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CICLO XXXI

GEOMETRIC INTEGRAL INEQUALITIES ON HOMOGENEOUS SPACES

Coordinatore: Ch.mo Prof. Martino Bardi
Supervisore: Ch.mo Prof. Paolo Ciatti

Dottorando: Roberto Bramati

Riassunto

In questa tesi studiamo alcune stime integrali su gruppi di Lie e loro spazi omogenei.
Nella prima parte della tesi sviluppiamo una strategia generale per ottenere stime multilineari di tipo Brascamp-Lieb su spazi omogenei compatti e la applichiamo ai casi del toro e della sfera unitaria reale. Otteniamo anche delle disuguaglianze di tipo Brascamp-Lieb nel contesto non compatto dei gruppi di Lie stratificati.
Nella seconda parte della tesi, come conseguenza di stime integrali per armoniche sferiche quaternioniche, dimostriamo alcuni limiti dal basso per le norme (L^p, L^2) degli operatori di proiezione sugli spazi delle armoniche sferiche sulla sfera quaternionica, per $p \in [1, 2]$. Inoltre, in analogia con risultati di J. Duoandikoetxea sulla sfera unitaria reale, dimostriamo alcune stime non dipendenti dalla dimensione per armoniche sferiche bigradate sulle sfere unitarie complessa e quaternionica.

Abstract

This thesis is devoted to the study of some integral inequalities on Lie groups and their homogeneous spaces.
In the first part of the thesis we provide a general strategy to obtain multilinear inequalities of Brascamp-Lieb type on compact homogeneous spaces and we apply it to the case of the torus and of the real unit sphere. We also obtain some Brascamp-Lieb type inequalities in the noncompact context of stratified Lie groups.
In the second part of the thesis, as a consequence of integral bounds for quaternionic spherical harmonics, we prove some bounds from below for the (L^p, L^2) norm of the harmonic projection operators on the quaternionic sphere, for $p \in [1, 2]$. Moreover, in analogy with some earlier results by J. Duoandikoetxea on the real unit sphere, we prove some dimension free bounds for bigraded spherical harmonics on the complex and quaternionic unit spheres.

Contents

Introduction

In this thesis we study integral inequalities of different types in the context of Lie groups and their homogeneous spaces.
In the first part of the thesis we develop a general approach to obtain certain multilinear inequalities on compact homogeneous spaces and provide some applications. We also obtain similar inequalities in the noncompact context of stratified Lie groups.
In the second part of the thesis we prove some bounds from below for the operator norm of the harmonic projection operators on the quaternionic sphere, and sharp dimensions free L^p bounds for bigraded spherical harmonics on complex and quaternionic spheres.
In the rest of the introduction we describe in detail the topics treated in the thesis.

Part I

Many well-known multilinear inequalities commonly used in analysis, such as multilinear Hölder's inequality, Loomis–Whitney inequality and the sharp Young convolution inequality, can be seen as instances of a broader family of estimates: the so called Brascamp–Lieb inequalities. These are inequalities of the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(B_j x) dx \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{n_j})}, \tag{1}$$

where $p_j \in [1, \infty]$, $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ are linear surjective maps and the functions $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+$ are measurable, for $j = 1, \dots, m$. The constant C in (1) is the smallest constant, either finite or infinite, over all measurable inputs f_j for which (1) holds. This constant depends on the maps B_j and on the exponents p_j and is called the Brascamp–Lieb constant.
These inequalities were extensively studied in the last years, starting from the works of Rogers [49] Brascamp, Lieb and Luttinger [12] and Brascamp and Lieb [11], where the authors studied the rank-one case, that is the case where $n_j = 1$ for all j, using rearrangement techniques. In particular they proved that the Brascamp–Lieb constant is the same if one restricts the inputs to Gaussians, a result known as Lieb's Theorem. This result was then extended to the higher rank case by Lieb in [41], then Barthe gave an alternative proof using transportation of mass techniques in [2].
Another approach to the problem was introduced by Carlen, Lieb, Loss who used heat flow methods to prove Lieb's Theorem in the rank one case in [18]. This approach was rediscovered independently and used by Bennett, Carbery, Christ and Tao to prove Lieb's Theorem in the general case in [7]. In particular they were able to prove the following theorem.

Theorem 0.0.1 ([7]). *The constant C in (1) is finite if and only if the scaling condition*

$$\sum_{j=1}^{m} p_j^{-1} n_j = n \tag{2}$$

and the dimension condition

$$\dim(V) \leq \sum_{j=1}^{m} p_j^{-1} \dim(B_j V), \tag{3}$$

for all subspaces $V \subseteq \mathbb{R}^n$, are satisfied.

The heat flow technique consists in studying the monotonicity properties of a certain function, depending on a parameter that can be thought of as time, that is related to the heat evolution of some functions. Comparing this function at different times is a way of producing inequalities. For example in [7] the authors study, among other things, the case of the so called geometric Brascamp–Lieb inequality (already studied by Ball in [1] and Barthe in [2]), in which the linear maps B_j are such that B_j^* is an isometry and the condition

$$\sum_{j=1}^{m} p_j^{-1} B_j^* B_j = \mathrm{Id}_{\mathbb{R}^n} \tag{4}$$

holds. They show that for nonnegative Schwartz functions f_j the quantity

$$Q(t) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} u_j(t,x) dx \tag{5}$$

is nondecreasing, where $u_j(t,x)^{p_j}$ is the solution of the heat equation in $\mathbb{R}^n$ with initial datum $f_j^{p_j} \circ B_j$. Inequality (1) is then obtained by comparing $\lim_{t\to 0} Q(t)$ which gives the left-hand side of it with $\lim_{t\to\infty} Q(t)$ which gives the right-hand side.

In this thesis we will interpret inequality (1) in the following way: we are given a family of functions $f_j \circ B_j$, each one having some degree of symmetry (indeed, these functions are constant on the fibers of the maps B_j, that are affine subspaces parallel to $\ker B_j$) and we want to find exponents p_j for which the inequality holds with a finite constant C for all choices of functions. Theorem 0.0.1 gives a complete answer to this question in the Euclidean setting, relating the exponents p_j to the geometry of the maps B_j and to the scale invariant structure of $\mathbb{R}^n$.

An interesting issue is to extend inequality (1) to other settings. This problem was already addressed in the works [18, 19] where some inequalities were obtained in the case of spheres and of the permutation group on d elements S_d (see also [4, 3] for further comments).

In particular in [18, Theorem 1.1] the authors proved that for nonnegative measurable functions f_i on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ depending only on one variable x_i (that are functions $\tilde{f}$ defined on the interval $[-1,1]$ and pulled-back to the sphere by the projection on the i-th variable $\pi_i : \mathbb{S}^{n-1} \to [-1,1]$), the estimate

$$\int_{\mathbb{S}^{n-1}} \prod_{j=1}^{n} \tilde{f}_j(\pi_j x) d\sigma \leq \prod_{j=1}^{n} \|\tilde{f}_j \circ \pi_j\|_{L^p(\mathbb{S}^{n-1})} \tag{6}$$

holds, with $p \geq 2$, where $d\sigma$ is the normalized uniform measure on the sphere and

$$\|f\|^p_{L^p(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} |f(u)|^p \, d\sigma(u)\,.$$

The authors also proved that the inequality is sharp, in the sense that there exist n functions in $L^p(\mathbb{S}^{n-1})$ for $p < 2$, each depending on a different variable, for which the right-hand side of (6) is finite and the left-hand side diverges.
Inequality (6) can be interpreted in two ways:

- as a Hölder type inequality, but with the sum of the reciprocal of the exponents bigger than one, a condition that cannot be achieved for general functions just by multilinear Hölder's inequality and continuous embeddings of Lebesgue spaces on the sphere;
- as a Brascamp–Lieb type inequality, plugging in it the estimate $\|\tilde{f}_j \circ \pi_j\|_{L^p} \lesssim \|\tilde{f}_j\|_{L^p([-1,1])}$.

The proof of inequality (6) is based on the heat flow method and relies on the fact that the sphere is a compact homogeneous space. Indeed $\mathbb{S}^{n-1} = SO(n-1)\backslash SO(n)$, where $SO(n)$ is the group of real orthogonal matrices with determinant 1.
Following the ideas of [18] in the first chapter of this thesis we find inequalities similar to (6) in the setting of general compact homogeneous spaces of type $M = K\backslash G$, where G is a connected, unimodular Lie group and K is a closed subgroup of G. We endow M with the unique normalized measure $d\mu$ induced by the Haar measure on G. We fix a finite set of vector fields $\mathcal{I}$ in the Lie algebra of left invariant vector fields $\mathfrak{g}$ of G satisfying Hörmander's bracket generating condition and we construct the sum of squares sub-Laplacian L, which is a symmetric, negative, essentially self-adjoint, hypoelliptic operator acting on smooth functions defined on G and on its quotient M. By means of the heat semigroup $\{e^{tL}\}_{t>0}$, we consider the nonlinear heat flow

$$v(t,x) = \left(e^{tL} f^p\right)^{1/p},$$

where $p \geq 1$ and $f \in C^\infty(M)$, which is the solution of the nonlinear equation

$$\partial_t v(t,x) = (p-1)\frac{|\nabla_{\mathcal{I}} v(t,x)|^2}{v(t,x)} + Lv(t,x),$$

where $\nabla_{\mathcal{I}}$ is the gradient with respect to the vector fields in $\mathcal{I}$.
Taking m different nonnegative functions $f_i \in C^\infty(M)$ and considering their nonlinear evolutions $v_i(t,x)$ we will prove that the function

$$\phi(t) = \int_M \prod_{j=1}^m v_i(t,x) d\mu \tag{7}$$

is nondecreasing for $p \geq m$. By a comparison between $\lim_{t\to 0}\phi(t)$ and $\lim_{t\to\infty}\phi(t)$, it will then follow that

$$\int_M \prod_{j=1}^m f_i d\mu \leq \prod_{j=1}^m \|f_i\|_{L^p(M)}, \tag{8}$$

for $p \geq m$. In the case of general functions f_i the estimate (8) can also be obtained as a straightforward consequence of multilinear Hölder's inequality and continuous embeddings of Lebesgue spaces on M. This is not surprising, since with generic functions f_i one cannot expect to improve on Hölder's exponents. Nevertheless, if we let the functions involved have some kind of symmetries, we will obtain better exponents not directly deducible by Hölder's inequality and continuous embeddings.
The relevant symmetries in our analysis are those that can be described by means of subsets $\mathcal{A}$ of $\mathcal{I}$ made of vector fields that commute with L. We call a subset $\mathcal{A}$ of $\mathcal{I}$ maximal if $\langle\mathcal{A}\rangle \cap \mathcal{I} = \mathcal{A}$, where $\langle\mathcal{A}\rangle$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing $\mathcal{A}$. We say that a

function $f \in C^\infty(M)$ is $\mathcal{A}$-symmetric if $Xf = 0$ for all vector fields X in $\mathcal{A}$. Functions that are $\mathcal{A}$-symmetric are constant on certain nonintersecting submanifolds that cover the manifold M. The commutation property with L of the vector fields in $\mathcal{A}$ ensures that the symmetry is preserved by the heat flow.
The main result of the first chapter, contained in Theorem 1.6.1, says that taking m functions, each $\mathcal{A}_i$-symmetric for some maximal subset $\mathcal{A}_i$ of $\mathcal{I}$ the function (7) is nondecreasing for p greater than or equal to a critical $\tilde{p}$ that depends on the combinatorics of the sets $\mathcal{A}_i$. In Theorem 1.6.2 we obtain an analogous result, but we let each f_i evolve under a nonlinear heat flow with a different p_i. We prove that the function

$$\phi(t) = \int_M \prod_{j=1}^{m} \left(e^{tL} f^{p_i}\right)^{1/p_i} d\mu \tag{9}$$

is nondecreasing if each p_i is greater than or equal to a critical $\tilde{p}_i$ that again depends on the combinatorics of the sets $\mathcal{A}_i$.
As a first application of this machinery, in Chapter 1 we study the (abelian) case of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$; here the Hörmander system $\mathcal{I} = \{\partial_{x_i} : i = 1, \ldots, n\}$ is associated to an orthonormal basis of $\mathbb{R}^n$. By means of Theorem 1.6.1 we are able to recover a result by Calderon in [15] and a family of local geometric Brascamp–Lieb inequalities associated to projections to collections of coordinate variables.
In Chapter 2 we apply the results of Chapter 1 to the case of the sphere

$$\mathbb{S}^{n-1} = SO(n-1)\backslash SO(n)$$

in $\mathbb{R}^n$; here $\mathcal{I}$ is given by the vector fields

$$L_{i,j} = x_i\partial_{x_j} - x_j\partial_{x_i},$$

for $1 \leq i < j \leq n$. These vector fields form a basis of $\mathfrak{so}(n)$ and so they verify Hörmander's condition. In the case of the sphere we are able to classify all possible maximal subsets of $\mathcal{I}$, thus getting an easy algorithm to produce multilinear inequalities involving functions with special symmetries. With this language we recover the result of [18] for functions depending on one variable and extend it to functions depending on k variables, for $1 \leq k \leq n-1$, describing these properties of the functions as $\mathcal{A}$-symmetries for specific maximal subsets $\mathcal{A}$. A function f of k variables can be understood as a function $\tilde{f}$ defined on the k-dimensional unit ball B_k of $\mathbb{R}^k$ and pulled-back to the sphere by the projection $\pi : \mathbb{S}^{n-1} \to B_k$ on the k variables involved.
Let $C(n,k) = \binom{n}{k}$. We prove that if $f_1, \ldots, f_{C(n,k)}$ are nonnegative measurable functions, each depending on a different collection of k variables, denoted with x_{ω_i}, where $\omega_i \subset \{1, \ldots, n\}$, $|\omega_i| = k$, the inequality

$$\int_{\mathbb{S}^{n-1}} f_1(x_{\omega_1}) \ldots f_{C(n,k)}(x_{\omega_{C(n,k)}}) d\sigma \leq \prod_{i=1}^{C(n,k)} \|f_i\|_{L^p(\mathbb{S}^{n-1})} \tag{10}$$

holds for

$$p \geq \tilde{p} = \binom{n}{k} - \binom{n-2}{k}.$$

Moreover we prove that this inequality is sharp in the sense of [18]. Since for a function f depending on k variables, $\|f\|_{L^p} = \|\tilde{f} \circ \pi\|_{L^p} \lesssim \|\tilde{f}\|_{L^p(B_k)}$, we can interpret (10) as a

Brascamp–Lieb type inequality.
Inequality (10) is first proved for a small range of exponents that is then extended by interpolation. In Chapter 2 we study in what range of exponents the inequality can hold, obtaining some optimal result.
If we add an additional symmetry to the functions, requiring that each function depends radially on k variables we prove an (again sharp) improvement of inequality (10), obtaining a lower critical exponent

$$\tilde{q} = 2\binom{n-2}{k-1}.$$

We also address the case of functions each having a different kind of symmetry, providing an algorithm to compute the critical exponents and showing that they are sharp in some examples. We conjecture that the exponents obtained by this method are always sharp. We plan to treat this problem in future work.
In the remaining part of Chapter 2 we discuss some possible applications of our inequalities. In particular we provide some estimates in Lebesgue spaces with mixed angular-radial norms and some local Brascamp–Lieb inequalities for maps B_j associated to projections on collections of variables (see [6, 61] for possible applications of these local inequalities).
Finally, by transferring the Carlen–Lieb–Loss inequality on $\mathbb{S}^{n-1}$ to the Euclidean space $\mathbb{R}^{n-1}$ via stereographic projection, we obtain a family of weighted nonlinear Brascamp–Lieb inequalities of the form

$$\int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n} \phi_i(X_1, \dots, X_{n-1}) \frac{dX_1 \dots dX_{n-1}}{(X_1^2 + \dots + X_{n-1}^2 + 4)^2} \lesssim \prod_{i=1}^{n} \|\phi_i\|_{L^2(\mathbb{R}^{n-1})},$$

where the functions ϕ_i are constant on certain nonintersecting $(n-2)$-dimensional spheres that cover the whole space $\mathbb{R}^{n-1}$.
In Chapter 3 we extend the methods of [7] to stratified groups, that are nilpotent Lie groups whose Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r,$$

with

$$[\mathfrak{g}_l, \mathfrak{g}_1] = \mathfrak{g}_{l+1}, \quad l = 1, \dots, r-1.$$

These groups G can be identified with their Lie algebra via the Baker-Campbell-Hausdorff formula (see [28]). They are equipped with a family of anisotropic dilations δ_s, for $s > 0$, acting diagonally on the subspaces $\mathfrak{g}_1, \dots, \mathfrak{g}_r$. The quantity $Q = \dim \mathfrak{g}_1 + 2 \dim \mathfrak{g}_2 + \dots + r \dim \mathfrak{g}_r$ is called the homogeneous dimension of G and is related to the volume growth of balls. We consider projections $\pi^{(a)} : G \to M^{(a)} = H^{(a)} \backslash G$, where the $H^{(a)}$ are homogeneous subgroups (with respect to the same dilations) of G of homogeneous dimension $\bar{Q}^{(a)}$. Moreover we require that these projections commute with the dilations, making the quotient $M^{(a)}$ a homogeneous space in the sense of Coifman and Weiss of homogeneous dimension $Q^{(a)} = Q - \bar{Q}^{(a)}$. The standard hypoelliptic negative sub-Laplacian L on G is the sum of the squares of vector fields $\{X_1, \dots, X_n\}$ forming an orthonormal basis of the first layer $\mathfrak{g}_1$. Recall that, as in the Euclidean space, the heat kernel of L enjoys the homogeneity property

$$p_t(g) = t^{-Q/2} P(\delta_{t^{-1/2}} g),$$

for $t > 0$, where P is a strictly positive Schwartz function. The sub-Laplacian also acts on the quotient, via the push-forward related to $\pi_{(a)}$, so one can consider the heat equation of the sub-Laplacian on the homogeneous spaces $M^{(a)}$ and obtain the following result. We suppose that

$$p_1 Q^{(1)} + \cdots + p_l Q^{(l)} = Q. \tag{11}$$

Then the inequality

$$\int_G \bar{f}^{(1)}\big(\pi^{(1)}(g)\big)^{p_1} \dots \bar{f}^{(l)}\big(\pi^{(l)}(g)\big)^{p_l} dg \le I \left(\int_{M^{(1)}} \bar{f}^{(1)}(g_1) dg_1\right)^{p_1} \dots \left(\int_{M^{(l)}} \bar{f}^{(l)}(g_l) dg_l\right)^{p_l} \tag{12}$$

holds on G. The constant I appearing in this estimate is given by

$$I = \int_G P^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big)^{p_1} \dots P^{(l)}\big(\pi^{(l)}(e)\pi^{(l)}(g)\big)^{p_l} dg$$

and is finite under appropriate assumptions on the maps $\pi^{(a)}$. Condition (11) is the analog in the stratified groups context of condition (2) in the Euclidean case. The proof of this result, mutatis mutandis, follows the monotonicity approach of [7, Proposition 2.8], where the functions appearing in the integral in the definition of ϕ evolve under the heat flow of the sub-Laplacian L.

As an application we study the case of Hölder's inequality and Young convolution inequality. Finally we deduce a family of inequalities for stratified groups where the subgroups $H^{(a)}$ are given by the flows of the vector fields X_a in the first layer $\mathfrak{g}_1$ in $\mathfrak{g}$. Applying this inequality on the Heisenberg group $\mathbb{H}_1$ we are able to prove an inequality of Gagliardo-Nirenberg type, giving for a Schwartz function f on $\mathbb{H}_1$ the estimate

$$\|f\|_{L^{4/3}} \lesssim \|\nabla f\|_{L^1},$$

where ∇ denotes the horizontal gradient. By standard arguments (see [48, 16, 17]) this estimate leads to a sub-Riemannian isoperimetric inequality.

In this thesis we do not address the problems of extremisers (that should be related to the heat kernels on G) and extremisability, leaving it to a future work.

Part II

The final chapter is devoted to the proof of some sharp bounds (some of them depending on the dimension and some not) for bigraded spherical harmonics on complex and quaternionic spheres, in the spirit of some earlier work by C. Sogge and J. Duoandikoetxea. These estimates have been successfully applied to different problems in harmonic analysis, like Strichartz estimates for solutions of the Schrödinger equation [13, 14, 23], L^p summability of Bochner–Riesz means [52, 22], unique continuation problems [36, 53].

More precisely, denoting by $\mathbb{S}^{dn-1}$ the unit sphere in $\mathbb{R}^{dn}$, $d = 1, 2, 4$, we start from the well-known direct sum decomposition of the space of square-integrable functions on $\mathbb{S}^{dn-1}$, that is,

$$L^2(\mathbb{S}^{dn-1}) = \bigoplus_{\tau \in \mathcal{F}} V^\tau, \tag{13}$$

where

$$\tau = \begin{cases} \ell & \text{if } d = 1 \\ (\ell, \ell') & \text{if } d = 2, 4, \end{cases}$$

and

$$\mathcal{F} = \begin{cases} \mathbb{N} & \text{if } d = 1 \\ \mathbb{N} \times \mathbb{N} & \text{if } d = 2 \\ \{(\ell, \ell') \in \mathbb{N} \times \mathbb{N} \,:\, \ell \geq \ell'\} & \text{if } d = 4. \end{cases}$$

The spaces V^τ are formed by the so called spherical harmonics. If $d = 1$, it is well known that the V^τ are eigenspaces for the Laplace-Beltrami operator $\Delta_{\mathbb{S}^{n-1}}$, corresponding to the eigenvalues $\ell(\ell + n - 2)$. If $d = 2$ or 4, then the V^τ turn out to be eigenspaces both for the Laplace-Beltrami operator $\Delta_{\mathbb{S}^{dn-1}}$ to the eigenvalues $(\ell + \ell')(\ell + \ell' + dn - 2)$, and for a suitably defined sub-Laplacian L to the eigenvalues $\lambda_{\ell\ell'}$, given by $2\ell\ell' + (n-1)(\ell + \ell')$ for $d = 2$ and by $4(\ell\ell' + (n-1)\ell + n\ell')$ for $d = 4$. For this reason, the elements of V^τ when $d = 2$ or 4 are sometimes called bigraded joint spherical harmonics.
We recall that sharp estimates, depending on n, for the projections mapping $L^2(\mathbb{S}^{dn-1})$ onto V^τ, are already known in many cases. To be more precise, C. Sogge proved them in the real case, not only on the unit sphere, but also in the more general framework of Riemannian manifolds [51]. In the complex case, analogous bounds were proved in [20]. In collaboration with Casarino and Ciatti we recently started to study the quaternionic case, proving some bounds from below for the (L^p, L^2) norm of the quaternionic harmonic projectors $\pi_{\ell\ell'}$, mapping the space $L^2(\mathbb{S}^{4n-1})$ onto the eigenspace consisting of joint spherical harmonics of bidegree (ℓ, ℓ'), for $p \in [1, 2]$.
To prove these kind of inequalities, we are led to study the L^q norms of the functions $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, for $q \geq 2$, since

$$\|\pi_{\ell\ell'}\|_{(p,2)} \geq \frac{\|Y_{\ell\ell'}\|_{L^q}}{\|Y_{\ell\ell'}\|_{L^2}}, \tag{14}$$

for $q \geq 2$ and $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, due to the fact that the transposed operator $\pi^*_{\ell\ell'} : \mathcal{H}^{\ell\ell'} \to L^q(\mathbb{S}^{4n-1})$ is the inclusion operator (here $1/p + 1/q = 1$).
Our bounds are therefore strictly related to the problem of size concentration of the spherical harmonics. In the real framework, Sogge highlighted the existence of two classes of spherical harmonics with competing behaviors, the highest weight vectors and the zonal functions, playing an essential role in the analysis of the real harmonic projectors and also in some related applications (we refer to [51, 14] and especially to [13] for an application to Strichartz estimates). Analogously, both the complex highest weight vectors and the complex zonal functions turn out to be the key to understand the behavior of complex harmonic projectors. The quaternionic context is slightly different, since we identify three classes of spherical harmonics with competing behaviors, giving rise, in light of (14), to different estimates from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ on three subintervals of $p \in [1, 2]$. In fact, for p close to 2, the bounds are more sensitive to a sparse concentration along the Equator; in this case, we obtain bounds from below by considering the highest weight spherical harmonics, since these functions spread out in a small neighborhood around the Equator. When p is close to 1, exactly as in the real and complex case, the bounds for $\|\pi_{\ell\ell'}\|_{(p,2)}$ turn out to be sensitive to a high pointwise concentration. Thus we obtain bounds from below by considering the quaternionic zonal functions $\mathbb{Z}_{\ell\ell'}$, which are highly concentrated at the North Pole. Anyway, in a third interval inside $[1, 2]$, more precisely when $p \in (4/3, 2(4n-3)/(4n-1))$, we obtain better bounds from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, by considering a third class of spherical harmonics.
In Chapter 4 we will discuss these features of $\mathcal{H}^{\ell\ell'}$, which have no analog in the real or complex case and are related to representation-theoretic questions on $\mathbb{S}^{4n-1}$. It is worth mentioning that all the aforementioned bounds on $\mathbb{S}^{dn-1}$, $d = 1, 2, 4$, strongly depend on the dimension n and may indeed be considered a discrete version of the restriction estimates of Stein and

Tomas (we refer to [54] for a thorough discussion of this point).
Shortly after Sogge's estimates appeared, anyway, Duoandikoetxea proved some dimension free bounds for spherical harmonics on the real sphere [24, 25]. More precisely, on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ endowed with normalized Lebesgue measure, he showed that a generic spherical harmonic Y_k of degree k satisfies

$$\|Y_k\|_{L^p} \leq (p-1)^{k/2} \|Y_k\|_{L^2}, \tag{15}$$

for all $p \geq 2$. In addition, he proved that (15) is sharp, in the sense that no inequality like (15) may hold with an exponent lower than $k/2$. Recently, Duoandikoetxea's estimates were successfully applied in an algebraic context. Indeed, G. Blekherman used some bounds from [24] to compare the size of compact sections of the cones of nonnegative polynomials, of sums of squares and of sums of even powers of linear forms [8, 5].
Inspired, in particular, by this recent application, in the final section of Chapter 4 we prove sharp dimension-free estimates for joint complex and quaternionic spherical harmonics.
The free-dimensional approach proposed by Duoandikoetxea, indeed, may be easily adapted, finally covering the case of all spheres $\mathbb{S}^{dn-1}$, $d = 1, 2, 4$. In Section 4.2, in fact, we prove analogous estimates for bigraded spherical harmonics on the unit complex and quaternionic sphere. Our focus is mainly on the sharpness.

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Part I

Geometric inequalities related to the heat flow

CHAPTER 1

A general framework

1.1 Homogeneous spaces

In this chapter we provide a general framework to obtain a family of multilinear inequalities. Let G be a connected, unimodular Lie group, with bi-invariant Haar measure $d\mu$ and let K be a closed subgroup so that the homogeneous space $M = K\backslash G$ is compact and has no boundary. Denote by $\pi : G \to K\backslash G$ the canonical projection. Recall (see [33, Theorem 4.2, Ch. II]) that M is defined as the space of right cosets

$$M = \{Kg : g \in G\}$$

and has an analytic structure. We will sometimes write $[g]$ for a representative of the coset space Kg.
If $d\mu$ is the Haar measure on G we have a unique (up to scalars) bi-invariant measure on $K\backslash G$ (cfr. [33, Theorem 1.7, Ch. X]), which we will still denote by $d\mu$, defined as the push-forward of $d\mu$ by means of the projection π. We assume that this measure is normalized, i.e. $d\mu(M) = 1$.
The left translation on the group G is the map

$$\tau : G \to \mathrm{End}(G),$$

defined by

$$\tau_g(h) = gh.$$

There is an action of the group G on the set $C^\infty(G)$ given by left translations:

$$\begin{aligned} G \times C^\infty(G) &\to C^\infty(G) \\ (g, f) &\longmapsto f \circ \tau_g. \end{aligned}$$

Abusing notation we still denote this action by τ, writing $\tau_g f(h) = f(gh)$.
A left invariant vector field is a first order differential operator X that commutes with all left translation, i.e. such that

$$X(\tau_g f) = (\tau_g)(Xf)$$

for all $g \in G$ and $f \in C^\infty(G)$. The Lie algebra $\mathfrak{g}$ of the Lie group G is the vector space of all left invariant vector fields on G, endowed with the Lie algebra structure given by the bracket

$$[X, Y] = XY - YX,$$

for all $X, Y \in \mathfrak{g}$.
Recall that the exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the identity in G (see [58]). Since the vector fields are left invariant, the map exp parametrizes a small neighborhood of every point in G providing an atlas for the group. Recall that left invariant vector fields are defined by

$$Xf(g) = \frac{d}{dt} f(g \exp(t\bar{X}))|_{t=0}$$

for $\bar{X} \in T_eG$, where T_eG denotes the tangent space to G at the neutral element e, and $f \in C^\infty(G)$. For a vector field X, its adjoint ${}^tX$ is the vector field satisfying

$$\int_G (Xf) g d\mu = \int_G f({}^tXg) d\mu,$$

for all $f, g \in C_c^\infty(G)$, where $C_c^\infty(G)$ is the space of compactly supported smooth functions. We have the following proposition.

Proposition 1.1.1. *For $X \in \mathfrak{g}$ we have that ${}^tX = -X$.*

Proof. Let $f, g \in C_c^\infty(G)$. By the bi-invariance of the Haar measure we have

$$\int_G f(x \exp(tX)) g(x \exp(tX)) d\mu = \int_G f g d\mu.$$

Differentiating in t and evaluating at $t = 0$ both sides we obtain

$$\int_G f\,(Xg)\,d\mu + \int_G (Xf)\,g d\mu = 0,$$

which proves the proposition. $\square$

There is a one-to-one correspondence between smooth real-valued functions f on the quotient space $M = K\backslash G$ and smooth functions $\tilde{f}$ on G that are constant on coset spaces, i.e.

$$C^\infty(K; G) := \{\tilde{f} \in C^\infty(G) : \tilde{f}(g) = \tilde{f}(kg), \text{ for all } k \in K\}.$$

We denote this correspondence by

$$\Psi : C^\infty(M) \to C^\infty(K; G),$$

where $(\Psi f)(g) := f([g])$, and by

$$\Psi^{-1} : C^\infty(K; G) \to C^\infty(M)$$

with $(\Psi^{-1}\tilde{f})([g]) = \tilde{f}(g)$, its inverse. Note that, for $f \in C^\infty(M)$, (Ψf) is indeed a function in $C^\infty(K; G)$, since, taking $k \in K$ we have $(\Psi f)(kg) = f([kg]) = f([g]) = (\Psi f)(g)$. Analogously, for $\tilde{f} \in C^\infty(K; G)$, $(\Psi^{-1}\tilde{f})$ is well defined as a function in $C^\infty(M)$, since taking $[g'] = [g] \in M$, we have that $g' = kg$, which implies $(\Psi^{-1}\tilde{f})([g']) = \tilde{f}(g') = \tilde{f}(kg) = \tilde{f}(g) = (\Psi^{-1}\tilde{f})([g])$.
Let $f \in C^\infty(M)$ and $\tilde{f} = \Psi(f)$. A left invariant vector field $X \in \mathfrak{g}$ acts on smooth functions on M via the pushforward of the map π (that we denote with $T\pi$):

$$X\tilde{f} = \Psi(T\pi(X)f).$$

The same argument can be extended to left invariant differential operator in the universal enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{g})$. We write X instead of $T\pi(X)$ for vector fields of $\mathfrak{g}$ acting on $C^\infty(M)$.

Remark 1.1.2. Note that Proposition 1.1.1 yields an integration by parts formula on M:

$$\int_M (Xf)gd\mu = -\int_M f(Xg)d\mu$$

for $X \in \mathfrak{g}$ and $f, g \in C^\infty(M)$. The boundary terms are absent due to the compactness of the quotient.

1.2 Hörmander systems

Fix a finite subset $\mathcal{I} = \{X_1, \dots, X_l\}$ of $\mathfrak{g}$.

Definition 1.2.1. We say that $\mathcal{I}$ is a Hörmander system if $\langle \mathcal{I} \rangle = \mathfrak{g}$, where $\langle \mathcal{I} \rangle$ is the smallest Lie subalgebra containing $\mathcal{I}$.

We now define some differential operators on $M = K \backslash G$ adapted to a Hörmander system $\mathcal{I}$. First of all we define a gradient:

Definition 1.2.2. Let $f \in C^\infty(M)$. The $\mathcal{I}$-gradient of f is defined as

$$\nabla_{\mathcal{I}} f(x) = (X_1 f(x), \dots, X_l f(x)) \tag{1.1}$$

for all $x \in M$.

Next we define a divergence operator:

Definition 1.2.3. Let $F \in (C^\infty(M))^l$. The $\mathcal{I}$-divergence of F is defined as

$$\operatorname{div}_{\mathcal{I}} F(x) = \sum_{i=1}^{l} X_i F_i(x) \tag{1.2}$$

for all $x \in M$.

Remark 1.2.4. We call the operator just defined $\mathcal{I}$-divergence since it is the adjoint of the $\mathcal{I}$-gradient. Indeed, for $f \in C^\infty(M)$ and $G \in (C^\infty(M))^l$ we have

$$\begin{aligned}\int_M (\nabla_{\mathcal{I}} f) \cdot G d\mu &= \int_M \sum_{i=1}^{l} (X_i f)\, G_i d\mu \\ &= -\int_M f \left(\sum_{i=1}^{l} (X_i G_i) \right) d\mu = -\int_M f \operatorname{div}_{\mathcal{I}} G d\mu,\end{aligned}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^l$ and where we used Proposition 1.1.1.

Finally we define a sum of squares operator which we will sometimes refer to as sub-Laplacian.

Definition 1.2.5. Let $f \in C^\infty(M)$. The $\mathcal{I}$-sub-Laplacian of f is defined as

$$L_{\mathcal{I}} f(x) = \sum_{i=1}^{l} X_i^2 f(x) \tag{1.3}$$

for all $x \in M$.

Remark 1.2.6. We call the operator $L_\mathcal{I}$ defined in (1.3) sub-Laplacian since it can be understood as $\operatorname{div}_\mathcal{I} \nabla_\mathcal{I}$. Indeed, for $f \in C^\infty(M)$ we have

$$\operatorname{div}_\mathcal{I} \nabla_\mathcal{I} f = \sum_{i=1}^{l} X_i \left(\nabla_\mathcal{I} f\right)_i = \sum_{i=1}^{l} X_i^2 f.$$

Consider the Hilbert space $L^2(M)$ with respect to the measure $d\mu$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(M)$. The operator $L_\mathcal{I}$ is initially defined in the subspace $C^\infty(M)$, which is dense in $L^2(M)$ (recall that M is compact).

Proposition 1.2.7. *The operator* $-L_\mathcal{I}$ *is symmetric and positive.*

Proof. Let $f, g \in C^\infty(M)$. It suffices to prove that each operator $-X_i^2$ is symmetric and positive. By the integration by parts formula we have

$$-\langle X_i^2 f, g\rangle = \langle X_i f, X_i g\rangle = -\langle f, X_i^2 g\rangle,$$

so $-X_i^2$ is symmetric. Moreover

$$-\langle X_i^2 f, f\rangle = \langle X_i f, X_i f\rangle = \|X_i f\|^2_{L^2(M)} \geq 0,$$

hence $-L_\mathcal{I}$ is positive. □

Since the vector fields in $\mathcal{I}$ satisfy Condition 1.2.1 the operator $L_\mathcal{I}$ is hypoelliptic by Hörmander's theorem. By Nelson's theorem (see [46]) we conclude that the operator $L_\mathcal{I}$ is essentially self-adjoint. Moreover, since M is compact $-L_\mathcal{I}$ has a real discrete nonnegative spectrum $\Sigma \subset \mathbb{R}^+$ with eigenvalues, counted with multiplicity,

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \ldots$$

with $\lambda_k \to \infty$ as $k \to \infty$.
The associated L^2-normalized eigenfunctions φ_i form a complete orthonormal system for $L^2(M)$. Since the operator $L_\mathcal{I}$ is hypoelliptic, the eigenfunctions are $C^\infty(M)$ and in particular they are bounded. Note that λ_0 has multiplicity 1 and $\varphi_0 = 1$.
The spectral theorem provides a functional calculus for the operator $-L_\mathcal{I}$: if $m \in L^\infty(\Sigma)$, the operator $m(-L_\mathcal{I})$ defined by

$$m(-L_\mathcal{I})f = \sum_{i=0}^{\infty} m(\lambda_i)\langle f, \varphi_i\rangle \varphi_i$$

is bounded on $L^2(M)$.

1.3 The heat flow

We consider the Cauchy problem for the heat equation on M associated to $L_\mathcal{I}$ with initial datum f,

$$\begin{cases} \partial_t u(t,x) = L_\mathcal{I} u(t,x) & (t,x) \in \mathbb{R}^+ \times M \\ u(0,x) = f(x) & x \in M. \end{cases} \tag{1.4}$$

It is known (see [59]) that for all $t > 0$ the solution at time t of the heat equation with initial datum $f \in C^\infty(M)$ is obtained by applying the heat semigroup $e^{tL_\mathcal{I}}$, which is given by the multiplier $e^{-t(\cdot)} : \Sigma \to \mathbb{R}^+$. Explicitly we have

$$e^{tL_\mathcal{I}} f = \sum_{i=0}^{\infty} e^{-t\lambda_i}\langle f, \varphi_i\rangle \varphi_i. \tag{1.5}$$

Remark 1.3.1. Expression (1.5) makes sense also for $f \in L^2$. Indeed one can prove, by Lebesgue's dominated convergence theorem, that $e^{tL_{\mathcal{I}}} f$ is $C^\infty(M)$ for all $t > 0$. If f is continuous, the initial datum is recovered in a limit sense:

$$\lim_{t \to 0} e^{tL_{\mathcal{I}}} f(x) = f(x),$$

for all $x \in M$.

