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On Properties of *p*-adic Cohomologies

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Abstract

In this thesis we study various properties of p-adic cohomology theories. We construct a duality between the kernel and cokernel of the monodromy operator on the Hyodo-Steenbrink double complex associated to a semistable log scheme when the log scheme admits a well-behaved lift, and prove that it is perfect by comparison to Poincaré duality in rigid cohomology. In addition we prove a conjecture of Flach and Morin computing the cone of the monodromy operator on log-crystalline cohomology as rigid cohomology in the case of a family over a curve, using the techniques of Chiarellotto and Tsuzuki in their proof of the Clemens-Schmid exact sequence in characteristic p. Finally, we prove the well-definedness in the general case of a Hodge-type filtration on rigid cohomology encountered in the context of syntomic cohomology.

Sommario

In questa tesi andiamo a studiare alcune proprietà delle coomologie p-adiche. In primis, per un log-schema semistabile, costruiamo una dualità tra il ker e il coker dell'operatore di monodromia che agisce sul complesso doppio di Hyodo-Steenbrink: questo nel caso il log-schema abbia un opportuno lifting. Proviamo che tale dualità è perfetta usando una interpretazione via la dualità di Poincaré in ambito rigido. In seguito proviamo un caso particolare della congettura di "Flach-Morin": questa congettura lega il cono dell'operatore di monodromia con la coomologia rigida di uno log-schema semistabile (in ch.p). Proviamo la congettura nel caso il nostro log-schema appaia come la fibra speciale di una famiglia sopra una curva: le tecniche utilizzate sono quelle di Chiarellotto-Tsuzuki nella loro dimostrazione della esattezza della sequenza di Clemens-Schmid. Infine diamo una definizione di una filtrazione "a la Hodge" sulla coomologia rigida e mostriamo la sua indipendenza dalle scelte: questa filtrazione era apparsa nell'ambito della coomologia sintomica.

Contents

Introduction

This thesis orbits around structures on p-adic cohomology theories of schemes defined in characteristic p and over discrete valuation rings of mixed characteristic $(0,p)$, specifically log-crystalline cohomology and rigid cohomology. For this introduction, let k be a complete field of characteristic p and X be a semistable k log-scheme over the log point.

The Complex Monodromy Operator and Mixed Hodge Structures

The first two chapters are concerned with the p-adic monodromy operator on the log-crystalline cohomology groups $H^i_{\text{log-crys}}(X)$, also known as Hyodo-Kato cohomology, introduced by Hyodo and Kato in [HK94]. The motivation and inspiration for the construction ultimately stems from the classical case of the monodromy operator for a complex semistable family over a disk. Let Δ denote the complex unit disk and let $f : \mathfrak{X} \to \Delta$ be a proper flat holomorphic map of relative dimension n smooth outside $0 \in \Delta$ and such that $\mathfrak{X}_0 = f^{-1}(0)$ is a normal crossing divisor. Fix a smooth fiber $\mathfrak{X}_t$ (there is nothing exceptional about this choice since, by Ehresmann's fibration theorem, under these conditions the family is a locally trivial fibration). We can transport the cohomology groups $H^i(\mathfrak{X}_t)$ along loops $h \in \pi_1(\Delta^*)$, and this gives us a representation

$$\mathbb{Z} \cong \pi_1(\Delta^*) \to \mathrm{Aut}(H^i(\mathfrak{X}_t))$$

which is called the monodromy representation.

It was proven by Landman [Lan73] and later by Grothedieck using ℓ-adic techniques that action of the generator T of $\pi_1(\Delta^*) \cong \mathbb{Z}$ is unipotent; in fact, even if $f : \mathfrak{X} \to \Delta$ is not a semistable degeneration but simply a degeneration the operator T remains quasi-unipotent. In particular, in the semistable case this theorem allows us to define a well-defined logarithm

$$N = \log T := (T - I) - \frac{1}{2}(T - I)^2 + \frac{1}{3}(T - I^3) + \cdots$$

which is nilpotent. We call this the *monodromy operator* on $H^i(\mathfrak{X}_t)$.

It happens that this monodromy operator tells us a tremendous amount about the relationship between the cohomologies of a generic fiber $\mathfrak{X}_t$ and the central fiber $\mathfrak{X}_0$. An illustrative example is the following:

Theorem 0.1. (Clemens-Schmid (simplified)) *There are morphisms α, i^*, and β of $\mathbb{C}$-vector spaces such that the sequence*

$$\cdots \to H_{2n+2-m}(\mathfrak{X}) \xrightarrow{\alpha} H^m(\mathfrak{X}_0) \xrightarrow{i^*} H^m(\mathfrak{X}_t) \xrightarrow{N} H^m(\mathfrak{X}_t) \xrightarrow{\beta} H_{2n-m}(\mathfrak{X}) \xrightarrow{\alpha} H^{m+2}(\mathfrak{X}_0) \to \dots$$

is exact.

Here the morphism i^* is induced by the inclusion $i : \mathfrak{X}_t \hookrightarrow \mathfrak{X}$ and the fact that retraction provides an isomorphism $H^m(\mathfrak{X}) \cong H^m(\mathfrak{X}_0)$. The exactness of the snippet

$$H^m(\mathfrak{X}_0) \xrightarrow{i^*} H^m(\mathfrak{X}_t) \xrightarrow{N} H^m(\mathfrak{X}_t)$$

is already an interesting result, called the *local invariant cycle theorem.* It says that the cocycles which are invariant under the monodromy action are precisely those that come from the central fiber $\mathfrak{X}_0$.

But we can say much more. Deligne [Del71; Del74] extended the 'classical' Hodge structure on a compact Kähler manifold to arbitrary separated schemes X of finite type over $\mathbb{C}$ by associating *functorially* to every such scheme a *mixed Hodge structure*, whose ingredients are a weight filtration $H^n(X, \mathbb{Q})$ and a Hodge filtration $H^n(X, \mathbb{C})$. The weight filtration is trivial on compact smooth manifolds, in which case the mixed Hodge structure degenerates into the standard Hodge structure.

As such, the cohomology of the singular fiber $H^m(\mathfrak{X}_0)$ has a nontrivial mixed Hodge structure and the cohomology of the smooth fiber $H^m(\mathfrak{X}_t)$ has its standard Hodge structure. It turns out, however, that with the default filtrations the morphism $i^* : H^m(\mathfrak{X}_0) \to H^m(\mathfrak{X}_t)$ is not a morphism of mixed Hodge structures. Steenbrink [Ste76] remedied this deficiency by defining a *limit mixed Hodge structure* on $H^m(\mathfrak{X}_t, \mathbb{C})$ which captures the the behavior of the Hodge decomposition as t tends to 0. He proved that $H^m(\mathfrak{X}_t)$ equipped with this new mixed Hodge structure, which we denote by $H^m_{\text{lim}}(\mathfrak{X}_t)$, the morphism $i^* : H^m(\mathfrak{X}_0) \to H^m_{\text{lim}}(\mathfrak{X}_t)$ is a morphism of mixed Hodge structures.

The full force of the Clemens-Schmid exact sequence is that it even respects the respective filtrations:

Theorem 0.2. (Clemens-Schmid) *The maps α, i^*, N, and β are morphisms of weighted vector spaces, and the sequence*

$$\cdots \to H_{2n+2-m}(\mathfrak{X}) \xrightarrow{\alpha} H^m(\mathfrak{X}_0) \xrightarrow{i^*} H^m_{\text{lim}}(\mathfrak{X}_t) \xrightarrow{N} H^m_{\text{lim}}(\mathfrak{X}_t) \xrightarrow{\beta} H_{2n-m}(\mathfrak{X}) \xrightarrow{\alpha} H^{m+2}(\mathfrak{X}) \to \dots$$

is an exact sequence of weighted vector spaces.

ℓ- and p-adic analogues of monodromy and mixed Hodge structures

These sorts of results directed the search for similar structures and results on their ℓ- and p-adic cohomological analogues. The closed disk is replaced in general by a Henselian (or, simply, complete) trait (S, s, η) where $s = \mathrm{Spec}(k)$ denotes the closed point and $\eta = \mathrm{Spec}(K)$ the generic point, and the family $f : \mathfrak{X} \to S$ is replaced with a morphism $f : X \to S$.

In ℓ-adic cohomology one compares the cohomology of the (geometric) special and generic fibers of X using the formalism of nearby and vanishing cycles (as described by Deligne in [DK73, p. XIII]). Choose a ring of coefficients $\Lambda \in \{\mathbb{Z}/\ell^k\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell\}$ where $k \geq 1$ and ℓ is a prime not equal to p; for brevity we write $D(-)$ for $D(-, \Lambda)$. The theory associates to every complex $K \in D^+(X_\eta)$ a so-called *nearby cycles complex* $R\Psi_f(K)$ and *vanishing cycles complex* $R\Phi_f(K)$, and for f proper we obtain an exact sequence

$$\cdots \to H^{i-1}(X_{\tilde{s}}, R\Phi_X(K)) \to H^i(X_{\tilde{s}}, K) \xrightarrow{\mathrm{sp}} H^i(X_{\overline{\eta}}, K) \to H^i(X_{\tilde{s}}, R\Phi_X(K)) \to \cdots$$

where sp is the specialization map.

The role of the fundamental group is replaced by inertia group of I of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. When f is projective, the Galois group G_K acts on the geometric generic fiber $H^i(X_{\overline{\eta}}, \Lambda)$ and by restriction we obtain an action by I. Grothendieck proved the *ℓ-adic monodromy theorem* which states that, analogous to the complex case, this action is quasi-unipotent:

Theorem 0.3. (Grothendieck, SGA 7 t.I) *There exists an open subgroup $I_1 \subseteq I$ such that, for all $g \in I_1$ and all $i \in \mathbb{Z}$, g acts unipotently on $H^i(X_{\overline{\eta}}, \Lambda)$.*

Later on the p-adic side, when the special fiber X_s of X is semistable, *Hyodo-Kato* or *log-crystalline* cohomology was conjectured by Jannsen and Fontaine and defined by Hyodo [Hyo91] and Kato [Kat89] (see also [HK94]) to be a mixed characteristic analogue of the limit mixed Hodge structure. Let $S = \mathrm{Spec}(\mathcal{V})$, write $V^\times$ denote $\mathrm{Spec}(\mathcal{V})$ with the trivial log structure, and let (X_s, M_s) denote X_s equipped with the log structure M_s given by the semistable structure of X_s. Then the log-crystalline cohomology $H^i_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times)$ is defined over the ring of Witt vectors $W(k)$ and was designed so that $D = H^i_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times) \otimes \mathrm{Frac}(W(k))$ would be a finite-dimensional vector space over $K_0 := \mathrm{Frac}(W(k))$ equipped with

- A bijective Frobenius-semilinear endomorphism $\varphi : D \to D$ called the *Frobenius*;

- A nilpotent operator $\mathcal{N} : D \to D$ called the *monodromy operator* satisfying $\mathcal{N}\varphi = p\varphi\mathcal{N}$;
- a K-isomorphism with de-Rham cohomology

$$D \otimes_{K_0} K \xrightarrow{\sim} H^m_{dR}(X_K/K)$$

To reach the analogy between the ℓ-adic and classical cases we need to extend our perspective to see this cohomology theory as an object in the category of filtered (φ, N)-modules, which are essentially characterized as those K_0-vector spaces with these listed properties, and use the tools and framework of p-adic Hodge theory.

The formalism of p-adic Hodge theory associates to every such (φ, N)-module a well-behaved representation of the Galois group, called a *semi-stable representation* because of the above connection to semistable schemes. They are the p-adic equivalent of the unipotent representations we saw in the context of the ℓ-adic monodromy theorem, and one can in fact show [Ber01] that semi-stable representations correspond to unipotent p-adic differential equations.

The widest class of p-adic representations are the so-called *de Rham representations*: it was conjectured and proved that all representations coming 'from geometry' are de Rham (the class of all Galois representations is much too large, and many p-adic Galois representations are not de Rham; on the other hand, there are de Rham representations that do not come from geometry). The p-adic analogue to the ℓ-adic fact that the monodromy operator is quasi-unipotent would, then, be Fontaine's conjecture that every de Rham representation is semi-stable after a finite extension of the base field K. It was proven by Berger [Ber01] that this follows Crew's conjecture regarding the quasi-unipotency of differential modules over the Robba ring endowed with a Frobenius structure, which in turn was proven independently by André [And02], Mebkhout [Meb02], and by Kedlaya [Ked04]. This result is the p-adic analogue of the classical monodromy theorem over $\mathbb{C}$ and Grothendieck's ℓ-adic monodromy theorem and is known as the *p-adic monodromy theorem.*

Chapter 1 of this thesis study the properties of the monodromy operator $\mathcal{N}$ on log-crystalline cohomology.

If instead of a mixed characteristic family $f : X \to S$ over a Henselian trait we are in the setting of a family $f : X \to C$ over a curve, the Clemens-Schmid exact sequence involving the limit mixed Hodge structure also has a close p-adic analogue. Chiarellotto and Tsuzuki [CT03] constructed a sequence

$$\ldots \to H_{\mathrm{rig}}(X_s) \xrightarrow{\gamma} H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times) \otimes K \xrightarrow{N_m}$$
$$H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{V}^\times) \otimes K(-1) \xrightarrow{\delta} H^{m+2}_{X_s,\mathrm{rig}}(X) \xrightarrow{\alpha} H^{m+2}_{\mathrm{rig}}(X_s) \to \ldots$$

and proved its exactness, where N_m is the monodromy operator at level m and $H^m_{X_s,\mathrm{rig}}(X)$ denotes the rigid cohomology of X with support in X_s, which in this case is the dual of rigid cohomology by Poincaré duality [Ber97]. The proof of the exactness of this sequence requires sophisticated techniques linked to the coincidence of the so-called monodromy and weight filtrations on log-crystalline cohomology, which is a deep question in the field. In Chapter 2 we link this result to a motivic conjecture of Flach and Morin [FM18] and use similar techniques to prove the conjecture in this case of a family over a curve.

Chapter 3 of the thesis may be considered independent of the rest, going in another direction that one can take in studying p-adic cohomology theories. It deals not with the logarithmic cohomology of semistable varieties but with a filtration on rigid cohomology of general varieties. Our interest in the filtration stems from constructions surrounding the *syntomic cohomology* of a scheme defined over a mixed characteristic discrete valuation ring.

This cohomology theory has several, overlapping definitions (see as a sampler [CCM13, Definition 5.3.2], [Gro94, Definition 2.1], and [Bes00, Definition 8.4]) but the idea is that for a scheme $X \to \mathrm{Spec}(\mathcal{V})$ over a DVR one searches for a filtration that entwines the Hodge filtration on the cohomology of the generic fiber $H^*_{\mathrm{dR}}(X_K)$ with the Frobenius on the p-adic cohomology of the special fiber. From a broader point of view, it is the p-adic analogue of Deligne-Beilinson cohomology.

Instead of a traditional Hodge filtration, Gros's definition of syntomic cohomology in the proper and smooth case used a distinct filtration [Gro94, (3.2)] on the rigid cohomology arising from the characteristic p special fiber. This filtration has also been contextualized by Besser [Bes00, §9], who showed that it was related to his definition of syntomic cohomology. Just as the construction of rigid cohomology involves the choice of an embedding $X_k \subseteq Y_k$ followed by a choice of compactification $Y \subseteq P$, the definition of this filtration involves same choices. An essential part of the construction of rigid cohomology is that the resulting complex $R\Gamma_{\mathrm{rig}}(X_k)$ is independent of the choices. The same independence has been suggested as proven by both Gros and Besser but no such proof in fact exists, leaving a hole in the literature: our goal in Chapter 3 is to prove that the filtration is independent of the choice of frame. This filtration turns out to be trivial in in the smooth and affine case (corresponding to Monsky-Washnitzer cohomology), but there are potential links to prismatic cohomology and the Nygaard filtration.

Outline

In Chapter 1 we study the monodromy operator on log-crystalline cohomology. Originally defined in [HK94] as an analogue to the complex-analytic monodromy operator, Mokrane [Mok93] and later Große-Klonne [Gro07] use techniques introduced by Steenbrink [Ste76] to define a double complex $A^{\bullet,\bullet}$ given a semistable log-scheme X whose cohomology computed log-crystalline cohomology, and on which exists an endomorphism ν which induced the monodromy operator in cohomology

$$N_i : H^i_{\text{log-crys}}(X) \to H^i_{\text{log-crys}}(X)$$

for any i. This operator ν is neither surjective nor injective on $A^{\bullet,\bullet}$, and hence the link between the kernel and cokernel of the monodromy operators N_i and the kernel and cokernel of ν is subtle. This motivates the study of the kernel and cokernel of the operator ν, and our goal is to define a natural duality between the kernel and cokernel of ν at the level of complexes which induces a perfect pairing in cohomology, and to provide a possible geometric interpretation.

In Section 1.3 we prove general results regarding induced pairings on total complexes. Namely, given pairings $A^\bullet \times B^\bullet \to I^\bullet$ and $C^\bullet \times D^\bullet \to I^\bullet$ of complexes, along with morphisms $C^\bullet \to A^\bullet$ and $B^\bullet \to D^\bullet$, we prove the necessary form of a pairing between $\mathrm{Tot}(C^\bullet \to A^\bullet)$ and $\mathrm{Tot}(B^\bullet \to D^\bullet)$ under general circumstances (Proposition 1.3.1). In Section 1.4 we use this calculation to show that two natural pairings involving the hom functor on complexes of sheaves over a scheme or rigid-analytic space coincide in cohomology. Namely, if $f : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is a surjective morphism of complexes and $i : \ker f \hookrightarrow \mathscr{F}^\bullet$ is the inclusion (a quasi-isomorphism in the derived category), we show that in the derived category we have a compatibility of pairings

$$\begin{array}{ccccc}
(\underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet))_s & \times & (\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s & \longrightarrow & I^\bullet \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle i^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\underline{\mathrm{Hom}}^\bullet(\ker f, \mathscr{I}^\bullet) & \times & \ker f & \longrightarrow & I^\bullet
\end{array}$$

where the top pairing is induced by the composition pairing on $\mathscr{F}$ and $\mathscr{G}$ respectively as in Proposition 1.3.1 and bottom pairing is the standard composition pairing.

We specialize to the setting of rigid-analytic spaces in Section 1.5 where we establish natural functorialities of pairings of overconvergent sheaves. The previous results are used to show that the pairing of sheaves which induces Poincaré duality in [Ber97] in fact coincides with a natural hom pairing, and we use the fact that Poincaré duality is perfect to conclude that this hom pairing is perfect as well (Corollary 1.5.5).

Finally, in Section 1.6 we use a pairing defined by El Zein [El 83] to define a pairing between the kernel and cokernel of the monodromy operator on the Hyodo-Steenbrink double complex. In the case that our semistable scheme Y/k has a good lifting to a proper scheme over $\mathcal{V}$, we show that the perfect hom pairing in Section 1.4 can be interpreted via El Zein's methods as a perfect pairing between the cohomology of the kernel and the cokernel of the monodromy operator on the Hyodo-Steenbrink double complex.

In Chapter 2 we continue our study of the monodromy operator on log-crystalline cohomology by looking at a conjecture of Flach and Morin [FM18, Conjecture 7.15]. Inspired by motivic ideas, it posits that for a semistable log-scheme Y over k the cones of the monodromy operator N_i can be described purely in terms of the rigid cohomology of Y and its dual. Namely, they conjecture the existence of an exact triangle

$$R\Gamma_{\text{rig}}(Y^{\varnothing}/\mathcal{O}_F^{\varnothing}) \to \left[R\Gamma_{\text{log-crys}}(Y) \xrightarrow{N} R\Gamma_{\text{log-crys}}(Y)(-1)\right] \to R\Gamma^*_{\text{rig}}(Y^{\varnothing}/\mathcal{O}_F^{\varnothing})(-n-1)[-2n-1] \to$$

in the derived category of φ-modules.

We prove the conjecture in the case where Y can be viewed as the fiber over a k-rational point s of a proper family $X \to C$ where C is a curve. In this setting Poincaré Duality [Ber97] provides a canonical dual to rigid cohomology. Our proof is modeled on Chiarellotto and Tsuzuki's construction and proof of the p-adic Clemens-Schmid exact sequence [CT03]

$$\begin{aligned} \ldots \to H_{\text{rig}}(X_s) \xrightarrow{\gamma} H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{O}_F^{\varnothing}) \otimes K \xrightarrow{N_m} \\ H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{O}_F^{\varnothing}) \otimes K(-1) \xrightarrow{\delta} H^{m+2}_{X_s,\text{rig}}(X) \xrightarrow{\alpha} H^{m+2}_{\text{rig}}(X_s) \to \ldots \end{aligned}$$

whose resemblance to the conjectured exact triangle is clear in light of Poincaré duality identifying cohomology with support in a closed subset as the canonical dual of rigid cohomology.

In Chapter 3 we pivot from the monodromy operator on log-crystalline cohomology and explore a Hodge-type filtration on rigid cohomology. Recall that if X is a k-scheme then rigid cohomology respect to a frame $(X \subseteq Y \subseteq P)$ is defined to be

$$H^n_{\text{rig}}(X) := H^n(]Y[_P, j_X^\dagger \Omega^\bullet_{]Y[_P})$$

It is a classical and foundational result that this definition is independent of the choice of frame $(X \subseteq Y \subseteq P)$ in the derived category of overconvergent sheaves. Using the fact that the rigid generic fiber Y_K^{rig} is a closed subset of $]Y[_P$, say with ideal of definition I, we define a filtration

$$\mathrm{Fil}^s_{X,Y,P} := j_X^\dagger (I^{s-\bullet} \otimes \Omega^\bullet_{]Y[_P})$$

of $j_X^\dagger \Omega^\bullet_{]Y[_P}$ and an induced filtration

$$\mathrm{Fil}^s H^n_{\mathrm{rig}}(X) := \mathrm{Im}(H^n(\mathrm{Fil}^s_{X,Y,P}) \to H^n_{\mathrm{rig}}(X)).$$

on rigid cohomology. The question we answer in this chapter is the independence of this filtration on the chosen frame $(X \subseteq Y \subseteq P)$. To be more precise, we need to show that given a morphism of frames

$$\begin{tikzcd} & Y' \arrow[r] \arrow[dd, "g"] & P' \arrow[dd, "u"] \\ X \arrow[ur] \arrow[dr] & & \\ & Y \arrow[r] & P \end{tikzcd}$$

where g is proper and u is smooth, we have a quasi-isomorphism

$$\mathrm{Fil}^s_{X,Y,P} \cong Ru_{K*} \, \mathrm{Fil}^s_{X,Y',P'}\,.$$

Our technique of proof, which follows Berthelot's idea for the independence of rigid filtration detailed in [Le 07, Chapter 6], is detailed in 3.2. We reduce the general situation to two cases: that of a proper étale morphism of frames

$$\begin{tikzcd} & Y' \arrow[r] \arrow[dd, "g"] & P' \arrow[dd, "u"] \\ X \arrow[ur] \arrow[dr] & & \\ & Y \arrow[r] & P \end{tikzcd}$$

i.e. where g is proper and u is étale, which is the content of Section 3.3, and to a smooth morphism of frames

$$\begin{tikzcd} & & P' \arrow[d, "u"] \\ X \arrow[r] & Y \arrow[r] \arrow[ur] & P \end{tikzcd}$$

where u is smooth, the content of Section 3.4. Recent suggestion made by Kedlaya may indicate some possible connection between this filtration and some filtrations (namely the Nyaagard filtration) arising in the framework of prismatic cohomology: we plan to investigate this suggestion.

