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The p-adic BSD conjecture for critical stabilisation of modular forms

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Abstract

In this thesis, we consider three cuspidal modular forms

$$f \in S_k(N_f, \chi_f), \quad g \in S_l(N_g, \chi_g), \quad h \in S_m(N_h, \chi_h)$$

of weight $k \geq 2$ and l, $m \geq 1$ with conductor N_f, N_g, N_h and characters χ_f, χ_g and χ_h respectively. We assume that $\chi_f \cdot \chi_g \cdot \chi_h = 1$. We fix a rational prime p and suppose that $p \geq 3$. We assume that the root α_f of the p-th Hecke polynomial of f has positive p-adic valuation. We denote by f_{α_f} the α_f stabilisation of f. Similarly, for $\xi = g$, h, the p-th Hecke polynomial has roots α_ξ and β_ξ. We consider the p-stabilisations g_{α_g} and h_{α_h}. Under some assumptions, there are unique Coleman families $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ passing through f_{α_f}, g_{α_g} and h_{α_h} respectively. We fix a finite extension L of $\mathbb{Q}_p$ containing all the Fourier coefficients of f, g, h, the roots α_ξ and also the N-th roots of unity, where N is the least common multiple of N_f, N_g, N_h. For $\boldsymbol{\xi} = \boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, we denote by $U_{\boldsymbol{\xi}}$ the L-rational open disc centered at $u \in \mathbb{Z}_{\geq 1}$ in the weight space $\mathcal{W}_L$ defined over L. Let $\Lambda_{\boldsymbol{\xi}}$ be the ring of bounded by 1 analytic functions over $U_{\boldsymbol{\xi}}$ with respect to Gauss norm and we set $\mathcal{O}_{\boldsymbol{\xi}} = \Lambda_{\boldsymbol{\xi}}[1/p]$ and $\mathcal{O}_{\boldsymbol{fgh}} = \mathcal{O}_{\boldsymbol{f}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{h}}$. We attach to ξ a Galois representation $V(\xi)$ and consider

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h)((4 - k - l - m)/2)$$

Following [63], [22, Section 2] and [13, Section 3], we can define a three variable p-adic height pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} \colon H^1(V(f, g, h), D(f, g, h)^+) \times H^1(V(f, g, h), D(f, g, h)^+) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega = (k, l, m)$, $H^1(V(f, g, h), D(f, g, h)^+)$ is the extended selmer group of $V(f, g, h)$ which is constructed by Benois [10] and [11]. In [1], Andreatta and Iovita construct a triple product p-adic L-function $\mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for finite slope families. In [44], Huang investigates the reciprocity law of the AI p-adic L-function. Combining the calculation in [71] and [13] for the p-adic height pairing and the reciprocity law proved by Huang, we can generalize [22, Theorem 3.2] to our setting.

In the end, we formulate a p-adic BSD conjecture for AI p-adic L-function. We consider the elliptic curve E defined over $\mathbb{Q}$ with good ordinary reduction at p. Furthermore, we suppose that E does not have complex multiplication. From the modularity theorem, we can attached to E a weight 2 cuspidal modular form f. The p-th Hecke polynomial of f has a root α_f with positive p-adic valuation. We assume that f is non-critical, then there is a unique Coleman family $\boldsymbol{f}$ specializing to the α_f-stabilisation of f at weight 2, which is the critical stabilisation of f. Suppose that we have two odd irreducible Artin representations ϱ_1 and ϱ_2 of the absolute Galois group of $\mathbb{Q}$. Let $\varrho = \varrho_1 \otimes \varrho_2$. From the work of Khare and Winterberg, we can associate these two Artin representations to weight 1 cuspidal modular forms g and h respectively. Similarly, under some assumptions, we have two unique Coleman families $\boldsymbol{g}$ and $\boldsymbol{h}$ specializing to the p-stabilisation of g and h respectively. We set $L_p(E, \varrho) = \mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^2$, which interpolation the value of complex L-function $L(f \otimes g \otimes h, s)$ at the central point $s = 1$. We can define $E(K_\varrho)^\varrho$ the Mordell–Weil group of E twisted by ϱ. We restrict our p-adic height pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ to $E(K_\varrho)^\varrho$, then we obtain

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} \colon E(K_\varrho)^\varrho \times E(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2,$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega_0 = (2, 1, 1)$. Using this p-adic height pairing, we can define the regulator $R_p(E, \rho)$. We then give a conjectural relation between the p-adic L-function $L_p(E, \varrho)$ and the regulator $R_p(E, \rho)$.

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Contents

CHAPTER 1

Introduction

In [22], Bertolini, Seveso and Venerucci consider the cuspidal modular forms

$$f \in S_2(N_f), \quad g \in S_1(N_g, \chi_g) \quad h \in S_1(N_h, \chi_h)$$

satisfying $\chi_g \cdot \chi_h = 1$. Furthermore, they suppose that $\mathrm{ord}_p(N_f) \leq 1$ and the p-th Hecke polynomial of ξ has a root α_ξ which is a p-adic unit. Hence there is a Hida family $\boldsymbol{f}$ passing through the ordinary p-stabilisation f_{α_f}. Assume that $p \nmid N_g N_h$ and g, h are classical p-regular weight one newforms without p-split real multiplication. Then there are also Hida families $\boldsymbol{g}$, $\boldsymbol{h}$ passing through the p-stabilisation g_{α_g} and h_{α_h} respectively. We fix a finite extension L of $\mathbb{Q}_p$ which contains the Fourier coefficients of f, g, h, the roots α_f, α_g and α_h and the N-th roots of unity, where N is the least common multiple of N_f, N_g and N_h. For $\boldsymbol{\xi} = \boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, we let $U_{\boldsymbol{\xi}}$ be an L-rational open disc centered at $u \in \mathbb{Z}_{\geq 1}$ in the weight space $\mathcal{W}_L$ which is a rigid analytic space defined over L. Let $\Lambda_{\boldsymbol{\xi}}$ be the ring of bounded by 1 analytic functions over $U_{\boldsymbol{\xi}}$ with respect to Gauss norm. Set $\mathcal{O}_{\boldsymbol{\xi}} = \Lambda_{\boldsymbol{\xi}}[1/p]$ and we denote by $\mathcal{O}_{\boldsymbol{fgh}} = \mathcal{O}_{\boldsymbol{f}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{h}}$. We can attach each modular form ξ a Galois representation $V(\xi)$ and consider

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h)$$

In [22, Section 2.5], they construct a three variable p-adic height pairing

$$\langle\langle \cdot, \cdot \rangle\rangle_{\boldsymbol{fgh}} \colon \widetilde{H}^1_f(V(f, g, h)) \times \widetilde{H}^1_f(V(f, g, h)) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

Here $\mathscr{I}$ is the ideal of $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega = (2, 1, 1)$ and $\widetilde{H}^1_f(V(f, g, h))$ is the extended Selmer group of $V(f, g, h)$ whose construction is introduced by Nekovar in [63]. They prove the following theorem in [22, Section 3.4].

Theorem 1.0.1. *Assume that the complex Garrett L-function $L(f \otimes g \otimes h, 1) = 0$ and (f, g, h) is not exceptional. Then*

$$\frac{1 - \dfrac{\alpha_g \alpha_h}{\alpha_f}}{1 - \dfrac{\alpha_f}{p \alpha_g \alpha_h}} \langle\langle \kappa(f, g, h), y \rangle\rangle_{\boldsymbol{fgh}} = \log^{++}(\mathrm{res}_p(y)) \cdot \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \mod \mathscr{I}^2$$

for $y \in \mathrm{Sel}(\mathbb{Q}, V(f, g, h))$.

Here $\mathrm{Sel}(\mathbb{Q}, V(f, g, h))$ is the Bloch–Kato Selmer group of $V(f, g, h)$, $\kappa(f, g, h)$ is the diagonal class constructed in [23, Section 3], we recall its construction in Section 7.1.1. The map $\log^{++}$ is from $H^1_f(\mathbb{Q}_p, V(f, g, h))$ to L, where $H^1_f(\mathbb{Q}_p, V(f, g, h))$ is the Bloch–Kato finite subspace(See [22, Section 3.1] for the definition) and $\mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is the Hida–Garrett p-adic L-function constructed in [35].

In this thesis, we would like to generalize the Theorem 1.0.1 to Coleman families and more general weights. We let

$$f \in S_k(N_f, \chi_f), \quad g \in S_l(N_g, \chi_g), \quad h \in S_m(N_h, \chi_h).$$

Here $k \geq 2$, l and $m \geq 1$ and their conductors are N_f, N_g and N_h respectively. We suppose that

Assumption 1.0.2. $\chi_f \cdot \chi_g \cdot \chi_h = 1$.

We fix p a rational prime satisfying $p \geq 3$ and $p \nmid N_f N_g N_h$. For $\xi = f$, g or h, we write u for the weight of ξ respectively. We assume that ξ satisfies following assumptions

Assumption 1.0.3. 1. If $u \geq 2$, ξ is p-regular and non-critical. Furthermore, the p-stabilisation of ξ is not θ-critical.

2. If $u = 1$, ξ is a classical p-regular cuspidal weight one newform without p-split real multiplication.

We refer to Section 3.1 and Section 3.5 for more details on these definitions. Since ξ is p-regular, the p-th Hecke polynomial of ξ has two distinct roots α_ξ and β_ξ. We assume that α_f has positive p-adic valuation. Furthermore, we assume that $\mathrm{ord}_p(\alpha_\xi) < u-1$. Under our Assumption 1.0.3, there exist unique Coleman families passing through the p-stabilisations ξ_{α_ξ}(See Lemma 3.5.5). We denote by $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ the corresponding Coleman families respectively. As before, we fix a finite extension L of $\mathbb{Q}_p$ containing all the Fourier coefficients of f, g, h, the roots α_ξ and also the N-th roots of unity. For $\boldsymbol{\xi} = \boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, we still denote by $U_{\boldsymbol{\xi}}$ the L-rational open disc centered at $u \in \mathbb{Z}_{\geq 1}$ in the weight space $\mathcal{W}_L$. Let $\Lambda_{\boldsymbol{\xi}}$ be the ring of bounded by 1 analytic functions over $U_{\boldsymbol{\xi}}$ with respect to Gauss norm and we set $\mathcal{O}_{\boldsymbol{\xi}} = \Lambda_{\boldsymbol{\xi}}[1/p]$ and $\mathcal{O}_{\boldsymbol{fgh}} = \mathcal{O}_{\boldsymbol{f}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{h}}$. We attach to ξ a Galois representation $V(\xi)$ and consider

$$V(f,g,h) = V(f) \otimes_L V(g) \otimes_L V(h)((4-k-l-m)/2)$$

Following [63], [22, Section 2] and [13, Section 3], we can define a three variable p-adic height pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} \colon H^1(V(f,g,h), D(f,g,h)^+) \times H^1(V(f,g,h), D(f,g,h)^+) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega = (k,l,m)$, $H^1(V(f,g,h), D(f,g,h)^+)$ is the extended selmer group which is constructed by Benois [10] and [11] generalizing the original ideas of Nekovar. The construction of $H^1(V(f,g,h), D(f,g,h)^+)$ relies on the p-adic Hodge theory and the theory of (φ, Γ)-modules. We refer to Section 2 and 4 for more details.

Under the Assumption 1.0.2, the complex Garrett L-function $L(f \otimes g \otimes h, s)$ has an analytic continuation to the whole $\mathbb{C}$ and it has a functional equation which relates the values at s and $k+l+m-2-s$. The sign of the functional equation is always $\{\pm 1\}$. We assume that $(N_g, N_h, N_f) = 1$. This assumption guarantees that the sign in the above functional equation is $+1$ if $k \geq l+m$. We recall these facts in Section 6.1. In [1], Andreatta and Iovita construct a triple product p-adic L-function $\mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for finite slope families. Under our Assumption 1.0.3, the p-adic L-function $\mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ interpolates the central critical values of the complex L-functions $L(f_k \otimes g_l \otimes h_m, (k+l+m-2)/2)$ at triples of classical weights $\omega = (k,l,m)$ such that $k \geq l+m$ (see Equation 6.2.1). In [44], Huang investigates the reciprocity law of the AI p-adic L-function which generalizes the reciprocity law in [23, Theorem A] to the finite slope case. Huang's reciprocity law is

$$\mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \mathscr{L}_{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))).$$

Here $\mathscr{L}_{\boldsymbol{f}} \colon H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathcal{O}_{\boldsymbol{fgh}}$ is the Perrin-Riou big logarithm(See Section 5 for more details) and $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is the big diagonal class constructed in [23, Section 8.1], which is an element in $H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$. We recall the construction of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in Section 7.1.2. This reciprocity law is similar to the one proved by Bertolini, Seveso and Venerucci in [23].

We can define a map

$$\log^{++} \colon H^1_f(\mathbb{Q}_p, V(f,g,h)) \longrightarrow L,$$

using the Bloch–Kato logarithm. We refer to Section 8.1 for the definition of $\log^{++}$. Combining the calculation in [71] and [13] for the p-adic height pairing and the reciprocity law proved by Huang, we prove the following theorem with the strategy in the proof [22, Theorem 3.2].

Theorem 1.0.4. *Assume that the complex Garrett L-function $L(f \otimes g \otimes h, (k+l+m-2)/2) = 0$ and ω is not exceptional. Then*

$$\frac{1 - \dfrac{\alpha_g \alpha_h}{\alpha_f}}{1 - \dfrac{\alpha_f}{p\alpha_g \alpha_h}} \langle\!\langle \kappa(f,g,h), y \rangle\!\rangle_{\boldsymbol{fgh}} = \log^{++}(\mathrm{res}_p(y)) \cdot \mathscr{L}_p^{AI}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \mod \mathscr{I}^2$$

for $y \in \mathrm{Sel}(\mathbb{Q}, V(f,g,h))$.

Remark 1.0.5. 1. Here we only prove the result when $\mathrm{ord}_p(\alpha_\xi) < u-1$. It is interesting to prove the result when $\mathrm{ord}_p(\alpha_\xi) = u-1$, which we would like to consider in the future.

2. We deal with the case that the (f, g, h) is not exceptional. We also would like to discuss the exceptional case in the future.

This work follows closely the ideas, arguments and techniques developed by Bertolini–Seveso–Venerucci, especially in [22] and [23] and Bertolini–Darmon–Venerucci [20]; the notation and exposition follow the ones in loc.cit. with mild adaptions. Our contribution consists of replacing in these arguments the p-adic L-function in loc. cit. with that constructed by Andreatta–Iovita in [1], the p-adic height in loc. cit. with that constructed by Benois in [11] and the explicit reciprocity law in loc. cit. with that of [21]; see also Huang [44].

We conclude this introduction by offering a p-adic equivariant BSD conjecture in the case of critical stabilisation. This is similar to [22, Conjecture 1.1] in the case of ordinary p-stabilisation.

Suppose we have two odd Artin representations ϱ_1, ϱ_2 of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which are two dimensional. Furthermore, we assume that ϱ_1, ϱ_2 are irreducible and that the self-duality hypothesis is satisfied

$$\det(\varrho_1)\cdot\det(\varrho_2) = 1.$$

We set $\varrho \colon = \varrho_1 \otimes \varrho_2$. Suppose that E has good ordinary reduction at p and does not have complex multiplication. We define by

$$E(K_\varrho)^\varrho \colon = (E(K_\varrho)\otimes_{\mathbb{Z}} V_\varrho)^{\mathrm{Gal}(K_\varrho/\mathbb{Q})}$$

the Mordell–Weil group of E twisted by ϱ.

Using the modularity theorem, we may consider (f, g, h) the three cuspidal modular forms associated to $(E, \varrho_1, \varrho_2)$. Since E has good ordinary reduction at p, we know that one of the roots of the p-th Hecke polynomial of f has valuation 1. We denote by α_f this roots. We consider the p-stabilisation f_{α_f} of f with respect to α_f, which is called of critical slope(See Section 3.1 for more details). We can associate to f_{α_f} a Coleman family $\boldsymbol{f}$ and we can also associate to (g, h) the p-adic families of cuspidal modular forms. We set $L_p(E, \varrho) \colon = \mathscr{L}_p^{\mathrm{AI}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^2$. We restrict the p-adic height pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ to $E(K_\varrho)^\varrho$, then we obtain

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} \colon E(K_\varrho)^\varrho \times E(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2, \tag{1.0.1}$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega_0 = (2, 1, 1)$. See Section 9 for more details. We set

$$r(E, \varrho) = \dim_{\mathbb{Q}(\varrho)} E(K_\varrho)^\varrho.$$

Using the p-adic height pairing in Equation 1.0.1, we can define the regulator $R_p(E, \varrho)$(see Equation 9.0.1). From the definition of the regulator $R_p(E, \varrho)$, we know it is non-zero if the p-adic height pairing is non degenerate. We say that a non zero element F of $\mathcal{O}_{\boldsymbol{fgh}}$ has order of vanishing $r \in \mathbb{Z}_{\geq 0}$ at $\omega_0 = (2, 1, 1)$ if it belongs to $\mathscr{I}^r - \mathscr{I}^{r+1}$, and denote by F^* the image of F in the quotient $\mathscr{I}^r/\mathscr{I}^{r+1}$. Then we formulate the following conjecture.

Conjecture 1.0.6. *The p-adic L-function $L_p(E, \varrho)$ has vanishing order $r(E, \varrho)$ at $\omega_0 = (2, 1, 1)$ and then the following equality holds up to $(\mathbb{Q}(\varrho)^2)^\times$.*

$$L_p(E, \varrho)^* = R_p(E, \varrho) \mod \mathscr{I}^{r(E,\varrho)}/\mathscr{I}^{r(E,\varrho)+1}.$$

In particular, the p-adic height pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ is non-degenerate.

CHAPTER 2

The theory of (φ, Γ)-modules

2.1 Period rings

In this section, we recall the constructions of Fontaine's rings of p-adic periods in [37]. We also recall the period rings in [33] and [14]. We also refer to [40].

We always let p be a prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$. Choose a valuation $\nu_p\colon \mathbb{C}_p^\times \longrightarrow \mathbb{R}\cup\{+\infty\}$ satisfy that $\nu_p(p)=1$. We denote by $G_{\mathbb{Q}_p}\colon = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the absolute Galois groups of $\mathbb{Q}_p$. One chooses a system of primitive roots of unity $\varepsilon\colon = (\zeta_{p^n})_{n\geq 1}$, such that $\zeta_1 = 1$, $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for any $n\geq 0$ and $\zeta_p\neq 1$. Let $\mathbb{Q}_n\colon = \mathbb{Q}_p(\zeta_{p^n})$ and $\mathbb{Q}_\infty\colon = \bigcup\limits_{n\geq 1}\mathbb{Q}_n$. Let χ_{cyc} be the p-adic cyclotomic character

$$\chi_{\mathrm{cyc}}\colon G_{\mathbb{Q}_p}\longrightarrow \mathbb{Z}_p^\times. \tag{2.1.1}$$

We denote its kernel H which can be identified with $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_\infty)$. We put $\Gamma\colon = G_{\mathbb{Q}_p}/H$ and we identify it with $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}_p)$. We still use the same symbol to denote the map $\chi_{\mathrm{cyc}}\colon \Gamma\longrightarrow\mathbb{Z}_p^\times$ which is induced from Equation 2.1.1. We normalized the Hodge–Tate weight of $\mathbb{Q}_p(1)$ to be $+1$.