Proposition 1.3.2. *Let $u : \mathbb{R}^+ \times M \to \mathbb{C}$ be a solution of (1.4) with initial datum $f \in C^\infty(M)$. Then the following properties hold.*

1. *If f is nonnegative, $u(t,x)$ is strictly positive for $t > 0$.*
2. *The total mass of the solution is preserved at each time $t > 0$:*

$$\int_M u(t,x) d\mu = \int_M f d\mu.$$

3. *The operators $e^{tL_{\mathcal{I}}}$ enjoy the semigroup property, i.e. $e^{(t+s)tL_{\mathcal{I}}} = e^{tL_{\mathcal{I}}} e^{sL_{\mathcal{I}}}$ for $t, s > 0$ or, equivalently, $u(t+s,x) = u(s, u(t,x))$.*
4. *The solution at each point $x \in M$ converges to the mean of the initial datum, i.e.*

$$\lim_{t \to \infty} u(t,x) = \int_M f d\mu.$$

Proof. Properties (1) and (2) follow from Hunt's theorem (see [34]) for the group G and pass to the quotient M (see [44, Section 2.5]). Property (3) is obvious. Property (4) follows from the fact that 0 is an eigenvalue for $L_{\mathcal{I}}$ with constant eigenfunction $\varphi_0 = 1$ and that (1.5) converges to $\langle f, \varphi_0 \rangle = \int_M f d\mu$ as $t \to \infty$. □

In particular, notice by Property (4) that if f is nonnegative, we have

$$\lim_{t \to \infty} u(t,x) = \|f\|_{L^1(M)}.$$

Now fix $p \geq 1$ and consider the nonlinear evolution for a nonnegative $f \in C^\infty(M)$ given by

$$v(t,x) = \left(e^{tL_{\mathcal{I}}} f^p \right)^{1/p}(x). \tag{1.6}$$

We say that $v(t,x)$ is a nonlinear evolution because when $p > 1$ it satisfies a nonlinear equation, indeed

$$\begin{aligned} \partial_t v(t,x) &= \partial_t \left(e^{tL_{\mathcal{I}}} f^p \right)^{1/p} = \frac{1}{p} \left(e^{tL_{\mathcal{I}}} f^p \right)^{\frac{1-p}{p}} L_{\mathcal{I}} e^{tL_{\mathcal{I}}} f^p \\ &= \frac{1}{p} v^{1-p}(t,x) L_{\mathcal{I}} v^p(t,x). \end{aligned}$$

Since $L_{\mathcal{I}} = \mathrm{div}_{\mathcal{I}}(\nabla_{\mathcal{I}})$, we have that

$$\nabla_{\mathcal{I}} v^p(t,x) = p v^{p-1}(t,x) \nabla_{\mathcal{I}} v(t,x),$$

so that

$$\begin{aligned} L_{\mathcal{I}} v^p = \operatorname{div}_{\mathcal{I}}(\nabla_{\mathcal{I}} v^p) &= p \nabla_{\mathcal{I}} v^{p-1} \cdot \nabla_{\mathcal{I}} v + p v^{p-1} L_{\mathcal{I}} v \\ &= p(p-1) v^{p-2} |\nabla_{\mathcal{I}} v|^2 + p v^{p-1} L_{\mathcal{I}} v, \end{aligned}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^l$ and $|\cdot|$ the associated norm. Hence we see that $v(t,x)$ solves the nonlinear equation

$$\partial_t v(t,x) = (p-1) \frac{|\nabla_{\mathcal{I}} v(t,x)|^2}{v(t,x)} + L_{\mathcal{I}} v(t,x). \tag{1.7}$$

We have an analog of Proposition 1.3.2 for the nonlinear evolution (1.6).

Proposition 1.3.3. *Let $v : \mathbb{R}^+ \times M \to \mathbb{C}$ be a solution of equation* (1.6) *with initial datum $f \in C^\infty(M)$. Then the following properties hold.*

1. *If f is nonnegative, $v(t,x)$ is strictly positive for every $t > 0$.*
2. *The L^p mass of the solution is preserved at each time $t > 0$:*
$$\int_M v(t,x)^p d\mu = \int_M f^p d\mu.$$
3. *The operators $\left(e^{tL_{\mathcal{I}}}(\cdot)^p\right)^{1/p}$ enjoy the semigroup property, i.e. $v(t+s,x) = v(s, v(t,x))$.*
4. *The solution converges to the $L^p(M)$ norm of the initial datum at each point $x \in M$, i.e.*
$$\lim_{t\to\infty} v(t,x) = \left(\int_M f^p d\mu\right)^{1/p}.$$

Proof. All the properties are easy consequences of the properties stated in Proposition 1.3.2 applied to the function v^p, which is a solution of the linear heat equation (1.4) with initial datum f^p. $\square$

1.4 A monotonicity result

Fix $p \geq 1$ and $m \in \mathbb{N}$. Consider a set $\{f_1, \ldots, f_m\}$ of nonnegative smooth functions on M and the associated nonlinear flows

$$v_i(t,x) = \left(e^{tL_{\mathcal{I}}} f_i^p\right)^{1/p}(x). \tag{1.8}$$

For fixed $t > 0$ consider the function of t given by the integration over M of the product of the nonlinear evolutions $v_i(t,x)$:

$$\phi(t) = \int_M \prod_{i=1}^m v_i(t,x) d\mu. \tag{1.9}$$

Lemma 1.4.1. *If the function* (1.9) *is nondecreasing, the following inequality holds:*

$$\int_M \prod_{i=1}^m f_i d\mu \leq \prod_{i=1}^m \|f_i\|_{L^p(M)}. \tag{1.10}$$

Proof. Since $\phi(t)$ is nondecreasing we have

$$\lim_{t\to 0} \phi(t) \leq \lim_{t\to\infty} \phi(t). \tag{1.11}$$

By Remark 1.3.1, on the left-hand side we obtain the integral of the product of the initial data. For the right-hand side, by Property (4) of Proposition 1.3.3, each $v_i(t,x)$ converges to $\|f_i\|_{L^p(M)}$ and the result follows since $\int_M 1 d\mu = 1$. □

Remark 1.4.2. Since the space M is compact, the best constant in inequality (1.10) is 1 and is attained for constant functions, since for a nonnegative constant $a \in \mathbb{R}$, $\|a\|_{L^p(M)} = a$ for all $1 \leq p \leq \infty$, by the fact that $d\mu(M) = 1$.

Remark 1.4.3. At first sight the estimate (1.10) looks like Hölder's inequality, but it is not, since in (1.10) there are in general no constraints on the exponent p. In particular $\sum_{i=1}^m p_i^{-1}$ need not be 1. In fact, proving the monotonicity of ϕ under certain assumptions that will be made clear later on, we will get exponents that do not satisfy Hölder's condition.

Let us first find an explicit formula for the time derivative of the function $\phi(t)$. Note that ϕ is differentiable in time. By Property (1) of Proposition 1.3.3, each $v_i(t,x)$ is strictly positive. We define, for $t > 0$ and $i = 1, \dots, m$,

$$\tilde{v}_i(t,x) = \log(v_i(t,x))$$

and

$$G(t,x) = \prod_{i=1}^m v_i(t,x).$$

Proposition 1.4.4. *Under the assumptions above we have*

$$\begin{aligned}\frac{d}{dt}\phi(t) &= (p-1)\sum_{i=1}^m \int_M \sum_{j=1}^l \left(X_j \tilde{v}_i(t,x)\right)^2 G(t,x) d\mu \\ &\quad - \sum_{i=1}^m \sum_{j=1}^l \sum_{\substack{k=1\\k\neq i}}^m \int_M \left(X_j \tilde{v}_i(t,x) X_j \tilde{v}_k(t,x)\right) G(t,x) d\mu.\end{aligned} \tag{1.12}$$

Proof. By the Leibniz rule and (1.7) we have

$$\begin{aligned}\frac{d}{dt}\phi(t) &= \frac{d}{dt}\int_M \prod_{i=1}^m v_i(t,x) d\mu \\ &= \int_M \partial_t \prod_{i=1}^m v_i(t,x) d\mu = \sum_{i=1}^m \int_M \partial_t v_i(t,x) \prod_{\substack{j=1\\j\neq i}}^m v_j(t,x) d\mu \\ &= \sum_{i=1}^m \int_M \left((p-1)\frac{|\nabla_{L_\mathcal{I}} v_i(t,x)|^2}{v_i(t,x)} + L_\mathcal{I} v_i(t,x)\right) \prod_{\substack{j=1\\j\neq i}}^m v_j(t,x) d\mu.\end{aligned} \tag{1.13}$$

We split each integral in the sum into two pieces:

$$\begin{aligned}\int_M (p-1)\frac{|\nabla_{L_\mathcal{I}} v_i(t,x)|^2}{v_i(t,x)} \prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)d\mu + \int_M L_\mathcal{I} v_i(t,x) \prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)d\mu \\ =: I_i(t) + II_i(t).\end{aligned}$$

For $I_i(t)$ we have

$$\begin{aligned}I_i(t) &= (p-1)\int_M \frac{|\nabla_{L_\mathcal{I}} v_i(t,x)|^2}{v_i(t,x)} \prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)d\mu \\ &= (p-1)\int_M \frac{|\nabla_{L_\mathcal{I}} v_i(t,x)|^2}{v_i(t,x)^2} G(t,x)d\mu = (p-1)\int_M |\nabla_{L_\mathcal{I}} \tilde{v}_i(t,x)|^2 G(t,x)d\mu \\ &= (p-1)\int_M \sum_{j=1}^l (X_j\tilde{v}_i(t,x))^2 G(t,x)d\mu.\end{aligned}$$

For $II_i(t)$, integrating by parts, we obtain:

$$\begin{aligned}II_i(t) &= \int_M L_\mathcal{I} v_i(t,x) \prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)d\mu \\ &= \sum_{j=1}^l \int_M X_j^2 v_i(t,x) \prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)d\mu \\ &= -\sum_{j=1}^l \int_M X_j v_i(t,x) X_j \left(\prod_{\substack{j=1\\ j\neq i}}^m v_j(t,x)\right) d\mu,\end{aligned}$$

which, using again the Leibniz rule, gives

$$\begin{aligned}II_i(t) &= -\sum_{j=1}^l \int_M X_j v_i(t,x) \sum_{\substack{k=1\\ k\neq i}}^m \left(X_j v_k(t,x) \prod_{\substack{k'=1\\ k'\neq i,k}}^m v_{k'}(t,x)\right) d\mu \\ &= -\sum_{j=1}^l \sum_{\substack{k=1\\ k\neq i}}^m \int_M \frac{X_j v_i(t,x)}{v_i(t,x)} \frac{X_j v_k(t,x)}{v_k(t,x)} \left(\prod_{k'=1}^m v_{k'}(t,x)\right) d\mu \\ &= -\sum_{j=1}^l \sum_{\substack{k=1\\ k\neq i}}^m \int_M X_j\tilde{v}_i(t,x) X_j \tilde{v}_k(t,x) G(t,x)d\mu.\end{aligned}$$

Finally, taking the sum,

$$\sum_{i=1}^m \left(I_i(t) + II_i(t)\right),$$

we obtain the result. □

Remark 1.4.5. The time derivative of ϕ can be equivalently written as

$$\frac{d}{dt}\phi(t) = (p-1)\sum_{i=1}^{m}\int_M \sum_{j=1}^{l} \left(X_j\tilde{v}_i(t,x)\right)^2 G(t,x)d\mu$$
$$-2\sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{k=1}^{i-1}\int_M \left(X_j\tilde{v}_i(t,x)X_j\tilde{v}_k(t,x)\right)G(t,x)d\mu. \tag{1.14}$$

We observe that this expression contains all possible square type terms $(X_j\tilde{v}_i)^2$, and all possible double products $2X_j\tilde{v}_iX_j\tilde{v}_k$ for $j=1,\dots,l$ and $i,k=1,\dots,m$, with $i<k$.

One could allow each nonnegative $f_i \in C^\infty(M)$ to evolve with a different nonlinear evolution. Indeed, one could choose a different $p_i \geq 1$ for each f_i and define

$$v_i(t,x) = \left(e^{tL_{\mathcal{I}}}f^{p_i}\right)^{1/p_i}(x).$$

Concerning this point, we state a simple modification of Proposition 1.4.4.

Proposition 1.4.6. *In the hypotheses above we have*

$$\frac{d}{dt}\phi(t) = \sum_{i=1}^{m}\int_M (p_i-1)\sum_{j=1}^{l}\left(X_j\tilde{v}_i(t,x)\right)^2 G(t,x)d\mu$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq i}}^{m}\int_M \left(X_j\tilde{v}_i(t,x)X_j\tilde{v}_k(t,x)\right)G(t,x)d\mu. \tag{1.15}$$

Proof. The proof is the same as for Proposition 1.4.4, once noted that each $v_i(t,x)$ solves the equation

$$\partial_t v_i(t,x) = (p_i-1)\frac{|\nabla_{\mathcal{I}}v_i(t,x)|^2}{v_i(t,x)} + L_{\mathcal{I}}v_i(t,x).$$

□

As a simple corollary of Proposition 1.4.4 we obtain multilinear Hölder's inequality for a restricted range of exponents.

Corollary 1.4.7 (Multilinear Hölder's inequality)**.** *Let $f_i \in C^\infty(M)$ be nonnegative functions, for $i=1,\dots,m$. Then we have*

$$\int_M \prod_{i=1}^{m} f_i d\mu \leq \prod_{i=1}^{m}\|f_i\|_{L^m(M)}.$$

Proof. Fix $p \geq 1$. Let $v_i(t,x)$ be the evolution under the nonlinear flow of the functions $f_1,\dots,f_m$ as in (1.6), with the same exponent p for all $i=1,\dots,m$. Consider the function $\phi(t)$ defined in (1.9). From (1.14) we see that, for fixed X_j there are $m(m-1)/2$ double product terms $2X_j\tilde{v}_iX_j\tilde{v}_k$, with $i<k$. In particular, each term $X_j\tilde{v}_i$ appears $m-1$ times in the double products. If we take $p=m$ we have

$$\frac{d}{dt}\phi(t) = \sum_{i=1}^{m}\sum_{j=1}^{l}\int_M (m-1)\left(X_j\tilde{v}_i(t,x)\right)^2 G(t,x)d\mu$$

$$-2\sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{k=1}^{i-1}\int_M \left(X_j\tilde{v}_i(t,x)X_j\tilde{v}_k(t,x)\right)G(t,x)d\mu$$
$$=\sum_{k<i}\int_M\sum_{j=1}^{l}\left(X_j\tilde{v}_i-X_j\tilde{v}_k\right)^2G(t,x)d\mu\geq 0,$$

since $G(t,x)\geq 0$. So for $p=m$ the function ϕ is nondecreasing. The conclusion follows from Lemma 1.4.1. $\square$

More generally, looking at the proof of the result above, we note that, allowing the exponents p_i to be different, as in Proposition 1.4.6, and taking $p_i\geq m$, the time derivative of function (1.9) can be arranged in the form

$$\begin{aligned}\frac{d}{dt}\phi(t)&=\sum_{i=1}^{m}\sum_{j=1}^{p}\int_M(p_i-m)(X_j\tilde{v}_i(t,x))^2G(t,x)d\mu\\&+\sum_{i=1}^{m}\sum_{j=1}^{l}\int_M(m-1)\left(X_j\tilde{v}_i(t,x)\right)^2G(t,x)d\mu\\&-2\sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{k=1}^{i-1}\int_M\left(X_j\tilde{v}_i(t,x)X_j\tilde{v}_k(t,x)\right)G(t,x)d\mu\\&=\sum_{i=1}^{m}\sum_{j=1}^{l}\int_M(p_i-m)(X_j\tilde{v}_i(t,x))^2G(t,x)d\mu\\&+\sum_{k<i}\int_M\sum_{j=1}^{l}\left(X_j\tilde{v}_i-X_j\tilde{v}_k\right)^2G(t,x)d\mu\geq 0,\end{aligned}$$

since both summands are nonnegative.
Hence we can formulate the following immediate corollary.

Corollary 1.4.8. *Let $f_i\in C^\infty(M)$ be nonnegative functions, for $i=1,\ldots,m$, and $p_i\geq m$ for all i. Then we have*

$$\int_M\prod_{i=1}^{m}f_id\mu\leq\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(M)}.$$

Remark 1.4.9. Corollary 1.4.8 does not come as a surprise. Indeed, since M has finite measure, we have continuous embeddings of $L^p(M)$ spaces as p grows. Precisely we have that

$$L^p(M)\hookrightarrow L^q(M)$$

whenever $p\leq q$, with operator norm 1, since the measure $d\mu$ is normalized. Taking into account this remark, Corollary 1.4.8 is a straightforward consequence of Corollary 1.4.7.

We give the following definition which will be useful in what follows.

Definition 1.4.10. Let $f_1,\ldots,f_m$ be nonnegative measurable functions and $p_i\geq 1$ for $i=1,\ldots,m$. We say that the inequality

$$\int_M\prod_{i=1}^{m}f_id\mu\leq\prod_{i=1}^{m}\|f_i\|_{L^{p_i}}$$

is nontrivial if $\sum_{i=1}^{m}p_i^{-1}>1$, i.e. if it does not follow directly from Hölder's inequality and continuous embeddings of Lebesgue spaces.

1.5 Functions with symmetries

As the proof of Corollary 1.4.7 suggests, the choice of the exponent p depends in a combinatorial fashion on the number of vector fields and on the number of functions. Corollary 1.4.7 from this point of view represents the worst case, in which one considers all vector fields of the family $\mathcal{I}$ applied to all functions. In what follows we will investigate the cases where some of the functions are annihilated by a subset of the vector fields in the family $\mathcal{I}$.

Definition 1.5.1. Let $\mathcal{A} \subseteq \mathfrak{g}$. We say that a function $f \in C^\infty(M)$ is $\mathcal{A}$-symmetric if $Xf = 0$ for all $X \in \mathcal{A}$. We denote with $C^\infty_{\mathcal{A}}(M)$ the space of $\mathcal{A}$-symmetric functions, which is also an algebra with respect to pointwise multiplication.

Remark 1.5.2. By the simple observation that if X, Y are such that $Xf = 0$ and $Yf = 0$, then also $[X, Y]f = 0$, we see that if a function is $\mathcal{A}$-symmetric, then it is also $\langle\mathcal{A}\rangle$-symmetric, where $\langle\mathcal{A}\rangle$ is the smallest subalgebra of $\mathfrak{g}$ containing $\mathcal{A}$. In particular, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of $\mathfrak{g}$ such that $\langle\mathcal{A}_1\rangle = \langle\mathcal{A}_2\rangle$, we have that $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$.

Functions that are $\mathcal{A}$-symmetric enjoy invariance properties on the group G, and hence on the manifold M.

Definition 1.5.3. Let $G' < G$ be a Lie subgroup of G. We say that $f \in C^\infty(G)$ is G'-invariant if $f(gg') = f(g)$ for all $g' \in G'$.

Lemma 1.5.4. *Let $G' < G$ be a Lie subgroup of G and $\mathfrak{g}'$ its Lie algebra, which is a Lie subalgebra of $\mathfrak{g}$. Let $f \in C^\infty(G)$. Then $f \in C^\infty_{\mathfrak{g}'}(G)$ if and only if f is G'-invariant.*

Proof. Suppose that f is G'-invariant. Let $X \in \mathfrak{g}'$. Observing that $\exp(tX) \in G'$, we have

$$\begin{aligned} Xf(x) &= \frac{d}{dt} f(x \exp(tX))|_{t=0} \\ &= \frac{d}{dt} f(x)|_{t=0} = 0. \end{aligned}$$

Conversely, let $f \in C^\infty_{\mathfrak{g}'}$. Then, for t small enough, we have

$$f(x \exp(tX)) = f(x),$$

for all $X \in \mathfrak{g}'$. The result follows since exp is a local diffeomorphism between $\mathfrak{g}'$ and G' and G' is connected. □

Definition 1.5.5. Let $\mathcal{A} \subset \mathfrak{g}$ and $\mathcal{I}$ be a Hörmander system. We say that $\mathcal{A}$ is an $\mathcal{I}$-set if $L_{\mathcal{I}}$ commutes with all the elements in $\mathcal{A}$, i.e. $[L_{\mathcal{I}}, X] = L_{\mathcal{I}}X - XL_{\mathcal{I}}=0$.

In this definition the bracket is extended to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by the Poincaré–Birkhoff–Witt Theorem.

Remark 1.5.6. If $\mathcal{A}$ is a $\mathcal{I}$-set, then so is $\langle\mathcal{A}\rangle$, so it is enough to test the commutativity on the set $\mathcal{A}$ rather than on the whole generated subalgebra $\langle\mathcal{A}\rangle$. Indeed, if $L_{\mathcal{I}}$ commutes with $X, Y \in \mathcal{A}$, we have

$$\begin{aligned} [L_{\mathcal{I}}, [X, Y]] &= L_{\mathcal{I}}[X, Y] - [X, Y]L_{\mathcal{I}} \\ &= L_{\mathcal{I}}XY - L_{\mathcal{I}}YX - [X, Y]L_{\mathcal{I}} \\ &= XL_{\mathcal{I}}Y - YL_{\mathcal{I}}X - [X, Y]L_{\mathcal{I}} \end{aligned}$$

$$
\begin{aligned}
&= XYL_{\mathcal{I}} - YXL_{\mathcal{I}}YX - [X,Y]L_{\mathcal{I}} \\
&= 0.
\end{aligned}
$$

Alternatively, this can be seen as a consequence of the Jacobi identity for the bracket product extended to the universal enveloping algebra.

For a vector field that commutes with the operator $L_{\mathcal{I}}$ we have the following lemma.

Lemma 1.5.7. *Let $X \in \mathfrak{g}$ such that $[L_{\mathcal{I}}, X] = 0$, then $e^{tL_{\mathcal{I}}}X = Xe^{tL_{\mathcal{I}}}$ for all $t > 0$.*

Proof. Let $f \in C^\infty(M)$. Consider the map

$$\psi(s) = e^{sL_{\mathcal{I}}} X e^{(t-s)L_{\mathcal{I}}} f.$$

Computing the derivative in s we get

$$\frac{d}{ds}\psi(s) = e^{sL_{\mathcal{I}}}(L_{\mathcal{I}}X - XL_{\mathcal{I}})e^{(t-s)L_{\mathcal{I}}} f = 0.$$

Thus $\psi(0) = \psi(t)$, which yields

$$Xe^{tL_{\mathcal{I}}} f = e^{tL_{\mathcal{I}}} X f.$$

□

We will be interested in functions that are $\mathcal{A}$-symmetric, with $\mathcal{A}$ some $\mathcal{I}$-set. For these functions we have the following proposition.

Proposition 1.5.8. *Let $\mathcal{A} \subseteq \mathfrak{g}$ be an $\mathcal{I}$-set and let $f \in C^\infty_{\mathcal{A}}(M)$. Then $e^{tL_{\mathcal{I}}} f \in C^\infty_{\mathcal{A}}(M)$ for all $t > 0$.*

Proof. This is a direct consequence of Lemma 1.5.7. □

Remark 1.5.9. It is easy to see that an analog of Proposition 1.5.8 also holds for the nonlinear flow $\left(e^{tL_{\mathcal{I}}} f^p\right)^{1/p}$, for $p > 1$, with $f \in C^\infty_{\mathcal{A}}(M)$ nonnegative. Indeed, under the assumptions above, if $X \in \mathcal{A}$, we have

$$X\left(e^{tL_{\mathcal{I}}} f^p\right)^{1/p} = \frac{1}{p}\left(e^{tL_{\mathcal{I}}} f^p\right)^{\frac{1-p}{p}} X e^{tL_{\mathcal{I}}} f^p = 0,$$

since $[X, e^{tL_{\mathcal{I}}}] = 0$ and $f^p \in C^\infty_{\mathcal{A}}(M)$. Then $\left(e^{tL_{\mathcal{I}}} f^p\right)^{1/p} \in C^\infty_{\mathcal{A}}(M)$.

Thus the heat flow preserves the symmetry: if $\mathcal{A}$ is an $\mathcal{I}$-set and if the initial datum is $\mathcal{A}$-symmetric, then so is its evolution, either linear or nonlinear, under the heat equation.
Given a subalgebra $\mathcal{A} \subseteq \mathfrak{g}$ we can consider the vector space of functions $C^\infty_{\mathcal{A}}(M)$ that are annihilated by all vector fields in $\mathcal{A}$. The Lie algebra $\mathfrak{g}$ has a (nonunique) direct sum decomposition as a vector space:

$$\mathfrak{g} = \mathcal{A} \oplus \mathcal{B},$$

where $\mathcal{B}$ is a vector subspace of $\mathfrak{g}$.
As we saw in Proposition 1.4.4, for the task of proving the inequalities we are interested in, it suffices to take into account only the action of vector fields in the system $\mathcal{I}$. So we can only consider $\mathcal{I} \cap \mathcal{A}$ or we may as well consider subalgebras generated by subsets of vectors in $\mathcal{I}$. Different subsets of $\mathcal{I}$ could generate the same subalgebra and we do not want to make a distinction between them. This leads us to the following definition.

Definition 1.5.10. Let $\mathcal{A} \subseteq \mathcal{I}$. We say that $\mathcal{A}$ is maximal in $\mathcal{I}$ if for every $\mathcal{A}' \subseteq \mathcal{I}$ such that $\langle \mathcal{A} \rangle = \langle \mathcal{A}' \rangle$, we have that $\mathcal{A}' \subseteq \mathcal{A}$.

Lemma 1.5.11. *Let $\mathcal{A} \subseteq \mathcal{I}$. Then $\langle \mathcal{A} \rangle \cap \mathcal{I}$ is maximal in $\mathcal{I}$.*

Proof. The proof is straightforward, since $\langle \mathcal{A} \rangle \cap \mathcal{I}$ contains all possible linear combinations and brackets of elements in $\mathcal{A}$ that are in $\mathcal{I}$. □

From the point of view of functions, for each function $f \in C^\infty(M)$ we have a maximal subset of $\mathcal{I}$, which we denote by $\mathcal{A}_f$, such that for $X \in \mathcal{A}_f$, $Xf = 0$, and for $Y \in \mathcal{I} \backslash \mathcal{A}_f$, $Yf \neq 0$. With this observation we can reduce the symmetry property to a matter of subsets of the Hörmander system $\mathcal{I}$. Indeed, for every function f, $\mathcal{I}$ decomposes as the disjoint union

$$\mathcal{I} = \mathcal{A}_f \sqcup \mathcal{A}_f^c,$$

where the complement is taken with respect to $\mathcal{I}$.
Let us introduce a notation. If $A_1, \dots, A_m$ are finite sets, for a multi-index $\boldsymbol{j} = (j_1, \dots, j_m) \in \{0,1\}^m$ we denote by

$$\bigcap_{\boldsymbol{j}} A_i = \bigcap_{i\,:\,j_i=1} A_i$$

the intersection of the sets A_i such that $j_i = 1$.

1.6 Inequalities for functions with symmetries

We are now ready to state the main result of this chapter.

Theorem 1.6.1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be maximal subsets of $\mathcal{I}$ that are $\mathcal{I}$-sets. Let $f_i \in C^\infty_{\mathcal{A}_i}(M)$ be nonnegative functions, for $i = 1, \dots, m$. The following inequality holds*

$$\int_M \prod_{i=1}^m f_i d\mu \leq \prod_{i=1}^m \|f_i\|_{L^p(M)}$$

for $p \geq \tilde{p}$, where $\tilde{p}$ is the number of occurrences of the most recurrent element among the finite sets $\mathcal{A}_i^c$, i.e.

$$\tilde{p} = \max_{a \in \cup_i \mathcal{A}_i^c} \max_{\boldsymbol{j}\,:\,\cap_{\boldsymbol{j}} \mathcal{A}_i^c \ni a} |\boldsymbol{j}|.$$

Proof. Since the functions f_i are $\mathcal{A}_i$-symmetric, and the sets $\mathcal{A}_i$ are $\mathcal{I}$-sets, by Proposition 1.5.8 the nonlinear evolutions, defined in (1.8), are also $\mathcal{A}_i$-symmetric. By Proposition 1.4.4 all possible double products of the form $X_i \tilde{v}_j X_i \tilde{v}_k$, with $i < k$ and $X_i \in \mathcal{A}_j^c \cap \mathcal{A}_k^c$, will appear in the time derivative of ϕ. Recall that $\tilde{p} - 1$ depends on how many square type elements are needed to complete the squares. In order to have positive derivative, we will need $\tilde{p} - 1$ to be at least as big as the number of occurrences of the most recurrent vector field among the $\mathcal{A}_i^c$. □

It is easy to deduce the following sufficient condition to obtain a nontrivial inequality in the sense of Definition 1.4.10.

Proposition 1.6.1. *Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be maximal subsets of $\mathcal{I}$ that are $\mathcal{I}$-sets. To obtain a nontrivial inequality from the nonlinear heat flow associated to $L_{\mathcal{I}}$ the condition*

$$\bigcap_{i=1}^m \mathcal{A}_i^c = \emptyset \tag{1.16}$$

must be fulfilled. Condition (1.16) *is equivalent to*

$$\bigcup_{i=1}^{m} \mathcal{A}_i = \mathcal{I}. \tag{1.17}$$

Proof. The nontriviality condition when all exponents are the same reads

$$\sum_{i=1}^{m} p^{-1} > 1,$$

which implies that $p < m$. By Theorem 1.6.1 we have a nontrivial inequality if $\tilde{p} < m$ and this happens if no elements of $\mathcal{I}$ appear in all the sets $\mathcal{A}_i^c$, yielding condition (1.16). □

So far we considered the case where all functions evolve under the same nonlinear flow, i.e. with the same exponent $p \geq 1$. If we consider different exponents p_i for different evolutions we have the following result.

Theorem 1.6.2. *Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be maximal subsets of $\mathcal{I}$ that are $\mathcal{I}$-sets. Let $f_i \in C^\infty_{\mathcal{A}_i}(M)$ be nonnegative functions, for $i = 1, \dots, m$. The following inequality holds*

$$\int_M \prod_{i=1}^{m} f_i d\mu \leq \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(M)}$$

for $p_i \geq \tilde{p}_i$, where $\tilde{p}_i$ is the number of occurrences of the most recurrent element of $\mathcal{A}_i^c$ among the finite sets $\mathcal{A}_k^c$, i.e.

$$\tilde{p}_i = \max_{a \in \mathcal{A}_i^c} \max_{\boldsymbol{j} : \cap_j \mathcal{A}_k^c \ni a} |\boldsymbol{j}|.$$

Proof. The proof follows by the same argument as the proof of Theorem 1.6.1. □

1.7 The abelian case

As a first example, in this section we analyze the inequalities discussed in the previous section when the Lie group is $(\mathbb{R}^n, +)$. We fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and consider the corresponding Cartesian coordinates $(x_1, \dots, x_n)$. We take the quotient by the discrete subgroup $\mathbb{Z}^n$,

$$\mathbb{R}^n / \mathbb{Z}^n = \mathbb{T}^n,$$

where $\mathbb{T}^n$ is the n-dimensional torus, which can be understood as the cube $[0,1]^n$ in $\mathbb{R}^n$ with identifications of opposite sides.
The Lie algebra of $\mathbb{R}^n$ is generated by the vector fields

$$X_i = \partial_{x_i}$$

for $i = 1, \dots, n$. Clearly

$$[X_i, X_j] = 0$$

for all $i, j = 1, \dots, n$.
In this setting a Hörmander system of vector fields must necessarily contain a basis for the Lie algebra, since all commutators are trivial. So let

$$\mathcal{I} = \{Y_1, \dots, Y_l\}$$

with $l \geq n$ and let $\{Y_1, \dots, Y_n\}$ be a basis for the Lie algebra. Obviously each Y_j can be written as

$$Y_j = \sum_{k=1}^{n} a_{j,k} X_k, \tag{1.18}$$

with $a_{j,k} \in \mathbb{R}$. We denote by a^j the vector consisting of the components of Y_j in the basis $\{X_1, \dots, X_n\}$, i.e.

$$a^j = (a_{j,1}, \dots, a_{j,n})$$

as in (1.18). For vectors $b_1, \dots, b_n \in \mathbb{R}^n$, we denote by

$$\det(b_1, \dots, b_n)$$

the determinant of the matrix with b_i as the i-th column. Finally, for vectors $c_1, \dots, c_k$ we denote by

$$\operatorname{rank}(c_1, \dots, c_k)$$

the rank of the matrix with c_i as i-th column.
Note that by the assumption that $\{Y_1, \dots, Y_n\}$ is a basis for $\mathfrak{g}$, we have

$$\det(a^1, \dots, a^n) \neq 0.$$

In this abelian setting all subsets $\mathcal{A} \subseteq \mathcal{I}$ are $\mathcal{I}$-sets, since every two vectors commute. So we can pick any subset of $\mathcal{I}$ and we have the following proposition.

Proposition 1.7.1. *A subset $\mathcal{A} = \{Y_{i_1}, \dots, Y_{i_s}\}$ of $\mathcal{I}$, with $1 \leq i_1 < \cdots < i_s \leq l$, is maximal if and only if, for all $X \in \mathcal{I} \setminus \mathcal{A}$,*

$$\operatorname{rank}(Y_{i_1}, \dots, Y_{i_s}, X) \neq \operatorname{rank}(Y_{i_1}, \dots, Y_{i_s}). \tag{1.19}$$

Proof. We know that $\langle \mathcal{A} \rangle \cap \mathcal{I}$ is maximal. Since $\mathfrak{g}$ is abelian, vectors in $\langle \mathcal{A} \rangle \cap \mathcal{I}$ that are not in $\mathcal{A}$ are vectors of $\mathcal{I}$ that are linearly dependent from the vectors in $\mathcal{A}$. Condition (1.19) ensures that $\mathcal{A}$ already contains these vectors. □

Let us treat the case $l = n$, i.e. when $\mathcal{I}$ is a basis for $\mathfrak{g}$. In this case all subsets of $\mathcal{I}$ are maximal, so we have 2^n possible maximal subsets to which we can apply Theorem 1.6.1. If $\mathcal{A}$ is any subset, the vector space sum decomposition

$$\mathfrak{g} = \langle \mathcal{A} \rangle \oplus \langle \mathcal{A}^c \rangle$$

is also a Lie subalgebras decomposition, meaning that $[\langle \mathcal{A} \rangle, \langle \mathcal{A}^c \rangle] = \{0\}$. All subsets have maximal complement and we can directly consider the complements of the annihilating sets. Let us discuss the case where $\mathcal{I} = \{X_1, \dots, X_n\}$. Consider a subset $\mathcal{A} \subseteq \mathcal{I}$ given by $\mathcal{A} = \{X_{i_1}, \dots, X_{i_s}\}$ with $s \leq n$. The Lie subalgebra $\langle \mathcal{A} \rangle$ is just the vector subspace of $\mathfrak{g}$ given by

$$\operatorname{span}(X_{i_1}, \dots, X_{i_s}),$$

which corresponds to the Lie subgroup $\tilde{\mathcal{A}}$ given by

$$\operatorname{span}(e_{i_1}, \dots, e_{i_s}).$$

A nonnegative function

$$f = f(x_1, \dots, x_n)$$

on the torus $\mathbb{T}^n$ which is $\mathcal{A}$-symmetric is constant on translates by vectors in the Lie subgroup $\langle\mathcal{A}\rangle$, i.e.

$$f(x+v) = f(x),$$

for all $v \in \tilde{\mathcal{A}}$. In other words, the function f does not depend on the variables $x_{i_1}, \ldots, x_{i_s}$ and we can think of it as a function of the remaining $n-s$ variables. Suppose for simplicity that $i_j = j$, for $j = 1, \ldots, s$, then f can be identified with a function

$$F : \mathbb{R}^n/\mathbb{R}^s \simeq \mathbb{R}^{n-s} \to \mathbb{R}^+$$

such that $F(x_{s+1}, \ldots, x_n) = f(x_1, \ldots x_s, x_{s+1} \ldots, x_n)$ for any s-tuple $(x_1, \ldots, x_s)$. Equivalently we can write

$$F = f \circ \pi,$$

where $\pi : \mathbb{T}^n \to \mathbb{T}^{n-s}$ is the linear projection

$$\pi(x_1, \ldots, x_n) = (x_{s+1}, \ldots, x_n).$$

We follow the notation of [15], denoting with ω finite subsets of $\{1, \ldots, n\}$ and with x_ω the set of variables $\{x_{i_1}, \ldots, x_{i_{|\omega|}}\}$, where $\omega = \{i_1, \ldots, i_{|\omega|}\}$. We denote with f_ω a function only depending on x_ω. Note that

$$\int_{\mathbb{T}^{|\omega|}} f_\omega(x_\omega)dx_\omega = \int_{\mathbb{T}^n} f_\omega(x_\omega)dx_1 \ldots dx_n,$$

from which we get that

$$\|f_\omega\|_{L^p(\mathbb{T}^{|\omega|})} = \|f_\omega\|_{L^p(\mathbb{T}^n)}, \tag{1.20}$$

for all $p \geq 1$.
Let $C(n,k) := \binom{n}{k}$. We have the following proposition.

Proposition 1.7.2. *Let $\omega_1, \ldots, \omega_{C(n,k)}$ be the possible k-tuples of elements in $\{1, \ldots, n\}$, and let f_{ω_i} be nonnegative measurable functions only depending on the collection of variables ω_i. The inequality*

$$\int_{\mathbb{T}^n} \prod_{i=1}^{C(n,k)} f_{\omega_i}(x_{\omega_i})dx \leq \prod_{i=1}^{C(n,k)} \|f_{\omega_i}\|_{L^p(\mathbb{T}^n)} \tag{1.21}$$

holds, for

$$p \geq \tilde{p} = \binom{n-1}{k-1}.$$

Proof. In the language developed in this chapter, the sets $\mathcal{A}_i^c$ are given by $\{X_{i_1}, \ldots, X_{i_k}\}$ and they are in correspondence with the collection of variables x_{ω_i}, where $\omega_i = \{i_1, \ldots, i_k\}$. By Theorem 1.6.1 it suffices to check which is the number of occurrences of the most recurrent element among the $\mathcal{A}_i^c$, or, equivalently, the most recurrent variable x_l among the collections x_{ω_i}. It is easy to see that in this case every variable x_l appears exactly $\binom{n-1}{k-1}$ times. □

Remark 1.7.3. Proposition 1.7.2 is a local version of a result due to A. P. Calderón in [15] (see also the work of H. Finner [27] for further results). In the notation above, Calderón proved the inequality

$$\int_{\mathbb{R}^n} \prod_{i=1}^{C(n,k)} f_{\omega_i}(x_{\omega_i})dx \leq \prod_{i=1}^{C(n,k)} \|f_{\omega_i}\|_{L^{\tilde{p}}(\mathbb{R}^{|\omega_i|})},$$

with $\tilde{p} = \binom{n-1}{k-1}$, by induction on the cardinality k of the subsets ω_i. If the functions f_{ω_i} are supported in the unit cubes of $\mathbb{R}^{|\omega_i|}$, Calderón inequality becomes

$$\int_{\mathbb{T}^n} \prod_{i=1}^{C(n,k)} f_{\omega_i}(x_{\omega_i})dx \leq \prod_{i=1}^{C(n,k)} \|f_{\omega_i}\|_{L^{\tilde{p}}(\mathbb{T}^{|\omega_i|})},$$

which by (1.20) is equivalent to (1.21).

Remark 1.7.4. The case $k = n - 1$ is a local version of Loomis–Whitney inequality (see [43]).

All the estimates above (Calderón inequalities, Loomis–Whitney inequality and their local versions) can be proved by a smart iteration of Hölder's inequality.
Another way of proving this kind of inequalities is the heat flow method used in [7]. Recall that a geometric Brascamp–Lieb inequality is an estimate of the type

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(B_i x)dx \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^{n_i})}, \tag{1.22}$$

where $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are surjective linear maps such that B_i^* is an isometry, i.e. $B_i B_i^* = \mathrm{Id}_{\mathbb{R}^{n_i}}$, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$ are measurable functions, and the relation

$$\sum_{i=1} p_i^{-1} B_i^* B_i = \mathrm{Id}_{\mathbb{R}^n} \tag{1.23}$$

is satisfied. In [7] the authors prove that under condition (1.23), inequality (1.22) holds with $C = 1$. Restricting the supports of the functions to unit cubes in $\mathbb{R}^{n_i}$ this local version of the inequality obviously holds

$$\int_{\mathbb{T}^n} \prod_{i=1}^{m} f_i(B_i x)dx \leq \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{T}^{n_i})}, \tag{1.24}$$

under the same assumption (1.23).
Let us consider the case $m = C(n,k)$, $B_i = \pi_{\omega_i}$ being the projection onto the set of variables x_{ω_i}. The maps B_i^* are isometries, being inclusion maps. We have to check for what exponents p_i assumption (1.23) is satisfied.
It is easy to see that $B_i^* B_i$ is given by a diagonal matrix such that $(B_i^* B_i)_{jj} = 1$ if and only if $j \in \omega_i$, for $j = 1, \ldots, n$. Hence condition (1.23) requires that

$$\sum_{i:\ \omega_i \ni j} p_i^{-1} = 1$$

for $j = 1, \ldots, n$. Each sum is made by $\binom{n-1}{k-1}$ terms, so that condition (1.23) is certainly satisfied if $p_i^{-1} = \binom{n-1}{k-1}$ for all $i = 1, \ldots, C(n,k)$.
We note that the general condition (1.23) gives rise to exponents that are not covered by Proposition 1.7.2.