Notation

For reference we collect the notation that we will use throughout the thesis.

k: This will denote a perfect field of characteristic $p > 0$ unless specified otherwise.

$\mathcal{V}$: A complete or Henselian discrete valuation ring of mixed characteristic $(p, 0)$. We denote the residue field of $\mathcal{V}$ by k as above.

K: The fraction field of a complete discrete valuation ring $\mathcal{V}$ or the field of fractions of the ring of Witt vectors $W(k)$.

$(X \subseteq Y \subseteq P)$: Let $\mathcal{V}$ be a complete discrete valuation ring with residue field k and let X be an algebraic variety over k. A frame is a sequence of embeddings

$$X \hookrightarrow Y \hookrightarrow P$$

where $X \hookrightarrow Y$ is an open immersion into another algebraic variety over k and $Y \hookrightarrow P$ is a closed immersion of Y into a formal $\mathcal{V}$-scheme P. A frame is denoted $(X \subseteq Y \subseteq P)$.

$\underline{\Gamma}^{\dagger}_{T}$: The sheaf of overconvergent sections with support in $]Z[_P$. Suppose $h : T \hookrightarrow V$ is an inclusion of an admissible open subset into a rigid analytic variety, it induces a canonical pair of adjoint functors (h^{-1}, h_*) on topoi. The functor h^{-1} has a *left* adjoint $h_!$, and the sheaf of overconvergent sections with support in T is given by

$$\underline{\Gamma}^{\dagger}_{T}\mathcal{E} := h_! h^{-1}\mathcal{E}.$$

It fits into a short exact sequence

$$0 \to \underline{\Gamma}^{\dagger}_{T}\mathcal{E} \to \mathcal{E} \to j^{\dagger}_{V \setminus T}\mathcal{E} \to 0.$$

([Le 07, Proposition 5.2.4])

$\underline{\Gamma}_{V \setminus T}$: The sheaf of sections with support in $V \setminus T$. Suppose $h : T \hookrightarrow V$ is again an inclusion of an admissible open subset into a rigid analytic variety and let $i : V \setminus T \hookrightarrow V$ be the embedding of the closed complement. It induces a canonical pair of adjoint functors (i^{-1}, i_*). The functor i_* has a right adjoint $i^!$, and the sheaf of sections with support in $V \setminus T$ is given by

$$\underline{\Gamma}_{V\setminus T}\mathcal{E} := i_* i^! \mathcal{E}.$$

If $(X \subseteq Y \subseteq P)$ is a frame, then $\underline{\Gamma}_{]X[_P}$ can be computed as

$$\underline{\Gamma}_{]X[_P}\mathcal{E} = \ker(\mathcal{E} \to h_* h^{-1}\mathcal{E}).$$

([Le 07, Proposition 5.2.14])

$R\Gamma_{Z,\mathrm{rig}}(X)$: Rigid cohomology of X with support in Z. Let X be a k-scheme and $Z \subseteq X$ a closed subscheme. Fix a frame $(X \subseteq \overline{X} \subseteq P)$. Then the rigid cohomology of X with support in Z is defined to be

$$R\Gamma_{Z,\mathrm{rig}}(X) := R\Gamma(]\overline{X}[_P, \underline{\Gamma}^{\dagger}_{]Z[_P} j^{\dagger}_X \Omega^{\bullet}_{]\overline{X}[_P})$$

$R\Gamma_{c,\mathrm{rig}}(X)$: Rigid cohomology of X with compact support. Fix a frame $(X \subseteq \overline{X} \subseteq P)$. Then the rigid cohomology of X with compact support is defined to be

$$R\Gamma_{c,\mathrm{rig}}(X) := R\Gamma(]\overline{X}[_P, R\underline{\Gamma}_{]X[_P} \Omega^{\bullet}_{]\overline{X}[_P})$$

$\mathcal{O}_F$: If k is a field of characteristic of $p > 0$, we denote by $\mathcal{O}_F$ the ring of Witt vectors of k. We will typically denote by K its field of fractions.

$R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing})$: If Y is a semistable k-log scheme, then we denote by $R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing})$ the cohomology

$$R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing}) := R\Gamma_{\text{log-crys}}(Y/\mathcal{O}_F^{\varnothing}) \otimes K.$$

$M(d)$: If M is an object with a Frobenius action, say K or log-crystalline cohomology $H^i_{\text{log-crys}}(X)$, we denote by $M(d)$ the same object M with d-twisted Frobenius.

S^0: The log point $(\operatorname{Spec} k, 1 \mapsto 0)$.

$\mathfrak{W}$: For $\mathcal{V}$ a complete discrete valuation ring, $\mathfrak{W} := (\operatorname{Spf}(\mathcal{V}), \mathrm{triv})$ where triv is the trivial log structure.

$\mathfrak{S}^0$: $(\operatorname{Spf}(\mathcal{V}), 1 \mapsto 0)$.

$\mathfrak{S}$: $(\operatorname{Spf}(\mathcal{V}[t]), (\mathbb{N} \to \mathcal{V}[t], 1 \mapsto t))$.

Chapter 1

Pairing on the Kernel and Cokernel of the Monodromy Operator in Log-Crystalline Cohomology

1.1 Introduction

Let $\mathcal{V}$ be a complete discrete valuation ring and consider a semistable scheme $X_{\mathcal{V}}$ with special fiber X. In this chapter we try to contribute to the study of the monodromy operator

$$N_i : H^i_{\text{log-crys}}(X) \to H^i_{\text{log-crys}}(X)$$

This operator, which is the arithmetic analogue of the monodromy operator over $\mathbb{C}$, is the basis of several results and open conjectures about the arithmetic of Hyodo-Kato cohomology. One of the central conjectures is the p-adic weight-monodromy conjecture [Mok93, Conjecture 3.27], which states the so-called *monodromy filtration* induced by the operators N_i coincide with the *weight filtration*, related to the Frobenius structure on $H^i_{\text{log-crys}}$. Another conjecture along these lines is the *invariant cycles conjecture* (see [Chi99, Corollary 4.8]) which relates the kernel $\ker N_i$ of the monodromy operator to the rigid cohomology $H^i_{\text{rig}}(X)$.

In this chapter we try to contribute to the study of the monodromy operator via a study of an appropriate variation of the Hyodo-Steenbrink double complex $A^{\bullet,\bullet}$ for rigid cohomology. This double complex has an endomorphism ν which induces the monodromy operators N_* in cohomology. With the hope of understanding the kernel and cokernel $\ker N_i$ and $\operatorname{coker} N_i$ on cohomology, we study the cohomology of $\ker\nu$ and $\operatorname{coker}\nu$ and their relationship to one another. Our main result concerning this is Theorem 1.6.7, which states that when X has a nice formal model $\mathfrak{Y}$ then there exists a perfect pairing

$$H^i(\mathfrak{Y}_K, \ker\nu) \times H^{2n+2-i}(\mathfrak{Y}_K, \operatorname{coker}\nu) \to K$$

where $n = \dim Y$.

1.2 Overview

We briefly review Berthelot's construction [Ber97] of Poincaré duality for rigid cohomology. As usual, let $\mathcal{V}$ be a complete discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field K. Let X be a k-scheme, $Z \subseteq X$ a closed subscheme, and let $U := X \setminus Z$ be its open complement. Choose a compactification $\overline{X}$ over k and suppose we have a closed immersion into a p-adic formal scheme P over $\mathcal{V}$. In this situation Berthelot defines (e.g. in [Ber97]) the rigid cohomology with support in X (resp. with compact support) as

$$R\Gamma_{Z,\mathrm{rig}}(X) := R\Gamma(]\overline{X}[_P, \underline{\Gamma}^\dagger_{]Z[_P}\Omega^\bullet_{]\overline{X}[_P}) \cong R\Gamma(]\overline{X}[_P, (j^\dagger_X\Omega^\bullet_{]\overline{X}[_P} \to j^\dagger_U\Omega^\bullet_{]\overline{X}[_P})_s)$$

$$R\Gamma_{c,\mathrm{rig}}(Z) := R\Gamma(]\overline{X}[_P, \underline{\Gamma}_{]Z[_P}\Omega^\bullet_{]\overline{X}[_P}) \cong R\Gamma(]\overline{X}[_P, (\underline{\Gamma}_{]U[_P}(\mathscr{I}^\bullet) \to \underline{\Gamma}_{]X[_P}(\mathscr{I}^\bullet)_s)$$

where $\mathscr{I}^\bullet$ is a resolution of $\Omega^\bullet_{]\overline{X}[_P}$ and $(-)_s$ denotes the total complex. This notation is somewhat ambiguous, since the degrees of the total complex depend on the embedding of the constituent complexs as rows or columns in a double complex; here the columns of $(j^\dagger_X\Omega^\bullet_{]\overline{X}[_P} \to j^\dagger_U\Omega^\bullet_{]\overline{X}[_P})_s)$ are embedded as rows with $j^\dagger_X\mathcal{O}_{]\overline{X}[_P}$ in degree $(0,0)$, and similarly for $(\underline{\Gamma}_{]U[_P}(\mathscr{I}^\bullet) \to \underline{\Gamma}_{]X[_P}(\mathscr{I}^\bullet)_s)$ but with $\underline{\Gamma}_{]X[_P}(\mathscr{I}^0)$ in degree $(0,0)$.

Now let $\mathscr{I}^\bullet$ be an injective resolution of $\Omega^\bullet_{]\overline{X}[_P}$ which extends the canonical pairing. It is easy to show that this can be extended to a pairing

$$j^\dagger_?\mathscr{I}^\bullet \otimes_K \underline{\Gamma}_{]?[_P}\mathscr{I}^\bullet \to \underline{\Gamma}_{]?[_P}\mathscr{I}^\bullet.$$

where $? = X, U$ (see [Ber97, Lemme 2.1] or [Le 07, Corollary 5.3.6]). This pairing for X and U combine in the manner of §1.3 to produce a pairing

$$R\Gamma_{Z,\mathrm{rig}}(X) \otimes_K R\Gamma_{c,\mathrm{rig}}(Z) \to R\Gamma_{c,\mathrm{rig}}(X)$$

and composed with the trace map

$$H^{2n}_{c,\mathrm{rig}}(X) \to K$$

we obtain a pairing

$$R\Gamma_{Z,\mathrm{rig}}(X) \to R\operatorname{Hom}_K(R\Gamma_{c,\mathrm{rig}}(Z), K)[-2n]$$

which Berthelot proves is perfect [Ber97, Théoréme 2.4].

There is another natural pairing is latent in this construction. It is well-known (see §1.4 for details) by general homological principles that, because the following map f is surjective, there is a quasi-isomorphism

$$(j_X^\dagger \Omega^\bullet_{]\overline{X}[_P} \xrightarrow{f} j_U^\dagger \Omega^\bullet_{]\overline{X}[_P})_s \xleftarrow{i} \ker f.$$

In addition, letting $\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(-,-)$ denote the hom functor in the category of $\Omega^\bullet_{]\overline{X}[_P}$-modules, we have an exact sequence

$$0 \to \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(j_U^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(j_X^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(\ker f, \mathscr{I}^\bullet) \to 0$$

since $\mathscr{I}^\bullet$ is injective. Combined with the canonical isomorphism

$$\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(j_?^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet) \cong \underline{\Gamma}_{]?[_P} \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(\mathscr{I}^\bullet, \mathscr{I}^\bullet) \cong \underline{\Gamma}_{]?[_P} \mathscr{I}^\bullet$$

where $? = U, X$, (see for example [Le 07, Proposition 5.3.5]) we obtain a quasi-isomorphism

$$(\underline{\Gamma}_{]U[_P} \mathscr{I}^\bullet \to \underline{\Gamma}_{]X[_P} \mathscr{I}^\bullet)_s \xrightarrow{\varphi} \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(\ker f, \mathscr{I}^\bullet).$$

Thus it's natural to ask in what way the Poincaré pairing is related to the natural Hom pairing

$$\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[_P}}(\ker f, \mathscr{I}^\bullet) \otimes_K \ker f \to \mathscr{I}^\bullet$$

Our main result which we prove in §1.5 is that these two pairings coincide:

Corollary 2.1. (c.f. Corollary 1.5.4) *Let i^{-1} denote the inverse of i in the derived category. The pairings*

$$\begin{array}{ccccc}
\left(\underline{\Gamma}_{]U[} \mathscr{I}^\bullet \to \underline{\Gamma}_{]X[} \mathscr{I}^\bullet\right)_s & \times & \left(j_X^\dagger \mathscr{I}^\bullet \xrightarrow{f} j_U^\dagger \mathscr{I}^\bullet\right)_s & \longrightarrow & \mathscr{I}^\bullet \\
\downarrow \varphi & & \downarrow i^{-1} & & \downarrow = \\
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\ker f, \mathscr{I}^\bullet) & \times & \ker f & \longrightarrow & \mathscr{I}^\bullet
\end{array}$$

are compatible in cohomology. ◁

In §1.6 we use this result to study the kernel and cokernel of the monodromy operator on the Hyodo-Steenbrink double complex when X_k has a 'good' lifting to $\mathcal{V}$. Following [Chi99] we explicitly compute the kernel and cokernel of the monodromy operator on the Hyodo-Steenbrink double complex, and as a corollary of the above we will obtain

Theorem 2.2. *Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme of dimension n which admits a proper formal lift $(\mathcal{Z}, \mathcal{Y})$. Then there exists a perfect pairing*

$$H^i(\mathfrak{Y}_K, \ker \nu) \times H^{2n+2-i}(\mathfrak{Y}_K, \operatorname{coker} \nu) \to K$$

where $\mathfrak{Y} = \widehat{\mathcal{Y}}$.

Notation 2.3. For the sake of consistency we will adopt the notational conventions of the Stacks project. In particular we adopt once and for all the conventions in Section 0FNB for double complexes and their associated total complexes. ◁

1.3 Induced Pairings of Total Complexes

Suppose we have pairings $A^\bullet \times B^\bullet \to I^\bullet$ and $C^\bullet \times D^\bullet \to I^\bullet$ along with morphisms $f : C^\bullet \to A^\bullet$ and $g : B^\bullet \to D^\bullet$. Under what circumstances can we define a pairing

$$(C^\bullet \xrightarrow{f} A^\bullet)_s \times (B^\bullet \xrightarrow{g} D^\bullet)_s \to I^\bullet?$$

Here $(-)_s$ denotes the total complex of each respective morphism interpreted as a double complex.

We choose to embed complexes as rows as in Stacks Project, Remark 0G6B with A^n and B^n being in position $(n, 0)$ of the double complexes $C^\bullet \xrightarrow{f} A^\bullet$ and $B^\bullet \xrightarrow{g} D^\bullet$, respectively; note that $C^\bullet$ is in the lower half plane but $D^\bullet$ is in the upper!

First we expand what it means for $A^\bullet \times B^\bullet \to I^\bullet$ to be a pairing. Essentially a shorthand for a morphism $\langle \cdot, \cdot \rangle : A^\bullet \otimes B^\bullet \to I^\bullet$ of complexes, it means that in degree n the morphism

$$\bigoplus_{p+q=n} A^p \otimes B^q \to I^n$$
$$\sum_{p+q=n} a_p \otimes b_q \mapsto \langle a_p, b_q \rangle$$

is compatible with the differentials on each side (of course that's not a general element of the tensor product, but by linearity these considerations extend to a general representative).

Following the aforementioned conventions for the differentials of the total complex of a double complex, the differentials of the tensor product are given by

$$\sum_{p+q=n} a_p \otimes b_q \mapsto \sum_{p+q=n} (da_p \otimes b_q + (-1)^p a_p \otimes db_q)$$

Thus compatibility with the differentials concretely means that

$$d_I \left(\sum_{p+q=n} \langle a_p, b_q \rangle \right) = \sum_{p+q=n} \langle da_p, b_q \rangle + (-1)^p \langle a_p, db_q \rangle$$

and of course setting $a_i = b_j = 0$ for $(i,j) \neq (p,q)$ gives the necessary and sufficient compatibility

$$d_I \langle a_p, b_q \rangle = \langle da_p, b_q \rangle + (-1)^p \langle a_p, db_q \rangle$$

for all $a_p \in A^p$ and $b_q \in B^q$.

When there's no possibility of confusion we'll go back and forth between thinking of a pairing of complexes as a tensor product and as a family of bilinear maps $\langle \cdot, \cdot \rangle : A^p \times B^q \to I^{p+q}$.

The following gives one condition under which the above pairing is well-defined.

Proposition 3.1. *Assume the pairing of total complexes has the form*

$$\begin{aligned} &(C^\bullet \xrightarrow{f} A^\bullet)^p_s \times (B^\bullet \xrightarrow{g} D^\bullet)^q_s \to I^{p+q} \\ &\qquad (c_{p+1}, a_p) \times (b_q, d_{q-1}) \mapsto k(p,q)\langle a_p, b_q \rangle + \ell(p,q)\langle c_{p+1}, d_{q-1} \rangle \end{aligned}$$

with $k(p,q), \ell(p,q) \in \mathbb{Z} - \{0\}$. Then this pairing is well-defined if and only if there exists an integer α such that

$$\langle f(c_p), b_q \rangle = \alpha \langle c_p, g(b_q) \rangle,$$

and in this case, up to a constant, $k(p,q) = 1$ and $\ell(p,q) = (-1)^{q+1}\alpha$ for all p, q. $\lhd$

Proof. First some rote computation. We have

$$\begin{aligned} d((c_{p+1}, a_p) \otimes (b_q, d_{q-1})) &= d(c_{p+1}, a_p) \otimes (b_q, d_{q-1}) + (-1)^p (c_{p+1}, a_p) \otimes d(b_q, d_{q-1}) \\ &= (dc_{p+1}, (-1)^{p+1} f^{p+1}(c_{p+1}) + da_p) \otimes (b_q, d_{q-1}) \\ &\quad + (-1)^p ((c_{p+1}, a_p) \otimes (db_q, (-1)^q g^q(b_q) + d(d_{q-1}))) \end{aligned}$$

which gets mapped via our pairing to

$$k(p+1,q)\langle(-1)^{p+1}f^{p+1}(c_{p+1})+da_p,b_q\rangle+\ell(p+1,q)\langle dc_{p+1},d_{q-1}\rangle$$
$$+(-1)^p(k(p,q+1)\langle a_p,db_q\rangle+\ell(p,q+1)\langle c_{p+1},(-1)^q g^q(b_q)+d(d_{q-1})\rangle) \tag{1.1}$$

Compatibility with the differentials means that this must equal

$$d\left(k(p,q)\langle a_p,b_q\rangle+\ell(p,q)\langle c_{p+1},d_{q-1}\rangle\right)$$

which by our previous computation is equal to

$$k(p,q)\left(\langle da_p,b_q\rangle+(-1)^p\langle a_p,db_q\rangle\right)+\ell(p,q)\left(\langle dc_{p+1},d_{q-1}\rangle+(-1)^{p+1}\langle c_{p+1},d(d_{q-1})\right) \tag{1.2}$$

Set $c_{p+1}=d_{q-1}=0$. The required equality reduces to

$$k(p,q)d\langle a_p,b_q\rangle=k(p+1,q)\langle da_p,b_q\rangle+(-1)^p k(p,q+1)\langle a_p,db_q\rangle$$

which implies [$\star\star\star$ Assuming $k(p,q)$ are never zero and the pairing is... reasonable?] that $k(p,q)=k(p+1,q)=k(p,q+1)$ for all p,q. So $k(p,q)=k$ where k is constant.

Similarly, set $a_p=b_q=0$. Then the equality reduces to

$$\ell(p,q)d\langle c_{p+1},d_{q-1}\rangle=\ell(p+1,q)\langle dc_{p+1},d_{q-1}\rangle+(-1)^p\ell(p,q+1)\langle c_{p+1},d(d_{q-1})\rangle$$

Since

$$d\langle c_{p+1},d_{q-1}\rangle=\langle dc_{p+1},d_{q-1}\rangle+(-1)^{p+1}\langle c_{p+1},d_q\rangle$$

this demands that $\ell(p,q)=\ell(p+1,q)$ and $\ell(p,q)=-\ell(p,q+1)$ for all p,q. These two conditions together imply that $\ell(p,q)=(-1)^q\ell(0,0)$ for all p,q.

These conditions on k and ℓ provide necessary conditions for the pairing to be well-defined. Under that extra conditions are they sufficient? Substituting these values into equations (1.1) and (1.2) and simplifying, we get

$$k\langle f^{p+1},b_q\rangle=-\ell(0,0)\langle c_{p+1},g^q(b_q)\rangle$$

So given the above formulas for k and ℓ such an equality is necessary and sufficient for the pairing to be well-defined. Setting $\alpha=-\ell(0,0)/k$ we get our stated result. □

In particular, in the typical case

$$\langle f(c_p),b_q\rangle=\langle c_p,g(b_q)\rangle,$$

the pairing takes the form

$$(c_{p+1},a_p)\times(b_q,d_{q-1})\mapsto\langle a_p,b_q\rangle+(-1)^{q+1}\langle c_{p+1},d_{q-1}\rangle$$

1.4 Pairings of Hom Complexes

Let Z be any scheme or a rigid-analytic space over a field. Suppose $f : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is a surjective morphism of complexes of Ω^*_Z-modules and let $\mathscr{I}^\bullet$ be an injective complex (that is, an injective object in the category of complexes of sheaves). Let $i : \ker f \to \mathscr{F}^\bullet$ be the inclusion. By injectivity we have an exact sequence

$$0 \to \underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \xrightarrow{H(f)} \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet) \xrightarrow{H(i)} \underline{\mathrm{Hom}}^\bullet(\ker f, \mathscr{I}^\bullet) \to 0.$$

By general theory there is a quasi-isomorphism

$$\varphi : \left(\underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet)\right)_s \xrightarrow{\sim} \underline{\mathrm{Hom}}^\bullet(\ker f, \mathscr{I}^\bullet)$$

given by $(\psi_{\mathscr{G}}, \psi_{\mathscr{F}}) \mapsto H(i)(\psi_{\mathscr{F}})$; here $\underline{\mathrm{Hom}}^n(\mathscr{F}^\bullet, \mathscr{I}^\bullet)$ is taken to be in degree $(n, 0)$ in the double complex.

There is also a quasi-isomorphism $\ker f \xrightarrow{\sim} (\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s$ given by inclusion, where this time $\mathscr{F}^n$ is in degree $(n, 0)$. It is easy to show that the map is injective. The proof that it is surjective provides the preimage of an element in the target so we give it here. Let $[(a_{\mathscr{F}}, a_{\mathscr{G}})] \in h^q(\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s$. The usual conventions state that

$$d(a_{\mathscr{F}}, a_{\mathscr{G}}) = (da_{\mathscr{F}}, (-1)^q f(a_{\mathscr{F}}) + da_{\mathscr{G}}) \tag{1.3}$$

so being in the kernel means that $da_{\mathscr{G}} + (-1)^q f(a_{\mathscr{G}}) = 0$ and $da_{\mathscr{F}} = 0$. Since f is surjective, there exists a (not necessarily unique) $b_{\mathscr{F}} \in \mathscr{F}^{q-1}$ such that $f(b_{\mathscr{F}}) = a_{\mathscr{G}}$. Then

$$\begin{aligned} d(b_{\mathscr{F}}, 0) &= (db_{\mathscr{F}}, (-1)^{q-1} f(b_{\mathscr{F}})) \\ &= (db_{\mathscr{F}}, (-1)^{q-1} a_{\mathscr{G}}) \end{aligned}$$

so

$$\begin{aligned} (a_{\mathscr{F}}, a_{\mathscr{G}}) &\sim (a_{\mathscr{F}}, a_{\mathscr{G}}) + (-1)^q d(b_{\mathscr{F}}, 0) \\ &= (a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}, 0). \end{aligned}$$

This element is in the kernel of f since

$$\begin{aligned} f(a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}) &= f(a_{\mathscr{F}}) + (-1)^q f(db_{\mathscr{F}}) \\ &= f(a_{\mathscr{F}}) + (-1)^q d(f(b_{\mathscr{F}})) \\ &= f(a_{\mathscr{F}}) + (-1)^q da_{\mathscr{G}} \\ &= 0 \end{aligned}$$

by Equation (1.3). Hence $i([a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}]) = [(a_{\mathscr{F}}, a_{\mathscr{G}})]$ as desired.

