1.23}$$

The *total ancestor potential* is

$$\mathcal{A}^X = \sum_{g=0}^{\infty} \varepsilon^{2g} \overline{F}_g^X. \tag{1.24}$$

Proposition 1.8. We have $\mathcal{A}^X \in \Lambda[\![\mathbf{s}, t, \varepsilon]\!]$.

Dilaton shift. Let $\mathbf{q} = (q_0, q_1, \dots) \in H_X^{\oplus\infty}$ and regard $\mathbf{q}(z) = \sum_{i\geq 0} \sum_{\alpha=0}^{N} q_i^\alpha \phi_\alpha z^i \in H_X[\![z]\!]$. Similarly, write $\mathbf{t}(z) = \sum_{i\geq 0} \sum_{\alpha=0}^{N} t_i^\alpha \phi_\alpha z^i$. The affine coordinate transformation

$$t_i^\alpha = q_i^\alpha + \delta^{\alpha 0}\delta_{i1} \quad \Leftrightarrow \quad \mathbf{t}(z) = \mathbf{q}(z) + \phi_0 z \tag{1.25}$$

is called the *dilaton shift*. It re-centres the descendant potential around the formal neighbourhood of $-\phi_0 z \in H_X[\![z]\!]$.

1.5 Cohomological field theory

Cohomological field theories (CohFTs) are families of cohomology classes on moduli spaces of curves satisfying natural compatibility conditions, inspired by the structure of the tautological ring and the properties of Gromov–Witten classes. They can be viewed as a natural generalization of Gromov–Witten theory. The notion of a CohFT was introduced by Kontsevich and Manin [KM94]. In many cases, the genus-zero sector of a CohFT determines a Frobenius manifold structure.

1.5.1 CohFT correlators

Let V be a finite-dimensional $\mathbb{C}$-vector space with basis $\{e_1, \dots, e_N\}$, where e_1 is the distinguished unit element. Equip V with a nondegenerate symmetric bilinear form $\eta\colon V \times V \to \mathbb{C}$.

A *cohomological field theory* is a family of maps

$$c_{g,n}\colon V^{\otimes n} \longrightarrow H^*(\overline{M}_{g,n}, \mathbb{C}), \qquad v_1 \otimes \cdots \otimes v_n \longmapsto c_{g,n}(v_1, \dots, v_n), \tag{1.26}$$

defined for all $2g - 2 + n > 0$, such that $c_{g,n}$ satisfies the following axioms:

i. Each class $c_{g,n}$ lies in the even cohomology.

ii. The classes are graded-symmetric under the action of the symmetric group S_n, acting simultaneously on $V^{\otimes n}$ and on $\overline{M}_{g,n}$ by relabeling the marked points. For $s \in S_n$, let $\sigma_s\colon \overline{M}_{g,n} \to \overline{M}_{g,n}$ be the induced isomorphism. Then

$$c_{g,n}(v_{s(1)}, \dots, v_{s(n)}) = (\sigma_s^{-1})^* \, c_{g,n}(v_1, \dots, v_n).$$

iii. For the gluing morphism

$$\rho\colon \overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \longrightarrow \overline{M}_{g, n},$$

where $g = g_1 + g_2$ and $n = n_1 + n_2$, let S_1 (resp. S_2) index the markings lying on the first (resp. second) component, with $S_1 \sqcup S_2 = \{1, \dots, n\}$. Then

$$\rho^*(c_{g,n}(v_1, \dots, v_n)) = c_{g_1,n_1+1}\big(e_\alpha \otimes \bigotimes_{i\in S_1} v_i\big)\eta^{\alpha\beta} c_{g_2,n_2+1}\big(e_\beta \otimes \bigotimes_{i\in S_2} v_i\big),$$

where $\eta^{\alpha\beta}$ is the inverse matrix to $\eta_{\alpha\beta} := \eta(e_\alpha, e_\beta)$.

iv. For the morphism

$$\sigma\colon \overline{M}_{g-1, n+2} \longrightarrow \overline{M}_{g, n},$$

we require

$$\sigma^* c_{g,n}(v_1,\dots,v_n) = c_{g-1,n+2}(v_1,\dots,v_n,e_\alpha,e_\beta)\eta^{\alpha\beta}.$$

v. For the forgetful morphism

$$\pi\colon \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n},$$

we have

$$\pi^* c_{g,n}(v_1,\dots,v_n) = c_{g,n+1}(v_1,\dots,v_n,e_1),$$

and the metric η is given by

$$\eta(v_1,v_2) = c_{0,3}(v_1,v_2,e_1).$$

The *descendant correlators* of a CohFT are defined by

$$\langle \tau_{i_1}(e_{\alpha_1})\cdots\tau_{i_n}(e_{\alpha_n})\rangle_{g,n} := \int_{\overline{M}_{g,n}} c_{g,n}(e_{\alpha_1},\dots,e_{\alpha_n})\psi_1^{i_1}\cdots\psi_n^{i_n}, \tag{1.27}$$

where $\psi_k \in H^2(\overline{M}_{g,n},\mathbb{C})$ is the cotangent-line class at the k-th marking.

A *topological field theory (TFT)* is a triple (V,η,ω) where $\omega = (\omega_{g,n})_{2g-2+n>0}$ consists of the degree-zero components of a CohFT:

$$\omega_{g,n} := [c_{g,n}]^0 \in H^0(\overline{M}_{g,n},\mathbb{C}) \otimes (V^*)^{\otimes n}.$$

If the underlying CohFT has a unit, we say the TFT also has a unit.

Example 4.

i. The *trivial TFT*: $V = \mathbb{C}$, $\eta = 1$, and $\omega_{g,n} = 1$. The Witten–Kontsevich intersection numbers define a trivial CohFT.

ii. The classes $\Omega_{g,n} := \exp(2\pi^2\kappa_1)$ define a CohFT, which has been used in the study of Weil–Petersson volumes.

iii. For any smooth projective variety X, the Gromov–Witten classes $I_{g,n,d}$ form a canonical example of a CohFT.

iv. Witten's r-spin theory defines a CohFT known as the *Witten r-spin class*. The genus-zero sector was constructed in [DM93]; the higher-genus theory was later computed in [PPZ15] via Givental's formalism.

1.5.2 CohFT potentials

Let (V,η,c) be a CohFT, and fix a basis $\{e_\alpha\}_{\alpha=0}^N$ of V. We equip

$$V[\![z]\!] := V \otimes \mathbb{C}[\![z]\!]$$

with coordinates

$$\mathbf{q} = \big(q_i^\alpha \in \mathbb{C} \,:\, i \geq 0,\ 0 \leq \alpha \leq N\big).$$

For a fixed $\delta \in V$, we define the *dilaton-shifted coordinates*

$$\mathbf{t}(z) := \mathbf{q}(z) + \delta\, z.$$

We write $\mathbb{C}[\![\mathbf{t}]\!]$ for the ring of formal power series in the t_i^α with valuation

$$v(t_i^\alpha) := i+1.$$

The *descendant correlators* of c are

$$\langle \tau_{i_1}(e_{\alpha_1}) \cdots \tau_{i_n}(e_{\alpha_n}) \rangle_{g,n} := \int_{\overline{M}_{g,n}} c_{g,n}(e_{\alpha_1}, \dots, e_{\alpha_n}) \psi_1^{i_1} \cdots \psi_n^{i_n},$$

where $\psi_k \in H^2(\overline{M}_{g,n}, \mathbb{C})$ is the cotangent-line class at the k-th marking.

The *genus-g CohFT potential* is

$$F_g := \sum_{n, i_* \geq 0} \sum_{\alpha_*=0}^{N} \frac{\langle \tau_{i_1}(e_{\alpha_1}) \cdots \tau_{i_n}(e_{\alpha_n}) \rangle_{g,n}}{n!} t_{i_1}^{\alpha_1} \cdots t_{i_n}^{\alpha_n} \quad \in \quad \mathbb{C}[\![\mathbf{t}]\!].$$

We assume F_0 is analytic in the variables t_0^α.

The *total descendant potential* is

$$\mathcal{F} := \sum_{g=0}^{\infty} \varepsilon^{2g} F_g. \tag{1.28}$$

With respect to the dilaton shift $\mathbf{t}(z) = \mathbf{q}(z) + \delta z$, the function $\mathcal{F}$ is defined on a formal neighbourhood of the point $-\delta z \in V[\![z]\!]$.

We say that $\mathcal{F}$ is *tame* if $F_g \in \mathbb{C}[\![\mathbf{t}]\!]$ satisfies

$$\frac{\partial^n}{\partial t_{i_1}^{\alpha_1} \cdots \partial t_{i_n}^{\alpha_n}}\bigg|_{\mathbf{t}=0} F_g = 0 \quad \text{whenever} \quad i_1 + \cdots + i_n > 3g - 3 + n, \tag{1.29}$$

The tameness condition (1.29) is a direct expression of the dimension constraint on $\overline{M}_{g,n}$, which forces correlators to vanish whenever the total ψ-degree exceeds $\dim \overline{M}_{g,n} = 3g - 3 + n$. As a consequence, if $\mathcal{F}$ is tame, then for every $g \geq 1$ the potential F_g depends only on descendants of order at most $3g - 2$. Equivalently, F_g is independent of the variables t_i^α with $i \geq 3g - 1$, and its dependence on t_{3g-2}^α is at most linear. This finite–jet consequence is usually referred to as the $3g-2$ *jet property*. In particular, for each fixed g, F_g involves only finitely many of the variables t_i^α, and thus lies in $\mathbb{C}[\![\mathbf{t}]\!]$.

Proposition 1.9. We have $\mathcal{F} \in \mathbb{C}[\![\mathbf{t}, \varepsilon]\!]$.

The *Fock space* associated to (V, δ) is

$$\mathfrak{Fock}(V, \delta) := \left\{ \mathcal{F} \in \mathbb{C}[\![\mathbf{t}, \varepsilon]\!] \;\middle|\; \mathcal{F} \text{ associated to a tame CohFT} \right\}.$$

Proposition 1.10. Let (V, η, c) be a CohFT. If $\mathcal{F}$ is tame and F_0 satisfies the *string equation*

$$\frac{\partial F_0}{\partial t_0^1} = \frac{1}{2} \sum_{\alpha,\beta=0}^{N} \eta_{\alpha\beta}\, t_0^\alpha t_0^\beta + \sum_{i \geq 1} \sum_{\alpha=0}^{N} t_{i+1}^\alpha \frac{\partial F_0}{\partial t_i^\alpha},$$

then $F_0|_{\{t_i^\alpha = 0\ \forall i > 0\}}$ defines the potential function of a Frobenius manifold structure on a neighbourhood of $0 \in V$.

1.6 Frobenius structure

Frobenius manifolds are smooth manifolds encoding integrability structures, developed in Dubrovin's program, with deep connections to Gromov–Witten theory. In many cases, the genus-zero generating function of a Gromov–Witten theory or a cohomological field theory defines a Frobenius manifold; conversely, given a Frobenius manifold, one can reconstruct such a theory. We follow [Dub96].

1.6.1 Frobenius algebras

A *(commutative) Frobenius algebra* over $\mathbb{C}$ is a tuple $(V, \eta, \cdot, \mathbf{1})$, where $(V, \cdot, \mathbf{1})$ is a $\mathbb{C}$-algebra and η is a non-degenerate symmetric bilinear form satisfying $\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$.

Any CohFT (V, η, c) defines a *quantum product* by

$$\eta(v_1 \cdot v_2, v_3) = c_{0,3}(v_1, v_2, v_3).$$

If the CohFT has a unit $\mathbf{1}$, then $\eta(v_1, v_2) = c_{0,3}(v_1, v_2, \mathbf{1})$, so $\mathbf{1}$ is the identity for $\cdot$. A TFT with quantum product is a commutative Frobenius algebra, and conversely, a Frobenius algebra defines a TFT whose classes are determined from the product via pullback relations.

1.6.2 Frobenius manifolds

A *Frobenius manifold* is a tuple $(M, \eta, \cdot, \mathbf{1})$, where M is a smooth complex manifold and $(T_pM, \eta_p, \cdot|_p, \mathbf{1}_p)$ is a commutative Frobenius algebra for each $p \in M$, such that:

i. η is flat.

ii. The unit vector field $\mathbf{1}$ is η-flat.

iii. The tensor $c(X, Y, Z) = \eta(X \cdot Y, Z)$ is such that $(\nabla_W c)(X, Y, Z)$ is symmetric in all four entries.

The *Dubrovin connection* is defined by

$$\nabla_{z,X} Y = \nabla_X Y + \frac{1}{z} X \cdot Y.$$

If η is flat, there exist local *flat coordinates* $(t^\alpha)_{\alpha=1}^n$ with $\partial_{t^1} = \mathbf{1}$, and a holomorphic *potential* F such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}.$$

The following are equivalent:

i. $\cdot$ is associative.

ii. F satisfies the *Witten–Dijkgraaf–Verlinde–Verlinde (WDVV)* equations

$$\sum_{\mu,\nu} c_{\alpha\beta\mu} \eta^{\mu\nu} c_{\nu\gamma\kappa} = \sum_{\mu,\nu} c_{\alpha\gamma\mu} \eta^{\mu\nu} c_{\nu\beta\kappa}.$$

iii. ∇_z is flat for all z.

An *Euler vector field* E acts conformally on η and rescales the product, with $\nabla(\nabla E) = 0$:

$$\mathcal{L}_E(\eta) = \alpha\, \eta, \quad \mathcal{L}_E(\cdot) = \beta\, \cdot, \quad \mathcal{L}_E(\mathbf{1}) = \gamma\, \mathbf{1}.$$

A *conformal Frobenius manifold* is one with an Euler vector field, which in flat coordinates takes the form

$$E = \sum_{i=1}^n (\alpha_i t^i + \beta_i) \partial_{t^i}, \quad \mathcal{L}_E(\eta) = (2 - \delta)\eta,$$

where δ is the *conformal dimension*.

1.6.3 Semisimple Frobenius manifolds

A *semisimple Frobenius algebra* is one whose underlying algebra has vanishing Jacobson radical. A CohFT is *semisimple* if its quantum product is semisimple at a generic point. In this case, for an n-dimensional vector space V there exists a basis of *idempotents* $(\phi_i)_{i=1}^n$ such that

$$\phi_i \cdot \phi_j = \delta_{ij}\,\phi_i, \qquad \eta(\phi_i, \phi_j) = \Delta_i^{-1}\,\delta_{ij},$$

for some nonzero scalars $\Delta_i \in \mathbb{C} \setminus \{0\}$ (the *norms* of the idempotents).

A *semisimple point* $p \in M$ is one where T_pM is semisimple; M is *semisimple* if such points are dense. For a semisimple M, there are local *canonical coordinates* (u^i) with orthogonal idempotent vector fields

$$\partial_{u^i} \cdot \partial_{u^j} = \delta_{ij}\,\partial_{u^i}, \qquad \eta(\partial_{u^i}, \partial_{u^j}) = 0 \ \ (i \neq j).$$

If $\Delta^i := \eta(\partial_{u^i}, \partial_{u^i})^{-1}$, then

$$\eta = \sum_{i=1}^{n} (\Delta^i)^{-1}\, du^i \otimes du^i, \qquad c = \sum_{i=1}^{n} (\Delta^i)^{-1}\, du^i \otimes du^i \otimes du^i.$$

If M is conformal, then in normalized canonical coordinates

$$E = \sum_{i=1}^{n} u^i\,\partial_{u^i}.$$

The *orthonormal frame* (∂_{v^i}) is given by $\partial_{v^i} = \sqrt{\Delta^i}\,\partial_{u^i}$, with $\eta(\partial_{v^i}, \partial_{v^j}) = \delta_{ij}$. The *transition matrix* Ψ between flat and normalized canonical frames satisfies

$$\partial_{t^\alpha} = \sum_{i=1}^{n} \Psi^i_\alpha \partial_{v^i}, \quad \Psi^i_\alpha = \eta(\partial_{v^i}, \partial_{t^\alpha}).$$

For a semisimple point u, the equation $\nabla_z S = 0$ has a fundamental solution $S(z) = \Psi R(z) e^{U/z}$, where $U = \mathrm{diag}(u^1, \ldots, u^n)$, Ψ is the transition matrix, and $R(z) = \sum_{i \geq 0} R_i z^i$ satisfies $R_0 = \mathrm{Id}$ and $R^*(-z)R(z) = \mathrm{Id}$. The series $R(z)$ is unique up to right multiplication by $\exp(\sum_{i\geq 0} a_{2i+1} z^{2i+1})$ with diagonal a_{2i+1}. If M is conformal, $R(z)$ is uniquely fixed by the homogeneity condition

$$(z\partial_z + \sum_i u^i \partial_{u^i}) R(z) = 0.$$

1.7 Givental–Teleman's reconstruction formalism

We present Givental's group action from two complementary perspectives, the differential operator approach [Giv01b; Giv01a; Giv02; Giv04] and the graph-theoretic approach [PPZ15]. As this material is now classical, we omit proofs and refer to the literature for details.

The Givental action reconstructs the higher-genus part of a Gromov–Witten potential from its genus-zero data, which is encoded in the structure of a semisimple Frobenius manifold. Semisimplicity ensures the existence of an orthogonal idempotent coordinate system, with multiplication coming from quantum cohomology. In this framework, the action is expressed via geometric quantization and differential operators, and Teleman's theorem [Tel12] shows that it applies in full generality to semisimple cohomological field theories.

1.7.1 Quantization of symplectic transformations

Let V be a $\mathbb{C}$-vector space with basis $(\phi_\alpha)_{\alpha=0}^N$, equipped with a non-degenerate symmetric bilinear form $\eta : V \times V \to \mathbb{C}$. Let $(\phi^\alpha)_{\alpha=0}^N$ be the dual basis with respect to η. Define

$$\mathcal{V} = V[z, z^{-1}] = \left\{ \sum_{i\in\mathbb{Z}} a_i z^i \;\middle|\; a_i \in V, \text{ finitely many } a_i \neq 0 \right\}.$$

Givental's symplectic form. The *Givental symplectic form* ω on $\mathcal{V}$ is

$$\omega(f,g) = \frac{1}{2\pi i} \oint_{S^1} \eta(f(-z), g(z))\, dz, \quad f, g \in \mathcal{V}.$$

The symplectic vector space $(\mathcal{V}, \omega)$ is called the *Givental space*.

Polarization. We polarize $\mathcal{V}$ as

$$\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_- \simeq T^*\mathcal{V}_+,$$

where

$$\mathcal{V}_+ = V[z] = \left\{ \mathbf{q}(z) = \sum_{i\geq 0} \left(\sum_{\alpha=0}^N q_i^\alpha \phi_\alpha \right) z^i \right\},$$
$$\mathcal{V}_- = z^{-1}V[z^{-1}] = \left\{ \mathbf{p}(z) = \sum_{i\geq 0} \left(\sum_{\alpha=0}^N p_i^\alpha \phi^\alpha \right) (-z)^{-i-1} \right\}.$$

Both $\mathcal{V}_+$ and $\mathcal{V}_-$ are Lagrangian with respect to ω. The coordinates (q_i^α, p_i^α) are *Darboux coordinates*, since

$$\omega = \sum_{i\geq 0} \sum_{\alpha=0}^N dp_i^\alpha \wedge dq_i^\alpha.$$

An element $f \in \mathcal{V}$ is written as

$$f(z) = \sum_{i\geq 0} q_i z^i + \sum_{i\geq 0} p_i (-z)^{-i-1}.$$

Infinitesimal symplectic transformations. Let $\mathrm{End}(\mathcal{V})$ be the space of $\mathbb{C}$-linear maps $\mathcal{V} \to \mathcal{V}$. A map $A \in \mathrm{End}(\mathcal{V})$ has the form

$$A(z) = \sum_{i\in\mathbb{Z}} A_i z^i, \quad A_i \in \mathrm{End}(V).$$

It is *infinitesimal symplectic* if

$$\eta((Af)(-z), g(z)) + \eta(f(-z), (Ag)(z)) = 0,$$

equivalently,

$$\omega(Af, g) + \omega(f, Ag) = 0.$$

Denote the space of such maps by

$$\mathcal{S} = \{ A \in \mathrm{End}(V)[z, z^{-1}] \mid \omega(Af, g) + \omega(f, Ag) = 0 \}.$$

With the commutator $[A, B] = A \circ B - B \circ A$, $(\mathcal{S}, [\cdot, \cdot])$ is a Lie algebra.

Quadratic polynomials and Poisson bracket. Let $\mathcal{P}$ be the space of quadratic polynomials in $(\mathbf{p}, \mathbf{q})$:

$$\mathcal{P} = \left\{ \sum \left(a^{ij}_{\alpha\beta} p^\alpha_i p^\beta_j + b^{ij}_{\alpha\beta} q^\alpha_i q^\beta_j + c^{ij}_{\alpha\beta} p^\alpha_i q^\beta_j \right) \mid a^{ij}_{\alpha\beta}, b^{ij}_{\alpha\beta}, c^{ij}_{\alpha\beta} \in \mathbb{C} \right\}.$$

Equip $\mathcal{P}$ with the Poisson bracket

$$\{f, g\} = \sum_{i\geq 0} \sum_{\alpha=0}^{N} \frac{\partial f}{\partial p^\alpha_i} \frac{\partial g}{\partial q^\alpha_i} - \frac{\partial g}{\partial p^\alpha_i} \frac{\partial f}{\partial q^\alpha_i}.$$

The map $\mathcal{S} \to \mathcal{P}$, $A \mapsto H_A$ defined by

$$H_A = \frac{1}{2}\,\omega(A\cdot, \cdot), \quad H_A(f) = \frac{1}{2}\omega(Af, f)$$

is a Lie algebra isomorphism:

$$H_{[A,B]} = \{H_A, H_B\}.$$

Quantization. The *quantization* map $\widehat{\cdot} : \mathcal{S} \to \widehat{\mathcal{S}}$ is given by

$$A \mapsto H_A \mapsto \hat{A},$$

where $\hat{A}$ is obtained from H_A via the rules:

$$p^\alpha_i p^\beta_j \mapsto \varepsilon \sum_{\tilde{\alpha}, \tilde{\beta}} \eta^{\alpha\tilde{\alpha}} \eta^{\beta\tilde{\beta}} \partial q^{\tilde{\alpha}}_i \partial q^{\tilde{\beta}}_j, \qquad q^\alpha_i q^\beta_j \mapsto \varepsilon^{-1} q^\alpha_i q^\beta_j,$$
$$p^\alpha_i q^\beta_j \mapsto q^\beta_j \sum_{\tilde{\alpha}} \eta^{\alpha\tilde{\alpha}} \partial q^{\tilde{\alpha}}_i, \qquad q^\beta_j p^\alpha_i \mapsto q^\beta_j \sum_{\tilde{\alpha}} \eta^{\alpha\tilde{\alpha}} \partial q^{\tilde{\alpha}}_i.$$

Then $(\widehat{\mathcal{S}}, [\cdot, \cdot])$ is a Lie algebra, and

$$\{H_A, H_B\}^\wedge = [\hat{A}, \hat{B}] + \mathcal{C}(H_A, H_B),$$

where the cocycle $\mathcal{C}$ satisfies

$$\mathcal{C}(p^\alpha_i p^\beta_j, q^\alpha_i q^\beta_j) = 1 + \delta^{\alpha\beta}\delta_{ij}, \quad -\mathcal{C}(q^\alpha_i q^\beta_j, p^\alpha_i p^\beta_j) = 1 + \delta^{\alpha\beta}\delta_{ij},$$

and $\mathcal{C} = 0$ for all other quadratic Darboux monomials.

1.7.2 Action via differential operators

We reuse the notation from the previous section: $(\mathcal{V}, \omega)$ is the Givental space, and (V, η) is an $(N+1)$-dimensional $\mathbb{C}$–vector space equipped with a non-degenerate symmetric bilinear form. Let $(\phi^\alpha)_{\alpha=0}^N$ be the basis of V dual to $(\phi_\alpha)_{\alpha=0}^N$ with respect to η.

Twisted loop group. The *twisted loop group* is

$$L^{(2)}GL(V) = \left\{ M(z) = \sum_{i\in\mathbb{Z}} M_i z^i \in \mathrm{End}(V)((z)) \;\middle|\; M^\dagger(-z)M(z) = \mathrm{Id} \right\},$$

where $M_i \in \mathrm{End}(V)$, the adjoint is defined by

$$M^\dagger(z) = \eta^{-1} M^t(z) \eta,$$

and M^t denotes the transpose. The condition $M^\dagger(-z)M(z) = \mathrm{Id}$ is the *symplectic (unitary) condition*, ensuring that M defines a symplectic transformation on $\mathcal{V}$.

Triangular subgroups. The group $L^{(2)}GL(V)$ is generated by:

i. *Upper triangular subgroup*

$$G_+ = \left\{ R(z) = \mathrm{Id} + \sum_{i>0} R_i z^i \in \mathrm{End}(V)[\![z]\!] \;\middle|\; R^\dagger(-z)R(z) = \mathrm{Id} \right\}.$$

ii. *Lower triangular subgroup*

$$G_- = \left\{ S(z) = \mathrm{Id} + \sum_{i>0} S_i z^{-i} \in \mathrm{End}(V)[\![z^{-1}]\!] \;\middle|\; S^\dagger(-z)S(z) = \mathrm{Id} \right\}.$$

Infinitesimal symplectic transformations. If $R \in G_+$, write

$$R = \exp(\log R), \quad r := \log R.$$

The symplectic condition implies $r^\dagger(-z) + r(z) = 0$, so r is an infinitesimal symplectic transformation. The same applies to $S \in G_-$.

We define the corresponding Lie algebras:

$$\mathfrak{g}_+ = \left\{ r(z) = \sum_{l>0} r_l z^l \in \mathrm{End}(V)[\![z]\!] \;\middle|\; r^\dagger(-z) + r(z) = 0 \right\},$$
$$\mathfrak{g}_- = \left\{ s(z) = \sum_{l>0} s_l z^{-l} \in \mathrm{End}(V)[\![z^{-1}]\!] \;\middle|\; s^\dagger(-z^{-1}) + s(z^{-1}) = 0 \right\}.$$

These are Lie algebras under the commutator $[\cdot,\cdot]$.

Quantization. Since $r \in \mathfrak{g}_+$ and $s \in \mathfrak{g}_-$ are infinitesimal symplectic transformations, we can define their quantizations $\hat{r}$ and $\hat{s}$ by

$$\hat{R} := (\exp r)\hat{} = \exp(\hat{r}), \quad \hat{S} := (\exp s)\hat{} = \exp(\hat{s}).$$

In applications to cohomological field theories, G_- acts as a change of coordinates, so we focus primarily on G_+.

The quantization maps $\widehat{\cdot} : \mathfrak{g}_\pm \to \widehat{\mathfrak{g}}_\pm$ are Lie algebra isomorphisms. Explicitly, for $r(z) = \sum_{l>0} r_l z^l \in \mathfrak{g}_+$ and $s(z) = \sum_{l>0} s_l z^{-l} \in \mathfrak{g}_-$:

$$\hat{r} = \sum_{l>0,\, i\geq 0} \sum_{\alpha,\beta=0}^{N} (r_l)^\beta_\alpha \, q_i^\alpha \, \partial_{q^\beta_{i+l}} + \frac{\varepsilon}{2} \sum_{i,j\geq 0} \sum_{\alpha,\beta=0}^{N} (-1)^{i+1} (r_{i+j+1})^{\alpha\beta} \, \partial_{q_i^\alpha} \partial_{q_j^\beta}, \tag{1.30}$$

$$\hat{s} = \sum_{l>0,\, i\geq 0} \sum_{\alpha,\beta=0}^{N} (s_l)^\beta_\alpha \, q^\alpha_{i+l} \, \partial_{q_i^\beta} + \frac{1}{\varepsilon} \sum_{i,j\geq 0} \sum_{\alpha,\beta=0}^{N} (-1)^{i} (s_{i+j+1})^{\alpha\beta} \, q_i^\alpha q_j^\beta, \tag{1.31}$$

where $(r_l)^{\alpha\beta} = \sum_\rho \eta^{\alpha\rho} (r_l)^\beta_\rho$, and similarly for $(s_l)^{\alpha\beta}$.

For any $R, P \in G_+$ and $\mathcal{F} \in \mathfrak{Fock}(V, \delta)$,

$$(RP)\hat{}\mathcal{F} = \hat{R}\hat{P}\mathcal{F}.$$

Simplified action. For $R \in G_+$ and $\mathcal{F} \in \mathfrak{Fock}(V, \delta)$, we have

$$\hat{R}\mathcal{F} = \exp\left(\sum_{l>0, i\geq 0}\sum_{\alpha,\beta=0}^{N} (r_l)^{\beta}_{\alpha} q^{\alpha}_i \frac{\partial}{\partial q^{\beta}_{i+l}}\right) \exp\left(\frac{\varepsilon}{2}\sum_{i,j\geq 0}\sum_{\alpha,\beta=0}^{N} (E_{ij})^{\alpha\beta} \frac{\partial^2}{\partial q^{\alpha}_i \partial q^{\beta}_j}\right) \mathcal{F},$$

where E_{ij} is defined by

$$\sum_{i,j\geq 0} (-1)^{i+j} E_{ij}\, w^i z^j = \frac{R^{\dagger}(w)R(z) - \mathrm{Id}}{w+z}.$$

This matches Givental's original formula [Giv01a, p. 9].

Following Givental's and Teleman's convention for R^{-1}, we make the following definition.

Definition 1.11. Let $R \in G_+$. The *(upper) Givental differential action* is the operator

$$\hat{R} : \mathfrak{Fock}(V, \delta) \to \mathfrak{Fock}(V, \delta)$$

defined by

$$\hat{R}\mathcal{F}(\mathbf{q}) = \left(\exp\left(\frac{\varepsilon}{2}\Delta\right)\mathcal{F}\right)(R^{-1}\mathbf{q}),$$

where $R^{-1}\mathbf{q}$ denotes the product of formal series $R^{-1}(z)\mathbf{q}(z)$, and the *propagator* Δ is

$$\Delta = \sum_{i,j\geq 0}\sum_{\alpha,\beta=0}^{N} (E_{ij})^{\alpha\beta} \frac{\partial^2}{\partial q^{\alpha}_i \partial q^{\beta}_j}, \tag{1.32}$$

with coefficients determined by

$$\sum_{i,j\geq 0} (-1)^{i+j} (E_{ij})^{\alpha\beta} w^i z^j = \eta\left(\phi^{\alpha}, \frac{R^{\dagger}(w)R(z) - \mathrm{Id}}{w+z}\phi^{\beta}\right).$$

Theorem 1.12 ([Giv02, p. 17])**.** *The Givental differential action in Definition 1.11 is well-defined and defines a left group action of G_+ on the Fock space $\mathfrak{Fock}(V, \delta)$.*

1.7.3 Action via stable graphs

Let (V, η) be an $(N+1)$-dimensional $\mathbb{C}$-vector space with a non-degenerate symmetric bilinear form, and let $(\phi_{\alpha})_{\alpha=0}^{N}$ be a basis of V. Let G_+ denote the upper-triangular subgroup of $\mathrm{End}(V)[\![z]\!]$. For $v \in V$ and $R \in G_+$,

$$R(z)v = v + \sum_{i\geq 0} R_i(v) z^i \in V[\![z]\!].$$

Denote by $\mathrm{G}_{g,n}$ the (finite) set of stable graphs of genus g with n legs, up to isomorphism. For $\Gamma \in \mathrm{G}_{g,n}$, write:

- $\mathrm{V}(\Gamma)$: vertices of Γ,
- $\mathrm{E}(\Gamma)$: edges of Γ,
- $\mathrm{L}(\Gamma)$: legs of Γ.

The associated gluing morphism is

$$\mathrm{gl}_{\Gamma} : \overline{M}_{\Gamma} = \prod_{v\in \mathrm{V}(\Gamma)} \overline{M}_{g(v),n(v)} \longrightarrow \overline{M}_{g,n}.$$

We let $H_{g,n}$ be the even cohomology $H^{\mathrm{even}}(\overline{M}_{g,n}, \mathbb{C})$.

Graph action. Given a CohFT (V, η, c) and $R \in G_+$, the *Givental graph action* is the map

$$R : \text{CohFTs} \longrightarrow \text{CohFTs}, \quad c \longmapsto Rc,$$

where

$$(Rc)_{g,n} = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \text{Cont}_\Gamma,$$

and $\text{Cont}_\Gamma \in H_{g,n} \otimes (V^*)^{\otimes n}$ is defined by

$$\text{Cont}_\Gamma = (\text{gl}_\Gamma)_* \left(\prod_{v \in V(\Gamma)} \text{Cont}_v \prod_{e \in E(\Gamma)} \text{Cont}_e \prod_{l \in L(\Gamma)} \text{Cont}_l \right).$$

Decorations.

- *Legs:* For $l \in \mathrm{L}(\Gamma)$,

$$\text{Cont}_l = R^{-1}(\psi_l) \in \text{End}(V)[\![\psi_l]\!],$$

  where ψ_l is the ψ-class at the marked point l.

- *Edges:* For $e = \{e_1, e_2\} \in \mathrm{E}(\Gamma)$,

$$\text{Cont}_e = \frac{\eta^{-1} - R^{-1}(\psi_{e_1})\, \eta^{-1}\, R^{-1}(\psi_{e_2})^t}{\psi_{e_1} + \psi_{e_2}}$$

  equivalently,

$$\text{Cont}_e = \frac{\text{Id} \otimes \text{Id} - R^{-1}(\psi_{e_1}) \otimes R^{-1}(\psi_{e_2})}{\psi_{e_1} + \psi_{e_2}} \eta^{-1}.$$

- *Vertices:* For $v \in \mathrm{V}(\Gamma)$,

$$\text{Cont}_v = c_{g(v),n(v)} \in H_{g(v),n(v)} \otimes (V^*)^{\otimes n(v)},$$

  with ψ-powers pulled out of the vertex classes, so for $\gamma_i \in V$, we set:

$$\text{Cont}_v(\ldots, \psi_i^{d_i} \gamma_i, \ldots) = \psi_i^{d_i} \text{Cont}_v(\ldots, \gamma_i, \ldots).$$

Given $\gamma_1 \otimes \cdots \otimes \gamma_n \in V^{\otimes n}$,

$$\text{Cont}_\Gamma(\gamma_1, \ldots, \gamma_n) = (\text{gl}_\Gamma)_* \left(\prod_{v \in V(\Gamma)} \text{Cont}_v \left(\{\text{Cont}_{h_e}\}_{e \in E(v)}, \{\text{Cont}_{l_i}(\gamma_i)\}_{l_i \in L(v)} \right) \right).$$

where Cont_{h_e} is one part of the bivector Cont_e.

Theorem 1.13 ([PPZ15, p. 21])**.** *For any $R \in G_+$, the above defines a well-defined left group action on CohFTs.*

Translation action. Given $T(z) = \sum_{i \geq 2} T_i z^i \in z^2 V[\![z]\!]$, define

$$(Tc)_{g,n}(\gamma_1, \ldots, \gamma_n) = \sum_{m \geq 0} \frac{1}{m!} (\pi_m)_* \left(c_{g,n+m}(\gamma_1, \ldots, \gamma_n, T(\psi_{n+1}), \ldots, T(\psi_{n+m})) \right),$$

where $\pi_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ forgets the last m markings.

Proposition 1.14 ([PPZ15, p. 24])**.** The translation action is an abelian group action on CohFTs.

Unit-preserving action. If $(V, \eta, c, \mathbf{1})$ is a CohFT with unit, define

$$R \cdot c := R\,Tc,$$

where

$$T(z) = z(\mathbf{1} - R(z)\mathbf{1}) \in z^2 V[\![z]\!].$$

Theorem 1.15 ([PPZ15, p. 30])**.** *This defines a left group action of G_+ on CohFTs with unit.*

1.7.4 Givental–Teleman's theorem

The reconstruction of higher genus data from genus zero input was first developed by Givental in the context of Gromov–Witten theory, and later proved in full generality for semisimple CohFTs by Teleman. Their results can be formulated as follows.

Theorem 1.16 ([Tel12])**.** *Let $\omega = (c_{0,n})_n$ be a genus–zero, homogeneous, semisimple CohFT with unit. Then there exists a* unique *CohFT $c = (c_{g,n})_{g,n}$ with unit extending ω to all genera. Moreover, this unique extension is obtained by applying Givental's R–matrix action to the degree–zero (topological) part of ω.*

Teleman's theorem provides a complete classification of semisimple CohFTs and has led to important applications, including the derivation of tautological relations in the cohomology of moduli spaces of curves [PPZ15]. In this framework, Givental's formalism, initially derived via geometric quantization and expressed in terms of differential operators, extends naturally to the setting of CohFTs, where it can also be described in terms of decorated stable graphs.