CHAPTER 2

The case of the sphere

In this chapter we will find inequalities for functions with some degree of symmetry on spheres of all dimensions $n > 2$. We consider the Euclidean space $\mathbb{R}^n$, with the standard scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis and $(x_1, \dots, x_n)$ the associated coordinates. The $(n-1)$-dimensional unit sphere is the set

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\},$$

which we endow with the normalized uniform measure $d\sigma$.
The sphere $\mathbb{S}^{n-1}$ can be seen as a homogeneous space of the special orthogonal group

$$\mathbb{S}^{n-1} = SO(n-1) \backslash SO(n),$$

where $SO(n-1)$ is intended as a closed subgroup of $SO(n)$ that fixes one direction. The measure $d\sigma$ is, up to normalization, the push-forward through the projection map on the quotient $\mathbb{S}^{n-1}$ of the bi-invariant Haar measure on $SO(n)$.

2.1 Functions depending on k variables

In what follows we will use cartesian coordinates to describe points on the sphere. In particular we will often write $f(x_1, \dots, x_n)$ for functions $f : \mathbb{S}^{n-1} \to \mathbb{R}$, implicitly assuming the condition $x_1^2 + \cdots + x_n^2 = 1$.
We will consider functions on the sphere depending on k variables, with $1 \le k \le n-1$. Let $\omega = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n\}$, with $|\omega| = k$ and let $x_\omega = (x_{i_1}, \dots, x_{i_k})$. We will use the notation $\omega^c = \{i_{k+1}, \dots, i_n\}$ for the complement of ω in $\{1, \dots, n\}$.
Consider the projection

$$\pi_\omega : \mathbb{S}^{n-1} \to \mathbb{R}^k$$

that maps $(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_k})$. The image of the map π_ω is the closed unit ball B_k in $\mathbb{R}^k$.

Definition 2.1.1. We say that a function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ depends on k variables, for $1 \le k \le n-1$ if there exists a function $\tilde{f} : B_k \to \mathbb{R}$ such that

$$f = \tilde{f} \circ \pi_\omega$$

for some subset ω of $\{1, \dots, n\}$, with $|\omega| = k$.

By abuse of notation we will often write $f(x_\omega)$ for a function on the sphere depending on k variables, meaning $\tilde{f}(x_\omega)$.
Functions on the sphere depending on k variables enjoy special symmetry properties. Indeed, they are constant on $(n-k-1)$-dimensional subspheres of the original sphere. Indeed, the fiber of a point $y \in B_k$ is given by

$$\pi_\omega^{-1}(y) = \{(x_1, \dots, x_n) \in \mathbb{S}^{n-1} : x_{i_j} = y_j \text{ for } j = 1, \dots, k\},$$

so the set $\pi_\omega^{-1}(y)$ is the intersection of the sphere $\mathbb{S}^{n-1}$ with the $(n-k)$-dimensional affine subspace of equation

$$\begin{cases} x_{i_1} = y_1 \\ \dots \\ x_{i_k} = y_k. \end{cases}$$

Thus, the fiber of $y \in B_k$ is the set of points satisfying the equation

$$x_{i_{k+1}}^2 + \dots + x_{i_n}^2 = 1 - y_1^2 - \dots - y_k^2,$$

which describes a sphere $\mathbb{S}^{n-k-1}$ of radius $1 - y_1^2 - \dots - y_k^2$ contained in $\mathbb{S}^{n-1}$. Note that for fixed ω, $\pi_\omega^{-1}(y) \neq \pi_\omega^{-1}(y')$ if $y \neq y'$, so that the subspheres $\pi_\omega^{-1}(y)$ indexed by $y \in B_k$ do not intersect one each other and cover the whole $\mathbb{S}^{n-1}$.

Example 2.1.2. A function on $\mathbb{S}^2$ in $\mathbb{R}^3$ depending on one variable, say x_3, is constant on circles lying on planes orthogonal to the x_3 direction.

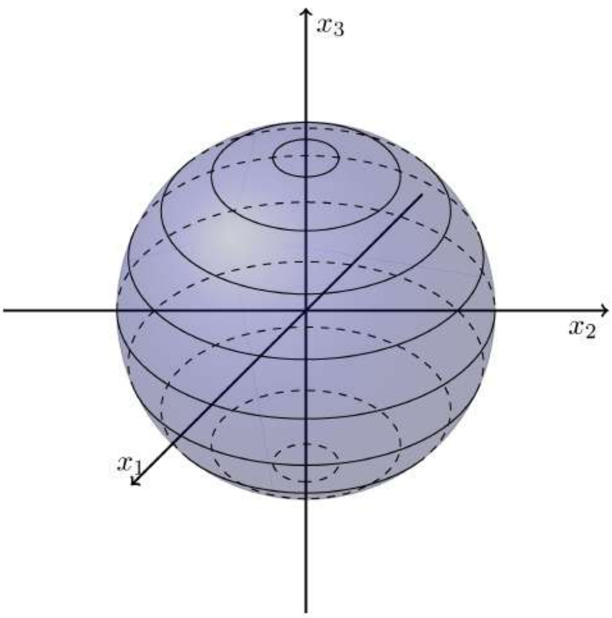


Figure 2.1: Level sets for a function on $\mathbb{S}^2$ depending on the variable x_3.

Remark 2.1.3. Functions on $\mathbb{S}^{n-1}$ depending on $n-1$ variables, say $x_1, \dots, x_{n-1}$, are essentially generic functions. Indeed, they are constant on 0-dimensional spheres, i.e. couples of points symmetric with respect to the hyperplane $x_n = 0$. In other words, these functions take the same value on two opposite hemispheres. It is easy to see that any generic function

$f : \mathbb{S}^{n-1} \to \mathbb{R}$ on the sphere can be written as a sum of two functions each depending on $n-1$ variables

$$\begin{aligned} f(x_1, \dots, x_n) &= f\left(x_1, \dots, x_{n-1}, (1 - x_1^2 - \dots - x_{n-1}^2)^{1/2}\right) \chi_{\{x_n \geq 0\}} \\ &\quad + f\left(x_1, \dots, x_{n-1}, -(1 - x_1^2 - \dots - x_{n-1}^2)^{1/2}\right) \chi_{\{x_n < 0\}} =: f_1 + f_2, \end{aligned}$$

where χ_A is the characteristic function of a set $A \subseteq \mathbb{R}^n$.

For a generic function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ we have

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} f(x_1, \dots, x_n) d\sigma = c_n \int_{B_{n-1}} \left[f\left(x_1, \dots, x_{n-1}, (1 - x_1^2 - \dots - x_{n-1}^2)^{1/2}\right) \right. \\ &\left. + f\left(x_1, \dots, x_{n-1}, -(1 - x_1^2 - \dots - x_{n-1}^2)^{1/2}\right)\right] \left(1 - x_1^2 - \dots - x_{n-1}\right)^{-1/2} dx_1 \dots dx_{n-1}, \end{aligned} \tag{2.1}$$

where B_{n-1} is the unit ball in $\mathbb{R}^{n-1}$. The constant c_n only depends on the dimension.
For convenience of the reader, in the following proposition we derive a similar and well known integration formula for functions depending on k variables.

Proposition 2.1.4. *Let $\omega = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n\}$ with $|\omega| = k$, for $1 \leq k \leq n-1$. Let $f : \mathbb{S}^{n-1} \to \mathbb{R}$ be a function depending on the k variables x_ω. The following integration formula holds:*

$$\int_{\mathbb{S}^{n-1}} f(x_\omega) d\sigma = c_{n,k} \int_{B_k} f(x_\omega)(1 - x_{i_1}^2 - \dots - x_{i_k}^2)^{\frac{n-k-2}{2}} dx_\omega. \tag{2.2}$$

The constant $c_{n,k}$ depends only on the dimension n and on the number of variables k.

Proof. Consider the parametrization of $\mathbb{S}^{n-1}$ in spherical coordinates

$$\begin{cases} x_{i_1} = \cos\theta_1 \\ x_{i_2} = \sin\theta_1 \cos\theta_2 \\ \dots \\ x_{i_k} = \sin\theta_1 \sin\theta_2 \dots \sin\theta_{k-1} \cos\theta_k \\ \dots \\ x_{i_{n-1}} = \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \cos\varphi \\ x_{i_n} = \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2} \sin\varphi \end{cases} \tag{2.3}$$

where $\theta_i \in [0, \pi)$ for $i = 1, \dots, n-2$ and $\varphi \in [0, 2\pi)$. It is well known (see for instance [31]) that in these coordinates

$$\Omega_n d\sigma = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin^2\theta_{n-3} \sin\theta_{n-2} d\theta_1 \dots d\theta_{n-2} d\varphi,$$

where Ω_n is the $(n-1)$-dimensional Hausdorff measure of the sphere $\mathbb{S}^{n-1}$. For our function f we have:

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} f(x_\omega) d\sigma &= \Omega_n^{-1} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} f(\cos\theta_1, \dots, \sin\theta_1 \sin\theta_2 \dots \sin\theta_{k-1} \cos\theta_k) \\ &\quad \times \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin^2\theta_{n-3} \sin\theta_{n-2} d\theta_1 \dots d\theta_{n-2} d\varphi \\ &= c_{n,k} \int_0^\pi \dots \int_0^\pi f(\cos\theta_1, \dots, \sin\theta_1 \sin\theta_2 \dots \sin\theta_{k-1} \cos\theta_k) \end{aligned}$$

$$\times \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin^{n-k-1}\theta_k d\theta_1 \dots d\theta_k,$$

where the constant $c_{n,k}$ is given by

$$c_{n,k} = 2\pi\Omega_n^{-1} \int_0^\pi \dots \int_0^\pi \sin^{n-k-2}\theta_{k+1} \dots \sin\theta_{n-2} d\theta_{k+1} \dots d\theta_{n-2}.$$

It follows from (2.3) that

$$\cos\theta_j = \frac{x_{i_j}}{\sqrt{1 - x_{i_1}^2 - \dots - x_{i_{j-1}}^2}},$$

so that

$$\sin\theta_j = \frac{\sqrt{1 - x_{i_1}^2 - \dots - x_{i_j}^2}}{\sqrt{1 - x_{i_1}^2 - \dots - x_{i_{j-1}}^2}}.$$

Note that $-1 \le x_{i_j} \le 1$ for $j = 1, \dots, k$ and that

$$x_{i_1}^2 + x_{i_2}^2 + \dots + x_{i_k}^2 = 1 - \sin^2\theta_1 \sin^2\theta_2 \dots \sin^2\theta_k \le 1,$$

so that the points $(x_{i_1}, \dots, x_{i_k})$ lie in the unit ball B_k of $\mathbb{R}^k$. Moreover

$$dx_1 \dots dx_k = \sin^k\theta_1 \sin^{k-1}\theta_2 \dots \sin^2\theta_{k-1} \sin\theta_k d\theta_1 \dots d\theta_k.$$

In conclusion we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} f(x_\omega) d\sigma &\sim \int_{B_k} f(x_{i_1}, \dots, x_{i_k}) (\sin\theta_1 \sin\theta_2 \dots \sin\theta_k)^{n-k-2} dx_{i_1} \dots dx_{i_k} \\ &= \int_{B_k} f(x_{i_1}, \dots, x_{i_k}) \left(1 - x_{i_1}^2 - \dots - x_{i_k}^2\right)^{\frac{n-k-2}{2}} dx_{i_1} \dots dx_{i_k}.\end{aligned}$$

□

Remark 2.1.5. Note that for functions on the sphere $\mathbb{S}^{n-1}$ depending on one variable, i.e. the case $k = 1$, we recover the well known integration formula:

$$\int_{\mathbb{S}^{n-1}} f(x_i) d\sigma \sim \int_{-1}^1 f(x_i) \left(1 - x_i^2\right)^{\frac{n-3}{2}} dx_i.$$

For functions depending on $n-1$ variables, i.e. the case $k = n-1$, we recover formula (2.1):

$$\int_{\mathbb{S}^{n-1}} f(x_{i_1}, \dots, x_{i_{n-1}}) d\sigma \sim \int_{B^{n-1}} f(x_{i_1}, \dots, x_{i_{n-1}}) \left(1 - x_{i_1}^2 - \dots - x_{i_{n-1}}^2\right)^{-\frac{1}{2}} dx_{i_1} \dots dx_{i_{n-1}}.$$

Remark 2.1.6. Let ω be as above, with $|\omega| = k$, for $1 \le k \le n-2$, and let $f : \mathbb{S}^{n-1} \to \mathbb{R}$ be a function depending on k variables. Since

$$\left(1 - x_{i_1}^2 - \dots - x_{i_k}^2\right)^{\frac{n-k-2}{2}} \le 1,$$

we have the trivial inequality

$$\int_{\mathbb{S}^{n-1}} f(x_\omega) d\sigma \lesssim \int_{B^k} f(x_\omega) dx_\omega. \tag{2.4}$$

In this way, we obtain a family of continuous immersions

$$L^p(B^k, dx_\omega) \hookrightarrow L^p(\mathbb{S}^{n-1}, d\sigma),$$

for $1 \le p \le \infty$.

2.2 The Lie algebra of the special orthogonal group

Recall that a basis for $\mathfrak{so}(n)$, the Lie algebra of $SO(n)$, in the coordinates $(x_1, \dots, x_n)$ is given by the vector fields

$$L_{i,j} = x_i \partial_{x_j} - x_j \partial_{x_i},$$

for $1 \le i < j \le n$. Obviously $L_{j,i} = -L_{i,j}$. The dimension of $\mathfrak{so}(n)$ is therefore $n(n-1)/2$. Let $\delta_{i,j}$ be the Kronecker delta. The bracket of two basis elements $L_{i,j}$ and $L_{k,l}$ is given by

$$\begin{aligned}
[L_{i,j}, L_{k,l}] &= (x_i\partial_{x_j} - x_j\partial_{x_j})(x_k\partial_{x_l} - x_l\partial_{x_k}) - (x_k\partial_{x_l} - x_l\partial_{x_k})(x_i\partial_{x_j} - x_j\partial_{x_j}) \\
&= x_i\delta_{j,k}\partial_{x_l} - x_j\delta_{i,k}\partial_{x_l} - x_i\delta_{j,l}\partial_{x_k} + x_j\delta_{i,l}\partial_{x_k} \\
&- \left(x_k\delta_{l,i}\partial_{x_j} - x_k\delta_{l,i}\partial_{x_i} - x_l\delta_{k,i}\partial_{x_j} + x_l\delta_{k,j}\partial_{x_i}\right) \\
&= (x_l\partial_{x_k} - x_j\partial_{x_l})\,\delta_{i,k} + (x_i\partial_{x_l} - x_l\partial_{x_i})\,\delta_{j,k} \\
&+ \left(x_j\partial_{x_k} - x_k\partial_{x_j}\right)\delta_{i,l} + (x_k\partial_{x_i} - x_i\partial_{x_k})\,\delta_{l,k} \\
&= L_{l,j}\delta_{i,k} + L_{i,l}\delta_{j,k} + L_{j,k}\delta_{i,l} + L_{k,i}\delta_{l,k}.
\end{aligned} \tag{2.5}$$

Hence the commutator of two elements of the basis $\{L_{i,j}\}_{i<j}$, if not trivial, is again an element of the basis. Indeed, the right hand side of (2.5) consists of at most one element, since getting two or more summands in (2.5) would force three or all indices among i, j, k, l to be equal, making the identity trivial. Note that if the indices i, j, k, l are pairwise different the vector fields commute.
This basis of $\mathfrak{so}(n)$ will be our Hörmander system $\mathcal{I}$. The corresponding sum of squares operator is given by

$$L = \sum_{i<j} L_{i,j}^2. \tag{2.6}$$

Proposition 2.2.1. *The operator L defined in* (2.6) *commutes with all the vector fields $L_{i,j}$.*

Proof. Fix a vector field $L_{i,j}$. As we noticed, $L_{i,j}$ commutes with all vector fields $L_{k,l}$ when the indices i, j, k, l are pairwise different, and so it commutes also with $L_{k,l}^2$. It obviously also commutes with itself. The remaining terms can be arranged in the following way:

$$\begin{aligned}
\left[\sum_{k<l} L_{k,l}^2, L_{i,j}\right] &= \sum_{k<i}[L_{k,i}^2, L_{i,j}] + \sum_{\substack{l>i \\ l\neq j}}[L_{i,l}^2, L_{i,j}] \\
&+ \sum_{\substack{k<j \\ k\neq i}}[L_{k,j}^2, L_{i,j}] + \sum_{l>j}[L_{j,l}^2, L_{i,j}].
\end{aligned}$$

The previous sum can be written as

$$\begin{aligned}
\left[\sum_{k<l} L_{k,l}^2, L_{i,j}\right] &= \sum_{l>j}\left([L_{i,l}^2, L_{i,j}] + [L_{j,l}^2, L_{i,j}]\right) \\
&+ \sum_{k<i}\left([L_{k,i}^2, L_{i,j}] + [L_{k,j}^2, L_{i,j}]\right) + \sum_{i<s<j}\left([L_{i,s}^2, L_{i,j}] + [L_{s,j}^2, L_{i,j}]\right).
\end{aligned} \tag{2.7}$$

For vector fields A, B, we have

$$\begin{aligned}
[A^2, B] &= AAB - BAA = A[A,B] + ABA - [B,A]A - ABA \\
&= A[A,B] + [A,B]A.
\end{aligned} \tag{2.8}$$

Let us compute a term in the first sum of (2.7). From (2.8) we get

$$\begin{aligned}[L_{i,l}^2, L_{i,j}] + [L_{j,l}^2, L_{i,j}] &= L_{i,l}[L_{i,l}, L_{i,j}] + [L_{i,l}, L_{i,j}]L_{i,l} + L_{j,l}[L_{j,l}, L_{i,j}] + [L_{j,l}, L_{i,j}]L_{j,l} \\ &= L_{i,l}L_{j,l} + L_{j,l}L_{i,j} - L_{j,l}L_{i,j} - L_{i,j}L_{j,l} = 0,\end{aligned}$$

where we used (2.5). Similarly, by using (2.8) and (2.5) one can see that each summand in (2.7) is zero, proving the proposition. □

Proposition 2.2.1 does not come as a surprise. Indeed, the operator L is the quadratic Casimir operator, which is an element of the center of the universal enveloping algebra $U(\mathfrak{so}(n))$, so it commutes with all left invariant differential operators.
Note also that L is the Laplace–Beltrami operator on $SO(n)$ with the Riemannian metric induced by the Killing form $B : \mathfrak{so}(n) \times \mathfrak{so}(n) \to \mathbb{R}$, given by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y),$$

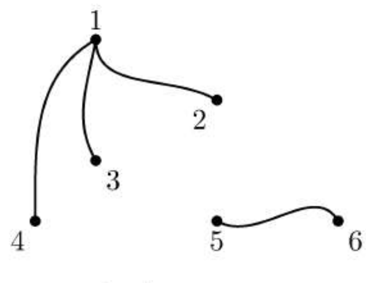
where $\mathrm{ad}\, X : \mathfrak{so}(\mathfrak{n}) \to \mathfrak{so}(\mathfrak{n})$, for fixed $X \in \mathfrak{so}(\mathfrak{n})$, is the Lie endomorphism defined by

$$\mathrm{ad}\, X(\cdot) = [X, \cdot].$$

2.3 Structure of maximal subsets

We now discuss the structure of maximal subsets of the system $\mathcal{I} = \{L_{i,j}\}_{i<j}$.
In order to visualize the subsets of $\mathcal{I}$ we associate to the vector fields $\{L_{i,j}\}_{i<j}$ the set of pairs $\{(i, j)\}_{i<j}$. Consider a subset $\mathcal{A} \subseteq \{(i, j)\}_{i<j}$. We can relate to this subset an undirected simple graph $G_{\mathcal{A}} = (V, E)$ where the set of vertices V is given by $\{1, \dots, n\}$ and the edges E are given by the (unordered) pairs $(i, j) \in \mathcal{A}$, so that we can identify $\mathcal{A}$ with E.

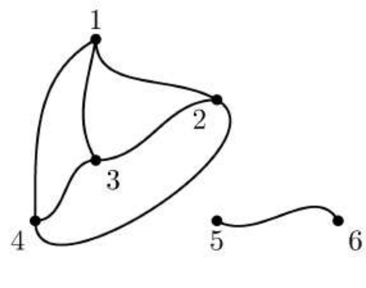
Example 2.3.1. In $\mathfrak{so}(6)$, consider the set $\mathcal{A} = \{(1, 2), (1, 3), (1, 4), (5, 6)\}$. The associated graph $G_{\mathcal{A}}$ has $V = \{1, \dots, 6\}$ and $E = \mathcal{A}$.



The set $\mathcal{A}$ is not maximal. Indeed, by (2.5) we see that

$$\langle \mathcal{A} \rangle \cap \mathcal{I} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (5, 6)\},$$

with associated graph



Note that in this graph each connected component is complete. Indeed, the following proposition holds.

Proposition 2.3.2. *Let $\mathcal{A}$ be a subset of $\mathcal{I}$. $\mathcal{A}$ is a maximal subset if and only if the associated graph $G_{\mathcal{A}} = (V, E)$ splits in complete connected components.*

Proof. First of all note that if $(a, b) \in E$ and $(b, c) \in E$, then $(a, c) \in E$ by the maximality assumption on the subset $\mathcal{A}$ and (2.5). Since connected components are path connected, each connected component of a graph associated to a maximal subset is complete. The converse is straightforward, again by (2.5). □

We also have the following result.

Proposition 2.3.3. *Let $\mathcal{A}$ be a maximal subset of $\mathcal{I}$ and $G_{\mathcal{A}} = (V, E)$ its associated graph. Each complete connected component $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, identifying $\widetilde{E}$ with a subset of $\mathcal{I}$, has the property that $\langle \widetilde{E} \rangle \simeq \mathfrak{so}(|\widetilde{V}|)$.*

Proof. Let $\widetilde{V} = \{i_1, \dots, i_k\}$, with $i_1 < \dots < i_k$, so that $|\widetilde{V}| = k$. Since $\widetilde{G}$ is complete, $\widetilde{E}$ contains all the edges in E with vertices in $\widetilde{V}$. It is easy to see that, by property (2.5), the map

$$\langle \widetilde{E} \rangle \to \mathfrak{so}(k)$$

that maps $L_{i_j, i_l} \mapsto L_{j,l}$, for $i_j < i_l$, is a Lie algebra isomorphism. Moreover the set $\widetilde{E}$ is a basis for $\langle \widetilde{E} \rangle$. □

Let us introduce some notation. Let $\alpha = (\alpha^1, \dots, \alpha^n) \in \{0, 1\}^n$ be a multi-index and denote by $|\alpha| = \alpha^1 + \dots + \alpha^n$ its length. The scalar product $\alpha \cdot \beta = \alpha^1\beta^1 + \dots + \alpha^n\beta^n$ indicates the number of 1's in common between α and β, so that two multi-indices are orthogonal if they do not have 1's in common.
We will denote with $\mathfrak{so}_\alpha$ the Lie algebra isomorphic to $\mathfrak{so}(|\alpha|)$ generated by the set $\{L_{k,l} : \alpha^k = \alpha^l = 1\}$. For example, if $n = 5$ and $\alpha = (1, 0, 1, 1, 0)$, the algebra $\mathfrak{so}_\alpha$ is the algebra isomorphic to $\mathfrak{so}(3)$ spanned by $\{L_{1,3}, L_{1,4}, L_{3,4}\}$.
We can now deduce the following theorem describing the structure of subalgebras generated by maximal subsets associated to basis systems of $\mathfrak{so}(n)$.

Theorem 2.3.1. *Let $\mathcal{A}$ be a maximal subset of $\mathcal{I} = \{L_{i,j}\}_{i<j}$. Then there exist multi-indices $\alpha_1, \dots, \alpha_N$ pairwise orthogonal, with $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_N|$ and $|\alpha_1| + \dots + |\alpha_N| \leq n$, such that*

$$\langle \mathcal{A} \rangle = \bigoplus_{k=1}^{N} \mathfrak{so}_{\alpha_k}, \tag{2.9}$$

where on the right-hand side we have a direct sum of Lie algebras, i.e. each subalgebra commutes with the others.

Proof. By Proposition 2.3.2 and Proposition 2.3.3 the graph associated to $\mathcal{A}$ splits in N, say, complete connected components $G_{\alpha_i} = (V_{\alpha_i}, E_{\alpha_i})$, where $i = 1, \dots, N$, each describing a graph associated to a basis system of a Lie algebra of type $\mathfrak{so}(k)$ for some k. Without loss of generality we can assume that $|V_{\alpha_1}| \geq \dots \geq |V_{\alpha_N}|$ so that $|\alpha_1| \geq \dots \geq |\alpha_N|$. The multi-indices are pairwise orthogonal since the graphs G_{α_i} are disconnected so that $V_{\alpha_i} \cap V_{\alpha_j} = \emptyset$ for $i \neq j$. It is clear that $|\alpha_1| + \dots + |\alpha_N| = |V| \leq n$. Finally the sum in (2.9) is direct by (2.5). □

Remark 2.3.4. To the splitting of the subalgebra $\langle\mathcal{A}\rangle$ we can associate a partition of the finite set $\mathcal{A}$ into basis systems of each subalgebra $\mathfrak{so}_{\alpha_i}$. Each basis system will have cardinality $\binom{|\alpha_i|}{2}$.

We now study the properties of functions annihilated by maximal subsets of vectors in $\mathcal{I} = \{L_{i,j}\}_{i<j} \subset \mathfrak{so}(n)$. First we consider the case of a singleton, i.e. $\mathcal{A} = \{L_{i,j}\}$.

Lemma 2.3.5. *Let $f : \mathbb{S}^{n-1} \to \mathbb{R}$, with $f \in C^\infty(\mathbb{S}^{n-1})$ and $L_{i,j}$ be as above. Then $L_{i,j}f(x) = 0$ for all $x \in \mathbb{S}^{n-1}$ if and only if there exists a function $\tilde{f}$ such that*

$$f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = \tilde{f}(x_i^2 + x_j^2, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n), \tag{2.10}$$

for all $(x_1, \dots, x_n) \in \mathbb{S}^{n-1}$, where by $\hat{x}_i$ we mean that the variable x_i is not appearing.

Proof. Clearly, (2.10) implies

$$\begin{aligned} L_{i,j}f(x_1, \dots, x_n) &= L_{i,j}\tilde{f}(x_i^2 + x_j^2, x_1, \dots, x_n) \\ &= 2x_ix_jD_1\tilde{f} - 2x_jx_iD_1\tilde{f} = 0 \end{aligned}$$

for all $x \in \mathbb{S}^{n-1}$, where D_1 denotes the partial derivative with respect to the first variable of $\tilde{f}$.
Conversely, suppose that f satisfies $L_{i,j}f(x) = 0$ for all $x \in \mathbb{S}^{n-1}$. Since $L_{i,j}$ is the infinitesimal generator of rotations in the x_ix_j-plane, it fixes circles of the type $x_i^2 + x_j^2 = r^2$ for some $r \geq 0$. Hence, f, being annihilated by $L_{i,j}$, is constant on these circles, thus it depends on x_i and x_j through $x_i^2 + x_j^2$. □

An analogous property holds if we consider functions annihilated by a maximal subset $\mathcal{A}$ of $\mathcal{I}$ whose generated Lie algebra is isomorphic to $\mathfrak{so}(k)$.

Lemma 2.3.6. *Let $f : \mathbb{S}^{n-1} \to \mathbb{R}$, with $f \in C^\infty(\mathbb{S}^{n-1})$ and $\mathcal{A}$ be a maximal subset of $\mathcal{I}$ such that $\langle\mathcal{A}\rangle \simeq \mathfrak{so}(k)$ for some $k \leq n$, i.e. $\mathcal{A} = \{L_{i_j,i_l}\}_{i_j<i_l}$ with $i_j, i_l \in \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. Then $L_{i_j,i_l}f(x) = 0$ for all $x \in \mathbb{S}^{n-1}$ and $L_{i_j,i_l} \in \mathcal{A}$ if and only if there exists a function $\tilde{f}$ such that*

$$f(x_1, \dots, x_{i_1}, \dots, x_{i_k}, \dots, x_n) = \tilde{f}(x_{i_1}^2 + \dots + x_{i_k}^2, x_1, \dots, \hat{x}_{i_1}, \dots, \hat{x}_{i_k}, \dots, x_n), \tag{2.11}$$

for all $(x_1, \dots, x_n) \in \mathbb{S}^{n-1}$.

Proof. The assertion is proved arguing as in Lemma 2.3.5, once noted that the subalgebra $\langle A\rangle$ generates the rotations in the k-plane related to the coordinates $x_{i_1}, \dots, x_{i_k}$. □

By abuse of notation we will just write f in place of $\tilde{f}$.

Remark 2.3.7. If a function $f \in C^\infty(\mathbb{S}^{n-1})$ is $\mathcal{A}$-symmetric with respect to a maximal subset $\mathcal{A}$ of $\mathcal{I}$ such that $\langle\mathcal{A}\rangle \simeq \mathfrak{so}_\alpha$ for some multi-index α with $|\alpha| = k$, the function f is a function of $n - k$ variables in the sense of Definition (2.1.1). Without loss of generality, assume $\alpha^i = 1$ for $i = 1, \dots, k$ and zero otherwise. By Lemma 2.3.6 we have that

$$f(x_1, \dots, x_n) = f(x_1^2 + \dots + x_k^2, x_{k+1}, \dots, x_n).$$

But, since $x_1^2 + \dots + x_k^2 = 1 - x_{k+1}^2 - \dots - x_n^2$,

$$f(x_1^2 + \dots + x_k^2, x_{k+1}, \dots, x_n) = f(1 - x_{k+1}^2 - \dots - x_n^2, x_{k+1}, \dots, x_n),$$

so that f is a function of the $n - k$ variables $x_{k+1}, \dots, x_n$.

A generic maximal subset $\mathcal{A}$ of $\mathcal{I}$ splits by Theorem 2.3.1 into N disjoint subsets, labeled by a family of multi-indices α_i. Each of these subsets generates a subalgebra of $\mathfrak{so}(n)$ isomorphic to $\mathfrak{so}(|\alpha_i|)$. In Theorem 2.3.1 we ordered these subsets by cardinality. We will interpret the splitting in the following way: the subalgebra related to the multi-index α_1 tells us on how many variables the functions annihilated by $\mathcal{A}$ depend, as explained in Remark 2.3.7; the subalgebras related to the multi-indices α_i, for $2 \leq i \leq N$ give instead information concerning radiality in the variables. To be more precise, functions in $C^\infty(\mathbb{S}^{n-1})$ that are $\mathcal{A}$-symmetric depend on the $n - |\alpha_1|$ variables x_i for which $\alpha_1^i = 0$, and depend radially on the collections of $|\alpha_i|$ variables associated to each multi-index α_i (that are disjoint by the orthogonality of the multi-indices).

Example 2.3.8. Consider the maximal system in $\mathfrak{so}(7)$ given by

$$\mathcal{A} = \{L_{5,6}, L_{5,7}, L_{6,7}, L_{1,2}, L_{3,4}\},$$

that splits as

$$\{L_{5,6}, L_{5,7}, L_{6,7}\} \sqcup \{L_{1,2}\} \sqcup \{L_{3,4}\},$$

and whose generated subalgebra $\langle \mathcal{A} \rangle$ splits therefore as

$$\mathfrak{so}_{\alpha_1}(3) \oplus \mathfrak{so}_{\alpha_2}(2) \oplus \mathfrak{so}_{\alpha_3}(2),$$

with $\alpha_1 = (0,0,0,0,1,1,1)$, $\alpha_2 = (1,1,0,0,0,0,0)$, $\alpha_3 = (0,0,1,1,0,0,0)$. A function $f \in C^\infty(\mathbb{S}^{n-1})$ that is $\mathcal{A}$-symmetric will depend on the $n - |\alpha_1| = 7 - 3 = 4$ variables x_1, x_2, x_3, x_4 and will be radial in the collections of variables x_1, x_2 associated to α_2 and x_3, x_4 associated to α_3. So it will be written as

$$f(x_1^2 + x_2^2, x_3^2 + x_4^2).$$

Remark 2.3.9. We stick to the convention of ordering the subsets by cardinality. We remark that all orderings are equivalent. Indeed, in Example 2.3.8 one could have considered instead the splitting

$$\mathfrak{so}_{\alpha_2}(2) \oplus \mathfrak{so}_{\alpha_1}(3) \oplus \mathfrak{so}_{\alpha_3}(2).$$

In this point of view, a function $f \in C^\infty(\mathbb{S}^{n-1})$ that is $\mathcal{A}$-symmetric is a function of the $n - |\alpha_2| = 7 - 2 = 5$ variables x_3, x_4, x_5, x_6, x_7, radial in the collections of variables x_5, x_6, x_7 associated to α_1 and x_3, x_4 associated to α_3. So it can be written as

$$f(x_3^2 + x_4^2, x_5^2 + x_6^2 + x_7^2),$$

but since $x_5^2 + x_6^2 + x_7^2 = 1 - x_1^2 - x_2^2 - x_3^2 - x_4^2$, we can reduce the dependence to $x_3^2 + x_4^2$ and $x_1^2 + x_2^2$, thus obtaining the same numerology as in Example 2.3.8.

In the rest of the chapter we will study on the sphere $\mathbb{S}^{n-1}$ some interesting instances of multilinear inequalities of the type (1.10) related to the system $\mathcal{I} = \{L_{i,j}\}_{i<j}$ described above. We will obtain nontrivial inequalities in the sense of Definition 1.4.10. As we saw, functions involved in the inequalities have symmetry properties determined by the maximal system $\mathcal{A}$ that annihilates them. We will also be able to show for some of the inequalities that the exponents $\tilde{p}$ found by means of Theorem 1.6.1 are sharp in a sense that we make precise with the following definition.

Definition 2.3.10. We will say that the exponent $\tilde{p}$ is sharp if the inequality

$$\int_{\mathbb{S}^{n-1}} \prod_{i=1}^m f_i d\mu \leq \prod_{i=1}^m \|f_i\|_{L^p(\mathbb{S}^{n-1})},$$

holds for $p = \tilde{p}$ and is false for $p < \tilde{p}$, i.e. there exist functions f_i for which the right-hand side is finite and the left-hand side diverges.

2.4 Carlen–Lieb–Loss inequality

The first inequality we discuss was discovered by Carlen, Lieb and Loss in [18]. This is an inequality for n functions each depending on a single different variable. In the terminology developed so far we have n maximal subsets $\mathcal{A}_1, \dots, \mathcal{A}_n$. The subalgebra generated by each maximal subset $\mathcal{A}_i$ is isomorphic to $\mathfrak{so}(n-1)$. Note that the splitting of $\langle \mathcal{A}_i \rangle$ given by Theorem 2.3.1 in this case has just one direct summand. The multi-index associated to this direct summand is $\alpha_i = (1, \dots, 1, 0, 1, \dots, 1)$ with 0 only in the i-th place. It is clear that

$$|\mathcal{A}_i| = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$$

for all $i = 1, \dots, n$. Each maximal subset $\mathcal{A}_i$ contains the vector fields $L_{k,l}$ in $\mathcal{I}$ with both $k, l \neq i$. The complement $\mathcal{A}_i^c$ of $\mathcal{A}_i$ in $\mathcal{I}$ is made of the vector fields $L_{k,l}$ with $k = i$ or $l = i$, and

$$|\mathcal{A}_i^c| = \binom{n}{2} - \binom{n-1}{2} = n-1$$

for all $i = 1, \dots, n$. By Remark 2.3.7 a function $f \in C^\infty(\mathbb{S}^{n-1})$ that is $\mathcal{A}_i$-symmetric is a function of the variable x_i only. As we saw we can think of functions depending on one variable as functions defined on the one-dimensional unit ball, i.e. the interval $[-1, 1]$, pulled back to the sphere $\mathbb{S}^{n-1}$ via the projections $\pi_i : \mathbb{S}^{n-1} \to [-1, 1]$ mapping $x \in \mathbb{S}^{n-1}$ to its i-th coordinate. We will write $f(x_i)$ for $f(\pi_i(x))$, with $x \in \mathbb{S}^{n-1}$.
We have the following theorem.

Theorem 2.4.1 ([18]). *Let $f_1, \dots, f_n$ be nonnegative measurable functions, $f_i : [-1, 1] \to \mathbb{R}^+$. The inequality*

$$\int_{\mathbb{S}^{n-1}} f_1(x_1) \dots f_n(x_n) d\sigma \leq \prod_{i=1}^{n} \|f_i\|_{L^p(\mathbb{S}^{n-1})} \tag{2.12}$$

holds for $p \geq \tilde{p} = 2$. Moreover inequality (2.12) *is sharp in the sense of Definition 2.3.10.*

Remark 2.4.1. We will give a proof of Theorem 2.4.1 based on Theorem 1.6.1, which is in the spirit of the original proof of [18]. Our proof is however written in a more abstract language to open up the way to our later discussion. Note that the exponent $\tilde{p} = 2$ found in this case is independent of the dimension and that the inequality obtained is nontrivial in the sense of Definition 1.4.10 for $n \geq 3$.