The morphism i is not bijective in general so i^{-1} is not well-defined as a morphism, but being a quasi-isomorphism it is well-defined in the derived category; the above proof provides an explicit inverse equivalence class. So when we speak of i^{-1} it will always be the inverse in the derived category.

These morphisms fit into the below diagram in the derived category

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet)\right)_s \arrow[d, "\varphi"] & \times & (\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s \arrow[r] \arrow[d, "i^{-1}"] & I^\bullet \arrow[d, "\mathrm{id}"] \\
\underline{\mathrm{Hom}}^\bullet(\ker f, \mathscr{I}^\bullet) & \times & \ker f \arrow[r] & I^\bullet
\end{tikzcd}$$

where the top pairing is induced by the composition pairing on $\mathscr{F}$ and $\mathscr{G}$ respectively as in Proposition 1.3.1 and bottom pairing is the standard composition pairing.

Proposition 4.1. *The above diagram commutes in cohomology.* $\lhd$

More formally, we're saying that the diagram

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet)\right)_s \otimes (\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s \arrow[r] \arrow[d, "\varphi\otimes i^{-1}"] & I^\bullet \arrow[d, "\mathrm{id}"] \\
\underline{\mathrm{Hom}}^\bullet(\ker f, \mathscr{I}^\bullet) \otimes \ker f \arrow[r] & I^\bullet
\end{tikzcd}$$

commutes in cohomology.

Proof. Fix $n \in \mathbb{Z}^{>0}$ and integers p, q such that $p + q = n$ and fix $(\psi_{\mathscr{G}}, \psi_{\mathscr{F}}) \in (\underline{\mathrm{Hom}}^\bullet(\mathscr{G}^\bullet, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet))_s^p$ and $(a_{\mathscr{F}}, a_{\mathscr{G}}) \in (\mathscr{F}^\bullet \to \mathscr{G}^\bullet)_s^q$.

If $\langle \cdot, \cdot \rangle_{\mathscr{F}} : \underline{\mathrm{Hom}}^\bullet(\mathscr{F}^\bullet, \mathscr{I}^\bullet) \times \mathscr{F}^\bullet \to I^\bullet$ is the composition pairing, with a similar definition for $\mathscr{G}$, then

$$\langle H(f)(\psi_{\mathscr{G}}), a_{\mathscr{F}} \rangle_{\mathscr{F}} := \psi_{\mathscr{G}}(f(a_{\mathscr{F}})) =: \langle \psi_{\mathscr{G}}, f(a_{\mathscr{F}}) \rangle_{\mathscr{G}}$$

so by Proposition 1.3.1 the top pairing is given by

$$(\psi_{\mathscr{G}}, \psi_{\mathscr{F}}) \times (a_{\mathscr{F}}, a_{\mathscr{G}}) \mapsto \psi_{\mathscr{F}}(a_{\mathscr{F}}) + (-1)^{q+1} \psi_{\mathscr{G}}(a_{\mathscr{G}}).$$

By definition $\varphi(\psi_{\mathscr{G}}, \psi_{\mathscr{F}}) = \psi_{\mathscr{F}}$, and as we explicitly calculated when proving the surjectivity of the map $\ker f \to (\mathscr{F}^\bullet \to \mathcal{G}^\bullet)_s$ we have $i^{-1}(a_{\mathscr{F}}, a_{\mathscr{G}}) = a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}$ where $f(b_{\mathscr{F}}) = a_{\mathscr{G}}$. Hence the top pairing maps to the pairing

$$\psi_{\mathscr{F}} \times (a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}) \mapsto \psi_{\mathscr{F}}(a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}).$$

We apply the conventions in Section 0A8H for hom complexes. Since we're working in cohomology, $d(\psi_{\mathscr{G}}, \psi_{\mathscr{F}}) = (d\psi_{\mathscr{G}}, d\psi_{\mathscr{F}} + (-1)^{p+1} H(f)(\psi_{\mathscr{G}})) =$

0. Writing $\psi_{\mathscr{G}} = (\psi_{\mathscr{G},j})_j \in \prod_j \underline{\mathrm{Hom}}(\mathscr{G}^j, \mathscr{I}^{j+p+1})$ and $\psi_{\mathscr{F}} = (\psi_{\mathscr{F},j})_j \in \prod_j \underline{\mathrm{Hom}}(\mathscr{F}^j, \mathscr{I}^{j+p})$ we have

$$(d\psi_{\mathscr{G}})_j = d_I \circ \psi_{\mathscr{G},j} + (-1)^p \psi_{\mathscr{G},j+1} \circ d_{\mathscr{G}} \in \prod_j \underline{\mathrm{Hom}}(\mathscr{G}^j, \mathscr{I}^{j+p+2})$$

and

$$\begin{aligned}(d\psi_{\mathscr{F}} + (-1)^{p+1} H(f)(\psi_{\mathscr{F}}))_j = (d_I \circ \psi_{\mathscr{F},j} &+ (-1)^{p+1}\psi_{\mathscr{F},j+1} \circ d_{\mathscr{F}} \\ &+ (-1)^{p+1}\psi_{\mathscr{G},j} \circ f_j)_j \in \prod_j \underline{\mathrm{Hom}}(\mathscr{F}^j, \mathscr{I}^{j+p+1}).\end{aligned}$$

In particular setting $j = q-1$ in the latter equation and inputting $b_{\mathscr{F}}$ we obtain

$$0 = d_I(\psi_{\mathscr{F},q-1}(b_{\mathscr{F}})) + (-1)^{p+1}\psi_{\mathscr{F},q}(d_{\mathscr{F}} b_{\mathscr{F}}) + (-1)^{p+1}\psi_{\mathscr{G},q-1}(f_j(b_{\mathscr{F}}))$$

so

$$\begin{aligned}\psi_{\mathscr{F}}(db_{\mathscr{F}}) &= -\psi_{\mathscr{G}}(f_j(b_{\mathscr{F}})) + (-1)^p d_I(\psi_{\mathscr{F},q-1}(b_{\mathscr{F}})) \\ &= -\psi_{\mathscr{G}}(a_{\mathscr{G}}) + (-1)^p d_I(\psi_{\mathscr{F},q-1}(b_{\mathscr{F}})).\end{aligned}$$

Thus

$$\begin{aligned}\psi_{\mathscr{F}} \times (a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}) &\mapsto \psi_{\mathscr{F}}(a_{\mathscr{F}} + (-1)^q db_{\mathscr{F}}) \\ &= \psi_{\mathscr{F}}(a_{\mathscr{F}}) + (-1)^q \psi_{\mathscr{F}}(db_{\mathscr{F}}) \\ &= \psi_{\mathscr{F}}(a_{\mathscr{F}}) + (-1)^{q+1}\psi_{\mathscr{G}}(a_{\mathscr{G}}) + (-1)^{p+q} d_I(\psi_{\mathscr{F},q-1}(b_{\mathscr{F}})) \\ &\sim \psi_{\mathscr{F}}(a_{\mathscr{F}}) + (-1)^{q+1}\psi_{\mathscr{G}}(a_{\mathscr{G}})\end{aligned}$$

which is what we needed to show. □

1.5 Pairings of Overconvergent Complexes

Before we start, we prove an expected functoriality.

Lemma 5.1. *Let $(X \subseteq Y \subseteq P)$ be a frame, V an admissible open subset of $]Y[_P$, and $\mathcal{A}$ a sheaf of rings on V. If $\mathcal{E}$ and $\mathscr{F}$ are two $\mathcal{A}$-modules, the canonical isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) \cong \underline{\Gamma}_{]X[_P} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$$

from [Le 07, Proposition 5.3.5] is functorial in $\mathcal{E}$ and $\mathcal{F}$, and if $U \subseteq X$ is open then the natural diagram

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_U^\dagger \mathcal{E}, \mathcal{F}) & \cong & \underline{\Gamma}_{]U[_P} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) & \cong & \underline{\Gamma}_{]X[_P} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})
\end{array}$$

commutes. ◁

Proof. We start with the functoriality for $U \subseteq X$ open. From the proof of [Le 07, Proposition 5.2.5] it is clear that if V is a rigid analytic variety, $T \subseteq V$ an admissible open subset, $u : V' \to V$ a morphism of rigid analytic varieties, T' an admissible open subset of V containing $u^{-1}(T)$, and $\mathcal{E}$ a sheaf on V, then the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & u^{-1}\underline{\Gamma}_T^\dagger \mathcal{E} & \longrightarrow & u^{-1}\mathcal{E} & \longrightarrow & u^{-1} j_{V \setminus T}^\dagger \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & \underline{\Gamma}_{T'}^\dagger u^{-1}\mathcal{E} & \longrightarrow & u^{-1}\mathcal{E} & \longrightarrow & j_{V' \setminus T'}^\dagger u^{-1}\mathcal{E} & \longrightarrow & 0
\end{array}$$

where all of the nontrivial morphisms are the natural ones arising from adjunction.

In the given setting, set $V' = V$, $u = \mathrm{id}$, and $T' = V \setminus (V \cap]U[_P)$. Then the above diagram is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\Gamma}_Z^\dagger \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & j_X^\dagger \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & \underline{\Gamma}_{Z'}^\dagger \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & j_U^\dagger \mathcal{E} & \longrightarrow & 0
\end{array}$$

where $Z = Y \setminus X$ and $Z' = Y \setminus U$ are the closed complements of X and U, respectively. If we let $h : V \setminus]X[_P \hookrightarrow V$ and $h' : V \setminus]U[_P \hookrightarrow V$ denote the inclusions as usual, we obtain as in the proof of the isomorphism a morphism of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) & \longrightarrow & h_* h^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \\
& & \uparrow & & \uparrow{=} & & \uparrow \\
0 & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{A}}(j_U^\dagger \mathcal{E}, \mathcal{F}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) & \longrightarrow & h'_* h'^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})
\end{array}$$

It then follows from general principles that we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) \arrow[r, "\cong"] & \ker\left(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to h_* h^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})\right) \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_U^\dagger \mathcal{E}, \mathcal{F}) \arrow[r, "\cong"] \arrow[u] & \ker\left(\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \to h'_* h'^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})\right) \arrow[u]
\end{tikzcd}$$

Finally, noting that in general the morphism $\underline{\Gamma}_{]X'[_P}\mathcal{E} \to \underline{\Gamma}_{]X[_P}\mathcal{E}$ for $X' \subseteq X$ open is induced via the identification with the kernel as in the proof of [Le 07, Proposition 5.2.16], this translates to a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) \arrow[r, "\cong"] & \underline{\Gamma}_{]X[_P} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(j_U^\dagger \mathcal{E}, \mathcal{F}) \arrow[r, "\cong"] \arrow[u] & \underline{\Gamma}_{]U[_P} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \arrow[u]
\end{tikzcd}$$

as desired.

Next we check functoriality in $\mathcal{E}$. It follows from [Bou+06, Exposé IV, 14] that the short exact sequence

$$0 \to \underline{\Gamma}_Z^\dagger \mathcal{E} \to \mathcal{E} \to j_X^\dagger \mathcal{E} \to 0$$

is functorial in $\mathcal{E}$. So if $\mathcal{E}'$ is another $\mathcal{A}$-module and $\mathcal{E} \to \mathcal{E}'$ is a morphism then we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_Z^\dagger \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & j_X^\dagger \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\Gamma}_Z^\dagger \mathcal{E}' \arrow[r] & \mathcal{E}' \arrow[r] & j_X^\dagger \mathcal{E}' \arrow[r] & 0
\end{tikzcd}$$

and the remainder of the argument is the same as in the above proof of functoriality for $U \subseteq X$ open.

Finally, suppose $\mathcal{F}'$ is another $\mathcal{A}$-module and let $\mathcal{F} \to \mathcal{F}'$ be a morphism. Then by functoriality of $\underline{\mathrm{Hom}}$ we have a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}) \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \arrow[r] & h_* h^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \\
0 \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{A}}(j_X^\dagger \mathcal{E}, \mathcal{F}') \arrow[r] \arrow[u] & \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}') \arrow[r] \arrow[u] & h_* h^{-1} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}') \arrow[u]
\end{tikzcd}$$

and again the remainder of the argument is identical to the end of the proof of functoriality for $U \subseteq X$ open. $\square$

Let $X \subseteq \overline{X} \subseteq P$ be a frame and fix an injective resolution $\varphi : \Omega^\bullet_{]\overline{X}[} \to \mathscr{I}^\bullet$ with a pairing $\mathscr{I}^\bullet \times \mathscr{I}^\bullet \to \mathscr{I}^\bullet$ lifting that of $\Omega^\bullet_{]\overline{X}[}$. Such an injective resolution is equivalent to the data of an injective resolution in the category of $\Omega^\bullet_{]\overline{X}[}$-modules, which has enough injectives by [HL71, Proposition 2.4]. This pairing induces a pairing $j_X^\dagger \mathscr{I}^\bullet \times \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \to \mathscr{I}^\bullet$, corresponding to the morphism

$$\underline{\Gamma}_{]X[}\mathscr{I}^\bullet \to \underline{\Gamma}_{]X[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\mathscr{I}^\bullet, \mathscr{I}^\bullet) \cong \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet)$$

where the first morphism is the restriction of the original pairing on $\mathscr{I}^\bullet$.

Consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[r, "\cong"] & \underline{\Gamma}_{]X[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[r, "\cong"] & \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \\
\times & & \times \\
j_X^\dagger \Omega^\bullet_{]\overline{X}[} \arrow[rr, "j_X^\dagger \varphi"] \arrow[d] & & j_X^\dagger \mathscr{I}^\bullet \arrow[d] \\
\mathscr{I}^\bullet & & \mathscr{I}^\bullet
\end{tikzcd}$$

The left pairing is the composition pairing and the right pairing is the one described above.

Lemma 5.2. *These two pairings are compatible.* $\lhd$

Proof. By adjunction, it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[d, "\mathrm{id}"] & \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \arrow[l, "\cong"'] \arrow[d] \\
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) & \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet) \arrow[l]
\end{tikzcd}$$

The fact that the pairing on $\mathscr{I}^\bullet$ extends that of $\Omega_{]\overline{X}[}$ means that the $\Omega_{]\overline{X}[}$-module structure on $\mathscr{I}^\bullet$ is compatible with the pairing on $\mathscr{I}^\bullet$, in the sense that the $\Omega_{]\overline{X}[}$-action is induced by the composition

$$\Omega^\bullet_{]\overline{X}[} \xrightarrow{\varphi} \mathscr{I}^\bullet \to \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\mathscr{I}^\bullet, \mathscr{I}^\bullet).$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) & \mathscr{I}^\bullet \arrow[l, "\cong"'] \arrow[d] \\
& \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\mathscr{I}^\bullet, \mathscr{I}^\bullet) \arrow[ul]
\end{tikzcd}$$

Alongside the functoriality from Lemma 1.5.1 we can glean that the following commutes too:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) & \underline{\Gamma}_{]X[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[l, "\cong"'] & \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \arrow[l, "\cong"'] \arrow[d] \\
& & \underline{\Gamma}_{]X[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\mathscr{I}^\bullet, \mathscr{I}^\bullet) \arrow[ul] \arrow[d] \\
& & \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \mathscr{I}^\bullet, \mathscr{I}^\bullet) \arrow[uull]
\end{tikzcd}$$

but this is precisely what we wanted to show. $\square$

Next, note that the natural morphism $\underline{\Gamma}_{]U[}\mathscr{I}^\bullet \to \underline{\Gamma}_{]X[}\mathscr{I}^\bullet$ coincides with the natural morphism $\underline{\mathrm{Hom}}^\bullet(j_U^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet)$. That is to say, the following diagram commutes in the category of $\Omega^\bullet_{]\overline{X}[}$-modules:

$$\begin{tikzcd}
\underline{\Gamma}_{]U[}\mathscr{I}^\bullet \arrow[r] \arrow[d, "\cong"] & \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \arrow[d, "\cong"] \\
\underline{\Gamma}_{]U[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[r] \arrow[d, "\cong"] & \underline{\Gamma}_{]X[}\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_U^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \arrow[r] & \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger \Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet)
\end{tikzcd}$$

The commutativity of the top square comes from the functoriality of the morphism

$$h'_* h'^{-1}\mathcal{E} \to h_* h^{-1}\mathcal{E}$$

in the proof of [Le 07, Proposition 5.2.16]. The bottom square commutes by Lemma 1.5.1; indeed, from its proof we know that the bottom morphism is the one derived from the morphism $j_X^\dagger \Omega^\bullet_{]\overline{X}[} \to j_U^\dagger \Omega^\bullet_{]\overline{X}[}$ so the diagram above coincides with the diagram in Lemma 1.5.1.

Finally, if $\psi : \underline{\Gamma}_{]U[}\mathscr{I}^\bullet \to \underline{\Gamma}_{]X[}\mathscr{I}^\bullet$ and $f : j_X^\dagger \mathscr{I}^\bullet \to j_U^\dagger \mathscr{I}^\bullet$ are the usual morphisms, the commutativity of the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{]U[}\mathscr{I}^\bullet \arrow[rr, "\psi"] \arrow[d, "\cong"] & & \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_U^\dagger\mathscr{I}^\bullet, \mathscr{I}^\bullet) \arrow[rr, "H(f)"] & & \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger\mathscr{I}^\bullet, \mathscr{I}^\bullet)
\end{tikzcd}$$

(which is of course proven in exactly the same way) implies the adjoint relation

$$\langle\psi(a), b\rangle_X = \langle a, f(b)\rangle_U$$

where we denote by $\langle\cdot,\cdot\rangle_X : \underline{\Gamma}_{]X[}\mathscr{I}^\bullet \times j_X^\dagger\mathscr{I}^\bullet \to \mathscr{I}^\bullet$ the usual pairing and similarly for U.

We infer from all of this the following:

Proposition 5.3. *There is a compatibility of pairings*

$$\begin{tikzcd}
\left(\underline{\Gamma}_{]U[}\mathscr{I}^\bullet \to \underline{\Gamma}_{]X[}\mathscr{I}^\bullet\right)_s \arrow[d, "\cong"] & \times & \left(j_X^\dagger\mathscr{I}^\bullet \to j_U^\dagger\mathscr{I}^\bullet\right)_s \arrow[r] \arrow[d, "="] & \mathscr{I}^\bullet \arrow[d, "="] \\
\left(\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_U^\dagger\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet) \to \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(j_X^\dagger\Omega^\bullet_{]\overline{X}[}, \mathscr{I}^\bullet)\right)_s & \times & \left(j_X^\dagger\mathscr{I}^\bullet \to j_U^\dagger\mathscr{I}^\bullet\right)_s \arrow[r] & \mathscr{I}^\bullet
\end{tikzcd}$$

where the pairings are given by Proposition 1.3.1. ◁

Hence by Proposition 1.4.1 we've proven

Corollary 5.4. *Let $f : j_X^\dagger\mathscr{I}^\bullet \to j_U^\dagger\mathscr{I}^\bullet$ denote the usual map. Then the pairings*

$$\begin{tikzcd}
\left(\underline{\Gamma}_{]U[}\mathscr{I}^\bullet \to \underline{\Gamma}_{]X[}\mathscr{I}^\bullet\right)_s \arrow[d, "\varphi"] & \times & \left(j_X^\dagger\mathscr{I}^\bullet \xrightarrow{f} j_U^\dagger\mathscr{I}^\bullet\right)_s \arrow[r] \arrow[d, "i^{-1}"] & \mathscr{I}^\bullet \arrow[d, "="] \\
\underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\ker f, \mathscr{I}^\bullet) & \times & \ker f \arrow[r] & \mathscr{I}^\bullet
\end{tikzcd}$$

are compatible in cohomology. ◁

Since the cohomology of the top pairing is Poincaré duality, we piggyback on Berthelot's result to obtain

Corollary 5.5. *Suppose X is a smooth k-scheme. Then the pairing*

$$R\Gamma\left(]\overline{X}[, \underline{\mathrm{Hom}}^\bullet_{\Omega_{]\overline{X}[}}(\ker f, \mathscr{I}^\bullet)\right) \times R\Gamma\left(]\overline{X}[, \ker f\right) \to R\Gamma_{c,\mathrm{rig}}(X) \to K$$

induces a perfect pairing. ◁

1.6 Poincaré Duality and the Monodromy Operator

We now apply Corollary 1.5.5 to the study of the kernel and cokernel of the monodromy operator on the rigid-analytic analogue of the Hyodo-Steenbrink double complex.

Our spaces live over the logarithmic extensions of the standard base schemes in rigid-analytic geometry.

Notation 6.1. We define

$$
\begin{aligned}
S^0 &:= (\operatorname{Spec} k, 1 \mapsto 0) \\
\mathfrak{W} &:= (\operatorname{Spf}(\mathcal{V}), \mathrm{triv}) \\
\mathfrak{S}^0 &:= (\operatorname{Spf}(\mathcal{V}), 1 \mapsto 0) \\
\mathfrak{S} &:= (\operatorname{Spf}(\mathcal{V}[t]), (\mathbb{N} \to \mathcal{V}[t], 1 \mapsto t))
\end{aligned}
$$

◁

The construction requires the additional assumption that, essentially, our model can be lifted to a family in characteristic 0 as well as in characteristic p.

Definition 6.2. Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme.

1. An *admissible lift* is a log-scheme $(\mathcal{Z}, \mathcal{N})$ defined over $\mathcal{V}[t]$ such that
   (a) $\mathcal{Z}$ is smooth over $\mathcal{V}$;
   (b) $(Y, \mathcal{N}_Y) \cong (\mathcal{Z}, \mathcal{N}) \times_{\mathfrak{S}} S^0$;
   (c) the scheme $\mathcal{Y} := V(t)$ is a normal crossing divisor in $\mathcal{Z}$ and $\mathcal{N}$ is the log structure corresponding to this divisor.

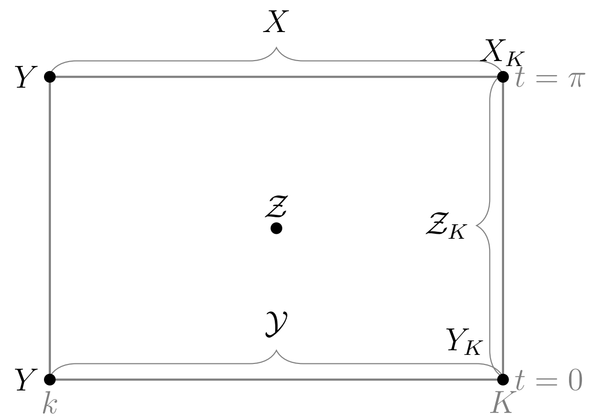
   We say that an admissible lift is *proper* if $\mathcal{Z}$ is proper over $\mathcal{V}$.