Theorem 1.17 ([Giv01a; Tel12])**.** *Let M be a complex, conformal Frobenius manifold of dimension N with semisimple origin, and let $R \in G_+$ be the R–matrix determined by the Euler vector field of M. Then the total descendant potential $\mathcal{F}^M$ associated to the genus–zero potential $\mathcal{F}_0$ of M is given by*

$$\mathcal{F}^M = \hat{\Psi}_1\,\hat{R}\,\hat{\Psi}_2\,\mathcal{T}, \qquad \mathcal{T} = \underbrace{\mathcal{F}^{\mathrm{WK}} \cdots \mathcal{F}^{\mathrm{WK}}}_{N\ \text{factors}}$$

where $\mathcal{T}$ is the N–fold product of Witten–Kontsevich tau–functions, and $\hat{\Psi}_1$, $\hat{\Psi}_2$ are quantized symplectic transformations associated to changes of basis between flat and canonical coordinates.

Together, Theorems 1.16 and 1.17 establish that for semisimple Frobenius manifolds (and hence semisimple CohFTs), all higher–genus data is uniquely and explicitly determined by the genus–zero structure.

1.8 Integrable hierarchies of topological type

Integrable systems are evolutionary partial differential equations constrained by special relations such as compatibility conditions, symmetries, or conserved quantities. These constraints make the equations exactly solvable, hence the term *integrable*. An *integrable hierarchy* is a natural extension, instead of a finite number of equations, one considers infinitely many commuting evolutionary PDEs organized in a hierarchical structure. The term *integrable hierarchies of topological type* refers to a modern class of hierarchies that can be constructed from cohomological field theories (CohFTs). In this framework, the geometry of the moduli space of curves provides the input data, while the associated hierarchy encodes intersection theory in analytic form. A central example is the Witten–Kontsevich theorem, which identifies the KdV hierarchy with the generating function of ψ-class intersections. In this section we introduce the notation that will be used throughout and give a brief overview of the subject. For a more elaborate overview, see the surveys [Ros17; Bur17].

1.8.1 Hamiltonian systems

Fix $N \geq 1$. We define the ring of *differential polynomials* as

$$\mathcal{A} = \mathbb{C}[\![u^\alpha]\!][u^\alpha_j][\![\varepsilon]\!],$$

equipped with an additional ∂_x-gradation

$$\deg_{\partial_x} : \mathcal{A} \to \mathbb{Z},$$

where:

- u^α_j are formal variables indexed by $\alpha = 1, \ldots, N$ and $j \in \mathbb{Z}_{\geq 0}$, with $\deg_{\partial_x}(u^\alpha_j) = j$;
- ε is the dispersion parameter with $\deg_{\partial_x}(\varepsilon) = -1$.

Total derivative. We have the following even differential operator on $\mathcal{A}$ of $\deg_{\partial_x}$-degree $+1$:

$$\partial_x = \sum_{i \geq 0} u^\alpha_{i+1} \frac{\partial}{\partial u^\alpha_i}.$$

In particular, $u^\alpha_j = \partial^j_x u^\alpha$.

The variables u^α may be interpreted as maps $S^1 \to V$, $x \mapsto u^\alpha(x)$, where V is an N-dimensional vector space. Thus, an evolutionary PDE of the form

$$\partial_t u^\beta = F^\beta\big((u^\alpha_j)_{\alpha=1,\ldots,N;\, j\geq 0}\big).$$

can be heuristically regarded as a vector field on the infinite-dimensional space of all loops $u(x) = (u^1(x), \ldots, u^N(x))$.

Local functionals. The space of *local functionals* is defined as the quotient

$$\hat{\mathcal{A}} := \mathcal{A}/\big(\mathrm{Im}(\partial_x) \oplus \mathbb{C}[\![\varepsilon]\!]\big),$$

where $\mathbb{C}$ denotes the subspace of constants in the variables u^α_j. The projection operator from $\mathcal{A}$ modulo constants to $\hat{\mathcal{A}}$ is

$$\begin{array}{ccc} \mathcal{A}/\mathbb{C}[\![\varepsilon]\!] & \longrightarrow & \hat{\mathcal{A}} \\ f & \longmapsto & \overline{f} := \int f\, dx \end{array}$$

Fourier variables. We often use another set of formal variables p^α_k related to the u-variables by a Fourier series expansion:

$$u^\alpha = \sum_{k \in \mathbb{Z}} p^\alpha_k e^{\mathrm{i}kx}. \tag{1.33}$$

This embeds elements of $\mathcal{A}$ into the auxiliary ring

$$\mathbb{C}[\![p^\alpha_{k>0}]\!][p^\alpha_{k\leq 0}][\![\varepsilon]\!][e^{\mathrm{i}x}, e^{-\mathrm{i}x}],$$

where p^α_k are formal variables indexed by $\alpha = 1, \ldots, N$ and $k \in \mathbb{Z}$.

Poisson structure. For any $f, g \in \hat{\mathcal{A}}$, the *Poisson bracket* is defined as

$$\{\overline{f}, \overline{g}\}_K = \int \frac{\delta \overline{f}}{\delta u^\alpha} K^{\alpha\beta} \frac{\delta \overline{g}}{\delta u^\beta}\, dx,$$

where:

- The *variational derivative operators* $\frac{\delta}{\delta u^\alpha} : \hat{\mathcal{A}} \to \mathcal{A}$ are defined by

$$\frac{\delta \overline{f}}{\delta u^\alpha} = \sum_{j\geq 0} (-\partial_x)^j \frac{\partial f}{\partial u^\alpha_j}.$$

- The *Poisson operator* $K = (K^{\alpha\beta})$ is an $N \times N$ matrix of differential operators:

$$K^{\alpha\beta} = \sum_{j\geq 0} K^{\alpha\beta}_j \partial^j_x, \quad \text{with } K^{\alpha\beta}_j \in \mathcal{A}.$$

- The bracket $\{\cdot,\cdot\}_K$ is antisymmetric and satisfies the Jacobi identity.

  This definition extends to $f \in \mathcal{A}$ and $\overline{g} \in \hat{\mathcal{A}}$ via

$$\{f, \overline{g}\}_K = \sum_{j\geq 0} \frac{\partial f}{\partial u^\alpha_j} \partial^j_x K^{\alpha\beta} \frac{\delta \overline{g}}{\delta u^\beta},$$

so that

$$\{\overline{f}, \overline{g}\}_K = \int \{f, \overline{g}\}_K \, dx.$$

Example 5. The *standard Poisson operator* is $K = \eta\partial_x$, where η is a symmetric non-degenerate complex matrix. In this case, the Poisson bracket has a particularly simple form in the p^α_j variables:

$$\{p^\alpha_i, p^\beta_j\}_{\eta\partial_x} = ij\, \eta^{\alpha\beta}\, \delta_{i+j,0}.$$

Hamiltonian hierarchies. A *Hamiltonian integrable hierarchy* in one spatial variable x with times t^α_j is a system of evolutionary PDEs of the form

$$\frac{\partial u^\alpha}{\partial t^\beta_d} = \{u^\alpha, \overline{h}_{\beta,d}\}_K = K^{\alpha\mu} \frac{\delta \overline{h}_{\beta,d}}{\delta u^\mu}, \tag{1.34}$$

where:

- $\alpha, \beta = 1, \ldots, N$ and $d \geq 0$;
- $\overline{h}^\beta_d \in \hat{\mathcal{A}}^{[0]}$ are *Hamiltonians* satisfying the integrability condition

$$\{\overline{h}^\alpha_i, \overline{h}^\beta_j\}_K = 0.$$

1.8.2 Tau–functions

Assume we have a Hamiltonian hierarchy as in Equation (1.34). Suppose the Hamiltonian $\overline{h}_{1,0}$ generates spatial translations:

$$K^{\alpha\mu} \frac{\delta \overline{h}_{1,0}}{\delta u^\mu} = u^\alpha_1.$$

A *tau–structure* for a Hamiltonian hierarchy is a family of differential polynomials $h_{\beta,d} \in \mathcal{A}^{[0]}$, indexed by $1 \leq \beta \leq N$ and $d \geq -1$, satisfying the following conditions:

- The Hamiltonians $\overline{h}_{\beta,-1} = \int h_{\beta,-1}\, dx$ are Casimirs of K:

$$K^{\alpha\mu} \frac{\delta \overline{h}_{\beta,-1}}{\delta u^\mu} = 0.$$

- The elements $\overline{h}_{\beta,-1}$ are linearly independent for all $\beta = 1, \dots, N$.
- For $d \geq 0$, we have $\overline{h}_{\beta,d} = \int h_{\beta,d}\, dx$.
- Tau–symmetry holds:

$$\{h_{\alpha,i-1}, \overline{h}_{\beta,j}\}_K = \{h_{\beta,j-1}, \overline{h}_{\alpha,i}\}_K, \qquad 1 \leq \alpha, \beta \leq N, i, j \geq 0.$$

A Hamiltonian hierarchy equipped with a tau–structure is often called a *tau–symmetric hierarchy*. A *tau–function* is a function associated to any solution of a tau–symmetric hierarchy. From the properties of a tau–structure, there exists a unique polynomial

$$\Omega_{\alpha,i;\beta,j} \in \hat{\mathcal{A}}^{[0]}$$

such that

$$\Omega_{\alpha,i;\beta,j} = \frac{\partial h_{\alpha,i-1}}{\partial t^{\beta}_{j}}, \qquad \Omega_{\alpha,i;\beta,j}\big|_{(u^{\alpha}_{j})_{\alpha=1,\dots,N;\, j\geq 0}=0} = 0,$$

and moreover

$$\Omega_{\alpha,i;\beta,j} = \Omega_{\beta,j;\alpha,i}.$$

For a given solution

$$u\big(x, (t^{\alpha}_{j})_{\alpha=1,\dots,N;\, j\geq 0}, \varepsilon\big) = \big(u^{\alpha}(x, (t^{\gamma}_{j})_{\gamma=1,\dots,N;\, j\geq 0}, \varepsilon)\big)_{\alpha=1,\dots,N}$$

of the Hamiltonian hierarchy, assume

$$u\big(x, (t^{\alpha}_{j})_{\alpha,j}, \varepsilon\big)\Big|_{x=(t^{\alpha}_{j})_{\alpha,j}=\varepsilon=0} = 0$$

(for convergence reasons). Then the tau–structure implies the existence of a function

$$T \in \varepsilon^{-2}\mathbb{C}[\![(t^{\alpha}_{j})_{\alpha=1,\dots,N;\, j\geq 0}, \varepsilon]\!]$$

such that

$$\Omega_{\alpha,i;\beta,j}\big(u\big(x, (t^{\alpha}_{j})_{\alpha,j}, \varepsilon\big)\big)\Big|_{x=0} = \varepsilon^2 \frac{\partial^2 T}{\partial t^{\alpha}_{i} \partial t^{\beta}_{j}}, \qquad 1 \leq \alpha, \beta \leq N, i, j \geq 0.$$

The exponent

$$\tau = e^T$$

is called the *tau–function* associated to the solution

$$u\big(x, (t^{\alpha}_{j})_{\alpha,j}, \varepsilon\big)$$

with respect to the given tau–structure. In practice, one usually studies the function T, since it compactly encodes the evolution along a particular solution for all the Hamiltonians.

Example 6. The most famous example of a tau–symmetric hierarchy is the *Korteweg–de Vries (KdV) hierarchy*. Here, $V = \mathbb{C}$, $\eta = 1$, and $K = \partial_x$. The *primary Hamiltonian* is

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_2 \right) dx,$$

which yields the KdV equation

$$\partial_t u = u u_1 + \frac{\varepsilon^2}{24} u_3.$$

The remaining Hamiltonians $\overline{h}_i = \int h_i \, dx$ can be determined recursively [BR16b] by

$$\partial_x \left(\sum_{i \geq 0} (i+1) u_i \frac{\partial}{\partial u_i} - 1 \right) h_{i+1} = \{h_i, \overline{h}_1\},$$

starting from $h_{-1} = u$. The first few Hamiltonian densities (see [Sin24] for the computational code) are

$$\begin{aligned}
h_0 &= \frac{u^2}{2} + \frac{\varepsilon^2}{24} u_2, \\
h_1 &= \frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_2 + \frac{\varepsilon^4}{1152} u_4, \\
h_2 &= \frac{u^4}{24} + \varepsilon^2 \frac{u^2 u_2}{48} + \varepsilon^4 \left(\frac{7u_2^2}{5760} + \frac{u u_4}{1152} \right) + \varepsilon^6 \frac{u_6}{82944}.
\end{aligned}$$

A tau–structure $\{g_d\}_{d \geq 0}$ can then be obtained by setting $g_d = \frac{\delta \overline{h}_{d+1}}{\delta u}$. In this case $\overline{g}_d = \overline{h}_d$, and tau–symmetry follows.

1.8.3 Dubrovin–Zhang hierarchy

The Dubrovin–Zhang hierarchy is the canonical dispersive integrable hierarchy associated to a semisimple CohFT. Built as a deformation of the principal hierarchy for a Frobenius manifold, it is uniquely characterized by tau-symmetry together with the two polynomiality conditions. Its canonical string solution reproduces the CohFT free energy through its tau-function, and in the semisimple case the hierarchy is equivalent to the DR hierarchy, providing a deep link between the intersection theory of moduli spaces and integrable systems. Moreover, the Dubrovin–Zhang framework provides the natural language for generalizing Witten's conjecture. In particular, the classical statement relating the Gromov–Witten theory of a point to the KdV hierarchy extends, within this framework, to arbitrary semisimple CohFTs.

Let $(V, \eta, \mathbf{1}, \Omega)$ be a CohFT with state space V, flat metric η, and unit $\mathbf{1}$. The total descendant potential (Equation (1.28)) admits the genus expansion

$$\mathcal{F}\big((t_i^\alpha)_{\alpha,i}; \varepsilon\big) = \sum_{g \geq 0} \varepsilon^{2g} F_g\big((t_i^\alpha)_{\alpha,i}\big), \quad \Omega_{\alpha,i;\beta,j}\big((t_k^\gamma)_{\gamma,k}; \varepsilon\big) := \frac{\partial^2 \mathcal{F}}{\partial t_i^\alpha \, \partial t_j^\beta}.$$

Here ε is the genus parameter and t_i^α are descendant variables.

Dispersionless limit (principal hierarchy). The genus-0 sector of the CohFT determines a Frobenius manifold structure on V. From this structure, Dubrovin's *principal hierarchy* is obtained, a system of evolutionary PDEs for fields $v^\alpha(x)$, whose Hamiltonian densities are

$$h_{\alpha,i}\big((v^\gamma)_{\gamma=1}^N\big) = \Omega^{[0]}_{\alpha,i;1,1}\big(t_0^\gamma = v^\gamma, \, t_{j>0}^\gamma = 0\big),$$

and whose Poisson operator is $\eta^{\alpha\beta}\partial_x$. This hierarchy is commutative and tau-symmetric, and its defining relations follow from the basic geometry of $\overline{\mathcal{M}}_{0,4}$.

Dispersive deformation (DZ hierarchy). Dubrovin–Zhang extended the principal hierarchy to incorporate higher-genus contributions. The resulting *DZ hierarchy* is a dispersive deformation in the parameter ε, with Hamiltonians $H_{\alpha,i}$ defined relative to a quasi-Miura transformation of the fields:

$$w^{\alpha}\big((v^{\gamma})_{\gamma};\varepsilon\big) = v^{\alpha} + \sum_{g\geq 1} \varepsilon^{2g} \frac{\partial^2 F_g\big(v_0^{\gamma},\ldots,v_{3g-2}^{\gamma}\big)}{\partial t_0^{\alpha}\,\partial t_0^{1}},$$

where the genus expansions of the descendant potentials $F_g((t_i^{\alpha})_{\alpha,i})$ are known to be expressible in terms of the finitely many variables $v_0^{\gamma},\ldots,v_{3g-2}^{\gamma}$ (see [BPS12]). The Hamiltonians are then given by

$$H_{\alpha,i}\big((w^{\gamma})_{\gamma};\varepsilon\big) = h_{\alpha,i}\big((v^{\gamma}(w;\varepsilon))_{\gamma}\big) + \sum_{g\geq 1} \varepsilon^{2g} \frac{\partial^2 F_g\big((v^{\gamma}(w;\varepsilon))_{\gamma}\big)}{\partial t_{i+1}^{\alpha}\,\partial t_0^{1}},$$

so that the densities are formal ε-deformations of their genus-0 counterparts. The distinguished *string solution* $v^{\alpha}(x,(t^{\gamma})_{\gamma},\varepsilon)$, with initial condition $v^{\alpha}|_{(t^{\gamma})=0} = \delta_{\alpha,1}x$, recovers the CohFT free energy via the standard topological solution recipe.

Polynomiality conditions. The Dubrovin–Zhang construction is uniquely characterized by tau-symmetry together with two polynomiality requirements:

1. **First polynomiality condition.** Each $\Omega_{\alpha,i;\beta,j}$ must be a differential polynomial in the fields w^{α} and their x-derivatives, with coefficients depending polynomially on ε. This guarantees that the Hamiltonians are well-defined local functionals and close under the evolutionary flows.

2. **Second polynomiality condition.** The deformed Poisson operator takes the form

$$K^{\alpha\beta}{=}\eta^{\alpha\beta}\,\partial_x{+}\textstyle\sum_{g\geq 1}\varepsilon^{2g}\,K^{\alpha\beta}_{[g]}\big((w^{\gamma}_j)_{\gamma,j};\partial_x\big),$$

   where each $K^{\alpha\beta}_{[g]}$ is a finite-order differential operator with coefficients in the polynomial algebra of the fields w^{α} and their x-derivatives. This ensures that the Poisson bracket of two local functionals remains local, preserving the Hamiltonian formalism under dispersive deformation.

Semisimplicity and DR/DZ equivalence. When the Frobenius manifold of the CohFT is *semisimple*, the DZ hierarchy exists as the canonical tau-symmetric integrable system attached to the theory. Semisimplicity allows diagonalization in canonical coordinates and guarantees compatibility of the deformation. Moreover, in this case the DZ hierarchy is equivalent, under a normal Miura transformation, to the *Double Ramification (DR) hierarchy*. This strong DR/DZ equivalence provides a unifying framework, with classical examples such as KdV (trivial CohFT) and extended Toda (Gromov–Witen of $\mathbb{P}^1$).

1.9 Quantum integrability and quasimodularity

There is by now a broad expectation that quasimodularity lies at the intersection of enumerative geometry for elliptic targets and integrable systems. This appears in several settings, the quasimodularity of generating series of Hurwitz numbers on elliptic curves, which also form KP/Toda tau-functions [Dij95; KZ95; OP06a]; the appearance of quasimodular forms in Gromov–Witten theory of elliptic curves and related targets [SZ17; MRS15; OP18]; and, more recently, connections between quasimodularity and quantum deformations, in particular the quantum KdV hierarchy [IR24; IR25]. These examples suggest that quasimodularity naturally accompanies enumerative problems with elliptic structures and quantum integrable systems, though often only implicitly. In Chapter 4 we make this connection explicit by constructing a quantum double ramification hierarchy with coefficients in the ring of quasimodular forms, focusing on the case where the target of the Gromov–Witten theory is an elliptic curve.

1.9.1 Deformation quantization

In the integrable setting, deformation quantization amounts to constructing a deformed multiplication that yields a new family of commuting Hamiltonians. In many practical cases the Moyal product is the canonical choice, for instance, the classical KdV hierarchy admits a quantization via the Moyal product, leading to the quantum KdV. In Chapter 4 we present the construction of a quantum double ramification hierarchy, viewed as the quantization of the classical double ramification hierarchy.

Formal deformation. Let A be an associative algebra over $\mathbb{C}$. A *formal deformation* of A is a $\mathbb{C}[\![\hbar]\!]$-bilinear product

$$m_\hbar : A[\![\hbar]\!] \otimes_{\mathbb{C}[\![\hbar]\!]} A[\![\hbar]\!] \longrightarrow A[\![\hbar]\!]$$

on the space of formal power series in $\hbar$, of the form

$$m_\hbar(a,b)=a\cdot b+m_1(a,b)\,\hbar+m_2(a,b)\,\hbar^2+\cdots,\quad a,b\in A,$$

where $a\cdot b$ is the original multiplication on A. The associativity of $m_\hbar$ is equivalent to the condition

$$m_\hbar(m_\hbar(a,b),c)=m_\hbar(a,m_\hbar(b,c)),\qquad a,b,c\in A.$$

If A is commutative, the first-order term of the commutator defines a Poisson bracket:

$$\{a,b\} := \frac{1}{2\hbar}\big(m_\hbar(a,b)-m_\hbar(b,a)\big)\,\mathrm{mod}\,\hbar = \frac{1}{2}\big(m_1(a,b)-m_1(b,a)\big).$$

Thus, a formal deformation of A induces the structure of a Poisson algebra on A. In physical terms, the parameter $\hbar$ is interpreted as the quantum parameter (Planck's constant), and the Poisson algebra $(A,\{\cdot,\cdot\})$ arises as the quasi-classical limit of the deformed algebra $A[\![\hbar]\!]$. The *deformation quantization problem* asks, conversely, to construct a formal deformation whose quasi-classical limit recovers a given Poisson structure on A.

The Hochschild complex. The main tool for studying deformations of an algebra A is the Hochschild complex

$$0\longrightarrow C^0(A,A)\xrightarrow{d}\cdots\xrightarrow{d}C^n(A,A)\xrightarrow{d}C^{n+1}(A,A)\xrightarrow{d}\cdots,$$

where $C^n(A,A):=\mathrm{Hom}(A^{\otimes n},A)$ is the space of n-linear maps $f(a_1,\dots,a_n)$ with values in A. The differential is defined by

$$\begin{aligned}(df)(a_1,\dots,a_{n+1}) := a_1 f(a_2,\dots,a_{n+1})&\\ +\sum_{i=1}^{n}(-1)^i f(a_1,\dots,a_{i-1},\, a_i a_{i+1},\, a_{i+2},\dots,a_{n+1})&\\ +(-1)^{n+1} f(a_1,\dots,a_n)a_{n+1}.&\end{aligned}$$

The associated cohomology is

$$H^*(A,A):=\ker d\,/\,\mathrm{im}\,d,$$

and encodes the deformation theory of A.

The Gerstenhaber bracket. The Hochschild complex carries a natural graded Lie bracket,

$$[\cdot,\cdot]: C^m(A,A)\otimes C^n(A,A)\longrightarrow C^{m+n-1}(A,A),$$

called the *Gerstenhaber bracket*. This bracket makes the Hochschild cochain complex into a differential graded Lie algebra, and provides the algebraic structure underlying deformation theory.

Associativity and the Gerstenhaber bracket. The following basic property may be viewed as the foundation of deformation theory. A formal multiplication

$$m_\hbar(a,b) = m_0(a,b) + m_1(a,b)\,\hbar + m_2(a,b)\,\hbar^2 + \cdots, \qquad a,b \in A,$$

is associative precisely when the Gerstenhaber bracket satisfies $[m_\hbar, m_\hbar] = 0$. Indeed, one computes

$$[m_\hbar, m_\hbar](a,b,c) = 2\big(m_\hbar(m_\hbar(a,b),c) - m_\hbar(a, m_\hbar(b,c))\big),$$

so that the condition $[m_\hbar, m_\hbar] = 0$ is equivalent to the associativity of $m_\hbar$.

In particular, the original multiplication $m_0(a,b) := a \cdot b$ is associative, hence $[m_0, m_0] = 0$. The commutator with m_0 then defines a differential on the Hochschild complex, namely

$$df = [f, m_0],$$

which coincides with the Hochschild differential introduced earlier. Thus, the Gerstenhaber bracket provides the natural framework in which associativity and deformations can be expressed.

Deformation quantization. In a geometric or physical setting, the deformation quantization problem asks whether a given Poisson algebra of functions can be deformed into a noncommutative algebra in such a way that the deformation recovers the Poisson bracket in the classical limit. A celebrated result due to Kontsevich provides an answer to this problem.

Theorem 1.18 ([Kon03])**.** *Every Poisson manifold $(M, \{\cdot,\cdot\})$ admits a deformation quantization. In other words, there exists a formal associative product*

$$f \star g = m_\hbar(f,g) = fg + m_1(f,g)\,\hbar + m_2(f,g)\,\hbar^2 + \cdots, \qquad f,g \in C^\infty(M),$$

on the algebra $A = C^\infty(M)$ of smooth functions, where each m_i is a bidifferential operator on M. The product $\star$ is associative and, in the quasi-classical limit, reproduces the Poisson bracket:

$$\frac{1}{2}\big(m_1(f,g) - m_1(g,f)\big) = \{f,g\}.$$

Physics motivation. In classical mechanics, the algebra of observables is the commutative Poisson algebra $C^\infty(P)$ of smooth functions on a symplectic manifold (phase space) P, with dynamics generated by a Hamiltonian $h \in C^\infty(P)$ through the Poisson bracket,

$$\frac{df}{dt} = \{f,h\}, \qquad f \in C^\infty(P).$$

Quantum mechanics, on the other hand, replaces $C^\infty(P)$ with a noncommutative algebra of operators $\mathcal{D} \subset \mathrm{End}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$. The time evolution of an observable $O_t \in \mathcal{D}$ in the Heisenberg picture is governed by

$$\frac{dO_t}{dt} = \frac{1}{\hbar}[O_t, H],$$

where $[\,\cdot\,,\cdot\,]$ is the commutator and H is the Hamiltonian operator. The correspondence principle asserts that, in the classical limit $\hbar \to 0$, this quantum dynamics reduces to the classical Hamiltonian flow.

Deformation quantization is a technique to formalize this correspondence, it realizes the quantum algebra of observables as a deformation of the commutative Poisson algebra, so that the commutator reproduces the Poisson bracket in the quasi-classical limit. From this perspective, a quantum integrable hierarchy is naturally viewed as a deformation of a classical integrable hierarchy, where a family of commuting Hamiltonians is deformed into a family of commuting quantum Hamiltonians.

Example 7. Consider the algebra $\mathbb{C}[q,p]$ of polynomials in two variables, endowed with the Poisson bracket
$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}.$$
A star product on $\mathbb{C}[q,p]$ is given explicitly by

$$f \star g = f \exp\left(\hbar \overleftarrow{\tfrac{\partial}{\partial p}} \overrightarrow{\tfrac{\partial}{\partial q}}\right) g = \sum_{k\geq 0} \frac{\hbar^k}{k!}\frac{\partial^k f}{\partial q^k}\frac{\partial^k g}{\partial p^k}.$$

To interpret this construction, note that q acts as the operator of multiplication by q, while p corresponds to the operator $\hbar\frac{\partial}{\partial q}$, so that
$$[p,q] = \hbar.$$
Normal ordering is obtained by placing powers of q to the left and powers of p to the right, which yields a well-defined correspondence between polynomials in (q,p) and differential operators. The star product $f \star g$ is then recovered by transporting the usual product of operators back to $\mathbb{C}[q,p]$ under this correspondence.

1.9.2 Quasimodular forms

Modular forms arise as functions on the upper half-plane that transform with precise weight under the action of $SL(2,\mathbb{Z})$, the symmetry group of elliptic curves. They are fundamental objects in number theory, geometry, and physics, as they encode invariants of elliptic curves and appear naturally in contexts ranging from arithmetic problems to partition functions in conformal field theory. A quasimodular form transforms almost like a modular form, but acquires a polynomial correction in $(c\tau+d)^{-1}$ under the $SL(2,\mathbb{Z})$-action. The simplest and most important example is the Eisenstein series $E_2(\tau)$, which fails to be modular but generates, together with $E_4(\tau)$ and $E_6(\tau)$, the full ring of quasimodular forms.

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ be the upper half-plane. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *quasimodular form of weight* k for $\Gamma = SL(2,\mathbb{Z})$ if for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ one has

$$\frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) + p_f\left(\frac{c}{c\tau+d}\right),$$

where $p_f(x)$ is a polynomial with coefficients in $\mathbb{C}$. If $p_f \equiv 0$, the function f is an ordinary modular form of weight k.

Applying this condition to the generators

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $SL(2,\mathbb{Z})$, one finds that a modular form f of weight k satisfies

$$f(\tau+1) = f(\tau), \qquad f\left(-\tfrac{1}{\tau}\right) = \tau^k f(\tau).$$

The modular transformation properties imply that modular forms are periodic of period 1, and therefore admit a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \qquad q = e^{2\pi i\tau}.$$

The condition of holomorphy as $\Im(\tau) \to \infty$ can be restated by requiring that the coefficients a_n are rational and of polynomial growth. We denote by Mod and QMod the spaces of modular and quasi-modular forms, respectively. Both are graded rings under multiplication of q-series, with grading determined by the weight.

Eisenstein series. For $k \in \mathbb{Z}_{>0}$ one defines

$$\mathsf{G}_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{d=1}^{\infty} d^{2k-1} \frac{q^d}{1-q^d},$$

where B_{2k} are the Bernoulli numbers determined by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$

The series G_{2k} is a modular form of weight $2k$ for $k > 1$, while G_2 is quasimodular of weight 2. It is often convenient to introduce the *normalized Eisenstein series*

$$E_{2k}(q) := -\frac{4k}{B_{2k}} \mathsf{G}_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{d=1}^{\infty} d^{2k-1} \frac{q^d}{1-q^d}, \qquad k \in \mathbb{N}_{>0}.$$

The first three play a central role and are traditionally denoted in Ramanujan's notation by

$$P = E_2, \qquad Q = E_4, \qquad R = E_6.$$

Here Q and R are modular forms, while P is quasimodular with $p_P(x) = \frac{6x}{2\pi i}$.

It is a classical fact that the rings of modular and quasimodular forms of the full modular group have the explicit structure

$$\mathrm{Mod} = \mathbb{C}[Q, R], \qquad \mathrm{QMod} = \mathbb{C}[P, Q, R].$$

A fundamental property of the ring QMod is that it is closed under the differential operator

$$D_q = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.$$

In particular, using Ramanujan's relations one finds

$$D_q(P) = \frac{P^2 - Q}{12}, \qquad D_q(Q) = \frac{PQ - R}{3}, \qquad D_q(R) = \frac{PR - Q^2}{2}.$$

The operator D_q acts as a derivation of degree $+2$, meaning that it increases the weight of a quasi-modular form by 2.

Chapter 2

Geometry and integrable structures of the double ramification cycle

2.1 Double ramification cycle

The double ramification cycle describes the locus of curves that admit a meromorphic function with prescribed zero and pole orders. Its construction is naturally formulated via the moduli spaces of relative and rubber stable maps, which provides the geometric framework for defining and analyzing its fundamental class in cohomology.

2.1.1 Moduli space of relative maps

This section follows the exposition in [Kat07]. Let X be a projective manifold and $D \subset X$ a divisor. Fix integers $g, n, n_1, n_2 > 0$ with $n = n_1 + n_2$. We aim to construct the moduli space of stable maps to X relative to D, denoted by

$$\overline{M}_{g,n}(X/D, \Gamma),$$

where Γ encodes the relevant topological data of the maps.

Expanded targets. Given a tuple of positive integers $\mu = (\mu_1, \ldots, \mu_{n_1})$, consider a marked pre-stable curve

$$(C, p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2})$$

together with a map $f : C \to X$ such that

$$f^*D = \sum_{i=1}^{n_1} \mu_i p_i.$$

Let $L = N_{D/X}$ be the normal bundle of D in X, and set

$$\overline{\mathbb{P}} := \mathbb{P}_D(L \oplus 1_D),$$

the projective completion of L. The zero and infinity sections of L determine divisors D_0 and D_∞ in $\overline{\mathbb{P}}$. For $k \geq 1$, define the expanded target

$$X_k := X \sqcup \overline{\mathbb{P}}_1 \sqcup \cdots \sqcup \overline{\mathbb{P}}_k,$$

where the copies $\overline{\mathbb{P}}_i$ are glued by identifying

$$D \subset X \text{ with } D_\infty \subset \overline{\mathbb{P}}_1, \qquad D_0 \subset \overline{\mathbb{P}}_i \text{ with } D_\infty \subset \overline{\mathbb{P}}_{i+1}, \quad i = 1, \ldots, k-1.$$

The singular locus of X_k is thus the disjoint union of $k-1$ copies of D.

Finally, note that the group

$$\mathrm{Aut}(X_k) \cong (\mathbb{C}^*)^k$$

acts on X_k by scaling the fibers of the $\mathbb{P}^1$-bundle $\overline{\mathbb{P}} \to D$. Concretely, if we denote homogeneous coordinates on the fiber of $\overline{\mathbb{P}}$ by $[u : v]$, the standard $\mathbb{C}^*$–action is given by

$$\lambda \cdot [u : v] = [\lambda u : v], \qquad \lambda \in \mathbb{C}^*,$$

which scales the L–direction while fixing the trivial direction. In particular, the divisors $D_0 = \{u = 0\}$ and $D_\infty = \{v = 0\}$ are fixed under this action. On X_k, each copy $\overline{\mathbb{P}}_i$ admits such an action independently, yielding the product action of $(\mathbb{C}^*)^k$.

Relative topological type. We now introduce the discrete data that controls maps into such expansions.

A *relative topological type* Γ is a combinatorial object encoding the discrete data of a relative stable map to (X, D). Concretely, it consists of:

- a finite set of vertices $V(\Gamma)$, each representing an irreducible component of the source curve;
- a genus assignment $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$;
- a degree assignment $d : V(\Gamma) \to H_2(X, \mathbb{Z})$, prescribing the curve class of each component;
- two collections of marked points:
  - relative markings $R = \{1, \ldots, n_1\}$ mapping into $D_0 \subset \overline{\mathbb{P}}_k \subset X_k$, together with an incidence map $a_R : R \to V(\Gamma)$ and multiplicities $\mu : R \to \mathbb{Z}_{\geq 1}$ recording their contact orders with D;
  - interior markings $I = \{1, \ldots, n_2\}$ mapping to $X \setminus D$, with incidence map $a_I : I \to V(\Gamma)$.

In other words, Γ encodes the combinatorial skeleton of a relative stable map, recording both the topology of the domain curve and the distribution of marked points.

Maps of type Γ. Given a relative topological type Γ, a *map of type* Γ consists of a marked curve

$$(C, p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2})$$

together with a morphism $f : C \to X_k$ such that:

- C is a disjoint union of pre-stable curves C_v indexed by the vertices $v \in V(\Gamma)$;
- each component C_v is connected of arithmetic genus $g(v)$;
- the class of the map on each component satisfies

  $$(\pi \circ f|_{C_v})_*[C_v] = d(v);$$

  where $\pi : X_k \to X$ is the map that is the identity on X and projects each $\overline{\mathbb{P}}_i$ to D.
- the relative markings p_j lie on the component $C_{a_R(j)}$;
- the interior markings q_j lie on the component $C_{a_I(j)}$;
- the pullback of the divisor D is given by

  $$f^* D = \sum_{i \in R} \mu(i)\, p_i.$$

Hence, a map of type Γ realizes this skeleton geometrically by specifying the curve, its decomposition, and the placement of marked points with tangency conditions along D.

Pre-deformability and stability. To obtain a proper moduli space, one must impose conditions ensuring good behavior under degeneration.

A map $f : C \to X_k$ is said to be *pre-deformable* if for each singular divisor $D_i \subset X_k$ the preimage $f^{-1}(D_i)$ consists only of nodes of C, such that at every node $p \in f^{-1}(D_i)$ the two branches of C are mapped to different irreducible components of X_k and meet D_i with equal contact order. A pre-deformable map $f : C \to X_k$ is called *stable* if it admits only finitely many automorphisms. This is the usual finiteness condition ensuring that the resulting moduli space is a Deligne–Mumford stack.

The moduli stack. Thus, for a fixed relative topological type Γ, the data above assembles into a Deligne–Mumford stack

$$\overline{M}_{g,n}(X/D,\Gamma),$$

called the *moduli space of relative stable maps*, which parameterizes isomorphism classes of stable pre-deformable maps to X_k for varying k. This provides the natural compactification of the space of maps to X with prescribed tangency along D. The moduli space carries a virtual fundamental class of complex dimension

$$\operatorname{vdim}\overline{M}_{g,n}(X/D,\Gamma) = \sum_{v\in V(\Gamma)} \big((\dim X-3)(1-g(v)) + \langle c_1(TX)-D, d(v)\rangle\big) + n,$$

2.1.2 Moduli space of rubber maps

In the construction of $\overline{M}_{g,n}(X/D,\Gamma)$ we considered maps to expansions X_k containing chains of $\mathbb{P}^1$–bundles. It is often useful to further quotient by the natural $\mathbb{C}^*$–action on these fibers, leading to the notion of *rubber maps*. Intuitively, one studies maps to a chain of $\mathbb{P}^1$–bundles but only up to rescaling in the fiber direction.

Rubber targets. Let X be a projective manifold and L a line bundle on X. Set

$$\overline{\mathbb{P}} = \mathbb{P}_X(L\oplus 1_X),$$

with distinguished divisors X_0 and X_∞ denoting the zero and infinity sections. We consider stable maps to $\overline{\mathbb{P}}$ relative to X_0 and X_∞, where we quotient by the $\mathbb{C}^*$–factor dilating the fibers of $\overline{\mathbb{P}} \to X$.