Proof. By Theorem 1.6.1, the exponent $\tilde{p}$ is given by the number of occurrences of the most recurrent element among the finite sets $\mathcal{A}_i^c$, for $i = 1, \dots, n$. As we saw the elements of $\mathcal{A}_i^c$ are vector fields of type $L_{i,j}$ for $j > i$ or $L_{k,i}$ for $k < i$. An element of type $L_{i,j}$ belongs only to the sets $\mathcal{A}_i^c$ and $\mathcal{A}_j^c$ and an element of type $L_{k,i}$ belongs only to the sets $\mathcal{A}_i^c$ and $\mathcal{A}_k^c$. As a result, every element of $\mathcal{A}_i^c$ occurs exactly twice among the sets $\mathcal{A}_l$. This implies, by Theorem 1.6.1, that (2.12) holds for $\tilde{p} = 2$. This, together with continuous embeddings of Lebesgue spaces on $\mathbb{S}^{n-1}$, concludes the first part of the proof.
To prove that the exponent $\tilde{p} = 2$ is sharp we give an explicit counterexample, taken from [18]. We assume $n \geq 3$, the case $n = 2$ being trivial by Hölder's inequality.
Let $f_i : [-1, 1] \to \mathbb{R}^+$ be the functions given by

$$f_i(x_i) = |x_i|^{-\gamma} + (1 - x_i^2)^{-\frac{(n-1)\gamma}{2}}, \tag{2.13}$$

with $\gamma > 0$ to be determined. Let $p \geq 1$. Thanks to the integration formula (2.2) we have that

$$\begin{aligned}
\|f_i\|^p_{L^p(\mathbb{S}^{n-1})} &= \int_{\mathbb{S}^{n-1}} \left(|x_i|^{-\gamma} + (1-x_i^2)^{-\frac{(n-1)\gamma}{2}}\right)^p d\sigma \\
&\lesssim \int_{\mathbb{S}^{n-1}} |x_i|^{-\gamma p} d\sigma + \int_{\mathbb{S}^{n-1}} (1-x_i^2)^{-\frac{(n-1)\gamma p}{2}} d\sigma \\
&\lesssim \int_{-1}^{1} |x_i|^{-\gamma p}(1-x_i^2)^{\frac{n-3}{2}} dx_i + \int_{-1}^{1} (1-x_i^2)^{-\frac{(n-1)\gamma p}{2}+\frac{n-3}{2}} dx_i \\
&\lesssim \int_{-1}^{1} |x_i|^{-\gamma p} dx_i + \int_{-1}^{1} (1-x_i^2)^{\frac{-(n-1)\gamma p+n-3}{2}} dx_i,
\end{aligned}$$

which is finite for $-\gamma p > -1$, i.e. $\gamma p < 1$, and $-(n-1)\gamma p + n - 3 > -2$, i.e. $\gamma p < 1$, again. So the left-hand side of (2.12) with our functions f_i is finite if $\gamma p < 1$.
Now we want to make the right-hand side unbounded. We have that

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{n} f_i(x_i) d\sigma &\geq \int_{\mathbb{S}^{n-1}} \left(\prod_{i=1}^{n-1} |x_i|^{-\gamma}\right) (1-x_n^2)^{-\frac{(n-1)\gamma}{2}} d\sigma \\
&\sim \int_{B_{n-1}} \left(\prod_{i=1}^{n-1} |x_i|^{-\gamma}\right) (x_1^2 + \dots + x_{n-1}^2)^{-\frac{(n-1)\gamma}{2}} \frac{dx_1 \dots dx_{n-1}}{\left(1 - x_1^2 - \dots - x_{n-1}^2\right)^{1/2}},
\end{aligned}$$

where we ignored many nonnegative summands in the product $\prod_{i=1}^n f_i(x_i)$ and used once more the integration formula (2.2). Using the fact that $|x_i| \leq \left(x_1^2 + \dots + x_{n-1}^2\right)^{1/2}$ for $i = 1, \dots, n-1$, and passing to polar coordinates in the ball B_{n-1} we get that

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{n} f_i(x_i) d\sigma &\gtrsim \int_{B_{n-1}} (x_1^2 + \dots + x_{n-1}^2)^{-(n-1)\gamma} \frac{dx_1 \dots dx_{n-1}}{\left(1 - x_1^2 - \dots - x_{n-1}^2\right)^{1/2}} \\
&\sim \int_0^1 \rho^{-2(n-1)\gamma} \rho^{n-2} (1-\rho^2)^{-1/2} d\rho \\
&\sim \int_0^1 \rho^{-2(n-1)\gamma + n - 2} (1-\rho^2)^{-1/2} d\rho,
\end{aligned}$$

which diverges for $-2(n-1)\gamma + n - 2 \leq -1$, i.e. $\gamma \geq 1/2$.
Pick $\gamma = 1/2$. Since $\gamma p < 1$, this provides a counterexample to (2.12) whenever

$$p < \frac{1}{\gamma} = 2 = \tilde{p},$$

thus proving the result. □

Remark 2.4.2. One could ask if it is possible to obtain an inequality of type (2.12) with a different p_i for each f_i. Of course this is possible as a direct consequence of Theorem 2.4.1 , by continuous embeddings of $L^p(\mathbb{S}^{n-1})$ spaces, as long as each $p_i \geq \tilde{p} = 2$. Anyway a use of Theorem 1.6.2 in this particular case is not effective. Indeed, even if one allows different p_i's in the nonlinear heat evolution associated to the operator L defined in (2.6), the presence of functions of one variable for each variable $x_1, \dots, x_n$ forces all the exponents p_i to be equal, since each element of each set $\mathcal{A}_i^c$ occurs exactly twice in the the sets $\mathcal{A}_k^c$, for $k = 1, \dots, n$.

Remark 2.4.3. Since constant functions are trivially functions of one variable, inequality (2.12) also holds for $m \leq n$ functions depending on one variable, with m different variables. The inequality is nontrivial for $m \geq 3$, since the case $m = 2$ is just Hölder's inequality and the case $m = 1$ follows from continuous embeddings of Lebesgue spaces on the sphere. Note that also in this case an application of Theorem 1.6.2 is not effective, since for each $\mathcal{A}_i^c$ there is at least one vector field with two occurrences in the sets $\mathcal{A}_k^c$, for $k = 1, \ldots, m$, i.e. the vector field $L_{i,k}$ if $i < k$ or $L_{k,i}$ if $k < i$.

2.5 Inequalities for functions depending on k variables

In this section we will generalize the result of [18] to functions depending on $1 \leq k \leq n-1$ variables.

The case of functions depending on $n-1$ variables is the easiest one and we treat it separately. In this case we have $\binom{n}{n-1} = n$ possible $(n-1)$-tuples of variables, which correspond to empty maximal systems $\mathcal{A}_i$, for which $\langle \mathcal{A}_i \rangle = \{0\}$. Indeed, functions depending on $n-1$ variables are almost generic functions, as explained at the beginning of the chapter, and there is no hope to obtain something better than Hölder's inequality, i.e. $\tilde{p} = n$. This is confirmed by Theorem 1.6.1, since each element in each $\mathcal{A}_i^c = \mathcal{I}$ occurs in all $\mathcal{A}_k^c$, for $k = 1, \ldots, n$.

Let us now consider the case of functions depending on $1 \leq k \leq n-2$ variables. We have $\binom{n}{k} := C(n,k)$ possible choices ok k-tuples out of the set $\{1, \ldots, n\}$. We will label them as $\omega_1, \ldots, \omega_{C(n,k)}$ following the notation introduced at the beginning of the chapter. To each collection of variables $\omega_i = \{i_1, \ldots, i_k\}$ corresponds a maximal subset $\mathcal{A}_i$ which contains the vector fields $L_{j,l}$ for which $j, l \neq i_s$ for all $s = 1, \ldots, k$.

The subalgebra generated by each maximal subset $\mathcal{A}_i$ is isomorphic to $\mathfrak{so}(n-k)$ and the splitting of $\langle \mathcal{A}_i \rangle$ given by Theorem 2.3.1 has just one direct summand $\mathfrak{so}_{\alpha_i}$, with α_i a multi-index such that $\alpha_i^j = 0$ if $j \in \omega_i$, for $j = 1, \ldots, n$.

It is clear that

$$|\mathcal{A}_i| = \binom{n-k}{2},$$

for all $i = 1, \ldots, C(n,k)$. The complement $\mathcal{A}_i^c$ of $\mathcal{A}_i$ in $\mathcal{I}$ is made of the vector fields $L_{k,l}$ for which either j or l, or both, are in ω_i, and

$$|\mathcal{A}_i^c| = \binom{n}{2} - \binom{n-k}{2},$$

for all $i = 1, \ldots, C(n,k)$. By Remark 2.3.7 a function $f \in C^\infty(\mathbb{S}^{n-1})$ that is $\mathcal{A}_i$-symmetric is a function of the variable x_{ω_i} in $\mathbb{R}^k$. As we saw we can think of a function depending on x_{ω_i} as a function defined on the k-dimensional unit ball $B_k \subset \mathbb{R}^k$, pulled back to the sphere $\mathbb{S}^{n-1}$ via the projection $\pi_{\omega_i} : \mathbb{S}^{n-1} \to B_k$, mapping a point $x \in \mathbb{S}^{n-1}$ to x_{ω_i}. We will write $f(x_{\omega_i})$ for $f(\pi_{\omega_i}(x))$, with $x \in \mathbb{S}^{n-1}$.

We have the following theorem.

Theorem 2.5.1. *Let $f_1, \ldots, f_{C(n,k)}$ be nonnegative measurable functions, $f_i : B_k \to \mathbb{R}^+$. The inequality*

$$\int_{\mathbb{S}^{n-1}} f_1(x_{\omega_1}) \ldots f_{C(n,k)}(x_{\omega_{C(n,k)}}) d\sigma \leq \prod_{i=1}^{C(n,k)} \|f_i\|_{L^p(\mathbb{S}^{n-1})} \tag{2.14}$$

holds for

$$p \geq \tilde{p} = \binom{n}{k} - \binom{n-2}{k}.$$

Moreover inequality (2.14) *is sharp in the sense of Definition 2.3.10.*

Remark 2.5.1. For $k = 1$ we recover the result of [18], since

$$\tilde{p} = \binom{n}{1} - \binom{n-2}{1} = 2.$$

Note that inequality (2.14) is nontrivial in the sense of Definition 1.4.10 for $n \geq 3$, since $\tilde{p} < C(n, k)$.

Proof. By Theorem 1.6.1, the exponent $\tilde{p}$ is given by the number of occurrences of the most recurrent element among the sets $\mathcal{A}_i^c$, for $i = 1, \dots, C(n, k)$. As we said, the elements of $\mathcal{A}_i^c$ are vector fields of type $L_{j,l}$ with either j or l or both j, l in ω_i. So an element $L_{j,l}$ will occur in all $\mathcal{A}_i^c$ apart from those for which $j, l \notin \omega_i$. The number of sets ω_i made of k elements taken from $\{1, \dots, n\}$ that do not contain two fixed elements j, l is $\binom{n-2}{k}$. This means that each vector field will occur in exactly

$$\tilde{p} = \binom{n}{k} - \binom{n-2}{k}$$

sets $\mathcal{A}_i^c$, proving the first half of the theorem.
To show that $\tilde{p}$ is sharp we construct a counterexample. We consider functions $f_i : B_k \to \mathbb{R}^+$, where B_k is the unit ball in $\mathbb{R}^k$, of the form

$$f_i(x_{\omega_i}) = (|x_{i_1}||x_{i_2}| \dots |x_{i_k}|)^{-\gamma/k} + (1 - x_{i_1}^2)^{-\gamma\delta/2} + \dots + (1 - x_{i_k}^2)^{-\gamma\delta/2}, \tag{2.15}$$

where γ, δ are positive constants to be determined. This seems to be a natural guess, since for $k = 1$ this set of functions reduces to the counterexample in [18].
The right-hand side of inequality (2.14) must be finite. We first compute the L^p norm of these functions. Without loss of generality we focus on the case $\omega = \{1, 2, \dots, k\}$ and work with $f(x_1, \dots, x_k)$.
Let $p \geq 1$. We have

$$\begin{aligned}
\|f\|_{L^p(\mathbb{S}^{n-1})}^p &= \int_{\mathbb{S}^{n-1}} \left[(|x_1||x_2| \dots |x_k|)^{-\gamma/k} + (1 - x_1^2)^{-\gamma\delta/2} + \dots + (1 - x_k^2)^{-\gamma\delta/2} \right]^p d\sigma \\
&\lesssim \int_{\mathbb{S}^{n-1}} \left[(|x_1||x_2| \dots |x_k|)^{-\gamma p/k} + (1 - x_1^2)^{-\gamma\delta p/2} + \dots + (1 - x_k^2)^{-\gamma\delta p/2} \right] d\sigma \\
&= \int_{\mathbb{S}^{n-1}} (|x_1||x_2| \dots |x_k|)^{-\gamma p/k} d\sigma + \sum_{i=1}^k \int_{\mathbb{S}^{n-1}} (1 - x_i^2)^{-\gamma\delta p/2} d\sigma \\
&=: I_0 + \sum_{i=1}^k I_i.
\end{aligned}$$

For the first term I_0 we have

$$\begin{aligned}
I_0 &\lesssim \int_{B_k} (|x_1| \dots |x_k|)^{-\gamma p/k} (1 - x_1^2 - \dots - x_k^2)^{(n-k-2)/2} dx_1 \dots dx_k \\
&\leq \prod_{i=1}^k \int_{-1}^1 |x_i|^{-\gamma p/k} dx_i
\end{aligned}$$

where we used the integration formula (2.2), the fact that $(1-x_1^2-\cdots-x_k^2)^{(n-k-2)/2}\leq 1$ in B_k, since $k\leq n-2$, and also that $B_k\subset[-1,1]^k$. So I_0 is finite if $\gamma p<k$.
For each of the terms I_i we have

$$I_i=\int_{-1}^{1}(1-x_i^2)^{-\gamma\delta p/2}(1-x_i^2)^{(n-3)/2}dx_i,$$

by the integration formula (2.2). So I_i is finite whenever $\gamma\delta p<(n-1)$.
We conclude that the right-hand side of (2.14) is finite if

$$\gamma p<\min\left\{k,\frac{n-1}{\delta}\right\}. \tag{2.16}$$

To estimate the left-hand side of (2.14) we pass to polar coordinates in the hyperplane $\mathbb{R}^{n-1}$ with coordinates $x_1,\dots,x_{n-1}$; on the sphere $\mathbb{S}^{n-1}$ the variable $|x_n|$ will then just be $(1-\rho^2)^{1/2}$.
There are $\binom{n-1}{k}$ functions not involving the x_n variable, and $\binom{n-1}{k-1}$ involving it. For the functions not depending on x_n we select the first summand of (2.15), for those depending on x_n we select the summand $(1-x_n^2)^{-\gamma\delta/2}$.
So for the left-hand side we have:

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}}\prod_{i=1}^{C(n,k)}f_i(x_{\omega_i})d\sigma\\
&\geq\int_{\mathbb{S}^{n-1}}\left[\prod_{\omega_i\not\ni x_n}|x_{i_1}|\dots|x_{i_k}|\right]^{-\frac{\gamma}{k}}(1-x_n^2)^{-\frac{\gamma\delta}{2}\binom{n-1}{k-1}}d\sigma\\
&\geq\int_0^1\left(\rho^k\right)^{-\frac{\gamma}{k}\binom{n-1}{k}}\left(\rho^2\right)^{-\frac{\gamma\delta}{2}\binom{n-1}{k-1}}\rho^{n-2}d\sigma\\
&=\int_0^1\rho^{-\left(\gamma\binom{n-1}{k}+\gamma\delta\binom{n-1}{k-1}\right)+n-2}\frac{d\rho}{\sqrt{1-\rho^2}},
\end{aligned}$$

where we used the trivial fact that $|x_i|\leq\left(x_1^2+\cdots+x_{n-1}^2\right)^{1/2}$, for $i=1,\dots,n-1$.
The left-hand side of (2.14) diverges when $-\left[\gamma\binom{n-1}{k}+\gamma\delta\binom{n-1}{k-1}\right]+n-2\leq-1$, i.e.

$$\gamma\geq(n-1)\left[\binom{n-1}{k}+\delta\binom{n-1}{k-1}\right]^{-1}. \tag{2.17}$$

Comparing (2.16) and (2.17) we see that, in order to make the right-hand side finite and the left-hand side divergent, we must have

$$\begin{aligned}
p&<\gamma^{-1}\min\left\{k,\frac{n-1}{\delta}\right\}\\
&\leq(n-1)^{-1}\min\left\{k,\frac{n-1}{\delta}\right\}\left[\binom{n-1}{k}+\delta\binom{n-1}{k-1}\right]\\
&=:g(\delta)\leq\max_{\delta>0}g(\delta).
\end{aligned}$$

Easy computations show that g attains its maximum at $\delta=\frac{n-1}{k}$, for which we have

$$p<g\left(\frac{n-1}{k}\right)=\binom{n-2}{k-1}+\binom{n-1}{k-1}=\tilde{p},$$

thus proving the sharpness of the exponent $\tilde{p}$. $\square$

Remark 2.5.2. One could ask as in the case of functions of one variable if it is possible to obtain an inequality like (2.12) which is not a consequence of embeddings of $L^p(\mathbb{S}^{n-1})$ spaces with a different p_i for each f_i. In this case also an application of Theorem 1.6.2 is not effective. Indeed, even if one allows different p_i's in the nonlinear heat evolution associated to the operator L defined in (2.6), the presence of functions of k variables for each k-tuple of elements from $x_1, \dots, x_n$ forces all exponents p_i to be equal, since by symmetry each element of each finite set $\mathcal{A}_i^c$ has the same number $\tilde{p}$ of occurrences among the sets $\mathcal{A}_k^c$, for $k = 1, \dots, C(n,k)$.

Remark 2.5.3. Since constant functions are trivially functions of k variables, inequality (2.12) also holds for $m \le C(n,k)$ functions of m different k-tuples of variables. The inequality is non-trivial for $m > \tilde{p}$, since when $m \le \tilde{p}$ a direct application of multilinear Hölder's inequality gives a better outcome in terms of exponents. Note that in this case an application of Theorem 1.6.2 could be effective. For example consider functions on the sphere $\mathbb{S}^4$ depending on 2 variables, for which $\tilde{p} = 7$. There are 10 possible pairs of variables in the set $\{1, \dots, 5\}$. Take just $m = 8$ functions, say those associated to the pairs $(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4)$. We denote with f_{ij} the nonnegative function depending on variables x_i, x_j and with $\mathcal{A}_{i,j}^c$ the associated maximal set. It is easy to see that the vector field $L_{1,2}$ lies in all $\mathcal{A}_{i,j}^c$ except for $\mathcal{A}_{3,4}^c$. So the exponent associated to all functions except f_{34} will be $\tilde{p} = 7$. It is also easy to check that each element of $\mathcal{A}_{3,4}^c$ occurs at most six times among all the complements of the maximal sets. So an application of Theorem 1.6.2 shows that the inequality

$$\begin{aligned}\int_{\mathbb{S}^4} & f_{12} f_{13} f_{14} f_{15} f_{23} f_{24} f_{25} f_{34} d\sigma \\ & \le \|f_{12}\|_7 \|f_{13}\|_7 \|f_{14}\|_7 \|f_{15}\|_7 \|f_{23}\|_7 \|f_{24}\|_7 \|f_{25}\|_7 \|f_{34}\|_6\end{aligned}$$

holds. This inequality is nontrivial and is not a direct consequence of Theorem 2.5.1.

Remark 2.5.4. Thanks to Formula (2.4) it is possible to rewrite equation (2.14) in the form

$$\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(\pi_{\omega_i} x) d\sigma \lesssim \prod_{i=1}^{C(n,k)} \|f_i\|_{L^p(B_k)}, \tag{2.18}$$

which has the structure of a Brascamp–Lieb inequality.

2.6 Inequalities for radial functions on k variables

In this section we improve on Theorem 2.5.1 by adding an additional symmetry. We consider functions of k variables, i.e. functions that are defined on a k-dimensional unit ball and pulled-back to the sphere by means of a projection, that are radial with respect to the variables in the k-dimensional ball, for $1 \le k \le n-1$. Given a subset $\omega_i = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, we will use the notation $r(x_{\omega_i})$ to denote the radius $(x_{i_1}^2 + \dots + x_{i_k}^2)^{1/2}$. A functions depending radially on the variables x_ω is a function $f : [0,1] \to \mathbb{R}$ pulled back to the sphere via the composition $r \circ \pi_{\omega_i}$. We will write $f(r(x_{\omega_i}))$ for $f((r(\pi_{\omega_i}(x))))$, with $x \in \mathbb{S}^{n-1}$.
We have $\binom{n}{k} := C(n,k)$ possible choices of k-tuples out of the set $\{1, \dots, n\}$, as in the generic case of functions depending on k variables. We will label the tuples by $\omega_1, \dots, \omega_{C(n,k)}$, as in the previous section. To each collection of variables $\omega_i = \{i_1, \dots, i_k\}$ corresponds a maximal subset $\mathcal{A}_i$ which contains all the vector fields $L_{h,l}$ for which $h, l \notin \omega_i$, but also the vector fields $L_{h,l}$ for which both $h, l \in \omega_i$, by the radiality assumption.

The subalgebra generated by each maximal subset $\mathcal{A}_i$ is isomorphic to the direct sum $\mathfrak{so}(n-k) \oplus \mathfrak{so}(k)$ and has the form

$$\langle \mathcal{A}_i \rangle = \mathfrak{so}_{\alpha_i} \oplus \mathfrak{so}_{\beta_i},$$

where α_i is a multi-index such that $\alpha_i^j = 0$ if $j \in \omega_i$ and $\beta_i = (1, 1, \dots, 1) - \alpha_i$.
Note that by the convention of Theorem 2.3.1 the splitting should be ordered by the cardinality of multi-indices. We can reduce to the cases where $k \leq \lfloor \frac{n}{2} \rfloor$. Indeed, consider a function f that depends radially on the k variables $\{x_{i_1}, \dots, x_{i_k}\}$ and let $\{x_{i_{k+1}}, \dots, x_{i_n}\}$ be the remaining $n-k$ variables. It is straightforward that

$$f(x_{i_1}^2 + \dots + x_{i_k}^2) = f(1 - (x_{i_{k+1}}^2 + \dots + x_{i_n}^2)) = g(x_{i_{k+1}}^2 + \dots + x_{i_n}^2),$$

for some function g. There is a correspondence between functions that depend radially on k variables and functions that depend radially on $n-k$ variables. Indeed, the number of possible choices of k-tuples and $(n-k)$-tuples is the same, since $\binom{n}{k} = \binom{n}{n-k}$, for $k \leq \lfloor \frac{n}{2} \rfloor$. Moreover the splittings of the corresponding associated maximal subsets is the same up to change in the order of the direct summands.
We will stick to the convention that the first direct summand is related to the longest multi-index, so it suffices to look at the case $k \leq \lfloor \frac{n}{2} \rfloor$. The case of n even and $k = n/2$ is a bit different and will be treated separately at the end of the section. In the other cases, i.e. when $k < n/2$, we have that

$$|\mathcal{A}_i| = \binom{n-k}{2} + \binom{k}{2},$$

so that

$$|\mathcal{A}_i^c| = \binom{n}{2} - \binom{n-k}{2} - \binom{k}{2}$$

for $i = 1, \dots, C(n,k)$.
We have the following theorem.

Theorem 2.6.1. *Let $k < n/2$. Let $f_1, \dots, f_{C(n,k)}$ be nonnegative measurable functions, $f_i : [0,1] \to \mathbb{R}^+$. The inequality*

$$\int_{\mathbb{S}^{n-1}} f_1(r(x_{\omega_1})) \dots f_{C(n,k)}(r(x_{\omega_{C(n,k)}})) d\sigma \leq \prod_{i=1}^{C(n,k)} \|f_i\|_{L^p(\mathbb{S}^{n-1})} \tag{2.19}$$

holds for

$$p \geq \tilde{p} = 2\binom{n-2}{k-1}.$$

Moreover inequality (2.19) *is sharp in the sense of Definition 2.3.10.*

Remark 2.6.1. The result of [18] is again recovered, since functions that depend radially on one variable are just even functions of one variable. Indeed, for $k = 1$ we have $\tilde{p} = 2$. Note that the exponent $\tilde{p}$ obtained for this type of functions is smaller than that obtained for generic functions of k variables. This in particular implies that inequality (2.19) is nontrivial in the sense of Definition 1.4.10.

Proof. By Theorem 1.6.1, the exponent $\tilde{p}$ is given by the number of occurrences of the most recurrent element among the sets $\mathcal{A}_i^c$, for $i = 1, \dots, C(n,k)$. The elements of $\mathcal{A}_i^c$ are vector fields of type $L_{h,l}$ with exactly one among h and l in ω_i. So an element $L_{h,l}$ will occur in all

$\mathcal{A}_i^c$ associated to subsets ω_i containing either h but not l, which are $\binom{n-2}{k-1}$, or l but not h, which are again $\binom{n-2}{k-1}$. Altogether, each vector field $L_{h,l}$ will occur $2\binom{n-2}{k-1}$ times among the $\mathcal{A}_i^c$, yielding the exponent $\tilde{p}$.
To prove that $\tilde{p}$ is sharp we construct an explicit counterexample. Consider functions $f_i : [0,1] \to \mathbb{R}^+$, of the form

$$f_i(r(x_{\omega_i})) = (x_{i_1}^2 + \cdots + x_{i_k}^2)^{-\gamma/2} + (1 - x_{i_1}^2 - \cdots - x_{i_k}^2)^{-\gamma\delta/2}, \tag{2.20}$$

where γ, δ are positive constants to be determined. One could proceed with an unknown δ and then optimize as we did in the proof of Theorem 2.5.1, but to simplify the proof we will take the optimal δ, that is $\frac{n-k}{k}$, from the outset.
The right-hand side of inequality (2.19) must be finite. We start computing the L^p norm of such functions. Without loss of generality we focus on the case $\omega = \{1, \ldots, k\}$ and work with $f(x_1^2 + \cdots + x_k^2)$.
Let $p \geq 1$. We have

$$\begin{aligned}
\|f\|_{L^p(\mathbb{S}^{n-1})}^p &= \int_{\mathbb{S}^{n-1}} \left[(x_1^2 + \cdots + x_k^2)^{-\gamma/2} + (1 - x_1^2 - \cdots - x_k^2)^{-\frac{\gamma(n-k)}{2k}}\right]^p d\sigma \\
&\lesssim \int_{\mathbb{S}^{n-1}} \left[(x_1^2 + \cdots + x_k^2)^{-\gamma p/2} + (1 - x_1^2 - \cdots - x_k^2)^{-\frac{\gamma(n-k)p}{2k}}\right] d\sigma \\
&\lesssim \int_{B_k} \Big[(x_1^2 + \cdots + x_k^2)^{-\gamma p/2} \\
&\quad + (1 - x_1^2 - \cdots - x_k^2)^{-\frac{\gamma(n-k)p}{2k}}\Big] (1 - x_1^2 - \cdots - x_k^2)^{(n-k-2)/2} dx_1 \ldots dx_k \\
&\lesssim \int_0^1 \rho^{-\gamma p + k - 1}(1-\rho^2)^{(n-2-k)/2} + \rho^{k-1}(1-\rho^2)^{-\frac{\gamma(n-k)p}{2k} + \frac{n-k-2}{2}} d\rho,
\end{aligned}$$

where we used the integration formula (2.2) and then passed to polar coordinates. This integral is finite if $\gamma p < k$.
We control the left-hand side of (2.19) by the following trivial bounds:

$$(x_1^2 + \cdots + x_k^2)^{-\gamma} \geq (x_1^2 + \cdots + x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2)^{-\gamma},$$

with $\gamma > 0$, for terms not involving the x_n variable, and

$$(1 - x_1^2 - \cdots - x_n^2)^{-\gamma} \geq (1 - x_n^2)^{-\gamma},$$

for terms involving the x_n variable. We make this distinction since we want to pass to polar coordinates in the hyperplane $\mathbb{R}^{n-1}$ with coordinates $x_1, \ldots, x_{n-1}$; on the sphere $\mathbb{S}^{n-1}$, $|x_n|$ will then just be $(1-\rho^2)^{1/2}$.
There are $\binom{n-1}{k}$ terms not involving x_n, and $\binom{n-1}{k-1}$ involving it. For the functions not depending on x_n we select the first summand of (2.20), for those depending on x_n we select the second one.
For the left-hand side we have:

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(r(x_{\omega_i})) d\sigma \\
&\geq \int_{\mathbb{S}^{n-1}} (x_1^2 + \cdots + x_{n-1}^2)^{-\frac{\gamma}{2}\binom{n-1}{k}} (1 - x_n^2)^{-\frac{\gamma}{2}\frac{n-k}{k}\binom{n-1}{k-1}} d\sigma
\end{aligned}$$

$$\geq \int_0^1 \rho^{-(\gamma\binom{n-1}{k}+\gamma\frac{n-k}{k}\binom{n-1}{k-1})+n-2}\frac{d\rho}{\sqrt{1-\rho^2}}.$$

This integral diverges for

$$\gamma \geq (n-1)\left[\binom{n-1}{k}+\frac{n-k}{k}\binom{n-1}{k-1}\right]^{-1}. \tag{2.21}$$

Comparing the condition $\gamma p < k$ and (2.21) we see that, in order to make finite the right-hand side and divergent the left-hand side, we must have

$$\begin{aligned} p &< k\gamma^{-1} \\ &\leq \frac{k}{n-1}\left[\binom{n-1}{k}+\frac{n-k}{k}\binom{n-1}{k-1}\right] \\ &= 2\binom{n-2}{k-1} = \tilde{p}, \end{aligned}$$

thus proving the optimality of the exponent $\tilde{p}$. $\square$

In the case of n even and functions depending radially on $k = n/2$ variables, the splitting associated to a maximal subset is of type $\mathfrak{so}(n/2) \oplus \mathfrak{so}(n/2)$ so that there are two possible orderings. This corresponds to the fact that, given a subset $\omega_i = \{i_i, \ldots, i_{n/2}\}$ of $\{1, \ldots, n\}$, the set $\{i_{(n/2)+1}, \ldots, i_n\}$ being its complement, a function radial in the variables of ω_i is also radial in the variables of its complement, but in this case both sets have cardinality $n/2$. So one needs to consider a family of (different) k-tuples ω_i, for $i = 1, \ldots, C(n,k)/2$, with $\omega_i \cap \omega_j \neq \emptyset$ for all i, j. Different choices of families of subsets ω_i give equivalent types of functions. In this setting we have that $|\mathcal{A}_i| = \frac{n}{2}\left(\frac{n}{2}-1\right)$, so that $|\mathcal{A}_i^c| = \frac{n^2}{4}$, for all $i = 1, \ldots, C(n,k)/2$.
We have the following theorem.

Theorem 2.6.2. *Let $n > 3$ be even and $k = n/2$. Let ω_i be a family of $C(n,k)/2$ different k-tuples such that $\omega_i \cap \omega_j \neq \emptyset$ for all i, j. Let $f_1, \ldots, f_{C(n,k)/2}$ be nonnegative measurable functions, $f_i : [0,1] \to \mathbb{R}^+$. The inequality*

$$\int_{\mathbb{S}^{n-1}} f_1(r(x_{\omega_1})) \ldots f_{C(n,k)/2}(r(x_{\omega_{C(n,k)/2}}))d\sigma \leq \prod_{i=1}^{C(n,k)/2} \|f_i\|_{L^p(\mathbb{S}^{n-1})} \tag{2.22}$$

holds for

$$p \geq \tilde{p} = \binom{n-2}{k-1}.$$

Moreover inequality (2.22) *is sharp in the sense of Definition 2.3.10.*

Proof. We need to understand how many the occurrences of each vector field $L_{h,l}$ among the sets $\mathcal{A}_i^c$ are. As before we must consider sets ω_i containing either h or l, but not both. To a k-tuple σ containing h and not l corresponds a unique k-tuple τ containing l and not h such that $\sigma \cap \tau = \emptyset$. By the assumptions on the ω_i, one and just one between σ and τ is among the sets ω_i. Thus, each vector field $L_{h,l}$ occurs $\binom{n-2}{k-1}$ times among the $\mathcal{A}_i^c$. This provides the exponent $\tilde{p}$.
To prove the sharpness we use essentially the same argument as in Theorem 2.6.1. Without loss of generality, upon renaming variables, we consider a family of ω_i such that $n \notin \omega_i$ for all $i = 1, \ldots, C(n,k)/2$. We consider functions $f_i : [0,1] \to \mathbb{R}^+$ of the form

$$f_i(r(x_{\omega_i})) = (x_{i_1}^2 + \cdots + x_{i_k}^2)^{-\gamma/2} \tag{2.23}$$

with $\gamma > 0$ to be determined. The right-hand side is finite for $\gamma p < k$, by the same computations as in the previous proof. For the left-hand side we have

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)/2} f_i(r(x_{\omega_i}))d\sigma \\
&\geq \int_{\mathbb{S}^{n-1}} (x_1^2 + \cdots + x_{n-1}^2)^{-\frac{\gamma}{2}\frac{C(n,k)}{2}} d\sigma \\
&\geq \int_0^1 \rho^{-\frac{\gamma C(n,k)}{2}+n-2} \frac{d\rho}{\sqrt{1-\rho^2}},
\end{aligned}$$

which diverges for

$$\gamma \geq \frac{2(n-1)}{C(n,k)}. \tag{2.24}$$

Comparing the conditions $\gamma p < k$ and (2.24) we see that, in order to make the right-hand side finite and the left-hand side divergent, we must have

$$\begin{aligned}
p &< k\gamma^{-1} \\
&\leq \frac{k}{2(n-1)}\binom{n}{k} = \frac{n}{2(n-k)}\binom{n-2}{k-1} \\
&= \binom{n-2}{k-1} = \tilde{p},
\end{aligned}$$

since $(n-k) = n/2$, thus proving the sharpness of the exponent $\tilde{p}$. □

Remark 2.6.2. In the case n even and $k = n/2$ one could also apply Theorem 2.6.1 and work with all k-tuples ω_i of elements in $\{1, \ldots, k\}$. This means counting twice functions that have the same subalgebra of annihilating vector fields, upon commutating the direct summands. Anyway one could understand the product of function $f(x_{i_1}^2 + \cdots + x_{i_k}^2)g(x_{i_{k+1}}^2 + \cdots + x_{i_n}^2)$ as a function $F(x_{i_1}^2 + \cdots + x_{i_k}^2)$. Since the exponent $\tilde{p}$ in Theorem 2.6.1 is the double of that of Theorem 2.6.2, from Theorem 2.6.2 applied to the $C(n,k)/2$ functions F_i just defined we can deduce the result of Theorem 2.6.1 by applying Cauchy-Schwarz inequality. Indeed, let ω_i be a family of subsets satisfying the hypotheses of Theorem 2.6.2 and let ω_i^c be their complements. We have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)/2} f_i(r(x_{\omega_i}))g_i(r(x_{\omega_i^c}))d\sigma &= \int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)/2} F_i(r(x_{\omega_i}))d\sigma \\
\leq \prod_{i=1}^{C(n,k)/2} \|F_i\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})} &= \prod_{i=1}^{C(n,k)/2} \|f_i g_i\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})} \\
\leq \prod_{i=1}^{C(n,k)/2} \|f_i\|_{L^{2\tilde{p}}(\mathbb{S}^{n-1})}\|g_i\|_{L^{2\tilde{p}}(\mathbb{S}^{n-1})}&,
\end{aligned}$$

which is exactly the estimate in Theorem 2.6.1.

2.7 Inequalities for different exponents

In Remark 2.5.2 of Theorem 2.5.1 we saw that an application of Theorem 1.6.1 always yields the same exponent for all functions. In this section we want to understand for what exponents

$p_1, \dots, p_{C(n,k)}$ an inequality of the type

$$\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i}) d\sigma \leq \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{p_i}(\mathbb{S}^{n-1})} \tag{2.25}$$

may hold. Each p_i can vary between 1 and ∞, so that p_i^{-1} varies in $[0,1]$. A point in the unit cube $\mathcal{Q} = [0,1]^{C(n,k)}$ in $\mathbb{R}^{C(n,k)}$ identifies a choice of exponents. The inequality holds for points $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \in \mathcal{Q}$ for which

$$\sum_{i=1}^{C(n,k)} p_i^{-1} \leq 1$$

by Hölder's inequality and continuous embeddings of Lebesgue spaces. By Theorem 2.5.1 we know that (2.25) also holds for $(\tilde{p}^{-1}, \dots, \tilde{p}^{-1}) \in \mathcal{Q}$, with $\tilde{p} = \binom{n}{k} - \binom{n-2}{k}$. Then inequality (2.25) holds for points $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \in \mathcal{Q}$ for which

$$\sum_{i=1}^{C(n,k)} p_i^{-1} \leq \frac{C(n,k)}{\tilde{p}}$$

with $p_i > \tilde{p}$ for all $i = 1, \dots, C(n,k)$, by using again continuous embeddings of Lebesgue spaces on the sphere.
Nevertheless we can extend the range of exponents for which (2.25) is valid by interpolation. We now state a result that we will use for this purpose.

Theorem 2.7.1 (Multilinear interpolation). *Let (X,μ) be a measure space. Let $1 < p_k, q_k < \infty$, with $k = 1, \dots, n$, and for $\theta \in [0,1]$ let*

$$\frac{1}{r_k} = \frac{\theta}{p_k} + \frac{1-\theta}{q_k}.$$

Suppose that a multilinear map T satisfies

$$|T(f_1, \dots, f_n)| \leq A_1 \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(X)},$$

and

$$|T(f_1, \dots, f_n)| \leq A_2 \prod_{i=1}^{n} \|g_i\|_{L^{q_i}(X)},$$

for $f_i \in L^{p_i}(X)$ and $g_i \in L^{q_i}(X)$. Then we have

$$|T(f_1, \dots, f_n)| \leq A_1^{\theta} A_2^{1-\theta} \prod_{i=1}^{n} \|f_i\|_{L^{r_i}(X)},$$

for $f_i \in L^{r_i}(X)$.

A proof of Theorem 2.7.1 can be found in [47] (see also [31]).
In our case the measure space is $\big(B_k, (1-|x|^2)^{(n-k-2)/2} dx\big)$, where dx is the Lebesgue measure in $\mathbb{R}^k$ and the functional is

$$T(f_1, \dots, f_{C(n,k)}) = \int_{\mathbb{S}^{n-1}} \prod f_i(\pi_{\omega_i} x) d\sigma.$$

Recall that thanks to the integration formula (2.2) we have

$$\|f_i \circ \pi_{\omega_i}\|_{L^p(\mathbb{S}^{n-1})} \simeq \|f_i\|_{L^p(B_k,(1-|x|^2)^{(n-k-2)/2}dx)}.$$

From Theorem 2.7.1, by interpolating the exponents that verify Hölder's condition with the point $(\tilde{p}^{-1}, \dots, \tilde{p}^{-1})$, we obtain the following corollary.

Corollary 2.7.1. *Let $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \in \mathcal{Q}$ be such that*

$$\sum_{i=1}^{C(n,k)} p_i^{-1} \leq 1.$$

Then the inequality

$$\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i}) d\sigma \leq \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{r_i}(\mathbb{S}^{n-1})}$$

holds for all exponents r_i such that

$$\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{\tilde{p}}$$

for all $i = 1, \dots, C(n,k)$ and $\theta \in [0,1]$.

So (2.25) holds in the convex hull $\mathcal{R}$ of the region $\sum p_i^{-1} \leq 1$ and the point $(\tilde{p}^{-1}, \dots, \tilde{p}^{-1})$. We conjecture that outside $\mathcal{R}$ the inequality (2.25) is false, or in other words that for all points $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \notin \mathcal{R}$ there are functions f_i such that the right-hand side of (2.25) is finite, while the left-hand side diverges. Theorem 2.5.1 excludes the points $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \in \mathcal{Q}$ such that $p_i < \tilde{p}$ for all i, that are not in $\mathcal{R}$, but do not exhaust the complement of $\mathcal{R}$ in $\mathcal{Q}$. Unfortunately we do not have a complete proof of the conjecture. We have however the following partial result for points in the hyperplane in $\mathcal{Q}$ given by the equation

$$\sum_{i=1}^{C(n,k)} p_i^{-1} = \frac{C(n,k)}{\tilde{p}},$$

to which the point $(\tilde{p}^{-1}, \dots, \tilde{p}^{-1})$ belongs.

Theorem 2.7.2. *Let $(p_1^{-1}, \dots, p_{C(n,k)}^{-1}) \in \mathcal{Q}$ be such that*

$$\sum_{i=1}^{C(n,k)} p_i^{-1} = \frac{C(n,k)}{\tilde{p}}. \tag{2.26}$$

For any l consider the set J_l consisting of the indices j such that $l \in \omega_j$ (then $|J_l| = \binom{n-1}{k-1}$). If there is l such that

$$\sum_{j \in J_l} p_j^{-1} > \frac{\binom{n-1}{k-1}}{\tilde{p}}, \tag{2.27}$$

then the inequality (2.25) is false.

