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Classification of representations of finite groups of Lie type and the Langlands framework

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Abstract

This thesis is devoted to the representation theory of finite groups of Lie type and of p-adic groups, with a particular emphasis on their interplay. The central theme is the classification of irreducible representations over an algebraically closed field of characteristic zero, approached from both finite and p-adic perspectives.

Each chapter develops a self-contained study within this broad framework. To preserve autonomy, every chapter begins with its own introduction, situating the problem in the literature and highlighting the main results. The present introduction provides a concise overview of the topics addressed, while the reader is referred to the individual chapter introductions for more technical and detailed expositions.

The first part of the thesis concerns the representation theory of finite groups of Lie type.

Chapter I extends the classical theory of parabolically induced cuspidal representations from connected algebraic groups to the case of disconnected groups. In the connected setting, the associated endomorphism algebras are known to be finite extended Hecke algebras. We show that this remains true when the ambient group is disconnected through extension by an automorphism stabilizing both a Levi subgroup and a cuspidal representation. In addition, the chapter establishes an analogue of the classical comparison theorem, and provides a systematic analysis of the restriction of characters from the rational points of a disconnected group to those of its connected components. These structural results are not only of independent interest, but also provide the necessary groundwork for the developments of Chapter II.

Building on this foundation, Chapter II addresses the compatibility of Lusztig's non-abelian Fourier transform for unipotent representations of finite classical groups of types B, C, and D with parabolic induction. Its main contribution is a detailed combinatorial proof of this compatibility, with particular focus on disconnected groups of type D_n, where both split and non-split forms occur. Treating these inner twists simultaneously simplifies the combinatorial framework underlying Lusztig's construction, and this perspective arises naturally when considering connections to the representation theory of p-adic groups.

The second part of the thesis turns to the relationship between the representation theory of finite groups of Lie type and that of p-adic groups, within a Langlands-theoretic framework. The guiding problem is to establish parameterizations of irreducible representations of finite groups of Lie type in a manner compatible with the (tame) local Langlands correspondence for p-adic groups.

Chapter III develops this connection for the general linear group. We review the tame local Langlands correspondence for GL_n over a p-adic field, together with the Macdonald correspondence for GL_n over a finite field, which may be viewed as a Langlands-type parameterization in the finite setting. Both constructions yield bijections between isomorphism classes of irreducible representations of the appropriate group and a suitable set of parameters. We then show that parahoric restriction to maximal compact subgroups provides a link between these two parametrizations, thereby establishing the expected compatibility. Although this study was carried out independently, the main result is not original, as we became aware upon completion that it had already been established by Silberger and Zink. We include our approach here because it is presented in a form that is convenient for generalization to other cases, and in particular it serves as the basis for Chapter IV.

Chapter IV extends the analysis to the special linear group. We introduce a parameterization for the irreducible representations of the finite special linear group, which we call the Macdonald–Vogan correspondence, analogous in spirit to the Macdonald correspondence for GL_n. This construction is motivated by a conjecture originally formulated by Vogan in 2016, which has been very recently formalized by Imai and Vogan and subsequently established in full generality for all finite groups of Lie type by the first author. In contrast to the case of GL_n, the Macdonald–Vogan correspondence, like the local Langlands correspondence for SL_n, is surjective but not bijective. The representations in each fiber are parametrized by irreducible representations of a certain component group depending on the parameter, mirroring the p-adic situation. Confirming a prediction of Vogan, we prove that the Macdonald–Vogan correspondence is compatible with the tame local Langlands correspondence for SL_n via parahoric restriction, linking together the finite and local theories.

Chapter I

Hecke Algebras for disconnected groups

Introduction

The subject of this chapter is ordinary (i.e. characteristic 0) representation theory of finite groups of Lie type. Its main goal is to extend the study of the endomorphism algebra of a parabolically induced cuspidal representation of a finite group of Lie type to the case in which the underlying algebraic group is disconnected.

In the setting in which the defining algebraic group is connected, the endomorphism algebra of a parabolically induced cuspidal representation is a finite (extended) Hecke algebra [31, 22].

Here, we examine the next easiest possible situation: that is, if the enveloping algebraic group is extended by an outer automorphism that stabilizes both a Levi subgroup and a cuspidal representation defined on the Levi subgroup's rational points. In this context, we show that the corresponding endomorphism algebra is still a finite extended Hecke algebra, which can be viewed as a split extension of the endomorphism algebra of the representation obtained parabolically inducing the same cuspidal representation to the rational points of the identity component of the enveloping algebraic group.

After a brief review of the classical theory in Section 1.1.1, we study the structure of the endomorphism algebra in the disconnected setting in Section 1.2. The main result is Theorem 1.2.17. The arguments we use to prove it follow closely the arguments available in the literature for the connected case.

Moreover, in Section 1.2.3, we show that an analogue of the comparison theorem of Howlett and Lehrer [32, Theorem 5.9] still applies in this case, reducing the problem of the decomposition into irreducible constituents of parabolically induced (or, equivalently, restricted) representations to analogous questions about induced (respectively restricted) representations of extended Weyl groups. The relevant result in this regard is Theorem 1.2.24.

Finally, Section 1.2.4 is devoted to the study of the restriction of characters of the group of the rational points of the disconnected algebraic group to the cosets of the group of the rational points of the identity component. This restriction exhibits a remarkably ordered behaviour, and in particular we show that studying this behaviour can be reduced to an analogous problem on an extended Weyl group (see Theorem 1.2.41).

For us, the interest of this subject lies in the use that we will make of these results in chapter II.

Notation

- ⋄ For any finite set S, we write $\mathbb{C}[S]$ for the complex finite-dimensional vector space having as basis the elements of S. We endow it with the Hermitian scalar product defined by requiring S to be an orthonormal basis.

- ⋄ For any finite group H, we will denote by $Irr(H)$ the set of isomorphism classes of irreducible complex representations of H. We denote by $R(H)$ the space of complex class functions of H, that is the same as the complexification of the

Grothendieck group of the representations of H. More in general, if A is a semisimple algebra over $\mathbb{C}$, we denote by $Irr(A)$ the set of isomorphism classes of simple modules over A. If $A := \mathbb{C}H$ is the group algebra of some finite group H, we have $Irr(\mathbb{C}H) = Irr(H)$.

- ◇ Let H be a finite group and let $K \leqslant H$. If π is a representation of H, we write Res^H_K for the restriction of π to K.
  If γ is a representation of K, we write $Ind^H_K\gamma$ for the induced representation. More specifically, it is the representation given by the left multiplication action of H on $\mathbb{C}H \otimes_{\mathbb{C}K} V_\gamma$, where V_γ is the representation space of γ. More in general, if A is an algebra over $\mathbb{C}$ and B is a subalgebra, we denote by Res^A_B the restriction of scalars from A-modules to B-modules, and by Ind^B_A the extension of scalars $A \otimes_B -$ from B-modules to A-modules.

  Let N be a normal subgroup of H. If π is a representation of H, we denote by π^N the H/N representation on the N-fixed subspace of π. If γ is a representation of H/N, we denote by $Infl^H_{H/N}\gamma$ the inflated representation. More specifically, writing the idempotent element in $\mathbb{C}H$ associated to N as $e_N = \frac{1}{|N|}\sum_{n\in N} n$, it is the representation given by the left multiplication action of H on $\mathbb{C}He_N \otimes_{\mathbb{C}H/N} V_\gamma$, where V_γ is the representation space of γ.

- ◇ Let p be a prime number. Let G be a connected reductive linear algebraic group over $\overline{\mathbb{F}}_p$. Let q be a power of p and assume G to be defined over $\mathbb{F}_q$ with a Frobenius morphism F. We denote by G^F the finite group of Lie type given by the F-fixed points in G.

- ◇ For any F-stable Levi subgroup L of some parabolic subgroup P of G we denote by $R^G_{L\subset P} : R(L^F) \to R(G^F)$ the parabolic induction functor and by ${}^*R^G_{L\subset P} : R(G^F) \to R(L^F)$ the parabolic restriction functor [20, Definition 5.1.7]. In the following we will be mostly concerned with the case in which P is F-stable, in which case we define explicitly the parabolic induction as $R^G_{L\subset P} = Ind^{G^F}_{P^F} \circ Infl^{P^F}_{L^F}$ and the parabolic restriction as ${}^*R^G_{L\subset P} = (Res^{G^F}_{P^F})^{U^F}$, where U denotes the unipotent radical of P. In this case, $R^G_{L\subset P}$ and ${}^*R^G_{L\subset P}$ do not depend on the parabolic, so we denote them by $R^{G^F}_{L^F}$ and ${}^*R^{G^F}_{L^F}$ respectively.

1.1 Finite Hecke algebras

Let (W, S) be a finite Coxeter system. Denote by l the length function on W relative to S, and let $\{x_s | s \in S\}$ be a set of indeterminates such that $x_s = x_{s'}$ if $s, s' \in S$ are conjugate in W.

Let Ω be a finite group acting on W preserving S, and set $\widetilde{W} := W \rtimes \Omega$. We allow the case $\Omega = 1$. Let $\mu : \widetilde{W} \times \widetilde{W} \to \mathbb{C}^*$ be a cocycle that is trivial on $W \times W$. The extended twisted generic finite Hecke algebra $\mathcal{H}_\mu(\widetilde{W}, \{x_s\}_{s\in S})$ is the associative algebra over the free module over $\mathbb{C}[\{x_s\}_{s\in S}]$ with basis $\{b_x \mid x \in \widetilde{W}\}$, and with the following relations: for $x, y \in W$ and $\omega, \nu \in \Omega$

$$\begin{aligned}
b_x b_y &= b_{xy}, && \text{if } l(x)+l(y)=l(xy)\\
b_s^2 &= (x_s^2-1)b_s + x_s^2 && s \in S\\
b_x b_\omega &= \mu(x,\omega) b_{x\omega}\\
b_\omega b_x &= \mu(\omega,x) b_{\omega x}\\
b_\omega b_\nu &= \mu(\omega,\nu) b_{\omega\nu}
\end{aligned}$$

When the cocycle μ is the trivial cocycle 1, we drop it in the notation, and we call $\mathcal{H}(\widetilde{W}, \{x_s\}_{s\in S}) = \mathcal{H}_1(\widetilde{W}, \{x_s\}_{s\in S})$ an extended generic finite (untwisted) Hecke algebra.

If $\Omega = 1$, we call $\mathcal{H}(W, \{x_s\}_{s\in S})$ a generic finite Hecke algebra.

For a $\mathbb{C}$-algebras morphism $f : \mathbb{C}[\{x_s\}_{s\in S}] \to \mathbb{C}$ the specialization of a extended twisted generic Hecke algebra $\mathcal{H}_{\mu(\widetilde{W}, \{x_s\}_{s\in S})}$ by f is

$$\mathcal{H}_\mu(\widetilde{W}, \{f(x_s)\}_{s\in S}) := \mathcal{H}_\mu(\widetilde{W}, \{x_s\}_{s\in S}) \otimes_f \mathbb{C},$$

where $\mathbb{C}$ is regarded as a $\mathbb{C}[\{x_s\}_{s\in S}]$-module via the morphism f. We call it a (specialized) extended twisted Hecke algebra.

Let $\mathbb{C}_\mu[\widetilde{W}]$ denote the twisted group algebra of $\widetilde{W}$ determined by the cocycle μ, that is the algebra with $\mathbb{C}$-basis $\{\overline{x} \mid x \in \widetilde{W}\}$ and multiplication given by $\overline{x}\ \overline{y} = \mu(x,y)\overline{xy}$ for any two elements of the basis. This is a semisimple algebra, and it is the specialization of $\mathcal{H}_\mu(\widetilde{W}, \{x_s\}_{s\in S})$ by the morphism given by the assignment $f_1 : x_s \mapsto 1$ for any $s \in S$.

By Tits' deformation theorem, the following theorem holds.

Theorem 1.1.1. *[31, Theorem 5.3] For any $\mathbb{C}$-algebra morphism $f : \mathbb{C}[\{x_s\}_{s\in S}] \to \mathbb{C}$ such that the finite extended twisted Hecke algebra $\mathcal{H}_\mu(\widetilde{W}, \{f(x_s)\}_{s\in S})$ is semisimple, it holds $\mathcal{H}_\mu(\widetilde{W}, \{f(x_s)\}_{s\in S}) \cong \mathbb{C}_\mu\widetilde{W}$*

In particular when $\mu = 1$ this theorem states that any semisimple finite extended Hecke algebra $\mathcal{H}(\widetilde{W}, \{c_s\}_{s\in S})$, with $c_s \in \mathbb{C}$, is isomorphic to the group algebra of $\widetilde{W}$.

1.1.1 Finite Hecke algebras as endomorphism algebras

An irreducible representation $\sigma \in Irr(G^F)$ is said to be cuspidal if it does not appear as an irreducible component of any representation parabolically induced from some proper F-stable Levi subgroup of an F-stable parabolic subgroup of G. In other words, σ is cuspidal if for any F-stable Levi subgroup L of a proper F-stable parabolic subgroup, and for any $\rho \in Irr(L^F)$, it holds

$$\langle \sigma, R_{L^F}^{G^F} \rho \rangle = 0.$$

Since an F-stable Levi subgroup L of a F-stable parabolic subgroup of G is a connected reductive algebraic group over $\overline{\mathbb{F}}_p$ with $\mathbb{F}_q$-structure given by the restriction to L of the Frobenius morphism F, the same definition of cuspidality can be given

for irreducible representations of L^F.
Let L be an F-stable Levi subgroup F-stable parabolic subgroup P of G such that L^F has a cuspidal representation σ. Then we say that (L,σ) is a cuspidal pair, and we denote by $\mathcal{E}^G(L,\sigma)$ the set of irreducible representations of G^F that appear in $R_{L^F}^{G^F}(\sigma)$ (this set is called an "Harish-Chandra series"). Two series $\mathcal{E}^G(L,\sigma)$, $\mathcal{E}^G(L',\sigma')$ are equal if (L,σ) and (L',σ') are conjugate by an element of G^F, and are disjoint otherwise [20, Theorem 5.3.7] . Hence the Harish-Chandra series yield a partition of the set $Irr(G^F)$:

$$Irr(G^F) = \bigsqcup_{(L,\sigma)} \mathcal{E}^G(L,\sigma)$$

where the disjoint union runs through G^F-conjugacy classes of cuspidal pairs (L,σ). The relative Weyl group of the cuspidal pair (L,σ) is given by

$$W_{G^F}(L,\sigma) := {\{g \in G^F \mid gLg^{-1} = L\ ,\ \sigma \circ ad(g) = \sigma\}}\big/{L^F}.$$

The group $W_{G^F}(L,\sigma)$ decomposes as $W_{G^F}(L,\sigma) = W' \rtimes \Omega$ [31, Lemma 2.7], see also [49, Theorem 3.1.8], where W' is a Weyl group and Ω is a finite group acting on it. If G has connected centre or σ is unipotent (in the sense of [14, Section 12.1]), then Ω is trivial, i.e. $W_{G^F}(L,\sigma)$ is a Weyl group [49, Theorem 3.2.5].

Let T be an F-stable maximal torus in G contained in an F-stable Borel subgroup B such that $T \leqslant L$. We realize the Weyl group of G as $W := {N_G(T)}\big/{T}$, and we let S be the set of simple reflections of W corresponding to B. The G^F-conjugacy class of L corresponds to an F-stable subset $I \subset S$ of simple reflections. Let W_I be the subgroup of W generated by I. Then the relative Weyl group $W_{G^F}(L,\sigma)$ can be identified with a subgroup of W^F through an appropriate choice of representative thanks to the following lemma.

Lemma 1.1.2. *[20, Lemma 6.1.7] In the notation above, it holds*

$${N_{G^F}(L)}\big/{L^F} \cong N_I^F := \{w \in W^F | {}^wI = I,\ w \text{ has minimal length in } wW_I\}$$

The relative Weyl group plays a central role in the study of the representations of G^F, thanks to the following theorem.

Theorem 1.1.3. *[49, Theorem 3.2.5] Let (L,σ) be a cuspidal pair. Then the endomorphism algebra $End_{G^F}(R_{L^F}^{G^F}(\sigma))$ is isomorphic to the finite extended Hecke algebra $\mathcal{H}(W_{G^F}(L,\sigma),\{q_s\}_{s\in S})$ for some suitable parameters, and*

$$End(R_{L^F}^{G^F}(\sigma)) \cong \mathbb{C}W_{G^F}(L,\sigma). \tag{1.1}$$

The isomorphism (1.1) holds by Theorem 1.1.1 applied to the case in which Ω and μ are trivial. Indeed $End_{G^F}(R_{L^F}^{G^F}(\sigma))$ is a semisimple algebra because $R_{L^F}^{G^F}(\sigma)$ is semisimple.

For any cuspidal pair (L,σ), there is a bijection

$$\begin{aligned}\mathcal{E}^G(L,\sigma) &\to Irr(End_{G^F}(R_{L^F}^{G^F}(\sigma)))\\ \rho &\mapsto Hom_{G^F}(\rho, R_{L^F}^{G^F}(\sigma))\end{aligned}$$

and by Theorem 1.1.3, there is canonical isomorphism $Irr(End_{G^F}(R_{L^F}^{G^F}(\sigma))) \cong Irr(W_{G^F}(L,\sigma))$, given by specialization of characters values. It follows that there is a canonical bijection

$$Rep_{G^F} : \mathcal{E}^G(L,\sigma) \to Irr(W_{G^F}(L,\sigma)). \tag{1.2}$$

Extending by linearity we obtain invertible linear maps

$$Rep_{G^F} : \mathbb{C}[\mathcal{E}^G(L,\sigma)] \to R(W_{G^F}(L,\sigma))]. \tag{1.3}$$

The isomorphism Rep_{G^F} is an isometry, since it maps the orthonormal base of $\mathbb{C}[\mathcal{E}^G(L,\sigma)]$ given by $\mathcal{E}^G(L,\sigma)$ to the orthonormal base of $R(W_{G^F}(L,\sigma))$ given by $Irr(W_{G^F}(L,\sigma))$. The following result, due to Howlett and Lehrer and known as the comparison theorem, describes parabolic induction in terms of induction of representations in the relative Weyl groups.

Theorem 1.1.4. *([32, Theorem 5.9], see also [49, Theorem 3.2.7]). Let M be an F-stable Levi subgroup of an F-stable parabolic subgroup of G. Let L be an F-stable Levi subgroup of an F-stable parabolic subgroup of M, and let σ be a cuspidal representation of L. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathbb{C}[\mathcal{E}^M(L,\sigma)] & \xrightarrow{Rep_{M^F}} & \mathbb{C}[Irr(W_{M^F}(L,\sigma))] \\
\big\downarrow R_{M^F}^{G^F} & & \big\downarrow Ind_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \\
\mathbb{C}[\mathcal{E}^G(L,\sigma)] & \xrightarrow{Rep_{G^F}} & \mathbb{C}[Irr(W_{G^F}(L,\sigma))]
\end{array}$$

where $Ind_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)}$ denotes the extension by linearity on $\mathbb{C}[Irr(W_{M^F}(L,\sigma))]$ of the usual induction of representations of finite groups.

Corollary 1.1.5. *Let M be an F-stable Levi subgroup of an F-stable parabolic subgroup of G. Let L be an F-stable Levi subgroup of an F-stable parabolic subgroup of G, and let σ be a cuspidal representation of L. Then*

- ◇ *If L is contained in M, the following diagram commutes*

$$\begin{array}{ccc}
\mathbb{C}[\mathcal{E}^G(L,\sigma)] & \xrightarrow{Rep_{G^F}} & \mathbb{C}[Irr(W_{G^F}(L,\sigma))] \\
\big\downarrow {}^*R_{M^F}^{G^F} & & \big\downarrow Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \\
\mathbb{C}[\mathcal{E}^M(L,\sigma)] & \xrightarrow{Rep_{M^F}} & \mathbb{C}[Irr(W_{M^F}(L,\sigma))]
\end{array}$$

  where $Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)}$ denotes the extension by linearity to $\mathbb{C}[Irr(W_{M^F}(L,\sigma))]$ of the usual restriction of representations of finite groups.

- ◇ *If no G^F-conjugate to L is contained in M, then ${}^*R_{M^F}^{G^F} = 0$ on $\mathcal{E}^G(L,\sigma)$.*

Proof. The last statement follows from the disjointedness of the Harish-Chandra series. Indeed, suppose there is $\chi \in \mathcal{E}^G(L,\sigma)$ such that ${}^*R_{M^F}^{G^F}(\chi) \neq 0$. Then there is a Levi subgroup L' contained in M supporting a cuspidal representation σ' such that $\langle {}^*R_{L'}^{M} \circ^* R_{M^F}^{G^F}\chi, \sigma'\rangle_{L'} \neq 0$, and so, by transitivity of restriction and Frobenius reciprocity, it holds $\chi \in \mathcal{E}^G(L', \sigma')$. So (L, σ) and (L', σ') are conjugate in G^F.

By the same argument, the image through ${}^*R_{M^F}^{G^F}$ of $\mathbb{C}[\mathcal{E}^G(L, \sigma)]$ is contained in $\mathbb{C}[\mathcal{E}^M(L, \sigma)]$. We check that for any $\psi \in \mathcal{E}^G(L, \sigma)$, the irreducible constituents of $(Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)}) \circ Rep_{G^F}(\psi))$ and $(Rep_{M^F} \circ {}^*R_{M^F}^{G^F}(\psi))$ are the same.
For any $\pi \in Irr(W_{M^F}(L, \sigma))$, it holds $\pi = Rep_M(\rho)$ for some $\rho \in \mathcal{E}^M(L, \sigma)$. Then

$$
\begin{aligned}
&\langle Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \circ Rep_{G^F}(\psi), \pi\rangle_{W_{M^F}(L,\sigma)} \\
=&\langle Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \circ Rep_{G^F}(\psi), Rep_{M^F}(\rho)\rangle_{W_{M^F}(L,\sigma)} && \text{adjunction of } Ind_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \text{ and } Res_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \\
=&\langle Rep_{G^F}(\psi), Ind_{W_{M^F}(L,\sigma)}^{W_{G^F}(L,\sigma)} \circ Rep_{M^F}(\rho)\rangle_{W_{M^F}(L,\sigma)} && \text{Theorem 1.1.4} \\
=&\langle Rep_{G^F}(\psi), Rep_{G^F} \circ R_{M^F}^{G^F}(\rho)\rangle_{W_{G^F}(L,\sigma)} && Rep_{G^F} \text{ is an isometry} \\
=&\langle \psi, R_{M^F}^{G^F}(\rho)\rangle_{G^F} && \text{adjunction of } R_{M^F}^{G^F} \text{ and } {}^*R_{M^F}^{G^F} \\
=&\langle {}^*R_{M^F}^{G^F}(\psi), \rho\rangle_{M^F} && Rep_{M^F} \text{ is an isometry} \\
=&\langle Rep_{M^F} \circ {}^*R_{M^F}^{G^F}(\psi), Rep_{M^F}(\rho)\rangle_{W_{M^F}(L,\sigma)} \\
=&\langle Rep_{M^F} \circ {}^*R_{M^F}^{G^F}(\psi), \pi\rangle_{W_{M^F}(L,\sigma)}.
\end{aligned}
$$

□

1.2 Endomorphism algebras, the disconnected case

The main goal of the rest of this chapter is to generalize Theorem 1.1.3 and Theorem 1.1.4 in a case in which the underlying algebraic group is not connected. In particular, we focus on the case of a split extension of the group G given by a finite cyclic group. To the knowledge of the author, these results have not been published anywhere at the time of writing.

1.2.1 General setting

Let δ be an automorphism of order d acting on G and commuting with F. Then δ acts on the finite group of Lie type G^F, and consequently induces a bijection δ^* on $Irr(G^F)$. If M is an F-stable and δ-stable Levi subgroup of a parabolic subgroup of G, then δ acts on the finite group of Lie type M^F, and consequently induces an action δ^* on $Irr(M^F)$.

Lemma 1.2.1. *Let M be an F-stable and δ-stable Levi subgroup of an F-stable and δ-stable parabolic subgroup P of G, and let U be the unipotent radical of P. Then:*

1. *The bijections δ^* on $Irr(G^F)$ and $Irr(M^F)$ are compatible with the parabolic restriction ${}^*R_{M^F}^{G^F}$ and the parabolic induction $R_{M^F}^{G^F}$, that is*

$$
\begin{aligned}
{}^*R_{M^F}^{G^F} \circ \delta^*\chi &= \delta^* \circ {}^*R_{M^F}^{G^F}\chi && \textit{for any } \chi \in Irr(G^F), \\
R_{M^F}^{G^F} \circ \delta^*\phi &= \delta^* \circ {}^*R_{M^F}^{G^F}\phi && \textit{for any } \phi \in Irr(M^F).
\end{aligned}
$$

2. *The bijection δ^* maps cuspidal representations to cuspidal representations*

3. *The bijection δ^* maps unipotent representations to unipotent representations.*

In particular, if G is simple of classical type, the action δ^* fixes unipotent cuspidal representations.

Proof. 1. We show that δ^* is compatible with the parabolic restriction ${}^*R_{M^F}^{G^F}$: for any $\chi \in Irr(G^F)$ and for any $m \in M^F$

$$\begin{aligned}
{}^*R_{M^F}^{G^F} \circ \delta^*\chi(m) &= \frac{1}{|U^F|}\sum_{u \in U^F} \delta^*\chi(mu) \\
&= \frac{1}{|U^F|}\sum_{u \in U^F} \chi(\delta^{-1}(mu)) \\
&= \frac{1}{|U^F|}\sum_{u \in U^F} \chi(\delta^{-1}(m)\delta^{-1}(u)) \qquad \delta \text{ is a bijection on } U^F \\
&= \frac{1}{|U^F|}\sum_{v \in U^F} \chi(\delta^{-1}(m)v) \\
&=^* R_{M^F}^{G^F}\chi(\delta^{-1}(m)) \\
&= \delta^* \circ^* R_{M^F}^{G^F}\chi(m).
\end{aligned}$$

It follows that the action of δ^* is also compatible with parabolic induction: for any $\chi \in Irr(G^F)$ and for any $\phi \in Irr(M^F)$, it holds

$$\begin{aligned}
\langle R_{M^F}^{G^F} \circ \delta^*\phi \ , \ \chi\rangle_{G^F} = \langle \delta^*\phi \ , \ {}^*R_{M^F}^{G^F}\chi\rangle_{M^F} = \langle \phi \ , \ (\delta^*)^{-1} \circ {}^*R_{M^F}^{G^F}\chi\rangle_{M^F} \\
= \langle \phi \ , \ {}^*R_{M^F}^{G^F} \circ (\delta^*)^{-1}\chi\rangle_{M^F} = \langle R_{M^F}^{G^F}\phi \ , \ (\delta^*)^{-1}\chi\rangle_{G^F} = \langle \delta^* \circ R_{M^F}^{G^F}\phi \ , \ \chi\rangle_{G^F}
\end{aligned}$$

In the above calculation we used the fact that δ^* is an isometry because it is induced by an automorphism, so it maps irreducible representations to irreducible representations, and we used the adjunction between parabolic induction and parabolic restriction.

2. Follows from (1).

3. We claim that for any F-stable maximal torus T of G, it holds

$$\delta^* R_T^G 1 = R_{\delta^{-1}(T)}^G 1 \tag{1.4}$$

where $R_T^G 1$ is the Deligne-Lusztig generlized character as in [14, 7.2]. Indeed, let B be a Borel subgroup of G containing T and let U denote the unipotent radical of B; let

$$X_U := \{x \in G \mid x^{-1}F(x) \in F(U)\} \big/ U \cap F(U),$$

$$X_{\delta^{-1}(U)} := \{x \in G \mid x^{-1}F(x) \in F(\delta^{-1}(U))\} \big/ \delta^{-1}(U) \cap F(\delta^{-1}(U))$$

be the Deligne-Lusztig varieties associated to T and $\delta^{-1}(T)$ respectively, with left G^F action by multiplication. Then, since δ and F commute, $\delta(X_{\delta^{-1}(U)}) = X_U$ and the following diagram commutes for any $g \in G^F$:

$$\begin{array}{ccc} X_{\delta^{-1}(U)} & \xrightarrow{\delta} & X_U \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \delta(g)} \\ X_{\delta^{-1}(U)} & \xrightarrow{\delta} & X_U \end{array} \tag{1.5}$$

Hence the following diagram is commutative:

$$\begin{array}{ccc} H_c^*(X_U) & \xrightarrow{\delta} & H_c^*(X_{\delta^{-1}(U)}) \\ \downarrow{\scriptstyle \delta(g)} & & \downarrow{\scriptstyle g} \\ H_c^*(X_U) & \xrightarrow{\delta} & H_c^*(X_{\delta^{-1}(U)}) \end{array} \tag{1.6}$$

where H_c^* denotes l-adic cohomology, with l prime different from p, and the arrows are the maps on the cohomology spaces induced from the corresponding ones in the above diagram [14, 7.1.3]. Therefore

$$\delta^* \circ^* R_T^G 1(g) = Tr(\delta(g) \mid H_c^*(X_U)) = Tr(g \mid H_c^*(X_{\delta^{-1}(U)})) = R_{\delta^{-1}(T)}^G 1(g).$$

Hence, if G^F affords a cuspidal unipotent representation σ, then $\delta^*\sigma$ is still unipotent and cuspidal. In particular, if G is simple of classical type, G^F affords at most one cuspidal unipotent representation, so this has to be fixed by δ^*.

□

We now consider the group $G^F \rtimes \langle\delta\rangle$.

For each F-stable Levi subgroup M of an F-stable parabolic subgroup P of G, with unipotent radical U, we define the parabolic induction to $G^F \rtimes \langle\delta\rangle$ as

$$R_{M^F}^{G^F \rtimes\langle\delta\rangle} := Ind_{G^F}^{G^F \rtimes\langle\delta\rangle} \circ R_{M^F}^{G^F} = Ind_{P^F}^{G^F \rtimes\langle\delta\rangle} \circ Infl_{M^F}^{P^F}. \tag{1.7}$$

and the parabolic restriction as the adjoint functor

$${}^*R_{M^F}^{G^F \rtimes\langle\delta\rangle} := {}^*R_{M^F}^{G^F} \circ Res_{G^F}^{G^F \rtimes\langle\delta\rangle} = (Res_{P^F}^{G^F \rtimes\langle\delta\rangle})^{U^F} \tag{1.8}$$

where $(\cdot)^{U^F}$ denotes the projection to the U^F-invariant subspace.

Moreover, if M and P are δ-stable, we set

$$R_{M^F \rtimes\langle\delta\rangle}^{G^F \rtimes\langle\delta\rangle} := Ind_{P^F \rtimes\langle\delta\rangle}^{G^F \rtimes\langle\delta\rangle} \circ Infl_{M^F \rtimes\langle\delta\rangle}^{P^F \rtimes\langle\delta\rangle} \tag{1.9}$$

and denote by ${}^*R_{M^F \rtimes\langle\delta\rangle}^{G^F \rtimes\langle\delta\rangle}$ the adjoint functor

$${}^*R_{M^F \rtimes\langle\delta\rangle}^{G^F \rtimes\langle\delta\rangle} := (Res_{P^F \rtimes\langle\delta\rangle}^{G^F \rtimes\langle\delta\rangle})^{U^F}. \tag{1.10}$$

If L is an F-stable and δ-stable Levi subgroup of an F-stable and δ-stable parabolic subgroup of M, we write $R_{L^F}^{M^F \rtimes\langle\delta\rangle}$ and $R_{L^F \rtimes\langle\delta\rangle}^{M^F \rtimes\langle\delta\rangle}$ for the functors analogously defined plugging in M in place of G and L in place of M.

Remark 1.2.2. Let N be a finite group, $H \leqslant K \leqslant Aut(N)$. Then it holds

$$Ind_{N\rtimes H}^{N\rtimes K} \circ Infl_{H}^{N\rtimes H} = Infl_{K}^{N\rtimes K} \circ Ind_{H}^{K}. \tag{1.11}$$

Indeed for any $\chi \in Irr(H)$, $n \in N, k \in K$, it holds

$$\begin{aligned}(Ind_{N\rtimes H}^{N\rtimes K} Infl_{H}^{N\rtimes H}\chi)(nk) &= \frac{1}{|N||K|} \sum_{\substack{n'\in N, k'\in K \\ (n'k')^{-1}nkn'k'\in N\rtimes H}} Infl_{H}^{N\rtimes H}\chi((n'k')^{-1}nkn'k') \\ &= \frac{1}{|N||K|} \sum_{\substack{n'\in N, k'\in K \\ (k')^{-1}kk'\in H}} Infl_{H}^{N\rtimes H}\chi((k')^{-1}kk') \\ &= \frac{1}{|K|} \sum_{\substack{k'\in K \\ (k')^{-1}kk'\in H}} \chi((k')^{-1}kk') \\ &= Ind_{H}^{K}\chi(k) = (Infl_{K}^{N\rtimes K} Ind_{H}^{K}\chi)(nk).\end{aligned}$$

Lemma 1.2.3. *Let M be an F-stable and δ-stable Levi subgroup of an F-stable and δ-stable parabolic subgroup P of G. Then*

$$R_{M^F}^{G^F\rtimes\langle\delta\rangle} = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{M^F}^{M^F\rtimes\langle\delta\rangle} \tag{1.12}$$

Proof. The unipotent radical U of P is F-stable and δ-stable, and so U^F is a normal subgroup in $P^F \rtimes \langle\delta\rangle$ yielding a projection $P^F \rtimes \langle\delta\rangle \twoheadrightarrow M^F \rtimes \langle\delta\rangle$. Applying (1.11) to U^F, M^F and $M^F \rtimes \langle\delta\rangle$ we get

$$Ind_{P^F}^{P^F\rtimes\langle\delta\rangle} \circ Infl_{M^F}^{P^F} = Infl_{M^F\rtimes\langle\delta\rangle}^{P^F\rtimes\langle\delta\rangle} \circ Ind_{M^F}^{M^F\rtimes\langle\delta\rangle}.$$

It follows that

$$R_{M^F}^{G^F\rtimes\langle\delta\rangle} = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{M^F}^{M^F\rtimes\langle\delta\rangle}$$

where $R_{M^F}^{M^F\rtimes\langle\delta\rangle} = Ind_{M^F}^{M^F\rtimes\langle\delta\rangle}$, compatibly with (1.9) for M viewed as an F-stable Levi subgroup of itself. □

Corollary 1.2.4. *Let M and L be F-stable and δ-stable Levi subgroups of F-stable and δ-stable parabolic subgroups P_L and P_M of G, respectively. Assume that $P_L \leqslant P_M$ and $L \leqslant M$. Then*

$$R_{L^F}^{G^F\rtimes\langle\delta\rangle} = R_{M^F}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F} = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F\rtimes\langle\delta\rangle}. \tag{1.13}$$

Proof. The first equality follows directly from the definition of $R_{L^F}^{G^F\rtimes\langle\delta\rangle}$ and transitivity of parabolic induction:

$$R_{L^F}^{G^F\rtimes\langle\delta\rangle} = Ind_{G^F}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{G^F} = Ind_{G^F}^{G^F\rtimes\langle\delta\rangle} \circ R_{M^F}^{G^F} \circ R_{L^F}^{M^F} = R_{M^F}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F}.$$

The second equality then follows from (1.12) and transitivity of parabolic induction:

$$R_{L^F}^{G^F\rtimes\langle\delta\rangle} = R_{M^F}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F} = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{M^F}^{M^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F} = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F\rtimes\langle\delta\rangle}.$$

□

The notion of Harish-Chandra series can be extended to the group $G^F \rtimes \langle\delta\rangle$: if L is an F-stable Levi subgroup of an F-stable parabolic subgroup of G such that L^F affords a cuspidal representation σ, we set

$$\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma) := \{\chi \in Irr(G^F \rtimes \langle\delta\rangle) \mid \langle \chi, R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma\rangle_{G^F \rtimes\langle\delta\rangle} \neq 0\}.$$

Lemma 1.2.5. *Let L and L' be δ-stable and F-stable Levi subgroups of δ-stable and F-stable parabolic subgroups of G. Assume that L^F and L'^F afford δ^*-stable cuspidal representations σ and σ' respectively.*
Then

$$\langle R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma \ , \ R_{L'^F}^{G^F \rtimes\langle\delta\rangle}\sigma'\rangle_{G^F \rtimes\langle\delta\rangle} = d\langle R_{L^F}^{G^F}\sigma \ , \ R_{L'^F}^{G^F}\sigma'\rangle_{G^F}. \tag{1.14}$$

In particular the Harish-Chandra series $\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ and $\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L',\sigma')$ are equal if (L,σ) and (L',σ') are G^F-conjugate, and they are disjoint otherwise.

Proof. Frobenius reciprocity gives

$$\begin{aligned}&\langle R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma \ , \ R_{L'^F}^{G^F \rtimes\langle\delta\rangle}\sigma'\rangle_{G^F \rtimes\langle\delta\rangle} = \\ &\langle Ind_{G^F}^{G^F \rtimes\langle\delta\rangle} R_{L^F}^{G^F}\sigma \ , \ Ind_{G^F}^{G^F \rtimes\langle\delta\rangle} R_{L'^F}^{G^F}\sigma'\rangle_{G^F \rtimes\langle\delta\rangle} = \\ &\langle R_{L^F}^{G^F}\sigma \ , \ Res_{G^F}^{G^F \rtimes\langle\delta\rangle} Ind_{G^F}^{G^F \rtimes\langle\delta\rangle} R_{L'^F}^{G^F}\sigma'\rangle_{G^F}.\end{aligned} \tag{1.15}$$

Since $\{\delta^i | i = 0, \ldots, d-1\}$ is a set of representatives for the double cosets of G^F in $G^F \rtimes \langle\delta\rangle$, and G^F is δ-stable, the Mackey formula gives

$$Res_{G^F}^{G^F \rtimes\langle\delta\rangle} Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}(R_{L'^F}^{G^F}\sigma') = \sum_{i=0}^{d-1}(\delta^i)^* \ R_{L'^F}^{G^F}\sigma'. \tag{1.16}$$

By Remark 1.2.1 we have $\delta^*(R_{L'^F}^{G^F}\sigma') = R_{L'^F}^{G^F}(\delta^*\sigma')$. Combining with (1.15) gives

$$\begin{aligned}&\langle R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma \ , \ R_{L'^F}^{G^F \rtimes\langle\delta\rangle}\sigma'\rangle_{G^F \rtimes\langle\delta\rangle} = \\ &\langle R_{L^F}^{G^F}\sigma \ , \sum_{i=0}^{d-1} \ R_{L'^F}^{G^F}(\delta^i)^*\sigma'\rangle_{G^F} = \\ &\langle R_{L^F}^{G^F}\sigma \ , \sum_{i=0}^{d-1} \ R_{L'^F}^{G^F}\sigma'\rangle_{G^F} = d\langle R_{L^F}^{G^F}\sigma \ , \ R_{L'^F}^{G^F}\sigma'\rangle_{G^F}.\end{aligned}$$

The statement on Harish-Chandra series follows since if (L,σ) and (L',σ') are not G^F conjugate, then $\langle R_{L^F}^{G^F}\sigma \ , \ R_{L'^F}^{G^F}\sigma'\rangle_{G^F} = 0$, so

$$\langle R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma \ , \ R_{L'^F}^{G^F \rtimes\langle\delta\rangle}\sigma'\rangle_{G^F \rtimes\langle\delta\rangle} = 0$$

and hence $R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma$ and $R_{L'^F}^{G^F \rtimes\langle\delta\rangle}\sigma'$ don't have any common irreducible constituents. On the other hand, if (L,σ) and (L',σ') are G^F-conjugate, then $R_{L^F}^{G^F}\sigma \cong R_{L'^F}^{G^F}\sigma'$, therefore inducing to $G^F \rtimes \langle\delta\rangle$ yields isomorphic representations. □

Corollary 1.2.6. *Retaining notation from Lemma 1.2.5, there holds*

$$\langle R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma\ ,\ R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma\rangle_{G^F\rtimes\langle\delta\rangle} = |W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)|.$$

Proof. By [20, Corollary 5.3.8]

$$\langle R_{L^F}^{G^F}\sigma\ ,\ R_{L^F}^{G^F}\sigma\rangle_{G^F} = |W_{G^F}(L,\sigma)|$$

and Lemma 1.2.5 gives

$$\langle R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma\ ,\ R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma\rangle_{G^F\rtimes\langle\delta\rangle} = d\langle R_{L^F}^{G^F}\sigma\ ,\ R_{L^F}^{G^F}\sigma\rangle_{G^F}.$$

Since $W_{G^F\rtimes\langle\delta\rangle}(L,\sigma) = W_{G^F}(L,\sigma)\rtimes\langle\delta\rangle$, with δ of order d, it holds

$$|W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)| = d|W_{G^F}(L,\sigma)|.$$

□

Let L be an F-stable Levi subgroup of an F-stable parabolic subgroup P of G. We set

$$N_{G^F}(L) := \{g\in G^F \mid gLg^{-1} = L\} \qquad W_{G^F}(L) := N_{G^F}(L)\big/L^F$$
$$N_{G^F\rtimes\langle\delta\rangle}(L) := \{g\in G^F\rtimes\langle\delta\rangle \mid gLg^{-1} = L\} \quad W_{G^F\rtimes\langle\delta\rangle}(L) = N_{G^F\rtimes\langle\delta\rangle}(L)\big/L^F.$$

If L^F affords a cuspidal representation σ, we set

$$N_{G^F}(L,\sigma) := \{g\in G^F \mid gLg^{-1} = L\ ,\ \sigma\circ ad(g)\cong\sigma\} \tag{1.17}$$
$$W_{G^F}(L,\sigma) := N_{G^F}(L,\sigma)\big/L^F \tag{1.18}$$
$$N_{G^F\rtimes\langle\delta\rangle}(L,\sigma) := \{g\in G^F\rtimes\langle\delta\rangle \mid gLg^{-1} = L\ ,\ \sigma\circ ad(g)\cong\sigma\} \tag{1.19}$$
$$W_{G^F\rtimes\langle\delta\rangle}(L,\sigma) = N_{G^F\rtimes\langle\delta\rangle}(L,\sigma)\big/L^F. \tag{1.20}$$

If L is δ-stable and σ is δ^*-stable, then δ stabilizes the subgroup $N_{G^F}(L,\sigma)$, and $N_{G^F\rtimes\langle\delta\rangle}(L,\sigma)\cong N_{G^F}(L,\sigma)\rtimes\langle\delta\rangle$.

Taking quotient by L^F, we have

$$W_{G^F}(L,\sigma)\rtimes\langle\delta\rangle\cong W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$$

where on the left hand side the action of δ on $W_{G^F}(L,\sigma)$ is induced by the action of δ on $N_{G^F}(L,\sigma)$.

If M is an F-stable and δ-stable Levi subgroup of G containing L, we write $N_{M^F}(L,\sigma)$, $W_{M^F}(L,\sigma), N_{M^F\rtimes\langle\delta\rangle}(L,\sigma)$, $W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)$ for the groups analogous to the ones defined above for G but relative to M.

1.2.2 The structure of the endomorphism algebra

Until the end of chapter I, we assume the centre of G to be connected.

Moreover we assume that δ stabilizes a pair (T,B) consisting of an F-stable maximal torus T contained in an F-stable Borel subgroup B, and we fix such a pair. Note that up to composing δ with an inner automorphism of G^F, a stable pair always exists. Indeed, since δ commutes with F, if (T, B) is a pair consisting of an F-stable maximal torus T contained in an F-stable Borel subgroup B, then $(F(\delta(T)), F(\delta(B))) = (\delta(F(T)), \delta(F(B))) = (\delta(T), \delta(B))$, that is the pair $(\delta(T), \delta(B))$ is F-stable. All pairs consisting of an F-stable maximal torus contained in an F-stable Borel are G^F-conjugate [20, Corollary 4.2.15], so there exists $x \in G^F$ such that $(\delta(T), \delta(B)) = ({}^xT, {}^xB)$. Hence composing δ with the inner automorphism given by conjugation by x we obtain an automorphism stabilizing the pair (T, B).

From now until the end of Section 1.2.2, L is a δ-stable and F-stable Levi subgroup of a δ-stable and F-stable parabolic subgroup P of G containing T, and U denotes the unipotent radical of P. We assume that L^F has a δ-stable cuspidal representation σ.

We show that the endomorphism algebra $End_{G^F \rtimes \langle\delta\rangle}(L, \sigma)$ is a finite extended Hecke algebra. Firstly, we construct a basis for $End_{G^F \rtimes\langle\delta\rangle}(R_L^{G^F \rtimes\langle\delta\rangle}(\sigma))$ and then we exhibit a set of relations for this algebra, in the spirit of [31]. The exposition follows closely [20, Section 6].

Let V be the $\mathbb{C}$-vector space affording the representation σ. For any $n \in N_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$, there is a linear operator t_n on V, determined up to a scalar multiple, such that for any $l \in L^F$

$$\sigma(n^{-1}ln) = t_n^{-1}\sigma(l)t_n. \tag{1.21}$$

A choice for the t_n, with $n \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$, determines a two-cocycle

$$\begin{aligned} \widehat{\lambda} : N_{G^F \rtimes\langle\delta\rangle}(L, \sigma) \times N_{G^F \rtimes\langle\delta\rangle}(L, \sigma) &\to \mathbb{C} \\ (n, m) &\mapsto t_n t_m t_{nm}^{-1} \end{aligned} \tag{1.22}$$

The cocycle property follows from the associativity of multiplication of the linear operators t_n. For any $x \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$, we fix a choice of a representative $\dot{x} \in N_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$. Then, for any choice of the linear operator $t_{\dot{x}}$ and for any $l, l' \in L^F$ it holds

$$\begin{aligned} \sigma((l'\dot{x})^{-1}ll'\dot{x}) &= \sigma(\dot{x}^{-1}(l')^{-1}ll'\dot{x}) = t_{\dot{x}}^{-1}\sigma((l')^{-1}ll')t_{\dot{x}} \\ &= t_{\dot{x}}^{-1}\sigma(l')^{-1}\sigma(l)\sigma(l')t_{\dot{x}} = (\sigma(l')t_{\dot{x}})^{-1}\sigma(l)\sigma(l')t_{\dot{x}}. \end{aligned} \tag{1.23}$$

For any $n \in N_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ there exists an $x \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ and an $l' \in L^F$ such that $n = l'\dot{x}$, and by (1.23) the linear operator t_n can be chosen to satisfy

$$t_{l'\dot{x}} := \sigma(l')t_{\dot{x}}. \tag{1.24}$$

Since σ is multiplicative, it follows that for any $n \in N_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ and any $l \in L^F$ it holds

$$t_{ln} := \sigma(l)t_n. \tag{1.25}$$

Remark 1.2.7. Given a choice of the linear operators t_n satisfying (1.25), the cocycle (1.22) induces a well defined cocycle λ on $W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$.
Indeed, for any $l, l' \in L^F$ and for any $n, m \in N_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ there holds

$$t_{ln}t_{l'm} = \widehat{\lambda}(ln, l'm)t_{lnl'm} = \widehat{\lambda}(ln, l'm)\sigma(l)t_{(nl'n^{-1})nm} = \widehat{\lambda}(ln, l'm)\sigma(l)\sigma(nl'n^{-1})t_{nm}$$

$$t_{ln}t_{l'm} = \sigma(l)t_n\sigma(l')t_m = \sigma(l)t_n\sigma(l')t_n^{-1}t_nt_m = \sigma(l)\sigma(nl'n^{-1})\widehat{\lambda}(n, m)t_{nm}.$$

Therefore $\widehat{\lambda}(ln, l'm) = \widehat{\lambda}(n, m)$, and so

$$\begin{aligned}\lambda : W_{G^F \rtimes \langle\delta\rangle}(L,\sigma) \times W_{G^F \rtimes \langle\delta\rangle}(L,\sigma) &\to \mathbb{C}\\ (x,y) &\mapsto \widehat{\lambda}(\dot{x},\dot{y})\end{aligned}$$

is well defined.

Moreover the linear maps $t_{\dot{x}}$ with $x \in W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ can be chosen so that the cocycle λ is trivial on $W_{G^F}(L,\sigma)$, [20, Theorem 6.1.9]. We fix such a choice.

We write $\mathbb{C}(G^F \rtimes \langle\delta\rangle)$ for the complex group algebra of the group $G^F \rtimes \langle\delta\rangle$, and set

$$e_U = \frac{1}{|U^F|}\sum_{u\in U^F} u \in \mathbb{C}(G^F \rtimes \langle\delta\rangle).$$

Note that since L^F normalizes U^F, the idempotent e_U commutes with any $l \in L^F$. The representation $R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma$ can be realised as the $G^F \rtimes \langle\delta\rangle$-module

$$\mathbb{C}(G^F \rtimes \langle\delta\rangle)e_U \otimes_{\mathbb{C}L^F} V$$

where $G^F \rtimes \langle\delta\rangle$ acts by left multiplication.
For any $n \in N_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ define

$$\begin{aligned}B_n : \mathbb{C}(G^F \rtimes \langle\delta\rangle)e_U \otimes_{\mathbb{C}L^F} V &\to \mathbb{C}(G^F \rtimes \langle\delta\rangle)e_U \otimes_{\mathbb{C}L^F} V\\ ge_U \otimes v &\mapsto ge_U n^{-1} e_U \otimes t_n(v).\end{aligned} \tag{1.26}$$

These maps are well-defined:

$$\begin{aligned}B_n(ge_U l \otimes v) &= ge_U l n^{-1} e_U \otimes t_n(v) = ge_U n^{-1}(nln^{-1})e_U \otimes v\\ &= ge_U n^{-1} e_U (nln^{-1}) \otimes v = ge_U n^{-1} e_U \otimes \sigma(nln^{-1}) t_n v\\ &= ge_U n^{-1} e_U \otimes t_n \sigma(l) v = B_n(ge_U \otimes \sigma(l)v).\end{aligned}$$

By construction the maps B_n commute with the action of $G^F \rtimes \langle\delta\rangle$, so they belong to $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$.

Remark 1.2.8. For any $n \in N_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$, for any $l \in L^F$ and $ge_U \otimes v \in \mathbb{C}(G^F \rtimes \langle\delta\rangle)e_U \otimes_{\mathbb{C}L^F} V$ it holds

$$\begin{aligned}B_{ln}(ge_U \otimes v) &= ge_U n^{-1} l^{-1} e_U \otimes t_{ln}(v)\\ &= ge_U n^{-1} e_U l^{-1} \otimes \sigma(l) t_n(v)\\ &= ge_U n^{-1} e_U \otimes t_n(v) = B_n(ge_U \otimes v).\end{aligned}$$

Therefore B_n depends only on the class of n in $W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$.

In consequence of Remark 1.2.8, in the following we will write B_x, with $x \in W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ in place of B_n, where x is the class of n in $W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$.

Remark 1.2.9. The representation $R_{L^F}^{G^F}\sigma$ can be realized as $\mathbb{C}G^F e_U \otimes_{\mathbb{C}L^F} V$, where G^F acts by left multiplication. By [20, Lemma 6.1.6] a basis for $End_{G^F}(R_{L^F}^{G^F}\sigma)$ is given by the endomorphisms

$$\begin{aligned} \widetilde{B}_x : R_{L^F}^{G^F}\sigma &\to R_{L^F}^{G^F}\sigma \\ ge_U &\mapsto ge_U\dot{x}^{-1}e_U \otimes t_{\dot{x}}(v) \end{aligned}$$

for x running in $W_{G^F}(L,\sigma)$. Note that $\widetilde{B}_x$ is the restriction of B_x to $\mathbb{C}G^F e_U \otimes_{\mathbb{C}L^F} V$, and that $R_{L^F}^{G^F \rtimes \langle\delta\rangle} = \mathbb{C}(G^F \rtimes \langle\delta\rangle) \otimes_{G^F} R_{L^F}^{G^F}$, so

$$B_x = id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{B}_x. \tag{1.27}$$

Now we prove that the intertwiners $\{B_x \mid x \in W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)\}$ are a basis for $End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)$. We will need the following lemma.

Lemma 1.2.10. *The map*

$$W_{G^F \rtimes \langle\delta\rangle}(L) \to P^F \backslash G^F \rtimes \langle\delta\rangle / P^F$$

induced by the inclusion $N_{G^F \rtimes \langle\delta\rangle}(L) \hookrightarrow G^F \rtimes \langle\delta\rangle$ *is injective*

Proof. By [66, Proposition 2.20], the map

$$\begin{aligned} W_{G^F}(L) &\hookrightarrow P^F \backslash G^F / P^F \\ \omega &\mapsto P^F \omega P^F \end{aligned} \tag{1.28}$$

induced through natural projection by the inclusion $N_{G^F \rtimes \langle\delta\rangle}(L) \hookrightarrow G^F \rtimes \langle\delta\rangle$ is injective. In (1.28) we write $P^F\omega P^F$ instead of $P^F\dot{\omega}P^F$ with $\dot{\omega}$ a representative of ω in $N_{G^F}(L)$, because the P^F double coset $P^F\dot{\omega}P^F$ does not depend on the choice of the representative $\dot{\omega}$ in $N_{G^F}(L)$.

Since L is δ-stable, so are L^F and its normalizer in G^F, therefore δ induces an action on $W_{G^F}(L)$ and $W_{G^F \rtimes \langle\delta\rangle}(L) \cong W_{G^F}(L) \rtimes \langle\delta\rangle$. Moreover since P is δ-stable, so is P^F. Let $\omega_1\delta^{i_1}, \omega_2\delta^{i_2} \in W_{G^F \rtimes \langle\delta\rangle}(L)$ with $\omega_1, \omega_2 \in W_{G^F}(L)$, and $i_1, i_2 \in \{0, \ldots, d-1\}$. The corresponding P^F-double cosets in G^F are given respectively by $P^F\dot{\omega}_1P^F\delta^{i_1}$ and $P^F\dot{\omega}_2P^F\delta^{i_2}$. Two such cosets are equal if and only if $i_1 = i_2$ and $P^F\dot{\omega}_1P^F = P^F\dot{\omega}_2P^F$ and by the injectivity of (1.28), if and only if $\omega_1 = \omega_2$. So the P^F-double coset in G^F corresponding to $\omega_1\delta^{\epsilon_1}$ and $\omega_2\delta^{\epsilon_2} \in W_{G^F \rtimes \langle\delta\rangle}(L)$ are equal if and only if $\omega_1\delta^{i_1} = \omega_2\delta^{i_2}$. ☐

Proposition 1.2.11. *The set* $\{B_x \mid x \in W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)\}$ *is a* $\mathbb{C}$*-linear basis for* $End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)$.

Proof. We adopt the strategy of the proof of [66, Proposition 5.8].
By Corollary 1.2.6, it holds $dim_{\mathbb{C}} End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)) = |W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)|$, therefore

it is enough to show that the set $\{B_x \mid x \in W_{G^F \rtimes \langle \delta \rangle}(L, \sigma)\}$ is linearly independent. Assume that

$$\sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x B_x = 0$$

for some $c_x \in \mathbb{C}$. Then for $v \in V$ we get

$$\sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x B_x(e_U \otimes v) = \sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x(e_U \dot{x}^{-1} e_U \otimes t_{\dot{x}}(v)) = 0.$$

As a vector space, $\mathbb{C}(G^F \rtimes \langle \delta \rangle)e_U \otimes_{\mathbb{C}L^F} V$admits a decomposition given by

$$\mathbb{C}(G^F \rtimes \langle \delta \rangle)e_U \otimes_{\mathbb{C}L^F} V = \bigoplus_{gP^F \in {G^F \rtimes \langle \delta \rangle}/{P^F}} ge_U \otimes_{\mathbb{C}L^F} V. \tag{1.29}$$

For $\dot{y} \in N_{G^F \rtimes \langle \delta \rangle}(L, \sigma)$, the coset $\dot{y}P^F \in {G^F \rtimes \langle \delta \rangle}/{P^F}$ depends only on the class $y \in N_{G^F \rtimes \langle \delta \rangle}(L, \sigma)$, so we denote it by yP^F. Let π_y be the projection of $\mathbb{C}(G^F \rtimes \langle \delta \rangle)e_U \otimes_{\mathbb{C}L^F} V$ on $\dot{y}^{-1}e_U \otimes_{\mathbb{C}L^F} V$ along the decomposition(1.29). Then

$$\begin{aligned} 0 &= \pi_y\Big(\sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x B_x(e_U \otimes v)\Big) \\ &= \pi_y\Big(\sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x\big(\frac{1}{|U^F|}\sum_{u \in U^F} u\dot{x}^{-1}e_U \otimes t_{\dot{x}}(v)\big)\Big) \\ &= \frac{1}{|U^F|}\sum_{\substack{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma) \\ u \in U^F \\ u\dot{x}^{-1} \in \dot{y}^{-1}P^F}} c_x u\dot{x}^{-1}e_U \otimes t_{\dot{x}}(v). \end{aligned}$$

If $u\dot{x}^{-1} \in \dot{y}^{-1}P^F$ for some $u \in U^F$ then $\dot{x} \in P^F \dot{y} P^F$, so Lemma 1.2.10 gives $x = y$. Therefore

$$0 = \pi_y\Big(\sum_{x \in W_{G^F \rtimes \langle \delta \rangle}(L,\sigma)} c_x B_x(e_U \otimes v)\Big) = \frac{1}{|U^F|} c_y \sum_{\substack{u \in U^F \\ \dot{y}u\dot{y}^{-1} \in P^F}} u\dot{y}^{-1}e_U \otimes t_{\dot{y}}(v)$$

Since $\dot{y} \in N_{G^F \rtimes \langle \delta \rangle}(L^F)$, the conditions $u \in U^F$ and $\dot{y}u\dot{y}^{-1} \in P^F$ force $\dot{y}u\dot{y}^{-1} \in U^F$, because $yPy^{-1} \cap P = (yUy^{-1} \cap U) \rtimes L$, so $yUy^{-1} \cap P \subseteq U \cap yUy^{-1}$. Then all of the terms in the last sum are equal, since they satisfy $u\dot{y}^{-1}e_U \otimes t_{\dot{y}}(v) = \dot{y}^{-1}e_U \otimes t_{\dot{y}}(v)$. Moreover there is at least one non-zero summand (take $u = 1$). Hence $c_y = 0$, for any $y \in W_{G^F \rtimes \langle \delta \rangle}(L, \sigma)$. □

In order to exploit the results known for connected groups, we choose an appropriate scaling of the basis $\{B_x \mid x \in W_{G^F \rtimes \langle \delta \rangle}(L, \sigma)\}$, which allows us to express the multiplicative relations in a favourable way. Since the center of G is connected, the group $W_{G^F}(L, \sigma)$ is a Weyl group [49, Theorem 3.2.5], and it is identified with a reflection subgroup of the finite Weyl group W_G^F of G^F as recalled in Lemma 1.1.2. Here $W_G = {N_G(T)}/{T}$ with T the δ-stable and F-stable maximal torus fixed at the

beginning of this section.
Let $\mathcal{S}$ be the set of simple reflections of W_G with respect to the elements in the base of the root system determined by the δ-stable and F-stable Borel B fixed at the beginning of this section. Let S be the set of simple reflections of $W_{G^F}(L,\sigma)$ with respect to the elements in the base of the root system associated with $W_{G^F}(L,\sigma)$ as in [31, Lemma 2.7]. We denote by l the length function in W_G with respect $\mathcal{S}$, and by $\widetilde{l}(\omega)$ the length of ω in $W_{G^F}(L,\sigma)$ with respect to S. For any $\omega \in W_{G^F}(L,\sigma)$, we write $l(\omega)$ for the length of the element in W_G corresponding to ω through the identification in Lemma 1.1.2.

The basis for the endomorphism algebra $End_{G^F}(R_{L^F}^{G^F}\sigma)$ given by

$$\widetilde{T}_\omega := q^{l(\omega)}\widetilde{B}_\omega$$

for any $\omega \in W_{G^F}(L,\sigma)$, with $\widetilde{B}_\omega$ as in Remark 1.2.9, satisfies [20, Proposition 6.1.16], i.e.

$$\begin{aligned} \widetilde{T}_{\omega_1}\widetilde{T}_{\omega_2} &= \widetilde{T}_{\omega_1\omega_2} && \text{if } \widetilde{l}(\omega_1)+\widetilde{l}(\omega_2)=\widetilde{l}(\omega_1\omega_2) \\ \widetilde{T}_s^2 &= (q^{c_s}-1)\widetilde{T}_s + q^{c_s} && s \in S \end{aligned} \tag{1.30}$$

where the $c_s \in \mathbb{N}$ are defined as in [20, Proposition 6.1.16(ii)].

The assumption of connectedness on the centre of G ensures that $W_{G^F}(L,\sigma)$ is a reflection group [49, Theorem 3.2.5], and since S is a set of simple reflections in $W_{G^F}(L,\sigma)$ the relations in (1.30) generate a complete set of relations for the endomorphism algebra $End_{G^F}(R_{L^F}^{G^F}\sigma)$.

For each $\omega \in W_{G^F}(L,\sigma)$ and $i \in \{1,\dots,d\}$, set $l(\omega\delta^i) = l(\omega)$. Note that since δ stabilizes the pair (T,B), it induces an automorphism on W_G^F stabilizing the set of the simple reflections, hence it preserves the length l of the elements. In particular $l(\delta x) = l(x) = l(x\delta)$ for any $x \in W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. For any $x \in W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$, set

$$T_x := q^{l(x)}B_x.$$

Note that by Remark 1.2.9 for $\omega \in W_{G^F}(L,\sigma)$, it holds

$$T_\omega = q^{l(\omega)}B_\omega = q^{l(\omega)}id_{\mathbb{C}(G^F\rtimes\langle\delta\rangle)}\otimes\widetilde{B}_\omega = id_{\mathbb{C}(G^F\rtimes\langle\delta\rangle)}\otimes\widetilde{T}_\omega.$$

Proposition 1.2.12. *Let S be the set of simple reflections for $W_{G^F}(L,\sigma)$, and denote by $\widetilde{l}$ the associated length function. A complete set of relations for the basis $\{T_x \mid x \in W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)\}$ of $End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)$ can be deduced from the following fundamental relations: for any $x,y \in W_{G^F}(L,\sigma), i \in \{0,\dots,d-1\}$ it holds*

$$\begin{aligned} T_xT_y &= T_{xy}, && \text{if } \widetilde{l}(x)+\widetilde{l}(y)=\widetilde{l}(xy) \\ T_s^2 &= (q^{c_s}-1)T_s + q^{c_s}, && s \in S \\ T_{\delta^i}T_x &= \lambda(\delta^i,x)T_{\delta^i x}, \\ T_xT_{\delta^i} &= \lambda(x,\delta^i)T_{x\delta^i}, \\ T_{\delta^i}T_{\delta^j} &= \lambda(\delta^i,\delta^j)T_{\delta^{i+j}} \end{aligned} \tag{1.31}$$

where $c_s \in \mathbb{N}$ is defined as in [20, Proposition 6.1.16(ii)], and $\lambda : W_{G^F\rtimes\langle\delta\rangle}(L,\sigma) \times W_{G^F\rtimes\langle\delta\rangle}(L,\sigma) \to \mathbb{C}$ is the 2-cocycle on $W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ defined in Remark 1.2.7.

Proof. First we focus on the first two relations. For any $x \in W_{G^F}(L,\sigma)$

$$T_x = id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{T}_x,$$

hence exploiting the relations 1.30 we have

$$T_xT_y = (id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{T}_x)(id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{T}_y) = id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{T}_{xy} = T_{xy}$$

if $\widetilde{l}(x) + \widetilde{l}(y) = \widetilde{l}(xy)$ and

$$T_s^2 = (id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes \widetilde{T}_s)^2 = id_{\mathbb{C}(G^F \rtimes \langle\delta\rangle)} \otimes ((q^{c_s}-1)\widetilde{T}_s + q^{c_s}) = (q^{c_s}-1)T_s + q^{c_s}$$

for any $s \in S$.

Now we focus on the relations involving T_δ. Note that since δ induces a bijection on U^F, it holds $\delta(e_U) = e_U$. For any $g \in G^F \rtimes \langle\delta\rangle$, $x \in W_{G^F}(L,\sigma)$ and $v \in V$,

$$\begin{aligned}
T_{\delta^i}T_x(ge_U \otimes v) &= T_{\delta^i}(q^{l(x)}ge_U\dot{x}^{-1}e_U \otimes t_{\dot{x}}(v)) \\
&= q^{l(x)}ge_U\dot{x}^{-1}e_U\delta^{-i}e_U \otimes t_{\delta^i}(t_{\dot{x}}(v)) \\
&= q^{l(x)}ge_U\dot{x}^{-1}\delta^{-i}e_U \otimes \lambda(\delta^i,x)t_{\delta^i\dot{x}}(v) \\
&= \lambda(\delta^i,x)q^{l(\delta^i x)}ge_U(\delta^i\dot{x})^{-1}e_U \otimes t_{\delta^i\dot{x}}(v) = \lambda(\delta^i,x)T_{\delta^i x}(ge_U \otimes v), \\
T_xT_{\delta^i}(ge_U \otimes v) &= T_x(ge_U\delta^{-i}e_U \otimes t_{\delta^i}(v)) \\
&= q^{l(x)}ge_U\delta^{-i}e_U\dot{x}^{-1}e_U \otimes t_{\dot{x}}(t_{\delta^i}(v)) \\
&= q^{l(x)}ge_U\delta^{-i}\dot{x}^{-1}e_U \otimes \lambda(x,\delta^i)t_{\dot{x}\delta^i}(v) \\
&= \lambda(x,\delta^i)q^{l(x\delta^i)}ge_U(\dot{x}\delta^i)^{-1}e_U \otimes t_{\dot{x}\delta^i}(v) = \lambda(x,\delta^i)T_{x\delta^i}(ge_U \otimes v), \\
T_{\delta^i}T_{\delta^j}(ge_U \otimes v) &= T_{\delta^i}(ge_U\delta^{-j}e_U \otimes t_{\delta^j}(v)) \\
&= ge_U\delta^{-j}e_U\delta^{-i}e_U \otimes t_{\delta^i}(t_{\delta^j}(v)) \\
&= ge_U\delta^{-j}\delta^{-i}e_U \otimes \lambda(\delta^i,\delta^j)t_{\delta^{i+j}}v = \lambda(\delta^i,\delta^j)T_{\delta^{i+j}}(ge_U \otimes v).
\end{aligned}$$

This shows that the algebra $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$ is a quotient of the algebra $\widetilde{E}$ generated by $\{b_x,\ x \in W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)\}$ with relations as in (1.31). Now we show that $dim_{\mathbb{C}}(\widetilde{E}) = dim_{\mathbb{C}} End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$.

For any pair of elements $x\delta^i, y\delta^j \in W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$, with $x,y \in W_{G^F}(L,\sigma)$ and $i,j \in \{0,\dots,d-1\}$ the product $b_{x\delta^i}b_{x\delta^j}$ can be computed in terms of the relations (1.31):

$$\begin{aligned}
b_{x\delta^i}b_{y\delta^j} &= \lambda(x,\delta^i)^{-1}\lambda(y,\delta^j)^{-1}b_xb_{\delta^i}b_yb_{\delta_j} \\
&= \lambda(x,\delta^i)^{-1}\lambda(y,\delta^j)^{-1}\lambda(\delta^i,y)b_xb_{\delta^i y}b_{\delta^j} \\
&= \lambda(x,\delta^i)^{-1}\lambda(y,\delta^j)^{-1}\lambda(\delta^i,y)b_xb_{\delta^i(y)\delta^i}b_{\delta^j} \\
&= \lambda(x,\delta^i)^{-1}\lambda(y,\delta^j)^{-1}\lambda(\delta^i,y)\lambda(\delta^i(y),\delta^i)^{-1}b_xb_{\delta^i(y)}b_{\delta^i}b_{\delta^j} \\
&= \lambda(x,\delta^i)^{-1}\lambda(y,\delta^j)^{-1}\lambda(\delta^i,y)\lambda(\delta^i(y),\delta^i)^{-1}\lambda(\delta^i,\delta^j)b_xb_{\delta^i(y)}b_{\delta^{i+j}}.
\end{aligned}$$

Then the product $b_xb_{\delta(y)}$ can be expressed as a linear combination of $\{b_z, z \in W_{G^F}(L,\sigma)\}$ using the first two relations in (1.31), and the product of any b_z with

$z \in W_{G^F}(L,\sigma)$ with $b_{\delta^{i+j}}$ is given by the second to last relation in (1.31). It follows that $\{b_x, x \in W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)\}$ is a $\mathbb{C}$-linear basis for the algebra $\widetilde{E}$. Hence $dim(\widetilde{E}) = dim(End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$, so the two algebras are isomorphic and (1.31) is a complete set of relations.

□

Remark 1.2.13. The parameters c_s for $s \in S$ appearing in Proposition 1.2.11 are determined locally. By [49, Proposition 3.1.29] and the discussion above, there exists an F-stable Levi subgroup M of a parabolic subgroup of G containing L and such that $W_M(L,\sigma) = \langle s\rangle$. Then q^{c_s} is the quotient of the dimensions of the two irreducible constituents of $R_{L^F}^{M^F}(\sigma)$.

Corollary 1.2.14. *Retain notation from Proposition 1.2.12. The algebra* $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$ *is the finite extended twisted Hecke algebra* $\mathcal{H}_\lambda(W_{G^F}(L,\sigma), \{q^{c_s}\}_{s\in S})$*. Therefore,*

$$End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma) \cong \mathbb{C}_\lambda W_{G^F}(L,\sigma).$$

Proof. The first statement follows by comparing the relations in Proposition 1.2.12 and the definition of finite twisted extended Hecke algebra in Section 1.1. The isomorphism with the twisted group algebra of $W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ follows from Theorem 1.1.1, since $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$ is semisimple. □

Lemma 1.2.15. *The cocycle λ as in Remark 1.2.7 is a coboundary.*

Proof. The cocycle λ is a coboundary if and only if $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$ has a 1-dimensional representation, that is, if and only if $R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma$ has an irreducible constituent with multiplicity 1, [31, Lemma 6.5]. It is known that $R_{L^F}^{G^F}\sigma$ always has an irreducible constituent χ_{sgn} of multiplicity 1 that is uniquely determined by its degree [22, Theorem 1]. Therefore χ_{sgn} is δ-stable. We claim that if ρ is another irreducible consitituent of $R_{L^F}^{G^F}\sigma$, the representations $Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\chi_{sgn}$ and $Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\rho$ have no irreducible constituent in common. Indeed by Frobenius reciprocity and the Mackey formula,

$$\langle Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\chi_{sgn}, Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\rho\rangle_{G^F \rtimes\langle\delta\rangle} = \langle Res_{G^F}^{G^F \rtimes\langle\delta\rangle}Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\chi_{sgn}, \rho\rangle_{G^F} = \langle\sum_{i=0}^{d-1}(\delta^i)^*(\chi_{sgn}), \rho\rangle_{G^F} = d\langle\chi_{sgn}, \rho\rangle_{G^F} = 0.$$

Moreover since $G^F \rtimes \langle\delta\rangle$ is the semidirect product of G^F with a cyclic group, the induction of a representation in $Irr(G^F)$ to $G^F \rtimes \langle\delta\rangle$ is multiplicity free [21, Theorem 9.12]. Therefore all the irreducible constituents of $Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\chi_{sgn}$ have multiplicity 1 in $R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma$, since they appear with multiplicity 1 in $Ind_{G^F}^{G^F \rtimes\langle\delta\rangle}\chi_{sgn}$ and cannot appear in the induction to $G^F \rtimes \langle\delta\rangle$ of any other constituent of $R_{L^F}^{G^F}\sigma$. It follows that $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)$ has a representation of degree 1, and therefore the cocycle λ is a coboundary. □

Lemma 1.2.16. *There exists a choice of the linear maps t_n for $n \in N_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ satisfying* (1.25) *and such that λ is trivial.*

Proof. By Lemma 1.2.15 the cocycle λ is a coboundary, so there exists a 1-cocycle $\alpha : W_{G^F \rtimes \langle\delta\rangle}(L,\sigma) \to \mathbb{C}$ such that $\lambda(x,y) = \alpha(xy)\alpha(x)^{-1}\alpha(y)^{-1}$. Let $\widetilde{\alpha} : N_{G^F \rtimes \langle\delta\rangle}(L,\sigma) \to \mathbb{C}$ be the lift of α through the quotient by L^F. We set $\widetilde{t}_n := \widetilde{\alpha}(n)t_n$ for any $n \in N_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$. Then for any n and $m \in N_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ whose class in $W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ is denoted respectively by $\overline{n}$ and $\overline{m}$, it holds

$$\widetilde{t}_n\widetilde{t}_m\widetilde{t}_{nm}^{-1} = \widetilde{\alpha}(n)\widetilde{\alpha}(m)\widetilde{\alpha}(nm)^{-1}\widehat{\lambda}(n,m) = \alpha(\overline{n})\alpha(\overline{n})\alpha(\overline{nm})^{-1}\lambda(\overline{n},\overline{m}) = 1.$$

Moreover for any $l \in L^F$ and $n \in N_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ it holds

$$\widetilde{t}_{ln} := \widetilde{\alpha}(ln)t_{ln} = \widetilde{\alpha}(n)\sigma(l)t_n = \sigma(l)\widetilde{t}_n.$$

□

We now fix a scaling of the linear maps t_n for $n \in W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ for which the cocycle λ is trivial as in Lemma 1.2.16. In this way, the relations in Proposition 1.2.12 for the basis $\{T_x, x \in W_{G^F}(L,\sigma)\}$ become

$$\begin{aligned}
&T_xT_y = T_{xy}, && \text{if } \widetilde{l}(x) + \widetilde{l}(y) = \widetilde{l}(xy) \\
&T_s^2 = (q^{c_s}-1)T_s + q^{c_s}, && s \in S \\
&T_{\delta^i}T_x = T_{\delta^i x}, && \\
&T_xT_{\delta^i} = T_{x\delta^i}, && \\
&T_{\delta^i}T_\delta = T_{\delta^{i+1}} &&
\end{aligned} \tag{1.32}$$

and Corollary 1.2.14 simplifies as follows.

Theorem 1.2.17. *The algebra $End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)$ is isomorphic to the finite extended Hecke algebra $\mathcal{H}(W_{G^F}(L,\sigma), \{q^{c_s}\}_{s\in S})$, for some $c_s \in \mathbb{N}$. Therefore,*

$$End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma) \cong \mathbb{C}W_{G^F \rtimes \langle\delta\rangle}(L,\sigma).$$

Remark 1.2.18. Particular cases of Theorem 1.2.17 have already been studied by different authors. The case of $End_{G^F \rtimes \langle\delta\rangle}(R_{T^F}^{G^F \rtimes \langle\delta\rangle}1)$, where 1 denotes the trivial representation, was settled in [48, Bemerkung 1.2]. In [53, 2.9], the authors describe the structure of $End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)$ where G is of type D, δ is an outer automorphism of G corresponding to a graph automorphism and σ is unipotent.

1.2.3 Comparison theorem

In this section, we will always consider Levi and parabolic subgroups that are standard with respect to the based root datum of G corresponding to the δ and F-stable pair (T,B) fixed at the beginning of the Section 1.2.2. Let L be a δ-stable and F-stable Levi subgroup of δ-stable and F-stable parabolic subgroup P_L of G, such that L^F affords a δ-stable cuspidal representation σ. Let M be a δ-stable and F-stable Levi subgroup of a δ-stable and F-stable parabolic subgroup P_M of G

such that $L \leqslant M$ and $P_L \leqslant P_M$. We denote by U_L and U_M the unipotent radicals of P_L and P_M respectively.

By Corollary 1.2.4

$$R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma = R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle} \circ R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma = \mathbb{C}(G^F \rtimes \langle\delta\rangle)e_{U_M} \otimes_{\mathbb{C}(M^F\rtimes\langle\delta\rangle)} R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma,$$

where the last equality follows from the definition of $R_{M^F\rtimes\langle\delta\rangle}^{G^F\rtimes\langle\delta\rangle}$ in (1.9). Therefore there is a natural embedding

$$\begin{aligned} End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma) &\to End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma) \\ T &\mapsto id_{\mathbb{C}(G^F\rtimes\langle\delta\rangle)e_{U_M}} \otimes T \end{aligned}$$

Using the construction carried on in Section 1.2.2 but with M in place of G, we define a basis $\{T_x^M\ ,\ x \in W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)\}$ of $End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)$ analogous to the one defined for G. This basis satisfies relations analogous to the ones satisfied by $\{T_x\ ,\ x \in W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)\}$, and so $End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)$ is the finite extended Hecke algebra $\mathcal{H}(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma), \{q^{c_S^M}\}_{s\in S})$ for suitable $c_s^M \in \mathbb{N}$. It follows that $End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma) \cong \mathbb{C}W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)$.

The goal of this section is to show the analogue of Theorem 1.1.4 in the disconnected setting, using the explicit description of the endomorphism algebras of parabolically induced representations obtained in Section 1.2.2.

To lighten the presentation, we introduce the following notation. For any finite group H and for any representation π of H, we denote by $\mathcal{E}(H|\pi)$ the set of isomorphism classes of the irreducible constituents of π, i.e. $\mathcal{E}(H|\pi) := \{\rho \in Irr(H) \mid \langle\rho\ ,\ \pi\rangle_H \neq 0\}$. We recall the following result.

Lemma 1.2.19. *[32, Corollary 1.14] Let $J \leqslant K \leqslant H$ be finite groups and let $\chi \in Irr(J)$. Let*

$$id_K \otimes (-) : End_K(Ind_J^K\chi) \to End_H(Ind_J^H\chi)$$

and set

$$\theta_K = Hom(-, Ind_J^K\chi) : \mathcal{E}(K|Ind_J^K\chi) \to Irr(End_K(Ind_J^K\chi))$$

$$\theta_H = Hom(-, Ind_J^H\chi) : \mathcal{E}(K|Ind_J^K\chi) \to: Irr(End_H(Ind_J^H\chi))$$

where the module structures on the endomorphism algebras on the Hom spaces are given by composition. Then θ_K and θ_H are bijections, and their linear extensions make the following diagram commute:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}(H|Ind_J^H\chi)] & \xrightarrow{\theta_H} & \mathbb{C}[Irr(End_H(Ind_J^H\chi))] \\ \big\downarrow{\scriptstyle Res_K^H} & & \big\downarrow{\scriptstyle Res_{End_K(Ind_J^K\chi)}^{End_H(Ind_J^H\chi)}} \\ \mathbb{C}[\mathcal{E}(K|Ind_J^K\chi)] & \xrightarrow{\theta_K} & \mathbb{C}[Irr(End_K(Ind_J^K\chi))] \end{array} \tag{1.33}$$

where $Res_{End_K(Ind_J^K\chi)}^{End_H(Ind_J^H\chi)}$ denotes the restriction of scalars.

Corollary 1.2.20. *Retain notation from Lemma 1.2.19. The following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}(H|Ind_J^H\chi)] & \xrightarrow{\theta_K} & \mathbb{C}[Irr(End_H(Ind_J^H\chi))] \\ {\scriptstyle Ind^{End_H(Ind_J^H\chi)}_{End_K(Ind_J^K\chi)}}\uparrow & & {\scriptstyle Ind_H^K}\uparrow \\ \mathbb{C}[\mathcal{E}(K|Ind_J^K\chi)] & \xrightarrow{\theta_H} & \mathbb{C}[Irr(End_K(Ind_J^K\chi))] \end{array} \tag{1.34}$$

Proof. Endowing all the spaces in diagram (1.33) with the scalar product obtained by extending linearly the one given by the dimension of the Hom spaces on actual modules, the horizontal maps are isometries, because they are induced from bijections between the sets of simple objects. It follows that the commutativity of diagram (1.33) is equivalent to the commutativity of diagram (1.34). □

We set

$$\theta_{G^F\rtimes\langle\delta\rangle} = Hom(-, R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma) : \mathcal{E}^G(L,\sigma) \to Irr(End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)) \tag{1.35}$$

$$\theta_{M^F\rtimes\langle\delta\rangle} = Hom(-, R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma) : \mathcal{E}^M(L,\sigma) \to Irr(End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)) \tag{1.36}$$

We denote by the same symbols their linear extensions.

Proposition 1.2.21. ◇ *The maps $\theta_{G^F\rtimes\langle\delta\rangle}, \theta_{M^F\rtimes\langle\delta\rangle}$ are bijections.*

◇ *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)] & \xrightarrow{\theta_{G^F\rtimes\langle\delta\rangle}} & \mathbb{C}[Irr(End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma))] \\ {\scriptstyle R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle}}\uparrow & {\scriptstyle Ind^{End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)}_{End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)}} & \uparrow \\ \mathbb{C}[\mathcal{E}^{M^F\rtimes\langle\delta\rangle}(L,\sigma)] & \xrightarrow{\theta_{M^F\rtimes\langle\delta\rangle}} & \mathbb{C}[Irr(End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma))] \end{array} \tag{1.37}$$

Proof. Retain notation from the beginning of Section 1.2.3. We specialize Lemma 1.2.19 to the situation $J = P_L^F$, $K = P_M^F \rtimes \langle\delta\rangle$, $H = G^F \rtimes \langle\delta\rangle$ and $\chi = Infl_{L^F}^{P_L^F}\sigma$. Then $\theta_{G^F\rtimes\langle\delta\rangle}$ is bijective, since

$$\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma) = \mathcal{E}\big(G^F \rtimes \langle\delta\rangle | Ind_{P_L^F}^{G^F\rtimes\langle\delta\rangle}(Infl_{L^F}^{P_L^F}(\sigma))\big).$$

Since $\theta_{M^F\rtimes\langle\delta\rangle}$ is the analogous map for M, it is a bijection as well.