2. An *admissible formal lift* is a formal $\mathfrak{S}$ log-scheme $(\mathfrak{Z}, \mathcal{N}_{\mathfrak{Z}})$ such that
   (a) $\mathfrak{Z}$ is smooth over $\mathfrak{W}$ and flat over $\mathfrak{S}$;
   (b) $(Y, \mathcal{N}_Y) \cong (\mathfrak{Z}, \mathcal{N}_{\mathfrak{Z}}) \times_{\mathfrak{S}} S^0$;
   (c) The fiber $\mathfrak{Y} = V(t)$ is a normal crossing divisor on $\mathfrak{Z}$ and $\mathcal{N}_{\mathfrak{Z}}$ is the log structure corresponding to this divisor.

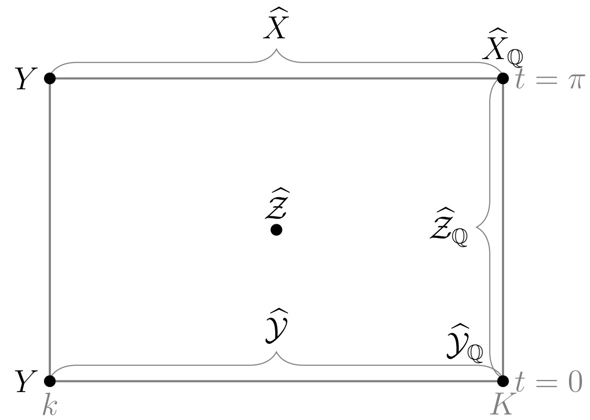
   We say that a formal admissible lift is *proper* if $\mathfrak{Z}$ is proper over $\mathfrak{W}$.

$\lhd$

Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme with a proper admissible lift $(\mathcal{Z}, \mathcal{N})$. We visualize the base change of $\mathcal{Z}$ to k and K as well as its fibers over $t = 0$ and $t = \pi$ as below:



The formal completion of the constitutent schemes can be visualized as follows:



Since $\mathcal{Y} = V(t)$ remains a normal crossing divisor after completion, this is the data of a proper admissible formal lift of $(Y, \mathcal{N}_Y)$.

Write $\mathfrak{Z} = \widehat{\mathcal{Z}}$ and $\mathfrak{Y} = \widehat{\mathcal{Y}}$. Since the lifting is proper, we have an isomorphism $\mathfrak{Z}_K \cong \mathcal{Z}_K^{\mathrm{an}}$ and we may take Z_k to be its own compatification, so that $(Y \subseteq \mathcal{Z}_k \subseteq \mathfrak{Z})$ is a frame; note that by construction we have $]\mathcal{Z}_k[_{\mathfrak{Z}} = \mathfrak{Z}_K$. Let $U = \mathcal{Z}_k \setminus Y$ be the open complement of Y in $\mathcal{Z}_k$ and let $\mathscr{I}^\bullet$ be an injective resolution of $\Omega^\bullet_{\mathfrak{Z}_K}$ with a pairing extending the wedge product. By Corollary 1.5.5 the pairing

$$H^i(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}((\Omega^\bullet_{\mathfrak{Z}_K} \to j_U^\dagger \Omega^\bullet_{\mathfrak{Z}_K})_s, \mathscr{I}^\bullet)) \times H^{2n+2-i}(\mathfrak{Z}_K, (\Omega^\bullet_{\mathfrak{Z}_K} \to j_U^\dagger \Omega^\bullet_{\mathfrak{Z}_K})_s) \to K$$

is perfect.

Our goal is to interpret this as a pairing between the kernel and cokernel of the monodromy operator on Grosse-Klonne's rigid analogue of the Hyodo-Steenbrink double complex. As a preparatory step we have a transmutation of this pairing.

Lemma 6.3. *Let $(Y, \mathcal{N})$ be a semistable k-log scheme of dimension n which admits a proper formal admissible lift $(\mathfrak{Z}, \mathfrak{Y})$. Then the hom pairing*

$$H^{i+1}(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}(\Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}, \mathscr{I}^\bullet)) \times H^{2n+2-i-1}(\mathfrak{Z}_K, \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}) \to K$$

is perfect. $\lhd$

Proof. First, it's observed in [CT03, §4] that there is a quasi-isomorphism

$$\Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle \xrightarrow{\cong} j_U^\dagger \Omega^\bullet_{\mathfrak{Z}_K}$$

fitting into the commutative diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathfrak{Z}_K} \arrow[r] \arrow[d, "="] & \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle \arrow[d, "\cong"] \\ \Omega^\bullet_{\mathfrak{Z}_K} \arrow[r] & j_U^\dagger \Omega^\bullet_{\mathfrak{Z}_K}. \end{tikzcd}$$

This induces a quasi-isomorphism

$$\left(\Omega^\bullet_{\mathfrak{Z}_K} \to \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle\right)_s \xrightarrow{\cong} \left(\Omega^\bullet_{\mathfrak{Z}_K} \to j_U^\dagger \Omega^\bullet_{\mathfrak{Z}_K}\right)_s.$$

Secondly, since $\Omega^\bullet_{\mathfrak{Z}_K} \to \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle$ is injective there is a standard quasi-isomorphism

$$(\Omega^\bullet_{\mathfrak{Z}_K} \to \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle)_s \xrightarrow{\cong} \mathrm{coker}(\Omega^\bullet_{\mathfrak{Z}_K} \to \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle)[-1] \cong \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}[-1]$$

given by projection. Substituting these isomorphisms into our perfect pairing we obtain a pairing

$$H^{i}(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}(\Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}[-1], \mathscr{I}^\bullet)) \times H^{2n+2-i}(\mathfrak{Z}_K, \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}[-1]) \to K,$$

that is,

$$H^{i+1}(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}(\Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}, \mathscr{I}^\bullet)) \times H^{2n+2-i-1}(\mathfrak{Z}_K, \Omega^\bullet_{\mathfrak{Z}_K}\langle\mathfrak{Y}_K\rangle/\Omega^\bullet_{\mathfrak{Z}_K}) \to K$$

which is what we wanted to show. $\square$

1.6.1 The Monodromy Operator and the rigid Hyodo-Steenbrink Double Complex

We briefly describe Grosse-Klonne's description of the rigid analogue of the Hyodo-Steenbrink double complex, described in [Gro07] and elaborated upon in [Gro14]. Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme. Admissible lifts exist étale locally on Y. Indeed, it is a feature of semistable k-log schemes that they étale locally have embeddings as normal crossing divisors into smooth k-schemes, say $Y \hookrightarrow Z$, which we may assume affine [Kat96, Proposition 11.3]. Smooth affine schemes can be lifted to formally smooth affine formal schemes, so we can find a $W(k)$-scheme $\mathfrak{Z}$ lifting Z. Then we lift the equations of Y in $\mathcal{O}_Z$ to equations in $\mathcal{O}_{\mathfrak{Z}}$, and we let $\mathfrak{Y}$ be the formal scheme defined by these equations.

Given this, let $Y = \bigcup_{h\in H} U_h$ be an open covering such that each U_h has an admissible lifting $(\mathfrak{Z}_h, \mathfrak{Y}_h)$. For each $G \subseteq H$, let $U_G = \bigcap_{h\in G} U_h$. Through blow-ups Grosse-Klonne constructs a canonical embedding

$$U_G \hookrightarrow \mathfrak{Y}_G \hookrightarrow \mathfrak{K}_G$$

where $U_G \hookrightarrow \mathfrak{Y}_G$ is an exact closed embedding of formal $\mathfrak{S}$-log schemes, $\mathfrak{K}_G$ is smooth over $\mathfrak{W}$, and $\mathfrak{Y}_G \hookrightarrow \mathfrak{K}_G$ is an embedding of a normal crossing divisor. We give $\mathfrak{K}_G$ the log structure provided by $\mathfrak{Y}_G$.

Denote by $\widetilde{\omega}^{\bullet}_{\mathfrak{K}_G}$ the log de Rham complex of $\mathfrak{K}_G \to \mathfrak{W}$, write $\theta = d\log t$, and let

$$\widetilde{\omega}^{\bullet}_{\mathfrak{Y}_G} := \widetilde{\omega}^{\bullet}_{\mathfrak{K}_G} \otimes \mathcal{O}_{\mathfrak{Y}_G}, \qquad \omega^{\bullet}_{\mathfrak{Y}_G} = \frac{\widetilde{\omega}^{\bullet}_{\mathfrak{Y}_G}}{\widetilde{\omega}^{\bullet-1}_{\mathfrak{Y}_G} \wedge \theta}$$

Let $\mathfrak{Y}_{G,K}$ be the generic fiber of $\mathfrak{Y}_G$ and let

$$\widetilde{\omega}^{\bullet}_{\mathfrak{Y}_{G,K}} := \widetilde{\omega}^{\bullet}_{\mathfrak{Y}_G} \otimes K \qquad \omega^{\bullet}_{\mathfrak{Y}_{G,K}} := \omega^{\bullet}_{\mathfrak{Y}_G} \otimes K;$$

when there's no risk of ambiguity, we'll denote by the same $\widetilde{\omega}^{\bullet}_{\mathfrak{Y}_{G,K}}$ and $\omega^{\bullet}_{\mathfrak{Y}_{G,K}}$ their respective restrictions to the tube $]U_G[_{\mathfrak{Y}_G}$. For $G_1 \subseteq G_2$ there is a natural transition map

$$]U_{G_2}[_{\mathfrak{Y}_{G_2}} \to]U_{G_1}[_{\mathfrak{Y}_{G_1}}$$

which provides the data for a simplicial rigid-analytic space $]U_\bullet[_{\mathfrak{Y}_\bullet}$, and it is easy to see that $\omega^{\bullet}_{\mathfrak{Y}_\bullet}$ and $\widetilde{\omega}^{\bullet}_{\mathfrak{Y}_\bullet}$ are sheaves on the associated site.

It follows from the definition of (log-)rigid cohomology [Gro14, §1] that

$$R\Gamma_{\mathrm{rig}}(Y) = R\Gamma(]U_\bullet[_{\mathfrak{Y}_\bullet}, \omega^{\bullet}_{\mathfrak{Y}_\bullet})$$

revealing that this complex can be seen as a rigid-analytic analogue of the complex $W_n\omega_Y{}^{\bullet}$ in [Mok93] which computes log-crystalline cohomology, in fact coinciding with it in the proper case.

The following construction of the double complex is essentially identical to that of [Mok93, §3.8] and [Ste76, §4]. For $j \geq 0$, let

$$P_j\widetilde{\omega}^k_{\mathfrak{K}_\bullet} := \mathrm{Im}(\widetilde{\omega}^j_{\mathfrak{K}_\bullet} \otimes \Omega^{k-j}_{\mathfrak{K}_\bullet} \to \widetilde{\omega}^k_{\mathfrak{K}_\bullet})$$

and let

$$P_j\widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet} := \frac{P_j\widetilde{\omega}^\bullet_{\mathfrak{K}_\bullet}}{\widetilde{\omega}^\bullet_{\mathfrak{K}_\bullet} \otimes \mathfrak{I}_{\mathfrak{Y}_\bullet}}$$

where $\mathfrak{I}_{\mathfrak{Y}_\bullet}$ is the ideal of definition of the embedding $\mathfrak{Y}_\bullet \hookrightarrow \mathfrak{K}_\bullet$. They are the sheaves of differentials on $\mathfrak{K}_\bullet$ and $\mathfrak{Y}_\bullet$, respectively, with poles of degree at most j. The graded part, then, can be computed as usual via the residue maps and in our setting we have a residue isomorphism

$$\mathrm{Gr}_j(\widetilde{\omega}^\bullet_{\mathfrak{Y}_{G,K}}) \cong \bigoplus_{\mathfrak{N} \in \Theta_{j,G}} \Omega^\bullet_{\mathfrak{N}_K}[-j]$$

where the sum runs through the set of all intersections $\mathfrak{N}$ of j distinct irreducible components of $\mathfrak{Y}_G$.

Finally, we define a double complex

$$A^{ij}_G := \frac{\widetilde{\omega}^{i+j+1}_{\mathfrak{Y}_{G,K}}}{P_j(\widetilde{\omega}^{i+j+1}_{\mathfrak{Y}_{G,K}})}$$

where the horizontal differentials $A^{ij}_G \to A^{(i+1)j}_G$ are induced by $(-1)^j d$ and the vertical differentials $A^{ij}_G \to A^{i(j+1)}_G$ are induced by $\omega \mapsto \omega \wedge \theta$. We denote its total complex by $A^\bullet_G$.

Following Mokrane and Steenbrink, we define an endomorphism $\nu : A^\bullet_G \to A^\bullet_G$ by

$$\nu^{ij}_G = (-1)^j \,\mathrm{proj} : A^{ij}_G \to A^{i-1,j+1}_G$$

where proj denotes the projection of the quotient. The choice of sign differs from Mokrane, who used a sign of $(-1)^{i+j+1}$; it was pointed out by Nakkajima in [Nak05, Remark 11.9], who attributes the observation to Große-Klönne, that Mokrane's choice of sign fails to define a morphism of double complexes and needs to be amended to the above. Interpreting everything as components of a simplicial object, we obtain a complex $A^\bullet$ on $\mathfrak{Y}_\bullet$.

The monodromy operator

$$N : H^*_{\mathrm{rig}}(Y) \to H^*_{\mathrm{rig}}(Y)$$

on rigid cohomology is defined to be the connecting homomorphism in cohomology associated to the exact sequence

$$0 \to \omega^\bullet_{\mathfrak{Y}_\bullet}[-1] \xrightarrow{\wedge\theta} \widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet} \to \omega^\bullet_{\mathfrak{Y}_\bullet} \to 0.$$

It is purely formal (see [Mok93, Proposition 3.18] or [Ste76, §4.22]) that the connecting homomorphism of [Gro14, §5.4] coincides with the action of the monodromy operator ν on $A^\bullet$. The cohomology of the kernel and cokernel of the monodromy operator on $A^\bullet$ can be described explicitly as follows:

Lemma 6.4. *With the above notation, we have*

$$H^i(\mathfrak{Y}_K, \ker \nu) \cong H^i(\mathfrak{Z}_K, \mathrm{Gr}_*(\widetilde{\omega}^\bullet_{\mathfrak{Z},K})[*])$$

and

$$H^i(\mathfrak{Y}_K, \operatorname{coker} \nu) \cong H^{i+1}\left(\mathfrak{Z}_K, \frac{\widetilde{\omega}^\bullet_{\mathfrak{Z},K}}{P_0(\widetilde{\omega}^\bullet_{\mathfrak{Z},K})}\right)$$

$\lhd$

Proof. The argument of Chiarellotto in [Chi99, Proposition 1.8] shows that

$$\ker \nu = (0 \to \mathrm{Gr}_1(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[1] \to \mathrm{Gr}_2(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[2] \to \mathrm{Gr}_3(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[3] \to \cdots).$$

Since all of the morphisms ν_G^{ij} which are nontrivial are surjective, we immediately have

$$\operatorname{coker} \nu = A^{\bullet 0} = \frac{\widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet}}{P_0(\widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet})}[1].$$

To apply Corollary 1.5.5 we specialize to the case in which Y globally has a proper admissible lift. In this case we have $G = \{*\}$ the singleton set, $U_G = Y$, $\mathfrak{Y}_G = \mathfrak{Y}$, and $\mathfrak{K}_G = \mathfrak{Z}$. Let $i : \mathfrak{Y} \hookrightarrow \mathfrak{Z}$ denote the closed immersion. Since we assume properness, we have $]U_G[_{\mathfrak{Y}_G} =]Y[_{\mathfrak{Y}} = \mathfrak{Y}_K$. By definition,

$$\tilde{\omega}^\bullet_{\mathfrak{Z},K} = \Omega^\bullet_{\mathfrak{Z}_K}\langle \mathfrak{Y}_K \rangle$$

Since i_* is an exact functor, for all $j > k$ we have

$$\begin{aligned} P_j\tilde{\omega}^\bullet_{\mathfrak{Z}}/P_k\tilde{\omega}^\bullet_{\mathfrak{Z}} &\cong \left(\frac{P_j\tilde{\omega}^\bullet_{\mathfrak{Z}}}{\tilde{\omega}^\bullet_{\mathfrak{Z}} \otimes I_{\mathfrak{Y}}}\right) / \left(\frac{P_k\tilde{\omega}^\bullet_{\mathfrak{Z}}}{\tilde{\omega}^\bullet_{\mathfrak{Z}} \otimes I_{\mathfrak{Y}}}\right) \\ &\cong i_*P_j\tilde{\omega}^\bullet_{\mathfrak{Y}}/i_*P_k\tilde{\omega}^\bullet_{\mathfrak{Y}} \\ &\cong i_*(P_j\tilde{\omega}^\bullet_{\mathfrak{Y}}/P_k\tilde{\omega}^\bullet_{\mathfrak{Y}}). \end{aligned}$$

This holds even after passing to the generic fiber. In particular

$$\mathrm{Gr}_j(\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}})[j] \cong i_*(\mathrm{Gr}_j(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[j])$$

for all j. It is standard that i_* preserves injectives since its left adjoint i^* is exact so we have an isomorphism

$$\begin{aligned} H^i(\mathfrak{Z}_K, \mathrm{Gr}_*(\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}})[*]) &\cong H^i(\mathfrak{Z}_K, i_*(\mathrm{Gr}_*(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[*])) \\ &\cong H^i(\mathfrak{Y}_K, \mathrm{Gr}_*(\tilde{\omega}^\bullet_{\mathfrak{Y},\mathbb{Q}})[*]) \\ &= H^i(\mathfrak{Y}_K, \ker\nu) \end{aligned}$$

Similarly, the surjectivity of ν on each $A^{p,q}$ except when $q = 0$ implies that

$$\operatorname{coker}\nu = (0 \to \tilde{\omega}^1_{\mathfrak{Y},\mathbb{Q}}/P_0\,\tilde{\omega}^1_{\mathfrak{Y},\mathbb{Q}} \to \tilde{\omega}^2_{\mathfrak{Y},\mathbb{Q}}/P_0\,\tilde{\omega}^2_{\mathfrak{Y},\mathbb{Q}} \to \tilde{\omega}^3_{\mathfrak{Y},\mathbb{Q}}/P_0\,\tilde{\omega}^3_{\mathfrak{Y},\mathbb{Q}} \to \cdots)$$

and the same isomorphisms show

$$H^i(\mathfrak{Z}_K, \tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}/P_0\,\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}) \cong H^i(\mathfrak{Y}_K, \operatorname{coker}\nu).$$

□

1.6.2 El Zein's Isomorphism

We now define a pairing between the cohomology groups of the previous subsection using El Zein's isomorphisms in [El 83].

For a moment we step into El Zein's settings and consider varieties over $\mathbb{C}$. Let X be a smooth and proper variety of dimension $n+1$ and Y a normal crossing divisor in X. For each i, let $Y^{(i)}$ denote the disjoint union of intersections of i distinct irreducible components of Y. There is a natural inclusion $\Pi : Y^{(i)} \to Y$ and canonical maps

$$\delta_j : Y^{(i)} \to Y^{(i-1)}$$

whose restrictions to the components are simply the inclusions

$$Y_{t_1} \cap \cdots \cap Y_{t_i} \to Y_{t_1} \cap \cdots \cap \widehat{Y}_{t_j} \cap \cdots \cap Y_{t_i}$$

where $\widehat{Y}_{t_j}$ denotes the fact that the component Y_{t_j} is omitted. One can then define maps $\rho_i : \Omega^\bullet_{Y^{(i)}} \to \Omega^\bullet_{Y^{(i+1)}}$ as the alternating sum

$$\rho_i = (-1)^i \sum_{1 \le j \le i+1} (-1)^{j+1} \delta_j^*.$$

We let $\Omega^*_{Y^{(\bullet)}}$ denote the natural complex

$$0 \to \Pi_*\Omega^*_{Y^{(0)}} \to \Pi_*\Omega^*_{Y^{(1)}} \to \ldots$$

induced by these maps.

El Zein's construction also requires the residual complex $K_X^\bullet$ of X, which we will identify with its image in the derived category where we will call it the dualizing complex of X. For completeness we briefly describe it here; Hartshorne's [Har66] is the standard reference for the constructions.

The dualizing complex of an arbitrary scheme X is a complex $R^\bullet \in D^b(X)$ of finite injective dimension such that for any coherent sheaf $F^\bullet \in D_c^b(X)$ the functorial isomorphism

$$F^\bullet \to R\underline{\mathrm{Hom}}^\bullet(R\underline{\mathrm{Hom}}^\bullet(F^\bullet, R^\bullet), R^\bullet)$$

of [Har66, Lemma V.1.2], which we think of as the 'double dual' of $F^\bullet$ with respect to $R^\bullet$, is an isomorphism. In the course of [Har66, Chapter V/VI] Hartshorne shows that dualizing complexes exist for a wide variety of schemes, not just varieties over $\mathbb{C}$; for example, any regular scheme of finite dimension and any scheme of finite type over a field admits a dualizing complex (see [Har66, §V.10]).

Hartshorne shows in [Har66, Proposition VI.1.1] that the image of a so-called *residual complex* on X is always a representative of the dualizing complex of X. In particular, for a regular scheme X the Cousin complex of the structure sheaf $\mathcal{O}_X$ is a residual complex [Har66, Example §VI.1.1]. From this we can use [Har66, Theorem V.3.1], which says that dualizing complexes are unique up to shifting of degrees and multiplication by an invertible sheaf, to produce other representatives for the dualizing complex. The particular residual complex which El Zein uses in his construction is the Cousin complex $K_X^\bullet$ of $\Omega_X^{n+1}[n+1]$; by general properties of the Cousin complex [Har66, §IV.2] is an injective resolution of $\Omega_X^{n+1}[n+1]$ consisting of quasi-coherent sheaves.

We now continue with El Zein's construction. Let $K_X^\bullet$ be the Cousin complex of $\Omega_X^{n+1}[n+1]$ and identify it with its image in the derived category. The variety X being smooth its sheaf of differentials is locally free, so we have an isomorphism

$$\underline{\mathrm{Hom}}_X(\Omega_X^i, K_X^\bullet) \cong K_X^\bullet \otimes (\Omega_X^i)^\vee.$$

Since K_X is the Cousin complex of $\Omega_X^{n+1}[n+1]$ it follows that this is an injective resolution of $\Omega_X^{n+1-i}[n+1]$, and in fact a Cousin complex of $\Omega_X^{n+1-i}[n+1]$. The temptation is to incorporate these into a morphism

$$\Omega_X^{n+1-\bullet}[n+1] \to \underline{\mathrm{Hom}}_X(\Omega_X^*, K_X^\bullet)$$

where the latter is the hom complex, but unfortunately this is not well-defined: since the differentials of $\Omega_X^\bullet$ are not $\mathcal{O}_X$-linear we have to strip it of its differential structure and imagine it as simply as a graded algebra to include it in the category of $\mathcal{O}_X$-modules. But while the hom complex construction is not well-defined, in [El 78, §II] El Zein proves that the temptation can be realized:

Proposition 6.5. ([El 78, Proposition 2.1.1]) *There exists on*

$$K_X^{\bullet,*} := \underline{\mathrm{Hom}}_X(\Omega_X^*, K_X^\bullet)$$

the structure of a double complex, and its total complex is a canonical injective resolution of the de Rham complex $\Omega_X^\bullet[-2n-2]$. $\lhd$

In addition to its structure as a complex, $K_X^{\bullet,*}$ has the structure of an $\Omega_X^\bullet$-module by multiplication on Ω_X^* from the right. The action is characterized by the relation

$$D\varphi + (-1)^{i+1}\varphi D = d\varphi$$

for $\varphi \in \Omega^i$ where D is the differential on $K_X^{\bullet,*}$. For general definitions and constructions about $\Omega_X^\bullet$-modules we refer the reader to [HL71].