Proof. Without loss of generality suppose that $l = n$ in the hypothesis. We label the $\binom{n-1}{k-1}$ sets ω_i for which $n \in \omega_i$ with $i = 1, \dots, \binom{n-1}{k-1}$. Consider functions $f_i : B_k \to \mathbb{R}^+$, where B_k is the unit ball in $\mathbb{R}^k$, of the form

$$f_i(x_{\omega_i}) = (|x_{i_1}||x_{i_2}| \dots |x_{i_k}|)^{-\gamma_i/k} + (1 - x_{i_1}^2)^{-\frac{\gamma_i(n-1)}{2k}} + \dots + (1 - x_{i_k}^2)^{-\frac{\gamma_i(n-1)}{2k}}. \tag{2.28}$$

From the proof of Theorem 2.5.1 we know that each f_i is in $L^{p_i}(\mathbb{S}^{n-1})$ if

$$\gamma_i p_i < k \tag{2.29}$$

for all $i = 1, \dots, C(n, k)$, so that under this assumption the right-hand side of (2.25) is finite. For the left-hand side we proceed as follows. For the functions not depending on the variable x_n we select the first summand in (2.28), for those depending on x_n we select the summand $(1 - x_n^2)^{-\frac{\gamma_i(n-1)}{2k}}$.
So the left-hand side satisfies:

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i}) d\sigma \\
&\geq \int_{\mathbb{S}^{n-1}} \prod_{i=\binom{n-1}{k-1}+1}^{C(n,k)} (|x_{i_1}| \dots |x_{i_k}|)^{-\frac{\gamma_i}{k}} \prod_{j=1}^{\binom{n-1}{k-1}} (1 - x_n^2)^{-\frac{\gamma_j(n-1)}{2k}} d\sigma \\
&\geq \int_0^1 \rho^{-\sum_i \gamma_i - \frac{n-1}{k} \sum_j \gamma_j + n - 2} \frac{d\rho}{\sqrt{1 - \rho^2}},
\end{aligned}$$

where we proceeded as in the proof of Theorem 2.5.1. We call I the set $\{\binom{n-1}{k-1} + 1, \dots, C(n, k)\}$ and J the set $\{1, \dots, \binom{n-1}{k-1}\}$. The left-hand side of (2.25) diverges if

$$-\sum_{i \in I} \gamma_i - \frac{n-1}{k} \sum_{j \in J} \gamma_j + n - 2 = -1,$$

that is

$$\sum_{i \in I} \gamma_i = (n-1) \left(1 - \frac{1}{k} \sum_{j \in J} \gamma_j \right).$$

Since by (2.29) $\sum_{i \in I} \gamma_i < k \sum_{i \in I} p_i^{-1}$, to make the left-hand side divergent we must have

$$\left((n-1) - k \sum_{i \in I} p_i^{-1} \right) < \frac{n-1}{k} \sum_{j \in J} \gamma_j.$$

Since by (2.29) $\sum_{j \in J} \gamma_j < k \sum_{j \in J} p_j^{-1}$, we must also have

$$\left(1 - \frac{k}{n-1} \sum_{i \in I} p_i^{-1} \right) < \frac{1}{k} \sum_{j \in J} \gamma_j < \sum_{j \in J} p_j^{-1}.$$

It is possible to choose γ_j so that $k^{-1} \sum_{j \in J} \gamma_j$ is squeezed between these two terms if

$$\left(1 - \frac{k}{n-1} \sum_{i \in I} p_i^{-1} \right) < \sum_{j \in J} p_j^{-1},$$

which by (2.26) becomes

$$1 - \frac{k}{n-1}\left(\frac{C(n,k)}{\tilde{p}} - \sum_{j\in J} p_j^{-1}\right) < \sum_{j\in J} p_j^{-1},$$

which is equivalent to

$$\sum_{j\in J} p_j^{-1} > \left(1 - \frac{k}{n-1}\right)^{-1}\left(1 - \frac{k}{n-1}\frac{C(n,k)}{\tilde{p}}\right),$$

from which one deduces the assumption in the theorem. □

Remark 2.7.2. Note that there are n and k for which the assumptions of Theorem 2.7.2 are not fulfilled. For example, consider $n = 4$ and $k = 2$, i.e. the case of functions of two variables on the sphere $\mathbb{S}^3$, for which $\tilde{p} = 5$. There are 6 possible tuples (i, j), with $1 \le i < j \le 4$. We will denote by p_{ij} the exponent corresponding to the pair (i, j). It is easy to check that choosing $p_{12} = p_{13} = p_{24} = p_{34} = 10$, $p_{23} = p_{14} = 5/2$, we have that $\sum_{i<j} p_{ij}^{-1} = 6/5$. Nevertheless, Theorem 2.7.2 cannot be applied, since for all l and all triples (j_1, j_2, j_3) we have $p_{lj_1} + p_{lj_2} + p_{lj_3} \le 3/5$, so that (2.27) is never satisfied.

2.8 The case $n = 3$ and $k = 1$

In this section we will discuss in more detail the case of functions of one variable on the sphere $\mathbb{S}^2$. We want to understand for which $(p_1^{-1}, p_2^{-1}, p_3^{-1}) \in \mathcal{Q} = [0, 1]^3$ the inequality

$$\int_{\mathbb{S}^2} f_1(x_1)f_2(x_2)f_3(x_3)d\sigma \le \|f_1\|_{L^{p_1}(\mathbb{S}^2)}\|f_2\|_{L^{p_2}(\mathbb{S}^2)}\|f_3\|_{L^{p_3}(\mathbb{S}^2)} \tag{2.30}$$

holds true for all measurable functions $f_i : [-1, 1] \to \mathbb{R}^+$, for $i = 1, 2, 3$. As explained in the previous section, the inequality holds in the region $\mathcal{R}$, which is the convex hull of the the Hölder's tetrahedron and the point $(1/2, 1/2, 1/2)$ given by Theorem 2.4.1.

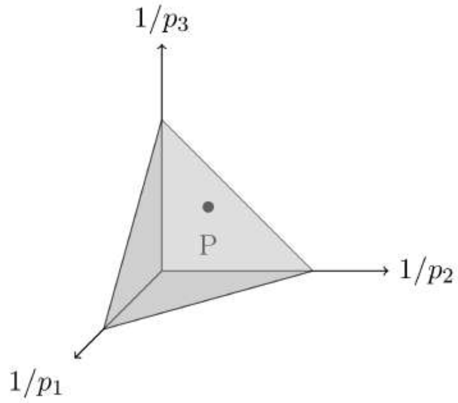


Figure 2.2: Hölder's tetrahedron and the point $P = (1/2, 1/2, 1/2)$.

Moreover in this case the assumptions of Theorem 2.7.2 are always fulfilled, since given any triple $(p_1^{-1}, p_2^{-1}, p_3^{-1}) \neq (1/2, 1/2, 1/2)$ such that $p_1^{-1} + p_2^{-1} + p_3^{-1} = 3/2$, by pigeonholing there must always be one $p_i > 1/2$. This implies that the point $(1/2, 1/2, 1/2)$ is the only point in

the hyperplane $p_1^{-1} + p_2^{-1} + p_3^{-1} = 3/2$ where inequality (2.30) holds.
From this we also deduce that inequality (2.30) cannot hold for points in $\mathcal{Q}$ such that $p_1^{-1} + p_2^{-1} + p_3^{-1} > 3/2$. Indeed, by interpolation with points in $\mathcal{R}$ this would yield points in the hyperplane $p_1^{-1}+p_2^{-1}+p_3^{-1} = 3/2$ for which the inequality holds, providing a contradiction. This goes in the direction of our conjecture, that the region $\mathcal{R}$ is the optimal region of validity for inequality (2.30).

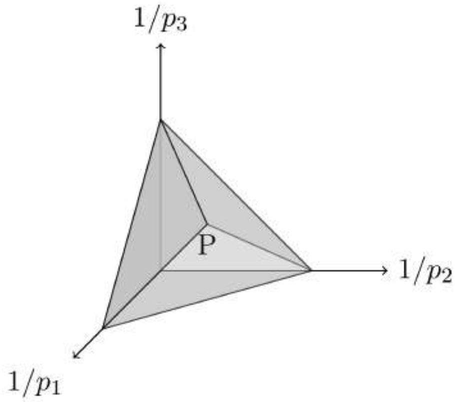


Figure 2.3: The conjectured sharp region $\mathcal{R}$.

The only points left are those outside of $\mathcal{R}$ for which $1 < p_1^{-1} + p_2^{-1} + p_3^{-1} < 3/2$. In this range we have the following proposition which leads to a partial improvement towards the sharpness.

Proposition 2.8.1. *Suppose that* $1 < p_1^{-1} + p_2^{-1} + p_3^{-1} < 3/2$ *and that the condition*

$$\frac{1}{p_a} + \frac{1}{p_b} > 2\left(1 - \frac{1}{p_c}\right) \tag{2.31}$$

holds for at least one choice of a, b, c *in* $\{1, 2, 3\}$ *with* a, b, c *pairwise distinct. Then inequality* (2.30) *is false.*

Proof. We make the usual construction. Assume for instance that $a = 1, b = 2, c = 3$. We let

$$f_i(x_i) = |x_i|^{-\gamma_i} + (1 - x_i^2)^{-\frac{(n-1)\gamma_i}{2}} = |x_i|^{-\gamma_i} + (1 - x_i^2)^{-\gamma_i},$$

for $i = 1, 2, 3$. As usual the integrability condition for the right-hand side of (2.30) is $\gamma_i p_i < 1$. For the left-hand side, taking the first summand for f_1 and f_2 and the second one for f_3, we get that

$$\begin{aligned}&\int_{\mathbb{S}^2} f_1(x_1) f_2(x_2) f_3(x_3) d\sigma \\ &\geq \int_0^1 \rho^{-\gamma_1-\gamma_2-2\gamma_3+1} \frac{d\rho}{\sqrt{1-\rho^2}},\end{aligned}$$

which diverges for $\gamma_1 + \gamma_2 + 2\gamma_3 = 2$, that is for

$$\gamma_3 = 1 - \frac{\gamma_1 + \gamma_2}{2}.$$

From the condition $\gamma_i p_i < 1$ we get that we need to have

$$2\left(1 - \frac{1}{p_3}\right) < \gamma_1 + \gamma_2 < \frac{1}{p_1} + \frac{1}{p_2}.$$

Clearly $\gamma_1 + \gamma_2$ can be in this range only when (2.31) holds. $\square$

Remark 2.8.2. To sum up, we do not know what happens in the range $1 < p_1^{-1} + p_2^{-1} + p_3^{-1} < 3/2$, outside of $\mathcal{R}$, where none of the conditions (2.31) is satisfied for any exponent p_i. An example of a point in this region is $(2/3, 2/3, 0)$.

2.9 Inequalities with other symmetries

In the last sections we saw applications of Theorem 1.6.1 in special cases, where the choices of the maximal subsets $\mathcal{A}_i$ of $\{L_{j,l}\}_{j<l}$ reflected particular symmetries of the functions involved. Nevertheless, Theorem 1.6.1 (and Theorem 1.6.2) can also be applied to other type of symmetries. Indeed, let $\mathcal{A}_i$ be maximal subsets for $i = 1, \dots, m$.
An easy algorithm to compute the exponent $\tilde{p}$ of Theorem 1.6.1 and the exponents $\tilde{p}_i$ of Theorem 1.6.2 is as follows. We consider the matrix of zeros and ones with m rows indexed by the m maximal subsets and $\binom{n}{2}$ columns indexed by the vector fields of the basis of $\mathfrak{so}(n)$. We set $a_{ij} = 1$ if the vector field corresponding to the j-th column is in $\mathcal{A}_i^c$ and zero otherwise. In this way the exponent $\tilde{p}$ of Theorem 1.6.1, being the number of occurrences of the most recurrent element among the $\mathcal{A}_i^c$, is just

$$\max_j \sum_{i=1}^{m} a_{ij}.$$

The exponent $\tilde{p}_i$ in Theorem 1.6.2, being the number of occurrences of the most recurrent element in $\mathcal{A}_i^c$, is given by

$$\max_{j\,:\,a_{ij}=1} \sum_{k=1}^{m} a_{kj},$$

where we take the maximum only over the columns j for which $a_{ij} = 1$, so that we check how many times the vector fields that are contained in $\mathcal{A}_i^c$ occur in the sets $\mathcal{A}_k^c$.
Here we show two examples. We remark that also in these examples the exponents given by Theorem 1.6.1 turn out to be sharp.

Example 2.9.1. On the sphere $\mathbb{S}^3$ consider three functions, f_1 depending on the variable x_1, f_2 depending on the variable x_2, and f_3 depending radially on x_1 and x_2 (or equivalently depending radially on x_3 and x_4). The maximal subset annihilating f_1 is $\mathcal{A}_1 = \{L_{2,3}, L_{2,4}, L_{3,4}\}$, with $\langle \mathcal{A}_1 \rangle \simeq \mathfrak{so}(3)$, so that $\mathcal{A}_1^c = \{L_{1,2}, L_{1,3}, L_{1,4}\}$. The maximal subset annihilating f_2 is $\mathcal{A}_2 = \{L_{1,3}, L_{1,4}, L_{3,4}\}$, with $\langle \mathcal{A}_2 \rangle \simeq \mathfrak{so}(3)$, so that $\mathcal{A}_2^c = \{L_{1,2}, L_{2,3}, L_{2,4}\}$. The maximal subset annihilating f_3 is $\mathcal{A}_3 = \{L_{1,2}, L_{3,4}\}$, with $\langle \mathcal{A}_3 \rangle \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(2)$, so that $\mathcal{A}_3^c = \{L_{1,3}, L_{1,4}, L_{2,3}, L_{2,4}\}$. Each $\mathcal{A}_i^c$ has an element that occurs twice among the sets $\mathcal{A}_k^c$, for $k = 1, 2, 3$, so by Theorem 1.6.1 we have $\tilde{p} = 2$. It follows that

$$\int_{\mathbb{S}^3} f_1(x_1) f_2(x_2) f_3(x_1^2 + x_2^2)\, d\sigma \leq \|f_1\|_{L^2(\mathbb{S}^3)} \|f_2\|_{L^2(\mathbb{S}^3)} \|f_3\|_{L^2(\mathbb{S}^3)}.$$

Moreover this inequality is sharp. Indeed, consider the functions $f_i(x_i) = |x_i|^{-1/2}$ for $i = 1, 2$ and the function $f_3(x_1^2 + x_2^2) = (x_1^2 + x_2^2)^{-1/2}$. It is easy to see, proceeding in the same way as above, that $\|f_i\|_{L^p(\mathbb{S}^3)} < \infty$ with $p < 2$ for $i = 1, 2, 3$. Nonetheless we have that

$$\begin{aligned}\int_{\mathbb{S}^3} f_1 f_2 f_3 d\sigma &= \int_{\mathbb{S}^3} |x_1|^{-1/2}|x_2|^{-1/2}(x_1^2 + x_2^2)^{-1/2} d\sigma \\ &\sim \int_{B_2} |x_1|^{-1/2}|x_2|^{-1/2}(x_1^2 + x_2^2)^{-1/2}(1 - x_1^2 - x_2^2)^{(4-2-2)/2} dx_1 dx_2 \\ &\gtrsim \int_0^1 \rho^{-(1/2)-(1/2)-1}\rho d\rho = \int_0^1 \rho^{-1} d\rho,\end{aligned}$$

that diverges.

Example 2.9.2. On the sphere $\mathbb{S}^4$ we consider functions depending on $k = 3$ variables, with radial dependence on two of them. This corresponds to the case of maximal subsets $\mathcal{A}_i$ with two elements $L_{i,j}, L_{k,l}$ with i, j, k, l pairwise distinct so that the generated subalgebras have the form

$$\langle \mathcal{A}_i \rangle = \mathfrak{so}(2) \oplus \mathfrak{so}(2).$$

As we discussed above, the first subalgebra indicates the number of variables the functions depend on, in this case we have $n - k = 5 - 2 = 3$. The second subalgebra refers to radiality in two of the variables involved. The ambiguity in the order of the subalgebras is not a problem, since the two possibilities are equivalent in the following sense. If $\mathcal{A} = \{L_{1,2}, L_{3,4}\}$ we are considering a function f either of type $f(x_3^2 + x_4^2, x_5)$ or a function of type $f(x_1^2 + x_2^2, x_5)$ which are equivalent, since $x_3^2 + x_4^2 = 1 - x_1^2 - x_2^2 - x_5^2$.
There are $\binom{5}{2} = 10$ possible choices for $L_{i,j}$, and having fixed i and j we have $\binom{3}{2} = 3$ choices for $L_{k,l}$. By the aforementioned equivalence we have 15 possible maximal subsets.
It is easy to see that in this case the critical exponent given by Theorem 1.6.1 is $\tilde{p} = 12$ and it is sharp. Indeed, consider the functions

$$f_i^{jl}(x_i, x_j^2 + x_l^2) = |x_i|^{-1/12}(x_j^2 + x_l^2)^{-1/12} + (1 - x_i^2)^{-1/6}. \tag{2.32}$$

Note that the function f_i^{jl} is equivalent to the function $f_i^{j'l'}$ where $\{j', l'\}$ is the complement in $\{1, \dots, 5\}$ of the set $\{i, j, l\}$, so that the variable x_i is fixed but we can change j, l obtaining an equivalent function. Thanks to this remark we can choose functions in a way that the variable x_5 never appears in the radial part. The L^p norm of f_i^{jl} is controlled by

$$\begin{aligned}\|f_i^{jl}\|_{L^p(\mathbb{S}^4)}^p &\lesssim \int_{\mathbb{S}^4} \left(|x_i|^{-p/12}(x_j^2 + x_l^2)^{-p/12} + (1 - x_i^2)^{-p/6}\right) d\sigma \\ &\sim \int_{B_3} |x_i|^{-p/12}(x_j^2 + x_l^2)^{-p/2}(1 - x_i^2 - x_j^2 - x_l^2)^{\frac{5-3-2}{2}} dx_i dx_j dx_l \\ &+ \int_{-1}^1 (1 - x_i^2)^{-p/6+1} dx_i \\ &\lesssim \int_{-1}^1 |x_i|^{-p/12} dx_i \int_{-1}^1 \int_{-1}^1 (x_j^2 + x_l^2)^{-p/12} dx_j dx_l + \int_{-1}^1 (1 - x_i^2)^{-p/6+1} dx_i,\end{aligned}$$

which is finite for $p < 12$. On the left-hand side of the inequality we take for $i \neq 5$ the first term in (2.32) and for $i = 5$ the second. Hence we choose the first term 12 times and the second one 3 times. We use the estimate $|x_i|^{-\gamma}(x_j^2+x_l^2)^{-\gamma} \geq (\sqrt{x_1^2+x_2^2+x_3^2+x_4^2})^{-\gamma-2\gamma}$, whenever $i,j,l \neq 5$, obtaining

$$\begin{aligned}\int_{\mathbb{S}^4} \prod f_i^{jl} d\sigma &\geq \int_{\mathbb{S}^4} \left(\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\right)^{-3\frac{1}{12}12} (1-x_5^2)^{-\frac{1}{6}3} d\sigma \\ &\sim \int_0^1 \rho^{-3}\rho^{-1}\rho^3 \frac{d\rho}{\sqrt{1-\rho^2}} \sim \int_0^1 \rho^{-1}\frac{d\rho}{\sqrt{1-\rho^2}},\end{aligned}$$

which diverges, thus proving the sharpness of $\tilde{p}=12$.

2.10 Mixed norm inequalities

As an application of the inequalities found in this chapter we prove some inequalities in mixed norm spaces. We introduce, for a nonnegative function f, defined on $\mathbb{R}^n$, the mixed norms

$$\begin{aligned}\|f\|_{L^p_{\mathrm{rad}}L^q_{\mathbb{S}}} &= \left(\int_0^\infty \left(\int_{\mathbb{S}^{n-1}} f(\rho x')^q d\sigma(x')\right)^{\frac{p}{q}} \rho^{n-1}d\rho\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \|f(\rho\cdot)\|^p_{L^q(\mathbb{S}^{n-1})}\rho^{n-1}d\rho\right)^{\frac{1}{p}},\end{aligned}$$

where $\mathbb{R}^n \ni x = \rho x'$ with $x' \in \mathbb{S}^{n-1}$, and in this case the measure $d\sigma$ is not normalized. Using the same notation as above to denote k-tuples of variables, by applying Theorem 2.5.1 we obtain the following proposition.

Proposition 2.10.1. *Let $f_i : \mathbb{R}^k \to \mathbb{R}^+$ for $i=1,\dots,C(n,k)$ and let $\tilde{q} = \binom{n}{k} - \binom{n-2}{k}$. The inequality*

$$\int_{\mathbb{R}^n} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i})dx \lesssim \prod_{i=1}^{C(n,k)} \|f_i\|_{L^p_{rad}L^q_{\mathbb{S}}}$$

holds, with $\sum p_i^{-1} = 1$ and $q_i \geq \tilde{q}$.

These estimates are obtained observing that a function $f : \mathbb{R}^k \to \mathbb{R}^+$ restricted to a sphere of radius ρ gives rise to a function which is defined on the sphere and depends on k variables in the sense of Definition 2.1.1.

Proof. The proof is just an application of Theorem 2.5.1 and Hölder's inequality. Indeed, passing to spherical coordinates we have

$$\begin{aligned}\int_{\mathbb{R}^n} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i})dx &= \int_0^\infty \Omega_{n-1}\int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(\rho x'_{\omega_i})\frac{d\sigma}{\Omega_{n-1}}\rho^{n-1}d\rho \\ &\lesssim \Omega_{n-1}^{1-\sum q_i^{-1}} \int_0^\infty \prod_{i=1}^{C(n,k)} \|f_i(\rho\cdot)\|_{L^{q_i}(\mathbb{S}^{n-1})}\rho^{n-1}d\rho \\ &\lesssim \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{p_i}_{\mathrm{rad}}L^{q_i}_{\mathbb{S}}},\end{aligned}$$

where the normalizations factors are introduced in order to apply Theorem 2.5.1 and in the last line we used Hölder's inequality with exponents p_i. $\square$

Observe that an analogous result can be obtained by considering other types of symmetries and applying Theorem 1.6.1 or Theorem 1.6.2 in their general form.

Remark 2.10.2. If we integrate the product $\prod_{i=1}^{C(n,k)} f_i(x_{\omega_i})$ over a ball $B_n(0,R)$ of $\mathbb{R}^n$, centered at 0 and of radius R, we obtain a local Brascamp–Lieb inequality with a blow-up factor. Indeed, by using (2.18), we see that

$$\begin{aligned}\int_{B_n(0,R)} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i})dx &\sim \int_0^R \int_{\mathbb{S}^{n-1}} \prod_{i=1}^{C(n,k)} f_i(\rho x'_{\omega_i})d\sigma\rho^{n-1}d\rho \\ &\lesssim \int_0^R \prod_{i=1}^{C(n,k)} \|f_i(\rho\cdot)\|_{L^{q_i}(B_k(0,1))}\rho^{n-1}d\rho \\ &\sim \int_0^R \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{q_i}(B_k(0,\rho))}\rho^{-k\sum q_i^{-1}}\rho^{n-1}d\rho \\ &\lesssim \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{q_i}(\mathbb{R}^k)} \int_0^R \rho^{-k\sum q_i^{-1}+n-1}d\rho,\end{aligned}$$

for $q_i \geq \tilde{q}$. Observing that $n-1-\left(k\sum q_i^{-1}\right) \geq n-1-k\binom{n}{k}\tilde{q}^{-1}$, and that

$$n-1-k\binom{n}{k}\tilde{q}^{-1} = \frac{(n-1)n}{k-2n+1}+n-1 \geq 0,$$

for $k=1,\dots,n-2$, we finally obtain that

$$\int_{B_n(0,R)} \prod_{i=1}^{C(n,k)} f_i(x_{\omega_i})dx \lesssim R^{n-k\sum q_i^{-1}} \prod_{i=1}^{C(n,k)} \|f_i\|_{L^{q_i}(\mathbb{R}^k)}.$$

The same type of inequalities can be proved on any spherically symmetric manifold. These are Riemannian manifolds M_ψ that topologically coincide with $\mathbb{R}^n$ and are endowed with a metric that in spherical coordinates can be written as $g = d\rho^2 + \psi^2(\rho)g_{\mathbb{S}^{n-1}}$, where ψ is a positive smooth function on $\mathbb{R}^+$ such that $\psi(0)=0$ and $\psi'(0)=1$ and $g_{\mathbb{S}^{n-1}}$ is the standard metric on the sphere $\mathbb{S}^{n-1}$ (see [32] for further details). The parameter ρ coincides with the Riemannian distance. The case $\psi(\rho)=\rho$ corresponds to the Euclidean metric on $\mathbb{R}^n$. Here as an example we treat the case $\psi(\rho)=\tanh(\rho/2)$ that corresponds to the hyperbolic space $\mathbb{H}^n$ with the hyperbolic metric. In spherical coordinates the Riemannian measure of $\mathbb{H}^n$ is given by

$$d\eta(x) = \sinh^{n-1}\rho\, d\sigma_{\mathbb{S}^{n-1}}(\omega)d\rho.$$

Define

$$H_R = \{\tanh(\rho/2)\omega : \rho \leq R, \omega \in \mathbb{S}^{n-1}\},$$

which is the geodesic ball of radius R around 0. For functions depending on two variables, using first Theorem 2.5.1 and then Hölder's inequality we obtain, for $1 \leq i < j \leq n$,

$$\begin{aligned}&\int_{H_R} \prod_{i<j} f_{ij}(y_i,y_j)d\eta(y) = \int_0^R \left(\int_{\mathbb{S}^{n-1}} \prod_{i<j} f_{ij}(\tanh(\rho/2)\omega_i, \tanh(\rho/2)\omega_j)d\sigma(\omega)\right)\sinh^{n-1}\rho\, d\rho \\ &\lesssim \int_0^R \prod_{i<j}\left(\int_{B_2(0,1)} f_{ij}(\tanh(\rho/2)\omega_i, \tanh(\rho/2)\omega_j)^{2n-3}d\omega_i d\omega_j\right)^{\frac{1}{2n-3}} \sinh^{n-1}\rho\, d\rho\end{aligned}$$

$$\lesssim \prod_{i<j}\left(\int_0^R\left(\int_{B_2(0,1)} f_{ij}(\tanh(\rho/2)\omega_i,\tanh(\rho/2)\omega_j)^{2n-3}d\omega_i d\omega_j\right)^{\frac{n(n-1)}{(4n-6)}}\sinh^{n-1}\rho d\rho\right)^{\frac{2}{n(n-1)}}.$$

Then by changing coordinates we find

$$\begin{aligned}
&\int_{H_R}\prod_{i<j} f_{ij}(y_i,y_j)d\eta(y)\\
&\lesssim \prod_{i<j}\left(\int_0^R\left(\int_{B_2(0,\tanh(\rho/2))} f_{ij}(x_i,x_j)^{2n-3}dx_i dx_j\right)^{\frac{n(n-1)}{(4n-6)}}\frac{\sinh^{n-1}\rho}{\tanh^{\frac{n(n-1)}{(2n-3)}}(\rho/2)}d\rho\right)^{\frac{2}{n(n-1)}}\\
&\lesssim \prod_{i<j}\left(\int_0^R\left(\int_{B_2(0,\tanh(\rho/2))} f_{ij}(x_i,x_j)^{2n-3}dx_i dx_j\right)^{\frac{n(n-1)}{(4n-6)}}\right.\\
&\qquad\qquad \left.\times\cosh^{\frac{3(n-1)^2}{2n-3}}\rho\,\sinh^{\frac{(n-1)(n-3)}{2n-3}}\rho\,d\rho\right)^{\frac{2}{n(n-1)}}\\
&\lesssim \int_0^{2R}\cosh^{\frac{3(n-1)^2}{2n-3}}\rho\,\sinh^{\frac{(n-1)(n-3)}{2n-3}}\rho\,d\rho\prod_{i<j}\left(\int_{\mathbb{R}^2} f_{ij}(x_i,x_j)^{2n-3}dx_i dx_j\right)^{\frac{1}{2n-3}}\\
&\lesssim C(R)\prod_{i<j}\|f_{ij}\|_{L^{2n-3}(\mathbb{R}^2)}
\end{aligned}$$

with $C(R)$ diverging exponentially as $R\to\infty$.

2.11 Some weighted nonlinear Brascamp–Lieb inequalities

In this section we show how to derive from Carlen–Lieb–Loss original inequality, i.e. Theorem 2.4.1, with $n=3$, an inequality for functions on the plane that are constant on certain curves. The idea is to use stereographic projection to transfer the inequality from the sphere to the plane.

Consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ centered at the origin. Let $N=(0,0,1)$ and $S=(0,0,-1)$. Let $P=(x,y,z)\in\mathbb{S}^2$ and $Q=(0,0,z)$. Let π be the tangent plane to $\mathbb{S}^2$ at S, which we endow with cartesian coordinates with origin S and axes X and Y parallel respectively to x and y. We call $P'=(X,Y)$ the point of π which lies on the line joining N and P. We notice that the triangle of vertices N, P, Q and that of vertices N, S, P' are similar, so that, since $NQ=1-z$ e $NS=2$, we have

$$\frac{P'S}{PQ}=\frac{2}{1-z}.$$

Moreover,

$$\frac{X}{x}=\frac{Y}{y}=\frac{P'S}{PQ},$$

so that

$$X=\frac{2x}{1-z},\quad Y=\frac{2y}{1-z}.$$

We also note that the triangle of vertices N, P, S and that of vertices N, S, P' are similar, so that

$$\frac{PN}{NS}=\frac{NS}{P'N},$$

that is $(PN)(P'N) = (NS)^2$, thus

$$\frac{x}{X} = \frac{PQ}{P'S} = \frac{PN}{P'N} = \frac{NS^2}{P'N^2}.$$

Since $(P'N)^2 = X^2 + Y^2 + 4$, we find that

$$\frac{x}{X} = \frac{NS^2}{P'N^2} = \frac{4}{X^2+Y^2+4}$$

that is

$$x = \frac{4}{X^2+Y^2+4}X;$$

analogously

$$y = \frac{4}{X^2+Y^2+4}Y.$$

Finally, plugging the two expressions for x and y just obtained in the equation of the sphere, it follows that

$$|z| = \frac{|X^2+Y^2-4|}{X^2+Y^2+4}.$$

In particular the points of the circle that are the intersection between $\mathbb{S}^2$ and the plane $x = x_0$, for $0 < x_0 < 1$, satisfy

$$x_0(X^2+Y^2+4) = 4X,$$

that can be written as

$$\left(X - \frac{2}{x_0}\right)^2 + Y^2 = \frac{4}{x_0^2}(1 - x_0^2), \tag{2.33}$$

that is a circle in the plane π with center $\left(\frac{2}{x_0}, 0\right)$ and radius $\frac{2}{|x_0|}\sqrt{1-x_0^2}$.

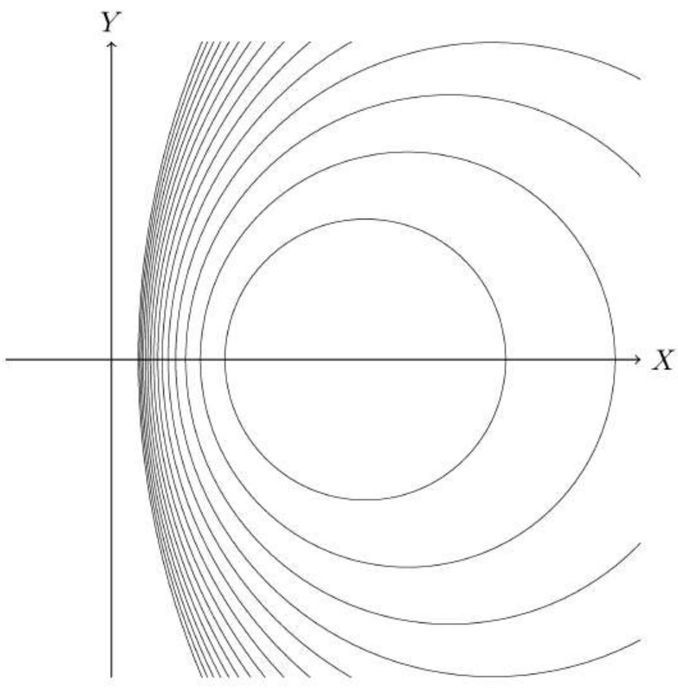


Figure 2.4: Stereographic projection of the circles $x = x_0$ in $\mathbb{S}^2$ for $0 < x_0 < 1$.

Analogously the points of the circle that is the intersection between $\mathbb{S}^2$ and the plane $y = y_0$, $0 < y_0 < 1$, originate a circle in the plane π with equation

$$X^2 + \left(Y - \frac{2}{y_0}\right)^2 = \frac{4}{y_0^2}(1 - y_0^2). \tag{2.34}$$

Finally, points that are on the circle that is the intersection between $\mathbb{S}^2$ and the plane $z = z_0$, for $|z_0| < 1$, give rise to a circle on π centered at the origin and with radius $\sqrt{\frac{4(1+z_0)}{1-z_0}}$.
Let us consider three functions $f_i : [-1,1] \to \mathbb{R}^+$. By Theorem 2.4.1 we have

$$\int_{\mathbb{S}^2} f_1(x)f_2(y)f_3(z)d\sigma(x,y,z) \leq \|f_1\|_{L^2([-1,1])}\|f_2\|_{L^2([-1,1])}\|f_3\|_{L^2([-1,1])}.$$

We define the functions

$$\phi_1(X,Y) = f_1\left(\frac{4}{X^2+Y^2+4}X\right),$$

$$\phi_2(X,Y) = f_2\left(\frac{4}{X^2+Y^2+4}Y\right)$$

and

$$\phi_3(X,Y) = f_3\left(\frac{X^2+Y^2-4}{X^2+Y^2+4}\right).$$

The function ϕ_1 is constant on the circles of π given by equation (2.33) (see Figure 2.4), where it is equal to $f_1(x_0)$. Analogously ϕ_2 is constant on the circles of π given by equation (2.34), where it is equal to $f_2(y_0)$. Finally ϕ_3 is constant on the circles

$$X^2 + Y^2 = \frac{4(1+z_0)}{(1-z_0)},$$

where it is equal to $f_3(z_0)$.
We have

$$\begin{aligned} d\sigma(x,y,z) &= \frac{dxdy}{\sqrt{1-x^2-y^2}} = \frac{dxdy}{|z(x,y)|} \\ &= \frac{X^2+Y^2+4}{|4-X^2-Y^2|}dxdy. \end{aligned}$$

Computing the Jacobian we find out that in the coordinates (X,Y) we have

$$\begin{aligned} d\sigma(x,y,z) &= \frac{X^2+Y^2+4}{|4-X^2-Y^2|}dxdy \\ &= 16\frac{X^2+Y^2+4}{|4-X^2-Y^2|}\frac{|4-X^2-Y^2|}{(X^2+Y^2+4)^3}dXdY \\ &= 16\frac{dXdY}{(X^2+Y^2+4)^2}. \end{aligned}$$

The left hand side of the inequality becomes therefore

$$\int_{\mathbb{S}^2} f_1(x)f_2(y)f_3(z)d\sigma(x,y,z)$$

$$= 16 \int_{\mathbb{R}^2} \phi_1(X,Y)\phi_2(X,Y)\phi_3(X,Y)\frac{dXdY}{(X^2+Y^2+4)^2}.$$

Thus we obtain

$$\begin{aligned}
&16 \int_{\mathbb{R}^2} \phi_1(X,Y)\phi_2(X,Y)\phi_3(X,Y)\frac{dXdY}{(X^2+Y^2+4)^2} \\
&= \int_{\mathbb{S}^2} f_1(x)f_2(y)f_3(z)d\sigma(x,y,z) \\
&\leq \|f_1\|_{L^2([-1,1])}\|f_2\|_{L^2([-1,1])}\|f_3\|_{L^2([-1,1])}.
\end{aligned} \tag{2.35}$$

We now look for a relation between $\|f_1\|_{L^2([-1,1])}$ and

$$\int_{\mathbb{R}^2} f_1\left(\frac{4X}{X^2+Y^2+4}\right)^2 dXdY.$$

Writing

$$x(X,Y) = \frac{4X}{X^2+Y^2+4},$$

the fundamental theorem of calculus and Fubini's theorem give

$$\begin{aligned}
\int_{\mathbb{R}^2} f_1\left(x(X,Y)\right)^2 dXdY &= \int_{\mathbb{R}^2}\left(\int_{-\infty}^{x(X,Y)} \frac{d}{dt}(f_1(t)^2)dt\right)dXdY \\
&= \int_{\mathbb{R}} \frac{d}{dt}(f_1(t)^2)\left(\int_{\{(X,Y):x(X,Y)>t\}} dXdY\right)dt \\
&= \int_{\mathbb{R}} \frac{d}{dt}(f_1(t)^2)\lambda_x(t)dt,
\end{aligned}$$

where

$$\lambda_x(t) = |\{(X,Y) : x(X,Y) > t\}|.$$

Integrating by parts we find that

$$\begin{aligned}
\int_{\mathbb{R}^2} f_1\left(x(X,Y)\right)^2 dXdY &= \int_{\mathbb{R}} \frac{d}{dt}(f_1(t)^2)\lambda_x(t)dt \\
&= -\int_{\mathbb{R}} f_1(t)^2\frac{d}{dt}\lambda_x(t)dt.
\end{aligned} \tag{2.36}$$

So we need to compute $\frac{d}{dt}\lambda_x(t)$. Note that

$$-1 \leq x(X,Y) = \frac{4X}{X^2+Y^2+4} \leq 1.$$

Suppose first $t > 0$. The region where

$$t < x(X,Y) = \frac{4X}{X^2+Y^2+4}$$

is the region where (X,Y) is bounded by

$$\left(X - \frac{2}{t}\right)^2 + Y^2 < \frac{4}{t^2} - 4.$$

This inequality defines a disk of radius $R(t) = 2\left(\frac{1}{t^2} - 1\right)^{\frac{1}{2}}$ and center $\left(\frac{2}{t}, 0\right)$; hence

$$\lambda_x(t) = \pi R^2(t) = 4\pi\left(\frac{1}{t^2} - 1\right)$$

and

$$\frac{d}{dt}\lambda_x(t) = -8\pi\frac{1}{t^3}.$$

The case $t < 0$ can be treated analogously.
Substituting this in (2.36), we find

$$\begin{aligned}\int_{\mathbb{R}^2} f_1\left(x(X,Y)\right)^2 dXdY &= -\int_{-1}^{1} f_1(t)^2\frac{d}{dt}\lambda_x(t)dt\\ &= 8\pi\int_{-1}^{1} f_1(t)^2\frac{1}{t^3}dt.\end{aligned}$$

Being $1 \geq |t|$ we finally obtain

$$\begin{aligned}\|\phi_1\|^2_{L^2(\mathbb{R}^2)} &= \|f_1\circ x\|^2_{L^2(\mathbb{R}^2)}\\ &= \int_{\mathbb{R}^2} f_1\left(x(X,Y)\right)^2 dXdY\\ &= 8\pi\int_{-1}^{1} f_1(t)^2\frac{1}{|t|^3}dt\\ &\geq 8\pi\int_{-1}^{1} f_1(t)^2 dt = 8\pi\|f_1\|^2_{L^2(-1,1)}.\end{aligned}$$