We first define the ring

$$\widetilde{\mathbf{E}}^+\colon = \varprojlim_{x\longmapsto x^p}\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}.$$

This ring is perfect of characteristic p and its residue field is $\overline{\mathbb{F}}_p$. We refer to [40, Proposition 5.2 and Theorem 5.5]. The elements in $\widetilde{\mathbf{E}}^+$ can be written as $(x_0,\, x_1, \cdots, x_n,\cdots)$ such that $x_n\in\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ and $x_{n+1}^p = x_n$ for all n. For each n, we choose a lifting $\widehat{x_n}\in\mathcal{O}_{\mathbb{C}_p}$ of x_n. For every $m\geq 0$, the sequence $\{\widehat{x}_{m+n}^{p^n}\}$ converges to $x^{(m)}\in\mathcal{O}_{\mathbb{C}_p}$, which does not depend on the choice of $\widehat{x}_n$. This can be seen from the proof of [40, Proposition 5.3]. We then define the valuation $\nu_{\widetilde{\mathbf{E}}^+}$ on $\widetilde{\mathbf{E}}^+$ by $\nu_{\widetilde{\mathbf{E}}^+}(x) = \nu_p(x^{(0)})$. This valuation makes $\widetilde{\mathbf{E}}^+$ a complete valuation ring, see [40, Theorem 5.5] for more details. We denote by $\widetilde{\mathbf{E}}\colon = \mathrm{Frac}(\widetilde{\mathbf{E}}^+)$ the fraction field of $\widetilde{\mathbf{E}}^+$. The valuation $\nu_{\widetilde{\mathbf{E}}^+}$ can be extended to a valuation $\nu_{\widetilde{\mathbf{E}}}$ on $\widetilde{\mathbf{E}}$. We have a natural continuous action of $G_{\mathbb{Q}_p}$ on $\widetilde{\mathbf{E}}^+$ given by

$$g(x)\colon = (gx^{(n)})_n.$$

We can view ε as an element in $\widetilde{\mathbf{E}}^+$ and we denote $\pi = \varepsilon - 1$. We can show

$$\nu_{\widetilde{\mathbf{E}}}(\pi) = \frac{p}{p-1}.$$

Furthermore, the action of $g\in G_{\mathbb{Q}_p}$ on ε and π is given by

$$g(\varepsilon) = \varepsilon^{\chi_{\mathrm{cyc}}(g)},\qquad g(\pi) = (1+\pi)^{\chi_{\mathrm{cyc}}(g)} - 1.$$

See more details in [40, Lemma 5.11]. Now we let $\widetilde{\mathbf{A}}^+\colon = W(\widetilde{\mathbf{E}}^+)$, $\widetilde{\mathbf{A}}\colon = W(\widetilde{\mathbf{E}})$ be the rings of Witt vectors of $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{E}}$ respectively, which are naturally equipped with actions of φ and $G_{\mathbb{Q}_p}$. More explicitly, if we write the element in $\widetilde{\mathbf{A}}^+$(resp. $\widetilde{\mathbf{A}}$) as $\sum_{k=0}^{\infty} p^k[x_k]$ for $x_k \in \widetilde{\mathbf{E}}^+$(resp. $x_k \in \widetilde{\mathbf{E}}$), we have

$$\varphi\left(\sum_{k=0}^{\infty} p^k[x_k]\right) = \sum_{k=0}^{\infty} p^k[x_k^p] \text{ and } g\left(\sum_{k=0}^{\infty} p^k[x_k]\right) = \sum_{k=0}^{\infty} p^k[g(x_k)]$$

for any $g \in G_{\mathbb{Q}_p}$. We know that φ commutes with the action of $G_{\mathbb{Q}_p}$, i.e., $\varphi(ga) = g\varphi(a)$ for any $g \in G_{\mathbb{Q}_p}$ and $a \in \widetilde{\mathbf{A}}^+$(resp. $a \in \widetilde{\mathbf{A}}$). We define

$$\widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[1/p] = \left\{ \sum_{k \gg -\infty} p^k[x_k] \mid \ x_k \in \widetilde{\mathbf{E}}^+ \right\}$$

and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$. The Frobenius of $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}^+$ induce the Frobenius on $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$. We have a natural $G_{\mathbb{Q}_p}$-equivariant surjective ring homomorphism $\theta\colon \widetilde{\mathbf{B}}^+ \longrightarrow \mathbb{C}_p$ such that

$$\theta\left(\sum_{k\geq 0} p^k[x_k]\right) := \sum_{k\geq 0} p^k x_k^{(0)}$$

The construction of this map, we refer to [40, Section 6.2.1 and Section 6.2.2]. We consider $\widetilde{P} := (P^{(n)})$ to be an element in $\widetilde{\mathbf{E}}^+$, where $P^{(0)} = p$. We can check that $\ker(\theta) = ([\widetilde{P}] - p)$, see [40, Proposition 6.12]. We define $\mathbf{B}_{\mathrm{dR}}^+ := \varprojlim_{n\geq 0} \widetilde{\mathbf{B}}^+/\ker(\theta)^n$ and an element

$$t := \log([\varepsilon]) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}([\varepsilon]-1)^n \in \mathbf{B}_{\mathrm{dR}}^+.$$

The ring $\mathbf{B}_{\mathrm{dR}}^+$ is a discrete valuation ring with the maximal ideal $t\mathbf{B}_{\mathrm{dR}}^+$ and the residue field $\mathbb{C}_p$. We have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for any $\gamma \in \Gamma$. We put

$$\mathbf{B}_{\mathrm{dR}} := \mathrm{Frac}(\mathbf{B}_{\mathrm{dR}}^+) = \mathbf{B}_{\mathrm{dR}}^+[1/t].$$

We know that $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{dR}}$ are equipped with the action of $G_{\mathbb{Q}_p}$. There exists a filtration on $\mathbf{B}_{\mathrm{dR}}$ given by $\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}}) := t^i\mathbf{B}_{\mathrm{dR}}^+$ for $i \in \mathbb{Z}$.

We suppose that $\mathbf{A}_{\max}$ is the separate completion of $\widetilde{\mathbf{A}}^+[\frac{[\widetilde{P}]}{p}]$ with respect to the p-adic topology. We define by $\mathbf{B}_{\max}^+ = \mathbf{A}_{\max}[1/p]$ and by $\mathbf{B}_{\max} = \mathbf{B}_{\max}^+[1/t]$. The ring $\mathbf{B}_{\max}$ is canonically contained in $\mathbf{B}_{\mathrm{dR}}$, then it is equipped with Galois action of $G_{\mathbb{Q}_p}$, the filtrations induced from $\mathbf{B}_{\mathrm{dR}}$ and the Frobenius φ induced from that of $\widetilde{\mathbf{A}}^+$. We define $\mathbf{B}_e := \mathbf{B}_{\max}^{\varphi=1}$. The series $\log(\pi^{(0)}) + \log([\pi]/\pi^{(0)})$ converges to an element $\log([\pi])$ in $\mathbf{B}_{\mathrm{dR}}^+$ which is transcendent over $\mathbf{B}_{\max}$, then we define $\mathbf{B}_{\mathrm{st}} := \mathbf{B}_{\max}[\log([\pi])]$. We define the following rings for $r > 0$

$$\widetilde{\mathbf{A}}^{(0,r]} := \left\{ x = \sum_{k=0}^{\infty} p^k[x_k] \in \widetilde{\mathbf{A}} \mid \lim_{k\longrightarrow\infty}\left(\nu_{\widetilde{\mathbf{E}}}(x_k) + \frac{k}{r}\right) = +\infty \right\}.$$

This is a subring of $\widetilde{\mathbf{A}}$ stable under the action of $G_{\mathbb{Q}_p}$, φ induces a ring isomorphism

$$\widetilde{\mathbf{A}}^{(0,r]} \xrightarrow{\sim} \widetilde{\mathbf{A}}^{(0,p^{-1}r]}$$

and the action of φ and $G_{\mathbb{Q}_p}$ on $\widetilde{\mathbf{A}}^{(0,r]}$ are continuous. We refer to [33, Corollaire 5.5, Proposition 5.10] for more details. There is a valuation $\nu^{(0,r]}$ on $\widetilde{\mathbf{A}}^{(0,r]}$ which is given by

$$\nu^{(0,r]}(x) = \inf_{k\geq 0}\left(\nu_{\widetilde{\mathbf{E}}}(x_k) + \frac{k}{r}\right).$$

We define the ring $\widetilde{\mathbf{A}}^r := \widetilde{\mathbf{A}}^{(0,\frac{p-1}{pr}]}$ and then the subring $\widetilde{\mathbf{A}}^{\dagger,r}$ of $\widetilde{\mathbf{A}}^r$ as following

$$\widetilde{\mathbf{A}}^{\dagger,r} := \{x = \sum_{k=0}^{\infty} p^k[x_k] \in \widetilde{\mathbf{A}}^r \mid \nu_{\widetilde{\mathbf{E}}}(x_k) + \frac{pr}{p-1}k \geq 0 \text{ for every } k \geq 0\}.$$

We define $\widetilde{\mathbf{B}}^{\dagger,r} := \widetilde{\mathbf{A}}^{\dagger,r}[1/p]$ and we set $\widetilde{\mathbf{B}}^{\dagger} := \cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}$. We consider the complete discrete valuation ring $\mathbf{A}_{\mathbb{Q}_p}$ consisting of the Laurent series of the form $\sum_{k\in\mathbb{Z}} a_k\pi^k$, where $a_k \in \mathbb{Z}_p$ and a_k tends 0 if k goes to $-\infty$. Set $\mathbf{B}_{\mathbb{Q}_p} := \mathbf{A}_{\mathbb{Q}_p}[1/p]$. Then $\mathbf{B}_{\mathbb{Q}_p}$ is stable under φ and $G_{\mathbb{Q}_p}$. See [29, Section 2] for more details. We denote by $\mathbf{B}$ is the p-adic completion of the maximal unramified extension of $\mathbf{B}_{\mathbb{Q}_p}$ in $\widetilde{\mathbf{B}}$. Then we define $\mathbf{B}^{\dagger,r} := \mathbf{B}\cap\widetilde{\mathbf{B}}^{\dagger,r}$ and $\mathbf{B}^{\dagger} = \cup_{r\geq 0}\mathbf{B}^{\dagger,r}$. We define the following ring

$$\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} := \bigcup_{r\geq 0}\bigcap_{s\geq r}\widetilde{\mathbf{A}}^{+}\left\{\frac{p}{[\pi]^r}, \frac{[\pi]^s}{p}\right\}\left[\frac{1}{p}\right], \tag{2.1.2}$$

where if A is a p-adic complete ring, $A\{X,Y\}$ denotes the p-adic completion of $A[X,Y]$.

2.2 p-adic Hodge theory

We call an L-representation of $G_{\mathbb{Q}_p}$ is a finite dimensional L-vector space equipped with a continuous L-linear action of $G_{\mathbb{Q}_p}$. For an L-representation V of $G_{\mathbb{Q}_p}$, we can associate several different L-vector spaces introduced by Fontaine in [38].

1. We can define a L-vector space $\mathbf{D}_{\mathrm{dR}}(V) := (\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. There is a L-linear filtration on $\mathbf{D}_{\mathrm{dR}}(V)$ given by $\mathrm{Fil}^i\,\mathbf{D}_{\mathrm{dR}}(V) := (\mathrm{Fil}^i\,\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. We call V de Rham if it satisfies $\dim_L \mathbf{D}_{\mathrm{dR}}(V) = \dim_L(V)$.

2. We define a L-vector space $\mathbf{D}_{\mathrm{st}}(V) := (\mathbf{B}_{\mathrm{st}}\otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. The L-vector space $\mathbf{D}_{\mathrm{st}}(V)$ is a filtered (φ, N)-modules which means that it is a finite dimensional E-vector space equipped with L-linear Frobenius φ, filtrations by L-vector subspaces induced from $\mathbf{D}_{\mathrm{dR}}(V)$ and a E-linear operator N called monodromy operator satisfying $N\varphi = p\varphi N$. We call V is semi-stable if it satisfies $\dim_L \mathbf{D}_{\mathrm{st}}(V) = \dim_L(V)$.

3. We define a L-vector space $\mathbf{D}_{\mathrm{cris}}(V) := (\mathbf{B}_{\max}\otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. The module $\mathbf{D}_{\mathrm{cris}}(V)$ is equipped with L-linear Frobenius φ and filtrations by L-vector subspaces. We call V is crystalline if it satisfies $\dim_L \mathbf{D}_{\mathrm{cris}}(V) = \dim_L V$. It is known that V is crystalline if and only if that it is semi-stable and $N = 0$ on $\mathbf{D}_{\mathrm{st}}(V)$.

2.3 The theory of (φ, Γ)-module over Robba ring

In this section, we recall the basic properties of (φ,Γ)-modules. The notion of (φ,Γ)-modules was introduced by Fontaine in his fundamental paper [39].

Let L be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_L$ its ring of integers. We denote the p-adic valuation on L by ν_L. For each $0 \leq r < 1$, we denote by $\mathrm{ann}(r,1)$ the p-adic annulus

$$\mathrm{ann}(r,1) = \left\{x\in\mathbb{C}_p \mid p^{-1/r} \leq |x|_p < 1\right\},$$

here $|x|_p = p^{-\nu_p(x)}$. We define $\mathscr{R}_L^{(r)}$ over L to be the ring of power series $f(X) = \sum_{n\in\mathbb{Z}} a_nX^n$, where $a_n \in L$ such that $f(X)$ converges on $\mathrm{ann}(r,1)$. The Robba ring $\mathscr{R}_L$ is defined to be $\mathscr{R}_L = \bigcup_{0\leq r<1}\mathscr{R}_L^{(r)}$. The power series $t' = \log(1+X)$ belongs to the $\mathscr{R}_L$. The Robba ring $\mathscr{R}_L$ is endowed with a Frobenius map φ given by $(\varphi f)(X) = f((1+X)^p - 1)$. The Robba ring $\mathscr{R}_L$ is also endowed with an action of Γ given by $(af)(X) = f((1+X)^{\chi_{\mathrm{cyc}}(a)} - 1)$ and this action commutes with φ. We can see that $\varphi(t') = pt'$ and $a(t') = at'$. Moreover, we have

$$\begin{aligned}\tau(\mathscr{R}_L^{(r)}) &= \mathscr{R}_L^{(r)}, \qquad \tau\in\Gamma,\\ \varphi(\mathscr{R}_L^{(r)}) &= \mathscr{R}_L^{(pr)}.\end{aligned}$$

We also define the ring $\mathcal{E}_L$, which is the subring of $\mathscr{R}_L$ consisting of the power series $f(X) = \sum_{n\in\mathbb{Z}} a_nX^n$ for which the sequence $\{a_n\}$ is bounded. We denote by $\mathcal{E}_{\mathcal{O}_L}$ the subring of $\mathcal{E}_L$ consisting of those $f(X) = \sum_{n\in\mathbb{Z}} a_nX^n$ for which $\nu_L(a_n) \leq 1$.

2.3.1 The φ-modules over $\mathscr{R}_L$

We recall Kedlaya's slope theory for φ-modules over $\mathscr{R}_L$. We first give the definition of φ-modules over $\mathscr{R}_L$.

Definition 2.3.1. We say D is a φ-module over $\mathscr{R}_L$ if it is a free $\mathscr{R}_L$-module of finite rank d with a φ-semi-linear map φ_D such that the matrix $\mathrm{Mat}(\varphi_D)$ for some basis of D is in $\mathrm{GL}_d(\mathscr{R}_L)$.

If $s \in \mathbb{Q}$, then we write $s = a/h$ for a, $h \in \mathbb{Z}$ and $(a, h) = 1$. We say that a φ-module over $\mathscr{R}_L$ is pure of slope s if it is rank ≥ 1 and has a basis in which $\mathrm{Mat}(p^{-a}\varphi^h) \in \mathrm{GL}_d(\mathcal{E}_{\mathcal{O}_L})$. A φ-module over $\mathscr{R}_L$ which is pure of certain slope is said to be isoclinic. We have the following theorem from [50, Theorem 6.10].

Theorem 2.3.2. *If D is a φ-module over $\mathscr{R}_L$, then there exists a unique filtration*

$$\{0\} = D_0 \subset D_1 \subset \cdots \subset D_\ell = D$$

of D by sub-φ-modules such that

1. *for all $i \geq 1$, D_i/D_{i-1} is an isoclinic φ-module.*
2. *if s_i is the slope of D_i/D_{i-1}, then $s_1 < s_2 < \cdots < s_\ell$.*

2.3.2 The (φ, Γ)-modules over $\mathscr{R}_L$

Definition 2.3.3. We say that D is a (φ, Γ)-module over $\mathscr{R}_L$ if

1. D is a free $\mathscr{R}_L$-module of finite rank d,
2. D is equipped with a φ-semi-linear map $\varphi_D \colon D \longrightarrow D$ such that the matrix $\mathrm{Mat}(\varphi_D)$ of φ_D in some basis belongs to $\mathrm{GL}_d(\mathscr{R}_L)$,
3. D is equipped with a continuous semi-linear action of Γ which commutes with φ_D. Here semi-linear means that $\gamma(ax) = \gamma(a)\gamma(x)$ for any $a \in \mathscr{R}_L$, $x \in D$ and $\gamma \in \Gamma$.

We denote by $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_L}$ the category of (φ, Γ)-modules over $\mathscr{R}_L$.

Definition 2.3.4. We say that a (φ, Γ)-module over $\mathscr{R}_L$ is étale if it has a basis in which $\mathrm{Mat}(\varphi_D) \in \mathrm{GL}_d(\mathcal{E}_{\mathcal{O}_L})$.

Remark 2.3.5. We can see that étale is equivalent to being pure of slope zero.

We denote $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_L,\text{étale}}$ the category of étale (φ, Γ)-modules over $\mathscr{R}_L$. We denote by $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$ the category of L-representation of $G_{\mathbb{Q}_p}$. If D is a (φ, Γ)-module over $\mathscr{R}_L$, then $V(D) = (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathscr{R}_L} D)^{\varphi=1}$ is an L-vector space, equipped with the action of $G_{\mathbb{Q}_p}$ given by $g(x \otimes e) = g(x) \otimes [\chi_{\mathrm{cyc}}(g)](e)$. From the results of Fontaine [39, Theorem 3.4.3], Cherbonnier–Colmez [28, Corollary III.5.2] and Kedlaya [51, Theorem 6.3.3], we have the following result.

Theorem 2.3.6. *If D is an étale (φ, Γ)-module of rank d over $\mathscr{R}_L$, then $V(D)$ is an L-linear representation of dimension d of $G_{\mathbb{Q}_p}$ and the resulting functor from $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_L,\text{étale}}$ to $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$ is an equivalence of categories. We denote by $\mathbf{D}^{\dagger}_{rig}(V)$ the inverse functor from $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$ to $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_L,\text{étale}}$.*

Remark 2.3.7. From the discussion in [17, Section 2.4], the condition that D is étale is the right one for $V(D)$ to be finite dimensional L-vector space.

2.4 Relation to the p-adic Hodge theory

We want to recover the classical Fontaine's functors $\mathbf{D}_{\mathrm{dR}}(V)$, $\mathbf{D}_{\mathrm{st}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ from $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. The work of Berger [14] and [16] gives the solution to this problem. In this section, we recall some results of Berger.

We write $\mathbb{Q}_\infty[[t']]$ for the ring of power series. We have a natural action of Γ given by

$$\gamma\left(\sum_{i\in\mathbb{Z}} a_i t'^i\right) = \sum_{i\in\mathbb{Z}} \gamma(a_i)\chi_{\mathrm{cyc}}(\gamma)^i t'^i.$$

We define $n(r)$ is the smallest integer n such that $p^{n-1}(p-1) \geq r$. We know that there exists an integer $r(K)$ such that if $p^{n-1}(p-1) \geq r \geq r(K)$, for any $n \geq n(r)$, we have a well defined injective homomorphism

$$\iota_n \colon \mathscr{R}_L^{(n)} \longrightarrow \mathbb{Q}_\infty[[t']] \otimes_{\mathbb{Q}_p} E.$$

See more details from [61, Section 2.1]. Let D be a (φ,Γ)-module over $\mathscr{R}_L$. We have the following theorem.

Theorem 2.4.1. *We let D is a (φ,Γ)-module over $\mathscr{R}_L$ and there exists $r(D) > 0$ such that for all $r \geq r(D)$, there is a unique $\mathscr{R}_L^{(r)}$-module $D^{(r)}$ of D satisfying*

1. $D = \mathscr{R}_L \otimes_{\mathscr{R}_L^{(r)}} D^{(r)}$.
2. *If $s > r$, then* $D^{(s)} \cong D^{(r)} \otimes_{\mathscr{R}_L^{(r)}} \mathscr{R}_L^{(s)}$.
3. $D^{(rp)} \cong \mathscr{R}_L^{(rp)} \otimes_{\mathscr{R}_L^{(r)},\varphi} D^{(r)}$.

Proof. See [16, Théorèm I.3.3]. □

Hence we can write $D = D^{(r)} \otimes_{\mathscr{R}_L^{(r)}} \mathscr{R}_L$ for $r \geq r(D)$. We denote

$$\mathscr{D}^+_{\mathrm{dif}}(D) \colon = D^{(r)} \otimes_{\mathscr{R}_L^{(r)},\,\iota_n} \mathbb{Q}_\infty[[t']] \otimes_{\mathbb{Q}_p} L$$

for $r \geq r(D)$ and for all n such that $p^n(p-1) \geq r$ and $\mathscr{D}_{\mathrm{dif}}(D) \colon = \mathscr{D}^+_{\mathrm{dif}}(D)[1/t']$. We then define

$$\mathscr{D}^+_{\mathrm{dR}}(D) \colon = \mathscr{D}^+_{\mathrm{dif}}(D)^\Gamma \text{ and } \mathscr{D}_{\mathrm{dR}}(D) \colon = \mathscr{D}_{\mathrm{dif}}(D)^\Gamma$$

This definition does not depend on the choice of r and n. We know that $\mathscr{D}_{\mathrm{dR}}(D)$ is a finite dimensional L-vector space such that

$$\dim_L \mathscr{D}_{\mathrm{dR}}(D) \leq \mathrm{rank}_{\mathscr{R}_L}(D).$$

Moreover, $\mathscr{D}_{\mathrm{dR}}(D)$ is equipped with a filtration

$$\mathrm{Fil}^i\, \mathscr{D}_{\mathrm{dR}}(D) \colon = \mathscr{D}_{\mathrm{dR}}(D) \cap t'^i \mathscr{D}^+_{\mathrm{dif}}(D),$$

which also does not depend on the choice of r and n. Now we let $\mathscr{R}_L[\log X]$ denote the ring of power series in variable $\log X$ with coefficients in $\mathscr{R}_L$. We can extend the actions of φ and Γ to $\mathscr{R}_L[\log X]$, which is given by

$$\varphi(\log X) = p\log X + \log\left(\frac{\varphi(X)}{X^p}\right)$$
$$\gamma(\log X) = \log X + \log\left(\frac{\gamma(X)}{X}\right), \text{ for } \gamma \in \Gamma.$$

Here $\log\left(\frac{\varphi(X)}{X^p}\right)$ and $\log\left(\frac{\gamma(X)}{X}\right)$ converge in $\mathscr{R}_L$. See [14, Section 2.6]. We can also define a monodromy operator $N \colon \mathscr{R}_L[\log X] \longrightarrow \mathscr{R}_L[\log X]$ by

$$N = -\left(1 - \frac{1}{p}\right)^{-1} \frac{d}{d\log X}.$$

Definition 2.4.2. For a (φ,Γ)-module D over $\mathscr{R}_L$, we define

$$\mathscr{D}_{\mathrm{crys}}(D) \colon = (D \otimes_{\mathscr{R}_L} \mathscr{R}_L[1/t'])^\Gamma, \qquad \mathscr{D}_{\mathrm{st}}(D) \colon = (D \otimes_{\mathscr{R}_L} \mathscr{R}_L[\log X, 1/t'])^\Gamma.$$

Then $\mathscr{D}_{\mathrm{st}}(D)$ is a finite dimensional L-vector space equipped with natural actions of φ and N such that $N\varphi = p\varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration. On the other hand, $\mathscr{D}_{\mathrm{cris}}(D) = \mathscr{D}_{\mathrm{st}}(D)^{N=0}$ and

$$\dim_L \mathscr{D}_{\mathrm{cris}}(D) \leq \dim_L \mathscr{D}_{\mathrm{st}}(D) \leq \dim_L \mathscr{D}_{\mathrm{dR}}(D).$$

Definition 2.4.3. We say that D is crystalline (resp., semistable, resp., de Rham) if

$$\dim_L \mathscr{D}_{\mathrm{cris}}(D) = \mathrm{rank}_{\mathscr{R}_L}(D)$$

(resp., $\dim_L \mathscr{D}_{\mathrm{st}}(D) = \mathrm{rank}_{\mathscr{R}_L}(D)$, resp., $\dim_L \mathscr{D}_{\mathrm{dR}}(D) = \mathrm{rank}_{\mathscr{R}_L}(D)$).