The main result we use, which is a variation of [El 83, Theorem II.2.1] and whose proof we omit for brevity, is

Theorem 6.6. ([El 83, Theorem II.2.1]) *Let X be a smooth and proper variety, Y a normal crossing divisor in X. There is a quasi-isomorphism*

$$\varphi : \Omega_{Y(\bullet)}^* \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\Omega_X^*}^\bullet(\Omega_X^*\langle Y\rangle/\Omega_X^*, K_X^{\bullet,*})[-2n-1]$$

The basic idea is that first by a change of rings formula we have an isomorphism

$$\underline{\mathrm{Hom}}_{\Omega_X^*}^\bullet(\Omega_X^*\langle Y\rangle, K_X^{\bullet,*}) \cong \underline{\mathrm{Hom}}_X^\bullet(\Omega_X^*\langle Y\rangle, K_X^\bullet).$$

El Zein then defines a morphism

$$\Omega_{X-Y(\bullet)}^* \to \underline{\mathrm{Hom}}_X^\bullet(\Omega_X^*\langle Y\rangle, K_X^\bullet)[2n-2]$$

where $\Omega_{X-Y(\bullet)}^*$ denotes the resolution $\Omega_X^* \to \Omega_{Y(\bullet)}^*$ by defining for each $0 \le \lambda, m \le n+1$ a morphism

$$\Omega_{Y_{j_0}\cap\cdots\cap Y_{j_m}}^\lambda \to \underline{\mathrm{Hom}}_X\left(\Omega_X^{n+1-\lambda}\langle Y\rangle, H_{y_{j_0\cdots j_m}}^{m+1}(\Omega_X^{n+1})\right).$$

Here $y_{j_0\cdots j_m}$ denotes the generic point of the intersection of components $Y_{j_0} \cap \cdots \cap Y_{j_m}$ and $H_{y_{j_0\cdots j_m}}^{m+1}(\Omega_X^{n+1})$ denotes the local cohomology with support on $y_{j_0\cdots j_m}$, which can be computed as

$$H_{y_{j_0\cdots j_m}}^{m+1}(\Omega_X^{n+1}) \cong \underline{H}_{Y_{j_0}\cap\cdots\cap Y_{j_m}}^{m+1}(\Omega_X^{n+1})_{y_{j_0\cdots j_m}}$$

(see [Har66, §IV.1]). El Zein imposes a filtration on both sides of this morphism and shows that this is a bifiltered quasi-isomorphism by reducing this morphism to Poincaré duality at each graded level. The cited theorem is this isomorphism after applying the same change of rings formula and truncating the filtration.

While El Zein works over $\mathbb{C}$, his arguments and constructions are algebraic and transfer seamlessly to the algebraic setting of schemes over K. Using this result, we prove the following:

Theorem 6.7. *Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme of dimension n which admits a proper formal lift $(\mathcal{Z}, \mathcal{Y})$. Then there exists a perfect pairing*

$$H^i(\mathfrak{Y}_K, \ker \nu) \times H^{2n+2-i}(\mathfrak{Y}_K, \operatorname{coker} \nu) \to K$$

where $\mathfrak{Y} = \widehat{\mathcal{Y}}$.

Proof. By El Zein's theorem, we have a quasi-isomorphism

$$\Omega^*_{\mathcal{Y}_K^{(\bullet)}} \xrightarrow{\sim} \underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K}\langle \log \mathcal{Y}_K\rangle / \Omega^*_{\mathcal{Z}_K}, K^{\bullet;*}_{\mathcal{Z}_K})[-2n-1]$$

We now transfer this quasi-isomorphism into the rigid-analytic setting via analytification. Since the admissible lift is assumed to be proper, we have identifications

$$\mathcal{Z}_K^{\mathrm{an}} \cong \mathfrak{Z}_K, \qquad \mathcal{Y}_K^{\mathrm{an}} \cong \mathfrak{Y}_K$$

as well as canonical identifications

$$(\Omega^\bullet_{\mathcal{Z}_K})^{\mathrm{an}} \cong \Omega^\bullet_{\mathfrak{Z},K}, \qquad (\Omega^\bullet_{\mathcal{Z}_K}\langle \log \mathcal{Y}_K\rangle)^{\mathrm{an}} \cong \Omega^\bullet_{\mathfrak{Z},K}\langle \mathfrak{Y}_K\rangle$$

In addition, there is a natural isomorphism

$$\left(\underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K}\langle \log \mathcal{Y}_K\rangle / \Omega^*_{\mathcal{Z}_K}, K^{\bullet;*}_{\mathcal{Z}_K})\right)^{\mathrm{an}} \cong \underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}(\Omega^\bullet_{\mathfrak{Z},K}\langle \mathfrak{Y}_K\rangle / \Omega^\bullet_{\mathfrak{Z},K}, (K^{\bullet;*}_{\mathcal{Z}_K})^{\mathrm{an}}).$$

To see this, first note that there is a change of rings formula

$$\underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K}\langle \log \mathcal{Y}_K\rangle / \Omega^*_{\mathcal{Z}_K}, K^{\bullet;*}_{\mathcal{Z}_K}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K}\langle \mathcal{Y}_K\rangle, K^\bullet_{\mathcal{Z}_K})$$

(see [El 83, §II.1.2] or the proof of [HL71, Proposition 2.9 (3)]). Analytification on $\mathcal{O}_{\mathcal{Z}_K}$-modules is a fully faithful functor when the first component is of finite presentation, and in particular when it is coherent. Hence we can deduce an isomorphism

$$\left(\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K}\langle \mathcal{Y}_K\rangle, K^\bullet_{\mathcal{Z}_K})\right)^{\mathrm{an}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z},K}\langle \mathfrak{Y}_K\rangle / \Omega^*_{\mathfrak{Z},K}, (K^\bullet_{\mathcal{Z}_K})^{\mathrm{an}}).$$

Then by the same change of rings formula we have

$$\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z},K}\langle\mathfrak{Y}_K\rangle,(K^\bullet_{\mathcal{Z}_K})^{\mathrm{an}})\cong\underline{\mathrm{Hom}}^\bullet_{\Omega_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z},K}\langle\mathfrak{Y}_K\rangle/\Omega^*_{\mathfrak{Z},K},\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z},K},(K^\bullet_{\mathcal{Z}_K})^{\mathrm{an}})).$$

But again by fully faithfulness we have

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{O}^{\mathrm{an}}_{\mathfrak{Z},K}}(\Omega^*_{\mathfrak{Z},K},(K^\bullet_{\mathcal{Z}_K})^{\mathrm{an}})&\cong\Big(\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{Z}_K}}(\Omega^*_{\mathcal{Z}_K},K^\bullet_{\mathcal{Z}_K})\Big)^{\mathrm{an}}\\&=(K^{\bullet,*}_{\mathcal{Z}_K})^{\mathrm{an}}\end{aligned}$$

and substituting these isomorphisms into each other we obtain the desired isomorphism.

The analytification functor is exact (for example by [Con06, Example 2.2.11]) so we may apply it in the derived category as its own derived functor. Thus we have a quasi-isomorphism

$$\Omega^*_{\mathfrak{Y}^{(\bullet)}_K}\xrightarrow{\sim}R\underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z}_K}\langle\log\mathfrak{Y}_K\rangle/\Omega^*_{\mathfrak{Z}_K},(K^{\bullet,*}_{\mathcal{Z}_K})^{\mathrm{an}})[-2n-1]$$

However, it is not clear that after analytification the resolution

$$(\Omega^\bullet_{\mathcal{Z}_K})^{\mathrm{an}}\xrightarrow{\sim}(K^{\bullet,*}_{\mathcal{Z}_K})^{\mathrm{an}}$$

is an *injective* resolution. To get around this, let $\Omega^\bullet_{\mathfrak{Z}_K}\to\mathscr{I}^\bullet$ be an injective resolution extending the pairing on $\Omega^\bullet_{\mathfrak{Z}_K}$. We then have quasi-isomorphisms

$$\mathscr{I}^\bullet\xleftarrow{\sim}\Omega^\bullet_{\mathfrak{Z}_K}\xrightarrow{\sim}(K^{\bullet,*}_{\mathcal{Z}_K})^{\mathrm{an}}[2n+2]$$

and via their composition we deduce a pairing

$$\Omega^*_{\mathfrak{Y}^{(\bullet)}_K}\xrightarrow{\sim}R\underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathfrak{Z}_K}}(\Omega^*_{\mathfrak{Z}_K}\langle\log\mathfrak{Y}_K\rangle/\Omega^*_{\mathfrak{Z}_K},\mathscr{I}^\bullet)[1]$$

By definition (it's just a different choice of notation; see [CT03, pp.997]) we have

$$\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}/P_0\,\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}=\Omega^*_{\mathfrak{Z}_K}\langle\log\mathfrak{Y}_K\rangle/\Omega^*_{\mathfrak{Z}_K}$$

so that the above reads

$$\Omega^*_{\mathfrak{Y}^{(\bullet)}_K}\cong\underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathfrak{Z}_K}}(\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}/P_0\,\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}},\mathscr{I}^\bullet)[1]$$

Recall from §1.6.1 that the residue map gives an isomorphism

$$\mathrm{Gr}_j(\widetilde{\omega}^\bullet_{\mathfrak{Y}_{G,K}})[j]\cong\Omega^\bullet_{\mathfrak{Y}^{(j)}_K}.$$

As in [Ste76] and [Mok93] this isomorphism is defined in such a way that, with the standard transition morphisms, we have an isomorphism

$$\mathrm{Gr}_*(\widetilde{\omega}^\bullet_{\mathfrak{Y}_{G,K}})[*] \cong \Omega^\bullet_{\mathfrak{Y}_K^{(\bullet)}}$$

of complexes. Substituting into Lemma 1.6.4 we obtain

$$H^i(\mathfrak{Y}_K, \ker\nu) \cong H^{i+1}(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathfrak{Z}_K}}(\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}/P_0\,\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}, \mathscr{I}^\bullet)).$$

Since

$$H^{2n+2-i}(\mathfrak{Z}_K, \operatorname{coker}\nu) \cong H^{2n+2-i+1}(\mathfrak{Z}_K, \widetilde{\omega}^\bullet_{\mathfrak{Z},K}/P_0(\widetilde{\omega}^\bullet_{\mathfrak{Z},K}))$$

by the same lemma, the hom pairing

$$H^{i+1}(\mathfrak{Z}_K, \underline{\mathrm{Hom}}^\bullet_{\Omega^*_{\mathfrak{Z}_K}}(\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}/P_0\,\tilde{\omega}^\bullet_{\mathfrak{Z},\mathbb{Q}}, \mathscr{I}^\bullet)) \times H^{2n+2-i+1}(\mathfrak{Z}_K, \widetilde{\omega}^\bullet_{\mathfrak{Z},K}/P_0(\widetilde{\omega}_{\mathfrak{Z},K})) \to K$$

provides a pairing

$$H^i(\mathfrak{Y}_K, \ker\nu) \times H^{2n+2-i}(\mathfrak{Y}_K, \operatorname{coker}\nu) \to K.$$

The hom pairing is perfect by Lemma 1.5.5 so we have our claim. □

A corollary of the proof could be of independent interest:

Corollary 6.8. *Let $(Y, \mathcal{N}_Y)$ be a semistable k-log scheme of dimension n which admits a proper formal lift $(\mathcal{Z}, \mathcal{Y})$. Then we have isomorphisms*

$$H^i(\mathfrak{Y}_K, \ker\nu) \cong H^i_{\mathrm{rig}}(Y)$$
$$H^i(\mathfrak{Y}_K, \operatorname{coker}\nu) \cong H^i_{Y,\mathrm{rig}}(\mathfrak{Z}_k)$$

◁

There is another way to arrive at an isomorphism

$$H^i(\mathfrak{Y}_K, \ker\nu) \cong H^i_{\mathrm{rig}}(Y).$$

Namely, there is a Poincaré Lemma for the complex $\widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet}$ [Gro14, §5.2 (3)]

$$R\Gamma(]U_\bullet[_{\mathfrak{Y}_\bullet}, \mathrm{Gr}_j(\widetilde{\omega}^\bullet_{\mathfrak{Y}_\bullet})[j]) \cong R\Gamma_{\mathrm{rig}}(Y^{(j)}/K_0)$$

and hence if Y is proper we have, by proper descent [Tsu03], a canonical isomorphism

$$R\Gamma(]U_\bullet[_{\mathfrak{Y}_\bullet}, \ker\nu) \cong R\Gamma_{\mathrm{rig}}(Y).$$

But it is not clear that the isomorphism in Corollary 1.6.8, which passes through El Zein's isomorphism instead, coincides with this one.

Chapter 2

On a Conjecture of Flach and Morin for a Semistable Family over a Curve

2.1 Introduction

Notation 1.1. Throughout this chapter, k will be a finite field of characteristic $p > 0$ and $\mathcal{O}_F := W(k)$ the ring of Witt vectors of k with fraction field K. For any scheme X, we denote by $X^{\varnothing}$ the scheme X with the trivial log structure. By abuse of notation, when A is a ring we denote by $A^{\varnothing}$ the log-scheme $(\operatorname{Spec} A)^{\varnothing}$.

For a semistable k-log scheme Y, denote by $R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing})$ the cohomology with values in K-vector spaces $R\Gamma_{\text{log-crys}}(Y/\mathcal{O}_F^{\varnothing}) \otimes K$.

$\lhd$

Let k be a finite field of characteristic p and Y a proper semistable log scheme of dimension n over k. Let $\mathcal{O}_F := W(k)$ be the Witt ring of k. A conjecture of Flach and Morin [FM18, Conjecture 7.15] suggests the existence of an exact triangle in category of φ-modules

$$R\Gamma_{\mathrm{rig}}(Y) \to \left[R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(Y/\mathcal{O}_F^{\varnothing})(-1)\right] \to R\Gamma^*_{\mathrm{rig}}(Y)(-n-1)[-2n-1] \to$$

The motivation for this sequence is motivic. Here $R\Gamma^*_{\mathrm{rig}}$ denotes the motivic dual of rigid cohomology.

Rigid cohomology is finite-dimensional, so an abstract dual always exists, but it is noncanonical in general. However, when Y can be viewed as the closed subscheme of a proper k-scheme X, Poincaré duality [Ber97]

$$R\Gamma_{Y,\mathrm{rig}}(X) \cong R\Gamma_{\mathrm{rig}}(Y)(-n-1)[-2n-2],$$

where $R\Gamma_{Y,\mathrm{rig}}$ denotes rigid cohomology with support in Y, provides a geometric and canonical dual to rigid cohomology and the prospect of stating and proving the conjecture of Flach-Morin in this case.

One cannot help but see the resemblance between the Flach-Morin conjecture and the p-adic Clemens-Schmid exact sequence

$$\begin{aligned} \ldots \to H_{\mathrm{rig}}(X_s) \xrightarrow{\gamma} H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{O}_F^{\varnothing}) \otimes K \xrightarrow{N_m} \\ H^m_{\text{log-crys}}((X_s, M_s)/\mathcal{O}_F^{\varnothing}) \otimes K(-1) \xrightarrow{\delta} H^{m+2}_{X_s,\mathrm{rig}}(X) \xrightarrow{\alpha} H^{m+2}_{\mathrm{rig}}(X_s) \to \ldots \end{aligned}$$

whose existence and exactness Chiarellotto and Tsuzuki proved in [CT03] for a proper family $f : X \to C$ over a curve and $X_s = f^{-1}(s)$ the fiber of a k-rational point $s \in C$. In this chapter we follow their general method to prove the Flach-Morin conjecture in this setting. Namely, we will link the localization triangle for rigid cohomology with respect to the closed subscheme X_s with the canonical exact triangle for the monodromy operator in log-crystalline cohomology, using the fact that the rigid cohomology of an open complement of a closed subscheme can be computed using logarithmic structures.

Remark 1.2. As noted in [CT03, §2], there is no loss of generality in working with the ring of Witt vectors of k as opposed to the more general setting of a complete discrete valuation ring F with residue field k. Indeed all of our constructions are rational, that is, all of our constructions over the Witt ring are tensored by the fraction field K. Hence ramification does not affect the constructions or results and we can work under the assumption that F is absolutely unramified. $\lhd$

2.2 The Flach-Morin Conjecture for a Semistable Family over a Curve

Let k be a field of characteristic $p > 0$ and let

$$f : X \to C$$

be a proper and flat morphism over k where C is a smooth curve and X is a smooth variety of dimension $n+1$. Assume that for some k-rational point $s \in C$ the fiber X_s is a normal crossing divisor in X. We endow X with the log structure corresponding to the divisor X_s and to X_s itself the pullback log structure. Our goal is to prove the following variant of the conjecture of Flach and Morin [FM18, Conjecture 7.15]:

Proposition 2.1. *There is an exact triangle*

$$\begin{tikzcd}
 & R\Gamma^*_{\mathrm{rig}}(X_s)(-n-1)[-2n-1] \arrow[dl] & \\
R\Gamma_{\mathrm{rig}}(X_s) \arrow[rr] & & \left[R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing})(-1)\right] \arrow[ul]
\end{tikzcd}$$

in the derived category of φ-modules. $\lhd$

To describe the dual $R\Gamma^*_{\mathrm{rig}}(X_s)$ we recall the definition of cohomology with support in a closed subset (see [Ber97] or [Le 07, Definition 6.3.1]). For the moment we ignore log structures. Recall the following definition:

Definition 2.2. Let X be a k-scheme, $Z \subseteq X$ a closed subscheme, and fix a frame $(X \subseteq \overline{X} \subseteq P)$. Then the *rigid cohomology of X with support in Z* is defined to be

$$R\Gamma_{Z,\mathrm{rig}}(X) = R\Gamma(]\overline{X}[_P, \underline{\Gamma}^\dagger_{]Z[_P} j^\dagger_X \Omega^\bullet_{]\overline{X}[_P}).$$

$\lhd$

Its relation to standard rigid cohomology is the following. By abstract nonsense the functor $\underline{\Gamma}^\dagger_{]Z[_P}$ satisfies

$$0 \to \underline{\Gamma}^\dagger_{]Z[_P} \mathcal{E} \to \mathcal{E} \to j^\dagger_X \mathcal{E} \to 0$$

for any sheaf $\mathcal{E}$ on $]\overline{X}[_P$. In particular, if we denote by $U = X \setminus Z$ the open complement of X in Z and use the fact that $j^\dagger_U \circ j^\dagger_X = j^\dagger_U$ (see [Le 07, Proposition 5.1.11]) we obtain a natural exact sequence

$$0 \to \underline{\Gamma}^\dagger_{]Z[_P} j^\dagger_X \Omega^\bullet_{]\overline{X}[_P} \to j^\dagger_X \Omega^\bullet_{]\overline{X}[_P} \to j^\dagger_U \Omega^\bullet_{]\overline{X}[_P} \to 0.$$

This induces in cohomology the *localization triangle*

$$R\Gamma_{Z,\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(U) \to .$$

On the other hand, by definition [HK94, §1.6] the monodromy operator on $R\Gamma_{\text{log-crys}}(Z/\mathcal{O}_F^{\varnothing})$ is defined to be limit of the connecting homomorphisms of the exact sequence

$$0 \to W_n\omega^\bullet_Z[-1] \xrightarrow{\wedge\theta} W_n\widetilde{\omega}^\bullet_Z \to W_n\omega^\bullet_Z \to 0.$$

Log-crystalline cohomology is defined to be

$$R\Gamma_{\text{log-crys}}(Z/\mathcal{O}_F^{\varnothing}) \overset{\text{def}}{\cong} \varprojlim R\Gamma(W_n\omega^\bullet_Z) \qquad R\Gamma_{\text{log-crys}}(Z^{\varnothing}/\mathcal{O}_F^{\varnothing}) \overset{\text{def}}{\cong} \varprojlim R\Gamma(W_n\widetilde{\omega}^\bullet_Z)$$

where $Z^\varnothing$ denotes Z with the trivial log structure [HK94, §1.3] But for Z proper the log-crystalline cohomology over the trivial log structure coincides with convergent cohomology, so that we obtain an exact triangle

$$R\Gamma_{\mathrm{conv}}(Z) \to R\Gamma_{\mathrm{HK}}(Z/\mathcal{O}_F^\varnothing) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(Z/\mathcal{O}_F^\varnothing)(-1) \to .$$

Our proof of Proposition 2.2.1 consists of combining these two exact triangles via results connecting the cohomologies of X, X_s, and $X \setminus X_s$.

Proof of Proposition 2.2.1. Note that since X and X_s are proper, rigid cohomology coincides with convergent cohomology. Our setting is the derived category of φ-modules, and it will be implicit that all of the quasi-isomorphisms below are compatible with the Frobenius when the objects have a nontrivial φ-module structure.

As a first step, Chiarellotto and Tsuzuki show ([CT03, Proposition 4.1]) that the mapping fiber

$$R\Gamma_{X_s,\mathrm{rig}} \cong [R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\mathrm{rig}}(U)]$$

corresponding to the localization triangle can be interpreted as

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong [R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\mathrm{almostconv}}((X \setminus X_s))]$$

where $R\Gamma_{\mathrm{almostconv}}((X \setminus X_s))$, denoted by $R\Gamma_{\mathrm{rig}}((X \setminus X_s, X))$ in [CT03, §4], is overconvergent along X_s and convergent along the rest of X.