Analogous computations hold also for f_2 and f_3, so (2.35) implies

$$\begin{aligned}\int_{\mathbb{R}^2} &\phi_1(X,Y)\phi_2(X,Y)\phi_3(X,Y)\frac{dXdY}{(X^2+Y^2+4)^2}\\ &\sim \int_{\mathbb{S}^2} f_1(x)f_2(y)f_3(z)d\sigma(x,y,z)\\ &\lesssim \|f_1\|_{L^2([-1,1])}\|f_2\|_{L^2([-1,1])}\|f_3\|_{L^2([-1,1])}\\ &\lesssim \|\phi_1\|_{L^2(\mathbb{R}^2)}\|\phi_2\|_{L^2(\mathbb{R}^2)}\|\phi_3\|_{L^2(\mathbb{R}^2)},\end{aligned}$$

from which we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\phi_1(X,Y)\phi_2(X,Y)\phi_3(X,Y)\frac{dXdY}{(X^2+Y^2+4)^2}\\ &\lesssim \|\phi_1\|_{L^2(\mathbb{R}^2)}\|\phi_2\|_{L^2(\mathbb{R}^2)}\|\phi_3\|_{L^2(\mathbb{R}^2)},\end{aligned}\tag{2.37}$$

which indeed can be interpreted as a nonlinear weighted Brascamp–Lieb inequality, holding however for very special functions.
We can extend the previous argument to other cases. Consider the stereographic projection from the sphere $\mathbb{S}^{n-1}$ onto the $(n-1)$-dimensional hyperplane $\pi : x_n = -1$ tangent to $\mathbb{S}^{n-1}$ at the south pole $S = (0,\dots,0,-1)$. With the same notation as above we have, for a point $P = (x_1,\dots,x_n) \in \mathbb{S}^{n-1}$, and $P' = (X_1,\dots,X_{n-1})$ defined as the intersection of the line passing through N and P with the hyperplane π, that the following relations hold

$$x_i = \frac{4}{X_1^2+\cdots+X_{n-1}^2+4}X_i \tag{2.38}$$

for $i = 1, \dots, n-1$, and by the sphere condition we also obtain

$$|x_n| = \frac{|X_1^2 + \cdots + X_{n-1}^2 - 4|}{X_1^2 + \cdots + X_{n-1}^2 + 4}.$$

Let us consider intersections of the sphere with hyperplanes of the type $x_i = x_{i,0}$, with $|x_{i,0}| < 1$. We saw in the previous sections that this intersection is a $(n-2)$-dimensional sphere Σ_i inside $\mathbb{S}^{n-1}$. By stereographic projection this sphere maps to

$$x_{i,0}(X_1^2 + \cdots + X_{n-1}^2 + 4) = 4X_i,$$

that is the $(n-2)$-dimensional sphere $\widetilde{\Sigma}_i$ on π given by the equation

$$X_1^2 + \cdots + \left(X_i - \frac{2}{x_{i,0}}\right)^2 + \cdots + X_{n-1}^2 = 4\left(\frac{1}{x_{i,0}^2} - 1\right),$$

with center in π given by $(0, \dots, 2/x_{i,0}, \dots, 0)$ (where the only nonzero coordinate is in the i-th place) and radius $\frac{2}{|x_{i,0}|}\sqrt{1 - x_{i,0}^2}$, for $i = 1, \dots, n-1$. For $x_n = x_{n,0}$, $|x_{n,0}| < 1$, the sphere is instead given by

$$X_1^2 + \cdots + X_{n-1}^2 = \frac{4(1 + x_{n,0})}{(1 - x_{n,0})},$$

with center $(0, \dots, 0)$ and radius $\sqrt{\frac{4(1+x_{n,0})}{1-x_{n,0}}}$.
We consider functions $f_i : [-1, 1] \to \mathbb{R}^+$ and define

$$\phi_i(X_1, \dots, X_{n-1}) = f_i\left(\frac{4X_i}{X_1^2 + \cdots + X_{n-1}^2 + 4}\right),$$

for $i = 1, \dots, n-1$ and

$$\phi_n(X_1, \dots, X_{n-1}) = f_n\left(\frac{X_1^2 + \cdots + X_{n-1}^2 - 4}{X_1^2 + \cdots + X_{n-1}^2 + 4}\right).$$

Functions ϕ_i are constant on the $(n-2)$-dimensional spheres $\widetilde{\Sigma}_i$.
It is easy to see that

$$d\sigma(x_1, \dots, x_n) = 16\frac{dX_1 \dots dX_{n-1}}{(X_1^2 + \cdots + X_{n-1}^2 + 4)^2},$$

so that

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n} \phi_i(X_1, \dots, X_{n-1}) \frac{dX_1 \dots dX_{n-1}}{(X_1^2 + \cdots + X_{n-1}^2 + 4)^2} \\ &\sim \int_{\mathbb{S}^{n-1}} \prod_{i=1}^{n} f_i(x_i) d\sigma \leq \prod_{i=1}^{n} \|f_i\|_{L^2([-1,1])}. \end{aligned}$$

Hence, we need a relation between $\|f_i\|_{L^2([-1,1])}$ and

$$\int_{\mathbb{R}^{n-1}} f_i\left(\frac{4X_i}{X_1^2 + \cdots + X_{n-1}^2 + 4}\right) dX_1 \dots dX_n,$$

for $i = 1, \ldots, n-1$ and an analogous formula for f_n. Arguing as in the case $n = 3$ it is easy to see that $\|f_i\|_{L^2([-1,1])} \lesssim \|\phi_i\|_{L^2(\mathbb{R}^{n-1})}$. Hence we get another family of nonlinear weighted Brascamp–Lieb inequalities

$$\int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n} \phi_i(X_1, \ldots, X_{n-1}) \frac{dX_1 \ldots dX_{n-1}}{(X_1^2 + \cdots + X_{n-1}^2 + 4)^2} \lesssim \prod_{i=1}^{n} \|\phi_i\|_{L^2(\mathbb{R}^{n-1})}. \tag{2.39}$$

Remark 2.11.1. Note that inequalities (2.39) and (2.37) are not scale invariant. We obtained these estimates by stereographic projection starting from inequalities on the unit sphere. The fact that the inequalities are not scale invariant is due to the fact that after scaling, the level sets are not anymore circles coming from a stereographic projection.
Considering the case $n = 3$, one could extend inequality (2.37) to the sphere about the origin of arbitrary radius a. The corresponding inequality is then

$$\int_{\mathbb{R}^2} \phi_1(X,Y)\phi_2(X,Y)\phi_3(X,Y) \frac{dXdY}{(X^2+Y^2+4a^2)^2} \lesssim a^{-4} \|\phi_1\|_{L^2(\mathbb{R}^2)} \|\phi_2\|_{L^2(\mathbb{R}^2)} \|\phi_3\|_{L^2(\mathbb{R}^2)},$$

where the functions ϕ_i are constant on the circles in the plane tangent to $a\mathbb{S}^2$ at the point $(0, \ldots, 0, -a)$, which are stereographic projections of the intersections of the sphere $a\mathbb{S}^2$ with hyperplanes $x_i = x_{i,0}$ with $-a < x_{i,0} < a$.
The inequality is invariant under the transformations $\Psi_{a,b}$, that send the circles related to $a\mathbb{S}^2$ to the circles related to $b\mathbb{S}^2$, i.e. $\Psi_{a,b} = \mathcal{S}_b D_{b/a} \mathcal{S}_a^{-1}$, where $\mathcal{S}_r : r\mathbb{S}^2 \to \mathbb{R}^2$ is the stereographic projection associated to the sphere $r\mathbb{S}^2$ onto the plane tangent to its south pole, and D_r is the usual isotropic dilation in $\mathbb{R}^3$.

Remark 2.11.2. With the same argument as above one could obtain analogous results starting from inequalities involving functions that depend on more than one variable, transferring inequality (2.14) through the stereographic projection $\mathcal{S} : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ onto the hyperplane $\pi : x_n = -1$. In this case the $(n-k-1)$-dimensional subspheres of $\mathbb{S}^{n-1}$ will be mapped to nonintersecting $(n-k-1)$-dimensional spheres covering the hyperplane π.

CHAPTER 3

Brascamp–Lieb inequalities on stratified groups

3.1 Preliminaries

Let G be a connected, simply connected nilpotent Lie group. We can and will identify G with its Lie algebra $\mathfrak{g}$ by means of the exponential map. In the exponential coordinates the Haar measure on G coincides with the Lebesgue measure, dg, on $\mathfrak{g}$.
The convolution on G is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy.$$

Setting $u = y^{-1}x$ and using the right and left invariance of the measure, the convolution may be also written as

$$f * g(x) = \int_G f(xu^{-1})g(u)du = \int_G f(xy)g(y^{-1})dy, \tag{3.1}$$

where the last identity is a consequence of $d(y^{-1}) = dy$.
We assume that G is stratified, meaning that $\mathfrak{g}$ decomposes as vector space into a direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

and

$$[\mathfrak{g}_l, \mathfrak{g}_1] = \mathfrak{g}_{l+1}, \quad l = 1, \ldots, r-1.$$

Therefore the maps $\{\delta_s\}_{s>0}$, defined by

$$\delta_s X = s^l X$$

if $X \in \mathfrak{g}_l$ and extended to $\mathfrak{g}$ by linearity, are automorphisms. We assume moreover that $\mathfrak{g}_1$ is endowed with an inner product $\langle \cdot, \cdot \rangle$, that extended by translations to the entire group defines a sub-Riemannian metric. Stratified groups are also called Carnot groups. The number $Q = \dim \mathfrak{g}_1 + 2\dim \mathfrak{g}_2 + \cdots + r \dim \mathfrak{g}_r$ is the homogeneous dimension of G. We have

$$t^Q \int_G f(\delta_t g)dg = \int_G f(g)dg. \tag{3.2}$$

for all integrable f and $t > 0$.
Fix an orthonormal basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}_1$ and define the corresponding sub-Laplacian

$$L = X_1^2 + \cdots + X_n^2,$$

which is a second order, negative, symmetric, differential operator. Since the vector fields $\{X_1, \cdots, X_n\}$ generate the entire Lie algebra L, by the Hörmander theorem, is a hypoelliptic operator. It is well known that (see [28]), for a Schwartz function f on G, the solution the the problem

$$\begin{cases} \partial_t u(t,g) - Lu(t,g) = 0 \quad t > 0 \\ u(0,g) = f(g) \end{cases}$$

is given by

$$u(t,g) = u_t(g) = f * p_t(g) = \int_G f(x) p_t(x^{-1}g) dx,$$

where the heat kernel $\{p_t\}_{t>0}$ associated to L, may be written as (see [26])

$$p_t(g) = t^{-\frac{Q}{2}} P\big(\delta_{t^{-\frac{1}{2}}} g\big), \tag{3.3}$$

with $P = p_1$ is a (strictly) positive Schwartz function that satisfies

$$\int P(g) dg = 1.$$

It follows in particular that $\{p_t\}_{t>0}$ is an approximation of the identity, in fact, by (3.2) we have

$$\int p_t(g) dg = 1 \quad \text{for all } t > 0.$$

In this regard we recall the classical estimates for the heat kernel on a stratified group (see, for instance, [59] or [26]), according to which

$$p_t(g) \le C t^{-\frac{Q}{2}} e^{-c\frac{|g|^2}{t}}, \tag{3.4}$$

where $|\cdot|$ is any homogeneous norm on G satisfying $|g_1 g_2| \lesssim (|g_1| + |g_2|)$ and $C, c > 0$ are constants.
Recall also that p_t is a symmetric function of g, that is

$$p_t(g^{-1}) = p_t(g)$$

for all $g \in G$ and $t > 0$.
We now introduce a family $\{H^{(1)}, \dots, H^{(l)}\}$ of not necessarily normal subgroups of G with Lie algebras $\{\mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(l)}\}$ respectively. We define $M^{(a)} = H^{(a)} \backslash G$ and denote by $\pi^{(a)} : G \to M^{(a)}$ the canonical projection (which is a group homomorphism when $H^{(a)}$ is a normal subgroup). The Haar measure of G decomposes for all a according to the formula

$$\int_G f(g) dg = \int_{H^{(a)} \backslash G} \left(\int_{H^{(a)}} f(h^{(a)} g) dh^{(a)} \right) dg^{(a)}, \tag{3.5}$$

for all measurable f on G, where $dh^{(a)}$ is the Haar measure on $H^{(a)}$ (see [60, Chapter 2]). The inner integral

$$\bar{f}^{(a)}(g^{(a)}) = \int_{H^{(a)}} f(h^{(a)}g)dh^{(a)}$$

defines a function on $H^{(a)}\backslash G$, since for $k^{(a)} \in H^{(a)}$, by the invariance of $dh^{(a)}$ we have

$$\int_{H^{(a)}} f(h^{(a)}k^{(a)}g)dh^{(a)} = \int_{H^{(a)}} f(h^{(a)}g)dh^{(a)}.$$

We fix global sections, $\sigma^{(a)} : M^{(a)} \to G$, of the principal bundles $(G, \pi^{(a)}, M^{(a)})$. Recall that this means that $\pi^{(a)} \circ \sigma^{(a)}$ is the identity on $M^{(a)}$, which, in particular implies, that

$$\pi^{(a)}\big(h^{(a)}\sigma^{(a)}(g^{(a)})\big) = g^{(a)},$$

for $g^{(a)} \in M^{(a)}$ and $h^{(a)} \in H^{(a)}$.
We make the following assumptions on the groups $H^{(a)}$.

- We assume for any a that the restriction of δ_s to $H^{(a)}$ makes it a stratified group of, say, homogeneous dimension $\bar{Q}^{(a)}$. It follows in particular that $\delta_s H^{(a)} \subset H^{(a)}$ for all s and that the map defined by

$$\delta_s^{(a)} \circ \pi^{(a)} = \pi^{(a)} \circ \delta_s, \quad s > 0, \tag{3.6}$$

  yields a one parameter group of dilations on $M^{(a)}$, making it a space of homogeneous type in the sense of Coifman and Weiss with homogeneous dimension $Q^{(a)} = Q - \bar{Q}^{(a)}$. It follows in particular that

$$\sigma^{(a)} \circ \delta_s^{(a)} = \delta_s \circ \sigma^{(a)}, \quad \text{for all } s > 0.$$

- We assume that the layers of $H^{(a)}$ are given by $\mathfrak{h}_i^{(a)} = \mathfrak{h}^{(a)} \cap \mathfrak{g}_i$, so that

$$\mathfrak{h}^{(a)} = \mathfrak{h}_1^{(a)} \oplus \cdots \oplus \mathfrak{h}_l^{(a)}.$$

The push-forwards $\pi_*^{(a)} X_j$, of the vector fields X_j are denoted by $X_j^{(a)}$. The push-forward,

$$L^{(a)} = \left(X_1^{(a)}\right)^2 + \cdots + \left(X_n^{(a)}\right)^2 = \pi_*^{(a)}(L),$$

of L is a sub-Laplacian on $M^{(a)}$. It is well known (see, for instance, [44, Formula (2.11)]) that the heat kernel, $\bar{p}_t^{(a)}$, of $L^{(a)}$ is given by

$$\bar{p}_t^{(a)}\big(x^{(a)}, y^{(a)}\big) = \bar{p}_t^{(a)}\big(\pi^{(a)}(x), \pi^{(a)}(y)\big) = \int_{H^{(a)}} p_t\big(y^{-1}h^{(a)}x\big)dh^{(a)}. \tag{3.7}$$

Note that, since by the left and right invariance of $dh^{(a)}$ we have

$$\begin{aligned}\bar{p}_t^{(a)}\big(k_1^a\pi^{(a)}(x), k_2^a\pi^{(a)}(y)\big) &= \int_{H^{(a)}} p_t\big(y^{-1}(k_2^a)^{-1}h^{(a)}k_1^a x\big)dh^{(a)} \\ &= \int_{H^{(a)}} p_t\big(y^{-1}h^{(a)}x\big)dh^{(a)},\end{aligned}$$

$\bar{p}_t^{(a)}\big(x^{(a)}, y^{(a)}\big)$ makes sense as a function on $M^{(a)} \times M^{(a)}$.
Formula (3.7) can be proved showing that for each Schwartz function $\bar{f}^{(a)}$ on $M^{(a)}$ the function

$$\bar{u}^{(a)}(t, x^{(a)}) = \int_{M^{(a)}} \bar{p}_t^{(a)}\big(x^{(a)}, y^{(a)})\big) \bar{f}^{(a)}(y^{(a)}) dy^{(a)} \quad t > 0,$$

solves the problem

$$\begin{cases} \partial_t \bar{u}^{(a)}(t, x^{(a)}) + L^{(a)} \bar{u}^{(a)}(t, x^{(a)}) = 0, & t > 0 \\ \bar{u}^{(a)}(0, x^{(a)}) = \bar{f}^{(a)}(x^{(a)}), \end{cases} \tag{3.8}$$

which has a unique solution.
To do that, first observe that

$$\begin{aligned} X_x^{(a)} \bar{p}_t^{(a)}(x^{(a)}, y^{(a)}) &= \frac{d}{ds} \int_{H^{(a)}} p_t\big(y^{-1} h^{(a)} x \exp(sX)\big) dh^{(a)} \Big|_{s=0} \\ &= \int_{H^{(a)}} (X p_t)\big(y^{-1} h^{(a)} x\big) dh^{(a)} \\ &= X_x \int_{H^{(a)}} p_t\big(y^{-1} h^{(a)} x\big) dh^{(a)}, \end{aligned}$$

where we introduced the index x in $X_x^{(a)}$ and X_x to indicate that the vector field acts on the variable x. Therefore,

$$\begin{aligned} L_x^{(a)} \bar{p}_t^{(a)}(x^{(a)}, y^{(a)}) &= \int_{H^{(a)}} L p_t \left(y^{-1} h^{(a)} x\right) dh^{(a)} \\ &= L_x \int_{H^{(a)}} p_t \left(y^{-1} h^{(a)} x\right) dh^{(a)}, \end{aligned}$$

which implies

$$\begin{aligned} \partial_t \bar{u}^{(a)}(t, x^{(a)}) &= \int_{M^{(a)}} \partial_t \bar{p}_t^{(a)}\big(x^{(a)}, y^{(a)})\big) \bar{f}^{(a)}(y^{(a)}) dy^{(a)} \\ &= \int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)}) \left(\int_{H^{(a)}} \partial_t p_t\big(y^{-1} h^{(a)} x\big) dh^{(a)} \right) dy^{(a)} \\ &= \int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)}) \left(\int_{H^{(a)}} L p_t\big(y^{-1} h^{(a)} x\big) dh^{(a)} \right) dy^{(a)} \\ &= \int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)}) \left(L \int_{H^{(a)}} p_t\big(y^{-1} h^{(a)} x\big) dh^{(a)} \right) dy^{(a)} \\ &= \int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)}) L^{(a)} \bar{p}_t^{(a)}(x^{(a)}, y^{(a)}) dy^{(a)} \\ &= L^{(a)} \int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)}) \bar{p}_t^{(a)}(x^{(a)}, y^{(a)}) dy^{(a)} \\ &= L^{(a)} \bar{u}^{(a)}(t, x^{(a)}). \end{aligned}$$

From (3.3) and (3.7) it follows that

$$\begin{aligned} \bar{p}_t^{(a)}(x^{(a)}, y^{(a)}) &= \int_{H^{(a)}} p_t\big(y^{-1} h^{(a)} x\big) dh^{(a)} \\ &= t^{-\frac{Q}{2}} \int_{H^{(a)}} P\big(\delta_{t^{-\frac{1}{2}}}(y^{-1} h^{(a)} x)\big) dh^{(a)} \end{aligned}$$

$$\begin{aligned}
&= t^{-\frac{Q}{2}} \int_{H^{(a)}} P\big(\delta_{t^{-\frac{1}{2}}}(y^{-1})(\delta_{t^{-\frac{1}{2}}}h^{(a)})(\delta_{t^{-\frac{1}{2}}}x)\big)dh^{(a)} \\
&= t^{-\frac{Q}{2}} t^{\frac{\tilde{Q}^{(a)}}{2}} \int_{H^{(a)}} P\big((\delta_{t^{\frac{1}{2}}}y)^{-1}k^{(a)}(\delta_{t^{-\frac{1}{2}}}x)\big)dk^{(a)} \\
&= t^{-\frac{Q_a}{2}} \int_{H^{(a)}} P\big((\delta_{t^{\frac{1}{2}}}y)^{-1}k^{(a)}(\delta_{t^{-\frac{1}{2}}}x)\big)dk^{(a)} \\
&= t^{-\frac{Q_a}{2}} \bar{P}^a\big(\delta^{(a)}_{t^{-\frac{1}{2}}}x^{(a)}, \delta^{(a)}_{t^{-\frac{1}{2}}}y^{(a)}\big),
\end{aligned} \tag{3.9}$$

where we replaced $\delta_{t^{-\frac{1}{2}}}h^{(a)}$ with $k^{(a)}$, and $\bar{P}^a$ is the function on $M^{(a)} \times M^{(a)}$ defined by

$$\bar{P}^a\big(\pi^{(a)}(x), \pi^{(a)}(y)\big) = \int_{H^{(a)}} P\big(y^{-1}h^{(a)}x\big)dh^{(a)}. \tag{3.10}$$

Since P is a positive, symmetric, Schwartz function, it is easy to see that $\bar{P}^a$ is also a positive symmetric, i.e. $P^a(x^{(a)}, y^{(a)}) = P^a(y^{(a)}, x^{(a)})$, Schwartz function on $M^{(a)} \times M^{(a)}$. Moreover, it satisfies

$$\int_{M^{(a)}} \bar{P}^a(x^{(a)}, y^{(a)})dy^{(a)} = 1,$$

identically. Indeed, by (3.5) we have

$$\begin{aligned}
\int_{M^{(a)}} \bar{P}^a(x^{(a)}, y^{(a)})dy^{(a)} &= \int_{M^{(a)}} \left(\int_{H^{(a)}} P\big(y^{-1}h^{(a)}x\big)dh^{(a)}\right) dy^{(a)} \\
&= \int_G P\big(y^{-1}x\big)dy \\
&= \int_G P(y)dy = 1.
\end{aligned} \tag{3.11}$$

3.2 Brascamp–Lieb inequalities

For any a consider a Schwartz function $f^{(a)} : G \to \mathbb{R}^+$ invariant under the left action of the group $H^{(a)}$,

$$f^{(a)}(h^{(a)}g) = f(g) \quad \text{for all } h^{(a)} \in H^{(a)}.$$

Clearly, $f^{(a)}$ is the pullback on G of a function on $M^{(a)}$, which will be denoted $\bar{f}^{(a)}$, that is

$$f^{(a)} = (\pi^{(a)})^* \bar{f}^{(a)} = \bar{f}^{(a)} \circ \pi^{(a)}.$$

The functions $f^{(a)}$ enjoy the following property, which is actually a characterization of functions which are invariant under the left action of $H^{(a)}$

$$f^{(a)}(g) = f^{(a)}\big(\sigma^{(a)}(\pi^{(a)}(g))\big) = f\big(h^{(a)}\sigma^{(a)}(\pi(g^{(a)}))\big) \quad \text{for all } g \in G \text{ and all } h^{(a)} \in H^{(a)}.$$

It is sometimes convenient to think of the functions $f^{(a)}$ as functions on the manifolds defined by

$$\Sigma^{(a)} = \{\sigma^{(a)}(g^{(a)}) : g^{(a)} \in M^{(a)}\}$$

and, for any fixed $h^{(a)} \in H^{(a)}$, by

$$\Sigma^{(a)}_{h^{(a)}} = \{h^{(a)}\sigma^{(a)}(g^{(a)}) : g^{(a)} \in M^{(a)}\}.$$

Clearly, $\Sigma^{(a)}$ and $\Sigma^{(a)}_{h^{(a)}}$ are smooth submanifolds of G of the same dimension of $M^{(a)}$.
Recall that $T_g\pi^{(a)} : T_gG \to T_{\pi^{(a)}(g)}M^{(a)}$ is the linear map defined, for any given $X \in T_gG$, by

$$\big(T_g\pi^{(a)}(X)\bar{f}\big)\big(\pi^{(a)}(g)\big) = \big((\pi^a_* X)_g\bar{f}\big)\big(\pi^{(a)}(g)\big) = \frac{d}{dt}\bar{f}(\pi^{(a)}(\gamma(t)))\Big|_{t=0},$$

where $\bar{f}$ is any function in $\mathcal{C}^\infty(M^{(a)})$ and γ is a smooth curve satisfying $\gamma(0) = g$ and $\dot{\gamma}(0) = X$ (notice that $\pi^{(a)} \circ \gamma$ is a smooth curve in $H^{(a)}\backslash G$). We will use, according to our convenience, both the notation $T\pi$ and π_*.
To a given smooth curve γ in G based in g (meaning that $\gamma(0) = g$), we associate the curve

$$\bar{\gamma}^a_{h^{(a)}}(t) = h^{(a)}\sigma^{(a)}\big(\pi^{(a)}(\gamma(t))\big),$$

where $h^{(a)}$ is the element of $H^{(a)}$ defined by

$$g = h^{(a)}\sigma^{(a)}\big(\pi^{(a)}(g)\big).$$

It is clear that $\bar{\gamma}^a_{h^{(a)}}$ is a curve in $\Sigma^{(a)}_{h^{(a)}} \subset G$ based in $h^{(a)}\sigma^{(a)}\big(\pi^{(a)}(g)\big)$, meaning that $\gamma(0) = g = h^{(a)}\sigma^{(a)}(\pi^{(a)}(g))$.
Let $X = \dot{\gamma}(0) \in T_gG$. The vector

$$X^{(a)}_{hor} = \frac{d}{dt}\bar{\gamma}^a_{h^{(a)}}(t)\Big|_{t=0}$$

lies in the tangent space, $T_g\Sigma^{(a)}_{h^{(a)}}$, to $\Sigma^{(a)}_{h^{(a)}}$ at $g = h^{(a)}\sigma^{(a)}(g^{(a)})$. Obviously $X^{(a)}_{hor}$ depends linearly on X and is called the horizontal component of X with respect to $\pi^{(a)}$. It satisfies

$$\begin{aligned}
X^{(a)}_{hor} &= \frac{d}{dt}\bar{\gamma}^a_{h^{(a)}}(t)\Big|_{t=0} \\
&= \frac{d}{dt}h^{(a)}\sigma^{(a)}\big(\pi^{(a)}(\gamma(t))\big)\Big|_{t=0} \\
&= \frac{d}{dt}\tau_{h^{(a)}}\sigma^{(a)}\big(\pi^{(a)}(\gamma(t))\big)\Big|_{t=0} \\
&= T_g\tau_{h^{(a)}}\left(\frac{d}{dt}\sigma^{(a)}\big(\pi^{(a)}(\gamma(t))\big)\Big|_{t=0}\right) \\
&= T_g\tau_{h^{(a)}}T_{\pi^{(a)}(g)}\sigma^{(a)}\left(\frac{d}{dt}\pi^{(a)}(\gamma(t))\Big|_{t=0}\right) \\
&= T_{\sigma^{(a)}(\pi^{(a)}(g))}\tau_{h^{(a)}}T_{\pi^{(a)}(g)}\sigma^{(a)}T_g\pi^{(a)}(X),
\end{aligned} \tag{3.12}$$

(recall that τ_h denotes the left translation by h).
Since $\bar{\gamma}^a_{h^{(a)}}(t) = h^{(a)}\sigma^{(a)}\big(\pi^{(a)}(\gamma(t))\big)$ and $\pi^{(a)}(h^{(a)}\sigma^{(a)}(\pi^{(a)}(g))) = \pi^{(a)}(g)$, for all $g \in G$ and $h^{(a)} \in H^{(a)}$, we have

$$\begin{aligned}
T_g\pi^{(a)}\big(X^{(a)}_{hor}\big) &= \frac{d}{dt}\pi^{(a)}\big(\bar{\gamma}^a_{h^{(a)}}(t)\big)\Big|_{t=0} \\
&= \frac{d}{dt}\pi^{(a)}\big(h^{(a)}\sigma^{(a)}\big[\pi^{(a)}(\gamma(t))\big]\big)\Big|_{t=0} \\
&= \frac{d}{dt}\pi^{(a)}\big(\gamma(t)\big)\Big|_{t=0} = T_g\pi^{(a)}(X).
\end{aligned}$$

A vector $Y \in T_gG$ is said to be vertical with respect to $\pi^{(a)}$ if

$$T_g\pi^{(a)}(X) = 0.$$

Any tangent vector $X \in T_gG$ decomposes into its horizontal and vertical components (with respect to $\pi^{(a)}$), the latter being defined by

$$X = X^{(a)}_{hor} + X^{(a)}_{vert}.$$

We can prove the following proposition.

Proposition 3.2.1. *A smooth function f on G is invariant under the left action of $H^{(a)}$ if and only if it is annihilated by all vertical vectors with respect to $\pi^{(a)}$. Moreover, if f is invariant under the left action of $H^{(a)}$, then*

$$Xf = X^{(a)}_{hor}f, \tag{3.13}$$

for all tangent vectors X.

Proof. Suppose first that $Xf = 0$ for all vertical $X \in T_gG$ and all $g \in G$. Assume by contradiction that for some $g \in G$ and $h_1, h_2 \in H^{(a)}$ we have

$$f(h_1g) \neq f(h_2g).$$

Let $\gamma : (-1,1) \to G$ be a smooth curve satisfying $\gamma(-1/2) = gh_1$ and $\gamma(1/2) = gh_2$ and $\pi(\gamma(t)) = \pi(g)$ for $|t| < 1$. Then $\dot\gamma(t)$ is a vertical vector for all t, which implies that

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \big(\dot\gamma(t)f\big)(\gamma(t)) = 0,$$

but $f(\gamma(-1/2)) \neq f((\gamma(1/2))$, yielding a contradiction.
To prove the converse let $\bar f$ be a smooth function on $M^{(a)}$. Consider $X \in T_gG$ such that $\pi^{(a)}_*(X) = 0$. There are a smooth curve γ based in g and $\epsilon > 0$ so that $\pi^{(a)}(\gamma(t)) = \pi^{(a)}(g)$ for all $|t| < \epsilon$ and $\dot\gamma(0) = X$, hence,

$$\big(\pi^{(a)}_*(X)\bar f\big)(\pi^{(a)}(g)) = \frac{d}{dt}\bar f(\pi^{(a)}(\gamma(t))) = \frac{d}{dt}\bar f(g) = 0.$$

To conclude the proof it suffices now to notice that (3.13) is equivalent to the first part of the assertion. ☐

We introduce the functions $u^{(a)} : \mathbb{R}^+ \times G \to \mathbb{R}$, which are defined as the unique solutions of the Cauchy problems

$$\begin{cases} \partial_t u^{(a)}(t,g) - Lu^{(a)}(t,g) = 0, \quad t > 0 \\ u^{(a)}(0,g) = \bar f^{(a)}\big(\pi^{(a)}(g)\big). \end{cases} \tag{3.14}$$

These functions are left invariant under the action of the groups $H^{(a)}$; we state this observation as a lemma.

Lemma 3.2.2. *The functions $u^{(a)}$ are invariant under the left action of $H^{(a)}$, that is*

$$u^{(a)}(t, h^{(a)}g) = u^{(a)}(t,g)$$

for all $h^{(a)} \in H^{(a)}$, all $g \in G$ and all $t \geq 0$.

Proof. Let a be fixed. Because of the left invariance of the sub-Laplacian L we have

$$\partial_t(\tau_{h^{(a)}}u^{(a)}) - L(\tau_{h^{(a)}}u^{(a)}) = \tau_{h^{(a)}}\big(\partial_t u^{(a)} - Lu^{(a)}\big) = 0.$$

Since we also have

$$(\tau_{h^{(a)}}u^{(a)})(0,g) = u^{(a)}(0,h^{(a)}g) = f^{(a)}\big(\pi^{(a)}(h^{(a)}g)\big) = f^{(a)}\big(\pi^{(a)}(g)\big),$$

the functions $\tau_{h^{(a)}}u^{(a)}$ satisfy the same Cauchy problem (3.14) satisfied by $u^{(a)}$. Therefore $\tau_{h^{(a)}}u^{(a)} = u^{(a)}$, proving the lemma. □

The main result of this section is a consequence of the theorem that follows. The proof of it is essentially the same as that of [7, Proposition 2.8], which is in turn based on Lemma 2.6 in that paper. To do that and also to highlight the analogy with the proof of Proposition 2.8 in [7], we introduce a bit more notation.
Let, for $g \in G$,

$$B_g^{(a)} = T_g\pi^{(a)} \tag{3.15}$$

and

$$\big(B_g^{(a)}\big)^* = T_{\sigma^{(a)}(\pi^{(a)}(g))}\tau^R_{h^{(a)}} \circ T_{\pi^{(a)}(g)}\sigma^{(a)}, \tag{3.16}$$

(here τ_h^R denotes the right translation by h).
Let $u^{(a)} : \mathbb{R}^+ \times G \to \mathbb{R}^+$ be the functions introduced in (3.14), for $a = 1, \dots, l$. Set

$$v_i^{(a)}(t,g) = \frac{X_i u^{(a)}(t,g)}{u^{(a)}(t,g)} = X_i \log u^{(a)}(t,g),$$

which, in vector notation, becomes

$$\begin{aligned} v^{(a)}(t,g) &= \nabla^{(a)} \log u^{(a)}(t,g) \\ &= \big(X_1 \log u^{(a)}(t,g)\big)X_1(g) + \cdots + \big(X_n \log u^{(a)}(t,g)\big)X_n(g). \end{aligned}$$

With this notation the first equation in (3.14) becomes

$$0 = \partial_t u^{(a)} - \left(X_1^2 + \cdots + X_n^2\right)u^{(a)} = \partial_t u^{(a)} - \operatorname{div}\big(u^{(a)}v^{(a)}\big),$$

where, for $v = v_1 X_1 + \cdots + v_n X_n$, we set

$$\operatorname{div}(v) = \operatorname{div}(v_1 X_1 + \cdots + v_n X_n) = X_1(v_1) + \cdots + X_n(v_n).$$

Theorem 3.2.3. *Fix a set of positive numbers $\{p_1, \dots, p_l\}$. Assume, for all $g \in G$, that*

$$p_1 Q^{(1)}(B_g^{(1)})^* B_g^{(1)} + \cdots + p_l Q^{(l)}(B_g^{(l)})^* B_g^{(l)} = I_g, \tag{3.17}$$

where I_g is the identity on T_gG. Define, for $t \geq 0$, the function

$$\Phi(t) = \int_G u^{(1)}(t,g)^{p_1} \dots u^{(l)}(t,g)^{p_l}\, dg,$$

then

$$\Phi'(t) \geq 0, \tag{3.18}$$

for all $t > 0$.

Proof. The proof of (3.18) is based on [7, Lemma 2.6]. That lemma, which in [7] is stated in $\mathbb{R}^n$, works also in our context by the same proof.
We start noticing that condition (11), which is required in Lemma 2.6 of [BCCT] is automatically satisfied by (3.14). Then we set

$$v = p_1 Q^{(1)} v^{(1)} + \cdots + p_l Q^{(l)} v^{(l)},$$

so that also condition (12) in Lemma 2.6 of [BCCT] is fulfilled.
It remains to verify condition (13) in the same lemma. To accomplish that task, we recall that by (3.12),

$$X^{(a)}_{hor} = T_{\sigma^{(a)}(\pi^{(a)}(g))} \tau_{h^{(a)}} T_{\pi^{(a)}(g)} \sigma^{(a)} T_g \pi^{(a)}(X),$$

which by the definitions (3.15) and (3.16) may be written as

$$(B_g^{(a)})^* B_g^{(a)} X = (B_g^{(a)})^* B_g^{(a)} X_{hor} = X^{(a)}_{hor}, \tag{3.19}$$

for all $X \in T_g G$. This formula by (3.13) implies

$$v_j(g) = (B_g^{(a)})^* B_g^{(a)} v_j(g).$$

Observe also that $B_g (B_g^{(a)})^*$ is a projection from $T_g G$ onto $T_{\pi^{(a)}(g)}(M^{(a)})$. As in the proof of Lemma 2.6 in [BCCT] it follows from (3.17) and (3.19) that (13) (in that paper) is also satisfied, proving (3.18). □

Theorem 3.2.4. *Assume the hypotheses above and suppose moreover that*

$$p_1 Q^{(1)} + \cdots + p_l Q^{(l)} = Q. \tag{3.20}$$

Then the inequality

$$\int_G \bar{f}^{(1)}(\pi^{(1)}(g))^{p_1} \ldots \bar{f}^{(l)}(\pi^{(l)}(g))^{p_l} dg \le I \left(\int_{M^{(1)}} \bar{f}^{(1)}(g_1) dg_1 \right)^{p_1} \cdots \left(\int_{M^{(l)}} \bar{f}^{(l)}(g_l) dg_l \right)^{p_l} \tag{3.21}$$

holds on G. The constant I appearing in this estimate is given by

$$I = \int_G P^{(1)}(\pi^{(1)}(e), \pi^{(1)}(g))^{p_1} \ldots P^{(l)}(\pi^{(l)}(e)\pi^{(l)}(g))^{p_l} dg$$

and is finite if there is $A > 0$ satisfying

$$|\sigma^{(1)}(\pi^{(1)}(g))| + \cdots + |\sigma^{(l)}(\pi^{(l)}(g))| > A|g|$$

for all $g \in G$, where $|\cdot|$ is any homogeneous norm.

Remark 3.2.5. Observe that in the abelian case, taking into account (3.11), the constant I reduces to the constant obtained in [7].