Proposition 2.4.4. *Let V be an L-representation of $G_{\mathbb{Q}_p}$. Then*

$$\mathbf{D}_*(V) \cong \mathscr{D}_*(\mathbf{D}^\dagger_{rig}(V)), \quad * \in \{st,\ cris\}.$$

In particular, V is crystalline(resp., semistable) if and only if $\mathbf{D}^\dagger_{rig}(V)$ is.

Proof. See [14, Theorem 0.2]. ☐

2.5 Families of (φ,Γ)-module

Let A be an affinoid algebra over L. We define

$$\mathscr{R}^{(r)}_{L,A} = A\widehat{\otimes}_L \mathscr{R}^{(r)}_L, \qquad \mathscr{R}_{L,A} = \bigcup_{0\leq r<1} \mathscr{R}^{(r)}_{L,A}.$$

The actions of φ and Γ on $\mathscr{R}_L$ extend to $\mathscr{R}_{L,A}$ by linearity.

Definition 2.5.1. 1. A (φ,Γ)-module over $\mathscr{R}^{(r)}_{L,A}$ is a finite generated projective $\mathscr{R}^{(r)}_{L,A}$-module $\mathbf{D}^{(r)}$ equipped with the following structures

(a) A φ-semi-linear map

$$\varphi_D \colon \mathbf{D}^{(r)} \longrightarrow \mathbf{D}^{(r)} \otimes_{\mathscr{R}^{(r)}_{L,A}} \mathscr{R}^{(pr)}_{L,A}$$

such that the induced linear map

$$\varphi_D \otimes \mathrm{id}_{\mathscr{R}^{(r)}_{L,A}} \colon \mathbf{D}^{(r)} \otimes_{\mathscr{R}^{(r)}_{L,A},\,\varphi} \mathscr{R}^{(pr)}_{L,A} \longrightarrow \mathbf{D}^{(r)} \otimes_{\mathscr{R}^{(r)}_{L,A}} \mathscr{R}^{(pr)}_{L,A}$$

is an isomorphism of $\mathscr{R}^{(pr)}_{L,A}$-modules.

(b) A semi-linear continuous action of Γ on $\mathbf{D}^{(r)}$.

2. $\mathbf{D}$ is a (φ,Γ)-module over $\mathscr{R}_{L,A}$ if $\mathbf{D} = \mathbf{D}^{(r)} \otimes_{\mathscr{R}^{(r)}_{L,A}} \mathscr{R}_{L,A}$ for some (φ,Γ_K)-module $\mathbf{D}^{(r)}$ over $\mathscr{R}^{(r)}_{L,A}$.

A p-adic representation of $G_{\mathbb{Q}_p}$ with coefficients in an affinoid algebra A over L is a finitely generated projective A-module equipped with a continuous A-linear action of $G_{\mathbb{Q}_p}$. Let $\mathbf{Rep}_A(G_{\mathbb{Q}_p})$ denote the tensor category of p-adic representations with coefficients in A and $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{L,A}}$ denote the tensor category of (φ,Γ)-modules over $\mathscr{R}_{L,A}$. We can generalize the functor $\mathbf{D}^\dagger_{\mathrm{rig}}$ in Theorem 2.3.6 to these more general representations. More precisely, from the work [18, Théorèm A] and [48, Theorem 0.1], we know there is a fully faithful exact functor

$$\mathbf{D}^\dagger_{\mathrm{rig},A} \colon \mathbf{Rep}_A(G_K) \longrightarrow \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{L,A}}$$

which commutes with base change. We recall the explicit description of functor $\mathbf{D}^\dagger_{\mathrm{rig},A}$. We let A be a Banach algebra over L and the valuation Val_A on A, which satisfies

1. $\mathrm{Val}_A(f) = +\infty \Longleftrightarrow f = 0$.

2. $\mathrm{Val}_A(fg) \geq \mathrm{Val}_A(f) + \mathrm{Val}_A(g)$
3. $\mathrm{Val}_A(f+g) \geq \inf\{\mathrm{Val}_A(f), \mathrm{Val}_A(g)\}$

for all $f,\ g \in A$. Let $\mathcal{O}_A$ be the ring consisting of the element whose valuation is greater or equal to 0. Now we consider V is a p-adic representations of $G_{\mathbb{Q}_p}$ with coefficients in A of dimension d. From [18, Théorèm 4.2.9], there exists an $s(V,A) \geq 0$, such that for any $s \geq s(V,A)$, there is a $A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger,s}$-module $\mathbf{D}_A^{\dagger,s}(V)$ which is locally free of rank d. The module $\mathbf{D}_A^{\dagger,s}(V)\otimes_{A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger,s}}(A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger,s})$ is isomorphic to $V\otimes_A(A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger,s})$. Then we define

$$\mathbf{D}_A^{\dagger}(V) := \mathbf{D}_A^{\dagger,s}(V)\otimes_{A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger,s}} A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger}$$

for any such s. Our functor $\mathbf{D}^{\dagger}_{\mathrm{rig},A}$ is given by

$$\mathbf{D}^{\dagger}_{\mathrm{rig},A}(V) := \mathbf{D}_A^{\dagger}(V)\otimes_{A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger}} A\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig}}.$$

Remark 2.5.2. When A is a finite extension L of $\mathbb{Q}_p$, then the functor $\mathbf{D}^{\dagger}_{\mathrm{rig},L}$ is the same as the functor $\mathbf{D}^{\dagger}_{\mathrm{rig}}$ in Theorem 2.3.6.

Let $\mathscr{X} = \mathrm{Spm}(A)$. Then for every point x in $\mathscr{X}$, we denote by $\mathfrak{m}_x$ the maximal ideal of A associated to x, then we denote $E_x := A/\mathfrak{m}_x$. Suppose that V(resp. $\mathbf{D}$) is an object of $\mathbf{Rep}_A(G_{\mathbb{Q}_p})$(resp. of $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{L,A}}$), we set $V_x = V\otimes_A E_x$(resp. $\mathbf{D}_x = \mathbf{D}\otimes_A E_x$). We have the following commutative diagram

$$\begin{array}{ccc} \mathbf{Rep}_A(G_{\mathbb{Q}_p}) & \xrightarrow{\mathbf{D}^{\dagger}_{\mathrm{rig},A}} & \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{L,A}} \\ \downarrow{\scriptstyle \otimes E_x} & & \downarrow{\scriptstyle \otimes E_x} \\ \mathbf{Rep}_{E_x}(G_{\mathbb{Q}_p}) & \xrightarrow{\mathbf{D}^{\dagger}_{\mathrm{rig}}} & \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{L,E_x}}, \end{array}$$

i.e. we have $\mathbf{D}^{\dagger}_{\mathrm{rig},A}(V)_x \cong \mathbf{D}^{\dagger}_{\mathrm{rig},E_x}(V_x)$. We refer to [18, Théorèm A] and [48, Theorem 0.1] for more details.

CHAPTER 3

Big Galois representation

3.1 p-stabilization of modular forms

Let $f = \sum a_n q^n$ be a newform of level $\Gamma_1(N)$ and of weight $k+2 \geq 2$ with nebentype χ_f. We assume that $p \nmid N$. We then have $T_p f = a_p f$ and $\langle p \rangle f = \chi_f(p) f$. We denote α_f and β_f the roots of the polynomial

$$X^2 - a_p X + \chi_f(p) p^{k+1}.$$

Definition 3.1.1. We call f is p-regular if $\alpha_f \neq \beta_f$.

Remark 3.1.2. It is conjectured that f is always p-regular, but we only know for f of weight 2. See [30].

We let $\Gamma \colon = \Gamma_1(N) \cap \Gamma_0(p)$. We assume that f is p-regular. Hence there are two normalized modular forms for Γ of weight $k+2$ defined by

$$f_{\alpha_f}(z) = f(z) - \beta_f f(pz),$$

$$f_{\beta_f}(z) = f(z) - \alpha_f f(pz).$$

As is well known, these two modular forms are eigenforms for the Hecke operators T_ℓ, for $\ell \nmid pN$ and the diamond operators with the same eigenvalues as f. In addition, they are also eigenforms for the Atkin–Lehner operator U_p satisfying $U_p f_{\alpha_f} = \alpha_f f_{\alpha_f}$ and $U_p f_{\beta_f} = \beta_f f_{\beta_f}$.

We say that the modular form f_{β_f} is of critical slope if $\mathrm{ord}_p(\beta_f) = k+1$. We have the following classification.

Proposition 3.1.3. *When f is a cuspidal newform, then there exists a f_{β_f} of critical slope in and only in the following cases:*

1. *f is not CM and* $\mathrm{ord}_p(a_p) = 0$.
2. *f is CM for a quadratic imaginary field K where p is split.*

Proof. See [5, Proposition 2.13]. □

Definition 3.1.4. 1. We call f_{β_f} is critical if $\mathrm{ord}_p(\beta_f) > 0$ and the representation $\rho_f|_{G_{\mathbb{Q}_p}}$ is the direct sum of two characters. Here ρ_f is the unique Galois representation $\rho_f \colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ such that ρ_f is unramified outside Np and satisfies

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell$$

for all $\ell \nmid Np$.

2. We call f_{β_f} is θ-critical if it is in the image of the operator $\theta_k \colon M^\dagger_{-k}(\Gamma) \longrightarrow M^\dagger_{k+2}(\Gamma)$, which acts on q-expansion by $(q\dfrac{d}{dq})^{k+1}$. Here $M^\dagger_i(\Gamma)$ denote the space of overconvergent modular form for Γ of weight i.

Remark 3.1.5. The usual definition of non-critical is given by $\mathrm{ord}_p(\beta_f) < k-1$, which is different from the definition we give here. The condition $\mathrm{ord}_p(\beta_f) < k-1$ is called numerically non-critical in our case and it can be shown that the numerically non-critical implies the non-critical, but the converse is false in general. See [6, Remark 2.4.6(ii)] for more details.

From [5, Proposition 2.11], when f is cuspidal and f_{β_f} is of critical slope, then f_{β_f} critical is equivalent to that f_{β_f} θ-critical.

Remark 3.1.6.
1. It is conjectured that every cuspidal non CM form is non-critical(c.f. [5, Remark 2.14]). Hence in the first case in Proposition 3.1.3, we expect that f_{β_f} of critical slope is non-critical, hence not θ-critical.
2. In the second case in Proposition 3.1.3, the f_{β_f} of critical slope is always critical and hence θ-critical.

3.2 Locally analytic functions and distributions

We always assume that N is an integer and $N \geq 4$. We also assume $p \geq 3$. We let $\Lambda\colon=\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwasawa algebra. We write $\mathfrak{M}\colon=\mathrm{Spf}(\Lambda)$. Now we define $\mathcal{W}$ the rigid analytic space over $\mathbb{Q}_p$ associated to the formal scheme $\mathfrak{M}$. We let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and maximal ideal $\mathfrak{m}=\pi\mathcal{O}_L$. We base change $\mathcal{W}$ over L, which we denote by $\mathcal{W}_L$. If $y \in \mathcal{W}_L(L)$, we denote by $\kappa_y\colon \mathbb{Z}_p^\times \longrightarrow L^\times$ the associated character. We consider $\mathbb{Z}$ as a subset of $\mathcal{W}_L(L)$ identifying $k \in \mathbb{Z}$ with the character $t \longmapsto t^{k-2}$. We call the point in $\mathbb{Z}$ the classical point. We set

$$\mathcal{W}_L^*\colon=\{y \in \mathcal{W}_L(L) \mid v_p(\kappa_y(a)^{p-1}-1) > \frac{1}{p-1}, \text{ for all } a \in \mathbb{Z}_p^\times\}.$$

If $U \subset \mathcal{W}_L^*$ is either an affinoid disk or a wide open disk, we write Λ_U for the ring of analytic functions on U that are bounded by 1 with respect to Gauss norm which is an $\mathcal{O}_L$-algebra isomorphic to $\mathcal{O}_L[[T]]$. The $\mathcal{O}_L$-algebra Λ_U is complete with respect to $\mathfrak{m}_U$-adic topology for $\mathfrak{m}_U$ the maximal ideal of Λ_U. See [2, Section 3.1] for more details. Set $\mathcal{O}_U = \Lambda_U[1/p]$ for the ring of bounded analytic functions on U. We denote by $\kappa_U\colon \mathbb{Z}_p^\times \longrightarrow \Lambda_U^\times$ the universal weight character given by $\kappa_U\colon \mathbb{Z}_p^\times \hookrightarrow \mathcal{O}_L[[\mathbb{Z}_p^\times]]^\times \longrightarrow \Lambda_U^\times$.

We set $\mathsf{T}\colon=\mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathsf{T}'\colon=p\mathbb{Z}_p \times \mathbb{Z}_p^\times$. Let

$$\Sigma_0(p) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p) \mid a \in \mathbb{Z}_p^\times, c \in p\mathbb{Z}_p\right\}$$

and

$$\Sigma_0'(p) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p) \mid d \in \mathbb{Z}_p^\times, c \in p\mathbb{Z}_p\right\}.$$

These two semi-groups act on $\mathbb{Z}_p^2$ by right multiplication and preserve the subset T and T' respectively. Hence T and T' are preserved by scalar multiplication by $\mathbb{Z}_p^\times$ and right multiplication by the Iwahori subgroup $\Gamma_0(p\mathbb{Z}_p) = \Sigma_0(p) \cap \Sigma_0'(p)$.

Definition 3.2.1. Let $U \subset \mathcal{W}_L^*$ be an affinoid or wide open disc.

1. Let $\mathcal{A}(U)^\circ$ denote the space of functions $f\colon \mathsf{T} \longrightarrow \Lambda_U$ satisfying
   (a) $f(at) = \kappa_U(a)f(t)$, for all $a \in \mathbb{Z}_p^\times$.
   (b) The function $f(1,z)$ can be written in the form $\sum_{n=0}^{\infty} c_n z^n$, where $c_n \in \Lambda_U$ and c_n goes to zero when $n \longrightarrow \infty$ with respect to the $\mathfrak{m}_U$-adic topology.
2. Let $\mathcal{A}'(U)^\circ$ denote the space of functions $f\colon \mathsf{T}' \longrightarrow \Lambda_U$ satisfying
   (a) $f(at) = \kappa_U(a)f(t)$, for all $a \in \mathbb{Z}_p^\times$.
   (b) The function $f(pz,1)$ can be written in the form $\sum_{n=0}^{\infty} c_n z^n$, where $c_n \in \Lambda_U$ and c_n goes to zero when $n \longrightarrow \infty$ with respect to the $\mathfrak{m}_U$-adic topology.

Set $\mathcal{D}^{\cdot}(U)^{\circ} := \mathrm{Hom}_{\Lambda_U,\mathrm{cont}}(\mathcal{A}^{\cdot}(U)^{\circ}, \Lambda_U)$, $\mathcal{A}^{\cdot}(U) = \mathcal{A}^{\cdot}(U)^{\circ}[1/p]$ and $\mathcal{D}^{\cdot}(U) := \mathcal{D}^{\cdot}(U)^{\circ}[1/p]$ for $\cdot = \{\emptyset, \prime\}$. The Λ_U-module $\mathcal{A}^{\cdot}(U)^{\circ}$ is preserved by the left action of $\Sigma_0^{\cdot}(p)$ i.e. if $f\colon \mathsf{T}^{\cdot} \longrightarrow \Lambda_U$ is an element in $\mathcal{A}^{\cdot}(U)^{\circ}$, then $\gamma \cdot f(t) = f(t \cdot \gamma)$ for every $\gamma \in \Sigma_0^{\cdot}(p)$ and $t \in \mathsf{T}^{\cdot}$. See [55, Proposition 4.2.5] for more details. This gives the structure of a $\Lambda_U[\Sigma_0^{\cdot}(p)]$-module on $\mathcal{A}^{\cdot}(U)^{\circ}$ which induces a structure of a right $\Lambda_U[\Sigma_0^{\cdot}(p)]$-module on $\mathcal{D}^{\cdot}(U)^{\circ}$.

We suppose that $r \in \mathcal{W}_L(L)$. We have the following similar definition.

Definition 3.2.2. 1. Let $\mathcal{A}_r^{\circ}$ denote the space of functions $f\colon \mathsf{T} \longrightarrow \mathcal{O}_L$ satisfying

(a) $f(at) = r(a)f(t)$, for all $a \in \mathbb{Z}_p^{\times}$.

(b) The function $f(1,z)$ can be written in the form $\sum_{n=0}^{\infty} c_n z^n$, where $c_n \in \mathcal{O}_L$ and c_n goes to zero when $n \longrightarrow \infty$.

2. Let $\mathcal{A}_r^{\prime;\circ}$ denote the space of functions $f\colon \mathsf{T}' \longrightarrow \mathcal{O}_L$ satisfying

(a) $f(at) = r(a)f(t)$, for all $a \in \mathbb{Z}_p^{\times}$.

(b) The function $f(pz,1)$ can be written in the form $\sum_{n=0}^{\infty} c_n z^n$, where $c_n \in \mathcal{O}_L$ and c_n goes to zero when $n \longrightarrow \infty$.

We can also set $\mathcal{D}_r^{\cdot;\circ} := \mathrm{Hom}_{\mathcal{O}_L,\mathrm{cont}}(\mathcal{A}_r^{\cdot;\circ}, \mathcal{O}_L)$, $\mathcal{A}_r^{\cdot} = \mathcal{A}_r^{\cdot;\circ}[1/p]$ and $\mathcal{D}_r^{\cdot} := \mathcal{D}_r^{\cdot;\circ}[1/p]$ for $\cdot = \{\emptyset, \prime\}$.

Let $k = r + 2 \in U$ and $\pi_k \in \Lambda_U$ be a uniformizer at $k-2$. From [2, Proposition 3.11], we have short exact sequences of $\Sigma_0^{\cdot}(p)$-modules

$$0 \longrightarrow \mathcal{A}^{\cdot}(U) \xrightarrow{\pi_k} \mathcal{A}^{\cdot}(U) \xrightarrow{\rho_k} \mathcal{A}_r^{\cdot} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{D}^{\cdot}(U) \xrightarrow{\pi_k} \mathcal{D}^{\cdot}(U) \xrightarrow{\rho_k} \mathcal{D}^{\cdot}_{r} \longrightarrow 0$$

We can make the morphisms ρ_k more precise:

$$\rho_k(f)(x,y) = f(x,y)(k) \quad \text{and} \quad \rho_k(\mu)(\gamma) = \mu(\gamma_U)(k)$$

for every $f \in \mathcal{A}^{\cdot}(U)$, $(x,y) \in \mathsf{T}^{\cdot}$, $\mu \in \mathcal{D}^{\cdot}(U)$ and $\gamma \in \mathcal{A}_r^{\cdot}$, where

$$\gamma_U = \begin{cases} \kappa_U(x) \cdot \gamma(1, y/x), & \text{if } \mathsf{T}' = \mathsf{T} \\ \kappa_U(y) \cdot \gamma(x/y, 1), & \text{if } \mathsf{T}^{\cdot} = \mathsf{T}'. \end{cases}$$

3.3 Slope decompositions

Let $\mathscr{B}$ be a L-Banach algebra and $h \in \mathbb{Q}_{\geq 0}$. We say that a polynomial $P(t)$ in $\mathscr{B}[t]$ has slope $\leq h$ if every edge of its Newton polygon has slope $\leq h$. Let $\mathscr{B}[t]^{\leq h}$ be the set of polynomials in $\mathscr{B}[t]$ of slope $\leq h$ and whose leading coefficient is a multiplicative unit. For every $P(t) \in \mathscr{B}[t]$, we write $P^*(t) = t^{\deg(P)}P(1/t)$. Now we consider a $\mathscr{B}$-module N and let $\boldsymbol{u}$ be a $\mathscr{B}$-linear endomorphism of N. Following [4], we say that N admits a slope $\leq h$ decomposition with respect to $\boldsymbol{u}$ if there exists a direct sum decomposition $N = N^{\leq h} \oplus N^{>h}$ into $\mathscr{B}[\boldsymbol{u}]$-modules satisfying

1. $N^{\leq h}$ is finite generated over $\mathscr{B}$.
2. There exists $P(t) \in \mathscr{B}[t]^{\leq h}$ such that $P^*(\boldsymbol{u})$ kills $N^{\leq h}$.
3. For every $P(t) \in \mathscr{B}[t]^{\leq h}$, the endomorphism $P^*(\boldsymbol{u})$ of $N^{>h}$ is an isomorphism.

We let U be a connected wide open disc of $\mathcal{W}_L$ containing an element $k = r + 2$. From the discussion in [23, Section 4.1.4], there is a L-Banach algebra structure on $\mathcal{O}_U$. We recall how to give the L-Banach algebra norm on $\mathcal{O}_U$. We know that $\mathcal{O}_U$ is isomorphic to the L-module $L[[T]]^{\circ}$ of power series in $L[[T]]$ with bounded Gauss norm. If s is a real number satisiying $0 < s < 1$, we define $|\cdot|_s\colon L[[T]]^{\circ} \longrightarrow \mathbb{R}_{\geq 0}$ by $|\sum_{n\geq 0} a_n \cdot T^n|_s = \sup_{n\geq 0} s^n \cdot |a_n|_p$. The $|\cdot|_s$ is an L-Banach algebra norm on $L[[T]]^{\circ}$, which is independent of s, hence this corresponds to a L-Banach algebra norm on $\mathcal{O}_U$. We set

$$\mathcal{T}_r = \left\{(L, \mathcal{A}_r^{\cdot}, U_p^{\cdot}), (L, \mathcal{D}_r^{\cdot}, U_p^{\cdot}), (L, S_r(L), U_p^{\cdot}), (L, L_r(L), U_p)\right\}$$

where $S_r(L)$ is the set of two variable homogeneous polynomials of degree i in $L[x_1, x_2]$ and $L_r(L)$ is the L-linear dual of $S_r(L)$ and we set

$$\mathcal{T}_U = \left\{(\mathcal{O}_U, \mathcal{A}^{\cdot}(U), U_p^{\cdot}), (\mathcal{O}_U, \mathcal{D}^{\cdot}(U), U_p^{\cdot})\right\}.$$

Then we can discuss slope $\leq h$ decomposition of each triple $(\mathscr{B}, M, \boldsymbol{u})$ in $\mathcal{T}_r \cup \mathcal{T}_U$. From the work of Coleman [32] and Ash–Stevens [4], we have the following proposition. See also [2] and [23, Proposition 4.2].