A result of Shiho [Shi02, Proposition 2.4.4] says that the non-logarithmic object $R\Gamma_{\mathrm{almostconv}}((X \setminus X_s))$ can be interpreted in terms of logarithmic structures. Intuitively, it says that the poles introduced by the overconvergence over X_s are, in cohomology, exactly those captured by the logarithmic structure on X induced by the normal crossing divisor X_s. More precisely, it implies an isomorphism

$$R\Gamma_{\mathrm{rig}}((X \setminus X_s, X)) \cong R\Gamma_{\text{log-conv}}(X/\mathcal{O}_F^\varnothing)$$

so that the rigid cohomology with support in X_s is computed as the mapping cone

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong [R\Gamma_{\mathrm{conv}}(X) \to R\Gamma_{\text{log-conv}}(X/\mathcal{O}_F^\varnothing)]$$

Finally, in their remarks after [CT03, Lemma 4.6] Chiarellotto and Tsuzuki prove that the respective cohomologies in the mapping fiber can be computed on their restrictions to the divisor X_s, i.e.

$$R\Gamma_{\mathrm{conv}}(X) \cong R\Gamma_{\mathrm{conv}}(X_s)$$
$$R\Gamma_{\text{log-conv}}(X/\mathcal{O}_F^\varnothing) \cong R\Gamma_{\text{log-conv}}(X_s/\mathcal{O}_F^\varnothing)$$

so that in the end we are rewarded with an exact triangle

$$R\Gamma_{X_s,\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(X_s) \to R\Gamma_{\mathrm{conv}}(X_s). \tag{2.1}$$

As we remarked previously, by definition there is an exact triangle

$$R\Gamma_{\mathrm{conv}}(X_s) \to R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing})(-1) \to$$

for the monodromy operator on log-crystalline cohomology; in other words, there is an isomorphism

$$R\Gamma_{\mathrm{conv}}(X_s) \cong [R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing})(-1)]$$

computing convergent cohomology of X_s as the mapping cone of the monodromy operator. Plugging into (2.1) we obtain an exact triangle

$$R\Gamma_{X_s,\mathrm{rig}}(X) \to R\Gamma_{\mathrm{rig}}(X_s) \to [R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing})(-1)] \to$$

Finally, Poincaré duality [Ber97, Théoréme 2.4] (see [CL99, §2.1] for details on the Frobenius action) provides a canonical isomorphism

$$R\Gamma_{X_s,\mathrm{rig}}(X) \cong R\Gamma_{\mathrm{rig}}(X_s)^*(-n-1)[-2n-2]$$

so after substitution and shifting the triangle we obtain an exact triangle

$$\begin{tikzcd}
 & R\Gamma_{\mathrm{rig}}(X_s)^*(-n-1)[-2n-1] \arrow[dl] & \\
R\Gamma_{\mathrm{rig}}(X_s) \arrow[rr] & & \left[R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing}) \xrightarrow{N} R\Gamma_{\mathrm{HK}}(X_s/\mathcal{O}_F^{\varnothing})(-1)\right] \arrow[ul]
\end{tikzcd}$$

of φ-modules, as desired. □

Remark 2.3. It is important to notice that the Clemens-Schmid exact sequence is significantly stronger than the Flach-Morin conjecture in this setting. The two exact triangles we worked with above can be fitted together into the Clemens-Schmid exact sequence just as we did in the above proof. However, the Flach-Morin conjecture proven above is simply an intertwining of two exact triangles, while the Clemens-Schmid exact sequence involves the cohomology groups directory. In particular, the *exactness* of the resulting complex requires much deeper ideas, including comparisons between the weight and monodromy filtration on the relative cohomology of X over the curve; see for example [CT03, Theorem 5.1]. So while the use of the curve C may have a more formal character, the proof of exactness concretely requires the use of the curve C and the embedding of X_s as the fiber over a point. ◁

Remark 2.4. Of course a scheme Y in characteristic p can be embedded as a closed subscheme in other ways and these suggest further directions for research. For example, Y may be the fiber of a 1-dimensional *arithmetic* family, such as a discrete valuation ring of mixed or equal characteristic. In general the Clemens-Schmid exact sequence is connected only with logarithmic objects in characteristic p. It may be interesting to consider Y as as the special fiber of a scheme X over $\mathcal{V}$ where $\mathcal{V}$ is a complete discrete valuation ring with residue field k and to give meaning to the cohomology

$$R\Gamma_Y(X)$$

with support in the special fiber. Since the above proof, in contrast to the exactness of the Clemens-Schmid exact sequence, uses the curve C in a more formal way it may be possible to extend the proof to this situation if such a cohomology is defined. One could also study a family over a discrete valuation ring of equicharacteristic p, for example when Y is the special fiber of a scheme X over $k[\![t]\!]$. Rigid cohomology over such a Laurent series has a more analytic flavor. It is defined to be a functor

$$X \mapsto H^*_{\mathrm{rig}}(X/\mathscr{E}_K)$$

taking values in graded vector spaces over the Amice ring

$$\mathscr{E}_K := \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in K[\![t, t^{-1}]\!] \,:\, \sup_i |a_i| < \infty, a_i \to 0 \text{ as } i \to -\infty \right\}.$$

To study these objects one can study instead rigid cohomology over the bounded Robba ring $X \mapsto H^*_{\mathrm{rig}}(X/\mathscr{E}_K^\dagger)$, where

$$\mathscr{E}_K^\dagger = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in K[\![t, t^{-1}]\!] \,:\, \sup_i |a_i| < \infty, \exists \eta < 1 \,\text{s.\,t.}\, |a_i|\eta^i \to 0 \text{ as } i \to -\infty \right\}$$

The bounded Robba ring has the additional virtue that it is a Henselian discrete valued field with residue field $k((t))$. This cohomology theory is constructed so that when we base change to $\mathscr{E}_K$ one recovers $\mathscr{E}_K$-valued rigid cohomology (see [LP16, §2.2] for details). This is a direction for further research. ◁

Chapter 3

A Hodge-Type Filtration on Rigid Cohomology

3.1 Introduction

Let $\mathcal{V}$ be a discrete valuation ring with a perfect residue field k of characteristic p and fraction field K. For any algebraic k-variety X_k, Berthelot [Ber86] defines the rigid cohomology groups $H^n_{\mathrm{rig}}(X_k)$ by embedding X_k within a K-frame

$$X_k \hookrightarrow Y_k \hookrightarrow P_{\mathcal{V}}$$

where $X_k \hookrightarrow Y_k$ is an open immersion and $Y_k \hookrightarrow P_{\mathcal{V}}$ is a closed immersion of Y_k into a formal $\mathcal{V}$-scheme $P_{\mathcal{V}}$, smooth in a neighborhood of X_k. In this context he constructs the complex of overconvergent differential forms $j^\dagger_{X_k}\Omega^\bullet_{]Y[_P}$ and defines

$$H^n_{\mathrm{rig}}(X_k) := H^n(]Y_k[_P, j^\dagger_{X_k}\Omega^\bullet_{]Y[_P}).$$

This construction is shown to be independent of the choice of K-frame.

Following a construction introduced in [Gro94] in the crystalline setting we define a filtration $\mathrm{Fil}^s \subseteq j^\dagger_X\Omega^\bullet_{]Y[_P}$ of complexes of $j^\dagger_X\mathcal{O}_{]Y[_P}$-modules, which we call the *Gros filtration*, and define an induced filtration

$$F^s H^n_{\mathrm{rig}}(X_k) := \mathrm{Im}(H^n(]Y_k[_P, \mathrm{Fil}^s) \to H^n_{\mathrm{rig}}(X_k)) \subseteq H^n_{\mathrm{rig}}(X_k)$$

on rigid cohomology. Our goal in this chapter is to prove the independence of the filtration Fil^s in the derived category.

3.2 The Gros Filtration

Fix an algebraic k-variety X_k. Locally we may embed it inside a K-frame $X_k \to Y_k \to \widehat{P}$ arising from an *algebraic $\mathcal{V}$-frame*, i.e. a sequence of embeddings

$$X_{\mathcal{V}} \hookrightarrow Y_{\mathcal{V}} \hookrightarrow P_{\mathcal{V}}$$

where $X_{\mathcal{V}}, Y_{\mathcal{V}}$, and $P_{\mathcal{V}}$ are $\mathcal{V}$-schemes, both $Y_{\mathcal{V}}$ and $P_{\mathcal{V}}$ are proper, and $Y_{\mathcal{V}}$ is closed in $P_{\mathcal{V}}$. For brevity, we'll write $]X[_P :=]X_k[_{\widehat{P}_K}$ and $]Y[_P :=]Y_k[_{\widehat{P}_K}$ when there's no risk of confusion. Let $I = I_{X,Y,P}$ be the ideal defining the closed subspace $\widehat{Y}_K$ in $]Y[_P$, that is, the kernel

$$0 \to I \to \mathcal{O}_{]Y[_P} \to i_* \mathcal{O}_{\widehat{Y}_K} \to 0.$$

Finally, recall that the dagger functor $j_X^\dagger : \mathrm{Sh}(]Y[_P) \to \mathrm{Sh}(]Y[_P)$ where $\mathrm{Sh}(]Y[_P)$ can denote sheaves of sets, abelian groups, or $\mathcal{O}_{]Y[_P}$-modules (see, for example, [Le 07, §5.1]) is defined to be

$$j_X^\dagger \mathcal{E} = \varinjlim j_{V*} j_V^{-1} \mathcal{E}$$

where the injective limit runs over all strict neighborhoods of $]X[_P$ in $]Y[_P$.

The sheaf computing rigid cohomology is the overconvergent de Rham complex $j_X^\dagger \Omega^\bullet_{]Y[_P}$, and rigid cohomology is defined to be

$$H^n_{\mathrm{rig}}(X_k) := H^n(]Y[_P, j_X^\dagger \Omega^\bullet_{]Y[_P})$$

Our goal in this chapter is to study the following filtrations on $j_X^\dagger \Omega^\bullet_{]Y[_P}$ and the corresponding filtration in cohomology:

Definition 2.1.

1. The *Gros filtration* on $j_X^\dagger \Omega^\bullet_{]Y[_P}$ is given by

   $$\mathrm{Fil}^s = \mathrm{Fil}^s_{X,Y,P} := j_X^\dagger (I^s \to I^{s-1}\Omega^1_{]Y[_P} \to I^{s-2}\Omega^2_{]Y[_P} \to \dots) = j_X^\dagger (I^{s-\bullet} \otimes \Omega^\bullet_{]Y[_P})$$

2. The induced filtration

   $$F^s H^n_{\mathrm{rig}}(X_k) := \mathrm{Im}(H^n(]Y_k[_P, \mathrm{Fil}^s) \to H^n_{\mathrm{rig}}) \subseteq H^n_{\mathrm{rig}}(X_k)$$

   we call the *Hodge-type filtration* on rigid cohomology.

$\triangleleft$

The term Hodge-type filtration on rigid cohomology is justified in part by the following special cases.

Example 2.2. Suppose X_k admits a frame of the form $(X \subseteq Y \subseteq P) = (X_k \subseteq Y_k \subseteq \widehat{Y}_{\mathcal{V}})$ where $Y_{\mathcal{V}}$ is a $\mathcal{V}$-model for Y_k and $\widehat{Y}_{\mathcal{V}}$ denotes its p-adic completion. Then this filtration collapses into the usual "naive" filtration. Indeed, if such a frame exists then with respect to this frame we have

$$]Y[_P \cong \widehat{Y}_K$$

so that $I_{X,Y,P} = 0$. Then

$$\begin{aligned}(\mathrm{Fil}^s_{X,Y,P})^i &= j_X^\dagger (I^{s-i} \otimes \Omega^i_{]Y[_P}) \\ &= \begin{cases} 0 & \text{when } i \leq s \\ j_X^\dagger \Omega^i_{]Y[_P} & \text{when } i > s \end{cases}\end{aligned}$$

which is the filtration inducing the "naive" filtration.

This happens for instance X_k is affine smooth or when it has a proper and smooth model $X_\mathcal{V}$. In the latter case the trivial $\mathcal{V}$-frame

$$X_\mathcal{V} \hookrightarrow X_\mathcal{V} \hookrightarrow X_\mathcal{V}$$

satisfies the hypothesis. If $X_k \cong \mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{a})$ is smooth and affine, in which case rigid cohomology coincides with Monsky-Washnitzer cohomology, X_k has a smooth lifting

$$X_\mathcal{V} \cong \mathrm{Spec}(\mathcal{V}[x_1, \ldots, x_n]/\tilde{\mathfrak{a}}).$$

Then if $Y_\mathcal{V} = \overline{X_\mathcal{V}} \subseteq \mathbb{P}^n_\mathcal{V}$ denotes the closure of $X_\mathcal{V}$ in $\mathbb{P}^n_\mathcal{V}$ then the algebraic $\mathcal{V}$-frame

$$X_\mathcal{V} \hookrightarrow Y_\mathcal{V} = Y_\mathcal{V}$$

induces a frame satisfying the condition. $\triangleleft$

Berthelot shows that $H^n_{\mathrm{rig}}(X_k)$ is independent of the frame $(X_k \subseteq Y_k \subseteq P_\mathcal{V})$ in the derived category. In order to use this filtration to induce a well-defined filtration on rigid cohomology, we need to show that this filtration is independent of our choices in the derived category as well. More precisely, our goal is to show the following:

Theorem 2.3. *Suppose we have a morphism of algebraic $\mathcal{V}$-frames*

$$\begin{tikzcd} & \widetilde{Y}_\mathcal{V} \arrow[r, hook] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] \\ X_k \arrow[ur, hook] \arrow[dr, hook] & & \\ & Y_\mathcal{V} \arrow[r, hook] & P \end{tikzcd}$$

where g is proper and separated and u is smooth. Then the filtration induced by these frames are isomorphic, that is, the natural base change map

$$\mathrm{Fil}^s_{X,Y,P} \cong Ru_{K*}\,\mathrm{Fil}^s_{X,\tilde{Y},\tilde{P}}$$

is an isomorphism.

This result should be thought of as a variant of the independence of rigid cohomology proven by Berthelot and detailed by Le Stum in [Le 07]:

Proposition 2.4. (Le Stum) *Let*

$$\begin{tikzcd}
 & \widetilde{Y}_{\mathcal{V}} \arrow[r, hook] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] \\
X_k \arrow[ur, hook] \arrow[dr, hook] & & \\
 & Y_{\mathcal{V}} \arrow[r, hook] & P
\end{tikzcd}$$

be a proper smooth morphism of S-frames. If E is a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module with an integrable conection over S_K, the base change map is an isomorphism

$$v* : E \otimes_{]Y[_P} \Omega^\bullet_{]Y[_P/S_K} \cong Ru_{K*}u^\dagger E \otimes_{\mathcal{O}_{]\tilde{Y}[_{\widetilde{P}}}} \Omega^\bullet_{]\tilde{Y}[_{\tilde{P}}/S_K}.$$

$\lhd$

Notation. In the above scenarios the notation $j_X^\dagger$ is ambiguous since $j_X^\dagger := j^\dagger_{]X[_P}$ depends on the formal embedding $X \hookrightarrow P$. When there's risk of confusion we'll write $j_P^\dagger := j^\dagger_{]X[_P}$ for the dagger functor corresponding to a frame $(X \subseteq Y \subseteq P)$.

Our argument for the independence of the filtration follows closely Berthelot's argument for the above. It is a series of reductions concluding in an explicit computation of the filtration; namely, one reduces the general result to the case where (1) g is proper and u is étale, and (2) $g = \mathrm{id}_Y$ and u is smooth. Likewise, two special cases to which our result is reduced are the following:

Lemma 2.5. *Suppose*

$$\begin{tikzcd}
 & \tilde{Y} \arrow[r, hook] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] \\
X_k \arrow[ur, hook] \arrow[dr, hook] & & \\
 & Y \arrow[r, hook] & P
\end{tikzcd}$$

is a proper étale morphism of frames (recall that this hypothesis means that g is proper and u is étale). Then

$$\mathrm{Fil}^s_{X,Y,P} \xrightarrow{\sim} Ru_* \, \mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}} \, .$$

$\lhd$

Theorem 2.6. (Global Filtered Poincaré Lemma) *Let*

$$\begin{tikzcd} & & P' \arrow[dd, "u"] \\ X \arrow[r, hook, "\circ"] & Y \arrow[ur, hook, "\vert"] \arrow[dr, hook, "-"'] & \\ & & P \end{tikzcd}$$

be a smooth morphism of frames, and let u_K denote the the induced map of tubes $u_K :]Y[_{P'} \to]Y[_P$. Then there is a quasi-isomorphism

$$j^\dagger_P I^\ell_{X,Y,P} \xrightarrow{\sim} Ru_{K*} j^\dagger_{P'} (I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P}).$$

Moreover, $\mathrm{Fil}^s_{X,Y,P} \cong Ru_* \, \mathrm{Fil}^s_{X,Y,P'}$.

We'll return to these special cases in §3.3 and §3.4, respectively. For now we'll prove the main result supposing that they are given.

Proof of Theorem 3.2.3. By Chow's lemma [GR71, Corollary 5.7.14] we may blow up a closed subvariety of Y' outside X in P', and similarly for P', and obtain diagram

$$\begin{tikzcd} & \widetilde{Y}' \arrow[r, hook] \arrow[d, "f"] & \widetilde{P}' \arrow[d, "v"] \\ X \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \widetilde{Y} \arrow[r, hook] \arrow[d, "g"] & \widetilde{P} \arrow[d, "u"] \\ & Y \arrow[r, hook] & P \end{tikzcd}$$

where $f : \widetilde{Y}' \to \widetilde{Y}$ and $v : \widetilde{P}' \to \widetilde{P}$ are blow-ups and $g \circ f$ is projective. Because g is separated, it follows from [Sta22, Lemma 0C4Q] that f is projective. This reduces the theorem to the case where g is projective and u is smooth. Indeed, if the theorem holds in this special case then since $g \circ f$ and f are projective in the above diagram we can infer isomorphisms

$$\mathrm{Fil}^s_{X,Y,P} \cong R(u \circ v)_* \, \mathrm{Fil}^s_{X,\widetilde{Y}',\widetilde{P}'} \qquad \text{and} \qquad \mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}} \cong Rv_* \, \mathrm{Fil}^s_{X,\widetilde{Y}',\widetilde{P}'}$$

for their respective frames. The result for our original frame then follows immediately by composing the latter with Ru_*.

Having reduced it to the case where g is projective, we can further reduce it to the case that u is the projection $u : \mathbb{P}^n_P \to P$ as follows. From the projectivity of the map $g : \widetilde{Y} \to Y$ we get a map $\varphi : \widetilde{Y} \to \mathbb{P}^n_P$ for some n. If we embed $\widetilde{Y} \hookrightarrow \mathbb{P}^n_P \times P'$ diagonally using φ and the given morphism $\widetilde{Y} \to \widetilde{P}$ we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^n_P \times \widetilde{P} \arrow[d, "\pi_2"] \arrow[dr, "\pi_1"] & \\
 & \widetilde{Y} \arrow[ur, hook] \arrow[r, hook] \arrow[rr, bend right] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] & \mathbb{P}^n_P \arrow[ddl, "\pi_P"] \\
X_k \arrow[ur, hook] \arrow[dr, hook] & & & \\
 & Y \arrow[r, hook] & P &
\end{tikzcd}$$

By the Global Filtered Poincaré Lemma (Theorem 3.2.6) we have

$$\begin{aligned}
\mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}} &\cong R\pi_{2*} \mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P \times \widetilde{P}} \\
\mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P} &\cong R\pi_{1*} \mathrm{Fil}_{X,\widetilde{Y},\mathbb{P}^n_P \times \widetilde{P}}
\end{aligned}$$

and hence

$$\begin{aligned}
Ru_* \mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}} &\cong R(u \circ \pi_2)_* \mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P \times \widetilde{P}} \\
&\cong R(\pi_P \circ \pi_1)_* \mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P \times \widetilde{P}} \\
&\cong R\pi_{P*} \mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P} .
\end{aligned}$$

Thus to show that $\mathrm{Fil}^s_{X,Y,P} \cong Ru_* \mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}}$ it suffices to prove that $\mathrm{Fil}^s_{X,Y,P} \cong R\pi_{P*} \mathrm{Fil}^s_{X,\widetilde{Y},\mathbb{P}^n_P}$.

Thus we find ourselves in the setting

$$\begin{tikzcd}
 & \tilde{Y} \arrow[r, hook] \arrow[dd, "g"] & \mathbb{P}^n_P \arrow[dd, "u"] \\
X_k \arrow[ur, hook] \arrow[dr, hook] & & \\
 & Y \arrow[r, hook] & P
\end{tikzcd}$$

where g is projective and u is the canonical projection (smooth around X_k).

We may further find a closed subscheme $\widetilde{Y}' \subseteq \widetilde{Y}$ and $\widetilde{P}'$ such that the composition of frames

$$\begin{tikzcd}
 & \widetilde{Y}' \arrow[r] \arrow[d, hook, "/" marking] & \widetilde{P}' \arrow[d, hook, "/" marking] \\
X_k \arrow[ur, hook, "\circ" marking] \arrow[r, hook, "\circ" marking] \arrow[dr, hook, "\circ" marking] & \widetilde{Y} \arrow[r] \arrow[d, "g"] & \mathbb{P}^n_P \arrow[d, "u"] \\
 & Y \arrow[r, hook, "/" marking] & P
\end{tikzcd}$$

is proper étale. Indeed, note that the closed immersion $X \hookrightarrow p^{-1}(X) = \mathbb{P}^n_X$ is a section of the canonical projection which is smooth. It follows that there exists a covering of $\mathbb{P}^n_X$ by open sets U defined in X by a regular sequence $(\tilde{t}_1, \ldots, \tilde{t}_d)$, induced by sections $t_1, \ldots, t_d \in \Gamma(\mathbb{P}^n_P, \mathcal{O}(n))$. We may assume that $U = D^+(s) \cap u^{-1}(Y_k)$. It then suffices to take $\widetilde{P}' := V(\tilde{t}_1, \ldots, \tilde{t}_d)$ and $\widetilde{Y}' := \widetilde{Y} \cap \widetilde{P}'$.

The theorem holds for the upper morphisms of frames by a combination of Theorem 3.2.6 and Lemma 3.2.5. Therefore, as before, to prove the theorem for the lower morphism of frames it suffices to prove it for the outermost one. Thus we've reduced our theorem to a morphism

$$\begin{tikzcd}
 & \tilde{Y} \arrow[r, hook, "/" marking] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] \\
X_k \arrow[ur, hook, "\circ" marking] \arrow[dr, hook, "\circ" marking] & & \\
 & Y \arrow[r, hook, "/" marking] & P
\end{tikzcd}$$

which is proper étale. But this is precisely the content of Lemma 3.2.5. $\square$

3.3 The Proper Étale Case

In this section we prove Lemma 3.2.5 [how to repeat theorem numbers??]:

Lemma 3.1. *Suppose*

$$\begin{tikzcd} & \tilde{Y} \arrow[r, hook] \arrow[dd, "g"] & \widetilde{P} \arrow[dd, "u"] \\ X_k \arrow[ur, hook] \arrow[dr, hook] & & \\ & Y \arrow[r, hook] & P \end{tikzcd}$$

is a proper étale morphism of frames (recall that this hypothesis means that g is proper and u is étale). Then

$$\mathrm{Fil}^s_{X,Y,P} \xrightarrow{\sim} Ru_* \,\mathrm{Fil}^s_{X,\widetilde{Y},\widetilde{P}}\,.$$

$\lhd$

We first prove a special case:

Lemma 3.2. *Given a morphism of algebraic $\mathcal{V}$-frames*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y' \arrow[r] \arrow[d, hook, "g"] & P \arrow[dl] \\ & Y & \end{tikzcd}$$

where g is a closed immersion, the natural open immersion of tubes $]Y'[_P \hookrightarrow\,]Y[_P$ induces an isomorphism $j_X^\dagger \mathcal{O}_{]Y'[_P} \cong j_X^\dagger \mathcal{O}_{]Y[_P}$. Moreover, it gives an isomorphism of complexes

$$\mathrm{Fil}^s_{X,Y,P} \xrightarrow{\sim} \mathrm{Fil}^s_{X,Y',P}$$

$\lhd$

Proof. The claim is local on P and X, so we may suppose that P is affine and X_k is the complement of a hypersurface of Y_k defined by a function f on Y_k. Of course since $X_k \subseteq Y'_k$ we have $Y_k = Y'_k \cup (Y_k \setminus X_k)$. Take a lifting $\tilde{f}$ of f. We then have that

$$\widehat{Y}_K = \widehat{Y}'_K \cup V(\tilde{f})$$

as a closed analytic subspace of the tube $]Y[_P$.

On the other hand, recall from [Le 07, Corollary 3.3.3] that the admissible open subsets

$$V^\lambda :=]Y[_P \backslash \{|\tilde{f}| < \lambda\}$$

for $\lambda \xrightarrow{<} 1$ form a cofinal system of strict neighborhoods of $]X[_P$ in P_K. Furthermore, by [Le 07, Proposition 3.3.4] we have that $\{V^\lambda \cap]Y'[_P\}_\lambda$ still form a cofinal system of strict neighborhoods of $]X[_P$ in P_K (in both $]Y'[_P$ and $]Y[_P$).