For $k \geq 0$, define the expanded rubber target

$$P_k := \overline{\mathbb{P}}_0 \sqcup_X \overline{\mathbb{P}}_1 \sqcup_X \cdots \sqcup_X \overline{\mathbb{P}}_k,$$

by gluing $X_\infty \subset \overline{\mathbb{P}}_i$ to $X_0 \subset \overline{\mathbb{P}}_{i+1}$ for $i = 0,\dots,k-1$. The resulting space P_k has two distinguished divisors:

$$D_\infty = X_\infty \subset \overline{\mathbb{P}}_0, \qquad D_0 = X_0 \subset \overline{\mathbb{P}}_k.$$

Its automorphism group is

$$\operatorname{Aut}(P_k) \cong (\mathbb{C}^*)^{k+1},$$

acting by fiberwise scalings on each copy of $\overline{\mathbb{P}}$.

Rubber topological type. A *rubber topological type* Γ consists of:

- a finite set of vertices $V(\Gamma)$, representing components of the source curve;
- a genus assignment $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$;
- a degree assignment $d : V(\Gamma) \to H_2(X,\mathbb{Z})$;
- two sets of relative markings:

  - $R_0 = \{1, \dots, r_0\}$ indexing markings mapping to D_0,
  - $R_\infty = \{1, \dots, r_\infty\}$ indexing markings mapping to D_∞,

  together with incidence maps

$$a_0 : R_0 \to V(\Gamma), \qquad a_\infty : R_\infty \to V(\Gamma),$$

  and multiplicity assignments

$$\mu_0 : R_0 \to \mathbb{Z}_{\geq 1}, \qquad \mu_\infty : R_\infty \to \mathbb{Z}_{\geq 1};$$

- a set of interior markings $I = \{1, \dots, n_0\}$ with assignment

$$a_I : I \to V(\Gamma).$$

We require that the total number of markings satisfies

$$n = r_0 + r_\infty + n_0.$$

In other words, a rubber topological type records the same discrete data as in the relative case, but now with two boundary divisors D_0 and D_∞, together with interior markings, such that the total number of markings is fixed.

Maps of rubber type. Let Γ be a rubber topological type. A *map of type* Γ consists of a marked curve

$$(C, p_1^0, \dots, p_{r_0}^0, p_1^\infty, \dots, p_{r_\infty}^\infty, q_1, \dots, q_{n_0})$$

together with a morphism $f : C \to P_k$ such that:

- C is a disjoint union of pre-stable curves C_v indexed by $v \in V(\Gamma)$;
- each component C_v has arithmetic genus $g(v)$;
- the class of the map on each component satisfies

$$(\pi \circ f|_{C_v})_*[C_v] = d(v);$$

  where $\pi : P_k \to X$ is the natural projection, restricting on each copy $\overline{\mathbb{P}}_i$ to the bundle projection $\overline{\mathbb{P}} \to X$.
- the interior markings q_j lie on the component $C_{a_I(j)}$;
- the boundary markings p_i^0 lie on $C_{a_0(i)}$, while the markings p_i^∞ lie on $C_{a_\infty(i)}$;
- the pullback of the boundary divisors satisfies

$$f^*D_0 = \sum_{i \in R_0} \mu_0(i)\, p_i^0, \qquad f^*D_\infty = \sum_{i \in R_\infty} \mu_\infty(i)\, p_i^\infty.$$

Thus, maps of rubber type realize the combinatorial data of Γ while recording prescribed tangency conditions along both D_0 and D_∞.

Pre-deformability and stability. The notion of pre-deformability is defined exactly as in the relative case. Nodes mapping to singular divisors must join components with equal contact order. Stability is also analogous, requiring finitely many automorphisms.

The rubber moduli stack. For a fixed rubber topological type Γ, there exists a proper Deligne–Mumford stack

$$\overline{M}^{\sim}_{g,n}(X,\Gamma),$$

called the *moduli space of rubber stable maps*, which parameterizes isomorphism classes of stable pre-deformable maps to P_k for varying k. This stack carries a virtual fundamental class of expected dimension

$$\operatorname{vdim} \overline{M}^{\sim}_{g,n}(X,\Gamma) = \sum_{v\in V(\Gamma)} \big((\dim X - 2)(1-g(v)) + \langle c_1(TX), d(v)\rangle\big) + r_0 + r_\infty + n_0 - 1.$$

Specialization to $\mathbb{P}^1$. We have described the general construction of the moduli space of relative and rubber maps to a pair (X,D). A particularly important case arises when we restrict to $X = \mathbb{P}^1$ with boundary divisors $D_0 = \{0\}$ and $D_\infty = \{\infty\}$. Here the expanded targets are chains of $\mathbb{P}^1$'s glued at 0 and ∞, and rubber maps correspond to stable maps into such expansions, modulo rescaling in the $\mathbb{C}^*$–direction. This specialization therefore yields a moduli space

$$\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+),$$

which parameterizes stable relative maps of connected genus g curves to the rubber $\mathbb{P}^1$, with prescribed ramification profiles A_- and A_+ over 0 and ∞, together with n_0 additional interior markings.

In the general setting, one works with a fixed relative or rubber topological type Γ, which specifies the decomposition of the domain curve into components, their genera, degrees, and the incidence of marked points. In the case of $\mathbb{P}^1$, however, it is more natural to drop the Γ–notation and describe the moduli space directly in terms of ramification data. This is because the ramification profile $A = (a_1,\ldots,a_n)$ already determines the global degree of the map. Moreover, the stack $\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)$ automatically incorporates all possible degenerations of the source curve into multiple components, so that the summation over admissible topological types Γ is built into its definition.

2.1.3 Fundamental class and its properties

Let $A = (a_1,\ldots,a_n) \in \mathbb{Z}^n$ be a tuple such that $\sum_i a_i = 0$. Let A_+ denote a partition of A consisting of all the positive integers in A, and let A_- denote a partition of A consisting of all the negative integers in A, and let n_0 be the number of a_i's that are equal to 0. Let $\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)$ be the moduli space of stable relative maps of connected genus g curves to the rubber, with ramification profiles A_- and A_+ over the points $0,\infty \in \mathbb{P}^1$, respectively. Let

$$\mathsf{src}\colon \overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+) \to \overline{M}_{g,n}$$

be the source map that forgets the stable relative map and retains the stabilization of the source curve. Moreover, the space $\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)$ is endowed with the virtual fundamental class, whose Poincaré dual is denoted by

$$\left[\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)\right]^{\mathrm{vir}} \in H_{2(2g-3+n)}(\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)).$$

Define the double ramification cycle $\mathrm{DR}_g(A)$ for a given tuple A as:

$$\mathrm{DR}_g(A) := \mathsf{src}_* \left[\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)\right]^{\mathrm{vir}} \in H^{2g}(\overline{M}_{g,n}).$$

Basic properties. The double ramification cycle satisfies a number of basic identities which illustrate its structure. In genus zero, the cycle is trivial:

$$\mathrm{DR}_0(a_1,\ldots,a_n) = 1 \in H^0(\overline{M}_{0,n},\mathbb{Q}).$$

Moreover, let

$$\pi : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$$

be the forgetful map dropping the last marking. Then DR_g is compatible with forgetting marked points in the sense that

$$\mathrm{DR}_g(a_1,\ldots,a_n,0)=\pi^*\mathrm{DR}_g(a_1,\ldots,a_n).$$

Finally, in the degenerate case where all entries of A vanish, one obtains a λ–class:

$$\mathrm{DR}_g(0,\ldots,0) = (-1)^g\lambda_g \in H^{2g}(\overline{M}_{g,n},\mathbb{Q}),$$

where λ_g denotes the top Chern class of the Hodge bundle over $\overline{M}_{g,n}$.

Further properties. The double ramification cycle can be seen as a *partial CohFT*. It defines a cohomological field theory with respect to the infinite-dimensional $\mathbb{C}$–vector space V generated by $\{e_i\}_{i\in\mathbb{Z}}$, equipped with the metric

$$\eta(e_i,e_j) = \delta_{i+j,0}, \qquad \text{and unit } e_0,$$

via the identification

$$\mathrm{DR}_g(a_1,\ldots,a_n) = c_{g,n}(e_{a_1}\otimes\cdots\otimes e_{a_n}).$$

Here the structure satisfies all CohFT axioms except for the loop constraint, hence a partial CohFT.

A fundamental result is that the cycle depends polynomially on the ramification data $A=(a_1,\ldots,a_n)$. More precisely, the restriction of $\mathrm{DR}_g(A)$ to the compact type locus $\overline{M}^{\mathrm{ct}}_{g,n}$ is a homogeneous polynomial of degree $2g$ in the variables $a_1,\ldots,a_n$, with coefficients in $H^{2g}(\overline{M}^{\mathrm{ct}}_{g,n})$. In general, $\mathrm{DR}_g(A)$ is an even polynomial of degree $2g$ in the variables $a_1,\ldots,a_n$, with coefficients in $H^{2g}(\overline{M}_{g,n})$.

An additional property, proved in [Bur+15], describes the behavior of $\mathrm{DR}_g(A)$ under pushforward along a forgetful morphism. Let

$$\pi : \overline{M}_{g,n+g} \longrightarrow \overline{M}_{g,n}$$

be the map forgetting the last g marked points. Then one has

$$\pi_*\big(\mathrm{DR}_g(a_1,\ldots,a_{n+g})\big)=g!\,a_{n+1}^2\cdots a_{n+g}^2\,[\overline{M}_{g,n}].$$

Explicit formulas for the double ramification cycle will be presented in the following sections, both via Hain's description on the locus of curves of compact type, and through the general formula of Pixton.

2.2 Hain's divisor

On the locus of curves of compact type $M^{\mathrm{ct}}_{g,n}\subset\overline{M}_{g,n}$ (curves without non-separating nodes), the double ramification cycle admits a remarkably explicit form due to Hain [Hai12]. The formula reads

$$\mathrm{DR}_g(A)|_{M^{\mathrm{ct}}_{g,n}} = \frac{1}{g!}\left(\sum_{i=1}^{n}\frac{a_i^2}{2}\psi_i - \sum_{\substack{0\le h\le g\\ S\subset\{1,\ldots,n\}}}\frac{a_S^2}{4}\,\delta_{h,S}\right)^g,$$

where $a_S=\sum_{i\in S}a_i$. Here ψ_i denotes the ψ–class associated to the i–th marked point, and $\delta_{h,S}$ is the boundary divisor corresponding to a separating node, one component of genus h carries the markings indexed by S, and the complementary component of genus $g-h$ carries the remaining markings.

Hain's formula is especially useful for computations involving λ_g. Since the top Hodge class λ_g vanishes outside $M^{\mathrm{ct}}_{g,n}$, intersection numbers of the form $\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_g(\cdots)$ reduce immediately to compact type, where the explicit expression above is valid.

Example 8. As an illustration, consider the integral appearing in [OP23, p. 20], which will be used in later chapters. Let $A = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$ and let

$$\pi : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$$

be the forgetful map that forgets the first marked point. Then one has

$$\int_{\overline{M}_{g,n}} \pi_*(\mathrm{DR}_g(A))\,\lambda_g\lambda_{g-1}\prod_{i=1}^{n}\psi_i\sum_{i=1}^{n}\frac{1}{\psi_i} = (-1)^g\frac{B_{2g}}{4g}\frac{(2g-1)!}{(2g-2+n)!}\frac{a_1^2}{2^{2g-2}}\sum_{\sum b_i=g-1}\prod_{i=1}^{n+1}\frac{a_i^{2b_i}}{(2b_i+1)!}.$$

The key simplification comes from Hain's formula, together with the fact that $\lambda_g\lambda_{g-1}\psi_1\cdots\psi_{n-1}$ vanishes on all boundary divisors of $\overline{M}_{g,n}$. As a result, almost all terms in the pushforward $\pi_*\mathrm{DR}_g(a_1, \ldots, a_{n+1})$ disappear, leaving only three types of contributions:

i. terms with at least one factor of ψ_1, which push forward to κ–classes,

ii. pure ψ–class contributions,

iii. terms involving rational tails divisors, which are contracted by the pushforward.

Thus, the evaluation of the integral reduces to computing pairings of ψ– and κ–classes with $\lambda_g\lambda_{g-1}$, and this is accomplished by the standard socle evaluation formula of Faber [GP98].

2.3 Pixton's formula

Pixton proposed a conjectural formula for the double ramification cycle as a tautological class on $\overline{M}_{g,n}$, expressed via weighted sums over stable graphs. This conjecture was later established in [Jan+17], providing a powerful combinatorial description of $\mathrm{DR}_g(A)$ within the tautological ring. We now present a general expression for the double ramification cycle in terms of stable graphs and ψ–classes, following [Jan+17].

The result. For $g \geq 0$ and ramification data $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum_i a_i = 0$, the double ramification cycle is given by

$$\mathrm{DR}_g(A) = 2^{-g}\,P_g^g(A) \in H^{2g}(\overline{M}_{g,n}),$$

where $P_g^g(A)$ denotes the degree-g part of a tautological class $P_g(A)$ defined via a sum over stable graphs of genus g with n legs.

The formula. Let Γ be a stable graph of genus g with n legs. Denote by $V(\Gamma)$ its set of vertices, $H(\Gamma)$ its half-edges, and $E(\Gamma)$ its edges. Then the degree-d component of Pixton's class is defined by

$$P_g^{d,r}(A) = \sum_{\Gamma}\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{w\in W_{\Gamma,r}}(\mathrm{gl}_\Gamma)_*\left(\prod_{i=1}^{n}e^{a_i^2\psi_{\ell_i}}\cdot\prod_{e=(h,h')\in E(\Gamma)}\frac{1-e^{-w(h)w(h')(\psi_h+\psi_{h'})}}{\psi_h+\psi_{h'}}\right).$$

Weightings. For a given graph Γ, a *weighting* with respect to A is a map

$$w : H(\Gamma) \longrightarrow \mathbb{Z}$$

satisfying:

- for each leg ℓ_i corresponding to the i-th marking, $w(\ell_i) = a_i$;
- for each edge $e = (h, h')$, $w(h) + w(h') = 0$;
- for each vertex v, the sum of weights of all incident half-edges vanishes.

Since Γ may carry infinitely many weightings, Pixton introduces a regularization, one works modulo a positive integer r, defining

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\},$$

with all conditions interpreted modulo r. The resulting finite set of weightings is denoted $W_{\Gamma,r}$. Pixton proves that for fixed g, A, d, the class $P_g^{d,r}(A)$ is polynomial in r for all sufficiently large r, and defines

$$P_g^d(A) := P_g^{d,r}(A)\big|_{r=0},$$

i.e. the constant term of this polynomial.

Graph contributions. Each summand comes from pushing forward classes from

$$\overline{M}_\Gamma = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)}$$

via the gluing map gl_Γ. The contribution factors are:

- Legs: for each marking i, the factor $e^{a_i^2 \psi_{\ell_i}}$ arises;
- Edges: for each edge $e = (h, h')$, the factor

  $$\frac{1 - e^{-w(h)w(h')(\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}}$$

  encodes the interaction between the two half-edges.

The denominator is well-defined since it formally divides the numerator, and the expression is symmetric in h, h'.

Polynomiality. The key structural result is Pixton's polynomiality theorem, which states that for fixed g, A, and d, the class $P_g^{d,r}(A)$ is polynomial in r (for r sufficiently large). The tautological class $P_g^d(A)$ is then defined as its evaluation at $r = 0$. Thus Pixton's formula provides a universal expression for $\mathrm{DR}_g(A)$ as a tautological class, encoded entirely in the combinatorics of stable graphs and ψ–classes.

2.4 Buryak–Shadrin–Spitz–Zvonkine splitting formula

The double ramification cycle admits a recursive structure under degenerations of curves, formalized in the splitting formula of Buryak–Shadrin–Spitz–Zvonkine [Bur+15]. This relation expresses the product of ψ–classes with $\mathrm{DR}_g(A)$ in terms of double ramification cycles of lower genus, glued along relative marked points.

The result. Let $A = (a_1, \ldots, a_n)$ be integers with $\sum_i a_i = 0$ and assume $a_s \neq 0$ for some $s \in \{1, \ldots, n\}$. Then the splitting formula reads

$$a_s \psi_s \, \mathrm{DR}_g(A) = \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ p \geq 1 \\ g_1, g_2 \geq 0}} \sum_{\substack{k_1,\ldots,k_p > 0 \\ \sum k_i = \sum_{i \in I} a_i \\ g_1 + g_2 + p - 1 = g}} \frac{\rho}{r} \cdot \frac{\prod_{i=1}^{p} k_i}{p!} \mathrm{DR}_{g_1}(A_I, -k_1, \ldots, -k_p) \boxtimes \mathrm{DR}_{g_2}(A_J, k_1, \ldots, k_p),$$

where the terms are described below.

Description of terms.

- The partition $I \sqcup J = \{1, \ldots, n\}$ satisfies $\sum_{i \in I} a_i > 0$, ensuring that the balancing condition holds.
- The integers $k_1, \ldots, k_p$ record new ramification orders inserted at the nodes, chosen such that

$$\sum_{i \in I} a_i - \sum_{j=1}^{p} k_j = 0.$$

- The genera g_1, g_2 correspond to the two components, with the condition $g_1 + g_2 + p - 1 = g$ ensuring that the total genus is preserved after gluing.
- The coefficient ρ depends on the side containing the marking s, and is defined by

$$\rho = \begin{cases} r'' & \text{if } s \in I, \\ -r' & \text{if } s \in J, \end{cases}$$

  where $r' = 2g_1 - 2 + |I| + p$ and $r'' = 2g_2 - 2 + |J| + p$ are the numbers of additional branch points on each component.
- The factor $r = 2g - 2 + n$ is the total number of branch points for the initial cycle $\mathrm{DR}_g(A)$.

Interpretation. The formula expresses the effect of multiplying a double ramification cycle by a ψ–class at a nonzero marking. The right-hand side involves only double ramification cycles of strictly simpler type (either lower genus or fewer markings), glued along new points with positive ramification orders. Thus the splitting formula provides a recursive tool for computing intersection numbers involving $\mathrm{DR}_g(A)$.

Example 9. For any $g \geq 2$ and $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\neq 0}$, consider the integral

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(a_1, \ldots, a_n) \, \lambda_g \lambda_{g-2} \prod_{i=1}^{n} \psi_i \sum_{i=1}^{n} \frac{1}{\psi_i}.$$

Let $a := \sum_{i=1}^n a_i$, then the splitting formula provides the recursion

$$a_1 \cdot \int_{\overline{\mathcal{M}}_{g,n+1}} \mathrm{DR}_g(a_1,\dots,a_n,-a)\,\lambda_g\lambda_{g-2}\prod_{i=1}^n \psi_i =$$

$$\sum_{\substack{S_1\sqcup S_2=\{2,\dots,n\}\\ S_2\neq\emptyset}} \frac{-(2g-1+|S_2|)|S_1|}{2g-1+n}\cdot\left(\sum_{i\in S_2} a_i\right)\cdot$$

$$\int_{\overline{\mathcal{M}}_{g,|S_2|+1}} \mathrm{DR}_g\left(\{a_j\}_{j\in S_2},-\sum_{i\in S_2}a_i\right)\lambda_g\lambda_{g-2}\prod_{i=1}^{|S_2|}\psi_i$$

$$+\sum_{\substack{S_1\sqcup S_2=\{2,\dots,n\}\\ g_1+g_2=g\\ g_1,g_2\geq 1}} \frac{2g_1+|S_1|}{2g-1+n}\cdot\left(a_1+\sum_{i\in S_2}a_i\right)\cdot$$

$$\int_{\overline{\mathcal{M}}_{g_1,|S_1|+2}} \mathrm{DR}_{g_1}\left(\{a_j\}_{j\in S_1},a_1+\sum_{i\in S_2}a_i,-a\right)\lambda_{g_1}\lambda_{g_1-1}\prod_{i=1}^{|S_1|}\psi_i\cdot$$

$$\int_{\overline{\mathcal{M}}_{g_2,|S_2|+2}} \mathrm{DR}_{g_2}\left(\{a_j\}_{j\in S_2},a_1,-a_1-\sum_{i\in S_2}a_i\right)\lambda_{g_2}\lambda_{g_2-1}\prod_{i=1}^{|S_2|}\psi_i.$$

This recursion reduces the computation of the integral to simpler terms,

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \mathrm{DR}_g(a_1,\dots,a_n,-a)\,\lambda_g\lambda_{g-1}\prod_{i=1}^{n-1}\psi_i.$$

which can be evaluated explicitly using Hain's formula together with Faber's socle intersection formula. By iterating the recursion, one eventually obtains a closed expression, although in a complicated but computable form. As a concrete illustration, for $n=2$ one finds

$$\int_{\overline{\mathcal{M}}_{g,2}} \mathrm{DR}_g(a,-a)\,\lambda_g\lambda_{g-2}\psi_1 = \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} \frac{2g_1}{2g}\cdot\int_{\overline{\mathcal{M}}_{g_1,2}} \mathrm{DR}_{g_1}(a,-a)\,\lambda_{g_1}\lambda_{g_1-1}\cdot\int_{\overline{\mathcal{M}}_{g_2,2}} \mathrm{DR}_{g_2}(a,-a)\,\lambda_{g_2}\lambda_{g_2-1}.$$

2.5 Relations from DR/DZ equivalence

In this section, we review a further property of the double ramification cycle in connection with λ_g. This property arises in the proof of the DR/DZ equivalence conjecture [BGR19; BLS24a], which relates double ramification hierarchies to Dubrovin–Zhang hierarchies. Although the results will not be used in later chapters, we include them for completeness and to highlight new perspectives on relations and computations involving double ramification cycles.

Let $A=(a_1,\dots,a_n,-a)\in\mathbb{Z}^{n+1}$ with $a=\sum_{i=1}^n a_i$. Define

$$S(A)=\left\{\sum_{i=1}^n s_ia_i \;\middle|\; s_i\in\{0,1\}\right\}.$$

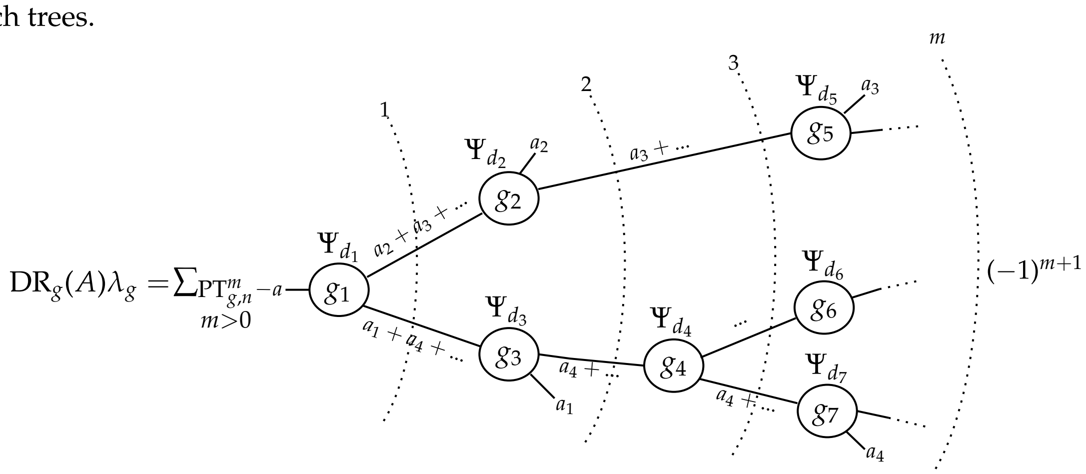
Let $\mathrm{PT}_{g,n}^m$ denote the set of rooted planar trees Γ satisfying the following conditions:

- Vertices: The set of vertices $\mathrm{V}(\Gamma)$ is equipped with a genus function $g:\mathrm{V}(\Gamma)\to\mathbb{Z}_{\geq 0}$, such that

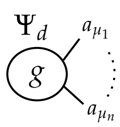
$$\sum_{v\in\mathrm{V}(\Gamma)} g(v)=g.$$

- Legs: The set of legs $\mathrm{L}(\Gamma)$ is labeled by a map $\tilde{a} : \mathrm{L}(\Gamma) \to \{a_1, \dots, a_n, -a\}$, with the root vertex v_{root} carrying at least one leg decorated by $-a$.
- Edges: For each vertex v, let $e_{\mathrm{in}}(v)$ denote the unique ingoing edge to its parent, and $e_{\mathrm{out}}(v)$ the set of outgoing edges to its children.
- Edge decorations: Each edge is assigned a weight $a : \mathrm{E}(\Gamma) \to S(A)$, subject to the balance condition
$$a(e_{\mathrm{in}}(v)) = \sum_{e \in e_{\mathrm{out}}(v)} a(e) + \sum_{l \in \mathrm{L}(v)} \tilde{a}(l),$$
where $\mathrm{L}(v) \subseteq \mathrm{L}(\Gamma)$ is the set of legs attached to v.
- Vertex decorations: Each vertex $v \in \mathrm{V}(\Gamma)$ carries a cohomology class $\Psi_{d(v)} \in H^{2d(v)}(\overline{M}_{g(v),n(v)})$, encoded by a degree function $d : \mathrm{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$.
- Level structure: A level function $l : \mathrm{V}(\Gamma) \to \{1, \dots, m\}$ satisfies:
  - $l(v_{\mathrm{root}}) = 1$;
  - if $e = (v_1, v_2)$ with v_1 the parent of v_2, then $l(v_1) \leq l(v_2)$;
  - the maximum level is $m = \max_{v \in \mathrm{V}(\Gamma)} l(v)$;
  - for each k, let $L_k = \{v \in \mathrm{V}(\Gamma) \mid l(v) = k\} \neq \emptyset$. Then $\{L_1, \dots, L_m\}$ partitions $\mathrm{V}(\Gamma)$.
- Level constraints: For every $k < m$,
$$\sum_{\substack{v \in \mathrm{V}(\Gamma) \\ l(v) \leq k}} d(v) \leq 2 \sum_{\substack{v \in \mathrm{V}(\Gamma) \\ l(v) \leq k}} g(v) - 1,$$
while for the final level,
$$\sum_{\substack{v \in \mathrm{V}(\Gamma) \\ l(v) \leq m}} d(v) = 2g.$$

Let $\lambda_g = c_g(\mathbb{E}) \in H^{2g}(\overline{M}_{g,n})$ denote the top Chern class of the Hodge bundle $\mathbb{E}$ on $\overline{M}_{g,n}$. It was shown in [BGR19; BLS24a] that the product $\mathrm{DR}_g(A) \cdot \lambda_g$ admits an explicit expansion as a sum over such trees.



where for a generic node:



The decoration at a vertex is given by the degree-d part of

$$\Psi_d = \left(\frac{1}{\prod_{i=1}^{n}(1 - a_{\mu_i}\psi_i)} \right)^{[d]} \in H^{2d}(\overline{M}_{g,n}).$$

To apply this construction concretely, we introduce a computational framework. Fix $m \in \mathbb{Z}_{>0}$ and let $N = \{1, \ldots, n\}$. Consider a partition

$$N = I_1 \sqcup \ldots \sqcup I_m,$$

with $n_\alpha = |I_\alpha|$ for each $\alpha \in \{1, \ldots, m\}$.

Intersection numbers. For $\alpha \in \{1, \ldots, m-1\}$, define

$$\langle \lambda_{g_\alpha} \rangle_{I_\alpha} = \int_{\overline{M}_{g_\alpha, n_\alpha+2}} \lambda_{g_\alpha}\, \psi_1 \cdots \psi_{n_\alpha} \left(\prod_{\substack{i=1 \\ \mu_i \in I_\alpha}}^{n_\alpha} \frac{1}{1 - a_{\mu_i}\psi_i} \cdot \frac{1}{1 - a_{I_{\alpha+1}\sqcup\cdots\sqcup I_m}\psi_{n+1}} \right)^{[2g_\alpha - 1]} a_{I_{\alpha+1}\sqcup\cdots\sqcup I_m},$$

where we use the shorthand $a_S = \sum_{i \in S} a_i$. For the final block $\alpha = m$, we set

$$\langle \tilde{\lambda}_{g_m - 1} \rangle_{I_m} = \int_{\overline{M}_{g_m, n_m+1}} \lambda_{g_m - 1}\, \psi_1 \cdots \psi_{n_m - 1} \left(\prod_{\substack{i=1 \\ \mu_i \in I_m}}^{n_m} \frac{1}{1 - a_{\mu_i}\psi_i} \right)^{[2g_m]},$$

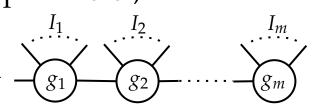
and

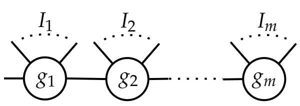
$$\langle \tilde{\lambda}_{g_m - 2} \rangle_{I_m} = \int_{\overline{M}_{g_m, n_m+1}} \lambda_{g_m - 2}\, \psi_1 \cdots \psi_{n_m} \left(\prod_{\substack{i=1 \\ \mu_i \in I_m}}^{n_m} \frac{1}{1 - a_{\mu_i}\psi_i} \right)^{[2g_m]}.$$

Associated polynomials. For $A = (a_1, \ldots, a_n, -a) \in \mathbb{Z}^{n+1}$, define

$$P_{g,n} = \int_{\overline{M}_{g,n+1}} \mathrm{DR}_g(A)\, \lambda_g \lambda_{g-1}\, \psi_1 \cdots \psi_{n-1} + \delta \int_{\overline{M}_{g,n+1}} \mathrm{DR}_g(A)\, \lambda_g \lambda_{g-2}\, \psi_1 \cdots \psi_n,$$

where δ is an auxiliary parameter, and

$Q_{g,n} = \sum_{\substack{g_1+\ldots+g_m=g \\ I_1\sqcup\ldots\sqcup I_m=N \\ m>0}}$  $(-1)^{m-1}\prod_{\alpha=1}^{m-1} \langle \lambda_{g_\alpha} \rangle_{I_\alpha} \langle \tilde{\lambda}_{g_m-1} \rangle_{I_m}$

$+\ \delta \sum_{\substack{g_1+\ldots+g_m=g \\ I_1\sqcup\ldots\sqcup I_m=N \\ m>0}}$  $(-1)^{m-1}\prod_{\alpha=1}^{m-1} \langle \lambda_{g_\alpha} \rangle_{I_\alpha} \langle \tilde{\lambda}_{g_m-2} \rangle_{I_m}$

We now state the following result.

Proposition 2.1. The polynomials satisfy the identity

$$P_{g,n} = Q_{g,n}.$$

Proof. We prove the equality for the second term, namely $[\delta]P_{g,n} = [\delta]Q_{g,n}$; the argument for the first term is analogous. Using the relation for $\mathrm{DR}_g(A)\lambda_g$, we obtain

$$\int_{\overline{M}_{g,n+1}} \mathrm{DR}_g(A)\,\lambda_g\lambda_{g-2}\psi_1\cdots\psi_n = \int_{\overline{M}_{g,n+1}} \sum_{\substack{\Gamma\in\mathrm{PT}^m_{g,n}\\ m>0}} \Gamma\cdot\lambda_{g-2}\,\psi_1\cdots\psi_n.$$

Consider the case where Γ is a planar rooted tree whose root vertex has two outgoing edges, each connecting to a child vertex. Suppose the root vertex is decorated with n_1-2 legs, while the two child vertices carry n_2 and n_3 legs, respectively. The contribution of this tree Γ is given by the integral

$$\int_{\overline{M}_{g,n+1}} \Gamma\cdot\lambda_{g-2}\psi_1\cdots\psi_n.$$

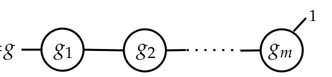
The optimal contribution from the root vertex in this configuration is

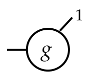
$$\int_{\overline{M}_{g_1,n_1+1}} \lambda_{g_1}\psi_1\cdots\psi_{n_1-2}\Psi_{2g_1-1}.$$

By dimension counting, this integral vanishes. The same reasoning applies when the root vertex has more than two children, in each case, the number of ψ–classes is insufficient to produce a top-dimensional form, and the integral is zero. This argument extends to any vertex of the tree, not just the root. Hence, only planar rooted trees with a bamboo-like structure (a single chain of vertices) contribute nontrivially. This proves the claim. □

Example 10. For $n=1$, the polynomials are given by

$$\begin{aligned}\left[\delta^0\right] P_{g,1} &= a_1^{2g}\int_{\overline{M}_{g,2}} \frac{1}{g!\,2^g}\,\lambda_g\lambda_{g-1}(\psi_1+\psi_2)^g\\ &= a_1^{2g}\left(\frac{1}{2^g}\sum_{b_1+b_2=g}\frac{1}{b_1!\,b_2!}\int_{\overline{M}_{g,2}}\lambda_g\lambda_{g-1}\psi_1^{b_1}\psi_2^{b_2}\right).\end{aligned}$$

$\left[\delta^0\right] Q_{g,1} = \sum_{\substack{g_1+\ldots+g_m=g\\ m>1}}$  $(-1)^{m-1}\prod_{\alpha=1}^{m-1}\langle\lambda_{g_\alpha}\rangle\langle\tilde{\lambda}_{g_m-1}\rangle_{\{1\}}$

$+$  $\langle\tilde{\lambda}_{g-1}\rangle_{\{1\}}$

We can simplify this expression to obtain

$$\left[\delta^0\right] Q_{g,1} = a_1^{2g}\left(\int_{\overline{M}_{g,2}}\lambda_{g-1}\psi_1^{2g} + \sum_{\substack{g_1+\cdots+g_m=g\\ m>1}}\prod_{\alpha=1}^{m-1}\int_{\overline{M}_{g_\alpha,2}}\lambda_{g_\alpha}\psi_1^{2g_\alpha-1}\cdot\int_{\overline{M}_{g_m,2}}\lambda_{g_m-1}\psi_1^{2g_m}\right).$$

Thus, we obtain the relation between intersection numbers

$$[\delta^0 a_1^{2g}]P_{g,1}{=}[\delta^0 a_1^{2g}]Q_{g,1}.$$

2.6 Double ramification hierarchy

The double ramification hierarchy is a system of commuting Hamiltonians on an infinite-dimensional phase space, which can be viewed heuristically as the loop space of a fixed vector space. It arises naturally from the geometry of the double ramification cycle and provides an algebraic counterpart to constructions in symplectic field theory, leading to a quantum integrable system formulated in the language of cohomological field theories.

2.6.1 Commutator and local functionals

We reuse the notation and construction in Section 1.8. Fix $N \geq 1$ and consider the N-dimensional vector space V with basis $\{e_\alpha\}_{\alpha=1}^N$. We introduce the formal variables u^α_j, $\alpha = 1, \dots, N$, $j \geq 0$, with $u^\alpha_0 = u^\alpha$, corresponding heuristically to the derivatives $\partial_x^j u^\alpha(x)$ of a formal loop $u : S^1 \to V$. The ring of *differential polynomials* is

$$\mathcal{A} := \mathbb{C}[\![u^\alpha]\!][u^\alpha_{j>0}][\![\varepsilon]\!],$$

and the space of *local functionals* is the image of the quotient map,

$$\mathcal{A} \to \hat{\mathcal{A}} := \mathcal{A}/\big(\mathrm{Im}(\partial_x) \oplus \mathbb{C}\big), \qquad f \mapsto \overline{f} := \int f\, dx.$$

Fourier variables. It is often convenient to work with another set of formal variables $\{p^\alpha_k\}_{k\in\mathbb{Z}}$ defined by the Fourier expansion

$$u^\alpha(x) = \sum_{k\in\mathbb{Z}} p^\alpha_k e^{\mathrm{i}kx}.$$

This change of variables embeds $\mathcal{A}$ into the auxiliary ring

$$\mathbb{C}[\![p^\alpha_{k>0}]\!][p^\alpha_{k\leq 0}][\![\varepsilon]\!][e^{\mathrm{i}x}, e^{-\mathrm{i}x}],$$

and local functionals $\overline{f}$ correspond to the constant Fourier mode of f.

Poisson structure. The space $\hat{\mathcal{A}}$ carries a natural Poisson bracket associated to the Hamiltonian operator $K = \eta\partial_x$, where η is a nondegenerate symmetric bilinear form on V. In terms of the Fourier variables, this bracket is given by

$$\{p^\alpha_i, p^\beta_j\}_{\eta\partial_x} = ij\,\eta^{\alpha\beta}\,\delta_{i+j,0}.$$

2.6.2 Hamiltonian densities

Consider a CohFT $c_{g,n} : V^{\otimes n} \to H^{\mathrm{even}}(\overline{M}_{g,n}, \mathbb{C})$ with unit $e_1 \in V$. Let ψ_1 be the cotangent line class at the first marking and $\lambda_j = c_j(\mathbb{E}) \in H^{2j}(\overline{M}_{g,n})$ the Chern classes of the Hodge bundle $\mathbb{E}$ over $\overline{M}_{g,n}$. The Hamiltonians of the double ramification hierarchy are defined by

$$\overline{g}_{\alpha,d} := \sum_{\substack{g\geq 0,\, n\geq 2\\ 2g-2+n>0}} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{A=(a_1,\dots,a_n)\in\mathbb{Z}^n\\ a_1+\cdots+a_n=0}} \int_{\overline{M}_{g,n+1}} \mathrm{DR}_g(0,A)\lambda_g\,\psi_1^d\,c_{g,n+1}\Big(e_\alpha \otimes \bigotimes_{i=1}^n e_{\alpha_i} p^{\alpha_i}_{a_i}\Big), \tag{2.1}$$

for $\alpha = 1, \dots, N$ and $d \geq 0$.