Proposition 3.3.1. *Let $(\mathscr{B}, M, \boldsymbol{u})$ be a triple in $\mathcal{T}_r \cup \mathcal{T}_U$. Shrinking U if necessary, the $\mathscr{B}$-module $H^1(\Gamma, M)$ admits a slope $\leq h$ decomposition with respect to $\boldsymbol{u}$. Moreover, for $\cdot = \emptyset, \prime$, the specialisation maps ρ_k in Equation 3.2 induce Hecke equivariant isomorphisms*

$$\rho_k\colon H^1(\Gamma, \mathcal{A}^{\cdot}(U))^{\leq h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, \mathcal{A}_r^{\cdot})^{\leq h}$$

and

$$\rho_k\colon H^1(\Gamma, \mathcal{D}^{\cdot}(U))^{\leq h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, \mathcal{D}_r^{\cdot})^{\leq h}$$

Lemma 3.3.2. *Let U is a wide open disc containing an integer $k > h+1$, we have a morphism of $G_{\mathbb{Q}}$-modules*

$$s_U\colon H^1(\Gamma, \mathcal{A}(U))^{\leq h} \longrightarrow H^1(\Gamma, \mathcal{D}'(U))^{\leq h}(-\boldsymbol{\kappa}_U),$$

where $\boldsymbol{\kappa}_U\colon G_{\mathbb{Q}} \longrightarrow \Lambda_U^{\times}$ is defined by $\boldsymbol{\kappa}_U(g) = \kappa_U(\chi_{cyc}(g))$ for every $g \in G_{\mathbb{Q}}$.

Proof. This is from [23, Equation (83)]. See [24] for more details. □

3.4 Local description of the eigencurve

Now we briefly recall the local pieces of the eigencurve constructed in [12, Section 5.3], which is a variant of Bellaiche's construction in [5, Section 3.4]. The construction of eigencurve is first given by Coleman and Mazur in [31] and later generalized by Buzzard in [27].

For any W the affinoid disc of $\mathcal{W}_L$, we set $\mathcal{H}_W := \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_W$. We fix a nice affinoid disk W of the weight space $\mathcal{W}_L$ See [5, Definition 3.5]. We also set W to be ν-adapted(c.f. [5, Section 3.2.4]). We denote by $M^{\dagger}(\Gamma, W)$ the Coleman's space of overconvergent modular forms of level Γ and weight in W. See [32, Section B.4] for more details. The $M^{\dagger}(\Gamma, W)^{\leq \nu}$ is the $\mathcal{O}_W$-submodule of $M^{\dagger}(\Gamma, W)$ on which U_p acts with slope at most ν. Similarly, we denote by $S^{\dagger}(\Gamma, W)$ the space of cuspidal overconvergent modular forms of level Γ and weights in W. We also define the $\mathcal{O}_W$-submodule $S^{\dagger}(\Gamma, W)^{\leq \nu}$ in the same way. Let $\mathbb{T}_{W,\nu}$ denote the image of $\mathcal{H}_W$ in $\mathrm{End}_{\mathcal{O}_W}(M^{\dagger}(\Gamma, W)^{\leq \nu})$ and $\mathbb{T}_{W,\nu}^{\mathrm{cusp}}$ denote the image of $\mathcal{H}_W$ in $\mathrm{End}_{\mathcal{O}_W}(S^{\dagger}(\Gamma, W)^{\leq \nu})$. Then we define $\mathcal{C}_{W,\nu} := \mathrm{Spm}(\mathbb{T}_{W,\nu})$ and $\mathcal{C}_{W,\nu}^{\mathrm{cusp}} := \mathrm{Spm}(\mathbb{T}_{W,\nu}^{\mathrm{cusp}})$. We know that $\mathcal{C}_{W,\nu}$ is an open affinoid subspace of the Coleman–Mazur–Buzzard eigencurve $\mathcal{C}$ and $\mathcal{C}_{W,\nu}^{\mathrm{cusp}}$ is an open affinoid subspace of the cuspidal eigencurve $\mathcal{C}^{\mathrm{cusp}}$. See more details in [5, Section 2.1.1, Section 2.1.3]. If $x \in \mathcal{C}_{W,\nu}^{\mathrm{cusp}}$, then $\mathcal{C}_{W,\nu}^{\mathrm{cusp}}$ and $\mathcal{C}_{W,\nu}$ are locally isomorphic at the point x. See [5, Corollary 2.17]. We denote by $\kappa\colon \mathcal{C}_{W,\nu} \longrightarrow W$ the weight map which is finite and flat. See [5, Section 2.1.1].

We denote by $H_c^i(\Gamma, -)$ the cohomology groups with compact support of Γ, which is defined to be $H^{i-1}(\Gamma, I(\cdot))$ the $i-1$-th cohomology group of Γ with values in the Γ-module

$$I(\cdot) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), \cdot),$$

where $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is the abelian group of divisors of degree 0 on $\mathbb{P}^1(\mathbb{Q})$. We set

$$H_c^1(\Gamma, \mathcal{D}(U)_W^{\pm, \leq \nu} := H_c^1(\Gamma, \mathcal{D}(U))^{\pm, \leq \nu} \otimes_{\mathcal{O}_U} \mathcal{O}_W,$$
$$H_c^1(\Gamma, \mathcal{A}'(U)_W^{\pm, \leq \nu} := H_c^1(\Gamma, \mathcal{A}'(U))^{\pm, \leq \nu} \otimes_{\mathcal{O}_U} \mathcal{O}_W.$$

Here $H_c^1(\Gamma, \cdot)^{\pm, \leq \nu}$ is the ± 1 eigenspace of the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we let

$$r_1^{\pm}\colon \mathcal{H}_W \longrightarrow \mathrm{End}_{\mathcal{O}_W}(H_c^1(\Gamma, \mathcal{D}(U))_W^{\pm, \leq \nu}), \quad r_2^{\pm}\colon \mathcal{H}_W \longrightarrow \mathrm{End}_{\mathcal{O}_W}(H_c^1(\Gamma, \mathcal{A}'(U))_W^{\pm, \leq \nu}).$$

We define the ideal $I_1^{\pm} := \ker(r_1^{\pm})$ and $I_2^{\pm} := \ker(r_2^{\pm})$.

Definition 3.4.1. We define $\mathcal{C}^{\pm}_{W,\nu} := \mathrm{Spm}(\mathcal{H}_W/I_1^{\pm})$ and $\mathcal{C}'^{,\pm}_{W,\nu} := \mathrm{Spm}(\mathcal{H}_W/I_2^{\pm})$

We have the following properties.

Theorem 3.4.2. *1. $\mathcal{C}^{\pm}_{W,\nu} \cong \mathcal{C}'^{,\pm}_{W,\nu}$ and $\mathcal{C}_{W,\nu} = \mathcal{C}^{+}_{W,\nu} \cup \mathcal{C}^{-}_{W,\nu}$.*

2. Suppose $x_0 \in \mathcal{C}_{W,\nu}$ is cuspidal and $k_0 = \kappa(x_0)$. For sufficiently small wide open neighborhood U of k_0 and an affinoid $W \subset U$ containing k_0, we have $x_0 \in \mathcal{C}^{+}_{W,\nu} \cap \mathcal{C}^{-}_{W,\nu}$. Furthermore,

(a) x_0 belongs to a unique connected component $\mathcal{X}$ of $\mathcal{C}_{W,\nu}$.

(b) Both $\mathcal{O}_{\mathcal{X}}$-modules $H^1_c(\Gamma, \mathcal{D}(U))^{\leq \nu}_W \otimes_{\mathcal{H}_W} \mathcal{O}_{\mathcal{X}}$ and $H^1_c(\Gamma, \mathcal{A}'(U))^{\leq \nu}_W \otimes_{\mathcal{H}_W} \mathcal{O}_{\mathcal{X}}$ are free of rank 2.

Proof. See [12, Theorem 5.6, Corollary 5.7] and [5, Theorem 3.30]. □

We have the following theorem given by Bellaiche for the description of $\mathcal{X}$.

Theorem 3.4.3. *After shrinking W and enlarge L, for $x_0 \in \mathcal{C}^{cusp}_{W,\nu}$, there exist integers $r \geq 1$, we write $\mathcal{O}_W = L\langle Y/p^{re}\rangle$, where e is the degree of ramification of the weight map κ at the point x_0. Furthermore there exists an affinoid neighborhood $\mathcal{X}$ of x_0, we have an isomorphism of $\mathcal{O}_W$-modules*

$$\mathcal{O}_W[X]/(X^e - Y) \cong \mathcal{O}_{\mathcal{X}}.$$

Proof. We can prove the theorem by combining [5, Proposition 4.11 and Corollary 2.17], see [12, Proposition 5.3]. □

Remark 3.4.4. We know that $e \geq 2$ if f_{k_0} is a critical slope CM cuspidal form. It is conjectured that $e = 2$ in this case. When f_{k_0} is critical slope non CM cuspidal form, it is conjectured that $e = 1$. See [5, Remark 1] for more details.

3.5 Coleman families

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and let $\mathcal{U} \subset \mathcal{W}^*_L$ be a wide open disc containing a classical point.

Definition 3.5.1. A Coleman family $\boldsymbol{\xi}$ over $\mathcal{U}$ of tame level $N_{\boldsymbol{\xi}}$ with character $\chi_{\boldsymbol{\xi}}$ is a formal power series

$$\sum_{n=1}^{\infty} a_n(\boldsymbol{\xi}) q^n \in \Lambda_{\mathcal{U}}[[q]]$$

satisfying the following properties

1. $a_1(\boldsymbol{\xi}) = 1$ and $a_p(\boldsymbol{\xi}) \in \mathcal{O}_{\mathcal{U}}^{\times}$.

2. For all but finitely many classical weights u contained in $\mathcal{U}$, the specialization

$$\boldsymbol{\xi}_u = \sum_{n=1}^{\infty} a_n(\boldsymbol{\xi})(u) q^n \in \mathcal{O}_L[[q]]$$

is the q-expansion of a classical modular form of weight $k+2$ and level $\Gamma_1(N) \cap \Gamma_0(p)$ with character $\chi_{\boldsymbol{\xi}_u} : (\mathbb{Z}/N_{\boldsymbol{\xi}}\mathbb{Z})^{\times} \longrightarrow L^{\times}$ that is a normalized eigenform for the Hecke operators.

3. For all but finitely many classical weights k contained in $\mathcal{U}$, the specialization of $\chi_{\boldsymbol{\xi}}$ at k coincides with the nebentypus $\chi_{\boldsymbol{\xi}_k}$ of $\boldsymbol{\xi}_k$.

For a Coleman family $\boldsymbol{\xi}$, if $\boldsymbol{\xi}_u$ is old at p, it is a p-stabilisation of a newform ξ_u of level $\Gamma_1(N_{\boldsymbol{\xi}})$.

Definition 3.5.2. We say that a cuspidal eigenform f of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $k+2$ is noble if

1. f is the p-stabilization of normalized cuspidal newform f' of level $\Gamma_1(N)$ such that f' is p-regular.

2. f' is non-critical.

We make the following assumptions.

Assumption 3.5.3. We call a classical point u is good if

1. p does not divide the conductor of $\boldsymbol{\xi}_u$.
2. $\boldsymbol{\xi}$ satisfies one of the following conditions at u:
   (a) when $u \geq 2$, $\boldsymbol{\xi}_u$ is noble and $\boldsymbol{\xi}_u$ is not θ-critical.
   (b) When $u = 1$, then $\boldsymbol{\xi}_1$ is a p-stabilization of a classical p-regular cuspidal weight one newform without p-split real multiplication. Here $\boldsymbol{\xi}_1$ has p-split real multiplication means that it is the weight one theta series attached to a ray class character of a real quadratic field in which p splits.

Remark 3.5.4. The Assumption 3.5.3 (2) makes sure that the eigenform $\boldsymbol{\xi}_u$ is an étale point of $\kappa^{\mathrm{cusp}} \colon \mathcal{C}^{\mathrm{cusp}}_{W,\nu} \longrightarrow W$. We talk about these two assumptions for more details.

1. In the case of Assumption 3.5.3 (2) (a), we see κ^{cusp} is étale at $\boldsymbol{\xi}_u$ from [5, Proposition 2.11].
2. The main result of [7] proves that κ^{cusp} is étale at $\boldsymbol{\xi}_1$ in the case of Assumption 3.5.3 (2) (b).

Lemma 3.5.5. *Let f be a noble eigenform of weight $k+2$ or f is the p-stabilization of a classical p-regular cuspidal weight one newform without p-split real multiplication. Then for any sufficiently small $U_{\boldsymbol{f}}$ containing k, there is a unique Coleman family $\boldsymbol{f}$ over $U_{\boldsymbol{f}}$ such that $\boldsymbol{f}_k = f$.*

Proof. We only need that κ^{cusp} is étale at the point corresponding to f. This follows from Remark 3.5.4. See the discussion in [55, Theorem 4.6.4]. □

Remark 3.5.6. If f corresponds to a point x_0 on $\mathcal{C}^{\mathrm{cusp}}_{W,\nu}$, then we have the affine neighborhood $\mathcal{X}$ of x_0 as in Theorem 3.4.2. It is convenient to identify $\mathcal{X}$ with the Coleman family $\boldsymbol{f}$ through f. So the overconvergent eigenform f_x that corresponds to a point $x \in \mathcal{X}(L)$ is simply the specialisation $\boldsymbol{f}_{\kappa(x)}$ of $\boldsymbol{f}$.

3.6 Big Galois representations

We consider three Coleman families $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ which are of tame levels $N_{\boldsymbol{f}}$, $N_{\boldsymbol{g}}$, $N_{\boldsymbol{h}}$ with characters $\chi_{\boldsymbol{f}}$, $\chi_{\boldsymbol{g}}$, $\chi_{\boldsymbol{h}}$ respectively. These three families are parametrised by connected affinoid dics $U_{\boldsymbol{f}}$, $U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ of $\mathcal{W}_L$ centered at $k \geq 1$, $l \geq 1$ and $m \geq 1$ respectively. We assume k, l and m are good points. Write $\boldsymbol{\xi}$ to be one of them. Let $\mathcal{U}_{\boldsymbol{\xi}}$ be a wide open subset containing $U_{\boldsymbol{\xi}}$ in $\mathcal{W}^*$. Set $\Gamma_{\boldsymbol{\xi}} = \Gamma_1(N_{\boldsymbol{\xi}}) \cap \Gamma_0(p)$ and let $\mathcal{D}'(\mathcal{U}_{\boldsymbol{\xi}})$ and $\mathcal{D}(\mathcal{U}_{\boldsymbol{\xi}})$ as in Section 3.2. Define

$$V^*(\boldsymbol{\xi}) \hookrightarrow H^1_c(\Gamma_{\boldsymbol{\xi}}, \mathcal{D}(\mathcal{U}_{\boldsymbol{\xi}}))^{\leq h}(-\boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}}) \otimes_{\mathcal{O}_{\mathcal{U}_{\boldsymbol{\xi}}}} \mathcal{O}_{U_{\boldsymbol{\xi}}}$$

to be the maximal $\mathcal{O}_{U_{\boldsymbol{\xi}}}$-submodule of $H^1_c(\Gamma_{\boldsymbol{\xi}}, \mathcal{D}'(\mathcal{U}_{\boldsymbol{\xi}}))^{\leq h}(-\boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}}) \otimes_{\mathcal{O}_{\mathcal{U}_{\boldsymbol{\xi}}}} \mathcal{O}_{U_{\boldsymbol{\xi}}}$ on which the Hecke operators T_ℓ, T_p and $\langle d \rangle$ act as multiplication by $a_\ell(\boldsymbol{\xi})$, $a_p(\boldsymbol{\xi})$ and $\chi_{\boldsymbol{f}}(d)$ for each prime $\ell \nmid Np$ and every $d \in (\mathbb{Z}/N_{\boldsymbol{\xi}}\mathbb{Z})^\times$. Here $\boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}} \colon G_{\mathbb{Q}_p} \longrightarrow \Lambda^\times_{U_{\boldsymbol{\xi}}}$ which is given by $\boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}} = \kappa_{U_{\boldsymbol{\xi}}} \circ \chi_{\mathrm{cyc}}$, $\kappa_{U_{\boldsymbol{\xi}}} \colon \mathbb{Z}_p^\times \longrightarrow \Lambda^\times_{U_{\boldsymbol{\xi}}}$. Dually, define

$$H^1(\Gamma_{\boldsymbol{\xi}}, \mathcal{D}'(\mathcal{U}_{\boldsymbol{\xi}}))^{\leq h}(1) \otimes_{\mathcal{O}_{\mathcal{U}_{\boldsymbol{\xi}}}} \mathcal{O}_{U_{\boldsymbol{\xi}}} \twoheadrightarrow V(\boldsymbol{\xi})$$

to be the maximal $\mathcal{O}_{U_{\boldsymbol{\xi}}}$ quotient of $H^1(\Gamma_{\boldsymbol{\xi}}, \mathcal{D}'(\mathcal{U}_{\boldsymbol{\xi}}))^{\leq h}(1) \otimes_{\mathcal{O}_{\mathcal{U}_{\boldsymbol{\xi}}}} \mathcal{O}_{U_{\boldsymbol{\xi}}}$ on which the dual Hecke operators T'_ℓ, T'_p and $\langle d \rangle'$ act as multiplication by $a_\ell(\boldsymbol{\xi})$, $a_p(\boldsymbol{\xi})$ and $\chi_{\boldsymbol{f}}(d)$ for each prime $\ell \nmid Np$ and every $d \in (\mathbb{Z}/N_{\boldsymbol{\xi}}\mathbb{Z})^\times$. For the definition of Hecke operators, dual Hecke operators, diamond operators and dual diamond operators, we refer to [3, Section 1.1] and [23, Section 4.1.1 and Section 4.1.2].

If u is good point of $\boldsymbol{\xi}$, we have $V^{\cdot}(\boldsymbol{\xi})$ is free $\mathcal{O}_{U_{\boldsymbol{\xi}}}$-module of rank 2. There is a perfect duality of $\mathcal{O}_{\boldsymbol{\xi}}$-modules

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}} \colon V(\boldsymbol{\xi}) \times V^*(\boldsymbol{\xi}) \longrightarrow \mathcal{O}_{U_{\boldsymbol{\xi}}}.$$

See [55, Theorem 4.6.6], Theorem 3.4.2 and [24].

3.7 Galois representations of modular forms

We recall the definition of the Galois representations for $\boldsymbol{\xi}_u$ and ξ_u.

3.7.1 Galois representation I

Let $u \geq 2$ be a classical point in $U_{\boldsymbol{\xi}}$. We first construct the Galois representations for $\boldsymbol{\xi}_u$. We consider the affine modular curve $Y = Y_1(N_{\boldsymbol{\xi}}, p)$ of level $\Gamma_1(N_{\boldsymbol{\xi}}) \cap \Gamma_0(p)$ over $\mathbb{Q}$. Assume $N_{\boldsymbol{\xi}}p \geq 5$. Let $\pi \colon E \longrightarrow Y$ be the universal elliptic curve. We consider the p-adic sheaves

$$\mathscr{L}_{u-2} := \mathrm{TSym}^{u-2}\mathbf{R}^1\pi_*\mathbb{Z}_p(1) \text{ and } \mathscr{S}_{u-2} := \mathrm{Symm}^{u-2}\mathbf{R}^1\pi_*\mathbb{Z}_p$$

on Y. Here $\mathbf{R}^1\pi_*$ is the first right derivative of π_*, $\mathrm{TSym}^i\cdot$ and $\mathrm{Symm}^i\cdot$ denote the submodule of symmetric tensors and the symmetric quotient of the i-th tensor power of $\cdot$ respectively. Set $Y_{\bar{\mathbb{Q}}} := Y \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$. Let $L/\mathbb{Q}_p$ be a finite extension containing the Fourier coefficients of $\boldsymbol{\xi}_u$, the values of $\chi_{\boldsymbol{\xi}_u}$ and $\mathcal{O}_L$ be its ring of integers. Then we define

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \mathscr{L}_{u-2}(1)) \otimes L \twoheadrightarrow V(\boldsymbol{\xi}_u)$$

to be the maximal L-quotient on which the dual Hecke operator T'_ℓ and the diamond operator $\langle d\rangle'$ act as multiplication by $a_\ell(\boldsymbol{\xi}_u)$ and $\chi_{\boldsymbol{\xi}_u}(d)$ respectively, for all $\ell \nmid N_{\boldsymbol{\xi}}p$ and $d \in (\mathbb{Z}/N_{\boldsymbol{\xi}}p\mathbb{Z})^\times$. Dually let

$$V^*(\boldsymbol{\xi}_u) \hookrightarrow H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \mathscr{S}_{u-2}) \otimes L$$

be the maximal L-submodule on which the Hecke operators T_ℓ and $\langle d\rangle$ act as multiplication by $a_\ell(\boldsymbol{\xi}_u)$ and $\chi_{\boldsymbol{\xi}_u}(d)$ respectively, for every prime $\ell \nmid N_{\boldsymbol{\xi}}p$ and unit $d \in (\mathbb{Z}/N_{\boldsymbol{\xi}}p\mathbb{Z})^\times$. For the definition of Hecke operator, dual Hecke operator, the diamond operator and dual diamond operator, we refer to [23, Section 2.3 and Section 2.3.1] for more details. There are canonical specialisation isomorphisms

$$\rho_u \colon V^{\cdot}(\boldsymbol{\xi}) \otimes_u L \cong V^{\cdot}(\boldsymbol{\xi}_u),$$

where $\cdot \otimes_u L$ denotes base change along evaluation at u on $\mathcal{O}_{\boldsymbol{\xi}}$. See [12, Proposition 6.6] for more details. For ξ_u, we working with $Y_1(N_{\boldsymbol{\xi}})$ and we assume that $N_{\boldsymbol{\xi}} \geq 4$. Similarly we can define $V(\xi_u)$ and $V^*(\xi_u)$. For $h = \boldsymbol{\xi}_u, \xi_u$, the morphism $\mathscr{L}_{u-2} \otimes \mathscr{S}_{u-2} \longrightarrow \mathbb{Z}_p$ arising from the relative Weil pairing induces a pairing

$$\langle \cdot, \cdot \rangle_h \colon V(h) \otimes_L V^*(h) \longrightarrow L,$$

which is perfect by Poincaré duality [57, Chapter VI]. If $\boldsymbol{\xi}_u$ is p-old, from [20, Section 2.2], we have a map

$$\Pi_{\boldsymbol{\xi}_u,*} \colon H^1_{\text{ét}}(Y_1(N_{\boldsymbol{\xi}}, p), \mathscr{L}_{u-2}) \longrightarrow H^1_{\text{ét}}(Y_1(N_{\boldsymbol{\xi}}), \mathscr{L}_{u-2})$$

induces an isomorphism between $V(\boldsymbol{\xi}_u)$ and $V(\xi_u)$. Its adjoint $\Pi^*_{\boldsymbol{\xi}_u}$ with respect to $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ and $\langle \cdot, \cdot \rangle_{\xi_u}$, yields an isomorphism between $V^*(\xi_u)$ and $V^*(\boldsymbol{\xi}_u)$.