Now we show how to deduce the commutativity of the diagram (1.37) from Corollary 1.2.20. In this case, $Ind_J^H(\chi) = Ind_{P_L^F}^{G^F\rtimes\langle\delta\rangle}(Infl_{L^F}^{P_L^F}\sigma) = R_{L^F}^{G^F\rtimes\langle\delta\rangle}(\sigma)$ and $Ind_J^K(\chi) = Ind_{P_L^F}^{P_M^F\rtimes\langle\delta\rangle}(Infl_{L^F}^{P_L^F}\sigma)$.

We apply Remark 1.2.2 with $N = U_M^F$, $H = P_L^F \cap M^F$ and $K = M^F \rtimes \langle\delta\rangle$. Since $P_L^F = U_M^F \rtimes (P_L^F \cap M^F)$ and $P_M^F \rtimes \langle\delta\rangle = U_M^F \rtimes (M^F \rtimes \langle\delta\rangle)$ we have

$$Ind_{P_L^F}^{P_M^F\rtimes\langle\delta\rangle} \circ Infl_{P_L^F\cap M^F}^{P_L^F} = Infl_{M^F\rtimes\langle\delta\rangle}^{P_M^F\rtimes\langle\delta\rangle} \circ Ind_{P_L^F\cap M^F}^{M^F\rtimes\langle\delta\rangle}.$$

Composing both terms with $Infl_{L^F}^{P_L^F \cap M^F}$ on the right yields

$$Ind_{P_L^F}^{P_M^F \rtimes \langle\delta\rangle} \circ Infl_{L^F}^{P_L^F} = Infl_{M^F \rtimes \langle\delta\rangle}^{P_M^F \rtimes \langle\delta\rangle} \circ R_{L^F}^{M^F \rtimes \langle\delta\rangle}.$$

Then

$$End_{P_M^F \rtimes \langle\delta\rangle}(Ind_{P_L^F}^{P_M^F \rtimes \langle\delta\rangle} \circ Infl_{L^F}^{P_L^F} \sigma) = End_{P_M^F \rtimes \langle\delta\rangle}(Infl_{M^F \rtimes \langle\delta\rangle}^{P_M^F \rtimes \langle\delta\rangle} \circ R_{L^F}^{M^F \rtimes \langle\delta\rangle} \sigma).$$

To lighten the notation, we temporarily set $R_{L^F}^{P_M^F \rtimes \langle\delta\rangle} \sigma := Infl_{M^F \rtimes \langle\delta\rangle}^{P_M^F \rtimes \langle\delta\rangle} \circ R_{L^F}^{M^F \rtimes \langle\delta\rangle}(\sigma)$. Specializing diagram (1.34) to this case yields

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}(G^F \rtimes \langle\delta\rangle \,| R_{L^F}^{G^F \rtimes \langle\delta\rangle}(\sigma))] & \xrightarrow{\theta_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}(\sigma)))] \\ \Big\uparrow Ind_{P_M^F \rtimes \langle\delta\rangle}^{G^F \rtimes \langle\delta\rangle} & & \Big\uparrow Ind_{End_{P_M^F \rtimes \langle\delta\rangle}(R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma)}^{End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}(\sigma))} \\ \mathbb{C}[\mathcal{E}(P_M^F \rtimes \langle\delta\rangle) | R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma] & \xrightarrow[\theta_{P_M^F \rtimes \langle\delta\rangle}]{} & \mathbb{C}[Irr(End_{P_M^F \rtimes \langle\delta\rangle}(R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma))] \end{array} \tag{1.38}$$

A linear endomorphism is an intertwiner for an inflated representation if and only if it is an intertwiner for the original representation, so

$$End_{P_M^F \rtimes \langle\delta\rangle}(R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma) = End_{M^F \rtimes \langle\delta\rangle}(R_{L^F}^{M^F \rtimes \langle\delta\rangle}(\sigma))$$

and therefore the following diagram commutes and all the arrows are bijections:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}(P_M^F \rtimes \langle\delta\rangle | R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma)] & \xrightarrow{\theta_{P_M^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(End_{P_M^F \rtimes \langle\delta\rangle}(R_{L^F}^{P_M^F \rtimes \langle\delta\rangle}\sigma))] \\ \Big\uparrow Infl_{M^F \rtimes \langle\delta\rangle}^{P_M^F \rtimes \langle\delta\rangle} & & \Big\uparrow id \\ \mathbb{C}[\mathcal{E}^{M^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{\theta_{M^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(End_{M^F \rtimes \langle\delta\rangle}(R_{L^F}^{M^F \rtimes \langle\delta\rangle}\sigma))] \end{array} \tag{1.39}$$

So composing vertically the diagrams (1.39), (1.38) we get commutativity of (1.37) □

Corollary 1.2.22. *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{\theta_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma))] \\ \Big\downarrow {}^*R_{M^F \rtimes \langle\delta\rangle}^{G^F \rtimes \langle\delta\rangle} & & \Big\downarrow Res_{End_{M^F \rtimes \langle\delta\rangle}(R_{L^F}^{M^F \rtimes \langle\delta\rangle}\sigma)}^{End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)} \\ \mathbb{C}[\mathcal{E}^{M^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{\theta_{M^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(End_{M^F \rtimes \langle\delta\rangle}(R_{L^F}^{M^F \rtimes \langle\delta\rangle}\sigma))] \end{array} \tag{1.40}$$

where $Res_{End_{M^F \rtimes \langle\delta\rangle}(R_{L^F}^{M^F \rtimes \langle\delta\rangle}\sigma)}^{End_{G^F \rtimes \langle\delta\rangle}(R_{L^F}^{G^F \rtimes \langle\delta\rangle}\sigma)}$ *denotes the restriction of scalars.*

Proof. Endowing all the spaces in (1.37) with the scalar product given by the dimension of the *Hom* spaces, the horizontal maps are isometries, because they are induced from bijections between the simple objects. Then, commutativity of diagram (1.37) is equivalent to the commutativity of (1.40) replacing the vertical maps with their right adjoint. □

Proposition 1.2.23. *There exist bijections*

$$S_{G^F\rtimes\langle\delta\rangle} : Irr(End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)) \to Irr(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)) \tag{1.41}$$

$$S_{M^F\rtimes\langle\delta\rangle} Irr(End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)) \to Irr(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma) \tag{1.42}$$

whose linear extensions make the following diagram commutative

$$\begin{array}{ccc} \mathbb{C}[Irr(End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma))] & \xrightarrow{S_{G^F\rtimes\langle\delta\rangle}} & \mathbb{C}[Irr(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma))] \\ \Big\downarrow Res^{End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)}_{End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)} & & \Big\downarrow Res^{W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)}_{W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)} \\ \mathbb{C}[Irr(End_{M^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma))] & \xrightarrow{S_{M^F\rtimes\langle\delta\rangle}} & \mathbb{C}[Irr(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma))] \end{array} \tag{1.43}$$

Proof. Let S be the set of simple reflections of $W_G(L,\sigma)$. Let $f : \mathbb{C}[x_s|s \in S] \to \mathbb{C}$ be the $\mathbb{C}$-algebra morphism defined by $f(x_s) = q^{c_s}$ and $g : \mathbb{C}[x_s|s \in S] \to \mathbb{C}$ be the $\mathbb{C}$-algebra morphism defined by $g(x_s) = 1$. Let f^* and g^* be respectively the extensions of f and g to the integral closure of $\mathbb{C}[x_s]$ in $\overline{\mathbb{C}(x_s)}$, the algebraic closure of $\mathbb{C}(x_s)$. Such extensions exists by [16, Lemma 68.16]. By Theorem 1.2.17, the algebra $End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)$ is the specialization at f of the finite generic extended Hecke algebra $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S})$, for some $c_s \in \mathbb{N}$. Moreover the specialization of the same algebra at g yields the group algebra $\mathbb{C}W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. By [16, Corollary 68.20], the character of any simple module of $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S}) \otimes_{\mathbb{C}[x_s|s\in S]} \overline{\mathbb{C}(x_s|s \in S)}$ takes values in the integral closure of $\mathbb{C}[x_s]$.

Let χ be the character of a simple module of $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S}) \otimes_{\mathbb{C}[x_s|s\in S]} \overline{\mathbb{C}(x_s|s \in \mathbb{S})}$. We define

$$\chi^f(T_x) := f^*(\chi(b_x)), \qquad \chi^g(x) := g^*(\chi(b_x))$$

where b_x for $x \in W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ is an element of the basis of $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S})$ as in Section 1.1. By [16, Corollary 68.20], the assignment $\chi \to \chi^f$ defines a bijection between the set of simple modules of $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S}) \otimes_{\mathbb{C}[x_s|s\in S]} \overline{\mathbb{C}(x_s|s \in \mathbb{S})}$ and the set of simple modules of $End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{G^F\rtimes\langle\delta\rangle}\sigma)$, while the assignment and $\chi \to \chi^g$ define a bijection between the set of simple modules of $\mathcal{H}(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma),\{x_s\}_{s\in S}) \otimes_{\mathbb{C}[x_s|s\in S]} \overline{\mathbb{C}(x_s|s \in \mathbb{S})}$ and the set of irreducible representations of $W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. Therefore we have a bijection

$$\begin{aligned} S_{G^F\rtimes\langle\delta\rangle} : Irr(End_{G^F\rtimes\langle\delta\rangle}(R_{L^F}^{M^F\rtimes\langle\delta\rangle}\sigma)) &\to Irr(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)) \\ \chi^f &\mapsto \chi^g \end{aligned} \tag{1.44}$$

We can reproduce the same argument with M in place of G. Since M is standard, the set J of simple reflections of $W_M(L,\sigma)$ is a subset of the set S of simple reflections of $W_G(L,\sigma)$ [32, Proposition 4.3, (i)], so we can take as specialization morphism the restrictions of f and g to $\mathbb{C}[x_s|s \in J]$. Indeed by Remark 1.2.13 the parameters $c_s^M, s \in J$ for $End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}(\sigma))$ are determined locally, so are the same as the parameters $c_s, s \in J$ for $End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}(\sigma))$. Hence $End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}(\sigma))$ is the specialization at f of the generic extended Hecke algebra $\mathcal{H}(W_{M^F \rtimes\langle\delta\rangle}(L,\sigma), \{x_s\}_{s\in J})$. Therefore also in this case we have a bijection

$$S_{M^F \rtimes\langle\delta\rangle} : Irr(End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}\sigma)) \to Irr(W_{M^F \rtimes\langle\delta\rangle}(L,\sigma)) \qquad (1.45)$$
$$\chi^f \mapsto \chi^g$$

Now since the algebra $\mathcal{H}(W_{M^F \rtimes\langle\delta\rangle}(L,\sigma), \{x_s\}_{s\in S})$ is semisimple, the restriction of χ to $\mathcal{H}(W_{M^F \rtimes\langle\delta\rangle}(L,\sigma), \{x_s\}_{s\in S})$ decomposes as $\bigoplus_{i=1}^N m_i\chi_i$, with $N \in \mathbb{N}$, where χ_i are characters of simple modules and $m_i \in \mathbb{N}$ are their multiplicities.

It follows that for any $\omega \in W_{M^F \rtimes\langle\delta\rangle}(L,\sigma)$

$$Res_{W_{M^F \rtimes\langle\delta\rangle}}^{W_{G^F \rtimes\langle\delta\rangle}}(\chi^g)(w) = \chi^g(\omega) = g(\chi(b_\omega)) = g(\sum_{i=1}^N m_i\chi_i(b_\omega)) = \sum_{i=1}^N m_i\chi_i^g.$$

Similarly

$$Res_{End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}\sigma)}^{End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)}(\chi^f) = \sum_{i=1}^N m_i\chi_i^f.$$

Therefore

$$\begin{aligned}
Res_{End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}\sigma))}^{End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)}(S_{G^F \rtimes\langle\delta\rangle}(\chi^g)) &= Res_{End_{M^F \rtimes\langle\delta\rangle}(R_{L^F}^{M^F \rtimes\langle\delta\rangle}\sigma)}^{End_{G^F \rtimes\langle\delta\rangle}(R_{L^F}^{G^F \rtimes\langle\delta\rangle}\sigma)}\chi^f \\
&= \sum_{i=1}^N m_i\chi_i^f \\
&= S_{M^F \rtimes\langle\delta\rangle}(\sum_{i=1}^N m_i\chi_i^g) \\
&= S_{M^F \rtimes\langle\delta\rangle}(Res_{W_{M^F \rtimes\langle\delta\rangle}}^{W_{G^F \rtimes\langle\delta\rangle}}(\chi^g)).
\end{aligned}$$

□

We are now in a position to state and prove the main result of this section.

Theorem 1.2.24. *There are bijections*

$$\mathcal{E}^{M^F \rtimes\langle\delta\rangle}(L,\sigma) \xrightarrow{Rep_{M^F \rtimes\langle\delta\rangle}} Irr(W_{M^F \rtimes\langle\delta\rangle}(L,\sigma)) \qquad (1.46)$$

$$\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma) \xrightarrow{Rep_{G^F \rtimes\langle\delta\rangle}} Irr(W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)) \qquad (1.47)$$

such that their linear extensions fit in the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{G^F \rtimes \langle\delta\rangle}(L,\sigma))] \\ {}^*R^{G^F \rtimes \langle\delta\rangle}_{M^F \rtimes \langle\delta\rangle}\big\downarrow & & Res^{W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)}_{W_{M^F \rtimes \langle\delta\rangle}(L,\sigma)}\big\downarrow \\ \mathbb{C}[\mathcal{E}^{M^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{M^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{M^F \rtimes \langle\delta\rangle}(L,\sigma))] \end{array} \tag{1.48}$$

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{G^F \rtimes \langle\delta\rangle}(L,\sigma))] \\ R^{G^F \rtimes \langle\delta\rangle}_{M^F \rtimes \langle\delta\rangle}\big\uparrow & & Ind^{W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)}_{W_{M^F \rtimes \langle\delta\rangle}(L,\sigma)}\big\uparrow \\ \mathbb{C}[\mathcal{E}^{M^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{M^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{M^F \rtimes \langle\delta\rangle}(L,\sigma))] \end{array} \tag{1.49}$$

Proof. We define

$$Rep_{G^F \rtimes \langle\delta\rangle} := (S_{G^F \rtimes \langle\delta\rangle} \circ \theta_{G^F \rtimes \langle\delta\rangle})$$

$$Rep_{M^F \rtimes \langle\delta\rangle} := (S_{M^F \rtimes \langle\delta\rangle} \circ \theta_{M^F \rtimes \langle\delta\rangle}).$$

Composing horizontally the commutative diagrams (1.43) and (1.40), we get

$$Res^{W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)}_{W_{M^F \rtimes \langle\delta\rangle}(L,\sigma)} \circ S_{G^F \rtimes \langle\delta\rangle} \circ \theta_{G^F \rtimes \langle\delta\rangle} = S_{M^F \rtimes \langle\delta\rangle} \circ \theta_{M^F \rtimes \langle\delta\rangle} \circ {}^*R^{G^F \rtimes \langle\delta\rangle}_{M^F \rtimes \langle\delta\rangle},$$

giving commutativity of (1.48).

The diagram (1.49) is obtained from (1.48) by substituting the vertical maps with their adjoint maps. Equipping all the spaces with the scalar product given by the dimension of the Hom spaces, the horizontal maps are isometries, because they map simple modules to irreducible representations. So commutativity of (1.49) follows. □

Remark 1.2.25. It is well-known that $End_{G^F}(R^{G^F}_{L^F}\sigma) \cong \mathbb{C}W_{G^F}(L,\sigma)$, [31]. Replacing $M^F \rtimes \langle\delta\rangle$ with G^F in this section, we obtain the analogue of Theorem 1.2.24, i.e., the following diagrams commute:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{G^F \rtimes \langle\delta\rangle}(L,\sigma))] \\ \big\downarrow Res^{G^F \rtimes \langle\delta\rangle}_{G^F} & & \big\downarrow Res^{W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)}_{W_{G^F}(L,\sigma)} \\ \mathbb{C}[\mathcal{E}^{G^F}(L,\sigma)] & \xrightarrow{Rep_{G^F}} & \mathbb{C}[Irr(W_{G^F}(L,\sigma))] \end{array} \tag{1.50}$$

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{G^F \rtimes \langle\delta\rangle}} & \mathbb{C}[Irr(W_{G^F \rtimes \langle\delta\rangle}(L,\sigma))] \\ Ind^{G^F \rtimes \langle\delta\rangle}_{G^F}\big\uparrow & & Ind^{W_{G^F \rtimes \langle\delta\rangle}(L,\sigma)}_{W_{G^F}(L,\sigma)}\big\uparrow \\ \mathbb{C}[\mathcal{E}^{G^F}(L,\sigma)] & \xrightarrow{Rep_{G^F}} & \mathbb{C}[Irr(W_{G^F}(L,\sigma))] \end{array} \tag{1.51}$$

1.2.4 Restriction to cosets

1.2.4.1 Restriction of characters to cosets

Let H be a finite group, let $\gamma \in Aut(H)$ be an automorphism of H of order d, and let $H \rtimes \langle\gamma\rangle$ be the extension of H by the cyclic group generated by γ. Then for any $i = 0, \dots, d-1$ we consider the space $Cl(H\gamma^i)$ of complex functions on $Cl(H\gamma^i)$ that are constant on $H \rtimes \langle\gamma\rangle$ conjugacy classes and denote by

$$\pi_{H,i} : R(H \rtimes \langle\gamma\rangle) \to Cl(H\gamma^i)$$

the restriction of class functions to the coset $H\gamma^i$.

The space $Cl(H\gamma^i)$ can be endowed with a non-degenerate sesquilinear form defined by

$$\langle f \ , \ g\rangle_{H\gamma^i} = \frac{1}{|H|} \sum_{h \in H} g(h\gamma^i)\overline{f(h\gamma^i)}.$$

In this section we prove that with respect to this sesquilinear form, an orthonormal basis for $Cl(H\gamma)$ is given by the restrictions to $H\gamma$ of irreducible characters of $H \rtimes \langle\gamma\rangle$ that have irreducible and distinct restrictions to H.

We denote by ζ_γ a generator for the character group of $\langle\gamma\rangle$, and we still denote by ζ_γ its inflation to $H \rtimes \langle\gamma\rangle$. We set $\zeta := \zeta_\gamma(\gamma)$, so ζ is a primitive d^{th} root of 1.

Lemma 1.2.26. *Let $\chi \in Irr(H \rtimes \langle\gamma\rangle)$ be such that $Res_H^{H\rtimes\langle\gamma\rangle}\chi$ is not irreducible. Then $\pi_{H,1}(\chi) = 0$.*

Proof. By Clifford theory, for $\chi \in Irr(H \rtimes \langle\gamma\rangle)$ it holds $Res_H^{H\rtimes\langle\gamma\rangle}\chi \in Irr(H)$ if and only if $\chi \otimes \zeta_\gamma^i \neq \chi$ for any $i = 1, \dots, d-1$. Indeed

$$\langle Res_H^{H\rtimes\langle\gamma\rangle}\chi, Res_H^{H\rtimes\langle\gamma\rangle}\chi\rangle_H = \langle \chi, Ind_H^{H\rtimes\langle\gamma\rangle} Res_H^{H\rtimes\langle\gamma\rangle}\chi\rangle_{H\rtimes\langle\gamma\rangle} = \langle \chi, \sum_{i=0}^{d-1} \chi \otimes \zeta_\gamma^i\rangle_{H\rtimes\langle\gamma\rangle}.$$

Then if $Res_H^{H\rtimes\langle\gamma\rangle}\chi$ is not irreducible, there exists some $1 \leqslant j \leqslant d-1$ such that $\chi = \chi \otimes \zeta_\gamma^j$. Hence for any $x \in H\gamma$ it holds $\chi(x) = \zeta^j\chi(x)$, and since $\zeta^j \neq 1$ for $1 \leqslant j \leqslant d-1$ this implies $\chi(x) = 0$ for any $x \in H\gamma$. □

Corollary 1.2.27. *The set*

$$\{\pi_{H,1}(\chi) \mid \chi \in Irr(H \rtimes \langle\gamma\rangle)\} \ , \ Res_H^{H\rtimes\langle\gamma\rangle}\chi \in Irr(H)\} \tag{1.52}$$

is a generating set for $Cl(H\gamma)$.

Proof. The coset $H\gamma^i$ is stable under the conjugation action of $H \rtimes \langle\gamma\rangle$, therefore any $H \rtimes \langle\gamma\rangle$-invariant function on $H\gamma^i$ can be extended by 0 to a class function on $H \rtimes \langle\gamma\rangle$. It follows that the restriction map $\pi_{H,i}$ is surjective, and therefore the set of non-zero elements in $\mathcal{K}_i = \{\pi_{H,i}(\chi) \mid \chi \in Irr(H \rtimes \langle\gamma\rangle)\}$ is a generating set for $Cl(H\gamma^i)$. By Lemma 1.2.26, this is the set consisting of all $\pi_{H,1}(\chi)$ with $\chi \in Irr(H \rtimes \langle\gamma\rangle))$ and $Res_H^{H\rtimes\langle\gamma\rangle}\chi \in Irr(H)$. □

Lemma 1.2.28. *Let $\chi, \chi' \in Irr(H \rtimes \langle\gamma\rangle)$ with irreducible restrictions to H and such that $Res_H^{H\rtimes\langle\gamma\rangle}\chi = Res_H^{H\rtimes\langle\gamma\rangle}\chi'$. Then $\pi_{H,1}(\chi)$ and $\pi_{H,1}(\chi')$ are linearly dependent.*

Proof. We have

$$1 = \langle Res_H^{H\rtimes\langle\gamma\rangle}\chi, Res_H^{H\rtimes\langle\gamma\rangle}\chi'\rangle_H = \langle\chi, Ind_H^{H\rtimes\langle\gamma\rangle}Res_H^{H\rtimes\langle\gamma\rangle}\chi'\rangle_{H\rtimes\langle\gamma\rangle} = \langle\chi, \sum_{i=0}^{d-1}\chi'\otimes\zeta_\gamma^i\rangle_{H\rtimes\langle\gamma\rangle},$$

and so $\chi' = \chi\otimes\zeta^j$ for some $0 \leqslant j \leqslant d-1$.

It follows that $\pi_{H,1}(\chi') = \zeta^j\pi_{H,1}(\chi)$. □

Corollary 1.2.29. *Let $\{\chi_i\}_{i=1}^r$ be a maximal subset of $Irr(H \rtimes \langle\gamma\rangle)$ such that the χ_i's have irreducible and distinct restictions to H. Then*

$$\{\pi_{H,1}(\chi_i)\ |i = 1,\dots,r\} \tag{1.53}$$

is a generating set for $Cl(H\gamma)$.

Proof. By Lemma 1.2.28, it is enough to take in (1.52) one element in each fiber $\pi_{H,0}^{-1}(\pi_{H,0}(\chi))$, where $\chi \in Irr(H \rtimes \langle\gamma\rangle))$ with $Res_H^{H\rtimes\langle\gamma\rangle}\chi \in Irr(H)$. □

Proposition 1.2.30. *Let $\{\chi_i\}_{i=1}^r$ be a maximal subset of $Irr(H \rtimes \langle\gamma\rangle)$ such that the χ_i have irreducible and distinct restictions to H, as in Corollary 1.2.29. Then $\{\chi_i\}_{i=1}^r$ is an orthonormal basis for $Cl(H\gamma)$, i.e. ,*

$$\begin{aligned} \langle\pi_{H,1}(\chi_i), \pi_{H,1}(\chi_i)\rangle_{H_\gamma} &= 1, && 1 \leqslant i \leqslant r, \\ \langle\pi_{H,1}(\chi_i), \pi_{H,1}(\chi_k)\rangle_{H\gamma} &= 0, && 1 \leqslant i \neq k \leqslant r. \end{aligned} \tag{1.54}$$

Proof. Let

$$y_j^{ik} := \langle\pi_{H,j}(\chi_i), \pi_{H,j}(\chi_k)\rangle_{H\gamma^j}$$

For any $1 \leqslant i, k \leqslant r$ we have

$$\begin{aligned} \sum_{j=0}^{d-1} y_j^{ik} &= \sum_{j=0}^{d-1}\langle\pi_{H,j}(\chi_i), \pi_{H,j}(\chi_k)\rangle_{H\gamma^j} \\ &= \sum_{j=0}^{d-1}\Big(\frac{1}{|H|}\sum_{h\in H}\chi_i(h\gamma^j)\overline{\chi_k(h\gamma^i)}\Big) \\ &= \frac{1}{|H|}\sum_{x\in H\rtimes\langle\gamma\rangle}\chi_i(x)\overline{\chi_k(x)} \\ &= \frac{d}{|H\rtimes\langle\gamma\rangle|}\sum_{x\in H\rtimes\langle\gamma\rangle}\chi_i(x)\overline{\chi_k(x)} \\ &= d\langle\chi_i, \chi_k\rangle_{H\rtimes\langle\gamma\rangle} = d\delta_{i,k} \end{aligned} \tag{1.55}$$

where the last equality comes from the first orthogonality relations of irreducible characters for $H \rtimes \langle\gamma\rangle$. Similarly, for any $1 \leqslant i, k \leqslant r$ and for any $1 \leqslant l \leqslant d-1$

$$\sum_{j=0}^{d-1}\zeta^{lj}y_j^{ik} = d\langle\chi_i\otimes\zeta_\gamma^l, \chi_k\rangle_{H\rtimes\langle\gamma\rangle}. \tag{1.56}$$

Since $Res_H^{H\rtimes\langle\gamma\rangle}\chi_i$ is irreducible, by Clifford theory $\chi \neq \chi\otimes\zeta_\gamma^l$ for any $l = 1, \ldots, d-1$. On the other hand, if $i \neq k$, then χ_i and χ_k have different restrictions to H, and so $\chi_i \neq \chi \otimes_k \zeta_\gamma^l$ for any $l = 1, \ldots, d-1$. Hence by orthogonality $\langle \chi_i \otimes \zeta_\gamma^l, \chi_k \rangle_{H\rtimes\langle\gamma\rangle} = 0$ for any $1 \leqslant i \neq k \leqslant r$ and for any $1 \leqslant l \leqslant d-1$. Therefore equation (1.56) gives

$$\sum_{j=0}^{d-1} \zeta^{lj} y_j^{ik} = 0. \tag{1.57}$$

For any pair i, k the equations (1.55) together with (1.57) where l runs from 1 to $d-1$ form a system of d equations and d indeterminates $y_j^{i,k}$ with $j = 1, \ldots, d$. The coefficient matrix is the following Vandermonde matrix:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{d-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta^{d-1} & \cdots & \zeta^{(d-1)^2} \end{pmatrix}$$

Its determinant $\prod_{1\leqslant j<l\leqslant d}(\zeta^l - \zeta^j)$ is non-zero because ζ is a primitive d^{th} root of 1. Therefore, for any $1 \leqslant i, k \leqslant r$ the system has a unique solution. Since $y_j^{ik} = \delta_{ik}$ for $j = 1, \ldots, d$ is a solution, by uniqueness we obtain

$$\langle \pi_{H,1}(\chi_i), \pi_{H,1}(\chi_k) \rangle_{H\gamma} = y_1^{ik} = \delta_{ik},$$

giving (1.54). ☐

Corollary 1.2.31. *It holds*

$$Ker(\pi_{H,1}) = span_{\mathbb{C}}\{\zeta^j\chi - \chi \otimes \zeta_\gamma^j \mid \chi \in Irr(H \rtimes \langle\gamma\rangle), j = 1, \ldots, d-1\}$$

Proof. For any $\chi \in Irr(H \rtimes \langle\gamma\rangle)$, for any $j \in \{0, \ldots, d-1\}$ and for any $h \in H$ we have $(\zeta^j\chi - \chi \otimes \zeta_\gamma^j)(h\gamma) = \zeta^j\chi(h\gamma) - \zeta_\gamma^j(\gamma)\chi(h\gamma) = 0$. Therefore

$$\mathcal{K} := span_{\mathbb{C}}\{\zeta^j\chi - \chi \otimes \zeta_\gamma^j \mid \chi \in Irr(H \rtimes \langle\gamma\rangle), i = 0, \ldots, d-1\} \subseteq Ker(\pi_{H,1}).$$

To prove equality, we show that $\mathcal{K}$ and $Ker(\pi_{H,1})$ have the same dimension over $\mathbb{C}$. Let

$$\begin{aligned} A_{irr} &= \{\chi \in Irr(H \rtimes \langle\gamma\rangle) \mid Res_H^{Irr(H\rtimes\langle\gamma\rangle)}\chi \in Irr(H)\} \\ &= \{\chi \in Irr(H \rtimes \langle\gamma\rangle) \mid \chi \otimes \zeta_\gamma^j \neq \chi \text{ for any } i = 1, \ldots, d\} \\ A_{red} &= \{\chi \in Irr(H \rtimes \langle\gamma\rangle) \mid Res_H^{Irr(H\rtimes\langle\gamma\rangle)}\chi \notin Irr(H)\} \\ &= \{\chi \in Irr(H \rtimes \langle\gamma\rangle) \mid \chi \otimes \zeta_\gamma^j = \chi \text{ for some } i \in \{1, \ldots, d\}\}. \end{aligned}$$

Since $Irr(H \rtimes \langle\gamma\rangle) = A_{irr} \bigsqcup A_{red}$, it holds $dim_{\mathbb{C}}R(H \rtimes \langle\gamma\rangle) = |A_{irr}| + |A_{red}|$.

Let $\{\chi_i\}_{i=1}^r$ be the orthonormal basis of $Cl(H\gamma)$ in Proposition 1.2.30, given by a maximal set of irreducible representations of $H \rtimes \langle\gamma\rangle$ having irreducible and distinct restrictions to H. Then $A_{irr} = \{\chi_i \otimes \zeta_\gamma^j \mid 1 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant d-1\}$, therefore

$|A_{irr}| = dr$. Since $\pi_{H,1}$ is surjective and $\{\chi_i\}_{i=1}^r$ is a basis of $Cl(H\gamma)$, it holds $dim_{\mathbb{C}}(Im(\pi_{H,1})) = dim_{\mathbb{C}}(Cl(H\gamma)) = r$. It follows that

$$\begin{aligned} dim_{\mathbb{C}}(Ker(\pi_{H,1})) &= dim_{\mathbb{C}} R(H \rtimes \langle\gamma\rangle) - (dim_{\mathbb{C}}(Im(\pi_{H,1})) \\ &= |A_{irr}| + |A_{red}| - r = |A_{red}| + (d-1)r. \end{aligned} \tag{1.58}$$

Now we prove that $dim_{\mathbb{C}}(\mathcal{K}) = |A_{red}| + (d-1)r$
We have $span_{\mathbb{C}}(A_{red}) \subseteq \mathcal{K}$: indeed for any $\chi \in A_{red}$, there exists $j \in \{1, \ldots, d-1\}$ such that $\chi = \chi \otimes \zeta_\gamma^j$, therefore

$$\chi = \frac{1}{(\zeta^j - 1)}(\zeta^j - 1)\chi = \frac{1}{(\zeta^j - 1)}(\zeta^j \chi - \chi) = \frac{1}{(\zeta^j - 1)}(\zeta^j \chi - \chi \otimes \zeta_\gamma^j) \in \mathcal{K}.$$

Since $R(H \rtimes \langle\gamma\rangle) = span_{\mathbb{C}}(A_{irr}) \bigoplus span_{\mathbb{C}}(A_{red})$, it holds $\mathcal{K} = (span_{\mathbb{C}}(A_{irr}) \cap \mathcal{K}) \bigoplus span_{\mathbb{C}}(A_{red})$ and so

$$dim_{\mathbb{C}}(\mathcal{K}) = |A_{red}| + dim_{\mathbb{C}}(span_{\mathbb{C}}(A_{irr}) \cap \mathcal{K}).$$

By (1.58), we need to prove that $dim_{\mathbb{C}}(span_{\mathbb{C}}(A_{irr}) \cap \mathcal{K}) = (d-1)r$. We have

$$span_{\mathbb{C}}(A_{irr}) \cap \mathcal{K} = span_{\mathbb{C}}\{\zeta^j \chi_i - \chi_i \otimes \zeta_\gamma^j \mid i \in \{1, \ldots, r\}, j \in \{1, \ldots, d-1\}\ \}.$$

Since the characters $\chi_i \otimes \zeta_\gamma^j$ for $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, d\}$ are linearly independent, the set $\{\zeta^j \chi_i - \chi_i \otimes \zeta_\gamma^j \mid i \in \{1, \ldots, r\}, j \in \{1, \ldots, d-1\}\ \}$ is linearly independent. It follows that

$$dim_{\mathbb{C}}(span_{\mathbb{C}}(A_{irr}) \cap \mathcal{K}) = |\{\zeta^j \chi_i - \chi_i \otimes \zeta_\gamma^j\ ;\ i \in \{1, \ldots, r\}, j \in \{1, \ldots, d-1\}\ \}| = (d-1)r,$$

giving the claim. □

Remark 1.2.32. Let $j \in \{1, \ldots, d-1\}$ be coprime with d. Then all the arguments in this section hold for $\pi_{H,j}$ and $Cl(H\gamma^j)$ in place of $\pi_{H,1}$ and $Cl(H\gamma)$, because γ^j is a generator for $\langle\gamma\rangle$.

Let H' be a γ-stable subgroup of H. We define

$$Ind_{H'\gamma^i}^{H\gamma^i} : Cl(H'\gamma^i) \to Cl(H\gamma^i)$$

by

$$Ind_{H'\gamma^i}^{H\gamma^i}(f)(x) = \frac{1}{|H'|} \sum_{\substack{h \in H \\ h^{-1}xh \in H'\gamma^i}} f(h^{-1}xh)$$

for any $f \in Cl(H'\gamma^i)$ and $x \in H\gamma^i$. We define

$$Res_{H\gamma^i}^{H'\gamma^i} : Cl(H'\gamma^i) \to Cl(H\gamma^i)$$

by

$$Res_{H'\gamma^i}^{H\gamma^i}(f)(x) = f(x)$$

for any $f \in Cl(H\gamma^i)$ and $x \in H'\gamma^i$.

Proposition 1.2.33. *With the above notation, for any $f \in Cl(H'\gamma^i)$ and $g \in Cl(H\gamma^i)$ it holds*

$$\langle Ind_{H'\gamma^i}^{H\gamma^i} f, g\rangle_{H\gamma^i} = \langle f, Res_{H\gamma^i}^{H'\gamma^i} g\rangle_{H'\gamma^i}$$

Proof. It holds

$$\begin{aligned}
\langle Ind_{H'\gamma^i}^{H\gamma^i} f, g\rangle_{H\gamma^i} &= \frac{1}{|H|}\sum_{x\in H\gamma^i} Ind_{H'\gamma^i}^{H\gamma^i} f(x)\overline{g(x)} \\
&= \frac{1}{|H|}\sum_{x\in H\gamma^i}\frac{1}{|H'|}\sum_{\substack{h\in H\\ h^{-1}xh\in H'\gamma^i}} f(h^{-1}xh)\overline{g(x)} \\
&= \frac{1}{|H|}\frac{1}{|H'|}\sum_{\substack{h\in H\\ x\in H\gamma^i\\ h^{-1}xh\in H'\gamma^i}} f(h^{-1}xh)\overline{g(x)} \\
&= \frac{1}{|H|}\frac{1}{|H'|}\sum_{\substack{h\in H\\ y\in H'\gamma^i}} f(y)\overline{g(hyh^{-1})} \\
&= \frac{1}{|H|}\frac{1}{|H'|}\sum_{\substack{h\in H\\ y\in H'\gamma^i}} f(y)\overline{g(y)} \\
&= \frac{1}{|H'|}\sum_{y\in H'\gamma^i} f(y)\overline{g(y)} = \langle f, Res_{H'\gamma^i}^{H\gamma^i}\rangle_{H'\gamma^i}.
\end{aligned}$$

□

Let N be a γ-stable normal subgroup of H. Then γ induces an automorphism on H/N, that we still denote by γ. We consider the group $H/N \rtimes \langle\gamma\rangle$. For $i = 1, \dots, d$ we define

$$Infl_{H/N\gamma^i}^{H\gamma^i} : Cl(H/N\gamma^i) \to Cl(H\gamma^i)$$

by

$$Infl_{H/N\gamma^i}^{H\gamma^i}(f)(h) = f(hN)$$

for any $f \in Cl(H/N\gamma^i)$ and $h \in H\gamma^i$, where $hN \in H/N$.

We also define the map

$$(-)^N : Cl(H\gamma^i) \to Cl(H/N\gamma^i)$$

by

$$(f)^N(xN) = \frac{1}{|N|}\sum_{n\in N} f(xn)$$

for any $f \in Cl(H\gamma^i)$ and $x \in H\gamma^i$.

Proposition 1.2.34. *With the above notation, for any* $f \in Cl({}^{H}\!/_{N}\gamma^i)$ *and* $h \in Cl(H\gamma^i)$ *it holds*

$$\langle Infl_{H/_{N}\gamma^i}^{H\gamma^i}(f), g\rangle_{H\gamma^i} = \langle f, (g)^N\rangle_{H/_{N}\gamma^i}$$

Proof. It holds

$$\begin{aligned}
\langle Infl_{H/N\gamma^i}^{H\gamma^i}(f), g\rangle_{H\gamma^i} &= \frac{1}{|H|}\sum_{x\in H\gamma^i} Infl_{H/N\gamma^i}^{H\gamma^i} f(x)\overline{g(x)} \\
&= \frac{1}{|H|}\sum_{x\in H\gamma^i} f(xN)\overline{g(x)} \\
&= \frac{1}{|H|}\sum_{\substack{yN\in H/N\gamma^i \\ n\in N}} f(yN)\overline{g(yn)} \\
&= \frac{|N|}{|H|}\sum_{yN\in H/N\gamma^i} f(yN)\left(\frac{1}{|N|}\sum_{n\in N}\overline{g(yn)}\right) \\
&= \frac{1}{|H/N|}\sum_{yN\in H/N\gamma^i} f(yN)\overline{(g)^N(yN)} = \langle f, (g)^N\rangle_{H/N\gamma^i}.
\end{aligned}$$

□

1.2.4.2 Comparison theorem on cosets

We use the same notation as in Section 1.2.3.

Let ζ_δ be a generator of the group of irreducible characters of $\langle\delta\rangle$. We denote by $\widetilde{\zeta_\delta^i}$ the inflation of this character to $W_{G^F}(L,\sigma) \rtimes \langle\delta\rangle \cong W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. With abuse of notation, we denote by $\widetilde{\zeta_\delta^i}$ also the inflation of this character to $G^F \rtimes \langle\delta\rangle$.

Lemma 1.2.35. *With notation as above, for any* $\pi \in \mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ *it holds*

$$Rep_{G^F\rtimes\langle\delta\rangle}(\pi\otimes\widetilde{\zeta_\delta^i}) = Rep_{G^F\rtimes\langle\delta\rangle}(\pi)\otimes\widetilde{\zeta_\delta^i}^{-1}$$

for $i \in \{0,\dots,d-1\}$.

Proof. By definition, $Rep_{G^F\rtimes\langle\delta\rangle} = S_{G^F\rtimes\langle\delta\rangle}\circ\theta_{G^F\rtimes\langle\delta\rangle}$, where $S_{G^F\rtimes\langle\delta\rangle}$ and $\theta_{G^F\rtimes\langle\delta\rangle}$ are respectively as in Proposition 1.2.21 and Proposition 1.2.23 . Let $\pi \in \mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. Then $\theta_{G^F\rtimes\langle\delta\rangle}(\pi)$ is the $End_{G^F\rtimes\langle\delta\rangle}(R_L^{G^F\rtimes\langle\delta\rangle}\sigma)$-module $Hom(\pi, R_L^{G^F\rtimes\langle\delta\rangle}\sigma)$, where the module structure is given by composition. Let $I_{\zeta_\delta^i}$ be the linear map on the representation space of $R_L^{G^F\rtimes\langle\delta\rangle}\sigma$ defined as the linear extension of

$$\mathbb{C}(G^F\rtimes\langle\delta\rangle)e_U\otimes_{\mathbb{C}L^F} V \to \mathbb{C}(G^F\rtimes\langle\delta\rangle)e_U\otimes_{\mathbb{C}L^F} V,$$
$$ge_U\otimes v \mapsto \widetilde{\zeta_\delta^i}^{-1}(g)ge_U\otimes v$$

where V is the representation space of σ. Note that this map is not an $G^F \rtimes \langle\delta\rangle$-endomorphism. Indeed for any $h \in G^F \rtimes \langle\delta\rangle$, we have

$$I_{\zeta_\delta^i}(hge_U \otimes v) = \widetilde{\zeta_\delta^i}^{-1}(h)hI_{\zeta_\delta^i}(ge_U \otimes v).$$

The composition with $I_{\zeta_\delta^i}$ defines a $\mathbb{C}$-linear isomorphism between $Hom_{G^F \rtimes\langle\delta\rangle}(\pi, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$ and $Hom_{G^F \rtimes\langle\delta\rangle}(\pi \otimes \widetilde{\zeta_\delta^i}, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$. Indeed let $\iota \in Hom_{G^F \rtimes\langle\delta\rangle}(\pi, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$. For any w in the representation space of π and for any $g \in G^F \rtimes \langle\delta\rangle$ it holds $\iota(\pi(g)w) = g\iota(w)$. It follows that

$$\begin{aligned}
I_{\zeta_\delta^i} \circ \iota\big((\pi \otimes \widetilde{\zeta_\delta^i})(g)w\big) &= \widetilde{\zeta_\delta^i}(g)I_{\zeta_\delta^i} \circ \iota(\pi(g)w) \\
&= \widetilde{\zeta_\delta^i}(g)I_{\zeta_\delta^i}(\iota(\pi(g)w)) \\
&= \widetilde{\zeta_\delta^i}(g)I_{\zeta_\delta^i}(g\iota(w)) \\
&= \widetilde{\zeta_\delta^i}(g)\widetilde{\zeta_\delta^i}^{-1}(g)gI_{\zeta_\delta^i}(\iota(w)) = gI_{\zeta_\delta^i} \circ \iota(w),
\end{aligned}$$

that is $I_{\zeta_\delta^i} \circ \iota \in Hom_{G^F \rtimes\langle\delta\rangle}(\pi \otimes \widetilde{\zeta_\delta^i}, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$.

For $x \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$, let $T_x \in End_{G^F \rtimes\langle\delta\rangle}(R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$ be as defined in Section 1.2.2. Then

$$\widetilde{\zeta_\delta^i}(x)T_x \circ I_{\zeta_\delta^i} = I_{\zeta_\delta^i} \circ T_x. \tag{1.59}$$

Indeed for any $ge_U \otimes v$ in the representation space of $R_L^{G^F \rtimes\langle\delta\rangle}\sigma$ it holds

$$\begin{aligned}
I_{\zeta_\delta^i} \circ T_x(ge_U \otimes v) &= I_{\zeta_\delta^i}(q^{l(x)}ge_Ux^{-1}e_U \otimes t_x(v)) \\
&= \widetilde{\zeta_\delta^i}^{-1}(gx^{-1})q^{l(x)}ge_Ux^{-1}e_U \otimes t_x(v) \\
&= \widetilde{\zeta_\delta^i}(x)(\widetilde{\zeta_\delta^i}^{-1}(g)q^{l(x)}ge_Ux^{-1}e_U \otimes t_x(v)) \\
&= \widetilde{\zeta_\delta^i}(x)(\widetilde{\zeta_\delta^i}^{-1}(g)T_x(ge_U \otimes v)) = \widetilde{\zeta_\delta^i}(x)T_x \circ I_{\zeta_\delta^i}(ge_U \otimes v).
\end{aligned}$$

Let χ_π be the character of $\theta_{G^F \rtimes\langle\delta\rangle}(\pi)$, that is, the character of the $End_{G^F \rtimes\langle\delta\rangle}(R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$-module $Hom(\pi, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$, and similarly let $\chi_{\pi\otimes\widetilde{\zeta_\delta^i}}$ be the character of $\theta_{G^F \rtimes\langle\delta\rangle}(\pi\otimes\widetilde{\zeta_\delta^i})$. Since $I_{\zeta_\delta^i}$ is an isomorphism between $Hom(\pi, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$ and $Hom(\pi \otimes \widetilde{\zeta_\delta^i}, R_L^{G^F \rtimes\langle\delta\rangle}\sigma)$, by Equation (1.59) for any $x \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ it holds

$$\chi_\pi(T_x) = \widetilde{\zeta_\delta^i}(x)\chi_{\pi\otimes\widetilde{\zeta_\delta^i}}(T_x).$$

Identifying representations with their characters, by definition of $S_{G^F \rtimes\langle\delta\rangle}$ for any $x \in W_{G^F \rtimes\langle\delta\rangle}(L, \sigma)$ it holds

$$\begin{aligned}
S_{G^F \rtimes\langle\delta\rangle}(\chi_\pi)(x) &= \chi_\pi(T_x), \\
S_{G^F \rtimes\langle\delta\rangle}(\chi_{\pi\otimes\widetilde{\zeta_\delta^i}})(x) &= \chi_{\pi\otimes\widetilde{\zeta_\delta^i}}(T_x).
\end{aligned}$$

Therefore

$$S_{G^F \rtimes\langle\delta\rangle}(\chi_\pi)(x) = \chi_\pi(T_x) = \widetilde{\zeta_\delta^i}(x)\chi_{\pi\otimes\widetilde{\zeta_\delta^i}}(T_x) = \widetilde{\zeta_\delta^i}(x)S_{G^F \rtimes\langle\delta\rangle}(\chi_{\pi\otimes\widetilde{\zeta_\delta^i}})(x),$$

so

$$S_{G^F \rtimes \langle\delta\rangle}(\chi_\pi) = \widetilde{\zeta_\delta^i} \otimes S_{G^F \rtimes\langle\delta\rangle}(\chi_{\pi\otimes\widetilde{\zeta_\delta^i}}).$$

This shows that

$$\widetilde{\zeta_\delta^i} \otimes S_{G^F \rtimes\langle\delta\rangle} \circ \theta_{G^F \rtimes\langle\delta\rangle}(\pi \otimes \widetilde{\zeta_\delta^i}) = S_{G^F \rtimes\langle\delta\rangle} \circ \theta_{G^F \rtimes\langle\delta\rangle}(\pi)$$

that is

$$Rep_{G^F \rtimes\langle\delta\rangle}(\pi \otimes \widetilde{\zeta_\delta^i}) = \widetilde{\zeta_\delta^i}^{-1} \otimes Rep_{G^F \rtimes\langle\delta\rangle}(\pi).$$

□

Using the notation introduced in Section 1.2.4.1, we denote the restriction maps

$$\begin{aligned} \pi_{G^F,1} : R(G^F \rtimes \langle\delta\rangle) &\to Cl(G^F\delta) \\ f &\mapsto f|_{G^F\delta} \\ \pi_{W_{G^F}(L,\sigma),-1} : R(W_{G^F \rtimes\langle\delta\rangle}(L,\sigma)) &\to Cl(W_{G^F}(L,\sigma)\delta^{-1}) \\ f &\mapsto f|_{W_{G^F}(L,\sigma)\delta^{-1}} \end{aligned}$$

We denote by $\mathcal{E}^{G^F\delta}(L,\sigma)$ the image of $\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ through $\pi_{G^F,1}$.

Lemma 1.2.36.

1. *It holds*

$$\begin{aligned} Ker(\pi_{G^F,1}) \cap \mathbb{C}[\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)] &= \\ span_{\mathbb{C}}\{\zeta_\delta(\delta^j)\chi - \chi \otimes \zeta_\delta^j | \chi \in \mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma), 1 \leqslant j \leqslant d-1\}&. \end{aligned}$$

2. *A basis for $\mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)]$ is given by $\pi_{G^F,1}(X_{(L,\sigma)})$ where $X_{(L,\sigma)} \subseteq \mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ is a maximal subset in $\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ consisting of irreducible representations with distinct and irreducible restriction to G^F .*

Proof. 1. We set

$$\mathcal{K}_{(L,\sigma)} := span_{\mathbb{C}}\{\zeta_\delta(\delta)^j\chi - \chi \otimes \zeta_\delta^j \mid \chi \in Irr(\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)),\ 1 \leqslant j \leqslant d-1\}$$

and

$$\mathcal{K}^\perp_{(L,\sigma)} := span_{\mathbb{C}}\{\zeta_\delta(\delta)^j\chi - \chi \otimes \zeta_\delta^j \mid \chi \in Irr(G^F \rtimes \langle\delta\rangle) \backslash Irr(\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)),\ 1 \leqslant j \leqslant d-1\}.$$

If $\chi \in \mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$, by Clifford theory $\chi \otimes \zeta_\delta^j \in \mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)$ for any $j = 1\ldots,d-1$. Hence $\mathcal{K}_{(L,\sigma)} \subseteq \mathbb{C}[\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)]$, and $\mathcal{K}^\perp_{(L,\sigma)}$ lies in the orthogonal complement of $\mathbb{C}[\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)]$. Moreover by Corollary 1.2.31

$$Ker(\pi_{G^F,1}) = span_{\mathbb{C}}\{\zeta_\delta(\delta)^j\chi - \chi \otimes \zeta_\delta^j \mid \chi \in Irr(G^F \rtimes \langle\delta\rangle),\ j = 1,\ldots,d-1\},$$

so the subspaces $\mathcal{K}_{(L,\sigma)}$ and $\mathcal{K}^\perp_{(L,\sigma)}$ yield an orthogonal decomposition of $Ker(\pi_{G^F,1})$. It follows that

$$Ker(\pi_{G^F,1}) \cap \mathbb{C}[\mathcal{E}^{G^F \rtimes\langle\delta\rangle}(L,\sigma)] = \mathcal{K}_{(L,\sigma)}$$

2. By the definition of $\mathcal{E}^{G^F\delta}(L,\sigma)$, the set $\pi_{G^F,1}(\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ is a generating set for $\mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)]$. By Lemma 1.2.26, the elements in $\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ with reducible restriction to G^F have 0 restriction to $G^F\delta$, and by Lemma 1.2.28 the elements in $\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ with the same restriction to G^F have linearly dependent restrictions to $G^F\delta$. It follows that the set $X_{(L,\sigma)}$ is a generating set for $\mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)]$.

   If $\chi \notin \mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$, then the restriction of χ to G^F differs from the restriction of any other character in $\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)$. Therefore $X_{(L,\sigma)}$ can be completed to be a maximal set X of irreducible representations of $G^F \rtimes \langle\delta\rangle$ with distinct and irreducible restrictions of G^F. By Proposition 1.2.30 the set $\pi_{G^F,1}(X)$ is linearly independent, and therefore the subset $\pi_{G^F,1}(X_{(L,\sigma)})$ is linearly independent.

□

Lemma 1.2.37. *There exists a linear isomorphism*

$$Rep_{G^F\delta} : \mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)] \to Cl(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)\delta^{d-1})$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{G^F\rtimes\langle\delta\rangle}} & R(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)) \\ \downarrow{\scriptstyle \pi_{G^F,1}} & & \downarrow{\scriptstyle \pi_{W_{G^F}(L,\sigma),-1}} \\ \mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)] & \xrightarrow{Rep_{G^F\delta}} & Cl(W_{G^F}(L,\sigma)\delta^{-1}) \end{array} \tag{1.60}$$

Proof. Let $\pi^{(L,\sigma)}_{G^F,1}$ denote the restriction of $(\pi_{G^F,1})$ to $\mathbb{C}[\mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma)]$. By Lemma 1.2.36 1 we obtain

$$Ker(\pi^{(L,\sigma)}_{G^F,1}) = span_{\mathbb{C}}\{\zeta_\delta(\delta^j)\chi - \chi\otimes\zeta_\delta^j | \chi \in \mathcal{E}^{G^F\rtimes\langle\delta\rangle}(L,\sigma), j = 1,\ldots,d-1\}.$$

Similarly, by Corollary 1.2.31, substituting $\delta^{d-1} = \delta^{-1}$ to δ as in Remark 1.2.32,

$$Ker(\pi_{W_{G^F}(L,\sigma),-1}) = span_{\mathbb{C}}\{\zeta_\delta(\delta^{-j})\chi - \chi\otimes\zeta_\delta^j | \chi \in Irr(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)), j = 1,\ldots,d-1\}.$$

By Lemma 1.2.35, it holds

$$\begin{aligned} Rep_{G^F\rtimes\langle\delta\rangle}(\zeta_\delta(\delta^j)\chi - \chi\otimes\zeta_\delta^j) &= \zeta_\delta(\delta^j)Rep_{G^F\rtimes\langle\delta\rangle}(\chi) - Rep_{G^F\rtimes\langle\delta\rangle}(\chi\otimes\zeta_\delta^j) = \\ &\zeta_\delta(\delta^j)Rep_{G^F\rtimes\langle\delta\rangle}(\chi) - Rep_{G^F\rtimes\langle\delta\rangle}(\chi)\otimes\zeta_\delta^{-j}. \end{aligned}$$

So the map $Rep_{G^F\rtimes\langle\delta\rangle}$ is an isomorphism respecting the kernels of the projections $\pi^{(L,\sigma)}_{G^F,1}$ and $\pi_{W_{G^F}(L,\sigma),-1}$. Hence, setting for any $\chi \in Irr(W_{G^F\rtimes\langle\delta\rangle}(L,\sigma))$

$$Rep_{G^F\delta}(\pi_{G^F,1}(\chi)) := \pi_{W_{G^F}(L,\sigma),-1}(Rep_{G^F\rtimes\langle\delta\rangle}(\chi)$$

gives a well-defined isomorphism between $\mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)]$ and $Cl(W_{G^F}(L,\sigma)\delta^{-1})$. □

Corollary 1.2.38. *The map $Rep_{G^F\delta}$ is an isometry with respect to the scalar product of class functions on cosets: for any $f,h \in \mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)]$ it holds*

$$\langle f,h\rangle_{G^F\delta} = \langle Rep_{G^F\delta}(f), Rep_{G^F\delta}(h)\rangle_{W_{G^F}(L,\sigma)\delta^{-1}}$$

Proof. Let $\{\chi_i\}_{i=1}^r \subseteq \mathcal{E}^{G^F \rtimes \langle\delta\rangle}(L,\sigma)$ be a maximal subset of irreducible representations with distinct and irreducible restriction to G^F. By Lemma (1.2.36) 2, $\{\pi_{G^F,1}(\chi_i)\}_{i=1}^r$ is an orthonormal basis of $\mathbb{C}[\mathcal{E}^{G^F}(L,\sigma)]$.

We claim that $\{Rep_{G^F\delta}(\pi_{G^F,1}(\chi_i))\}_{i=1}^r$ is an orthonormal basis for $R(W_{G^F}(L,\sigma)\delta^{-1})$. By Lemma 1.2.37, $Rep_{G^F\delta}(\pi_{G^F,1}(\chi_i)) = \pi_{W_{G^F}(L,\sigma),1}(Rep_{G^F\rtimes\langle\delta\rangle}(\chi_i))$. The maps $Rep_{G^F\rtimes\langle\delta\rangle}$, Rep_{G^F} restricted to irreducible representations are bijections, and by diagram (1.50) they are compatible with restriction. So the set $\{Rep_{G^F\rtimes\langle\delta\rangle}(\chi_i)\}_{i=1}^r$ is a set of irreducible representations of $W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)$ with irreducible and distinct restrictions to $W_{G^F}(L,\sigma)$. By Proposition 1.2.30, $\{Rep_{G^F\delta}(\pi_{G^F,1}(\chi_i))\}_{i=1}^r$ is an orthonormal basis for $Cl(W_{G^F}(L,\sigma)\delta^{-1})$.

This proves that $Rep_{G^F\delta}$ is an isometry, since it maps an orthonormal basis to an orthonormal basis. □

Let M be an F-stable and δ-stable standard Levi subgroup of an F-stable and δ-stable standard parabolic subgroup P_M of G, such that M contains L. We define the restriction maps

$$\pi_{M^F,1} : R(M^F \rtimes \langle\delta\rangle) \to Cl(M^F\delta),$$
$$\pi_{W_{M^F}(L,\sigma),-1} : R(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)) \to Cl(W_{M^F}(L,\sigma)\delta^{-1}).$$

We define $\mathcal{E}^{M^F\delta}(L,\sigma)$ similarly to how we defined it for G. As for G, there exists a linear isomorphism

$$Rep_{M^F\delta} : \mathbb{C}[\mathcal{E}^{M^F\delta}(L,\sigma)] \to Cl(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)\delta^{d-1})$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{M^F\rtimes\langle\delta\rangle}(L,\sigma)] & \xrightarrow{Rep_{M^F\rtimes\langle\delta\rangle}} & R(W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)) \\ \downarrow{\scriptstyle \pi_{M^F,1}} & & \downarrow{\scriptstyle \pi_{W_{M^F}(L,\sigma),-1}} \\ \mathbb{C}[\mathcal{E}^{M^F\delta}(L,\sigma)] & \xrightarrow{Rep_{M^F\delta}} & Cl(W_{M^F}(L,\sigma)\delta^{-1}) \end{array} \tag{1.61}$$

For $f \in Cl(M^F\delta)$, we define

$$R^{G^F\delta}_{M^F\delta} f := Ind^{G^F\delta}_{P^F\delta}(Infl^{P^F\delta}_{M^F\delta}(f)) \in Cl(G^F\delta), \tag{1.62}$$

that is for any $x \in G^F\delta$:

$$R^{G^F\delta}_{M^F\delta} f = \frac{1}{|P^F|} \sum_{\substack{g\in G^F \\ g^{-1}xg\in P^F\delta}} Infl^{P^F\delta}_{M^F\delta}(f)(g^{-1}xg).$$

For any $f \in Cl(G^F\delta)$ we define

$$^*R^{G^F\delta}_{M^F\delta} f := (Res^{G^F\delta}_{P^F\delta}(f))^{U_M^F} \in Cl(M^F\delta), \tag{1.63}$$

that is for any $m \in M^F\delta$,

$$^*R^{G^F\delta}_{M^F\delta} f = \frac{1}{|U_M^F|} \sum_{u\in U_M^F} f(mu)$$

Proposition 1.2.39. *With the above notation, for any $f \in Cl(M^F\delta)$ and $h \in Cl(G^F\delta)$ it holds*

$$\langle R^{G^F\delta}_{M^F\delta} f, h\rangle_{G^F\delta} = \langle f, {}^*R^{G^F\delta}_{M^F\delta} h\rangle_{M^F\delta}$$

Proof. We have

$$\begin{aligned}
\langle R^{G^F\delta}_{M^F\delta} f, h\rangle_{G^F\delta} &= \langle Ind^{G^F\delta}_{P^F\delta}(Infl^{P^F\delta}_{M^F\delta}(f)), h\rangle_{G^F\delta} && \text{Proposition 1.2.33}\\
&= \langle Infl^{P^F\delta}_{M^F\delta}(f), Res^{G^F\delta}_{P^F\delta}(h)\rangle_{P^F\delta} && \text{Proposition 1.2.34}\\
&= \langle f, (Res^{G^F\delta}_{P^F\delta})^{U_M^F}(h)\rangle_{M^F\delta} = \langle f, {}^*R^{G^F\delta}_{M^F\delta}(h)\rangle_{M^F\delta}.
\end{aligned}$$

□

Lemma 1.2.40. *The following diagrams commute:*

$$\begin{array}{ccc}
R(M^F \rtimes \langle\delta\rangle) & \xrightarrow{R^{G^F \rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle}} & R(G^F \rtimes \langle\delta\rangle) \\
\downarrow{\scriptstyle \pi_{M^F,1}} & & \downarrow{\scriptstyle \pi_{G^F,1}} \\
Cl(M^F\delta) & \xrightarrow{R^{G^F\delta}_{M^F\delta}} & Cl(G^F\delta)
\end{array}
\qquad
\begin{array}{ccc}
R(G^F \rtimes \langle\delta\rangle) & \xrightarrow{{}^*R^{G^F \rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle}} & R(M^F \rtimes \langle\delta\rangle) \\
\downarrow{\scriptstyle \pi_{G^F,1}} & & \downarrow{\scriptstyle \pi_{M^F,1}} \\
Cl(G^F\delta) & \xrightarrow{{}^*R^{G^F\delta}_{M^F\delta}} & Cl(M^F\delta)
\end{array}
\tag{1.64}$$

Proof. We first check the commutativity of the diagram on the left.

For any $f \in R(M^F \rtimes \langle\delta\rangle)$ and $x \in G^F\delta$, by (1.9)

$$\begin{aligned}
R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(x) &= \frac{1}{|P^F \rtimes \langle\delta\rangle|} \sum_{\substack{g\in G^F\rtimes\langle\delta\rangle \\ g^{-1}xg\in P^F\rtimes\langle\delta\rangle}} Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(g^{-1}xg)\\
&= \frac{1}{d|P^F|}\sum_{i=0}^{d} \sum_{\substack{g\delta^i\in G^F\delta^i \\ (g\delta^i)^{-1}xg\delta^i\in P^F\rtimes\langle\delta\rangle}} (Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f)(g^{-1}xg)
\end{aligned}$$

The condition $(g\delta^i)^{-1}xg\delta^i = \delta^{-i}(g^{-1}xg) \in P^F \rtimes \langle\delta\rangle$ holds if and only if $g^{-1}xg \in P^F \rtimes \langle\delta\rangle$, and since $x \in G^F\delta$, it is equivalent to $g^{-1}xg \in P^F\delta$. Moreover $Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f$ is a class function, so $(Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f)(\delta^{-i}g^{-1}xg\delta^i) = (Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f)(g^{-1}xg)$. Therefore

$$R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(x) = \frac{1}{|P^F|} \sum_{\substack{g\in G^F \\ g^{-1}xg\in P^F\delta}} Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(g^{-1}xg).$$

For any $x\delta \in P^F\delta$, writing $x\delta = mu\delta$ with $m \in M^F$, $u \in U_M^F$, it holds

$$Infl^{P^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(x\delta) = f(m\delta) = \pi_{M^F,1}(f)(m\delta) = Infl^{P^F\delta}_{M^F\delta}(\pi_{M^F,1}(f))(x\delta).$$

Therefore

$$R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} f(x) = R^{G^F\delta}_{M^F\delta} \circ \pi_{M^F,1} f(x)$$

for any $x \in G^F\delta$, that is

$$\pi_{G^F,1} \circ R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle} = R^{G^F\delta}_{M^F\delta} \circ \pi_{M^F,1}.$$

The commutativity of the diagram on the right follows immediately from the definition of ${}^*R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle}$ and ${}^*R^{G^F\delta}_{M^F\delta}$. Indeed for any $f \in R(G^F \rtimes \langle\delta\rangle)$, and any $x \in M^F\delta$ it holds

$$\pi_{M^F,1} \circ {}^*R^{G^F\rtimes\langle\delta\rangle}_{M^F\rtimes\langle\delta\rangle}(f)(x) = \frac{1}{|U^F_M|}\sum_{u\in U^F_M} f(xu) = {}^*R^{G^F\delta}_{M^F\delta} \circ \pi_{G^F,1}(f)(x).$$

□

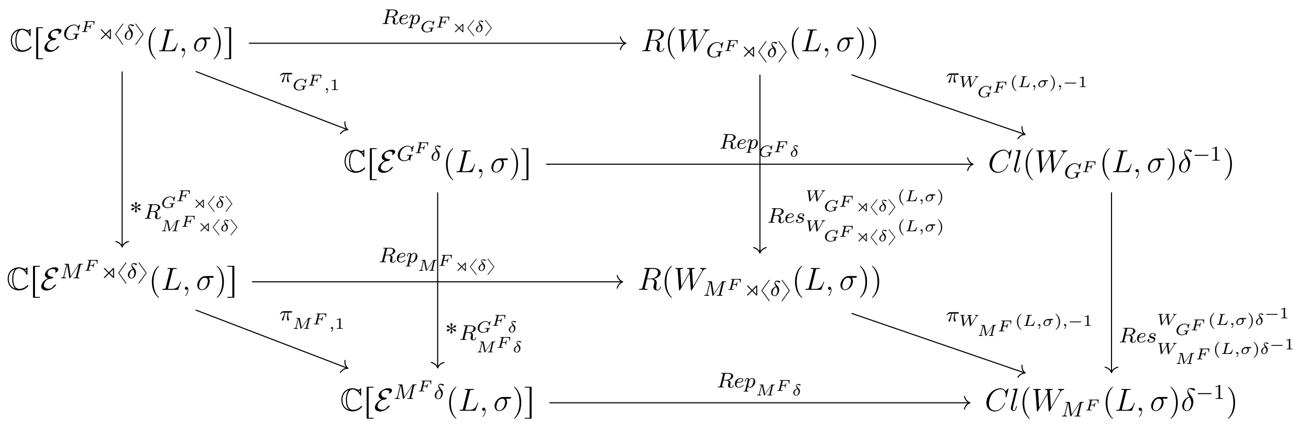
We now have all the necessary setup for the main result of this section.

Theorem 1.2.41. *The following diagrams commute:*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)] & \xrightarrow{Rep_{G^F\delta}} & Cl(W_{G^F}(L,\sigma)\delta^{-1}) \\ \downarrow {}^*R^{G^F\delta}_{M^F\delta} & & \downarrow Res^{W_{G^F}(L,\sigma)\delta^{-1}}_{W_{M^F}(L,\sigma)\delta^{-1}} \\ \mathbb{C}[\mathcal{E}^{M^F\delta}(L,\sigma)] & \xrightarrow{Rep_{M^F\delta}} & Cl(W_{M^F}(L,\sigma)\delta^{-1}) \end{array} \tag{1.65}$$

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F\delta}(L,\sigma)] & \xrightarrow{Rep_{G^F\delta}} & Cl(W_{G^F}(L,\sigma)\delta^{-1}) \\ R^{G^F\delta}_{M^F\delta}\uparrow & & Ind^{W_{G^F}(L,\sigma)\delta^{-1}}_{W_{M^F}(L,\sigma)\delta^{-1}}\uparrow \\ \mathbb{C}[\mathcal{E}^{M^F\delta}(L,\sigma)] & \xrightarrow{Rep_{M^F\delta}} & Cl(W_{M^F}(L,\sigma)\delta^{-1}) \end{array} \tag{1.66}$$

Proof. We first show the commutativity of the first diagram. Consider the following diagram:



(1.67)

- ◇ the back face commutes by Proposition 1.48,
- ◇ the top and the bottom faces commute by Lemma 1.2.37,
- ◇ the left side face commutes by Lemma (1.2.40),
- ◇ the right side face commutes because $Res^{W_{G^F}(L,\sigma)\delta}_{W_{M^F}(L,\sigma)\delta} \circ \pi_{W_{G^F}(L,\sigma),-1}$ and $\pi_{W_M} \circ Res^{W_{G^F\rtimes\langle\delta\rangle}(L,\sigma)}_{W_{M^F\rtimes\langle\delta\rangle}(L,\sigma)}$ are both just the restriction of functions to $W_{M^F}(L,\sigma)\delta$.

Since all the diagonal arrows are surjective, it follows that the front face commutes as well.

The commutativity of the second diagram is equivalent to the commutativity of the first one. Indeed for any $f \in \mathbb{C}[\mathcal{E}^{M^F\delta}(L,\sigma)]$ and any $\phi \in Cl(W_{M^F}(L,\sigma)\delta^{-1})$ we have

$$\begin{aligned}
&\langle Ind_{W_{M^F}(L,\sigma)\delta^{-1}}^{W_{G^F}(L,\sigma)\delta^{-1}} \circ Rep_{M^F\delta}(f), \phi\rangle_{W_{G^F}(L,\sigma)\delta^{-1}} && \text{Proposition 1.2.33}\\
&= \langle Rep_{M^F\delta}(f), Res_{W_{M^F}(L,\sigma)\delta^{-1}}^{W_{G^F}(L,\sigma)\delta^{-1}}(\phi)\rangle_{W_{M^F}(L,\sigma)\delta^{-1}} && \text{Corollary 1.2.38}\\
&= \langle f, Rep_{M^F\delta}^{-1} \circ Res_{W_{M^F}(L,\sigma)\delta^{-1}}^{W_{G^F}(L,\sigma)\delta^{-1}}(\phi)\rangle_{M^F\delta} && \text{Diagram (1.66)}\\
&= \langle f, {}^*R_{M^F\delta}^{G^F\delta} \circ Rep_{G^F\delta}^{-1}(\phi)\rangle_{M^F\delta} && \text{Proposition (1.2.39)}\\
&= \langle R_{M^F\delta}^{G^F\delta}(f), Rep_{G^F\delta}^{-1}(\phi)\rangle_{G^F\delta} && \text{Corollary 1.2.38}\\
&= \langle Rep_{G^F\delta} \circ R_{M^F\delta}^{G^F\delta}(f), \phi\rangle_{W_{G^F}(L,\sigma)\delta^{-1}}
\end{aligned}$$

□

Chapter II

Lusztig's Fourier transform and parabolic induction

Introduction

The main aim of this chapter is to give a detailed proof of the compatibility of Lusztig's non-abelian Fourier transform with parabolic induction for unipotent representations of finite groups of classical type B, C and D. The relevant theorems in this regard are Theroem 2.4.12, where we deal with groups of rational points of disconnected algebraic groups whose identity component is a simple group of type D_n (e.g. finite orthogonal groups in spaces of even dimension), Theorems 2.5.16 and 2.5.17, where we deal with the split and non-split rational forms of groups of type D_n (e.g. the two forms of the finite special orthogonal groups in spaces of even dimension), and Theorem 2.6.8, where we deal with finite groups of Lie type of type B_n and C_n (e.g. finite special orthogonal group in odd dimension or finite symplectic groups).

The Lusztig non-abelian Fourier transform [42, Section 4] is a map on the space of unipotent class functions on a finite group of Lie type. Lusztig's conjecture [44] claims that this map gives the change of basis between two natural bases: the one of unipotent characters of representations and the one of the characteristic functions of character sheaves. Lusztig's conjecture has been proved for connected groups with a connected center in [62, 63], and subsequently extended to more general cases (see [23] and the references therein).

We believe that the content of this chapter is in large part already known to experts (see [1, 2, 40]). Nevertheless, we think that a detailed exposition of a combinatorial proof of the compatibility can be of some relevance. In particular, we believe there is some novelty in our exposition on disconnected groups of type D, whose content we are now going to explain in some detail.

Groups of type D_n admit two forms over a finite field, a split and a non-split one. If G is a simple algebraic group of type D_n over $\overline{\mathbb{F}}_q$, the disconnected algebraic group obtained by extending G by an outer automorphism δ corresponding to the Dynkin-diagram automorphism of order 2 has two inner twists over the finite field $\mathbb{F}_q$, as defined in Secion 2.1.2.3. The subgroups of these inner twists corresponding to the identity component of $G \rtimes \langle\delta\rangle$ are respectively the split and the non-split form of a simple group of type D_n. The Lusztig's Fourier transforms for the split and the non-split groups of type D_n [42, Section 4] can be regarded as the restriction of an involution $\widetilde{\mathcal{R}}$ defined on the sum of the spaces of unipotent class functions on the two inner twists of $G \rtimes \langle\delta\rangle$. The definition of the involution $\widetilde{\mathcal{R}}$ was given in [40]. A similar involution $\mathcal{F}$ was defined in [53], see Remark 2.5.6. The compatibility of $\mathcal{F}$ with parabolic induction, that is the analogue of our Theorem 2.4.12 with $\mathcal{F}$ in place of $\widetilde{\mathcal{R}}$, was already stated in [53, 2.11], but there it is deduced admitting an extension of Lusztig's conjecture for disconnected groups, which was subsequently proved in [69]. We prove Theorem 2.4.12 making use just of elementary combinatorial techniques.

Working with disconnected groups comes up naturally in the classification theory of finite groups of Lie type, when dealing with the classification of non-unipotent representations of groups with disconnected center, see for instance [19]. Considering the split and non-split form of a group together gives a natural setting that simplifies the combinatorial description of Lusztig's non-abelian Fourier transform.

Moreover, in [3] the authors define an "elliptic Fourier transform" for p-adic groups, that is an involution on the space of unipotent representations of inner twists of a p-adic group. This map is conjectured to be compatible with Lusztig's non-abelian Fourier transform for finite groups of Lie type via parahoric restriction on the maximal compact subgroups [3, Conjecture 9.7]. The parahoric restriction to a maximal compact subgroup of unipotent representations of inner twists of a p-adic group lands in the space generated by unipotent representations of the inner forms of a possibly disconnected algebraic group over a finite field. Therefore, from this point of view it is natural to consider the disconnected setting and account for all the inner twists together.

The chapter is organized as follows.

In Section 2.1 we review the parameterization of unipotent representations of finite groups of Lie type of classical type A, B and C via partitions and bipartitions obtained using the parametrization of Harish-Chandra series by irreducible representations of the relative Weyl group and the combinatorial description of Weyl groups of type A and B [14, 13.8]. For groups of type D, we exploit the same strategy to parametrize together the irreducible representations of the two inner forms of the disconnected group via bipartitions and a sign, exploiting the results known in the connected setting [14, 13.8] and the results of Chapter I, in particular Theorem 1.2.24. We then recall how to describe combinatorially the parabolic restriction of a representation to a rational Levi subgroup of a rational parabolic subgroup, exploiting the Murnaghan-Nakayama rule for Weyl groups of type B.

Sections 2.2 and 2.3 are devoted to introducing the formalism of Symbols, due to Lusztig [41]. We show how ordered Symbols parametrize irreducible unipotent representations of the inner twists of disconnected groups of type D, and how the parabolic restriction translates to these combinatorial objects.

In Section 2.4 we introduce the combinatorial map $\widetilde{\mathcal{R}}$ and review its main combinatorial properties. We then show how this map induces an involution on the sum of the spaces of unipotent class functions on the two inner twists of a disconnected group of type D_n. In Theorem 2.4.12 we exploit the combinatorial result to prove that this map is compatible with parabolic induction.

Section 2.5 is devoted to showing how the results of Section 2.4 can be used to deduce compatibility of the Fourier Transform defined by Lusztig in [42, 4.6, 4.15] and in [42, 4.18] with parabolic induction.

The final section is an account of the analogous results for groups of type B.

We retain notation from Chapter I

2.1 Parameterization of unipotent representations of finite groups of Lie type

We are interested in unipotent representations of a finite group of Lie type [14, Section 12.1]. Let G be a connected reductive linear algebraic group defined over $\mathbb{F}_q$, and let F be a Frobenius morphism for G. We will denote by $Irr_u(G^F)$ the subset of $Irr(G^F)$ consisting of unipotent representations, and by $R_u(G^F)$ the subspace of

$R(G^F)$ generated by $Irr_u(G^F)$.

2.1.1 Reduction to simple groups

The classification of unipotent representations of a finite group of Lie type can be reduced to the classification of unipotent representations of finite groups of Lie type that are rational points of a simple algebraic group of adjoint type. We recall the reduction argument from [49, Remark 4.2.1].
The parameterization of unipotent representations of G^F is independent of the isogeny class of the group G. Indeed, for any reductive group G, let

$$\pi : G \to {}^{G}\!/_{Z(G)} \cong G_{ad}$$

be the natural projection to the adjoint group G_{ad}, and let $\overline{F} : G_{ad} \to G_{ad}$ be the Frobenius morphism such that $\overline{F} \circ \pi = \pi \circ F$.
The projection π restricts to a group morphism,

$$\pi_F : G^F \to G_{ad}^{\overline{F}}.$$

that is not necessarily surjective, and it induces by pullback a map π_F^* from representations of $G_{ad}^{\overline{F}}$ to representations of G^F. By [18, Theorem 7.10], the restriction of π_F^* to unipotent representations of $G_{ad}^{\overline{F}}$ yields a bijection

$$\begin{aligned} \pi_F^* : Irr_u(G_{ad}^{\overline{F}}) &\to Irr_u(G^F). \\ \chi &\to \chi \circ \pi_F \end{aligned}$$

Therefore, it suffices to study unipotent representations of $G_{ad}^{\overline{F}}$. The group G_{ad} admits always a decomposition as a direct product

$$G_{ad} = G_1 \times G_2 \times \cdots \times G_k$$

for some $k \in \mathbb{N}$, with G_i an adjoint simple algebraic group for any $i = 1, \dots, n$.
If $\overline{F}$ stabilizes each of the simple factors G_i, the unipotent representations of $G_{ad}^{\overline{F}}$ are given by tensor products of the unipotent representations of the $G_i^{\overline{F}}$.
Otherwise, $\overline{F}$ induces a permutation of the simple factors, and collecting together those in the same $\overline{F}$-orbits, we can write

$$G_{ad} = H_1 \times H_2 \times \cdots \times H_r$$

where each H_j is a direct product of those G_i that are cyclically permuted by $\overline{F}$.
If m_j is the number of simple factors in H_j, then any simple factor G_i in H_j is preserved by F^{m_j} and there is an injective homomorphism of algebraic groups

$$\begin{aligned} i_j : G_i &\to H_j \\ g &\to g\overline{F}(g) \cdots \overline{F}^{m_j - 1}(g) \end{aligned}$$

that induces an isomorphism $G_i^{\overline{F}^{m_j}} \cong H_j^{\overline{F}}$. Furthermore, this isomorphism induces a bijection between unipotent representations of $G_i^{\overline{F}^{m_j}}$ and of $H_j^{\overline{F}}$. Therefore the irreducible unipotent representations of $G_{ad}^{\overline{F}}$ are given by tensor products of the unipotent representations of finite Lie groups that are rational points of simple algebraic groups of adjoint type (i.e. the various $G_i^{\overline{F}_{m_j}}$).
All the maps we used for the reduction argument are compatible with parabolic induction, hence, in order to study unipotent cuspidal representations and the parameterization given by the map (1.2) for connected reductive groups, it is enough to study the case of adjoint simple algebraic groups. In this case, the cuspidal representations are completely classified,[14, Section 13.7] (see also [49, Theorem 4.4.28]).

In the next section, we recall the combinatorial parameterization for groups of classical type B, C and D.

2.1.2 Classical groups

We recall a classical combinatorial parameterization of the Harish-Chandra series of groups of classical type, by making use of the map (1.2). We follow [14, Section 13.8].

Let $\mathscr{P}^+ = \{(\alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathbb{N}^l | l \in \mathbb{N}, \alpha_i \leqslant \alpha_{i-1}\}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathscr{P}^+$ we call l the length of α and denote it by $l(\alpha)$, and we call the rank of α the number

$$\varrho(\alpha) := \sum_{i=1}^{l} \alpha_i.$$

We also set

$$\mathscr{P}_n^+ = \{\alpha \in \mathscr{P}^+ | \varrho(\alpha) = n\},$$

i.e. $\mathscr{P}_n^+$ is the set of "partitions admitting 0s" of n.

We set $\mathscr{P} := \{(\alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathbb{N}^l | l \in \mathbb{N}, 0 < \alpha_i \leqslant \alpha_{i-1}\}$ and set $\mathscr{P}_n := \mathscr{P}_n^+ \cap \mathscr{P}$, that is, $\mathscr{P}_n$ is the set of partitions of n. We denote by $\mathscr{P}_0$ the set of 1 element consisting of the empty partition.

We denote by $\mathscr{B}_n$ the set of the bipartitions of $n \in \mathbb{N}$. A bipartition is a pair of partitions $(\alpha, \beta) \in \mathscr{P} \times \mathscr{P}$, and $(\alpha, \beta) \in \mathscr{B}_n$ if $\varrho(\alpha) + \varrho(\beta) = n$. We denote by $\mathscr{B}_0$ the set of 1 element consisting of the empty bipartition.

Let $(\alpha, \beta) \in \mathscr{B}_n$. If $\alpha \neq \beta$, we set

$$[[(\alpha, \beta)]] := \{(\alpha, \beta), (\beta, \alpha)\} \tag{2.1}$$

and we call $[[(\alpha, \beta)]]$ a non degenerate unordered bipartition. If $\alpha = \beta$, we say that (α, α) is a degenerate bipartition of n, and we formally associate to it two degenerate unordered bipartitions of n, that we denote by $[[(\alpha, \alpha)]]_+$ and $[[(\alpha, \alpha)]]_-$. We set

$$\mathscr{D}_n := \{[[(\alpha, \beta)]] | (\alpha, \beta) \in \mathscr{B}_n, \alpha \neq \beta\} \cup \{[[(\alpha, \alpha)]]_+, [[(\alpha, \alpha)]]_- | (\alpha, \alpha) \in \mathscr{B}_n\},$$

and we call it the set of unordered bipartitions of n. Inside $\mathbb{C}[\mathscr{D}_n]$, we set $[[(\alpha, \alpha)]] := [[(\alpha, \alpha)]]_+ + [[(\alpha, \alpha)]]_-$.

We will exploit some well-known parametrizations for representations of classical Weyl groups in terms of partitions and bipartitions. We recall them here.

Let $n \in \mathbb{N}$.