Polynomiality. The integrals appearing in (2.1) are homogeneous polynomials of degree $2g$ in the integers $a_1, \dots, a_n$. Concretely, one can write

$$\int_{\overline{M}_{g,n+1}} \mathrm{DR}_g(0,A)\lambda_g\,\psi_1^d\,c_{g,n+1}\Big(e_\alpha \otimes \bigotimes_{i=1}^n e_{\alpha_i}\Big) = \sum_{\substack{b_1,\dots,b_n\geq 0\\ b_1+\cdots+b_n=2g\\ \overline{\alpha}=(\alpha,\alpha_1,\dots,\alpha_N)}} P^{b_1,\dots,b_n}_{\overline{\alpha},d;g,n}\, a_1^{b_1}\cdots a_n^{b_n},$$

with coefficients $P^{b_1,\dots,b_n}_{\overline{\alpha},d;g,n}$ depending only on the CohFT and tautological classes on $\overline{M}_{g,n+1}$. Thus the Hamiltonian densities can also be expressed in the u-variables as

$$g_{\alpha,d} = \sum_{\substack{g\geq 0,\, n\geq 2\\ 2g-2+n>0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{b_1,\dots,b_n\geq 0\\ b_1+\dots+b_n=2g\\ \overline{\alpha}=(\alpha,\alpha_1,\dots,\alpha_N)}} P^{b_1,\dots,b_n}_{\overline{\alpha},d;g,n}\, u^{\alpha_1}_{b_1}\cdots u^{\alpha_n}_{b_n}.$$

The corresponding Hamiltonians are

$$\overline{g}_{\alpha,d} = \int g_{\alpha,d}\, dx.$$

Casimirs. The definition extends naturally to $d=-1$, by setting

$$g_{\alpha,-1} := \eta_{\alpha\mu}u^{\mu}, \qquad \overline{g}_{\alpha,-1} = \int \eta_{\alpha\mu}u^{\mu}\, dx,$$

which correspond to Casimir elements for the Poisson bracket.

Integrability. For all $\alpha,\beta = 1,\dots,N$ and $p,q\geq -1$ one has

$$\{\overline{g}_{\alpha,p},\overline{g}_{\beta,q}\}_{\eta\partial_x} = 0.$$

The Hamiltonians $\overline{g}_{\alpha,d}$ define an integrable hierarchy of evolutionary PDEs, as shown initially in [Bur15]. Thus the double ramification hierarchy is a Hamiltonian integrable system associated to arbitrary CohFTs.

The proof of commutativity relies on the geometric behavior of double ramification cycles under boundary degenerations. The main input is the splitting formula of [Bur+15], which expresses products of DR cycles on boundary strata of $\overline{M}_{g,n}$ as sums of glued DR cycles with explicit combinatorial coefficients. When inserted into the definition of the Hamiltonians, this splitting identity matches exactly the algebraic expression for the Poisson brackets of the generating functions. In particular, the positive and negative contributions in the splitting formula cancel after summing over all boundary components, which yields the vanishing of the Poisson brackets. This geometric mechanism provides the machinery needed to prove that the Hamiltonians $\overline{g}_{\alpha,d}$ commute, and hence establishes integrability of the DR hierarchy.

2.6.3 Recursion relations and tau–structure

An important structural property of the double ramification hierarchy is that its Hamiltonian densities satisfy universal recursion relations. These relations are obtained from the intersection theory of ψ–classes with the double ramification cycle (the statement can be found in [Bur+15]), and play a fundamental role in reconstructing the hierarchy from a minimal amount of initial data [BR16b].

Recursion relations. For all $\alpha = 1,\dots,N$ and $p\geq -1$, let $g_{\alpha,-1} = \eta_{\alpha\mu}u^{\mu}$. Then the Hamiltonian densities of the double ramification hierarchy satisfy

$$\partial_x\left(\varepsilon\frac{\partial}{\partial\varepsilon} + \sum_{s\geq 0} u^{\alpha}_s\frac{\partial}{\partial u^{\alpha}_s} - 1\right) g_{\alpha,p+1} = \{g_{\alpha,p},\overline{g}_{1,1}\}, \tag{2.2}$$

Equation (2.2) is remarkable in that it allows the entire hierarchy to be reconstructed from the single Hamiltonian $\overline{g}_{1,1}$. Such a universal recursion had not appeared previously in the theory of integrable systems, and it differs fundamentally from the reconstruction procedures known for classical hierarchies such as KdV. Its independent of the underlying CohFT which indicates that this structure is not accidental, but rather an intrinsic property underlying a wide class of integrable hierarchies. In particular, it highlights the geometric origin of the DR hierarchy, where the interaction of ψ–classes with the DR cycle translates directly into recursion at the level of Hamiltonians.

Tau-structure. The double ramification hierarchy is tau-symmetric, and therefore admits a natural tau-structure. Concretely, the tau-structure is given by the Hamiltonian densities

$$h^{\mathrm{DR}}_{\alpha,p} := \frac{\delta g_{\alpha,p+1}}{\delta u^1},$$

which ensure the compatibility condition required for tau-symmetry.

Let $\tilde{u}^\alpha = \eta^{\mu\nu} h^{\mathrm{DR}}_{\mu,-1}$ be the normal coordinates of the hierarchy, and consider the topological solution defined by the initial condition

$$\tilde{u}^\alpha(x,0;\varepsilon) = x\delta^\alpha_1.$$

The tau-function associated to this solution is expressed as the generating series

$$\mathcal{F}^{\mathrm{DR}}((t^\alpha_i)_{\alpha,i};\varepsilon) = \sum_{g\geq 0} \varepsilon^{2g} F^{\mathrm{DR}}_g((t^\alpha_i)_{\alpha,i}),$$

where

$$F^{\mathrm{DR}}_g((t^\alpha_i)_{\alpha,i}) = \sum_{\substack{n\geq 0\\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n\geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle^{\mathrm{DR}}_g \prod_{i=1}^n t^{\alpha_i}_{d_i}.$$

The series $\mathcal{F}^{\mathrm{DR}}$ is often referred to as the *DR partition function*. Contrary to the Dubrovin–Zhang hierarchy, where the tau-function coincides with the CohFT partition function and its correlators are intersection numbers on $\overline{M}_{g,n}$, the correlators

$$\left\langle \prod_{i=1}^n \tau_{d_i}(e_{\alpha_i}) \right\rangle^{\mathrm{DR}}_g$$

arising from the DR hierarchy are not *a priori* defined as intersection numbers of tautological classes. Thus the DR partition function carries a more indirect geometric meaning, for now, its viewed as an algebraic object determined by the hierarchy rather than as a direct enumerative generating function. Nonetheless, $\mathcal{F}^{\mathrm{DR}}$ encodes the complete integrable structure of the hierarchy and serves as the natural analogue of the topological tau-function in the Dubrovin–Zhang setting. Moreover, in the semisimple case, the DR and DZ hierarchies are equivalent, their tau-functions coincide up to a Miura-type transformation, establishing a strong equivalence between the two theories.

Remark 3. Several important instances of the double ramification hierarchy have been explicitly constructed and shown how to coincide with their Dubrovin–Zhang counterparts [Bur+18; BGR19]. Examples include the trivial CohFT, where the DR hierarchy reduces to the KdV hierarchy (see Section 6); the full Hodge class; Witten's 3-, 4-, and 5-spin classes, yielding the corresponding r–spin hierarchies; and the Gromov–Witten theory of $\mathbb{P}^1$.

Example 11. Following Proposition 10.1 of [Bur+18], let X be a smooth projective variety with $\dim X > 0$ and non-positive first Chern class. Then the associated double ramification hierarchy is

$$\overline{g} = \overline{g}^{[0]} + \frac{\varepsilon^2}{48}\chi(X)\int u^1 u^1_2\, dx,$$

and

$$g_{\alpha,p} = g^{[0]}_{\alpha,p} + \delta_{\alpha,1}\frac{\varepsilon^2}{24}\frac{\chi(X)}{p!}(u^1)^p u^1_2,$$

where $\chi(X)$ is the Euler characteristic of X and u^1 is the variable associated to the unit class.

We sketch the main ideas of the proof. For the CohFT given by the Gromov–Witten classes of X, the degree condition on correlators shows that the potential $\overline{g}$ vanishes unless the number of unit insertions satisfies

$$a = \sum_{i=1}^{M} b_i\Big(\tfrac{\deg\theta_i}{2} - 1\Big) + g(\dim X - 1) - (\dim X - 3) - \langle c_1(X), d\rangle,$$

where $\{1, \theta_1, \ldots, \theta_M\}$ is a homogeneous basis of $H^{\mathrm{even}}(X, \mathbb{Q})$ with $\deg\theta_i \geq 2$. If $\dim X \geq 2$, one checks that $a > g$ for $g \geq 1$, so all contributions vanish, and the only nontrivial case is $g = 1$, $n = 2$. Using the known push-forward formula for $\mathrm{DR}_1(a, -a)$ together with $\lambda_1 = \frac{1}{24}\delta_{\mathrm{irr}}$, one obtains

$$\int_{\mathrm{DR}_1(a,-a)} \lambda_1\, c_{1,2}(e_1^2) = \delta_{a,0}\, \frac{\chi(X)}{24}\, a^2,$$

which produces the correction term in the Hamiltonians.

In the case of target curves ($\dim X = 1$), a simple degeneration argument due to Pandharipande shows that positive degree contributions vanish, since λ_g vanishes on boundary divisors with non-separating nodes. This extends the result to this case as well.

Finally, the Hamiltonian densities $g_{\alpha,p}$ can be computed using the recursion relation [BR16b],

$$\partial_x(D-1)g_{\alpha,p} = \{g_{\alpha,p-1}, (D-2)\overline{g}\}, \qquad D := \sum_{k\geq 0}(k+1)u^\alpha_k \frac{\partial}{\partial u^\alpha_k},$$

which reconstructs the full hierarchy from the genus-zero data together with the correction term above.

Chapter 3

Faber's socle intersection numbers via Gromov–Witten theory of elliptic curves

3.1 Overview

Faber's formula [Fab99] for the socle intersection numbers in the tautological ring of $\mathcal{M}_g$ has now several proofs, with quite different geometric ideas behind them. It can be stated in terms of the Deligne–Mumford compactification of the moduli spaces of curves $\overline{M}_{g,n}$, $g \geq 1$, $n \geq 1$, as

$$\int_{\overline{M}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{d_i} = \frac{(-1)^{g-1} B_{2g} (2g-3+n)!}{2^{2g-1} \cdot (2g)!} \prod_{i=1}^{n} \frac{1}{(2d_i - 1)!!}, \tag{3.1}$$

for all $d_1, \ldots, d_n \geq 0$ such that $d_1 + \cdots + d_n = g - 2 + n$. Here B_{2g} denote the Bernoulli numbers. Originally it was conjectured in [Fab99] and by now it is a well-established statement with the following five proofs based on a variety of quite different ideas:

- Virasoro constraints for the projective plane, [GP98; Giv01a].
- Mumford's formula and related combinatorics, [LX09].
- Intersections with double ramification cycles, [BS11].
- The 3-spin relations and related combinatorics, [Pix13].
- The half-spin relations and related combinatorics, [Gar+19; GZ22].

The purpose of this chapter is to give yet another proof that is based on a new tautological relation that follows from a recent work of Oberdieck–Pixton on the Gromov–Witten theory of the elliptic curve [OP18; OP23].

Although the utility of a new proof of a well-known statement might be questionable, it involves a completely new set of powerful ideas and allows us to exhibit a new tautological relation, which are all to be used in the analysis of the quantum integrable systems of Buryak–Rossi [BR16a].

3.2 New tautological relation

We consider a subset N of the set of stable graphs corresponding to the strata in $\overline{M}_{g,m}$, $g \geq 1$, $m \geq 1$, which have the form of a necklace:

- There are m vertices and to each of them we attach exactly one leaf labeled by i, $i = 1, \ldots, m$.
- There are m edges, and they connect the vertices in one cycle of length m.

There are $(m-1)!/2$ graphs of this shape for $m \geq 2$ (for $m = 2$ it is just one graph, but then with the automorphism group of order 2). The graphs $\Gamma \in N$ are further equipped by the genus function

$g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$, where $V(\Gamma) = \{v_1, \dots, v_m\}$ is the set of vertices, and we assume that the leaf i is attached to v_i, $i = 1, \dots, m$. The genus condition reads $\sum_{i=1}^m g(v_i) = g - 1$.

It will be a bit more convenient to introduce an orientation on these necklace graphs and consider the set of wheels $\vec{N}$ (necklaces with an added choice of orientation); then we have exactly $(m-1)!$ wheels.

Proposition 3.1. We have the following relation in $H^*(\overline{M}_{g,m})$:

$$\frac{1}{(m-1)!} \sum_{\substack{\Gamma \in \vec{N} \\ g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}}} (\mathsf{b}_{\Gamma,g})_* \left(\bigotimes_{i=1}^m \mathrm{DR}_{g(v_i)}(0,1,-1)\, \lambda_{g(v_i)} \right) = \frac{2 \cdot (2g)!}{(-1)^{g-1} B_{2g} (2g-2+m)!} \lambda_g \lambda_{g-1} \prod_{i=1}^m \psi_i \sum_{i=1}^m \frac{1}{\psi_i}. \tag{3.2}$$

Here $\mathsf{b}_{\Gamma,g}\colon \prod_{i=1}^m \overline{M}_{g(v_i),3} \to \overline{M}_{g,m}$ is the boundary map, $\mathrm{DR}_{g(v_i)}(0,1,-1)$ is the double ramification cycle on $\overline{M}_{g(v_i),3}$ assigned to v_i, where the multiplicity 0 is assigned to the marked point corresponding to the leaf.

Proof. This relation comes from two independent computations of the class $c_{g,m}(p^{\otimes m})\lambda_{g-1}$, $g \geq 1$, $m \geq 1$ in [OP23], where $c_{g,m}(p^{\otimes m}) \in H^*(\overline{M}_{g,m}) \otimes \mathbb{C}[[q]]$ are the classes of the cohomological field theory associated to the Gromov–Witten theory of the elliptic curve, where all primary fields are the classes of a point in the target curve, and q is the variable that controls the degree of the stable maps.

Let QMod $= \mathbb{C}[\mathsf{G}_2, \mathsf{G}_4, \mathsf{G}_6]$ be the algebra of quasimodular forms, where $\mathsf{G}_k(q)$ is the k-weighted Eisenstein series is given by

$$\mathsf{G}_k(q) = -\frac{B_k}{2k} + \sum_{n \geq 1} q^n \sum_{d|n} d^k, \qquad k = 2, 4, 6, \dots. \tag{3.3}$$

It is graded by nonnegative even integers, $\mathrm{QMod} = \bigoplus_{\ell=0}^\infty \mathrm{QMod}_{2\ell}$, where the grading is given by the weight of the modular forms $\mathrm{wt}(\mathsf{G}_k) := k$, $k = 2, 4, 6$. Note that for all even $k \geq 2$, $\mathrm{QMod}_k \ni \mathsf{G}_k$. Note also that the operator $q\partial_q$ acts on QMod increasing the weight by 2.

Using the holomorphic anomaly equation, Oberdieck and Pixton prove [OP23, Proposition 6.8] that

$$c_{g,m}(p^{\otimes m})\lambda_{g-1} = \frac{2 \cdot (2g)!}{(-1)^{g-1} B_{2g} (2g-2+m)!} \lambda_g \lambda_{g-1} \prod_{i=1}^m \psi_i \sum_{i=1}^m \frac{1}{\psi_i} \times (q\partial_q)^{m-1} \mathsf{G}_{2g}(q). \tag{3.4}$$

On the other hand, in [OP18, Proof of Theorem 5] they give a formula for $c_{g,m}(p^{\otimes m})$ as a sum over graphs:

$$\begin{aligned} c_{g,m}(p^{\otimes m}) = \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_k (\mathsf{b}_\Gamma)_* \left(\bigotimes_{i=1}^m \Delta_{g_i}\big((k(h))_{h \in v_i}\big) \right) \\ \times \prod_{\substack{e = \{h,h'\} \\ e \text{ is a loop}}} 2(-1)^{k(h)} \left(\frac{B_{k(h)+k(h')+2}}{2\,(k(h)+k(h')+2)} + \mathsf{G}_{k(h)+k(h')+2} \right) \\ \times \sum_w \prod_{\substack{e = \{h,h'\} \\ e \text{ not a loop}}} \frac{(-1)^{k(h')} w(h)^{\,k(h)+k(h')+1}}{1 - q^{w(h)}}. \end{aligned} \tag{3.5}$$

Here

- Γ is a stable graph representing a boundary stratum in $\overline{M}_{g,n}$ and b_Γ is the corresponding boundary map. We think of edges and vertices as subsets of the set $H(\Gamma)$ of the half-edges of Γ.

- The m legs of Γ labeled by $1,\dots,m$ are attached to m pairwise different vertices $v_1,\dots,v_m$ of genera $g_1,\dots,g_m \geq 0$, and there are no further additional vertices.
- Each edge of Γ can be included in a cycle in the graph, that is, if we cut any edge, the graph remains connected. Loops are allowed.
- The sum over k is the sum over all possible maps $k\colon H(\Gamma) \to \mathbb{Z}_{\geq 0}$ such that for every loop $e=(h,h')$ the sum $k(h)+k(h')$ is even and $k(h)=0$ if h is a leaf.
- The sum over w is the sum over all possible systems of "kissing weights" on the set of half-edges of Γ, which are the functions $w\colon \mathrm{H}(\Gamma) \to \mathbb{Z}$ such that
  - $w(h)+w(h')=0$ for every edge $e=\{h,h'\}$;
  - $\sum_{h\in v} w(h) = 0$ for every vertex v.
  - $w(h)=0$ whenever h is a leaf or if it belongs to a loop.
- If $e=(h,h')$ is not a loop, then it is assumed that h is attached to a vertex with smaller index than h'. This gives an orientation on each edge that we refer to as *index orientation* below.
- The classes $\Delta_{\tilde g}(k_1,\dots,k_{\tilde m})$ are defined in such a way that for any $a_1,\dots,a_{\tilde m}\in\mathbb{Z}$, $a_1+\cdots+a_{\tilde m}=0$, we have

$$\mathrm{DR}_{\tilde g}(a_1,\dots,a_{\tilde m}) = \sum_{k_1,\dots,k_{\tilde m}\in\mathbb{Z}_{\geq 0}} \Delta_{\tilde g}(k_1,\dots,k_{\tilde m}) \prod_{i=1}^{\tilde m} a_i^{k_i}. \tag{3.6}$$

  and the latter expression is a symmetric polynomial in a_i's (such expression exists by [Pix; Spe24]).

The series (5.5) is well-defined as a formal power series in q. Each factor $(1-q^{w(h)})^{-1}$ is expanded in positive powers of q, including the case $w(h)<0$, where we use

$$\frac{1}{1-q^{-m}} = q^m + q^{2m} + \cdots .$$

Since the degree in q grows with the absolute values of $w(h)$, and the balancing conditions imply that only finitely many weightings w exist with all $|w(h)|$ bounded, only finitely many terms contribute to each fixed degree in q. This ensures that (5.5) converges for $|q|<1$. By [OP18, Corollary 1], the series (5.5) is a quasimodular form of pure weight $2g-2+2m$, though the individual summands on the right hand side of (5.5) might have terms of lower weight that cancel out.

When we multiply this expression by λ_{g-1}, we get contributions only from the necklace graphs for $m\geq 2$, and the contributions from the graph with one vertex and no edges and the graphs with one vertex and one loop for $m=1$. The class assigned to the vertex v_i is subsequently multiplied by λ_{g_i}.

In $H^*(\overline{M}_{g_i,3})[a,b]/(a+b)$, we have

$$\lambda_{g_i}\mathrm{DR}_{g_i}(0,a,b) = \frac{a^{2g_i}}{2}\lambda_{g_i}\mathrm{DR}_{g_i}(0,1,-1) + \frac{b^{2g_i}}{2}\lambda_{g_i}\mathrm{DR}_{g_i}(0,1,-1), \tag{3.7}$$

since $\lambda_{g_i}\mathrm{DR}_{g_i}(0,a,-a) = a^{2g_i}\lambda_{g_i}\mathrm{DR}_{g_i}(0,1,-1)$. Therefore, we use only the following classes

$$\Delta_{g_i}(0,0,2g_i)\lambda_{g_i} = \Delta_{g_i}(0,2g_i,0)\lambda_{g_i} = \frac{1}{2}\mathrm{DR}_{g_i}(0,1,-1)\lambda_{g_i}, \tag{3.8}$$

to decorate the vertices (we assume here that the first index is for $k(h)=0$, where h is the leaf attached to v_i). Finally, we know that the resulting formula should be a quasimodular form of pure weight $2g-2+2m$ (as prescribed by Equation (3.4)), which allows us to drastically simplify the computation of the coefficients of the relevant graphs. To this end it might be seen as a refinement of [OP23, Proof of Lemma 6.6].

In the case $m = 1$, the graph with no edges doesn't contribute anything of weight other than 0 (in fact, it contributes a constant term that offsets the $B_{2g}/4g$ constant in the second line of (5.5) in the one-loop graph). Thus, the final result is simply given by

$$c_{g,1}\,(p)\,\lambda_{g-1} = \frac{1}{2}(\mathrm{b}_\Gamma)_*(\mathrm{DR}_{g-1}(0,1,-1)\lambda_{g-1})\cdot 2\mathsf{G}_{2g}(q), \tag{3.9}$$

where Γ is the one-vertex graph with a single loop (and its group of automorphisms has order 2). This expression matches exactly the left-hand side of (3.2) for $m = 1$.

In the case $m \geq 2$, there are $(m-1)!/2$ necklace graphs (for $m = 2$ it is just one graph, whose automorphism group has order 2), but we prefer to consider them as $(m-1)!$ oriented necklace graphs, and then divide the resulting expression by 2, and this way it works equally well for $m = 2$, with an additional choice of the genus labels $g_1, \dots, g_m$ for the vertices. The vertices are decorated by $\mathrm{DR}_{g_i}(0,1,-1)\lambda_{g_i}$, and the kissing weights are fully determined by the choice of one kissing weight $a \in \mathbb{Z}$, say, for the half-edge attached to the vertex v_1 that points in the direction prescribed by the orientation of the necklace.

The coefficient for each necklace is then given by

$$\sum_{a\in\mathbb{Z}_{\neq 0}} a^{2g-2}\left(\frac{a}{1-q^a}\right)^{j_+}\left(\frac{-a}{1-q^{-a}}\right)^{j_-}, \tag{3.10}$$

where j_+ (resp., j_-) is the number of edges, whose index orientation agrees (resp., disagrees) with the orientation of the necklace. Note that $j_+ + j_- = m$ and with our conventions $j_+ \geq 1$. The latter coefficient can then be computed as

$$\widetilde{\lim_{p\to 1}}\left(\sum_{a\in\mathbb{Z}_{\neq 0}} (p\partial_p)^{2g-2} P_{j_+,j_-}(p\partial_p, q\partial_q)\frac{a}{1-q^a}p^a\right), \tag{3.11}$$

where P_{j_+,j_-} is some polynomial, the series expansion is assumed to be taken in the ring $0 < |q| < |p| < 1$, and the operation $\widetilde{\lim}_{p\to 1}$ takes the limit of the expression that is regularized, if necessary, by removing the principal part of the singularity at $p = 1$ (as we see below, the principal part doesn't depend on q).

In order to compute (3.11), we rely on the following observations (see e.g. [OP18]):

- In the ring $0 < |q| < |p| < 1$ the sum

$$\sum_{a\in\mathbb{Z}_{\neq 0}} \frac{a}{1-q^a}p^a,$$

is equal to the shifted by constant Weierstraß function $\overline{\wp}_\tau(z) := \wp_\tau(z) + 2\mathsf{G}_2(q)$, where $q = \exp(2\pi\mathbf{i}\tau)$ and $p = \exp(2\pi\mathbf{i}z)$.

- The shifted Weierstraß function $\overline{\wp}_\tau(z)$ expands at $z = 0$ as

$$\frac{1}{(2\pi\mathbf{i}z)^2} + 2\sum_{\ell=0}^{\infty}\mathsf{G}_{2\ell+2}(q)\frac{(2\pi\mathbf{i}z)^{2\ell}}{(2\ell)!}. \tag{3.12}$$

- The limit $p \to 1$ in the ring $0 < |q| < |p| < 1$ in (3.11) is equal to the limit $z \to 0$ for $0 < \mathrm{Im}(z) < \mathrm{Im}(\tau)$ in terms of the shifted Weierstraß function regularized by removing the principal part at $z = 0$.

- Note that the principal part of $(\frac{1}{2\pi\mathbf{i}}\partial_z)^{2g-2}P_{j_+,j_-}(\frac{1}{2\pi\mathbf{i}}\partial_z, q\partial_q)\overline{\wp}_\tau(z)$ at $z = 0$ is independent of τ.

Now, consider (3.11) in terms of $\overline{\wp}_\tau(z)$. Note that with respect to the weight grading, $\mathrm{wt}(p\partial_p) = 1$ and $\mathrm{wt}(q\partial_q) = 2$. Moreover, for arbitrary j_+, j_-, the polynomial

$$P_{j_+,j_-}(p\partial_p, q\partial_q) = \frac{(q\partial_q)^{m-1}}{(m-1)!} + (\text{lower order terms with respect to the weight grading}). \tag{3.13}$$

So, since we are only interested in the pure weight grading $2g-2+2m$ and it appears to be the top weight grading in Equation (3.11), we only have to apply $\lim_{z\to 0}$ to

$$\frac{1}{(m-1)!}\left(q\partial_q\right)^{m-1}\left(\frac{1}{2\pi\mathbf{i}}\partial_z\right)^{2g-2}\overline{\wp}_\tau(z), \tag{3.14}$$

where the latter expression is regular at $z=0$ and the limit gives $\frac{2}{(m-1)!}(q\partial_q)^{m-1}\mathsf{G}_{2g}(q)$.

Thus, for $m \geq 2$, for the pure weight $2g-2+2m$, we obtain the sum over all oriented necklace graphs, whose vertices are decorated by $\mathrm{DR}_{g_i}(0,1,-1)$, with the coefficient $\frac{1}{2}\cdot\frac{2}{(m-1)!} = \frac{1}{(m-1)!}$. This is exactly the expression that we have on the left-hand side of (3.2). As a final remark, using the explicit expression in Proposition 3.2, a direct combinatorial check shows that the lower-weight quasimodular contributions of each necklace cancel out collectively. □

An alternative to the computation above is given by the following proposition:

Proposition 3.2. For all integers $g, m > 0$ such that $j_+ + j_- = m$, with the convention $j_+ \geq 1$, we have:

$$\sum_{a\in\mathbb{Z}_{\neq 0}} a^{2g-2}\left(\frac{a}{1-q^a}\right)^{j_+}\left(\frac{-a}{1-q^{-a}}\right)^{j_-} = 2\sum_{l=0}^{m-1} c_{m,l,j_-}(q\partial_q)^l\mathsf{G}_{2g+m-1-l} + \delta_{j_-,0}\frac{B_{2g+m-1}}{2g+m-1}$$

where the sum is over all l such that $m-1-l$ is even, and the constants c_{m,l,j_-} are defined by:

$$\binom{b-j_-+m-1}{m-1} = \sum_{l=0}^{m-1} c_{m,l,j_-}b^l$$

for a formal variable b.

Proof. For $j_- > 0$, we compute for positive integers a:

$$\begin{aligned}
\sum_{a>0} a^{2g-2}\left(\frac{a}{1-q^a}\right)^m q^{j_-a} &= \sum_{a>0} a^{2g-2+m}\sum_{i\geq 0}\binom{i+m-1}{m-1}q^{(i+j_-)a} \\
&= \sum_{n\geq 1} q^n \sum_{n=ab} a^{2g-2+m}\binom{b-j_-+m-1}{m-1} \\
&= \sum_{l=0}^{m-1} c_{m,l,j_-}\sum_{n\geq 1} q^n \sum_{n=ab} a^{2g-2+m}b^l \\
&= \sum_{l=0}^{m-1} c_{m,l,j_-}\sum_{n\geq 1} q^n \sum_{a|n} a^{2g-2+m-l}n^l \\
&= \sum_{l=0}^{m-1} c_{m,l,j_-}(q\partial_q)^l\mathsf{G}_{2g+m-1-l}.
\end{aligned}$$

Combining with the contributions from negative integers a, the terms with $m-1-l$ odd cancel, yielding:

$$2\sum_{l=0}^{m-1} c_{m,l,j_-}(q\partial_q)^l\mathsf{G}_{2g+m-1-l}$$

with the sum over l such that $m-1-l$ is even.

In the case $j_- = 0$, the constant term of the Eisenstein series G_{2g+m-1} acts as the regularization parameter to ensure convergence:

$$-\frac{B_{2g+m-1}}{2g+m-1} = \zeta(-2g+2-m) = \sum_{a>0} a^{2g-2+m}.$$

□

3.3 Intersection numbers with the double ramification cycles

In order to intersect Equation (3.2) with ψ-classes, we need certain specializations of the following general formula:

Proposition 3.3. For any $a_1, a_2 \in \mathbb{Z}$ and for any $g \geq 0$ we have:

$$\int_{\overline{M}_{g,3}} \mathrm{DR}_g(-a_1-a_2, a_1, a_2)\lambda_g\psi_1^g = \sum_{j=0}^{g} \frac{(2j-1)!!}{(2g+1)!!(2j)!!} \frac{(a_1+a_2)^{2j}(a_1^2-a_1a_2+a_2^2)^{g-j}}{12^g}. \tag{3.15}$$

Proof. Noting that this integral is a (homogenous) polynomial in a_1, a_2 [Bur15, Lemma 3.2], it is sufficient to compute it for $a_1, a_2 > 0$. To this end, we use the recursion relation for the ψ-class on a double ramification cycle in [Bur+15, Theorem 4], which implies (after we multiply the corresponding expression by λ_g and cancel the terms vanishing for the dimensional reasons) that

$$\begin{aligned}(a_1+a_2)(2g+1)\int_{\overline{M}_{g,3}} \mathrm{DR}_g(-a_1-a_2, a_1, a_2)\lambda_g\psi_1^g = (a_1+a_2)\int_{\overline{M}_{g,2}} \mathrm{DR}_g(-a_1-a_2, a_1+a_2)\lambda_g\psi_1^{g-1} + \\ 2\int_{\overline{M}_{g-1,3}} \mathrm{DR}_{g-1}(-a_1-a_2, a_1, a_2)\lambda_{g-1}\psi_1^{g-1}\left(a_1\int_{\overline{M}_{1,2}} \mathrm{DR}_1(-a_1, a_1)\lambda_1 + a_2\int_{\overline{M}_{1,2}} \mathrm{DR}_1(-a_2, a_2)\lambda_1\right).\end{aligned} \tag{3.16}$$

This recursion relation is exactly the same as the one used in [Sha06] for the intersection numbers with the cycles of admissible covers. Another reincarnation of this relation is the recursion of Buryak–Rossi for the Hamiltonian densities of the KdV hierarchy in [BR16b].

The integrals $\int_{\overline{M}_{g,2}} \mathrm{DR}_g(-b,b)\lambda_g\psi_1^{g-1}$ can either be computed by the same type of recursion, or, otherwise, using the bamboo formula for $\mathrm{DR}_g(-b,b)\lambda_g$ conjectured in [BHS22, Conjecture 2.1] and proved in [BS24, Theorem 2.2] it is straightforward to see that

$$\int_{\overline{M}_{g,2}} \mathrm{DR}_g(-b,b)\lambda_g\psi_1^{g-1} = b^{2g}\int_{\overline{M}_{g,2}} \psi_1^{3g-1} = \frac{b^{2g}}{24^g g!}. \tag{3.17}$$

Thus Equation (3.16) is equivalent to the following recursion relation:

$$\begin{aligned}(2g+1)\int_{\overline{M}_{g,3}} \mathrm{DR}_g(-a_1-a_2, a_1, a_2)\lambda_g\psi_1^g = \frac{(a_1+a_2)^{2g}}{24^g g!} + \\ \frac{a_1^2-a_1a_2+a_2^2}{12}\int_{\overline{M}_{g-1,3}} \mathrm{DR}_{g-1}(-a_1-a_2, a_1, a_2)\lambda_{g-1}\psi_1^{g-1}.\end{aligned} \tag{3.18}$$

It then follows directly that Equation (3.15) satisfies this recursion. □

3.4 A new proof of the socle intersection numbers

A direct corollary of Propositions 3.1 and 3.3 is the following:

Corollary 3.4. Equation (5.32) holds for all $g \geq 1$, $n \geq 1$, $d_1, \ldots, d_n \geq 0$ such that $d_1 + \cdots + d_n = g - 2 + n$.

Proof. Using the string equation [Wit91], we see that it is sufficient to prove (5.32) for $n = m + 1$, $m \geq 1$, with $d_{m+1} = 0$ and $d_1, \ldots, d_m \geq 1$. Then, using the string equation again, we see that under these assumptions

$$\frac{2 \cdot (2g)!}{(-1)^{g-1} B_{2g} (2g-3+n)!} \int_{\overline{M}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{d_i} = \int_{\overline{M}_{g,m}} \left(\frac{2 \cdot (2g)!}{(-1)^{g-1} B_{2g} (2g-2+m)!} \lambda_g \lambda_{g-1} \prod_{i=1}^{m} \psi_i \sum_{i=1}^{m} \frac{1}{\psi_i} \right) \prod_{i=1}^{m} \psi_i^{d_i - 1}. \tag{3.19}$$

By Proposition 3.1 the latter expression is equal to

$$\frac{1}{(m-1)!} \sum_{\substack{\Gamma \in \vec{N} \\ g : V(\Gamma) \to \mathbb{Z}_{\geq 0}}} \prod_{i=1}^{m} \int_{\overline{M}_{g(v_i),3}} \mathrm{DR}_{g(v_i)}(0, 1, -1)\, \lambda_{g(v_i)}\, \psi_1^{d_i - 1}. \tag{3.20}$$

For the dimensional reason, the integrals in this expression are non-trivial if and only if $g(v_i) = d_i - 1$. Thus the genus function g on the vertices is uniquely determined for every graph Γ, and there are exactly $(m-1)!$ identical summands in the sum. Thus, the expression is equal to

$$\prod_{i=1}^{m} \int_{\overline{M}_{d_i - 1,3}} \mathrm{DR}_{d_i - 1}(0, 1, -1) \lambda_{d_i - 1} \psi_1^{d_i - 1}, \tag{3.21}$$

which is equal to

$$\prod_{i=1}^{m} \frac{1}{(2d_i - 1)!! 2^{2d_i - 2}} = \frac{1}{2^{2g-2}} \prod_{i=1}^{m} \frac{1}{(2d_i - 1)!!}, \tag{3.22}$$

by Proposition 3.3. □

Chapter 4

Quantum hierarchy for the Gromov–Witten theory of elliptic curves

4.1 Overview

In this chapter, we construct the quantum double ramification (DR) hierarchy associated with the Gromov-Witten theory of elliptic curves. We use results of Oberdieck and Pixton [OP18; OP23] on the intersection numbers of the double ramification cycle, the Gromow-Witten classes of the elliptic curve and the Hodge class λ_{g-1}, together with vanishing results for λ_{g-2} to produce a closed, modular expression for the resulting integrable hierarchy. It is the first explicit nontrivial example of a quantum integrable hierarchy from a (non-semisimple) cohomological field theory containing fermionic fields, which correspond to the odd classes in the cohomology of the elliptic curve.

4.2 (Super) quantum double ramification hierarchy

The purpose of this section is to remind a definition of the quantum double ramification hierarchy [BR16a] indicating the needed adjustments in the case of an additional $\mathbb{Z}_2$-grading on the target space.

4.2.1 (Super) cohomological field theories

Let V be a finite-dimensional $\mathbb{C}$-vector space with basis $\{e_1, \dots, e_N\}$, where e_1 denotes the distinguished unit. Assume that V is $\mathbb{Z}_2$–graded, i.e. $V = V_0 \oplus V_1$, where vectors in V_0 are called *even* and those in V_1 are called *odd*, with $e_1 \in V_0$. We use the degree map $\deg_{\mathbb{Z}_2} \colon V \to \mathbb{Z}_2$ sending even vectors to 0 and odd vectors to 1. Equip V with a non-degenerate bilinear form $\eta : V \times V \to \mathbb{C}$ that is even and graded-symmetric, meaning:

- η is symmetric on $V_0 \otimes V_0$,
- η is skew-symmetric on $V_1 \otimes V_1$,
- η vanishes on $V_0 \otimes V_1$ and $V_1 \otimes V_0$.