3.7.2 Galois representation II

Now we assume that $u_0 = 1$, we set

$$V^*(\boldsymbol{\xi}_1) = V^*(\boldsymbol{\xi}) \otimes_1 L, \qquad V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}) \otimes_1 L,$$

here $\cdot \otimes_1 L$ denotes the base change along evaluation at 1 on $\mathcal{O}_{\boldsymbol{\xi}}$. We denote by

$$\rho_1 \colon V^*(\boldsymbol{\xi}) \longrightarrow V^*(\boldsymbol{\xi}_1)$$

the projection map. The weight one specialisation of the pairing $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$ yields a canonical perfect duality

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_1} \colon V(\boldsymbol{\xi}_1) \otimes_L V^*(\boldsymbol{\xi}_1) \longrightarrow L.$$

From [20, Proposition 2.2], we know $V^*(\boldsymbol{\xi}_1)$ and $V(\boldsymbol{\xi}_1)$ is the Deligne–Serre Artin representation of $G_{\mathbb{Q}}$ associated with $\boldsymbol{\xi}_1$ and its dual respectively.

3.7.3 p-adic Hodge theory

We denote $\bullet$ = cris, st, dR, $\cdot = \emptyset, *$ and $h = \boldsymbol{\xi}_u$, ξ_u, then we define $V^{\cdot}_{\bullet}(h) = \mathbf{D}_{\bullet}(V^{\cdot}(h))$ in Section 2.2. Since p does not divide the conductor of h, $V^{\cdot}(h)$ is semistable at p, we can identify $V^{\cdot}_{\mathrm{st}}(h)$ with $V^{\cdot}_{\mathrm{dR}}(h)$ (cf. [40, Proposition 8.11]), hence we can equip $V^{\cdot}_{\mathrm{dR}}(h)$ with the action of semistable Frobenius φ. We also have the following perfect pairing

$$\langle \cdot, \cdot \rangle_h : V_{\bullet}(h) \otimes V^*_{\bullet}(h) \longrightarrow L.$$

If we extend L to contain a primitive $N_{\boldsymbol{\xi}}$-th root of unit, then from the comparison theorem of Faltings [36] and Tsuji [70], we have

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(h) = S_u(\Gamma_1(N_{\boldsymbol{\xi}p^r}), \chi_h)_{h^\omega} \quad \text{and } \mathrm{Fil}^1 V^*_{\mathrm{dR}}(h) = S_u(\Gamma_1(N_{\boldsymbol{\xi}p^r}), \chi_h)_h,$$

here $r = 1$, if $h = \boldsymbol{\xi}_u$ and $r = 0$ if $\boldsymbol{\xi}_u$ is p-old and $h = \xi_u$ and h^ω is the image of h under the Atkin–Lehner operator. We refer to [23, Section 2.5] for more details. We choose ω_h is the element of $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(h)$ corresponding to h. At the good point, we know the $V(h)$ is crystalline, then $V^*_{\mathrm{dR}}(h) = V^*_{\mathrm{cris}}(h)$. Hence we have the decomposition

$$V_{\mathrm{dR}}(h) = \mathrm{Fil}^1 V^*_{\mathrm{dR}}(h) \oplus V^*_{\mathrm{dR}}(h)^{\varphi=\alpha_h}$$

here $\alpha_h = a_p(h)$. So we can define $\eta^\alpha_h \in V^*_{\mathrm{dR}}(h)^{\varphi=\alpha_h}$ to be the unique element such that for each ξ in $S_u(N_{\boldsymbol{\xi}p^r}, \chi_h)_h$, we have

$$\langle \eta^\alpha_f, \omega_\xi \rangle_f = \frac{\langle \xi^\omega, f^\omega \rangle_f}{\langle f^\omega, f^\omega \rangle_f}.$$

.

3.8 Triangulation

We now recall the definition of trianguline representations which is first introduced in Colmez paper [34, Section 0.4]. We also refer to [17] and [54].

Definition 3.8.1. If V is a p-adic representation of $G_{\mathbb{Q}_p}$ with coefficients in A of rank d, then we say that V is trianguline if the (φ, Γ)-module $\mathbf{D}^\dagger_{\mathrm{rig},A}(V)$ has a filtration

$$0 = \mathrm{Fil}^0(\mathbf{D}^\dagger_{\mathrm{rig},A}(V)) \subset \mathrm{Fil}^1(\mathbf{D}^\dagger_{\mathrm{rig},A}(V)) \subset \cdots \subset \mathrm{Fil}^{d-1}(\mathbf{D}^\dagger_{\mathrm{rig},A}(V)) \subset \mathrm{Fil}^d(\mathbf{D}^\dagger_{\mathrm{rig},A}(V)) = \mathbf{D}^\dagger_{\mathrm{rig},A}(V)$$

by (φ, Γ)-submodules over $\mathscr{R}_A$ such that

$$\mathrm{Fil}^i(\mathbf{D}^\dagger_{\mathrm{rig},A}(V)) / \mathrm{Fil}^{i-1}(\mathbf{D}^\dagger_{\mathrm{rig},A}(V))$$

is rank 1 (φ, Γ)-module over $\mathscr{R}_A$.

3.8.1 In the case of big Galois representations

Let $\mathscr{R}_{\boldsymbol{\xi}} = \mathscr{R}_L \widehat{\otimes}_L \mathcal{O}_{U_{\boldsymbol{\xi}}}$. We write $D^{\cdot}(\boldsymbol{\xi}) := \mathbf{D}^\dagger_{\mathrm{rig},\mathcal{O}_{\boldsymbol{\xi}}}(V^{\cdot}(\boldsymbol{\xi})|_{G_{\mathbb{Q}_p}})$ the associated (φ, Γ)-modules over $\mathscr{R}_{\boldsymbol{\xi}}$.

Definition 3.8.2. We write $\mathscr{R}_{\boldsymbol{\xi}}(\alpha)$ for the free rank 1 (φ, Γ)-module over $\mathscr{R}_{\boldsymbol{\xi}}$ with basis vector e such that $\varphi(e) = \alpha \cdot e$ and $\gamma e = e$ for all $\gamma \in \Gamma$.

From the work of [54, Theorem 1.13], we know that in the case of Assumption 3.5.3 (1) and (2) (a), one can find an affinoid disc $U_{\boldsymbol{\xi}}$ containing u and we have the following exact sequence

$$0 \longrightarrow \mathbf{D}^{\cdot}(\boldsymbol{\xi})^+ \longrightarrow \mathbf{D}^{\cdot}(\boldsymbol{\xi}) \longrightarrow \mathbf{D}^{\cdot}(\boldsymbol{\xi})^- \longrightarrow 0$$

of (φ, Γ)-module over $\mathscr{R}_{\boldsymbol{\xi}}$. More precisely, we have the following descriptions of $\mathbf{D}(\boldsymbol{\xi})^{\pm}$:

1. $\mathbf{D}(\boldsymbol{\xi})^+ \cong \mathscr{R}_{\boldsymbol{\xi}}(a_p(\boldsymbol{\xi}) \cdot \chi_{\boldsymbol{\xi}}(p)^{-1})[1 + \boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}}]$
2. $\mathbf{D}(\boldsymbol{\xi})^- \cong \mathscr{R}_{\boldsymbol{\xi}}(a_p(\boldsymbol{\xi})^{-1})$.

See [55, Theorem 6.3.2] for more details. We can see that the duality $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$ between $V(\boldsymbol{\xi})$ and $V^*(\boldsymbol{\xi})$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}} \colon \mathbf{D}(\boldsymbol{\xi}) \otimes_{\mathscr{R}_{\boldsymbol{\xi}}} \mathbf{D}^*(\boldsymbol{\xi}) \longrightarrow \mathscr{R}_{\boldsymbol{\xi}}$$

on the associated (φ, Γ)-modules, which is perfect between $\mathbf{D}(\boldsymbol{\xi})^{\pm}$ and $\mathbf{D}^*(\boldsymbol{\xi})^{\mp}$. See [20, Section 2.3.2].

If $u = 1$ and the condition assumption 3.5.3 (2) (b) is satisfied, then $\boldsymbol{\xi}$ is ordinary and the restriction of $V(\boldsymbol{\xi})$ to $G_{\mathbb{Q}_p}$ is nearly-ordinary: there exists a short exact sequence

$$\Delta_{\boldsymbol{\xi}} \colon V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-$$

of $\mathcal{O}_{U_{\boldsymbol{\xi}}}[G_{\mathbb{Q}_p}]$-modules, where $V(\boldsymbol{\xi})^+$ is the submodule on which $G_{\mathbb{Q}_p}$ acts via the character

$$\chi_{\boldsymbol{\xi}} \cdot \boldsymbol{\kappa}_{\xi} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1} \colon G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\boldsymbol{\xi}}^*,$$

and $V(\boldsymbol{\xi})^- = V(\boldsymbol{\xi})/V(\boldsymbol{\xi})^+$ is unramified. Here $\breve{a}_p(\boldsymbol{\xi})$ is the unramified character sending the arithmetic Frobenius to $a_p(\boldsymbol{\xi})$. The étaleness of the cuspidal eigencurve $\mathscr{C}^{\mathrm{cusp}}(N_{\boldsymbol{\xi}}) \longrightarrow \mathcal{W}_L$ at $\boldsymbol{\xi}_1$ guarantees that the $G_{\mathbb{Q}_p}$-modules $V(\boldsymbol{\xi})^{\pm}$ are free of rank one over $\mathcal{O}_{\boldsymbol{\xi}}$. Applying $\mathbf{D}^{\dagger}_{\mathrm{rig}, \mathscr{R}_{\boldsymbol{\xi}}}$ to $V(\boldsymbol{\xi})^{\pm}$, we have the triangulation of $V(\boldsymbol{\xi})$. See [20, Section 2.3.2] for more details.

3.8.2 In the case of modular forms

Since the representation $V^{\cdot}(h)|_{G_{\mathbb{Q}_p}}$ is semistable, the representations $V^{\cdot}(h)|_{G_{\mathbb{Q}_p}}$ are trianguline. See [17, Theorem 3.3.4]. Using the functor $\mathbf{D}^{\dagger}_{\mathrm{rig}}$ in Theorem 2.3.6, we define

$$D^{\cdot}(h) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^{\cdot}(h)|_{G_{\mathbb{Q}_p}}),$$

There exists a short exact sequence for $D^{\cdot}(h)$.

$$0 \longrightarrow D^{\cdot}(h)^+ \longrightarrow D^{\cdot}(h) \longrightarrow D^{\cdot}(h)^- \longrightarrow 0$$

of (φ, Γ)-modules over $\mathscr{R}_L$. The perfect Poincaré duality $\langle \cdot, \cdot \rangle_h$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_h \colon D(h) \otimes_{\mathscr{R}_L} D^*(h) \longrightarrow \mathscr{R}_L,$$

from which $D(h)^{\pm}$ is perfect to $D^*(h)^{\mp}$. The base change of $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$ along evaluation at a good point u corresponds to the pairing $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$. See [20, Section 2.3.2] for more details.

3.8.3 Specialization

We have the specialisation map

$$\rho_u \colon \mathbf{D}^{\cdot}(\boldsymbol{\xi}) \otimes_u L \cong D^{\cdot}(\boldsymbol{\xi}_u).$$

See the discussion in Section 2.5. We have

$$\mathbf{D}^{\cdot}(\boldsymbol{\xi})^{\pm} \otimes_u L \cong D^{\cdot}(\boldsymbol{\xi}_u)^{\pm}$$

at the good point. This is the result of [49, Proposition 6.4.5].

3.9 Differentials

We have the following theorem from [55, Theorem 6.4.1 and Corollary 6.4.3], when it satisfies Assumption 3.5.3 (2) (a).

Theorem 3.9.1. *After shrinking $U_{\boldsymbol{\xi}}$, there is a canonical $\mathcal{O}_{U_{\boldsymbol{\xi}}}$-basis vector*

$$\eta_{\boldsymbol{\xi}} \in (\mathbf{D}(\boldsymbol{\xi})^-)^{\Gamma=1}$$

such that for every classical weight $u \geq 0$ in $U_{\boldsymbol{\xi}}$, the specialisation of $\eta_{\boldsymbol{\xi}}$ at u coincides with the image of the differential form η_{ξ_u} attached to the normalized eigenform ξ_u. There is a $\mathcal{O}_{U_{\boldsymbol{\xi}}}$-basis vector

$$\omega_{\boldsymbol{\xi}} \in (\mathbf{D}(\boldsymbol{\xi})^+(-1 - \boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}} - \chi_{\boldsymbol{\xi}}(p)))^{\Gamma=1}$$

such that for every classical weight $u \geq 0$ in $U_{\boldsymbol{\xi}}$, the specialisation of $\omega_{\boldsymbol{\xi}}$ at u coincides with the image of the differential form ω_{ξ_u} attached to the normalized eigenform ξ_u.

If $u = 1$, then under the assumption 3.5.3 (2) (b), we refer to [23, Section 5] for more details. We define $\omega_{\boldsymbol{\xi}_1}$ and $\eta_{\boldsymbol{\xi}_1}$ in $V^*_{\mathrm{dR}}(\boldsymbol{\xi}_1)$ to be the weight one specialisations of $\omega_{\boldsymbol{\xi}}$ and $\eta_{\boldsymbol{\xi}}$ respectively. In this case, we set $\eta^{\alpha}_{\boldsymbol{\xi}_1} = \eta_{\boldsymbol{\xi}_1}$.

CHAPTER 4

Selmer complexes

4.1 Some basic of derived category

4.1.1 The complex category and derived category

We always assume that R is a commutative ring. Let $\mathscr{K}(R)$ be the category of complexes of R-modules. We say that a complex $C^\bullet = (C^n, d^n_{C^\bullet})$ is cohomologically finitely generated if $H^*(C^\bullet)$ are finitely generated modules over R for all $i \in \mathbb{Z}$. We denote by $\mathscr{K}_{\mathrm{ft}}(R)$ the subcategory of $\mathscr{R}(R)$ consisting of all cohomologically finitely generated complexes. We write $\mathscr{D}(R)$ and $\mathscr{D}_{\mathrm{ft}}(R)$ for the corresponding derived categories. We write $[\cdot]\colon \mathscr{K}_*(R) \longrightarrow \mathscr{D}_*(R)$ for the obvious functors when $* \in \{\emptyset, \mathrm{ft}\}$. We denote by $\mathscr{K}^{[a,b]}_{\mathrm{ft}}(R)$ the subcategory of $\mathscr{K}_{\mathrm{ft}}(R)$ consisting of objects whose cohomology groups are concentrated in degrees $[a, b]$. Consider the complex of R-modules of the form

$$0 \longrightarrow P_a \longrightarrow P_{a+1} \longrightarrow \cdots \longrightarrow P_b \longrightarrow 0,$$

where each P_i is a finitely generated projective R-module. We call these complexes of R-modules perfect. If R is noetherian, we denote by $\mathscr{D}^{[a,b]}_{\mathrm{perf}}$ the full subcategory of $\mathscr{D}_{\mathrm{ft}}(R)$ consisting of objects quasi-isomorphic to perfect complexes concentrated in degree $[a, b]$.

Now we let $C^\bullet = (C^n, d^n_{C^\bullet})_{n\in\mathbb{Z}}$ be a complex of R-modules. For any $m \in \mathbb{Z}$, we denote by $C^\bullet[m]$ the complex defined by $C^\bullet[m]^n = C^{n+m}$ and $d^n_{C^\bullet[m]}(x) = (-1)^m d_{C^\bullet}(x)$. For each m, the truncation $\tau_{\geq m} C^\bullet$ of $C^\bullet$ is the complex

$$0 \longrightarrow \operatorname{coker}(d^{m-1}_{C^\bullet}) \longrightarrow C^{m+1} \longrightarrow C^{m+2} \longrightarrow \cdots .$$

Hence from the definition

$$H^i(\tau_{\geq m} C^\bullet) = \begin{cases} 0, & \text{if } i < m, \\ H^i(C^\bullet), & \text{if } i \geq m. \end{cases}$$

4.1.2 The mapping cone

If $f\colon A^\bullet \longrightarrow B^\bullet$ is a morphism of complexes, the cone of f is defined to be the complex

$$\operatorname{Cone}(f) = A^\bullet[1] \oplus B^\bullet,$$

with differentials

$$d^n_{\operatorname{Cone}(f)}(a_{n+1}, b_n) = (-d^{n+1}_{A^\bullet}(a_{n+1}), f(a_{n+1}) + d^n_{B^\bullet}(b_n)).$$

We have a canonical distinguished triangle

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \operatorname{Cone}(f) \longrightarrow A^\bullet[1].$$

4.1.3 The complex $T^\bullet(A^\bullet)$

Let $A^\bullet = (A^n, d^n)$ be a complex equipped with a morphism $\varphi\colon A^\bullet \longrightarrow A^\bullet$. The total complex

$$T^\bullet(A^\bullet) = \mathrm{Tot}(A^\bullet \xrightarrow{\varphi-1} A^\bullet)$$

is given by $T^n(A^\bullet) = A^{n-1} \oplus A^n$ with differential

$$d^n_{T^\bullet(A^\bullet)}(a_{n-1}, a_n) = (d^{n-1}a_{n-1} + (-1)^n(\varphi-1)a_n, d^n a_n)$$

for $(a_{n-1}, a_n) \in T^n(A^\bullet)$.

4.2 Cohomology of (φ, Γ)-module

We recall that $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Let Δ be the p-torsion subgroup of Γ, we know it is trivial if $p \neq 2$. We choose a topological generator $\gamma \in \Gamma$. For a (φ, Γ)-module D over $\mathscr{R}_L$, we can define

$$C^\bullet_\gamma(D)\colon D \xrightarrow{\gamma-1} D,$$

where the first term is placed in degree 0. We denote by $H^i_\gamma(D)$ its cohomology group. we define a complex $C^\bullet_{\varphi,\gamma}(D)$ concentrated in degree $[0,2]$ by

$$C^\bullet_{\varphi,\gamma}(D)\colon = [D \xrightarrow{d_1} D \oplus D \xrightarrow{d_2} D].$$

with $d_1(x)\colon = ((\gamma-1)x, (\varphi-1)x)$ and $d_2(x,y)\colon = (\varphi-1)x - (\gamma-1)y$. These definitions are independent of the choice of γ. This complex conincides with the complex of Fontaine–Herr [42], [43]. The cohomology of D is defined by

$$H^i(D) = H^i(C^\bullet_{\varphi,\gamma}(D)).$$

For (φ, Γ)-modules D_1, D_2 over $\mathscr{R}_L$, we can define a cup product pairing:

$$H^{q_1}(D_1) \times H^{q_2}(D_2) \longrightarrow H^{q_1+q_2}(D_1 \otimes D_2).$$

When $(q_1, q_2) = (0,1)$, $(1,1)$, the pairing is defined by

$$H^0(D_1) \times H^1(D_2) \longrightarrow H^1(D_1 \otimes D_2), \quad (a, [x,y]) \longmapsto [a \otimes x, a \otimes y],$$

and

$$H^1(D_1) \times H^1(D_2) \longrightarrow H^2(D_1 \otimes D_2), \quad ([x,y],[x',y']) \longmapsto [y \otimes \varphi(x') - x \otimes \gamma(y')].$$

See [61, Section 2.2]. The following theorem is proved in [53, Theorem 0.2].

Theorem 4.2.1. *Let D be a (φ, Γ)-module over $\mathscr{R}_L$. Then $H^i(D)$ satisfies:*

1. *$H^i(D) = 0$ if $i \neq 0$, 1, 2.*
2. *$H^i(D)$ are all finite-dimensional L-vector spaces.*
3. *$\sum_{i=0}^{2} (-1)^i \dim_L H^i(D) = -\operatorname{rank} D$.*
4. *There is a pairing $\langle \cdot, \cdot \rangle$ given by*

$$\langle \cdot, \cdot \rangle : H^i(D) \times H^{2-i}(D^*(1)) \longrightarrow L,$$

This pairing is perfect for $i = 0$, 1 and 2.

We also have the following theorem.

Theorem 4.2.2. *Let V be an L-representation of $G_{\mathbb{Q}_p}$. Then we have the following isomorphisms*

$$H^i(\mathbf{D}^\dagger_{rig}(V)) = H^i(\mathbb{Q}_p, V)$$

which are functorial in V and compatible with cup products.