- ([49, Example 4.1.2]) Let $W(A_n) = \mathbb{S}_{n+1}$ be the Weyl group of type A_n, i.e. the symmetric group on $n+1$ elements. Then there exists a bijection

$$\kappa_A : Irr(W(A_n)) \to \mathscr{P}_{n+1} \tag{2.2}$$

which is uniquely determined as follows. For any $\alpha = (\alpha_1, \dots \alpha_l) \in \mathscr{P}_{n+1}$ write $\mathbb{S}_\alpha$ for the standard Young subgroup $\mathbb{S}_{\alpha_1} \times \dots \times \mathbb{S}_{\alpha_l}$ of $\mathbb{S}_n$. Let $\rho \in Irr(W(A_n))$. Then there is a unique $\alpha \in \mathscr{P}_{n+1}$ such that ρ is an irreducible constituent of both the representations $ind_{\mathbb{S}_\alpha}^{\mathbb{S}_n} sign$, where $sign$ denotes the $sign$ representation, and $ind_{\mathbb{S}_{\alpha'}}^{\mathbb{S}_n} 1$, where α' denotes the transpose of α and 1 denotes the trivial representation [24, Theorem 5.4.7]. We set $\kappa_A(\rho) = \alpha$.

This parameterization is obtained from the one in [49, Example 4.1.2] by tensor product with the sign representation [24, Corollary 5.4.9]. We choose this parameterization since it will be convenient in Chapter III.

- ([49, Example 4.1.3]), [24, 5.5.1, 5.5.3] Let $W(B_n)$ be the Weyl group of type B_n. Then there is a bijection

$$\kappa_B : Irr(W(B_n)) \to \mathscr{B}_n \tag{2.3}$$

which is uniquely determined as follows. For any $m \in \mathbb{N}$, the group $W(B_m)$ decomposes as $W(D_m) \rtimes \mathbb{Z}_2$, where $W(D_m)$ denotes a Weyl group of type D_m. Let ε'_m be $Infl_{\mathbb{Z}_2}^{W(B_m)} sign$. Let $\rho \in Irr(W(B_n)$. Then there exists a unique $(\alpha, \beta) \in \mathscr{B}_n$ such that

$$\rho = Ind_{W(B_{\varrho(\alpha)}) \times W(B_{\varrho(\beta)})}^{W(B_n)} (Infl_{\mathbb{S}_{\varrho(\alpha)}}^{W(B_{\varrho(\alpha)})} (\kappa_A^{-1}(\alpha) \otimes \varepsilon'_a) \boxtimes Infl_{\mathbb{S}_{\varrho(\beta)}}^{W(B_{\varrho(\beta)})} \kappa_A^{-1}(\beta)),$$

[24, 5.5.4]. We set $\kappa_B(\rho)) = (\alpha, \beta)$.

This parameterization is obtained from the one in [49, Example 4.1.3] by tensor product with the character $sign$ [24, Theorem 5.5.6 (c)]. We choose it this way to be consistent with the choice of κ_A.

We assume $W(B_0)$ to be the group with 1 element, and in this case the map $\kappa_B : Irr(W(B_0)) \to \mathscr{B}_0$ is the unique one.

- [49, Example 4.1.4], [24, 5.6.1] Let $W(D_n)$ be Weyl group of type D_n. Then there is a bijection

$$\kappa_D : Irr(W(D_n)) \to \mathscr{D}_n \tag{2.4}$$

which is uniquely determined as follows. We identify the group $W(D_n)$ with the kernel of the character ε'_n of $W(B_n)$. Let $(\alpha, \beta) \in \mathscr{B}_n$. If $[[(\alpha, \beta)]]$ is non-degenerate, then $Res_{W(D_n)}^{W(B_n)} \kappa_B^{-1}(\alpha, \beta) \in Irr(W(D_n))$, and we set $\kappa_D(\rho) = [[(\alpha, \beta)]]$. If $[[(\alpha, \alpha)]]$ is degenerate, then $Res_{W(D_n)}^{W(B_n)} \kappa_B^{-1}(\alpha, \alpha) = \rho^+ + \rho^-$ with $\rho^\pm \in Irr(W(D_n)$, and we set $\kappa_D(\rho^\pm) = [[(\alpha, \beta)]]_\pm$. The representations $\rho^\pm$

form an orbit under the $\mathbb{Z}_2$-action on $\mathbb{D}_n$ corresponding to a graph automorphism of order 2. To distinguish between the two irreducible constituents of $Res^{W(B_n)}_{W(D_n)}\kappa_B^{-1}(\alpha, \alpha)$ and for a concrete choice of which sign assign to each one of them, see [24, Proposition 5.6.3].

This parameterization is obtained from the one in [49, Example 4.1.4] by tensor product with the character *sign* [24, Theorem 5.5.6 (c)]. We choose it this way to be consistent with the choice of κ_B. In particular, for any $\rho \in Irr(W(B_n))$, the maps κ_B, κ_D satisfy

$$[[\kappa_B(\rho)]] = \kappa_D \circ Res^{W(B_n)}_{W(D_n)}(\rho), \tag{2.5}$$

where $[[\kappa_B(\rho)]] \in \mathscr{D}_n$ is the unordered bipartition associated to the bipartition $\kappa_B(\rho)$, as in (2.1).

We will use the same notation to denote the isomorphisms obtained by linear extension of the bijections above.

2.1.2.1 Type A

Let G be of type A_n, and F be a split Frobenius morphism. Then G^F admits a unipotent cuspidal representation if and only if $n = 0$, that is if the group is a torus, and in this case it is the trivial representation. Therefore in a split group of type A_n the only Levi subgroup affording a unipotent cuspidal representation is the maximal split torus T, and all the unipotent representations of G lie in the principal series. The relative Weyl group $W_{G^F}(T, 1_T)$ is the Weyl group of G, i.e. the symmetric group $\mathbb{S}_{n+1}$. Its representations parametrize via (1.2) the Harish Chandra series $\mathcal{E}^{G^F}(T, 1_T)$:

$$Rep_{G^F} : \mathcal{E}^{G^F}(T, 1_T) \to Irr(W_{G^F}(T, 1_T)) = Irr(\mathbb{S}_{n+1}).$$

The map κ_A as in (2.2) gives a bijection

$$\kappa_A : Irr(\mathbb{S}_{n+1}) \to \mathscr{P}_{n+1}.$$

Composing these bijections yields a bijection

$$\kappa_A \circ Rep_{G^F} : \mathcal{E}^{G^F}(T, 1_T) \to \mathscr{P}_{n+1},$$

whose linear extension is an isomorphism

$$\kappa_A \circ Rep_{G^F} : R_u(G^F) = \mathbb{C}[\mathcal{E}^{G^F}(T, 1_T)] \xrightarrow{\sim} \mathbb{C}[\mathscr{P}_{n+1}].$$

2.1.2.2 Type B and C

Let G be of type B_n (or C_n). We use the convention that $B_1 = A_1$. Then G^F admits a unipotent cuspidal representation if and only if $n = s^2 + s$ for some $s \in \mathbb{N}$. If this is the case, such a representation is unique.
Therefore in a group of type B_n (respectively C_n) the Levi subgroups affording a unipotent cuspidal representation are either the maximal split torus T or reductive

of type B_{s^2+s} (respectively C_{s^2+s}), with $s^2+s \leqslant n$. We denote by L_s the (standard) Levi subgroup of type B_{s^2+s} (respectively C_{s^2+s}), and by σ_s the unipotent cuspidal representation of L_s^F. The relative Weyl group $W_{G^F}(L_s, \sigma_s)$ is of type $B_{n-(s^2+s)}$, and its representations parameterize via (1.2) the Harish Chandra series $\mathcal{E}^{G^F}(L_s, \sigma_s)$:

$$Rep_{G^F} : \mathcal{E}^{G^F}(L_s, \sigma_s) \to Irr(W_{G^F}(L_s, \sigma_s)) = Irr(W(B_{n-(s^2+s)})).$$

With an abuse of notation, we allow $s = 0$ and treat the principal series together with the other series, setting $L_0 = T$ and $\sigma_0 = 1_T$, the trivial character of T. In this case $W_{G^F}(L_0, \sigma_0) = W$ is of type B_n.
For any $s \in \mathbb{N}$ with $s^2+s \leqslant n$, the map κ_B as in Lemma (2.3) gives a bijection

$$\kappa_B : Irr(W(B_{n-(s^2+s)})) \to \mathscr{B}_{n-(s^2+s)}.$$

Composing κ_B with Rep_{G^F} yields a bijection

$$\kappa_B \circ Rep_{G^F} : \mathcal{E}^{G^F}(L_s, \sigma_s) \to \mathscr{B}_{n-(s^2+s)}.$$

2.1.2.3 Type D

Let $n \in \mathbb{N}$, and let G be of type D_n. We use the convention that $D_2 = A_1 \times A_1$, $D_3 = A_3$, and that D_1 is the empty root system .

Let F_0 be a split Frobenius morphism, and let δ be an automorphism of G corresponding to a graph automorphism of order 2. Let $F_1 := \delta F_0$, that is, a non-split Forbenius morphism. We will sometimes write F_s with $s \in \mathbb{N}$, where s stands for its equivalence class modulo 2.

The rational forms of $G \rtimes \langle\delta\rangle$ over $\mathbb{F}_q$ are classified by $H^1(\mathbb{F}_q, Aut(G \rtimes \langle\delta\rangle))$, with the split form $G^{F_0} \rtimes \langle\delta\rangle$ corresponding to the trivial cocycle. A rational form of $G \rtimes \langle\delta\rangle$ over $\mathbb{F}_q$ is called an inner twist if it corresponds to an element of $H^1(\mathbb{F}_q, Inn(G \rtimes \langle\delta\rangle))$, where $Inn(G \rtimes \langle\delta\rangle)$ denotes the group of the inner automorphisms of $G \rtimes \langle\delta\rangle$ [3, Section 6]. By Lang Steinberg theorem, we have (see [61, III, 2.4, Corollary 3]),

$$H^1(\mathbb{F}_q, Inn(G \rtimes \langle\delta\rangle)) \cong H^1(\mathbb{F}_q, G \rtimes \langle\delta\rangle) \cong H^1(\mathbb{F}_q, {}^{G \rtimes \langle\delta\rangle}\!/_{G}) \cong H^1(\mathbb{F}_q, \langle\delta\rangle) \cong \langle\delta\rangle,$$

where the last isomorphism follows from the fact that F_0 and δ commute.

Therefore the group $G \rtimes \langle\delta\rangle$ has two inner twists over $\mathbb{F}_q$, one corresponding to the identity and the other corresponding to δ, given respectively by

$$\widetilde{G}^0 := G^{F_0} \rtimes \langle\delta\rangle \qquad \widetilde{G}^1 := G^{F_1} \rtimes \langle\delta\rangle. \tag{2.6}$$

We will sometimes write $\widetilde{G}^s$ with $s \in \mathbb{N}$, where s stands for its equivalence class modulo 2.

We will make use of the results of Chapter I, in particular of Theorem 1.2.24, to study the combinatorics of the unipotent representations of the groups $\widetilde{G}^i$ for $i \in \{0, 1\}$.

We say that an irreducible representation of $\widetilde{G}^i$ is unipotent if it appears as an irreducible constituent of the induction of some unipotent representation of G^{F_i}.

The group G^{F_0} (respectively G^{F_1}) is of type D_n (respectively ${}^2D_n$). The Levi subgroups L_s of G such that $L_s^{F_0}$ (respectively $L_s^{F_1}$) affords a unipotent cuspidal representation σ_s are the maximal split (respectively quasi-split) torus, that we denote by L_0 (respectively L_1), and Levi subgroups of type D_{s^2}, with $s > 0$ such that $s^2 \leqslant n$ and s is even (respectively odd). For any $s > 0$ with $s^2 \leqslant n$, both the Levi subgroup L_s and the unipotent cuspidal representation σ_s are δ-stable. The relative Weyl group $W_{G^{F_s}}(L_s, \sigma_s)$ is isomorphic to $W(B_{n-s^2})$ if $s \geqslant 0$, while $W_{G^{F_0}}(L_0, \sigma_0)$ is isomorphic to $W(D_n)$. For any $s > 0$, composing the map κ_B with the map $Rep_{G^{F_s}}$ yields a bijection

$$\kappa_B \circ Rep_{G^{F_s}} : \mathcal{E}^{G^{F_s}}(L_s, \sigma_s) \to \mathscr{B}_{n-s^2}.$$

For $s = 0$, composing the map κ_D with the map $Rep_{G^{F_0}}$ yields a bijection

$$\kappa_D \circ Rep_{G^{F_0}} : \mathcal{E}^{G^{F_0}}(L_0, \sigma_0) \to \mathscr{D}_n.$$

The group automorphism δ corresponds to the graph automorphism of $W(D_n)$ of order 2, therefore

$$\begin{aligned} &W_{\widetilde{G}^s}(L_s, \sigma_s) \cong W(B_{n-s^2}) \times \langle\delta\rangle \cong W(B_{n-s^2}) \times \mathbb{Z}_2 \qquad \text{for any } s > 0 \\ &W_{\widetilde{G}^0}(L_0, \sigma_0) \cong W(D_n) \rtimes \langle\delta\rangle \cong W(B_n). \end{aligned}$$

For any $s > 0$, we have the following canonical identification

$$Irr(W(B_{n-s^2}) \times \mathbb{Z}_2) \cong Irr(W(B_{n-s^2})) \times Irr(\mathbb{Z}_2).$$

We denote by $ev_{-1} : Irr(\mathbb{Z}_2) \to \{\pm 1\}$ the bijection given by evaluating the characters of $Irr(\mathbb{Z}_2)$ on the non trivial element of $\mathbb{Z}_2$.

Then for any $s > 0$, composing the map $(\kappa_B \times ev_{-1})$ with the map $Rep_{\widetilde{G}^s}$ as in (1.47) yields a bijection

$$(\kappa_B \times ev_{-1}) \circ Rep_{\widetilde{G}^s} : \mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s) \to \mathscr{B}_{n-s^2} \times \{\pm 1\}.$$

For $s = 0$, composing the map κ_B with $Rep_{\widetilde{G}^0}$ yields a bijection

$$\kappa_B \circ Rep_{\widetilde{G}^0} : \mathcal{E}^{\widetilde{G}^0}(L_0, \sigma_0) \to \mathscr{B}_n.$$

2.1.3 Parabolic restriction and removing hooks

We present in this section how the parabolic restriction of unipotent representations can be described in a combinatorial way for groups of classical type B, C and D. The comparison theorem allows to reduce this problem to the computation of restrictions of representations of Weyl groups. In turn, this problem can be solved with combinatorial methods, with the main ingredient being the Murnaghan–Nakayama rule [36, 2.4.7]. This strategy is well known in literature, see for instance [49, Section 4.6]. In this section, we explicitly apply it to the cases in which we need to compute the restrictions.

To our scope, by transitivity of parabolic restriction, it will be sufficient to focus on the case in which we restrict a Levi subgroup that is maximal among the F-stable ones (see Remarks 2.4.10, 2.6.4). Moreover we can, and will, assume the Levi subgroup to be standard, since all the F-stable Levi subgroups of an F-stable parabolic subgroup are G^F conjugate to a standard one.

By Corollary 1.1.5, computing the parabolic restriction of an irreducible representation of a finite group of Lie type reduces to computing the restriction of the corresponding representation in the appropriate relative Weyl Group.

Let G be a simple group of type B_n (or C_n). A maximal standard Levi subgroup M is of type $A_{r-1} \times B_{n-r}$ (repectively $A_{r-1} \times C_{n-r}$) for some $1 \leqslant r \leqslant n$.

Lemma 2.1.1. *Let G be a simple group of type B_n. Let $r, s \in \mathbb{N}$ be such that $1 \leqslant r \leqslant n$ and $s^2+s \leqslant n-r$. Let M be a standard Levi subgroup of type $A_{r-1}\times B_{n-r}$. Then the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{G^F}} & R(W(B_{n-(s^2+s)})) \\ \Big\downarrow {}^*R^{G^F}_{M^F} & & \Big\downarrow Res^{W(B_{n-(s^2+s)})}_{\mathbb{S}_r \times W(B_{n-(s^2+s)-r})} \\ \mathbb{C}[\mathcal{E}^{M^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{M^F}} & R(\mathbb{S}_r \times W(B_{n-(s^2+s)-r})) \end{array}$$

Proof. This is Corollary 1.1.5 specialized to a simple group of type B_n. □

In Theorem 1.2.24, we showed that computing the parabolic restriction translates into computing the restriction in the appropriate relative Weyl groups also in the case in which the finite group of Lie type has a disconnected underlying algebraic group.

Let G be of type D_n and retain the notation from Section 2.1.2.3. Let M be a maximal δ-stable standard Levi subgroup of G of type $A_{r-1} \times D_{n-r}$ for some $1 \leqslant r \leqslant n-1$. Then the automorphism δ stabilizes M and the restriction of δ to M induces an automorphism of M corresponding to a graph automorphism of order 2 on D_{n-r}. It follows that $M \rtimes \langle\delta\rangle$ has two inner twists: $M^{F_0} \rtimes \langle\delta\rangle$, that is of type $A_{r-1} \times D_{n-r}$, and $M^{F_1} \rtimes \langle\delta\rangle$, that is of type $A_{r-1} \times {}^2D_{n-r}$. We set $\widetilde{M}^i := M^{F_i} \rtimes \langle\delta\rangle$ for $i \in \{0,1\}$.

The Levi subgroup M is reductive but not semisimple in general. Nevertheless, by the reduction argument in 2.1 the unipotent representations of $\widetilde{M}^0$ (respectively $\widetilde{M}^1$) are the same as the unipotent representations of the group given by the direct product of a group of untwisted type A_{n-r} and the split (respectively non-split) inner twist over $\mathbb{F}_q$ of a disconnected group with identity component of type D_{n-r} and component group given by $\langle\delta\rangle$. Therefore a parameterization for unipotent representations of $\widetilde{M}^i$ is obtained by combining the results for groups of type A from Section 2.1.2.1 with the results for inner twists of type D from Section 2.1.2.3.

In the limit case $r = n-1$, the Levi subgroup M is of type $A_{n-2} \times D_1$. Since D_1 corresponds to a torus, in this case the irreducible unipotent representations of $\widetilde{M}^0$ and of $\widetilde{M}^1$ are both classified by $Irr(\mathbb{S}_{n-1} \times \mathbb{Z}_2)$. This case can be treated, as we will in the following of the chapter, with the same formalism as the others, keeping the conventions introduced in Section 2.1.2 for $W(B_0)$ and $\mathcal{B}_0$.

Lemma 2.1.2. *Let G be a simple group of type D_n. Let $r, s \in \mathbb{N}$ be such that $1 \leqslant r \leqslant n-1$ and $s^2 \leqslant n-r$. Let M be a standard Levi subgroup of type $A_{r-1} \times D_{n-r}$. Then the following diagrams commute*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s,\sigma_s)] & \xrightarrow{Rep_{\widetilde{G}^s}} & R(W(B_{n-s^2}) \times \mathbb{Z}_2) \\ \downarrow {}^*R^{\widetilde{G}^s}_{\widetilde{M}^s} & & \downarrow Res^{W(B_{n-s^2})\times\mathbb{Z}_2}_{\mathbb{S}_r \times W(B_{n-s^2-r})\times\mathbb{Z}_2} \\ \mathbb{C}[\mathcal{E}^{\widetilde{M}^s}(L_s,\sigma_s)] & \xrightarrow{Rep_{\widetilde{M}^s}} & R(\mathbb{S}_r \times W(B_{n-s^2-r}) \times \mathbb{Z}_2) \end{array}$$

for $s > 0$, and

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{\widetilde{G}^0}(L_0,\sigma_0)] & \xrightarrow{Rep_{\widetilde{G}^0}} & R(W(B_n)) \\ \downarrow {}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} & & \downarrow Res^{W(B_n)}_{\mathbb{S}_r \times W(B_{n-r})} \\ \mathbb{C}[\mathcal{E}^{\widetilde{M}^0}(L_0,\sigma_0)] & \xrightarrow{Rep_{\widetilde{M}^0}} & R(\mathbb{S}_r \times W(B_{n-r})) \end{array}$$

Proof. This is Theorem 1.2.24 specialized to a simple group of type D_n extended by a graph automorphism of order 2. ☐

Remark 2.1.3. For any $n, s \in \mathbb{N}$ such that $n \geqslant s^2$, we have $R(W(B_{n-s^2}) \times \mathbb{Z}_2) \cong R(W(B_{n-s^2})) \otimes R(\mathbb{Z}_2)$, and the following diagram commutes

$$\begin{array}{ccc} R(W(B_{n-s^2}) \times \mathbb{Z}_2) & \xrightarrow{\sim} & R(W(B_{n-s^2})) \otimes R(\mathbb{Z}_2) \\ \downarrow Res^{W(B_{n-s^2})\times\mathbb{Z}_2}_{\mathbb{S}_r \times W(B_{n-s^2-r})\times\mathbb{Z}_2} & & \downarrow Res^{W(B_{n-s^2})}_{\mathbb{S}_r \times W(B_{n-s^2-r})} \otimes id \\ R(\mathbb{S}_r \times W(B_{n-s^2-r}) \times \mathbb{Z}_2) & \xrightarrow{\sim} & R(\mathbb{S}_r \times W(B_{n-s^2-r})) \otimes R(\mathbb{Z}_2) \end{array}$$

Therefore, to describe combinatorially the parabolic restriction in the above situations it is enough to describe combinatorially the restriction from $W(B_n)$ to $\mathbb{S}_r \times W(B_{n-r})$, with $1 \leqslant r \leqslant n$.

2.1.3.1 Removing hooks on bipartitions

Let $\alpha \in \mathscr{P}_n$ be a partition. We quickly recall the definition of hooks, leg length, and removing hooks following [54].

An r-hook for α is a pair (i, j) with $1 \leqslant i \leqslant l(\alpha)$ and $1 \leqslant j \leqslant \alpha_j$, i.e., a box in the Young digram of α, that satisfies $\alpha_i - j + (\alpha')_j - i + 1 = r$, where (α') denotes the transpose partition. The number r is called the hook length of the hook (i, j). The leg length of a hook (i, j) is given by $l(i, j) = (\alpha')_j - i$. Removing the r-hook (i, j) from the partition α yields the partition $\alpha \smallsetminus (i, j)$ of $n-r$ corresponding to the Young diagram obtained by removing all the boxes (i, j') with $j' \geqslant j$ and (i', j) with $i' \geqslant i$, and translating any box (h, k) with $h > i$ and $k > j$ in position $(h-1, k-1)$. We write $H_r(\alpha)$ for the set of the r-hooks of α.

Example 2.1.4. *Here below we draw on the left the Young diagram of the partition $(4, 3, 1) \in \mathscr{P}_8$ with each box filled with the hook length of the corresponding hook,* *and on the right the same Young diagram with each box filled with the leg length of the corresponding hook*

6	4	3	1
4	2	1	
1			

2	1	1	0
1	0	0	
0			

Removing the 4*-hook* $(1,2)$*, one obtains the following Young diagram*

☐	☐
☐	
☐	

i.e. the partition $(2,1,1) \in \mathscr{P}_4$*.*

Let $(\alpha,\beta) \in \mathscr{B}_n$ be a bipartition of n. An r-hook for (α,β) is $(i,j) \in H_r(\alpha) \cup H_r(\beta)$, i.e. an r-hook for α or β. We define the removing r-hooks maps for bipartitions as follows:

$$\begin{aligned} \mathcal{H}_r : \mathbb{C}[\mathscr{B}_n] &\longrightarrow \mathbb{C}[\mathscr{B}_{n-r}] \\ (\alpha,\beta) &\mapsto \sum_{(i,j)\in H_r(\alpha)} (-1)^{l(i,j)}(\alpha \smallsetminus (i,j),\beta) + \sum_{(i,j)\in H_r(\beta)} (-1)^{l(i,j)}(\alpha,\beta \smallsetminus (i,j)). \end{aligned} \tag{2.7}$$

Let $\gamma = (\gamma_1,\dots,\gamma_k) \in \mathscr{P}_r$. We consider the composition

$$\mathcal{H}_\gamma = \mathcal{H}_{\gamma_k} \circ \cdots \circ \mathcal{H}_{\gamma_1} : \mathbb{C}[\mathscr{B}_n] \to \mathbb{C}[\mathscr{B}_{n-r}].$$

Let $r \in \mathbb{N}$. The conjugacy class of any element of $\mathbb{S}_r$ is uniquely determined by its cycle type, thus by a partition $\gamma \in \mathscr{P}_r$. We denote by $\chi_\gamma \in R(\mathbb{S}_r)$ the characteristic function of the conjugacy class labelled by γ in $\mathbb{S}_r$, that is

$$\chi_\gamma(x) = \begin{cases} 1 & \text{if } x \text{ has cycle type } \gamma \\ 0 & \text{if } x \text{ has cycle type } \alpha \in \mathscr{P}_n \smallsetminus \{\gamma\} \end{cases}$$

The following Proposition is an application of the Murnaghan–Nakayama rule to the cases we are interested in.

Proposition 2.1.5. *Let* $n \in \mathbb{N}$ *and* $r \in \mathbb{N}$*, with* $r \leqslant n$*. Then*

$$(\kappa_A \boxtimes k_B) \circ Res^{W(B_n)}_{\mathbb{S}_r \times W(B_{n-r})} = \Big(\sum_{\gamma \in \mathscr{P}_r} \kappa_A(\chi_\gamma) \boxtimes \mathcal{H}_\gamma\Big) \circ \kappa_B. \tag{2.8}$$

Proof. Let $(\alpha,\beta) \in \mathscr{B}_n$. Then $\kappa_B^{-1}(\alpha,\beta)$ is the character of $W(B_n)$ parametrized by the bipartition (α,β). Let $x' \in \mathbb{S}_r$ be an r-cycle and let $y \in W(B_{n-r})$. Then $x'y \in \mathbb{S}_r \times W(B_{n-r}) \leqslant W(B_n)$. By the Murnaghan–Nakayama rule [49, Proposition 4.6.3] [36, 2.4.7] it holds

$$(\kappa_B^{-1}(\alpha,\beta))(x'y) = (\kappa_B^{-1} \circ \mathcal{H}_r(\alpha,\beta))(y). \tag{2.9}$$

Now let $x \in \mathbb{S}_r$ be of cycle type $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathscr{P}_r$, and let $y \in B_{n-r}$. We now show that for any $(\alpha, \beta) \in \mathscr{B}_n$ it holds

$$(\kappa_B^{-1}(\alpha, \beta))(xy) := (\kappa_B^{-1} \circ \mathcal{H}_\gamma(\alpha, \beta))(y). \tag{2.10}$$

Since we are just interested in character values, it is enough to consider elements up to conjugacy, and therefore we might assume that x lies in $\mathbb{S}_{\gamma_1} \times \mathbb{S}_{\gamma_2} \times \cdots \times \mathbb{S}_{\gamma_k}$, the standard Young subgroup of $\mathbb{S}_r$. Then we argue by induction on the number of disjoint cycles of x:

- ◇ If $k = 1$, this is the Murnaghan–Nakayama rule, i.e. Equation (2.9).
- ◇ Assume that the result holds for $k-1$. Write $x = x_1 x_2$, with $x_1 \in \mathbb{S}_{\gamma_1}$ a γ_1-cycle, and $x_2 \in \mathbb{S}_{\gamma_2} \times \cdots \times \mathbb{S}_{\gamma_k}$ of cycle type $(\gamma_2, \dots, \gamma_k)$. Then for any $y \in W(B_{m-r})$

$$(\kappa_B^{-1}(\alpha, \beta))(xy) = (\kappa_B^{-1}(\alpha, \beta))(x_1 x_2 y).$$

Since x_1 is a γ_1-cycle and $x_2 y \in W(B_{n-\gamma_1})$, applying (2.9) yields

$$\kappa_B^{-1}(\alpha, \beta))(x_1 x_2 y) = (\kappa_B^{-1} \circ \mathcal{H}_{\gamma_1}(\alpha, \beta))(x_2 y).$$

Since x_2 has cycle type $(\gamma_2, \dots, \gamma_k)$, it factors in $k-1$ cycles. By induction hypothesis, we have

$$\kappa_B^{-1}(\mathcal{H}_{\gamma_1}(\alpha, \beta))(x_2 y) = \kappa_B^{-1} \circ \mathcal{H}_{\gamma_k} \circ \cdots \circ (\mathcal{H}_{\gamma_1}(\alpha, \beta))(y),$$

so it holds

$$\kappa_B^{-1}(\alpha, \beta)(xy) = \kappa_B^{-1} \circ \mathcal{H}_{\gamma_k} \circ \cdots \circ \mathcal{H}_{r_2} \circ \mathcal{H}_{r_1}(\alpha, \beta)(y) = (\kappa_B^{-1} \circ \mathcal{H}_\gamma(\alpha, \beta))(y).$$

By (2.10), we have thus

$$Res^{W(B_n)}_{\mathbb{S}_r \times W(B_{n-r})}(\kappa_B^{-1}(\alpha, \beta)) = \sum_{\gamma \in \mathscr{P}_r} \chi_\gamma \boxtimes \kappa_B^{-1}(\mathcal{H}_\gamma(\alpha, \beta)).$$

Indeed evaluating both sides on any element $xy \in \mathbb{S}_r \times W(B_{n-r})$, with $x \in \mathbb{S}_r$ of cycle type γ and $y \in B_{n-r}$, we obtain $\kappa_B^{-1}(\mathcal{H}_\gamma(\alpha, \beta))(y)$.

It follows that

$$Res^{W(B_n)}_{\mathbb{S}_r \times W(B_{n-r})} \circ \kappa_B^{-1} = (\kappa_A \boxtimes \kappa_B)^{-1} \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\chi_\gamma) \boxtimes \mathcal{H}_\gamma,$$

whence (2.8) holds. □

For $r \in \mathbb{N}$, we set

$$Res_r := \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\chi_\gamma) \boxtimes \mathcal{H}_\gamma. \tag{2.11}$$

By Proposition 2.1.5, this map encodes a combinatoria description via bipartitions of the restriction of representations of Weyl groups of type B, as we record in the following corollary.

Corollary 2.1.6. *For any $n, r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$ the following diagram commutes:*

$$\begin{array}{ccc} R(W(B_n)) & \xrightarrow{\kappa_B} & \mathbb{C}[\mathscr{B}_n] \\ \big\downarrow {\scriptstyle Res^{W(B_n)}_{\mathbb{S}_r \times W(B_{n-r})}} & & \big\downarrow {\scriptstyle Res_r} \\ R(\mathbb{S}_r \times W(B_{n-r})) & \xrightarrow{\kappa_A \boxtimes \kappa_B} & \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\mathscr{B}_{n-r}] \end{array} \tag{2.12}$$

Proof. This is a restatement of Lemma 2.1.5 □

2.2 β-sets and symbols

In order to parametrize the unipotent characters belonging to the same group, it is useful to introduce Symbols [41], see also [49, Section 4.4]. These are combinatorial objects that encode a bipartition and a natural number, allowing to keep track at once of the Harish-Chandra series to which a character belongs and of the representation of the relative Weyl group corresponding to that character.

Before defining symbols and their relation to bipartitions, it is convenient to define the analogous objects for partitions, that is given by β-sets.

2.2.1 β-sets

In this section, we recall the formalism of β-sets, and the translation of hooks for partitions into this formalism as given in [54].

For $k \in \mathbb{N}$, the shift of $\alpha \in \mathscr{P}^+$ is the sequence $\alpha^{\to k} \in \mathscr{P}^+$ obtained adding k zeros at the end of α, that is $\alpha^{\to k} = (\alpha_1, \dots, \alpha_l, 0, \dots, 0)$. Then the shift determines a relation on $\mathscr{P}^+$ given by $\alpha \sim \alpha^{\to k}$ for any $k \in \mathbb{N}$. The symmetric and transitive closure of this relation is an equivalence relation on $\mathscr{P}^+$ that preserves the rank. We denote by $[\alpha]$ the equivalence class of $\alpha \in \mathscr{P}^+$ with respect to this relation. The inclusion $\mathscr{P}_n \hookrightarrow \mathscr{P}_n^+$ induces a bijection $\mathscr{P}_n \xrightarrow{\sim} \mathscr{P}_n^+ \big/ \sim$. In other words, any equivalence class in $\mathscr{P}_n^+$ has a representative in $\mathscr{P}_n$.

Let $\mathscr{P}(\mathbb{N})$ be the power set of $\mathbb{N}$, and let $\mathscr{P}(\mathbb{N})^{<\infty} = \{A \in \mathscr{P}(\mathbb{N}) \mid |A| < \infty\}$. We define the rank of $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ to be

$$\varrho(A) = \sum_{a \in A} a - \frac{|A|(|A|-1)}{2}$$

and set

$$\mathscr{P}(\mathbb{N})^{<\infty}_n = \{A \in \mathscr{P}(\mathbb{N})^{<\infty} | \varrho(A) = n\}.$$

For any $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ and for any $k \in \mathbb{N}$, the shift of A by k is the set $A^{\to k} = \{0, 1, \dots, k-1\} \cup \{a + k \mid a \in A\}$. This defines a relation on $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ given by $A \sim A^{\to k}$ for any $k \in \mathbb{N}$. We denote by $[A]$ the equivalence class of $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ with respect of the equivalence relation given by its symmetric and transitive closure.

Lemma 2.2.1. *For any $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ and any $k \in \mathbb{N}$, it holds $\varrho(A) = \varrho(A^{\to k})$.*

Proof. Let $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ and $k \in \mathbb{N}$. Then

$$\begin{aligned}
\varrho(A^{\to k}) &= \sum_{a \in A^{\to k}} a - \frac{|A^{\to k}|(|A^{\to k}|-1)}{2} \\
&= \sum_{i=1}^{k-1} i + \sum_{a \in A} (a+k) - \frac{(|A|+k)(|A|+k-1)}{2} \\
&= \frac{k(k-1)}{2} + \sum_{a \in A} a + |A|k - \frac{|A|(|A|-1)}{2} - \frac{k(k-1)}{2} - k|A| \\
&= \sum_{a \in A} a - \frac{|A|(|A|-1)}{2} = \varrho(A).
\end{aligned}$$

□

We define the following map

$$\begin{aligned} \mathcal{B} : \mathscr{P}^+ &\to \mathscr{P}(\mathbb{N})^{<\infty} \\ \alpha &\mapsto \{\alpha_i + (l(\alpha) - i) \mid i = 1, \dots, l(\alpha)\} \end{aligned} \tag{2.13}$$

The map $\mathcal{B}$ is a bijection: for any $A \in \mathscr{P}(\mathbb{N})^{<\infty}$, there always exists a unique monotone decreasing function $a : \{1, \dots, |A|\} \to A$, and then the map defined by the assignment $A \to \alpha(A)$ with $\alpha(A) \in \mathscr{P}^+$ such that $l(\alpha(A)) = |A|$ and $\alpha(A)_i := a(i) + i - |A|$ for any $i = 1, \dots |A|$ yields an inverse of $\mathcal{B}$.

Example 2.2.2. *We have* $\mathscr{P}_3 = \{(3), (2,1), (1,1,1)\}$, *and*

$$\mathcal{B}(3) = \{3\}, \qquad \mathcal{B}(2,1) = \{1,3\}, \qquad \mathcal{B}(1,1,1) = \{1,2,3\} \tag{2.14}$$

Lemma 2.2.3. *Let* $\alpha \in \mathscr{P}^+$ *and* $k \in \mathbb{N}$. *Then* $\mathcal{B}(\alpha^{\to k}) = \mathcal{B}(\alpha)^{\to k}$ *and* $\varrho(\alpha) = \varrho(\mathcal{B}(\alpha))$.

Proof. ◇ For the first equality, $l(\alpha^{\to k}) = l(\alpha) + k$, and so

$$\begin{aligned} \mathcal{B}(\alpha^{\to k}) &= \{(\alpha^{\to k})_i + (l(\alpha) + k - i) | i = 1, \dots, l(\alpha) + k\} \\ &= \{\alpha_i + (l(\alpha) - i) + k | i = 1, \dots, l(\alpha)\} \bigsqcup \{0 + (l(\alpha) + k - i) | i = l(\alpha), \dots, l(\alpha) + k\} \\ &= \{0, 1, \dots, k-1\} \bigsqcup \{a + k | a \in \mathcal{B}(\alpha)\} = \mathcal{B}(\alpha)^{\to k}. \end{aligned}$$

◇ We prove the second equality.

$$\begin{aligned} \varrho(\mathcal{B}(\alpha)) &= \sum_{a \in \mathcal{B}(\alpha)} a - \frac{|\mathcal{B}(\alpha)|(|\mathcal{B}(\alpha)| - 1)}{2} \\ &= \sum_{i=1}^{l(\alpha)} (\alpha_i + i - 1) - \frac{l(\alpha)(l(\alpha) - 1)}{2} \\ &= \sum_{i=1}^{l(\alpha)} \alpha_i + \sum_{i=1}^{l(\alpha)} (i - 1) - \frac{l(\alpha)(l(\alpha) - 1)}{2} \\ &= \sum_{i=1}^{l(\alpha)} \alpha_i + \frac{l(\alpha)(l(\alpha) - 1)}{2} - \frac{l(\alpha)(l(\alpha) - 1)}{2} = \sum_{i=1}^{l(\alpha)} \alpha_i = \varrho(\alpha). \end{aligned}$$

□

By Lemma 2.2.3, for any $n \in \mathbb{N}$ the map $\mathcal{B}$ induces a bijection

$$\begin{aligned} [\mathcal{B}] : \mathscr{P}_n^+ \big/ \sim &\to \mathscr{P}(\mathbb{N})_n^{<\infty} \big/ \sim \\ \alpha &\mapsto [\mathcal{B}(\alpha)] \end{aligned} \tag{2.15}$$

We say that any element in the equivalence class $[\mathcal{B}(\alpha)]$ for $\alpha \in \mathscr{P}_n$ is a β-set associated to α.

2.2.2 Ordered Symbols

We recall that we denote by $\mathscr{B}_n$ the set of bipartitions of n.

The analogue of β-sets for bipartitions is given by Symbols. Symbols were introduced in [41]. We follow the notation in [40], see also [49, Section 4.4].

Definition 2.2.4. *An array is a pair $(X^0, X^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$. The sets X^0 and X^1 are called respectively the top row and the bottom row of the array. The defect and the rank of an array $X = (X^0, X^1)$ are respectively*

$$def(X) := |X^0| - |X^1|, \tag{2.16}$$

$$\rho(X) := \sum_{x^0 \in X^0} x^0 + \sum_{x^1 \in X^1} x^1 - \left\lfloor \left(\frac{|X^0| + |X^1| - 1}{2} \right)^2 \right\rfloor \tag{2.17}$$

Example 2.2.5. *Let $X^0 := \{1, 2\}$ and $X^1 := \{0, 2\}$, and let $X = (X^0, X^1)$. We adopt the following notation:*

$$X = \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

In this case,

$$def(X) = 0, \qquad \rho(X) = 1 + 2 + 0 + 2 - \left\lfloor \frac{9}{4} \right\rfloor = 3.$$

Lemma 2.2.6. *Let $X = (X^0, X^1)$ be an array. Then*

$$\rho(X) = \varrho(X^0) + \varrho(X^1) - \left\lfloor \frac{1 - def(X)^2}{4} \right\rfloor.$$

Proof. The proof is a direct computation. Indeed

$$\begin{aligned}
\rho(X) :&= \sum_{x^0 \in X^0} x^0 + \sum_{x^1 \in X^1} x^1 - \left\lfloor \left(\frac{|X^0| + |X^1| - 1}{2} \right)^2 \right\rfloor \\
&= \varrho(X^0) + \frac{|X^0|(|X^0| - 1)}{2} + \varrho(X^1) + \frac{|X^1|(|X^1| - 1)}{2} \\
&- \left\lfloor \frac{|X^0|^2 + |X^1|^2 + 1 - 2|X^0| - 2|X^1| + 2|X^0||X^1|}{4} \right\rfloor \\
&= \varrho(X^0) + \frac{|X^0|(|X^0| - 1)}{2} + \varrho(X^1) + \frac{|X^1|(|X^1| - 1)}{2} \\
&- \left\lfloor \frac{|X^0|(|X^0| - 1)}{2} + \frac{|X^1|(|X^1| - 1)}{2} + \frac{-|X^0|^2 - |X^1|^2 + 2|X^0||X^1| + 1}{4} \right\rfloor \\
&= \varrho(X^0) + \varrho(X^1) - \left\lfloor \frac{1 - |X^0|^2 - |X^1|^2 + 2|X^0||X^1|}{4} \right\rfloor \\
&= \varrho(X^0) + \varrho(X^1) - \left\lfloor \frac{1 - (|X^0| - |X^1|)^2}{4} \right\rfloor = \varrho(X^0) + \varrho(X^1) - \left\lfloor \frac{1 - def(X)^2}{4} \right\rfloor.
\end{aligned}$$

□

Let $k \in \mathbb{N}$ and let $X = (X^0, X^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ be an array. The shift by k of X is the array

$$X^{\to k} := ((X^0)^{\to k}, (X^1)^{\to k})$$

and for any $k \in \mathbb{N}$ we define the shift operation

$$^{\to k} : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] \to \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}]$$

as the linear extension of the map associating with each array its shift by k. The shift determines a relation on $\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ given by $X \sim X^{\to k}$ for any $k \in \mathbb{N}$, and we consider the equivalence relation given by its symmetric and transitive closure.

Definition 2.2.7. *Let $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ be an array. We denote by $[X]$ the equivalence class of X with respect to the shift equivalence relation, and we call such a class an ordered symbol. We denote the set of all ordered symbols by $\widetilde{\mathcal{S}}$.*

Lemma 2.2.8. *The rank and the defect of an array are invariant by shift: for any $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ and for any $k \in \mathbb{N}$ it holds*

$$def(X^{\to k}) = def(X), \qquad\qquad \rho(X^{\to k}) = \rho(X).$$

Proof. Let $X = (X^0, X^1)$. Then $X^{\to k} := ((X^0)^{\to k}, (X^1)^{\to k})$, and for any $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ it holds $|A^{\to k}| = |A| + k$. Therefore

$$def(X^{\to k}) = |(X^0)^{\to k}| - |(X^1)^{\to k}| = |X^0| + k - (|X^1| + k) = |X^0| - |X^1| = def(X).$$

Regarding the rank, by Lemma 2.2.6 and the above equality we have

$$\rho(X^{\to k}) = \varrho((X^0)^{\to k}) + \varrho((X^1)^{\to k}) - \left\lfloor \frac{1 - def(X)^2}{4} \right\rfloor.$$

It follows from Lemma 2.2.1 that

$$\rho(X^{\to k}) = \varrho(X^0) + \varrho(X^1) - \left\lfloor \frac{1 - def(X)^2}{4} \right\rfloor = \rho(X).$$

□

By Lemma 2.2.8, we may define the rank and defect of an ordered symbol as the rank and the defect of any array representing it.

For $n \in \mathbb{N}$ we set

$$\widetilde{\mathcal{S}}_n := \{[X] \text{ ordered symbols of rank } n\} \subset \widetilde{\mathcal{S}},$$

and for $d \in \mathbb{Z}$

$$\widetilde{\mathcal{S}}^d := \{[X] \text{ ordered symbols of defect } d\} \subset \widetilde{\mathcal{S}}$$

and we set $\widetilde{\mathcal{S}}^d_n := \widetilde{\mathcal{S}}_n \cap \widetilde{\mathcal{S}}^d$. Following [40] we set

$$\begin{aligned} \widetilde{\mathcal{S}}^{od,0} &:= \bigcup_{d \equiv 1 \bmod 4} \widetilde{\mathcal{S}}^d, & \widetilde{\mathcal{S}}^{od,1} &:= \bigcup_{d \equiv 3 \bmod 4} \widetilde{\mathcal{S}}^d, \\ \widetilde{\mathcal{S}}^{ev,0} &:= \bigcup_{d \equiv 0 \bmod 4} \widetilde{\mathcal{S}}^d, & \widetilde{\mathcal{S}}^{ev,1} &:= \bigcup_{d \equiv 2 \bmod 4} \widetilde{\mathcal{S}}^d. \end{aligned} \tag{2.18}$$

We now relate symbols with bipartitions. Let $n \in \mathbb{N}$ and $d \in \mathbb{Z}$. We define the map

$$\widetilde{\mathscr{S}}_{n,d} : \mathscr{B}_n \mapsto \tilde{\mathcal{S}} \tag{2.19}$$

$$(\alpha, \beta) \mapsto \begin{cases} [(\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)})] & \text{if } d \geqslant 0, \\ [(\mathcal{B}(\alpha)^{\to l(\beta)}, \mathcal{B}(\beta)^{\to l(\alpha)-d})] & \text{if } d < 0. \end{cases}$$

where $\mathcal{B}$ is as defined in (2.15).

Lemma 2.2.9. *For $n \in \mathbb{N}$ and $d \in \mathbb{Z}$, the map $\widetilde{\mathscr{S}}_{n,d}$ is a bijection between $\mathscr{B}_n$ and $\tilde{\mathcal{S}}^d_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}$.*

Proof. The map $\widetilde{\mathscr{S}}_{n,d}$ lands in $\tilde{\mathcal{S}}^d_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}$. Indeed for any $(\alpha, \beta) \in \mathscr{B}_n$ it holds

$$def(\widetilde{\mathscr{S}}_{n,d}(\alpha, \beta)) = \begin{cases} |\mathcal{B}(\alpha)^{\to l(\beta)+d}| - |\mathcal{B}(\beta)^{\to l(\alpha)}| = (l(\alpha) + l(\beta) + d) - (l(\beta) + l(\alpha)) = d & \text{if } d \geqslant 0, \\ |\mathcal{B}(\alpha)^{\to l(\beta)}| - |\mathcal{B}(\beta)^{\to l(\alpha)-d}| = (l(\alpha) + l(\beta)) - (l(\beta) + l(\alpha) - d) = d & \text{if } d < 0. \end{cases}$$

Moreover using that the rank of a β-set is invariant by shift (Lemma 2.2.8) and Lemma 2.2.3 we have

$$\begin{aligned} \rho(\widetilde{\mathscr{S}}_{n,d}(\alpha, \beta)) &= \varrho(\mathcal{B}(\alpha)) + \varrho(\mathcal{B}(\beta)) - \left\lfloor \frac{1 - def(\widetilde{\mathscr{S}}_{n,d}(\alpha, \beta))}{4} \right\rfloor \\ &= \varrho(\alpha) + \varrho(\beta) - \left\lfloor \frac{1-d^2}{4} \right\rfloor = n - \left\lfloor \frac{1-d^2}{4} \right\rfloor. \end{aligned}$$

To show that $\widetilde{\mathscr{S}}_{n,d}$ is a bijection we exhibit the inverse. Let $[(X^0, X^1)]$ be an ordered symbol of rank n. Let $\alpha^+ := \mathcal{B}^{-1}(X^0)$, $\beta^+ := \mathcal{B}^{-1}(X^1)$. Then $(\alpha^+, \beta^+) \in \mathscr{P}^+ \times \mathscr{P}^+$ satisfies $(X^0, X^1) = (\mathcal{B}(\alpha^+), \mathcal{B}(\beta^+))$ and $\varrho(\alpha^+) + \varrho(\beta^+) = n$. Then removing all the entries equal to 0 from α^+ and β^+ we obtain a bipartition $(\alpha, \beta) \in \mathscr{B}_n$. A straightforward computation using the commutativity of $\mathcal{B}$ with the shift shows that this algorithm gives the inverse of $\widetilde{\mathscr{S}}_{n,d}$. □

In virtue of Lemma 2.2.9, we denote by $\widetilde{\mathscr{S}}_{n,d}$ also the linear isomorphism

$$\widetilde{\mathscr{S}}_{n,d} : \mathbb{C}[\mathscr{B}_n] \mapsto \mathbb{C}[\tilde{\mathcal{S}}^d_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}]. \tag{2.20}$$

Example 2.2.10. *Under the bijection $\widetilde{\mathscr{S}}_{3,0}$, the bipartition $\big((1,1),(1)\big) \in \mathscr{B}_3$ corresponds to the array $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ from the Example 2.2.5*

2.2.2.1 The opposite involution

There is an involution on $\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ given by exchanging the rows:

$$\begin{aligned} {}^{op} : \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty} &\to \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty} \\ X = (X^0, X^1) &\to X^{op} = (X^1, X^0) \end{aligned}$$

Shifting a symbol by $k \in \mathbb{N}$ commutes with exchanging the rows.Therefore the assignment $[X] \mapsto [X^{op}]$ gives a well-defined involution on $\widetilde{\mathcal{S}}$. We write $[X]^{op}$ for $[X^{op}]$. An ordered symbol $[X]$ is said to be degenerate if $[X] = [X]^{op}$.

We now define an enhancement of the map $\widetilde{\mathscr{S}}_{n,d}$ as in (2.19) that allows to relate pairs consisting of bipartitions and a sign to symbols. For any $n, d \in \mathbb{N}$ with $d > 0$, we define

$$\widetilde{\mathscr{S}}^{\pm}_{n,d} : \mathscr{B}_n \times \{\pm 1\} \to \widetilde{\mathcal{S}}^{d}_{(n-\lfloor\frac{1-d^2}{4}\rfloor)} \bigsqcup \widetilde{\mathcal{S}}^{-d}_{(n-\lfloor\frac{1-d^2}{4}\rfloor)} \tag{2.21}$$

$$((\alpha,\beta),\varepsilon) \mapsto \begin{cases} \widetilde{\mathscr{S}}_{n,d}(\alpha,\beta) & if\ \varepsilon = 1, \\ (\widetilde{\mathscr{S}}_{n,d}(\alpha,\beta))^{op} & if\ \varepsilon = -1 \end{cases}$$

.

This is a bijection, since $\widetilde{\mathscr{S}}^{d}_{n}$ is a bijection between $\mathscr{B}_n$ and $\widetilde{\mathcal{S}}^{d}_{n-\lfloor\frac{1-d^2}{4}\rfloor}$, and $()^{op}$ is bijection between $\widetilde{\mathcal{S}}^{d}_{n-\lfloor\frac{1-d^2}{4}\rfloor}$ and $\widetilde{\mathcal{S}}^{-d}_{n-\lfloor\frac{1-d^2}{4}\rfloor}$.
For $d = 0$, we let

$$\widetilde{\mathscr{S}}^{\pm}_{n,0} := \widetilde{\mathscr{S}}_{n,0} : \mathscr{B}_n \to \widetilde{\mathcal{S}}^{0}_{n}. \tag{2.22}$$

For any $n, d \in \mathbb{N}$, we denote by $\widetilde{\mathscr{S}}^{\pm}_{n,d}$ also the isomorphism obtained extending linearly $\widetilde{\mathscr{S}}^{\pm}_{n,d}$.

Remark 2.2.11. Let $n \in \mathbb{N}$. For any $d > 0$, composing $\widetilde{\mathscr{S}}^{\pm}_{n+\lfloor\frac{1-d^2}{4}\rfloor,d}$ with $\kappa_B \times ev_{-1}$ yields a bijection

$$\widetilde{\ell^{n,d}} := \widetilde{\mathscr{S}}^{\pm}_{n+\lfloor\frac{1-d^2}{4}\rfloor,d} \circ (\kappa_B \times ev_{-1}) : Irr(W(B_{n+\lfloor\frac{1-d^2}{4}\rfloor}) \times \mathbb{Z}_2) \to \widetilde{\mathcal{S}}^{d}_{n} \bigsqcup \widetilde{\mathcal{S}}^{-d}_{n} \tag{2.23}$$

and for $d = 0$, composing $\widetilde{\mathscr{S}}_{n,0}$ with κ_B yields a bijection

$$\widetilde{\ell^{n,0}} : Irr(W(B_n)) \to \widetilde{\mathcal{S}}^{0}_{n}. \tag{2.24}$$

2.2.2.2 Parameterization of Unipotent representations of inner twists of type D_n

Let G be of type D_n and retain notation from Section 2.1.2.3.

For $i = 1, 2$, we denote by $R_u(\widetilde{G}^{i})$ the subspace of $R(\widetilde{G}^{i})$ generated by irreducible unipotent representations. We have

$$R_u(\widetilde{G}^{i}) = \bigoplus_{\substack{s \equiv i \ mod\ 2 \\ s^2 \leqslant n}} \mathcal{E}^{\widetilde{G}^{i}}(L_s, \sigma_s).$$

For any $s \geqslant 0$, the bijection $\widetilde{\ell^{n,2s}}$ as in Remark 2.2.11 is a bijection between $Irr(W_{\widetilde{G}^s}(L_s, \sigma_s))$ and $\bigsqcup_{d \in \pm 2s} \widetilde{\mathcal{S}}^{d}_{n}$, and the composition

$$\widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^s} : \mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s) \to \bigsqcup_{s' \in \pm s} \widetilde{\mathcal{S}}^{2s'}_{n}$$

is a bijection.

Then collecting all the series together and extending linearly yields the isomorphisms

$$\bigoplus_{\substack{s\in\mathbb{N}\ even\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^0} : R_u(\widetilde{G}^0) \to \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev,0}].$$

$$\bigoplus_{\substack{s\in\mathbb{N}\ odd\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^1} : R_u(\widetilde{G}^1) \to \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev,1}].$$

Collecting together the two inner twists we have an isomorphism

$$\widetilde{Symb}_D := \bigoplus_{\substack{s\in\mathbb{N}\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^s} : R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) \to \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}]. \tag{2.25}$$

Where

$$\bigoplus_{\substack{s\in\mathbb{N}\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^s} = (\bigoplus_{\substack{s\in\mathbb{N}\ even\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^0}) \oplus (\bigoplus_{\substack{s\in\mathbb{N}\ odd\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^1})$$

The map $\widetilde{Symb}_D$ is an isometry, since it maps the orthonormal basis of $R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1)$ consisting of irreducible unipotent characters onto the orthonormal basis of $\mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$ consisting of unordered symbols.

2.3 Parabolic restriction and removing hooks on symbols

In this section, we show how the combinatorial description of the parabolic restriction discussed in Section 2.1.3 translates in the language of Symbols.

2.3.1 Removing hooks on β-sets

We now recall, following [54], how the notion of hook, leg length, and removing hooks translates to β-sets.

Let $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ and $r \in \mathbb{N}$. An r-hook of A is an element $h \in A$ such that $r \leqslant h$ and $h - r \notin A$. We denote by $H_r(A)$ the set of the r-hooks of $A \in \mathscr{P}(\mathbb{N})^{<\infty}$. Let $A \in \mathscr{P}(\mathbb{N})^{<\infty}$. The leg length of a hook $h \in H_r(A)$ is the integer given by

$$l(h) := |\{a \in A \mid h - r < a < h\}|.$$

Let $h \in H_r(A)$. We define $A \smallsetminus_r (h) \in \mathscr{P}(\mathbb{N})^{<\infty}$ to be

$$A \smallsetminus_r (h) := A \smallsetminus \{h\} \cup \{h - r\}.$$

We say that $A \smallsetminus_r (h)$ is obtained from A removing the r-hook h. Removing a r-hook A diminishes the rank of a set by r, i.e. $\varrho(A \smallsetminus_r (h)) = \varrho(A) - r$.

Example 2.3.1. *Let* $\{1,4,6\} \in \mathscr{P}(\mathbb{N})^{<\infty}$*. Then* 6 *is a* 4*-hook, as* $4 < 6$ *and* $6-4 = 2 \notin \{1,4,6\}$*, and* $\{1,4,6\} \searrow_4 \{6\} = \{1,2,4\}$*.*

Lemma 2.3.2. *[54, Proposition 1.8] Let* $\alpha \in \mathscr{P}_n$*. The map*

$$\begin{aligned} H_r(\alpha) &\to H_r(\mathcal{B}(\alpha)) \\ (i,j) &\mapsto (\alpha_i + l(\alpha) - i) \end{aligned}$$

is a well-defined leg length preserving bijection. Moreover for any $(i,j) \in H_r(\alpha)$ *it holds*

$$\mathcal{B}(\alpha) \searrow_r (\alpha_i + l(\alpha) - i) = \mathcal{B}(\alpha \searrow (i,j))^{\to l(\alpha) - l(\alpha \searrow (i,j))},$$

where $\alpha \searrow (i,j) \in \mathscr{P}_{n-r}$ *is the partition obtained removing the* r*-hook* (i,j) *from the partition* α *as in Section 2.1.3.1.*

The following map on $\mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty}]$ is called the removing r-hooks map:

$$\begin{aligned} \mathcal{H}_r : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty}] &\longrightarrow \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty}] \\ A &\to \sum_{h \in H_r(A)} (-1)^{l(h)} A \searrow_r (h). \end{aligned}$$

We extend by linearity the shift by k on $\mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty}]$. The following observation will be used in Lemma 2.3.5.

Lemma 2.3.3. *Let* $A \in \mathscr{P}(\mathbb{N})^{<\infty}$ *and* $r, k \in \mathbb{N}$*. Then*

$$\mathcal{H}_r(A)^{\to k} = \mathcal{H}_r(A^{\to k})$$

Proof. For any $k \in \mathbb{N}$ there is a bijection

$$\begin{aligned} H_r(A) &\to H_r(A^{\to k}). \\ h &\mapsto h + k \end{aligned}$$

Moreover the leg length of $h+k$ in $A^{\to k}$ is the same as the leg length of h in A, since

$$\begin{aligned} l(h+k) &= |\{a \in A^{\to k} \mid h - r + k < a < h + k\}| \\ &= |\{a \in A \mid h - r + k < a + k < h + k\}| \\ &= |\{a \in A \mid h - r + < a < h\} = l(h). \end{aligned}$$

In addition, via this bijection, removing an r-hook is compatible with shifting. Indeed for any $h \in H_r(A)$

$$(A \searrow_r (h))^{\to k} = \{0,1,\dots k-1\} \bigsqcup \{a + k | a \in A \searrow \{h\} \cup \{h - r\}\} = A^{\to k} \searrow_r (h+k).$$

Then

$$\begin{aligned} \mathcal{H}_r(A^{\to k}) &= \sum_{h \in H_r(A^{\to k})} (-1)^{l(h)} A^{\to k} \searrow_r (h) \\ &= \sum_{h \in H_r(A)} (-1)^{l(h+k)} A^{\to k} \searrow_r (h+k) \\ &= \sum_{h \in H_r(A)} (-1)^{l(h)} (A \searrow_r (h))^{\to k} = \mathcal{H}_r(A)^{\to k}. \end{aligned}$$

□

2.3.2 Removing hooks on Symbols

Definition 2.3.4. *For $i \in \{0,1\}$, we say that (h,i) is a r-hook of the array $X = (X^0, X^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ if h is a r-hook of X^i. We denote by $H_r(X)$ the set of the r-hooks of X.*

Let X be an array. The leg length of a hook (h,i) of an array X, with $i \in \{0,1\}$, is the leg length of the hook h in X^i:

$$l(h,i) := |\{x \in X^i | h - r < x < h\}|.$$

Let $(h,i) \in H_r(X)$. We define the array $X \setminus_r (h,i)$ to be the array obtained from X by removing the r-hook h from X^i, and leaving the other row invariant. We say that $X \setminus_r (h,i)$ is the array obtained from X removing the r-hook (h,i). Removing a r-hook does not change the defect of an array, while decreases the rank by r. The removing r-hooks map is defined as follows:

$$\begin{aligned}\mathcal{H}_r : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] &\to \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] \\ X &\mapsto \sum_{(h,i) \in H_r(X)} (-1)^{l(h,i)} X \setminus_r (h,i).\end{aligned}$$

As for subsets, shift and removing hooks are compatible. For the sake of completeness, we record in the following Lemma the computation showing it.

Lemma 2.3.5. *Let X be an array and let $r, k \in \mathbb{N}$. Then*

$$\mathcal{H}_r(X)^{\to k} = \mathcal{H}_r(X^{\to k})$$

Proof. Using Lemma 2.3.3, it holds

$$\begin{aligned}\mathcal{H}_r(X^{\to k}) &= \sum_{(h,i) \in H_r(X^{\to k})} (-1)^{l(h,i)} X^{\to k} \setminus_r (h,i) \\ &= \sum_{(h,0) \in H_r(X^{\to k})} (-1)^{l(h,0)} X^{\to k} \setminus_r (h,0) + \sum_{(h,1) \in H_r(X^{\to k})} (-1)^{l(h,1)} X^{\to k} \setminus_r (h,1) \\ &= \sum_{h \in H_r(X^{0 \to k})} (-1)^{l(h)} (X^{0 \to k} \setminus_r (h), X^{1 \to k}) + \sum_{h \in H_r(X^{1 \to k})} (-1)^{l(h)} (X^{1 \to k}, X^{1 \to k} \setminus_r (h)) \\ &= \sum_{h \in H_r(X^0)} (-1)^{l(h)} (X^{0 \to k} \setminus_r (h+k), X^{1 \to k}) + \sum_{h \in H_r(X^1)} (-1)^{l(h)} (X^{1 \to k}, X^{1 \to k} \setminus_r (h+k)) \\ &= \sum_{h \in H_r(X^0)} (-1)^{l(h)} ((X^0 \setminus_r (h))^{\to k}, X^{1 \to k}) + \sum_{h \in H_r(X^1)} (-1)^{l(h)} (X^{1 \to k}, (X^1 \setminus_r (h))^{\to k}) \\ &= \sum_{(h,i) \in H_r(X)} (-1)^{l(h,i)} (X \setminus_r (h,i))^{\to k} = \mathcal{H}_r(X^{\to k}).\end{aligned}$$

□

Since for any $r \in \mathbb{N}$ the map $\mathcal{H}_r$ commutes with any shift, it induces a well-defined map

$$\mathcal{H}_r : \mathbb{C}[\widetilde{\mathcal{S}}] \to \mathbb{C}[\widetilde{\mathcal{S}}]. \tag{2.26}$$

Moreover, since $\mathcal{H}_r$ preserves the defect of a symbol and diminishes the rank by r, for any $n \in \mathbb{N}$ such that $n \geqslant r$ and $d \in \mathbb{Z}$ the map $\mathcal{H}_r$ restricts to a map, that we still denote by $\mathcal{H}_r$,

$$\mathcal{H}_r : \mathbb{C}[\widetilde{\mathcal{S}}_n^d] \to \mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^d]. \tag{2.27}$$

Lemma 2.3.6. *Let $n, r \in \mathbb{N}$ with $r \leqslant n$ and $d \in \mathbb{Z}$. Then the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\mathscr{B}_n] & \xrightarrow{\widetilde{\mathscr{S}}_{n,d}} & \mathbb{C}[\widetilde{S}^d_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}] \\ \downarrow{\scriptstyle \mathcal{H}_r} & & \downarrow{\scriptstyle \mathcal{H}_r} \\ \mathbb{C}[\mathscr{B}_{n-r}] & \xrightarrow{\widetilde{\mathscr{S}}_{n-r,d}} & \mathbb{C}[\widetilde{S}^d_{(n-r-\lfloor \frac{1-d^2}{4} \rfloor)}] \end{array} \tag{2.28}$$

Proof. Let $(\alpha, \beta) \in \mathscr{B}_n$. Assume $d \geqslant 0$. Then, using Lemma 2.3.3 we have

$$\begin{aligned} \mathcal{H}_r \circ \widetilde{\mathscr{S}}_{n,d}(\alpha,\beta) &= \mathcal{H}_r([(\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)})]) \\ &= \sum_{(h,i) \in H_r((\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)}))} (-1)^{l(h,i)} [(\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)}) \searrow_r (h,i)] \\ &= \sum_{h \in H_r(\mathcal{B}(\alpha)^{\to l(\beta)+d})} (-1)^{l(h)} [(\mathcal{B}(\alpha)^{\to l(\beta)+d} \searrow_r (h), \mathcal{B}(\beta)^{\to l(\alpha)})] \\ &+ \sum_{h \in H_r(\mathcal{B}(\beta)^{\to l(\alpha)})} (-1)^{l(h)} [(\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)} \searrow_r (h))] \\ &= \sum_{h \in H_r(\mathcal{B}(\alpha)} (-1)^{l(h+l(\beta)+d)} [(\mathcal{B}(\alpha)^{\to l(\beta)+d} \searrow_r (h+l(\beta)+d), \mathcal{B}(\beta)^{\to l(\alpha)})] \\ &+ \sum_{h \in H_r(\mathcal{B}(\beta))} (-1)^{l(h+l(\alpha))} [(\mathcal{B}(\alpha)^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)} \searrow_r (h+l(\alpha)))] \\ &= \sum_{h \in H_r(\mathcal{B}(\alpha)} (-1)^{l(h)} [((\mathcal{B}(\alpha) \searrow_r (h))^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)})] \\ &+ \sum_{h \in H_r(\mathcal{B}(\beta))} (-1)^{l(h)} [(\mathcal{B}(\alpha)^{\to l(\beta)+d}, (\mathcal{B}(\beta) \searrow_r (h))^{\to l(\alpha)})]. \end{aligned}$$

By Lemma 2.3.2, there is a bijection between the set of r-hooks of a partition and the set of r-hooks of the corresponding β-set, and it is compatible with removing hooks. Therefore the above linear combination becomes

$$
\begin{aligned}
&\sum_{(i,j)\in H_r(\alpha)} (-1)^{l(h)}[((\mathcal{B}(\alpha \setminus (i,j))^{\to l(\alpha)-l(\alpha\setminus(i,j))})^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha)})] \\
&+ \sum_{(i,j)\in H_r(\beta)} (-1)^{l(h)}[(\mathcal{B}(\alpha)^{\to l(\beta)+d}, (\mathcal{B}(\beta \setminus (i,j))^{\to l(\beta)-l(\beta\setminus(i,j))})^{\to l(\alpha)})] \\
&= \sum_{(i,j)\in H_r(\alpha)} (-1)^{l(h)}[(\mathcal{B}(\alpha \setminus (i,j))^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha\setminus(i,j))})^{\to l(\alpha)-l(\alpha\setminus(i,j))}] \\
&+ \sum_{(i,j)\in H_r(\beta)} (-1)^{l(h)}[(\mathcal{B}(\alpha)^{\to l(\beta\setminus(i,j))+d}, \mathcal{B}(\beta \setminus (i,j))^{\to l(\alpha)})^{\to l(\beta)-l(\beta\setminus(i,j))}] \\
&= \sum_{(i,j)\in H_r(\alpha)} (-1)^{l(h)}[(\mathcal{B}(\alpha \setminus (i,j))^{\to l(\beta)+d}, \mathcal{B}(\beta)^{\to l(\alpha\setminus(i,j))})] \\
&+ \sum_{(i,j)\in H_r(\beta)} (-1)^{l(h)}[(\mathcal{B}(\alpha)^{\to l(\beta\setminus(i,j))+d}, \mathcal{B}(\beta \setminus (i,j))^{\to l(\alpha)})] \\
&= \widetilde{\mathscr{S}}_{n-r,d}\Big(\sum_{(i,j)\in H_r(\alpha)} (-1)^{l(h)}((\alpha \setminus (i,j)), \beta) + \sum_{(i,j)\in H_r(\beta)} (-1)^{l(h)}(\alpha, (\beta \setminus (i,j)))\Big) \\
&= \widetilde{\mathscr{S}}_{n-r,d} \circ \mathcal{H}_r(\alpha, \beta).
\end{aligned}
$$

The case $d < 0$ is completely analogous. □

Lemma 2.3.7. *For any* $[X] \in \widetilde{\mathcal{S}}$ *and for any* $r \in \mathbb{N}$ *it holds* $\mathcal{H}_r([X])^{op} = \mathcal{H}_r([X^{op}])$.

Proof. Since the r-hooks for a symbol $[X]$ are defined as the r-hooks for some row in any array representing it, there is a bijection between $\mathcal{H}_r(X)$ and $\mathcal{H}_r(X^{op})$ given by the assignment $(h, i) \mapsto (h, |i - 1|)$. This bijection preserves the leg length of (h, i) and satisfies $(X \setminus_r (h, i))^{op} = X^{op} \setminus_r (h, |i - 1|)$. Then

$$
\begin{aligned}
\mathcal{H}_r([X^{op}]) &= \sum_{(h,i)\in H_r(X^{op})} (-1)^{l(h,i)}([X^{op} \setminus_r (h, i)]) \\
&= \sum_{(h,i)\in H_r(X)} (-1)^{l(h,i)}([X^{op} \setminus_r (h, |i - 1|)]) \\
&= \sum_{(h,i)\in H_r(X)} (-1)^{l(h,i)}[(X \setminus_r (h, i))^{op}] = H_r([X])^{op}.
\end{aligned}
$$

□

Lemma 2.3.8. *Let* $n \in \mathbb{N}$ *and* $d \in \mathbb{Z}$*. Then, for any* $(\alpha, \beta) \in \mathscr{B}_n$ *it holds*

$$\widetilde{\mathscr{S}}_{n,d}(\beta, \alpha) = (\widetilde{\mathscr{S}}_{n,-d}(\alpha, \beta))^{op}$$

Proof. Assume $d \geqslant 0$. Then,

$$(\widetilde{\mathscr{S}}_{n,d}(\beta, \alpha))^{op} = [(\mathcal{B}(\beta)^{\to l(\alpha)+d}, \mathcal{B}(\alpha)^{\to l(\beta)})^{op}] = [(\mathcal{B}(\alpha)^{\to l(\beta)}, \mathcal{B}(\beta)^{\to l(\alpha)-(-d)})] = \widetilde{\mathscr{S}}_{n,-d}(\alpha, \beta). \tag{2.29}$$

Applying ^{op} to both sides we get the statement for $d \geqslant 0$. If $d < 0$, then $-d > 0$ and (2.29) gives

$$(\widetilde{\mathscr{S}}_{n,-d}(\beta, \alpha))^{op} = \widetilde{\mathscr{S}}_{n,d}(\alpha, \beta).$$

□

Lemma 2.3.9. *Let $n, r, d \in \mathbb{N}$, with $r \leqslant n$ and $d > 0$. Then the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\mathscr{B}_n \times \{\pm 1\}] & \xrightarrow{\widetilde{\mathscr{S}}^{\pm}_{n,d}} & \mathbb{C}[\widetilde{S}^{d}_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}] \bigoplus \mathbb{C}[\widetilde{S}^{-d}_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}] \\ \downarrow {\scriptstyle \mathcal{H}_r \otimes id} & & \downarrow {\scriptstyle \mathcal{H}_r} \\ \mathbb{C}[\mathscr{B}_{n-r} \times \{\pm 1\}] & \xrightarrow{\widetilde{\mathscr{S}}^{\pm}_{n-r,d}} & \mathbb{C}[\widetilde{S}^{d}_{(n-r-\lfloor \frac{1-d^2}{4} \rfloor)}] \oplus \mathbb{C}[\widetilde{S}^{-d}_{(n-r-\lfloor \frac{1-d^2}{4} \rfloor)}] \end{array} \tag{2.30}$$

Proof. The vertical map $\mathcal{H}_r$ on the right hand side of (2.30) denotes the restriction of the map $\mathcal{H}_r$ as in (2.26) on the space $\mathbb{C}[\widetilde{S}^{d}_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}] \bigoplus \mathbb{C}[\widetilde{S}^{-d}_{(n-\lfloor \frac{1-d^2}{4} \rfloor)}]$. Since the map $\mathcal{H}_r$ preserves the defect of a symbol, it is the direct sum of the restriction of the map $\mathcal{H}_r$ on each of the direct summands. The space $\mathbb{C}[\mathscr{B}_n \times \{\pm 1\}]$ decomposes as $\mathbb{C}[\mathscr{B}_n \times \{+1\}] \bigoplus \mathbb{C}[\mathscr{B}_n \times \{-1\}]$. With respect to this decomposition, $\widetilde{\mathscr{S}}^{\pm}_{n,d} = (\widetilde{\mathscr{S}}_{n,d}) \oplus (()^{op} \circ \widetilde{\mathscr{S}}_{n,d})$. By Lemma 2.3.6, it holds

$$\mathcal{H}_r \circ \widetilde{\mathscr{S}}_{n,d} = \widetilde{\mathscr{S}}_{n-r,d} \circ \mathcal{H}_r,$$

so the restriction of the diagram (2.30) to $\mathbb{C}[\mathscr{B}_n \times \{1\}]$ commutes. Moreover, by Lemma 2.3.7, it holds

$$\mathcal{H}_r \circ ()^{op} = ()^{op} \circ \mathcal{H}_r,$$

therefore the restriction of the diagram (2.30) to $\mathbb{C}[\mathscr{B}_n \times \{-1\}]$ commutes as well. □

For any $r \in \mathbb{N}$ we define the analogue of (2.11) for symbols, setting

$$\begin{aligned} Res_r : \mathbb{C}[\widetilde{\mathcal{S}}] &\to \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\widetilde{\mathcal{S}}] \\ [X] &\mapsto \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\chi_\gamma) \boxtimes \mathcal{H}_\gamma([X]) \end{aligned} \tag{2.31}$$

where for any $\gamma \in \mathscr{P}_r$ the function χ_γ is the characteristic function of the conjugacy class labelled by $\gamma = (\gamma_1, \ldots, \gamma_k)$ in $\mathbb{S}_r$, and $\mathcal{H}_\gamma$ is the composition $\mathcal{H}_{\gamma_k} \circ \cdots \circ \mathcal{H}_{\gamma_1}$

Since for any $\gamma \in \mathscr{P}_r$ the map $\mathcal{H}_\gamma$ preserves the defect of an ordered symbol and lowers the rank by r, for any $n \in \mathbb{N}$ with $n \geqslant r$ and $d \in \mathbb{Z}$ the map Res_r restricts to a map

$$Res_r : \mathbb{C}[\widetilde{\mathcal{S}}^d_n] \to \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\widetilde{\mathcal{S}}^d_{n-r}].$$

In Proposition 2.3.11, we will show that this map is a combinatorial description of the parabolic restriction to a maximal Levi subgroup. The following Corollary 2.3.10 relates the map Res_r to restriction of representations of (extended) Weyl groups of type B, and it is an intermediate step toward Proposition 2.3.11.

Corollary 2.3.10. *Let $n, r \in \mathbb{N}$ with $r \leqslant n$. For any $d > 0$, let $m := n + \lfloor \frac{1-d^2}{4} \rfloor$. Then the following diagrams commute*

$$\begin{array}{ccc} R(W(B_m) \times \mathbb{Z}_2) & \xrightarrow{\widetilde{\ell^{n,d}}} & \mathbb{C}[\widetilde{\mathcal{S}}^d_n] \oplus \mathbb{C}[\widetilde{\mathcal{S}}^{-d}_n] \\ \downarrow {\scriptstyle Res^{W(B_m) \times \mathbb{Z}_2}_{\mathbb{S}_r \times W(B_m) \times \mathbb{Z}_2}} & & \downarrow {\scriptstyle Res_r} \\ R(\mathbb{S}_r \times W(B_{m-r}) \times \mathbb{Z}_2) & \xrightarrow{\kappa_A \boxtimes \widetilde{\ell^{n-r,d}}} & \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\widetilde{\mathcal{S}}^d_{n-r}] \oplus \mathbb{C}[\widetilde{\mathcal{S}}^{-d}_{n-r}] \end{array} \tag{2.32}$$

$$\begin{array}{ccc} R(W(B_n)) & \xrightarrow{\widetilde{\ell^{n,0}}} & \mathbb{C}[\widetilde{\mathcal{S}}_n^0] \\ \Big\downarrow Res^{W(B_n)}_{\mathbb{S}_r\times W(B_m)} & & \Big\downarrow Res_r \\ R(\mathbb{S}_r\times W(B_{n-r})) & \xrightarrow{\kappa_A\boxtimes\widetilde{\ell^{n-r,0}}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^0] \end{array} \tag{2.33}$$

where the map $\widetilde{\ell^{n,d}}$ *is as in* (2.23) *and the map* Res_r *is as in* (2.31).

Proof. The diagram (2.32) factors as

$$\begin{array}{ccccc} R(W(B_m)\times\mathbb{Z}_2) & \xrightarrow{\kappa_B\times ev_{-1}} & \mathbb{C}[\mathscr{B}_m\times\{\pm1\}] & \xrightarrow{\widetilde{\mathscr{S}}^{\pm}_{m,d}} & \mathbb{C}[\widetilde{\mathcal{S}}_n^d]\oplus\mathbb{C}[\widetilde{\mathcal{S}}_n^{-d}] \\ \Big\downarrow Res^{W(B_m)\times\mathbb{Z}_2}_{\mathbb{S}_r\times W(B_{m-r})\times\mathbb{Z}_2} & & \Big\downarrow Res_r\otimes id & & \Big\downarrow Res_r \\ R(\mathbb{S}_r\times W(B_{m-r})\times\mathbb{Z}_2) & \xrightarrow{\kappa_A\boxtimes(\kappa_B\times ev_{-1})} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathscr{B}_{m-r}\times\{\pm1\}] & \xrightarrow{id\boxtimes\widetilde{\mathscr{S}}^{\pm}_{m-r,d}} & \mathbb{C}[\mathscr{P}_r]\otimes(\mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^d]\oplus\mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^{-d}]) \end{array}$$

The left diagram commutes by Corollary 2.1.6 and Remark 2.1.3.

The right diagram commutes by Lemma 2.3.9. Indeed for any $((\alpha,\beta),\varepsilon)\in\mathscr{B}_m\times\mathbb{Z}_2$ and $j\in\mathbb{N}$ such that $j\leqslant m$ it holds

$$\mathcal{H}_j(\widetilde{\mathscr{S}}^{\pm}_{m,d}((\alpha,\beta),\varepsilon)) = \widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_j\otimes id((\alpha,\beta),\varepsilon) = \widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_j(\alpha,\beta),\varepsilon) \tag{2.34}$$

by the commutativity of (2.30). Then for any $\gamma=(\gamma_1,\ldots,\gamma_k)\in\mathscr{P}_r$, applying repeatedly (2.34) yields

$$\begin{aligned} \mathcal{H}_\gamma(\widetilde{\mathscr{S}}^{\pm}_{m,d}((\alpha,\beta),\varepsilon)) &= \mathcal{H}_{\gamma_k}\circ\cdots\circ\mathcal{H}_{\gamma_1}(\widetilde{\mathscr{S}}^{\pm}_{m,d}((\alpha,\beta),\varepsilon)) \\ &= \mathcal{H}_{\gamma_k}\circ\cdots\circ\mathcal{H}_{\gamma_2}(\widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_{\gamma_1}(\alpha,\beta),\varepsilon)) \\ &= \mathcal{H}_{\gamma_k}\circ\cdots\circ\mathcal{H}_{\gamma_3}(\widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_{\gamma_2}\circ\mathcal{H}_{\gamma_1}(\alpha,\beta),\varepsilon)) \\ &= \cdots = \widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_{\gamma_k}\circ\cdots\circ\mathcal{H}_{\gamma_1}(\alpha,\beta),\varepsilon)) = \widetilde{\mathscr{S}}^{\pm}_{m-j,d}(\mathcal{H}_\gamma(\alpha,\beta),\varepsilon). \end{aligned}$$

Therefore

$$\begin{aligned} Res_r\circ\widetilde{\mathscr{S}}^{\pm}_{m.d}((\alpha,\beta),\varepsilon) &= \sum_{\gamma\in\mathscr{P}_r}\kappa_A(\chi_\gamma)\boxtimes\mathcal{H}_\gamma(\widetilde{\mathscr{S}}^{\pm}_{m,d}((\alpha,\beta),\varepsilon)) \\ &= \sum_{\gamma\in\mathscr{P}_r}\kappa_A(\chi_\gamma)\boxtimes\widetilde{\mathscr{S}}^{\pm}_{m-r,d}(\mathcal{H}_\gamma(\alpha,\beta),\varepsilon) \\ &= (id\boxtimes\widetilde{\mathscr{S}}^{\pm}_{m-r,d})(\sum_{\gamma\in\mathscr{P}_r}\kappa_A(\chi_\gamma)\boxtimes(\mathcal{H}_\gamma(\alpha,\beta),\varepsilon)) \\ &= (id\boxtimes\widetilde{\mathscr{S}}^{\pm}_{m-r,d})(Res_r(\alpha,\beta),\varepsilon) \\ &= (id\boxtimes\widetilde{\mathscr{S}}^{\pm}_{m-r,d})\circ(Res_r\otimes id)((\alpha,\beta),\varepsilon) \end{aligned}$$

The diagram (2.33) factors as

$$\begin{array}{ccccc} R(W(B_n)) & \xrightarrow{\kappa_B} & \mathbb{C}[\mathscr{B}_n] & \xrightarrow{\widetilde{\mathscr{S}}^{\pm}_{n,0}} & \mathbb{C}[\widetilde{\mathcal{S}}_n^0] \\ \Big\downarrow Res^{W(B_n)}_{\mathbb{S}_r\times W(B_{n-r})} & & \Big\downarrow Res_r & & \Big\downarrow Res_r \\ R(\mathbb{S}_r\times W(B_{n-r})) & \xrightarrow{\kappa_A\boxtimes\kappa_B} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathscr{B}_{n-r}] & \xrightarrow{id\boxtimes\widetilde{\mathscr{S}}^{\pm}_{n-r,0}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^0] \end{array}$$

The left diagram commutes by Corollary 2.1.6, the right square commutes by Lemma 2.3.6

□

Proposition 2.3.11. *Let G be of type D_n and retain the notation from Section 2.1.2.3.*

Let $r \in \mathbb{N}_{\leqslant n}$ and let M be a maximal δ-stable standard Levi subgroup of G of type $A_{r-1} \times D_{n-r}$ with $1 \leqslant r \leqslant n-1$. Then the following diagram commutes:

$$\begin{array}{ccc}
R_u(\widetilde{G}^0) \bigoplus R_u(\widetilde{G}^1) & \xrightarrow{\bigoplus_{\substack{s\in\mathbb{N}\\ s^2\leqslant n}} \widetilde{\ell^{n,2s}} \circ Rep_{\widetilde{G}^s}} & \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] \\
\Big\downarrow {}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1} & & \Big\downarrow Res_r \\
R_u(\widetilde{M}^0) \bigoplus R_u(\widetilde{M}^1) & \xrightarrow{\bigoplus_{\substack{s\in\mathbb{N}\\ s^2\leqslant n-r}} (\kappa_A \boxtimes \widetilde{\ell^{n-r,2s}}) \circ Rep_{\widetilde{M}^s}} & \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\widetilde{\mathcal{S}}_{n-r}^{ev}]
\end{array}$$

Proof. For any $s \in \mathbb{N}$ satisfying $s^2 \leqslant n-r$ we have $L_s \subseteq \widetilde{M}^s$, since L_s and M are standard Levi subgroups of type D_{s^2} and $A_r \times D_{n-r}$ respectively.