Proof. Since all the functions $u^{(a)}$ are Schwartz on $M^{(a)}$, it is easy to see that the initial condition in (3.14) implies by monotone convergence that

$$\lim_{t \to 0+} \Phi(t) = \int_G f^{(1)}(\pi^{(1)}(g))^{p_1} \ldots f^{(l)}(\pi^{(l)}(g))^{p_l} dg. \tag{3.22}$$

Suppose that

$$p_1 Q^{(1)} + \dots + p_l Q^{(l)} = Q. \tag{3.23}$$

From this condition, using (3.6), we deduce

$$\begin{aligned}
\Phi(t) &= t^{-\frac{1}{2}\sum_{a=1}^{(l)} p_a Q_a} \int_G \left(\int_{M^{(1)}} f(g_1) \bar{P}^{(1)}\big(\delta^{(1)}_{t^{-\frac{1}{2}}} g_1, \delta^{(1)}_{t^{-\frac{1}{2}}} \pi^{(1)}(g)\big) dg_1 \right)^{p_1} \dots \\
&\dots \left(\int_{M^{(l)}} f(g_r) \bar{P}^{(l)}\big(\delta^{(l)}_{t^{-\frac{1}{2}}} g_r, \delta^{(l)}_{t^{-\frac{1}{2}}} \pi^{(l)}(g)\big) dg_r \right)^{p_l} dg \\
&= t^{-\frac{Q}{2}} \int_G \left(\int_{M^{(1)}} f(g_1) \bar{P}^{(1)}\big(\delta^{(1)}_{t^{-\frac{1}{2}}} g_1, \pi^{(1)}\big(\delta_{t^{-\frac{1}{2}}} g\big)\big) dg_1 \right)^{p_1} \dots \\
&\dots \left(\int_{M^{(l)}} f(g_r) \bar{P}^{(l)}\big(\delta^{(l)}_{t^{-\frac{1}{2}}} g_r, \pi^{(l)}\big(\delta_{t^{-\frac{1}{2}}} g\big)\big) dg_r \right)^{p_l} dg,
\end{aligned}$$

which, replacing $\delta_{t^{-\frac{1}{2}}} g$ by x, becomes

$$\begin{aligned}
\Phi(t) = &\int_G \left(\int_{M^{(1)}} f(g_1) \bar{P}^{(1)}\big(\delta^{(1)}_{t^{-\frac{1}{2}}} g_1, \pi^{(1)}(x)\big) dg_1 \right)^{p_1} \dots \\
&\dots \left(\int_{M^{(l)}} f(g_r) \bar{P}^{(l)}\big(\delta^{(l)}_{t^{-\frac{1}{2}}} g_r, \pi^{(l)}(x)\big) dg_r \right)^{p_l} dx.
\end{aligned}$$

Since $f^{(1)}, \dots, f^{(l)}$ are Schwartz functions we may apply Fatou's Lemma, obtaining

$$\begin{aligned}
\lim_{t\to\infty} \Phi(t) = &\left(\int_G \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big)^{p_1} \dots P^{(l)}\big(\pi^{(l)}(e), \pi^{(l)}(g)\big)^{p_l} dg \right) \\
&\times \left(\int_{M^{(1)}} f(g_1) dg_1 \right)^{p_1} \dots \left(\int_{M^{(l)}} f(g_r) dg_r \right)^{p_l}.
\end{aligned} \tag{3.24}$$

From (3.18), (3.22) and (3.24) we finally establish the following inequality

$$\begin{aligned}
&\int_G f^{(1)}\big(\pi^{(1)}(g)\big)^{p_1} \dots f^{(l)}\big(\pi^{(l)}(g)\big)^{p_l} dg \\
&\le \left(\int_G \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big)^{p_1} \dots \bar{P}^{(l)}\big(\pi^{(l)}(e), \pi^{(l)}(g)\big)^{p_l} dg \right) \\
&\times \left(\int_{M^{(1)}} f(g_1) dg_1 \right)^{p_1} \dots \left(\int_{M^{(l)}} f(g_r) dg_r \right)^{p_l}.
\end{aligned}$$

It remains to discuss the finiteness of the integral

$$I = \int_G \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big)^{p_1} \dots \bar{P}^{(l)}\big(\pi^{(l)}(e), \pi^{(l)}(g)\big)^{p_l} dg,$$

which by (3.10) is given by

$$\int_G \left(\int_{H^{(1)}} P\big(\sigma^{(1)}(\pi^{(1)}(g)) h^{(1)}\big) dh^{(1)} \right)^{p_1} \dots \left(\int_{H^{(l)}} P\big(\sigma^{(l)}(\pi^{(l)}(g)) h^{(l)}\big) dh^{(l)} \right)^{p_l} dg.$$

Since P is a Schwartz function, for any positive integers N there is a constant C_N such that

$$P\big(\sigma^{(a)}(\pi^{(a)}(g)) h^{(a)}\big) \le C_N \left(1 + |\sigma^{(a)}(\pi^{(a)}(g))| + |h^{(a)}| \right)^{-N},$$

for all a. Taking N sufficiently large ($N > \max\{Q^{(1)}, \dots, Q^{(l)}\}$ suffices), we find

$$
\begin{aligned}
I &\leq C'_N \int_G \Big(1 + |\sigma^{(1)}(\pi^{(1)}(g))|\Big)^{-Np_1} \dots \Big(1 + |\sigma^{(l)}(\pi^{(l)}(g))|\Big)^{-Np_l} dg \\
&\leq C''_N \int_G \Big(1 + |\sigma^{(1)}(\pi^{(1)}(g))| + \dots + |\sigma^{(l)}(\pi^{(l)}(g))|\Big)^{-N(p_1+\dots+p_l)} dg < \infty,
\end{aligned}
$$

since

$$
|\sigma^{(1)}(\pi^{(1)}(g))| + \dots + |\sigma^{(l)}(\pi^{(l)}(g))| > A|g|,
$$

concluding the proof. □

Example 3.2.6. In the case of Hölder's inequality we have $l = \dim G = d$,

$$
M^{(1)} = \dots = M^{(d)} = G,
$$

$$
p_1 + \dots + p_d = 1,
$$

and $\pi^{(a)}$ coincides with the identity on G for all $a \in \{1, \dots, d\}$. In particular, we have $P^{(1)} = \dots = P^{(d)} = P$, which implies

$$
\begin{aligned}
I &= \int_G \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big)^{p_1} \dots \bar{P}^{(d)}\big(\pi^{(d)}(e), \pi^{(d)}(g)\big)^{p_d} dg \\
&= \int_G P(g)^{p_1} \dots P(g)^{p_d} dg \\
&= \int_G P(g) dg = 1.
\end{aligned}
$$

Example 3.2.7. Young's convolution inequality on the group G is equivalent to

$$
\int_G \int_G f_1(x) f_2(y) f_3(x^{-1}y) dx dy \lesssim \Big(\int_G f_1\Big)^{p_1} \Big(\int_G f_2\Big)^{p_2} \Big(\int_G f_3\Big)^{p_3},
$$

(with $f_1, f_2, f_3 \geq 0$), where $p_1 + p_2 + p_3 = 2$.
Here, to apply our machinery we consider the direct product, $G_2 = G \times G$, of two copies of G. The group G_2 is endowed with the family of dilations $\{\delta_t \times \delta_t\}_{t>0}$. We consider the following subgroups of G_2:

$$
H^{(1)} = G \times \{e\}, \quad H^{(2)} = \{e\} \times G, \quad H^{(3)} = \{(g, g) : g \in G\}
$$

and the corresponding homogeneous spaces: $M^{(1)}, M^{(2)}, M^{(3)}$, with the projections

$$
\pi^{(1)}(x, y) = y, \quad \pi^{(2)}(x, y) = x, \quad \pi^{(3)}(x, y) = x^{-1}y.
$$

The heat kernel on G_2 is given in terms of the heat kernel on G, p_t, by the product $q_t(x, y) = p_t(x)p_t(y)$. Therefore we have

$$
\bar{q}_t^{(1)}(y) = \int_{H^{(1)}} q_t(x, y) dx = p_t(y),
$$

$$
\bar{q}_t^{(2)}(x) = \int_{H^{(2)}} q_t(x, y) dy = p_t(x),
$$

and

$$
\begin{aligned}
\bar{q}_t^{(3)}\big(\pi^{(3)}(e,e),\pi^{(3)}(x,y)\big) &= \int_G q_t\big((e,e)^{-1}(g,g)(x,y)\big)dg \\
&= \int_G q_t\big((g,g)(x,y)\big)dg \\
&= \int_G q_t\big((gx,gy)\big)dg \\
&= \int_G p_t(gx)p_t(gy)dg \\
&= \int_G p_t(g)p_t(gx^{-1}y)dg \\
&= \int_G p_t(g)p_t\big(gy^{-1}x\big)dg,
\end{aligned}
$$

by the invariance of Haar measure and the symmetry of the kernel. It follows that

$$
\begin{aligned}
I &= \int_{G\times G} \bar{P}^{(1)}\big((\pi^{(1)}(x,y)\big)^{p_1}\bar{P}^{(2)}\big(\pi^{(2)}(x,y)\big)^{p_2}\bar{P}^{(3)}\big(\pi^{(3)}(x,y)\big)^{p_3}dxdy \\
&= \int_{G\times G} P(y)^{p_1}P(x)^{p_2}\left(\int_G P(z)P\big(zx^{-1}y\big)dz\right)^{p_3} dxdy \\
&= \int_{G\times G} P(y)^{p_1}P(x)^{p_2}\left(\int_G P(z^{-1})P\big(zx^{-1}y\big)dz\right)^{p_3} dxdy \\
&= \int_{G\times G} P(y)^{p_1}P(x)^{p_2}\left(\int_G P(z)P\big(z^{-1}x^{-1}y\big)dz\right)^{p_3} dxdy \\
&= \int_{G\times G} P(y)^{p_1}P(x)^{p_2}(P*P)^{p_3}(x^{-1}y)dxdy,
\end{aligned}
$$

where to obtain the last expression we used the invariance of the measure and the symmetry of P. With the same tools we finally obtain

$$
\begin{aligned}
I &= \int_{G\times G} P(y)^{p_1}P(x)^{p_2}(P*P)^{p_3}(x^{-1}y)dxdy \\
&= \int_G P(y)^{p_1}\big(P^{p_2}*(P*P)^{p_3}\big)(y)dy \\
&= \int_G P(y^{-1})^{p_1}\big(P^{p_2}*(P*P)^{p_3}\big)(y)dy \\
&= \big(\big(P^{p_2}*(P*P)^{p_3}\big)*P^{p_1}\big)(e).
\end{aligned}
$$

Example 3.2.8. In this example we identify the Lie algebra of the group G with the tangent space at the identity T_eG. We denote by X^r and X^l the right and left invariant vector fields associated to $X \in T_eG$. Recall that $\dim \mathfrak{g}_1 = n$ and that $\{X_1,\dots,X_n\}$ is a basis of $\mathfrak{g}_1$.
We look for inequalities of the form

$$
\int_G \prod_{a=1}^{n}\left(\bar{f}^{(a)}\big(\pi^{(a)}(x)\big)\right)^p dx \lesssim \prod_{a=1}^{n}\left(\int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)})dy^{(a)}\right)^p, \tag{3.25}
$$

where $M^{(a)} = H^{(a)}\backslash G$, with

$$
H^{(a)} = \{\exp(tX_a) : t\in\mathbb{R}\}, \quad a = 1,\dots,n.
$$

Since the groups $H^{(a)}$ coincide with the flow associated to X_a, the functions $\bar{f}^{(a)}$ lift to functions $f^{(a)}$ which are right invariant along the flow of X_a. Therefore, when the functions $\bar{f}^{(a)}$ are smooth, their lifts satisfy $X_a^r f^{(a)} = 0$.
Since the vectors X_a lie in the first layer of the Lie algebra, the groups $H^{(a)}$ have homogeneous dimension 1 and hence the spaces $M^{(a)}$ have homogeneous dimension $Q-1$. Therefore, condition (3.20) now reads

$$p = \frac{Q}{(Q-1)n}.$$

By (3.10) we have

$$\bar{P}^a\big(\pi^{(a)}(x), \pi^{(a)}(y)\big) = \int_{\mathbb{R}} P\big(y^{-1}\exp(tX_a)x\big)dt,$$

which by the symmetry of P yields

$$\bar{P}^a\big(\pi^{(a)}(e), \pi^{(a)}(y)\big) = \int_{\mathbb{R}} P\big(\exp(-tX_a)y\big)dt,$$

from which we obtain

$$\begin{aligned} I &= \int_G \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(y)\big)^{\frac{Q}{(Q-1)n}} \cdots \bar{P}^{(l)}\big(\pi^{(l)}(e), \pi^{(l)}(y)\big)^{\frac{Q}{(Q-1)n}} dy \\ &= \int_G \left(\int_{\mathbb{R}} P\big(\exp(-tX_1)y\big)dt\right)^{\frac{Q}{(Q-1)n}} \cdots \left(\int_{\mathbb{R}} P\big(\exp(-tX_l)y\big)dt\right)^{\frac{Q}{(Q-1)n}} dy. \end{aligned} \tag{3.26}$$

One can show that the last integral is finite using the classical estimates (3.4) holding for P, which show that

$$P(g) \lesssim e^{-c|g|^2},$$

where $|\cdot|$ is a homogeneous norm on G and $c > 0$ (see [26]).

We have just proved the following inequality.

Theorem 3.2.9. *Let, for $a = 1, \cdots, n$, let $f^{(a)}$ be a Schwartz function on the space $M^{(a)}$. Then*

$$\int_G \prod_{a=1}^n \left(\bar{f}^{(a)}\big(\pi^{(a)}(x)\big)\right)^{\frac{Q}{(Q-1)n}} dx \leq I \prod_{a=1}^n \left(\int_{M^{(a)}} \bar{f}^{(a)}(y^{(a)})dy^{(a)}\right)^{\frac{Q}{(Q-1)n}}. \tag{3.27}$$

The constant I in (3.27) *is given by* (3.26).

Remark 3.2.10. Observe that condition $\frac{Q}{(Q-1)n} \leq 1$ is satisfied for $Q \geq n \geq 2$. In fact, writing $Q = \rho n$, we get

$$1 \geq \frac{Q}{(Q-1)n} = \frac{\rho n}{(\rho n - 1)n} = \frac{\rho}{\rho n - 1},$$

yielding

$$\rho \geq \frac{1}{n-1},$$

which is always satisfied when $n \geq 2$ and $\rho \geq 1$.

From inequality (3.27) we may deduce the isoperimetric inequality on G for any stratified Lie group. In the next section we consider the case of the three dimensional Heisenberg group, leaving the discussion of the general case of stratified groups to a forthcoming paper.

3.3 The Heisenberg group

In this section we specialize the example above to the Heisenberg group $\mathbb{H}_1$. Since now $n = 2$ and $Q = 4$, we have $p = \frac{Q}{(Q-1)n} = \frac{2}{3}$. A basis of right invariant fields of the Lie algebra is given by

$$X^r = \partial_x + \frac{1}{2} y \partial_z, \quad Y^r = \partial_y - \frac{1}{2} x \partial_z, \quad Z = \partial_z. \tag{3.28}$$

The corresponding left invariant fields are given instead by

$$X^l = \partial_x - \frac{1}{2} y \partial_z, \quad Y^l = \partial_y + \frac{1}{2} x \partial_z, \quad Z = \partial_z.$$

Adopting, as usual, exponential coordinates we write

$$g = \exp(xX + yY + zZ).$$

The subgroup $H^{(1)}$ in these coordinates is given by

$$H^{(1)} = \{\exp(tX) : t \in \mathbb{R}\} = \{(t, 0, 0) : t \in \mathbb{R}\}.$$

Hence, $(u, v, w) \in H^{(1)} g = H^{(1)} e^{xX+yY+zZ}$ if and only if

$$(u, v, w) = \left(x + t, y, z + \frac{yt}{2} \right).$$

We chose representatives for the classes $H^{(1)} g$ of the form $(0, \bar{y}, \bar{z}) = \exp(\bar{y} Y + \bar{z} Z)$ and, abusing notation, write $(\bar{y}, \bar{z}) = H^{(1)} e^{\bar{y} Y + \bar{z} Z}$. In this way, we identify $M^{(1)}$ with $\mathbb{R}^2$. The fiber over $(\bar{y}, \bar{z})$ is

$$\begin{aligned} \left(\pi^{(1)}\right)^{-1}(\bar{y}, \bar{z}) &= \{\exp(tX) \exp(\bar{y} Y + \bar{z} Z) : t \in \mathbb{R}\} \\ &= \left\{ \exp\left(tX + \bar{y} Y + \left(\bar{z} + \frac{1}{2} \bar{y} t \right) Z \right) : t \in \mathbb{R} \right\}. \end{aligned}$$

Similarly,

$$H^{(2)} = \{\exp(tY) : t \in \mathbb{R}\} = \{(0, t, 0) : t \in \mathbb{R}\}.$$

The representatives for the classes $H^{(2)} g$ of the form $(\bar{x}, 0, \bar{z}) = \exp(\bar{x} X + \bar{z} Z)$. We write $(\bar{x}, \bar{z}) = H^{(2)} e^{\bar{x} X + \bar{z} Z}$. and identify $M^{(2)}$ with $\mathbb{R}^2$. The fiber over $(\bar{x}, \bar{z})$ is

$$\begin{aligned} \left(\pi^{(2)}\right)^{-1}(\bar{x}, \bar{z}) &= \{\exp(tY) \exp(\bar{x} X + \bar{z} Z) : t \in \mathbb{R}\} \\ &= \left\{ \exp\left(\bar{x} X + tY + \left(\bar{z} - \frac{1}{2} x t \right) Z \right) : t \in \mathbb{R} \right\}. \end{aligned}$$

In the coordinates just described the measures $dg^{(1)}$ and $dg^{(2)}$ coincide with the Lebesgue measures, $d\bar{y} d\bar{z}$ and $d\bar{x} d\bar{z}$, on $\mathbb{R}^2$.

Theorem 3.3.1. *With the notation above we have the following inequality*

$$\begin{aligned} &\int_{\mathbb{H}_1} \bar{f}^{(1)} \left(\pi^{(1)}(g) \right)^{\frac{2}{3}} \bar{f}^{(2)} \left(\pi^{(2)}(g) \right)^{\frac{2}{3}} dg \\ &\quad \lesssim \left(\int_{M^{(1)}} \bar{f}^{(1)}(g^{(1)}) dg^{(1)} \right)^{\frac{2}{3}} \left(\int_{M^{(2)}} \bar{f}^{(2)}(g^{(2)}) dg^{(2)} \right)^{\frac{2}{3}}. \end{aligned} \tag{3.29}$$

Proof. We compute first the heat kernels on the homogeneous spaces $M^{(1)}$ and $M^{(2)}$. Since

$$\begin{aligned} g^{-1}\exp(tX) &= \exp(-xX - yY - zZ)\exp(tX) \\ &= \exp\left((t-x)X - yY - zZ - \frac{1}{2}ty[Y,X]\right) \\ &= \exp\left((t-x)X - yY + \left(\frac{1}{2}ty - z\right)Z\right), \end{aligned}$$

we have

$$\begin{aligned} \bar{P}^{(1)}\big(\pi^{(1)}(e), \pi^{(1)}(g)\big) &= \bar{P}^{(1)}\big(0, \pi^{(1)}(g)\big) \\ &= \int_{\mathbb{R}} P\big(g^{-1}\exp(tX)\big)dt \\ &= \int_{\mathbb{R}} P\left(\exp\left((t-x)X - yY + \left(\frac{1}{2}ty - z\right)Z\right)\right)dt \\ &= \int_{\mathbb{R}} P\left(t, -y, \frac{1}{2}(t+x)y - z\right)dt, \end{aligned}$$

where we slightly abused notation writing $P(u,v,w)$ for $P\left(\exp(uX + vY + wZ)\right)$. Now, using the the classical bounds holding for $P = p_1$ (see for instance [57, Prop. 2.8.2]), we obtain

$$\begin{aligned} \int_{\mathbb{R}} P\left(t, -y, \frac{1}{2}(t+x)y - z\right)dt &\lesssim \int_{\mathbb{R}} \exp\left(-c\big(y^2+t^2\big) - c\left|z - \frac{1}{2}(t+x)y\right|\right)dt \\ &\lesssim \exp\left(-c\,y^2\right)\int_{\mathbb{R}} \exp\left(-c\,t^2\right)\exp\left(-c\left|z - \frac{1}{2}xy - \frac{1}{2}yt\right|\right)dt, \end{aligned}$$

hence,

$$\begin{aligned} &\int_{\mathbb{R}} P\left(t, -y, \frac{1}{2}(t+x)y - z\right)dt \\ &\lesssim \exp\left(-c\,y^2\right)\exp\left(-c\left|z - \frac{1}{2}xy\right|\right)\int_{\mathbb{R}} \exp\left(-c\,t^2\right)\exp\left(\frac{c}{2}|yt|\right)dt \\ &\lesssim \exp\left(-\frac{15}{16}c\,y^2\right)\exp\left(-c\left|z - \frac{1}{2}xy\right|\right)\int_{\mathbb{R}} \exp\left(-c\left(t - \frac{1}{4}y\right)^2\right)dt \\ &\lesssim \exp\left(-\frac{15}{16}c\,y^2\right)\exp\left(-c\left|z - \frac{1}{2}xy\right|\right). \end{aligned}$$

Similarly, to obtain a formula for $\bar{P}^{(2)}$, we compute

$$\begin{aligned} g^{-1}\exp(tY) &= \exp(-xX - yY - zZ)\exp(tY) \\ &= \exp\left(-xX + (t-y)Y - \left(z + \frac{1}{2}tx\right)Z\right), \end{aligned}$$

which, using the same bounds on P as before, gives

$$\begin{aligned} \bar{P}^{(2)}\big(\pi^{(2)}(e), \pi^{(2)}(g)\big) &= \int_{\mathbb{R}} P\left(\exp\left(-xX + (t-y)Y - \left(z + \frac{1}{2}tx\right)Z\right)\right)dt \\ &\lesssim \int_{\mathbb{R}} \exp\left(-c\big(x^2+t^2\big) - c\left|z - \frac{1}{2}(t+y)x\right|\right)dt \end{aligned}$$

$$\lesssim \exp\left(-c\,x^2\right)\int_{\mathbb{R}} \exp\left(-c\,t^2\right)\exp\left(-c\left|z-\frac{1}{2}xy-\frac{1}{2}xt\right|\right)dt,$$

and finally

$$\begin{aligned}&\int_{\mathbb{R}} P\left(\exp\left(-xX+(t-y)Y-\left(z+\frac{1}{2}tx\right)Z\right)\right)dt\\ &\lesssim \exp\left(-\frac{15}{16}c\,x^2\right)\exp\left(-c\left|z-\frac{1}{2}xy\right|\right).\end{aligned}$$

These estimates imply that the constant in the inequality, which is given by

$$\begin{aligned}I \lesssim &\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}} P\left(t,-y,\frac{1}{2}(t+x)y-z\right)dt\right)^{\frac{2}{3}}\\ &\times\left(\int_{\mathbb{R}} P\left(-x,s,-z-\frac{1}{2}(s+y)x\right)ds\right)^{\frac{2}{3}}dxdydz,\end{aligned}$$

is finite. □

To obtain the isoperimetric inequality we start observing that for a Schwartz function f we have

$$\begin{aligned}X^l f\left(x+t,y,z+\frac{1}{2}ty\right)&=\frac{d}{dt}f\left(x+t,y,z+\frac{1}{2}ty\right),\\ Y^l f\left(x,y+t,z-\frac{1}{2}ty\right)&=\frac{d}{dt}f\left(x,y+t,z-\frac{1}{2}ty\right),\end{aligned}$$

for all $(x,y,z)\in\mathbb{H}_1$. The first of these relations, by the fundamental theorem of calculus, implies that

$$\begin{aligned}f(x,y,z)&=-\int_0^\infty \frac{d}{dt}f\left(x+t,y,z+\frac{1}{2}ty\right)dt\\ &=-\int_0^\infty X^l f\left(x+t,y,z+\frac{1}{2}ty\right)dt,\end{aligned}$$

from which it follows that

$$\begin{aligned}|f(x,y,z)|&\le\int_0^\infty\left|X^l f\left(x+t,y,z+\frac{1}{2}ty\right)\right|dt\\ &\le\frac{1}{2}\int_{-\infty}^\infty\left|X^l f\left(x+t,y,z+\frac{1}{2}ty\right)\right|dt\\ &=\frac{1}{2}\int_{-\infty}^\infty\left|X^l f\left(u,y,z-\frac{1}{2}xy+\frac{1}{2}uy\right)\right|du\\ &=\phi\left(y,z-\frac{1}{2}xy\right).\end{aligned}\tag{3.30}$$

Similarly we obtain

$$\begin{aligned}|f(x,y,z)| &\le \frac{1}{2}\int\limits_{-\infty}^{\infty}\left|Yf\left(x,v,z-\frac{1}{2}(v-y)x\right)\right|\,dv\\ &= \psi\left(x,z+\frac{1}{2}xy\right).\end{aligned} \tag{3.31}$$

We note that by (3.28) we have

$$X^r\phi = 0, \quad Y^r\psi = 0.$$

This property by Proposition 3.2.1 means that there are two smooth functions $\bar{\phi} : M^{(1)} \to \mathbb{R}$ and $\bar{\psi} : M^{(2)} \to \mathbb{R}$, satisfying

$$\phi\left(y,z-\frac{1}{2}xy\right) = \bar{\phi}\circ\pi^{(1)}(x,y,z), \quad \psi\left(x,z+\frac{1}{2}xy\right) = \bar{\psi}\circ\pi^{(2)}(x,y,z).$$

Therefore, we may apply (3.27) to obtain

$$\begin{aligned}\int_{\mathbb{H}_1}\bar{\phi}(\pi^{(1)}(g))^{\frac{2}{3}}\bar{\psi}(\pi^{(2)}(g))^{\frac{2}{3}}dg &\lesssim \left(\int_{M^{(1)}}\bar{\phi}(g^{(1)})dg^{(1)}\right)^{\frac{2}{3}}\left(\int_{M^{(2)}}\bar{\psi}(g^{(2)})dg^{(2)}\right)^{\frac{2}{3}}\\ &= \left(\int_{\mathbb{R}^2}\bar{\phi}(\bar{y},\bar{z})d\bar{y}d\bar{z}\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^2}\bar{\psi}(\bar{x},\bar{z})d\bar{x}d\bar{z}\right)^{\frac{2}{3}},\end{aligned} \tag{3.32}$$

recalling that the measures $dg^{(1)}$ and $dg^{(2)}$ coincide with the Lebesgue measure on $\mathbb{R}^2$. From (3.32) we obtain

$$\begin{aligned}\int_{\mathbb{H}_1}|f|^{\frac{4}{3}}dxdydz &= \int_{\mathbb{H}_1}|f|^{\frac{2}{3}}|f|^{\frac{2}{3}}dxdydz\\ &\lesssim \int_{\mathbb{H}_1}\phi\left(y,z-\frac{1}{2}xy\right)^{\frac{2}{3}}\psi\left(x,z+\frac{1}{2}xy\right)^{\frac{2}{3}}dxdydz\\ &\lesssim \left(\int_{\mathbb{R}^2}\bar{\phi}(\bar{y},\bar{z})d\bar{y}d\bar{z}\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^2}\bar{\psi}(\bar{x},\bar{z})d\bar{x}d\bar{z}\right)^{\frac{2}{3}}.\end{aligned}$$

Therefore, we have the bound

$$\begin{aligned}\|f\|_{L^{\frac{4}{3}}(\mathbb{H}_1)} &= \left(\int_{\mathbb{H}_1}|f|^{\frac{4}{3}}dxdydz\right)^{\frac{3}{4}}\\ &\lesssim \left(\int_{\mathbb{R}^2}\bar{\phi}(y,z)dydz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\bar{\psi}(x,z)dxdz\right)^{\frac{1}{2}}\\ &\le \frac{1}{2}\left(\int_{\mathbb{R}^3}\left|Xf(x,y,z)\right|dxdydz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left|Yf(x,y,z)\right|dxdydz\right)^{\frac{1}{2}}\\ &\le \frac{1}{4}\left(\int_{\mathbb{R}^3}\left|Xf(x,y,z)\right|dxdydz + \int_{\mathbb{R}^3}\left|Yf(x,y,z)\right|dxdydz\right)\\ &\le \frac{1}{4}\int_{\mathbb{R}^3}\left|\nabla f(x,y,z)\right|dxdydz\\ &= \frac{1}{4}\|\nabla f\|_{L^1(\mathbb{H}_1)},\end{aligned}$$

where we used the arithmetic-geometric mean inequality.
We have thus proved the following result.

Theorem 3.3.2 (Gagliardo–Nirenberg inequality for $\mathbb{H}_1$). *On $\mathbb{H}_1$ Schwartz functions obey the estimate*

$$\|f\|_{L^{\frac{4}{3}}(\mathbb{H}_1)} \leq \frac{1}{4}\|\nabla f\|_{L^1(\mathbb{H}_1)}. \tag{3.33}$$

The isoperimetric inequality, relating the horizontal perimeter (for the definition see for instance [17]) of a set to its volume, was first proved on $\mathbb{H}_1$ by P. Pansu in [48] and then extended to general stratified groups by L. Capogna, D. Danielli, N. Garofalo in [16] (see also [29] and recent works [40], [45]). It is well known that this inequality may be obtained as a consequence of the Gagliardo–Nirenberg inequality (see for instance [17] or [50]), therefore, as a corollary of the theorem above we obtain the following result.

Corollary 3.3.3. *Let E be a measurable bounded subset of $\mathbb{H}_1$. Then*

$$|E|^{\frac{3}{4}} \lesssim P(E), \tag{3.34}$$

where $P(E)$ denotes the horizontal perimeter of E.

By essentially the same argument one can show that on any stratified group the Gagliardo–Nirenberg and the isoperimetric inequalities are a consequence of (3.27).

Part II

A discrete restriction theorem on the quaternionic sphere

CHAPTER 4

L^p joint eigenfunctions bounds on spheres

This final chapter is devoted to the study of some sharp bounds (some of them depending on the dimension and some not) for bigraded spherical harmonics on complex and quaternionic spheres, in the spirit of some earlier work by C. Sogge and J. Duoandikoetxea. These estimates have been successfully applied to different problems in harmonic analysis, like Strichartz estimates for solutions of the Schrödinger equation [13, 14, 23], L^p summability of Bochner Riesz means [52, 22], unique continuation problems [36, 53].
Throughout this chapter we use the notation $A \sim B$ to indicate that the ratio of the two sides is bounded above and below.

4.1 Estimates for quaternionic harmonic projection operators

In this section we prove some bounds from below for the (L^p, L^2) norm of the projection operators mapping the space of square integrable functions defined on the quaternionic unit sphere $\mathbb{S}^{4n-1}$ in $\mathbb{H}^n$, where $\mathbb{H}$ is the skew-field of quaternions, onto certain subspaces of quaternion spherical harmonics into which $L^2(\mathbb{S}^{4n-1})$ decomposes. The results contained in this section were obtained in collaboration with V. Casarino and P. Ciatti and are published in [10].

4.1.1 Notation and preliminaries

We denote by $\mathbb{H}$ the skew field of quaternions $q = x_0 + x_1 i + x_2 j + x_3 k$ over $\mathbb{R}$, where x_0, x_1, x_2, x_3 are real numbers and the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1, ij = -ji = k, ik = -ki = -j, jk = -kj = i$. The conjugate $\overline{q}$ and the modulus $|q|$ are defined by $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and $|q|^2 = q\overline{q} = \sum_{j=0}^{3} x_j^2$, respectively. For $n \geq 1$ the symbol $\mathbb{H}^n$ will denote the n-dimensional vector space over $\mathbb{H}$. By abuse of notation, we write q also to denote $(q_1, \dots, q_n) \in \mathbb{H}^n$. Sometimes we will adopt a complex notation, writing $q = (z_1 + jz_{n+1}, \dots, z_n + jz_{2n})$, with $z_1, \dots, z_{2n} \in \mathbb{C}$.
Let $\mathbb{S}^{4n-1}$ be the unit sphere in $\mathbb{H}^n$, that is,

$$\mathbb{S}^{4n-1} = \{q = (q_1, \dots, q_n) \in \mathbb{H}^n : \langle q, q \rangle = 1\},$$

where the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ is defined as $\langle q, q' \rangle = q_1 \overline{q_1'} + \dots + q_n \overline{q_n'}$, with $q, q' \in \mathbb{H}^n$.
The sphere $\mathbb{S}^{4n-1}$ may be identified with the homogeneous space (with quotient on the left)

$$K/M = \mathrm{Sp(n)} \times \mathrm{Sp(1)}/\mathrm{Sp(n-1)} \times \mathrm{Sp(1)},$$

where Sp(n) denotes the group of $n \times n$ matrices A with quaternionic entries, such that $\overline{A^T}A = A\overline{A^T} = I_n$.
We introduce on $\mathbb{S}^{4n-1}$ the coordinate system

$$\begin{cases} q_1 = \cos\theta(\cos t + \tilde{q}\sin t) \\ q_s = \sigma_s \sin\theta, \qquad s = 2, \dots, n, \end{cases} \tag{4.1}$$

where $\theta \in [0, \pi/2]$, $t \in [0, \pi]$, $\sigma_s \in \mathbb{H}$ with $\sum_{s=2}^n |\sigma_s|^2 = 1$. Moreover, for $\tilde{q} \in \mathbb{H}$ with $|\tilde{q}|^2 = 1$ and $\Re\tilde{q} = 0$, we will write $\tilde{q} = \cos\psi i + \sin\psi\cos\varphi j + \sin\psi\sin\varphi k$, with $\psi \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.
We remark that $(\sin t \sin\psi\sin\varphi, \sin t\sin\psi\cos\varphi, \sin t\cos\psi, \cos t)$ yields a coordinate system for Sp(1).
The normalized invariant measure $d\sigma = d\sigma_{\mathbb{S}^{4n-1}}$ on $\mathbb{S}^{4n-1}$ with respect to the spherical coordinates (4.1) is, up to a constant depending only on the dimension n,

$$\sin^{4n-5}\theta\cos^3\theta d\theta \sin^2 t dt d\sigma_{\mathbb{S}^{4n-5}} d\sigma(\tilde{q}), \tag{4.2}$$

$d\sigma(\tilde{q})$ denoting the measure on the unit sphere in $\mathbb{R}^3$.
By $L^2(\mathbb{S}^{4n-1})$ we denote the Hilbert space of square integrable functions on $\mathbb{S}^{4n-1}$, with respect to the inner product

$$(f, g)_{L^2} = \int_{\mathbb{S}^{4n-1}} f(q)\overline{g(q)} d\sigma.$$

Johnson and Wallach, starting from some earlier work by Kostant [39], proved in [37] that this space may be decomposed as

$$L^2(\mathbb{S}^{4n-1}) = \bigoplus_{\ell \geq \ell' \geq 0} \mathcal{H}^{\ell\ell'}, \tag{4.3}$$

where each subspace $\mathcal{H}^{\ell\ell'}$

1. is irreducible under K;
2. is generated under K by the "highest weight vector"

$$\mathrm{P}_{\ell,\ell'}(z, \bar{z}) = \bar{z}_{n+1}^{\ell-\ell'}(z_1\bar{z}_{n+2} - z_2\bar{z}_{n+1})^{\ell'}; \tag{4.4}$$

3. is finite dimensional.

We will denote by $I_{\mathbb{S}}$ the set of indices $\{(\ell, \ell') \in \mathbb{N} \times \mathbb{N} : 0 \leq \ell' \leq \ell\}$.
In [21] the authors studied the $L^p - L^2$ norm of the joint spectral projectors $\pi_{\ell\ell'}$, $(\ell, \ell') \in I_{\mathbb{S}}$, mapping $L^p(\mathbb{S}^{4n-1})$ onto $\mathcal{H}^{\ell\ell'}$, $1 \leq p \leq 2$. They proved sharp bounds for these norms under the additional assumptions $\ell - \ell' \leq c_0$ or $\ell' \leq c_1$, for some positive constants c_0, c_1. Here, we prove some crucial estimates from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ in the general case.
In the following subsections we will prove the following result.

Theorem 4.1.1. *Let $n \geq 2$, $1 \leq p \leq 2$. Set $p_n = 2(4n-3)/(4n-1)$. Then the following estimate holds*

$$||\pi_{\ell\ell'}||_{(p,2)} \gtrsim (1+\ell)^{\alpha(\frac{1}{p},n)}(1+\ell')^{\beta(\frac{1}{p},n)}(\ell - \ell' + 1)^{\gamma(\frac{1}{p},n)}, \tag{4.5}$$

where

$$\alpha(\frac{1}{p},n) := 2(n-1)\big(\frac{1}{p}-\frac{1}{2}\big) \text{ for all } 1 \le p \le 2,$$

$$\beta(\frac{1}{p},n) := \begin{cases} 2(n-1)\big(\frac{1}{p}-\frac{1}{2}\big)-\frac{1}{2} & \text{if } 1 \le p \le p_n \\ \frac{1}{2}\big(\frac{1}{2}-\frac{1}{p}\big) & \text{if } p_n \le p \le 2, \end{cases}$$

and

$$\gamma(\frac{1}{p},n) := \begin{cases} 3\big(\frac{1}{p}-\frac{1}{2}\big)-\frac{1}{2} & \text{if } 1 \le p \le \frac{4}{3} \\ \frac{1}{p}-\frac{1}{2} & \text{if } \frac{4}{3} \le p \le 2, \end{cases}$$

for all $(\ell,\ell') \in I_{\mathbb{S}}$, such that $\ell-\ell'$ and ℓ' are sufficiently large. The implicit constants depend only on the dimension n and on the exponent p.

4.1.2 Estimates for zonal functions

We call *zonal function of bidegree* (ℓ,ℓ') with pole $e_1=(1,0\ldots,0)$ a M-invariant function in $\mathcal{H}^{\ell\ell'}$, i.e. a function f on $\mathbb{S}^{4n-1}$ such that $f(x)=f(mx)$ for all $m\in M$, $x\in\mathbb{S}^{4n-1}$, where x has to be interpreted as an element of the quotient K/M.
An explicit formula for the zonal function $\mathrm{Z}_{\ell\ell'}$ with pole e_1 is given for all $(\ell,\ell')\in I_{\mathbb{S}}$ by

$$\mathrm{Z}_{\ell\ell'}(\theta,t) = \frac{d_{\ell\ell'}}{\omega_{4n-1}}\frac{\sin\big((\ell-\ell'+1)t\big)}{(\ell-\ell'+1)\sin t}(\cos\theta)^{\ell-\ell'}\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}, \tag{4.6}$$

where $t\in[0,\pi], \theta\in[0,\frac{\pi}{2}]$, ω_{4n-1} denotes the surface area of $\mathbb{S}^{4n-1}$, $P_{\ell'}^{(2n-3,\ell-\ell'+1)}$ is a Jacobi polynomial and $d_{\ell\ell'}$ is the dimension of $\mathcal{H}^{\ell\ell'}$, given by

$$d_{\ell\ell'} = (\ell+\ell'+2n-1)(\ell-\ell'+1)^2\frac{(\ell+2n-2)!}{(\ell+1)!(2n-3)!}\frac{(\ell'+2n-3)!}{\ell'!(2n-1)!},\ \ell\ge\ell'\ge 0. \tag{4.7}$$

We recall the Mehler–Heine formula for the so-called disk polynomials, proved in [9, p. 10]. The symbol J_α denotes the Bessel function of the first kind of order α.

Proposition 4.1.1. *Fix $n\in\mathbb{N}$. Let $j,k\in\mathbb{N}$, $j\le k$. Then*

$$\lim_{\substack{j\to+\infty\\k\to+\infty}}\Big(\cos(\frac{\theta}{\sqrt{jk}})\Big)^{k-j}\frac{P_j^{(2n-3,k-j)}\big(\cos(\frac{2\theta}{\sqrt{jk}})\big)}{P_j^{(2n-3,k-j)}(1)} = \Gamma(2n-2)\frac{J_{2n-3}(2\theta)}{\theta^{2n-3}},$$

where Γ denotes the gamma function. This limit holds uniformly in every compact interval.