Proof. See [53, Theorem 0.1]. □

4.3 Analogues of Bloch–Kato subspaces

We first recall the generalization of the Bloch–Kato subspaces in term of (φ,Γ)-modules. We define

$$\begin{aligned} H^1_f(D)\colon &= \ker(H^1(D) \longrightarrow H^1_\gamma(D\otimes_{\mathscr{R}_L}\mathscr{R}_L[1/t'])),\\ H^1_g(D)\colon &= \ker(H^1(D) \longrightarrow H^1_\gamma(\mathscr{D}_{\mathrm{dif}}(D))). \end{aligned}$$

We have the following theorem.

Theorem 4.3.1. *For $*=\{f,g\}$ and V is a L-representation of $G_{\mathbb{Q}_p}$, we have*

$$H^1_*(\mathbb{Q}_p,V) = H^1_*(\mathbf{D}^\dagger_{rig}(V)).$$

Proof. See [8, Proposition 1.4.2]. □

We can also extend the definitions of Bloch–Kato maps to the (φ,Γ)-modules.

Theorem 4.3.2. *If L is a finite extension of $\mathbb{Q}_p$ and D is a de Rham (φ,Γ)-module over $\mathscr{R}_L$, then there are Bloch–Kato exponential and dual exponential maps*

$$\exp\colon \mathscr{D}_{dR}(D)/\operatorname{Fil}^0\mathscr{D}_{dR}(D) \longrightarrow H^1(D), \qquad \exp^*\colon H^1(D)\longrightarrow \operatorname{Fil}^0\mathscr{D}_{dR}(D).$$

Proof. See [61, Section 2.3, Section 2.4]. □

4.4 The cohomology of families of (φ,Γ)-modules

Let $\mathbf{D}$ be a (φ,Γ)-module over $\mathscr{R}_{L,A}$, here $\mathscr{R}_{L,A}$ is defined in Section 2.5. We define

$$C^\bullet_\gamma(\mathbf{D})\colon \mathbf{D}\xrightarrow{\gamma-1}\mathbf{D},$$

where the first term is placed in degree 0. We denote its cohomology group $H^i_\gamma(\mathbf{D})$. We consider the total complex

$$C^\bullet_{\varphi,\gamma}(\mathbf{D}) = \operatorname{Tot}\left(C^\bullet_\gamma(\mathbf{D})\xrightarrow{\gamma-1}C^\bullet_\gamma(\mathbf{D})\right).$$

More explicitly,

$$C^\bullet_{\varphi,\gamma}(\mathbf{D})\colon = [\mathbf{D}\xrightarrow{d_1}\mathbf{D}\oplus\mathbf{D}\xrightarrow{d_2}\mathbf{D}].$$

with $d_1(x)\colon = ((\gamma-1)x,(\varphi-1)x)$ and $d_2(x,y)\colon = (\varphi-1)x-(\gamma-1)y$. This complex conincides with the complex of Fontaine–Herr [42], [43]. If $\mathbf{D}_1$ and $\mathbf{D}_2$ are (φ,Γ)-modules over $\mathscr{R}_A$, we denote by

$$\cup_\gamma\colon C^\bullet_\gamma(\mathbf{D}_1)\otimes C^\bullet_\gamma(\mathbf{D}_2)\longrightarrow C^\bullet_\gamma(\mathbf{D}_1\otimes\mathbf{D}_2)$$

the bilinear map

$$\cup_\gamma(x_n\otimes y_m) = \begin{cases} x_n\otimes\gamma^n(y_m), & \text{for } x_n\in C^n_\gamma(\mathbf{D}_1), y_m\in C^m_\gamma(\mathbf{D}_2) \text{ and } n+m=0 \text{ or } 1,\\ 0, & \text{if } n+m\geq 2.\end{cases}$$

Then we have a bilinear map induced by $\cup_\gamma$.

$$\cup_{\varphi,\gamma}\colon C^\bullet_{\varphi,\gamma}(\mathbf{D}_1)\otimes C^\bullet_{\varphi,\gamma}(\mathbf{D}_2)\longrightarrow C^\bullet_{\varphi,\gamma}(\mathbf{D}_1\otimes\mathbf{D}_2).$$

Explicitly, the map is given by

$$\cup_{\varphi,\gamma}((x_{n-1},x_n)\otimes(y_{m-1},y_m)) = (x_n\cup_\gamma y_{m-1}+(-1)^m x_{n-1}\cup_\gamma\varphi(y_m), x_n\cup_\gamma y_m),$$

where $(x_{n-1},x_n)\in C^n_{\varphi,\gamma}(\mathbf{D}_1)$ and $(y_{m-1},y_m)\in C^m_{\varphi,\gamma}(\mathbf{D}_2)$.

For a (φ,Γ)-module $\mathbf{D}$ over $\mathscr{R}_{L,A}$, we denote by $\mathbf{R}\Gamma(\mathbf{D}) = [C^\bullet_{\varphi,\gamma}(\mathbf{D})]$ the corresponding object in the derived category $\mathscr{D}(A)$. The cohomology of $\mathbf{D}$ is defined by

$$H^i(\mathbf{D})\colon = \mathbf{R}^i\Gamma(\mathbf{D}) = H^i(C^\bullet_{\varphi,\gamma}(\mathbf{D})).$$

From [49, Theorem 4.4.3 (2)], we have the following base change in term of the complex $C^\bullet_{\varphi,\gamma}(\mathbf{D})$: if $f\colon A \longrightarrow B$ is a morphism of L-affinoid algebras, then

$$C^\bullet_{\varphi,\gamma}(\mathbf{D}) \otimes^{\mathbf{L}}_{\mathscr{R}_{L,A}} \mathscr{R}_{L,B} \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(\mathbf{D}\widehat{\otimes}_{\mathscr{R}_{L,A}}\mathscr{R}_{L,B}).$$

In particular, if $x \in \operatorname{Spm}(A)$ and $E_x = A/m_x$, where m_x the maximal ideal in $\operatorname{Spm}(A)$ corresponding to x, then

$$C^\bullet_{\varphi,\gamma}(\mathbf{D}) \otimes^{\mathbf{L}}_{\mathscr{R}_{L,A}} E_x \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(\mathbf{D}_x).$$

Theorem 4.4.1. *Let $\mathbf{D}$ be a (φ,Γ)-module over $\mathscr{R}_{L,A}$ where A is an affinoid algebra over L.*

1. *We have $\mathbf{R}\Gamma(\mathbf{D}) \in \mathscr{D}^{[0,2]}_{perf}(A)$.*
2. *we have the Euler–Poincaré formula, which means that*

$$\sum_{i=0}^{2}(-1)^i \dim_A H^i(\mathbf{D}) = -\operatorname{rank}_{\mathscr{R}_{L,A}} \mathbf{D}.$$

3. *We have the isomorphism in the derived category $\mathscr{D}(A)$*

$$\mathbf{R}\Gamma(\mathbf{D}^*(1)) \cong \mathbf{R}\operatorname{Hom}_A(\mathbf{R}\Gamma(\mathbf{D}), A)[-2].$$

In particular, we have

$$\cup\colon H^i(\mathbf{D}) \otimes H^{2-i}(\mathbf{D}^*(1)) \longrightarrow A.$$

Proof. See [49, Theorem 4.4.5]. □

From [67, Theorem 2.8], we have the following theorem similar to Theorem 4.2.2.

Theorem 4.4.2. *Let V be a p-adic representation of $G_{\mathbb{Q}_p}$ with coefficients in A. Then there are functorial isomorphisms*

$$H^i(\mathbb{Q}_p, V) \cong H^i(\mathbf{D}^\dagger_{rig,A}(V))$$

4.5 The complex $K^\bullet(V)$

In this section, we recall the construction of complex $K^\bullet(V)$ following [10, Section 1.5] and [11, Section 2.5.1].

Suppose that M is a topological $G_{\mathbb{Q}_p}$-module. We denote by $C^\bullet(G_{\mathbb{Q}_p}, M)$ the complex of continuous cochains with coefficients in M. Let V be a p-adic representation of $G_{\mathbb{Q}_p}$ with coefficients in an affinoid algebra A over L and $\mathbf{D}$ be $\mathbf{D}^\dagger_{\mathrm{rig},A}(V)$. We have the ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ constructed in Equation 2.1.2 and we set $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},A} := \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\widehat{\otimes}_{\mathbb{Q}_p}A$. We have the following exact sequence from [11, Equation 32]

$$0 \longrightarrow A \longrightarrow \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},A} \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},A} \longrightarrow 0. \tag{4.5.1}$$

We set $V^\dagger_{\mathrm{rig}} := V \otimes_A \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},A}$ and consider the the continuous cochain $C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})$. Equation 4.5.1 gives an exact sequence

$$0 \longrightarrow C^\bullet(G_{\mathbb{Q}_p}, V) \longrightarrow C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \xrightarrow{\varphi-1} C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \longrightarrow 0.$$

From this exact sequence, we define

$$K^\bullet(V) = \operatorname{Tot}(C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \xrightarrow{\varphi-1} C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})),$$

here Tot means the total complex.

We consider the map

$$\alpha_V\colon C^\bullet_\gamma(\mathbf{D}) \longrightarrow C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})$$

given by

$$\begin{cases} \alpha_V(x_0) = x_0, \quad x_0 \in C^0_\gamma(\mathbf{D}) \\ \alpha_V(x_1)(g) = \dfrac{g|_\Gamma - 1}{\gamma - 1}(x_1), \quad x_1 \in C^1_\gamma(\mathbf{D}), \end{cases}$$

where $g \in G_{\mathbb{Q}_p}$. We can check that α_V is a morphism of complexes which commutes with φ. Then using functorial, we have a morphism, which we still denote by α_V

$$\alpha_V : C^\bullet_{\varphi,\gamma}(\mathbf{D}) \longrightarrow K^\bullet(V).$$

At the same time, we have a morphism

$$\xi_V : C^\bullet(G_{\mathbb{Q}_p}, V) \longrightarrow K^\bullet(V),$$
$$x_n \longmapsto (0, x_n),$$

Here $(0, x_n) \in C^{n-1}(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \oplus C^n(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})$. We have the following proposition about α_V and ξ_V.

Proposition 4.5.1. *The map α_V and ξ_V are quasi-isomorphisms.*

Proof. See [9, Proposition A.3]. □

4.6 Selmer complexes

Let $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ be three Coleman families with tame characters satisfying the relation $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}} = 1$ and let a_f, a_g and a_h be their respective slopes. We set $\mathcal{O}_{\boldsymbol{fgh}} := \mathcal{O}_{\boldsymbol{f}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{g}} \widehat{\otimes}_L \mathcal{O}_{\boldsymbol{h}}$. Consider the character $\Xi\colon G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}^\times_{\boldsymbol{fgh}}$ defined by

$$\Xi := (4 - \kappa_{\boldsymbol{f}} - \kappa_{\boldsymbol{g}} - \kappa_{\boldsymbol{h}}) \circ \chi^{1/2}_{\mathrm{cyc}}.$$

Here $\chi^{1/2}_{\mathrm{cyc}} = \langle \chi_{\mathrm{cyc}} \rangle^{1/2} \omega^{1/2}$, ω is the Teichmuller character and $\langle \chi_{\mathrm{cyc}} \rangle$ is the restriction of χ_{cyc} to $1 + p\mathbb{Z}_p$. We set

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) := V(\boldsymbol{f}) \widehat{\otimes}_L V(\boldsymbol{g}) \widehat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathcal{O}_{\boldsymbol{fgh}}} \Xi,$$

which is Kummer self-dual(See [23, Section 7.2] for more details).

For three modular forms $f \in S_k(N_f, \chi_f)$, $g \in S_l(N_g, \chi_g)$ and $h \in S_m(N_h, \chi_h)$ satisfying $\chi_f \cdot \chi_g \cdot \chi_h = 1$, we set

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h)((4 - k - l - m)/2).$$

This is also Kummer self-dual. Furthermore, for all $\omega = (k, l, m)$ a triple weights, the specialisation maps induce isomorphisms

$$\rho_\omega : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_\omega L \cong V(f_k, g_l, h_m).$$

See [23, Section 7.2].

4.6.1 Definition of Selmer complexes

We set a finite set of primes S such that $p \in S$. Let $G_{\mathbb{Q},S}$ denote the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside S and ∞. For each prime ℓ, we fix a decomposition group at ℓ which we identify with $G_\ell = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ and denote by I_ℓ the inertia subgroup of G_ℓ. We will write V_p for the restriction of V on G_p. Let $\mathscr{R}_{\boldsymbol{fgh}} : = \mathscr{R}_L \otimes \mathcal{O}_{\boldsymbol{fgh}}$. We set $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) := \mathbf{D}^\dagger_{\mathrm{rig}, \mathscr{R}_{\boldsymbol{fgh}}}(V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_p)$ and $D(f, g, h) := \mathbf{D}^\dagger_{\mathrm{rig}, \mathscr{R}_L}(V(f, g, h)_p)$. Then $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \mathbf{D}(\boldsymbol{f}) \otimes_{\mathscr{R}_L} \mathbf{D}(\boldsymbol{g}) \otimes_{\mathscr{R}_L} \mathbf{D}(\boldsymbol{h}) \otimes_{\mathscr{R}_{\boldsymbol{fgh}}} \mathscr{R}_{\boldsymbol{fgh}}(\Xi)$ and

$$D(f, g, h) = D(f) \otimes_{\mathscr{R}_L} D(g) \otimes_{\mathscr{R}_L} D(h) \otimes_{\mathscr{R}_L} \mathscr{R}_L(\chi^{(4-k-l-m)/2}_{\mathrm{cyc}}).$$

We denote by $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^+ : = \mathbf{D}(\boldsymbol{f})^+ \otimes_{\mathscr{R}_L} \mathbf{D}(\boldsymbol{g}) \otimes_{\mathscr{R}_L} \mathbf{D}(\boldsymbol{h}) \otimes_{\mathscr{R}_{\boldsymbol{fgh}}} \mathscr{R}_{\boldsymbol{fgh}}(\Xi)$. Similarly, we define $D(f, g, h)^+$. We then write $\mathbf{D}^+ = \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^+$ or $D(f, g, h)^+$, $\mathbf{V} = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ or $V(f, g, h)$ and $\mathbf{D} = \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ or $D(f, g, h)$. Set $U^+_p(\mathbf{V}, \mathbf{D}^+) = C^\bullet_{\varphi,\gamma}(\mathbf{D}^+)$. We compose the quasi-isomorphism α_V in Proposition 4.5.1 with the canonical morphism $U^+_p(\mathbf{V}, \mathbf{D}^+) \longrightarrow C^\bullet_{\varphi,\gamma}(\mathbf{D})$ to obtain a map

$$i^+_p : U^+_p(\mathbf{V}, \mathbf{D}) \longrightarrow K^\bullet_p(\mathbf{V}).$$

For each $\ell \in S \backslash \{p\}$, we set

$$U^+_\ell(\mathbf{V}) := \left[\mathbf{V}^{I_\ell} \overset{\mathrm{Frob}_\ell - 1}{\longrightarrow} \mathbf{V}^{I_\ell} \right],$$

where Frob_ℓ denotes the geometric Frobenius. Set $K_\ell^\bullet(\mathbf{V}) := C^\bullet(G_\ell, \mathbf{V})$. We then define a morphism

$$i_\ell^+ : U_\ell^+(\mathbf{V}) \longrightarrow K_\ell^\bullet(\mathbf{V})$$

by

$$\begin{aligned} i_\ell^+(x) &= x && \text{in degree } 0,\\ (i_\ell^+(x))(\mathrm{Frob}_\ell) &= x && \text{in degree } 1. \end{aligned}$$

We then set

$$U_S^+(\mathbf{V}, \mathbf{D}^+) = U_p^+(\mathbf{V}, \mathbf{D}^+) \oplus \left(\bigoplus_{\ell \in S \setminus \{p\}} U_\ell^+(\mathbf{V}) \right)$$

and

$$K_S^\bullet(\mathbf{V}) = K_p(\mathbf{V}) \oplus \left(\bigoplus_{\ell \in S \setminus \{p\}} K_\ell^\bullet(\mathbf{V}) \right).$$

Then we have a diagram

$$\begin{array}{ccc} C^\bullet(G_{\mathbb{Q},S}, V) & \xrightarrow{\mathrm{res}_S} & K_S^\bullet(\mathbf{V}) \\ & & \uparrow{\scriptstyle i_S^+} \\ & & U_S^+(\mathbf{V}, \mathbf{D}^+), \end{array}$$

where $i_S^+ = (i_\ell^+)_{\ell \in S}$ and res_S denotes the localization map.

Definition 4.6.1. The selmer complex associated to these data is defined as

$$C^\bullet(\mathbf{V}, \mathbf{D}^+) = \mathrm{Cone}\left[C^\bullet(G_{\mathbb{Q},S}, \mathbf{V}) \oplus U_S^+(\mathbf{V}, \mathbf{D}^+) \overset{\mathrm{res}_S - i_S^+}{\longrightarrow} K_S^\bullet(\mathbf{V}) \right][-1].$$

Each element $x_f \in C^i(\mathbf{V}, \mathbf{D}^+)$ can be written as a triple

$$x_f = (x, (x_\ell^+)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}),$$

where $x \in C^i(G_{\mathbb{Q},S}, \mathbf{V})$, $(x_\ell^+)_{\ell \in S} \in U_S^+(\mathbf{V}, \mathbf{D}^+)^i$ and $(\lambda_\ell)_{\ell \in S} \in K_S^{i-1}(\mathbf{V})$. Furthermore, x_f is a cocycle if and only if

$$d(x) = 0, \quad d((x_\ell^+)_{\ell \in S}) = 0, \quad i_S((x_\ell^+)_{\ell \in S}) = \mathrm{res}_S(x) + d(\lambda_\ell)_{\ell \in S}.$$

Definition 4.6.2. We denote by $\mathbf{R}\Gamma(\mathbf{V}, \mathbf{D}^+)$ the class of $C^\bullet(\mathbf{V}, \mathbf{D}^+)$ in the derived category of A-modules, here $A = \mathcal{O}_{\boldsymbol{fgh}}$ or L and define $H^i(\mathbf{V}, \mathbf{D}^+) := \mathbf{R}^i\Gamma(\mathbf{V}, \mathbf{D}^+)$. For each cocycle $x \in C^i(\mathbf{V}, \mathbf{D}^+)$, we write $[x]$ for the class of x in $H^i(\mathbf{V}, \mathbf{D}^+)$.

We define

$$Z^\bullet := \mathrm{Cone}\left(\tau_{\geq 2} C^\bullet(G_{\mathbb{Q},S}, A(1)) \overset{\mathrm{res}_S}{\longrightarrow} \tau_{\geq 2} K_S^\bullet(A(1)) \right)[-1].$$

We then have a canonical quasi-isomorphism

$$r_S \colon Z^\bullet \cong A[-3].$$

See [63, Section 5.4.1] for more details. We can define a morphism of complexes

$$\cup_{K,p} \colon K_p^\bullet(\mathbf{V}) \otimes K_p^\bullet(\mathbf{V}) \longrightarrow K_p^\bullet(A(1)).$$

See more details from [11, Proposition 1.1.5] and [13, Section 2.7]. We have the following map

$$C^\bullet(\mathbf{V}, \mathbf{D}^+) \otimes_A C^\bullet(\mathbf{V}, \mathbf{D}^+) \longrightarrow Z^\bullet \tag{4.6.1}$$

which is induced by the cup product

$$(x, (x_\ell^+), (\lambda_\ell)) \cup_{K,p} (y, (y_\ell^+), (\mu_\ell)) = (x \cup_{K,p} y, (\lambda_\ell \cup_{K,p} i_S^+(y_\ell^+) + (-1)^{\deg(x)} \mathrm{res}_S(x) \cup_{K,p} \mu_\ell)).$$

See [63, Proposition 1.3.2]. Following Nekovar, we define a pairing

$$\cup_{\mathbf{V},\mathbf{D}^+} \colon C^\bullet(\mathbf{V}, \mathbf{D}^+) \otimes_A C^\bullet(\mathbf{V}, \mathbf{D}^+) \longrightarrow A[-3]$$

to be the composition map in Equation 4.6.1 and $r_S \colon Z^\bullet \longrightarrow A[-3]$. We then have a morphism

$$\cup_{\mathbf{V},\mathbf{D}^+} \colon H^1(\mathbf{V}, \mathbf{D}^+) \otimes_A H^2(\mathbf{V}, \mathbf{D}^+) \longrightarrow A. \tag{4.6.2}$$

4.7 p-adic height pairing

We construct a p-adic heights following [11, Section 3.2]. We also refer to [13, Section 3, Section 4.3] for more details.