Since $|\tilde{f}| \geq \lambda > 0$ on V^λ, we have that $V(\tilde{f}) \cap V^\lambda = \varnothing$ and hence

$$\widehat{Y}_K \cap V^\lambda = \widehat{Y}'_K \cap V^\lambda$$

for all λ. Intersecting with $]Y'[_P$, we have

$$\widehat{Y}_K \cap (V^\lambda \cap]Y'[_P) = \widehat{Y}'_K \cap (V^\lambda \cap]Y'[_P) \hookrightarrow V^\lambda \cap]Y'[_P.$$

But the ideals of the immersions

$$\widehat{Y}_K \cap (V^\lambda \cap]Y'[_P) \hookrightarrow V^\lambda \cap]Y'[_P$$
$$\widehat{Y}'_K \cap (V^\lambda \cap]Y'[_P) \hookrightarrow V^\lambda \cap]Y'[_P$$

are respectively the restrictions of $I_{X,Y,P}$ and $I_{X,Y',P}$ to the strict neighborhood $V^\lambda \cap]Y'[_P$. That these two immersions are identical means that

$$I_{X,Y,P|(V^\lambda \cap]Y'[_P)} = I_{X,Y',P|(V^\lambda \cap]Y'[_P)}$$

Since $j_X^\dagger$ is the limit over the cofinal system $(V^\lambda \cap]Y'[_P)_\lambda$, we have our lemma. □

In addition, we provide for completion a proof of a fact that is well-known:

Fact 3.3. Let $(X \subseteq Y \subseteq P)$ be a frame and let $f : \mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet$ be a morphism of complexes of overconvergent sheaves in $D(]Y[_P)$. If there exists a strict neighborhood V of $]X[_P$ in $]Y[_P$ such that $f|_V$ is a quasi-isomorphism, then f is a quasi-isomorphism. ⊲

Proof. The category of overconvergent modules has enough injectives and the construction of injective resolutions is functorial, so there exist injective resolutions $\mathcal{E}_1^\bullet \xrightarrow{\sim} \mathcal{I}_1^\bullet$ and $\mathcal{E}_2^\bullet \xrightarrow{\sim} \mathcal{I}_2^\bullet$ and a morphism $g : \mathcal{I}_1^\bullet \to \mathcal{I}_2^\bullet$ such that

$$\begin{tikzcd}
\mathcal{E}_1^\bullet \arrow[r, "f"] \arrow[d, "\cong"] & \mathcal{E}_2^\bullet \arrow[d, "\cong"] \\
\mathcal{I}_1^\bullet \arrow[r, "g"] & \mathcal{I}_2^\bullet
\end{tikzcd}$$

commutes.

If a sheaf $\mathcal{F}$ is overconvergent then it trivially satisfies the universal property of $j_X^\dagger \mathcal{F}$ so $\mathcal{F} = j_X^\dagger \mathcal{F}$. Hence we have $\mathcal{I}_1^\bullet = j_X^\dagger \mathcal{I}_1^\bullet$ and similarly for $\mathcal{I}_2^\bullet$. Furthermore, restriction to V is functorial and we can check that a morphism φ is a quasi-isomorphism by checking that $R\Gamma(W, \varphi)$ is a quasi-isomorphism for all affinoid open subsets of $]Y[_P$.

Putting this together, let W be an affinoid open, which in particular is a quasi-compact admissible open subset of $]Y[_P$. We have

$$
\begin{tikzcd}
R\Gamma(W, \mathcal{E}_1^\bullet) \arrow[r, "R\Gamma(f)"] \arrow[d, "\cong"] & R\Gamma(W, \mathcal{E}_2^\bullet) \arrow[d, "\cong"] \\
\Gamma(W, \mathcal{I}_1^\bullet) \arrow[r, "R\Gamma(g)"] \arrow[d, "="] & \Gamma(W, \mathcal{I}_2^\bullet) \arrow[d, "="] \\
\Gamma(W, j_X^\dagger \mathcal{I}_1^\bullet) \arrow[r] \arrow[d, "="] & \Gamma(W, j_X^\dagger \mathcal{I}_2^\bullet) \arrow[d, "="] \\
\varinjlim_{V' \subset]Y[_P} \Gamma(W \cap V', \mathcal{I}_1^\bullet) \arrow[r] \arrow[d, "="] & \varinjlim_{V' \subset]Y[_P} \Gamma(W \cap V', \mathcal{I}_2^\bullet) \arrow[d, "="] \\
\varinjlim_{V' \subset]Y[_P} \Gamma(W \cap V', \mathcal{I}_{1|V}^\bullet) \arrow[r, "\cong"] & \varinjlim_{V' \subset]Y[_P} \Gamma(W \cap V', \mathcal{I}_{2|V}^\bullet)
\end{tikzcd}
$$

where the last morphism (equal to $R\Gamma(W, g|_V)$) is a quasi-isomorphism since $f|_V$ is a quasi-isomorphism by assumption. This gives the result. $\square$

Proof of Lemma 3.2.5. We first show that it suffices to provide strict neighborhoods V' and V of Y' and Y, respectively, such that u_K restricts to a morphism $u_K : V' \to V$ and the vertical morphisms in the diagram

$$
\begin{tikzcd}
\widehat{Y}'_K \cap V' \arrow[r, hook] \arrow[d, "u_K"] & V' \arrow[d, "u_K"] \\
\widehat{Y}_K \cap V \arrow[r, hook] & V
\end{tikzcd}
$$

are isomorphisms. To see this, note the top (resp. bottom) closed immersion is the restriction to V' (resp. V) of the closed immersion

$$
\widehat{Y}'_K \hookrightarrow]Y'[_{P'}
$$
$$
(\text{resp. } \widehat{Y}_K \hookrightarrow]Y[_P)
$$

whose ideals of definition is $I_{P'} := I_{X,Y',P'}$ (resp. $I_{P'} := I_{X,Y',P'}$). [Le 07, Proposition 6.2.2] tells us that

$$
Ru_{K*} \operatorname{Fil}^s_{X,Y',P'} =: Ru_{rig}(j^\dagger_{P'} I^{s-\bullet}_{P'})
$$

is overconvergent and that

$$
(Ru_{K*} \operatorname{Fil}^s_{X,Y',P'})|_V = Ru_{K*}(j^\dagger_{P'} I^{s-\bullet}_{P'} \otimes \Omega^\bullet_{]Y'[_{P'}})|_{V'}.
$$

But u_K is an isomorphism so in particular its own derived functor, and restiction commutes with dagger functors ([Le 07, Proposition 5.1.5]) so

$$(Ru_{K*}\operatorname{Fil}^s_{X,Y',P'})|_V = u_{K*}(j^\dagger_{P'}(I^{s-\bullet}_{P'}|_{V'}) \otimes \Omega^\bullet_{V'})$$

But the fact that the above vertical maps are isomorphisms means precisely that $I^{s-\bullet}_P|_V \cong u_{K*}(I^{s-\bullet}_{P'}|_{V'})$, since the closed immersions are the restrictions to V and V' of the closed immersions defining the ideals I_P and $I_{P'}$, respectively.

This implies that

$$\begin{aligned}\operatorname{Fil}^s_{X,Y,P}|_V &= j^\dagger_P(I^{s-\bullet}_P|_V) \otimes \Omega^\bullet_V \\ &\cong u_{K*}(j^\dagger_{P'}(I^{s-\bullet}_{P'}|_{V'}) \otimes \Omega^\bullet_{V'}) \\ &\cong (Ru_{K*}\operatorname{Fil}^s_{X,Y',P'})|_V\end{aligned}$$

and by Fact 3.3.3 this is enough to conclude the lemma.

Now we move onto proving the claim. Proper maps are in particular closed, so we have a factorization

$$\begin{tikzcd}
 & Y' \arrow[r, hook] \arrow[d, two heads, "g"] & P' \arrow[d, "u"] \\
X_k \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \operatorname{Im}(g) \arrow[r, hook] \arrow[d, hook] & P \\
 & Y \arrow[ur, hook] &
\end{tikzcd}$$

where $\operatorname{Im}(g)$ is the schematic image of g. We know from Lemma 3.3.2 that

$$\operatorname{Fil}^\bullet_{X,Y,P} = \operatorname{Fil}^\bullet_{X,\operatorname{Im}(g),P}$$

so we may replace Y by $\operatorname{Im}(g)$ and thus suppose that g is proper surjective.

We know from the proof of [Le 07, Proposition 3.4.12] and [Le 07, Proposition 3.3.11] that given a proper étale morphism of frames we may find isomorphic stright neighborhoods $V' \xrightarrow{u_K} V$ as follows. [Le 07, Proposition 3.3.11] tells us that for a fixed sequence $\eta_n \xrightarrow{<} 1$ we can find $\delta_n \xrightarrow{<} 1$ and $\lambda_n \xrightarrow{<} 1$ such that, for each $n \in \mathbb{N}$, the morphism u_K induces an isomorphism

$$V'_n := u_K^{-1}([Y]_{P\eta_n}) \cap V'^{\lambda_n}_{\eta_n} = [Y']_{P'\delta_n} \cap u_K^{-1}(V^{\lambda_n}_{\eta_n}) \cong V^{\lambda_n}_{\eta_n}$$

and that the induced morphism $u_K : V' := \cup_n V'_n \to \cup_n V^{\lambda_n}_{\eta_n} =: V$ is an isomorphism of strict neighborhoods. We show that these strict neighborhoods suffice.

Since $g : Y' \to Y$ is proper surjective, it follows from base change that $g_K : \widehat{Y}'_K \to \widehat{Y}_K$ is surjective. We may identify $\widehat{Y}'_K$ and $\widehat{Y}_K$ with their images in $]Y'[_{P'}$ and $]Y[_P$, respectively, and by commutativity this identification is compatible with u_K. Hence for each n we have a surjection

$$u_K : \widehat{Y}'_K \cap u_K^{-1}(V_{\eta_n}^{\lambda_n}) \twoheadrightarrow \widehat{Y}_K \cap V_{\eta_n}^{\lambda_n}.$$

Since $\widehat{Y}'_K \subset [Y']_{P'\delta}$ for all δ, we have

$$\widehat{Y}'_K \cap [Y']_{P'\delta_n} \cap u_K^{-1}(V_{\eta_n}^{\lambda_n}) = \widehat{Y}'_K \cap u_K^{-1}(V_{\eta_n}^{\lambda_n})$$

so this is identical to the map

$$u_K : \widehat{Y}'_K \cap [Y']_{P'\delta_n} \cap u_K^{-1}(V_{\eta_n}^{\lambda_n}) \twoheadrightarrow \widehat{Y}_K \cap V_{\eta_n}^{\lambda_n}.$$

Taking the union over all n, we find that

$$u_K : \widehat{Y}'_K \cap V' \twoheadrightarrow \widehat{Y}_K \cap V$$

is surjective.

In addition, this map is clearly a closed immersion. Being a surjective closed immersion does not guarantee in itself that u_K is an isomorphism. It does follow, however, that the ideal corresponding to this closed immersion is locally nilpotent. Indeed, locally it is a morphism of the form

$$u_K : \mathrm{Spm}(B/I) \twoheadrightarrow \mathrm{Spm}(B)$$

for B an affinoid K-algebra and I an ideal. Algebraically this means that every maximal ideal of B contains I, so that I is contained in the intersection of all maximal ideals of B. But all affinoid algebras are Jacobson, so I is nilpotent.

But we have the additional fact that Y is a reduced scheme - the schematic image of a closed map is the topological image endowed with the reduced induced closed subscheme structure. It would suffice, then, to show that the rigid-analytic variety $\widehat{Y}_K$ associated to the reduced scheme Y is itself reduced: if $\widehat{Y}_K$ is reduced then the admissible open subset $\widehat{Y}_K \cap V$ is reduced as well, and it would follow that the locally nilpotent ideal sheaf corresponding to the closed immersion u_K is in fact zero.

This is done in the work of Conrad and De Jong that we now describe, and completes the proof of the lemma. □

Since the question of whether the ideal of definition is zero is local, we may suppose $Y = \mathrm{Spec}(A)$ for a finite-generated reduced $\mathcal{V}$-algebra A.

The work of Conrad and De Jong in [Con99] and [Jon95] allows us to transport the reducedness of Y to the corresponding rigid-analytic variety

$\widehat{Y}_K$. As a technical point, the passage hinges on the fact that A is excellent: this is guaranteed for A by the fact that Noetherian complete local rings are excellent and that excellence is stable under passage to a finitely-generated algebra (see [Mat80, §34]). It is proven in [Gro65, 7.8.3 (v)] that an excellent local ring R is reduced if and only if its completion $\hat{R}$ with respect to its maximal ideal is reduced, so we can conclude that the p-adic completion $\widehat{A}$ of A is reduced.

We are now in the setting to evoke a general fact established by De Jong which transfers reducedness of formal schemes to their associated rigid-analytic varieties.

Definition 3.4. We denote by $\mathrm{FS}_{\mathcal{O}}$ the category consisting of Noetherian adic formal schemes $\mathfrak{X}$ whose reduction $\mathfrak{X}_{\mathrm{red}}$ via its biggest ideal of definition is a k-scheme locally of finite type. $\lhd$

Any formal scheme studied here is an examples of an object of $\mathrm{FS}_{\mathcal{O}}$ since they are explicitly chosen to be formal models of k-schemes locally of finite type.

Let $(-)^{\mathrm{rig}}$ be the functor $\mathrm{FS}_{\mathcal{O}} \to \mathrm{Rig}_K$ defined by Berthelot in [Ber96, §0.2.2, 0.2.6]. The link between the reducedness of formal schemes and their rigid-analytification is

Proposition 3.5. (De 95, Proposition 7.2.4 (c)) *Let $\mathfrak{X} \in \mathrm{Ob}(\mathrm{FS}_{\mathcal{O}})$. If $\mathfrak{X}$ is a formally reduced formal scheme (see definition following) then $\mathfrak{X}^{\mathrm{rig}}$ is reduced.* $\lhd$

Here a formal scheme is said to be *formally reduced* if it can be covered by affine formal schemes $\mathrm{Spf}(B)$ where B is reduced.

It is taken for granted, but not obvious, that if B is a reduced ring then $\mathrm{Spf}(B)$ is reduced as a formal scheme. This is clarified by Conrad, who proved the following. Let P denote any of the following standard homological properties of noetherian rings: reduced, normal, regular, Gorenstein, Cohen-Macaulay, complete intersection. If A is a Noetherian ring, we denote by $P(A)$ the statement that P is true for the ring A; if X is a rigid space, scheme, or formal scheme, we define by $P(X)$ the statement that the *non-P locus*

$$\{x \in X \,:\, P(\mathcal{O}_{X,x}) \text{ fails}\}$$

is empty. Then we have

Lemma 3.6. (Con99, Lemma 1.2.1) *Suppose R is a complete discrete valuation ring and let $\mathfrak{X}$ be a formal scheme over R covered by affines of type* $\mathrm{Spf}(A)$ *where A is a quotient of an R-algebra of the form*

$$R\langle\langle x_1, \ldots, x_n\rangle\rangle[\![y_1, \ldots, y_m]\!].$$

1. *The non-P locus in $\mathfrak{X}$ is a Zariski-closed set.*

2. *If $\mathfrak{X} \cong \operatorname{Spf}(A)$ is affine, then $P(\mathfrak{X})$ is equivalent to $P(A)$.*

$\lhd$

In particular, $\operatorname{Spf}(A) \in \mathrm{FS}_{\mathcal{O}}$ is reduced if and only if A is a reduced ring.

Putting it together, from the fact that $Y = \operatorname{Spec}(A)$ is reduced and A is excellent, we can infer that $\widehat{A}$ is reduced; Conrad assures us that $\widehat{Y} = \operatorname{Spf}(\widehat{A})$ is reduced; and finally De Jong's result ensures that $\widehat{Y}_K$ is reduced.

3.4 The Global Filtered Poincaré Lemma

Here we prove the remaining case, which is a variant of the Global Poincaré Lemma for rigid cohomology [Le 07, Lemma 6.5.5].

Theorem 4.1. (Global Poincaré Lemma) *Let*

$$\begin{tikzcd} & & P' \arrow[dd, "u"] \\ X \arrow[r, hook, "\circ"] & Y \arrow[ur, hook] \arrow[dr, hook] & \\ & & P \end{tikzcd}$$

be a smooth morphism of frames, and let u_K denote the the induced map of tubes $u_K :]Y[_{P'} \to]Y[_P$. Then there is a quasi-isomorphism

$$j^\dagger I^\ell_{X,Y,P} \xrightarrow{\sim} Ru_{K*} j^\dagger (I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P})$$

Moreover, $\mathrm{Fil}^s_{X,Y,P} \cong Ru_* \mathrm{Fil}^s_{X,Y,P'}$.

The first simplification we can make is the following:

Lemma 4.2. *The theorem is local on X and on P, and we may assume that P and P' are affine.* $\lhd$

Proof. The theorem is local on X by [Le 07, Proposition 5.2.8]. Furthermore, by [Le 07, Proposition 6.2.9] we can reduce to the case that $P = \operatorname{Spec} A$ is affine; in this case, since all closed subschemes of affine schemes are affine, we have $Y = \operatorname{Spec}(A/I)$ for some ideal $I \subseteq A$.

Our morphism of frames thus looks like

$$
\begin{tikzcd}
 & & P' \arrow[d, "u"] \\
X \arrow[r, hook, "\circ" marking] & \operatorname{Spec}(A/I) \arrow[r, hook, "/" marking] \arrow[ur, hook, "/" marking] & \operatorname{Spec} A
\end{tikzcd}
$$

Since we know that the result is local on X we will omit it from the below diagrams for brevity. Choose an affine open subset $\operatorname{Spec} B \cong U \subseteq P'$ and the intersection $Y \cap U$; they fit into a diagram

$$
\begin{tikzcd}
 & & U \cong \operatorname{Spec} B \arrow[d, hook, "\circ" marking] \\
 & & P' \arrow[d, "u"] \\
Y \cap U \arrow[r, hook, "\circ" marking] \arrow[uurr, hook, "/" marking] & \operatorname{Spec}(A/I) \arrow[r, hook, "/" marking] \arrow[ur, hook, "/" marking] & \operatorname{Spec} A
\end{tikzcd}
$$

Let $D(\overline{f})$ be a principal open subset of the open subscheme $Y \cap U$ of $\operatorname{Spec}(A/I)$, where $f \in A$ and $\overline{f} \in A/I$ is the reduction. Since localization commutes with quotients, we have a commutative diagram

$$
\begin{tikzcd}
 & & & U \cong \operatorname{Spec} B \arrow[d, hook, "\circ" marking, "\iota"] \\
 & & & P' \arrow[d, "u"] \\
 & Y \cap U \arrow[r, hook, "\circ" marking] \arrow[uurr, hook, "/" marking] & \operatorname{Spec}(A/I) \arrow[r, hook, "/" marking] \arrow[ur, hook, "/" marking] & \operatorname{Spec} A \\
D(\overline{f}) \arrow[ur, hook, "\circ" marking] \arrow[rr, hook, "/" marking] & & D(f) \arrow[ur, hook, "\circ" marking] &
\end{tikzcd}
$$

Again by [Le 07, Proposition 6.2.9] we may consider the base change of our morphism of frames with respect to the open Cartesian open subframe induced by $D(f) \hookrightarrow \operatorname{Spec} A$. Observe that $Y \cap D(f) = D(\overline{f})$; indeed if $i : \operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$ is the closed immersion and $\pi : A \twoheadrightarrow A/I$ the corresponding projection, then

$$Y \cap D(f) = i^{-1}(D(f)) = D(\pi(f)) = D(\overline{f}).$$

Hence we obtain

$$
\begin{tikzcd}
 & U \cap u^{-1}(D(f)) \arrow[d, hook, "\iota"] \\
 & u^{-1}(D(f)) \arrow[d, "u"] \\
D(\overline{f}) \arrow[ur, hook] \arrow[r, hook] & D(f)
\end{tikzcd}
$$

By the above diagram, the morphism $D(\overline{f}) \hookrightarrow u^{-1}(D(f))$ factors through $U \cap u^{-1}(D(f))$. In addition, if $\rho : A \to B$ is the morphism corresponding to the morphism of affine schemes $u \circ \iota : U = \operatorname{Spec} B \to \operatorname{Spec} A$, then by the same reasoning as above, $U \cap u^{-1}(D(f)) \cong D(\rho(f))$. We end up with a diagram

$$
\begin{tikzcd}
 & D(\rho(f)) \arrow[d, hook, "\iota"] \\
 & u^{-1}(D(f)) \arrow[d, "u"] \\
D(\overline{f}) \arrow[uur, hook] \arrow[ur, hook] \arrow[r, hook] & D(f)
\end{tikzcd}
$$

Since the tube $]Y[_{P'}$ depends only on an open neighborhood of Y we may replace $u^{-1}(D(f))$ with $D(\rho(f))$. Thus by restriction we may assume that P and P' are both affine.

To complete the proof, it suffices to show that Y can be covered by affine open subsets of this form. If (U_i) be an affine open covering of P', then $(Y \cap U_i)$ is an open covering of Y. For any principal subset $D(\overline{f}) \subseteq Y$ we can choose the U_i such that $D(\overline{f}) \subseteq Y \cap U_i$, and this provides the data needed to construct the open immersion of frames. $\square$

As a technicality, we also need the following:

Lemma 4.3. *In the situation of Theorem 3.2.6, the following are equivalent:*

1. *The base change map*

$$j^\dagger I^{\ell}_{X,Y,P} \xrightarrow{\sim} Ru_* j^\dagger (I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P})$$

is an isomorphism.

2. *For any (finite) locally free sheaf $\mathcal{M}$ of $\mathcal{O}_{]Y[_P}$-modules the base change map*

$$j^\dagger I^{\ell}_{X,Y,P} \otimes \mathcal{M} \xrightarrow{\sim} Ru_* j^\dagger (I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} u^*\mathcal{M} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P})$$

is an isomorphism.

$\triangleleft$

Proof. This is an immediate consequence of the following projection formula in the derived category [Sta22, Lemma 0B54]. Let $f : X \to Y$ is a morphism of ringed spaces and let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect (see [Sta22, Section 08CL]), then

$$Rf_*E \otimes^{\mathbb{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbb{L}} Lf^*K).$$

Note that the functor u^* is exact and locally free sheaves are both perfect complexes and their own projective resolutions. Using [Le 07, Proposition 5.3.2] we can infer that

$$\begin{aligned} Ru_*j^\dagger(I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} u^*\mathcal{M} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P}) &\cong Ru_*((j^\dagger(I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P}) \otimes_{\mathcal{O}_{]Y[_{P'}}} u^*\mathcal{M}) \\ &\cong Ru_*j^\dagger(I^{\ell-\bullet}_{X,Y,P'} \otimes_{\mathcal{O}_{]Y[_{P'}}} \Omega^\bullet_{]Y[_{P'}/]Y[_P}) \otimes \mathcal{M}. \end{aligned}$$

Hence we can infer (2) from (1) by tensoring each side with $\mathcal{M}$. $\square$

Proof of Global Filtered Poincaré Lemma. For brevity, let $A =]Y[_{P'}$, $B =]Y[_P$, and write $I_A := I_{X,Y,P'}$ and $I_B := I_{X,Y,P}$. Consider the double complex

$$C^{a,b}(s) := j^\dagger_A(I^{s-a-b}_A \otimes_A u^*\Omega^a_B \otimes_A \Omega^b_{A/B})$$

where the horizontal (resp. vertical) differential is induced by that of $\Omega^\bullet_B$ (resp. $\Omega^\bullet_{A/B}$). Its total complex is given by

$$\begin{aligned} \mathrm{Tot}(C^{\bullet,\bullet}(s))^k &\cong \bigoplus_{a+b=k} j^\dagger_A(I^{s-a-b}_A \otimes_A u^*\Omega^a_B \otimes_A \Omega^b_{A/B}) \\ &\cong j^\dagger_A I^{s-k}_A \otimes_A \left(\bigoplus_{a+b=k} u^*\Omega^a_B \otimes_A \Omega^b_{A/B} \right) \\ &\cong j^\dagger_A I^{s-k}_A \otimes_A \Omega^k_A \\ &\cong j^\dagger_A(I^{s-k}_A \otimes_A \Omega^k_A) \\ &\cong (\mathrm{Fil}^s_{X,Y,P'})^k \end{aligned}$$

so we have a natural isomorphism $\mathrm{Tot}(C^{\bullet,\bullet}(s)) \cong \mathrm{Fil}^s_{X,Y,P'}$.