We also recall (see [9, p. 12]) that for all $j,k\in\mathbb{N}$, $j\le k$,

$$\sup_{\theta\in[0,\pi/2]}\Big|(\cos\theta)^{k-j}\frac{P_j^{(2n-3,k-j)}\big(\cos(2\theta)\big)}{P_j^{(2n-3,k-j)}(1)}\Big| \le 1. \tag{4.8}$$

For $q\ge 2$ set

$$\mathcal{I}_q = \Big(\int_0^{\pi/2}\Big|\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}(\cos\theta)^{\ell-\ell'}\Big|^q(\sin\theta)^{4n-5}(\cos\theta)^3 d\theta\Big)^{1/q}. \tag{4.9}$$

Lemma 4.1.2. *For all $q \geq 2$ and for all $(\ell, \ell') \in I_{\mathbb{S}}$ such that ℓ' is sufficiently large, we have*

$$\begin{aligned}\frac{\mathcal{I}_q}{\mathcal{I}_2} &\gtrsim (\ell')^{(2n-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \ell^{(2n-2)(\frac{1}{2}-\frac{1}{q})} \\ &\times \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q([0,1];\theta^{4n-5}d\theta)}.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}(\mathcal{I}_q)^q &\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \right|^q (\sin\theta)^{4n-5} (\cos\theta)^3 d\theta \\ &= \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'+\frac{3}{q}} \right|^q (\sin\theta)^{4n-5} d\theta \\ &\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'+1} \right|^q (\sin\theta)^{4n-5} d\theta,\end{aligned}$$

where the last inequality follows from the fact that $\theta \in (0, 1/\sqrt{\ell\ell'})$, with $1/\sqrt{\ell\ell'} \leq 1/2$, so that $\cos\theta \sim 1$, independently of ℓ, ℓ'. Then, after a change of variables we get

$$\begin{aligned}(\mathcal{I}_q)^q &\gtrsim \int_0^1 \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right|^q (\sin(\theta/\sqrt{\ell\ell'}))^{4n-5} \frac{d\theta}{\sqrt{\ell\ell'}} \\ &\sim \int_0^1 \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right|^q (\theta/\sqrt{\ell\ell'})^{4n-5} d\theta/(\sqrt{\ell\ell'}) \\ &\sim (\ell\ell')^{-(2n-2)} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|^q_{L^q([0,1];\theta^{4n-5}d\theta)}. \qquad (4.10)\end{aligned}$$

It is convenient to set

$$F_{\ell\ell'}(\theta) := \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1}.$$

For $q = 2$ we obtain a more precise estimate. Indeed, from standard properties of zonal harmonics it follows that $||\mathrm{Z}_{\ell\ell'}||_2 \sim (d_{\ell\ell'})^{1/2}$ which, by means of (4.6), yields

$$\begin{aligned}d_{\ell\ell'} &\sim (d_{\ell\ell'})^2 \int_0^\pi \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^2 \sin^2 t dt \\ &\times \int_0^{\pi/2} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \right|^2 (\sin\theta)^{4n-5} (\cos\theta)^3 d\theta.\end{aligned}$$

Since

$$\int_0^\pi \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^2 \sin^2 t dt \sim (\ell-\ell'+1)^{-2}, \qquad (4.11)$$

we have

$$(\mathcal{I}_2)^2 \sim (\ell-\ell'+1)^2 (d_{\ell\ell'})^{-1}. \qquad (4.12)$$

Then, combining (4.10) and (4.12), we get for all $q > 2$

$$\begin{aligned}\frac{\mathcal{I}_q}{\mathcal{I}_2} &\gtrsim (\ell-\ell'+1)^{-1}(d_{\ell\ell'})^{1/2}(\ell\ell')^{-(2n-2)/q}\|F_{\ell\ell'}(\theta)\|_{L^q([0,1];\theta^{4n-5}d\theta)}\\ &\gtrsim (\ell')^{(2n-3)/2}\ell^{(2n-2)/2}(\ell\ell')^{-(2n-2)/q}\|F_{\ell\ell'}(\theta)\|_{L^q([0,1];\theta^{4n-5}d\theta)}\\ &\gtrsim (\ell')^{(2n-2)(\frac12-\frac1q)-\frac12}\ell^{(2n-2)(\frac12-\frac1q)}\|F_{\ell\ell'}(\theta)\|_{L^q([0,1];\theta^{4n-5}d\theta)}.\end{aligned}$$

□

To state the following lemma we set, for $q \geq 2$,

$$\mathcal{J}_q = \Big(\int_0^\pi \Big|\frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t}\Big|^q \sin^2 t dt\Big)^{1/q}. \tag{4.13}$$

Lemma 4.1.3. *For all $q \geq 2$ and for all $(\ell,\ell') \in I_{\mathbb{S}}$ such that $\ell-\ell'$ is sufficiently large, we have*

$$\frac{\mathcal{J}_q}{\mathcal{J}_2} \sim \begin{cases} (\ell-\ell'+1)^{1-3/q} & \textit{for all } q>3\\ (\log(\ell-\ell'))^{1/3} & \textit{for all } q=3\\ 1 & \textit{for all } q<3.\end{cases}$$

Proof. We start recalling that

$$\frac{\sin((\ell-\ell'+1)t)}{\sin t} = O((\ell-\ell'+1)^{1/2})P_{\ell-\ell'}^{(\frac12,\frac12)}(\cos t),$$

[56, p.60]. Thus, using a known asymptotic integral estimate (see [56, p.391]), we see that

$$(\mathcal{J}_q)^q \sim \int_0^{\pi/2} \Big|\frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t}\Big|^q \sin^2 t dt \sim (\ell-\ell'+1)^{-3}, \tag{4.14}$$

for $q > 3$ and $\ell-\ell'$ sufficiently large. Combining (4.11) and (4.14) we get the expected estimate for $\mathcal{J}_q/\mathcal{J}_2$ for all $q > 3$. Similarly, the other two cases follow from [56, p.391], and (4.11). □

Combining Lemma 4.1.2 and Lemma 4.1.3 we obtain a bound from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, with $1 \leq p \leq 2$.

Proposition 4.1.4. *Fix $n \geq 2$. For all $(\ell,\ell') \in I_{\mathbb{S}}$ such that ℓ' and $\ell-\ell'$ are sufficiently large, and for all $q \geq 2$ we have*

$$\frac{||\mathrm{Z}_{\ell\ell'}||_q}{||\mathrm{Z}_{\ell\ell'}||_2} \gtrsim \begin{cases} (\ell-\ell'+1)^{1-3/q}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \textit{for all } q>3\\ (\log(\ell-\ell'))^{1/3}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \textit{for } q=3\\ (\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2} & \textit{for all } q<3.\end{cases} \tag{4.15}$$

Proof. For $q > 3$, as a consequence of Lemma 4.1.2 we have

$$\begin{aligned}\frac{||\mathrm{Z}_{\ell\ell'}||_q}{||\mathrm{Z}_{\ell\ell'}||_2} &\gtrsim (\ell-\ell'+1)^{1-3/q}\mathcal{I}_q/\mathcal{I}_2\\ &\sim (\ell-\ell'+1)^{1-3/q}(\ell\ell')^{(2n-2)(1/2-1/q)}(\ell')^{-1/2}\|F_{\ell\ell'}(\theta)\|_{L^q(\theta^{4n-5}d\theta,[0,1])}.\end{aligned}$$

Then the first inequality in (4.15) follows from a slight variation of Proposition 4.1.1, (4.8) and some trivial asymptotics for the Bessel function. The proof of the other two inequalities is similar. □

4.1.3 Estimates for the highest weight spherical harmonics.

In this subsection we will estimate the norm of the highest weight spherical harmonics $\mathrm{P}_{\ell,\ell'}$ in $\mathcal{H}^{\ell\ell'}$, defined in (4.4).
In [21, Lemma 5.3] the authors proved that for all $\zeta_1 \in \mathbb{R}$, $\zeta_1 > 0$, and for all $\zeta_2 \in \mathbb{N}$ one has

$$\int_{\mathbb{S}^{4n-1}} |\bar{z}_{n+1}|^{2\zeta_1} |z_1\bar{z}_{n+2} - z_2\bar{z}_{n+1}|^{2\zeta_2} d\sigma = \frac{c_n\, \Gamma(\zeta_1+\zeta_2+2)\Gamma(\zeta_2+1)}{\Gamma(\zeta_1+2\zeta_2+2n)(\zeta_1+1)}. \tag{4.16}$$

They also proved that as a consequence of (4.16) the following bound holds

$$\|\mathrm{P}_{\ell,\ell'}\|_2 \sim \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\ (\ell-\ell'+1)}\right)^{\frac{1}{2}}. \tag{4.17}$$

Proposition 4.1.5. *Let $\mathrm{P}_{\ell\ell'}$ be the highest weight vector defined by (4.4). For all $q \geq 2$ we have*

$$\limsup_{\ell' \to +\infty} \Big(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\ (\ell-\ell'+1)}\Big)^{\frac{1}{2}-\frac{1}{q}} \frac{\|\mathrm{P}_{\ell,\ell'}\|_q}{\|\mathrm{P}_{\ell,\ell'}\|_2} > 0. \tag{4.18}$$

Proof. Fix any $q \geq 2$ and let $(\ell,\ell') \in I_{\mathbb{S}}$. First of all, we choose $2\zeta_1 = (\ell-\ell')q$. Then, if $\ell' q \in 2\mathbb{N}$, (4.16) applied to $\mathrm{P}_{\ell\ell'}$ with $2\zeta_2 = \ell' q$ yields

$$\|\mathrm{P}_{\ell,\ell'}\|_q^q = \frac{c_n\ \Gamma(\frac{q}{2}\ell+2)\ \Gamma(\frac{q}{2}\ell'+1)}{\Gamma(\frac{q}{2}(\ell+\ell')+2n)\ (\frac{q}{2}(\ell-\ell')+1)}.$$

Now a standard application of Stirling's estimate leads to

$$\|\mathrm{P}_{\ell,\ell'}\|_q \sim \frac{(\frac{q}{2}\ell+1)^{\frac{1}{2}\ell+(1+\frac{1}{2})/q}(\frac{q}{2}\ell'+1)^{\frac{1}{2}\ell'+1/(2q)}}{(\frac{q}{2}(\ell+\ell')+2n-1)^{\frac{1}{2}(\ell+\ell')+(2n-1+\frac{1}{2})/q}\ (\frac{q}{2}(\ell-\ell')+1)^{1/q}},$$

which, combined with (4.17), yields

$$\frac{\|\mathrm{P}_{\ell,\ell'}\|_q}{\|\mathrm{P}_{\ell,\ell'}\|_2} \sim \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\ (\ell-\ell'+1)}\right)^{\frac{1}{q}-\frac{1}{2}}. \tag{4.19}$$

This proves the assertion under the assumption $\ell' q \in 2\mathbb{N}$.
If $q = \frac{m_0}{n_0}$, for some $m_0,\ n_0 \in \mathbb{N}^*$, it suffices to replace ℓ' with $2n_0\ell'$ and then choose $\zeta_2 = m_0\ell'$. By considering $(\ell,\ell') \in I_{\mathbb{S}}$ such that $\ell \geq 2n_0\ell'$, we get an estimate analogous to (4.19) for $\|\mathrm{P}_{\ell,2n_0\ell'}\|_q$, yielding (4.18).
Finally, if q is not rational, the desired estimate follows from the continuity of the L^q norms and the previous arguments for rational values of q. □

4.1.4 Estimates for mixed spherical harmonics.

We consider the function $\mathrm{Q}_{\ell\ell'}$, given by

$$\mathrm{Q}_{\ell\ell'}(\theta,\varphi,t) = \big(\sin t \sin\psi e^{i\varphi}\big)^{\ell-\ell'} (\cos\theta)^{\ell-\ell'} \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}, \tag{4.20}$$

for all $(\ell,\ell') \in I_{\mathbb{S}}$, with $t,\psi \in [0,\pi]$, $\varphi \in [0,2\pi]$, $\theta \in [0,\frac{\pi}{2}]$. Observe that $\mathrm{Q}_{\ell\ell'}$ is obtained replacing the factor $\sin((\ell-\ell'+1)t)/\big((\ell-\ell'+1)\sin t\big)^{-1}$ in (4.6) with the highest weight

spherical harmonic of degree $\ell - \ell'$ in Σ^3, the unit sphere in $\mathbb{R}^4$. For a discussion about the role of Σ^3 (or, equivalently, of $\mathrm{Sp}(1)$) in our analysis we refer to [21, Remark 2.3].
We recall here that $\mathcal{H}^{\ell\ell'}$ is a joint eigenspace for the spherical Laplacian $\Delta_{\mathbb{S}^{4n-1}}$ and for an operator Γ, which essentially coincides with the Casimir operator on $\mathrm{Sp}(1)$ and in our coordinates reads as

$$\Gamma = \frac{1}{\sin^2 t}\frac{\partial}{\partial t}\sin^2 t\frac{\partial}{\partial t} + \frac{1}{\sin^2 t \sin\psi}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi} + \frac{1}{\sin^2 t}\frac{1}{\sin^2\psi}\frac{\partial^2}{\partial^2\varphi}.$$

We refer to [37] and [35, p. 696] for a discussion about the role of this operator. Then it is easily seen that $\mathrm{Q}_{\ell\ell'}$ belongs to $\mathcal{H}^{\ell\ell'}$, since it is an eigenvector for both $\Delta_{\mathbb{S}^{4n-1}}$ and Γ.

Proposition 4.1.6. *Fix $n \geq 2$. For all $(\ell, \ell') \in I_{\mathbb{S}}$, such that ℓ' and $\ell - \ell'$ are sufficiently large, and for all $q > 2$ we have*

$$\frac{||\mathrm{Q}_{\ell\ell'}||_q}{||\mathrm{Q}_{\ell\ell'}||_2} \gtrsim (\ell - \ell' + 1)^{1/2-1/q}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2}.$$

Proof. The estimate in the assertion follows from Lemma 4.1.2, Proposition 4.1.1 and some basic bounds for the spherical harmonics in Σ^3 (see [51, Theorem 4.1]). □

Remark 4.1.7. A comparison between Proposition 4.1.4, Proposition 4.1.5 and Proposition 4.1.6 leads to the estimates of Theorem 4.1.1, thus proving it.

Remark 4.1.8. The proof of the reverse inequality of (4.5), which involves both real and complex interpolation arguments, multiplier theorems for $\Delta_{\mathbb{S}^{4n-1}}$, Γ and for $\mathcal{L}$, and a very detailed analysis of the Jacobi polynomials, is quite long and involved. This work is already under way.

4.2 Dimension free estimates for bigraded spherical harmonics

In this section we prove some dimension free estimates for bigraded spherical harmonics on complex and quaternionic spheres, inspired by an earlier paper by Duoandikoetxea [24]. Our focus is on the sharpness of these estimates

4.2.1 Some dimension free estimates

We start recalling some well-known decompositions of the spaces of square-integrable functions on spheres. More precisely, we recall that the space $L^2(\mathbb{S}^{n-1})$ may be written as

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{\ell\in\mathbb{N}} V_\ell, \tag{4.21}$$

where each subspace V_ℓ, formed by the spherical harmonics of degree ℓ, is invariant under the action of the orthogonal group $\mathrm{O}(n)$ [55]. If $\mathbb{S}^{n-1}$ is replaced by the complex unit sphere in $\mathbb{C}^n$, $\mathbb{S}^{2n-1} \simeq \mathrm{U}(n)/\mathrm{U}(n-1)$, Vilenkin (see [38]) proved a finer decomposition of $L^2(\mathbb{S}^{2n-1})$ as direct sum of invariant subspaces $H_{\ell,\ell'}$, irreducible under the action of the unitary group $\mathrm{U}(n)$, that is

$$L^2(\mathbb{S}^{2n-1}) = \bigoplus_{(\ell,\ell')\in\mathbb{N}\times\mathbb{N}} H_{\ell,\ell'}. \tag{4.22}$$

The relationship between (4.21) and (4.22) is given by

$$V_k = \bigoplus_{\ell+\ell'=k} H_{\ell,\ell'}. \tag{4.23}$$

Moreover, by identifying the unit sphere $\mathbb{S}^{4n-1}$ in the quaternionic vector space of dimension n, $\mathbb{H}^n$, with $(\mathrm{Sp}(n)\times\mathrm{Sp}(1))/(\mathrm{Sp}(n-1)\times\mathrm{Sp}(1))$, as we saw in Section 4.1, Kostant proved that

$$L^2(\mathbb{S}^{4n-1}) = \bigoplus_{\ell\geq\ell'\geq 0} \mathcal{H}_{\ell\ell'}, \tag{4.24}$$

where each subspace $\mathcal{H}_{\ell\ell'}$ is irreducible under the action of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ (see [39]).
It is also well known that

$$V_k = \bigoplus_{\ell+\ell'=k} \mathcal{H}_{\ell\ell'}. \tag{4.25}$$

We refer to [37] for a unified representation-theoretic approach to these decompositions.
From the spectral side, it is well known that the V_ℓ are eigenspaces for the Laplace–Beltrami operator $\Delta_{\mathbb{S}^{n-1}}$, corresponding to the eigenvalue $\ell(\ell+n-2)$. In the complex and in the quaternionic case, $\mathcal{H}_{\ell\ell'}$ and $H_{\ell\ell'}$ turn out to be eigenspaces both for the Laplace–Beltrami operators on $\mathbb{S}^{2n-1}$ and $\mathbb{S}^{4n-1}$ corresponding to the eigenvalues $(\ell+\ell')\,(\ell+\ell'+2n-2)$ and $(\ell+\ell')\,(\ell+\ell'+4n-2)$, respectively, and for a suitably defined sub-Laplacian L to the eigenvalue $\lambda_{\ell\ell'}$, given by $2\ell\ell'+(n-1)(\ell+\ell')$ in the complex case and by $4(\ell\ell'+(n-1)\ell+n\ell')$ in the quaternionic case (see [21]). For this reason, the elements of $\mathcal{H}_{\ell\ell'}$ and $H_{\ell\ell'}$ are usually called bigraded spherical harmonics.
On $\mathbb{S}^{2n-1}\subset\mathbb{C}^n$ we consider the standard coordinate system (see [38, 11.1.4])

$$\begin{cases} \xi_1 = e^{i\varphi_1}\sin\theta_{n-1}\dots\sin\theta_1 \\ \xi_2 = e^{i\varphi_2}\sin\theta_{n-1}\dots\cos\theta_1 \\ \quad\vdots \\ \xi_{n-1} = e^{i\varphi_{n-1}}\sin\theta_{n-1}\cos\theta_{n-2} \\ \xi_n = e^{i\varphi_n}\cos\theta_{n-1}, \end{cases} \tag{4.26}$$

with $\theta_j\in[0,\pi/2]$, $j\in\{1,\dots,n-1\}$, $\varphi_k\in[0,2\pi]$, $k\in\{1,\dots,n\}$, and the corresponding normalized Lebesgue measure, given by

$$\frac{1}{|\mathbb{S}^{2n-1}|}d\sigma = \frac{1}{|\mathbb{S}^{2n-1}|}\Pi_{j=1}^{n-1}\sin^{2j-1}\theta_j\cos\theta_j d\theta_j \Pi_{k=1}^{n} d\varphi_k,$$

where $|\mathbb{S}^{2n-1}| = \frac{2\pi^n}{\Gamma(n)} = \frac{2\pi^n}{(n-1)!}$ (see [38, 11.1.8]).
On $\mathbb{S}^{4n-1}\subset\mathbb{H}^n$ we adopt a system of coordinates different from those used in Section 4.1. These coordinates were introduced in [38, 11.7.1] and are given by

$$\begin{cases} q_1 = u_1\sin\theta_{n-1}\sin\theta_{n-2}\dots\sin\theta_2\sin\theta_1 \\ q_2 = u_2\sin\theta_{n-1}\sin\theta_{n-2}\dots\sin\theta_2\cos\theta_1 \\ \quad\vdots \\ q_{n-1} = u_{n-1}\sin\theta_{n-1}\cos\theta_{n-2} \\ q_n = u_n\cos\theta_{n-1}, \end{cases} \tag{4.27}$$

where

$$u_k = e^{i\varphi_k} \cos \omega_k + e^{i\psi_k} \sin \omega_k j,$$

with $\theta_j \in [0, \pi/2]$, $j \in \{1, \dots, n-1\}$, $\varphi_k, \psi_k \in [0, 2\pi]$, $\omega_k \in [0, \pi/2]$, $k \in \{1, \dots, n\}$. The corresponding normalized Lebesgue measure is

$$\begin{aligned} d\sigma = & \frac{1}{|\mathbb{S}^{4n-1}|} \sin^{4n-5} \theta_{n-1} \cos^3 \theta_{n-1} \sin^{4n-9} \theta_{n-2} \cos^3 \theta_{n-2} \dots \sin^7 \theta_2 \cos^3 \theta_2 \sin^3 \theta_1 \cos^3 \theta_1 \\ & \times d\theta_{n-1} d\theta_{n-2} \dots d\theta_2 d\theta_1 \\ & \times \sin \omega_1 \cos \omega_1 \dots \sin \omega_n \cos \omega_n d\omega_1 \dots d\omega_n d\varphi_1 \dots d\varphi_n d\psi_1 \dots d\psi_n, \end{aligned}$$

where

$$|\mathbb{S}^{4n-1}| = \frac{2\pi^{2n}}{\Gamma(2n)}. \tag{4.28}$$

If $\mathbb{S}$ denotes $\mathbb{S}^{2n-1}$ or $\mathbb{S}^{4n-1}$ endowed with the normalized Lebesgue measure, we define

$$\|f\|_p^p = \int_{\mathbb{S}} |f(u)|^p d\sigma(u).$$

The following result was proved in the case of the real sphere in [24]. In the complex and quaternionic framework it is a straightforward consequence of [24, Theorem 1].

Proposition 4.2.1. *Let $Y_{\ell\ell'}$ be any bigraded spherical harmonic in $H_{\ell\ell'}$ or $\mathcal{H}_{\ell\ell'}$, with $(\ell, \ell') \in \mathbb{N} \times \mathbb{N}$. If $Y_{\ell\ell'} \in \mathcal{H}_{\ell\ell'}$, we assume in addition that $\ell \geq \ell'$. Then*

$$\|Y_{\ell\ell'}\|_p \leq C(\ell, \ell')(p-1)^{(\ell+\ell')/2} \|Y_{\ell\ell'}\|_2 \text{ for all } p \geq 2. \tag{4.29}$$

Proof. The proof is very similar to that of [24, Theorem 1], as long as we replace the standard Hecke's identity for spherical harmonics by means of Hecke's identity for bigraded spherical harmonics (for a proof of this identity the complex case, see for instance [30] or [57]; the quaternionic case may be treated analogously). □

4.2.2 Sharpness of the results

In the following, we shall focus on the sharpness of the bound found in the last subsection.

Proposition 4.2.2. *Estimate (4.29) is sharp, in the sense that there exist spherical harmonics in $H_{\ell\ell'}$ or $\mathcal{H}_{\ell\ell'}$ such that the reverse inequality holds with the same exponent $(\ell + \ell')/2$.*

Proof. We consider the complex and the quaternionic case separately.
The complex case. Consider the "highest weight spherical harmonic" $Q_{\ell\ell'} = z_1^\ell \bar{z}_2^{\ell'}$. From now on we shall assume $\ell \geq \ell'$. We have

$$\|Q_{\ell\ell'}\|_p^p = \frac{1}{|\mathbb{S}^{2n-1}|} (2\pi)^n J_n(p, \ell, \ell') \times J(p, \ell, \ell'), \tag{4.30}$$

where

$$\begin{aligned} J_n(p, \ell, \ell') = & \int_0^{\pi/2} \left(\sin \theta_{n-1}\right)^{(\ell+\ell')p+2n-3} \cos \theta_{n-1} d\theta_{n-1} \\ & \times \int_0^{\pi/2} \left(\sin \theta_{n-2}\right)^{(\ell+\ell')p+2n-5} \cos \theta_{n-2} d\theta_{n-2} \times \dots \times \int_0^{\pi/2} \left(\sin \theta_2\right)^{(\ell+\ell')p+3} \cos \theta_2 d\theta_2 \end{aligned}$$

and

$$J(p,\ell,\ell')=\int_0^{\pi/2}(\sin\theta_1)^{\ell p+1}(\cos\theta_1)^{\ell' p+1}d\theta_1=\frac{1}{2}\frac{\Gamma\big(\ell p/2+1\big)\Gamma\big(\ell' p/2+1\big)}{\Gamma\big((\ell+\ell')p/2+2\big)}.$$

From Stirling's formula we easily get

$$J(p,\ell,\ell')^{1/p}\sim\frac{1}{2^{1/p}}(2\pi)^{1/(2p)}e^{1/p}\frac{(\ell)^{\ell/2+1/(2p)}(\ell')^{\ell'/2+1/(2p)}}{(\ell+\ell'+2/p)^{\ell/2+\ell'/2+3/(2p)}},$$

so that

$$\begin{aligned}\frac{(J(p,\ell,\ell'))^{1/p}}{(J(2,\ell,\ell'))^{1/2}}&\sim 2^{\frac{1}{2}-\frac{1}{p}}(2\pi)^{\frac{1}{2p}-\frac{1}{4}}e^{\frac{1}{p}-\frac{1}{2}}\Big(\frac{\ell}{\ell+\ell'+2/p}\Big)^{\frac{1}{2p}-\frac{1}{4}}\Big(\frac{\ell'}{\ell+\ell'+2/p}\Big)^{\frac{1}{2p}-\frac{1}{4}}\\&\quad\times(\ell+\ell'+2/p)^{1/4-1/(2p)}\\&\gtrsim(\ell+\ell'+2/p)^{1/4-1/(2p)},\end{aligned}\tag{4.31}$$

for all $p\geq 2$.
Another standard application of Stirling's formula shows that, for $m,s\in\mathbb{N}$, $m\geq 3$ sufficiently large, $s\geq 1$, $p\in\mathbb{R}$ and $p\geq 2$, we have

$$\frac{\Gamma(m/2)\Gamma(sp/2+1)}{\Gamma((sp+m)/2)}\sim\sqrt{\pi}\big(m-2\big)^{m/2-1/2}\big(sp\big)^{sp/2+1/2}\Big(sp+m-2\Big)^{-(sp+m)/2+1/2}.\tag{4.32}$$

When $m=2$, we have in particular

$$\frac{\Gamma(m/2)\Gamma(sp/2+1)}{\Gamma((sp+m)/2)}=1.\tag{4.33}$$

Then (4.32), combined with (4.33), yields

$$\begin{aligned}\frac{(J_n(p,\ell,\ell'))^{1/p}}{(J_n(2,\ell,\ell'))^{1/2}}&\sim\frac{\big((\ell+\ell')p/2+1\big)^{1/p}}{\big(\ell+\ell'+1\big)^{1/2}}\big(2^{n-2}\Gamma(n)\big)^{\frac{1}{2}-\frac{1}{p}}\pi^{\frac{1}{2p}-\frac{1}{4}}(2n-2)^{(\frac{1}{p}-\frac{1}{2})(n-\frac{1}{2})}\\&\quad\times\big(\ell+\ell'\big)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}p^{(\ell+\ell')/2+1/(2p)}2^{-(\ell+\ell')/2-1/4}\\&\quad\times\frac{\big((\ell+\ell')2+(2n-2)\big)^{(\ell+\ell')/2+n/2-1/4}}{\big((\ell+\ell')p+(2n-2)\big)^{(\ell+\ell')/2+n/p-1/(2p)}}.\end{aligned}\tag{4.34}$$

As a consequence of (4.30), (4.31), and (4.34) we finally get

$$\begin{aligned}\frac{\|Q_{\ell\ell'}\|_p}{\|Q_{\ell\ell'}\|_2}&=\frac{(2\pi)^{n(\frac{1}{p}-\frac{1}{2})}}{|\mathbb{S}^{2n-1}|^{1/p-1/2}}\;\frac{\big(J_n(p,\ell,\ell')\big)^{1/p}}{(J_n(2,\ell,\ell'))^{1/2}}\;\frac{(J(p,\ell,\ell')))^{1/p}}{(J(2,\ell,\ell'))^{1/2}}\\&\sim p^{(\ell+\ell')/2}2^{1/p-1/2}\;\frac{\big((\ell+\ell')p/2+2\big)^{1/p}}{\big(\ell+\ell'+2\big)^{1/2}}\pi^{\frac{1}{2p}-\frac{1}{4}}\;\big(\ell+\ell'\big)^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}2^{-(\ell+\ell')/2-1/4}\\&\quad\times\frac{(2n-2)^{\frac{1}{p}(n-\frac{1}{2})}}{\big((\ell+\ell')p+(2n-2)\big)^{\frac{1}{p}(n-\frac{1}{2})}}\;\frac{\big((\ell+\ell')2+(2n-2)\big)^{\frac{1}{2}(n-\frac{1}{2})}}{(2n-2)^{\frac{1}{2}(n-\frac{1}{2})}}\\&\quad\times\frac{\big((\ell+\ell')2+(2n-2)\big)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+(2n-2)\big)^{(\ell+\ell')/2}}\pi^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}2^{1/4}\frac{\ell^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}\ell'^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}}{\big(\ell+\ell'+1\big)^{\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}}\end{aligned}$$

$$
\begin{aligned}
&\sim p^{(\ell+\ell')/2} 2^{-1/2}\ p^{1/p} \frac{\big((\ell+\ell')+4/p\big)^{1/p}}{\big(\ell+\ell'+2\big)^{1/2}} \pi^{\frac{1}{p}-\frac{1}{2}}\ 2^{-(\ell+\ell')/2} \\
&\times \frac{\big((\ell+\ell')2+(2n-2)\big)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+(2n-2)\big)^{(\ell+\ell')/2}} \frac{\ell^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}\ell'^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}}{\big(\ell+\ell'+1\big)^{\frac{1}{p}-\frac{1}{2}}},
\end{aligned}
$$

where we used the fact that

$$
\frac{(2n-2)^{\frac{1}{p}(n-\frac{1}{2})}}{\big((\ell+\ell')p+(2n-2)\big)^{\frac{1}{p}(n-\frac{1}{2})}}\ \frac{\big((\ell+\ell')2+(2n-2)\big)^{\frac{1}{2}(n-\frac{1}{2})}}{(2n-2)^{\frac{1}{2}(n-\frac{1}{2})}} \sim 1.
$$

Since, moreover,

$$
\frac{\big((\ell+\ell')+4/p\big)^{1/p}}{\big(\ell+\ell'+2\big)^{1/2}}\ \frac{1}{\big(\ell+\ell'+1\big)^{\frac{1}{p}-\frac{1}{2}}} \geq c
$$

for some positive constant c, we have

$$
\begin{aligned}
\frac{\|Q_{\ell\ell'}\|_p}{\|Q_{\ell\ell'}\|_2} &\geq p^{(\ell+\ell')/2}\ 2^{-1/2}\ p^{1/p}\pi^{\frac{1}{p}-\frac{1}{2}}\ 2^{-(\ell+\ell')/2} \\
&\times \frac{\big((\ell+\ell')2+(2n-2)\big)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+(2n-2)\big)^{(\ell+\ell')/2}} \ell^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}\ell'^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})} \\
&\geq C(\ell,\ell')p^{(\ell+\ell')/2} \times \frac{\big((\ell+\ell')2+(2n-2)\big)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+(2n-2)\big)^{(\ell+\ell')/2}} \ell^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}\ell'^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})} \\
&\geq C(\ell,\ell')p^{(\ell+\ell')/2} \times (\ell\ell')^{-\frac{1}{4}} \\
&\geq C(\ell,\ell')p^{(\ell+\ell')/2},
\end{aligned}
$$

proving the assertion in the complex framework.

The quaternionic case. By convention, we shall write $q \in \mathbb{H}^n$ as $q = (z_1 + jz_{n+1}, z_2 + jz_{n+2}, \dots, z_n + jz_{2n})$, $z_1, \dots, z_{2n} \in \mathbb{C}$. Then consider the highest weight spherical harmonic

$$
P_{\ell,\ell'}(z,\bar{z}) = \bar{z}_{n+1}^{\ell-\ell'}\big(z_1\bar{z}_{n+2} - z_2\bar{z}_{n+1}\big)^{\ell'} \tag{4.35}
$$

(we refer to [42, p. 2999] for an explicit computation). Then

$$
\begin{aligned}
\|P_{\ell\ell'}\|_p^p &= \frac{1}{\mathbb{S}^{4n-1}} \int_{\mathbb{S}^{4n-1}} \big|\sin\omega_1\big|^{(\ell-\ell')p+1}\big|e^{i\varphi_1}e^{-i\psi_2}\cos\omega_1\sin\omega_2 - e^{i\varphi_2}e^{-i\psi_1}\cos\omega_2\sin\omega_1\big|^{\ell' p} \\
&\times \big(\sin\theta_{n-1}\big)^{(\ell+\ell')p+4n-5}\cos^3\theta_{n-1} \\
&\times \big(\sin\theta_{n-2}\big)^{(\ell+\ell')p+4n-9}\cos^3\theta_{n-2}\dots\dots\big(\sin\theta_2\big)^{(\ell+\ell')p+7}\cos^3\theta_2 \\
&\times \big(\sin\theta_1\big)^{\ell p+3}\big(\cos\theta_1\big)^{\ell' p+3} \\
&\times \Pi_{j=1}^{n-1} d\theta_j\ \cos\omega_1\dots\sin\omega_n\cos\omega_n\Pi_{k=1}^n d\omega_k\ d\varphi_k d\psi_k \\
&= \frac{1}{\mathbb{S}^{4n-1}}\ \frac{1}{2^{n-2}}(2\pi)^{2n-4}\big(J^{(q)}_{\ell,\ell',p}\big)\frac{1}{2}B\big((\ell+\ell')p/2+2n-2,2\big) \\
&\times \frac{1}{2}B\big((\ell+\ell')p/2+2n-4,2\big)\times\dots\dots\times\frac{1}{2}B\big((\ell+\ell')p/2+4,2\big) \\
&\times \frac{1}{2}B\big((\ell p)/2+2,(\ell' p)/2+2\big),
\end{aligned}
$$

where B denotes the beta function and

$$J^{(q)}_{\ell,\ell',p}=\int_0^{\pi/2}\dots\int_0^{2\pi}\left|e^{i\varphi_1}e^{-i\psi_2}\cos\omega_1\sin\omega_2-e^{i\varphi_2}e^{-i\psi_1}\cos\omega_2\sin\omega_1\right|^{\ell' p}\left|\sin\omega_1\right|^{(\ell-\ell')p+1}$$
$$\cos\omega_1 d\omega_1 d\omega_2 d\varphi_1 d\varphi_2 d\psi_1 d\psi_2.$$

As a consequence of (4.28), we finally obtain

$$\begin{aligned}\|P_{\ell\ell'}\|_p^p&=\frac{1}{4\pi^4}\big(J^{(q)}_{\ell,\ell',p}\big)\Gamma(2n)\times\frac{\Gamma\big((\ell p)/2+2\big)\Gamma\big((\ell' p)/2+2\big)}{\Gamma((\ell+\ell')p/2+2n)}\\&\sim\frac{1}{4\pi^4}\big(J^{(q)}_{\ell,\ell',p}\big)\frac{\Gamma\big((\ell p)/2+2\big)\Gamma\big((\ell' p)/2+2\big)}{\Gamma((\ell+\ell')p/2+4)}\times((\ell+\ell')p/2)^4\frac{\Gamma((\ell+\ell')p/2)\Gamma(2n)}{\Gamma((\ell+\ell')p/2+2n)}\\&\sim\big(\tilde J^{(q)}_{\ell,\ell',p}\big)\times B\Big((\ell+\ell')p/2,2n\Big),\end{aligned}$$

where

$$\tilde J^{(q)}_{\ell,\ell',p}=\frac{1}{4\pi^4}\big(J^{(q)}_{\ell,\ell',p}\big)\frac{\Gamma\big((\ell p)/2+2\big)\Gamma\big((\ell' p)/2+2\big)}{\Gamma((\ell+\ell')p/2+4)}\;((\ell+\ell')p/2)^4.$$

Finally, Stirling's approximation yields

$$\|P_{\ell\ell'}\|_p\sim\big(\tilde J^{(q)}_{\ell,\ell',p}\big)^{1/p}\times\frac{p^{(\ell+\ell')/2-1/(2p)}(\ell+\ell')^{(\ell+\ell')/2-1/(2p)}(2n)^{(2n-1/2)/p}}{2^{(\ell+\ell')/2-1/(2p)}\big((\ell+\ell')p/2+2n\big)^{(\ell+\ell')/2+(2n-1/2)/p}},$$

and

$$\begin{aligned}\frac{\|P_{\ell\ell'}\|_p}{\|P_{\ell\ell'}\|_2}&\sim\big(\tilde J^{(q)}_{\ell,\ell',p}\big)^{1/p}\big(\tilde J^{(q)}_{\ell,\ell',2}\big)^{-1/2}\times\frac{p^{(\ell+\ell')/2-1/(2p)}(\ell+\ell')^{(\ell+\ell')/2-1/(2p)}(2n)^{(2n-1/2)/p}}{2^{(\ell+\ell')/2-1/(2p)}\big((\ell+\ell')p/2+2n\big)^{(\ell+\ell')/2+(2n-1/2)/p}}\\&\times\frac{2^{(\ell+\ell')/2-1/4}(\ell+\ell'+2n)^{(\ell+\ell')/2+n-1/4}}{2^{(\ell+\ell')/2-1/4}(\ell+\ell')^{(\ell+\ell')/2-1/4}(2n)^{n-1/4}}\\&\sim\big(\tilde J^{(q)}_{\ell,\ell',p}\big)^{1/p}\big(\tilde J^{(q)}_{\ell,\ell',2}\big)^{-1/2}\times\frac{p^{(\ell+\ell')/2-1/(2p)}(\ell+\ell')^{\frac14-\frac{1}{2p}}}{2^{(\ell+\ell')/2-1/(2p)}}\\&\times\frac{2^{(\ell+\ell')/2}(\ell+\ell'+2n)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+4n\big)^{(\ell+\ell')/2}}\times\Big(\frac{\ell+\ell'+2n}{2n}\Big)^{n-1/4}\times\frac{(2n)^{(2n-\frac12)\frac1p}}{\big((\ell+\ell')p/2+2n\big)^{(2n-1/2)/p}}.\end{aligned}$$

Since

$$\Big(\frac{\ell+\ell'+2n}{2n}\Big)^{n-\frac14}\sim e^{\frac{n-1/4}{2n}(\ell+\ell')},$$

and

$$\Big(\frac{2n}{(\ell+\ell')p/2+2n}\Big)^{(2n-\frac12)/p}\sim e^{-\frac{n-1/4}{2n}(\ell+\ell')},$$

one has

$$\begin{aligned}\frac{\|P_{\ell\ell'}\|_p}{\|P_{\ell\ell'}\|_2}&\sim\big(\tilde J^{(q)}_{\ell,\ell',p}\big)^{1/p}\big(\tilde J^{(q)}_{\ell,\ell',2}\big)^{-1/2}\times\frac{p^{(\ell+\ell')/2-1/(2p)}(\ell+\ell')^{\frac14-\frac{1}{2p}}}{2^{-1/(2p)}},\\&\times\frac{(\ell+\ell'+2n)^{(\ell+\ell')/2}}{\big((\ell+\ell')p+4n\big)^{(\ell+\ell')/2}}.\end{aligned}$$

Note that for n large

$$\begin{aligned}\Big(\frac{\ell+\ell'+2n}{p(\ell+\ell')+4n}\Big)^{(\ell+\ell')/2} &= 2^{-(\ell+\ell')/2}\Big(\frac{2(\ell+\ell')+4n}{p(\ell+\ell')+4n}\Big)^{(\ell+\ell')/2} = 2^{-(\ell+\ell')/2}e^{\frac{\ell+\ell'}{2}\log\left(\frac{2(\ell+\ell')+4n}{p(\ell+\ell')+4n}\right)}\\ &= 2^{-(\ell+\ell')/2}e^{\frac{\ell+\ell'}{2}\log\left(1-\frac{(p-2)(\ell+\ell')}{p(\ell+\ell')+4n}\right)} \sim 2^{-(\ell+\ell')/2}e^{-\frac{\ell+\ell'}{2}\frac{(p-2)(\ell+\ell')}{p(\ell+\ell')+4n}}\\ &\geq 2^{-(\ell+\ell')/2}e^{-\frac{\ell+\ell'}{2}\frac{(p-2)(\ell+\ell')}{p(\ell+\ell')+4}},\end{aligned}$$

and this latter function is bounded as a function of p. Thus

$$\frac{\|P_{\ell\ell'}\|_p}{\|P_{\ell\ell'}\|_2} \geq f\big(\ell,\ell',p\big)\times p^{(\ell+\ell')/2},$$

proving the assertion in the quaternionic case. $\square$

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