In this section, we denote $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by $\mathbf{V}$, $V(f,g,h)$ by V, $\mathbf{D}^+(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by $\mathbf{D}^+$ and $D(f,g,h)^+$ by D^+. The specialisation map ρ_{ω_0} gives isomorphisms

$$\rho_{\omega_0}\colon \mathbf{R}\Gamma(\mathbf{V},\mathbf{D}^+)\otimes^{\mathbf{L}}_{\mathcal{O}_{\boldsymbol{fgh}},\omega_0} L \cong \mathbf{R}\Gamma(V,D^+) \tag{4.7.1}$$

and

$$\rho_{\omega_0}\otimes \mathrm{id}\colon \mathbf{R}\Gamma(\mathbf{V},\mathbf{D}^+)\otimes^{\mathbf{L}}_{\mathcal{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2 \cong \mathbf{R}\Gamma(V,D^+)\otimes_L \mathscr{I}/\mathscr{I}^2. \tag{4.7.2}$$

We consider the exact triangle

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \mathcal{O}_{\boldsymbol{fgh}}/\mathscr{I}^2 \longrightarrow L \longrightarrow \mathscr{I}/\mathscr{I}^2[1],$$

in $\mathscr{D}_{\mathrm{ft}}(\mathcal{O}_{\boldsymbol{fgh}})$. Applying $\mathbf{R}\Gamma(\mathbf{V},\mathbf{D}^+)\otimes^{\mathbf{L}}_{\mathcal{O}_{\boldsymbol{fgh}}}$ in $\mathscr{D}_{\mathrm{ft}}(\mathcal{O}_{\boldsymbol{fgh}})$, we have a morphism

$$\mathbf{R}\Gamma(\mathbf{V},\mathbf{D}^+)\otimes^{\mathbf{L}}_{\mathcal{O}_{\boldsymbol{fgh}},\omega_0} L \longrightarrow \mathbf{R}\Gamma(\mathbf{V},\mathbf{D}^+)\otimes^{\mathbf{L}}_{\mathcal{O}_{\boldsymbol{fgh}},\omega_0} \mathscr{I}/\mathscr{I}^2[1].$$

Using Equation 4.7.1 and 4.7.2, we get the map

$$\widetilde{\boldsymbol{\beta}}_{\boldsymbol{fgh}}\colon \mathbf{R}\Gamma(V,D^+) \longrightarrow \mathbf{R}\Gamma(V,D^+)[1]\otimes_L \mathscr{I}/\mathscr{I}^2.$$

Furthermore, we have the following map

$$\widetilde{\beta}_{\boldsymbol{fgh}}\colon H^1(V,D^+) \longrightarrow H^2(V,D^+)\otimes_L \mathscr{I}/\mathscr{I}^2.$$

We then define the p-adic height pairing

$$\langle\!\langle \cdot,\cdot\rangle\!\rangle_{\boldsymbol{fgh}}\colon H^1(V,D^+)\otimes_L H^1(V,D^+) \longrightarrow \mathscr{I}/\mathscr{I}^2 \tag{4.7.3}$$

to be the composition of the cup product pairing in Equation 4.6.2

$$\cup_{V,D^+}\otimes \mathscr{I}/\mathscr{I}^2\colon H^2(V,D^+)\otimes_L H^1(V,D^+)\otimes_L \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{I}/\mathscr{I}^2$$

with the morphism

$$\widetilde{\beta}_{\boldsymbol{fgh}}\otimes \mathrm{id}\colon H^1(V,D^+)\otimes_L H^1(V,D^+) \longrightarrow H^2(V,D^+)\otimes_L H^1(V,D^+)\otimes_L \mathscr{I}/\mathscr{I}^2.$$

4.8 The extended Selmer group

We set $D(f,g,h)^- := D(f,g,h)/D(f,g,h)^+$.

Lemma 4.8.1. *We have the following exact sequence*

$$0 \longrightarrow H^0(D(f,g,h)^-) \longrightarrow H^1(V(f,g,h),D(f,g,h)^+) \longrightarrow \mathrm{Sel}(\mathbb{Q},V(f,g,h)) \longrightarrow 0.$$

Proof. We follow the strategy in [10, Proposition 11]. By the definition, $H^1(V(f,g,h),D(f,g,h)^+)$ is the kernel of the map

$$H^1_S(\mathbb{Q},V(f,g,h))\oplus\left(\bigoplus_{v\in S-\{p\}} H^1_f(\mathbb{Q}_v,V(f,g,h))\right)\oplus H^1(D(f,g,h)^+) \longrightarrow \bigoplus_{v\in S} H^1(\mathbb{Q}_v,V(f,g,h)).$$

We need to prove the following two things

1. $H^0(D(f,g,h)^-) = \ker\left(H^1(D(f,g,h)^+) \longrightarrow H^1(\mathbb{Q}_p,V(f,g,h)\right)$.
2. $H^1_f(\mathbb{Q}_p,V(f,g,h)) = \mathrm{Im}\left(H^1(D(f,g,h)^+) \longrightarrow H^1(\mathbb{Q}_p,V(f,g,h))\right)$.

Since the Hodge–Tate weight of $D(f,g,h)^+$ is negative and $D(f,g,h)^-$ is non-negative and that we prove

$$\mathbf{D}_{\mathrm{cris}}(V(f,g,h))^{\varphi=1}=0,$$

in the proof of Lemma 7.1.2, we prove the result from [68, Lemma 2.3.2]. □

Definition 4.8.2. We call $H^1(V(f,g,h),D(f,g,h)^+)$ to be the extended Selmer group in this case.

CHAPTER 5

Perrin-Riou big logarithm

5.1 Balanced Selmer group

We follow the strategy in [23, Section 7.2] to define the balanced Selmer group. We also refer to [44, Section 4.3]. We recall that $\boldsymbol{D}(\boldsymbol{\xi}) = \mathbf{D}^{\dagger}_{\mathrm{rig},\mathcal{O}_{\boldsymbol{\xi}}}(V(\boldsymbol{\xi}))$. There is a filtration on $\boldsymbol{D}(\boldsymbol{\xi})$ by $F^0\boldsymbol{D}(\boldsymbol{\xi}) = \boldsymbol{D}(\boldsymbol{\xi})$, $F^1\boldsymbol{D}(\boldsymbol{\xi}) = \boldsymbol{D}(\boldsymbol{\xi})^+$, $F^2\boldsymbol{D}(\boldsymbol{\xi}) = 0$. These filtrations induce a filtration on $\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ defined by

$$F^n\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) := \left[\sum_{p+q+r=n} F^p\boldsymbol{D}(\boldsymbol{f})\widehat{\otimes}_{\mathscr{R}_L}F^q\boldsymbol{D}(\boldsymbol{g})\widehat{\otimes}_{\mathscr{R}_L}F^r\boldsymbol{D}(\boldsymbol{h})\right]\otimes_{\mathscr{R}_{\boldsymbol{fgh}}}\mathscr{R}_{\boldsymbol{fgh}}(\Xi).$$

From the definition, we know $F^4\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = 0$ and $F^0\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. We set

$$\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \boldsymbol{D}(\boldsymbol{f})^-\widehat{\otimes}_{\mathscr{R}_L}\boldsymbol{D}(\boldsymbol{g})^+\widehat{\otimes}_{\mathscr{R}_L}\boldsymbol{D}(\boldsymbol{h})^+\otimes_{\mathscr{R}_{\boldsymbol{fgh}}}\mathscr{R}_{\boldsymbol{fgh}}(\Xi),$$

and defines similarly $\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g$ and $\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h$, then

$$\mathrm{gr}^2\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\oplus\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g\oplus\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h.$$

Definition 5.1.1. We define the balanced Selmer group of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to be

$$H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) := H^1(F^2\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$$

Lemma 5.1.2. *The map*

$$H^1_{bal}(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$$

induced by $F^2\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ *is injective.*

Proof. We consider the short exact sequence

$$0 \longrightarrow F^2\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow 0,$$

which induces the morphisms of cohomology groups

$$H^0(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2) \longrightarrow H^1(F^2\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})). \tag{5.1.1}$$

From the Theorem 4.2.2 and the definition of balanced Selmer group, the Equation 5.1.1 becomes

$$H^0(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2) \longrightarrow H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})).$$

We then study the cohomology group $H^0(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2)$. We can write $\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ as

$$\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\oplus\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g\oplus\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h\oplus\mathbf{D}(\boldsymbol{f})^-\widehat{\otimes}_{\mathscr{R}_L}\mathbf{D}(\boldsymbol{g})^-\widehat{\otimes}_{\mathscr{R}_L}\mathbf{D}(\boldsymbol{h})^-\widehat{\otimes}_{\mathscr{R}_{\boldsymbol{fgh}}}\mathscr{R}_{\boldsymbol{fgh}}(\Xi).$$

We only deal with $\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$, the other cases are similar. We know that

$$\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \mathscr{R}_{\boldsymbol{fgh}}(\chi_{\boldsymbol{f}}^{-1}(p)\cdot a_p(\boldsymbol{f})a_p(\boldsymbol{g})^{-1}a_p(\boldsymbol{h})^{-1})[(6+\kappa_{\boldsymbol{f}}-\kappa_{\boldsymbol{g}}-\kappa_{\boldsymbol{h}})\circ\chi_{\mathrm{cyc}}^{1/2}].$$

The Γ-action on $\boldsymbol{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is given by $(6+\kappa_{\boldsymbol{f}}-\kappa_{\boldsymbol{g}}-\kappa_{\boldsymbol{h}})\circ\chi_{\mathrm{cyc}}^{1/2}$, this can not be 1. Hence from the definition of the cohomology group, we have $H^0(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/F^2) = 0$. So we prove the result. □

We have the following lemma.

Lemma 5.1.3. *If $\omega = (k, l, m)$ is a balanced classical triple, i.e. $k < l+m$, $l < k+m$ and $m < k+l$, then*

$$H^1_{bal}(\mathbb{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) = H^1_f(\mathbb{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)),$$

where $H^1_f(\mathbb{Q}_p, \cdot)$ is the Bloch–Kato finite subspace.

Proof. See [44, Lemma 4.21]. □

5.2 Perrin-Riou's big logarithms

If $\omega = (k, l, m)$ satisfying any sum of two is greater than the other, we can construct an element

$$\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m} \in \mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$$

as in Section 3.7.3 and we refer to [21, Section 3.1.3] for more details. We have the Bloch–Kato logarithm

$$\log_p \colon H^1_f(\mathbb{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \xrightarrow{\sim} [\mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))]^{\vee}.$$

Lemma 5.1.3 enables us to have

$$\log_{p,f} \colon H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow L$$

as the composition of $\log_p$ and evaluation at $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. On the other hand, if $\omega = (k, l, m)$ satisfying $k \geq l + m$, we have the Bloch–Kato dual exponential map

$$\exp^*_p \colon H^1(\mathbb{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow \mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)).$$

We also let $\exp^*_{p,f}$ be the composition of $\exp^*_p$ with the valuation at $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. Here we need to identify $\mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ as a subspace of $[\mathrm{Fil}^0\, \mathbf{D}_{\mathrm{dR}}(V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))]^{\vee}$ via the Poincaré duality. See [23, Section 7.3] for more details.

Now we let $\omega = (k, l, m)$ be a classical weight, we write $\alpha_{\boldsymbol{f}_k} = a_p(\boldsymbol{f})_k$ and $\beta_{\boldsymbol{f}_k} = \chi_{\boldsymbol{f}}(p)p^{k-1}/\alpha_{\boldsymbol{f}_k}$ and similarly for $\boldsymbol{g}$ and $\boldsymbol{h}$. We also set $c_\omega \colon= (k + l + m - 2)/2$.

Proposition 5.2.1. *There is a unique $\mathcal{O}_{\boldsymbol{fgh}}$-module morphism*

$$\mathscr{L}_f \colon H^1_{bal}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathcal{O}_{\boldsymbol{fgh}}$$

such that, for all $\omega = (k, l, m) \in \Sigma^{cl}$ with $\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z} \neq p^{c_\omega}$ and $Z \in H^1_{bal}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$,

$$\mathscr{L}_f(Z)_\omega = (p-1)\alpha_{\boldsymbol{f}_k} \cdot \frac{1 - \dfrac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^{c_\omega}}}{1 - \dfrac{\alpha_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^{c_\omega}}} \cdot \begin{cases} \dfrac{(-1)^{c_\omega - k}}{(c_\omega - k)!} \log_{p,f}(Z_\omega) & \text{if } \omega \in \Sigma_{bal} \\ (k - c_\omega - 1)! \exp^*_{p,f}(Z_\omega) & \text{if } \omega \in \Sigma_f, \end{cases}$$

where the subscription ω means the specialization at ω. Moreover, we have

$$\begin{array}{ccc} H^1_{bal}(\mathbb{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) & \xrightarrow{\mathscr{L}_p} & \mathcal{O}_{\boldsymbol{fgh}} \\ \downarrow & \nearrow & \\ H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) & & \end{array}$$

we still denote the dot map by $\mathscr{L}_p$.

Proof. In the non-ordinary case, it is constructed in [55]. We also refer to [44, Proposition 4.22] for more details. □

CHAPTER 6

The triple product of p-adic L-function for finite slope families

6.1 Garrett–Rankin triple product L-function

We recall the Garrett–Rankin triple product L-function in this section for reader's convenience. We use the same notation as in [35, Section 4.1].

Let $f \in S_k(N_f, \chi_f)$, $g \in S_l(N_g, \chi_g)$ and $h \in S_m(N_h, \chi_h)$ be three normalized primitive cuspidal eigenforms which satisfy that

$$\chi_f \cdot \chi_g \cdot \chi_h = 1.$$

We denote by $\mathbb{Q}_{f,g,h}$ the field generated by $a_n(f)$, $a_n(g)$, $a_n(h)$ over $\mathbb{Q}$. We write $N := \mathrm{lcm}(N_f, N_g, N_h)$.

The Garrett–Rankin triple product L-function $L(f \otimes g \otimes h, s)$ is defined by the Euler product

$$L(f \otimes g \otimes h, s) = \prod_p L^{(p)}(f \otimes g \otimes h, p^{-s})^{-1}.$$

For the primes $p \nmid N$, the local factor $L^{(p)}(f \otimes g \otimes h, T)$ is a polynomial

$$\begin{aligned} L^{(p)}(f \otimes g \otimes h, T) :=& (1 - \alpha_f \alpha_g \alpha_h T) \times (1 - \alpha_f \alpha_g \beta_h T) \\ & \times (1 - \alpha_f \beta_g \alpha_h T) \times (1 - \alpha_f \beta_g \beta_h T) \\ & \times (1 - \beta_f \alpha_g \alpha_h T) \times (1 - \beta_f \alpha_g \beta_h T) \\ & \times (1 - \beta_f \beta_g \alpha_h T) \times (1 - \beta_f \beta_g \beta_h T). \end{aligned}$$

In [66], Piatetski-Shapiro and Rallis give the local Euler factors $L^{(p)}(f \otimes g \otimes h, s)$ for the primes $p|N$. They also showed that there exists an Archimedean factor $L^{(\infty)}(f \otimes g \otimes h, s)$ such that the completed L-function

$$\Lambda(f, g, h, s) = L(f \otimes g \otimes h, s) \cdot L^{(\infty)}(f \otimes g \otimes h, s)$$

has the functional equation

$$\Lambda(f, g, h, s) = \varepsilon(f, g, h) \Lambda(f, g, h, k + l + m - 2 - s),$$

where $\varepsilon(f, g, h) \in \{\pm 1\}$. The sign $\varepsilon(f, g, h)$ can be written as the product of local root number $\prod_q \varepsilon_q(f, g, h)$, here $\varepsilon_q(f, g, h) \in \{\pm 1\}$ for all the places q of $\mathbb{Q}$.

Assumption 6.1.1. The local sign $\varepsilon_q(f, g, h) = +1$ for all finite prime.

We know it is satisfied when $(N_f, N_g, N_h) = 1$. From [69], we have

$$\varepsilon(f, g, h) = \begin{cases} -1, & \text{if } (k, l, m) \text{ are balanced,} \\ +1, & \text{if } (k, l, m) \text{ are unbalanced.} \end{cases}$$

Here (k, l, m) is balanced means that any one of them is smaller than the sum of the other two and otherwise, (k, l, m) is unbalanced .

We can relate the central critical value on $L(f \otimes g \otimes h, s)$ to certain trilinear period integrals. We assume that (k, l, m) is unbalanced such that $k = l + m + 2t$ with $t \geq 0$. We consider $\phi \in S_k(N, \chi_\phi)$, we write ϕ^* for the dual form obtained by twisting ϕ by the character χ_f^{-1}. We denote by π_ϕ the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by ϕ. If N is a multiple of N_f, we let $S_k(N, \chi_\phi)[\pi_\phi]$ denote the ϕ-isotypic subspace of $S_k(N, \chi_\phi)$ attached to π_ϕ. We give the definition of the trilinear period.

Definition 6.1.2. The trilinear period attached to

$$(\widetilde{f}, \widetilde{g}, \widetilde{h}) \in S_k(N, \chi_f)[\pi_f] \times S_l(N, \chi_g)[\pi_g] \times S_m(N, \chi_h)[\pi_h]$$

is

$$I(\widetilde{f}, \widetilde{g}, \widetilde{h}) \colon = (\widetilde{f}^*, \delta_l^t \widetilde{g} \times \widetilde{h})_N,$$

here δ_l is the Shimura–Maass operators.

From [35, Theorem 4.2], we have the following result, which is first proved by Harris and Kudla in [41] and refined by Ichino [46] and Watson [72].

Theorem 6.1.3. *Let (f, g, h) be the triple of modular forms with unbalanced weights (k, l, m) with $k = l + m + 2t$ for $t \geq 0$. Then there exists*

1. *holomorphic modular forms $\widetilde{f} \in S_k(N, \chi_f)[\pi_f]$, $\widetilde{g} \in S_l(N, \chi_g)[\pi_g]$ and $\widetilde{h} \in S_m(N, \chi_h)[\pi_h]$.*
2. *The local constant $C_v \in \mathbb{Q}_{f,g,h}$ for $v|N\infty$ only depend on the local components at v of $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$.*

such that

$$\frac{\prod_{v|N\infty} C_v}{\pi^{2k}} \cdot L(f, g, h, \frac{k + l + m - 2}{2}) = |I(\widetilde{f}, \widetilde{g}, \widetilde{h})|^2.$$

There is a choice of $(\widetilde{f}, \widetilde{g}, \widetilde{h})$ such that the constants C_q are all non-zero.

6.2 The triple product p-adic L-function

We use the notations and the assumptions for f, g and h as in the last section. Furthermore, we assume that f has finite slope a. We further assume that $a < 1$ if $k = 2$ and $2a < k - 1$ if $k > 2$.

Remark 6.2.1. In [1, Section 5.2.1], Andreatta and Iovita explain that the assumption on the slope is to make sure that the interpolation formula is valid. The interpolation formula is valid when the point correspond to the p-stabilisation of f is an étale point on the eigencurve.

We let $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$ be as in Theorem 6.1.3 such that all constants C_q are non-zero. We denote by K a finite extension of $\mathbb{Q}_p$ which contains $\mathbb{Q}_{f,g,h}$ and all the values of χ_f, χ_g, χ_h. We denote by $\mathcal{O}_K$ the ring of integer of K. We denote by $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ the overconvergent families of modular forms deforming f, g and h respectively. We also denote by $\widetilde{\boldsymbol{f}}$, $\widetilde{\boldsymbol{g}}$ and $\widetilde{\boldsymbol{h}}$ the overconvergent families of modular forms deforming $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$ respectively as in [35, Section 2.6.1]. We make it more precise. We set $\xi = f,\ g,\ h$, $\boldsymbol{\xi} = \boldsymbol{f},\ \boldsymbol{g},\ \boldsymbol{h}$ and $\widetilde{\xi} = f,\ g,\ h$. There are scalars $\lambda_d \in K$ indexed by the divisors of N/N_ξ and satisfying

$$\widetilde{\xi} = \sum_{d|(N/N_\xi)} \lambda_d \cdot \xi(q^d).$$

The p-stabilization of the modular form $\xi(q^d)$ is the weight k specialization of the formal q-series

$$\boldsymbol{\xi}(q^d) \colon = \sum a_n(\boldsymbol{\xi})(q^d).$$

We then set

$$\widetilde{\boldsymbol{\xi}}(q) \colon = \sum_{d|(N/N_\xi)} \lambda_d \cdot \boldsymbol{\xi}(q^d).$$

We denote by $U_{\boldsymbol{\xi}}$ the rational open subsets centered at u defined over K for these families respectively and as before we write $\Lambda_{\boldsymbol{\xi}} = \Lambda_{U_{\boldsymbol{\xi}}}$ for the bounded by 1 analytic functions on $U_{\boldsymbol{\xi}}$. We write

$$k_{\boldsymbol{f}} \colon \mathbb{Z}_p^\times \longrightarrow \Lambda_{\boldsymbol{f}}^\times, \quad k_{\boldsymbol{g}} \colon \mathbb{Z}_p^\times \longrightarrow \Lambda_{\boldsymbol{g}}^\times, \quad k_{\boldsymbol{h}} \colon \mathbb{Z}_p^\times \longrightarrow \Lambda_{\boldsymbol{h}}^\times$$

are the weights of these families respectively.

Assumption 6.2.2. 1. Suppose that the weights $k_{\boldsymbol{f}}$, $k_{\boldsymbol{g}}$, $k_{\boldsymbol{h}}$ are such that $k_{\boldsymbol{f}} - k_{\boldsymbol{g}} - k_{\boldsymbol{h}}$ is even, which means that there is a weight $u \colon \mathbb{Z}_p^\times \longrightarrow \Lambda_u^\times$ such that $2u = k_{\boldsymbol{f}} - k_{\boldsymbol{g}} - k_{\boldsymbol{h}}$.

2. we can write the weights $k_{\boldsymbol{g}}$, u as $k_{\boldsymbol{g}}(t) = t^a \cdot \chi_{\boldsymbol{g}}(t) \cdot \exp(u_g \log(t))$ and $u(t) = t^b \cdot \epsilon(t) \cdot \exp(u_s \log(t))$, where a and b are integers, $\chi_{\boldsymbol{g}}$, ϵ are finite order cahracters and $\chi_{\boldsymbol{g}}$ is an even character and $u_g \in p\Lambda_g$, $u_s \in p\Lambda_u$.

Remark 6.2.3. The second assumption can be removed using refinement of the vector bundles with marked sections in [59] and [47].