A *(super) cohomological field theory (CohFT)* is a family of maps $\{c_{g,n}\}_{2g-2+n>0}$, where

$$c_{g,n} \colon V^{\otimes n} \to H^*(\overline{M}_{g,n}, \mathbb{C}), \qquad v_1 \otimes \cdots \otimes v_n \mapsto c_{g,n}(v_1 \otimes \cdots \otimes v_n). \tag{4.1}$$

We can also identify $c_{g,n}$ as elements of $H^*(\overline{M}_{g,n}, \mathbb{C}) \otimes (V^*)^{\otimes n}$, and we refer to them as 'classes'. The classes $c_{g,n}$ are required to satisfy the following properties:

i. The classes $c_{g,n}$ are even, i.e., the maps (4.1) preserve the $\mathbb{Z}_2$-grading.

ii. The classes $c_{g,n}$ are graded equivariant under the action of the symmetry group S_n by permutations on the factors of $V^{\otimes n}$ and by relabeling of the marked points on $\overline{M}_{g,n}$. This means that for a given permutation $s \in S_n$ we denote $\sigma_s\colon \overline{M}_{g,n} \to \overline{M}_{g,n}$ to be the isomorphism induced by the relabeling of the marked points according to s, and then

$$\pm c_{g,n}(v_{s(1)} \otimes \ldots \otimes v_{s(n)})) = (\sigma_s^{-1})^* c_{g,n}(v_1 \otimes \ldots \otimes v_n),$$

where the sign $\pm$ is the Koszul sign corresponding to the reordering of the homogeneous vectors v_i by the permutation s.

iii. Consider the gluing map of the first type,

$$\mathsf{gl}_1\colon \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n},$$

where $g = g_1 + g_2$ and $n = n_1 + n_2$, and we assume that the points with the labels $i \in S_1$ (resp., $i \in S_2$) lie on the first (resp., second) component of the degenerated curve, so $S_1 \sqcup S_2 = \{1, \ldots, n\}$ and $|S_1| = n_1$ and $|S_2| = n_2$. Then its pullback on $c_{g,n}$ is given by

$$\mathsf{gl}_1{}^*(c_{g,n}(v_1, \ldots, v_n)) = \pm c_{g_1,n_1+1}\big(e_\alpha \otimes \bigotimes_{i\in S_1} v_i\big)\eta^{\alpha\beta} c_{g_2,n_2+1}\big(e_\beta \otimes \bigotimes_{i\in S_2} v_i\big),$$

where the sign $\pm$ is the Koszul sign.

iv. For the gluing map of the second type,

$$\mathsf{gl}_2\colon \overline{M}_{g-1,n+2} \to \overline{M}_{g,n},$$

its corresponding pullback on $c_{g,n}$ is expressed as

$$\mathsf{gl}_2{}^* c_{g,n}(v_1 \otimes \cdots \otimes v_n)) = c_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\alpha \otimes e_\beta)\eta^{\alpha\beta}.$$

Note that there is no Koszul sign in this case since η is even.

v. For the forgetful map that forgets the last marked point,

$$\mathsf{fg}\colon \overline{M}_{g,n+1} \to \overline{M}_{g,n},$$

its corresponding pullback on $c_{g,n}$ is expressed as

$$\mathsf{fg}^*(c_{g,n}(v_1 \otimes \cdots \otimes v_n)) = c_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes e_1)$$

Note that there is no Koszul sign in this case since e_1 is even. We also demand that

$$\eta(v_1, v_2) = c_{0,3}(v_1 \otimes v_2 \otimes e_1).$$

4.2.2 Double ramification cycle

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum_i a_i = 0$, and let A_+ and A_- denote the positive and negative parts of A, with n_0 the number of zero entries. The double ramification cycle is defined as

$$\mathrm{DR}_g(A) := \mathsf{src}_* \big[\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)\big]^{\mathrm{vir}} \in H^{2g}(\overline{M}_{g,n}),$$

where $\overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+)$ is the moduli space of relative stable maps to the rubber, and

$$\mathsf{src}\colon \overline{M}^{\sim}_{g,n_0}(\mathbb{P}^1, A_-, A_+) \longrightarrow \overline{M}_{g,n}$$

is the source map forgetting the relative map and stabilizing the domain curve.

The double ramification cycle $\mathrm{DR}_g(A)$ can be described as a restriction to the hyperplane $\sum_i a_i = 0$ an even polynomial of degree $2g$ in the parameters a_i with the coefficients in the tautological classes in $H^{2g}(\overline{M}_{g,n})$. This description is not unique (as we can add an arbitrary polynomial that vanishes on the hyperplane $\sum_i a_i = 0$); so we fix it in the way that doesn't depend a particular variable a_j, $j = 1, \dots, n$

$$\mathrm{DR}_g(A) = \sum_{\substack{K=(k_1,\dots,k_n)\\ k_j=0\\ k_i\in\mathbb{Z}_{\geq 0}, i\neq j\\ k_1+\dots+k_n\leq 2g}} \mathfrak{D}_g^{(j)}(K)\prod_{i=1}^n a_i^{k_i}.$$

The average of these expressions over $j = 1, \dots, n$ gives us another expression for the double ramification cycle, denoted by

$$\mathrm{DR}_g(A) = \sum_{\substack{K=(k_1,\dots,k_n)\\ k_i\in\mathbb{Z}_{\geq 0}\\ k_1+\dots+k_n\leq 2g}} \mathfrak{D}_g^{\mathrm{sym}}(K)\prod_{i=1}^n a_i^{k_i}, \tag{4.2}$$

which is a polynomial symmetric all variables $a_1, \dots, a_n$:

4.2.3 Quantum commutator and local functionals

Let V be the N-dimensional graded $\mathbb{C}$-vector space considered in Section 4.2.1. We associate to it the ring of *quantum differential polynomials* defined as the $\mathbb{Z}_2$-graded ring

$$\mathcal{A} = \mathbb{C}[\![u^\alpha]\!][u_j^\alpha][\![\varepsilon, \hbar]\!],$$

equipped with an additional ∂_x-gradation $\deg_{\partial_x} : \mathcal{A} \to \mathbb{Z}$, where

- u_j^α are formal variables indexed by $\alpha = 1, \dots, N$ and $j \in \mathbb{Z}_{\geq 0}$ with the ∂_x-degree $\deg_{\partial_x} u_j^\alpha = j$;
- ε is the dispersion parameter with ∂_x-degree $\deg_{\partial_x}(\varepsilon) = -1$;
- $\hbar$ is the quantization parameter with ∂_x-degree $\deg_{\partial_x}(\hbar) = -2$.

The $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ descents from the $\mathbb{Z}_2$-grading on V. That is, we assume that the basis $\{e_\alpha\}$ of V is homogeneous with respect to the $\mathbb{Z}_2$-grading, and then defined $\deg_{\mathbb{Z}_2} : \mathcal{A} \to \mathbb{Z}_2$ as a ring morphism with the following initial values of $\deg_{\mathbb{Z}_2}$ on the generators:

- $\deg_{\mathbb{Z}_2}(u_j^\alpha) := \deg_{\mathbb{Z}_2}(e_\alpha)$;
- $\deg_{\mathbb{Z}_2}(\epsilon) := 0$;
- $\deg_{\mathbb{Z}_2}(\hbar) := 0$.

We have the following even differential operator on $\mathcal{A}$ of $\deg_{\partial_x}$-degree $+1$:

$$\partial_x = \sum_{i\geq 0} u_{i+1}^\alpha \frac{\partial}{\partial u_i^\alpha}.$$

Note that in particular $u_j^\alpha = \partial_x^j u^\alpha$. We define the space of *quantum local functionals* as the quotient space

$$\mathcal{A}/(\mathrm{Im}(\partial_x) \oplus \mathbb{C}[\![\varepsilon, \hbar]\!]),$$

where $\mathbb{C}$ is the subspace of constants in u_j^α in $\mathcal{A}$. The projection operator from the ring of differential polynomials modulo the constants to the space of local functionals is given by:

$$\mathcal{A}/\mathbb{C}[\![\varepsilon, \hbar]\!] \to \mathcal{A}/\left(\mathrm{Im}(\partial_x) \oplus \mathbb{C}[\![\varepsilon, \hbar]\!]\right);$$
$$f \mapsto \overline{f} := \int f\, dx.$$

We often use another set of formal variables p^α_k that are related to the u-variables via a Fourier series expansion,

$$u^\alpha = \sum_{k\in\mathbb{Z}} p^\alpha_k e^{\mathbf{i}kx}. \tag{4.3}$$

This allows us to consider elements of the ring $\mathcal{A}$ inside the auxiliary ring

$$\mathbb{C}[[p^\alpha_{k>0}]][p^\alpha_{k\le 0}][[\varepsilon,\hbar]][e^{\mathbf{i}x},e^{-\mathbf{i}x}].$$

Here p^α_k are formal variables indexed by $\alpha = 1,\dots,N$ and $k\in\mathbb{Z}$, and $\deg_{\mathbb{Z}_2} p^\alpha_k := \deg_{\mathbb{Z}_2} u^\alpha$. This auxiliary ring is used for construction of the $\star$ product and the quantum commutator as follows. For any $f,g\in\mathcal{A}$ homogeneous with respect to the $\mathbb{Z}_2$-grading, the *quantum commutator* is given in terms of the star product defined on $\mathcal{A}$:

$$[f,g] := f\star g - \pm g\star f \tag{4.4}$$

where the sign $\pm$ is the Koszul sign (which is equal in this case to $(-1)^{\deg_{\mathbb{Z}_2}(f)\cdot\deg_{\mathbb{Z}_2}(g)}$) and the $\star$ product is defined as

$$\begin{aligned} f\star g &= f\,\exp\left(\sum_{k\ge 0}\mathbf{i}\hbar k\eta^{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial p^\alpha_k}\frac{\overrightarrow{\partial}}{\partial p^\beta_{-k}}\right) g\\ &= \sum_{\substack{n\ge 0\\ k_1,k_2,\dots,k_n\ge 0}} \pm\frac{\hbar^n}{n!}k_1\eta^{\alpha_1\beta_1}\dots k_n\eta^{\alpha_n\beta_n}\frac{\partial^n f}{\partial p^{\alpha_1}_{k_1}\dots\partial p^{\alpha_n}_{k_n}}\frac{\partial^n g}{\partial p^{\beta_1}_{-k_1}\dots\partial p^{\beta_n}_{-k_n}}, \end{aligned}$$

where $\pm$ is the Koszul sign. As a consequence, in the auxiliary ring we obtain the standard commutation rules for creation and annihilation operators,

$$\frac{1}{\mathbf{i}\hbar}[p^\alpha_k,p^\beta_j] = k\eta^{\alpha\beta}\delta_{k+j,0}.$$

The quantum commutator can be explicitly expressed in the u-variables, as shown in [BR16a]. It defines a structure of the $\mathbb{Z}_2$-graded Lie algebra, that is, it is $\mathbb{Z}_2$-graded skew-symmetric and satisfies the $\mathbb{Z}_2$-graded Jacobi identity.

4.2.4 Hamiltonian densities

The Hamiltonians $\overline{G}_{\alpha,d}$ of the quantum double ramification hierarchy are defined in terms of their densities,

$$\overline{G}_{\alpha,d} = \int G_{\alpha,d}(x)\,dx,$$

and the latter ones can be expressed in terms of the p- and u-variables, respectively, as

$$\begin{aligned} G_{\alpha,d} &= \sum_{\substack{g,n\in\mathbb{Z}_{\ge 0}\\ 2g-1+n>0}}\frac{\hbar^g}{n!}\sum_{\substack{A=(a_1,\dots,a_n)\\ a_i\in\mathbb{Z}}}\int_{\overline{M}_{g,n+1}}\mathrm{DR}_g\left(-\textstyle\sum_{i=1}^n a_i, A\right)\,\Lambda\big(\tfrac{-\varepsilon^2}{\mathbf{i}\hbar}\big)\,\psi_1^d\,c_{g,n+1}\Big(e_\alpha\otimes\bigotimes_{i=1}^n e_{\alpha_i}p^{\alpha_i}_{a_i}e^{\mathbf{i}a_i x}\Big)\\ &= \sum_{\substack{g,n\in\mathbb{Z}_{\ge 0}\\ 2g-1+n>0}}\frac{\hbar^g}{n!}\sum_{\substack{K=(0,k_1,\dots,k_n)\\ k_i\in\mathbb{Z}_{\ge 0}\\ k_1+\dots+k_n\le 2g}}\int_{\overline{M}_{g,n+1}}\mathfrak{D}^{(1)}_g(K)\,\Lambda\big(\tfrac{-\varepsilon^2}{\mathbf{i}\hbar}\big)\,\psi_1^d\,c_{g,n+1}\Big(e_\alpha\otimes\bigotimes_{i=1}^n e_{\alpha_i}u^{\alpha_i}_{k_i}\Big) \end{aligned}$$

Here $\Lambda(\frac{-\varepsilon^2}{\mathbf{i}\hbar}) = \sum_{j=0}^g(\frac{-\varepsilon^2}{\mathbf{i}\hbar})^j\lambda_j$, where $\lambda_j = c_j(\mathbb{E})\in H^{2j}(\overline{M}_{g,n+1})$ denote the j-th Chern class of the Hodge bundle $\mathbb{E}$ over $\overline{M}_{g,n+1}$, $j=1,\dots,g$. Also, in both cases we extend the linearity of $c_{g,n+1}$ with respect to coefficients in the auxiliary ring and the ring $\mathcal{A}$, respectively. In it not necessary as one can collect all variables in an extra factor, but it is convenient to do so since the products $e_\alpha p^\alpha_a$ and $e_\alpha u^\alpha_k$ are even and this way the Koszul sign doesn't enter the formula.

It is useful to separately define $G_{\alpha,-1} := \eta_{\alpha\mu}u^\mu$, which gives the Casimir elements of the quantum commutator, and the local functional

$$\overline{G} = \int \left[\sum_{\substack{g,n\in\mathbb{Z}_{\geq 0}\\ 2g-2+n>0}} \frac{(\mathbf{i}\hbar)^g}{n!} \sum_{\substack{K=(k_1,\dots,k_n)\\ k_i\in\mathbb{Z}_{\geq 0}\\ k_1+\cdots+k_n\leq 2g}} \int_{\overline{M}_{g,n}} \mathfrak{D}_g^{\mathrm{sym}}(K)\,\Lambda\big(\tfrac{-\varepsilon^2}{\mathbf{i}\hbar}\big)\, c_{g,n}\Big(\bigotimes_{i=1}^n e_{\alpha_i} u^{\alpha_i}_{k_i}\Big) \right] dx \tag{4.5}$$

which we refer to as the DR hierarchy potential. This last local functional plays a role towards the DR hierarchy analogous to the role of the Dubrovin-Frobenius potential towards the principal hierarchy of the corresponding Frobenius manifold. In particular, pushing fowards with respect to the first marked point and using the dilaton equation, we obtain immediately

$$\overline{G}_{1,1} = \Big(\varepsilon\frac{\partial}{\partial\varepsilon} + 2\hbar\frac{\partial}{\partial\hbar} + \sum_{s=0}^{\infty} u^\alpha_s \frac{\partial}{\partial u^\alpha_s} - 2\Big)\overline{G}. \tag{4.6}$$

All together, the quantum Hamiltonians form a quantum integrable system:

Proposition 4.1 ([BR16a]). For all $\alpha,\beta = 1,\dots,N$ and $p,q\in\mathbb{Z}_{\geq -1}$ we have:

$$[\overline{G}_{\alpha,q}, \overline{G}_{\beta,q}] = 0.$$

4.2.5 Recursion relation

We can reconstruct the entire quantum double ramification hierarchy from the knowledge of $\overline{G}_{1,1}$ (and hence from $\overline{G}$) alone:

Proposition 4.2 ([BR16a]). For all $\alpha = 1,\dots,N$ and $p\in\mathbb{Z}_{\geq -1}$ we have:

$$\partial_x\Big(\varepsilon\frac{\partial}{\partial\varepsilon} + 2\hbar\frac{\partial}{\partial\hbar} + \sum_{s=0}^{\infty} u^\alpha_s \frac{\partial}{\partial u^\alpha_s} - 1\Big)G_{\alpha,p+1} = \frac{1}{\hbar}[G_{\alpha,p}, \overline{G}_{1,1}].$$

Notice how, in light of Proposition 4.1, the right-hand side of the equation in Proposition 4.2 lies in the image of the operator ∂_x, and how the operator in parenthesis in the left-hand side of the above equation has densities of Casimirs as kernel. This is why Proposition 4.2 allows to renconstruct the entire integrable hierarchy.

4.3 Quantum hierarchy for elliptic curves

4.3.1 Gromov–Witten classes

Let E be a non-singular complex elliptic curve. Its cohomology $H^*(E)$ is a $\mathbb{Z}_2$-graded vector space spanned by $\{e_1, e_2, e_3, e_4\}$, where

- $e_1 \in H^0(E)$ is the unit.
- $e_2, e_3 \in H^1(E)$ are odd classes such that $\int_E e_2 \cup e_3 = 1$.
- $e_4 \in H^2(E)$ is the Poincaré dual of a point, $\int_E e_4 = 1$.

The moduli space of stable maps to E of degree d from the genus g curves with n marked points is denoted by $\overline{M}_{g,n}(E,d)$ and we assume that $2g-2+n>0$. There are natural maps

- $\mathrm{src}_d\colon \overline{M}_{g,n}(E,d) \to \overline{M}_{g,n}$ forgets the stable map and retains the stabilization of the source curve;
- $\mathrm{ev}_i\colon \overline{M}_{g,n}(E,d) \to E$ is the evaluation map at the i-th marked point, $i = 1,\dots,n$.

The cohomological field theory $\{c_{g,n}\}$ associated with the Gromov–Witten theory of E is defined over the ring of formal power series $\mathbb{C}[\![q]\!]$ by

$$c_{g,n}(e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n}) := \sum_{d\geq 0}(\mathsf{src}_d)_*\left([\overline{M}_{g,n}(E,d)]^{\mathrm{vir}}\textstyle\prod_{i=1}^n \mathsf{ev}_i^*(e_{\alpha_i})\right) q^d \in H^*(\overline{M}_{g,n})\otimes\mathbb{C}[\![q]\!].$$

4.3.2 Quasimodular forms

For even $k\geq 2$, the k-weighted Eisenstein series is given by

$$\mathsf{G}_k(q) = -\frac{B_k}{2k}+\sum_{n\geq 1}\sigma_{k-1}(n)q^n = -\frac{B_k}{2k}+\sum_{d\geq 1}d^{k-1}\frac{q^d}{1-q^d},$$

where B_k are the Bernoulli numbers and $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function. The algebra of quasimodular forms if defined as

$$\mathrm{QMod} = \mathbb{C}[\mathsf{G}_2,\mathsf{G}_4,\mathsf{G}_6].$$

It is graded by non-negative even integers, $\mathrm{QMod} = \bigoplus_{\ell=1}^{\infty}\mathrm{QMod}_{2\ell}$, where the grading (also called weight) is defined on the generators as $\deg_{\mathrm{QMod}}(\mathsf{G}_k) := k$, $k = 2,4,6$. Note that for all even $k\geq 2$, $\mathrm{QMod}_k \ni \mathsf{G}_k$. There are two differential operators that are often used in computations with quasimodular forms:

$$\frac{d}{d\mathsf{G}_2}\colon \mathrm{QMod}_k\to\mathrm{QMod}_{k-2};$$
$$D_q := q\frac{d}{dq}\colon \mathrm{QMod}_k\to\mathrm{QMod}_{k+2}.$$

Their commutator $\left[\frac{d}{d\mathsf{G}_2}, D_q\right]$ acts on QMod_k as the operator of multiplication by $-2k$.

4.3.3 Closed formula for the quantum DR potential

Proposition 4.2 and Equation (4.6) imply that it is enough to know the DR hierarchy potential $\overline{G}$ to reconstruct the full system of quantum Hamiltonians from the recursion relations.
From Equation (4.5), to write down $\overline{G}$ we need to compute the following integrals:

$$\int_{\overline{M}_{g,n}}\mathrm{DR}_g(A)\,\Lambda(\tfrac{-\varepsilon^2}{i\hbar})\,c_{g,n}(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}) \tag{4.7}$$

Let us collect some remarks that will restrict the number of non-trivial integrals we have to handle:

- First of all, since the double ramification cycle $\mathrm{DR}_g\big(-\sum_{i=1}^n a_i, A\big)$ is symmetric in $a_1,\ldots,a_n$, while $c_{g,n}$ is graded-S_n-equivariant we have non-trivial contributions to (4.7) only when $c_{g,n}\big(\bigotimes_{i=1}^n e_{\alpha_i}\big)$ contains at most one $\alpha_i = 2$ and at most one $\alpha_i = 3$.
- Second, by [Jan17], the number of α_i's equal to 2 should be the same as the number of α_i's equal to 3, otherwise $c_{g,n}\big(\bigotimes_{i=1}^n e_{\alpha_i}\big)$ vanishes.
- Third, by [OP23], the class $\lambda_g\, c_{g,n}\big(\bigotimes_{i=1}^n e_{\alpha_i}\big)$ vanishes for $g\geq 1$.
- Finally, the total cohomological degree of the class in (4.7) must be equal to $2(3g-3+n)$, and the cohomological degrees of the involved classes are given by

$$\dim \mathrm{DR}_g(A) = 2g,\quad \dim\lambda_j = 2j,\quad\text{and}\quad \dim c_{g,n}(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}) = 2g-2+\textstyle\sum_{i=1}^n \dim e_{\alpha_i}.$$

Thus the potentially nontrivial integrals (4.7) are reduced to:

$$\int_{\overline{M}_{0,n}} c_{0,n}(e_4 \otimes e_1^{\otimes(n-1)}), \qquad \int_{\overline{M}_{0,n}} c_{0,n}(e_2 \otimes e_3 \otimes e_1^{\otimes(n-2)}), \tag{4.8}$$

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_{g-1}\,c_{g,n}(e_1 \otimes e_4^{\otimes(n-1)}), \qquad \int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_{g-1}\,c_{g,n}(e_2 \otimes e_3 \otimes e_4^{\otimes(n-2)}), \tag{4.9}$$

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_{g-2}\,c_{g,n}(e_4^{\otimes n}). \tag{4.10}$$

By axiom (v) of cohomological field theories, the genus 0 integrals (4.8) vanish unless $n = 3$, and for $n = 3$ both integrals are equal to 1, see e.g. [Bur23]. The higher genera integrals with λ_{g-1} (4.9) are computed in [OP23]:

Proposition 4.3 ([OP23, Theorem 6.10])**.** For $g \geq 1, n \geq 1$ and $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we have

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_{g-1}\,c_{g,n}\left(e_1 \otimes e_4^{n-1}\right) = \frac{a_1^2}{2^{2g-2}} \sum_{\sum b_i = g-1} \prod_{i=1}^{n} \frac{a_i^{2b_i}}{(2b_i+1)!} D_q^{n-2}\mathsf{G}_{2g}$$

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\lambda_{g-1}\,c_{g,n}\left(e_2 \otimes e_3 \otimes e_4^{n-2}\right) = \frac{-a_1 a_2}{2^{2g-2}} \sum_{\sum b_i = g-1} \prod_{i=1}^{n} \frac{a_i^{2b_i}}{(2b_i+1)!} D_q^{n-2}\mathsf{G}_{2g}$$

Finally, the integral (4.10) vanishes by the following Proposition.

Proposition 4.4. Let $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum_i a_i = 0$ and $g \geq 2$. Then

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(a_1, \ldots, a_n)\,\lambda_{g-2}\,c_{g,n}\left(e_4^{\otimes n}\right) = 0.$$

Proof. If $a_1 = \cdots = a_n = 0$, then $\mathrm{DR}_g(a_1, \ldots, a_n) = (-1)^g \lambda_g$ and the desired vanishing follows from the vanishing of $\lambda_g\, c_{g,n}\left(\bigotimes_{i=1}^n e_{\alpha_i}\right)$ for $g \geq 1$ that we recall above.

Assume that not all a_i vanish. Then we prove a more refined statement, using the following interpretation of the product $\mathrm{DR}_g(a_1, \ldots, a_n)\; c_{g,n}\left(e_4^{\otimes n}\right)$. As before, let A_+ denote the subtuple of A consisting of all the positive integers in A, and let A_- denote the subtuple of A consisting of all the negative integers in A, and let n_0 be the number of a_i's that are equal to 0. Namely, let $\overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+)$ be the moduli space of stable relative maps of connected genus g curves to the rubber trivial bundle over E, whose projection to E has degree d. By relative we mean relative to the zero and infinity sections, with ramification profiles given by A_- and A_+, respectively. Let

$$\mathsf{src}\colon \overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+) \to \overline{M}_{g,n}$$

be the source map that forgets the stable relative map and retains the stabilization of the source curve. Then we have

$$\begin{aligned}&\int_{\overline{M}_{g,n}} \mathrm{DR}_g(a_1, \ldots, a_n)\lambda_{g-2}c_{g,n}\left(e_4^{\otimes n}\right)\\ &= \sum_{d=0}^{\infty} q^d \int_{\overline{M}_{g,n}} \mathsf{src}_*\left(\left[\overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+)\right]^{\mathrm{vir}} \prod_{i=1}^{n} \mathsf{ev}_i^*(e_4)\right)\lambda_{g-2},\end{aligned}$$

where $\mathsf{ev}_i\colon \overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+) \to E$ is the evaluation map at the i-th marked point for the relative stable map combined with the projection to E.

Now the statement of the proposition follows from the following lemma:

Lemma 4.5. Assume that not all a_i vanish. Then for any $g, d \geq 0$ we have

$$\left[\overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+)\right]^{\mathrm{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(e_4) = 0.$$

The proof of the lemma is based on the following general observation. Let $F\colon C_1 \to C_2$ be a holomorphic map of smooth compact curves and $\sum_{i=1}^n b_i p_i$ be a principal divisor on C_1. Let us prove that then $\sum_{i=1}^n b_i F(p_i)$ is a principal divisor on C_2. If F is constant map, then the statment is obvious. Assume F is non-constant and let $\sum_{i=1}^k b_i p_i = (f)$, where $f\colon C_1 \to \mathbb{P}^1$ is a meromorphic function. Then $\sum_{i=1}^k b_i F(p_i) = (g)$, with $g\colon C_2 \to \mathbb{P}^1$ defined as $g(q) := \prod_{p \in F^{-1}(q)} f(p)^{\mathrm{ord}_p F}$, where $\mathrm{ord}_p F$ denotes the local ramification degree of F at $p \in C_1$.

This observation implies that for a choice of points $p_1, \dots, p_n \in E$ such that $\sum_{i=1}^n a_i p_i$ is not a principal divisor on E, the geometric intersection $\cap_{i=1}^n \mathrm{ev}_i^{-1}(p_i)$ on $\overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+)$ is empty. Indeed, let $(C, x, \dots, x_n, F, f) \in \overline{M}^{\sim}_{g,n_0}(E \times \mathbb{P}^1, d, A_-, A_+)$ be a possibly nodal curve C with marked points $x_1, \dots, x_n$ whose normalization has l components $C_1, \dots, C_l$, equipped with a map F to E whose restriction to C_j is denoted by F_j, $j = 1, \dots, l$, and a relative stable map f to the rubber $\mathbb{P}^1$, whose restriction to C_j is denoted by f_j, $j = 1, \dots, l$. Note that

- The divisor of f_j on C_j is principal and supported on nodes and marked points. Applying the observation above to the map $F_j\colon C_j \to E$, we conclude that the image of the divisor of f_j under the map F_j is a principal divisor on E, $j = 1, \dots, l$.
- Each marked point x_i enters the divisor of exactly one f_j with coefficient a_i. Each node enters the divisors of exactly two f_j's, with the opposite coefficients. Hence, the sum of these principal divisors is equal to $\sum_{i=1}^n a_i F(x_i) = \sum_{i=1}^n a_i \mathrm{ev}_i(C, x, \dots, x_n, F, f)$.

Thus, $\cap_{i=1}^n \mathrm{ev}_i^{-1}(p_i)$ is empty if $\sum_{i=1}^n a_i p_i$ is not a principal divisor on E, which implies the statement of the lemma and, as a corollary, the statement of the proposition. □

Given two power series $f(\varepsilon) = \sum_{i \geq 0} a_i \varepsilon^i \in R[\![\varepsilon]\!]$ and $g(\varepsilon) = \sum_{i \geq 0} b_i \varepsilon^i \in M[\![\varepsilon]\!]$ with R a ring and M a left module over R, consider their Hadamard product $(f \odot g)(\varepsilon) := \sum_{i \geq 0} a_i b_i \varepsilon^i \in M[\![\varepsilon]\!]$. Moreover, for any power series $h(\varepsilon_1, \varepsilon_2) = \sum_{i,j \geq 0} c_{i,j} \varepsilon_1^i \varepsilon_2^j \in M[\![\varepsilon_1, \varepsilon_2]\!]$, we define its diagonal as the power series $\mathrm{diag}_{\varepsilon_1, \varepsilon_2} \{h(\varepsilon_1, \varepsilon_2)\} := \sum_{i \geq 0} c_{i,i} \varepsilon_1^i \in M[\![\varepsilon_1]\!]$.

Corollary 4.6. The quantum double ramification hierarchy for the Gromov–Witten theory of the elliptic curve is uniquely determined by the DR potential

$$\overline{G} = \int \Big[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \frac{\mathbf{i}\hbar}{\varepsilon^2}\Big(\sum_{\substack{g \geq 1 \\ n \geq 2}} \frac{\epsilon^{2g}}{2^{2g-2}(n-2)!} \sum_{\sum_{i=1}^n b_i = g-1} \Big(\frac{u^1_{2b_1+1}}{(2b_1+1)!} \frac{u^4_{2b_2+1}}{(2b_2+1)!} + \frac{u^2_{2b_1+1}}{(2b_1+1)!} \frac{u^3_{2b_2+1}}{(2b_2+1)!} \Big) \prod_{i=3}^{n} \frac{u^4_{2b_i}}{(2b_i+1)!} D_q^{n-2} \mathsf{G}_{2g}(q) \Big) \Big] dx.$$

or, equivalently,

$$\begin{aligned} \overline{G} &= \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \mathbf{i}\hbar \left(\left(\mathcal{S}_\varepsilon(u^1_x)\, \mathcal{S}_\varepsilon(u^4_x) + \mathcal{S}_\varepsilon(u^2_x)\, \mathcal{S}_\varepsilon(u^3_x) \right) \exp\left(\mathcal{S}_\varepsilon(u^4) D_q \right) \right) \odot \mathcal{G}(\varepsilon, q) \right] dx \\ &= \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \mathbf{i}\hbar\, \mathrm{diag}_{\varepsilon, \varepsilon'} \left\{ \left(\mathcal{S}_\varepsilon(u^1_x)\, \mathcal{S}_\varepsilon(u^4_x) + \mathcal{S}_\varepsilon(u^2_x)\, \mathcal{S}_\varepsilon(u^3_x) \right) \mathcal{G}\left(\varepsilon', q \exp\left(\mathcal{S}_\varepsilon(u^4) \right) \right) \right\} \right] dx, \end{aligned}$$

where

$$\mathcal{S}_{\uparrow} := \frac{\sinh(\frac{1}{2}\varepsilon\partial_x)}{\frac{1}{2}\varepsilon\partial_x}, \qquad \mathcal{G}(\varepsilon, q) := \sum_{g\geq 0} \varepsilon^{2g} \mathsf{G}_{2g+2}(q).$$

Proof. The genus 0 integrals (4.8) vanish unless $n = 3$, and for $n = 3$ both integrals are equal to 1 and by Proposition 4.3 (exploiting $a_1 = -\sum_{i=2}^n a_i$ in the first type of intersection numbers) and Proposition 4.4, we compute formula (4.5) as

$$\overline{G} = \int \Big[\frac{(u^1)^2u^4}{2} + u^1u^2u^3 + \frac{\mathbf{i}\hbar}{\varepsilon^2}\Big(\sum_{\substack{g\geq 1\\ n\geq 2}} \frac{\epsilon^{2g}}{2^{2g-2}(n-2)!} \sum_{\sum_{i=1}^n b_i = g-1} \Big(\frac{u^1_{2b_1+1}}{(2b_1+1)!}\frac{u^4_{2b_2+1}}{(2b_2+1)!} + \frac{u^2_{2b_1+1}}{(2b_1+1)!}\frac{u^3_{2b_2+1}}{(2b_2+1)!}\Big)\prod_{i=3}^n \frac{u^4_{2b_i}}{(2b_i+1)!} D_q^{n-2}\mathsf{G}_{2g}(q)\Big)\Big]dx.$$

Next we remark that, for a formal variable δ and for $1 \leq \alpha \leq 4$,

$$\sum_{b\geq 0} \frac{\delta^{2b} u^\alpha_{2b}}{2^{2b}(2b+1)!} = \mathcal{S}_\delta(u^\alpha),$$

so that

$$\begin{aligned}
&\sum_{\substack{g\geq 1\\ n\geq 2}} \frac{\epsilon^{2g-2}}{2^{2g-2}(n-2)!} \sum_{\sum_{i=1}^n b_i = g-1} \Big(\frac{u^1_{2b_1+1}}{(2b_1+1)!}\frac{u^4_{2b_2+1}}{(2b_2+1)!} + \frac{u^2_{2b_1+1}}{(2b_1+1)!}\frac{u^3_{2b_2+1}}{(2b_2+1)!}\Big)\prod_{i=3}^n \frac{u^4_{2b_i}}{(2b_i+1)!} D_q^{n-2}\mathsf{G}_{2g}(q)\\
&= \sum_{\substack{g\geq 1\\ n\geq 2}} \frac{\varepsilon^{2g-2}}{(n-2)!} \mathrm{Coeff}_{\delta^{2g-2}} \Big[\Big(\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x) + \mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)\Big)\,\Big(\mathcal{S}_\delta(u^4)\,D_q\Big)^{n-2}\Big]\,\mathsf{G}_{2g}(q)\\
&= \sum_{g\geq 1} \varepsilon^{2g-2}\,\mathrm{Coeff}_{\delta^{2g-2}} \Big[\Big(\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x) + \mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)\Big)\,\exp\Big(\mathcal{S}_\delta(u^4)D_q\Big)\Big]\,\mathsf{G}_{2g}(q).
\end{aligned}$$

The above power series is the Hadamard product

$$\Big[\Big(\mathcal{S}_\varepsilon(u^1_x)\,\mathcal{S}_\varepsilon(u^4_x) + \mathcal{S}_\varepsilon(u^2_x)\,\mathcal{S}_\varepsilon(u^3_x)\Big)\,\exp\Big(\mathcal{S}_\varepsilon\left(u^4\right)D_q\Big)\Big] \odot \mathcal{G}(\varepsilon, q).$$

In the second line of the statement, we have made use of the following identity, valid for any differential polynomial f:

$$e^{fD_q}\mathcal{G}(\varepsilon, q) = \mathcal{G}(\varepsilon, e^f q).$$

□

Remark 4. Notice that, since the power series whose coefficients are the Bernoulli numbers is divergent, so is the series $\mathcal{G}(\varepsilon, q)$ as a power series in ε. This means that, in the above formulae, $\mathcal{G}(\varepsilon, q)$ must be treated as a formal power series only. With the idea of finding an expression for $\overline{G}$ which avoids the use of divergent power series, let us recall some facts about the Weierstraß elliptic function and its relation with the Eisenstein series G_k, see for instance [OP18]. In the ring $0 < |q| < |p| < 1$, define the shifted Weierstraß elliptic function as

$$\overline{\wp}_\tau(z) := \sum_{a\in\mathbb{Z}_{\neq 0}} \frac{a}{1-q^a}p^a, \qquad \text{where } q = \exp(2\pi\mathbf{i}\tau), \quad p = \exp(2\pi\mathbf{i}z).$$

The shifted Weierstraß function $\overline{\wp}_\tau(z)$ expands at $z=0$ as

$$\frac{1}{(2\pi\mathbf{i}z)^2}+2\sum_{\ell=0}^{\infty}G_{2\ell+2}(q)\frac{(2\pi\mathbf{i}z)^{2\ell}}{(2\ell)!}.$$

The Weierstraß elliptic function $\wp_\tau(z)$ is then defefined by removing the constant term from $\overline{\wp}_\tau(z)$, i.e. $\overline{\wp}_\tau(z):=\wp_\tau(z)+2G_2(q)$.

Next, consider the following elementary fact about residues and the Laplace transform:

$$\frac{1}{2\pi\mathbf{i}}\oint_{|\zeta|=1}\int_{\delta\in\mathbb{R}_{\geq 0}}\delta^k\zeta^\ell e^{-\delta\zeta}d\delta\,d\zeta:=\delta_{k\ell}\ell!\,,\qquad k\in\mathbb{Z}_{\geq 0},\ell\in\mathbb{Z}.$$

We combine these two observations to rewrite the formula for $\overline{G}$ in the following equivalent way:

$$\overline{G}=\int\left[\frac{(u^1)^2u^4}{2}+u^1u^2u^3+\frac{\hbar}{4\pi}\oint\int\left(\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x)+\mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)\right)\overline{\wp}_{\tau+\frac{\mathcal{S}_\delta(u^4)}{2\pi\mathbf{i}}}\left(\frac{\zeta}{2\pi\mathbf{i}}\right)\frac{e^{-\delta\zeta/\varepsilon}}{\varepsilon}d\delta d\zeta\right]dx,$$

where in the choice of the contour one has to assume that either $|\zeta/\varepsilon|=1$ and $\delta\in\mathbb{R}_{\geq 0}$, or, alternatively, $|\zeta|=1$ and $\delta/\varepsilon\in\mathbb{R}_{\geq 0}$.