The restriction of the diagram (2.3.11) to the subspace $\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)]$ with $s^2 \leqslant n-r$ is the diagram

$$\begin{array}{ccc}
\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)] & \xrightarrow{Rep \circ \widetilde{\ell^{n,2s}}} & \bigoplus_{s' \in \{\pm s\}} \mathbb{C}[\mathcal{S}_n^{2s'}] \\
\Big\downarrow {}^*R^{\widetilde{G}^s}_{\widetilde{M}^s} & & \Big\downarrow Res_r \\
\mathbb{C}[\mathcal{E}^{\widetilde{M}^s}(L_s, \sigma_s)] & \xrightarrow{Rep \circ (\kappa_A \boxtimes \widetilde{\ell^{n-r,2s}})} & \mathbb{C}[\mathscr{P}_r] \otimes (\bigoplus_{s' \in \{\pm s\}} \mathbb{C}[\mathcal{S}_{n-r}^{2s'}])
\end{array}$$

which factors as:

$$\begin{array}{ccccc}
\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)] & \xrightarrow{Rep_{\widetilde{G}^s}} & R(W_{\widetilde{G}^s}(L_s, \sigma_s)) & \xrightarrow{\widetilde{\ell^{n,2s}}} & \bigoplus_{s' \in \{\pm s\}} \mathbb{C}[\mathcal{S}_n^{2s'}] \\
\Big\downarrow {}^*R^{\widetilde{G}^s}_{\widetilde{M}^s} & & \Big\downarrow Res^{W_{\widetilde{G}^s}(L_s,\sigma_s)}_{W_{\widetilde{M}^s}(L_s,\sigma_s)} & & \Big\downarrow Res_r \\
\mathbb{C}[\mathcal{E}^{\widetilde{M}^s}(L_s, \sigma_s)] & \xrightarrow{Rep_{\widetilde{M}^s}} & R(W_{\widetilde{M}^s}(L_s, \sigma_s)) & \xrightarrow{\kappa_A \boxtimes \widetilde{\ell^{n-r,2s}}} & \mathbb{C}[\mathscr{P}_r] \otimes (\bigoplus_{s' \in \{\pm s\}} \mathbb{C}[\mathcal{S}_{n-r}^{2s'}])
\end{array}$$

The left square commutes by Corollary 1.1.5. The right square commutes by Corollary 2.3.10, since

$$\begin{array}{ll}
W_{\widetilde{G}^s}(L_s, \sigma_s) \cong W(B_{n-s^2}) \times \mathbb{Z}_2 & \text{for } s > 0 \\
W_{\widetilde{M}^s}(L_s, \sigma_s) \cong \mathbb{S}_r \times W(B_{n-r-s^2}) \times \mathbb{Z}_2 & \text{for } s > 0 \\
W_{\widetilde{G}^0}(L_0, \sigma_0) \cong W(B_n) & \\
W_{\widetilde{M}^0}(L_0, \sigma_0) \cong \mathbb{S}_r \times W(B_{n-r}). &
\end{array}$$

Therefore the diagram (2.3.11) commutes on each subspace $\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)]$ with $s^2 \leqslant n-r$.

It commutes on each subspace $\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)]$ with $s^2 > n-r$ because the vertical maps are 0.

□

2.4 Fourier transform for inner twists of type D_n

In [42, Section 4] Lusztig defines a map on the space of unipotent class functions for any finite group of Lie type, the so-called Lusztig's Fourier transform. For groups of classical type, such a map has a combinatorial description.

In this section, following [40] we define the combinatorial map $\widetilde{\mathcal{R}}$ on the $\mathbb{C}$-span of arrays (2.36), that induces a map on the $\mathbb{C}$-span of.ordered symbols. By transport, the map $\widetilde{\mathcal{R}}$ gives an involution on the sum of the spaces of unipotent class functions on the inner twists of a disconnected group of type D as in (2.6), see (2.38). We call $\widetilde{\mathcal{R}}$ the Fourier transform (for disconnected groups of type D). It is closely related to Lusztig's Fourier transform for connected groups of type D, as we explain in detail in Section 2.5 (see Remarks 2.5.10 and 2.6.1) .

Given an array $X = (X^0, X^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$, we denote by

- $X^\cup := X^0 \cup X^1$, that is, the set of entries that appear in at least one of the rows of X;
- $X^\cap := X^0 \cap X^1$, that is, the set of entries that appear in both rows of X;
- $X^\ominus := X^0 \ominus X^1$, where $\ominus$ denotes the symmetric difference, i.e. $X^0 \ominus X^1 = X^\cup \backslash X^\cap$, so that $X^\ominus$ is the set of entries that appear in one row of X but not in both of them.

With these notations, for any $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$,

$$\rho(X) = \sum_{x \in X^\cup} x + \sum_{z \in X^\cap} z + \left\lfloor \left(\frac{|X^\cup| + |X^\cap| - 1}{2} \right)^2 \right\rfloor \tag{2.35}$$

Definition 2.4.1. *The similarity class of an array $X = (X^0, X^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ is the set of arrays*

$$Sim(X) := \{Y = (Y^0, Y^1) \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty} \mid Y^\cup = X^\cup, Y^\cap = X^\cap\}$$

In other words, $Sim(X)$ is the set of arrays with the same set of entries as X, with some multiplicity of occurrence.

All the arrays in a similarity class have the same rank, as one can see from (2.35). Moreover, the defects of all arrays in the same similarity class have the same parity: $def(X) = |X^0| - |X^1| \equiv |X^0| + |X^1| = |X^\cup| + |X^\cap| \ (mod\ 2)$.

Definition 2.4.2. *Let X be an array. The special array $X_{sp} \in Sim(X)$ is the array determined by the following conditions:*

- *If $def(X)$ is even:*
  - $\star$ *$def(X_{sp}) = 0$,*
  - $\star$ *Ordering the elements of $(X_{sp})^\ominus$ in increasing order, consecutive elements never occur in the same row of X_{sp}, and the lowest of these elements belongs to the bottom row of X_{sp}.*

- ◇ *If $def(X)$ is odd:*
  - ⋆ $def(X_{sp}) = 1$,
  - ⋆ *Ordering the elements of $(X_{sp})^{\ominus}$ in increasing order, consecutive elements never occur in the same row of X_{sp}, and the lowest of these entries belongs to the top row of X_{sp}.*

For any array X, we set

$$X^{\#} := X^1 \ominus X^1_{sp}.$$

In other words, $X^{\#}$ is the set of the entries by which the bottom row of X differs from the bottom row of X_{sp}. The assignment $X \mapsto X^{\#}$ defines a bijection between $Sim(X)$ and the power set of $X^{\ominus}$. It follows that $|Sim(X)| = 2^{|X^{\ominus}|}$.

We set

$$s(X) := 2^{\frac{|X^{\ominus}| - def(X_{sp})}{2}}$$

We are now in the position to give the combinatorial definition of the linear map that is the main object of this chapter. Appropriate restrictions of this map coincide with the Fourier transform defined by Lusztig in [42, Section 4]. The latter is a map on the space of unipotent class functions on groups of type D and B. We refer to Remarks 2.5.10 and 2.6.1 for a more precise statement. Let

$$\begin{aligned} \widetilde{\mathcal{R}} : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] &\to \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] \\ X &\mapsto \frac{1}{s(X)} \sum_{Y \in Sim(X)} (-1)^{|X^{\#} \cap Y^{\#}|} Y. \end{aligned} \tag{2.36}$$

Example 2.4.3. *Let $X = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$. Then*

$$Sim(X) = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ & 2 & \end{pmatrix}, \begin{pmatrix} & 2 & \\ 0 & 1 & 2 \end{pmatrix} \right\}.$$

In this case, $def(X) = 0$ and $X_{sp} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

We have $X^{\ominus} := \{0, 1\}$, and

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^{\#} = \varnothing, \quad \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}^{\#} = \{0, 1\}, \quad \begin{pmatrix} 0 & 1 & 2 \\ & 2 & \end{pmatrix}^{\#} = \{0\}, \quad \begin{pmatrix} & 2 & \\ 0 & 1 & 2 \end{pmatrix}^{\#} = \{1\}.$$

Computing the map $\widetilde{\mathcal{R}}$ on X yields

$$\widetilde{\mathcal{R}}(X) = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 1 & 2 \\ & 2 & \end{pmatrix} - \frac{1}{2} \begin{pmatrix} & 2 & \\ 0 & 1 & 2 \end{pmatrix}.$$

The following compatibility, first stated in [1],[2], will be crucial for our treatment.

Theorem 2.4.4. *[40, Theorem 3.1] The map $\widetilde{\mathcal{R}}$ commutes with $\mathcal{H}_r$ for any $r \in \mathbb{N}$:*

$$\widetilde{\mathcal{R}} \circ \mathcal{H}_r = \mathcal{H}_r \circ \widetilde{\mathcal{R}}.$$

We now prove some properties of the map $\widetilde{\mathcal{R}}$ that will be useful in the following: namely, the fact that the map $\widetilde{\mathcal{R}}$ induces an isometric involution on the space of symbols.

Lemma 2.4.5. *Let $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ and let $k \in \mathbb{N}$. Then*

1. *the map induced by the shift by k*

$$\begin{aligned}(\cdot)^{\to k} : Sim(X) &\to Sim(X^{\to k}) \\ Y &\mapsto Y^{\to k}\end{aligned}$$

is a bijection.

2. $(X^{\to k})^{\ominus} = \{x + k \mid x \in X^{\ominus}\}$

3. $(X^{\to k})_{sp} = (X_{sp})^{\to k}$

4. $(X^{\to k})^{\#} = \{x + k \mid x \in X^{\#}\}$

Proof. We first observe that by the definition of the shift, it holds

$$\begin{aligned}(X^{\to k})^{\cup} &= (X^{\to k})^0 \cup (X^{\to k})^1 = \{0, 1, \ldots, k-1\} \cup \{x + k | x \in X^{\cup}\} = (X^{\cup})^{\to k} \\ (X^{\to k})^{\cap} &= (X^{\to k})^0 \cap (X^{\to k})^1 = \{0, 1, \ldots, k-1\} \cup \{x + k | x \in X^{\cap}\} = (X^{\cap})^{\to k}\end{aligned}$$

Now we prove the assertions in the statement of this lemma.

1. Since $(\cdot)^{\to k}$ is injective , it is enough to check that the map $(\cdot)^{\to k} : Sim(X) \to Sim(X^{\to k})$ is well-defined and surjective. First we check that $Sim(X)^{\to k} \subseteq Sim(X^{\to k})$. For any $Y \in Sim(X)$, it holds

$$\begin{aligned}(Y^{\to k})^{\cup} &= (Y^{\cup})^{\to k} = (X^{\cup})^{\to k} = (X^{\to k})^{\cup} \\ (Y^{\to k})^{\cap} &= (Y^{\cap})^{\to k} = (X^{\cap})^{\to k} = (X^{\to k})^{\cap}\end{aligned}$$

so $Y^{\to k} \in Sim(X^{\to k})$.

Now we check that $Sim(X^{\to k}) \subseteq Sim(X)^{\to k}$. For any $Z = (Z^0, Z^1) \in Sim(X^{\to k})$, it holds $Z^{\cup} = (X^{\to k})^{\cup} = \{0, 1, \ldots, k-1\} \cup \{x + k | x \in X^{\cup}\}$ and $Z^{\cap} = (X^{\to k})^{\cap} = \{0, 1, \ldots, k-1\} \cap \{x + k | x \in X^{\cap}\}$. Hence setting $Y^i = \{z - k | z \in Z^i, z \geqslant k\}$ for $i = 1, 2$ and $Y = (Y^0, Y^1)$ we get $Z = Y^{\to k}$, and $Y \in Sim(X)$.

2. We have

$$\begin{aligned}(X^{\to k})^{\ominus} &= (X^{\to k})^{\cup} \backslash (X^{\to k})^{\cap} \\ &= (\{0, 1, \ldots, k-1\} \cup \{x + k | x \in X^{\cup}\}) \backslash (\{0, 1, \ldots, k-1\} \cup \{x + k | x \in X^{\cup}\}) \\ &= \{x + k \mid x \in X^{\cup} \backslash X^{\cap}\} = \{x + k \mid x \in X^{\ominus}\}\end{aligned}$$

3. If X_{sp} is the special array in $Sim(X)$, by the previous point, $X_{sp}^{\to k} \in Sim(X^{\to k})$. We have to check that $X_{sp}^{\to k}$ satisfies the conditions to be the special array in $Sim(X^{\to k})$. Shifting by k does not change the defect of an array, so $X_{sp}^{\to k}$ has the

right defect. Ordering the elements of $X_{sp}^{\ominus}$ in increasing order, two consecutive elements never occur in the same row and the lowest element occurs in the appropriate row according to the parity defect. Therefore, ordering elements of $(X^{\to k})^{\ominus} = \{x + k \mid x \in X^{\ominus}\}$ in increasing order, they satisfy the same property, since shifting preserves the relative order of elements in different rows.

4. It follows from (3) that

$$\begin{aligned}(X^{\to k})^{\#} &= (X^1)^{\to k} \ominus (X^{\to k})^1_{sp} = (X^1)^{\to k} \ominus (X^1_{sp})^{\to k} \\ &= (\{0, 1, \ldots, k-1\} \cup \{x + k \mid x \in X^1\}) \ominus (\{0, 1, \ldots, k-1\} \cup \{x + k \mid x \in X^1_{sp}\}) \\ &= \{x + k \mid x \in X^1 \ominus X^1_{sp}\} = \{x + k \mid x \in X^{\#}\}.\end{aligned}$$

□

Corollary 2.4.6. *The map $\widetilde{\mathcal{R}}$ defined in* (2.36)*, preserves the kernel of the projection*

$$[\cdot] : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] \to \mathbb{C}[\widetilde{\mathcal{S}}]$$

defined by $X \mapsto [X]$ for any $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$.

Proof. The kernel of the projection [ ] is spanned by the elements $X - X^{\to k}$ for $X \in \mathcal{P}(\mathbb{N})^{<\infty} \times \mathcal{P}(\mathbb{N})^{<\infty}$ and $k \in \mathbb{N}$. So it is enough to prove that for any $k \in \mathbb{N}$:

$$\widetilde{\mathcal{R}}(X^{\to k}) = \widetilde{\mathcal{R}}(X)^{\to k}.$$

We have

$$\widetilde{\mathcal{R}}(X^{\to k}) := \frac{1}{s(X^{\to k})} \sum_{Y \in Sim(X^{\to k})} (-1)^{|(X^{\to k})^{\#} \cap Y^{\#}|} Y.$$

We use the properties of invariance by shift collected in Lemma 2.4.5. First of all we prove that $s(X^{\to k}) = s(X)$. The assignment $x \mapsto x + k$ induces a bijection between $X^{\ominus}$ and $(X^{\to k})^{\ominus}$, so in particular $|(X^{\to k})^{\ominus}| = |X^{\ominus}|$. Moreover $(X^{\to k})_{sp} = (X_{sp})^{\to k}$, so

$$s(X^{\to k}) = 2^{\frac{|(X^{\to k})^{\ominus}| - def(X_{sp}^{\to k})}{2}} = 2^{\frac{|X^{\ominus}| - def(X_{sp})}{2}} = s(X).$$

Lemma 2.4.5 1 guarantees that $Y \in Sim(X)$ if and only if $Y^{\to k} \in Sim(X^{\to k})$. Then we have

$$\widetilde{\mathcal{R}}(X^{\to k}) := \frac{1}{s(X)} \sum_{Y \in Sim(X)} (-1)^{|(X^{\to k})^{\#} \cap (Y^{\to k})^{\#}|} Y^{\to k}.$$

Applying Lemma 2.4.5 4 we have

$$(X^{\to k})^{\#} \cap (Y^{\to k})^{\#} = \{x + k \mid x \in X^{\#}\} \cap \{y + k \mid y \in Y^{\#}\} = \{z + k \mid z \in X^{\#} \cap Y^{\#}\},$$

so

$$\widetilde{\mathcal{R}}(X^{\to k}) := \frac{1}{s(X)} \sum_{Y \in Sim(X)} (-1)^{|X^{\#} \cap Y^{\#}|} Y^{\to k} = \widetilde{\mathcal{R}}(X)^{\to k}.$$

□

As a consequence, the map $\widetilde{\mathcal{R}}$ induces a well-defined endomorphism of the vector space $\mathbb{C}[\widetilde{\mathcal{S}}]$, that we still denote by $\widetilde{\mathcal{R}}$, given by $\widetilde{\mathcal{R}}([X]) = [\widetilde{\mathcal{R}}(X)]$.

Since all the arrays belonging to the same family have defects of the same parity, the map $\widetilde{\mathcal{R}}$ preserves the spaces $\mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$ and $\mathbb{C}[\widetilde{\mathcal{S}}^{od}]$.

From now until the end of Section 2.5, we will be mostly concerned with the restriction of the map $\widetilde{\mathcal{R}}$ to the space $\mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$, since the latter is relevant for the parameterization of unipotent representations of groups of type D, as seen in (2.25).

The restriction of the map $\widetilde{\mathcal{R}}$ to the space $\mathbb{C}[\widetilde{\mathcal{S}}^{od}]$ will be relevant in Section 2.6, where we study groups of type B and C.

The following computation will be used in Lemmas 2.4.8 and 2.4.9.

Lemma 2.4.7. *Let* $[X] \in \widetilde{\mathcal{S}}$, *and let* $[Z] \in Sim(X)\backslash[X]$. *Then*

$$\sum_{Y \in Sim(X)} (-1)^{|Y^{\#} \cap (X^{\#} \ominus Z^{\#})|} = 0. \tag{2.37}$$

Proof. We recall that the assignment $Y \mapsto Y^{\#}$ is a bijection between $Sim(X)$ and the set of the subsets of $X^{\ominus}$. Since $Z \in Sim(X)\backslash\{X\}$, then $Z^{\#} \neq X^{\#}$, so there is some $a \in Z^{\#} \ominus X^{\#}$.

The assignment $Y^{\#} \mapsto Y^{\#} \ominus \{a\}$ determines a bijection without fixed points of the power set of $(X^{\ominus})$ into itself, By transport of structure, it determines a bijection of $Sim(X)$ into itself. Therefore there is some array $Y' \in Sim(X)$ such that $Y'^{\#} = Y^{\#} \ominus \{a\}$. By the choice of a, it holds $|Y^{\#} \cap (X^{\#} \ominus Z^{\#})| \equiv |Y'^{\#} \cap (X^{\#} \ominus Z^{\#})| + 1 \pmod 2$, so (2.37) follows. □

Lemma 2.4.8. *The map*

$$\widetilde{\mathcal{R}} : \mathbb{C}[\widetilde{\mathcal{S}}^{ev}] \to \mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$$

is an involution.

Proof. Let $[X] \in \widetilde{\mathcal{S}}^{ev}$. Let $Z \in Sim(X)\backslash\{X\}$. The coefficient of Z in $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}(X)$ is given by

$$\frac{1}{s(X)^2} \sum_{Y \in Sim(X)} (-1)^{|X^{\#} \cap Y^{\#}| + |Y^{\#} \cap Z^{\#}|} = \frac{1}{s(X)^2} \sum_{Y \in Sim(X)} (-1)^{|Y^{\#} \cap (X^{\#} \ominus Z^{\#})|} = 0$$

by Lemma 2.4.7. On the other hand, the coefficient of $[X]$ in $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}$ is given by

$$\frac{1}{s(X)^2} |Sim(X)| = \frac{2^{|X^{\ominus}|}}{2^{|X^{\ominus}| - def(X_{sp})}} = \frac{2^{|X^{\ominus}|}}{2^{|X^{\ominus}|}} = 1$$

□

Lemma 2.4.9. *The map*

$$\widetilde{\mathcal{R}} : \mathbb{C}[\widetilde{\mathcal{S}}^{ev}] \to \mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$$

is an isometry.

Proof. We prove that $\widetilde{\mathcal{R}}$ maps an orthonormal basis to an orthonormal basis.

Let $[X], [Z] \in \widetilde{\mathcal{S}}^{ev}$. Then

$$\langle \widetilde{\mathcal{R}}([X]), \widetilde{\mathcal{R}}([Z]) \rangle = \frac{1}{s(X)^2} \langle \sum_{Y \in Sim(X)} (-1)^{|X^\# \cap Y^\#|}[Y], \sum_{Y \in Sim(Z)} (-1)^{|Z^\# \cap Y^\#|}[Y] \rangle$$

If $Z \notin Sim(X)$, the two sums have no symbols in common, and hence this scalar product is 0. If $Z \in Sim(X)$,

$$\begin{aligned} \langle \widetilde{\mathcal{R}}([X]), \widetilde{\mathcal{R}}([Z]) \rangle &= \frac{1}{s(X)^2} \sum_{Y \in Sim(X)} (-1)^{|X^\# \cap Y^\#| + |Y^\# \cap Z^\#|} \\ &= \frac{1}{s(X)^2} \sum_{Y \in Sim(X)} (-1)^{|Y^\# \cap (X^\# \ominus Z^\#)|}. \end{aligned}$$

If $[Z] \neq [X]$ this is 0 by Lemma 2.4.7.

If $[Z] = [X]$, we have

$$\langle \widetilde{\mathcal{R}}([X]), \widetilde{\mathcal{R}}([X]) \rangle = \frac{1}{2^{|X^\ominus| - def(X_{sp})}} |Sim(X)| = 1.$$

□

2.4.1 Compatibility of parabolic induction and Fourier transform

Let G be of type D_n and retain the notation from Section 2.1.2.3. We define

$$\widetilde{\mathcal{R}}^G : R_u(\widetilde{G}^0) \bigoplus R_u(\widetilde{G}^1) \to R_u(\widetilde{G}^0) \bigoplus R_u(\widetilde{G}^1)$$

as the map making the following diagram commutative:

$$\begin{array}{ccc} R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) & \xrightarrow{\widetilde{\mathcal{R}}^G} & R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) \\ \downarrow \widetilde{Symb}_D & & \downarrow \widetilde{Symb}_D \\ \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] & \xrightarrow{\widetilde{\mathcal{R}}} & \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] \end{array} \tag{2.38}$$

This is well-posed since the vertical maps are linear isomorphisms.

By Lemmas 2.4.8 and 2.4.9, the map $\widetilde{\mathcal{R}}$ on $\mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}]$ is an involutive isometry. Therefore, since $\widetilde{Symb}_D$ as in (2.25) is an isometry, the map $\widetilde{\mathcal{R}}$ on $R_u(\widetilde{G}^0) \bigoplus R_u(\widetilde{G}^1)$ is an involutive isometry.

Let M be a maximal δ-stable standard Levi subgroup of G of type $A_{r-1} \times D_{n-r}$, with $1 \leqslant r \leqslant n-1$ and retain notation from Section 2.1.3. The Fourier transform on groups of type A is defined to be the identity [42, 4.4]. Therefore we define

$$\widetilde{\mathcal{R}}^M : R_u(\widetilde{M}^0) \bigoplus R_u(\widetilde{M}^1) \to R_u(\widetilde{M}^0) \bigoplus R_u(\widetilde{M}^1)$$

as the map making the following diagram commutative:

$$\begin{array}{ccc} R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1) & \xrightarrow{\widetilde{\mathcal{R}}^M} & R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1) \\ \Big\downarrow{\scriptstyle \oplus(\kappa_A\boxtimes\widetilde{\ell^{n-r,2s}})\circ Rep_{\widetilde{M}^s}} & & \Big\downarrow{\scriptstyle \oplus(\kappa_A\boxtimes\widetilde{\ell^{n-r,2s}})\circ Rep_{\widetilde{M}^s}} \\ \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}^{ev}_{n-r}] & \xrightarrow{id\boxtimes\widetilde{\mathcal{R}}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}^{ev}_{n-r}] \end{array} \tag{2.39}$$

As for G, the map $\widetilde{\mathcal{R}}$ on $R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1)$ is an involutive isometry.

Remark 2.4.10. Let L be a standard F_0-stable and δ-stable Levi subgroup of a standard F_0-stable and δ-stable parabolic subgroup. Then L is isogenous to a direct product of groups of type A and a group of type D_m for some $1 \leqslant m \leqslant n-1$. We include the limit cases in which the type A factor or the type D factor reduce to a torus, admitting the situations in which the former is a product of groups of type A_0 or the latter is a group of type D_1.

Hence L is a standard Levi subgroup of the maximal δ-stable Levi subgroup M of type $A_{n-m-1}\times D_m$. Similarly as we did for G and M, we set

$$\widetilde{L}^0 := L^{F_0} \rtimes \langle\delta\rangle, \qquad \widetilde{L}^1 := L^{F_1} \rtimes \langle\delta\rangle$$

Since the Fourier transform is the identity on groups of type A, it commutes with the map ${}^*R^{\widetilde{M}^0}_{\widetilde{L}^0} \oplus {}^*R^{\widetilde{M}^1}_{\widetilde{L}^1}$. It follows that by transitivity of parabolic restriction, to prove compatibility of the Fourier transform with parabolic restriction from G to standard F_0 and F_1 stable Levi subgroups L, it is enough to prove compatibility of the Fourier transform with parabolic restriction from G to a maximal δ-stable Levi subgroup.

Lemma 2.4.11. *Let $r \in \mathbb{N}_{\leqslant n}$, and let Res_r be the map defined in* (2.31)*. Then the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\widetilde{\mathcal{S}}^{ev}_n] & \xrightarrow{\quad\widetilde{\mathcal{R}}\quad} & \mathbb{C}[\widetilde{\mathcal{S}}^{ev}_n] \\ \Big\downarrow{\scriptstyle Res_r} & & \Big\downarrow{\scriptstyle Res_r} \\ \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}^{ev}_{n-r}] & \xrightarrow{id\boxtimes\widetilde{\mathcal{R}}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\widetilde{\mathcal{S}}^{ev}_{n-r}] \end{array}$$

Proof. For each $[X] \in \mathcal{S}^{ev}_n$, it holds

$$\begin{aligned} Res_r\circ\widetilde{\mathcal{R}}([X]) &= \sum_{\gamma\in\mathscr{P}_r}\kappa_A(\delta_\gamma)\otimes\mathcal{H}_\gamma(\widetilde{\mathcal{R}}([X])) && \text{by Theorem 2.4.4} \\ &= \sum_{\gamma\in\mathscr{P}_r}\kappa_A(\delta_\gamma)\otimes\widetilde{\mathcal{R}}(\mathcal{H}_\gamma([X])) \\ &= (id\boxtimes\widetilde{\mathcal{R}})(\sum_{\gamma\in\mathscr{P}_r}\kappa_A(\delta_\gamma)\otimes\mathcal{H}_\gamma(X)) = (id\boxtimes\widetilde{\mathcal{R}})\circ Res_r([X]). \end{aligned}$$

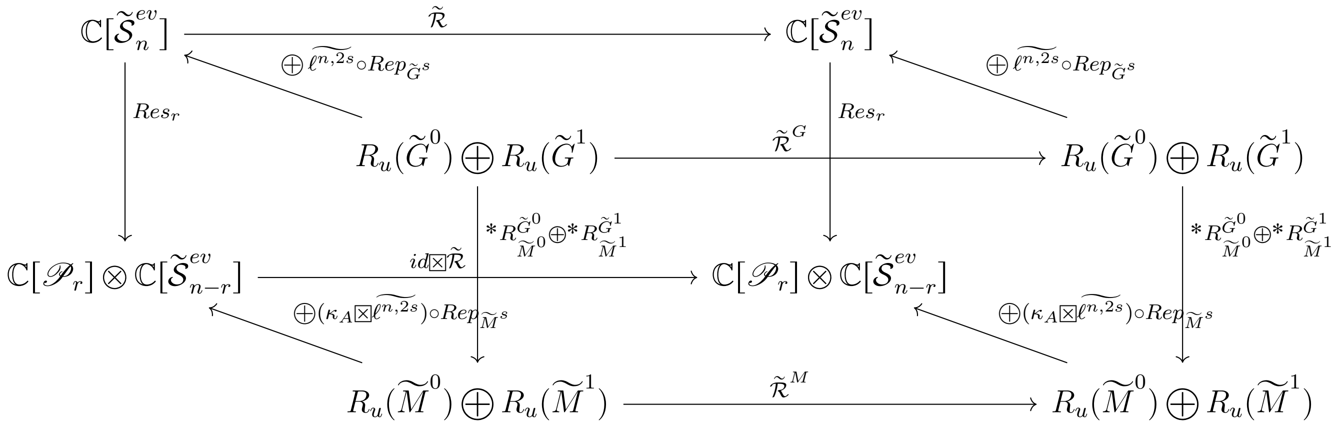
□

The following theorem (and the corollary below), stating the compatibility of the Fourier transform $\mathcal{R}$ for inner twists of disconnected groups of type D and the parabolic restriction (respectively induction) is the main result of this chapter.

Theorem 2.4.12. *We retain notation from the beginning of Section 2.4.1. The following diagram commutes:*

$$\begin{array}{ccc} R_u(\widetilde{G}^0)\bigoplus R_u(\widetilde{G}^1) & \xrightarrow{\widetilde{\mathcal{R}}^G} & R_u(\widetilde{G}^0)\bigoplus R_u(\widetilde{G}^1) \\ \Big\downarrow {}^*R^{\widetilde{G}^0}_{\widetilde{M}^0}\oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1} & & \Big\downarrow {}^*R^{\widetilde{G}^0}_{\widetilde{M}^0}\oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1} \\ R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1) & \xrightarrow{\widetilde{\mathcal{R}}^M} & R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1) \end{array} \tag{2.40}$$

Proof. We consider the following diagram.



The upper and lower faces commute by the definition of $\widetilde{\mathcal{R}}$ on the space of unipotent representations, that is, the diagrams (2.38) and (2.39). The side faces commute because of Proposition 2.3.11, the back face commutes because of Lemma 2.4.11.

The front face commutes because all the other faces commute and the diagonal arrows $\bigoplus_{s^2\leqslant n}\widetilde{\ell^{n,2s}}\circ Rep_{\widetilde{G}^s}$ and $\bigoplus_{s^2\leqslant n-r}(\kappa_A\boxtimes\widetilde{\ell^{n-r,2s}})\circ Rep_{\widetilde{M}^s}$ are isomorphisms. ☐

Corollary 2.4.13. *The following diagram commutes:*

$$\begin{array}{ccc} R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{M}^1) & \xrightarrow{\widetilde{\mathcal{R}}^M} & R_u(\widetilde{M}^0)\bigoplus R_u(\widetilde{G}^1) \\ \Big\downarrow R^{\widetilde{G}^0}_{\widetilde{M}^0}\oplus R^{\widetilde{G}^1}_{\widetilde{M}^1} & & \Big\downarrow R^{\widetilde{G}^0}_{\widetilde{M}^0}\oplus R^{\widetilde{G}^1}_{\widetilde{M}^1} \\ R_u(\widetilde{G}^0)\bigoplus R_u(\widetilde{G}^1) & \xrightarrow{\widetilde{\mathcal{R}}^G} & R_u(\widetilde{G}^0)\bigoplus R_u(\widetilde{G}^1) \end{array} \tag{2.41}$$

Proof. It follows from Theorem 2.4.12 by adjunction. Indeed, $\widetilde{\mathcal{R}}^G$ and $\widetilde{\mathcal{R}}^M$ are involutive isometries, hence are self-adjoint. ☐

2.5 Groups of type D_n and ${}^2D_n$

In this Section, we will show how our results can be applied to study connected groups of type D_n and ${}^2D_n$.

In particular, in Section 2.5.1 we will recall how to parametrize unipotent representations of groups of type D_n and ${}^2D_n$ via Symbols, relating it to the parameterization of the unipotent representations of inner twists of disconnected groups of type D_n described in Section 2.2.2.2.

In Section 2.5.2 we will show how to deduce from the map $\widetilde{\mathcal{R}}$ the Fourier transform on the space of unipotent class functions on connected groups of type D_n and ${}^2D_n$ defined by Lusztig in [42, Section 4]. In Section 2.5.3, we will exploit Theorem 2.4.12 to show compatibility of the maps defined in Section 2.5.2 with parabolic restriction and induction.

2.5.1 Unordered Symbols

Definition 2.5.1. *Let $[X] \in \widetilde{\mathcal{S}}$. We set*

$$[[X]] := \{[X], [X^{op}]\}$$

and call $[[X]]$ a (unordered) symbol.
If $X^0 = X^1$, we say that $[X]$ is degenerate and we introduce two degenerate unordered symbols denoted by $[[X]]_+$, $[[X]]_-$. We set

$$\mathcal{S} := \{[[X]] \mid [X] \in \widetilde{\mathcal{S}},\ X^0 \neq X^1\} \cup \{[[X]]_+, [[X]]_- \mid [X] \in \widetilde{\mathcal{S}},\ X^0 = X^1\},$$

and for any degenerate $[X] \in \widetilde{\mathcal{S}}$ we define $[[X]]$ to be the sum $[[X]]_+ + [[X]]_-$ in $\mathbb{C}[\mathcal{S}]$.

The rank and the defect of an unordered symbol $[[X]]$ are given respectively by

$$\rho([[X]]) := \rho([X]), \qquad def([[X]]) := |def([X])|.$$

They are well-posed since $def(X) = -def(X^{op})$ and $\rho([X]) = \rho([X]^{op})$, as one can see from the symmetry in (2.17). If $[X]$ is degenerate, we set $\rho([[X]]_\pm) := \rho([X])$ and $def([[X]]_\pm) = 0$.

We partition $\mathcal{S}$ according to the rank and the defect. For $n, d \in \mathbb{N}$, we denote by $\mathcal{S}_n$ the set of unordered symbols of rank n, and by $\mathcal{S}^d$ the set of unordered symbols of defect d. We put $\mathcal{S}_n^d := \mathcal{S}_n \cap \mathcal{S}^d$. Moreover we set

$$\mathcal{S}^{od} := \bigcup_{d \text{ odd}} \mathcal{S}^d, \qquad \mathcal{S}^{ev,0} := \bigcup_{d \equiv 0 \ mod\ 4} \mathcal{S}^d, \qquad \mathcal{S}^{ev,1} := \bigcup_{d \equiv 2 \ mod\ 4} \mathcal{S}^d,$$

and

$$\mathcal{S}_n^{od} = \mathcal{S}_n \cap \mathcal{S}^{od}, \qquad \mathcal{S}_n^{ev,0} = \mathcal{S}_n \cap \mathcal{S}^{ev,0}, \qquad \mathcal{S}_n^{ev,1} = \mathcal{S}_n \cap \mathcal{S}^{ev,1}.$$

We define the map

$$[[\ ]] : \mathbb{C}[\widetilde{\mathcal{S}}] \to \mathbb{C}[\mathcal{S}] \tag{2.42}$$

as the linear extension of the map given by the assignment $[X] \mapsto [[X]]$ for any $X \in \widetilde{\mathcal{S}}$.

For any $d \neq 0$, the restriction of $[[\ ]]$ to $\mathbb{C}[\widetilde{\mathcal{S}}^d]$ defines an isomorphism between $\mathbb{C}[\widetilde{\mathcal{S}}^d]$ and $\mathbb{C}[\mathcal{S}^{|d|}]$.

For $d = 0$ we denote by $\mathbb{C}[\mathcal{S}^0]^{nd}$ the image of $\mathbb{C}[\widetilde{\mathcal{S}}^0]$ in $\mathbb{C}[\mathcal{S}^0]$ through the map $[[\ ]]$, that is

$$\mathbb{C}[\mathcal{S}^0]^{nd} = span_{\mathbb{C}}\{[[X]] \mid [X] \in \widetilde{\mathcal{S}}^0\}.$$

We set

$$\mathbb{C}[\mathcal{S}]^C := span_{\mathbb{C}}\{[[X]]_+ - [[X]]_- \mid [X] \in \widetilde{\mathcal{S}}^0 \text{ degenerate } \},$$

so $\mathbb{C}[\mathcal{S}]^C \oplus \mathbb{C}[\mathcal{S}^0]^{nd} = \mathbb{C}[\mathcal{S}^0]$.

Moreover, we denote by $\mathbb{C}[\mathcal{S}]^{nd}$ the image of $\mathbb{C}[\widetilde{\mathcal{S}}]$ in $\mathbb{C}[\mathcal{S}]$ through the map $[[\ ]]$, that is, the subspace $\mathbb{C}[\mathcal{S}^0]^{nd} \oplus \left(\bigoplus_{d\geqslant 1} \mathbb{C}[\mathcal{S}^d]\right)$. Then we have

$$\mathbb{C}[\mathcal{S}] = \mathbb{C}[\mathcal{S}]^{nd} \oplus \mathbb{C}[\mathcal{S}^0]^C,$$

and

$$\mathbb{C}[\mathcal{S}]^{nd} \cong \mathbb{C}[\widetilde{S}] \Big/ span_{\mathbb{C}}\{[X] - [X^{op}] \mid [X] \in \widetilde{\mathcal{S}} \text{ non degenerate}\}.$$

Recall that $\mathbb{C}[\mathcal{S}]$ is endowed with the scalar product that makes $\mathcal{S}$ into an orthonormal basis. Then $\mathbb{C}[\mathcal{S}]^{nd}$ and $\mathbb{C}[\mathcal{S}^0]^C$ are orthogonal complements in $\mathbb{C}[\mathcal{S}]$.

We define the quotient space

$$\mathcal{A}^{ev} := \mathbb{C}[\widetilde{\mathcal{S}}^{ev}] \Big/ span_{\mathbb{C}}\{[X] + [X^{op}] \mid [X] \in \widetilde{\mathcal{S}}^{ev,0}\}$$

and denote by $\langle\!\langle\ \rangle\!\rangle$ the natural projection

$$\langle\!\langle\ \rangle\!\rangle : \mathbb{C}[\widetilde{\mathcal{S}}^{ev}] \to \mathcal{A}^{ev}. \tag{2.43}$$

We set

$$\mathcal{A}^{ev,0} = \langle\!\langle \mathbb{C}[\widetilde{\mathcal{S}}^{ev,0}] \rangle\!\rangle \cong \mathbb{C}[\widetilde{\mathcal{S}}^{ev,0}] \Big/ span_{\mathbb{C}}\{[X] + [X^{op}] \mid [X] \in \widetilde{\mathcal{S}}^{ev,0}\}, \tag{2.44}$$

$$\mathcal{A}^{ev,1} = \langle\!\langle \mathbb{C}[\widetilde{\mathcal{S}}^{ev,0}] \rangle\!\rangle \cong \mathbb{C}[\widetilde{\mathcal{S}}^{ev,1}] \Big/ span_{\mathbb{C}}\{[X] + [X^{op}] \mid [X] \in \widetilde{\mathcal{S}}^{ev,1}\}.$$

For any $n, d \in \mathbb{N}$, denote by $\mathcal{A}_n$ the subspace of $\mathcal{A}^{ev}$ that is the image of $\mathbb{C}[\widetilde{\mathcal{S}}_n]$ through the map $\langle\!\langle\ \rangle\!\rangle$, and denote by $\mathcal{A}^d$ the subspace of $\mathcal{A}^{ev}$ that is the image of $\mathbb{C}[\widetilde{\mathcal{S}}^d] \bigsqcup \mathbb{C}[\widetilde{\mathcal{S}}^{-d}]$ through the map $\langle\!\langle\ \rangle\!\rangle$. Moreover we set $\mathcal{A}_n^d := \mathcal{A}_n \cap \mathcal{A}^d$, that is the subspace of $\mathcal{A}^{ev}$ that is the image of the restriction of $\mathbb{C}[\widetilde{\mathcal{S}}_n^d] \bigsqcup \mathbb{C}[\widetilde{\mathcal{S}}_n^{-d}]$ through the map $\langle\!\langle\ \rangle\!\rangle$. We set

$$\mathcal{A}_n^{ev,0} := \bigoplus_{d\equiv 0\ mod 4} \mathcal{A}_n^d, \qquad \mathcal{A}_n^{ev,1} := \bigoplus_{d\equiv 2\ mod 4} \mathcal{A}_n^d.$$

They are respectively the images of $\mathbb{C}[\widetilde{\mathcal{S}}_n^{ev,0}]$ and $\mathbb{C}[\widetilde{\mathcal{S}}_n^{ev,1}]$ through the map $\langle\!\langle\ \rangle\!\rangle$.

The space $\mathbb{C}[\mathcal{A}^{ev}]$ is endowed with a scalar product defined by

$$\langle \langle\!\langle X \rangle\!\rangle\ ,\ \langle\!\langle Y \rangle\!\rangle \rangle = \delta_{[X],[Y]} - \delta_{[X],[Y]^{op}}$$

for any $[X], [Y] \in \widetilde{\mathcal{S}}^{ev}$.

Proposition 2.5.2. *The map*

$$\begin{aligned} \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] &\xrightarrow{([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)} \mathbb{C}[\mathcal{S}_n^{ev}]^{nd} \oplus \mathcal{A}_n^{ev} \\ [X] &\mapsto ([[X]], \langle\!\langle X \rangle\!\rangle) \end{aligned}$$

is an isomorphism. Moreover, if we endow the space on the right-hand side with orthogonal sum euclidean structure, the map $([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)$ *is a* 2*-dilation.*

Proof. The map $([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)$ is surjective. Indee,d the set $\{[[X]], \langle\!\langle X\rangle\!\rangle |\ [X] \in \widetilde{\mathcal{S}}_n^{ev}\}$ is a generating set for $\mathbb{C}[\mathcal{S}_n^{ev}]^{nd} \oplus \mathcal{A}_n^{ev}$, and for any $[X] \in \widetilde{\mathcal{S}}_n^{ev}$ it holds

$$([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)\left(\frac{[X] + [X^{op}]}{2}\right) = [[X]]$$
$$([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)\left(\frac{[X] - [X^{op}]}{2}\right) = \langle\!\langle X\rangle\!\rangle.$$

The map $([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)$ is injective since

$$\begin{aligned} Ker([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle) &= Ker(([[\ ]]) \cap Ker(\langle\!\langle\ \rangle\!\rangle) \\ &= span_{\mathbb{C}}\{[X] - [X^{op}] | [X] \in \widetilde{\mathcal{S}}_n^{ev}\} \cap span_{\mathbb{C}}\{[X] + [X^{op}] | [X] \in \widetilde{\mathcal{S}}_n^{ev}\}, \end{aligned}$$

and for any $[X], [Y] \in \widetilde{\mathcal{S}}_n^{ev}$ it holds

$$\langle [X] - [X^{op}], [Y] + [Y^{op}] \rangle = \delta_{[X],[Y]} + \delta_{[X],[Y^{op}]} - \delta_{[X^{op}],[Y]} - \delta_{[X^{op}],[Y^{op}]} = 0$$

since $\delta_{[X],[Y]} = \delta_{[X^{op}],[Y^{op}]}$ and $\delta_{[X],[Y^{op}]} = \delta_{[X^{op},[Y]}$. Therefore $Ker([[\ ]])$ and $Ker(\langle\!\langle\ \rangle\!\rangle)$ are orthogonal, and consequently have trivial intersection, giving injectivity.

Now we prove that the map is a 2-dilation. Let $[X], [Y] \in \widetilde{\mathcal{S}}_n^{ev}$. Then

$$\begin{aligned} \langle ([[X]], \langle\!\langle X\rangle\!\rangle), ([[Y]], \langle\!\langle Y\rangle\!\rangle) \rangle &= \langle [[X]], [[Y]] \rangle + \langle \langle\!\langle X\rangle\!\rangle, \langle\!\langle Y\rangle\!\rangle \rangle \\ &= \delta_{[X],[Y]} + \delta_{[X],[Y]^{op}} + \delta_{[X],[Y]} - \delta_{[X],[Y]^{op}} \\ &= 2\delta_{[X],[Y]} = 2\langle [X], [Y] \rangle. \end{aligned}$$

□

2.5.1.1 Parameterization of Unipotent representations of groups of type D_n and ${}^2D_n$

Let $s, n \in \mathbb{N}$ be such that $s^2 \leqslant n$. The linear extensions of the bijections $\widetilde{\ell^{n,2s}}$ as in Remark 2.2.11 are isomorphisms

$$\widetilde{\ell^{n,2s}} : R(W(B_{n-s^2}) \times \langle\delta\rangle) \to \mathbb{C}[\widetilde{\mathcal{S}}_n^{2s}] \oplus \mathbb{C}[\widetilde{\mathcal{S}}_n^{-2s}] \tag{2.45}$$
$$\widetilde{\ell^{n,0}} : R(W(D_n) \rtimes \langle\delta\rangle) \to \mathbb{C}[\widetilde{\mathcal{S}}_n^{0}] \tag{2.46}$$

where δ has order 2 and acts trivially for $s > 0$, while it acts as the automorphism of $W(D_n)$ corresponding to a graph automorphism for $s = 0$. Retaining notation from Section 1.2.4, for $m \in \mathbb{N}$ and $i = 0, 1$ we denote by $Cl(W(B_m)\delta^i)$, respectively $Cl(W(D_m)\delta^i)$, the space of complex functions on $W(B_m)\delta^i$ that are constant on $W(B_m) \times \langle\delta\rangle$-conjugacy classes, respectively on $W(D_m) \rtimes \langle\delta\rangle$-conjugacy classes.

We observe that $Cl(W(B_m)) = R(W(B_m))$, because δ acts trivially on $W(B_m)$, while $Cl(W(D_m))$ is the subspace of $R(W(D_m))$ generated by the irreducible characters of $W(D_m)$ that are stable by the action of δ, i.e., the characters of $Irr(W(D_m))$ labelled by non degenerate unordered bipartitions via κ_D. We set

$$R(W(D_m))^C := span_{\mathbb{C}}\{\kappa_D^{-1}([(\alpha,\alpha)]_+) - \kappa_D^{-1}([(\alpha,\alpha)]_-) \mid [(\alpha,\alpha)]_\pm \in \mathscr{B}_m \text{ degenerate}\}$$

so that

$$R(W(D_m)) = Cl(W(D_m)) \oplus R(W(D_m))^C.$$

The following proposition shows how to deduce a parameterization via symbols for the spaces $Cl(W(B_m)\delta^i)$ and $Cl(W(D_m)\delta^i)$, for $i = 0, 1$, that is compatible with the parameterization for $R(W(B_m) \times \langle\delta\rangle)$ and $R(W(D_m) \rtimes \langle\delta\rangle)$ in (2.45).

Proposition 2.5.3. *Let $s, n \in \mathbb{N}$ be such that $s^2 \leqslant n$. Let δ be as in (2.45). Then there exist unique linear isomorphisms*

$$\begin{aligned}
\ell^{n,2s} &: Cl(W(B_{n-s^2})) \to \mathbb{C}[\mathcal{S}_n^{2s}] && \textit{for } s > 0\\
\ell_\delta^{n,2s} &: Cl(W(B_{n-s^2})\delta) \to \mathcal{A}_n^{2s} && \textit{for } s > 0\\
\ell^{n,0} &: Cl(W(D_n)) \to \mathbb{C}[\mathcal{S}_n^0]^{nd} && \textit{for } s = 0\\
\ell_\delta^{n,0} &: Cl(W(D_n)\delta) \to \mathcal{A}_n^0 && \textit{for } s = 0
\end{aligned}$$

so that the following diagrams commute

$$\begin{array}{ccc}
R(W(B_{n-s^2} \times \mathbb{Z}_2)) & \xrightarrow{\pi_{W(B_{n-s^2}),0} \bigoplus \pi_{W(B_{n-s^2}),1}} & Cl(W(B_{n-s^2})) \bigoplus Cl(W(B_{n-s^2})\delta) \\
\Big\downarrow \widetilde{\ell^{n,2s}} & & \Big\downarrow \ell^{n,2s} \oplus \ell_\delta^{n,2s} \\
\mathbb{C}[\widetilde{\mathcal{S}}_n^{2s}] \bigoplus \mathbb{C}[\widetilde{\mathcal{S}}_n^{-2s}] & \xrightarrow{[[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle} & \mathbb{C}[\mathcal{S}_n^{2s}] \bigoplus \mathcal{A}_n^{2s}
\end{array} \tag{2.47}$$

$$\begin{array}{ccc}
R(W(D_n \times \mathbb{Z}_2)) & \xrightarrow{\pi_{W(D_{n,0}),0} \bigoplus \pi_{W(D_n),1}} & Cl(W(D_n)) \bigoplus Cl(W(D_n)\delta) \\
\Big\downarrow \widetilde{\ell^{n,0}} & & \Big\downarrow \ell^{n,0} \oplus \ell_\delta^{n,0} \\
\mathbb{C}[\widetilde{\mathcal{S}}_n^{0}] & \xrightarrow{[[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle} & \mathbb{C}[\mathcal{S}_n^{0}]^{nd} \bigoplus \mathcal{A}_n^{0}
\end{array} \tag{2.48}$$

Proof. Let $s > 0$. We set

$$\begin{aligned}
\ell^{n,2s} : Cl(W(B_{n-s^2})) &\to \mathbb{C}[\mathcal{S}_n^{2s}] \qquad (2.49)\\
\pi_{W(B_{n-s^2}),0}(f) &\mapsto [[\widetilde{\ell^{n,2s}}(f)]]\\
\ell_\delta^{n,2s} : Cl(W(B_{n-s^2})\delta) &\to \mathcal{A}_n^{2s} \qquad (2.50)\\
\pi_{W(B_{n-s^2}),1}(f) &\mapsto \langle\!\langle \widetilde{\ell^{n,2s}}(f) \rangle\!\rangle,
\end{aligned}$$

where $f \in R(W(B_{n-s^2}) \times \mathbb{Z}_2)$.

Since the restriction maps $\pi_{W(B_{n-s^2}),i}$ for $i = 0, 1$ are surjective, in order to prove that the maps $\ell^{n,2s}$ and $\ell_\delta^{n,2s}$ are well-defined we need to show that

$$Ker([[\ ]] \circ \widetilde{\ell^{n,2s}}) \supseteq Ker(\pi_{W(B_{n-s^2}),0}), \quad Ker(\langle\!\langle\ \rangle\!\rangle \circ \widetilde{\ell^{n,2s}}) \supseteq Ker(\pi_{W(B_{n-s^2}),1}). \quad (2.51)$$

We show the stronger assertion that

$$Ker([[\ ]] \circ \widetilde{\ell^{n,2s}}) = Ker(\pi_{W(B_{n-s^2}),0}), \quad Ker(\langle\!\langle\ \rangle\!\rangle \circ \widetilde{\ell^{n,2s}}) = Ker(\pi_{W(B_{n-s^2}),1}), \quad (2.52)$$

so the maps $\ell^{n,2s}$ and $\ell_\delta^{n,2s}$ are not only well-defined, but they are also injective. We have

$$\begin{aligned} Ker([[\ ]] \circ \widetilde{\ell^{n,2s}}) &= (\widetilde{\ell^{n,2s}})^{-1}(span_{\mathbb{C}}\{[X] - [X]^{op} \mid X \in \widetilde{S}_n^{2s}\}) \\ &= (\kappa_B \times ev_{-1})^{-1}(span_{\mathbb{C}}\{((\alpha, \beta), 1) - ((\alpha, \beta), -1) \mid (\alpha, \beta) \in \mathscr{B}_{n-s^2}\}) \\ &= span_{\mathbb{C}}\{\rho \boxtimes 1 - \rho \boxtimes sign \mid \rho \in Irr(W(B_{n-s^2}))\} \\ &= Ker(Res_{W(B_{n-s^2})}^{W(B_{n-s^2}) \times \langle\delta\rangle}) = Ker(\pi_{W(B_{n-s^2}),0}) \end{aligned}$$

and similarly

$$\begin{aligned} Ker(\langle\!\langle\ \rangle\!\rangle \circ \widetilde{\ell^{n,2s}}) &= (\widetilde{\ell^{n,2s}})^{-1}(span_{\mathbb{C}}\{[X] + [X]^{op} \mid X \in \widetilde{S}_n^{2s}\}) \\ &= (\kappa_B \times ev_{-1})^{-1}(span_{\mathbb{C}}\{((\alpha, \beta), 1) + ((\alpha, \beta), -1) \mid (\alpha, \beta) \in \mathscr{B}_{n-s^2}\}) \\ &= span_{\mathbb{C}}\{\rho \boxtimes 1 + \rho \boxtimes sign \mid \rho \in Irr(W(B_{n-s^2}))\} = Ker(\pi_{W(B_{n-s^2}),1}). \end{aligned}$$

The map $\ell^{n,2s} \oplus \ell_\delta^{n,2s}$ makes the diagram (2.47) commute by definition of $\ell^{n,2s}$ and $\ell_\delta^{n,2s}$ in (2.49). Moreover they are isomorphisms, since they are injective by (2.52) and they are surjective because $\widetilde{\ell}^{n,2s}$ is an isomorphism and the maps $[[\ ]]$ and $\langle\!\langle\ \rangle\!\rangle$ are surjective. This proves the statement for (2.47).

We turn to diagram (2.47). Let $s = 0$. We set

$$\begin{aligned} \ell^{n,0} : Cl(W(D_n)) &\to \mathbb{C}[\mathcal{S}_n^0]^{nd} & (2.53) \\ \pi_{W(D_n),0}(f) &\mapsto [[\widetilde{\ell^{n,2s}}(f)]] \\ \ell_\delta^{n,0} : Cl(W(D_n)\delta) &\to \mathcal{A}_n^0 & (2.54) \\ \pi_{W(D_n),1}(f) &\mapsto \langle\!\langle \widetilde{\ell^{n,0}}(f) \rangle\!\rangle, \end{aligned}$$

with $f \in R(W(B_n))$. Similarly to the case $s > 0$, in order to prove that the maps $\ell^{n,0}$ and $\ell_\delta^{n,0}$ are well-defined and injective we show that

$$Ker([[\ ]] \circ \widetilde{\ell^{n,0}}) = Ker(\pi_{W(D_n),0}), \qquad Ker(\langle\!\langle\ \rangle\!\rangle \circ \widetilde{\ell^{n,0}}) = Ker(\pi_{W(D_n),1}). \qquad (2.55)$$

By (2.5), for any $\rho \in Irr(W(B_n))$ it holds $[[\kappa_B(\rho)]] = \kappa_D \circ Res_{W(D_n)}^{W(B_n)}(\rho)$.

It follows that $Ker(Res_{W(D_n)}^{W(B_n)}) = \kappa_B^{-1}(span_{\mathbb{C}}\{(\alpha, \beta) - (\beta, \alpha) \mid (\alpha, \beta) \in \mathscr{B}_n\})$. Moreover Proposition (2.3.8) gives $\widetilde{\mathcal{S}}_n^0(\alpha, \beta) = (\widetilde{\mathcal{S}}_n^0(\beta, \alpha))^{op}$.

Then

$$\begin{aligned}
Ker([[\ ]] \circ \widetilde{\ell^{n,0}}) &= (\widetilde{\ell^{n,0}})^{-1}(span_{\mathbb{C}}\{[X] - [X]^{op} \mid X \in \widetilde{S}^0_n\}) \\
&= \kappa_B^{-1}(span_{\mathbb{C}}\{(\alpha,\beta) - (\beta,\alpha) \mid (\alpha,\beta) \in \mathscr{B}_n\}) \\
&= Ker(Res^{W(D_n)\rtimes\langle\delta\rangle}_{W(D_n)}) = Ker(\pi_{W(D_n),0}).
\end{aligned}$$

Let $\rho \in Irr(W(B_n))$. By (2.5), since $\kappa_B(\rho)$ and $\kappa_B(\rho \otimes sign)$ have the same restriction to $W(D_n)$, if $\kappa_B(\rho) = (\alpha,\beta)$ then $\kappa_B(\rho \otimes sign) = (\beta,\alpha)$, where $sign$ denotes the inflation of the $sign$ character of $\langle\delta\rangle$. It follows that

$$\widetilde{\ell^{n,0}}(\rho \otimes sign) = (\widetilde{\ell^{n,0}}(\rho))^{op}. \tag{2.56}$$

Equation (2.56) proves that $\widetilde{\ell^{n,0}}(Ker(\pi_{W(D_m),1})) = Ker(\langle\!\langle\ \rangle\!\rangle)$, and equality follows from the fact that $\widetilde{\ell^{n,0}}$ is a bijection between $Irr(W(B_n))$ and $\widetilde{\mathcal{S}}^0_n$. Then

$$\begin{aligned}
Ker(\langle\!\langle\ \rangle\!\rangle \circ \widetilde{\ell^{n,0}}) &= (\widetilde{\ell^{n,0}})^{-1}(span_{\mathbb{C}}\{[X] + [X]^{op} \mid X \in \widetilde{S}^0_n\}) \\
&= \kappa_B^{-1}(span_{\mathbb{C}}\{(\alpha,\beta) + (\beta,\alpha) \mid (\alpha,\beta) \in \mathscr{B}_n\}) \\
&= span_{\mathbb{C}}\{\rho \otimes 1 + \rho \otimes sign \mid \rho \in Irr(W(B_n))\} = Ker(\pi_{W(D_n),1}).
\end{aligned}$$

The maps $\ell^{n,0}$ and $\ell^{n,0}_\delta$ are isomorphisms and the map $\ell^{n,0} \oplus \ell^{n,0}_\delta$ makes the diagram (2.48) commute by arguments analogue to the ones used for the case $s > 0$.

□

Remark 2.5.4. For any $n, s \in \mathbb{Z}$ with $s^2 \leqslant n$, the maps $\ell^{n,2s}$ and $\ell^{n,2s}_\delta$ are isometries. Indeed the isomorphism $\widetilde{\ell^{n,2s}}$ is an isometry, and the maps $\pi_{W(B_{n-s^2}),0}$ (respectively $\pi_{W(D_n),0}$ for $s = 0$) and $[[\ ]]$ are orthogonal projections along subspaces that by (2.52) (respectively (2.55)) correspond to each other through $\widetilde{\ell^{n,2s}}$. Hence the map $\ell^{n,2s}$ is an isometry.

Similarly, the maps $\pi_{W(B_{n-s^2}),1}$ (respectively $\pi_{W(D_n),1}$ for $s = 0$) and $\langle\!\langle\ \rangle\!\rangle$ are orthogonal projections along subspaces that by (2.52) (respectively (2.55) correspond to each other through $\widetilde{\ell^{n,2s}}$. Hence, the map $\ell^{n,2s}_\delta$ is an isometry as well.

We extend the map

$$\ell^{n,0} : Cl(W(D_n)) \to \mathbb{C}[\mathcal{S}^0_n]^{nd}$$

to the whole $R(W(D_n))$ by setting

$$\ell^{n,0}(\kappa_D^{-1}([(\alpha,\alpha)]_+) - \kappa_D^{-1}([(\alpha,\alpha)]_-)) := [[\widetilde{\mathscr{S}}_{n,0}(\alpha,\alpha)]]_+ - [[\widetilde{\mathscr{S}}_{n,0}(\alpha,\alpha)]]_-$$

for any $(\alpha,\alpha) \in \mathscr{B}_n$. This map is still an isometry, since with this definition $\ell^{n,0}_D$ maps $R(W(D_n))^C$ isometrically in the orthogonal complement $\mathbb{C}[\mathcal{S}]^C$ of $\mathbb{C}[\mathcal{S}]^{nd}$.

We retain notation from Section 2.1.2.3, so that G is a simple group of type D_n and $\widetilde{G}^i$ for $i = 0, 1$ denote respectively the split and non-split inner twist of $G \rtimes \langle\delta\rangle$ as in (2.6). We denote by $Cl_u(G^i\delta)$ (respectively $Cl_u(G^i)$) the image of $R_u(\widetilde{G}^i)$ in $Cl(G^{F_i}\delta)$ (respectively $Cl(G^{F_i})$) through $\pi_{G^{F_i},1}$ (respectively $\pi_{G^{F_i},0}$), with notation as in Section 1.2.4.

For any $s \in \mathbb{N}$ with $s^2 \leqslant n$, composing the isomorphisms $\ell^{n,2s}$ (respectively $\ell_\delta^{n,2s}$) with $Rep_{G^{F_s}}$ (respectively $Rep_{G^{F_s}\delta}$) we obtain isomorphisms

$$Rep_{G^{F_s}} \circ \ell^{n,2s} : \mathbb{C}[\mathcal{E}^{G^{F_s}}(L_s, \sigma_s)] \to \mathbb{C}[\mathcal{S}_n^{2s}]$$
$$Rep_{G^{F_s}\delta} \circ \ell_\delta^{n,2s} : \mathbb{C}[\mathcal{E}^{G^{F_s}\delta}(L_s, \sigma_s)] \to \mathcal{A}_n^{2s}.$$

Collecting together all the series for $s \equiv 0 \ mod\ 2$, we get the isomorphisms

$$Symb_D^0 := \bigoplus_{s \equiv 0 \ mod 2} Rep_{G^{F_s}} \circ \ell^{n,2s} : R_u(G^{F_0}) \to \mathbb{C}[\mathcal{S}_n^{ev,0}]$$
$$Symb_D^{0,\delta} := \bigoplus_{s \equiv 0 \ mod 2} Rep_{G^{F_s}\delta} \circ \ell_\delta^{n,2s} : Cl_u(G^{F_0}\delta) \to \mathcal{A}_n^{ev,0}$$

Collecting together all the series for $s \equiv 1 \ mod\ 2$ we get the isomorphisms

$$Symb_D^1 := \bigoplus_{s \equiv 1 \ mod 2} Rep_{G^{F_s}} \circ \ell^{n,2s} : R_u(G^{F_1}) \to \mathbb{C}[\mathcal{S}_n^{ev,1}]$$
$$Symb_D^{1,\delta} := \bigoplus_{s \equiv 1 \ mod 2} Rep_{G^{F_s}\delta} \circ \ell_\delta^{n,2s} : Cl_u(G^{F_1}\delta) \to \mathcal{A}_n^{ev,1}$$

Let
$$\tau : Cl_u(G^{F_0}\delta) \oplus Cl_u(G^{F_1}) \to Cl_u(G^{F_1}) \oplus Cl_u(G^{F_0}\delta)$$
be the swap isomorphism.

Proposition 2.5.5. *The following diagram commutes*

$$\begin{array}{ccc} R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) & \xrightarrow{\pi_{G^{F_0},0} \oplus \tau \circ (\pi_{G^{F_0},1} \oplus \pi_{G^{F_1},0}) \oplus \pi_{G_1^{F_1}}} & Cl_u(G^{F_0}) \oplus Cl_u(G^{F_1}) \oplus Cl_u(G^{F_0}\delta) \oplus Cl_u(G^{F_1}\delta) \\ \downarrow \widetilde{Symb}_D & & \downarrow Symb_D^0 \oplus Symb_D^1 \oplus Symb_D^{0,\delta} \oplus Symb_D^{1,\delta} \\ \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] & \xrightarrow{([[\ ]] \oplus \langle\langle\ \rangle\rangle)} & \mathbb{C}[\mathcal{S}_n^{ev,0}]^{nd} \oplus \mathbb{C}[\mathcal{S}_n^{ev,1}] \oplus \mathcal{A}_n^{ev,0} \oplus \mathcal{A}_n^{ev,1} \end{array} \tag{2.57}$$

Proof. For any $s \in \mathbb{N}$ with $s^2 \leqslant n$, the restriction of the diagram to $\mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)]$ factors as

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{\widetilde{G}^s}(L_s, \sigma_s)] & \xrightarrow{\pi_{G^{F_s},0} \oplus \pi_{G^{F_s},1}} & \mathbb{C}[\mathcal{E}^{G^{F_s}}(L_s, \sigma_s)] \oplus \mathbb{C}[\mathcal{E}^{G^{F_s}\delta}(L_s, \sigma_s)] \\ \downarrow Rep_{\widetilde{G}^s} & & \downarrow Rep_{G^{F_s}} \oplus Rep_{G^{F_s}\delta} \\ Cl(W_{\widetilde{G}^s}(L_s, \sigma_s)) & \xrightarrow{\pi_{W_{G^{F_s}}(L_s,\sigma_s),0} \oplus \pi_{W_{G^{F_s}}(L_s,\sigma_s),1}} & Cl(W_{G^{F_s}}(L_s, \sigma_s)) \oplus Cl(W_{G^{F_s}}(L_s, \sigma_s)\delta) \\ \downarrow \widetilde{\ell^{n,2s}} & & \downarrow \ell^{n,2s} \oplus \ell_\delta^{n,2s} \\ \bigoplus_{s' \in \{\pm s\}} \mathbb{C}[\widetilde{\mathcal{S}}_n^{2s'}] & \xrightarrow{[[\ ]] \oplus \langle\langle\ \rangle\rangle} & \mathbb{C}[\mathcal{S}_n^{2s}]^{nd} \oplus \mathcal{A}_n^{2s,0} \end{array} \tag{2.58}$$

The upper square commutes by Remark 1.2.25 and Proposition 1.2.37. The lower square commutes by Proposition 2.5.3 □

2.5.2 Fourier transform

In this section, we show, following [40], how the Fourier transform for groups of type D_n, which was defined in [42, 4.6, 4.15], and the Fourier transform for groups of type and ${}^2D_n$, which was defined in [42, 4.18], are related to the map $\widetilde{\mathcal{R}}$. See Remark 2.5.10 for the precise statement.

For any $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$, let

$$d(X) := \frac{def(X) - def(X_{sp})}{2} \tag{2.59}$$

and, for $i = 0, 1$, let

$$Sim_i(X) := \{Y \in Sim(X) \mid d(Y) \equiv i \ mod \ 2\}. \tag{2.60}$$

The assignment $Y \mapsto Y^{\#}$ defines a bijection between $Sim_i(X)$ and the subsets of $X^{\ominus}$ of cardinality equivalent to i mod 2.

The map $\widetilde{\mathcal{R}}$ can be decomposed as $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_0 + \widetilde{\mathcal{R}}_1$, where $\widetilde{\mathcal{R}}_0, \widetilde{\mathcal{R}}_1$ are endomorphisms of $\mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}]$ defined as follows:

$$\widetilde{\mathcal{R}}_0(X) := \frac{1}{s(X)} \sum_{Y \in Sim_0(X)} (-1)^{|X^{\#} \cap Y^{\#}|} Y, \tag{2.61}$$

$$\widetilde{\mathcal{R}}_1(X) := \frac{1}{s(X)} \sum_{Y \in Sim_1(X)} (-1)^{|X^{\#} \cap Y^{\#}|} Y \tag{2.62}$$

The bijection given by the shift by k

$$\begin{aligned} (\cdot)^{\to k} : Sim(X) &\to Sim(X^{\to k}) \\ Y &\mapsto Y^{\to k} \end{aligned}$$

maps $Sim_i(X)$ to $Sim_i(X^{\to k})$ for $i = 1, 0$. Therefore The maps $\widetilde{\mathcal{R}}_0, \widetilde{\mathcal{R}}_1$ preserve the kernel of the projection

$$[\cdot] : \mathbb{C}[\mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}] \to \mathbb{C}[\widetilde{\mathcal{S}}]$$

defined by $X \mapsto [X]$ for any $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$. As a consequence, the maps $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ induce well-defined endomorphisms of the vector space $\mathbb{C}[\widetilde{\mathcal{S}}]$, still denoted by $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$, given by

$$\widetilde{\mathcal{R}}_0([X]) = [\widetilde{\mathcal{R}}_0(X)], \qquad \widetilde{\mathcal{R}}_1([X]) = [\widetilde{\mathcal{R}}_1(X)].$$

Remark 2.5.6. In [53] the authors define a map $\mathcal{F}$ on $\mathbb{C}[\mathcal{S}^{ev}]$ that is similar to our map $\widetilde{\mathcal{R}}$, that is, the map $\mathcal{F}$ is the combinatorial description of an extension of the Fourier transform defined in [42, Secton 4] for unipotent representations of a special orthogonal group to the space of unipotent representations of an inner form of the relative orthogonal group. The relation between $\mathcal{F}$ and $\widetilde{\mathcal{R}}$ is given by

$$\mathcal{F}([X]) = \widetilde{\mathcal{R}}_0([X]) + (-1)^{d([X])} \widetilde{\mathcal{R}}_1([X]).$$

Lemma 2.5.7. *Let* $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ *and let* $Y \in Sim(X)$*. Then*

$$|X^\# \cap (Y^{op})^\#| \equiv |X^\# \cap Y^\#| + d(X) \quad (mod\ 2). \tag{2.63}$$

Proof. Since $(\cdot)^{op}$ just interchanges the rows of an array, for any $Y \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$ it holds $Sim(Y) = Sim(Y^{op})$, and so $Y_{sp} = (Y^{op})_{sp}$ and $Y^{\ominus} = (Y^{op})^{\ominus}$. Moreover $(Y^{op})^1 \cap Y^{\ominus} = Y^0 \cap Y^{\ominus} = Y^{\ominus} \backslash (Y^{\ominus} \cap Y^1)$, so $(Y^{op})^\# = Y^{\ominus} \backslash Y^\#$. It follows that

$$|X^\# \cap (Y^{op})^\#| + |X^\# \cap Y^\#| = |X^\#|. \tag{2.64}$$

The following computation shows that $|X^\#| \equiv d(X)$ (mod 2):

$$\begin{aligned} |X^\#| &= |X^1_{sp} \ominus X^1| = |X^1_{sp} \cup X^1| - |X^1_{sp} \cap X^1| \\ &= |X^1_{sp}| + |X^1| - 2|X^1_{sp} \cap X^1| = 2|X^1_{sp}| + \frac{def(X) - def(X_{sp})}{2} - 2|X^1_{sp} \cap X^1|. \end{aligned}$$

□

Proposition 2.5.8. *Let* $X \in \mathscr{P}(\mathbb{N})^{<\infty} \times \mathscr{P}(\mathbb{N})^{<\infty}$*. Then for* $i = 0, 1$ *it holds*

$$\begin{aligned} \widetilde{\mathcal{R}}_i(X^{op}) &= (-1)^i \widetilde{\mathcal{R}}_i(X) \\ \widetilde{\mathcal{R}}_i(X)^{op} &= (-1)^{d(X)} \widetilde{\mathcal{R}}_{i+def(X)}(X) \end{aligned}$$

where $i + def(X)$ *denotes the residue mod 2.*

Proof. By definition of $\widetilde{\mathcal{R}}_i$, it holds

$$\widetilde{\mathcal{R}}_i(X^{op}) = \frac{1}{s(X^{op})} \sum_{Y \in Sim_i(X^{op})} (-1)^{|(X^{op})^\# \cap Y^\#|} Y.$$

We observe that $Sim_i(X) = Sim_i(X^{op})$ for $i = 0, 1$ and moreover

$$s(X^{op}) = 2^{\frac{|(X^{op})^{\ominus}| - def((X^{op})_{sp})}{2}} = 2^{\frac{|X^{\ominus}| - def(X_{sp})}{2}} = s(X).$$

So

$$\widetilde{\mathcal{R}}_i(X^{op}) = \frac{1}{s(X)} \sum_{Y \in Sim_i(X)} (-1)^{|(X^{op})^\# \cap Y^\#|} Y.$$

Lemma 2.5.7 gives $(-1)^{|(X^{op})^\# \cap Y^\#|} = (-1)^{|X^\# \cap Y^\#| + d(Y)}$, therefore

$$\begin{aligned} \widetilde{\mathcal{R}}_i(X^{op}) &= \frac{1}{s(X)} \sum_{Y \in Sim_i(X)} (-1)^{|X^\# \cap Y^\#| + i} Y \\ &= (-1)^i \frac{1}{s(X)} \sum_{Y \in Sim_i(X)} (-1)^{|X^\# \cap Y^\#|} Y = (-1)^i \widetilde{\mathcal{R}}_i(X), \end{aligned}$$

giving the first equality.

For the second one, we compute

$$\widetilde{\mathcal{R}}_i(X)^{op} = \frac{1}{s(X)} \sum_{Y \in Sim_i(X)} (-1)^{|X^\# \cap Y^\#|} Y^{op} = \frac{1}{s(X)} \sum_{Y^{op} \in (Sim_i(X))^{op}} (-1)^{|(X)^\# \cap Y^\#|} Y^{op}.$$

Since $d(Y^{op}) \equiv d(Y) + def(Y)\ (mod\ 2)$, we obtain $(Sim_i(X))^{op} = Sim_{i+def(X)}(X)$. Lemma 2.5.7 gives $(-1)^{|X^\# \cap Y^\#|} = (-1)^{|X^\# \cap (Y^{op})^\#| + d(X)}$, therefore

$$\widetilde{\mathcal{R}}_i(X)^{op} = \frac{1}{s(X)} \sum_{Y^{op} \in Sim_{i+def(X)}(X)} (-1)^{|X^\# \cap (Y^{op})^\#| + d(X)} Y^{op}$$
$$= (-1)^{d(X)} \frac{1}{s(X)} \sum_{Y \in Sim_{i+def(X)}} (-1)^{|X^\# \cap Y^\#|} Y = (-1)^{d(X)} \widetilde{\mathcal{R}}_{i+def(X)}(X).$$

□

Corollary 2.5.9. *1. The map $\widetilde{\mathcal{R}}_0$ preserves the kernel of*

$$[[\cdot]] : \mathbb{C}[\widetilde{\mathcal{S}}] \to \mathbb{C}[\mathcal{S}].$$

2. The map $\widetilde{\mathcal{R}}_1$ preserves the kernel of

$$\langle\!\langle \cdot \rangle\!\rangle : \mathbb{C}[\widetilde{\mathcal{S}}^{ev}] \to \mathcal{A}^{ev}.$$

3. If $[X] \in \widetilde{\mathcal{S}}^{ev,1}$, then $[[\widetilde{\mathcal{R}}_i([X])]] = 0$ for $i = 0, 1$.

4. If $[X] \in \widetilde{\mathcal{S}}^{ev,0}$, then $\langle\!\langle \widetilde{\mathcal{R}}_i([X]) \rangle\!\rangle = 0$ for $i = 0, 1$.

Proof. 1. The kernel of the map $[[\cdot]]$ is generated by $[X] - [X]^{op}$ with X running through $\widetilde{\mathcal{S}}$, so the statement follows from the identity

$$\mathcal{R}_0([X]^{op}) = \mathcal{R}_0([X])$$

in Proposition 2.5.8.

2. The kernel of the map $\langle\!\langle \cdot \rangle\!\rangle$ is generated by $[X] + [X^{op}]$ with X running through $\widetilde{\mathcal{S}}^{ev}$, so the statement follows from the identity

$$\mathcal{R}_1([X]^{op}) = -\mathcal{R}_1([X])$$

in Proposition 2.5.8.

3. If $[X] \in \widetilde{\mathcal{S}}^{ev,1}$, then by Proposition 2.5.8

$$[\widetilde{\mathcal{R}}_i([X])]^{op} = -[\widetilde{\mathcal{R}}_i([X])],$$

so

$$[[\widetilde{\mathcal{R}}_i([X])]] = -[[\widetilde{\mathcal{R}}_i([X])^{op}]] = -[[\widetilde{\mathcal{R}}_i([X])]]$$

hence

$$[[\widetilde{\mathcal{R}}_i([X])]] = 0.$$

4. If $[X] \in \widetilde{\mathcal{S}}^{ev,0}$, then by Proposition 2.5.8

$$[\widetilde{\mathcal{R}}_i([X])]^{op} = [\widetilde{\mathcal{R}}_i([X])],$$

so

$$\langle\!\langle \widetilde{\mathcal{R}}_i([X]) \rangle\!\rangle = \langle\!\langle \widetilde{\mathcal{R}}_i([X])^{op} \rangle\!\rangle = -\langle\!\langle \widetilde{\mathcal{R}}_i([X]) \rangle\!\rangle$$

hence

$$\langle\!\langle \widetilde{\mathcal{R}}_i([X]) \rangle\!\rangle = 0.$$

□

By Corollary 2.5.9 1, the map $\widetilde{\mathcal{R}}_0$ factors through $[[\ ]]$, and we denote the induced map by

$$\widetilde{\mathcal{R}}_0 : \mathbb{C}[\mathcal{S}]^{nd} \to \mathbb{C}[\widetilde{\mathcal{S}}].$$

We recall that the map $\widetilde{\mathcal{R}}_0$ is extended to the whole $\mathbb{C}[\mathcal{S}]$ by setting $\widetilde{\mathcal{R}}_0$ to be the identity on $\mathbb{C}[\mathcal{S}]^C$.

We set

$$\begin{aligned} \mathcal{R}_0 : \mathbb{C}[\mathcal{S}] &\to \mathbb{C}[\mathcal{S}]. \\ [[X]] &\mapsto [[\widetilde{\mathcal{R}}_0([X])]] \end{aligned} \tag{2.65}$$

The map $\mathcal{R}_0$ preserves the subspaces $\mathbb{C}[\mathcal{S}^{od}]$, $\mathbb{C}[\mathcal{S}^{ev,0}]$, while by Corollary 2.5.9 3 its restriction on $\mathbb{C}[\mathcal{S}^{ev,1}]$ is 0. We consider the restrictions

$$\mathcal{R}_0 : \mathbb{C}[\mathcal{S}^{ev,0}] \to \mathbb{C}[\mathcal{S}^{ev,0}], \qquad \mathcal{R}_0 : \mathbb{C}[\mathcal{S}^{od}] \to \mathbb{C}[\mathcal{S}^{od}] \tag{2.66}$$

We set

$$\begin{aligned} \mathcal{Q}_0 : \mathbb{C}[\mathcal{S}^{ev}] &\to \mathcal{A}^{ev,0} \\ [[X]] &\mapsto \langle\!\langle \widetilde{\mathcal{R}}_0([X]) \rangle\!\rangle \end{aligned}$$

By Corollary 2.5.9 4, the restriction of the map $\mathcal{Q}_0$ to $\mathbb{C}[\mathcal{S}^{ev,0}]$ is 0. We consider the restriction

$$\mathcal{Q}_0 : \mathbb{C}[\mathcal{S}^{ev,1}] \to \mathcal{A}^{ev,0}. \tag{2.67}$$

By Corollary 2.5.9 2, the map $\widetilde{\mathcal{R}}_1$ restricted to $\mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$ factors through $\langle\!\langle\ \rangle\!\rangle$, and we denote the induced map by

$$\widetilde{\mathcal{R}}_1 : \mathcal{A}^{ev} \to \mathbb{C}[\widetilde{\mathcal{S}}].$$

We set

$$\begin{aligned} \mathcal{R}_1 : \mathcal{A}^{ev} &\to \mathbb{C}[\mathcal{S}^{ev,1}] \\ \langle\!\langle X \rangle\!\rangle &\mapsto [[\widetilde{\mathcal{R}}_1([X])]]. \end{aligned}$$

By Corollary 2.5.9 3 the restriction of $\mathcal{R}_1$ to $\mathcal{A}^{ev,1}$ is 0. We consider the restriction

$$\mathcal{R}_1 : \mathcal{A}^{ev,0} \to \mathbb{C}[\mathcal{S}^{ev,1}]. \tag{2.68}$$

We set

$$\begin{aligned}\mathcal{Q}_1 : \mathcal{A}^{ev} &\to \mathcal{A}^{ev,1}\\ \langle\!\langle X\rangle\!\rangle &\mapsto \langle\!\langle \widetilde{\mathcal{R}}_1([X])\rangle\!\rangle\end{aligned}$$

By Corollary 2.5.9 4, the restriction of the map $\mathcal{Q}_1$ to $\mathcal{A}^{ev,0}$ is 0. We consider the restriction

$$\mathcal{Q}_1 : \mathcal{A}^{ev,1} \to \mathcal{A}^{ev,1}. \tag{2.69}$$

Remark 2.5.10. The map $\mathcal{R}_0$ on $\mathbb{C}[\mathcal{S}^{ev,0}]$ coincides with the Fourier transform defined by Lusztig in [42, 4.6, 4.15]. The map $\mathcal{R}_1$ coincides with the Fourier transform defined by Lusztig in [42, 4.18].

Let $n \in \mathbb{N}$ and let

$$\tau : \mathcal{A}_n^{ev,0} \oplus \mathbb{C}[\mathcal{S}_n^{ev,1}] \to \mathbb{C}[\mathcal{S}_n^{ev,1}] \oplus \mathcal{A}_n^{ev,0}$$

be the swap isomorphism.