Hence it suffices to prove that $Ru_* \mathrm{Tot}(C^{\bullet,\bullet}(s)) \cong \mathrm{Fil}^s_{X,Y,P}$. To do this, we may show that the direct image $Ru_*C^{k,\bullet}(s)$ of each column of the total complex is isomorphic to $(\mathrm{Fil}^s_{X,Y,P})^k$, i.e.,

$$j^\dagger_B(I^{s-k}_B \otimes_B \Omega^k_B) \cong Ru_*j^\dagger_A(I^{s-k-\bullet}_A \otimes u^*\Omega^k_B \otimes \Omega^\bullet_{A/B}). \quad (3.1)$$

By Lemma 3.4.3 it suffices to prove that

$$j_B^\dagger I_B^{s-k} \cong Ru_* j_A^\dagger (I_A^{s-k-\bullet} \otimes \Omega^\bullet_{A/B}).$$

In the course of reducing this quasi-isomorphism to an explicitly computable special case we'll need the following expected fact:

Lemma 4.4. *Consider a diagram of smooth morphisms of frames*

$$\begin{tikzcd} & & P' \arrow[d, "u"] \\ X_k \arrow[r, hook] & Y \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & P \arrow[d, "v"] \\ & & P'' \end{tikzcd}$$

If isomorphism (3.1) holds for the frames corresponding to v and u, respectively, then it holds for the frame corresponding to $v \circ u$. ◁

Proof. We keep the notation of the proof and further write $C =]Y[_{P''}$ and write $I_C := I_{X,Y,P''}$ for the ideal corresponding to the frame $(X \subseteq Y \subseteq P'')$. By assumption, we have

$$j_C^\dagger (I_C^{s-k} \otimes \Omega^s_C) \cong Rv_* j_B^\dagger (I_B^{s-k-\bullet} \otimes v^*\Omega^k_C \otimes \Omega^\bullet_{B/C})$$

On the other hand, by assumption and Lemma 3.4.3 we have

$$\begin{aligned} j_B^\dagger (I_B^{s-k-a} \otimes v^*\Omega^k_C \otimes \Omega^a_{B/C}) &\cong Ru_*(j_A^\dagger (I_A^{s-k-a-\blacktriangle} \otimes u^*(v^*\Omega^s_C \otimes \Omega^a_{B/C}) \otimes \Omega^\blacktriangle_{A/B})) \\ &\cong Ru_*(j_A^\dagger (I_A^{s-k-a-\blacktriangle} \otimes (u \circ v)^*\Omega^s_C \otimes u^*\Omega^a_{B/C} \otimes \Omega^\blacktriangle_{A/B})) \end{aligned}$$

and hence

$$j_B^\dagger (I_B^{s-k-\bullet} \otimes v^*\Omega^k_C \otimes \Omega^\bullet_{B/C})) \cong Ru_*(\operatorname{Tot} j_A^\dagger (I_A^{s-k-\bullet-\blacktriangle} \otimes (u \circ v)^*\Omega^s_C \otimes u^*\Omega^\bullet_{B/C} \otimes \Omega^\blacktriangle_{A/B}))$$

But the i^{th} component of the total complex on the right is

$$j_A^\dagger \left(I_A^{s-k-i} \otimes (u \circ v)^*\Omega^s_C \otimes \bigoplus_{\bullet+\blacktriangle=i} \left(u^*\Omega^\bullet_{B/C} \otimes \Omega^\blacktriangle_{A/B}\right) \right) \cong j_A^\dagger (I_A^{s-k-i} \otimes (u \circ v)^*\Omega^s_C \otimes \Omega^i_{A/C})$$

and so

$$j_B^\dagger (I_B^{s-k-\bullet} \otimes v^*\Omega^k_C \otimes \Omega^\bullet_{B/C})) \cong Ru_*(j_A^\dagger (I_A^{s-k-\bullet} \otimes (u \circ v)^*\Omega^s_C \otimes \Omega^\bullet_{A/C})).$$

Thus

$$\begin{aligned} j_C^\dagger (I_C^{s-k} \otimes \Omega^s_C) &\cong Rv_*(Ru_*(j_A^\dagger (I_A^{s-k-\bullet} \otimes (u \circ v)^*\Omega^s_C \otimes \Omega^\bullet_{A/C}))) \\ &\cong R(v \circ u)_* j_A^\dagger (I_A^{s-k-\bullet} \otimes (u \circ v)^*\Omega^s_C \otimes \Omega^\bullet_{A/C})) \end{aligned}$$

as desired. □

We go back to proving our theorem. By Lemma 3.4.2 we can assume that P and P' are affine. Further, since the question is local on X_k, we can assume that the complement of X_k in Y_k is a hyperplane. Finally, consider the diagram

$$\begin{tikzcd}
 & Y \times P' \arrow[r, hook] \arrow[d, hook] & P' \arrow[d, "u"] \\
X_k \arrow[ur, hook] \arrow[r, hook] & Y \arrow[ur, hook] \arrow[r, hook] & P
\end{tikzcd}$$

By Lemma 3.3.2 we have $\mathrm{Fil}^s_{X,Y,P} \cong \mathrm{Fil}^s_{X,Y\times P',P}$ and $\mathrm{Fil}^s_{X,Y,P'} = \mathrm{Fil}^s_{X,Y\times P',P'}$, hence we may even assume that the morphisms of frames is Cartesian.

By [Le 07, Proposition 3.3.13] we may assume, locally, that there is an étale morphism of affine frames

$$\begin{tikzcd}
 & & P' \arrow[dd, "u'"] \\
X \arrow[r, hook] & Y \arrow[ur, hook] \arrow[dr, hook] & \\
 & & \widehat{\mathbb{A}}^d_P
\end{tikzcd}$$

where Y is embedded into $\widehat{\mathbb{A}}^d_P$ using the zero section. We thus have a factorization

$$\begin{tikzcd}
 & & P' \arrow[d, "u'"] \arrow[dd, bend left, "u"] \\
X_k \arrow[r, hook] & Y \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \widehat{\mathbb{A}}^d_P \arrow[d, "\pi_P"] \\
 & & P
\end{tikzcd}$$

We already have the result for the upper morphism of frames by Lemma 3.2.5. Lemma 3.4.4 then reduces the theorem to the case

$$\begin{tikzcd}
 & & \widehat{\mathbb{A}}^d_P \arrow[dd, "u"] \\
X \arrow[r, hook] & Y \arrow[ur, hook] \arrow[dr, hook] & \\
 & & P
\end{tikzcd}$$

with u the canonical projection. Chaining together multiple instances of Lemma 3.4.4, we may further reduce ourself to the case $d = 1$. Since we may prove the result by checking it on $R\Gamma(W, -)$ for all affinoid subsets $W \subseteq]Y[_P$, it remains to prove the following variation of the Local Poincaré Lemma ([Le 07, Lemma 6.5.7]). $\square$

Lemma 4.5. (Local Filtered Poincaré Lemma) *Let $(X \subseteq Y \subseteq P)$ be a strictly local frame and*

$$p : \widehat{\mathbb{A}}^1_P \to P$$

be the projection. Consider the morphism of affine frames

$$\begin{tikzcd} & & \widehat{\mathbb{A}}^1_P \arrow[dd, "p"] \\ X_k \arrow[r, hook] & Y \arrow[ur, hook] \arrow[dr, hook] & \\ & & P \end{tikzcd}$$

Let $I_{\mathbb{A}^1}$ denote the ideal corresponding to the frame $(X \subseteq Y \subseteq \widehat{\mathbb{A}}^1_P)$ and I_P the ideal corresponding to the frame $(X \subseteq Y \subseteq P)$. If W is an affinoid subset of $]Y[_P$, there is a canonical isomorphism

$$\Gamma(W, j^\dagger I_P^\ell) \cong R\Gamma(W \times \mathbb{D}(0, 1^-), j^\dagger I_{\mathbb{A}^1}^\ell \xrightarrow{\partial/\partial t} j^\dagger I_{\mathbb{A}^1}^{\ell-1})$$

$\lhd$

We start by giving an explicit description of $I_{\mathbb{A}^1}^\ell$.

Lemma 4.6. *Let $W = \mathrm{Sp}(A) \subseteq]Y[_P$ be an affinoid subset and let $0 < \eta < 1$. We have*

$$\Gamma(W \times \mathbb{D}(0, \eta^+), I_{\mathbb{A}^1}^n) = \left\{ \sum_{i \geq 0} a_i t^i \in A\{t\}_\eta \;:\; a_i \in \Gamma(W, I_P^{n-i}) \right\}$$

for all $n \geq 0$, where

$$A\{t\}_\eta = \left\{ \sum_{i \geq 0} a_i t^i \;:\; a_i \in A, |a_i|\eta^i \to 0 \right\}.$$

$\lhd$

Proof. Let $\widehat{P}_K = \mathrm{Sp}(R)$. The diagram

$$\begin{tikzcd}
 & {]Y[_{\mathbb{A}^1_P}} \arrow[dd, "u"] \\
\widehat{Y}_K \arrow[ur, hook, "\iota_{\mathbb{A}^1}"] \arrow[dr, hook, "\iota_P"'] & \\
 & {]Y[_P}
\end{tikzcd}$$

corresponding to the ideals I_P and $I_{\mathbb{A}^1}$ induces on sheaves of sections the diagram

$$\begin{tikzcd}
 & (u_*\mathcal{O}_{]Y[_{\widehat{\mathbb{A}}^1}})(W) = \mathcal{O}_{]Y[_{\widehat{\mathbb{A}}^1}}(W \times D(0,1^-)) \cong A \otimes_R R\{t\} \cong A\{t\} \arrow[dl, "\iota^\flat_{\mathbb{A}^1}(W)"'] \\
\iota_{P*}\mathcal{O}_{\widehat{Y}_K}(W) & \\
 & \mathcal{O}_{]Y[_P}(W) = A \arrow[ul, "\iota^\flat_P(W)"] \arrow[uu, hook]
\end{tikzcd}$$

where the vertical arrow is the natural inclusion. Since Y is embedded into $\widehat{\mathbb{A}}^1_P$ by the zero section, $\iota^\flat_{\mathbb{A}^1}(W)$ is the map $t \mapsto 0$. By restriction, we get a similar diagram with the uppermost group being replaced by $\mathcal{O}_{]Y[^1_{\widehat{\mathbb{A}}}}(W \times D(0,\eta^+)) \cong A\{t\}_\eta$; as an abuse of notation we'll use the same notation for any $\eta > 0$.

Hence if $\Gamma(W, I_P) = \ker(\iota^\flat_P(W))$ is generated as an A-module by sections $(h_1, \ldots, h_s)$ (such a finite set of generators exists since I_P is coherent), then $\Gamma(W \times D(0,\eta^+), I_{\mathbb{A}^1})$ is generated as an $A\{t\}_\eta$-module by $(h_1, \ldots, h_s, t)$.

With this description it is easy to see that

$$\Gamma(W \times D(0,\eta^+), I_{\mathbb{A}^1}) = \left\{ \sum_{i \geq 0} a_i t^i \in A\{t\}_\eta \,:\, a_0 \in \Gamma(W, I_P) \right\}$$

which is precisely the lemma when $n = 1$. We'll prove the general case by induction.

For the rest of the proof, we write I^n_P and $I^n_{\mathbb{A}^1}$ for $\Gamma(W, I^n_P)$ and $\Gamma(W \times \mathbb{D}(0,\eta^+), I^n_{\mathbb{A}^1})$, respectively, with the implicit assumption that we're always working with sections.

Let

$$S^n := \left\{ \sum_{i \geq 0} a_i t^i \in A\{t\}_\eta \,:\, a_i \in I^{n-i}_P \right\}.$$

First note that this is an ideal. Indeed, if $\sum_{j\geq 0} b_j t^j \in A\{t\}_\eta$ is arbitrary and $\sum_{i\geq 0} a_i t^i \in S^n$ then

$$\left(\sum_{j\geq 0} b_j t^j\right)\left(\sum_{i\geq 0} a_i t^i\right) = \sum_{k\geq 0}\left(\sum_{i+j=k} a_i b_j\right) t^k$$

But $a_i b_j \in I_P^{n-i} \subseteq I_P^{n-k}$ for all such i and j, so this product is in S^n.

As our inductive hypothesis suppose $I_{\mathbb{A}^1}^{n-1} = S^{n-1}$.

- $S^n \subseteq I^n_{\mathbb{A}^1_P}$: Let $\sum a_i t^i \in S^n$ and view this sum as

$$\sum_{i\geq 0} a_i t^i = a_0 + t\sum_{i\geq 0} a_{i+1} t^i$$

  Since $a_{i+1} \in I_P^{n-i-1}$ for all i by assumption, we have by the inductive hypothesis that $\sum_{i\geq 0} a_{i+1} t^i \in I^{n-1}_{\mathbb{A}^1_P}$, and hence that $t\sum_{i\geq 0} a_{i+1} t^i \in I^n_{\mathbb{A}^1_P}$. Since also

$$a_0 \in I_P^n \subseteq I^n_{\mathbb{A}^1_P}$$

  we have that each summand is in $I^n_{\mathbb{A}^1_P}$, which is all we need.

- $I^n_{\mathbb{A}^1_P} \subseteq S^n$: By definition $I^n_{\mathbb{A}^1_P} = I^{n-1}_{\mathbb{A}^1_P} \cdot I_{\mathbb{A}^1_P}$. Applying the inductive hypothesis we see that this ideal consists of sums of elements of the form

$$\left(\sum_{i\geq 0} a_i t^i\right)\left(\sum_{j\geq 0} b_j t^j\right)$$

  where $a_i \in I_P^{n-i-1}$ for all i and $b_0 \in I_P$. We rearrange this as

$$\begin{aligned}\left(\sum_{i\geq 0} a_i t^i\right)\left(\sum_{j\geq 0} b_j t^j\right) &= \left(\sum_{i\geq 0} a_i t^i\right)\left(b_0 + t\left(\sum_{j\geq 1} b_j t^{j-1}\right)\right)\\ &= \left(\sum_{i\geq 0} b_0 a_i t^i\right) + t\left(\sum_{i\geq 0} a_i t^i\right)\left(\sum_{j\geq 1} b_j t^{j-1}\right)\\ &= \left(\sum_{i\geq 0} b_0 a_i t^i\right) + \left(\sum_{i\geq 1} a_{i-1} t^i\right)\left(\sum_{j\geq 1} b_j t^{j-1}\right)\end{aligned}$$

  We have $b_0 a_i \in I_P^{n-i}$ for all i so the first summand is in S^n. Likewise, $a_{i-1} \in I_P^{n-i}$ for all i so $\sum_{i\geq 1} a_{i-1} t^i \in S^n$, and since S^n is an ideal the second summand is in S^n also.

$\square$

Proof (of local filtered Poincaré lemma). Let V^λ be the usual cofinal system of strict neighborhoods of $]X[_P$ in $]Y[_P$ (see, e.g., [Le 07, Proposition 3.3.1]). If $M(\ell) := \Gamma(W, j_P^\dagger I_P^\ell)$, it follows from [Le 07, Proposition 5.1.12] that

$$M(\ell) = \varinjlim_\lambda M^\lambda(\ell)$$

where $M^\lambda(\ell) := \Gamma(W \cap V^\lambda, I_P^\ell)$.

Choose a sequence $\eta_k \xrightarrow{<} 1$ and define

$$M_k(\ell) := \Gamma(W \times \mathbb{D}(0, \eta_k^+), j_{\mathbb{A}^1}^\dagger I_{\mathbb{A}^1}^\ell)$$

The neighborhoods $V^\lambda \times \mathbb{D}(0, 1^-)$ form a cofinal system of strict neighborhoods of $]X[_{\widehat{\mathbb{A}}_P^1}$ in $]Y[_{\widehat{\mathbb{A}}_P^1}$, so again we have

$$M_k(\ell) = \varinjlim_\lambda M_k^\lambda(\ell)$$

where $M_k^\lambda(\ell) = \Gamma((W \cap V^\lambda) \times \mathbb{D}(0, \eta_k^+), I_{\mathbb{A}^1}^\ell)$. Note that Lemma 3.4.6 gives an explicit description of this module.

Since X can be assumed to be the complement of a hypersuface in Y it follows from [Le 07, Proposition 5.4.14] that the affinoid covering

$$W \times \mathbb{D}(0, 1^-) = \bigcup_k (W \times \mathbb{D}(0, \eta_k^+))$$

is acyclic for coherent modules. Hence we are in the setting of [Le 07, Lemma 6.5.10], which tells us that the cohomology

$$R\Gamma(W \times \mathbb{D}(0, 1^-), j^\dagger I_{\mathbb{A}^1}^\ell \xrightarrow{\partial/\partial t} j^\dagger I_{\mathbb{A}^1}^{\ell-1})$$

can be computed as the total complex of the double complex

$$\begin{array}{ccc}
\prod_k M_k(\ell) & \xrightarrow{\partial} & \prod_k M_k(\ell-1) \\
\downarrow d & & \downarrow d \\
\prod_k M_k(\ell) & \xrightarrow{\partial} & \prod_k M_k(\ell-1)
\end{array}$$

where $d(s_k) = (s_{k+1}|_{D(0,\eta_k^+)} - s_k)$ and $\partial(s_k) = (\partial/\partial t(s_k))$.

Let $i : M^\lambda(\ell) \hookrightarrow M_k^\lambda(\ell)$ denote the inclusion as the constant coefficient (see Lemma 3.4.6). We'll also make use of the integration map

$$M_k^\lambda(\ell) \xrightarrow{I} M_{k-1}^\lambda(\ell+1) \tag{3.2}$$

$$t^i \mapsto \frac{t^{i+1}}{i+1}. \tag{3.3}$$

This is well-defined: $|\frac{a_{i-1}}{i}|\eta_{k-1}^i \to 0$ if $|a_i|\eta_k^i \to 0$ given that $\eta_k \xrightarrow{<} 1$ is strictly increasing, and if $\sum a_i t^i \in M_k^\lambda(\ell)$ then

$$\sum_{i\geq 1} \frac{a_{i-1}}{i} t^i \in M_{k-1}^\lambda(\ell+1)$$

since $a_{i-1} \in I_P^{\ell+1-i}$ for all i.

Taking limits we can extend i and I to maps map $i : M(\ell) \hookrightarrow M_k(\ell)$ and $I : M_k(\ell) \to M_{k-1}(\ell+1)$, respectively, and taking the product we obtain maps

$$i : M(\ell)^{\mathbb{N}} \hookrightarrow \prod_k M_k(\ell) \tag{3.4}$$

$$I : \prod_k M_k(\ell) \to \prod_k M_k(\ell+1). \tag{3.5}$$

Finally, let $\delta : M(\ell) \to M(\ell)^{\mathbb{N}}$ be the diagonal embedding.

With these notations, the lemma is equivalent to the following claim:

Claim 4.7. The sequence

$$0 \to M(\ell) \xrightarrow{i\circ\delta} \prod_k M_k(\ell) \xrightarrow{(d,\partial)} \prod_k M_k(\ell) \oplus \prod_k M_k(\ell-1) \xrightarrow{\partial-d} \prod_k M_k(\ell-1) \to 0$$

is exact. $\lhd$

1. It's immediate that this is a complex and that the first map is injective.

2. To see that the last map is surjective, note that for all $s = (s_k) \in M_k(\ell-1)$ we have

   $$(s_k)_k \overset{I}{\mapsto} (I(s_{k+1}))_k \overset{\partial}{\mapsto} (s_{k+1})_k.$$

   Hence

   $$(s_k)_k = (s_{k+1})_k - (s_{k+1} - s_k)_k = \partial(I(s_k)) - d(s_k)$$

   that is, $(\partial - d)(I(s), s) = s$.

3. Next we show exactness in the second term. Let $(s_k) \in \prod_k M_k(\ell)$ and suppose that $d(s_k) = \partial(s_k) = 0$. Each s_k can be represented as a compatible system of power series $(\sum_i a_{\lambda,k,i} t^i)_\lambda \in \varinjlim_\lambda M_k^\lambda(\ell)$. The fact that $\partial(s_k) = 0$ means that $(\partial/\partial t(\sum_i a_{\lambda,k,i} t^i))_\lambda = 0$ for all k. The transition functions are simply restrictions of coefficients of power series so this means, as expected, that all of these power series are constant, i.e. $a_{\lambda,k,i} = 0$ for $i \geq 1$. On the other hand, $d(s_k) = 0$ means that, as systems of power series, $s_i = s_j$ for all $i, j \geq 0$. Thus

$$(s_k)_k = (s_0)_k = ((a_\lambda)_\lambda)_k = (i \circ \delta)((a_\lambda)_\lambda)$$

as needed.

4. Finally we show exactness in the third term. As an abuse of notation we'll imagine an element $(s_k) \in \prod_k M_k(\ell)$ to consist of power series $s_k = \sum a_{k,i} t^i$, with the implicit understanding that these are in actuality compatible sets of power series as in the previous step.

   Let $s = (s_k) = (\sum a_{k,i} t^i) \in \prod_k M_k(\ell)$ and $s' = (s'_k) = (\sum b_{k,j} t^j) \in \prod_k M_k(\ell - 1)$ with $\partial(s) = d(s')$. We're looking for $s'' \in \prod_k M_k(\ell)$ such that $d(s'') = s$ and $\partial(s'') = s'$. We have

$$\begin{aligned} I(s'_k)_k &= \left(\sum_{j \geq 1} \frac{b_{j-1,k+1}}{i} t^i \right)_k \\ \Rightarrow d(I(s'_k)) &= \left(\sum_{j \geq 1} \frac{b_{j-1,k+2} - b_{j-1,k+1}}{i} t^i \right)_k \\ &= I(s'_{k+1} - s'_k)_k \\ &= I(d(s')) \\ &= I(\partial(s)) \\ &= (s_{k+1} - a_{0,k+1})_k. \end{aligned}$$

   Hence

$$\begin{aligned} d(I(s') - s + (a_{0,k+1})) &= (s_{k+1} - a_{0,k+1})_k - (s_{k+1} - s_k)_k - (a_{0,k+1})_k \\ &= s \end{aligned}$$

   and

$$\begin{aligned} \partial(I(s') - s + (a_{0,k+1})_k) &= \partial(I(s') - s) \\ &= s' + d(s') - \partial(s) \\ &= s' + d(s') - d(s') \\ &= s' \end{aligned}$$

(see the computation in (2)), so $s'' = I(s') - s + (a_{0,k+1})_k$ does the trick.

$\square$

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