It is also interesting to compute the primary Hamiltonians of the DR hierarchy, $\overline{G}_{\alpha,0}=\frac{\partial\overline{G}}{\partial u^\alpha}$, $1\leq\alpha\leq 4$, as

$$\begin{aligned}
\overline{G}_{1,0}&=\int\left(u^1u^4+u^2u^3\right)dx,\\
\overline{G}_{2,0}&=\int\left(u^1u^3\right)dx,\\
\overline{G}_{3,0}&=\int\left(-u^1u^2\right)dx,\\
\overline{G}_{4,0}&=\int\left[\frac{(u^1)^2}{2}-\frac{\mathbf{i}\hbar}{8\pi^2}\oint\int\left(\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x)+\mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)\right)\partial_\tau\overline{\wp}_{\tau+\frac{\mathcal{S}_\delta(u^4)}{2\pi\mathbf{i}}}\left(\frac{\zeta}{2\pi\mathbf{i}}\right)\frac{e^{-\delta\zeta/\varepsilon}}{\varepsilon}d\delta\,d\zeta\right]dx.
\end{aligned}$$

The last Hamiltonian can be written as well as

$$\overline{G}_{4,0}=\int\left[\frac{(u^1)^2}{2}-\frac{\mathbf{i}\hbar}{8\pi^2}\oint\int\left(\Delta_\delta(u^1)\,\Delta_\delta(u^4)+\Delta_\delta(u^2)\,\Delta_\delta(u^3)\right)\partial_\tau\overline{\wp}_{\tau+\frac{\Delta_\delta(\partial_x^{-1}u^4)}{2\pi\mathbf{i}}}\left(\frac{\zeta}{2\pi\mathbf{i}}\right)\frac{e^{-\delta\zeta/\varepsilon}}{\varepsilon}d\delta\,d\zeta\right]dx,$$

where $\Delta_y(u^\alpha)(x)=\frac{2}{y}\left(u^\alpha(x+y/2)-u^\alpha(x-y/2)\right)$, $\tau=\frac{1}{2\pi\mathbf{i}}\log q$.

Remark 5. There are several interesting limits for the potential computed in Corollary 4.6. First, the *dispersionless limit* $\varepsilon\to 0$,

$$\overline{G}\Big|_{\varepsilon=0}=\int\left[\frac{(u^1)^2u^4}{2}+u^1u^2u^3+\mathbf{i}\hbar\,(u^1_xu^4_x+u^2_xu^3_x)\,\mathsf{G}_2\left(qe^{u_4}\right)\right]dx.$$

Second, the *trigonometric* $q\to 0$ limit, or $\tau\to+\mathbf{i}\infty$. Since $\mathsf{G}_{2g}(q)\to-\frac{B_{2g}}{4g}$ as $q\to 0$ we have

$$\lim_{\tau\to+\mathbf{i}\infty}\overline{\wp}_\tau\left(\frac{\varepsilon\zeta}{2\pi\mathbf{i}}\right)=-\frac{1}{4\sin^2\left(\frac{\varepsilon\zeta}{2\mathbf{i}}\right)}$$

and we obtain

$$\overline{G}\Big|_{q=0} = \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \frac{\hbar}{4\pi} \oint \int \frac{\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x) + \mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)}{4\sin^2\left(\frac{\varepsilon\zeta}{2\mathbf{i}}\right)} e^{-\delta\zeta} d\delta d\zeta \right] dx.$$

Third, the *double scaling classical limit* obtained by replacing

$$\begin{array}{lll} u^1_k \mapsto a^{-1} u^1_k, & u^2_k \mapsto a^{-1} u^2_k, & k \geq 0, \\ u^3_k \mapsto a u^3_k, & u^4_k \mapsto a u^4_k, & k \geq 0, \\ \tau \mapsto a\tau, & \varepsilon \mapsto a\varepsilon, & \hbar \mapsto a\mu^2 \end{array} \tag{4.11}$$

in the rescaled potential $a\overline{G}$, with μ a new formal variable, and then taking the $a \to 0^+$ limit, along the real axis, with τ fixed in the upper half plane. Because $\mathsf{G}_{2g}(e^{2\pi \mathbf{i} a\tau}) \sim -\frac{B_{2g}}{4g}(a\tau)^{-2g}$ as $a \to 0^+$, we have

$$\lim_{a\to 0^+} a^2 \overline{\wp}_{a\tau}\left(\frac{a\varepsilon\zeta}{2\pi\mathbf{i}}\right) = -\frac{1}{4\tau^2 \sin^2\left(\frac{\varepsilon\zeta}{2\mathbf{i}\tau}\right)}$$

and we obtain a new *classical* DR potential

$$\overline{h} = \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \frac{\mu^2}{4\pi} \oint \int \frac{\mathcal{S}_\delta(u^1_x)\,\mathcal{S}_\delta(u^4_x) + \mathcal{S}_\delta(u^2_x)\,\mathcal{S}_\delta(u^3_x)}{4\left(\tau + \frac{\mathcal{S}_\delta(u^4)}{2\pi\mathbf{i}}\right)^2 \sin^2\left(\frac{\varepsilon\zeta}{2\mathbf{i}\left(\tau + \frac{\mathcal{S}_\delta(u^4)}{2\pi\mathbf{i}}\right)}\right)} e^{-\delta\zeta} d\delta d\zeta \right] dx.$$

In particular, the Hamiltonian densities $h_{\alpha,d}$, $1 \leq \alpha \leq 4$, $d \geq -1$, obtained, in the $a \to 0$ limit, by the replacements (4.11) in the rescaled DR Hamiltonian densities $a^d G_{1,d}$, $a^d G_{2,d}$, $a^{d+2} G_{3,d}$, $a^{d+2} G_{4,d}$, respectively, satisfy the DR recursion

$$\partial_x (D-1) h_{\alpha,d+1} = \{h_{\alpha,d}, h_{1,1}\}, \qquad 1 \leq \alpha \leq 4, d \geq -1,$$

where $D = \mu\frac{\partial}{\partial\mu} + \varepsilon\frac{\partial}{\partial\varepsilon} + \sum_{s=0}^{\infty} u^\alpha_s \frac{\partial}{\partial u^\alpha_s}$, $h_{1,-1} = u^4$, $h_{2,-1} = u^3$, $h_{3,-1} = -u^2$, $h_{4,-1} = u^1$, $\{\cdot,\cdot\} = \left(\frac{1}{\hbar}[\cdot,\cdot]\right)\Big|_{\hbar=0}$ and $\overline{h}_{1,1} = (D-2)\overline{h}$.

Further sending $\varepsilon \to 0$ (i.e. taking the double scaling limit of the quantum dispersionless limit above) we obtain

$$\overline{h}\Big|_{\varepsilon=0} = \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + \frac{\mathbf{i}\pi^2\mu^2}{6} \frac{(u^1_x u^4_x + u^2_x u^3_x)}{(2\pi\mathbf{i}\tau + u^4)^2} \right] dx.$$

which produces, in particular, the Hamiltonians $\overline{h}_{4,0} = \int \left[\frac{(u^1)^2}{2} - \frac{\mathbf{i}\pi^2\mu^2}{3} \frac{(u^1_x u^4_x + u^2_x u^3_x)}{(2\pi\mathbf{i}\tau + u^4)^3} \right] dx$ and $\overline{h}_{1,1} = \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 - \frac{2\pi^3\mu^2\tau}{3} \frac{(u^1_x u^4_x + u^2_x u^3_x)}{(2\pi\mathbf{i}\tau + u^4)^3} \right] dx$.

Chapter 5

Quasimodular structures for the Gromov–Witten classes of elliptic curves

5.1 Overview

In this chapter, we investigate the quasimodular structure of the Gromov–Witten classes of elliptic curves paired with the Hodge class λ_{g-2}. Our main result is an explicit bi-cyclic expression for the pairing

$$c_{g,n}(e_4^{\otimes n})\,\lambda_{g-2},$$

where $c_{g,n}(e_4^{\otimes n})$ denotes the Gromov–Witten class of the elliptic curve with stationary insertions only. Together with the splitting formula of Buryak–Shadrin–Spitz–Zvonkine [Bur+15], this yields a recursive technique for computing descendant integrals of the form

$$\int_{\overline{M}_{g,n}} c_{g,n}(e_4^{\otimes n})\,\lambda_{g-2}\prod_{i=1}^{n}\psi_i^{d_i}. \tag{5.1}$$

Although the above integrals are of independent interest, they also play a role in the reconstruction of the Dubrovin–Zhang (DZ) hierarchy. The DZ hierarchy for the Gromov–Witten theory of the elliptic curve can be completely recovered from the stationary invariants (see [OP06a; Bur23])

$$\int_{\overline{M}_{g,n}} c_{g,n}(e_4^{\otimes n})\prod_{i=1}^{n}\psi_i^{d_i}.$$

Specializing (5.1) to the case $g = 2$ provides a computational framework for these invariants.

Finally, we present an alternative approach to evaluating (5.1), by expressing the pairing with λ_{g-2} through the Pixton–Zagier formula for the double ramification cycle. This leads to a more explicit expression for the same intersection numbers and allows for a direct comparison between two distinct computational frameworks. Such comparisons may shed light on new tautological relations on $\overline{M}_{g,n}$.

5.2 Holomorphic anomaly equation

We recall some results from Oberdieck and Pixton's work that we will use in the next sections.

Proposition 5.1 ([OP18, Theorem 2 and Corollary 1])**.** Consider the cohomological field theory $\{c_{g,n}\}$ associated to an elliptic curve E. For any $e_{\alpha_1},\ldots,e_{\alpha_n}$

$$c_{g,n}(e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n}) \in H^*(\overline{M}_{g,n})\otimes \mathrm{QMod},$$

and more specifically,

$$c_{g,n}(e_4^{\otimes n}) \in H^*(\overline{M}_{g,n})\otimes \mathrm{QMod}_{2g-2+2n} \tag{5.2}$$

Proposition 5.2 ([OP18, Theorem 3]). In the setup of the previous proposition, the following equation, called the *holomorphic anomaly equation (HAE)*, holds in $H^*(\overline{M}_{g,n}) \otimes \mathrm{QMod}$:

$$\begin{aligned}\frac{d}{d\mathsf{G}_2} c_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) = {} & (\mathsf{gl}_2)_* c_{g-1,n+2}\left(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n} \otimes e_1 \otimes e_1\right) \\ & + \sum_{\substack{g_1+g_2=g \\ S_1 \sqcup S_2 = \{1,\ldots,n\}}} \pm (\mathsf{gl}_1)_* \Big(c_{g_1,|S_1|+1}\big(e_1 \otimes \bigotimes_{i\in S_1} e_{\alpha_i}\big) \boxtimes c_{g_2,|S_2|+1}\big(e_1 \otimes \bigotimes_{i \in S_2} e_{\alpha_i}\big)\Big) \\ & - 2 \sum_{\substack{1\le i\le n \\ \alpha_i = 4}} c_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{i-1}} \otimes e_1 \otimes e_{\alpha_{i+1}} \otimes \cdots \otimes e_{\alpha_n})\psi_i.\end{aligned}$$

Here $\pm$ is the eventual Koszul sign.

In the special case of $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} = e_4^{\otimes n}$ the HAE reads:

$$\begin{aligned}\frac{d}{d\mathsf{G}_2} c_{g,n}(e_4^{\otimes n}) = {} & (\mathsf{gl}_2)_* c_{g-1,n+2}(e_4^{\otimes n} \otimes e_1^{\otimes 2}) \\ & + \sum_{\substack{g_1+g_2=g \\ S_1 \sqcup S_2 = \{1,\ldots,n\}}} (\mathsf{gl}_1)_* \Big(c_{g_1,|S_1|+1}\big(e_1 \otimes e_4^{\otimes |S_1|}\big) \boxtimes c_{g_2,|S_2|+1}\big(e_1 \otimes e_4^{\otimes |S_2|}\big)\Big) \\ & - 2\sum_{i=1}^n c_{g,n}(e_4^{\otimes (i-1)} \otimes e_1 \otimes e_4^{\otimes (n-i)})\psi_i.\end{aligned}$$

5.3 Rankin–Cohen brackets

The algebra of modular forms is defined as

$$\mathrm{Mod} = \mathbb{C}[\mathsf{G}_4, \mathsf{G}_6],$$

with grading inherited from the grading on QMod. For even integers $k, \ell > 0$ and even $n \geq 0$, the *n-th Rankin–Cohen bracket* is a bilinear map

$$[\,\cdot\,,\,\cdot\,]_n = [\,\cdot\,,\,\cdot\,]_n^{(k,\ell)} : \mathrm{Mod}_k \otimes \mathrm{Mod}_\ell \longrightarrow \mathrm{Mod}_{k+\ell+n},$$

defined by the formula

$$[f,g]_n(q) = \sum_{s=0}^n (-1)^s \binom{k+n-1}{n-s}\binom{\ell+n-1}{s} D_q^s f(q) \cdot D_q^{n-s} g(q).$$

For even integers $k > 0$ and $r \in \left\{0,1,2,\ldots,\left\lfloor \frac{k}{2}\right\rfloor\right\}$, we denote by $\mathrm{QMod}_k^{\le r}$ the algebra of quasimodular forms of weight k and *depth* at most r. The depth parameter governs the maximal power of the strictly quasimodular part. Concretely, a function $f \in \mathrm{QMod}_k^{\le r}$ can be written as

$$f(q) = \sum_{j=0}^r f_j(q)\, \mathsf{G}_2(q)^j, \qquad \text{with } f_j \in \mathrm{Mod}_{k-2j}.$$

Following [MR09], the Rankin–Cohen bracket admits an extension to quasimodular forms. For even integers $k, \ell > 0$, and depths $r \in \left\{0,1,2,\ldots,\left\lfloor \frac{k}{2}\right\rfloor\right\}$ and $t \in \left\{0,1,2,\ldots,\left\lfloor \frac{\ell}{2}\right\rfloor\right\}$, the *n-th Rankin–Cohen bracket of type* (r,t) is a bilinear map

$$[\,\cdot\,,\,\cdot\,]_n^{(r,t)} = [\,\cdot\,,\,\cdot\,]_n^{(r,t,k,\ell)} : \mathrm{QMod}_k^{\le r} \otimes \mathrm{QMod}_\ell^{\le t} \longrightarrow \mathrm{QMod}_{k+\ell+n}^{\le r+t},$$

defined by the formula

$$[f,g]_n^{(r,t)}(q) = \sum_{s=0}^{n} (-1)^s \binom{k-r+n-1}{n-s} \binom{\ell-t+n-1}{s} D_q^s f(q) \cdot D_q^{n-s} g(q).$$

The use of Rankin–Cohen brackets to construct algebraic relations among modular forms is well established. Beyond this, the entire family of n-th Rankin–Cohen brackets equips the algebra of modular forms with a formal deformation structure, originally proposed by Cohen, Manin, and Zagier [CMZ97] through its connection to the Moyal product. This perspective has since been developed in various directions; see, for example, [OS00; Yao07; Pev08]. An extension of this framework was carried out in [DR14], where formal deformations on the algebra of quasimodular forms were constructed. In Section 5.6, we heuristically derive a family of bilinear differential operators on the algebra of quasimodular forms that mimic the structure of Rankin–Cohen brackets, giving rise to a non-commutative product structure.

5.4 Derivatives of Eisenstein series and weighted Lambert series

Proposition 5.3. For $p \geq 1$, we have

$$\sum_{d \in \mathbb{Z}_{\neq 0}} d^k \left(\frac{d}{1-q^d}\right)^p = \begin{cases} \text{(no quasimodular terms)}, & \text{if } k \text{ is odd;} \\ \frac{2}{(p-1)!} D_q^{p-1} \mathsf{G}_{k+2}(q) + \text{(terms with lower weight)}, & \text{if } k \text{ is even and } k \geq 0; \\ \frac{2}{(p-1)!} D_q^{k+p} \mathsf{G}_k(q) + \text{(terms with lower weight)}, & \text{if } k \text{ is even and } k < 0. \end{cases}$$

Proof. Let $k = 2m$ for some integer $m \geq 0$. We write the Eisenstein series of weight $2m+2$ as

$$\mathsf{G}_{2m+2}(q) = -\frac{B_{2m+2}}{4m+4} + \sum_{d>0} d^{2m} \left(\frac{d}{1-q^d} - d\right). \tag{5.3}$$

Note that, with respect to the weight grading $\deg_{\mathrm{QMod}}(D_q) = 2$, the operator acts as

$$D_q^p \left(\frac{d}{1-q^d}\right) = p! \left(\frac{d}{1-q^d}\right)^{p+1} + \text{(terms with lower weight)}.$$

Applying the operator D_q^p to both sides of (5.3), we obtain

$$\frac{1}{p!} D_q^p \mathsf{G}_{2m+2}(q) = \sum_{d>0} d^{2m} \left(\frac{d}{1-q^d}\right)^{p+1} + \text{(terms with lower weight)},$$

combining with negative integers of d, we get a factor of 2, which establishes the second case.

For the third case, let $k = -2m$ for some integer $m > 0$. Then we have

$$\begin{aligned} \sum_{d>0} d^{-2m} \left(\frac{d}{1-q^d}\right)^{p+1} &= \frac{1}{p!} D_q^p \left(\sum_{d>0} d^{-2m} \frac{d}{1-q^d}\right) + \text{(terms with lower weight)}, \\ &= \frac{1}{p!} D_q^p \left(\sum_{n>0} \sum_{d|n} d^{-2m+1} q^n\right) + \text{(terms with lower weight)}. \end{aligned}$$

Using the involution $d \mapsto d' = n/d$, we arrive at

$$\sum_{d>0} d^{-2m} \left(\frac{d}{1-q^d}\right)^{p+1} = \frac{1}{p!} D_q^{p+1-2m} \mathsf{G}_{2m}(q) + \text{(terms with lower weight)},$$

combining with negative integers of d, we get a factor of 2, which proves the third case. Moreover, the first case follows from a straightforward observation. □

5.5 Quasimodularity of the classes

Denote by $\mathrm{QMod}^{E,d} \subset \mathrm{QMod}$, $d \geq 1$, the linear subspace spanned by the polynomials of degree at most d in the generators of the form $\{D_q^i \mathsf{G}_{2\ell} \mid i \geq 0, \ell \geq 1\}$.

Proposition 5.4. We have $c_{g,n}(e_4^{\otimes n})\lambda_{g-2} \in H^*(\overline{M}_{g,n}) \otimes \mathrm{QMod}^{E,2}$.

Proof. We briefly recall some results from [OP18; OP23]. Degenerate the elliptic curve E into the necklace of n projective lines $\mathbb{P}^1$ glued consecutively at points 0 and ∞. Then apply Li's degeneration formula [Li02] to $[\overline{M}_{g,n}(E,d)]^{\mathrm{vir}}$, and subsequently to $c_{g,n}(e_4^{\otimes n})$. The resulting expression is identified in [OP18] with the following formula:

$$c_{g,n}(e_4^{\otimes n}) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} (\mathrm{gl}_\Gamma)_* \left(\sum_{w} \prod_{e=\{h,h'\}} \frac{w(h)}{1-q^{w(h)}} \bigotimes_{i=1}^{n} \mathrm{DR}_{g_i}\left((w(h))_{h\in v_i}\right) \right). \tag{5.4}$$

Here

- Γ is a stable graph representing a boundary stratum in $\overline{M}_{g,n}$ and gl_Γ is the corresponding boundary map. We think of edges and vertices as subsets of the set $H(\Gamma)$ of the half-edges of Γ.
- The n legs of Γ labeled by $1,\dots,n$ are attached to n pairwise different vertices $v_1,\dots,v_n$ of genera $g_1,\dots,g_n \geq 0$, and there are no further vertices.
- Each edge of Γ can be included in a cycle in the graph, that is, if we cut any edge, the graph remains connected. Loops are allowed.
- The sum over w is the sum over all possible systems of "kissing weights" on the set of half-edges of Γ, which are the functions $w\colon \mathrm{H}(\Gamma) \to \mathbb{Z}$ such that
  - $w(h) + w(h') = 0$ for every edge $e = \{h,h'\}$;
  - $\sum_{h\in v} w(h) = 0$ for every vertex v.
  - $w(h) = 0$ whenever h is a leg.
- If $e = (h,h')$ is a loop, than $w(h)$ is assumed to be negative. Otherwise it is assumed that h is attached to a vertex with smaller index than h'.

Recall (4.2) for the double ramification cycles. In [OP18] it is used to substitute

$$\mathrm{DR}_{g_i}\left((w(h))_{h\in v_i}\right) = \sum_{k\colon \{h\in v_i\}\to\mathbb{Z}_{\geq 0}} \mathfrak{D}^{\mathrm{sym}}_{g_i}\left((k(h))_{h\in v_i}\right) \prod_{h\in v_i} w(h)^{k(h)}, \quad i = 1,\dots,n,$$

in Equation (5.4) and further rewrite it as

$$\begin{aligned} c_{g,n}\left(e_4^{\otimes n}\right) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{k} (\mathrm{gl}_\Gamma)_* \left(\bigotimes_{i=1}^{n} \mathfrak{D}^{\mathrm{sym}}_{g_i}\left((k(h))_{h\in v_i}\right) \right) \\ \times \prod_{\substack{e=\{h,h'\} \\ e \text{ is a loop}}} 2(-1)^{k(h)} \left(\tfrac{B_{k(h)+k(h')+2}}{2(k(h)+k(h')+2)} + \mathsf{G}_{k(h)+k(h')+2} \right) \\ \times \sum_{w} \prod_{\substack{e=\{h,h'\} \\ e \text{ is not a loop}}} \frac{(-1)^{k(h')} w(h)^{k(h)+k(h')+1}}{1-q^{w(h)}}. \end{aligned} \tag{5.5}$$

Here

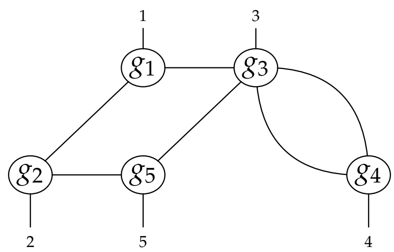


FIGURE 5.1: A graph with the first Betti number equal to 2 and one vertex of index 5.

- The sum over k is the sum over all possible maps $k\colon H(\Gamma) \to \mathbb{Z}_{\geq 0}$ such that for every loop $e = (h, h')$ the sum $k(h) + k(h')$ is even.
- As it follows from Equation (4.2), we assume that $\mathfrak{D}^{\mathrm{sym}}_{g_i}\big((k(h))_{h\in v_i}\big) = 0$ if $\sum_{h\in v_i} k(h) > 2g_i$.
- The sum over w is the sum over all possible systems of "kissing weights" on the set of half-edges of Γ that do not belong to loops, with the conditions exactly as above.

When we multiply Equation (5.5) by λ_{g-2}, the sum over graphs on the right hand side restricts to the graphs Γ with the first Betti number $b_1(\Gamma) \leq 2$. We have then the following possible cases:

Case 1 $b_1(\Gamma) = 0$, $n = 1$, that is, Γ is the graph with one vertex and no further edges. It means that the degree of the corresponding stable map is 0 (and thus n cannot be bigger than 1), and we obtain some class constant in q, which belongs to $\mathrm{QMod}^{E,0}$.

Case 2 $b_1(\Gamma) = 1$, $n = 1$, that is, Γ is the graph with one vertex and one loop. In this case the coefficient in (5.5) manifestly belongs to $\mathrm{QMod}^{E,1}$.

Case 3 $b_1(\Gamma) = 1$, $n \geq 2$. Then Γ is a necklace graph, and the coefficient in (5.5) belongs to $\mathrm{QMod}^{E,1}$ by the same argument as in [OP23, Lemma 6.6], see also [BSS25].

Case 4 $b_1(\Gamma) = 2$, there are exactly two loops. Then $n = 1$, that is, Γ is the graph with one vertex and two loops. In this case the coefficient in (5.5) manifestly belongs to $\mathrm{QMod}^{E,2}$.

Case 5 $b_1(\Gamma) = 2$, there is exactly one loop. Then $n \geq 2$ and Γ is a necklace graph with one loop attached to one of its vertices. We combine the argument of [OP23, Lemma 6.6] or rather its version in [BSS25] for the necklace graph and the explicit formula of the coefficient of the loop to see that in this case the coefficient in (5.5) belongs to $\mathrm{QMod}^{E,2}$.

Case 6 $b_1(\Gamma) = 2$, there are no loops, and there is one vertex of index 5 (and all other vertices are of index 3). Then $n \geq 3$ and Γ is of the form as in Figure 5.1. In this case we treat both necklaces independently, and get by the argument of [OP23, Lemma 6.6] or [BSS25] that the coefficient belongs to $\mathrm{QMod}^{E,2}$.

Case 7 $b_1(\Gamma) = 2$, there are no loops, and there are two vertices of index 4 (and all other vertices are of index 3). Then $n \geq 2$ and Γ is of the form as in Figure 5.2. Analysis of this type of graphs takes up the rest of the proof.

Remark 6. Note that Case 4, Case 5, and Case 6 can be combined together and covered by the same argument. The only reason to keep them separated is that in (5.5) the sum over possible kissing weights on the loops is already expressed explicitly in the second line of the formula.

Consider the graph Γ in Figure 5.2. The kissing weights are labeled by $\pm a, \pm b, \pm c \in \mathbb{Z}_{\neq 0}$, $a + b + c = 0$. Assume that there are exactly $i_\pm$ (resp., $j_\pm$, $k_\pm$) edges $e = (h, h')$ such that $w(h) = \pm a$ (resp.,

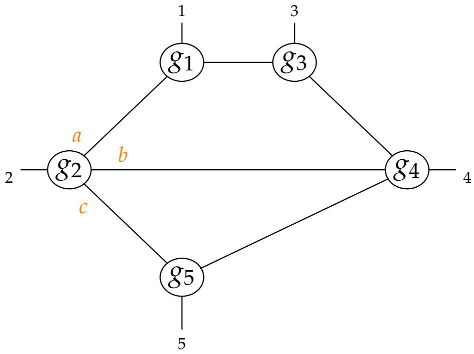


FIGURE 5.2: A graph with the first Betti number equal to 2 and two vertices of index 4.

$\pm b, \pm c$). Note that $i_+ + i_- + j_+ + j_- + k_+ + k_- = n + 1$ Up to a total sign, the coefficient of Γ in (5.5) is equal to

$$\sum_{\substack{a,b,c\in\mathbb{Z}_{\neq 0}\\ a+b+c=0}} a^I \left(\frac{a}{1-q^a}\right)^{i_+} \left(\frac{-a}{1-q^{-a}}\right)^{i_-} b^J \left(\frac{b}{1-q^b}\right)^{j_+} \left(\frac{-b}{1-q^{-b}}\right)^{j_-} c^K \left(\frac{c}{1-q^c}\right)^{k_+} \left(\frac{-c}{1-q^{-c}}\right)^{k_-}, \tag{5.6}$$

for some non-negative integer constants I, J, K such that $I + J + K = 2g - 4$. Our goal is to represent (5.6) as an element of QMod. As we show below, it is in general a non-homogeneous quasimodular form with the components of degree $\deg_{\mathrm{QMod}} \leq 2g - 2 + 2n$. From Equation (5.2) we know that we are interested only in its top weight term with $\deg_{\mathrm{QMod}} = 2g - 2 + 2n$. To this end, we use below $\approx$ for the equalities that hold modulo the terms of lower quasimodular weight. Expression (5.6) can be rewritten as

$$\sum_{\substack{a,b,c\in\mathbb{Z}_{\neq 0}\\ a+b+c=0}} P(a, b, c, \tfrac{1}{1-q^a}, \tfrac{1}{1-q^b}, \tfrac{1}{1-q^c}),$$

a polynomial P in a, b, c and $\frac{1}{1-q^a}$, $\frac{1}{1-q^b}$, and $\frac{1}{1-q^c}$. In order to keep track of the quasimodular weight, we define the grading $\deg_{\mathrm{QMod}}$ on the ring $\mathbb{C}[a, b, c, \frac{1}{1-q^a}, \frac{1}{1-q^b}, \frac{1}{1-q^c}]$ by assigning

$$\deg_{\mathrm{QMod}} a = \deg_{\mathrm{QMod}} b = \deg_{\mathrm{QMod}} c = \deg_{\mathrm{QMod}} \tfrac{1}{1-q^a} = \deg_{\mathrm{QMod}} \tfrac{1}{1-q^b} = \deg_{\mathrm{QMod}} \tfrac{1}{1-q^c} = 1$$

(this assignment is justified by the subsequent computation) and note that $\deg_{\mathrm{QMod}} P \leq 2g - 2 + 2n$ and

$$P(a, b, c, \tfrac{1}{1-q^a}, \tfrac{1}{1-q^b}, \tfrac{1}{1-q^c}) \approx a^{\tilde{I}} \left(\frac{1}{1-q^a}\right)^i b^{\tilde{J}} \left(\frac{1}{1-q^b}\right)^j c^{\tilde{K}} \left(\frac{1}{1-q^c}\right)^k,$$

where $\tilde{I} = I + i$, $\tilde{J} = J + j$, $\tilde{K} = K + k$, and $i = i_+ + i_-$, $j = j_+ + j_-$, $k = k_+ + k_-$. Note that

$$\frac{1}{1-q^b}\frac{1}{1-q^c} \approx -\frac{q^b}{1-q^b}\frac{1}{1-q^{-c}} = -\frac{1}{1-q^a}\frac{1}{1-q^b} + \frac{1}{1-q^a}\frac{1}{1-q^{-c}} \approx -\frac{1}{1-q^a}\frac{1}{1-q^b} - \frac{1}{1-q^a}\frac{1}{1-q^c}.$$

Applying this identity multiple times and using that $a + b + c = 0$, we see that

$$\begin{aligned} &P(a, b, c, \tfrac{1}{1-q^a}, \tfrac{1}{1-q^b}, \tfrac{1}{1-q^c}) \\ &\approx \sum_{\substack{s+p+t+r\\ =2g-2+2n\\ s,t\geq 0, p,r\geq 1}} B_{sptr} \cdot a^s \left(\frac{1}{1-q^a}\right)^p b^t \left(\frac{1}{1-q^b}\right)^r + C_{sptr} \cdot a^s \left(\frac{1}{1-q^a}\right)^p c^t \left(\frac{1}{1-q^c}\right)^r, \end{aligned}$$

where B_{sptr} and C_{sptr} are some explicit coefficients (we compute them below). By reparametrization we conclude that

$$\sum_{\substack{a,b,c\in\mathbb{Z}_{\neq 0}\\ a+b+c=0}} P\left(a,b,c,\tfrac{1}{1-q^a},\tfrac{1}{1-q^b},\tfrac{1}{1-q^c}\right) \approx \sum_{\substack{s+p+t+r\\ =2g-2+2n\\ s,t\geq 0,p,r\geq 1}} A_{sptr} \sum_{a\in\mathbb{Z}_{\neq 0}} a^s \left(\frac{1}{1-q^a}\right)^p \sum_{b\in\mathbb{Z}_{\neq 0}} b^t \left(\frac{1}{1-q^b}\right)^r,$$

where $A_{sptr} = B_{sptr} + C_{sptr}$. We conclude the proof by applying Proposition 5.3, which holds for $s \geq 0$ and $p \geq 1$.

$$\sum_{a\in\mathbb{Z}_{\neq 0}} a^s \left(\frac{1}{1-q^a}\right)^p \approx \begin{cases} 0 & s+p \text{ is odd;} \\ \frac{2}{(p-1)!} D_q^{p-1} \mathsf{G}_{s-p+2} & s+p \text{ is even, } s \geq p; \\ \frac{2}{(p-1)!} D_q^{s} \mathsf{G}_{p-s} & s+p \text{ is even, } s < p. \end{cases} \tag{5.7}$$

Note that Equation (5.7) also justifies that all the terms that we dropped throughout the computation of (5.6) using our extension of the quasimodular weight $\deg_{\mathrm{QMod}}$ are either vanishing or give the quasimodular terms of lower weight. □

5.6 Deformed Rankin–Cohen type brackets

Let $T = (I,J,K,i,j,k) \in \mathbb{Z}^6_{\geq 0}$ be a tuple of non-negative integers satisfying $i+j+k = n+1$ and $I+J+K = 2g-4$. We define two types of bilinear differential operators acting on the algebra of quasimodular forms, motivated by considerations in the following section.

Let γ be a formal variable. For each $s \in \{0,\dots,j-1\}$, define the coefficients of the bilinear operators by

$$f_s^T(\gamma) = (-1)^s \frac{\binom{s+k-1}{s}\binom{K+k}{\gamma+k+s-I-2}}{(n-j+s)!(j-1-s)!}.$$

For even non-negative integers $\alpha \in \{2,4,\dots,2g-2-J\}$, define the *bi-differential operator of type* $(T,1)$ by

$$[\mathsf{G}_\alpha, \mathsf{G}_{2g-\alpha}]^{(T,1)} = \sum_{s=0}^{j-1} f_s^T(\alpha)\, D_q^{n-j+s}\mathsf{G}_\alpha \cdot D_q^{j-1-s}\mathsf{G}_{2g-\alpha} \in \mathrm{QMod}_{2g-2+2n}.$$

Similarly, for even non-negative integers $\beta \in \{2,4,\dots,n-I-i\}$, define the *bi-differential operator of type* $(T,2)$ by

$$[\mathsf{G}_\beta, \mathsf{G}_{2g-2+\beta}]^{(T,2)} = \sum_{s=0}^{j-1} f_s^T(-\beta+2)\, D_q^{n-j-\beta+1+s}\mathsf{G}_\beta \cdot D_q^{j-1-s}\mathsf{G}_{2g-2+\beta} \in \mathrm{QMod}_{2g-2+2n}.$$

Example 12. We focus on operators of type $(T,1)$, choosing T such that $j = n$ and $J = 0$. To obtain non-trivial contributions, we require $i = 0$ and $k = 1$, and set $I = 2$ so that the bracket becomes antisymmetric when $n = 2$. Then, for $\alpha \in \{2,4,\dots,2g-2\}$, we obtain the bracket

$$[\mathsf{G}_\alpha, \mathsf{G}_{2g-\alpha}]^{(T,1)} = \sum_{s=0}^{n-1} (-1)^s \frac{\binom{s+k-1}{s}\binom{2g-5}{\alpha+s-3}}{s!(n-1-s)!} D_q^s\mathsf{G}_\alpha \cdot D_q^{n-1-s}\mathsf{G}_{2g-\alpha},$$

which satisfies the basic properties of a $(n-1)$-th Rankin–Cohen bracket on QMod, with modified coefficients.

We define a non-commutative product $*_T$ by specifying a left action of $\mathbb{C}[\mathsf{G}_2]$ on QMod. That is, for each $g \geq 2$, the product $\mathsf{G}_2 *_T \mathsf{G}_{2g-2}$ is defined, although $*_T$ is not extended to arbitrary pairs of quasimodular forms. (We emphasize that the product $*_T$ is not necessarily associative.) The explicit formula for the product is

$$\begin{aligned}\mathsf{G}_2 *_T \mathsf{G}_{2g-2} = &\binom{2K+2k}{K+k}\left(\sum_{\alpha=2}^{2g-2-J}[\mathsf{G}_\alpha,\mathsf{G}_{2g-\alpha}]^{(T,1)} + \sum_{\beta=2}^{n-I-i}[\mathsf{G}_\beta,\mathsf{G}_{2g-2+\beta}]^{(T,2)}\right)\\ &+(-1)^I\binom{2J+2j}{J+j}\left(\sum_{\alpha=2}^{2g-2-K}[\mathsf{G}_\alpha,\mathsf{G}_{2g-\alpha}]^{(\overline{T},1)} + \sum_{\beta=2}^{n-I-i}[\mathsf{G}_\beta,\mathsf{G}_{2g-2+\beta}]^{(\overline{T},2)}\right),\end{aligned} \tag{5.8}$$

where all sums range over even positive integers, and $\overline{T}$ denotes the tuple T with the components (K,k) and (J,j) exchanged.

5.7 Triple weighted Lambert series and quasimodularity

We compute the coefficients in Equation (5.6) and express them in terms of the brackets and products defined previously. As before, set $i = i_+ + i_-$, $j = j_+ + j_-$, $k = k_+ + k_-$.