The following proposition is a precise statement of the relation between the map $\widetilde{\mathcal{R}}$ and the maps $\mathcal{R}_i, \mathcal{Q}_i$ for $i = 0, 1$ defined above. In Proposition 2.5.13, it will be translated in terms of unipotent class functions on groups of type D.

Proposition 2.5.11. *The following diagram commutes*

$$\begin{array}{ccc}\mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] & \xrightarrow{[[\cdot]]\oplus\langle\!\langle\cdot\rangle\!\rangle} & \mathbb{C}[\mathcal{S}_n^{ev,0}]^{nd} \oplus \mathbb{C}[\mathcal{S}_n^{ev,1}] \oplus \mathcal{A}_n^{ev,0} \oplus \mathcal{A}_n^{ev,1}\\ \Big\downarrow{\scriptstyle \widetilde{\mathcal{R}}} & & \Big\downarrow{\scriptstyle \mathcal{R}_0\oplus\tau\circ(\mathcal{Q}_0\oplus\mathcal{R}_1)\oplus\mathcal{Q}_1}\\ \mathbb{C}[\widetilde{\mathcal{S}}_n^{ev}] & \xrightarrow{[[\cdot]]\oplus\langle\!\langle\cdot\rangle\!\rangle} & \mathbb{C}[\mathcal{S}_n^{ev,0}]^{nd} \oplus \mathbb{C}[\mathcal{S}_n^{ev,1}] \oplus \mathcal{A}_n^{ev,0} \oplus \mathcal{A}_n^{ev,1}\end{array} \tag{2.70}$$

Proof. We need to prove that for any $[X] \in \widetilde{\mathcal{S}}_n^{ev,0}$ it holds

$$([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)(\widetilde{\mathcal{R}}(X)) = \mathcal{R}_0([[X]]) + \mathcal{R}_1(\langle\!\langle X\rangle\!\rangle),$$

and that for any $[X] \in \widetilde{\mathcal{S}}_n^{ev,1}$ it holds

$$([[\widetilde{\mathcal{R}}([X])]], \langle\!\langle\widetilde{\mathcal{R}}(X)\rangle\!\rangle) = \mathcal{Q}_0([[X]]) + \mathcal{Q}_1(\langle\!\langle X\rangle\!\rangle)).$$

For any $[X] \in \widetilde{\mathcal{S}}_n^{ev}$, it holds

$$\begin{aligned}([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)(\widetilde{\mathcal{R}}(X)) &= [[\widetilde{\mathcal{R}}([X])]] + \langle\!\langle\widetilde{\mathcal{R}}(X)\rangle\!\rangle)\\ &= [[\widetilde{\mathcal{R}_0}(X) + \widetilde{\mathcal{R}}_1(X)]] + \langle\!\langle\widetilde{\mathcal{R}_0}(X) + \widetilde{\mathcal{R}}_1(X)\rangle\!\rangle\\ &= ([[\widetilde{\mathcal{R}_0}(X)]] + [[\widetilde{\mathcal{R}}_1(X)]], \langle\!\langle\widetilde{\mathcal{R}_0}(X)\rangle\!\rangle + \langle\!\langle\widetilde{\mathcal{R}}_1(X)\rangle\!\rangle)\\ &= \mathcal{R}_0([[X]]) + \mathcal{Q}_0([[X]]) + \mathcal{R}_1(\langle\!\langle X\rangle\!\rangle) + \mathcal{Q}_1(\langle\!\langle X\rangle\!\rangle).\end{aligned}$$

By Corollary 2.5.9(3), if $[X] \in \widetilde{\mathcal{S}}^{ev,1}$ then

$$\mathcal{R}_0([[X]]) = [[\widetilde{\mathcal{R}}_0([X])]] = 0, \qquad \mathcal{R}_1(\langle\!\langle X\rangle\!\rangle) = [[\widetilde{\mathcal{R}}_1([X])]] = 0.$$

By Corollary 2.5.9 (4) if $[X] \in \widetilde{\mathcal{S}}^{ev,0}$ then

$$\mathcal{Q}_0([[X]]) = \langle\!\langle\widetilde{\mathcal{R}}_0([X])\rangle\!\rangle = 0, \qquad \mathcal{Q}_1(\langle\!\langle X\rangle\!\rangle) = \langle\!\langle\widetilde{\mathcal{R}}_1([X])\rangle\!\rangle = 0$$

□

Corollary 2.5.12. *The maps*

$$\begin{aligned} \mathcal{R}_0 &: \mathbb{C}[\mathcal{S}^{ev,0}] \to \mathbb{C}[\mathcal{S}^{ev,0}] & \mathcal{R}_1 &: \mathcal{A}^{ev,0} \to \mathbb{C}[\mathcal{S}^{ev,1}] \\ \mathcal{Q}_0 &: \mathbb{C}[\mathcal{S}^{ev,1}] \to \mathcal{A}^{ev,0} & \mathcal{Q}_1 &: \mathcal{A}^{ev,1} \to \mathcal{A}^{ev,1} \end{aligned}$$

are isometries. Moreover the maps $\mathcal{R}_0$ and $\mathcal{Q}_1$ are involutions, while the maps $\mathcal{R}_1$ and $\mathcal{Q}_0$ are inverse to each other.

Proof. The map $\widetilde{\mathcal{R}}$ is an isometry by Lemma 2.6.3 and the map $([[\ ]] \oplus \langle\!\langle\ \rangle\!\rangle)$ is a 2-dilation by Lemma 2.5.2. By Proposition 2.5.11, it follows that the maps $\mathcal{R}_i, \mathcal{Q}_i$ for $i = 0, 1$ are isometries.

The map $\widetilde{\mathcal{R}}$ is an involution by Lemma 2.4.8, and so by Proposition 2.5.11 the maps $\mathcal{R}_0$ and $\mathcal{Q}_1$ are involutions and the maps $\mathcal{R}_1$ and $\mathcal{Q}_0$ are inverse to each other. □

2.5.3 Compatibility of parabolic induction and Fourier transform

In this section, we show how to deduce from Theorem 2.4.12 the compatibility of the Fourier transforms defined by Lusztig for groups of type D_n and ${}^2D_n$ (see Remark 2.5.10). The relevant results in this sense are Theorems 2.5.16 and 2.5.17. We discuss in Remark 2.5.18 the equivalence of these theorems with Theorem 2.4.12.

Retain notation from Section 2.5.1.1, so that G is a simple group of type D_n and $\widetilde{G}^i$ for $i = 0, 1$ denote respectively the split and non-split inner twist of $G \rtimes \langle\delta\rangle$. We define

$$\begin{aligned} \mathcal{R}_0^G &: R_u(G^{F_0}) \to R_u(G^{F_0}) \\ \mathcal{R}_1^G &: Cl_u(G^{F_0}\delta) \to R_u(G^{F_1}) \\ \mathcal{Q}_0^G &: R_u(G^{F_1}) \to Cl_u(G^{F_0}\delta) \\ \mathcal{Q}_1^G &: Cl_u(G^{F_1}\delta) \to Cl_u(G^{F_1}\delta) \end{aligned}$$

as the maps making the following diagrams commutative:

$$
\begin{CD}
R_u(G^{F_0}) @>{\mathcal{R}_0^G}>> R_u(G^{F_0}\delta)\\
@V{Symb_D^0}VV @VV{Symb_D^0}V\\
\mathbb{C}[\mathcal{S}_n^{ev,0}] @>{\mathcal{R}_0}>> \mathbb{C}[\mathcal{S}_n^{ev,0}]
\end{CD}
\qquad
\begin{CD}
R_u(G^{F_1}) @>{\mathcal{Q}_0^G}>> Cl_u(G^{F_0}\delta)\\
@V{Symb_D^1}VV @VV{Symb_D^{0,\delta}}V\\
\mathbb{C}[\mathcal{S}_n^{ev,1}] @>{\mathcal{Q}_0}>> \mathcal{A}_n^{ev,1}
\end{CD}
$$

$$
\begin{CD}
Cl_u(G^{F_0}\delta) @>{\mathcal{R}_1^G}>> R_u(G^{F_1})\\
@V{Symb_D^{0,\delta}}VV @VV{Symb_D^1}V\\
\mathcal{A}_n^{ev,1} @>{\mathcal{R}_1}>> \mathbb{C}[\mathcal{S}_n^{ev,1}]
\end{CD}
\qquad
\begin{CD}
Cl_u(G^{F_1\delta}) @>{\mathcal{Q}_1^G}>> Cl_u(G^{F_1\delta})\\
@V{Symb_D^{1,\delta}}VV @VV{Symb_D^{1,\delta}}V\\
\mathcal{A}_n^{ev,1} @>{\mathcal{Q}_1}>> \mathcal{A}_n^{ev,1}
\end{CD}
$$

The maps $\mathcal{R}_i^G$ and $\mathcal{Q}_i^G$ for $i = 0, 1$ are isometries, since they are compositions of isometries.

Let

$$\tau : Cl_u(G^{F_1}) \oplus Cl_u(G^{F_0}\delta) \to Cl_u(G^{F_0}\delta) \oplus Cl_u(G^{F_1})$$

be the swap isomorphism. The next proposition is a translation of Proposition 2.5.11 in terms of spaces of unipotent class functions on groups of type D, so it gives a precise statement of the relation between the Fourier transform on the sum of spaces of unipotent class functions for inner twists of a disconnected group of type D_n with the Fourier transforms of groups of type D_n and ${}^2D_n$ (see Remark 2.5.10).

Proposition 2.5.13. *The following diagram commutes*

$$
\begin{CD}
R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) @>{\pi_{G^{F_0},0} \oplus \pi_{G^{F_0},1} \oplus \pi_{G^{F_1},0} \oplus \pi_{G^{F_1},1}}>> Cl_u(G^{F_0}) \oplus Cl_u(G^{F_0}\delta) \oplus Cl_u(G^{F_1}) \oplus Cl_u(G^{F_1}\delta)\\
@V{\widetilde{\mathcal{R}}^G}VV @VV{\mathcal{R}_0^G \oplus \tau \circ (\mathcal{R}_1^G \oplus \mathcal{Q}_0^G) \oplus \mathcal{Q}_1^G}V\\
R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1) @>{\pi_{G^{F_0},0} \oplus \pi_{G^{F_0},1} \oplus \pi_{G^{F_1},0} \oplus \pi_{G^{F_1},1}}>> Cl_u(G^{F_0}) \oplus Cl_u(G^{F_0}\delta) \oplus Cl_u(G^{F_1}) \oplus Cl_u(G^{F_1}\delta)
\end{CD}
\tag{2.71}
$$

Proof. By Proposition 2.5.5, the commutativity of the diagram in the statement is equivalent to the commutativity of (2.70) ☐

Remark 2.5.14. We observe that substituting the map $\mathcal{Q}_1^G$ with the map $-\mathcal{Q}_1^G$ in (2.71), we obtain a commutative diagram analogue to (2.71) for the map $\mathcal{F}$ defined in [53] (see Remark 2.5.6) in place of the map $\widetilde{\mathcal{R}}$. It follows that the remainder of this section could be applied to the map $\mathcal{F}$ in place of $\widetilde{\mathcal{R}}$ with minor modifications.

Let M be a maximal δ-stable standard Levi subgroup of G of type $A_{r-1} \times D_{n-r}$ with $1 \leqslant r \leqslant n-1$, and retain notation from Section 2.2.2.2.

Since the Fourier transform is the identity on groups of type A, we define

$$
\begin{aligned}
\mathcal{R}_0^M &: R_u(M^{F_0}) \to R_u(M^{F_0})\\
\mathcal{R}_1^M &: Cl_u(M^{F_0}\delta) \to R_u(M^{F_1})\\
\mathcal{Q}_0^M &: R_u(M^{F_1}) \to Cl_u(M^{F_0}\delta)\\
\mathcal{Q}_1^M &: Cl_u(M^{F_1}\delta) \to Cl_u(M^{F_1}\delta)
\end{aligned}
$$

as the maps making the following diagrams commutative:

$$\begin{array}{ccc}
R_u(M^{F_0}) & \xrightarrow{\quad \mathcal{R}_0^M \quad} & R_u(M^{F_0}) \\
\Big\downarrow{\scriptstyle \bigoplus_{s\equiv 0}(k_A \otimes \ell^{n,2s})\circ Rep_{M^{F_0}}} & & \Big\downarrow{\scriptstyle \bigoplus_{s\equiv 0}(k_A \otimes \ell^{n,2s})\circ Rep_{M^{F_0}}} \\
\mathbb{C}[\mathscr{P}_r]\otimes \mathbb{C}[\mathcal{S}^{ev,0}_{n-r}] & \xrightarrow{id\boxtimes \mathcal{R}_0} & \mathbb{C}[\mathscr{P}_r]\otimes \mathbb{C}[\mathcal{S}^{ev,0}_{n-r}]
\end{array}$$

$$\begin{array}{ccc}
R_u(M^{F_1}) & \xrightarrow{\quad \mathcal{Q}_0^M \quad} & Cl_u(M^{F_0}\delta) \\
\Big\downarrow{\scriptstyle \bigoplus_{s\equiv 1}(k_A \otimes \ell^{n,2s})\circ Rep_{M^{F_1}}} & & \Big\downarrow{\scriptstyle \bigoplus_{s\equiv 0}(k_A \otimes \ell_\delta^{n,2s})\circ Rep_{M^{F_0}\delta}} \\
\mathbb{C}[\mathscr{P}_r]\otimes \mathbb{C}[\mathcal{S}^{ev,1}_{n}] & \xrightarrow{\mathcal{Q}_0} & \mathbb{C}[\mathscr{P}_r]\otimes \mathcal{A}^{ev,1}_{n}
\end{array}$$

$$\begin{array}{ccc}
Cl_u(M^{F_0}\delta) & \xrightarrow{\quad \mathcal{R}_1^M \quad} & R_u(M^{F_1}) \\
\Big\downarrow{\scriptstyle \bigoplus_{s\equiv 0}(k_A \otimes \ell_\delta^{n,2s})\circ Rep_{M^{F_0}\delta}} & & \Big\downarrow{\scriptstyle \bigoplus_{s\equiv 1}(k_A \otimes \ell^{n,2s})\circ Rep_{M^{F_1}}} \\
\mathbb{C}[\mathscr{P}_r]\otimes \mathcal{A}^{ev,1}_{n} & \xrightarrow{id\boxtimes \mathcal{R}_1} & \mathbb{C}[\mathscr{P}_r]\otimes \mathbb{C}[\mathcal{S}^{ev,1}_{n}]
\end{array}$$

$$\begin{array}{ccc}
Cl_u(M^{F_1}\delta) & \xrightarrow{\quad \mathcal{Q}_1^M \quad} & Cl_u(M^{F_1}\delta) \\
\Big\downarrow{\scriptstyle \bigoplus_{s\equiv 1}(k_A \otimes \ell_\delta^{n,2s})\circ Rep_{M^{F_1}\delta}} & & \Big\downarrow{\scriptstyle \bigoplus_{s\equiv 1}(k_A \otimes \ell^{n,2s})\circ Rep_{M^{F_1}\delta}} \\
\mathbb{C}[\mathscr{P}_r]\otimes \mathcal{A}^{ev,1}_{n} & \xrightarrow{id\boxtimes \mathcal{Q}_1} & \mathbb{C}[\mathscr{P}_r]\otimes \mathcal{A}^{ev,1}_{n}
\end{array}$$

Remark 2.5.15. Proposition 2.5.13 holds for M as well, since for $i = 0, 1$ the space $R_u(\widetilde{M}^i)$ of unipotent class functions is the tensor product of the space of unipotent class functions on a group of type A_r with the space of unipotent class function on a group of type D_{n-r} or ${}^2D_{n-r}$, and on the factor of type A all the maps in (2.71) are the identity maps.

The restrictions $\pi_{G^F,i}$ (respectively $\pi_{M^F,i}$), for $i = 0, 1$, are orthogonal projections on the space $R_u(\widetilde{G}^i)$ (respectively $R_u(\widetilde{M}^i)$) along the decomposition $Cl_u(G^{F_i}) \oplus Cl_u(G^{F_i}\delta)$ (respectively $Cl_u(M^{F_i}) \oplus Cl_u(M^{F_i}\delta)$). We now show that projecting diagram (2.40) using the appropriate restrictions of class functions, we obtain the compatibility of the Fourier transform maps with parabolic restriction.

By Proposition 2.5.13, we have

$$(\pi_{G^{F_0},0} \oplus 0) \circ \widetilde{\mathcal{R}}^G = \mathcal{R}_0^G \circ (\pi_{G^{F_0},0} \oplus 0) \tag{2.72}$$

$$(\pi_{G^{F_0},1} \oplus 0) \circ \widetilde{\mathcal{R}}^G = \mathcal{Q}_0^G \circ (0 \oplus \pi_{G^{F_1},0}) \tag{2.73}$$

$$(0 \oplus \pi_{G^{F_1},0}) \circ \widetilde{\mathcal{R}}^G = \mathcal{R}_1^G \circ (\pi_{G^{F_0},1} \oplus 0) \tag{2.74}$$

and similarly for M in place of G.

Moreover for $i \in \{0, 1\}$ it holds

$$\pi_{M^{F_i},0} \circ {}^*R^{\widetilde{G}^i}_{\widetilde{M}^i} = {}^*R^{G^{F_i}}_{M^{F_i}} \circ \pi_{G^{F_i},0} \qquad \text{Lemma 1.12}$$

$$\pi_{M^{F_i},1} \circ {}^*R^{\widetilde{G}^i}_{\widetilde{M}^i} = {}^*R^{G^{F_i}\delta}_{M^{F_i}\delta} \circ \pi_{G^{F_i},1} \qquad \text{Lemma 1.2.40}$$

Since ${}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}$ maps $R_u(\widetilde{G}^i)$ to $R_u(\widetilde{M}^i)$, it follows that

$$
\begin{aligned}
(\pi_{M^{F_0},0} \oplus 0) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) &= {}^*R^{G^{F_0}}_{M^{F_0}} \circ (\pi_{G^{F_0},0} \oplus 0) && (2.75)\\
(0 \oplus \pi_{M^{F_1},0}) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) &= {}^*R^{G^{F_1}}_{M^{F_1}} \circ (0 \oplus \pi_{G^{F_1},0}) && \text{Lemma 1.12}\\
(\pi_{M^{F_0},1} \oplus 0) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) &= {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta} \circ (\pi_{G^{F_0},1} \oplus 0) &&\\
(0 \oplus \pi_{M^{F_1},1}) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) &= {}^*R^{G^{F_i}\delta}_{M^{F_i}\delta} \circ (0 \oplus \pi_{G^{F_i},1}) && \text{Lemma 1.2.40}
\end{aligned}
$$

The following Theorem shows how to deduce the compatibility of Lusztig's Fourier transform for split groups of type D [42, 4.6, 4.15] and parabolic restriction (or induction) from Theorem 2.4.12.

Theorem 2.5.16. *The following diagrams commute:*

$$
\begin{array}{ccc}
R_u(G^{F_0}) & \xrightarrow{\mathcal{R}_0^G} & R_u(G^{F_0}) \\
\downarrow {}^*R^{G^{F_0}}_{M^{F_0}} & & \downarrow {}^*R^{G^{F_0}}_{M^{F_0}} \\
R_u(M^{F_0}) & \xrightarrow{\mathcal{R}_0^M} & R_u(M^{F_0})
\end{array} \qquad (2.76)
$$

$$
\begin{array}{ccc}
R_u(G^{F_0}) & \xrightarrow{\mathcal{R}_0^G} & R_u(G^{F_0}) \\
R^{G^{F_0}}_{M^{F_0}} \uparrow & & R^{G^{F_0}}_{M^{F_0}} \uparrow \\
R_u(M^{F_0}) & \xrightarrow{\mathcal{R}_0^M} & R_u(M^{F_0})
\end{array} \qquad (2.77)
$$

Proof. We prove the commutativity of (2.76). The one of (2.77) then follows from adjunction, since $\mathcal{R}_0^G$ and $\mathcal{R}_0^G$ is an involutive isometry.

The following diagram commutes

$$
\begin{array}{ccc}
Cl_u(G^{F_0}) & \xrightarrow{\mathcal{R}_0^G} & Cl_u(G^{F_0}) \\
\downarrow {}^*R^{G^{F_0}}_{M^{F_0}} & & \downarrow {}^*R^{G^{F_0}}_{M^{F_0}} \\
Cl_u(M^{F_0}) & \xrightarrow{\mathcal{R}_0^M} & Cl_u(M^{F_0}).
\end{array} \qquad (2.78)
$$

Indeed by (2.72) and (2.75), the diagram (2.78) is obtained from the diagram (2.40) in Theorem 2.4.12 by projecting $R_u(\widetilde{G}^0) \oplus R_u(\widetilde{G}^1)$ (respectively $R_u(\widetilde{M}^0) \oplus R_u(\widetilde{M}^1)$) along $(\pi_{G^{F_0},0} \oplus 0)$ (respectively $(\pi_{M^{F_0},0} \oplus 0)$).

Now we show the commutativity of the restriction of (2.76) to the orthogonal complement of $Cl_u(G^{F_0})$ in $R_u(G^{F_0})$.

Recall that for any $m \in \mathbb{N}$, we set

$$R(W(D_m))^C := span_{\mathbb{C}}\{(\kappa_D^{-1}([(\alpha,\alpha)]_+) - \kappa_D^{-1}([(\alpha,\alpha)]_-)) \mid [(\alpha,\alpha)] \in \mathscr{B}_m \ degenerate\}.$$

The orthogonal complement of $Cl_u(G^{F_0})$ in $R_u(G^{F_0})$ is given by $Rep^{-1}_{G^{F_0}}(R(W(D_n))^C)$, and the orthogonal complement of $Cl_u(M^{F_0})$ in $R_u(M^{F_0})$ is given by $R(\mathbb{S}_r) \otimes Rep^{-1}_{M^{F_0}}(R(W(D_{n-r}))^C)$.

By definition, the map $\mathcal{R}_0$ is the identity on both the spaces $Rep^{-1}_{G^{F_0}}(R(W(D_n))^C)$ and $R(\mathbb{S}_r) \otimes Rep^{-1}_{M^{F_0}}(R(W(D_{n-r}))^C)$. Then to prove that (2.76) commutes it is sufficient to show that

$$^*R^{G^{F_0}}_{M^{F_0}}(Rep^{-1}_{G^{F_0}}(R(W(D_n))^C)) \subseteq R(\mathbb{S}_r) \otimes Rep^{-1}_{M^{F_0}}(R(W(D_{n-r}))^C).$$

By Theorem 1.1.5, it is equivalent to prove that

$$Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(R(W(D_n))^C) \subseteq R(\mathbb{S}_r) \otimes R(W(D_{n-r}))^C \tag{2.79}$$

By the definition of κ_D [49, Example 4.1.4], for any $\rho \in Irr(W(B_n))$ the maps κ_B and $kappa_D$ satisfy (2.5), that is

$$[[\kappa_B(\rho)]] = \kappa_D \circ Res^{W(B_n)}_{W(D_n)}(\rho). \tag{2.80}$$

Since for any $m \in \mathbb{N}$ the space $R(W(D_m))^C$ is the orthogonal complement of $Cl(D_m) = span_{\mathbb{C}}\{\kappa_D^{-1}([[(\alpha,\beta)]]) \mid (\alpha,\beta) \in \mathscr{B}_n \; non \; degenerate\}$ in $R(W(D_m))$, to show (2.79) it is enough to prove that for any $f \in R(W(D_n))^C$ and $g \in R(\mathbb{S}_r) \otimes Cl(W(D_{n-r}))$ it holds

$$\langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(f), g\rangle_{\mathbb{S}_r \times W(D_{n-r})} = 0.$$

By definition of $R(W(D_n))^C$, it is enough to take $f = \kappa_D^{-1}(\overline{(\alpha,\alpha)_+} - \overline{(\alpha,\alpha)_-})$ with $(\alpha,\alpha) \in \mathscr{B}_n$ degenerate.

For any $m \in \mathbb{N}$, let $\delta_m : W(D_m) \to W(D_m)$ be the involution of $W(D_m)$ corresponding to a graph automorphism of the Dynkin diagram of type D_m. Let δ_m^* be the involution induced by δ_m on $Irr(W(D_m))$ and also the linear extension of δ_m^* on $R(W(D_m))$. This map is an isometry because it preserves the orthonormal basis $Irr(D_m)$. The involution δ_n on $W(D_n)$ preserves the subgroup $\mathbb{S}_r \times W(D_{n-r})$ for any $r < n-1$, and the restriction of δ_n to this subgroup is given by $id \times \delta_{n-r}$. Since the restriction is the pullback of the inclusion, we have

$$Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})} \circ \delta_n^* = (id \boxtimes \delta_{n-r}^*) \circ Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}. \tag{2.81}$$

The involution δ_n^* on $Irr(W(D_n))$ interchanges the characters $\kappa_D^{-1}(\overline{(\alpha,\alpha)_+})$ and $\kappa_D^{-1}(\overline{(\alpha,\alpha)_-})$ ([24, Proposition 5.6.3] and its proof) and fixes the other ones. Hence for any $g \in R(\mathbb{S}_r) \otimes Cl(W(D_{n-r}))$ it holds $(id \boxtimes \delta_{n-r}^*)(g) = g$ and

$$\begin{aligned}
&\langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_+}) - \kappa_D^{-1}(\overline{(\alpha,\alpha)_-})), g\rangle_{\mathbb{S}_r \times W(D_{n-r})} \\
&= \langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})} \circ \delta_n^*(\kappa_D^{-1}(\overline{(\alpha,\alpha)_-} - \kappa_D^{-1}(\overline{(\alpha,\alpha)_+})), g\rangle_{\mathbb{S}_r \times W(D_{n-r})} \\
&= \langle (id \boxtimes \delta_{n-r}^*) \circ Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_-} - \kappa_D^{-1}(\overline{(\alpha,\alpha)_+})), g\rangle_{\mathbb{S}_r \times W(D_{n-r})} \\
&= \langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_-} - \kappa_D^{-1}(\overline{(\alpha,\alpha)_+})), (id \boxtimes \delta_{n-r}^*)(g)\rangle_{\mathbb{S}_r \times W(D_{n-r})} = \\
&= \langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_-} - \kappa_D^{-1}(\overline{(\alpha,\alpha)_+})), g)\rangle_{\mathbb{S}_r \times W(D_{n-r})} \\
&= -\langle Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_+}) - \kappa_D^{-1}(\overline{(\alpha,\alpha)_-})), g\rangle_{\mathbb{S}_r \times W(D_{n-r})}.
\end{aligned}$$

Therefore the above scalar product vanishes, and so $Res^{W(D_n)}_{\mathbb{S}_r \times W(D_{n-r})}(\kappa_D^{-1}(\overline{(\alpha,\alpha)_+}) - \kappa_D^{-1}(\overline{(\alpha,\alpha)_-}))$ is orthogonal to $R(\mathbb{S}_r) \otimes Cl(W(D_{n-r}))$, so it lies in $R(\mathbb{S}_r) \otimes R(W(D_{n-r}))^C$.

□

The following Theorem shows how to deduce from Theorem 2.4.12 the compatibility of Lusztig's Fourier transform [42, 4.18] and parabolic restriction (or induction) for non split groups of type D.

Theorem 2.5.17. *The following diagrams commute:*

$$\begin{array}{ccc} R_u(G^{F_1}) & \xrightarrow{\mathcal{Q}_0^G} & Cl_u(G^{F_0}\delta) \\ \downarrow {\scriptstyle {}^*R^{G^{F_1}}_{M^{F_1}}} & & \downarrow {\scriptstyle {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta}} \\ R_u(M^{F_1}) & \xrightarrow{\mathcal{Q}_0^M} & Cl_u(M^{F_0}\delta) \end{array} \qquad \begin{array}{ccc} Cl_u(G^{F_0}\delta) & \xrightarrow{\mathcal{R}_1^G} & R_u(G^{F_1}) \\ \downarrow {\scriptstyle {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta}} & & \downarrow {\scriptstyle {}^*R^{G^{F_1}}_{M^{F_1}}} \\ Cl_u(M^{F_0}\delta) & \xrightarrow{\mathcal{R}_1^M} & R_u(M^{F_1}) \end{array} \tag{2.82}$$

$$\begin{array}{ccc} R_u(G^{F_1}) & \xrightarrow{\mathcal{Q}_0^G} & Cl_u(G^{F_0}\delta) \\ {\scriptstyle R^{G^{F_1}}_{M^{F_1}}} \uparrow & & {\scriptstyle R^{G^{F_0}\delta}_{M^{F_0}\delta}} \uparrow \\ R_u(M^{F_1}) & \xrightarrow{\mathcal{Q}_0^M} & Cl_u(G^{F_0}\delta) \end{array} \qquad \begin{array}{ccc} Cl_u(G^{F_0}\delta) & \xrightarrow{\mathcal{R}_1^G} & R_u(G^{F_1}) \\ {\scriptstyle R^{G^{F_0}\delta}_{M^{F_0}\delta}} \uparrow & & {\scriptstyle R^{G^{F_1}}_{M^{F_1}}} \uparrow \\ Cl_u(M^{F_0}\delta) & \xrightarrow{\mathcal{R}_1^M} & R_u(M^{F_1}) \end{array} \tag{2.83}$$

Proof. We prove the commutativity of the diagrams (2.82). The commutativity of the diagrams (2.83) follows by adjunction, since $\mathcal{Q}_0$ and $\mathcal{R}_1$ are mutually inverse isometries.

The commutativity of diagrams (2.82) follows from the commutativity of diagram (2.40) in Theorem 2.4.12.

Indeed projecting the space $R_u(\tilde{G^i})$ along the decomposition $Cl_u(G^{F_i}) \oplus Cl_u(G^{F_i}\delta)$, using (2.72) and (2.75) we get

$$\begin{aligned} {}^*R^{G^{F_1}}_{M^{F_1}} \circ \mathcal{R}_1^G \circ (\pi_{G^{F_0},1} \oplus 0) &= {}^*R^{G^{F_1}}_{M^{F_1}} \circ (0 \oplus \pi_{G^{F_1},0}) \circ \widetilde{\mathcal{R}}^G \\ &= (0 \oplus \pi_{M^{F_1},0}) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \circ \widetilde{\mathcal{R}}^G \\ &= (0 \oplus \pi_{M^{F_1},0}) \circ \widetilde{\mathcal{R}}^M \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \\ &= \mathcal{R}_1^M \circ (\pi_{M^{F_0},1\oplus 0}) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \\ &= \mathcal{R}_1^M \circ {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta} \circ (\pi_{G^{F_0},1} \oplus 0) \end{aligned}$$

and since $(\pi_{G^{F_0},1} \oplus 0)$ is surjective, it implies

$${}^*R^{\widetilde{G}^1}_{\widetilde{M}^1} \circ \mathcal{R}_1^G = \mathcal{R}_1^M \circ {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}.$$

Similarly

$$\begin{aligned} {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta} \circ \mathcal{Q}_0^G \circ (0 \oplus \pi_{G^{F_1},0}) &= {}^*R^{G^{F_0}\delta}_{M^{F_0}\delta} \circ (\pi_{G^{F_0},1} \oplus 0) \circ \widetilde{\mathcal{R}}^G \\ &= (\pi_{M^{F_0},1} \oplus 0) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \circ \widetilde{\mathcal{R}}^G \\ &= (\pi_{M^{F_0},1} \oplus 0) \circ \widetilde{\mathcal{R}}^M \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \\ &= \mathcal{Q}_0^M \circ (0 \oplus \pi_{M^{F_1},0}) \circ ({}^*R^{\widetilde{G}^0}_{\widetilde{M}^0} \oplus {}^*R^{\widetilde{G}^1}_{\widetilde{M}^1}) \\ &= \mathcal{Q}_0^M \circ {}^*R^{G^{F_1}}_{M^{F_1}} \circ (0 \oplus \pi_{G^{F_1},0}) \end{aligned}$$

and since $(0 \oplus \pi_{G^{F_1},0})$ is surjective, it implies

$$^*R^{G^{F_0}\delta}_{M^{F_0}\delta} \circ \mathcal{Q}^G_0 = \mathcal{Q}^M_0 \circ {}^*R^{G^{F_1}}_{M^{F_1}}.$$

□

Remark 2.5.18. Theorem 2.4.12 is equivalent to the commutativity of the diagram (2.78) in Theorem 2.5.16 and of the diagrams (2.82) in Theorem 2.5.17, together with the commutativity of the following diagram

$$\begin{array}{ccc} R_u(G^{F_1}\delta) & \xrightarrow{\mathcal{Q}^G_1} & Cl_u(G^{F_0}\delta) \\ \Big\downarrow {}^*R^{G^{F_1}\delta}_{M^{F_1}\delta} & & \Big\downarrow {}^*R^{G^{F_1}\delta}_{M^{F_1}\delta} \\ R_u(M^{F_1}\delta) & \xrightarrow{\mathcal{Q}^M_1} & Cl(M^{F_1}\delta) \end{array} \tag{2.84}$$

We showed how to prove Theorem 2.4.12 and deduce the commutativity of diagrams (2.78), (2.82), (2.84). The draw-back of this approach is that it only considers δ-stable Levi subgroups, so it does not take into account the maximal Levi subgroup of type A_{n-1} in G^{F_0}.

An alternative strategy is to prove the commutativity of the diagrams (2.76), (2.82), (2.84) and deduce the commutativity of (2.40), gluing them together via Lemma 2.5.13. To prove the commutativity of (2.78), (2.82), (2.84), the strategy is analogous to the one used to prove Theorem 2.4.12, exploiting the results in Section 1.2.4.2, in particular Theorem 1.2.41. Following this approach, since it consider any group individually, it is possible to include proving the commutativity of the diagram (2.76) the case in which M is of type A_{n-1}.

2.6 Groups of type B_n

In this section, we show results analogous to those of Section 2.5.3 for groups of type B (or C). More precisely, we show the compatibility of the Fourier transform for groups of type B, that we will denote by $\mathcal{R}_0$ [42, 4.5, 4.15], with parabolic induction. The strategy is similar to the one used in Section 2.4.1.

2.6.0.1 Parameterization of Unipotent representations of groups of type B_n

In the remainder of this chapter, G is of type B_n (or C_n). We retain notation from Section 2.1.2.2.

For any $s \geqslant 0$ such that $s^2 + s \leqslant n$, we have a bijection

$$\kappa_B : Irr(W(B_{n-(s^2+s)})) \to \mathscr{B}_{n-(s^2+s)}.$$

Composing it with the map $\widetilde{\mathscr{S}}_{n-s^2-s,2s+1}$ as in (2.19) yields a bijection

$$\mathscr{S}_{n-s^2-s,2s+1} \circ \kappa_B : Irr(W_{G^F}(L_s, \sigma_s)) \to \mathscr{B}_{n-(s^2+s)} \to \widetilde{\mathcal{S}}^{2s+1}_n.$$

Since $2s+1 \neq 0$ for any $s > 0$, the map [[ ]] induces a bijection between $\widetilde{\mathcal{S}}_n^{2s+1}$ and $\mathcal{S}_n^{2s+1}$, and we set

$$\ell^{n,2s+1} := [[]] \circ \mathscr{S}_{n-s^2-s,2s+1} \circ \kappa_B : Irr(W(B_{n-(s^2+s)})) \to \mathcal{S}_n^{2s+1}, \tag{2.85}$$

and we still denote by $\ell^{n,2s+1}$ the isomorphism obtained by linear extension of this bijection.

Since $Irr(W_{G^F}(L_s, \sigma_s)) \cong Irr(W(B_{n-(s^2+s)}))$ for any s such that $s^2 + s \leqslant n$, composing $\ell^{n,2s+1}$ with Rep_{G^F} yields a bijection

$$\ell^{n,2s+1} \circ Rep_{G^F} : \mathcal{E}^{G^F}(L_s, \sigma_s) \to \mathcal{S}_n^{2s+1}.$$

Collecting together all the series yields an isomorphism of vector spaces

$$Symb_B := \bigoplus_{\substack{s\in\mathbb{N} \\ s^2+s\leqslant n}} \ell^{n,2s+1} \circ Rep_{G^F} : R_u(G^F) \to \mathbb{C}[\mathcal{S}_n^{od}] \tag{2.86}$$

This map is an isometry, since it maps the orthonormal basis of $R_u(G^F)$ consisting of irreducible unipotent characters onto the orthonormal basis of $\mathbb{C}[\mathcal{S}_n^{od}]$ consisting of ordered symbols.

2.6.1 Compatibility of parabolic induction and Fourier transform

We define

$$\mathcal{R}_0^G : R_u(G^F) \to R_u(G^F)$$

as the map making the following diagram commutative:

$$\begin{array}{ccc} R_u(G^F) & \xrightarrow{\mathcal{R}_0^G} & R_u(G^F) \\ \Big\downarrow {\scriptstyle Symb_B} & & \Big\downarrow {\scriptstyle Symb_B} \\ \mathbb{C}[\mathcal{S}_n^{od}] & \xrightarrow{\mathcal{R}_0} & \mathbb{C}[\mathcal{S}_n^{od}] \end{array} \tag{2.87}$$

where $\mathcal{R}_0 : \mathbb{C}[\mathcal{S}_n^{od}] \to \mathbb{C}[\mathcal{S}_n^{od}]$ is as in (2.66).

This is well-posed since the vertical maps are linear isomorphism.

Remark 2.6.1. The map $\mathcal{R}_0$ on $\mathbb{C}[\mathcal{S}^{od}]$ coincides with the Fourier transform defined by Lusztig in [42, 4.5, 4.15].

The vertical maps $Symb_B$ in (2.87) are isometries. In Lemma 2.6.3 we will show that $\mathcal{R}_0^G$ is an involutive isometry.

We need a variant of Lemma 2.4.7

Lemma 2.6.2. *Let* $[X] \in \widetilde{\mathcal{S}}$ *and let* $Z \in Sim(X)\backslash\{X, X^{op}\}$. *Then*

$$\sum_{Y \in Sim_0(X)} (-1)^{|Y^\# \cap (X^\# \ominus Z^\#)|} = 0, \tag{2.88}$$

Proof. Since $Z \in Sim(X)\backslash\{X, X^{op}\}$, there exists some $a \in Z^{\#} \ominus X^{\#}$ and some $b \notin Z^{\#} \ominus X^{\#}$.

The assignment $Y \mapsto Y^{\#}$ is a bijection between $Sim_0(X)$ and the set of the subsets of $X^{\ominus}$ with even cardinality. The assignment $Y^{\#} \mapsto Y^{\#} \ominus \{a, b\}$ determines a bijection without fixed points of the set of subsets of even cardinality of $X^{\ominus}$ into itself. By transport of structure, it determines a bijection without fixed points of $Sim_0(X)$ into itself. Therefore there is some array $Y' \in Sim_0(X)$ such that ${Y'}^{\#} = Y^{\#} \ominus \{a, b\}$. By our choice of a and b, it holds $|Y^{\#} \cap (X^{\#} \ominus Z^{\#})| \equiv |Y'^{\#} \cap (X^{\#} \ominus Z^{\#})| + 1 \pmod 2$, and (2.88) follows. □

Lemma 2.6.3. *The map*

$$\mathcal{R}_0 : \mathbb{C}[\mathcal{S}^{od}] \to \mathbb{C}[\mathcal{S}^{od}]$$

is an isometry and an involution.

Proof. We begin by proving that the map $\mathcal{R}_0$ is an involution.

Let $[X] \in \widetilde{\mathcal{S}}^{od}$. If $Sim(X) = \{X, X^{op}\}$, then $\mathcal{R}_0([[X]]) = [[X]]$ and so $\mathcal{R}_0^2([[X]]) = [[X]]$. Assume $\{X, X^{op}\} \subsetneq Sim(X)$ and let $Z \in Sim(X)\backslash\{X, X^{op}\}$. The coefficient of Z in $\widetilde{\mathcal{R}}_0 \circ \widetilde{\mathcal{R}}_0(X)$ is given by

$$\frac{1}{s(X)^2} \sum_{Y \in Sim_0(X)} (-1)^{|X^{\#} \cap Y^{\#}| + |Y^{\#} \cap Z^{\#}|} = \frac{1}{s(X)^2} \sum_{Y \in Sim_0(X)} (-1)^{|Y^{\#} \cap (X^{\#} \ominus Z^{\#})|} = 0$$

by Lemma (2.6.2). It follows that the coefficient of $[[Z]]$ in $\mathcal{R}_0^2([[X]]) = [[\widetilde{\mathcal{R}}_0 \circ \widetilde{\mathcal{R}}_0([X])]]$ vanishes for any $[[Z]] \neq [[X]]$.

On the other hand, the coefficient of $[X]$ in $\widetilde{\mathcal{R}}_0 \circ \widetilde{\mathcal{R}}_0([X])$ is

$$\frac{1}{s(X)^2} |Sim_0(X)| = \frac{2^{|X^{\ominus}|-1}}{2^{|X^{\ominus}| - def(X_{sp})}} = \frac{2^{|X^{\ominus}|-1}}{2^{|X^{\ominus}|-1}} = 1$$

if $X \in Sim_0(X)$, and 0 otherwise. Similarly, the coefficient of $[X]^{op}$ in $\widetilde{\mathcal{R}}_0 \circ \widetilde{\mathcal{R}}_0$ is 1 if $X^{op} \in Sim_0(X)$, and 0 otherwise.

Since $[[X]] \in \mathcal{S}^{od}$, one and only one between X and X^{op} lies in $Sim_0(X)$, and so the coefficient of $[[X]]$ in $\mathcal{R}_0^2([[X]])$ is given by the sum of the coefficients of $[X]$ and $[X^{op}]$ in $\widetilde{\mathcal{R}}_0 \circ \widetilde{\mathcal{R}}_0([X])$, that is 1, whence $\mathcal{R}_0^2 = id$.

Now we prove that the map $\mathcal{R}_0$ is an isometry, showing that it maps an orthonormal basis to an orthonormal basis. For any $[[X]] \in \mathcal{S}^{od}$, it holds

$$\langle \mathcal{R}_0([[X]]), \mathcal{R}_0([[Z]]) \rangle = \frac{1}{s(X)^2} \langle \sum_{Y_1 \in Sim_0(X)} (-1)^{|X^{\#} \cap Y_1^{\#}|} [[Y_1]], \sum_{Y_2 \in Sim_0(Z)} (-1)^{|Z^{\#} \cap Y_2^{\#}|} [[Y_2]] \rangle.$$

If $Sim(X) \neq Sim(Z)$, the two sums have no symbols in common and hence this scalar product is 0.

If $Sim(Z) = Sim(X)$, since X has odd defect, any $Y \in Sim(X)$ has odd defect, hence $Y \in Sim_0(X)$ implies $Y^{op} \notin Sim_0(X)$. Therefore $[[Y_1]] = [[Y_2]]$ in the above sums if and only if $[Y_1] = [Y_2]$. It follows that

$$\begin{aligned}\langle \mathcal{R}_0([[X]]), \mathcal{R}_0([[Z]])\rangle &= \frac{1}{s(X)^2} \sum_{Y \in Sim_0(X)} (-1)^{|X^\# \cap Y^\#| + |Y^\# \cap Z^\#|} \\ &= \frac{1}{s(X)^2} \sum_{Y \in Sim_0(X)} (-1)^{|Y^\# \cap (X^\# \ominus Z^\#)|}.\end{aligned}$$

If $[[Z]] \neq [[X]]$, this sum vanishes by (2.88).

If $[[Z]] = [[X]]$, we have

$$\langle \mathcal{R}_0([[X]]), \mathcal{R}_0([[X]])\rangle = \frac{1}{2^{|X^\#|-1}} |Sim_0(X)| = 1.$$

□

Let M be a maximal standard Levi subgroup of G of type $A_{r-1} \times B_{n-r}$ (or $A_{r-1} \times C_{n-r}$) for some $1 \leqslant r \leqslant n$.

We define

$$\mathcal{R}_0^M : R_u(M^F) \to R_u(M^F)$$

as the map making the following diagram commutative:

$$\begin{array}{ccc} R_u(M^F) & \xrightarrow{\mathcal{R}_0^M} & R_u(M^F) \\ \Big\downarrow{\scriptstyle \oplus(\kappa_A \boxtimes \ell^{n-r,2s+1}) \circ Rep_{M^F}} & & \Big\downarrow{\scriptstyle \oplus(\kappa_A \boxtimes \ell^{n-r,2s+1}) \circ Rep_{M^F}} \\ \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\mathcal{S}_n^{od}] & \xrightarrow{id \otimes \mathcal{R}_0} & \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\mathcal{S}_n^{od}] \end{array} \tag{2.89}$$

As for G, the map $\mathcal{R}_0$ on $R_u(M^F)$ is an involutive isometry.

Remark 2.6.4. Since the Fourier transform on groups of type A is the identity, by an argument analogous to the one used in Remark 2.4.10, the compatibility of the Fourier transform with the parabolic restriction to a maximal standard Levi subgroup implies the compatibility of the Fourier transform with the parabolic restriction to any standard Levi subgroup.

Indeed if L is a standard F-stable Levi subgroup, then L is isogenous to a direct product of groups of type A and a group of type B_m (or C_m) for some $0 \leqslant m \leqslant n$. It follows that L is a standard Levi subgroup of the maximal Levi subgroup M of type $A_{n-m-1} \times B_m$ (or $A_{n-m-1} \times C_m$). Since the Fourier transform is the identity on groups of type A, it commutes with the map ${}^*R_{\widetilde{L}^0}^{\widetilde{M}^0} \oplus {}^*R_{\widetilde{L}^1}^{\widetilde{M}^1}$. By transitivity of parabolic restriction, to prove compatibility of the Fourier transform with parabolic restriction from G to a standard Levi subgroup L it is enough to prove compatibility of the Fourier transform with parabolic restriction from G to a maximal Levi subgroup.

By Lemma 2.3.7, for any $r \in \mathbb{N}$ the map $\mathcal{H}_r$ factors through $[[\ ]] : \mathbb{C}[\widetilde{\mathcal{S}}] \to \mathbb{C}[\mathcal{S}]$. It follows that the map Res_r as in (2.31) induces a well defined map

$$\begin{aligned} Res_r : \mathbb{C}[\mathcal{S}]^{nd} &\to \mathbb{C}[\mathscr{P}_r] \otimes \mathbb{C}[\mathcal{S}]^{nd} \\ [[X]] &\mapsto \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\chi_\gamma) \boxtimes \mathcal{H}_\gamma([[X]]) \end{aligned} \tag{2.90}$$

The following proposition shows that the map Res_r is a combinatorial description of the parabolic restriction to maximal Levi subgroups.

Proposition 2.6.5. *Let $s, r, n \in \mathbb{N}$ satisfy $s^2 + s \leqslant n - r$.*
Then the following diagram is commutative

$$\begin{array}{ccc} \mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{G^F}\circ \ell^{n,2s+1}} & \mathbb{C}[\mathcal{S}_n^{2s+1}] \\ \Big\downarrow {}^*R^{G^F}_{M^F} & & \Big\downarrow Res_r \\ \mathbb{C}[\mathcal{E}^{M^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{M^F}\circ(\kappa_A\boxtimes\ell^{n-r,2s+1})} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathcal{S}^{2s+1}_{n-r}] \end{array}$$

Proof. The diagram in the statement factors as

$$\begin{array}{ccccc} \mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{G^F}} & R(W(B_{n-(s^2+s)})) & \xrightarrow{\ell^{n,2s+1}} & \mathbb{C}[\mathcal{S}_n^{2s+1}] \\ \Big\downarrow {}^*R^{G^F}_{M^F} & & \Big\downarrow Res^{W(B_{n-(s^2+s)})}_{\mathbb{S}_r\times W(B_{n-(s^2+s)-r})} & & \Big\downarrow Res_r \\ \mathbb{C}[\mathcal{E}^{M^F}(L_s,\sigma_s)] & \xrightarrow{Rep_{M^F}} & R(\mathbb{S}_r\times W(B_{n-r-(s^2+s)})) & \xrightarrow{\kappa_A\boxtimes\ell^{n-r,2s+1}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathcal{S}^{2s+1}_{n-r}] \end{array}$$

The left square commutes by Corollary 1.1.5, the right square commutes by Corollary 2.1.6 and Lemma 2.3.6. ☐

Corollary 2.6.6. *The following diagram is commutative*

$$\begin{array}{ccc} R_u(G^F) & \xrightarrow{\bigoplus \ell^{n,2s+1}\circ Rep_{G^F}} & \mathbb{C}[\mathcal{S}_n^{od}] \\ \Big\downarrow {}^*R^{G^F}_{M^F} & & \Big\downarrow Res_r \\ R_u(M^F) & \xrightarrow{\bigoplus(\kappa_A\boxtimes\ell^{n-r,2s+1})\circ Rep_{G^F}} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathcal{S}^{od}_{n-r}] \end{array}$$

Proof. The restriction of diagram (2.6.6) to a subspace $\mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)]$ with s such that $s + s^2 \leqslant n - r$ commutes by Proposition 2.6.5. The restriction of diagram (2.6.6) to a subspace $\mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)]$ with s such that $s + s^2 > n - r$ commutes because the vertical maps are 0. Since the subspaces $\mathbb{C}[\mathcal{E}^{G^F}(L_s,\sigma_s)]$ give a direct sum decomposition of $R_u(G^F)$, diagram (2.6.6) commutes.

☐

Lemma 2.6.7. *Let $r \in \mathbb{N}_{\leqslant n}$. The following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}[\mathcal{S}_n^{od}] & \xrightarrow{\mathcal{R}_0} & \mathbb{C}[\mathcal{S}_n^{od}] \\ \Big\downarrow Res_r & & \Big\downarrow Res_r \\ \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathcal{S}^{od}_{n-r}] & \xrightarrow{id\boxtimes\mathcal{R}_0} & \mathbb{C}[\mathscr{P}_r]\otimes\mathbb{C}[\mathcal{S}^{od}_{n-r}] \end{array}$$

Proof. As a consequence of Theorem 2.4.4 the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{S}^{od}] & \xrightarrow{\mathcal{R}_0} & \mathbb{C}[\mathcal{S}^{od}] \\ \Big\downarrow \mathcal{H}_r & & \Big\downarrow \mathcal{H}_r \\ \mathbb{C}[\mathcal{S}^{od}] & \xrightarrow{\mathcal{R}_0} & \mathbb{C}[\mathcal{S}^{od}] \end{array} \tag{2.91}$$

Then for $[[X]] \in \mathcal{S}_n^{od}$, it holds

$$
\begin{aligned}
Res_r \circ \mathcal{R}_0([[X]]) &= \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\delta_\gamma) \otimes \mathcal{H}_\gamma(\mathcal{R}_0(X)) \\
&= \sum_{\gamma \in \mathscr{P}_r} \kappa_A(\delta_\gamma) \otimes \mathcal{R}_0(\mathcal{H}_\gamma(X)) \\
&= (id \boxtimes \mathcal{R}_0)\Big(\sum_{\gamma \in \mathscr{P}_r} \kappa_A(\delta_\gamma) \otimes \mathcal{H}_\gamma(X)\Big) = (id \boxtimes \mathcal{R}_0) \circ Res_r(X).
\end{aligned}
$$

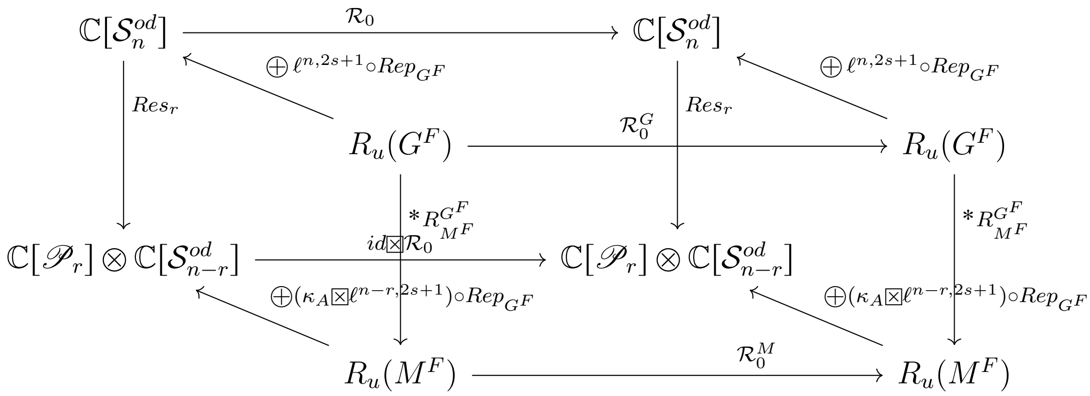
□

The following theorem (and the corollary below), stating the compatibility of the Lusztig's Fourier transform $\mathcal{R}_0$ for groups of type B and parabolic restriction (respectively induction) is the main result of this section.

Theorem 2.6.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
R_u(G^F) & \xrightarrow{\mathcal{R}_0^G} & R_u(G^F) \\
\downarrow {}^*R_{M^F}^{G^F} & & \downarrow {}^*R_{M^F}^{G^F} \\
R_u(M^F) & \xrightarrow{\mathcal{R}_0^M} & R_u(M^F)
\end{array}
$$

Proof. We consider the following diagram.



The upper and lower faces commute by definition of $\mathcal{R}_0$ on the spaces of class functions, the side faces commute because of Corollary 2.6.6, and the back face commutes because of Lemma 2.6.7.

The front face commutes because all the other faces commute and the diagonal arrows $\bigoplus \ell^{n,2s+1} \circ Rep_{G^F}$ and $\bigoplus(\kappa_A \boxtimes \ell^{n-r,2s+1}) \circ Rep_{G^F}$ are isomorphisms. □

Corollary 2.6.9. *The following diagram commutes:*

$$
\begin{array}{ccc}
R_u(M^F) & \xrightarrow{\mathcal{R}_0^M} & R_u(M^F) \\
\downarrow R_{M^F}^{G^F} & & \downarrow R_{M^F}^{G^F} \\
R_u(G^F) & \xrightarrow{\mathcal{R}_0^G} & R_u(G^F)
\end{array}
\tag{2.92}
$$

Proof. The statement follows from Theorem 2.6.8 by adjunction, because $\mathcal{R}_0^G$ and $\mathcal{R}_0^M$ are involutive isometries. □

Nomenclature

$\mathscr{P}^+$ The set of partitions admitting 0s, page 4

$\mathscr{P}_n^+$ The set of partitions of n admitting 0s, page 4

$\mathscr{P}$ The set of partitions, page 4

$\mathscr{P}_n$ The set of partitions of n, page 4

$l(\alpha)$ The length of the partition α, page 4

$\varrho(\alpha)$ The rank of the partition α, page 4

$\mathscr{B}_n$ The set of bipartitions of n, page 4

$\mathscr{D}_n$ The set of unordered bipartitions of n, page 4

κ_A The bijection between $IrrW(A_{n-1})$ and $\mathscr{P}_n$ as in (2.2), page 5

κ_B The bijection between $IrrW(B_n)$ and $\mathscr{B}_n$ as in (2.3), page 5

κ_D The bijection between $IrrW(D_n)$ and $\mathscr{D}_n$ as in (2.4), page 6

$H_r(\alpha)$ The set of r-hooks of the partition α, page 10

$l(i,j)$ The leg length of $(i,j) \in H_r(\alpha)$, page 10

$\mathcal{H}_r$ The removing r-hook map for bipartitions as in (2.7) , page 11

Res_r The map on bipartitions as in (2.11), page 12

$\rightarrow^k$ The shift operation, page 13

$\mathscr{P}(\mathbb{N})^{<\infty}$ The set of subsets of $\mathbb{N}$ of finite cardinality, page 13

$\varrho(A)$ The rank of $A \in \mathscr{P}(\mathbb{N})^{<\infty}$, page 13

$\mathcal{B}$ The bijection in (2.13), page 14

$def(X)$ The defect of the array X, page 15

$\rho(X)$ The rank of the array X, page 15

$\widetilde{\mathcal{S}}$ The set of ordered symbols, page 16

$\widetilde{\mathcal{S}}_n$ The set of ordered symbols of rank n, page 16

$\widetilde{\mathcal{S}}^d$ The set of ordered symbols of defect d, page 16

$\widetilde{\mathcal{S}}^{od,0}, \widetilde{\mathcal{S}}^{od,1}, \widetilde{\mathcal{S}}^{ev,0}, \widetilde{\mathcal{S}}^{ev,1}$ The subsets of $\widetilde{\mathcal{S}}$ as in (2.18), page 16

$\widetilde{\mathscr{S}}_{n,d}$ The bijection as in (2.19) and its linear extension., page 17

$\widetilde{\mathscr{S}}^{\pm}_{n,d}$ The bijection as in (2.21) (resp.(2.22) for $d = 0$) and its linear extension, page 18

$\widetilde{\ell^{n,d}}$ The bijection as in (2.23) (resp.(2.24) for $d = 0$) and its linear extension, page 18

$\widetilde{Symb_D}$ The isomorphism as in (2.25), page 19

$H_r(A)$ The set of the r-hooks of $A \in \mathscr{P}(\mathbb{N})^{<\infty}$., page 19

$l(h)$ The leg length of $h \in H_r(A)$., page 19

$H_r(X)$ The set of r-hooks of the array X, page 21

$l(h,i)$ The leg length of $(h,i) \in H - r(X)$., page 21

$\mathcal{H}_r$ The removing r-hooks map on Symbols as in (2.26), page 21

Res_r The map on $\mathbb{C}[\widetilde{\mathcal{S}}]$ as in (2.31), page 24

$X^{\ominus}$ The entries that appear in one row of the array X but not in both of them, page 27

$Sim(X)$ The similarity class of the array X, page 27

X_{sp} The special array in $Sim(X)$, page 28

$X^{\#}$ $X^1 \ominus X^1_{sp}$, page 28

$\widetilde{\mathcal{R}}$ The Lusztig's Fourier transform on ordered Symbols, page 28

$\mathcal{S}$ The set of unordered Symbols, page 35

$\mathcal{S}_n$ The set of unordered Symbols of rank n, page 35

$\mathcal{S}$ The set of unordered Symbols of defect d, page 35

$[[\ ]]$ The map from $\mathbb{C}[\widetilde{\mathcal{S}}]$ to $\mathbb{C}[\mathcal{S}]$ as in (2.42), page 35

$\mathbb{C}[\mathcal{S}]^{nd}$ The image of the map $[[\ ]]$, page 36

$\mathbb{C}[\mathcal{S}]^C$ the orthogonal complement of $\mathbb{C}[\mathcal{S}]^{nd}$ in $\mathbb{C}[\mathcal{S}]$, page 36

$\langle\langle\ \rangle\rangle$ The map from $\mathbb{C}[\widetilde{\mathcal{S}}^{ev}]$ to $\mathcal{A}^{ev}$ as in (2.43), page 36

$\mathcal{A}^{ev,0}\mathcal{A}^{ev,1}$ The subspaces of $\mathcal{A}^{ev}$ as in (2.44), page 36

$\mathcal{A}^{ev}_n$, $\mathcal{A}^d$ The image of $\mathbb{C}[\widetilde{\mathcal{S}}_n]$ and of $\mathbb{C}[\widetilde{\mathcal{S}}^d_n]\bigsqcup\mathbb{C}[\widetilde{\mathcal{S}}^{-d}_n]$ through $\langle\!\langle\ \rangle\!\rangle$, page 36

$R(W(D_n))^C$ The orthogonal complement of $Cl(W(D_n))$ in $R(W(D_n))$, page 38

$\ell^{n,2s}$, $\ell^{n,2s}_{\delta}$ The linear isomorphisms as in Proposition 2.5.3, page 38

$d(X)$ $\frac{def(X)-def(X_{sp})}{2}$, page 42

$Sim_i(X)$ The subsets of $Sim(X)$ as in (2.60) , page 42

$\widetilde{\mathcal{R}}_i$ The maps on $\mathbb{C}[\widetilde{\mathcal{S}}]$ as in (2.61), (2.62) for $i = 0, 1$, page 42

$\mathcal{R}_0$ The automorphism of $\mathbb{C}[\mathcal{S}]$ as in (2.65), page 45

$\mathcal{Q}_0$ The isomorphism from $\mathbb{C}[\mathcal{S}^{ev,1}]$ to $\mathcal{A}^{ev,0}$ as in (2.67), page 45

$\mathcal{R}_1$ The isomorphism from $\mathcal{A}^{ev,0}$ to $\mathbb{C}[\mathcal{S}^{ev,1}]$ as in (2.68), page 45

$\mathcal{Q}_1$ The automorphism of $\mathcal{A}^{ev,1}$ as in (2.69), page 46

$\ell^{n,2s+1}$ The bijection as in (2.85) and its linear extension , page 54

$Symb_B$ The isomorphism as in (2.86), page 54

Chapter III

Langlands and Macdonald Correspondence

Introduction

In this chapter, we begin by studying the representation theory of p-adic groups and its connection with that of finite groups of Lie type. In particular, we focus here on general linear groups, while in Chapter IV we address the case of special linear groups.

The research direction of this and the next chapter is inspired by ideas presented in [68]. We give a quick overview of these ideas in general terms before specializing to the case relevant for this chapter, namely that of GL_N.

Let F be a non-archimedean local field of characteristic 0, with residue field k_F. Let $\mathbb{G}$ be a connected, reductive, F-split linear algebraic group defined over F, and let $\mathbf{G} = \mathbb{G}(F)$ denote its group of F-points.

A (tame) Langlands parameter for $\mathbf{G}$ is a pair (ρ, E), where ρ is a continuous homomorphism from the Weil group of F into $\mathbb{G}^\vee(\mathbb{C})$, the complex dual group of $\mathbb{G}$ [5, Section 2.1], and E is a nilpotent element in the Lie algebra of $\mathbb{G}^\vee$ [5, Section 8.1], satisfying certain compatibility conditions.

The (tame) Langlands correspondence is conjectured to be a surjective map with finite fibers, called L-packets from the set of isomorphism classes of (depth-0) smooth admissible representations of $\mathbf{G}$ to the set of equivalence classes of (tame) Langlands parameters for $\mathbf{G}$. Within an L-packet, representations are conjecturally parametrized by irreducible representations of a (slightly modified) component group associated with the centralizer of the parameter. For a detailed overview, see [37].

The finite group of Lie type $\overline{\mathbf{G}} = \mathbb{G}(k_F)$, given by the k_F-points of $\mathbb{G}$, arises as a quotient of the hyperspecial maximal compact subgroup $\mathbf{K} = \mathbb{G}(\mathcal{O}_F)$. The most comprehensive and general parameterization for irreducible representations of finite groups of Lie type is due to Lusztig [42, 43]. While it resembles a Langlands parameterization in some respects, it differs in some important aspects. In [35], the relation between Lusztig's construction and a Langlands-type parameterization is discussed in detail, and [35, Conjecture 4.3] proposes a set of Langlands parameters for the group $\overline{\mathbf{G}}$, which can be viewed as equivalence classes of the tame Langlands parameters for $\mathbf{G}$. As in the p-adic case, the conjecture [35, Conjecture 4.3] predicts that for each parameter, the corresponding packet is parametrized by the irreducible representations of a component group derived from the parameter.

Additionally, in [68] a compatibility between this parameterization and the depth-0 Langlands correspondence for $\mathbf{G}$ is proposed: if an irreducible representation π of $\overline{\mathbf{G}}$ corresponds to the equivalence class of a tame Langland parameter (ρ, E), then its inflation to $\mathbf{K}$ should appear as a $\mathbf{K}$-subrepresentation of a representation in the L-packet associated to (ρ, E).

Furthermore, there is a natural map ι between the component groups of the centralizers in $\mathbb{G}^\vee(\mathbb{C})$ of (ρ, E), whose irreducible representations parametrize the L-packet for (ρ, E), and the component group of the centralizer in $\mathbb{G}^\vee(\mathbb{C})$ of the equivalence class of (ρ, E), whose irreducible representations parametrize the representations of $\overline{\mathbf{G}}$ associated to the equivalence class of (ρ, E). This map is expected to link the parametrizations of irreducible representations of $\overline{\mathbf{G}}$ associated to the equivalence class of (ρ, E) and the representations in the L-packet of $\mathbf{G}$

corresponding to (ρ, E): if π is associated with a representation ζ of the second component group, then the representation of $\mathbf{G}$ in the L-packet containing the inflation of π is the one associated with $\zeta \circ \iota$.

In the case $\mathbf{G} = \mathrm{GL}_N$, the local Langlands correspondence is established in [27, 30, 60], and an explicit description for depth-0 representations, i.e. for the tame Local Langland correspondence, is provided in [6, 7, 10]. We review this description in Sections 3.1.1 and 3.2.1.3. A parametrization for the irreducible representations of $\mathrm{GL}_N(k_F)$ consistent with the perspective in [35] was already constructed in [47], as we recall in Sections 3.1.2.1 and 3.2.2.

In this case, both parameterizations are bijections, simplifying the expected compatibility: each Langlands parameter corresponds to a unique irreducible admissible representation of $\mathrm{GL}_N(F)$, whose restriction to $\mathrm{GL}_N(\mathcal{O}_F)$ contains the inflation of the $\mathrm{GL}_N(k_F)$-representation associated with the equivalence class of the parameter. The proof of this compatibility is the main result of this chapter, and it is given by Theorem 3.2.32. The main steps toward establishing this theorem are given by Proposition 3.1.11, where the compatibility is proved in the case of cuspidal representations and irreducible parameters, and Theorem 3.2.24, where the compatibility is proved in the case of essentially square integrable representations and indecomposable parameters.

The work presented in this chapter was developed independently. However, after completing the main results, I became aware of [64, Appendix A], which proves the same compatibility. The argument there relies on the construction in [59], particularly on the "reduction to tempered types" map, which in the depth-0 setting coincides with the "head of the parahoric restriction" defined in Section 3.63.

The simplifications arising here by working entirely in the depth-0 setting allow this chapter to avoid the full machinery of the Bushnell–Kutzko theory of types (see [11]), which plays a major role in [59]. While we do rely on types for some key technical points (e.g., Proposition 3.2.15), the overall presentation here is more direct and self-contained. As such, despite the lack of originality in the main result, this chapter offers an independent, and in some respects simpler, account of the compatibility in the case of GL_N.

Notation

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- ◇ Let F be a non-archimedean local field of characteristic 0 with residue field k_F, the finite field with q elements. Let $\mathcal{O}_F$ be the valuation ring in F and $\mathfrak{p}_F$ be the maximal ideal of $\mathcal{O}_F$, and let $\mathcal{O}_F^*$ denote the units in $\mathcal{O}_F$. We denote by $\overline{k_F}$ the algebraic closure of k_F.

  For any E finite extension of F, we use analogous notation.

  We denote by W_F the Weil group of F. We denote by I_F and P_F respectively the inertia and the wild ramification subgroups of W_F, and by Fr a Frobenius element of W_F.

- ◇ Let H be a group. We denote by $\widehat{H}$ the character group of H, sometimes with $H^\wedge$ for notation reasons. With a slight abuse of notation, when $H = F^*$ or $H = W_F$, we write $\widehat{F^*}$ and $\widehat{W_F}$ for the groups of tamely ramified characters of F^* and W_F respectively.

  If H acts on a set X, for any $x \in X$ we write $Stab_H(x)$ for the subgroup of H fixing x

- ◇ Let H be a group and let $K \leqslant H$. If π is a representation of H, we write Res^H_K, or sometimes $\pi|_K$, for the restriction of π to K.
  If γ is a representation of K, we write $Ind^H_K\gamma$ for the induced representation and $\text{c-ind}^H_K \gamma$ for the compactly induced representation.
  For any $h \in H$, we write ${}^h\gamma$ for the representation of ${}^hK = hKh^{-1}$ defined by ${}^h\gamma(x) = \gamma(h^{-1}xh)$ for any $x \in {}^hK$.

  Let N be a normal subgroup of H. If π is a representation of H, we denote by π^N the H/N representation on the N-fixed subspace of π. If γ is a representation of H/N, we denote by $Infl^H_{H/N}\gamma$ the inflated representation.

- ◇ We denote by $G_n := \mathrm{GL}_n(F)$. We denote by $\Omega(G_n)_0$ the set of isomorphism classes of irreducible admissible representations of G_n of depth 0.

  We set K_n to be the hyperspecial maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ of G_n, and we denote by K_n^+ its pro-unipotent radical. We write $\overline{G}_n := K_n / K_n^+ \cong \mathrm{GL}_n(k_F)$.

  We write the parahoric restriction for G_n as

$$\begin{aligned} \mathcal{P}^{G_n}_{\overline{G}_n} : \Omega(G_n)_0 &\to Irr(\overline{G}_n) \\ \pi &\mapsto (Res^{G_n}_{K_n}\pi)^{K_n^+} \end{aligned} \tag{3.1}$$

- ◇ We denote the normalized parabolic induction functor by R. We will mainly need it for the general linear group, so we introduce a notation tailored on the situations in which we will use it. Let $n \in \mathbb{N}$ and $(m_1, \ldots, m_k)$ be a composition of n, and let π_j (respectively $\overline{\pi}_j$) be a representation of G_{m_j} (respectively $\overline{G}_{m_j}$) for $j \in \{1, \ldots, k\}$. Then we write $\mathrm{R}^{G_n}_{\prod_{j=1}^k G_{m_j}} \boxtimes_{j=1}^k \pi_j$ (respectively $\mathrm{R}^{\overline{G}_n}_{\prod_{j=1}^k \overline{G}_{m_j}} \boxtimes_{j=1}^k \overline{\pi}_j$) for the parabolic induction to G_n (respectively to $\overline{G}_n$) from the standard Levi subgroup sitting diagonally in G_n (respectively $\overline{G}_n$) with blocks of the prescribed dimensions, where the inflation is performed through the parabolic subgroup of block upper triangular matrices containing the aforementioned Levi subgroup.

3.1 The cuspidal - irreducible case

3.1.1 Explicit Local Langlands Correspondence for depth-0 supercuspidal representations

3.1.1.1 Admissible pairs

Let $N \in \mathbb{N}$. We recall the explicit description of the local Langlands correspondence for supercuspidal representations of G_N of depth 0 given in [6, 7, 10].

Definition 3.1.1. *[6, 0.3]Let E/F be a finite, tamely ramified field extension and let ζ be a character of E^*. The pair $(E/F, \zeta)$ is called admissible if it satisfies the following two conditions, where K ranges over intermediate fields, $F \subseteq K \subseteq E$:*

- ◇ *If ζ factors through the relative norm $N_{E/K}$, then $K = E$.*
- ◇ *If $\zeta|_{1+\mathfrak{p}_E}$ factors through $N_{E/K}$, then E/K is unramified.*

Two admissible pairs $(E/F, \zeta), (E/F, \zeta')$ are F-isomorphic if there exists an F-isomorphism $j : E \to E'$ such that $\zeta = \zeta' \circ j$. In the case $E = E'$ this corresponds to $\zeta' = \zeta^\sigma$ for some $\sigma \in Gal(E/F)$.

We denote by $P_N(F)_0$ the set of F-isomorphism classes of admissible pairs $(E/F, \zeta)$ such that E/F is of degree N and ζ is a tamely ramified character, i.e. $\zeta|_{1+\mathfrak{p}_E} = 1$. This implies that E/F is unramified, by the definition of admissible pairs.

Lemma 3.1.2. *Let E/F be an unramified extension of degree N and let ζ be a tamely ramified character of E^*. Then the following are equivalent*

1. *the pair $(E/F, \zeta)$ is admissible;*
2. *$\zeta \neq \zeta^\sigma$ for any non trivial $\sigma \in Gal(E/F)$;*
3. *$\zeta|_{\mathcal{O}_E^*} \neq \zeta^\sigma|_{\mathcal{O}_E^*}$ for any non trivial $\sigma \in Gal(E/F)$.*

Proof. When $N = 2$ this is [8, Lemma 19.1]. The proof given there actually applies to any N, we write it here for the reader's convenience. Conditions (2) and (3) are equivalent since E is an unramified extension, therefore $E^* = F^*\mathcal{O}_E^*$. Let K be an intermediate extension $F \subseteq K \subseteq E$. Then E/K is unramified and so it is cyclic, so by Hilbert's theorem 90 the kernel of $N_{E/K}$ contains all and only the elements of shape $x\sigma(x^{-1})$ with $x \in E^*$ for σ a generator of $Gal(E/K)$. Therefore ζ factorizes through $N_{E/K}$ if and only if $\zeta = \zeta^\sigma$ for some generator σ of $Gal(E/K)$. Since the subgroup generated by an element in $Gal(E/F)$ is the Galois group of some intermediate extension K, it follows that the first condition of admissibility is equivalent to condition (2). The second condition of admissibility is empty in this case, since ζ is unramified. □

Thus $P_N(F)_0$ is the set of F-equivalence classes of admissible pairs $(E/F, \zeta)$ where E/F is unramified of degree N and $\zeta \in E^*$ is tamely ramified, with $\zeta \neq \zeta^\sigma$

for any non trivial $\sigma \in Gal(E/F)$. Since the unramified extension of degree N of F is unique up to isomorphism, we can consider such an extension E to be fixed, and an F-equivalence class of admissible pairs corresponds to an orbit of the Galois group acting on the tamely ramified characters of E^*.

3.1.1.2 Parameterization of supercuspidal representations of depth 0

We denote by $Cusp(\overline{G}_N)$ the set of isomorphism classes of irreducible cuspidal representations of $\overline{G}_N$.

We denote by $Cusp(G_N)$ the set of isomorphism classes of irreducible supercuspidal representations of G_N, and by $Cusp(G_N)_0 = Cusp(G_N) \cap \Omega(G_N)_0$ the set of isomorphism classes of irreducible supercuspidal representations of G_N of depth 0. Following [6, 2.2], to an element of $P_N(F)_0$ we can associate an element of $Cusp(G_N)_0$ as follows. Let $(E/F, \zeta) \in P_N(F)_0$. Since ζ is tamely ramified, the character $\zeta|_{\mathcal{O}_E^*}$ is the inflation of a character $\overline{\zeta}$ of $\mathcal{O}_E^* \big/ 1 + \mathfrak{p}_E \cong k_E$ and by Lemma 3.1.2 the pair $(E/F, \zeta)$ is admissible if and only if $\zeta|_{\mathcal{O}_E^*} \neq \zeta^\sigma|_{\mathcal{O}_E^*}$ for any $\sigma \in Gal(E/F)$. Identifying $Gal(E/F) \cong Gal(k_E/k_F)$, this is equivalent to $\overline{\zeta} \neq \overline{\zeta}^\sigma$. In other words, there is a surjection

$$\begin{aligned} P_N(F)_0 &\to \{Gal(k_E/k_F) \text{ -orbits on } \widehat{k_E^*} \text{ of cardinality } N\} \\ \zeta &\to Gal(k_E/k_F)\overline{\zeta} \end{aligned} \tag{3.2}$$

By [26], there is a canonical bijection between $Gal(k_E/k_F)$-orbits of characters of k_E^* of cardinality N and cuspidal representations of $\overline{G}_N = GL_N(k_F)$:

$$\begin{aligned} \{Gal(k_E/k_F) \text{ - orbits on } \widehat{k_E^*} \text{ of cardinality } N\} &\to Cusp(\overline{G}_N) \\ Gal(k_E/k_F)\overline{\zeta} &\mapsto \lambda_{\overline{\zeta}} \end{aligned} \tag{3.3}$$

So let $\lambda_{\overline{\zeta}}$ be the cuspidal representation of $\overline{G}_N$ corresponding to $Gal(k_E/k_F)\overline{\zeta}$ through this bijection. We denote by $\widetilde{\lambda}_{\overline{\zeta}}$ the inflation of this representation to K_N, obtained letting K_N^+ act trivially. Then $\widetilde{\lambda}_{\overline{\zeta}}$ can be extended to a representation Λ_ζ of F^*K_N by requiring that $\Lambda_\zeta|_{F^*}$ is a scalar multiple of $\zeta|_{F^*}$. Finally, we set

$$_F\pi_\zeta := \text{c-ind}_{F^*K_N}^{G_N} \Lambda_\zeta.$$

Proposition 3.1.3. *[6, Proposition 2.2] For any $(E/F, \zeta) \in P_N(F)_0$ the representation $_F\pi_\zeta$ defined as above is an irreducible supercuspidal representation of G_N of depth 0, and the isomorphism class of $_F\pi_\zeta$ depends only on the F-equivalence class of the admissible pair $(E/F, \zeta)$. The map*

$$\begin{aligned} {}_F\pi : P_N(F)_0 &\to Cusp(G_N)_0 \\ (E/F, \zeta) &\mapsto {}_F\pi_\zeta \end{aligned} \tag{3.4}$$

is a canonical bijection.

3.1.1.3 Parameterization of tamely ramified representations of W_F

We write $\Phi^0(G_N)$ for the set of equivalence classes of irreducible smooth (complex) representations of W_F of dimension N.

By local class field theory, Artin reciprocity map

$$a_E : W_E \to E^*$$

yields by transposition a canonical isomorphism between $\widehat{E^*}$ and $\widehat{W_E}$ [8, 29.1, 29.2]. Hence any character ζ of E^* can be regarded as a character $\zeta \circ a_E$ of W_E. Note that since $a_E(I_E) = \mathcal{O}_E^*$ and $a_E(P_E) = 1 + \mathfrak{p}_E$, characters that are trivial on $\mathcal{O}_E^*$ correspond to characters that are trivial on I_E, and characters that are trivial on $1+\mathfrak{p}_E$ correspond to characters that are trivial on P_E [8, 29.2]. We denote by $\Phi^0(G_N)_0$ the subset of $\Phi^0(G_N)$ consisting of tamely ramified (or tame) representations of W_F, i.e., representations whose restriction to P_F is trivial.

Proposition 3.1.4. *The map*

$$\begin{aligned} \Sigma : P_N(F)_0 &\to \Phi^0(G_N)_0 \\ (E/F, \zeta) &\mapsto \Sigma_\zeta := Ind_{W_E}^{W_F}(\zeta \circ a_E) \end{aligned} \tag{3.5}$$

is a canonical bijection.

Proof. The map (3.5) is the restriction to $P_N(F)_0$ of the canonical bijection in [6, A.3]. So it is enough to prove that $\zeta \in E^*$ is trivial on $1 + \mathfrak{p}_E$ if and only if $Ind_{W_E}^{W_F}(\zeta \circ a_E)$ is trivial on P_F. Recall that ζ is trivial on $1 + \mathfrak{p}_E$ if and only if $(\zeta \circ a_E)$ is trivial on P_E.

Since it will be useful in the following, we first compute the restriction of $Ind_{W_E}^{W_F}\zeta$ to I_F. We apply Mackey's theorem [39], and since the restriction of any finite dimensional representation of W_F to I_F is semisimple we can write

$$Res_{I_F}^{W_F} \circ Ind_{W_E}^{W_F}(\zeta \circ a_E) = \bigoplus_{x \in I_F \diagdown W_F \diagup W_E} Ind_{I_F \cap {}^x W_E}^{I_F} Res_{I_F \cap {}^x W_E}^{W_E} {}^x(\zeta \circ a_E). \tag{3.6}$$

Since E/F is Galois, W_E is normal in W_F [8, Proposition 28.5 (2)] and since E/F is unramified, $I_F = I_E \subseteq W_E$. So any conjugate of I_E lies in W_E and hence

$$I_F \diagdown W_F \diagup W_E = {}^{W_F} \diagup_{W_E}.$$

Therefore

$$Res_{I_F}^{W_F} \circ Ind_{W_E}^{W_F}(\zeta \circ a_E) = \bigoplus_{x \in W_F/W_E} Ind_{I_F \cap {}^x W_E}^{I_F} Res_{I_F \cap {}^x W_E}^{W_E} {}^x(\zeta \circ a_E). \tag{3.7}$$

Moreover by normality of W_E in W_F it holds ${}^x W_E = W_E$ for any $x \in W_F/W_E$, and hence $I_F \cap {}^x W_E = I_F \cap W_E = I_F = I_E$ where the last two equalities follow from the fact that E/F is unramified. Hence

$$\begin{aligned} Res_{I_F}^{W_F} \circ Ind_{W_E}^{W_F}(\zeta \circ a_E) &= \bigoplus_{x \in W_F/W_E} Ind_{I_F}^{I_F} Res_{I_F}^{W_E}\ {}^x(\zeta \circ a_E) \\ &= \bigoplus_{x \in W_F/W_E} Res_{I_F}^{W_E}\ {}^x(\zeta \circ a_E). \end{aligned} \tag{3.8}$$

Since E/F is unramified, the ramifications groups of E and F are the same, i.e. $P_E = P_F$, so

$$Res_{P_F}^{W_F} \circ Ind_{W_E}^{W_F}(\zeta \circ a_E) = \oplus_{x \in W_F/W_E} Res_{P_F}^{I_F} Res_{I_F}^{W_E}\, {}^x(\zeta \circ a_E) = \oplus_{x \in W_F/W_E} Res_{P_E}^{W_E}\, {}^x(\zeta \circ a_E)$$

so $Res_{P_F}^{W_F} \circ Ind_{W_E}^{W_F}(\zeta \circ a_E)$ is trivial if and only if each term in the last direct sum is trivial, that is equivalent to the condition that $Res_{(1+\mathfrak{p}_E)}^{E^*}\zeta$ is trivial. □

3.1.1.4 Description of Langlands correspondence

Composing the inverse of the bijection (3.5) with the bijection (3.4) one obtains a bijection

$$\begin{aligned} {}_F\mathcal{N}_N : \Phi^0(G_N)_0 &\to Cusp(G_N)_0 \\ {}_F\Sigma_\zeta &\mapsto {}_F\pi_\zeta. \end{aligned} \tag{3.9}$$

Now let

$$\mathscr{L}_N^0 : \Phi^0(G_N)_0 \to Cusp(G_N)_0 \tag{3.10}$$

denote the tame local Langlands correspondence as in [7]. By [10, 5.2 Corollary 3], for any $(E/F, \zeta) \in P_N(F)_0$, there exists a character μ_ζ of E^* such that

$$\mathscr{L}_N^0(\Sigma_\zeta) = {}_F\pi_{\mu_\zeta \zeta}. \tag{3.11}$$

Moreover, in the case we are considering, it follows from [10, 5.2 Corollary 3, 8.2 Proposition 21] that μ_ζ is unramified and $\mu_\zeta(p_F) = (-1)^{N-1}$ where p_F is an uniformizer of $\mathcal{O}_F$. Note that since E/F is unramified, p_F is an uniformizer in $\mathcal{O}_E$ as well.