Proposition 5.5. Let $g, n > 1$. For any tuple $T = (I, J, K, i, j, k) \in \mathbb{Z}^6_{\geq 0}$ satisfying $i + j + k = n + 1$ and $I + J + K = 2g - 4$, the product $\mathsf{G}_2 *_T \mathsf{G}_{2g-2}$ equals the top quasimodular weight part of:

$$\frac{1}{4}\sum_{\substack{a,b,c\in\mathbb{Z}_{\neq 0}\\ a+b+c=0}} a^I\left(\frac{a}{1-q^a}\right)^{i_+}\left(\frac{-a}{1-q^{-a}}\right)^{i_-} b^J\left(\frac{b}{1-q^b}\right)^{j_+}\left(\frac{-b}{1-q^{-b}}\right)^{j_-} c^K\left(\frac{c}{1-q^c}\right)^{k_+}\left(\frac{-c}{1-q^{-c}}\right)^{k_-}.$$

Proof. As in Proposition 5.4, we write $\approx$ to denote equality modulo terms of lower quasimodular weight. Then Equation (5.6) simplifies to

$$\approx \frac{1}{4}\sum_{\substack{a,b,c\in\mathbb{Z}_{\neq 0}\\ a+b+c=0}} a^{\tilde{I}}\left(\frac{1}{1-q^a}\right)^i b^{\tilde{J}}\left(\frac{1}{1-q^b}\right)^j c^{\tilde{K}}\left(\frac{1}{1-q^c}\right)^k, \tag{5.9}$$

where $\tilde{I} = I + i$, $\tilde{J} = J + j$, $\tilde{K} = K + k$. Using the identity

$$\begin{aligned}\left(\frac{1}{1-q^b}\right)^j\left(\frac{1}{1-q^c}\right)^k = &\sum_{s=0}^{j-1}(-1)^s\binom{s+k-1}{s}\left(\frac{1}{1-q^a}\right)^{k+s}\left(\frac{1}{1-q^b}\right)^{j-s}\\ &+(-1)^j\sum_{s=0}^{k-1}\binom{s+j-1}{s}\left(\frac{1}{1-q^a}\right)^{j+s}\left(\frac{1}{1-q^c}\right)^{k-s},\end{aligned}$$

and substituting into Equation (5.9), we obtain:

$$\begin{aligned}\approx &\frac{1}{4}\sum_{\substack{s=0\\ r+t=\tilde{K}}}^{j-1}(-1)^s\binom{s+k-1}{s}\binom{\tilde{K}}{r,t}\sum_{a,b\in\mathbb{Z}_{\neq 0}} a^{\tilde{I}+r}b^{\tilde{J}+t}\left(\frac{1}{1-q^a}\right)^{i+k+s}\left(\frac{1}{1-q^b}\right)^{j-s}\\ &+\frac{(-1)^j}{4}\sum_{\substack{s=0\\ r+t=\tilde{J}}}^{k-1}(-1)^r\binom{s+j-1}{s}\binom{\tilde{J}}{r,t}\sum_{a,c\in\mathbb{Z}_{\neq 0}} a^{\tilde{I}+r}c^{\tilde{K}+t}\left(\frac{1}{1-q^a}\right)^{i+j+s}\left(\frac{1}{1-q^c}\right)^{k-s}.\end{aligned}$$

We focus on the first term, as the second follows analogously by exchanging $(\tilde{K}, k) \leftrightarrow (\tilde{J}, j)$. Applying Proposition 5.7 and working modulo lower-weight terms, we have:

$$\approx \sum_{\substack{s=0 \\ r+t=\tilde{K}}}^{j-1} (-1)^s \frac{\binom{s+k-1}{s}\binom{\tilde{K}}{r,t}}{(n-j+s)!(j-s-1)!} D_q^{n-j+s} \mathsf{G}_{I+r-k-s+2} D_q^{j-s-1} \mathsf{G}_{J+t+s+2}$$
$$+ \sum_{\substack{s=0 \\ r+t=\tilde{K}}}^{j-1} (-1)^s \frac{\binom{s+k-1}{s}\binom{\tilde{K}}{r,t}}{(n-j+s)!(j-s-1)!} D_q^{\tilde{I}+r} \mathsf{G}_{k+s-I-r} D_q^{j-s-1} \mathsf{G}_{J+t+s+2}.$$

The first sum corresponds to the case $I + r \geq k + s$, and the second to $I + r < k + s$. We encode these conditions by substituting $\alpha = I + r - k - s + 2$, $\beta = k + s - I - r$, and account for overcounting by including a factor of $\binom{2\tilde{K}}{\tilde{K}}$. After reindexing, we recover the first two terms of the product $\mathsf{G}_2 *_T \mathsf{G}_{2g-2}$. □

5.8 Bi-cyclic graph representations of the classes

Building on our analysis of the quasimodular behaviour of the pairing $c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$ in the previous sections, we can now derive an explicit formula for this pairing. We denote by Ne the subset of stable graphs corresponding to strata in $\overline{M}_{g,n}$, with $g \geq 2$ and $n \geq 1$, whose underlying graph has the shape of a *necklace*. Each $\Gamma \in$ Ne satisfies the following properties:

- It has n vertices $V(\Gamma) = \{v_1, \ldots, v_n\}$, each carrying exactly one leaf labeled by i, with leaf i attached to v_i.
- The n edges connect the vertices in a single cycle of length n.

For $n \geq 2$, there are $(n-1)!/2$ distinct graphs of this type (for $n = 2$, the unique such graph has an automorphism group of order 2). Each graph $\Gamma \in$ Ne is equipped with a genus function $g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$, satisfying the stability condition $\sum_{i=1}^{n} g(v_i) = g - 1$. We fix an orientation on these necklaces, then there exactly $(n-1)!$ oriented necklaces.

Next, we denote by $\mathrm{Bi}_{(m)}$ the subset of stable graphs corresponding to strata in $\overline{M}_{g,n}$, with $g \geq 2$ and $n \geq 1$, whose underlying graph has the shape of a *bi-cyclic* graph. Each $\Gamma \in \mathrm{Bi}_{(m)}$ satisfies the following properties:

- It has n vertices $V(\Gamma) = \{v_1, \ldots, v_n\}$, each carrying exactly one leaf labeled by i, with leaf i attached to v_i.
- When $m = 1$, the graph has $n + 1$ edges forming two cycles that meet at a single vertex. All vertices have index 3 except for the common vertex, which has index 5 (see Figure 5.1).
- When $m > 1$, the graph has $n + 1$ edges forming two cycles that share m common vertices, so that the cycles are glued along a chain of m consecutive vertices. All vertices have index 3 except for the two endpoints of the shared chain, each having index 4 (see Figure 5.2).

Each graph $\Gamma \in \mathrm{Bi}_{(m)}$ is equipped with a genus function $g\colon V(\Gamma) \to \mathbb{Z}_{\geq 0}$ satisfying the stability condition $\sum_{i=1}^{n} g(v_i) = g - 2$. We also introduce vertex-counting functions $n_1(\Gamma)$ and $n_2(\Gamma)$, recording the numbers of edges in the two cycles, such that $n_1(\Gamma) + n_2(\Gamma) + (m-1) = n + 1$. Analogously, let $g_1(\Gamma)$ and $g_2(\Gamma)$ denote the total genera along the two cycles, and $g_3(\Gamma)$ the total genus on the connecting chain, satisfying $g_1(\Gamma) + g_2(\Gamma) = g - g_3(\Gamma)$.

Proposition 5.6. Let $g \geq 2$ and $n \geq 1$, then we have

$$\begin{aligned} c_{g,n}\left(e_4^{\otimes n}\right)\lambda_{g-2} = & \frac{g}{(n-1)!} \sum_{\substack{\Gamma_1 \in \mathrm{Ne} \\ g\colon V(\Gamma_1)\to \mathbb{Z}_{\geq 0} \\ j=1,\ldots,n}} (\mathsf{gl}_{\Gamma_1,g})_* \\ & \left(\bigotimes_{v \in V(\Gamma_1)\setminus\{v_j\}} \mathrm{DR}_{g(v)}(0,-1,1)\lambda_{g(v)} \otimes \mathfrak{D}^{\mathrm{sym}}_{g(v_j)}(0,2g(v_j),0)\lambda_{g(v_j)-1} \right) D_q^{n-1}\mathsf{G}_{2g} \\ & + \frac{4}{n!} \sum_{\substack{\Gamma_2 \in \mathrm{Bi}_{(1)} \\ g: V(\Gamma_2)\to\mathbb{Z}_{\geq 0}}} \left(\sum_{n_1(\Gamma_2)\neq n_2(\Gamma_2)} + 2 \sum_{n_1(\Gamma_2)=n_2(\Gamma_2)} \right) \frac{1}{(n_1(\Gamma_2)-1)!(n_2(\Gamma_2)-1)!} \\ & \sum_{\substack{j=1,\ldots,n \\ h_1+h_2=2g(v_j)}} (\mathsf{gl}_{\Gamma_2,g})_* \left(\bigotimes_{v\in V(\Gamma_2)\setminus\{v_j\}} \mathrm{DR}_{g(v)}(0,-1,1)\lambda_{g(v)} \otimes \mathfrak{D}^{\mathrm{sym}}_{g(v_j)}(0,h_1,0,h_2,0)\lambda_{g(v_j)} \right) \\ & D_q^{n_1(\Gamma_2)-1}\mathsf{G}_{2(g_1(\Gamma_2)+h_1+1)} D_q^{n_2(\Gamma_2)-1}\mathsf{G}_{2(g_2(\Gamma_2)+h_2+1)} \\ & + \frac{1}{n!} \sum_{\substack{\Gamma_3 \in \mathrm{Bi}_{(m)} \\ g: V(\Gamma_3)\to\mathbb{Z}_{\geq 0}}} \left(\sum_{n_1(\Gamma_3)\neq n_2(\Gamma_3)} + 2 \sum_{n_1(\Gamma_3)=n_2(\Gamma_3)} \right) \sum_{\substack{j,k=1,\ldots,n \\ h_1+h_2+h_3=2g(v_j) \\ l_1+l_2+l_3=2g(v_k)}} (\mathsf{gl}_{\Gamma_3,g})_* \\ & \left(\bigotimes_{v\in V(\Gamma_3)\setminus\{v_j,v_k\}} \mathrm{DR}_{g(v)}(0,-1,1)\lambda_{g(v)} \otimes \mathfrak{D}^{(1)}_{g(v_j)}(0,h_1,h_2,h_3)\lambda_{g(v_j)} \otimes \mathfrak{D}^{(1)}_{g(v_k)}(0,l_1,l_2,l_3)\lambda_{g(v_k)} \right) \\ & \mathsf{G} *_{T(\Gamma_3)} \mathsf{G}_{2g-2}. \end{aligned} \tag{5.10}$$

Here, the boundary morphisms are given by

$$\mathsf{gl}_{\Gamma_1,g}\colon \prod_{v\in V(\Gamma_1)} \overline{M}_{g(v),3} \to \overline{M}_{g,n}, \quad \mathsf{gl}_{\Gamma_2,g}\colon \prod_{v\in V(\Gamma_2)\setminus\{v_j\}} \overline{M}_{g(v),3} \times \overline{M}_{g(v_j),5} \to \overline{M}_{g,n},$$

and

$$\mathsf{gl}_{\Gamma_3,g}\colon \prod_{v\in V(\Gamma_3)\setminus\{v_j,v_k\}} \overline{M}_{g(v),3} \times \overline{M}_{g(v_j),4} \times \overline{M}_{g(v_k),4} \to \overline{M}_{g,n}.$$

Moreover, $\mathrm{DR}_{g(v)}(0,1,-1)$ denotes the double ramification cycle on $\overline{M}_{g(v),3}$ assigned to the vertex v, where the multiplicity 0 corresponds to the marked point associated with the leaf. The classes $\mathfrak{D}^{\mathrm{sym}}_g$ and $\mathfrak{D}^{(1)}_g$ are the cohomology classes associated to the double ramification cycle defined in Equations (4.2) and (4.2.2), respectively. Finally, $*_{T(\Gamma_3)}$ denotes the product defined in Equation (5.8), where $T(\Gamma_3)$ is the tuple datum determined by the graph Γ_3.

Application. By pairing the bi-cyclic expression with ψ-classes and integrating over $\overline{M}_{g,n}$, we obtain an explicit algorithm for computing the integrals in (5.1). The resulting computation closely parallels that of Corollary 3.4, where we integrate over each vertex, with contributions involving either λ_g or λ_{g-1}, together with a double ramification cycle on each component. The only remaining step is the evaluation of intersection numbers involving the double ramification cycle at each individual vertex, which is addressed in the following section.

5.9 Intersection numbers with the double ramification cycle

To obtain an explicit evaluation of the pairing in Equation (5.1), we require specializations of the following results. We introduce the following notation:

$$\mathcal{P}^{(0)}_{g,n}(a_1,\dots,a_n) = \int_{\overline{M}_{g,n+1}} \mathrm{DR}_g\left(-\sum_{i=1}^n a_i, a_1,\dots,a_n\right)\lambda_g\psi_1^{g+n-2}. \tag{5.11}$$

Note that this integral is a homogeneous polynomial in $a_1,\dots,a_n$. We now present a formula that is recursive with respect to the genus and the number of marked points.

Proposition 5.7. For any $(a_1,\dots,a_n)\in\mathbb{Z}^n$, with $g\ge 0$ and $n>2$, we have:

$$\begin{aligned}\mathcal{P}^{(0)}_{g,n}(a_1,\dots,a_n) &= \frac{\left(\sum_{i=1}^n a_i^3\right)}{12\left(\sum_{i=1}^n a_i\right)(2g+n-1)}\mathcal{P}^{(0)}_{g-1,n}(a_1,\dots,a_n)\\ &\quad+\frac{1}{\left(\sum_{i=1}^n a_i\right)(2g+n-1)}\sum_{1\le i<j\le n}(a_i+a_j)\mathcal{P}^{(0)}_{g,n-1}(a_i+a_j,(a_k)_{k\in\{1,\dots,n\}\setminus\{i,j\}}).\end{aligned} \tag{5.12}$$

The case $n\le 2$ is given in [BSS25, Proposition 3.1].

Proof. We apply the recursion relation for the ψ-class on the double ramification cycle from [Bur+15, Theorem 4]. After pairing with λ_g and discarding terms that vanish for dimensional reasons, we obtain

$$\begin{aligned}\left(\sum_{i=1}^n a_i\right)(2g+n-1)\int_{\overline{M}_{g,n+1}}\mathrm{DR}_g\left(-\sum_{i=1}^n a_i, A\right)\lambda_g\psi_1^{g+n-2} &= \sum_{1\le i<j\le n}(a_i+a_j)\\ &\times\int_{\overline{M}_{g,n}}\mathrm{DR}_g\left(-\sum_{i=1}^n a_i, a_i+a_j,(a_k)_{k\in\{1,\dots,n\}\setminus\{i,j\}}\right)\lambda_g\psi_1^{g+n-3}\\ &+2\int_{\overline{M}_{g-1,n+1}}\mathrm{DR}_{g-1}\left(-\sum_{i=1}^n a_i, A\right)\lambda_{g-1}\psi_1^{g+n-3}\left(\sum_{i=1}^n a_i\int_{\overline{M}_{1,2}}\mathrm{DR}_1(-a_i,a_i)\lambda_1\right).\end{aligned} \tag{5.13}$$

Applying [Bur+15, Theorem 4] again (see also [BSS25, Proposition 3.1]), we obtain

$$\int_{\overline{M}_{g,2}}\mathrm{DR}_g(-b,b)\lambda_g = \frac{b^{2g}}{24^g g!}. \tag{5.14}$$

Substituting this into (5.13) and rearranging, we derive the desired recursion relation. □

Introduce the following notation:

$$\mathcal{P}^{(1)}_{g,n}(a_1,\dots,a_n) = \int_{\overline{M}_{g,n+1}}\mathrm{DR}_g\left(-\sum_{i=1}^n a_i, a_1,\dots,a_n\right)\lambda_{g-1}\psi_1^{g+n-1} \tag{5.15}$$

We now present a recursive formula to construct the above polynomial.

Proposition 5.8. For any $(a_1,\dots,a_n)\in\mathbb{Z}^n$ and $g,n\ge 1$, we have:

$$\begin{aligned}\mathcal{P}^{(1)}_{g,n}(a_1,\dots,a_n) &= \frac{\left(\sum_{i=1}^n a_i^3\right)}{12\left(\sum_{i=1}^n a_i\right)(2g+n-1)}\mathcal{P}^{(1)}_{g-1,n}(a_1,\dots,a_n)\\ &\quad+\frac{1}{\left(\sum_{i=1}^n a_i\right)(2g+n-1)}\sum_{1\le i<j\le n}(a_i+a_j)\mathcal{P}^{(1)}_{k,n-1}(a_i+a_j,(a_k)_{k\in\{1,\dots,n\}\setminus\{i,j\}})\\ &\quad+\frac{1}{2\left(\sum_{i=1}^n a_i\right)(2g+n-1)}\sum_{i=1}^n\sum_{\alpha=1}^{a_i}\alpha(a_i-\alpha)\mathcal{P}^{(0)}_{g-1,n+1}((a_j)_{j\in\{1,\dots,n\}\setminus\{i\}},\alpha,a_i-\alpha)\end{aligned} \tag{5.16}$$

Proof. We apply the recursion relation for the ψ-class on the double ramification cycle from [Bur+15, Theorem 4]. After pairing with λ_{g-1} and discarding terms that vanish for dimensional reasons, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{n} a_i\right)(2g+n-1)\int_{\overline{M}_{g,n+1}} \mathrm{DR}_g\left(-\sum_{i=1}^{n} a_i, A\right)\lambda_{g-1}\psi_1^{g+n-1} &= \sum_{1\le i<j\le n}(a_i+a_j)\\ &\times\int_{\overline{M}_{g,n}} \mathrm{DR}_g(a_i+a_j,(a_k)_{k\in\{1,\dots,n\}\setminus\{i,j\}})\lambda_{g-1}\psi_1^{g+n-2}\\ &+2\int_{\overline{M}_{g,n+1}} \mathrm{DR}_g\left(-\sum_{i=1}^{n} a_i, A\right)\lambda_{g-1}\psi_1^{g+n-2}\left(\sum_{i=1}^{n} a_i\int_{\overline{M}_{g,2}} \mathrm{DR}_1(-a_i,a_i)\lambda_1\right)\\ &+\frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{a_i}\alpha(a_i-\alpha)\int_{\overline{M}_{g,n+2}} \mathrm{DR}_g((a_j)_{j\in\{1,\dots,n\}\setminus\{i\}},\alpha,a_i-\alpha)\lambda_{g-1}\psi_1^{g+n-2}\end{aligned}\tag{5.17}$$

Substituting (5.14) into (5.17) and rearranging, we obtain the claim. □

Example 13. For $g=2$ and $n=3$, the previous propositions implies:

$$\mathcal{P}^{(0)}_{2,3}(a_1,a_2,a_3)=\frac{1}{1152}a_1^4+\frac{7}{2880}a_1^2a_2^2+\frac{1}{1152}a_2^4+\frac{7}{2880}a_1^2a_3^2+\frac{7}{2880}a_2^2a_3^2+\frac{1}{1152}a_3^4$$

and,

$$\begin{aligned}\mathcal{P}^{(1)}_{2,3}(a_1,a_2,a_3)=&\frac{1}{720}a_1^4+\frac{1}{240}a_1^2a_2^2+\frac{1}{720}a_2^4+\frac{1}{240}a_1^2a_3^2+\frac{1}{240}a_2^2a_3^2\\ &+\frac{1}{720}a_3^4-\frac{1}{576}a_1^2-\frac{1}{576}a_2^2-\frac{1}{576}a_3^2+\frac{1}{2880}\end{aligned}$$

5.10 Shaving down Pixton–Zagier formula

From this section onward, we present an alternative approach to computing (5.1). Our strategy is to analyze the integral (4.10) and use the vanishing result of Proposition 4.4. We recall the Pixton–Zagier formula for the double ramification cycle and we study its coupling with $c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$. As we shall see, this coupling gives rise to only a small number of non-vanishing contributions.

Pixton–Zagier formula

We use a variation of Pixton's formula [Jan+17, p. 9] for the double ramification cycle. The variation is based on a combination of Pixton's formula with Zagier's spanning tree formula [Pix]. Specifically, we follow the explicit formulation given in [BLS24b, Equation 3.1]. The notation and structure of the formula are explained in detail in [BLS24b, p. 10]. Let $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfy $\sum_{i=1}^n a_i=0$. The Pixton–Zagier formula is given by:

$$\begin{aligned}\mathrm{DR}_g(A)=\frac{1}{2^g}\sum_{\Gamma}\frac{(-2)^{|\mathrm{E}(\Gamma)|}}{|\operatorname{Aut}(\Gamma)|}\sum_{d:H(\Gamma)\to\mathbb{Z}_{\ge0}}&\left[(\mathrm{gl}_\Gamma)_*\left(\prod_{i=1}^{n}\psi_{\ell_i}^{d(\ell_i)}\prod_{e\in\mathrm{E}(\Gamma)}\psi_{h(e)}^{d(h(e))}\psi_{t(e)}^{d(t(e))}\right)\right]_g\\ &\times\prod_{i=1}^{n}\frac{a_i^{2d(\ell_i)}}{d(\ell_i)!}\prod_{e\in\mathrm{E}(\Gamma)}\frac{(2d(h(e))+2d(t(e))+1)!}{d(h(e))!d(t(e))!}\\ &\times\operatorname{Coeff}_{\left(\prod_{e\in\mathrm{E}(\Gamma)}x_e^{2d(h(e))+2d(t(e))+2}\right)}\sum_{\mathrm{T}\in\mathrm{SpTr}(\Gamma)}\prod_{f\in\mathrm{E}(\mathrm{T})}e^{a_{f,\mathrm{T}}x_f}\prod_{f\notin\mathrm{E}(\mathrm{T})}\frac{x_f}{e^{x_{f,\mathrm{T}}}-1}.\end{aligned}\tag{5.18}$$

Here we use the following notation and conventions:

- The first sum is over the stable graphs Γ of genus g with n leaves. The set of leaves (resp., edges, vertices) is denoted by $L(\Gamma)$ (resp., $E(\Gamma)$, $V(\Gamma)$). The leaves are labeled by the bijection $\ell : \{1,\ldots,n\} \to L(\Gamma)$, $i \mapsto \ell_i$. $|\mathrm{Aut}(\Gamma)|$ denotes the order of the automorphism group of Γ. On each edge of Γ we choose an orientation.
- Let $H(\Gamma)$ be the set of half-edges of Γ. The orientations on the edges allow us to define $h\colon E(\Gamma) \hookrightarrow H(\Gamma)$ and $t\colon E(\Gamma) \hookrightarrow H(\Gamma)$ (heads and tails), and we have $H(\Gamma) = L(\Gamma) \sqcup h(E(\Gamma)) \sqcup t(E(\Gamma))$.
- We apply the boundary map gl_Γ to the tensor product of the monomials of ψ-classes on the moduli spaces of curves corresponding to the vertices of Γ. Notation $[C]_g$ means that we take the homogeneous component of degree g of the resulting class $C \in R^*(\overline{M}_{g,n})$.
- The operator $\mathrm{Coeff}_{\prod_{i=1}^k x_i^{p_i}}$ extracts the coefficient of $\prod_{i=1}^k x_i^{p_i}$ from the Laurent series next to it. The latter Laurent series is the sum over the set $\mathrm{SpTr}(\Gamma)$ of the spanning trees of Γ.
- In the expression associated to a spanning tree T,
  - x_e's are the formal variables associated to the edges of Γ;
  - $a_{f,T}$ is defined as the sum of a_i's over all leaves attached to the vertices that are ahead of f in T with respect to the orientation of f.
  - $x_{f,T}$ is the sum of $\pm x_e$ over all edges e that enter the unique cycle formed by f and edges in T. The sign is positive if the orientation of e agrees with the orientation of f in the cycle, and negative otherwise.

Remark 7. There are several features of this expression that are important to mention. First, the ingredients of the formula depend on the choices of orientations of edges, but the whole formula does not depend on these choices. Second, the summands in the final sum over spanning trees are not formal power series in x_e, but the whole sum is.

Coupling to $c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$

Consider the product $\mathrm{DR}_g(A)c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$ and use Equation (5.18) for $\mathrm{DR}_g(A)$. The factorization properties of $c_{g,n}$ and λ_{g-2} imply that this product has a similar expression, where in addition to the monomials of ψ-classes the vertices v of stable graphs are also decorated by the classes $c_{g(v),n(v)}$ with some primary fields and λ_h with $g(v) - 2 \le h \le g(v)$, where $g(v)$ and $n(v)$ are the genus and the index of the vertex v. Thus we get a Pixton–Zagier type formula for $\mathrm{DR}_g(A)c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$.

Proposition 5.9. The only stable graphs Γ that non-trivially contribute to the Pixton–Zagier type formula for $\mathrm{DR}_g(A)c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$ are listed in Figures 5.3–5.8 below.

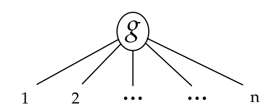


FIGURE 5.3: A graph with the first Betti number equal to 0 and one vertex of genus g and index n.

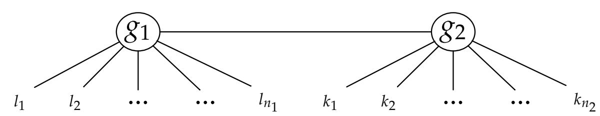


FIGURE 5.4: For each $L = \{l_1,\ldots,l_{n_1}\}$, $K = \{k_1,\ldots,k_{n_2}\}$ such that $I \sqcup K = \{1,\ldots,n\}$, and $g_1, g_2 \ge 0$ such that $g_1 + g_2 = g$, a graph with two vertices and the first Betti number equal to 0.

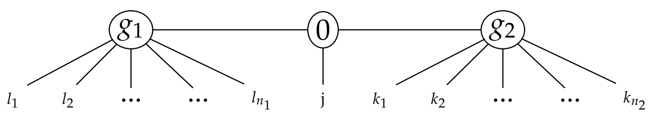


FIGURE 5.5: For each $j \in \{1, \dots, n\}$ and $L = \{l_1, \dots, l_{n_1}\}$, $K = \{k_1, \dots, k_{n_2}\}$ such that $\{j\} \sqcup I \sqcup K = \{1, \dots, n\}$, and $g_1, g_2 \geq 0$ such that $g_1 + g_2 = g$, a graph with three vertices and the first Betti number equal to 0.

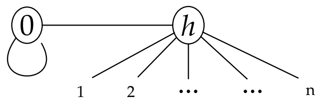


FIGURE 5.6: A graph with two vertices, one of genus 0 and index 3 with an attached loop, and the other one of genus $h = g - 1$ with all leaves attached to the second vertex.

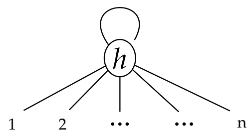


FIGURE 5.7: A graph with one vertex of genus $h = g - 1$ and the first Betti number equal to 1 (that is, a one-vertex graph with a loop).

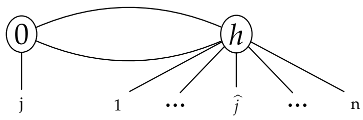


FIGURE 5.8: For each $j \in \{1, \dots, n\}$, a graph with two vertices connected by two edges, such that one vertex is of genus 0 and the other one is of genus $h = g - 1$. The notation $\widehat{j}$ indicates that the j-th leaf is not attached to the second vertex.

Proof. Recall some properties of the Gromov–Witten classes for elliptic curves, which we will use throughout this proof:

(i) The only non-trivial classes in genus 0 are $c_{0,n+1}(e_4 \otimes e_1^{\otimes n})$ and $c_{0,n+2}(e_2 \otimes e_3 \otimes e_1^{\otimes n})$.

(ii) $c_{g,n}(e_1^{\otimes n}) = 0$ for all $g \geq 0, n \geq 1, 2g - 2 + n > 0$.

(iii) $c_{g,n}(\otimes_{i=1}^{n} e_{\alpha_i})\lambda_g = 0$ for all $g \geq 1, n \geq 1$.

Statement (i) follows from the fact that, in genus zero, there are only degree zero stable maps from an algebraic curve to an elliptic curve, and constant maps cannot pass through two distinct points. Statements (ii) and (iii) are proven in [OP23, Lemma 6.1] and [OP23, Lemma 6.4], respectively. Moreover, we use the shorthand notation $c_{g,n}$ for the Gromov–Witten classes of the elliptic curve whenever we wish to suppress dependence on specific inputs.

To proceed with the argument, we temporarily disregard all structure of the Pixton–Zagier formula except for the stable graphs Γ that enter it, and consider the pull-backs $(\mathrm{gl}_\Gamma)^*(c_{g,n}(e_4^{\otimes n})\lambda_{g-2})$ that might be identified with the restictions of $c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$ to the corresponding strata in $\overline{M}_{g,n}$. Since λ_{g-2} vanishes on the strata represented by the stable graphs with the first Betti number greater than 2, the Pixton–Zagier type formula for $\mathrm{DR}_g(A)c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$ reduced to a sum over stable graphs Γ with first Betti number $b_1(\Gamma) = 0, 1, 2$. Below we discuss all these cases. Furthermore, we frequently use that the restruction of $(\mathrm{gl}_\Gamma)^*(c_{g,n}(e_4^{\otimes n}))$ factorizes into the product of $c_{g',n'}$ associated to the vertices of the graph, and the primary fields on the edges of the graph are prescribed by the pairing $\int_E e_1 \cup e_4 = \int_E e_2 \cup e_3 = 1$.

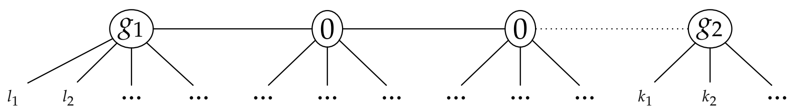
Consider stable graphs Γ with the first Betti number $b_1(\Gamma) = 0$. Recall that a stable graph Γ is decorated with a genus function $g\colon \mathrm{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$. Since $b_1(\Gamma) = 0$, the genus condition reads $\sum_{v\in \mathrm{V}(\Gamma)} g(v) = g$. Also, let $n(v)$ denote the index of a vertex $v \in \mathrm{V}(\Gamma)$. We have then the following possible cases:

Case 1 Assume $(\mathrm{gl}_\Gamma)^*\lambda_{g-2}$ factorizes to $\lambda_{g(v)-2}$ for some vertex $v \in \mathrm{V}(\Gamma)$ and hence to $\lambda_{g(w)}$ for every other vertex $w \in \mathrm{V}(\Gamma)$, $w \neq v$. By (iii), we are constrained by $g(w) = 0$ for $w \neq v$, hence $g(v) = g$. Furthermore, if there is at least one vertex w with $g(w) = 0$, then there should a vertex w connected with exactly one edge to the rest of the graph (a "terminal" vertex). On this vertex w of index $n(w) \geq 3$ the even part of the class obtained by the restriction of $c_{g,n}(e_4^{\otimes n})$ is equal to $c_{0,n(w)}(e_1 \otimes e_4^{\otimes n(w)-1})$ or $c_{0,n(w)}(e_4^{\otimes n(w)})$. Since $n(w) \geq 3$, by (i) we see that $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n}) = 0$ if there is at least one vertex w with $g(w) = 0$. As a result, the only non-trivial contribution is possible from the graph given in Figure 5.3.

Case 2 Assume $(\mathrm{gl}_\Gamma)^*\lambda_{g-2}$ factorizes to $\lambda_{g(v_1)-1}$ and $\lambda_{g(v_2)-1}$ for some vertices $v_1, v_2 \in \mathrm{V}(\Gamma)$ and hence to $\lambda_{g(w)}$ for every other vertex $w \in \mathrm{V}(\Gamma)$, $w \neq v_1, v_2$. Assume also that v_1 and v_2 are connected by an edge. Then repeating literally the argument of Case 1, we see that we obtain a non-vanishing contribution only from the graphs with $\mathrm{V}(\Gamma) = \{v_1, v_2\}$, which are the graph in Figure 5.4.

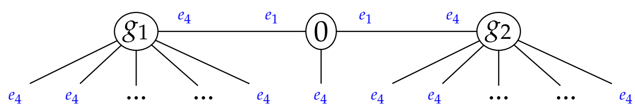
Case 3 In the same setup as in Case 2, assume that v_1 and v_2 are not connected by an edge. This means that the set of additional genus 0 vertices w is non-empty.

By the same argument as in Case 1 and Case 2, we know that in the non-vanishing restrictions of $c_{g,n}(e_4^{\otimes n})$ the genus 0 vertices cannot be "terminal", that is, they all must be in a chain of vertices between v_1 and v_2. Thus, we may have graphs only of the following shape:



where the number of vertices of genus 0 is ≥ 1. Analyzing $(\mathrm{gl}_\Gamma)^*(c_{g,n}(e_4^{\otimes n}))$ and insisting that we should get a class of even cohomological degree on each irreducible component, we see that the primary fields for the factorization of $c_{g,n}(e_4^{\otimes n})$ all must be e_1 or e_4 on all internal half-edges. Moreover, the shape of pairing implies that if we assign e_1 to $t(e)$, then we must assign e_4 to $h(e)$ for any $e \in \mathrm{E}(\Gamma)$, and vice versa.

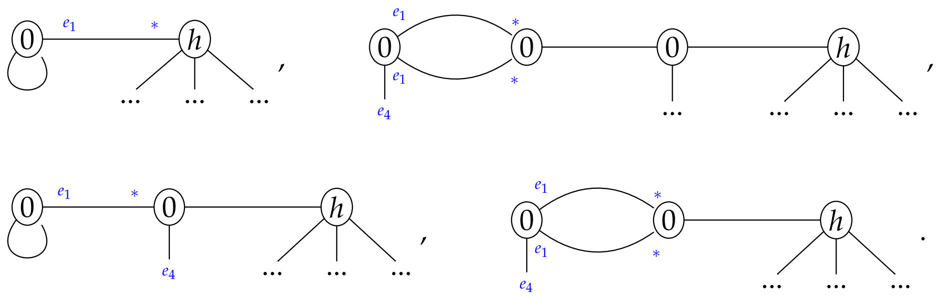
Thus, if there are two vertices of genus 0 connected by an edge, the primary field assigned to the half-edge incident to one of them is e_4. By the stability condition there is also at least one leaf incident to each genus 0 vertex, and the primary fields assigned to the leaves are always e_4. Thus, the factorization of $c_{g,n}(e_4^{\otimes n})$ contains a factor $c_{0,n(w)}(e_4^{m'} e_1^{n'})$, $m' \geq 2$, $n' \leq 1$, $m' + n' = n(w)$, which vanishes by (i). Therefore, there is exactly one vertex of genus 0. Moreover, (i) implies that this vertex must be of index 3, that is, there is exactly one leaf attached to it. Thus we obtain the graph as in Figure 5.5, and the primary fields of the factorization of $c_{g,n}(e_4^{\otimes n})$ are necessarily as follows:



Now consider stable graphs Γ with the first Betti number $b_1(\Gamma) = 1$. The condition on the genus function reads $\sum_{v\in \mathrm{V}(\Gamma)} g(v) = g - 1$. Moreover, $(\mathrm{gl}_\Gamma)^*\lambda_{g-2}$ factorizes in such a way that there is always a vertex $v_0 \in \mathrm{V}(\Gamma)$ where the associated class is $\lambda_{g(v_0)-1}$, and for all other vertices $v \neq v_0$ the corresponding factor in the factorization of λ_{g-2} is $\lambda_{g(v)}$. Note that this implies that every other vertex

$v \in V(\Gamma)$ is of genus 0, by the same argument as in Case 1, and $h := g(v_0) = g - 1$. Note also that $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n}) = 0$ on all graphs with two or more leaves attached to a genus 0 vertex, and by the same argument as in Case 3 we may not have chains of genus 0 vertices each with attached leaf of length more than 1. We then have the following possible cases:

Case 4 Assume the only nontrivial cycle in Γ does not pass through v_0. Then this non-trivial cycle consists of genus 0 vertices. Since there are no "terminal" genus zero vertices in the graph, all but one half-edges attached to the vertices in the cycle that do not belong to the edges of the cycle are leaves. Thus there are at most two vertices in the cycle. Using further the condition that we may not have two genus 0 vertices both with leaves attached to each other, we see that the only possible non-vanishing graphs are:



In these pictures, we have already marked the graphs with the primary fields that must occur there once we consider the factorization rules for $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n})$ and use (i) to ensure that the non-vanishing $c_{0,3}$ on the irreducible component corresponding to the leftmost vertex of genus 0 in each picture. Note that the structure of the pairing implies that the primary fields marked by $*$ on the second left vertex in each picture must be e_4. Thus, by (i) used once again, in all cases when the second left vertex is of genus 0, the class $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n})$ vanishes. Hence, the only non-trivial case is given by the first graph, which is the one given in Figure 5.6.

Case 5 Assume the only nontrivial cycle in Γ does pass through v_0. Since there are no "terminal" genus zero vertices in the graph, all vertices belong to this cycle, and all half-edges attached to the vertices in the cycle that do not belong to the edges of the cycle are leaves. Since we may not have chains of genus 0 vertices each with attached leaf of length more than 1, the cycle either consists of one vertex (then it is v_0 with a loop attached to it, as in Figure 5.7), or of two vertices. In the latter case the second vertex is of genus 0, with exactly one leaf attached to it, so we obtain the graph in Figure 5.8.

Finally, consider stable graphs Γ with the first Betti number $b_1(\Gamma) = 2$. The condition on the genus function reads $\sum_{v \in V(\Gamma)} g(v) = g - 2$. Moreover, $(\mathrm{gl}_\Gamma)^* \lambda_{g-2}$ factorizes in such a way that for all vertices $v\, V(\Gamma)$ the corresponding factor in the factorization of λ_{g-2} is $\lambda_{g(v)}$. This implies that all vertices $v \in V(\Gamma)$ are of genus 0. Recall that $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n}) = 0$ on all graphs with two or more leaves attached to a genus 0 vertex, and by the same argument as in Case 3 we may not have chains of genus 0 vertices each with attached leaf of length more than 1. This leaves a very small number of possible graphs, and for each of them $(\mathrm{gl}_\Gamma)^* c_{g,n}(e_4^{\otimes n})$ vanishes by the analysis of the primary fields in the same vein as we performed in Case 4 using (i). □

5.11 Intersection with the Pixton–Zagier formula

We use the graphs listed in Propoposition 5.9 in order to reduce the computation of the integral (4.10) considered as a polynomial $a_1, \ldots, a_n$ to a finite number of integrals.

Proposition 5.10. The intersection number

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\, \lambda_{g-2}\, c_{g,n}(e_4^{\otimes n}) \tag{5.19}$$

is equal to the sum of the following terms:

$$\frac{1}{2^g} \sum_{\substack{d_1,\dots,d_n\geq 0\\ d_1+\cdots+d_n=g}} \prod_{i=1}^n \frac{a_i^{2d_i}}{d_i!} \int_{\overline{M}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}\, \lambda_{g-2}\, c_{g,n}(e_4^{\otimes n}); \tag{5.20}$$

$$\begin{aligned} &\frac{1}{2^g} \sum_{\substack{g_1,g_2\geq 1\\ g_1+g_2=g\\ L\sqcup K=\{1,\dots,n\}}} \sum_{\substack{d_1,\dots,d_n,d',d''\geq 0\\ d_1+\cdots+d_n\\ +d'+d''=g-1}} \prod_{i=1}^n \frac{a_i^{2d_i}}{d_i!} \frac{(\sum_{i\in K} a_i)^{2d'+2d''+2}}{d'!d''!(2d'+2d''+2)} \\ &\quad\times \int_{\overline{M}_{g_1,|L|+1}} \prod_{i=1}^{|L|} \psi_i^{d_{\iota_L(i)}}\, \psi_{|L|+1}^{d'}\, \lambda_{g_1-1}\, c_{g_1,|L|+1}(e_4^{\otimes |L|}\otimes e_1) \\ &\quad\times \int_{\overline{M}_{g_2,|K|+1}} \prod_{i=1}^{|K|} \psi_i^{d_{\iota_K(i)}}\, \psi_{|K|+1}^{d''}\, \lambda_{g_2-1}\, c_{g_1,|K|+1}(e_4^{\otimes(|K|+1)}); \end{aligned} \tag{5.21}$$

$$\begin{aligned} &\frac{1}{2^{g+1}} \sum_{j=1}^n \sum_{\substack{g_1,g_2\geq 1\\ g_1+g_2=g\\ L\sqcup K\sqcup\{j\}=\{1,\dots,n\}}} \sum_{\substack{d_1,\dots,d_n,d',d''\geq 0\\ d_1+\cdots+d_n\\ +d'+d''=g-2\\ d_j=0}} \prod_{i=1}^n \frac{a_i^{2d_i}}{d_i!} \frac{(\sum_{i\in L} a_i)^{2d'+2}}{d'!(2d'+2)} \frac{(\sum_{i\in K} a_i)^{2d''+2}}{d''!(2d''+2)} \\ &\quad\times \int_{\overline{M}_{g_1,|L|+1}} \prod_{i=1}^{|L|} \psi_i^{d_{\iota_L(i)}}\, \psi_{|L|+1}^{d'}\, \lambda_{g_1-1}\, c_{g_1,|L|+1}(e_4^{\otimes(|L|+1)}) \\ &\quad\times \int_{\overline{M}_{g_2,|K|+1}} \prod_{i=1}^{|K|} \psi_i^{d_{\iota_K(i)}}\, \psi_{|K|+1}^{d''}\, \lambda_{g_2-1}\, c_{g_1,|K|+1}(e_4^{\otimes(|K|+1)}); \end{aligned} \tag{5.22}$$

$$\begin{aligned} &\frac{1}{2^g} \sum_{\substack{d_1,\dots,d_n,d',d''\geq 0\\ d_1+\cdots+d_n\\ +d'+d''=g-1}} \prod_{i=1}^n \frac{a_i^{2d_i}}{d_i!} \frac{(2d'+2d''+1)!B_{2d'+2d''+2}}{d'!d''!} \\ &\quad\times \int_{\overline{M}_{g-1,n+2}} \prod_{i=1}^n \psi_i^{d_i} \psi_{n+1}^{d'} \psi_{n+2}^{d''}\, \lambda_{g-2} \left(c_{g-1,n+2}(e_4^{\otimes(n+1)}\otimes e_1) + c_{g-1,n+2}(e_4^{\otimes(n)}\otimes e_2\otimes e_3)\right); \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} &\frac{1}{2^{g+1}} \sum_{j=1}^n \sum_{\substack{d_1,\dots,d_n,d',d''\geq 0\\ d_1+\cdots+d_n\\ +d'+d''=g-2\\ d_j=0}} \prod_{i=1}^n \frac{a_i^{2d_i}}{d_i!} \frac{(2d'+1)!(2d''+1)!}{d'!d''!} \mathrm{Coeff}_{((x')^{2d'+2}(x'')^{2d''+2})} \left(\frac{e^{a_j x'} x''}{e^{x''-x'}-1} + \frac{e^{a_j x''} x'}{e^{x'-x''}-1} \right) \\ &\quad\times \int_{\overline{M}_{g-1,n+1}} \prod_{i=1}^{n-1} \psi_i^{d_{\iota_{\hat{j}}(i)}}\, \psi_n^{d'} \psi_{n+1}^{d''}\, \lambda_{g-2}\, c_{g-1,n+1}(e_4^{\otimes(n+1)}). \end{aligned} \tag{5.24}$$

Here ι_L (resp., ι_K) is the monotonely increasing isomorphism $\{1,\dots,|L|\}\to L$ (resp., $\{1,\dots,|K|\}\to K$), $L,K\subseteq\{1,\dots,n\}$, and $\iota_{\hat{j}}$ is the monotonely increasing isormorphism $\{1,\dots,n-1\}\to\{1,\dots,n\}\setminus$

$\{j\}$. We use these maps to relable the indices. The symbol $B_{2d'+2d''+2}$ denotes the Bernoulli number,

$$B_{2d'+2d''+2} := \mathrm{Coeff}_{(x^{2d'+2d''+2})} \frac{x}{e^x - 1}. \tag{5.25}$$

Proof. We apply the Pixton–Zagier formula directly to the graphs shown in Figures 5.3–5.8, pairing each with $c_{g,n}(e_4^{\otimes n})\lambda_{g-2}$. The only exception is the graph in Figure 5.6, whose contribution vanishes under this pairing. □

Corollary 5.11. For $g \geq 2$, $n \geq 1$ and $d_1 + \cdots + d_n = g$, the integral

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, \lambda_{g-2} \, c_{g,n}(e_4^{\otimes n}), \tag{5.26}$$

can be expressed in terms of the following integrals:

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, \lambda_{g-1} \, c_{g,n}(e_4^{\otimes n}), \qquad d_1 + \cdots + d_n = g - 1; \tag{5.27}$$

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, \lambda_{g-1} \, c_{g,n}(e_4^{\otimes (n-1)} \otimes e_1), \qquad d_1 + \cdots + d_n = g; \tag{5.28}$$

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, \lambda_{g-1} \, c_{g,n}(e_4^{\otimes (n-2)} \otimes e_2 \otimes e_3), \qquad d_1 + \cdots + d_n = g. \tag{5.29}$$

Proof. The statement follows by applying the vanishing result of Proposition 4.4 to the relations established in Proposition 5.10, and reorganizing the terms according to their polynomial degree profiles. □

Remark. The three integrands appearing in Corollary 5.11, Equations (5.27)–(5.29), can be computed using the holomorphic anomaly equation [OP18], Faber's socle intersection number formula [Fab99], and a lemma proved in [OP23].

(See [OP23, Proposition 6.8].) We have

$$\lambda_{g-1} \, c_{g,n}(e_4^{\otimes n}) = \frac{(-1)^{g-1} 2 \cdot (2g)!}{B_{2g} \cdot (2g - 2 + n)!} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i \sum_{i=1}^{n} \frac{1}{\psi_i} D_q^{n-1} \mathsf{G}_{2g}. \tag{5.30}$$

As an example of computation, we obtain the following immediate consequence. The intersection number (5.27) is given by

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, \lambda_{g-1} \, c_{g,n}(e_4^{\otimes n}) = \frac{1}{2^{2g-2}} \prod_{i=1}^{n} \frac{1}{(2d_i + 1)!!} D_q^{n-1} \mathsf{G}_{2g}. \tag{5.31}$$

Indeed, we obtain this by using Faber's intersection number formula, which states that

$$\int_{\overline{M}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{\tilde{d}_i} = \frac{(-1)^{g-1} B_{2g} (2g - 3 + n)!}{2^{2g-1} \cdot (2g)!} \prod_{i=1}^{n} \frac{1}{(2\tilde{d}_i - 1)!!}, \tag{5.32}$$

for all $\tilde{d}_1, \ldots, \tilde{d}_n \geq 1$ satisfying $\tilde{d}_1 + \cdots + \tilde{d}_n = g - 2 + n$. (This formula was conjectured in [Fab99] and has since been proved in several ways; see [BSS25] for a recent new proof using methods similar to those of the present paper, as well as an overview of existing proofs.) The condition $\tilde{d}_i \geq 1$ can be relaxed using the string equation, allowing at most one $\tilde{d}_j = 0$. We then obtain

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \lambda_{g-1} c_{g,n}(e_4^{\otimes n}) = \frac{(-1)^{g-1} 2 \cdot (2g)!}{B_{2g} \cdot (2g-2+n)!} \int_{\overline{M}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{d_i+1} \sum_{i=1}^{n} \frac{1}{\psi_i} D_q^{n-1} \mathsf{G}_{2g} \tag{5.33}$$

$$= \frac{1}{2^{2g-2}} \prod_{i=1}^{n} \frac{1}{(2d_i+1)!!} D_q^{n-1} \mathsf{G}_{2g}. \tag{5.34}$$

Next, consider the intersection number with $c_{g,n}(e_4^{\otimes(n-1)} \otimes e_1)$ in the case $d_n = 0$. Then Equation (5.28) gives

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n-1} \psi_i^{d_i} \lambda_{g-1} c_{g,n}(e_4^{\otimes(n-1)} \otimes e_1) = \frac{2g+n-1-|\{j \,|\, d_j = 0\}|}{2^{2g-2} \prod_{i=1}^{n-1} (2d_i+1)!!} D_q^{n-2} \mathsf{G}_{2g}. \tag{5.35}$$

Indeed, applying the string equation gives

$$\int_{\overline{M}_{g,n}} \prod_{i=1}^{n-1} \psi_i^{d_i} \lambda_{g-1} c_{g,n}(e_4^{\otimes(n-1)} \otimes e_1) = \sum_{\substack{1 \le j \le n-1 \\ d_j \ge 1}} \int_{\overline{M}_{g,n-1}} \prod_{i=1}^{n-1} \psi_i^{d_i - \delta_{ij}} \lambda_{g-1} c_{g,n-1}(e_4^{\otimes(n-1)}) \tag{5.36}$$

$$= \frac{1}{2^{2g-2}} \prod_{i=1}^{n-1} \frac{1}{(2d_i+1)!!} \sum_{\substack{1 \le j \le n-1 \\ d_j \ge 1}} (2d_j+1) D_q^{n-2} \mathsf{G}_{2g},$$

and note that

$$\sum_{\substack{1 \le j \le n-1 \\ d_j \ge 1}} (2d_j+1) = \sum_{j=1}^{n-1} (2d_j+1) - |\{j \,|\, d_j = 0\}| = 2g+n-1-|\{j \,|\, d_j = 0\}|.$$

5.12 Conclusion and future work

Main result. The quantum double ramification hierarchy for the Gromov–Witten theory of the elliptic curve is uniquely determined by the DR potential

$$\overline{G} = \int \left[\frac{(u^1)^2 u^4}{2} + u^1 u^2 u^3 + i\hbar \left(\left(\mathcal{S}_\varepsilon(u_x^1)\, \mathcal{S}_\varepsilon(u_x^4) + \mathcal{S}_\varepsilon(u_x^2)\, \mathcal{S}_\varepsilon(u_x^3) \right) \exp\left(\mathcal{S}_\varepsilon(u^4) D_q \right) \right) \odot \mathcal{G}(\varepsilon, q) \right] dx$$

where $\odot$ denotes the Hadamard product, and

$$\mathcal{S}_\uparrow := \frac{\sinh(\frac{1}{2}\varepsilon\partial_x)}{\frac{1}{2}\varepsilon\partial_x}, \qquad \mathcal{G}(\varepsilon, q) := \sum_{g \ge 0} \varepsilon^{2g} \mathsf{G}_{2g+2}(q).$$

Generalizations and refinements. The construction developed in this chapter admits natural generalizations, though further work is required to establish them rigorously. In particular, our approach suggests that the quasimodularity of the classes extends beyond the cases proven here. We conjecture that

$$c_{g,n}(e_4^{\otimes n})\, \lambda_{g-l} \in H^*(\overline{M}_{g,n}) \otimes \mathrm{QMod}^{E,l},$$

for all $l = 3, \ldots, g-1$. The statement holds for $l = 1, 2$, and our arguments indicate that the same method may apply in general. This would provide an explicit cyclic representation of the pairings $c_{g,n}(e_4^{\otimes n})\lambda_{g-l}$, extending Equation (5.10). Moreover, a necessary refinement concerns obtaining closed formulas or developing a more effective treatment of the recursions introduced in Section 5.9.

Quadratic Hodge integrals. Assuming the above generalizations hold, an interesting application emerges. Using techniques analogous to those in Chapter 3, one can derive new tautological relations. In particular, combining the holomorphic anomaly equation with a generalized cyclic expression for $c_{g,n}(e_4^{\otimes n})\lambda_{g-l}$ yields an expression for $\lambda_g\lambda_{g-l}\prod_{i=1}^n \psi_i \sum_{i=1}^n \frac{1}{\psi_i}$ in terms of a sum over necklace graphs paired with λ_{g-l}. Following the approach of Chapter 3 and using the splitting formula of [Bur+15], one can then recursively compute quadratic Hodge integrals of the form

$$\int_{\overline{M}_{g,n}} \lambda_g\lambda_{g-l}\prod_{i=1}^n \psi_i^{d_i},$$

for $l = 2, \dots, n$. In particular, the case $l = 2$ can already be treated explicitly.

Relation to the Dubrovin–Zhang hierarchy. The precise relationship between the quantum double ramification (qDR) hierarchy and the Dubrovin–Zhang (DZ) hierarchy remains an open question. Nevertheless, our results suggest several connections where information from the qDR side can inform the construction of the DZ hierarchy.

For the case of the elliptic curve, it was observed in [OP06a] and made explicit in [Bur23] that all descended Gromov–Witten invariants can be expressed in terms of those with stationary insertions only (corresponding to e_4 in our notation). In particular, the DZ hierarchy can be fully reconstructed from the invariants

$$\int_{\overline{M}_{g,n}} c_{g,n}(e_4^{\otimes n})\prod_{i=1}^n \psi_i^{d_i}.$$

In this work, we studied explicit cohomological expressions for $c_{g,n}(e_4^{\otimes n})\lambda_{g-l}$ with $l = 1, 2$, which are relevant to the qDR hierarchy. Restricting these formulas to $g = 1$ and $g = 2$ respectively provides concrete ways to compute

$$\int_{\overline{M}_{1,n}} c_{1,n}(e_4^{\otimes n})\prod_{i=1}^n \psi_i^{d_i} \quad \text{and} \quad \int_{\overline{M}_{2,n}} c_{2,n}(e_4^{\otimes n})\prod_{i=1}^n \psi_i^{d_i}.$$

If the general pattern we observe extends to higher genus, a complete understanding of the classes $c_{g,n}(e_4^{\otimes n})\lambda_{g-l}$ would allow, in principle, the computation of

$$\int_{\overline{M}_{l,n}} c_{l,n}(e_4^{\otimes n})\prod_{i=1}^n \psi_i^{d_i}, \qquad l = 3, \dots, g-1.$$

Quantization of the Calabi–Yau case. Another interesting observation arises when extending and quantizing Example 11. Let X be a Calabi–Yau variety with cohomology ring $H^*(X, \mathbb{Q})$ spanned by even-degree classes $\{e_1, \dots, e_M\}$ and odd-degree classes $\{\phi_1, \dots, \phi_N\}$. In [Bur+18], it was shown that when restricted to even classes, the DR hierarchy for the Gromov–Witten theory of X is determined by the primary Hamiltonian density

$$\overline{g}_{1,1} = \overline{g}_{1,1}^{[0]} + \varepsilon^2 \frac{\chi(X)}{24}\int u^1 u^1_2\, dx,$$

where $\overline{g}_{1,1}^{[0]}$ is the genus-zero contribution, $\chi(X)$ is the Euler characteristic of X, and u^1 corresponds to the identity element e_1. Our goal is to promote this hierarchy to its quantum version.

To compute $\overline{G}_{1,1}$, one must evaluate intersection numbers of the form

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\,\Lambda(\epsilon)\, c_{g,n}\left(e_1^{b_1}\otimes\cdots\otimes e_M^{b_M}\otimes\phi_1^{c_1}\otimes\cdots\otimes\phi_N^{c_N}\right), \tag{5.37}$$

where $c_{g,n}$ is the Gromov–Witten class associated with X. These intersection numbers satisfy the dimensional constraint

$$2g-3+n = j + \dim X\,(g-1) + \sum_{i=1}^{M} b_i \frac{\deg e_i}{2} + \sum_{i=1}^{N} c_i \frac{\deg \phi_i}{2} - \langle c_1(X), d\rangle,$$

with j the degree of the Hodge class λ_g, $c_1(X)$ the first Chern class of X, and d the degree of stable maps.

Focusing on the case of a Calabi–Yau threefold, we find $b_1 \geq g$ for all $g > 0$. Using the push-forward relation

$$\pi_* \mathrm{DR}_g(a_1,\dots,a_n) = g!\, a_1^2 \cdots a_g^2\, [\overline{\mathcal{M}}_{g,n-g}],$$

for the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-g}$, one deduces that the intersection numbers in (5.37) vanish unless $b_1 = g$. This occurs only when $j = N = 0$ and $e_i \in H^2(X)$ for all $i \in \{1,\dots,M\}$. In this case, Equation (5.37) simplifies to

$$\int_{\overline{M}_{g,n}} \mathrm{DR}_g(A)\, c_{g,n}\left(e_1^{b_1} \otimes \cdots \otimes e_M^{b_M}\right) = g!\, a_1^2 \cdots a_g^2 \int_{\overline{M}_{g,n-g}} c_{g,n-g}\,(e_2^{b_2} \otimes \cdots \otimes e_M^{b_M}).$$

Consequently, the primary Hamiltonian density for the qDR hierarchy of a Calabi–Yau threefold is

$$\overline{G}_{1,1} = \overline{G}_{1,1}^{[0]} + \sum_{g \in \mathbb{Z}_{>0}} \hbar^g\, \overline{G}_{1,1}^{[2g]},$$

where

$$\overline{G}_{1,1}^{[2g]} = \sum_{\substack{n \in \mathbb{Z}_{>0} \\ \alpha_1,\dots,\alpha_{n-g}=2,\dots,M}} \frac{g!\,(2g-2+n)}{n!} \left(\int_{\overline{M}_{g,n-g}} c_{g,n-g}\left(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{n-g}}\right) \right) \int (u_2^1)^g u^{\alpha_1} \cdots u^{\alpha_{n-g}}\, dx,$$

with $e_{\alpha_i} \in H^2(X)$ for all $\alpha_i \in \{2,\dots,M\}$ and $i \in \{1,\dots,n-g\}$. This computation shows that, for Calabi–Yau threefolds, the Hamiltonians of the qDR hierarchy encode precisely the Gromov–Witten invariants required to construct the primary DZ hierarchy. The quantization from DR to qDR thus enhances the underlying enumerative structure, and a deeper understanding of this connection may clarify how the qDR hierarchy relates to the DZ framework.

Bibliography

[ALR07] Adem, Leida, and Ruan, *Orbifolds and string topology*. Cambridge University Press, 2007.

[BGR19] Buryak, Guéré, and Rossi, "DR/DZ equivalence conjecture and tautological relations". In: *Geometry Topology* 23.7 (2019), pp. 3537–3600. arXiv: 1705.03287.

[BHS22] Buryak, Hernández Iglesias, and Shadrin, "A conjectural formula for $\mathrm{DR}_g(a,-a)\lambda_g$". In: *Épijournal Géom. Algébrique* 6 (2022), Art. 8, 17.

[BLS24a] Blot, Lewanski, and Shadrin, *On the strong DR/DZ conjecture*. 2024. arXiv: 2405.12334.

[BLS24b] Blot, Lewański, and Shadrin, *Cohomological representations of quantum tau functions*. 2024. arXiv:2411.03499.

[BPS12] Buryak, Posthuma, and Shadrin, "On deformations of quasi-Miura transformations and the Dubrovin–Zhang bracket". In: *Journal of Geometry and Physics* 62.7 (2012), pp. 1639–1651. ISSN: 0393-0440. DOI: `10.1016/j.geomphys.2012.03.006`. arXiv: 1104.2722.

[BR16a] Buryak, and Rossi, "Double ramification cycles and quantum integrable systems". In: *Lett. Math. Phys.* 106.3 (2016), pp. 289–317. ISSN: 0377-9017. DOI: `10.1007/s11005-015-0814-6`. URL: `https://doi.org/10.1007/s11005-015-0814-6`.

[BR16b] Buryak, and Rossi, "Recursion Relations for Double Ramification Hierarchies". In: *Communications in Mathematical Physics* 342 (2016), pp. 533–568. ISSN: 0010-3616. DOI: `10.1007/s00220-015-2535-1`. arXiv: 1411.6797.

[BS11] Buryak, and Shadrin, "A new proof of Faber's intersection number conjecture". In: *Adv. Math.* 228.1 (2011), pp. 22–42. ISSN: 0001-8708. DOI: `10.1016/j.aim.2011.05.009`. URL: `https://doi.org/10.1016/j.aim.2011.05.009`.

[BS24] Buryak, and Shadrin, "Tautological relations and integrable systems". In: *Épijournal Géom. Algébrique* 8 (2024), Art. 12, 44.

[BSS25] Blot, Shadrin, and Singh, "Faber's socle intersection numbers via Gromov–Witten theory of elliptic curves". In: *Bulletin of the London Mathematical Society* 57.9 (2025), pp. 2698–2707. DOI: `10.1112/blms.70117`. arXiv: 2502.02297. URL: `https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70117`.

[Bur+15] Buryak, et al. "Integrals of -classes over double ramification cycles". In: *American Journal of Mathematics* 137.3 (2015), pp. 699–737. ISSN: 1080-6377. DOI: `10.1353/ajm.2015.0022`. URL: `http://dx.doi.org/10.1353/ajm.2015.0022`. arXiv: 1211.5273.

[Bur+18] Buryak, et al. "Tau-structure for the double ramification hierarchies". In: *Communications in Mathematical Physics* (2018), pp. 191–260. arXiv: 1602.05423.

[Bur15] Buryak, "Double ramification cycles and integrable hierarchies". In: *Communications in Mathematical Physics* (2015), pp. 1085–1107. arXiv: 1403.1719.

[Bur17] Buryak, "New approaches to integrable hierarchies of topological type". In: *Russian Mathematical Surveys* 72.6 (2017), pp. 1077–1090. DOI: `10.1070/RM9777`.

[Bur23] Buryak, "A formula for the Gromov-Witten potential of an elliptic curve". In: *Mosc. Math. J.* 23.3 (2023), pp. 309–317. ISSN: 1609-3321,1609-4514. DOI: `10.17323/1609-4514-2023-23-3-309-317`. arXiv: 2205.12777.

[CI15] Coates, and Iritani, "On the convergence of Gromov-Witten potentials and Givental's formula". In: *Michigan Math* 64.3 (2015). arXiv: 1203.4193.

[CI18] Coates, and Iritani, "A Fock Sheaf For Givental Quantization". In: *Kyoto J. Math.* 58.4 (2018), pp. 695–864. arXiv: 1411.7039.

[CK99] Cox, and Katz, *Mirror symmetry and algebraic geometry. Monografias de Matematica*. American Mathematical Society, 1999.

[CMZ97] Cohen, Manin, and Zagier, "Automorphic pseudodifferential operators". In: *Algebraic Aspects of Integrable Systems*. Vol. 26. Progress in Nonlinear Differential Equations and Their Applications. Boston, MA: Birkhäuser Boston, 1997, pp. 17–47.

[Dij95] Dijkgraaf, "Mirror symmetry and elliptic curves". In: *The moduli space of curves*. Vol. 129. Progress in Mathematics. Birkhäuser, 1995, pp. 149–163.

[DM69] Deligne, and Mumford, "The irreducibility of the space of curves of given genus". In: *Publications Mathématiques de l'IHES* 36 (1969), pp. 75–109.

[DM93] Deligne, and Mumford, "Algebraic geometry associated with matrix models of twodimensional gravity". In: *Topological methods in modern mathematics (Stony Brook, NY, 1991)* (1993), pp. 235–269.

[Dot+24] Dotsenko, et al. "Deformation theory of cohomological field theories". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 0.0 (2024). ISSN: 1435-5345. DOI: `10.1515/crelle-2023-0098`. URL: `http://dx.doi.org/10.1515/crelle-2023-0098`.

[DR14] Dumas, and Royer, "Poisson structures and star products on quasimodular forms". In: *Algebra amp; Number Theory* 8.5 (2014), pp. 1127–1149. ISSN: 1937-0652. DOI: `10.2140/ant.2014.8.1127`. URL: `http://dx.doi.org/10.2140/ant.2014.8.1127`. arXiv:1306.3634.

[Dub96] Dubrovin, "Geometry of 2D topological field theories". In: *Integrable systems and quantum groups* 1620 (1996), pp. 120–348. arXiv: hep-th/9407018.

[DZ01] Dubrovin, and Zhang, "Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants". In: *Advances in Mathematics* (2001), p. 189. arXiv: math/0108160.

[DZ98] Dubrovin, and Zhang, "Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation". In: *Communciations in Mathematical Physics* 198.2 (1998), pp. 311–361. arXiv: hep-th/9712232.

[Eyn18] Eynard, "Lectures notes on compact Riemann surfaces". In: (2018). arXiv: 1805.06405.

[Fab99] Faber, "A conjectural description of the tautological ring of the moduli space of curves". In: *Moduli of curves and abelian varieties*. Vol. E33. Aspects Math. Friedr. Vieweg, Braunschweig, 1999, pp. 109–129. ISBN: 3-528-03125-5. arXiv: math/9711218.

[Ful98] Fulton, *Intersection Theory*. 2nd ed. Vol. 3. Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin: Springer-Verlag, 1998. DOI: 10.1007/978-1-4612-1700-8.

[Gar+19] Garcia-Failde, et al. "Half-spin tautological relations and Faber's proportionalities of kappa classes". In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 15 (2019), Paper No. 080, 27. DOI: 10.3842/SIGMA.2019.080. URL: https://doi.org/10.3842/SIGMA.2019.080.

[Gat00] Gatto, *Intersection theory on moduli spaces of curves. Monografias de Matematica*. Instituto Nacional de Matematica Pure e Aplicada, 2000.

[Giv01a] Givental, "Gromov-Witten invariants and quantization of quadratic Hamiltonians". In: *Moscow Mathematics Journal* 1.4 (2001), pp. 551–568. arXiv: math/0108100v2.

[Giv01b] Givental, "Semisimple Frobenius structures at higher genus". In: *International Mathematics Research Notices* 23 (2001), pp. 1265–1286. arXiv: math/0008067.

[Giv02] Givental, "A_{n-1} singularities and nKdV hierarchies". In: *Moscow Mathematical Journal* (2002). arXiv: math/0209205.

[Giv04] Givental, "Symplectic geometry of Frobenius structures". In: *Frobenius manifolds. Quantum cohomology and singularities. Aspects of Mathematics* E.36 (2004), pp. 91–112. arXiv: math/0305409.

[GP98] Getzler, and Pandharipande, "Virasoro constraints and the Chern classes of the Hodge bundle". In: *Nuclear Physics B* (1998), pp. 701–714. arXiv: math/9805114.

[GZ22] Garcia-Failde, and Zagier, "A curious identity that implies Faber's conjecture". In: *Bull. Lond. Math. Soc.* 54.5 (2022), pp. 1839–1845. ISSN: 0024-6093. DOI: 10.1112/blms.12659. URL: https://doi.org/10.1112/blms.12659.

[Hai12] Hain, "Normal functions and the geometry of moduli spaces of curves". In: *Handbook of Moduli, Adv. Lect. Math.* I.24 (2012), pp. 527–578. arXiv: 1102.4031.

[HM91] Haris, and Morisson, *Moduli of curves. Graduate Texts in Mathematics*. Spring, 1991.

[IR24] Ittersum, van and Ruzza, "Quantum KdV hierarchy and quasimodular forms". In: *Communications in Number Theory and Physics* 18.2 (2024), pp. 405–439. ISSN: 1931-4531. DOI: 10.4310/cntp.2024.v18.n2.a4. URL: http://dx.doi.org/10.4310/CNTP.2024.v18.n2.a4.

[IR25] Ittersum, van and Ruzza, "Quantum KdV Hierarchy and Shifted Symmetric Functions". In: *International Mathematics Research Notices* 2025.9 (2025). ISSN: 1687-0247. DOI: 10.1093/imrn/rnaf102. URL: http://dx.doi.org/10.1093/imrn/rnaf102.

[Jan+17] Janda, et al. "Double ramification cycles on the moduli spaces of curves". In: *Publications mathématiques de l'IHÉS* 125.1 (2017), pp. 221–266. ISSN: 1618-1913. DOI: 10.1007/s10240-017-0088-x. URL: http://dx.doi.org/10.1007/s10240-017-0088-x. arXiv: 1602.04705.

[Jan17] Janda, "Gromov-Witten theory of target curves and the tautological ring". In: *Michigan Math. J.* 66.4 (2017), pp. 683–698. ISSN: 0026-2285,1945-2365. DOI: 10.1307/mmj/1508810814. arXiv: 1308.6182. URL: https://doi.org/10.1307/mmj/1508810814.

[Kat07] Katz, "An algebraic formulation of symplectic field theory". In: *Journal of Symplectic Geometry* 5.4 (2007), pp. 385–437. DOI: 10.4310/JSG.2007.v5.n4.a2.

[KM94] Kontsevich, and Manin, "Gromov-Witten classes, quantum cohomology, and enumerative geometry". In: *Communications in Mathematical Physics* 164.3 (1994), pp. 525–562. arXiv: hep-th/9402147.

[Kon03] Kontsevich, "Deformation Quantization of Poisson Manifolds". In: *Letters in Mathematical Physics* 66.3 (2003), pp. 157–216. ISSN: 1573-0530. DOI: 10.1023/b:math.0000027508.00421.bf. arXiv: 9709040. URL: http://dx.doi.org/10.1023/B:MATH.0000027508.00421.bf.

[Kon92] Kontsevich, "Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function". In: *Communications in Mathematical Physics* 147 (1992), pp. 1–23.

[KV07] Kock, and Vainsencher, *An invitation to quantum cohomology. Kontsevich's formula for rational plane curves*. Birkhäuser Boston, MA, 2007.

[KZ95] Kaneko, and Zagier, "A generalized Jacobi theta function and quasimodular forms". In: *The moduli space of curves*. Vol. 129. Progress in Mathematics. Birkhäuser, 1995, pp. 165–172.

[Li02] Li, "A degeneration formula of GW-invariants". In: *J. Differential Geom.* 60.2 (2002), pp. 199–293. ISSN: 0022-040X,1945-743X. DOI: 10.4310/jdg/1090351102. arXiv: math/0110113. URL: http://projecteuclid.org/euclid.jdg/1090351102.

[LX09] Liu, and Xu, "A proof of the Faber intersection number conjecture". In: *J. Differential Geom.* 83.2 (2009), pp. 313–335. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1261495334.

[MR09] Martin, and Royer, "Rankin-Cohen brackets on quasimodular forms". In: *Journal of the Ramanujan Mathematical Society* 24.3 (2009), pp. 213–233. ISSN: 0970-1249. URL: https://lmbp.uca.fr/~royer/publication/mr2568053/mr2568053.pdf. arXiv:0509653.

[MRS15] Milanov, Ruan, and Shen, "Gromov-Witten theory and cycle-valued modular forms". In: *Journal fur die Reine und Angewandte Mathematik* 2015 (2015). arXiv: 1206.3879.

[Nic11] Nicolaescu, "Intersection theory". In: (2011). ePrint: Lecture Notes.

[OP06a] Okounkov, and Pandharipande, "Gromov-Witten theory, Hurwitz theory, and completed cycles". In: *Annals of Mathematics* 163.2 (2006), pp. 517–560. arXiv: 0204305.

[OP06b] Okounkov, and Pandharipande, "The equivariant Gromov-Witten theory of $\mathbb{P}^1$". In: *Annals of Mathematics* 163.2 (2006), pp. 561–605. arXiv: 0207233.

[OP06c] Okounkov, and Pandharipande, "Virasoro constraints for target curves". In: *Inventiones Mathematicae* 163.1 (2006), pp. 47–108. arXiv: 0308097.

[OP18] Oberdieck, and Pixton, "Holomorphic anomaly equations and the Igusa cusp form conjecture". In: *Invent. Math.* 213.2 (2018), pp. 507–587. ISSN: 0020-9910. DOI: 10.1007/s00222-018-0794-0. URL: https://doi.org/10.1007/s00222-018-0794-0.

[OP23] Oberdieck, and Pixton, *Quantum cohomology of the Hilbert scheme of points on an elliptic surface*. 2023. arXiv: 2312.13188.

[OS00] Olver, and Sanders, "Transvectants, modular forms, and the Heisenberg algebra". In: *Advances in Applied Mathematics* 25.3 (2000), pp. 252–283. DOI: 10.1006/aama.2000.0672.

[Pev08] Pevzner, "Rankin–Cohen brackets and associativity". In: *Letters in Mathematical Physics* 85.2-3 (2008), pp. 195–202. DOI: 10.1007/s11005-008-0221-z.

[Pix] Pixton, *DR cycle polynomiality and related results*.

[Pix13] Pixton, *The tautological ring of the moduli space of curves*. ProQuest LLC, Ann Arbor, MI, 2013, p. 133. ISBN: 978-1303-09766-9. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:3562218.

[PPZ15] Pandharipande, Pixton, and Zvonkine, "Relations on $\overline{M}_{g,n}$ via 3-spin structures". In: *Journal of American Mathematics Society* (2015), pp. 279–309. arXiv: 1303.1043.

[Ros17] Rossi, "Integrability, Quantization and Moduli Spaces of Curves". In: *Symmetry, Integrability and Geometry: Methods and Applications* (2017). ISSN: 1815-0659. DOI: 10.3842/sigma.2017.060. arXiv: 1703.00232.

[Sch20] Schmitt, "The moduli space of curves". In: (2020). ePrint: Lecture Notes.

[Sha06] Shadrin, "Combinatorics of binomial decompositions of the simplest Hodge integrals". In: *Gromov-Witten theory of spin curves and orbifolds*. Vol. 403. Contemp. Math. Amer. Math. Soc., Providence, RI, 2006, pp. 153–165. DOI: 10.1090/conm/403/07600. URL: https://doi.org/10.1090/conm/403/07600.

[Sin24] Singh, *KdV Recursion Relations (Code Repository)*. https://github.com/IshanJaztar/DR-Recursion-Relations/blob/main/KdV-Recursion. 2024.

[Spe24] Spelier, *Polynomiality of the double ramification cycle*. 2024. arXiv: 2401.17421. URL: https://arxiv.org/abs/2401.17421.

[SZ17] Shen, and Zhou, "Ramanujan identities and quasi-modularity in Gromov–Witten theory". In: *Communications in Number Theory and Physics* 11.2 (2017), pp. 405–452. ISSN: 1931-4531. DOI: 10.4310/cntp.2017.v11.n2.a5. URL: http://dx.doi.org/10.4310/CNTP.2017.v11.n2.a5.

[Tel12] Teleman, "The structure of 2D semi-simple field theories". In: *Inventiones Mathematicae* 188.3 (2012), pp. 525–588. arXiv: 0712.0160.

[Wit91] Witten, "Two-dimensional gravity and intersection theory on moduli space". In: *Surveys in differential geometry* 1 (1991), pp. 243–310.

[Yao07] Yao, "Autour des déformations de Rankin–Cohen". PhD thesis. École Polytechnique, 2007. Available at http://pastel.archives-ouvertes.fr/pastel-00002414.

[Zvo14] Zvonkine, "An introduction to moduli spaces of curves and their intersection theory". In: (2014). ePrint: Lecture Notes.