Remark 3.1.5. By [6, Proposition 3.2], the map $\mathscr{L}_1^0$ is the restriction to tame characters of the bijection of local class field theory induced by the Artin reciprocity map [9, Section 29], and for any tame character $\chi \in \widehat{W_F}$ and any $\rho \in \Phi(G_N)_0$ it holds

$$\mathscr{L}_n^0(\chi\rho) = (\mathscr{L}_1^0(\chi) \circ det)\mathscr{L}_N^0(\rho). \tag{3.12}$$

3.1.2 Parahoric restriction and Macdonald's correspondence

Our aim in the following is to give a reduction of the Langlands correspondence over a finite field for the "supercuspidal-irreducible" case. In particular, we will recall a construction of Macdonald [47] that gives a correspondence between the restriction to the inertia subgroup of irreducible Weil representations of dimension N and cuspidal representations of $\overline{G}_N$, (3.15). We show that the compatibility between Macdonald's correspondence and Langlands correspondence is given by the Parahoric restriction functor, which will be defined in 3.1.3.

3.1.2.1 Macdonald correspondence for $\overline{G}_N$: the cuspidal-irreducible case

Following [47, Section 3] two irreducible smooth representations ρ_1, ρ_2 of W_F of dimension N are called I_F-equivalent if

$$\rho_1|_{I_F} \cong \rho_2|_{I_F}.$$

We denote by ${}^{\Phi^0(G_N)_0}\!\diagup_{\sim_{I_F}}$ the set of I_F-equivalence classes in $\Phi^0(G_N)_0$. A particular case of [47] is the following:

Proposition 3.1.6. *There is a canonical bijection*

$$\mathscr{M}_N^0 : {}^{\Phi^0(G_N)_0}\!\diagup_{\sim_{I_F}} \to Cusp(\overline{G}_N).$$

We recall briefly the description of the bijection given in [47] in this special case. Let $k = k_F$ and $\overline{Fr}$ be the Frobenius automorphism $x \mapsto x^q$ of $\overline{k}/k$. We denote by k_n the extension of k of degree n and let

$$\Gamma := \varinjlim \widehat{k_n^*}$$

where the direct system is defined by setting the transition morphisms to be the transposes of the norm maps. Since $\overline{Fr}$ acts on any $\widehat{k_n}$ and the action is compatible with the norm maps, the Frobenius $\overline{Fr}$ acts on Γ, and we can identify $\widehat{k_N^*}$ with $(\Gamma)^{\overline{Fr}^N}$. In particular regular characters of k_N^* are identified with elements of Γ belonging to $\overline{Fr}$-orbits in Γ of cardinality N, and so $\overline{Fr}$-orbits in Γ of cardinality N are in canonical bijection with $Gal(k_N/k)$-orbits in $\widehat{k_N^*}$ of cardinality N. Therefore the map (3.3) yields a canonical bijection

$$\begin{aligned} \{\overline{Fr}\text{-orbits in } \Gamma \text{ of cardinality } N\} &\to Cusp(\overline{G}_N) \\ \langle Fr \rangle \overline{\zeta} &\mapsto \lambda_{\overline{\zeta}}. \end{aligned} \tag{3.13}$$

The character group of ${}^{I_F}\!\diagup_{P_F}$ is canonically isomorphic to Γ, and the action of a Frobenius element Fr of W_F on ${}^{I_F}\!\diagup_{P_F}$ induced by conjugacy in W_F is the same as the action of $\overline{Fr}$ on Γ described above. Therefore there is a natural identification

$$\{\overline{Fr}\text{-orbits in } \Gamma \text{ of cardinality } N\} \cong \{Fr\text{-orbits in } {}^{I_F}\!\diagup_{P_F} \text{ of cardinality } N\}.$$

If $\rho \in \Phi^0(G_N)_0$, then $\rho|_{P_F} = 1$, so $\rho|_{I_F}$ is the inflation through P_F of a representation of ${}^{I_F}\!\diagup_{P_F}$. Moreover since ρ is an irreducible representation of W_F, the restriction $\rho|_{I_F}$ is the direct sum of all and only the characters belonging to a single Fr-orbit in Γ of cardinality N [47, Section 3]. Therefore there is a bijection

$$\begin{aligned} {}^{\Phi^0(G_N)_0}\!\diagup_{\sim_{I_F}} &\to \{Fr\text{-orbits in } {}^{I_F}\!\diagup_{P_F} \text{ of cardinality } N\} \\ \rho &\mapsto \rho|_{I_F}. \end{aligned} \tag{3.14}$$

Composing the maps (3.13) and (3.14) yields a description of the bijection

$$\mathscr{M}_N^0 : {}^{\Phi^0(G_N)_0}\!\diagup_{\sim_{I_F}} \to Cusp(\overline{G}_N). \tag{3.15}$$

Remark 3.1.7. We denote by F^{ur} a maximal unramified extension of F, and denote by F_n the unramified extension of F of degree n contained in F^{ur}. For any $n \in \mathbb{N}$, it holds $I_{F_n} = I_F$ and $P_{F_n} = P_F$, since we are taking into account unramified extensions. For any $d, m \in \mathbb{N}$ such that $d|m$, the Artin reciprocity maps fit into the following commutative diagram [8, 29.1, (4)]

$$\begin{array}{ccc} W_{F_m} & \xrightarrow{a_{F_m}} & F_m^* \\ \downarrow & & \downarrow N_{m/d} \\ W_{F_d} & \xrightarrow{a_{F_d}} & F_d^* \end{array} \tag{3.16}$$

where the left vertical map is the inclusion and $N_{m/d}$ denotes the norm map between F_m and F_d. Since for any $n \in \mathbb{N}$ it holds that $a_{F_n}(I_{F^n}) = \mathcal{O}_{F_n}^*$ and $a_{F_n}(P_{F^n}) = 1+\mathfrak{p}_{F_n}$, the diagram (3.16) induces the following commutative diagram:

$$\begin{array}{ccccc} I_F / P_F \cong I_{F_m} / P_{F_m} & \xrightarrow{\overline{a}_{F_m}} & \mathcal{O}_{F_m}^* / 1+\mathfrak{p}_{F_m} \cong k_m^* \\ \downarrow id & & \downarrow \overline{N}_{m/d} \\ I_F / P_F \cong I_{F_d} / P_{F_d} & \xrightarrow{\overline{a}_{F_d}} & \mathcal{O}_{F_d}^* / 1+\mathfrak{p}_{F_d} \cong k_d^* \end{array} \tag{3.17}$$

where k_m and k_d denote respectively the residue fields of F_m and F_d, and $\overline{N}_{m/d}$ denotes the map induced by the norm map on the residue fields, that is the norm map between k_m and k_d.

It follows that for any $n \in \mathbb{N}$ the transpose of the map $\overline{a}_{F_n}$ realizes the inclusion of $\widehat{k_n^*}$ into Γ considered as the character group of I_F / P_F.

Moreover a character χ of I_F / P_F factors through $\overline{a}_{F_n}$ if and only if it is Fr^n-stable. Therefore the transpose of the map $\overline{a}_{F_n}$ is an isomorphism between $\widehat{k_n^*}$ and the subgroup of Γ given by Fr^n-stable characters.

Remark 3.1.8. Retain the notation from Remark 3.1.7.

The character group of k_F^* acts on $Irr(\overline{G}_N)$, with a character χ acting by $\pi \mapsto \pi \otimes (\chi \circ det)$ for $\pi \in Irr(\overline{G}_N)$. Moreover $\widehat{k_F^*}$ acts on $\widehat{k_m^*}$ by $\gamma \mapsto (\chi \circ N_m)\gamma$ for any $m \in \mathbb{N}$, and hence it acts on Γ. The map (3.13) is equivariant with respect to these actions of k_F^* by [38, Proposition, Section 3].

By Remark 3.1.7, the transpose of the map $\overline{a}_F$ yields an isomorphism between $\widehat{k_F^*}$ and the set of Frobenius stable characters of I_F / P_F. Any Frobenius stable character χ of I_F / P_F can be extended to W_F by letting the Frobenius elements act trivially, and we still denote by χ the extended character. Then χ acts on $\widehat{W_F}$ and on $\widehat{I_F / P_F}$ by multiplication. The embedding of $\widehat{k_m^*}$ into $\widehat{I_F / P_F}$ given by the transpose of $\overline{a}_{F_m}$ is equivariant with respect to the $\widehat{k_F^*}$-actions because diagram (3.17) is commutative. So the identification of Γ with $\widehat{I_F / P_F}$ is equivariant with respect to the $\widehat{k_F^*}$-actions. It follows that the map (3.14) is equivariant.

Therefore the map $\mathscr{M}_N^0$ is equivariant with respect to the $\widehat{k_F^*}$-actions.

3.1.3 From Langlands to Macdonald correspondence

We write $\mathcal{P}^{G_N}_{\overline{G}_N}$ for the parahoric restriction with respect to the maximal compact subgroup K_N.The parahoric restriction is an exact functor: the restriction functor is exact, and taking the invariant subspace for a compact open subgroup is exact [8, 2.3 Corollary 1].

Proposition 3.1.9. *Let $(E/F, \zeta) \in P_N(F)_0$. Then*

$$\mathcal{P}^{G_N}_{\overline{G}_N}\ {}_F\pi_{\mu_\zeta \zeta} = \lambda_{\overline{\zeta}}$$

where μ_ζ is as in (3.11)*, ${}_F\pi_{\mu_\zeta\zeta}$ is the image of $(E/F, \zeta)$ through the bijection* (3.4) *and $\lambda_{\overline{\zeta}}$ is the image of ζ through the composition of the maps* (3.2) *and* (3.3)*.*

Proof. We use the same notation as in Subsection 3.1.1.2. We show that $\lambda_{\overline{\zeta}}$ is a constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}\ {}_F\pi_{\mu_\zeta\zeta}$. Using the definition of ${}_F\pi_{\mu_\zeta\zeta}$ and Mackey's formula, we have

$$\begin{aligned} Res^{G_N}_{K_N}\ {}_F\pi_{\mu_\zeta\zeta} &= Res^{G_N}_{K_N} \text{ c-ind}^{G_N}_{F^*K_N} \Lambda_{\mu_\zeta\zeta} \\ &= \bigoplus_{x \in K_N \diagdown G_N \diagup F^*K_N} Ind^{K_N}_{K_N \cap {}^xF^*K_N} Res^{F^*K_N}_{K_N \cap {}^xF^*K_N} {}^x\Lambda_{\mu_\zeta\zeta}. \end{aligned}$$

The summand corresponding to 1 in the double coset is $Res^{F^*K_N}_{K_N \cap F^*K_N} \Lambda_{\mu_\zeta\zeta}$, and by construction $\Lambda_{\mu_\zeta\zeta}$ is an extension to F^*K_N of $\widetilde{\lambda}_{\overline{\mu_\zeta\zeta}}$, the inflation to K_N of the cuspidal representation $\lambda_{\overline{\mu_\zeta\zeta}}$ of $\overline{G}_N$. Note that since μ_ζ is an unramified character, $\mu_\zeta\zeta|_{\mathcal{O}_E^*} = \zeta|_{\mathcal{O}_E^*}$, therefore $\overline{\mu_\zeta\zeta} = \overline{\zeta}$, and hence

$$\widetilde{\lambda}_{\overline{\zeta}} = \widetilde{\lambda}_{\overline{\mu_\zeta\zeta}}.$$

It follows that

$$Res^{F^*K_N}_{K_N \cap F^*K_N} \Lambda_{\overline{\mu_\zeta\zeta}} = \widetilde{\lambda}_{\overline{\mu_\zeta\zeta}} = \widetilde{\lambda}_{\overline{\zeta}}.$$

By construction $\widetilde{\lambda}_{\overline{\zeta}}|_{K_N^+} = 1$, and the action of $\overline{G}_N \cong {K_N}\diagup{K_N^+}$ is given by $\lambda_{\overline{\zeta}}$. So since $Res^{G_N}_{K_N}\ {}_F\pi_{\mu_\zeta\zeta}$ contains $\widetilde{\lambda}_{\overline{\zeta}}$ as K_N-subrepresentation, $\mathcal{P}^{G_N}_{\overline{G}_N}\ {}_F\pi_{\mu_\zeta\zeta}$ contains $\lambda_{\overline{\zeta}}$ as an irreducible component.
The stronger assertion that $\lambda_{\overline{\zeta}}$ is the unique constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}\ {}_F\pi_{\mu_\zeta\zeta}$ and it has multiplicity 1 is an application of [55, Proposition 6.12]. □

Proposition 3.1.10. *Let $(E/F, \zeta) \in P_N(F)_0$. Then*

$$Res^{W_F}_{I_F}\ {}_F\Sigma_\zeta = \bigoplus_{w \in Gal(k_E/k_F)} (Infl^{I_F}_{I_F \diagup P_F}({}^w\overline{\zeta}) \circ \overline{a}_E). \tag{3.18}$$

Proof. By (3.8), it holds

$$Res^{W_F}_{I_F}\ {}_F\Sigma_\zeta = \bigoplus_{w \in W_F/W_E} Res^{W_E}_{I_F} {}^w(\zeta \circ a_E).$$

Since E/F is unramified, $I_F = I_E$ and so is normal in W_E, and since $a_E(I_E) = \mathcal{O}_E^*$ it holds, as I_E-representations

$$Res_{I_E}^{W_E}\ {}^{w}(\zeta \circ a_E) = {}^{w}(Res_{\mathcal{O}_E^*}^{E}(\zeta) \circ a_E).$$

But ζ is unramified, so

$$Res_{\mathcal{O}_E^*}^{E}(\zeta) = Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}\ \overline{\zeta}$$

where $\overline{\zeta}$ is the representation of $k_E \cong \mathcal{O}_E^*/1+\mathfrak{p}_E$ given by $\zeta|_{\mathcal{O}_E^*}$.

For any w in W_F, let $\overline{w}$ denote its image in $Gal(\overline{k_F})$. Then the norm $\|w\|$ is defined by the equation $\overline{w}(x) = x^{\|w\|}$ for any element $x \in \overline{k_F}$. It holds $wxw^{-1} \equiv x^{\|w\|}$ mod P_F for any $x \in I_F$, therefore $a_E(wxw^{-1}) \equiv a_E(x)^{\|w\|}$ mod $1 + \mathfrak{p}_F$. For any $y \in \mathcal{O}_E^*$, we write $\overline{y} \in k_E$ for the image through the projection mod $1+\mathfrak{p}_F$. Then for any $x \in I_F$

$$\begin{aligned}
{}^{w}(Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}\ \overline{\zeta} \circ a_E)(x) &= (Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}\ \overline{\zeta})(a_E(w^{-1}xw)) \\
&= \overline{\zeta}(\overline{a_E(w^{-1}xw)}) \\
&= \overline{\zeta}(\overline{a_E(x)}^{\|w^{-1}\|}) \\
&= \overline{\zeta}(\overline{w^{-1}}\ \overline{a_E(x)}\ \overline{w^{-1}}) \\
&= {}^{\overline{w^{-1}}}\overline{\zeta}(\overline{a_E(x)}) \\
&= Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}({}^{\overline{w^{-1}}}\overline{\zeta})(a_E(x)) = Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}({}^{\overline{w^{-1}}}\overline{\zeta}) \circ a_E(x).
\end{aligned}$$

The assignment $w \mapsto \overline{w}$ establishes a well-defined bijection $W_F/W_E \to Gal(k_E/k_F)$, so in conclusion we have

$$Res_{I_F}^{W_F}\ {}_F\Sigma_\zeta = \bigoplus_{w \in W_F/W_E} {}^{w}(Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}\ \overline{\zeta} \circ a_E) = \bigoplus_{\overline{w} \in Gal(k_E/k_F)} Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}({}^{\overline{w}}\overline{\zeta}) \circ a_E.$$

Then defining $\overline{a}_E$ as in Remark 3.1.7, the following diagram is commutative

$$\begin{array}{ccc}
I_F = I_E & \xrightarrow{\quad a_E \quad} & \mathcal{O}_E^* \\
\downarrow & & \downarrow \\
I_F/P_F = I_E/P_E & \xrightarrow{\overline{a}_E} & \mathcal{O}_E^*/1+\mathfrak{p}_E \cong k_E^*
\end{array} \tag{3.19}$$

Therefore

$$Infl_{\mathcal{O}_E^*/1+\mathfrak{p}_E}^{\mathcal{O}_E^*}(\overline{\zeta}^{\overline{w}}) \circ a_E = Infl_{I_F/P_F}^{I_F}(\overline{\zeta}^{\overline{w}} \circ \overline{a}_E)$$

gives the statement.

∎

Proposition 3.1.11. *The following diagram commutes:*

$$\begin{array}{ccc}
\Phi^0(G_N)_0 & \xrightarrow{\mathscr{L}_N^0} & Cusp(G_N)_0 \\
\downarrow{\scriptstyle Res_{I_F}^{W_F}} & & \downarrow{\scriptstyle \mathcal{P}_{\overline{G}_N}^{G_N}} \\
\Phi^0(G_N)_0/\sim_{I_F} & \xrightarrow{\mathscr{M}_N^0} & Cusp(\overline{G}_N)
\end{array} \tag{3.20}$$

Proof. Let $(E/F, \zeta) \in P_N(F)_0$. By Proposition 3.1.10 the representation $Res^{W_F}_{I_F}\ {}_F\Sigma_\zeta$ is the direct sum of the inflation to I_F of the characters $\{({}^\sigma\overline{\zeta}) \circ \overline{a}_E \mid \sigma \in Gal(k_E/k_F)\}$ of $\Gamma \cong {}^{I_F}\!\diagup_{P_F}$, so the I_F-equivalence class of the representation ${}_F\Sigma_\zeta$ corresponds through the map (3.14) to the orbit by the Frobenius action in Γ of $\overline{\zeta} \circ \overline{a}_E$. By Remark 3.1.7, the character $\overline{\zeta} \circ \overline{a}_E$ of Γ is identified with the character $\overline{\zeta}$ of k_E, and since the map (3.3) gives the assignemnt $Gal(k_E/k_F)\overline{\zeta} \mapsto \lambda_{\overline{\zeta}}$, the map (3.13) maps the Frobenius-orbit in Γ of $\overline{\zeta} \circ \overline{a}_E$ to $\lambda_{\overline{\zeta}}$. Since the Macdonald correspondence $\mathscr{M}^0_N$ is obtained composing the bijections (3.14) and (3.13), it follows that

$$\mathscr{M}^0_N({}_F\Sigma_\zeta) = \lambda_{\overline{\zeta}}.$$

On the other hand, $\mathscr{L}^0_N(\Sigma_\zeta) = \pi_{\mu_\zeta \zeta}$ so the statament follows from Proposition 3.1.9. ☐

3.2 The non-cuspidal case

In this section we describe the compatibility between Langlands and Macdonald correspondence for non-cuspidal representations.

3.2.1 Local Langlands correspondence in the non-cuspidal case

3.2.1.1 Zelevensky parameterization

In this section we recall the classification of the irreducible admissible representations of G_n from supercuspidal ones from [70]. We denote by $\Omega(G_n)$ the set of irreducible admissible representations of G_n, and by $\Omega(G_n)_0$ the subset of irreducible representations of G_n with supercuspidal support of depth 0.

A segment $\Delta = \Delta(\pi, r)$ with $r \in \mathbb{N}$ and $\pi \in Cusp(G_m)_0$ for some $m \in \mathbb{N}$, is a subset of $Cusp(G_m)_0$ of the form

$$\Delta(\pi, r) := \{\pi \otimes |det(\cdot)|^i,\ i = 0, \ldots, r-1\}.$$

Two segments Δ_1, Δ_2 are said to be linked if none of neither is contained in the other and $\Delta_1 \cup \Delta_2$ is still a segment. The segment $\Delta_1 = \Delta(\pi_1, r_1)$ precedes the segment $\Delta_2 = \Delta(\pi_2, r_2)$ if Δ_1 and Δ_2 are linked and $\pi_2 = \pi_1 \otimes |\det()|^k$ for some $k \in \mathbb{N}$.
Let $r \in \mathbb{N}$ and $\pi \in \Omega(G_m)_0$ for $m \in \mathbb{N}$. We set

$$\rho(\Delta(\pi, r)) := \mathrm{R}^{G_{mr}}_{G^r_m}(\pi \boxtimes (\pi \otimes |det(\cdot)|) \boxtimes \cdots \boxtimes (\pi \otimes |det(\cdot)|^{r-1}))$$

where $\mathrm{R}^{G_{mr}}_{G^r_m}$ denotes the normalized parabolic induction functor with respect to the standard parabolic subgroup of G_{mr} containing G^r_m consisting of block upper triangular matrices.

Proposition 3.2.1. *[70, Proposition 2.10] For any segment Δ, the representation $\rho(\Delta)$ has a unique irreducible quotient $Q(\Delta)$ (and a unique irreducible subrepresentation $Z(\Delta)$).*

Remark 3.2.2. Let $r, m \in \mathbb{N}$ and $\pi \in Cusp(G_m)_0$. Let $\Delta = \Delta(\pi, r)$, and let $\chi \in \widehat{F^*}$. We set $\chi\Delta := \Delta(\pi \otimes (\chi \circ det), r)$. Then

$$Q(\chi\Delta) = Q(\Delta) \otimes (\chi \circ det). \tag{3.21}$$

Indeed $\rho(\chi\Delta) = \rho(\Delta) \otimes (\chi \circ det)$ because parabolic induction is compatible with twisting by a character, and (3.21) follows from uniqueness of the quotient.

Let $\{\Delta_1, \ldots, \Delta_k\}$ be a multiset (i.e. a set allowing repetitions of elements, see [70, Notation]) of segments, where $\Delta_i = \Delta(\pi_i, r_i)$ with $\pi_i \in Cusp(G_{m_i})_0$ and $m_i, r_i \in \mathbb{N}$ for $1 \leqslant i \leqslant k$ are such that $\sum_{i=1}^k m_i r_i = N$. We set

$$I(\Delta_1, \ldots \Delta_k) := \mathrm{R}^{G_N}_{\prod_{i=1}^k G_{m_i r_i}} (Q(\Delta_1) \boxtimes Q(\Delta_2) \boxtimes \cdots \boxtimes Q(\Delta_k)).$$

Theorem 3.2.3. *([70, Theorem 6.1], see also [58, Theorem 3]) Let $\{\Delta_1, \ldots \Delta_k\}$ be a multiset of segments such that Δ_i does not precedes Δ_j for any $i \leqslant j$, where $\Delta_i = \Delta(\pi_i, r_i)$ with $\pi_i \in Cusp(G_{m_i})_0$ and $m_i, r_i \in \mathbb{N}$ for $1 \leqslant i \leqslant k$ are such that $\sum_{i=1}^k m_i r_i = N$. Then the representation $I(\Delta_1, \Delta_2, \ldots, \Delta_k)$ has a unique irreducible quotient $Q(\Delta_1, \Delta_2, \ldots, \Delta_k)$.*
Any admissible irreducible representation of G_N is isomorphic to a representation of the form $Q(\Delta_1, \Delta_2, \ldots, \Delta_k)$ for some k and some uniquely determined (up to permutations) multiset $\{\Delta_1, \ldots \Delta_k\}$.

Remark 3.2.4. We retain notation from Remark 3.2.2. For any $\chi \in \widehat{F^*}$ it holds

$$Q(\chi\Delta_1, \ldots, \chi\Delta_k) = Q(\Delta_1, \ldots, \Delta_k) \otimes (\chi \circ det) \tag{3.22}$$

for $\{\Delta_1, \ldots, \Delta_k\}$ a multiset of segments as in Theorem 3.2.3. Indeed by the compatibility of parabolic induction with the twisting by a character

$$I(\chi\Delta_1, \ldots, \chi\Delta_k) = I(\Delta_1, \ldots, \Delta_k) \otimes (\chi \circ det)$$

and (3.2.4) follows from the uniqueness of the quotient.

An elementary move on a multiset $\{\Delta_1 \ldots \Delta_k\}$ of segments is the replacement of two linked segments Δ_i, Δ_j with $1 \leqslant i, j \leqslant k$ by the segments $\Delta_i \cup \Delta_j$ and $\Delta_i \cap \Delta_j$, omitting the intersection if it is trivial [70, Section 7.1].

Theorem 3.2.5. *([70, Theorem 9.7], see also [58, Proposition 13], [58, Theorem 5, Remark 7]) The representation $I(\Delta_1, \ldots, \Delta_k)$ is irreducible if and only if none of the segments Δ_i is linked to another.*

A representation $Q(\Delta'_1, \Delta'_2, \cdots, \Delta'_{k'}))$ is an irreducible subquotient of $I(\Delta_1, \ldots, \Delta_k)$ if and only if the multiset of segments $\{\Delta'_1, \ldots, \Delta'_{k'}\}$ can be obtained from the segments $\{\Delta_1, \ldots, \Delta_k\}$ by a sequence of elementary moves.

Remark 3.2.6. A representation of G_{mr} with $m, r \in \mathbb{N}$ is essentially square integrable, i.e., square integrable up to a twist by a character, if and only if it is of the form $Q(\Delta)$ for some segment $\Delta = \Delta(\pi, r)$, and it is square integrable if $\pi \otimes |det(\cdot)|^{\frac{r-1}{2}}$, i.e. the middle element of Δ, has unitary central character [70, Theorem 9.3]. A representation $Q(\Delta_1, \ldots, \Delta_k)$ of G_N is tempered if and only if $Q(\Delta_i)$ is square integrable for any $i = 1, \ldots, k$. Note that if $Q(\Delta_1, \ldots, \Delta_k)$ is tempered, any segment Δ_i has middle element with unitary central character, so it cannot be linked to another. It follows that for a tempered representation $Q(\Delta_1, \ldots, \Delta_k)$ there holds $Q(\Delta_1, \ldots, \Delta_k) = I(\Delta_1, \ldots, \Delta_k)$.

3.2.1.2 Weil-Deligne representations

The group G_n is split, and its complex dual group is $\mathrm{GL}_n(\mathbb{C})$. In this case, the set of Langlands parameters for G_n is given by equivalence classes of Weil-Deligne representations. We now define Weil-Deligne representations and explain the conventions we adopt.

The Weil-Deligne group of F is the group $W_F' := \mathbb{C} \rtimes W_F$, with $wxw^{-1} := \|w\|x$ for any $w \in W_F$ and $x \in \mathbb{C}$, where $\| \cdot \|$ denotes the norm in W_F. Let $n \in \mathbb{N}$. An n-dimensional complex representation of W_F' is an homomorphism

$$\phi : W_F' \to \mathrm{GL}_n(\mathbb{C})$$

that is trivial on an open subgroup of I_F. It can be described through a pair (ρ, u) where: ρ is a smooth n-dimensional representations of W_F and u is a unipotent element in $\mathrm{GL}_n(\mathbb{C})$ satisfying

$$\rho(w)u\rho(w)^{-1} = u^{\|w\|} \qquad \text{for any } w \in W_F.$$

The pair is obtained from Φ by setting $(\rho, u) = (\phi|_{W_F}, \phi((1, 1_{W_F}))$, where $1 \in \mathbb{C} \leqslant W_F'$. Moreover for $u \in \mathrm{GL}_n(\mathbb{C})$ unipotent let E_u be the nilpotent element in the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ satysfing $exp(E_u) = u$. Then, the assignment $(\rho, u) \mapsto (\rho, E_u)$ is a bijection between the set of n-dimensional representations of W_F' and the set of pairs given by a smooth n-dimensional representation ρ of W_F and a nilpotent element $E_u \in \mathfrak{gl}_n(\mathbb{C})$ satisfying

$$Ad(\rho(w))(E_u) = \|w\|E_u \tag{3.23}$$

for any $w \in W_F$.

From now on, by an n-dimesional Weil-Deligne representation we mean a pair (ρ, E) consisting of a smooth n-dimensional representation ρ of W_F and a nilpotent element $E \in \mathfrak{gl}_n(\mathbb{C})$ satisfying (3.23), as in [17, 3.1.1]. Two n-dimensional Weil-Deligne representations (ρ_i, E_i), for $i = 1, 2$ are equivalent if there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $A\rho_1 A^{-1} = \rho_2$ and $Ad(A)E_1 = E_2$.

Let (ρ_1, E_1), (ρ_2, E_2) be two Weil-Deligne representations of dimension respectively n_1, n_2. Their tensor product is defined as

$$(\rho_1, E_1) \otimes (\rho_2, E_2) = (\rho_1 \otimes \rho_2, E_1 \otimes I_{n_2} + I_{n_1} \otimes E_2).$$

A Weil-Deligne representation (ρ, E) is said to be semisimple if ρ is semisimple as a representation of W_F, or equivalently if the image $\rho(Fr)$ of a Frobenius element $Fr \in W_F$ is semisimple as an element of $\mathrm{GL}_n(\mathbb{C})$, [8, Proposition 28.7]. We denote by $\Phi(G_n)$ the set of equivalence classes of semisimple n-dimensional Weil-Deligne representations. From now on, we will always assume Weil-Deligne representations to be semisimple.

A Weil-Deligne representation φ is irreducible if and only if there is a $\rho \in \Phi^0(G_n)$, i.e. ρ is an irreducible n-dimesional smooth representation of W_F, such that $\varphi = (\rho, 0)$ [17, Proposition 3.1.3 (i)].

We denote by $Sp(n)$ the n-dimensional special Weil-Deligne representation defined in [17, 3.1.2]. By definition, $Sp(n)$ is given by the pair (ρ, E) such that, with respect of some basis for the representation space of ρ, it holds $\rho(w) = diag(1, \|w\|, \ldots, \|w\|^{n-1})$ for any $w \in W_F$ and E is an n-dimesional nilpotent Jordan block.

An n-dimesional Weil-Deligne representation ϕ is indecomposable if and only if there is a d diving n and an irreducible d-dimesional irreducible Weil-Deligne representation φ such that $\phi = \varphi \otimes Sp(\frac{n}{d})$ [17, Proposition 3.1.3 (ii)].
A Weil-Deligne representation (ρ, E) is said to be tamely ramified (or tame) if ρ is tamely ramified as a representation of W_F.

We denote by $\Phi(G_n)_0$ the set of equivalence classes of semisimple tamely ramified n-dimensional Weil-Deligne representations. In particular an irreducible tamely ramified Weil-Deligne representation is of the form $(\rho, 0)$, with ρ an irreducible tamely ramified n-dimensional smooth representation of W_F, i.e., $\rho \in \Phi^0(G_n)_0$. Note that the special Weil-Deligne representations are tamely ramified, so an indecomposable n-dimensional Weil-Deligne representation $\varphi \otimes Sp(\frac{n}{d})$ is tamely ramified if and only if the d-dimensional irreducible Weil-Deligne representation φ is tamely ramified.

3.2.1.3 Local Langlands correspondence

The Zelevinsky parameterization allows one to deduce the Langlands parameterization for irreducible representations of G_N from the Langlands parameterization of supercuspidal representations of G_n for any $n \leqslant N$.

For any $n \in \mathbb{N}$, the tamely ramified supescuspidal Langlands correspondence is the bijection (3.10). Any tamely ramified Weil-Deligne representation $\phi \in \Phi(G_N)_0$ is a sum of indecomposable ones, so

$$\phi = \oplus_{i=1}^{k} \varphi_i \otimes Sp(r_i)$$

where each $\varphi_i \in \Phi^0(G_{m_i})_0$ and $m_i, r_i \in \mathbb{N}$ are such that $\sum_{i=1}^{k} m_i r_i = N$. So for all i there is a $\rho_i \in \Phi^0(G_{m_i})_0$ such that $\varphi_i = (\rho_i, 0)$, and $\mathscr{L}^0_{m_i}(\rho_i)$ is a supercuspidal representation of G_{m_i} of depth 0. Therefore we can associate to ϕ the collection of segments $\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i = 1, \ldots, k\}$ and hence, by Theorem 3.2.3, we associate to ϕ the irreducible representation of G_N given by $Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i = 1, \ldots, k\})$. The tamely ramified local Langlands correspondence for N-dimesional representations is then given by the map:

$$\begin{aligned} \mathscr{L}_N : \Phi(G_N)_0 &\to \Omega(G_N)_0 \\ \oplus_{i=1}^{k} (\rho_i, 0) \otimes Sp(r_i) &\mapsto Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i = 1, \ldots, k\}). \end{aligned} \tag{3.24}$$

Remark 3.2.7. Let $\chi \in \widehat{W_F}$ be a tame character. The map $\mathscr{L}_N$ satisfies

$$\mathscr{L}_N(\chi\rho) = (\mathscr{L}_1(\chi) \circ det)\mathscr{L}_N(\rho) \tag{3.25}$$

Indeed for any $(\phi, E) = \oplus_{i=1}^{k} (\rho_i, 0) \otimes Sp(r_i)$ in $\Phi(G_N)_0$ it holds

$$\mathscr{L}_N(\chi\rho) = Q(\{\Delta(\mathscr{L}^0_{m_i}(\chi\rho_i, r_i) \mid i = 1, \ldots, k\}).$$

By Remark 3.1.5

$$\mathscr{L}^0_{m_i}(\chi\rho_i) = (\mathscr{L}_1(\chi) \circ det)\mathscr{L}_N(\rho),$$

and by Remark 3.2.4 it holds

$$Q(\{\Delta((\mathscr{L}_1(\chi)\circ det)\mathscr{L}_N(\rho), r_i) \mid 1 \leqslant i \leqslant k\}) = Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i, r_i) \mid 1 \leqslant i \leqslant k\})\otimes(\mathscr{L}_1(\chi)\circ det).$$

3.2.2 Macdonald correspondence: the general case

In this section, we briefly recall a construction from [47]. It builds on the correspondence recalled in Section 3.1.2.1 between I_F-equivalence classes of irreducible n-dimensional Weil representations and cuspidal representations of $\overline{G}_n$, with $n \leqslant N$, and gives a correspondence between I_F-equivalence classes of N-dimensional Weil-Deligne representations, that we are going to define next, and irreducible representations of $\overline{G}_N$.

3.2.2.1 I_F-equivalence

Two n-dimensional Weil-Deligne representations (ρ_i, E_i), for $i = 1, 2$ are called I_F-equivalent if there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that

$$A(\rho_1|_{I_F})A^{-1} = \rho_2|_{I_F}, \tag{a}$$

$$Ad(A)E_1 = E_2. \tag{b}$$

In this case, we write $(\rho_1, E_1) \sim_{I_F} (\rho_2, E_2)$. We denote by $\Phi(G_N)_0 \big/ \sim_{I_F}$ the set of I_F-equivalence classes in $\Phi(G_N)_0$, and by $(\rho, E)_{I_F}$ the I_F-equivalence class of $(\rho, E) \in \Phi(G_N)_0$.

The next Lemma will be useful in Chapter IV. For any $(\rho, E) \in \Phi(G_N)_0$, we denote by $C_{\mathrm{GL}_N(\mathbb{C})}(\rho|_{I_F}, E)$ the simultaneous centralizer of all the elements in $\rho(I_F)$ and E. We denote by $C^0_{\mathrm{GL}_N(\mathbb{C})}(\rho|_{I_F}, E)$ the connected component of the identity.

Lemma 3.2.8. *Let $(\rho_i, E_i) \in \Phi(G_N)_0$ for $i = 1, 2$ be such that $(\rho_1, E_1) \sim_{I_F} (\rho_2, E_2)$. Then for any $A \in \mathrm{GL}_N(\mathbb{C})$ satisfying* (a),(b) *it holds*

$$A(\rho_1(Fr)C^0_{\mathrm{GL}_N(\mathbb{C})}(\rho_1|_{I_F}, E_1))A^{-1} = \rho_2(Fr)C^0_{\mathrm{GL}_N(\mathbb{C})}(\rho_2|_{I_F}, E_2). \tag{c}$$

Proof. The centralizer in $\mathrm{GL}_N(\mathbb{C})$ of any subset $S \subseteq \mathrm{M}_N(\mathbb{C})$ is connected. Indeed it is a principal open subset of an affine space, namely the centralizer of S in $\mathrm{M}_N(\mathbb{C})$ [33, Section 1.2]. So in particular for $i = 1, 2$ it holds

$$C^0_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F}, E_i) = C_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F}, E_i).$$

If $A \in \mathrm{GL}_N(\mathbb{C})$ satisfies conditions (a) and (b), then

$$AC_{\mathrm{GL}_N(\mathbb{C})}(\rho_1|_{I_F}, E_1)A^{-1} = C_{\mathrm{GL}_n(\mathbb{C})}(A(\rho_1|_{I_F})A^{-1}, AE_1A^{-1}) = C_{\mathrm{GL}_N(\mathbb{C})}(\rho_2|_{I_F}, E_2).$$

Therefore A satisfies (c) if and only if

$$\rho_2(Fr)^{-1}A\rho_1(Fr)A^{-1} \in C_{\mathrm{GL}_n(\mathbb{C})}(\rho_2|_{I_F}, E_2).$$

The ρ_i's are tame representations, so they are representations of ${W_F}/{P_F} = {I_F}/{P_F} \rtimes \langle Fr \rangle$, where Fr acts by powering to the q every element in ${I_F}/{P_F}$. Therefore using condition (a) twice we get

$$\begin{aligned}
&\rho_2(Fr)^{-1}A\rho_1(Fr)A^{-1}\rho_2|_{I_F}A\rho_1(Fr)^{-1}A^{-1}\rho_2(Fr)\\
&= \rho_2(Fr)^{-1}A\rho_1(Fr)\rho_1|_{I_F}\rho_1(Fr)^{-1}A^{-1}\rho_2(Fr)\\
&= \rho_2(Fr)^{-1}A(\rho_1|_{I_F})^qA^{-1}\rho_2(Fr)\\
&= \rho_2(Fr)^{-1}(A\rho_1|_{I_F}A^{-1})^q\rho_2(Fr)\\
&= \rho_2(Fr)^{-1}(\rho_2|_{I_F})^q\rho_2(Fr)\\
&= \rho_2|_{I_F}.
\end{aligned}$$

Similarly, using condition (b) twice we get

$$\begin{aligned}
&Ad(\rho_2(Fr)^{-1}A\rho_1(Fr)A^{-1})E_2\\
&= Ad(\rho_2(Fr)^{-1}) \circ Ad(A) \circ Ad(\rho_1(Fr)) \circ Ad(A^{-1})E_2\\
&= Ad(\rho_2(Fr)^{-1}) \circ Ad(A) \circ Ad(\rho_1(Fr))E_1\\
&= Ad(\rho_2(Fr)^{-1}) \circ Ad(A)qE_1\\
&= qAd(\rho_2(Fr)^{-1})E_2 = qq^{-1}E_2\\
&= E_2.
\end{aligned}$$

This shows that $\rho_2(Fr)^{-1}A\rho_1(Fr)A^{-1} \in C_{\mathrm{GL}_N(\mathbb{C})}(\rho_2|_{I_F}, E_2)$. □

For any $(\rho, E) \in \Phi(G_N)_0$, we set

$$\overline{\rho(Fr)} := \rho(Fr)C_{\mathrm{GL}_N(\mathbb{C})}(\rho|_{I_F}, E).$$

3.2.2.2 Macdonald correspondence

Proposition 3.2.9. *[47] There is a canonical bijection*

$$\mathscr{M}_N : {\Phi(G_N)_0}/{\sim_{I_F}} \to Irr(\overline{G}_N). \tag{3.26}$$

We now introduce the notation needed to give a description of this map. All the claims in this section are restatements of results in [47].

Let $(\rho, E) \in \Phi(G_N)_0$. We write the decomposition into sum of indecomposable representations of (ρ, E) as follows:

$$(\rho, E) = \oplus_{i=1}^k (\rho_i, 0) \otimes Sp(r_i),$$

with uniquely determined $\rho_i \in \Phi^0(G_{m_i})_0$ and $m_i, r_i \in \mathbb{N}$ such that $\sum_{i=1}^k m_i r_i = N$.

Let $\mathscr{P}_r$ denote the set of the partitions of r for $r \in \mathbb{N}$, and let $\mathscr{P} := \bigsqcup_{r\in\mathbb{N}} \mathscr{P}_r$ denote the set of all partitions. For any $\lambda \in \mathscr{P}$, we write $|\lambda| = r$ if $\lambda \in \mathscr{P}_r$.

We define the partition valued function associated to (ρ, E) by:

$$\begin{aligned}
\Lambda_{(\rho,E)} : \bigsqcup_{m\in\mathbb{N}} {\Phi^0(G_m)_0}/{\sim_{I_F}} &\longrightarrow \mathscr{P} \\
\rho_0 &\mapsto (r_i \mid i \in \{1, \dots, k\} \ s.t. \ \rho_0|_{I_F} \cong \rho_i|_{I_F})
\end{aligned} \tag{3.27}$$

where $(r_i \mid i \in \{1, \ldots, k\}\ s.t.\ \rho_0|_{I_F} \cong \rho_i|_{I_F})$ is ordered to have decreasing entries. For any $\rho_0 \in \bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$, let $m(\rho_0)$ be the dimension of the representation space of ρ_0, i.e., $\rho_0 \in \Phi^0(G_{m(\rho_0)})_0$. Then since $(\rho, E) \in \Phi(G_N)_0$, the partition valued function $\Lambda_{(\rho,E)}$ satisfies $\sum_{\rho_0} m(\rho_0)|\Lambda(\rho_0)| = N$, where the ρ_0 runs through a set of representatives for $\bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$. We say that $\Lambda_{(\rho,E)}$ has degree N.

The assignment $(\rho, E) \mapsto \Lambda_{(\rho,E)}$ is constant on the I_F-equivalence classes, because $\Lambda_{(\rho,E)}$ depends only on $(\rho|_{I_F}, E)$. Therefore it defines a map from $\Phi(G_N)_0 / \sim_{I_F}$ to partition valued functions on $\bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$ of degree N.

This map is a bijection. It is injective because if two tame Weil-Deligne representations $(\rho_1, E_1), (\rho_2, E_2) \in \Phi(G_N)_0$ are associated to the same partition valued function, then their indecomposable constituent are pairwise I_F-equivalent and so $(\rho_1, E_1)_{I_F} = (\rho_2, E_2)_{I_F}$. It is surjective because any partition valued function Λ satisfying $\sum_{\rho_0} m(\rho_0)|\Lambda(\rho_0)| = N$ is the image of $\bigoplus_{\rho_0}(\bigoplus_{r\in\Lambda(\rho_0)} \rho_0 \otimes Sp(r))$, where ρ_0 runs through a set of representatives for $\bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$.

Let $m \in \mathbb{N}$ and $\tau \in Cusp(\overline{G}_m)$. For any $r \in \mathbb{N}$, the irreducible components in $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} \tau$ are in bijection with simple modules over a finite Hecke algebra with equal parameters $\mathcal{H}(\mathbb{S}_r, q^m)$, that is isomorphic to $\mathbb{C}[\mathbb{S}_r]$. In particular simple modules over $\mathcal{H}(\mathbb{S}_r, q^m)$ are parametrized by $\mathscr{P}_r$, and we normalize the parametrization in such a way that the partition (r) corresponds to the sign representation. As a consequence, irreducible components in $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} \tau$ are parametrized by partitions of r.
Let $\lambda \in \mathscr{P}_r$. Then there is a uniquely determined irreducible component in $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} \boxtimes_{i=1}^r \tau$ corresponding to λ . We denote it by π_τ^λ.

Let $\Lambda : \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m) \to \mathscr{P}$ be a partition valued function such that $\sum_\sigma m(\sigma)|\Lambda(\sigma)| = N$, where σ runs through $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ and $m(\sigma)$ is such that $\sigma \in Cusp(\overline{G}_{m(\sigma)})$. We say that N is the degree of Λ. We set

$$\pi_\Lambda := \mathrm{R}_{\boxtimes_\tau \overline{G}_{m(\tau)|\Lambda(\tau)|}}^{\overline{G}_N} (\boxtimes_\tau \pi_\tau^{\Lambda(\tau)}). \tag{3.28}$$

It is in an irreducible representation since it is the parabolic induction of pairwise not-isomorphic irreducible representations. Any irreducible representation of $\overline{G}_N$ corresponds to a partition-valued function as in (3.28).

We are now in the position to describe the map (3.26).

For any $m \in \mathbb{N}$, the map $\mathscr{M}_m^0$ in (3.15) is a bijection from $\Phi^0(G_m)_0 / I_F$ to $Cusp(\overline{G}_m)$. Collecting together these bijections for any m yields a bijection

$$\mathscr{M}^0 : \bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F} \to \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$$

satisfying $m(\mathscr{M}^0(\rho_0|_{I_F})) = m(\rho_0)$ for any $\rho_0 \in \bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$. It follows that the composition with $(\mathscr{M}^0)^{-1}$ yields a preserving degree bijection between partition valued functions on $\bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 / \sim_{I_F}$ and partition valued functions on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$.

Let $(\rho, E) \in \Phi(G_N)_0$, and let $\Lambda_{(\rho,E)}$ be the partition valued function on $\bigsqcup_{m\in\mathbb{N}} \Phi^0(G_m)_0 \diagup \sim_{I_F}$ associated to $(\rho, E)_{I_F}$ as in (3.27). Then $\Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}$ is a partition-valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N. The image of $(\rho, E)_{I_F}$ through the map $\mathscr{M}_N$ is defined by

$$\mathscr{M}_N((\rho, E)_{I_F}) := \pi_{\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}} \tag{3.29}$$

where $\pi_{\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}}$ is the irreducible representation associated with $\Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}$ as in (3.28).

The following observations will be useful in Chapter IV.

Lemma 3.2.10. *Let Λ be a partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N and let $\chi \in \widehat{k_F^*}$. Let $\chi\Lambda$ be the the partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N defined by*

$$\chi\Lambda(\tau) := \Lambda(\tau \otimes (\chi^{-1} \circ det))$$

for $\tau \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$. Then

$$\pi_{\chi\Lambda} := \pi_\Lambda \otimes (\chi \circ det). \tag{3.30}$$

Proof. The representation π_Λ is defined in (3.28) as $\pi_\Lambda = \mathrm{R}^{\overline{G}_N}_{\prod_\tau \overline{G}_{m(\tau)|\Lambda(\tau)|}}(\boxtimes_\tau \tau^{\Lambda(\tau)})$, with $\tau \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ and where $\tau^{\Lambda(\tau)}$ is the irreducible constituent of $\mathrm{R}^{\overline{G}_{m(\tau)|\Lambda(\tau)|}}_{\overline{G}^{|\Lambda(\tau)|}_{m(\tau)}}(\boxtimes_{i=1}^{|\Lambda(\tau)|} \tau)$ corresponding to the partition $\Lambda(\tau)$. It follows that

$$\pi_\Lambda\otimes(\chi\circ det) = \mathrm{R}^{\overline{G}_N}_{\prod_\tau \overline{G}_{m(\tau)|\Lambda(\tau)|}}(\boxtimes_\tau \tau^{\Lambda(\tau)})\otimes(\chi\circ\det) = \mathrm{R}^{\overline{G}_N}_{\prod_\tau \overline{G}_{m(\tau)|\Lambda(\tau)|}}(\boxtimes_\tau(\tau^{\Lambda(\tau)}\otimes(\chi\circ\det))). \tag{3.31}$$

On the other hand,

$$\pi_{\chi\Lambda} = \mathrm{R}^{\overline{G}_N}_{\prod_\tau \overline{G}_{m(\tau)|\chi\Lambda(\tau)|}}(\boxtimes_\tau \tau^{\chi\Lambda(\tau)}) = \mathrm{R}^{\overline{G}_N}_{\prod_\tau \overline{G}_{m(\tau)|\Lambda(\tau)|}}(\boxtimes_\tau(\tau\otimes(\chi\circ\det))^{\Lambda(\tau)}). \tag{3.32}$$

We show that for any $\tau \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ it holds

$$\tau^{\Lambda(\tau)} \otimes (\chi \circ \det) = (\tau \otimes (\chi \circ \det))^{\Lambda(\tau)}. \tag{3.33}$$

Let $m := m(\tau)$ and $n := |\Lambda(\tau)|$. We have

$$\mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^n_m}(\boxtimes_{i=1}^n \tau) \otimes (\chi \circ \det) = \mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^n_m}(\boxtimes_{i=1}^n (\tau \otimes (\chi \circ \det))).$$

It implies that the $End_{\overline{G}_{mn}}\big(\mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^n_d}\big(\boxtimes_{i=1}^n(\tau\otimes(\chi\circ\det))\big)\big)$ -module structure on $Hom_{\overline{G}_{mn}}\big(\mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^n_d}\big(\boxtimes_{i=1}^n(\tau\otimes(\chi\circ\det))\big), \tau^{\Lambda(\tau)}\otimes(\chi\circ\det)\big)$ is the same as the $End_{\overline{G}_{mn}}\big(\mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^{|\Lambda(\tau)|}_d}(\boxtimes_{i=1}^n\tau)\big)$ -module structure on $Hom_{\overline{G}_{mn}}\big(\mathrm{R}^{\overline{G}_{mn}}_{\overline{G}^{|\Lambda(\tau)|}_d}\big(\boxtimes_{i=1}^n\tau\big), \tau^{\Lambda(\tau)}\big)$. This proves (3.33). Comparing (3.31) and (3.32), it follows that $\pi_\Lambda \otimes (\chi \circ det) = \pi_{\chi\Lambda}$. □

Remark 3.2.11. We retain notation from Remark 3.1.8. Let $\chi \in \widehat{k_F^*}$. Then χ acts on $Irr(\overline{G}_N)$ by $\pi \mapsto \pi \otimes (\chi \circ det)$. Moreover, as in Remark 3.1.8 we identify χ with a Frobenius stable character of ${}^{I_F}\!/\!_{P_F}$ and lift it to W_F by letting the Frobenius elements act trivially. Then χ acts on $\Phi(G_N)_0$ by $(\rho, E) \mapsto (\rho \otimes \chi, E)$. Since $(\rho_1, E_1) \sim_{I_F} (\rho_2, E_2)$ implies $(\rho_1 \otimes \chi, E_1) \sim_{I_F} (\rho_2 \otimes \chi, E_2)$, this defines an action of the character group of k_F^* on ${}^{\Phi(G_N)_0}\!/\!_{\sim_{I_F}}$.

The Macdonald correspondence

$$\mathscr{M}_N : {}^{\Phi(G_N)_0}\!/\!_{\sim_{I_F}} \to Irr(\overline{G}_N)$$

is equivariant with respect to the actions of the character group of k_F^* just described [47, Proposition 1.3].

Indeed, for any $(\rho, E) = \oplus(\rho_i, 0) \otimes Sp(r_i)$ in $\Phi(G_N)_0$, retaining notation from Lemma 3.2.10, it holds

$$\Lambda_{(\chi\rho,E)} \circ (\mathscr{M}^0)^{-1} = \chi(\Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}),$$

because for any $\tau \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ we have

$$\begin{aligned}
&\Lambda_{(\chi\rho,E)} \circ (\mathscr{M}^0)^{-1}(\tau) \\
&= \big(r_i \mid i \in \{1, \dots, k\} \ s.t.\ (\mathscr{M}^{0^{-1}})(\tau)|_{I_F} \cong \rho_i|_{I_F}\big) && \text{by (3.27)} \\
&= \big(r_i \mid i \in \{1, \dots, k\} \ s.t.\ (\mathscr{M}^0)^{-1}(\tau)|_{I_F} \cong \chi\rho_i|_{I_F}\big) \\
&= \big(r_i \mid i \in \{1, \dots, k\} \ s.t.\ \chi^{-1}(\mathscr{M}^0)^{-1}(\tau)|_{I_F} \cong \rho_i|_{I_F}\big) && \text{by Remark 3.1.8} \\
&= \big(r_i \mid i \in \{1, \dots, k\} \ s.t.\ (\mathscr{M}^0)^{-1}(\tau \otimes (\chi^{-1} \circ det))|_{I_F} \cong \rho_i|_{I_F}\big) \\
&= \Lambda_{(\rho,E)}(\tau \otimes (\chi^{-1} \circ det)) = \chi\Lambda_{(\rho,E)}(\tau).
\end{aligned}$$

Hence by Lemma 3.2.10

$$\begin{aligned}
\mathscr{M}_N((\chi\rho, E)_{I_F}) &= \pi_{\Lambda_{(\chi\rho,E)}\circ(\mathscr{M}^0)^{-1}} = \pi_{\chi\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}} \\
&= \pi_{\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}} \otimes (\chi \circ det) = \mathscr{M}_N((\rho, E)_{I_F}) \otimes (\chi \circ det).
\end{aligned} \tag{3.34}$$

3.2.3 The indecomposable - essentially square integrable case

In this section, we assume $N = mr$ for $m, r \in \mathbb{N}$.

Now that we recalled how to build the tame Langlands correspondence and the Macdonald correspondence in the general case starting from the irreducible-supercuspidal one, we start addressing the problem of the compatibility of these bijections via parahoric restriction.

The following results were developed indipendently, but after their completion the author became aware of [59, 64], that get to the same conclusions (see the introduction of this chapter). The result in [64, Appendix A], is stated in terms of the "reduction to tempered type" map [59]. We are interested just in the depth-0 case, where this map can be described just in terms of the parahoric restriction functor, as we do in (3.63). One reason to adopt this description is that the "reduction to tempered type" map is not yet available for a general split reductive group, and giving statements in terms of parahoric restriction will make the extension of the results to the SL_N case in chapter IV more straightforward.

As first step toward proving the desired compatibility of the tame Langlands correspondence and the Macdonald correspondence, we consider indecomposable tamely ramified Weil-Deligne representations and essentially square integrable representations, that correspond to each other through the bijection (3.24). Indeed, if $\rho \in \Phi^0(G_m)_0$,

$$\mathscr{L}_{mr}((\rho,0)\otimes Sp(r)) = Q(\Delta(\mathscr{L}^0_m(\rho),r)). \tag{3.35}$$

So the aim of this section is to show that

$$\mathcal{P}^{G_{mr}}_{\overline{G}_{mr}}\mathscr{L}_{mr}((\rho,0)\otimes Sp(r)) = \mathscr{M}_{mr}((((\rho,0)\otimes Sp(r))_{I_F}). \tag{3.36}$$

In the notation of Section 3.2.2, by construction of the map (3.26)

$$\mathscr{M}_{mr}((((\rho,0)\otimes Sp(r))_{I_F}) = \pi^{(r)}_{\mathscr{M}^0_m(\rho|_{I_F})}, \tag{3.37}$$

where $\pi^{(r)}_{\mathscr{M}^0_m(\rho|_{I_F})}$ is the irreducible constituent of $\mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m}\mathscr{M}^0_m(\rho|_{I_F})$ parametrized by the partition (r), i.e.,corresponding to the sign representation in the corresponding Hecke algebra.

Indeed, the partition valued function $\Lambda_{(\rho,0)\otimes Sp(r)}$ associated with the indecomposable representation $(\rho,0)\otimes Sp(r)$ as in (3.27) is supported on the I_F-equivalence class of ρ, and it holds $\Lambda_{(\rho,0)\otimes Sp(r)}(\rho|_{I_F}) = (r)$. Therefore

$$\pi_{\Lambda_{(\rho,0)\otimes Sp(r)}\circ(\mathscr{M}^0)^{-1}} = \mathrm{R}^{\overline{G}_{mr}}_{\overline{G}_{mr}}\pi^{(r)}_{\mathscr{M}^0_m(\rho|_{I_F})} = \pi^{(r)}_{\mathscr{M}^0_m(\rho|_{I_F})},$$

and so (3.29) reduces in this case to (3.37).

Combining (3.35) with (3.37), in order to have (3.36) we need to prove that

$$\mathcal{P}^{G_{mr}}_{\overline{G}_{mr}}Q(\Delta(\mathscr{L}^0_m(\rho),r)) = \pi^{(r)}_{\mathscr{M}^0_m(\rho|_{I_F})}. \tag{3.38}$$

This is what we are going to show in the rest of this section.

We will need the following result about compatibility of parabolic induction and parahoric restriction.

Proposition 3.2.12. *Let σ_m be a supercuspidal representation of G_m. Then, the representation*

$$\mathcal{P}^{G_{mr}}_{\overline{G}_{mr}}Q(\Delta(\sigma_m,r))$$

is a subrepresentation of

$$\mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m}(\mathcal{P}^{G_m}_{\overline{G}_m}\sigma_m)^r.$$

Proof. By definition, $Q(\Delta(\sigma_m,r))$ is the unique irreducible quotient of

$$\mathrm{R}^{G_{mr}}_{G^r_m}(\sigma_m \boxtimes (\sigma_m\otimes|det(\cdot)|)\boxtimes\cdots\boxtimes(\sigma_m\otimes|det(\cdot)|^{r-1})$$

so, since the parahoric restriction functor $\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}}$ is exact,

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\sigma_m, r))$$

is a quotient of

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} \mathrm{R}_{G_m^r}^{G_{mr}} (\sigma_m \otimes (\sigma_m \otimes |det(\cdot)|) \cdots (\sigma_m \otimes |det(\cdot)|^{r-1})).$$

By [52, Proposition 2.1],

$$\begin{aligned} &\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} \mathrm{R}_{G_m^r}^{G_{mr}} (\sigma_m \boxtimes (\sigma_m \otimes |det(\cdot)|) \boxtimes \cdots \boxtimes (\sigma_m \otimes |det(\cdot)|^{r-1})) \\ &= \mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} \mathcal{P}_{\overline{G}_m^r}^{G_m^r} (\sigma_m \boxtimes (\sigma_m \otimes |det(\cdot)|) \boxtimes \cdots \boxtimes (\sigma_m \otimes |det(\cdot)|^{r-1})) \\ &= \mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} (\mathcal{P}_{\overline{G}_m}^{G_m} \sigma_m \boxtimes \mathcal{P}_{\overline{G}_m}^{G_m} (\sigma_m \otimes |det(\cdot)|) \boxtimes \cdots \boxtimes \mathcal{P}_{\overline{G}_m}^{G_m} (\sigma_m \otimes |det(\cdot)|^{r-1})). \end{aligned} \quad (3.39)$$

Note that

$$\mathcal{P}_{\overline{G}_m}^{G_m} (\sigma_m \otimes |det(\cdot)|^i) = \mathcal{P}_{\overline{G}_m}^{G_m} \sigma_m$$

for any $i = 1, \ldots, r-1$, as $|det(k)| = 1$ for any $k \in K_m$.

Therefore $\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\sigma_m, r))$ is a quotient of $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} (\mathcal{P}_{\overline{G}_m}^{G_m} \sigma_m)^r$. Since the category of representations of $\overline{G}_{mr}$ is semisimple, any quotient of a representation of $\overline{G}_{mr}$ is isomorphic to a subrepresentation. □

Corollary 3.2.13. *The representation*

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\mathscr{L}_m^0(\rho), r))$$

is a subrepresentation of

$$\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} ((\mathscr{M}_m^0(\rho|_{I_F}))^r).$$

Proof. It directly follows by combining Proposition 3.2.12 and Proposition 3.1.11. □

Therefore, to completely characterize $\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\mathscr{L}_m^0(\rho), r))$ it is enough to understand which irreducible constituents of $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}} ((\mathscr{M}_m^0(\rho|_{I_F}))^r)$ it contains and with which multiplicity. In the rest of this Section, we prove that the only irreducible constituent appearing is the one corresponding to the *sign* representation of $\mathcal{H}(\mathbb{S}_r, q^m)$, with multiplicity 1.

3.2.3.1 Bernstein blocks

Let G be a reductive p-adic group, and let M be a Levi subgroup of G. We denote by M^c the subgroup of M generated by all the compact subgroups of M. An unramified character of M is a character that is trivial on M^c, and the group of unramified characters of M is denoted by $X_{nr}(M)$. Two supercuspidal representations of M are called equivalent if they differ by a twist by an unramified character: if $\sigma \in Cusp(M)$ and $\chi \in X_{nr}(M)$ we write $\sigma \sim \sigma \otimes \chi$. We denote by $[M, \sigma]$ the equivalence class of σ.
We say that an irreducible representation π of G has supercuspidal support in $[M, \sigma]$

if π is a subquotient of a representation parabolically induced from $\sigma \otimes \chi$ for some $\chi \in X_{nr}(M)$. We denote by $\mathcal{B}^{[M,\sigma]}(G)$ the full subcategory of smooth representations of G whose irreducible subquotients have supercuspidal support in $[M, \sigma]$. The subcategory $\mathcal{B}^{[M,\sigma]}(G)$ is called a Bernstein block. The Bernstein blocks for M can be defined analogously, and in particular $\mathcal{B}^{[M,\sigma]}(M)$ is the full subcategory of representations of M whose irreducible subquotients are isomorphic to some unramified twist of σ.
We recall the construction of a projective generator of $\mathcal{B}^{[M,\sigma]}(G)$ as in [56, Section 1]. Let σ^c be an irreducible constituent of the semisimple representation $Res^M_{M^c}\sigma$. Then

$$\Pi_M^{[M,\sigma]} := \text{c-ind}_{M^c}^M \sigma^c$$

is a projective generator for $\mathcal{B}^{[M,\sigma]}(M)$, and

$$\Pi_G^{[M,\sigma]} := \mathrm{R}_M^G(\Pi_M^{[M,\sigma]})$$

is a projective generator of $\mathcal{B}^{[M,\sigma]}(G)$.

The projective generator $\Pi_G^{[M,\sigma]}$ of $\mathcal{B}^{[M,\sigma]}(G)$ induces an equivalence of categories

$$\begin{aligned} \mathcal{B}^{[M,\sigma]}(G) &\xrightarrow{Hom_G(\Pi_G^{[M,\sigma]},\cdot)} Mod - End_G(\Pi_G^{[M,\sigma]}) \\ V &\mapsto Hom_G(\Pi_G^{[M,\sigma]}, V). \end{aligned} \tag{3.40}$$

3.2.3.2 Compatibility of Parahoric restriction with Restriction to the finite Hecke Algebra

The aim of this section is to interpret parahoric restriction through the equivalence of categories (3.40).

The parahoric restriction functor $\mathcal{P}^{G_N}_{\overline{G}_N} = (Res^{G_N}_{K_N})^{K_N^+}$ has a left adjoint, namely

$$\mathcal{I}^{G_N}_{\overline{G}_N} := \text{c-ind}^{G_N}_{K_N} \circ Infl^{K_N}_{\overline{G}_N},$$

as we now show. Recall that $\overline{G}_N \cong {K_N}/{K_N^+}$. The inflation functor $Infl^{K_N}_{K_N/K_N^+}$ is left adjoint the fixed point functor $(\cdot)^{K_N^+}$, so for any representation τ_1 of ${K_N}/{K_N^+}$ and any representation τ_2 of K_N the identity induces a natural isomorphism

$$Hom_{K_N/K_N^+}(\tau_1, (\tau_2)^{K_N^+}) \cong Hom_{K_N}(Infl^{K_N}_{K_N/K_N^+}\tau_1, \tau_2). \tag{3.41}$$

On the other hand, c-ind$^{G_N}_{K_N}$ is left adjoint to $Res^{G_N}_{K_N}$. For any representation τ of K_N and any representation π of G_N, the natural isomorphism of the Hom-spaces can be explicitly written as follows [57, 1.1.1]:

$$\begin{aligned} Hom_{K_N}(\tau, Res^{G_N}_{K_N}\pi) &\xrightarrow{\sim} Hom_{G_N}(\text{c-ind}^{G_N}_{K_N}\tau, \pi) \\ \psi &\mapsto s_\psi \end{aligned} \tag{3.42}$$

where s_ψ for any $f \in \text{c-ind}^{G_N}_{K_N}\tau$ is given by

$$s_\psi(f) = \sum_{x \in G_N/K_N} \pi(x)\psi(f(x^{-1})). \tag{3.43}$$

Composing the isomorphisms (3.41) and (3.42), we have that for any representation τ of $\overline{G}_N$ and any smooth admissible representation π of G_N we have an isomorphism

$$\begin{gathered} Hom_{\overline{G}_N}(\tau, \mathcal{P}^{G_N}_{\overline{G}_N}\pi) \xrightarrow{\sim} Hom_{G_N}(\mathcal{I}^{G_N}_{\overline{G}_N}\tau, \pi). \\ \psi \mapsto s_\psi \end{gathered} \tag{3.44}$$

The left hand space is a right $End_{\overline{G}_N}(\tau)$-module with action given by composition, and the right hand space is a right $End_{G_N}(\mathcal{I}^{G_N}_{\overline{G}_N}\tau)$-module with action given by composition. The functor $\mathcal{I}^{G_N}_{\overline{G}_N}$ gives a map

$$\begin{gathered} End_{\overline{G}_N}(\tau) \to End_{G_N}(\mathcal{I}^{G_N}_{\overline{G}_N}\tau). \\ T \mapsto \mathcal{I}^{G_N}_{\overline{G}_N}(T) \end{gathered} \tag{3.45}$$

More explicitly, for any $f \in \mathcal{I}^{G_N}_{\overline{G}_N}\tau$,

$$\mathcal{I}^{G_N}_{\overline{G}_N}(T)(f) = T \circ f. \tag{3.46}$$

From this description it is easy to see that the map (3.45) is an injective morphism of algebras. We denote its image by $\mathcal{I}^{G_N}_{\overline{G}_N}(End_{\overline{G}_N}(\tau))$, so we have an algebra isomorphism

$$End_{\overline{G}_N}(\tau) \xrightarrow{\sim} \mathcal{I}^{G_N}_{\overline{G}_N}(End_{\overline{G}_N}(\tau)). \tag{3.47}$$

As any $End_{G_N}(\mathcal{I}^{G_N}_{\overline{G}_N}\tau)$-module is an $\mathcal{I}^{G_N}_{\overline{G}_N}(End_{\overline{G}_N}(\tau))$-module by restriction, it can be regarded as an $End_{\overline{G}_N}(\tau)$ module.

Lemma 3.2.14. *The map* (3.44) *is an isomorphism of right* $End_{\overline{G}_N}(\tau)$*-modules.*

Proof. Let τ be a representation of $\overline{G}_N$ and π a smooth admissible representation of G_N. We need to prove that for any $T \in End_{\overline{G}_N}(\tau)$, it holds

$$s_{\psi\circ T} = s_\psi \circ \mathcal{I}^{G_N}_{\overline{G}_N}(T)$$

Let $f \in \mathcal{I}^{G_N}_{\overline{G}_N}\tau$. Then

$$\begin{aligned} s_{\psi\circ T}(f) = &\sum_{x\in G_N/K_N} \pi(x)\psi \circ T(f(x^{-1})) = && (3.43) \\ &\sum_{x\in G_N/K_N} \pi(x)\psi(T \circ f(x^{-1})) = && (3.46) \\ &\sum_{x\in G_N/K_N} \pi(x)\psi(\mathcal{I}^{G_N}_{\overline{G}_N}(T)(f)(x^{-1})) = s_\psi \circ \mathcal{I}^{G_N}_{\overline{G}_N}(T)(f). && (3.43) \end{aligned}$$

∎

We now specialize to the case we are interested in: let $N = mr$, let $\sigma \in Cusp(G_m)_0$, and let $\overline{\sigma} = \mathcal{P}^{G_m}_{\overline{G}_m}\sigma$. By Proposition 3.1.9, $\overline{\sigma} \in Cusp(\overline{G}_m)$, and we write $\overline{\sigma}^r = \boxtimes_{i=1}^r \overline{\sigma}$.

Proposition 3.2.15. *In the notation above, it holds:*

$$\mathcal{I}^{G_{mr}}_{\overline{G}_{mr}} \circ \mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m} \overline{\sigma}^r = \Pi^{[G^r_m,\sigma^r]}_{G_{mr}}.$$

Proof. Let $\overline{P}_{m,r}$ be the parabolic subgroup of of $\overline{G}_{mr}$ of block upper triangular matrices containing $\overline{G}^r_m$. Let K_P be the preimage in K_{mr} of $\overline{P}_{m,r}$ through the projection $K_{mr} \to {}^{K_{mr}}\!\diagup_{K^+_{mr}} \cong \overline{G}_{mr}$. Let K^+_P denote the pro-unipotent radical of K_P. Then,

$$ {}^{K_P}\!\diagup_{K^+_{mr}} \cong \overline{P}_{m,r}, \qquad\qquad {}^{K_P}\!\diagup_{K^+_P} \cong \overline{G}^r_m.$$

Since the group K^+_{mr} is a normal subgroup of K_{mr} and K_P, we have

$$\begin{aligned} Infl^{K_{mr}}_{\overline{G}_{mr}} \circ Ind^{\overline{G}_{mr}}_{\overline{P}_{m,r}} &= Infl^{K_{mr}}_{K_{mr}\diagup K^+_{mr}} \circ \text{c-ind}^{K_{mr}\diagup K^+_{mr}}_{K_P\diagup K^+_{mr}} \\ &= \text{c-ind}^{K_{mr}}_{K_P} \circ Infl^{K_P}_{K_P\diagup K^+_{mr}} = \text{c-ind}^{K_{mr}}_{K_P} \circ Infl^{K_P}_{\overline{P}_{m,r}} \end{aligned}$$

where we used that that ${}^{K_{mr}}\!\diagup_{K_P}$ is a compact space, so there is no difference between induction and compact induction. Therefore

$$\begin{aligned} \mathcal{I}^{G_{mr}}_{\overline{G}_{mr}} \circ \mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m} \overline{\sigma}^r &= \text{c-ind}^{G_{mr}}_{K_{mr}} \circ Infl^{K_{mr}}_{\overline{G}_{mr}} \circ Ind^{\overline{G}_{mr}}_{\overline{P}_{m,r}} \circ Infl^{\overline{P}_{m,r}}_{\overline{G}^r_m} \overline{\sigma}^r \\ &= \text{c-ind}^{G_{mr}}_{K_{mr}} \circ \text{c-ind}^{K_{mr}}_{K_P} \circ Infl^{K_P}_{\overline{P}_{m,r}} \circ Infl^{\overline{P}_{m,r}}_{\overline{G}^r_m} \overline{\sigma}^r = \text{c-ind}^{K_{mr}}_{K_P} \circ Infl^{K_P}_{\overline{G}^r_m} \overline{\sigma}^r. \end{aligned} \tag{3.48}$$

The pair $(K^r_m, Infl^{K^r_m}_{\overline{G}^r_m}\overline{\sigma}^r)$ is a type for $\mathcal{B}^{[G^r_m,\sigma^r]}(M)$, in the sense of [13]. Indeed since $\overline{\sigma}^r = \mathcal{P}^{G^r_m}_{\overline{G}^r_m}\sigma^r$, the restriction of the supercuspidal representation σ^r to K^r_m contains $Infl^{K^r_m}_{\overline{G}^r_m}\overline{\sigma}^r$ and so $(K^r_m, Infl^{K^r_m}_{\overline{G}^r_m}\overline{\sigma}^r)$ is a type by [11, Theorem 6.2.3] .

Moreover $(K_P, Infl^{K_P}_{\overline{G}^r_m}\overline{\sigma}^r)$ is a cover for $(K^r_m, Infl^{K^r_m}_{\overline{G}^r_m}\overline{\sigma}^r)$, in the sense of [13, Defnition 8.1]: it is a particular case of [51, 3.8].
In particular, [4, Lemma B1] applies to this situation, and it yields

$$\text{c-ind}^{G_{mr}}_{K_P} \circ Infl^{K_P}_{G^r_m}\overline{\sigma}^r = \Pi^{[G^r_m,\sigma^r]}_{G_{mr}}.$$

Therefore (3.48) gives

$$\mathcal{I}^{G_{mr}}_{\overline{G}_{mr}} \circ \mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m} \overline{\sigma}^r = \Pi^{[G^r_m,\sigma^r]}_{G_{mr}}. \tag{3.49}$$

□

Proposition 3.2.16. *For any $\pi \in \mathcal{B}^{[G^r_m,\sigma^r]}(G)$, there is an isomorphism of right $End_{\overline{G}_{mr}}(\mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m}\overline{\sigma}^r)$-modules*

$$Hom_{\overline{G}_{mr}}(\mathrm{R}^{\overline{G}_{mr}}_{\overline{G}^r_m}\overline{\sigma}^r, \mathcal{P}^{G_{mr}}_{\overline{G}_{mr}}\pi) \xrightarrow{\sim} Hom_{G_{mr}}(\Pi^{[G^r_m,\sigma^r]}_{G_{mr}}, \pi).$$

Proof. By Proposition 3.2.15, the map (3.45) gives an embedding

$$End_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r) \to End_{G_{mr}}(\Pi_{G_{mr}}^{[G_m^r,\sigma^r]}) \tag{3.50}$$

and for any $\pi \in \mathcal{B}^{[G_m^r,\sigma^r]}(G)$ the map (3.44) gives an isomorphism

$$Hom_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r, \mathcal{P}_{\overline{G}_{mr}}^{G_{mr}}\pi) \xrightarrow{\sim} Hom_{G_{mr}}(\Pi_{G_{mr}}^{[G_m^r,\sigma^r]}, \pi). \tag{3.51}$$

By Lemma 3.2.14 the isomorphism (3.51) is an isomorphism of right $End_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r)$-modules, where the $End_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r)$-module structure on the right hand side is given by restriction to the image of the map (3.50). □

Remark 3.2.17. We already recalled that the algebra $End_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r)$ is isomorphic to the finite Hecke algebra $\mathcal{H}(\mathbb{S}_r, q^m)$. The endomorphism algebra $End_{G_{mr}}(\Pi_{G_{mr}}^{[G_m^r,\sigma^r]})$ is isomorphic to $\widetilde{\mathcal{H}}(r, q^m)$, the affine Hecke algebra relative to a based root datum of type GL_r [28, Proposition 1.14, Proposition 1.15, Theorem 7.7], with parameters equal to q^m [50, 8.2]. The algebra $\widetilde{\mathcal{H}}(r, q^m)$ is isomorphic to $\mathcal{H}(\mathbb{S}_r, q^m) \otimes \mathbb{C}[\mathbb{Z}^r]$ as a vector space, it contains $\mathcal{H}(\mathbb{S}_r, q^m)$ and $\mathbb{C}[\mathbb{Z}^r]$ as subalgebras, and satisfies cross relations as in [65, Definition 1.6]. The inclusion $End_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}\,\overline{\sigma}^r) \hookrightarrow End_{G_{mr}}(\Pi_{G_{mr}}^{[G_m^r,\sigma^r]})$ given by (3.45) in this case is the natural inclusion $\mathcal{H}(\mathbb{S}_r, q^m) \hookrightarrow \widetilde{\mathcal{H}}(r, q^m)$.

Then Proposition 3.2.16 can be restated as follows: given a smooth admissible representation π of G_{mr}, the $\mathcal{H}(\mathbb{S}_r, q^m)$-module structure on $Hom_{\overline{G}_{mr}}(\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_m r}\,\overline{\sigma}^r, \mathcal{P}_{\overline{G}_{mr}}^{G_{mr}}\pi)$ is given by restriction of the $\widetilde{\mathcal{H}}(r, q^m)$ module structure on $Hom_{G_{mr}}(\Pi_{Gmr}^{[G_m^r,\sigma^r]}, \pi)$.

3.2.3.3 Segments and Steinberg representation

In this section we describe explicitly the equivalence (3.40) for $G = G_{mr}$, $M = G_m^r$ and $\sigma = \boxtimes_{i=1}^r \sigma_m := (\sigma_m)^r$,with $\sigma_m \in Cusp(G_m)_0$.

In this case, $\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]} = \boxtimes_{i=1}^r \Pi_{G_m}^{[G_m,\sigma_m]}$, and therefore $End_{G_m^r}(\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]}) \cong \otimes_{i=1}^r End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]})$. By [57, Remark 1.6.1.3], it holds $End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]}) \cong \mathbb{C}[\mathbb{Z}]$. Following [57], this isomorphism can be realized as we now explain. Retaining notation from Section 3.2.3.1, we denote by G_m^c the subgroup of G_m generated by all the compact subgroups of G_m, and we denote by σ_m^c an irreducible constituent of the restriction of σ_m to G_m^c. Then we have $\Pi_{G_m}^{[G_m,\sigma_m]} := \text{c-ind}_{G_m^c}^{G_m}\sigma^c$.

Let

$$N_{G_m}(\sigma_m^c) = \{x \in G_m \mid {}^x\sigma_m^c \cong \sigma_m^c\}$$

be the stabilizer in G_m of σ_m^c. By [57, Remark 1.6.1.3], the semisimple representation $Res_{G_m^c}^{G_m}\sigma_m$ is multiplicity-free. It follows that $N_{G_m}(\sigma_m^c)$ stabilizes the representation space $V_{\sigma_m^c}$ of σ_m^c. We write $\widetilde{\sigma_m}$ for the representation of $N_{G_m}(\sigma_m^c)$ on $V_{\sigma_m^c}$ obtained by restriction of σ_m. Note that $Res_{G_m^c}^{N_{G_m}(\sigma_m^c)}\widetilde{\sigma_m} = \sigma_m^c$.
Let

$$\begin{aligned} H(G_m, \sigma_m^c) := \{f : G_m \to End(V_{\sigma^c}) \mid & f(gxg') = \sigma_m^c(g)f(x)\sigma^c(g'), \\ & supp(f) \text{ is a finite union of } G_m^c\text{-cosets}\}. \end{aligned}$$

The vector space $H(G_m, \sigma_m^c)$ is an algebra with convolution product

$$(f * h)(x) := \sum_{g \in G_m / G_m^c} f(g)h(g^{-1}x).$$

There is a canonical algebra isomorphism [57, 1.1.2]

$$End_{G_m^c}(\Pi_{G_m}^{[G_m,\sigma_m]}) \cong H(G_m, \sigma_m^c). \tag{3.52}$$

A $\mathbb{C}$-basis for $H(G_m, \sigma_m^c)$ is given by

$$\{\delta_{G_m^c n} \mid G_m^c n \in \{G_m^c\text{-coset in } \ N_{G_m}(\sigma_m^c)\}\}$$

with $\delta_{G_m^c n}$ supported on $G_m^c n$ and satisfying $\delta_{G_m^c n}(n) = \sigma_N(n)$. This yields an algerbra isomorphism between $H(G_m, \sigma_m^c)$ and the group algebra of $N_{G_m}(\sigma_m^c) / G_m^c$:

$$\begin{aligned} H(G_m, \sigma_m^c) &\xrightarrow{\sim} \mathbb{C}\big[N_{G_m}(\sigma_m^c) / G_m^c\big] \\ \delta_{G_m^c n} &\mapsto n \end{aligned} \tag{3.53}$$

Since any central element of G_m stabilizes σ_m^c, we have $Z(G_m)G_m^c \subseteq N_{G_m}(\sigma_m^c)$. So we can write the chain of inclusions $Z(G_m)G_m^c \subseteq N_{G_m}(\sigma_m^c) \subseteq G_m$, and the valuation of the determinant gives isomorphisms $G_m / G_m^c \cong \mathbb{Z}$ and $Z(G_m)G_m^c / G_m^c \cong m\mathbb{Z} \cong \mathbb{Z}$. This implies that $N_{G_m}(\sigma_m^c) / G_m^c \cong \mathbb{Z}$.
We determine $N_{G_m}(\sigma_m^c)$ when σ_m is a depth 0 supercuspidal representations of G_m.

Lemma 3.2.18. *Let $\sigma_m \in Cusp(G_m)_0$ and let σ_m^c be an irreducible subrepresentation of $Res_{G_m^c}^{G_m}\sigma$. Then $N_{G_m}(\sigma_m^c) = Z(G_m)G_m^c$.*

Proof. We need to prove that $N_{G_m}(\sigma_m^c) \subseteq Z(G_m)G_m^c$. Consider $Res_{G_m^c}^{G_m}\sigma_m$. Since σ_m is a supercuspidal representation of depth 0 , it holds $\sigma_m = \text{c-ind}_{Z(G_m)K_m}^{G_m} \Lambda$ for some representation Λ of $Z(G_m)K_m$. Since representations of G_m^c are semisimple, Mackey formula gives

$$Res_{G_m^c}^{G_m} \text{c-ind}_{Z(G_m)K_m}^{G_m} \Lambda = \bigoplus_{x \in G_m^c \backslash G_m / Z(G_m)K_m} \mathrm{R}_{G_m^c \cap {}^x Z(G_m)K_m}^{G_m^c} Res_{G_m^c \cap {}^x Z(G_m)K_m}^{{}^x Z(G_m)K_m} {}^x\Lambda.$$

Since $K_m \subseteq G_m^c$, and G_m^c is normal in G_m, there holds

$$G_m^c \backslash G_m / Z(G_m)K_m = G_m^c \backslash G_m / Z(G_m) = G_m / Z(G_m)G_m^c.$$

Moreover $G_m^c = {}^x G_m^c$ for any $x \in G_m$, hence

$$\begin{aligned} &\bigoplus_{x \in G_m^c \backslash G_m / Z(G_m)K_m} \mathrm{R}_{G_m^c \cap {}^x Z(G_m)K_m}^{G_m^c} Res_{G_m^c \cap {}^x Z(G_m)K_m}^{{}^x Z(G_m)K_m} {}^x\Lambda \\ = &\bigoplus_{x \in G_m / Z(G_m)G_m^c} {}^x(\mathrm{R}_{G_m^c \cap Z(G_m)K_m}^{G_m^c} Res_{G_m^c \cap Z(G_m)K_m}^{Z(G_m)K_m} \Lambda). \end{aligned}$$

Hence

$$Res^{G_m}_{G^c_m}\sigma = \bigoplus_{x\in {G_m}/{Z(G_m)G^c_m}} {}^x(\mathrm{R}^{G^c_m}_{G^c_m\cap Z(G_m)K_m}\, Res^{Z(G_m)K_m}_{G^c_m\cap Z(G_m)K_m}\Lambda). \tag{3.54}$$

Since σ^c_m is an irreducible subrepresentation of $Res^{G_m}_{G^c_m}\sigma$, it is an irreducible subrepresentation of ${}^y(\mathrm{R}^{G^c_m}_{G^c_m\cap Z(G_m)K_m}\, Res^{Z(G_m)K_m}_{G^c_m\cap Z(G_m)K_m}\Lambda)$ for some $y \in {G_m}/{Z(G_m)G^c_m}$. By [57, Remark 1.6.1.3] the representation $Res^{G_m}_{G^c_m}\sigma$ is multiplicity-free, so the direct summands in (3.54) don't have any irreducible subrepresentation in common. If $x \notin Z(G_m)G^c_m$, then σ^c_m and ${}^x\sigma^c_m$ will be irreducible subrepresentations of two different direct summands in (3.54), hence $x \notin N_{G_m}(\sigma^c_m)$. □

Lemma 3.2.19. *Let $\sigma_m \in Cusp(G_m)_0$, and let $|\det(\cdot)|^s$ with $s \in \mathbb{C}$ be an unramified character of G_m. Then the modules corrsponding to simple objects through the equivalence* (3.40) *for $\mathcal{B}^{[G_m,\sigma_m]}(G_m)$ are given by*

$$\begin{aligned}\mathcal{B}^{[G_m,\sigma_m]}(G_m) &\to Mod - \mathbb{C}[\mathbb{Z}]\\ \sigma_m \otimes |\det(\cdot)|^s &\mapsto \mathbb{C}_{(q^{ms})},\end{aligned}$$

where $\mathbb{C}_{(q^{ms})}$ is the unidimensional module over $\mathbb{C}[\mathbb{Z}]$ defined by

$$z.1 = q^{zms}$$

for any $z \in \mathbb{Z}$.

Proof. Since $\mathbb{C}[\mathbb{Z}]$ is abelian, we do not distinguish between left and right modules. It holds

$$\begin{aligned}Hom_{G_m}(\Pi^{[G_m,\sigma_m]}_{G_m}, \sigma_m \otimes |\det(\cdot)|^s) &= Hom_{G_m}(\text{c-ind}^{G_m}_{G^c_m}\, \sigma^c_m, \sigma_m \otimes |\det(\cdot)|^s)\\ &\cong Hom_{G^c_m}(\sigma^c_m, Res^{G_m}_{G^c_m}\sigma_m \otimes |\det(\cdot)|^s).\end{aligned} \tag{3.55}$$

The latter is a 1-dimesional $\mathbb{C}$-vector space because $Res^{G_m}_{G^c_m}(\sigma_m \otimes |\det(\cdot)|^s) = Res^{G_m}_{G^c_m}\sigma_m$ is multiplicity-free, [57, Remark 1.6.1.3].

There is a right $H(G_m, \sigma^c_m)$-action on $Hom_{G^c_m}(\sigma^c_m, Res^{G_m}_{G^c_m}\sigma_m \otimes |\det(\cdot)|^s)$ given by

$$t.f(v) = \sum_{g\in {G_m}/{G^c_m}} (\sigma_m \otimes |\det(\cdot)|^s)(g)t(f(g^{-1})v)$$

for any $f \in H(G_m, \sigma^c_m)$, $t \in Hom_{G^c_m}(\sigma^c_m, Res^{G_m}_{G^c_m}\sigma_m \otimes |\det(\cdot)|^s)$ and $v \in V_{\sigma^c_m}$. Note that this expression is well defined, since each summand on the right does not depend on the choice of representative for ${G_m}/{G^c_m}$ in G_m: for any $g \in G_m, g_1 \in G^c_m$ and $v \in V_{\sigma^c_m}$,

$$\begin{aligned}&(\sigma_m \otimes |\det(\cdot)|^s)(gg_1)t(f((gg_1)^{-1})v) && f \in H(G_m, \sigma^c_m)\\ =&(\sigma_m \otimes |\det(\cdot)|^s)(gg_1)t(\sigma^c_m(g_1^{-1})f(g)v) && t \in Hom_{G^c_m}(\sigma^c_m, Res^{G_m}_{G^c_m}\sigma_m \otimes |\det(\cdot)|^s)\\ =&(\sigma_m \otimes |\det(\cdot)|^s)(gg_1)(\sigma_m \otimes |\det(\cdot)|^s)(g_1^{-1})t(f(g)v)\\ =&(\sigma_m \otimes |\det(\cdot)|^s)(g)t(f(g^{-1})v).\end{aligned}$$

Under the isomorphism (3.52) $End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]}) \cong H(G_m,\sigma_m^c)$, the isomorphism (3.55) intertwines the $End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]})$ composition action on $Hom_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]},\sigma_m\otimes|\det(\cdot)|^s)$ with the $H(G_m,\sigma_m^c)$-action on $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}\sigma_m\otimes|\det(\cdot)|^s)$ by [57, Section 1.5.4].

In other words, the $End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]})$-module structure of $Hom_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]},\sigma_m\otimes|\det(\cdot)|^s)$ corresponds to the $H(G_m,\sigma_m^c)$-module structure of $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}\sigma_m\otimes|\det(\cdot)|^s)$.

Let ι be the canonical inclusion of $V_{\sigma_m^c}$ in the representation space of σ. Since $Res_{G_m^c}^{G_m}(\sigma_m\otimes|\det(\cdot)|^s) = Res_{G_m^c}^{G_m}\sigma_m$, the inclusion ι is a non zero element in $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}\sigma_m\otimes|\det(\cdot)|^s)$. For any $n\in N_{G_m}(\sigma_m^c)$ and $v\in V_{\sigma_m^c}$ it holds

$$(\iota.\delta_{G_m^c n})v = \sum_{g\in G_m/G_m^c}(\sigma_m\otimes|\det(\cdot)|^s)(g)\iota(\delta_{G_m^c n}(g^{-1})v) = (\sigma_m\otimes|\det(\cdot)|^s)(n^{-1})\iota(\widetilde{\sigma_m}(n)v).$$

By construction, $\widetilde{\sigma_m}$ is the representation of $N_{G_m}(\sigma_m^c)$ obtained by restriction of σ_m to $V_{\sigma_m^c}$, and ι is the canonical inclusion, so it holds

$$\iota(\widetilde{\sigma_m}(n)v) = \sigma_m(n)\iota(v).$$

Therefore

$$(\sigma_m\otimes|\det(\cdot)|^s)(n^{-1})\iota(\widetilde{\sigma_m}(n)v) = (\sigma_m\otimes|\det(\cdot)|^s)(n^{-1})\sigma_m(n)\iota(v) = |\det(n^{-1})|^s\iota(v)$$

So δ_n acts on $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}\sigma_m\otimes|\det(\cdot)|^s)$ by the scalar $|\det(n^{-1})|^s$.

Via the isomorphism (3.53), this is the same as saying that $N_{G_m}(\sigma_m^c)/G_m^c$ acts on $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}\sigma_m\otimes|\det(\cdot)|^s)$ by the character $|\det(\cdot)|^{-s}$.

Let $\nu : F\to\mathbb{Z}$ be the valuation of F. By Lemma 3.2.18, we have the isomorphism

$$\begin{aligned} N_{G_m}(\sigma_m^c)/G_m^c = Z(G_m)G_m^c/G_m^c &\to \mathbb{Z} \\ n &\mapsto \frac{\nu(det(n))}{m} \end{aligned} \tag{3.56}$$

For $z\in\mathbb{Z}$, let $n_z\in Z(G_m)G_m^c$ be such that $z=\frac{\nu(det(n_z))}{m}$. Through the isomorphism (3.56), $Hom_{G_m^c}(\sigma_m^c, Res_{G_m^c}^{G_m}(\sigma_m\otimes|\det(\cdot)|^s))$ is a $\mathbb{Z}$-module with action

$$z.\iota = \frac{\nu(det(n_z))}{m}.\iota = n_z.\iota = |det(n_z)|^{-s}i = q^{\nu(\det(n_z)s}\iota = q^{mzs}\iota$$

for $z\in\mathbb{Z}$. □

Remark 3.2.20. Note that $\mathcal{B}^{[G_m^r,\sigma_m^r]}(G_m^r) \cong \boxtimes_{i=1}^r\mathcal{B}^{[G_m,\sigma_m]}(G_m)$ and $End_{G_m^r}(\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]}) \cong \boxtimes_{i=1}^r End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]})$, and the equivalence (3.40) for $\mathcal{B}^{[G_m^r,\sigma_m^r]}(G_m^r)$ is given by

$$\boxtimes_{i=1}^r\mathcal{B}^{[G_m,\sigma_m]}(G_m) \xrightarrow{\boxtimes_{i=1}^r Hom_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]},\cdot)} \boxtimes_{i=1}^r End_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]})-Mod \tag{3.57}$$

As already recalled in Remark 3.2.17, the endomorphism algebra $End_{G_{mr}}(\Pi_G^{[G_m^r,\sigma_m^r]})$ is isomorphic to the affine Hecke algebra $\widetilde{\mathcal{H}}(r,q^m)$.

Proposition 3.2.21. *The representation $Q(\Delta(\sigma_m \otimes |det(\cdot)|^s, r)) \in \mathcal{B}^{[G_m^r,\sigma^r]}(G_{mr})$ corresponds to a (twisted) Steinberg representation of $\widetilde{\mathcal{H}}(r,q^m)$ through the equivalence* (3.40) *.*

Proof. For any $a \in \mathbb{C}$, let St_a denote the a-twisted Steinberg representation of $\widetilde{\mathcal{H}}(r,q^m)$. This representation is the unique irreducible quotient of the representation

$$Ind_{\mathbb{C}[\mathbb{Z}^r]}^{\widetilde{\mathcal{H}}(r,q^m)}\mathbb{C}_{(q^{ma})q^{m(\frac{1-r}{2},\frac{3-r}{2}\cdots\frac{r-1}{2})}}$$

where Ind denotes the extension of scalars, and $\mathbb{C}_{q^{ma}q^{m(\frac{1-r}{2},\frac{3-r}{2}\cdots\frac{r-1}{2})}}$ is the irreducible unidimensional module over $\mathbb{C}[\mathbb{Z}^r]$ defined by

$$(z_1,\ldots,z_r).1 = \prod_{i=1}^{r} q^{ma}q^{mz_i\frac{1-r+2i}{2}}$$

for $(z_1,\ldots,z_r) \in \mathbb{Z}^r$.

Since $\Pi_{G_{rm}}^{[G_m^r,\sigma^r]} = \mathrm{R}_{G_m^r}^{G_{mr}}\Pi_{G_m^r}^{[G_m^r,\sigma^r]}$, the algebra $End_{G_m^r}(\Pi_{G_m^r}^{[G_m^r,\sigma^r]})$ is naturally a subalgebra of $End_{G_{mr}}(\Pi_{G_{rm}}^{[G_m^r,\sigma^r]})$, and by [56, Theorem 5.3], the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}^{[G_m^r,\sigma^r]}(G_{mr}) & \xrightarrow{Hom(\Pi_{G_{mr}},\cdot)} & Mod - End_{G_{mr}}(\Pi_{G_{mr}}^{[G_m^r,\sigma^r]}) \\ \mathrm{R}_{G_m^r}^{G_{mr}}\Big\uparrow & & Ind\Big\uparrow \\ \mathcal{B}^{[G_m^r,\sigma^r]}(G_m^r) & \xrightarrow{Hom(\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]},\cdot)} & Mod - End_{G_m^r}(\Pi_{G_m^r}^{[G_m^r,\sigma^r]}) \end{array} \tag{3.58}$$

By Remark 3.2.20, the bottom horizontal map in (3.58) is $\boxtimes_{i=1}^r Hom_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]},\cdot)$, and by Lemma (3.2.19) $Hom_{G_m}(\Pi_{G_m}^{[G_m,\sigma_m]},\sigma_m \otimes |det(\cdot)|^{(s+i)}) \cong \mathbb{C}_{(q^{m(s+i)})}$ as $\mathbb{C}[\mathbb{Z}]-$module. Therefore

$$\begin{aligned} &Hom(\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]},(\sigma_m \otimes |det(\cdot)|^s)\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+1})\boxtimes\cdots\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+r-1}))\\ &\cong \boxtimes_{i=1}^{r} Hom(\Pi_{G_m}^{[G_m,\sigma_m]},\sigma_m \otimes |\det(\cdot)|^{s+i})\\ &= \boxtimes_{i=1}^{r}\mathbb{C}_{(q^{m(s+i)})}\\ &\cong \mathbb{C}_{q^{ms}q^{m(1,2,\ldots r)}}\\ &= \mathbb{C}_{q^{m(s+\frac{1+r}{2})}q^{m(\frac{1-r}{2},\frac{3-r}{2}\cdots\frac{r-1}{2})}} \end{aligned}$$

Since the diagram (3.58) commutes,

$$\begin{aligned} &Hom(\Pi_{G_{mr}}^{[G_m^r,\sigma_m^r]},\mathrm{R}_{G_m^r}^{G_{mr}}((\sigma_m \otimes |det(\cdot)|^s)\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+1})\boxtimes\cdots\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+r-1})))\\ =&Ind(Hom(\Pi_{G_m^r}^{[G_m^r,\sigma_m^r]},(\sigma_m \otimes |det(\cdot)|^s)\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+1})\boxtimes\cdots\boxtimes(\sigma_m \otimes |det(\cdot)|^{s+r-1})))\\ =&Ind_{\mathbb{C}[\mathbb{Z}^r]}^{\widetilde{\mathcal{H}}(d,q^m)}\mathbb{C}_{q^{m(s+\frac{1+r}{2})}q^{m(\frac{1-r}{2},\frac{3-r}{2}\cdots\frac{r-1}{2})}}. \end{aligned}$$

Moreover the top horizontal functor in the diagram (3.58) is an equivalence of categories, so it maps the unique irreducible quotient $Q(\Delta(\sigma_m \otimes |det(\cdot)|^s, r))$ of the representation $\mathrm{R}_{G_m^r}^{G_{mr}}((\sigma_m \otimes |det(\cdot)|^s) \otimes (\sigma_m \otimes |det(\cdot)|^{s+1}) \otimes \cdots \otimes (\sigma_m \otimes |det(\cdot)|^{s+r-1}))$ to the unique irreducible quotient $St_{s+\frac{1+r}{2}}$ of the module $Ind_{\mathbb{C}[\mathbb{Z}^r]}^{\widetilde{\mathcal{H}}(r,q^m)}\mathbb{C}_{q^{m(s+\frac{1+r}{2})}q^{m(\frac{1-r}{2},\frac{3-r}{2}\dots\frac{r-1}{2})}}$. □

Proposition 3.2.22. *Let $\sigma \in Cusp(G_m)_0$ and $r \in \mathbb{N}$. Then*

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\sigma, r)) = \pi^{(r)}_{\mathcal{P}_{\overline{G}_m}^{G_m}\sigma}. \tag{3.59}$$

Proof. By Proposition 3.2.21, the representation $Q(\Delta(\sigma, r))$ corresponds to a (twisted) Steinberg representation St_c. Then by Proposition 3.2.16 and Remark 3.2.17, the parahoric restriction $\mathcal{P}_{\overline{G}}^{G} Q(\Delta(\sigma, r))$ is the irreducible constituent of $\mathrm{R}_{\overline{G}_m^r}^{\overline{G}_{mr}}(\mathcal{P}_{\overline{G}_m}^{G_m}\overline{\sigma})^r$ corresponding to the restriction to $\mathcal{H}(\mathbb{S}_r, q^m)$ of St_a . The restriction of the representation St_a of $\widetilde{\mathcal{H}}(r, q^m)$ to the finite Hecke algebra $\mathcal{H}(\mathbb{S}_r, q^m)$ is the *sign* representation, that is parameterized by the partition (r). Hence we have identity (3.59). □

Proposition 3.2.23. *For any $\rho \in \Phi^0(G_m)_0$ it holds*

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\mathscr{L}_m^0(\rho), r)) = \pi^{(r)}_{\mathscr{M}_m^0(\rho|_{I_F})}.$$

Proof. By Proposition 3.2.22,

$$\mathcal{P}_{\overline{G}}^{G} Q(\Delta(\mathscr{L}_m^0(\rho), r)) = \pi^{(r)}_{\mathcal{P}_{\overline{G}_m}^{G_m}\mathscr{L}_m^0(\rho)}$$

and by Corollary 3.2.13

$$\pi^{(r)}_{\mathcal{P}_{\overline{G}_m}^{G_m}\mathscr{L}_m^0(\rho)} = \pi^{(r)}_{\mathscr{M}_m^0(\rho|_{I_F})}.$$

□

This concludes the proof of the identity (3.36).

Theorem 3.2.24. *For any indecomposable tamely ramified Weil-Deligne representation $(\rho, 0) \otimes Sp(r)$, with $\rho \in \Phi^0(G_m)_0$, it holds:*

$$\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} \mathscr{L}_{mr}((\rho, 0) \otimes Sp(r)) = \mathscr{M}_{mr}(((\rho, 0) \otimes Sp(r))_{I_F}). \tag{3.60}$$

Proof. By Proposition (3.2.23), the identity (3.38) is proved, so we can use the chain of identities at the beginning of this Section as follows:

$$\begin{aligned}
&\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}}(\mathscr{L}_{mr}((\rho, 0) \otimes Sp(r))) && \text{by } (3.35)\\
=&\mathcal{P}_{\overline{G}_{mr}}^{G_{mr}} Q(\Delta(\mathscr{L}_m^0(\rho), r)) && \text{by Proposition 3.2.23}\\
=&\pi^{(r)}_{\mathscr{M}_m^0(\rho|_{I_F})} && \text{by } (3.37)\\
=&\mathscr{M}_{mr}(((\rho, 0) \otimes Sp(r))_{I_F}).
\end{aligned}$$

□

3.2.4 The general case

In this section, we extend the compatibility via parahoric restriction to the whole tame Langlands corresponence and Macdonald correspondence, treating the general case of an N-dimensional tame semisimple Weil-Deligne representation and the corresponding smooth irreducible representation of G_N.

While the parahoric restriction of essentially square integrable representations is irreducible, as we have seen in the last section, this is not the case for a general smooth irreducible representation of G_N. In the beginning of this section, we describe which representations of $\overline{G}_N$ appear as irreducible constituents of the parahoric restriction of a smooth irreducible representation of G_N. This description allows us to introduce a way of selecting an irreducible constituent, that is given in (3.63). Therefore we obtain a truncated version of the parahoric restriction, that yields the desired compatibility between the tame Langlands correspondence and the Macdonald correspondence.

As already mentioned in the introduction of this chapter and at the beginning of Section 3.2.3, this compatibility has already been proved in [64, Appendix A]. The "Head of parahoric restriction" that we define in (3.63) coincides in the depth-0 case with the "reduction to tempered types" map defined in [59].

Lemma 3.2.25. *Let $\overline{\sigma}$ be a cuspidal representation of $\overline{G}_m$, and let $\sigma_1, \dots, \sigma_k \in Cusp(G_m)_0$ be such that $\mathcal{P}^{G_m}_{\overline{G}_m}\sigma_i = \overline{\sigma}$ for any $i \in \{1, \dots, k\}$. Let $\lambda = (r_1, \dots, r_k) \in \mathscr{P}_P$ with $P \in \mathbb{N}$, and let $n = Pm$. Let*

$$\pi = \mathrm{R}^{G_n}_{\prod_{i=1}^k G_{mr_i}}(Q(\Delta(\sigma_1, r_1)) \boxtimes \cdots \boxtimes Q(\Delta(\sigma_k, r_k))).$$

Then the irreducible constituents of $\mathcal{P}^{G_n}_{\overline{G}_n}\pi$ are the irreducible representations $\pi^{\lambda'}_{\overline{\sigma}}$ were λ' runs through the partitions of P such that $\lambda' \geqslant \lambda$ with respect to the dominance order. Moreover the representation $\pi^{\lambda}_{\overline{\sigma}}$ appears with multiplicity one.

Proof. By compatibility of parahoric restriction with parabolic induction, it holds

$$\begin{aligned}
\mathcal{P}^{G_n}_{\overline{G}_n}\pi &= \mathcal{P}^{G_n}_{\overline{G}_n}\mathrm{R}^{G_n}_{\prod_{i=1}^k G_{mr_i}}(Q(\Delta(\sigma_1, r_1)) \boxtimes \cdots \boxtimes Q(\Delta(\sigma_k, r_k))) \\
&= \mathrm{R}^{\overline{G}_n}_{\prod_{i=1}^k \overline{G}_{mr_i}} \mathcal{P}^{\prod_{i=1}^k G_{mr_i}}_{\prod_{i=1}^k \overline{G}_{mr_i}}(Q(\Delta(\sigma_1, r_1)) \boxtimes \cdots \boxtimes Q(\Delta(\sigma_k, r_k))) \\
&= \mathrm{R}^{\overline{G}_n}_{\prod_{i=1}^k \overline{G}_{mr_i}}(\mathcal{P}^{G_{mr_1}}_{\overline{G}_{mr_1}}(Q(\Delta(\sigma_1, r_1))) \boxtimes \cdots \boxtimes \mathcal{P}^{G_{m_kr_k}}_{\overline{G}_{m_kr_k}}(Q(\Delta(\sigma_k, r_k)))) \qquad (3.59) \\
&= \mathrm{R}^{\overline{G}_n}_{\prod_{i=1}^k \overline{G}_{mr_i}}(\pi^{(r_1)}_{\mathcal{P}^{G_m}_{\overline{G}_m}\sigma_1} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\mathcal{P}^{G_m}_{\overline{G}_m}\sigma_k}) \\
&= \mathrm{R}^{\overline{G}_n}_{\prod_{i=1}^k \overline{G}_{mr_i}}(\pi^{(r_1)}_{\overline{\sigma}} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\overline{\sigma}}).
\end{aligned}$$

By [32, Theorem 5.9], the irreducible constituents and multiplicities of the representation $\mathrm{R}^{\overline{G}_n}_{\overline{M}}(\pi^{(r_1)}_{\overline{\sigma}} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\overline{\sigma}})$ correspond to the irreducible constituents of the induced representation $Ind^{\mathbb{S}_P}_{\mathbb{S}_{r_1} \times \cdots \times \mathbb{S}_{r_k}} sign \boxtimes \cdots \boxtimes sign$ of the symmetric group $\mathbb{S}_P$. Recall that the parameterization of the representations of the symmetric group $\mathbb{S}_P$ is normalized in such a way that the highest partition (P) corresponds to the sign representation and the lowest partition (1^P) corresponds to the trivial representation. Hence, in the representation $Ind^{\mathbb{S}_P}_{\mathbb{S}_{r_1} \times \cdots \times \mathbb{S}_{r_k}} sign \boxtimes \cdots \boxtimes sign$ the irreducible constituents are parametrized by partitions $\lambda' \geqslant \lambda$ in the dominance order, and the irreducible constituent relative to $(r_1, r_2, \dots, r_k)$ appears with multiplicity 1. Hence the irreducible constituents of $\mathcal{P}^{G_n}_{\overline{G}_n}\pi$ are the irreducible representations $\pi^{\lambda'}_{\overline{\sigma}}$ with $\lambda' \geqslant \lambda$ and $\pi^{\lambda}_{\overline{\sigma}}$ has multiplicity 1. □

We retain notation from Section 3.2.1. Let $\{\Delta_1, \dots \Delta_k\}$ be a multiset of segments, with

$$\Delta_i = \Delta(\sigma_i, r_i), \qquad \text{for } \sigma_i \in Cusp(G_{m_i})_0,\ m_i r_i \in \mathbb{N}$$

and $\sum_{i=1}^k m_i r_i = N$.We recall that $I(\Delta_1, \dots, \Delta_k) := \mathrm{R}^{G_N}_{\prod_{i=1}^k G_{m_i r_i}}(Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_k))$.

We associate to the multiset of segments $\{\Delta_1, \dots \Delta_k\}$ the partition-valued function of degree N on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ defined by

$$\begin{aligned} \Lambda_{\{\Delta_1,\dots\Delta_k\}} : \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m) &\longrightarrow \mathscr{P} \\ \overline{\sigma} &\mapsto \left(r_i \mid i \in \{1\dots,k\} \ s.t.\ \mathcal{P}^{G_{m_i}}_{\overline{G}_{m_i}}\sigma_i = \overline{\sigma}\right) \end{aligned} \tag{3.61}$$

where $\left(r_i \mid i \in \{1\dots,k\} \ s.t.\ \mathcal{P}^{G_{m_i}}_{\overline{G}_{m_i}}\sigma_i = \overline{\sigma}\right)$ is ordered to have decreasing entries.

Let Λ, Λ' be two partition valued functions on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$. We say that $\Lambda \leqslant \Lambda'$ if for any $\overline{\sigma} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ it holds $\Lambda(\overline{\sigma}) \leqslant \Lambda'(\overline{\sigma})$, where $\leqslant$ denotes the dominance order on partitions.

Lemma 3.2.26. *Retain the notation above. Let Λ be a partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N. Then π_Λ as in* (3.28) *is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \dots, \Delta_k)$ if and only if $\Lambda \geqslant \Lambda_{\{\Delta_1,\dots,\Delta_k\}}$.*

Moreover the irreducible representation $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ appears with multiplicity 1.

Proof. By compatibility of parahoric restriction and parabolic induction, it holds

$$\begin{aligned} \mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \dots, \Delta_k) &= \mathcal{P}^{G_N}_{\overline{G}_N} \mathrm{R}^{G_N}_{\prod_{i=1}^k G_{m_i r_i}}(Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_k)) \\ &= \mathrm{R}^{\overline{G}_N}_{\prod_{i=1}^k \overline{G}_{m_i r_i}} \mathcal{P}^{\prod_{i=1}^k G_{m_i r_i}}_{\prod_{i=1}^k \overline{G}_{m_i r_i}}(Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_k)) \\ &= \mathrm{R}^{\overline{G}_N}_{\prod_{i=1}^k \overline{G}_{m_i r_i}}(\mathcal{P}^{G_{m_1 r_1}}_{\overline{G}_{m_1 r_1}}(Q(\Delta_1)) \boxtimes \cdots \boxtimes Q(\Delta_k)) \\ &= \mathrm{R}^{\overline{G}_N}_{\prod_{i=1}^k \overline{G}_{m_i r_i}}(\pi^{(r_1)}_{\mathcal{P}^{G_{m_1}}_{\overline{G}_{m_1}}\sigma_1} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\mathcal{P}^{G_{m_k}}_{\overline{G}_{m_k}}\sigma_k}) \end{aligned}$$

where the last equality is obtained using (3.59) on each factor. We decompose $\{1, 2, \dots, k\} = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_l$ so that $\mathcal{P}^{G_{m_i}}_{\overline{G}_{m_i}}\sigma_i \cong \mathcal{P}^{G_{m_j}}_{\overline{G}_{m_j}}\sigma_j$ if and only if i and j for $i, j \in \{1, 2, \dots, k\}$, belong to the same J_s, for some $s \leqslant l$, so $m_i = m_j$. For $i \in J_s$ we set $\overline{\sigma}_s := \mathcal{P}^{G_{m_i}}_{\overline{G}_{m_i}}\sigma_i$ and $\widetilde{m}_s := m_i$ for $i \in J_s$. Moreover we set $P_s = \sum_{i\in J_s} r_i$, and $\lambda_s = \Lambda_{\{\Delta_1,\dots\Delta_k\}}(\overline{\sigma}_s)$, that is the partition of P_s having as entries the lengths of the segments relative to the cuspidal representations whose parahoric restriction in $\overline{\sigma}_s$, ordered in decreasing order. We write the entries of λ_s as $\lambda_s = (\lambda_{s1}, \ldots, \lambda_{s|J_s|})$.

Transitivity of parabolic induction gives

$$\begin{aligned} &\mathrm{R}^{\overline{G}_N}_{\prod_{i=1}^k \overline{G}_{m_i r_i}} \big(\pi^{(r_1)}_{\mathcal{P}^{G_{m_1}}_{\overline{G}_{m_1}} \sigma_1} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\mathcal{P}^{G_{m_k}}_{\overline{G}_{m_k}} \sigma_k}\big) \\ &= \mathrm{R}^{\overline{G}_N}_{\prod_{s=1}^l \overline{G}_{\widetilde{m}_s P_s}} \Big(\boxtimes_{s=1}^{l} \mathrm{R}^{\overline{G}_{\widetilde{m}_s P_s}}_{\prod_{j\in J_s} \overline{G}_{\widetilde{m}_s r_j}} \big(\pi^{(\lambda_{s1})}_{\overline{\sigma}_s} \boxtimes \cdots \boxtimes \pi^{(\lambda_{s|J_s|})}_{\overline{\sigma}_s}\big)\Big). \end{aligned}$$

By Lemma 3.2.25, for any $s \in \{1, \ldots, l\}$ an irreducible representation is an irreducible constituent of $\mathrm{R}^{\overline{G}_{\widetilde{m}_s P_s}}_{\prod_{j\in J_s} \overline{G}_{\widetilde{m}_s r_j}} \pi^{(\lambda_{s1})}_{\overline{\sigma}_s} \boxtimes \cdots \boxtimes \pi^{(\lambda_{s|J_s|})}_{\overline{\sigma}_s}$ if and only if it is of the shape $\pi^{\lambda'}_{\overline{\sigma}_s}$ with $\lambda' \geqslant \lambda_s$, and the irreducible constituent $\pi^{\lambda_s}_{\overline{\sigma}_s}$ has multiplicity 1.

Therefore an irreducible representation of $\overline{G}_N$ is an irreducible constituent of $\mathrm{R}^{\overline{G}_N}_{\prod_{i=1}^k \overline{G}_{m_i r_i}} (\pi^{(r_1)}_{\mathcal{P}^{G_{m_1}}_{\overline{G}_{m_1}} \sigma_1} \boxtimes \cdots \boxtimes \pi^{(r_k)}_{\mathcal{P}^{G_{m_k}}_{\overline{G}_{m_k}} \sigma_k})$, that is, of $\mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \ldots, \Delta_k)$, if and only if it is an irreducible constituent of a representation of the shape

$$\mathrm{R}^{\overline{G}_N}_{\prod_{s=1}^l \overline{G}_{\widetilde{m}_s P_s}} (\pi^{\mu_1}_{\overline{\sigma}_1} \boxtimes \cdots \boxtimes \pi^{\mu_l}_{\overline{\sigma}_l}) \quad \text{with } \mu_s \geqslant \lambda_s \text{ for any s=1,...,l.} \tag{3.62}$$

We claim that a representation of $\overline{G}_N$ is of the form (3.62) if and only if it is a representation π_Λ as in (3.28) for Λ a partition-valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N satisfying $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$.

Indeed let Λ be a partition-valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N such that $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$. In particular it holds $|\Lambda(\overline{\sigma})| = |\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\sigma})|$ for any $\overline{\sigma} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$, that is, $|\Lambda(\overline{\sigma})| = P_s$ if $\overline{\sigma} = \overline{\sigma}_s$ for some $s = 1, \ldots, l$ and 0 otherwhise. Then π_Λ as in (3.28) is given by

$$\pi_\Lambda = \mathrm{R}^{\overline{G}_N}_{\prod_{s=1}^l \overline{G}_{\widetilde{m}_s P_s}} (\pi^{\Lambda(\overline{\sigma}_1)}_{\overline{\sigma}_1} \boxtimes \cdots \boxtimes \pi^{\Lambda(\overline{\sigma}_l)}_{\overline{\sigma}_l})$$

and since $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$ it holds $\Lambda(\overline{\sigma}_s) \geqslant \lambda_s$ for any $s = 1, \ldots, l$, that is, π_Λ is one of the representations as in (3.62).

Conversely let $\mathrm{R}^{\overline{G}_N}_{\prod_{s=1}^l \overline{G}_{\widetilde{m}_s P_s}} (\pi^{\mu_1}_{\overline{\sigma}_1} \boxtimes \cdots \boxtimes \pi^{\mu_l}_{\overline{\sigma}_l})$ be a representation as in (3.62). Then it is equal to π_Λ where Λ is the partition-valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ defined by $\Lambda(\overline{\sigma}) = \mu_s$ is $\overline{\sigma} = \sigma_s$ for some $s = 1, \ldots, l$ and 0 otherwise. Since $\mu_s \geqslant \lambda_s$ for any $s = 1, \ldots, l$ it follows that $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$.

Hence, any representation of the form (3.62) is irreducible, and the irreducible constituents of $\mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \ldots, \Delta_k)$ are exactly the π_Λ with $\Lambda \geqslant \Lambda_{\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}}$. Moreover the representation $\pi_{\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}} = \mathrm{R}^{\overline{G}_N}_{\prod_{s=1}^l \overline{G}_{\widetilde{m}_s P_s}} (\pi^{\lambda_1}_{\overline{\sigma}_1} \boxtimes \cdots \boxtimes \pi^{\lambda_l}_{\overline{\sigma}_l})$ has multiplicity 1, since any $\pi^{\lambda_s}_{\overline{\sigma}_s}$ for $s = 1, \ldots, l$ appears only once.

□

Corollary 3.2.27. *Let $\pi \in \Omega(G_N)_0$, let $\{\Delta_1, \ldots, \Delta_k\}$ be a multiset of segments such that $\pi = Q(\Delta_1, \ldots, \Delta_k)$ as in Theorem 3.2.3. Let Λ be a partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N. If π_Λ as in* (3.28) *is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N} Q(\Delta_1, \ldots, \Delta_k)$, then $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$.*

Proof. Since $Q(\Delta_1, \dots, \Delta_k)$ is a quotient of $I(\Delta_1, \dots, \Delta_k)$, the representation $\mathcal{P}^{G_N}_{\overline{G}_N} Q(\Delta_1, \dots, \Delta_k)$ is a subrepresentation of $\mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \dots, \Delta_k)$, by exactness of parahoric restriction and semisemplicity of the representations of $\overline{G}_N$. By Lemma 3.2.26, an irreducible representation of $\overline{G}_N$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N} I(\Delta_1, \dots, \Delta_k)$ if and only if it is of the form π_Λ for Λ a partition valued function such that $\Lambda \geqslant \Lambda_{\{\Delta_1,\dots,\Delta_k\}}$. Hence any irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N} Q(\Delta_1, \dots, \Delta_k)$ is of the form π_Λ for Λ a partition valued function such that $\Lambda \geqslant \Lambda_{\{\Delta_1,\dots,\Delta_k\}}$. □

Let $\pi \in \Omega(G_N)_0$. Then by Theorem 3.2.3, $\pi = Q(\Delta_1, \dots, \Delta_k)$ for $\{\Delta_1, \dots, \Delta_k\}$ a multiset of segments such that Δ_i does not precede Δ_j for any $i < j$.

We define the Head of the parabolic restriction of $Q(\Delta_1, \dots, \Delta_k)$ to be the irreducible representation of $\overline{G}_N$

$$\mathcal{H}\mathcal{P}^{G_N}_{\overline{G}_N}(Q(\Delta_1, \dots, \Delta_k)) := \pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}. \tag{3.63}$$

Corollary 3.2.28. *Let $\{\Delta_1, \dots, \Delta_k\}$ be a multiset of segments such that none of the segments Δ_i is linked to each other. Then $\mathcal{H}\mathcal{P}^{G_N}_{\overline{G}_N}(Q(\Delta_1, \dots, \Delta_k))$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}(Q(\Delta_1, \dots, \Delta_k))$ of multiplicity* 1.

Proof. Since none of the segments Δ_i is linked to each other, by Theorem 3.2.5 it holds

$$Q(\Delta_1, \dots, \Delta_k) = I(\Delta_1, \dots, \Delta_k).$$

The statement follows from Lemma 3.2.26. □

Remark 3.2.29. If $\pi \in \Omega(G_N)_0$ is a tempered representation, then by Remark 3.2.6 $\pi = Q(\Delta_1, \dots, \Delta_k)$ with no linked segments. It follows that if π is tempered, then the representation $\mathcal{H}\mathcal{P}^{G_N}_{\overline{G}_N}\pi$ is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G_N}_{\overline{G}_N}\pi$ by Corollary 3.2.28.

We now extend the result of Corollary 3.2.6 to representations in $\Omega(G_N)_0$. We will need the following combinatorial lemma.

Lemma 3.2.30. *Let $\{\Delta_1, \dots, \Delta_k\}$ be a multiset of segments, and let $\{\Delta'_1, \dots, \Delta'_{k'}\}$ be a multiset of segments obtained from $\{\Delta_1, \dots, \Delta_k\}$ by an elementary move. Then $\Lambda_{\{\Delta_1,\dots\Delta_k\}} < \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}}$.*

Proof. The multiset $\{\Delta'_1, \dots, \Delta'_{k'}\}$ is obtained from $\{\Delta_1, \dots \Delta_k\}$ by substituting two linked segments Δ_i, Δ_j by $\Delta_i \cup \Delta_j$ and $\Delta_i \cap \Delta_j$. Let $\Delta_i = \Delta(\sigma_i, r_i)$ and $\Delta_j = \Delta(\sigma_j, r_j)$. We are assuming Δ_i and Δ_j to be linked, so there exists $k \in \mathbb{N}_{>0}$ such that $\sigma_j = \sigma_i \otimes |det|^k$ with $k \leqslant r_i < k + r_j$. Then $\mathcal{P}^{G_N}_{\overline{G}_N}\sigma_i = \mathcal{P}^{G_N}_{\overline{G}_N}\sigma_j$, since $|det|$ is an unramified character, and it holds $\Delta_i \cup \Delta_j = \Delta(\sigma_i, k+r_j)$ and $\Delta_i \cap \Delta_j = \Delta(\sigma_j, r_i - k)$. Let $\overline{\sigma} := \mathcal{P}^{G_N}_{\overline{G}_N}\sigma_i = \mathcal{P}^{G_N}_{\overline{G}_N}\sigma_j$. Since the elementary move that brings $\{\Delta_1, \dots, \Delta_k\}$ to $\{\Delta'_1, \dots, \Delta'_{k'}\}$ involves only segments relative to supercuspidal representations whose parahoric restriction is $\overline{\sigma}$, for any $\tau \in \bigsqcup_{m \in \mathbb{N}} Cusp(\overline{G}_m)$ different from $\overline{\sigma}$ it holds $\Lambda_{\{\Delta_1,\dots,\Delta_k\}}(\tau) = \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}}(\tau)$.

To conclude we prove that $\Lambda_{\{\Delta_1,\dots\Delta_k\}}(\overline{\sigma}) > \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}}(\overline{\sigma})$.

Let $\lambda := \Lambda_{\{\Delta_1,\dots\Delta_k\}}(\overline{\sigma})$. We write $\lambda = (\lambda_1, \dots, \lambda_l)$ and denote by λ_s and λ_t the entries relative respectively to the segments Δ_i and Δ_j, namely r_i and r_j. Let

$\lambda' := \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}}(\overline{\sigma})$. We write $\lambda' = (\lambda'_1, \dots, \lambda'_{l'})$ and denote by $\lambda'_{s'}$ and $\lambda'_{t'}$ the entries relative respectively to the segments $\Delta_i \cup \Delta_j$ and $\Delta_i \cap \Delta_j$, namely $r_i - k$ and $r_j + k$.

So the partition λ' is obtained from the partition λ by substituting the entries $\{\lambda_s, \lambda_t\}$ by $\{\lambda'_{s'} = \lambda_s - k, \lambda'_{t'} = \lambda_t + k\}$, and leaving the other entries unchanged.

Let $d = \lambda_s - k$. Then $\lambda'_{s'} = d$ and $\lambda'_{t'} = \lambda_s + \lambda_t - d$. We observe that $d \geqslant 0$ since $k \leqslant r_i = \lambda_s$, and $\lambda_s - d = k > 0$ and $\lambda_t - d > 0$ since $r_j + k > r_i$. Recall also that the partitions are ordered in decreasing order. The following computation shows that $\lambda' > \lambda$ with respect to the dominance order, i.e. $\sum_{v=0}^{h} \lambda'_v - \lambda_v \geqslant 0$ for any $0 \leqslant h \leqslant l$, and there is at least one h for which the inequality is strict.

$$\sum_{v=0}^{h} \lambda'_v - \lambda_v = 0 \qquad 0 \leqslant h < t',$$

$$\sum_{v=0}^{h} \lambda'_v - \lambda_v = \lambda'_{t'} - \lambda_h = \lambda'_{t'} - \lambda'_{h+1} \geqslant 0 \qquad t' \leqslant h < min\{s,t\},$$

$$\sum_{v=0}^{h} \lambda'_v - \lambda_v = \lambda'_{t'} - \lambda_{min\{t,s\}} = \lambda_{max\{t,s\}} - d > 0 \qquad min\{s,t\} \leqslant h < max\{s,t\},$$

$$\sum_{v=0}^{h} \lambda'_v - \lambda_v = \lambda'_h - d = \lambda'_h - \lambda'_{s'} \geqslant 0 \qquad max\{s,t\} \leqslant h < s',$$

$$\sum_{v=0}^{h} \lambda'_v - \lambda_v = 0 \qquad h \geqslant s'.$$

□

Proposition 3.2.31. *Let $\pi \in \Omega(G_N)_0$. Then $\mathcal{HP}^{G_N}_{\overline{G}_N}(\pi)$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}(\pi)$ of multiplicity* 1.

Proof. By Theorem 3.2.3, there exists a multiset of segments $\{\Delta_1, \dots, \Delta_k\}$ such that Δ_i does not precede Δ_j for any $i \leqslant j$ for which $\pi = Q(\Delta_1, \dots, \Delta_k)$. Then $\mathcal{HP}^{G_N}_{\overline{G}_N}(\pi) = \pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$. By Lemma 3.2.26, $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G_N}_{\overline{G}_N}(I(\Delta_1, \dots, \Delta_k))$. Therefore $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ is an irreducible constituent of multiplicity 1 of the parahoric restriction of one of the irreducible subquotients of $I(\Delta_1, \dots, \Delta_k)$. Now we prove that $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ is not an irreducible constituent of the parahoric restriction of any irreducible subquotient of $I(\Delta_1, \dots, \Delta_k)$ different from $Q(\Delta_1, \dots \Delta_k)$, and so it follows that $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}} = \mathcal{HP}^{G_N}_{\overline{G}_N}(\pi)$ is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G_N}_{\overline{G}_N}(Q(\Delta_1, \dots, \Delta_k)) = \mathcal{P}^{G_N}_{\overline{G}_N}(\pi)$.

By Theorem 3.2.5, a representation in $\Omega(G_N)_0$ is an irreducible subquotient of $I(\Delta_1, \dots, \Delta_k)$ if and only if it is of the form $Q(\Delta'_1, \dots, \Delta'_{k'})$ for a multiset of segments $\{\Delta'_1, \dots, \Delta'_{k'}\}$ obtained from $\{\Delta_1, \dots, \Delta_k\}$ by a sequence of elementary moves. It follows that if $Q(\Delta'_1, \dots, \Delta'_{k'})$ is an irreducible constituent of $I(\Delta_1, \dots, \Delta_k)$ different from $Q(\Delta'_1, \dots, \Delta'_{k'})$, by Lemma 3.2.30 it holds $\Lambda_{\{\Delta_1,\dots\Delta_k\}} < \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}}$.

By Corollary 3.2.27, any irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N} Q(\Delta'_1, \dots, \Delta'_{k'})$ is of the form π_Λ for Λ a partition valued function such that $\Lambda \geqslant \Lambda_{\{\Delta'_1,\dots,\Delta'_{k'}\}} > \Lambda_{\{\Delta_1,\dots,\Delta_k\}}$. It follows that $\pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ is not a subrepresentation of $\mathcal{P}^{G_N}_{\overline{G}_N} Q(\Delta'_1, \dots, \Delta'_{k'})$ for any subquotient of $I(\Delta_1, \dots, \Delta_k)$ different from $Q(\Delta_1, \dots, \Delta_k)$. □

By Proposition 3.2.31, the head of the parabolic restriction $\mathcal{HP}^{G_N}_{\overline{G}_N}\pi$ is indeed a truncated version of the parahoric restriction $\mathcal{P}^{G_N}_{\overline{G}_N}\pi$, selecting a specific irreducible constituent of the latter.

Now we are ready to prove the main result of this chapter, namely the fact that the tame Langlands corresponence ad the Macdonald correspondence are compatible via the head of the parabolic restriction.

Theorem 3.2.32. *Let $(\rho, E) \in \Phi(G_N)_0$. Then*

$$\mathscr{M}_N((\rho, E)_{I_F}) = \mathcal{HP}^{G_N}_{\overline{G}_N}\mathscr{L}_N(\rho, E).$$

In particular $\mathscr{M}_N((\rho, E)_{I_F})$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}\mathscr{L}_N(\rho, E)$ of multiplicity 1*.*

Proof. With notation as in Section 3.2.2, by (3.29) we have

$$\mathscr{M}_N((\rho, E)_{I_F}) = \pi_{\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}}. \tag{3.64}$$

We write the decomposition into indecomposables of (ρ, E) as

$$(\rho, E) = \bigoplus_{i=1}^{k}(\rho_i, 0) \otimes Sp(r_i),$$

where $\rho_i \in \Phi^0(G_n)_0$ and $r_i \in \mathbb{N}$ for any $i \in \{1, 2, \ldots, k\}$.

Then by (3.27) for any $\overline{\sigma} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ it holds

$$\Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}(\overline{\sigma}) = \big(r_i \mid i = 1, \ldots k \text{ such that } \rho_i|_{I_F} = (\mathscr{M}^0)^{-1}(\overline{\sigma})\big),$$

that is

$$\Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}(\overline{\sigma}) = \big(r_i \mid i = 1, \ldots k \text{ such that } \mathscr{M}^0(\rho_i|_{I_F}) = \overline{\sigma}\big). \tag{3.65}$$

By (3.24), it holds

$$\mathscr{L}_N((\rho, E)) = \mathscr{L}_N(\oplus_{i=1}^k(\rho_i, 0)\otimes Sp(r_i)) = Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i = 1, \ldots, k\}). \tag{3.66}$$

By (3.63), we have

$$\mathcal{HP}^{G_N}_{\overline{G}_N}(Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i = 1, \ldots, k\})) := \pi_{\Lambda_{\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i=1,\ldots,k\}}}, \tag{3.67}$$

and by (3.61) for any $\overline{\sigma} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ it holds

$$\Lambda_{\{\Delta(\mathscr{L}^0_{m_i}(\rho_i), r_i) \mid i=1,\ldots,k\}}(\overline{\sigma}) = \big(r_i \mid i = 1, \ldots, k \text{ such that } \mathcal{P}^{G_{m_1}}_{\overline{G}_{m_i}}(\mathscr{L}^0_{m_i}(\rho_i)) = \overline{\sigma}\big). \tag{3.68}$$

For any $1 = 1, \ldots, k$, Proposition (3.1.11) gives

$$\mathcal{P}^{G_{m_i}}_{\overline{G}_{m_i}}\mathscr{L}^0_{m_i}(\rho_i) = \mathscr{M}^0_{m_i}(\rho_i|_{I_F}).$$

Therefore for any $\overline{\sigma} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ comparing (3.65) and (3.68) gives $\Lambda_{\{\Delta(\mathscr{L}^0_{m_i}(\rho_i),r_i) \ | \ i=1,\dots,k\}}(\overline{\sigma}) = \Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}(\overline{\sigma})$, and so

$$\Lambda_{\{\Delta(\mathscr{L}^0_{m_i}(\rho_i),r_i) \ | \ i=1,\dots,k\}} = \Lambda_{(\rho,E)} \circ (\mathscr{M}^0)^{-1}. \tag{3.69}$$

Therefore

$$\begin{aligned}
\mathscr{M}_N((\rho,E)_{I_F}) &= \pi_{\Lambda_{(\rho,E)}\circ(\mathscr{M}^0)^{-1}} && \text{by (3.64)}\\
&= \pi_{\Lambda_{\{\Delta(\mathscr{L}^0_{m_i}(\rho_i),r_i) \ | \ i=1,\dots,k\}}} && \text{by (3.69)}\\
&= \mathcal{HP}^{G_N}_{\overline{G}_N}(Q(\{\Delta(\mathscr{L}^0_{m_i}(\rho_i),r_i) \mid i=1,\dots,k\})) && \text{by (3.67)}\\
&= \mathcal{HP}^{G_N}_{\overline{G}_N}(\mathscr{L}_N((\rho,E)). && \text{by (3.66)}
\end{aligned}$$

It follows by Proposition 3.2.31 that $\mathscr{M}_N((\rho,E)_{I_F})$ is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G_N}_{\overline{G}_N}(\mathscr{L}_N((\rho,E)))$. □

Chapter IV

A reduction of the tame Langlands correspondence for SL_n over finite fields

Introduction

In this chapter, we extend the results of Chapter III to the case of SL_N.

In particular, we establish a Langlands-type parameterization of the irreducible representations of the finite group $\mathrm{SL}_N(k_F)$, thereby confirming [35, Conjecture 4.3] in this case. We then prove a compatibility result with the tame local Langlands correspondence for $\mathrm{SL}_N(F)$, in line with the prediction from [68] discussed in the introduction to Chapter III.

The first instance in the literature where a Langlands-type parameterization for representations of a finite group of Lie type along the lines of [35] had been established is the Macdonald correspondence in [47] for GL_N. After this chapter was completed, in [34] such a parameterization for any connected reductive groups, called there "finite Laglands correspondence", has been constructed, yielding a general proof of [35, Conjecture 4.3]. The construction in [34] relies on categorical equivalences (see [34, Proposition 4.1, Theorem 4.5]) and on deep results on the relation between representations and Character sheaves (e.g. [45, 46]). The SL_N case considered here allows for several simplifications and a more direct construction, which we exploit in our approach.

The results presented in this chapter have been recently published in [15], and the exposition here follows closely that of the article.

The key idea, following [35], is to refine the notion of I_F-equivalence by retaining partial information about the image of a Frobenius element. In Conjecture 4.2.5, we formulate a version of [35, Conjecture 4.3] specialized to our setting; see Remark 1 for a detailed comparison between the two formulations.

We define the Macdonald–Vogan correspondence $\mathcal{M}'_N$ in Equation (4.14), which provides the desired parameterization of the irreducible representations of $\mathrm{SL}_N(k_F)$. Theorem 4.3.3 proves that the fibers of this correspondence are in bijection with irreducible representations of the expected component groups, thus establishing Conjecture 4.2.5. Although the parameterization is not canonical, it becomes so upon fixing a "base point" representation. The proof of Theorem 4.3.3 draws inspiration from [25, Section 4], adapting the arguments to the finite field setting.

The remainder of the chapter is devoted to prove the compatibility between the tame local Langlands correspondence $\mathcal{L}'_N$ for $\mathrm{SL}_N(F)$ and the Macdonald-Vogan correspondence $\mathcal{M}'_N$ for $\mathrm{SL}_N(k_F)$ as conjecetured by Vogan. An explicit formulation of this conjecture in the case of SL_N is given in Conjecture 4.4.1. The key results to establish Conjecture 4.4.1 are Theorem 4.4.6, which establishes the compatibility between L-packets for $\mathrm{SL}_N(F)$ and the fibers of the Macdonald–Vogan correspondence for $\mathrm{SL}_N(k_F)$, and Theorem 4.4.16, which confirms Vogan's prediction regarding the interplay between the parameterizations of the fibers via irreducible representations of component groups and the restriction to maximal compact subgroups [68].

Notation

- ⋄ We retain notation from Chapter III

- ⋄ Let H be a group. Let π be a representation H and π' is a subrepresentation of π, not necessarily irreducible. We say that π' has multiplicity 1 in π if $Hom_H(\pi', \pi) = Hom_H(\pi', \pi')$.

- ⋄ Let $\mathbb{H} = \mathrm{GL}_n$ or PGL_n. If $X \subseteq \mathbb{H}(\mathbb{C})$, we write $C_{\mathbb{H}(\mathbb{C})}(X) = \{h \in \mathbb{H}(\mathbb{C}) | hXh^{-1} = X\}$ for the stabilizer of X in $\mathbb{H}$ with respect to the conjugation action. If f is a map to $\mathbb{H}(\mathbb{C})$, we write $C_{\mathbb{H}(\mathbb{C})}(f) = \{h \in \mathbb{H}(\mathbb{C}) | hxh^{-1} = x \text{ for any } x \in Im(f)\}$ for the centralizer of the image of f. For $x \in \mathfrak{h} = Lie(\mathbb{H}(\mathbb{C}))$, we write $C_{\mathbb{H}(\mathbb{C})}(x) = \{h \in \mathbb{H}(\mathbb{C}) | Ad(h)x = x\}$. We set $C_{\mathbb{H}(\mathbb{C})}(f, X, x) := C_{\mathbb{H}(\mathbb{C})}(f) \cap C_{\mathbb{H}(\mathbb{C})}(X) \cap C_{\mathbb{H}(\mathbb{C})}(x)$. Similarly, if $\mathcal{Y}$ is a collection of subsets of $\mathbb{H}(\mathbb{C})$, maps to $\mathbb{H}(\mathbb{C})$ and elements of $\mathfrak{h}$, we set $C_{\mathbb{H}(\mathbb{C})}(\mathcal{Y}) := \bigcap_{y \in \mathcal{Y}} C_{\mathbb{H}(\mathbb{C})}(y)$ and $C^0_{\mathbb{H}}(\mathcal{Y})$ for the identity component of $C_{\mathbb{H}}(\mathcal{Y})$. We denote by $A_{\mathbb{H}}(\mathcal{Y}) := C_{\mathbb{H}}(\mathcal{Y}) / C^0_{\mathbb{H}}(\mathcal{Y})$ the component group of $C_{\mathbb{H}}(\mathcal{Y})$.

- ⋄ We set $G'_n := \mathrm{SL}_n(F)$ and we denote by $\Omega(G'_n)_0$ the set of isomorphism classes of irreducible admissible representations of G_n of depth 0.

  We set K'_n to be the hyperspecial maximal compact subgroup $\mathrm{SL}_n(\mathcal{O}_F)$ of G'_n, and we denote by ${K'_n}^+$ its pro-unipotent radical. We set $\overline{G'}_n := K'_n / {K'_n}^+ \cong \mathrm{SL}_n(k_F)$.

  We write the parahoric restriction functor for G'_n as

$$\begin{aligned} \mathcal{P}^{G'_n}_{\overline{G'}_n} : \Omega(G'_n)_0 &\to Irr(\overline{G'}_n) \\ \pi &\mapsto (Res^{G'_n}_{K'_n} \pi)^{{K'_n}^+} \end{aligned} \tag{4.1}$$

- ⋄ In Chapter III we described the tame Local Langlands correspondence (3.24)

$$\mathscr{L}_N : \Phi(G_N)_0 \to \Omega(G_N)_0.$$

  We chose to define this maps rather than the inverse in order to align with [6, 7, 10].

  On the other hand, the tame Langlands correspondence for G'_N (see Section 4.1) is not a bijection, but a surjection from $\Omega(G'_N)_0$ to the set of Langlands parameters for G'_N. In order to maintain consistency and readability, we define

$$\mathcal{L}_N := \mathscr{L}_N^{-1} : \Omega(G_N)_0 \to \Phi(G_N)_0. \tag{4.2}$$

  Similarly, for the Macdonald correspondence (3.26) we set

$$\mathcal{M}_N := \mathscr{M}_N^{-1} : Irr(G_N) \to \Phi(G_N)_0 / \sim_{I_F}. \tag{4.3}$$

4.1 Local Langlands correspondence for $\mathrm{SL}_N(F)$

We recall the construction of the local Langlands correspondence for G'_N as carried out in [25]. The two "Working Hypothesis" assumed in [25] have been confirmed: the existence of a local Langlands correspondence for G_N was established independently in [27, 30, 60], and the restrictions of irreducible representations of G_N to G'_N have been proved to be multiplicity free in [67].

Since the dual group of SL_N is PGL_N, and G'_N is split over F, a tame Langlands parameter for G'_N is a pair (ρ, E) where $\rho : W_F \mapsto \mathrm{PGL}_N(\mathbb{C})$ is a continuous group morphism that is trivial on the wild ramification subgroup P_F and has semisimple image, and $E \in \mathfrak{sl}_N(\mathbb{C})$ is a nilpotent element satisfying $Ad(\rho(w))(E) = \|w\|E$ for any $w \in W_F$. The group $\mathrm{PGL}_N(\mathbb{C})$ acts on the set of tame Langlands parameters by adjoint action, and two tame Langlands parameters are equivalent if they are in the same orbit by this action. We denote by $\Phi(G'_N)_0$ the set of equivalence classes of tame Langlands parameters for G'_N.

The group $\widehat{F^*}$ acts on $\Omega(G_N)_0$, with a character χ acting by $\tilde{\pi} \mapsto \tilde{\pi} \otimes (\chi \circ det)$ for $\tilde{\pi} \in \Omega(G_N)_0$. The group $\widehat{W_F}$ acts on $\Phi(G_N)_0$ by multiplication. Identifying $\widehat{F^*}$ and $\widehat{W_F}$ by the bijection of local class field theory, the tame local Langlands correspondence $\mathscr{L}_N$ for G_N becomes equivariant with respect to these group actions by (3.12), and therefore the inverse $\mathcal{L}_N = \mathscr{L}_N^{-1}$ is equivariant with respect to these group actions.

Therefore, denoting the orbit-sets of the action of the tame character groups by ${}^{\Phi(G_N)_0}\!\diagup_{\widehat{W_F}}$ and ${}^{\Omega(G_N)_0}\!\diagup_{\widehat{F^*}}$, the local Langlands correspondence for G_N induces a bijection

$$\overline{\mathcal{L}}_N : {}^{\Omega(G_N)_0}\!\diagup_{\widehat{F^*}} \to {}^{\Phi(G_N)_0}\!\diagup_{\widehat{W_F}}.$$

Let $\eta : \mathrm{GL}_N(\mathbb{C}) \to \mathrm{PGL}_N(\mathbb{C})$ be the natural projection. Then the map

$$\begin{aligned} \eta^* : {}^{\Phi(G_N)_0}\!\diagup_{\widehat{W_F}} &\to \Phi(G'_N)_0 \\ \widehat{W_F}(\rho, E) &\mapsto (\eta \circ \rho, E) \end{aligned} \tag{4.4}$$

is a bijection.
For any $\tilde{\pi} \in \Omega(G_N)_0$, the restriction $Res^{G_N}_{G'_N}\tilde{\pi}$ is a direct sum of finitely many mutually inequivalent representations of G'_N. Moreover, the restrictions of $\tilde{\pi}_1, \tilde{\pi}_2 \in \Omega(G_N)_0$ to G'_N are either equal, and that happens if and only if $\tilde{\pi}_1 = \tilde{\pi}_2 \otimes \chi \circ det$ for some $\chi \in \widehat{F^*}$, or they do not have any irreducible constituent in common.

Let $\Omega(G'_N)_0$ be the set of isomorphism classes of irreducible smooth admissible representations of G'_N with supercuspidal support of depth 0. For any $\pi \in \Omega(G'_N)_0$, let $D(\pi) \in \Omega(G_N)_0$ be an irreducible representation of G_N containing π as G'_N subrepresentation. Then the map

$$\begin{aligned} \mathscr{D} : \Omega(G'_N)_0 &\to {}^{\Omega(G_N)_0}\!\diagup_{\widehat{F^*}} \\ \pi &\mapsto \widehat{F^*}D(\pi) \end{aligned} \tag{4.5}$$

is a well-defined surjection. We will usually denote by $D(\pi)$ a representative of the orbit $\mathscr{D}(\pi)$.

The local Langlands correspondence for G'_N is the surjection given by the composition

$$\mathcal{L}'_N : \Omega(G'_N)_0 \xrightarrow{\mathscr{D}} \Omega(G_N)_0 \big/ \widehat{F^*} \xrightarrow{\mathcal{L}_N} \Phi(G_N)_0 \big/ \widehat{W_F} \xrightarrow{\eta^*} \Phi(G'_N)_0. \tag{4.6}$$

The fibers of this surjection are called L-packets. For any $(\rho, E) \in \Phi(G'_N)_0$, the character group of the component group $A_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E)$ has a canonical simply transitive action on the L-packet ${\mathcal{L}'_N}^{-1}(\rho, E)$ [25, Theorem 4.3]. To lighten the notation, we will omit the group $\mathrm{PGL}_N(\mathbb{C})$ and write $A(\rho, E)$ for $A_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E)$.

4.2 The Macdonald-Vogan correspondence for $\mathrm{SL}_N(F)$

We set $\overline{G'}_N := \mathrm{SL}_N(k_F)$, so that $\overline{G'}_N \cong G'_N \big/ K'_N$. We denote by $Irr(\overline{G'}_N)$ the set of isomorphism classes of irreducible representations of $\overline{G'}_N$.

4.2.1 The I_F-equivalence classes

Definition 4.2.1. *The elements $(\rho_1, E_1), (\rho_2, E_2) \in \Phi(G'_N)_0$ are I_F-equivalent if there exists $A \in \mathrm{PGL}_N(\mathbb{C})$ such that*

$$A\rho_1|_{I_F}A^{-1} = \rho_2 \tag{a'}$$

$$Ad(A)E_1 = E_2 \tag{b'}$$

$$A\big(\rho_1(Fr)C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho_1|_{I_F}, E_1)\big)A^{-1} = \rho_2(Fr)C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho_2|_{I_F}, E_2). \tag{c'}$$

We denote by $\sim_{I_F}$ the I_F-equivalence relation and $(\rho, E)_{I_F}$ for the I_F-equivalence class of $(\rho, E) \in \Phi(G'_N)_0$

To lighten the notation, for any $(\rho, E) \in \Phi(G'_N)_0$ we set

$$\overline{\rho(Fr)} := \rho(Fr)C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E).$$

Remark 4.2.2. In view of Lemma 3.2.8, this definition of I_F-equivalence is actually analogous to the one given for G_N.

Remark 4.2.3. The equivalence relation on $\Phi(G'_N)_0$ in [47, Section 5] requires the conditions (a') and (b') only. The set of equivalence classes with respect to this relation is denoted by $\Phi^t_I(\mathrm{SL}_N)$. The I_F-equivalence relation from Definition 4.2.1 is an actual refinement of Macdonald's one. We give an example in which the two notions are different.
Let $N = 2$, and let q be odd. Let $(\rho_1, 0), (\rho_2, 0) \in \Phi(G'_2)_0$ be defined by $\rho_1(I_F) = \rho_2(I_F) = \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$ and $\rho_1(Fr) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\rho_2(Fr) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $(\rho_1, 0), (\rho_2, 0)$ are equivalent for the relation in [47], since they have the same restriction to the inertia subgroup, but they are not I_F-equivalent. Indeed, $C_{\mathrm{PGL}_2(\mathbb{C})}(\rho_i|_{I_F}, 0) = T \rtimes \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where T denotes the diagonal torus in $\mathrm{PGL}_2(\mathbb{C})$, and there is no $A \in \mathrm{PGL}_2(\mathbb{C})$ fixing $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and conjugating T to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)T$.

We will make use of the $\widehat{k_F^*}$-action on ${}^{\Phi(G_N)_0}\diagup_{\sim_{I_F}}$ introduced in Remark 3.2.11.

Lemma 4.2.4. *Let $\eta : \mathrm{GL}_N(\mathbb{C}) \to \mathrm{PGL}_N(\mathbb{C})$ be the natural projection. The assignment $(\rho, E) \mapsto (\eta \circ \rho, E)$, for $(\rho, E) \in \Phi(G_N)_0$, induces a well-defined bijection*

$$\eta^* : \left({}^{\Phi(G_N)_0}\diagup_{\sim_{I_F}}\right)\diagup_{\widehat{k_F^*}} \to {}^{\Phi(G'_N)_0}\diagup_{\sim_{I_F}}. \tag{4.7}$$

Proof. We first show that the map $(\rho, E) \mapsto (\eta \circ \rho, E)$ induces a well defined map

$$ {}^{\Phi(G_N)_0}\diagup_{\sim_{I_F}} \to {}^{\Phi(G'_N)_0}\diagup_{\sim_{I_F}}. \tag{4.8}$$

If $(\rho_i, E_i) \in \Phi(G_N)_0$, for $i = 1, 2$, are such that $(\rho_1, E_1) \sim_{I_F} (\rho_2, E_2)$, by Lemma 3.2.8 there exists $A \in \mathrm{GL}_N(\mathbb{C})$ satisfying conditions (a), (b), (c). Therefore applying η gives

$$\begin{aligned} \eta(A)(\eta \circ \rho_1|_{I_F})\eta(A^{-1}) &= \eta \circ A\rho_1|_{I_F}A^{-1} = \eta \circ \rho_2|_{I_F} && \text{condition (a')} \\ Ad(\eta(A))E_1 &= E_2 && \text{condition (b').} \end{aligned}$$

We check condition (c'). We claim that for $i = 1, 2$ it holds

$$\eta(C_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F}, E_i)) = C^0_{\mathrm{PGL}_N(\mathbb{C})}(\eta \circ \rho_i|_{I_F}, E_i). \tag{4.9}$$

Indeed the inclusion $\subseteq$ in equation (4.9) follows from connectedness of centralizers of $\mathrm{GL}_N(\mathbb{C})$, together with the fact that $\eta(C_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F}, E_i)) \subseteq C_{\mathrm{PGL}_N(\mathbb{C})}(\eta \circ \rho_i|_{I_F}, E_i)$. Now we show the inclusion $\supseteq$. For any $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(\eta\circ\rho_i|_{I_F}, E_i)$ we denote by $\widetilde{g}$ a lift of g in $\mathrm{GL}_N(\mathbb{C})$. Let x be an element in the image of $\rho_i|_{I_F}$. Since $g\eta(x)g^{-1} = \eta(x)$, there exists a scalar $\lambda \in \mathbb{C}^*$, not depending on the choice of the lift $\widetilde{g}$, such that $\widetilde{g}x\widetilde{g}^{-1} = \lambda x$, and equating the determinant in the last identity yields that λ is an N^{th} root of 1. Therefore the assignment $g \mapsto \widetilde{g}x\widetilde{g}^{-1}x^{-1}$ defines a continuous map from $C_{\mathrm{PGL}_N(\mathbb{C})}(\eta\circ\rho_i|_{I_F}, E_i)$ to the set of scalar matrices of order a divisor of N. Since the target set is finite, the connected component $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\eta \circ \rho_i|_{I_F}, E_i)$ has trivial image. Hence for any $y \in C^0_{\mathrm{PGL}_N(\mathbb{C})}(\eta \circ \rho_i|_{I_F}, E_i)$, any lift $\widetilde{y}$ satisfies $\widetilde{y}x\widetilde{y}^{-1} = x$ with x in the image of $\rho|_{I_F}$, and so $\widetilde{y} \in C_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F})$. Moreover $\widetilde{y}$ centralizes E_i, because y centralizes E_i and the adjoint action of $\mathrm{GL}_N(\mathbb{C})$ factorizes through as the adjoint action of $\mathrm{PGL}_N(\mathbb{C})$, so $\widetilde{y} \in C_{\mathrm{GL}_N(\mathbb{C})}(\rho|_{I_F}, E_i)$. By definition of lift we have $\eta(\widetilde{y}) = y$, and so $y \in \eta(C_{\mathrm{GL}_N(\mathbb{C})}(\rho_i|_{I_F}, E_i))$. This concludes the proof of equality (4.9).

By (4.9) there holds

$$\begin{aligned} \eta(A)\overline{\eta \circ \rho_1(Fr)}\eta(A^{-1}) &= \eta(A)\eta(\overline{\rho_1(Fr)})\eta(A^{-1}) \\ &= \eta(A\overline{\rho_1(Fr)})A^{-1}) \\ &= \eta(\overline{\rho_2(Fr)}) \\ &= \overline{\eta \circ \rho_2(Fr))} \end{aligned}$$

so $(\eta \circ \rho_1, E_1) \sim_{I_F} (\eta \circ \rho_2, E_2)$ in $\Phi(G'_N)_0$.

The map (4.8) is constant on $\widehat{k_F^*}$-orbits because $(\eta \circ (\rho \otimes \chi), E) = (\eta \circ \rho, E)$ for any $(\rho, E) \in \Phi(G_N)_0$, hence η^* is well defined.

We show that it is injective. Let $(\rho_1, E_1), (\rho_2, E_2) \in \Phi(G_N)_0$ be such that $(\eta \circ \rho_1, E_1) \sim_{I_F} (\eta \circ \rho_2, E_2)$ in $\Phi(G'_N)_0$. Then there exists $A \in \mathrm{PGL}_N(\mathbb{C})$ satisfying conditions (a'),(b') and (c').

Let $\widetilde{A} \in \mathrm{GL}_N(\mathbb{C})$ be such that $\eta(\widetilde{A}) = A$. Then $Ad(\widetilde{A})E_1 = E_2$, and $\widetilde{A}\rho_1|_{I_F}\widetilde{A}^{-1} = \chi\rho_2|_{I_F}$ for some character χ of I_F. Since ρ_1 and ρ_2 are tame, χ is a character of ${}^{I_F}\!\diagup_{P_F}$, so $(\rho_1, E_1) \sim_{I_F} (\rho_2 \otimes \chi, E_2)$ in $\Phi(G_N)_0$. In order to prove injectivity, it is therefore sufficient to show that χ is Frobenius stable. For $i = 1, 2$ it holds

$$\rho_i(Fr)\rho_i|_{I_F}\rho_i(Fr^{-1}) = (\rho_i|_{I_F})^q.$$

It follows that

$$\widetilde{A}\rho_1(Fr)(\rho_1|_{I_F})\rho_1(Fr^{-1})\widetilde{A}^{-1} = \widetilde{A}(\rho_1|_{I_F})^q\widetilde{A}^{-1} = (\widetilde{A}(\rho_1|_{I_F})\widetilde{A}^{-1})^q = (\chi\rho_2|_{I_F})^q = \chi^q(\rho_2|_{I_F})^q. \tag{4.10}$$

Moreover lifting to $\mathrm{GL}_N(\mathbb{C})$ the relation $A\overline{\eta \circ \rho_1(Fr)}A^{-1} = \overline{\eta \circ \rho_2(Fr)}$ we obtain

$$\widetilde{A}\overline{\rho_1(Fr)}\widetilde{A}^{-1} = \overline{\rho_2(Fr)}.$$

In particular there exists a $c \in C_{\mathrm{GL}_N(\mathbb{C})}(\rho_2|_{I_F}, E_2)$ such that

$$\widetilde{A}\rho_1(Fr)\widetilde{A}^{-1} = \rho_2(Fr)c.$$

Therefore

$$\begin{aligned} \widetilde{A}\rho_1(Fr)(\rho_1|_{I_F})\rho_1(Fr^{-1})\widetilde{A}^{-1} &= \widetilde{A}\rho_1(Fr)\widetilde{A}\widetilde{A}^{-1}(\rho_1|_{I_F})\widetilde{A}^{-1}\widetilde{A}\rho_1(Fr^{-1})\widetilde{A}^{-1} \\ &= \rho_2(Fr)c\chi(\rho_2|_{I_F})c^{-1}\rho_2(Fr^{-1}) \\ &= \chi\rho_2(Fr)(\rho_2|_{I_F})\rho_2(Fr^{-1}) \\ &= \chi(\rho_2|_{I_F})^q. \end{aligned} \tag{4.11}$$

Comparing (4.10) and (4.11) we get

$$\chi^q(\rho_2|_{I_F})^q = \widetilde{A}\rho_1(Fr)(\rho_1|_{I_F})\rho_1(Fr^{-1})\widetilde{A}^{-1} = \chi(\rho_2|_{I_F})^q,$$

giving $\chi^q = \chi$. Hence, χ is a Frobenius stable character of ${}^{I_F}\!\diagup_{P_F}$, i.e. it is an element of $\widehat{k_F^*}$. In other words, the I_F-equivalence classes in $\Phi(G_N)_0$ of (ρ_1, E_1) and (ρ_2, E_2) are in the same $\widehat{k_F^*}$-orbit.

It remains to show that (4.7) is surjective. Let $(\overline{\rho}, E) \in \Phi(G'_N)_0$. The existence of $(\rho, E) \in \Phi(G_N)_0$ such that $(\overline{\rho}, E) = (\eta \circ \rho, E)$ amounts to the existence of a semisimple Weil representation $\rho : W_F \to \mathrm{GL}_N(\mathbb{C})$ that is a lift of $\overline{\rho}$, which was proved in [29, Theorem 2.7]. □

4.2.2 Imai and Vogan's Conjecture

We are now in a position to state Imai and Vogan's conjecture [35, Conjecture 4.3] for $\overline{G'}_N$. We write

$$C((\rho, E)_{I_F}) := C_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, \overline{\rho(Fr)}, E)$$

and $C^0((\rho, E)_{I_F})$ for its identity component, and we denote by

$$A((\rho, E)_{I_F}) := {C((\rho, E)_{I_F})} \diagup {C^0((\rho, E)_{I_F})}$$

its component group.

Conjecture 4.2.5. There exists a surjective map

$$\mathcal{M}'_N : Irr(\overline{G'}_N) \to {\Phi(G'_N)_0} \diagup {\sim_{I_F}}$$

such that, for any $(\rho, E)_{I_F} \in {\Phi(G'_N)_0} \diagup {\sim_{I_F}}$, there is a bijection between ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$ and the character group $A((\rho, E)_{I_F})^\wedge$.

Remark 4.2.6.

1. Conjecture 4.2.5, stated here for SL_N, is a specialization of [35, Conjecture 4.3], which is formulated for reductive groups. In our version the representation field is taken to be $\mathbb{C}$ rather than $\mathbb{Q}_l$ as in [35, Conjecture 4.3]. For a detailed comparison between our notation and that of [35], see point (3) below. We prove Conjecture 4.2.5 in Theorem 4.3.3.

2. Aside from our result for SL_N, the only previous case in the literature where [35, Conjecture 4.3] has been proved is for GL_N, established in [47] via the construction of the Macdonald correspondence, that we recalled in 3.2.2. Indeed in the GL_N case by connectedness of centralizers (see Lemma 3.2.8) the conjecture amounts to the existence of a bijection between $Irr(\overline{G}_N)$ and ${\Phi(G_N)_0} \diagup {\sim_{I_F}}$.

   A general proof of the conjecture for reductive groups has recently appeared in [34].

3. Conjecture 4.2.5 is the version over $\mathbb{C}$ (rather than $\overline{\mathbb{Q}}_l$) of [35, Conjecture 4.3] for G'_N. Since we adopted some different notation, we show more in detail how the objects described in [35, Section 4] relate to ours.

   3.a **Conventions on Langlands parameters** The Frobenius semisimple L-parameters of Weil Deligne type φ defined in [35, Definition 4.1] for the group G'_N are in bijection with our tame Langlands parameters. Given $(\rho, E) \in \Phi(G'_N)_0$, the corresponding φ is defined by $\varphi(w, a) = (exp(aE)\rho(w), Fr^{\|w\|})$ for $w \in W_k$ and $a \in \mathbb{C}$, and this assignment defines a bijection. Note that this makes sense because the finite Weil group W_k defined in [35, Definition 3.1] is isomorphic to ${W_F} \diagup {P_F}$. We worked with the complex dual group rather that with the L-group because we are dealing with a split group, and, following Macdonald [47], we preferred the formalism of a nilpotent element E rather than using a morphism $\mathbb{C} \to \mathrm{PGL}_N(\mathbb{C})$.

   The notion of special parameters introduced in [35] does not play a role in the present setup since in type A all unipotent/nilpotent orbits are special.

   The I_F-equivalence relation on $\Phi(G'_N)_0$ corresponds under the bijection $(\rho, E) \mapsto \varphi$ above to the equivalence relation on Frobenius semisimple L-parameters of Weil Deligne type defined in [35, Section 4] . So the set ${\Phi(G'_N)_0} \diagup {\sim_{I_F}}$ is in bijection with the set denoted by $\Phi_{\mathbb{C}}(\mathrm{SL}_N)_{sp}$ in [35, Section 4].

3.b **Definition of the component groups** - Centralizers of semisimple elements in $\mathrm{PGL}_N(\mathbb{C})$ are direct products of groups of type A, and centralizers of unipotent elements are connected in type A. In particular, the notion of Lusztig's canonical quotient does not play any role in our situation. It follows that the groups denoted by $A(\varphi_0)$ and $\overline{A}(\varphi_0)$ in [35, Section 4] both coincide with the group that we denote by $A_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E)$, where (ρ, E) is the tame Langlands parameter corresponding to the L-parameter of Weil-Deligne type φ as in (3.a).

The group denoted by A_φ in [35, Section 4] is isomorphic to the group $A((\rho, E)_{I_F})$. Indeed using the equality $\overline{A}(\varphi_0) = A_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E)$, it follows from the definition of A_φ that quotienting by $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E)$ yields a projection

$$\pi : C((\rho, E)_{I_F}) = C_{C_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F},E)}\big(\rho(Fr)\big) \to A_\varphi$$

with $ker(\pi) = C^0_{\mathrm{PGL}_N(\mathbb{C}}(\rho|_{I_F}, E) \cap C((\rho, E)_{I_F})$. We claim that $ker(\pi) = C^0((\rho, E)_{I_F})$. The inclusion $\supseteq$ holds because A_φ is a finite set and the projection is continuous. The inclusion $\subseteq$ is obtained as follows. Since $\rho(Fr)$ normalizes $\rho|_{I_F}$ and acts by a scalar on E, it normalizes $C_{\mathrm{PGL}_N}(\rho|_{I_F}, E)$, and therefore $\rho(Fr)$ normalizes $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E)$. It follows that $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E) \subseteq C((\rho, E)_{I_F})$. By connectdness, $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E) \subseteq C^0((\rho, E)_{I_F})$, and so $ker(\pi) = C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E) \cap C((\rho, E)_{I_F}) \subseteq C^0((\rho, E)_{I_F})$.

Therefore the projection π induces a group isomorphism $A((\rho, E)_{I_F}) \to A_\varphi$.

4.2.3 The Macdonald-Vogan Correspondence for G'_N

We explain now how to construct a map $\mathcal{M}'_N$ as predicted by Conjecture 4.2.5.

We recall from Remark 3.2.11 that the group $\widehat{k_F^*}$ acts on $Irr(\overline{G}_N)$ with a character χ acting by $\pi \mapsto \pi \otimes (\chi \circ det)$ for $\pi \in Irr(\overline{G}_N)$. By (3.34), the Macdonald correspondence $\mathscr{M}_N$ (3.26) is equivariant with respect to the action of $\widehat{k_F^*}$ on $\Phi(G_N)_0 / \sim_{I_F}$ and $Irr(\overline{G}_N)$, so the inverse $\mathcal{M}_N$ is equivariant as well. It follows that $\mathcal{M}_N$ induces a bijection between orbit sets

$$\overline{\mathcal{M}}_N : Irr(\overline{G}_N) / \widehat{k_F^*} \to \left(\Phi(G_N)_0 / \sim_{I_F}\right) / \widehat{k_F^*}.$$

Since $\overline{G'}_N$ is normal in $\overline{G}_N$, the latter acts on $Irr(\overline{G'}_N)$ by conjugation, inducing an action of $\overline{G}_N / \overline{G'}_N \cong k_F^*$ on $Irr(\overline{G'}_N)$.

For any $\pi \in Irr(\overline{G'}_N)$ let $\overline{D}(\pi) \in Irr(\overline{G}_N)$ be such that π is a subrepresentation of $Res^{\overline{G}_N}_{\overline{G'}_N} \overline{D}(\pi)$. From Clifford theory it follows that the map

$$Irr(\overline{G'}_N) / k_F^* \to Irr(\overline{G}_N) / \widehat{k_F^*} \qquad (4.12)$$
$$k_F^* \pi \mapsto \widehat{k_F^*} \overline{D}(\pi)$$

is a bijection [47, Proposition 5.1]. In particular, the map

$$\begin{aligned} \overline{\mathscr{D}} : Irr(\overline{G'}_N) &\to Irr(\overline{G}_N) \big/ \widehat{k_F^*} \\ \pi &\mapsto \widehat{k_F^*}\overline{D}(\pi) \end{aligned} \tag{4.13}$$

is a well-defined surjection. In the following, we will denote by $\overline{D}(\pi)$ a representative of the orbit $\overline{\mathscr{D}}(\pi)$.

Definition 4.2.7. *We call the surjection*

$$\mathcal{M}'_N : Irr(\overline{G'}_N) \xrightarrow{\overline{\mathscr{D}}} Irr(\overline{G}_N) \big/ \widehat{k_F^*} \xrightarrow{\overline{\mathcal{M}}_N} \left(\Phi(G_N)_0 \big/ \sim_{I_F} \right) \big/ \widehat{k_F^*} \xrightarrow{\eta^*} \Phi(G'_N)_0 \big/ \sim_{I_F} \tag{4.14}$$

the Macdonald-Vogan correspondence.

Remark 4.2.8. The surjection built by Macdonald in [47, Section 5] can be recovered by composing the Macdonald-Vogan correspondence $\mathcal{M}'_N$ with the natural projection $\Phi(G'_N)_0 \big/ \sim_{I_F} \to \Phi_I^t(\mathrm{SL}_N)$, see Remark 4.2.3. We exploited the idea in [47] for the construction of the map $\mathcal{M}'_N$, but some caution is needed because the fibers in the surjection in loc. cit. are bigger than claimed there, i.e., larger than k_F^*-orbits. An example is given by the equivalence class in the sense of [47] defined in Remark 4.2.3, whose fiber by Macdonald's surjection contains two cuspidal and two principal series representations.

We refined the equivalence relation on $\Phi(G'_N)_0$ as proposed in [68] precisely to ensure that the fibers of the Macdonald-Vogan correspondence $\mathcal{M}'_N$ are the orbits for the action of k_F^*, see Equations (4.17) and (4.18).

4.3 Parametrization of the fibers of $\mathcal{M}'_N$

The goal of this Section is to establish Conjecture 4.2.5 in Theorem 4.3.3, which yields a parameterization of each fiber of $\mathcal{M}'_N$ in terms of irreducible representations of the suitable component group.

Lemma 4.3.1. *For $\pi \in Irr(\overline{G'}_N)$, let $\overline{D}(\pi) \in Irr(\overline{G}_N)$ be such that π is an irreducible G'_N subrepresentation of $\overline{D}(\pi)$. There is a canonical isomorphism*

$$k_F^* \big/ Stab_{k_F^*}(\pi) = (Stab_{\widehat{k_F^*}}(\overline{D}(\pi)))^\wedge. \tag{4.15}$$

where $(Stab_{\widehat{k_F^*}}(\overline{D}(\pi)))^\wedge$ denotes the character group of $Stab_{\widehat{k_F^*}}(\overline{D}(\pi))$.

Proof. We show that $Stab_{\widehat{k_F^*}}(\overline{D}(\pi)) = \left(k_F^* \big/ Stab_{k_F^*}(\pi)\right)^\wedge$, where the equality makes sense viewing $\chi \in \left(k_F^* \big/ Stab_{k_F^*}(\pi)\right)^\wedge$ as a character in $\widehat{k_F^*}$ that is trivial on $Stab_{k_F^*}(\pi)$. The statement will then follow by duality.

We start by proving that $\left(k_F^* \big/ Stab_{k_F^*}(\pi)\right)^\wedge \leqslant Stab_{\widehat{k_F^*}}(\overline{D}(\pi))$.

Let $\chi \in \widehat{k_F^*}$ such that $\chi|_{Stab_{k_F^*}(\pi)} = 1$. The determinant map gives an isomorphism $\overline{G}_N \diagup \overline{G'}_N \cong k_F^*$ and the action of k_F^* on $Irr(\overline{G'}_N)$ is induced by the conjugation action of $\overline{G}_N$. It follows that the determinant map induces a group isomorphism $\overline{G}_N \diagup Stab_{\overline{G}_N}(\pi) \cong k_F^* \diagup Stab_{k_F^*}(\pi)$, that in turn induces the group isomorphism $\chi \mapsto \chi \circ \det$ between the group of characters of k_F^* that are trivial on $Stab_{k_F^*}(\pi)$ and the group of characters of $\overline{G}_N$ that are trivial on $Stab_{\overline{G}_N}(\pi)$.

Let $\{g_1, \ldots, g_k\}$ be a set of representatives of the cosets of $Stab_{G_N}(\pi)$ in $\overline{G}_N$. Then $Res^{\overline{G}_N}_{\overline{G'}_N} \overline{D}(\pi) = \bigoplus_{i=1}^k {}^{g_i}\pi$. Let $A := \bigoplus_{i=1}^k \chi(\det(g_i^{-1}))$, that is, the linear map acting as the scalar $\chi(\det(g_i^{-1}))$ on the subspace on which $Res^{\overline{G}_N}_{\overline{G'}_N} \overline{D}(\pi)$ acts as ${}^{g_i}\pi$. Then

$$A\overline{D}(\pi)A^{-1} = \overline{D}(\pi) \otimes (\chi \circ \det).$$

Hence

$$\overline{D}(\pi) \otimes (\chi \circ \det) \cong \overline{D}(\pi)$$

that is, $\chi \in Stab_{\widehat{k_F^*}}(\overline{D}(\pi))$. So $\left(k_F^* \diagup Stab_{k_F^*}(\pi)\right)^\wedge \leqslant Stab_{\widehat{k_F^*}}(\overline{D}(\pi))$.

By Clifford theory, [47, Proposition 5.1]

$$|k_F^* \pi| = |Stab_{\widehat{k_F^*}}(\overline{D}(\pi))|.$$

Therefore

$$\left|\left(k_F^* \diagup Stab_{k_F^*}(\pi)\right)^\wedge\right| = \left|k_F^* \diagup Stab_{k_F^*}(\pi)\right| = |k_F^* \pi| = |Stab_{\widehat{k_F^*}}(\overline{D}(\pi))|$$

and hence, by cardinality,

$$\left(k_F^* \diagup Stab_{k_F^*}(\pi)\right)^\wedge = Stab_{\widehat{k_F^*}}(\overline{D}(\pi)). \tag{4.16}$$

Since $k_F^* \diagup Stab_{k_F^*}(\pi)$ is a finite abelian group, it is canonically isomorphic to its double character group. Dualizing (4.16) we get (4.15). □

We now describe the fibers of $\mathcal{M}'_N$.

Lemma 4.3.2. *Let $(\rho, E)_{I_F} \in \Phi(G'_N)_0 \diagup \sim_{I_F}$ and $(\tilde{\rho}, E)_{I_F} \in \Phi(G_N)_0 \diagup \sim_{I_F}$ be such that $(\eta \circ \tilde{\rho}, E)_{I_F} = (\rho, E)_{I_F}$, and let*

$$Stab_{\widehat{k_F^*}}((\tilde{\rho}, E)_{I_F}) = \{\chi \in \widehat{k_F^*} \mid (\chi \otimes \tilde{\rho}, E)_{I_F} = (\tilde{\rho}, E)_{I_F}\}.$$

Then the character group of $Stab_{\widehat{k_F^*}}((\tilde{\rho}, E)_{I_F})$ acts simply transitively on ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$.

Proof. Set $\tilde{\pi}_{(\tilde{\rho},E)} := \mathcal{M}_N^{-1}((\tilde{\rho}, E)_{I_F})$. By construction $\mathcal{M}'_N = \eta^* \circ \overline{\mathcal{M}}_N \circ \overline{\mathscr{D}}$. The maps $\overline{\mathcal{M}}_N$ and η^* are bijections, so $\widehat{k_F^*}\tilde{\pi}_{(\tilde{\rho},E)}$ is the unique $\widehat{k_F^*}$-orbit in $Irr(\overline{G}_N)$ satisfying $(\rho, E) = \eta^* \circ \overline{\mathcal{M}}_N(\widehat{k_F^* \tilde{\pi}_{(\tilde{\rho},E)}})$. Then

$${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F}) = \overline{\mathscr{D}}^{-1}(\widehat{k_F^* \tilde{\pi}_{(\tilde{\rho},E)}}). \tag{4.17}$$

The map $\overline{\mathscr{D}}$ factorizes as

$$Irr(\overline{G'}_N) \xrightarrow{/k_F^*} Irr(\overline{G'}_N)\big/ k_F^* \to Irr(\overline{G}_N)\big/ \widehat{k_F^*}$$

where the last map is the bijection defined in (4.12). Therefore if $\pi_{(\rho,E)} \in Irr(\overline{G'}_N)$ satisfies $\overline{D}(\pi_{(\rho,E)}) = \widehat{k_F^*}\widetilde{\pi}_{(\widetilde{\rho},E)}$, then

$$\overline{\mathscr{D}}^{-1}(\widehat{k_F^*}\widetilde{\pi}_{(\widetilde{\rho},E)}) = k_F^*\pi_{(\rho,E)}. \tag{4.18}$$

It follows that the group $k_F^*\big/ Stab_{k_F^*}(\pi_{(\rho,E)})$ has a canonical simply transitive action on ${\mathcal{M}'_N}^{-1}((\rho,E)_{I_F})$. Therefore by Lemma 4.3.1, the character group $(Stab_{\widehat{k_F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)}))^\wedge$ has a simply transitive action on ${\mathcal{M}'_N}^{-1}((\rho,E)_{I_F})$.

Since the map $\mathcal{M}_N$ is compatible with the $\widehat{k_F^*}$-actions,

$$Stab_{\widehat{k_F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)}) = Stab_{\widehat{k_F^*}}((\widetilde{\rho},E)_{I_F}) = \{\chi \in \widehat{k_F^*} \mid (\chi \otimes \widetilde{\rho}, E)_{I_F} = (\widetilde{\rho},E)_{I_F}\}$$

and dualizing

$$(Stab_{\widehat{k_F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)}))^\wedge = (Stab_{\widehat{k_F^*}}((\widetilde{\rho},E)_{I_F}))^\wedge = \{\chi \in \widehat{k_F^*} \mid (\chi \otimes \widetilde{\rho}, E)_{I_F} = (\widetilde{\rho},E)_{I_F}\}^\wedge.$$

□

Theorem 4.3.3. *Let $(\rho,E)_{I_F} \in \Phi(G'_N)_0\big/\sim_{I_F}$. Then $A((\rho,E)_{I_F})$ is a finite abelian group and its character group $A((\rho,E)_{I_F})^\wedge$ acts simply transitively on ${\mathcal{M}'_N}^{-1}((\rho,E)_{I_F})$.*

Proof. Let $(\widetilde{\rho},E)_{I_F} \in \Phi(G_N)_0\big/\sim_{I_F}$ be such that $(\eta \circ \widetilde{\rho},E)_{I_F} = (\rho,E)_{I_F}$. By Lemma 4.3.2 it is enough to provide a group isomorphism between $A((\rho,E)_{I_F})$ and $Stab_{\widehat{k_F^*}}((\widetilde{\rho},E)_{I_F})$. Since the latter is a subgroup of $\widehat{k_F^*}$, it is finite and abelian.

Let $g \in C((\rho,E)_{I_F})$, and let $\widetilde{g}$ be a representative of g in $\mathrm{GL}_N(\mathbb{C})$. We set

$$\chi_g := \widetilde{g}(\widetilde{\rho}|_{I_F})\widetilde{g}^{-1}(\widetilde{\rho}^{-1}|_{I_F}) : I_F \to \mathrm{GL}_N(\mathbb{C}). \tag{4.19}$$

We observe that χ_g does not depend on the choice of $\widetilde{g}$. Since in particular $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F})$, there holds

$$\eta(\chi_g)(\omega) = \eta(\widetilde{g}\widetilde{\rho}(\omega)\widetilde{g}^{-1}\widetilde{\rho}(\omega)^{-1}) = 1$$

for any $\omega \in I_F$, so $\chi_g(\omega)$ is a scalar for any $\omega \in I_F$. In this way we have a character

$$\chi_g : I_F \to \mathbb{C}^*.$$

We show that it is Frobenius stable. Since $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(\overline{\rho(Fr)})$, we have

$$c := \rho(Fr)^{-1}g\rho(Fr)g^{-1} \in C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E).$$

Lifting this identity to $\mathrm{GL}_N(\mathbb{C})$ yields

$$\widetilde{c} := \widetilde{\rho}(Fr)^{-1}\widetilde{g}\widetilde{\rho}(Fr)\widetilde{g}^{-1} \in C_{\mathrm{GL}_n(\mathbb{C})}(\widetilde{\rho}|_{I_F}, E). \tag{4.20}$$

Therefore, for any $\omega \in I_F$

$$\begin{aligned}
&\chi_g(Fr\ \omega\ Fr^{-1})\\
&= \widetilde{g}(\widetilde{\rho}(Fr\ \omega\ Fr^{-1}))\widetilde{g}^{-1}(\widetilde{\rho}(Fr\ \omega\ Fr^{-1})^{-1})\\
&= \widetilde{g}\widetilde{\rho}(Fr)(\widetilde{\rho}(\omega))\widetilde{\rho}(Fr^{-1})\widetilde{g}^{-1}\widetilde{\rho}(Fr^{-1})^{-1}(\widetilde{\rho}(\omega)^{-1})\widetilde{\rho}(Fr)^{-1}\\
&= \widetilde{g}\widetilde{\rho}(Fr)(\widetilde{\rho}(\omega))\widetilde{\rho}(Fr)^{-1}\widetilde{g}^{-1}\widetilde{\rho}(Fr)(\widetilde{\rho}(\omega)^{-1})\widetilde{\rho}(Fr)^{-1}\\
&= \widetilde{\rho}(Fr)\widetilde{c}\widetilde{g}(\widetilde{\rho}(\omega))\widetilde{g}^{-1}\widetilde{c}^{-1}\widetilde{\rho}(Fr)^{-1}\widetilde{\rho}(Fr)(\widetilde{\rho}(\omega)^{-1})\widetilde{\rho}(Fr)^{-1} && \widetilde{g}\widetilde{\rho}(Fr) = \widetilde{\rho}(Fr)\widetilde{c}\widetilde{g} \text{ by (4.20)}\\
&= \widetilde{\rho}(Fr)\widetilde{c}\widetilde{g}(\widetilde{\rho}(\omega))\widetilde{g}^{-1}\widetilde{c}^{-1}(\widetilde{\rho}(\omega)^{-1})\widetilde{\rho}(Fr)^{-1}\\
&= \widetilde{\rho}(Fr)\widetilde{c}\widetilde{g}(\widetilde{\rho}(\omega))\widetilde{g}^{-1}(\widetilde{\rho}(\omega)^{-1})\widetilde{c}^{-1}\widetilde{\rho}(Fr)^{-1} && \widetilde{c} \in C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}) \text{ by (4.20)}\\
&= \widetilde{\rho}(Fr)\widetilde{c}\chi_g(\omega)\widetilde{c}^{-1}\widetilde{\rho}(Fr)^{-1}\\
&= \chi_g(\omega). && \chi_g(\omega) \in \mathbb{C}^*
\end{aligned}$$

Hence χ_g is a Frobenius stable character of I_F. Moreover by definition $\chi_g|_{P_F} = 1$, hence we can regard χ_g as a character of k_F^*. By construction of χ_g it holds

$$\chi_g\widetilde{\rho}|_{I_F} = \widetilde{g}(\widetilde{\rho}|_{I_F})\widetilde{g}^{-1}.$$

In addition, $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(E)$ and hence $\widetilde{g} \in C_{\mathrm{GL}_N(\mathbb{C})}(E)$, so $(\widetilde{\rho}, E) \sim_{I_F} (\widetilde{\rho} \otimes \chi_g, E)$ in $\Phi(G_N)_0$.

Therefore the assignment $g \mapsto \chi_g$ gives a map

$$\widetilde{\Xi} : C((\rho, E)_{I_F}) \to Stab_{\widehat{k_F^*}}((\widetilde{\rho}, E)_{I_F}). \tag{4.21}$$

The map $\widetilde{\Xi}$ is a group morphism: for any $g_1, g_2 \in C((\rho, E)_{I_F})$

$$\chi_{g_1g_2}\widetilde{\rho}|_{I_F} = \widetilde{g_1}\widetilde{g_2}(\widetilde{\rho}|_{I_F})\widetilde{g_2}^{-1}\widetilde{g_1}^{-1} = \widetilde{g_1}(\chi_{g_2}\widetilde{\rho}|_{I_F})\widetilde{g_1}^{-1} = \chi_{g_2}\widetilde{g_1}(\widetilde{\rho}|_{I_F})\widetilde{g_1}^{-1} = \chi_{g_1}\chi_{g_2}\widetilde{\rho}|_{I_F}$$

hence $\chi_{g_1g_2} = \chi_{g_1}\chi_{g_2}$.

We prove the surjectivity of $\widetilde{\Xi}$. Let $\chi \in \widehat{k_F^*}$ be such that $(\chi \otimes \widetilde{\rho}, E)_{I_F} = (\widetilde{\rho}, E)_{I_F}$. There exists an element $\widetilde{A} \in \mathrm{GL}_N(\mathbb{C})$ satisfying conditions (a), (b) and (c) for $(\chi \otimes \widetilde{\rho}, E)_{I_F}$ and $(\widetilde{\rho}, E)_{I_F}$. This implies that $A := \eta(\widetilde{A}) \in C((\rho, E)_{I_F})$. From $\widetilde{A}(\widetilde{\rho}|_{I_F})\widetilde{A}^{-1} = \chi \otimes (\widetilde{\rho}|_{I_F})$ we obtain

$$\chi = \widetilde{A}(\widetilde{\rho}|_{I_F})\widetilde{A}^{-1}(\widetilde{\rho}|_{I_F})^{-1} = \chi_A = \widetilde{\Xi}(A).$$

Since the image of $\widetilde{\Xi}$ is a finite group, the morphism (4.21) factors through the component group

$$\begin{array}{ccc}
C((\rho, E)_{I_F}) & & \\
\downarrow & \searrow^{\widetilde{\Xi}} & \\
A((\rho, E)_{I_F}) & \xrightarrow{\ \Xi\ } & Stab_{\widehat{k_F^*}}((\widetilde{\rho}, E)_{I_F})
\end{array} \tag{4.22}$$

so the map Ξ in (4.22) is a surjective group morphism. We prove that it is also injective by showing that $ker(\widetilde{\Xi}) = C^0((\rho, E)_{I_F})$.

Let $g \in ker(\widetilde{\Xi})$ and let $\widetilde{g}$ be a representative of g in $\mathrm{GL}_N(\mathbb{C})$. Then

$$\widetilde{g}(\widetilde{\rho}|_{I_F})\widetilde{g}^{-1}(\widetilde{\rho}^{-1}|_{I_F}) = 1,$$

that is $\widetilde{g} \in C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F})$. Since $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(E)$, it holds $\widetilde{g} \in C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, E)$, and the latter is connected.
By Lemma 3.2.8, for any $\widetilde{g} \in C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, E)$ it holds $\widetilde{g}\overline{\widetilde{\rho}(Fr)}\widetilde{g}^{-1} = \overline{\widetilde{\rho}(Fr)}$, so

$$\widetilde{g} \in C_{\mathrm{GL}_N(\mathbb{C})}(\rho|_{I_F}, E) = C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, \overline{\widetilde{\rho}(Fr)}, E).$$

Then

$$g = \eta(\widetilde{g}) \in \eta\big(C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, \overline{\widetilde{\rho}(Fr)}, E)\big) \leqslant C((\rho, E)_{I_F}).$$

Since $C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, \overline{\widetilde{\rho}(Fr)}, E)$ is the same as $C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, E)$, it is connected. It follows that

$$\eta\big(C_{\mathrm{GL}_N(\mathbb{C})}(\widetilde{\rho}|_{I_F}, \overline{\widetilde{\rho}(Fr)}, E)\big) \leqslant C^0((\rho, E)_{I_F})$$

that is,

$$g \in C^0_{\mathrm{PGL}_N(\mathbb{C})}((\rho, E)_{I_F}),$$

giving injectivity of Ξ. ☐

4.3.1 Comparison of the fibers of $\mathcal{M}'_N$ and$\mathcal{L}'_N$

The aim of this section is to show that the fibers of the Langlands correspondence and the fibers of the Macdonald-Vogan correspondence may have different cardinality, and the natural map (4.24) between them is neither an injection nor a surjection in general.

Let $(\rho, E) \in \Phi(G'_N)_0$. By [25, Theorem 4.3] the component group $A(\rho, E)$ is finite and abelian and there is a canonical simply transitive action of its character group on ${\mathcal{L}'_N}^{-1}(\rho, E)$. Therefore there is a bijection between ${\mathcal{L}'_N}^{-1}(\rho, E)$ and the group $A(\rho, E)^\wedge$.

Similarly, by Theorem 4.3.3 there is a bijection between ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$ and the group $A((\rho, E)_{I_F})^\wedge$.

The inclusion $C_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E) \hookrightarrow C((\rho, E)_{I_F})$ induces a map between the component groups

$$\iota : A(\rho, E) \to A((\rho, E)_{I_F}) \tag{4.23}$$

that gives by duality a map

$$\hat{\iota} : A((\rho, E)_{I_F})^\wedge \to A(\rho, E)^\wedge. \tag{4.24}$$

This map is neither injective nor surjective in general. We give two examples to illustrate this.

We denote by T the diagonal torus in $\mathrm{PGL}_N(\mathbb{C})$.

1. **Failure of the surjectivity**: Let $(\rho, E) \in \Phi(G'_N)_0$ be given by

    - ◇ $\rho|_{I_F} = 1$;

◇ $\rho(Fr) = \begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{N-1} \end{pmatrix}$ where $\zeta \in \mathbb{C}$ a primitive N-th root of 1;

◇ $E = 0$.

In this case

$$A(\rho, E) = C_{\mathrm{PGL}_N(\mathbb{C})}(\rho(Fr)) \big/ C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho(Fr))$$

and

$$C_{\mathrm{PGL}_N(\mathbb{C})}(\rho(Fr)) = T \rtimes \left\langle \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \right\rangle.$$

Therefore $A(\rho, E)$ is isomorphic to the cyclic group of order N and $|{\mathcal{L}'_N}^{-1}(\rho, E)| = N$.

On the other hand, $C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E) = C^0_{\mathrm{PGL}_N(\mathbb{C})}(1, 0) = \mathrm{PGL}_N(\mathbb{C})$, hence

$$C((\rho, E)_{I_F}) = C_{\mathrm{PGL}_N(\mathbb{C})}(1, \mathrm{PGL}_N(\mathbb{C}), 0) = \mathrm{PGL}_N(\mathbb{C}) = C^0((\rho, E)_{I_F}).$$

Hence $|{\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})| = |A((\rho, E)_{I_F})^\wedge| = 1$.

In this case, the map

$$\hat{\iota} : 1 \cong A((\rho, E)_{I_F})^\wedge \to A(\rho, E)^\wedge$$

is injective but not surjective.

For the sake of completeness, we describe explicitly the representations in ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$ and ${\mathcal{L}'_N}^{-1}(\rho, E)$. Let $\chi_\zeta \in \widehat{F^*}$ be the unramified character of F^* that takes value ζ on any uniformizer of F^*. This corresponds by the bijection of local class field theory to the unramified character of W_F that takes value ζ at Fr. Then $\widetilde{\pi} := \mathrm{R}^{G_N}_{G_1^N}(\boxtimes_{i=0}^{n-1} \chi_\zeta^i)$ is an irreducible representation of G_N. The representations in ${\mathcal{L}'_N}^{-1}(\rho, E)$ are the N irreducible constituents of $Res^{G_N}_{G'_N} \widetilde{\pi}$.

On the other hand, ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$ contains only the trivial representation.

2. **Failure of injectivity:** Assume that $(q-1, N) \neq 1$. Let $e > 1$ be a common divisor of $q-1$ and N and let $\zeta \in \mathbb{C}$ be a primitive e-th root of 1.

   We consider $(\rho, E) \in \Phi(G'_N)_0$ such that

   ◇ $\rho(I_F) = \langle M \rangle$ with

   $$M := \begin{pmatrix} \mathbf{1}_{\frac{N}{e}} & & & \\ & \zeta \mathbf{1}_{\frac{N}{e}} & & \\ & & \ddots & \\ & & & \zeta^{e-1} \mathbf{1}_{\frac{N}{e}} \end{pmatrix}$$

   where $\mathbf{1}_{\frac{N}{e}}$ denotes the identity matrix of dimension $\frac{N}{e}$;

- $\diamond$ $\rho(Fr)$ is a regular element of T;
- $\diamond$ $E = 0$.

By construction, $M^q = M$, so a Weil-Deligne representation needs to map Fr to an element commuting with M, and $\rho(Fr)$ satisfies this condition. We have

$$C_{\mathrm{PGL}_N(\mathbb{C})}(M) \cong \left(\mathrm{GL}_{\frac{N}{e}}(\mathbb{C})^e \rtimes \mathbb{Z}/e\mathbb{Z}\right)/\mathbb{C}^*.$$

Moreover $C_{\mathrm{PGL}_N(\mathbb{C})}(\rho(Fr)) = T$. So

$$C_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E) = C_{\mathrm{PGL}_N(\mathbb{C})}(M, \rho(Fr)) = T = C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E).$$

Hence

$$A(\rho, E) = 1.$$

On the other hand,

$$C^0_{\mathrm{PGL}_N(\mathbb{C})}(\rho|_{I_F}, E) = C^0_{\mathrm{PGL}_N(\mathbb{C})}(M) = \mathrm{GL}_{\frac{N}{e}}(\mathbb{C})^e/\mathbb{C}^*.$$

This group contains T, and hence $\rho(Fr)$, so

$$C((\rho, E)_{I_F}) = C_{\mathrm{PGL}_N(\mathbb{C})}(M, C^0_{\mathrm{PGL}_N(\mathbb{C})}(M)) = C_{\mathrm{PGL}_N(\mathbb{C})}(M).$$

Hence,

$$A((\rho, E)_{I_F}) = C_{\mathrm{PGL}_N(\mathbb{C})}(M)/C^0_{\mathrm{PGL}_N(\mathbb{C})}(M) \cong \mathbb{Z}/e\mathbb{Z}.$$

Since we assumed $e > 1$, it follows that in this case, the map

$$\hat{\imath} : A((\rho, E)_{I_F})^\wedge \to A(\rho, E)^\wedge \cong 1$$

is surjective but not injective.

As in the previous example, we describe explicitly the representations in ${\mathcal{M}'_N}^{-1}((\rho, E)_{I_F})$ and ${\mathcal{L}'_N}^{-1}(\rho, E)$. For the sake of simplicity of the description, assume $\rho(Fr)$ to be an element of the compact torus, i.e. assume the entries of $\rho(Fr)$ to have all the same absolute value in $\mathbb{C}$.

Let $\widetilde{\rho}$ be a lift of ρ to $\mathrm{GL}_N(\mathbb{C})$ such that $\widetilde{\rho}(Fr) = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_N \end{pmatrix}$ with $|t_j| = 1$ for any $j = 1, \ldots, N$. Then $\widetilde{\rho} = \bigoplus_{j=1}^N \chi_j$ with $\chi_j \in \widehat{W_F}$. By the bijection of local class field theory, any χ_j can be regarded as a tame character of F^*. More explicitly, χ_j takes value t_j on any uniformizer of $\widehat{F^*}$, and $\chi_j|_{\mathcal{O}_F^*} = Infl^{\mathcal{O}_F^*}_{\mathcal{O}_F^*/1+\mathfrak{p}_F} \widehat{\zeta}^{\lfloor \frac{e(j-1)}{N} \rfloor}$, where $\widehat{\zeta}$ denotes a character of $k_F^* \cong \mathcal{O}_F^*/1+\mathfrak{p}_F$ of order e. Then $\widetilde{\pi} := \mathrm{R}^{G_N}_{G_1^N}(\boxtimes_{i=0}^{n-1} \chi_j)$ is a tempered irreducible representation of G_N. This representation remains irreducible upon restriction to G'_N, and $Res^{G_N}_{G'_N}\widetilde{\pi}$ is the unique element in ${\mathcal{L}'_N}^{-1}(\rho, E)$.

On the other hand, the I_F-equivalence class of $(\widetilde{\rho}, 0)$ corresponds, according to Macdonald's construction for $\overline{G}_N$, to the partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ taking value $(1^{\frac{N}{e}})$ on $\widehat{\zeta}^i$ for any $0 \leqslant i \leqslant e-1$, and 0 everywhere else. Hence $\mathcal{M}_N{}^{-1}((\widetilde{\rho}, 0)_{I_F}) = \mathrm{R}^{\overline{G}_N}_{\overline{G}^e_{\frac{N}{e}}}(\boxtimes_{i=0}^{e-1}(\widehat{\zeta}^i \circ det))$. This is an irreducible representation of $\overline{G}_N$, but when restricted to $\overline{G'}_N$ it splits into e inequivalent irreducible subrepresentations. These are exactly the representations in $\mathcal{M}'_N{}^{-1}((\rho, E)_{I_F})$.

4.4 Compatibility between $\mathcal{L}'_N$ and $\mathcal{M}'_N$

In Section 4.3.1 we observed that the natural map $\hat{\imath}$ in (4.24) is not injective nor surjective in general. Nevertheless, it allows to establish a compatibility between $\mathcal{L}'_N$ and $\mathcal{M}'_N$ along the lines of the one predicted in [68]. We denote by $\mathcal{P}^{G'_N}_{\overline{G'}_N}$ the parahoric restriction with respect to the hyperspecial maximal compact subgroup K'_N of G'_N. We now state a conjecture proposed by Vogan in [68] specialized to the case of SL_N.

Conjecture 4.4.1. Let $(\rho, E) \in \Phi(G'_N)_0$. Then there exist bijections

$$\mathcal{F}_{(\rho,E)} : A(\rho, E)^\wedge \to \mathcal{L}'_N{}^{-1}(\rho, E)$$

and

$$\overline{\mathcal{F}}_{(\rho,E)_{I_F}} : A((\rho, E)_{I_F})^\wedge \to \mathcal{M}'_N{}^{-1}(\rho, E)$$

such that for any $\psi \in A(\rho, E)^\wedge$ and any $\overline{\psi} \in A((\rho, E)_{I_F})^\wedge$ it holds

$$dim\big(Hom_{\overline{G'}_N}\big(\mathcal{P}^{G'_N}_{\overline{G'}_N}(\mathcal{F}_{(\phi,E)}(\psi)), \overline{\mathcal{F}}_{(\rho,E)_{I_F}}(\overline{\psi})\big)\big) = \delta_{\hat{\imath}^{-1}(\psi)}(\overline{\psi}) \qquad (4.25)$$

where $\delta_{\hat{\imath}^{-1}(\psi)}$ denotes the indicator function on $\hat{\imath}^{-1}(\psi)$.

Conjecture 4.4.1 will be established in Theorem 4.4.16.

4.4.1 Compatibility of the fibers

An intermediate result toward the proof of Conjecture 4.4.1 is given by Theorem 4.4.6, where we establish a preliminary compatibility of Langlands and Macdonald correspondences considering all the representations in the same fiber together.

4.4.1.1 The head of Parahoric Restriction

We retain notation from Section 3.2.4. Let $\widetilde{\pi} \in \Omega(G_N)_0$. By Theorem 3.2.3 there exists a multiset $\{\Delta_1, \dots, \Delta_k\}$ of segments such that $\widetilde{\pi} = Q(\Delta_1, \dots, \Delta_k)$. By Corollary 3.2.27 the irreducible constituents of the parahoric restriction $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi})$ are of the form π_Λ as in (3.28), with Λ a partition valued function on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ of degree N such that $\Lambda \geqslant \Lambda_{\{\Delta_1,\dots,\Delta_k\}}$. Moreover by Proposition 3.2.31, the irreducible representation $\mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi}) = \pi_{\Lambda_{\{\Delta_1,\dots,\Delta_k\}}}$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi})$ of multiplicity 1.

Let $\chi \in \widehat{F^*}$. Since χ is tame, the restriction $Res^{F^*}_{\mathcal{O}^*_F}\chi$ gives a character $\overline{\chi} \in \left(\mathcal{O}^*_F \diagup 1+\mathfrak{p}_F\right)^\wedge \cong \widehat{k^*_F}$. We recall from Lemma 3.2.10 that the character group $\widehat{k^*_F}$ acts on the set of partition valued functions on $\bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$ by

$$(\overline{\chi}\overline{\Lambda})(\overline{\tau}) := \overline{\Lambda}(\overline{\tau} \otimes (\overline{\chi}^{-1} \circ det)). \tag{4.26}$$

Lemma 4.4.2. *With the above notation, the irreducible constituents of* $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det))$ *are of the form* $\pi_{\overline{\chi}\Lambda}$ *with* $\Lambda \geqslant \Lambda_{\Delta_1,\ldots,\Delta_k}$.

Proof. As a first step, we show that $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det)) = \mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi}) \otimes (\overline{\chi} \circ \det)$. Indeed since $\chi|_{1+\mathfrak{p}_F} = 1$, there holds $\chi \circ det|_{K_N^+} = 1$, therefore $(\widetilde{\pi} \otimes (\chi \circ \det))^{K_N^+} = \widetilde{\pi}^{K_N^+} \otimes (\chi \circ \det)$. Moreover $det(K_N) \subseteq \mathcal{O}^*_F$, so

$$\begin{aligned}
\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det)) &= Res^{G_N}_{K_N}(\widetilde{\pi} \otimes (\chi \circ \det))^{K_N^+} \\
&= Res^{G_N}_{K_N}(\widetilde{\pi})^{K_N^+} \otimes Res^{G_N}_{K_N}(\chi \circ \det)^{K_N^+} \\
&= Res^{G_N}_{K_N}(\widetilde{\pi})^{K_N^+} \otimes (\overline{\chi} \circ \det) \\
&= \mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi}) \otimes (\overline{\chi} \circ \det).
\end{aligned}$$

Therefore the irreducible constituents of $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det))$ are of the form $\pi_\Lambda \otimes (\overline{\chi} \circ det)$ for $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$.

By Lemma 3.2.10,

$$\pi_\Lambda \otimes (\overline{\chi} \circ det) = \pi_{\chi\Lambda},$$

so the irreducible constituents of $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det))$ are of the form $\pi_{\chi\Lambda}$ for $\Lambda \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}$.

□

Corollary 4.4.3. *For any representation* $\widetilde{\pi} \in \Omega(G_N)_0$ *and for any tame character* $\chi \in \widehat{F^*}$*, it holds*

$$\mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det)) = \mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi}) \otimes (\overline{\chi} \circ \det)$$

where $\overline{\chi}$ *is the character of* k^*_F *obtained by restriction of* χ *to* $\mathcal{O}^*_F$.

Proof. We retain notation from Lemma 4.4.2. By Corollary 3.2.27 the head of the parahoric restriction $\mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi}\otimes(\chi\circ\det))$ is the irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi}\otimes(\chi\circ\det))$ corresponding to the minimal partition valued function appearing. Acting by the same character $\overline{\chi}$ preserves the relative order between partition valued functions, so the description of the irreducible constituents of $\mathcal{P}^{G_N}_{\overline{G}_N}(\widetilde{\pi}\otimes\chi\circ\det)$ in Lemma 4.4.2 gives

$$\mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi} \otimes (\chi \circ \det)) = \pi_{\overline{\chi}\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}}.$$

By Lemma 3.2.10

$$\pi_{\overline{\chi}\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}} = \pi_{\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}} \otimes (\overline{\chi} \circ det) = \mathcal{HP}^{G_N}_{\overline{G}_N}(\widetilde{\pi}) \otimes (\chi \circ \det).$$

□

For any $\pi \in \Omega(G'_N)_0$, we denote by $F^*\pi$ the orbit of π under the conjugation action of the group ${G_N}\diagup{G'_N} \cong F^*$. This orbit is a finite set [25, Theorem 4.1]. We set

$$\pi^{\oplus} := \bigoplus_{\pi' \in F^*\pi} \pi'. \tag{4.27}$$

Similarly, for any $\overline{\pi} \in Irr(\overline{G'}_N)$, we denote by $k_F^*\overline{\pi}$ the orbit of $\overline{\pi}$ under the conjugation action of the group ${\overline{G}_N}\diagup{\overline{G'}_N} \cong k_F^*$, and we set

$$\overline{\pi}^{\oplus} := \bigoplus_{\overline{\pi}' \in k_F^*\overline{\pi}} \overline{\pi}'. \tag{4.28}$$

Since the restriction of irreducible representations from G_N (respectively $\overline{G}_N$) to G'_N (respectively $\overline{G'}_N$) is multiplicity free, it holds

$$\pi^{\oplus} := Res_{G'_N}^{G_N} D(\pi) \qquad\qquad \overline{\pi}^{\oplus} := Res_{\overline{G'}_N}^{\overline{G}_N} \overline{D}(\overline{\pi}) \tag{4.29}$$

where $D(\pi)$ (respectively $\overline{D}(\overline{\pi})$) is an irreducible representation of G_N (respectively of $\overline{G}_N$) containing π (respectively $\overline{\pi}$) as G'_N-subrepresentation (respectively $\overline{G'}_N$-subrepresentation).

Moreover by definition of the maps $\mathcal{L}'_N$ and $\mathcal{M}'_N$, for any $\pi \in \Omega(G'_N)_0$ and for any $\overline{\pi} \in Irr(\overline{G'}_N)$ it holds

$$\pi^{\oplus} = \bigoplus_{\pi' \in {\mathcal{L}'_N}^{-1}(\mathcal{L}'_N(\pi))} \pi' \qquad \text{and} \qquad \overline{\pi}^{\oplus} = \bigoplus_{\overline{\pi}' \in {\mathcal{M}'_N}^{-1}(\mathcal{M}'_N(\overline{\pi}))} \overline{\pi}'. \tag{4.30}$$

We extend the notion of head of parahoric restriction from representations of G_N to representations of G'_N of the form $\pi^{\oplus}$ for some $\pi \in \Omega(G'_N)_0$ as follows.

Definition 4.4.4. *Let $\pi \in \Omega(G'_N)_0$ and let $D(\pi) \in \Omega(G_N)_0$ be a representation of G_N containing π as G'_N-subrepresentation. The head of the parahoric restriction of $\pi^{\oplus}$ is the $\overline{G'}_N$-representation*

$$\mathcal{HP}_{\overline{G'}_N}^{G'_N} \pi^{\oplus} := Res_{\overline{G'}_N}^{\overline{G}_N} \mathcal{HP}_{\overline{G}_N}^{G_N} D(\pi).$$

The definition above is well posed: by Corollary 4.4.3, the representation $\mathcal{HP}_{\overline{G'}_N}^{G'_N} \pi^{\oplus}$ does not depend on the choice of $D(\pi)$.

Lemma 4.4.5. *Let $\pi \in \Omega(G'_N)_0$. The representation $\mathcal{HP}_{\overline{G'}_N}^{G'_N} \pi^{\oplus}$ is a subrepresentation of $\mathcal{P}_{\overline{G'}_N}^{G'_N} \pi^{\oplus}$, and every irreducible constituent of $\mathcal{HP}_{\overline{G'}_N}^{G'_N} \pi^{\oplus}$ occurs exactly once in $\mathcal{P}_{\overline{G'}_N}^{G'_N} \pi^{\oplus}$.*

Proof. By [12, Lemma 1.11] , for any $\widetilde{\pi} \in \Omega(G_N)_0$ the K_N^+-fixed subspace and the ${K'_N}^+$-fixed subspace of the representations space of $\widetilde{\pi}$ coincide. It follows that

$$Res_{K'_N}^{K_N} (Res_{K_N}^{G_N} \widetilde{\pi})^{K_N^+} = (Res_{K'_N}^{G_N} \widetilde{\pi})^{{K'_N}^+}.$$

Hence

$$\begin{aligned}
\mathcal{P}^{G'_N}_{\overline{G'}_N}\pi^{\oplus} &= Res^{G'_N}_{K'_N}(\pi^{\oplus})^{{K'_N}^+} && \text{by (4.29)}\\
&= \left(Res^{G'_N}_{K'_N}(Res^{G_N}_{G'_N}D(\pi))\right)^{{K'_N}^+}\\
&= \left(Res^{G_N}_{K'_N}(D(\pi))\right)^{{K'_N}^+}\\
&= Res^{K_N}_{K'_N}(Res^{G_N}_{K_N}(D(\pi))^{K_N^+}) = Res^{\overline{G}_N}_{\overline{G'}_N}\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi)). && (4.31)
\end{aligned}$$

The representation $\mathcal{HP}^{G_N}_{\overline{G}_N}(D(\pi))$ is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$ by Theorem 3.2.26.
Therefore $\mathcal{HP}^{G'_N}_{\overline{G'}_N}\pi^{\oplus} = Res^{\overline{G}_N}_{\overline{G'}_N}\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)$ is a subrepresentation of $\mathcal{P}^{G'_N}_{\overline{G'}_N}\pi^{\oplus} = Res^{\overline{G}_N}_{\overline{G'}_N}\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$.

Since $\mathcal{HP}^{G'_N}_{\overline{G'}_N}\pi^{\oplus}$ is multiplicity free, to prove that all its irreducible constituents occur in $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$ exactly once it is enough to show that there are no irreducible constituents of $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$ different from $\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)$ and having its same restriction to $\overline{G'}_N$. Indeed the restrictions of irreducible representations of $\overline{G}_N$ to $\overline{G'}_N$ are either equal or don't have any common irreducible constituent.

Irreducible representations of $\overline{G}_N$ have the same restriction to $\overline{G'}_N$ only if they differ by the tensor product with a character induced by the determinant map. Hence it is sufficient to show that for any $\chi \in \widehat{k_F^*}$ the representation $\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)\otimes(\chi\circ det)$ is a subrepresentation of $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$ if and only if it is equal to $\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)$. Let $\{\Delta_1,\ldots,\Delta_k\}$ be a multiset of segments such that $D(\pi) = Q(\Delta_1,\ldots,\Delta_k)$ as in Theorem 3.2.3. By Corollary 4.4.3, $\mathcal{HP}^{G_N}_{\overline{G}_N}(D(\pi)\otimes(\chi\circ\det)) = \pi_{\chi\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}}$. If $\pi_{\chi\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}}$ is an irreducible constituent of $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$, then by Theorem 3.2.31 it holds $\chi\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau}) \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau})$ for any $\overline{\tau} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$, hence $\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{-1}\otimes\overline{\tau}) \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau})$. Iterating, we get $\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{i-1}\otimes\overline{\tau}) \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{i}\otimes\overline{\tau})$ for $i \geqslant 0$. By transitivity it implies

$$\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{-1}\otimes\overline{\tau}) \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau}) \geqslant \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{q-2}\otimes\overline{\tau}) = \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\chi^{-1}\otimes\overline{\tau}),$$

that is $\chi\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau}) = \Lambda_{\{\Delta_1,\ldots,\Delta_k\}}(\overline{\tau})$ for any $\overline{\tau} \in \bigsqcup_{m\in\mathbb{N}} Cusp(\overline{G}_m)$. Hence if $\mathcal{HP}^{G_N}_{\overline{G}_N}(D(\pi))\otimes(\chi\circ\det)$ is a subrepresentation of $\mathcal{P}^{G_N}_{\overline{G}_N}(D(\pi))$ it holds

$$\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)\otimes(\chi\circ\det) = \pi_{\chi\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}} = \pi_{\Lambda_{\{\Delta_1,\ldots,\Delta_k\}}} = \mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi).$$

□

4.4.1.2 Compatibility of $\mathcal{L}'_N$ and $\mathcal{M}'_N$

Let

$$\Sigma_N := \{\pi^{\oplus} \mid \pi \in \Omega(G'_N)_0\} \qquad \text{and} \qquad \overline{\Sigma}_N := \{\overline{\pi}^{\oplus} \mid \overline{\pi} \in Irr(\overline{G'}_N)\}. \tag{4.32}$$

where $\pi^{\oplus}$ and $\overline{\pi}^{\oplus}$ are respectively as in (4.27) and (4.28). By (4.30) these sets are in bijection with the sets of the fibers of $\mathcal{L}'_N$ and $\mathcal{M}'_N$ respectively.

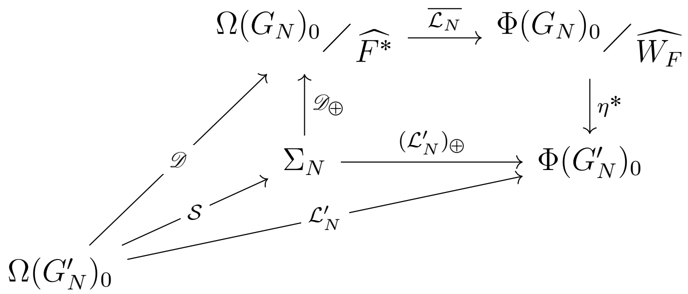
In other words, defining the maps

$$\mathcal{S} : \Omega(G'_N)_0 \to \Sigma_N \qquad\qquad \overline{\mathcal{S}} : Irr(\overline{G'}_N) \to \overline{\Sigma}_N, \tag{4.33}$$

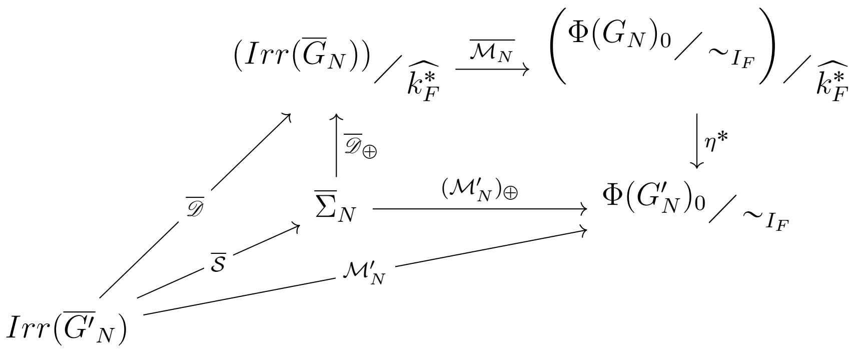
$$\pi \mapsto \pi^{\oplus} \qquad\qquad \overline{\pi} \mapsto \overline{\pi}^{\oplus} \tag{4.34}$$

we have that the maps $\mathscr{D}$ as in (4.5) and $\mathcal{L}'_N$ factor through $\mathcal{S}$, and we write $\mathscr{D} = \mathscr{D}_{\oplus} \circ \mathcal{S}$ and $\mathcal{L}'_N = (\mathcal{L}'_N)_{\oplus} \circ \mathcal{S}$, where $\mathscr{D}_{\oplus}$ and $(\mathcal{L}'_N)_{\oplus}$ are bijections. Analogously, the maps $\overline{\mathscr{D}}$ (4.13) and $\mathcal{M}'_N$ factor through $\overline{\mathcal{S}}$, and we write $\overline{\mathscr{D}} = \overline{\mathscr{D}}_{\oplus} \circ \overline{\mathcal{S}}$ and $\mathcal{M}'_N = (\mathcal{M}'_N)_{\oplus} \circ \overline{\mathcal{S}}$, where $\overline{\mathscr{D}}_{\oplus}$ and $(\mathcal{M}'_N)_{\oplus}$ are bijections.

With this notation, the following diagrams are commutative:



(4.35)

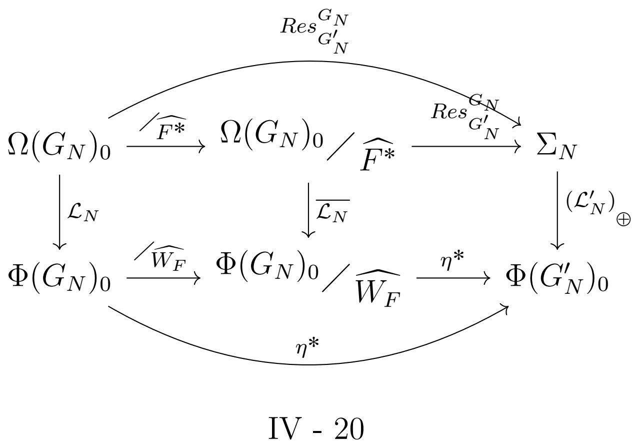


(4.36)

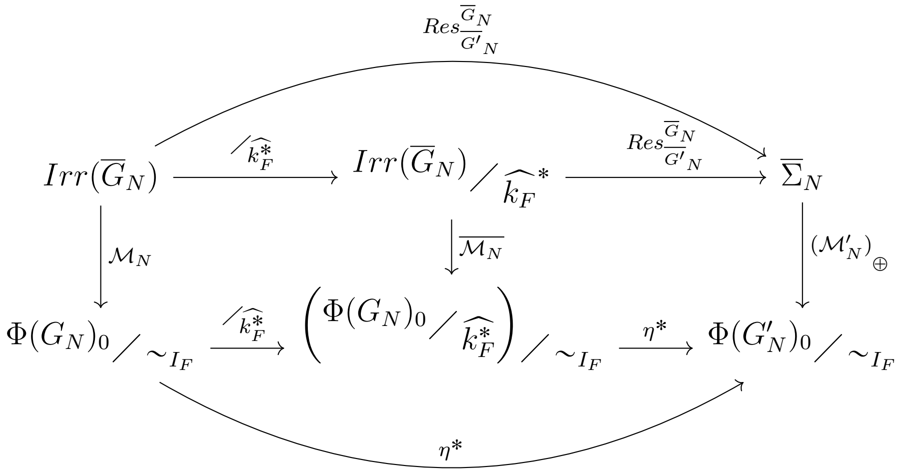
Theorem 4.4.6. *For any $\pi \in \Omega(G'_N)_0$, it holds*

$$((\mathcal{L}'_N)_{\oplus}\pi^{\oplus})_{I_F} = (\mathcal{M}'_N)_{\oplus}(\mathcal{HP}^{G'_N}_{\overline{G'}_N}\pi^{\oplus})$$

Proof. By (4.29), the restriction from G_N to G'_N (respectively from $\overline{G}_N$ to $\overline{G'}_N$) lands in Σ_N (respectively $\overline{\Sigma}_N$), and by (4.29) it factors through the inverse of the map $\mathscr{D}_{\oplus}$ (respectively $\overline{\mathscr{D}}_{\oplus}$). So the following diagrams are commutative:



(4.37)



(4.38)

and so for a representation $\pi \in \Omega(G'_N)_0$, and for any $D(\pi) \in \Omega(G_N)_0$ containing π as G'_N-subrepresentation, we have

$$\begin{aligned}
(\mathcal{M}'_N)_{\oplus}(\mathcal{HP}^{G'_N}_{\overline{G'}_N}\pi^{\oplus}) =&(\mathcal{M}'_N)_{\oplus}(Res^{\overline{G}_N}_{\overline{G'}_N}\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)) \\
&= \eta^* \circ \mathcal{M}_N(\mathcal{HP}^{G_N}_{\overline{G}_N}D(\pi)) && \text{by (4.38)} \\
&= \eta^* \circ (\mathcal{L}_N(D(\pi)))_{I_F} && \text{by Theorem 3.2.32} \\
&= (\eta^* \circ \mathcal{L}_N(D(\pi)))_{I_F} \\
&= ((\mathcal{L}'_N)_{\oplus}(Res^{G_N}_{G'_N}D(\pi)))_{I_F} && \text{by (4.37)} \\
&= ((\mathcal{L}'_N)_{\oplus}\pi^{\oplus})_{I_F}. && \text{by (4.29)}
\end{aligned}$$

□

Remark 4.4.7. Theorem 4.4.6 does not depend on the choice of the maximal compact subgroup K'_N of G'_N.
Indeed, any maximal compact subgroup J'_N of G'_N is of the form $J'_N = {}^gK'_N$ for some $g \in G_N$.

For any $\pi \in \Omega(G'_N)_0$, we have ${}^g\pi^{\oplus} = \pi^{\oplus}$ and so

$$(Res^{G'_N}_{J'_N}\pi^{\oplus})^{{J'_N}^+} = (Res^{G'_N}_{{}^gK'_N}\pi^{\oplus})^{{}^g{K'_N}^+} = ({}^gRes^{G'_N}_{K'_N}({}^{g^{-1}}\pi^{\oplus}))^{{}^g{K'_N}^+} = {}^g((Res^{G'_N}_{K'_N}\pi^{\oplus})^{{K'_N}^+}),$$

We have ${J'_N} / {{J'_N}^+} \cong {K'_N} / {{K'_N}^+} \cong \overline{G'}_N$ where the first isomorphism is induced by conjugation by g. Therefore $(Res^{G'_N}_{J'_N}\pi^{\oplus})^{{J'_N}^+}$ and $(Res^{G'_N}_{K'_N}\pi^{\oplus})^{{K'_N}^+}$ are isomorphic as $\overline{G'}_N$ representations.

4.4.2 Enhancement of the compatibility via the parametrization of the fibers

The central result of this section is Theorem 4.4.16, where Theorem 4.4.6 is refined in order to consider representations individually, rather than fibers, confirming Conjecture 4.4.1.

For the rest of this section, we fix $(\rho, E) \in \Phi(G'_N)_0$, and we let $\pi_{(\rho,E)} \in \mathcal{L}'^{-1}(\rho, E)$ and $\overline{\pi}_{(\rho,E)} \in \mathcal{M}'^{-1}((\rho, E)_{I_F})$. We fix $(\widetilde{\rho}, E) \in \Phi(G_N)_0$ such that $\eta \circ \widetilde{\rho} = \rho$, and we set $\widetilde{\pi}_{(\widetilde{\rho},E)} := \mathcal{L}_N^{-1}(\widetilde{\rho}, E)$ and $\widetilde{\overline{\pi}}_{(\widetilde{\rho},E)} := \mathcal{M}_N^{-1}((\widetilde{\rho}, E)_{I_F})$.

By [25, Theorem 4.3], the character group $A(\rho, E)^\wedge$ acts simply transitively on $\mathcal{L}'_N{}^{-1}(\rho, E)$, and by Theorem 4.3.3 the character group $A((\rho, E)_{I_F})^\wedge$ acts simply transitively on $\mathcal{M}'_N{}^{-1}(\rho, E)$. We now describe explicitly these actions.

For any $g \in C_{\mathrm{PGL}_N(\mathbb{C})}(\rho, E)$, let $\widetilde{g}$ be a lift of g in $\mathrm{GL}_N(\mathbb{C})$. Let $\chi_g : W_F \to \mathrm{GL}_N(\mathbb{C})$ be the map defined by $\chi_g(w) := \widetilde{g}\widetilde{\rho}(w)\widetilde{g}^{-1}\widetilde{\rho}(w)^{-1}$. This map does not depend on the choice of $\widetilde{g}$, and its image consist of scalar matrices. By [25, Theorem 4.3] $\chi_g \in \widehat{W_F}$ and the assignment $g \mapsto \chi_g$ induces an isomorphism

$$\begin{aligned} \Xi_F : A(\rho, E) &\to Stab_{\widehat{W_F}}(\widetilde{\rho}, E) \\ [g] &\mapsto \chi_g := \widetilde{g}\widetilde{\rho}\widetilde{g}^{-1}\widetilde{\rho}^{-1}. \end{aligned} \tag{4.39}$$

Remark 4.4.8. In [25, Theorem 4.3] the map Ξ_F is defined from $A(\rho, E)$ to the stabilizer in the whole character group of W_F. But a character χ of W_F stabilizing a tame Langlands parameter $(\rho, E) \in \Phi(G_N)_0$ is necessarily tame: indeed since (ρ, E) is tame, $\rho|_{P_F} = 1$, hence

$$\chi|_{P_F} = \chi|_{P_F} \otimes \rho|_{P_F} = (\chi \otimes \rho)|_{P_F} \cong \rho|_{P_F} = 1.$$

The bijection of class field theory induces an identification $Stab_{\widehat{W_F}}(\widetilde{\rho}, E) \cong Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})$. We still denote by Ξ_F the isomorphism $A(\rho, E) \cong Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})$ obtained using this identification. Hence $A(\rho, E)$ is a finite abelian group and Ξ_F induces the dual isomorphism

$$\widehat{\Xi}_F : Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})^\wedge \to A(\rho, E)^\wedge.$$

By [25, Corollary 2.2] it holds $Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)}) \cong \left({F^*}/{Stab_{F^*}(\pi_{(\rho,E)})}\right)^\wedge$. Dualizing we have a canonical isomorphism

$$\begin{aligned} \Psi : {F^*}/{Stab_{F^*}(\pi_{(\rho,E)})} &\to Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})^\wedge \\ x &\mapsto \psi_x \end{aligned} \tag{4.40}$$

defined by $\psi_x(\chi) := \chi(x)$ for any $\chi \in Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})$. Then $\widehat{\Xi}_F \circ \Psi$ is a canonical isomorphism between ${F^*}/{Stab_{F^*}(\pi_{(\rho,E)})}$ and $A(\rho, E)^\wedge$. The determinant induces an isomorphism $det : {G_N}/{Stab_{G_N}(\pi_{(\rho,E)})} \to {F^*}/{Stab_{F^*}(\pi_{(\rho,E)})}$. By abuse of notation we denote with the same symbol elements corresponding to each other through this isomorphism. For any $\psi \in A(\rho, E)^\wedge$, let $x_\psi = (\widehat{\Xi}_F \circ \Psi)^{-1}(\psi)$. The action of $A(\rho, E)^\wedge$ on $\mathcal{L}'^{-1}(\rho, E)$ is given by $\psi.\pi = {}^{x_\psi}\pi$.

The canonical isomorphism Ξ defined in Theorem 4.3.3 induces by duality the isomorphism

$$\widehat{\Xi} : Stab_{\widehat{k_{F^*}}}(\widetilde{\rho}, E)^\wedge \to A((\rho, E)_{I_F})^\wedge$$

and by Lemma 4.3.1 there is a canonical isomorphism

$$\overline{\Psi} : {k_F^*}\big/ {Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})} \to Stab_{\widehat{k_F^*}}(\widetilde{\overline{\pi}}_{(\widetilde{\rho},E)})^\wedge \qquad (4.41)$$
$$x \mapsto \overline{\psi}_x$$

defined by $\overline{\psi}_x(\overline{\chi}) := \overline{\chi}(x)$ for any $\overline{\chi} \in Stab_{\widehat{k_F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})$. As before, the determinant induces an isomorphism ${\overline{G}_N}\big/ {Stab_{\overline{G}_N}(\overline{\pi}_{(\rho,E)})} \cong {k_F^*}\big/ {Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})}$, and we identify corresponding elements in these groups. For any $\overline{\psi} \in A((\rho,E)_{I_F})^\wedge$, let $x_{\overline{\psi}} = (\widehat{\Xi} \circ \overline{\Psi})^{-1}(\overline{\psi})$. The action of $A((\rho,E)_{I_F})^\wedge$ is given by $\overline{\psi}.\pi = {}^{x_{\overline{\psi}}}\pi$.

Recall that the group morphism (4.23) induces by duality a canonical group morphism

$$\hat{\iota} : A((\rho,E)_{I_F})^\wedge \to A(\rho,E)^\wedge.$$

Lemma 4.4.9. *The following diagram commutes:*

$$\begin{array}{ccc} Stab_{\widehat{k_F^*}}(\widetilde{\overline{\pi}}_{(\widetilde{\rho},E)})^\wedge & \xrightarrow{(Res|_{\mathcal{O}_F^*})^\wedge} & Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})^\wedge \\ \downarrow \hat{\Xi} & & \downarrow \hat{\Xi}_F \\ A((\rho,E)_{I_F})^\wedge & \xrightarrow{\hat{\iota}} & A(\rho,E)^\wedge \end{array} \qquad (4.42)$$

Proof. The diagram below, where the vertical maps labelled by $\cong$ are the identifications given by local class field theory, commutes by definition of the maps Ξ and Ξ_F as in (4.21) and (4.39):

$$\begin{array}{ccc} Stab_{\widehat{k_F^*}}(\widetilde{\overline{\pi}}_{(\widetilde{\rho},E)}) & \xleftarrow[Res|_{\mathcal{O}_F^*}]{} & Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)}) \\ \cong \uparrow & & \cong \uparrow \\ Stab_{\widehat{k_F^*}}(\widetilde{\rho},E) & \xleftarrow[Res|_{I_F}]{} & Stab_{\widehat{W_F}}(\widetilde{\rho};E) \\ \Xi \uparrow & & \Xi_F \uparrow \\ A((\rho,E)_{I_F}) & \xleftarrow[\iota]{} & A(\rho,E) \end{array}$$

Dualizing, we obtain the statement. □

Any section of the projection $\mathcal{O}_F^* \to {\mathcal{O}_F^*}\big/ {1+\mathfrak{p}_F} \cong k_F$ induces a map

$$\widetilde{\mathcal{J}} : k_F^* \to {F^*}\big/ {Stab_{F^*}(\pi_{(\rho,E)})}.$$

The map $\widetilde{\mathcal{J}}$ does not depend on the choice of the section: in order to check that $(1+\mathfrak{p}_F) \leqslant Stab_{F^*}(\pi_{(\rho,E)})$ it is enough to check that $\Psi(1+\mathfrak{p}_F) = 1$, because Ψ is a group isomorphism. For any $y \in 1+\mathfrak{p}_F$ it holds $\Psi(y)(\chi) = \chi(y) = 1$ since any $\chi \in Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})$ is tame (see Remark 4.4.8).

Lemma 4.4.10. *With notation as above, the map* $\widetilde{\mathcal{J}}$ *induces a map*

$$\mathcal{J} : {k_F^*}/{Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})} \to {F^*}/{Stab_{F^*}(\pi_{(\rho,E)})} \tag{4.43}$$

that makes the following diagram commutative:

$$\begin{array}{ccc} {k_F^*}/{Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})} & \xrightarrow{\mathcal{J}} & {F^*}/{Stab_{F^*}(\pi_{(\rho,E)})} \\ \downarrow{\scriptstyle \overline{\Psi}} & & \downarrow{\scriptstyle \Psi} \\ Stab_{\widehat{k_F^*}}(\widetilde{\overline{\pi}}_{(\widetilde{\rho},E)})^\wedge & \xrightarrow{(Res|_{\mathcal{O}_F^*})^\wedge} & Stab_{\widehat{F^*}}(\widetilde{\pi}_{(\widetilde{\rho},E)})^\wedge \end{array} \tag{4.44}$$

Proof. Let $p : k_F^* \to {k_F^*}/{Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})}$ denote the natural projection. It holds

$$(Res|_{\mathcal{O}_{F^*}})^\wedge \circ \overline{\Psi} \circ p = \Psi \circ \widetilde{\mathcal{J}}. \tag{4.45}$$

Indeed for any $x \in k_F^*$ and for any $\chi \in \widehat{F^*}$

$$\begin{aligned}(Res|_{\mathcal{O}_F^*})^\wedge \circ \overline{\Psi} \circ p(x)(\chi) &= (Res|_{\mathcal{O}_F^*})^\wedge(\overline{\psi}_{p(x)})(\chi) = \overline{\psi}_{p(x)}(\chi|_{\mathcal{O}_F^*}) \\ &= \chi|_{\mathcal{O}_F^*}(x) = \chi(\widetilde{\mathcal{J}}(x)) = \psi_{\widetilde{\mathcal{J}}(x)}(\chi) = \Psi \circ \widetilde{\mathcal{J}}(x)(\chi).\end{aligned}$$

Therefore $Stab_{k_F^*}(\overline{\pi}_{(\rho,E)}) \leqslant Ker(\widetilde{\mathcal{J}})$ because Ψ is an isomorphism and $Stab_{k_F^*}(\overline{\pi}_{(\rho,E)}) \leqslant Ker(p)$. So $\widetilde{\mathcal{J}}$ induces a map $\mathcal{J}$ such that $\widetilde{\mathcal{J}} = \mathcal{J} \circ p$. Since p is an epimorphism, equation (4.45) yields

$$(Res|_{\mathcal{O}_{F^*}})^\wedge \circ \overline{\Psi} = \Psi \circ \mathcal{J}$$

that is the commutativity of (4.44). ☐

Proposition 4.4.11. *With notation as above, the following diagram commutes:*

$$\begin{array}{ccc} {k_F^*}/{Stab_{k_F^*}(\overline{\pi}_{(\rho,E)})} & \xrightarrow{\mathcal{J}} & {F^*}/{Stab_{F^*}(\pi_{(\rho,E)})} \\ \downarrow{\scriptstyle \widehat{\Xi}\circ\overline{\Psi}} & & \downarrow{\scriptstyle \widehat{\Xi}_F\circ\Psi} \\ A((\rho,E)_{I_F})^\wedge & \xrightarrow{\hat{\iota}} & A(\rho,E)^\wedge \end{array} \tag{4.46}$$

Proof. The statement is obtained stacking the commutative diagram (4.44) in Lemma 4.4.9 on top of the commutative diagram (4.42) in Lemma 4.4.10. ☐

Proposition 4.4.12. *Let* $\overline{\psi} \in A((\rho,E)_{I_F})^\wedge$. *If* $\overline{\pi}_{(\rho,E)}$ *is an irreducible constituent of* $\mathcal{P}_{\overline{G'}_N}^{G'_N}\pi_{(\rho,E)}$, *then* $\overline{\psi}.\overline{\pi}_{(\rho,E)}$ *is an irreducible constituent of* $\mathcal{P}_{\overline{G'}_N}^{G'_N}(\hat{\iota}(\overline{\psi}).\pi_{(\rho,E)})$.

Proof. Let $x = (\widehat{\Xi} \circ \overline{\Psi})^{-1}(\overline{\psi})$ so that $\overline{\psi}.\overline{\pi}_{(\rho,E)} = {}^x\overline{\pi}_{(\rho,E)}$. By Proposition 4.4.11

$$(\widehat{\Xi}_F \circ \Psi)^{-1}(\hat{\iota}(\overline{\psi})) = \mathcal{J}((\widehat{\Xi} \circ \overline{\Psi})^{-1}(\overline{\psi})) = \mathcal{J}(x)$$

and therefore $\hat{\iota}(\overline{\psi}).\pi_{(\rho,E)} = {}^{\mathcal{J}(x)}\pi_{(\rho,E)}$.

We need to prove that if $\overline{\pi}_{(\rho,E)}$ is an irreducible constituent of $\mathcal{P}_{\overline{G'}_N}^{G'_N}\pi_{(\rho,E)}$, then $\overline{\psi}.\overline{\pi}_{(\rho,E)} = {}^x\overline{\pi}_{(\rho,E)}$ is an irreducible constituent of $\mathcal{P}_{\overline{G'}_N}^{G'_N}(\hat{\iota}(\overline{\psi}).\pi_{(\rho,E)}) = \mathcal{P}_{\overline{G'}_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})$. Rephrasing it, we need to show that $Hom_{\overline{G'}_N}(\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}(\pi_{(\rho,E)})) \neq 0$ implies that $Hom_{\overline{G'}_N}({}^x\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})) \neq 0$.

The inflation by ${K'_N}^+$ is left adjoint to taking the ${K'_N}^+$-fixed subspace, so

$$Hom_{\overline{G'}_N}({}^x\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})) \cong Hom_{K'_N}(Infl_{\overline{G'}_N}^{K'_N}\, {}^x\overline{\pi}_{(\rho,E)}, Res_{K'_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})). \tag{4.47}$$

The lifting of the map $\mathcal{J}$ through the map det, that by abuse of notation we still denote by $\mathcal{J}$, is the map $\mathcal{J} : {\overline{G}_N}/{Stab_{\overline{G}_N}(\overline{\pi}_{(\rho,E)})} \to {G_N}/{Stab_{G_N}(\pi_{(\rho,E)})}$ induced by any section of the projection map $r : K_N \to {K_N}/{K_N^+} \cong \overline{G}_N$, so the element $\mathcal{J}(x) \in {G_N}/{Stab_{G_N}(\pi)}$ has a representative in $K_N \leqslant G_N$. In the following of this proof we fix such a representative. Note that ${}^{\mathcal{J}(x)}\pi_{(\rho,E)}$ does not depend on the choice of the representative of $\mathcal{J}(x)$ in G_N.

We observe that

$$Infl_{\overline{G'}_N}^{K'_N}({}^x\overline{\pi}_{(\rho,E)}) = {}^{\mathcal{J}(x)}Infl_{\overline{G'}_N}^{K'_N}\overline{\pi}_{(\rho,E)}, \tag{4.48}$$

where the right-hand side is a representation of K'_N by normality of K'_N in K_N. Indeed, since the natural projection $K'_N \to {K'_N}/{{K'_N}^+}$ is obtained by restriction of r, for any $k \in K'$

$$\begin{aligned} Infl_{\overline{G'}_N}^{K'_N}({}^x\overline{\pi}_{(\rho,E)})(k) &= \overline{\pi}_{(\rho,E)}(x^{-1}r(k)x) \\ &= \overline{\pi}_{(\rho,E)}(r(\mathcal{J}(x)^{-1}k\mathcal{J}(x))) = {}^{\mathcal{J}(x)}Infl_{\overline{G'}_N}^{K'_N}\overline{\pi}_{(\rho,E)}(k). \end{aligned}$$

Combining (4.47) and (4.48) gives

$$\begin{aligned} Hom_{\overline{G'}_N}({}^x\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})) &\cong Hom_{K'_N}({}^{\mathcal{J}(x)}Infl_{\overline{G'}_N}^{K'_N}\overline{\pi}_{(\rho,E)}, Res_{K'_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})) \\ &= Hom_{K'_N}(Infl_{\overline{G'}_N}^{K'_N}\overline{\pi}_{(\rho,E)}, {}^{\mathcal{J}(x)^{-1}}Res_{K'_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})). \end{aligned}$$

We rewrite ${}^{\mathcal{J}(x)^{-1}}Res_{K'_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)}) = Res_{\mathcal{J}(x)^{-1}K'_N}^{G'_N}(\pi_{(\rho,E)})$. By normality of K'_N in K_N, it holds $Res_{\mathcal{J}(x)^{-1}K'_N}^{G'_N}(\pi_{(\rho,E)}) = Res_{K'_N}^{G'_N}(\pi_{(\rho,E)})$. Then

$$\begin{aligned} Hom_{\overline{G'}_N}({}^x\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}({}^{\mathcal{J}(x)}\pi_{(\rho,E)})) &\cong Hom_{K'_N}(Infl_{\overline{G'}_N}^{K'_N}\overline{\pi}_{(\rho,E)}, Res_{K'_N}^{G'_N}\pi_{(\rho,E)}) \\ &\cong Hom_{\overline{G'}_N}(\overline{\pi}_{(\rho,E)}, \mathcal{P}_{\overline{G'}_N}^{G'_N}(\pi_{(\rho,E)})) \neq 0. \end{aligned}$$

□

Lemma 4.4.13. *For any $\overline{\boldsymbol{\pi}}_{(\rho,E)} \in \mathcal{M}'^{-1}((\rho,E)_{I_F})$ there exists a unique $\boldsymbol{\pi}_{(\rho,E)} \in \mathcal{L}'^{-1}(\rho,E)$ such that $\overline{\boldsymbol{\pi}}_{(\rho,E)}$ is an irreducible constituent of $\mathcal{P}_{\overline{G'}_N}^{G'_N}\boldsymbol{\pi}_{(\rho,E)}$. Moreover $\overline{\boldsymbol{\pi}}_{(\rho,E)}$ has multiplicity 1 in $\mathcal{P}_{\overline{G'}_N}^{G'_N}\boldsymbol{\pi}_{(\rho,E)}$.*

Proof. Denote $\pi^{\oplus}_{(\rho,E)} = \bigoplus_{\pi\in\mathcal{L}'^{-1}(\rho,E)}\pi$ and $\overline{\pi}^{\oplus}_{(\rho,E)} = \bigoplus_{\overline{\pi}\in\mathcal{M}'^{-1}(\rho,E)}\overline{\pi}$. By Theorem 4.4.6, it holds $\mathcal{HP}^{G'_N}_{\overline{G'}_N}\pi^{\oplus}_{(\rho,E)} = \overline{\pi}^{\oplus}_{(\rho,E)}$. Hence by Lemma 4.4.5, the representation $\overline{\pi}^{\oplus}_{(\rho,E)}$ is a subrepresentation of $\mathcal{P}^{G'_N}_{\overline{G'}_N}(\pi^{\oplus}_{(\rho,E)})$ and it satisfies:

$$Hom_{\overline{G'}_N}(\overline{\pi}^{\oplus}_{(\rho,E)}, \mathcal{P}^{G'_N}_{\overline{G'}_N}\pi^{\oplus}_{(\rho,E)}) = Hom_{\overline{G'}_N}(\overline{\pi}^{\oplus}_{(\rho,E)}, \overline{\pi}^{\oplus}_{(\rho,E)}).$$

Since $\overline{\boldsymbol{\pi}}_{(\rho,E)}$ is an irreducible constituent of $\overline{\pi}^{\oplus}_{(\rho,E)}$, it is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G'_N}_{\overline{G'}_N}(\pi^{\oplus}_{(\rho,E)}) = \bigoplus_{\pi\in\mathcal{L}'^{-1}(\rho,E)}(\mathcal{P}^{G'_N}_{\overline{G'}_N}\pi)$. Therefore it is an irreducible constituent of multiplicity 1 of $\mathcal{P}^{G'_N}_{\overline{G'}_N}\boldsymbol{\pi}_{(\rho,E)}$ for a uniquely determined $\boldsymbol{\pi}_{(\rho,E)} \in \mathcal{L}'^{-1}(\rho, E)$. □

From now on, we fix a representation $\overline{\boldsymbol{\pi}}_{(\rho,E)} \in \mathcal{M}'^{-1}((\rho,E)_{I_F})$ and we denote by $\boldsymbol{\pi}_{(\rho,E)}$ the representation in $\mathcal{L}'^{-1}(\rho, E)$ as in Lemma 4.4.13.

This choice determines the bijections

$$\begin{aligned}\mathcal{F}_{(\rho,E)} : A(\rho,E)^\wedge &\to {\mathcal{L}'_N}^{-1}(\rho,E),\\ \psi &\mapsto \psi.\boldsymbol{\pi}_{(\rho,E)}\\ \overline{\mathcal{F}}_{(\rho,E)_{I_F}} : A((\rho,E)_{I_F})^\wedge &\to {\mathcal{M}'_N}^{-1}(\rho,E).\\ \overline{\psi} &\mapsto \overline{\psi}.\overline{\boldsymbol{\pi}}_{(\rho,E)}\end{aligned}$$

For any $\psi \in A(\rho,E)^\wedge$, the preimage $\hat{\iota}^{-1}(\psi)$ is either empty or a coset of $Ker(\hat{\iota})$ in $A((\rho,E)_{I_F})^\wedge$. We define the head of parahoric restriction for the representations in $\mathcal{L}'^{-1}(\rho, E)$.

Definition 4.4.14. *With notation as above, we define*

$$\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)}) := \bigoplus_{\overline{\psi}\in\hat{\iota}^{-1}(\psi)} \overline{\psi}.\overline{\boldsymbol{\pi}}_{(\rho,E)}.$$

Definition 4.4.14 is compatible with the Definition 4.4.4 because

$$\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\pi^{\oplus}_{(\rho,E)}) = \bigoplus_{\psi\in A(\rho,E)^\wedge} \mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)}).$$

Remark 4.4.15. Let $\psi \in A(\rho,E)^\wedge$. By the definition of $\hat{\iota}$ and Frobenius reciprocity, the coset $\hat{\iota}^{-1}(\psi)$ in $A((\rho,E)_{I_F})^\wedge$ can be described as follows. If $\psi \in A(\rho,E)^\wedge$ does not factor through ι, then $\hat{\iota}^{-1}(\psi)$ is empty. Otherwise, there exists a character ξ of $\iota(A(\rho,E))$ such that $\xi\circ\iota = \psi$. Then $\hat{\iota}^{-1}(\psi)$ is the set of the irreducible constituents of $ind^{A((\rho,E)_{I_F})}_{\iota(A(\rho,E))}\xi$.

Theorem 4.4.16. *For any $\psi \in A(\rho,E)^\wedge$, the representation $\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$ is a subrepresentation of $\mathcal{P}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$, and every irreducible constituent of $\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$ occurs exactly once in $\mathcal{P}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$.*

Moreover if $\overline{\pi} \in {\mathcal{M}'_N}^{-1}((\rho,E)_{I_F})$ is an irreducible component of $\mathcal{P}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$, then $\overline{\pi}$ is an irreducible component of $\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)})$.

Proof. By Proposition 4.4.12, for any $\overline{\psi} \in \hat{\iota}^{-1}(\psi)$ the representation $\overline{\psi}.\overline{\boldsymbol{\pi}}_{(\rho,E)}$ is an irreducible constituent of $\mathcal{P}^{G'_N}_{\overline{G'}_N}\psi.\boldsymbol{\pi}_{(\rho,E)}$, and by Lemma 4.4.13 it has multiplicity 1. Hence $\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\boldsymbol{\pi}_{(\rho,E)}) = \bigoplus_{\overline{\psi}\in\hat{\iota}^{-1}(\psi)} \overline{\psi}.\overline{\boldsymbol{\pi}}_{(\rho,E)}$ is a subrepresentation of $\mathcal{P}^{G'_N}_{\overline{G'}_N}\psi.\boldsymbol{\pi}_{(\rho,E)}$ and all of its irreducible components appear exactly once.

If $\overline{\pi} \in \mathcal{M}'^{-1}((\rho,E)_{I_F})$, there exists $\xi \in A((\rho,E)_{I_F})^\wedge$ such that $\overline{\pi} = \xi.\overline{\boldsymbol{\pi}}_{(\rho,E)}$ by transitivity of the action. By Proposition 4.4.12, $\overline{\pi}$ is an irreducible constituent of $\mathcal{P}^{G'_N}_{\overline{G'}_N}\hat{\iota}(\xi).\boldsymbol{\pi}_{(\rho,E)}$. Then if $\overline{\pi} \leqslant \mathcal{P}^{G'_N}_{\overline{G'}_N}\psi.\boldsymbol{\pi}_{(\rho,E)}$, by the uniqueness statement in Lemma 4.4.13 it holds $\psi.\boldsymbol{\pi}_{(\rho,E)} = \hat{\iota}(\xi).\boldsymbol{\pi}_{(\rho,E)}$ and since the action of $A(\rho,E)^\wedge$ over $\mathcal{L}'^{-1}_N(\rho,E)$ is free, $\hat{\iota}(\xi) = \psi$. Therefore, $\xi \in \hat{\iota}^{-1}(\psi)$, and so $\overline{\pi} = \xi.\overline{\boldsymbol{\pi}}_{(\rho,E)}$ is an irreducible constituent of $\mathcal{HP}^{G'_N}_{\overline{G'}_N}(\psi.\pi_{(\rho,E)})$. □

Remark 4.4.17. Theorem 4.4.16 depends on the choice of the maximal compact subgroup K'_N of G'_N. When a different maximal compact subgroup J'_N is considered, the correspondence in Lemma 4.4.13 between a representation $\overline{\pi}_{(\rho,E)} \in \mathcal{M}'^{-1}_N(\rho,E)$ and $\pi_{(\rho,E)} \in \mathcal{L}'^{-1}_N(\rho,E)$ given by $Hom_{\overline{G'}_N}(\overline{\pi}_{(\rho,E)}, \mathcal{P}^{G'_N}_{\overline{G'}_N}\pi_{(\rho,E)}) \neq 0$, for $(\rho,E) \in \Phi(G'_N)_0$, varies as follows. Let $g \in G_N$ be such that $J_N = {}^gK'_N$. Then ${}^g\pi_{(\rho,E)} \in \mathcal{L}'^{-1}_N(\rho,E)$, and

$$(Res^{G_N}_{J'_N}({}^g\pi_{(\rho,E)}))^{J'^+_N} = (Res^{G_N}_{{}^gK'_N}({}^g\pi_{(\rho,E))}))^{{}^gK'^+_N} = {}^g((Res^{G_N}_{K'_N}\pi_{(\rho,E)})^{K'^+_N}).$$

We have $\overline{G'}_N \cong {K'_N}/{K'^+_N} \cong {J'_N}/{J'^+_N}$ where the last isomorphism is induced by conjugation by g. Hence $(Res^{G_N}_{J'_N}{}^g\pi_{(\rho,E)})^{J'^+_N}$ and $(Res^{G_N}_{K'_N}\pi_{(\rho,E)})^{K'^+_N}$ are isomorphic as $\overline{G'}_N$ representations. It follows that $Hom_{\overline{G'}_N}(\overline{\pi}_{(\rho,E)}, (Res^{G_N}_{J'_N}{}^g\pi_{(\rho,E)})^{J'^+_N}) \neq 0$. Therefore, considering the parahoric restriction with respect to J'_N rather than K'_N, Lemma 4.4.13 would associate to $\overline{\pi}_{(\rho,E)} \in \mathcal{M}'^{-1}_N(\rho,E)$ the representation ${}^g\pi_{(\rho,E)} \in \mathcal{L}'^{-1}_N(\rho,E)$ rather than $\pi_{(\rho,E)} \in \mathcal{L}'^{-1}_N(\rho,E)$.

However, the equivariance result in Proposition 4.4.12 still holds when considering parahoric restriction with respect to a different maximal parahoric subgroup.

We conclude that if we replace K'_N by $J'_N = {}^gK'_N$, then Theorem 4.4.16 holds replacing $\boldsymbol{\pi}_{(\rho,E)}$ with ${}^g\boldsymbol{\pi}_{(\rho,E)}$.

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