In [1], Andreatta and Iovita constructed a triple product p-adic L-function $\mathscr{L}_p(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})$ attached to the triple $(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})$ of p-adic families of modular forms, of which $\widetilde{\boldsymbol{f}}$ has finite slope at most h. This p-adic L-function is in $\mathcal{K}_f \widehat{\otimes}_{\mathcal{O}_K} \Lambda_g \widehat{\otimes}_{\mathcal{O}_K} \Lambda_h$, where if we write $k_f(t) = t^c \cdot \chi_f(t) \cdot \exp(u_f \log(t))$, and then $\mathcal{K}_f$ is obtained from Λ_f by inverting the elements $\{u_f - n \mid n \in \mathbb{N}\}$. Now let k, l and m are obtained by specializing k_f, k_g and k_h at integral weights in $\mathbb{Z}_{\geq 2}$ and there is a classical weight $t' \geq 0$ with $k - l - m = 2t'$. We assume that $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ are the specialization of $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ at k, l and m and also assume f_k, g_l and h_m are eigenforms of level $\Gamma_1(N)$ and nebentypus χ_f, χ_g, χ_h respectively. We set α_ξ and β_ξ to be the corresponding roots of the Hecke polynomials for the form ξ_u for $\xi_u = f_k$, g_l, h_m. We assume that ξ_u is p-regular. We write

$$L^{\mathrm{alg}}\left(f_k, g_l, h_m, \frac{k+l+m-2}{2}\right) := \frac{\left(\frac{\prod_{q \mid N\infty} C_q}{\pi^{2k}} L(f_k, g_l, h_m, \frac{k+l+m-2}{2})\right)^{\frac{1}{2}}}{\langle f_k^*, f_k^* \rangle}.$$

Following [35, Theorem 1.3], we define

$$\mathcal{E}(g_l, h_m, T) \colon = (1 - p^{t'} \alpha_g \alpha_h T^{-1})(1 - p^{t'} \alpha_g \beta_h T^{-1})(1 - p^{t'} \beta_g \alpha_h T^{-1})(1 - p^{t'} \beta_g \beta_h T^{-1}),$$

$$\mathcal{E}_1(g_l, h_m, T) \colon = 1 - p^{2t'} \alpha_g \beta_g \alpha_h \beta_h T^{-2}, \qquad \mathcal{E}_0(S, T) \colon = 1 - \frac{T}{S}$$

and

$$\mathcal{E}_2(T) \colon = 1 - \frac{\chi_f^{-2}(p) \bar{\alpha}_f T}{p^{k-1}(p+1)}.$$

Then we have the following interpolation formula for $\mathscr{L}_p(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})$ at unbalanced weights.

$$\mathscr{L}_p(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})(k, l, m) = \left(\frac{\mathcal{E}(g_l, h_m, \bar{\alpha}_f) \mathcal{E}_2(\bar{\beta}_f)}{\mathcal{E}_0(\bar{\alpha}_f, \bar{\beta}_f) \mathcal{E}_1(g_l, h_m, \bar{\alpha}_f)} + \frac{\mathcal{E}(g_l, h_m, \bar{\beta}_f) \mathcal{E}_2(\bar{\alpha}_f)}{\mathcal{E}_0(\bar{\beta}_f, \bar{\alpha}_f) \mathcal{E}_1(g_l, h_m, \bar{\beta}_f)} \right) L^{\mathrm{alg}}\left(f_k, g_l, h_m, \frac{k+l+m-2}{2}\right), \tag{6.2.1}$$

where $\bar{\alpha}_f$ and $\bar{\beta}_f$ is the complex conjugation of α_f and β_f respectively.

CHAPTER 7

Explicit reciprocity law

7.1 Diagonal class

The following construction is taken with no modifications from [23, Section 3, Section 8.1]. We reproduce it here for the reader's convenience with the same notation,.

7.1.1 Diagonal class for modular forms

Let $S_i(\mathbb{Z}_p)$ be the set of two-variable homogeneous polynomials of degree i in $\mathbb{Z}_p[x_1, x_2]$. We also let $L_i(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-linear dual of $S_i(\mathbb{Z}_p)$. There are $\mathrm{GL}_2(\mathbb{Z}_p)$-actions on these two sets. From the discussion in [23, Section 3], there is a natural functor $\cdot^{\text{ét}}$ from the category of p-adic representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ to the category of p-adic étale sheaves on $Y_1(N)$. Then one defines

$$\mathscr{L}_i(\mathbb{Z}_p) = L_i(\mathbb{Z}_p)^{\text{ét}} \quad \text{and} \quad \mathscr{S}_i(\mathbb{Z}_p) = S_i(\mathbb{Z}_p)^{\text{ét}}$$

on $Y_1(N)$. Let $\boldsymbol{r} = (r_1, r_2, r_3) \in \mathbb{N}^3$ be a triple of non-negative integers satisfying the following assumption

1. $r_1 + r_2 + r_3 = 2r$ with $r \in \mathbb{N}$.
2. For every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, one has $r_i + r_j > r_k$.

Let $S_{\boldsymbol{r}}$ denote the $\mathrm{GL}_2(\mathbb{Z}_p)$-representation $S_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} S_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} S_{r_3}(\mathbb{Z}_p)$. Then $S_{\boldsymbol{r}}$ is the module of six-variable polynomials in $\mathbb{Z}_p[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ which are homogeneous of degree r_1, r_2 and r_3 in the variables $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{z} = (z_1, z_2)$ respectively. We can check that

$$\mathrm{Det}_N^{\boldsymbol{r}} = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \cdot \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \cdot \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1}$$

is an element in $H^0(\mathrm{GL}_2(\mathbb{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r})$. We set $\mathscr{S}_{\boldsymbol{r}} := \mathscr{S}_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{S}_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{S}_{r_3}(\mathbb{Z}_p)$. Using the injection([23, Equation (40)])

$$H^0(\mathrm{GL}_2(\mathbb{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r}) \longrightarrow H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)),$$

we can view $\mathrm{Det}_N^{\boldsymbol{r}} \in H^0_{\text{ét}}(Y, \mathscr{S}_{\boldsymbol{r}}(r))$. Now we let $p_j \colon Y_1(N)^3 \longrightarrow Y_1(N)$ be the natural projections and we define

$$\mathscr{S}_{[\boldsymbol{r}]} := p_1^* \mathscr{S}_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} p_2^* \mathscr{S}_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} p_3^* \mathscr{S}_{r_3}(\mathbb{Z}_p)$$

and

$$W_{N,\boldsymbol{r}} := H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)).$$

Since $Y_1(N)_{\bar{\mathbb{Q}}}$ is a smooth affine curve over $\bar{\mathbb{Q}}$, we obtain $H^4_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) = 0$, which gives the morphism

$$\mathrm{HS} \colon H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \longrightarrow H^1(\mathbb{Q}, W_{N,\boldsymbol{r}}).$$

from the Hochschild–Serre spectral sequence

$$H^p(\mathbb{Q}, H^q_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{I}_{[\boldsymbol{r}]}(r+2))) \Longrightarrow H^{p+q}_{\text{ét}}(Y_1(N)^3, \mathscr{I}_{[\boldsymbol{r}]}(r+2)).$$

Let $d\colon Y_1(N) \longrightarrow Y_1(N)^3$ be the diagonal embedding. There is a natural isomorphism $d^*\mathscr{S}_{[\boldsymbol{r}]} \cong \mathscr{S}_{\boldsymbol{r}}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$([23, Section 3]). From [56, Theorem 16.1], the map d gives a pushforward map

$$d_*\colon H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)) \longrightarrow H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)),$$

and one defines

$$\tilde{\kappa}_{N,\boldsymbol{r}} = \text{HS} \circ d_*(\text{Det}^{\boldsymbol{r}}_N) \in H^1(\mathbb{Q}, W_{N,\boldsymbol{r}}).$$

to be the diagonal class of level N and weight (r_1+2, r_2+2, r_3+2).

Let $\mathbf{W}_{N,\boldsymbol{r}} = W_{N,\boldsymbol{r}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and let $H^1_g(\mathbb{Q}, \mathbf{W}_{N,\boldsymbol{r}})$ be the geometric Bloch–Kato Selmer group of $\boldsymbol{W}_{N,\boldsymbol{r}}$ over $\mathbb{Q}$. The classes in $H^1_g(\mathbb{Q}, \mathbf{W}_{N,\boldsymbol{r}})$ are unramified at every prime different from p and the restriction at p belong to the geometric subspace in the sense of Bloch–Kato(cf. [26, Section 3]). The result of [65, Theorem 5.9] yields the following fact about the diagonal class.

Proposition 7.1.1. *The class $\tilde{\kappa}_{N,\boldsymbol{r}}$ belongs to $H^1_g(\mathbb{Q}, \boldsymbol{W}_{N,\boldsymbol{r}})$.*

From [23, Equation (44)], we obtain an isomorphism of sheaves

$$\boldsymbol{s}_i\colon \mathscr{S}_i(\mathbb{Q}_p) \cong \mathscr{L}_i(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-i). \tag{7.1.1}$$

We set $V_{N,\boldsymbol{r}} \colon = H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{L}_{[\boldsymbol{r}]}(2-r))$ and $\boldsymbol{V}_{N,\boldsymbol{r}} \colon = V_{N,\boldsymbol{r}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The tensor product of $\boldsymbol{s}_{r_j}$ in Equation 7.1.1 gives an isomorphism $\boldsymbol{s}_{\boldsymbol{r}}\colon \boldsymbol{W}_{N,\boldsymbol{r}} \cong \boldsymbol{V}_{N,\boldsymbol{r}}$. Through this isomorphism, we get a class

$$\kappa_{N,\boldsymbol{r}} = \boldsymbol{s}_{\boldsymbol{r}*}(\tilde{\kappa}_{N,\boldsymbol{r}}) \in H^1_{\text{geo}}(\mathbb{Q}, \boldsymbol{V}_{N,\boldsymbol{r}}).$$

Now let $k = r_1+2$, $l = r_2+2$, $m = r_3+2$. From the Künneth decomposition and projection to the (f,g,h)-isotypic component, we have a morphism of $G_{\mathbb{Q}}$-modules

$$pr_{fgh}\colon \boldsymbol{V}_{N,\boldsymbol{r}} \otimes_{\mathbb{Q}_p} L \twoheadrightarrow V(f,g,h)$$

and then we can define the diagonal class associated to the triple (f,g,h) by

$$\kappa(f,g,h) = pr_{fgh*}(\kappa_{N,\boldsymbol{r}}) \in H^1_{\text{geo}}(\mathbb{Q}, V(f,g,h)).$$

Lemma 7.1.2. *We assume that*

1. $\text{ord}_p(N) \leq 1$.
2. *p does not divide the conductor of χ_f, χ_g and χ_h.*

For $\bullet$ is one of $\{g,f,e\}$, the Bloch–Kato local conditions

$$H^1_\bullet(\mathbb{Q}_p, V(f,g,h)) \hookrightarrow H^1(\mathbb{Q}_p, V(f,g,h))$$

are all equal.

Proof. Here we use the strategy in the proof of [23, Lemma 3.5] and [23, Lemma 9.1].

We know that $V(f,g,h)$ is Kummer self dual, hence from [26, Proposition 3.8], we only need to prove that $H^1_e(\mathbb{Q}_p, V(f,g,h)) = H^1_f(\mathbb{Q}_p, V(f,g,h))$. On the other hand, [26, Corollary 3.8.4] gives that $H^1_f(\mathbb{Q}_p, V(f,g,h))/H^1_e(\mathbb{Q}_p, V(f,g,h)) = \mathbf{D}_{\text{cris}}(V(f,g,h))/(\varphi-1)\mathbf{D}_{\text{cris}}(V(f,g,h))$. So it sufficient to show that

$$\mathbf{D}_{\text{cris}}(V(f,g,h))^{\varphi=1} = 0,$$

For $\xi = f,\ g,\ h$, we denote by ξ^{new} the newform of conductor $N_\xi | N$ and weight $u = k,\ l,\ m$ associated to ξ. We set

$$V = V(f^{\text{new}}) \otimes_L V(g^{\text{new}}) \otimes_L V(h^{\text{new}})((4-k-l-m)/2).$$

From [23, Section 2.4], we have that $V(\xi)$ is isomorphic to the direct sum of finite copies of $V(\xi^{\text{new}})$. Hence we only need to prove the statement after replacing $V(f,g,h)$ with V.

From the assumption here, we have $V(\xi^{\mathrm{new}})|_{G_{\mathbb{Q}_p}}$ is semi-stable, hence $V|_{\mathbb{Q}_p}$ is also semi-stable. One has

$$\mathbf{D}_{\mathrm{st}}(V(\xi^{\mathrm{new}})) = L\cdot \boldsymbol{a}_\xi \oplus L\cdot \boldsymbol{b}_\xi,$$

where $\boldsymbol{a}_\xi$ and $\boldsymbol{b}_\xi$ are φ-eigenvectors with eigenvalues $a_p(\xi^{\mathrm{new}})^{-1}$ and $p^{1-u}\chi_\xi(p)^{-1}a_p(\xi^{\mathrm{new}})$. See [23, Section 2.5]. We define the monodromy operator N_ξ on $\mathbf{D}_{\mathrm{st}}(V(\xi^{\mathrm{new}}))$ to be zero, if $p \nmid N_\xi$ and

$$N_\xi(\boldsymbol{a}_\xi) = \boldsymbol{b}_\xi \quad \text{and} \quad N_\xi(\boldsymbol{b}_\xi) = \boldsymbol{a}_\xi$$

if $p \parallel N_\xi$. We define

$$\mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}_{\mathrm{st}}(V(f^{\mathrm{new}}))\otimes_L \mathbf{D}_{\mathrm{st}}(V(g^{\mathrm{new}}))\otimes_L \mathbf{D}_{\mathrm{st}}(V(h^{\mathrm{new}}))\otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{st}}(\mathbb{Q}_p((4-k-l-m)/2)),$$

then we consider the set $\mathcal{B}_\omega$ which is an L-basis of φ-eigenvectors of $\mathbf{D}_{\mathrm{st}}(V)$. The elements in $\mathcal{B}_\omega$ are given by $\boldsymbol{a}^\xi_\omega$ and $\boldsymbol{b}^\xi_\omega$, where $\xi = \emptyset$, f, g and h. We refer to the proof of [23, Lemma 3.5] for the explicit form. The respective eigenvalues $\mathcal{E}_\omega = \{\alpha^\xi_\omega, \beta^\xi_\omega\}$ are given by

$$\alpha_\omega = \frac{p^{c_\omega - 1}}{a_p(f^{\mathrm{new}})a_p(g^{\mathrm{new}})a_p(h^{\mathrm{new}})}, \qquad \alpha^f_\omega = \frac{p^{c_\omega - k}a_p(f^{\mathrm{new}})}{\chi_f(p)a_p(g^{\mathrm{new}})a_p(h^{\mathrm{new}})}.$$

We can calculate α^g_ω and α^h_ω similarly. Furthermore, β^ξ_ω are defined by

$$p\cdot\alpha^\xi_\omega\cdot\beta^\xi_\omega = 1.$$

We set

$$\varepsilon_\xi = \begin{cases} 0 & \text{if } p \text{ divides the conductor } N_\xi \\ 1 & \text{if } p \text{ does not divide the conductor } N_\xi \end{cases}$$

and also set $\varepsilon_\omega = \varepsilon_f + \varepsilon_g + \varepsilon_h$. According to [58, Theorem 4.5.17 and Theorem 4.6.17], we have

$$|\alpha_\omega|_\infty = p^{(\varepsilon_\omega - 1)/2}, \qquad |\beta_\omega|_\infty = p^{(-\varepsilon_\omega - 1)/2}, \qquad |\alpha^\xi_\omega|_\infty = p^{(\varepsilon_\omega - 2\varepsilon_\xi - 1)/2}, \qquad |\beta^\xi_\omega|_\infty = p^{(2\varepsilon_\xi - \varepsilon_\omega - 1)/2}.$$

Here $|\cdot|_\infty$ denotes the complex absolute value. Now if $\varepsilon_\omega = 0$, we have

$$|\alpha_\omega|_\infty = |\beta_\omega|_\infty = |\alpha^\xi_\omega|_\infty = |\beta^\xi_\omega|_\infty = p^{-1/2}.$$

Hence $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ vanishes. When $\varepsilon_\omega = 2$, we say $\varepsilon_f = 0$, then

$$|\alpha_\omega|_\infty = p^{1/2}, \qquad |\beta_\omega|_\infty = p^{-3/2}$$
$$|\alpha^\xi_\omega|_\infty = \begin{cases} p^{-1/2} & \xi = g^{\mathrm{new}} \text{ and } h^{\mathrm{new}} \\ p^{1/2} & \xi = f^{\mathrm{new}}, \end{cases}$$
$$|\beta^\xi_\omega|_\infty = \begin{cases} p^{-1/2} & \xi = g^{\mathrm{new}} \text{ and } h^{\mathrm{new}} \\ p^{-3/2} & \xi = f^{\mathrm{new}}. \end{cases}$$

In this case, we also have $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ vanishes.

If $\varepsilon_\omega = 1$, we let $\varepsilon_f = 1$. Then

$$|\alpha_\omega|_\infty = 1, \qquad |\beta_\omega|_\infty = p^{-1}$$
$$|\alpha^\xi_\omega|_\infty = \begin{cases} 1 & \xi = g^{\mathrm{new}} \text{ and } h^{\mathrm{new}} \\ p^{-1/2} & \xi = f^{\mathrm{new}}, \end{cases}$$
$$|\beta^\xi_\omega|_\infty = \begin{cases} p^{-1} & \xi = g^{\mathrm{new}} \text{ and } h^{\mathrm{new}} \\ p^{-1/2} & \xi = f^{\mathrm{new}}. \end{cases}$$

Then we have $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ is contained in $L\cdot\boldsymbol{a}_\omega \oplus L\cdot\boldsymbol{a}^g_\omega \oplus L\cdot\boldsymbol{a}^h_\omega$. Considering the operator N on $\mathbf{D}_{\mathrm{st}}(V)$, we get

$$N(r\cdot\boldsymbol{a}_\omega + s\cdot\boldsymbol{a}^g_\omega + t\cdot\boldsymbol{a}^h_\omega) = r\cdot\boldsymbol{a}^f_\omega + s\cdot\boldsymbol{b}^h_\omega + t\cdot\boldsymbol{b}^g_\omega$$

Hence we know that $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1,N=0}$ vanishes.

Finally, if $\varepsilon_w = 3$, we have

$$|\alpha_\omega|_\infty = p, \qquad |\beta_\omega|_\infty = p^{-2},$$
$$|\alpha_\omega^\xi|_\infty = 1, \qquad |\beta_\omega^\xi|_\infty = p^{-1}.$$

So $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$ is contained in $\oplus_\xi L \cdot \boldsymbol{a}_\omega^\xi$. We know that

$$N(\boldsymbol{a}_\omega^\xi) = \boldsymbol{b}_\omega^{\xi'} + \boldsymbol{b}_\omega^{\xi''}$$

for each permutation $(\xi, \xi', \xi'') = (f, g, h)$. Hence $\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1,N=0} = 0$. We prove the argument. □

7.1.2 Big diagonal class

Let $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ be three Coleman families with slopes $a_{\boldsymbol{f}}$, $a_{\boldsymbol{g}}$ and $a_{\boldsymbol{h}}$. We set $\mathrm{ord}_p(a_{\boldsymbol{f}}) > 0$ and $\mathrm{ord}_p(a_{\boldsymbol{\xi}}) < u - 1$. For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ and $\cdot = \emptyset, \prime$, we set $\mathcal{A}^{\cdot}(\boldsymbol{\xi}) := \mathcal{A}^{\cdot}(U_{\boldsymbol{\xi}})$ and $\mathcal{D}^{\cdot}(\boldsymbol{\xi}) := \mathcal{D}^{\cdot}(U_{\boldsymbol{\xi}})$. From the discussion in [23, Section 4.2], we can associate $\mathcal{A}^{\cdot}(\boldsymbol{\xi})$ and $\mathcal{D}^{\cdot}(\boldsymbol{\xi})$ to the étale sheaves $\boldsymbol{\mathcal{A}}^{\cdot}(\boldsymbol{\xi})$ and $\boldsymbol{\mathcal{D}}^{\cdot}(\boldsymbol{\xi})$ on $Y_1(N,p)$. We recall that $\mathsf{T} = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathsf{T}' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$ as before. We let

$$(\mathsf{T} \times \mathsf{T})_0 = \{(t_1, t_2) \in \mathsf{T} \times \mathsf{T} \mid \det(t_1, t_2) \in \mathbb{Z}_p^\times\},$$

where $\det((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$. Let $(\mathsf{T} \times \mathsf{T})^0$ be the complement of $(\mathsf{T} \times \mathsf{T})_0$ in $\mathsf{T} \times \mathsf{T}$. From discussion in [23, Section 8.1], we know that $(\mathsf{T} \times \mathsf{T})_0$ and $(\mathsf{T} \times \mathsf{T})^0$ are open compact subsets of $\mathsf{T} \times \mathsf{T}$, preserved by the diagonal action of $\Gamma_0(p\mathbb{Z}_p)$. We denote by $\mathcal{A}(\boldsymbol{g})\widehat{\otimes}\mathcal{A}(\boldsymbol{h}) := \mathcal{A}(\boldsymbol{g})\widehat{\otimes}_{\mathcal{O}_L}\mathcal{A}(\boldsymbol{h})$ the space of locally analytic functions on $\mathsf{T} \times \mathsf{T}$. From [23, Remark 4.1], we have the orthonormal basis of $\mathcal{A}(\boldsymbol{g})\widehat{\otimes}\mathcal{A}(\boldsymbol{h})$ which gives a decomposition of $\Gamma_0(p\mathbb{Z}_p)$-modules

$$\mathcal{A}(\boldsymbol{g})\widehat{\otimes}\mathcal{A}(\boldsymbol{h}) = (\mathcal{A}(\boldsymbol{g})\widehat{\otimes}\mathcal{A}(\boldsymbol{h}))_0 \oplus (\mathcal{A}(\boldsymbol{g})\widehat{\otimes}\mathcal{A}(\boldsymbol{h}))^0.$$

where $(\mathcal{A}_{\boldsymbol{g}}\widehat{\otimes}\mathcal{A}_{\boldsymbol{h}})_0$ and $(\mathcal{A}_{\boldsymbol{g}}\widehat{\otimes}\mathcal{A}_{\boldsymbol{h}})^0$ consist in locally analytic functions supported on $(\mathsf{T} \times \mathsf{T})_0$ and $(\mathsf{T} \times \mathsf{T})^0$ respectively. Let $\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}}\widehat{\otimes}_{\mathcal{O}_L}\Lambda_{\boldsymbol{g}}\widehat{\otimes}_{\mathcal{O}_L}\Lambda_{\boldsymbol{h}}$, and we define $\kappa_{\boldsymbol{fgh}}^* : \mathbb{Z}_p^\times \longrightarrow \Lambda_{\boldsymbol{fgh}}^\times$ by

$$\kappa_{\boldsymbol{fgh}}^* = (\kappa_{\boldsymbol{f}} + \kappa_{\boldsymbol{g}} + \kappa_{\boldsymbol{h}} - 6)/2.$$

We define the characters $\kappa_{\boldsymbol{f}}^* : \mathbb{Z}_p^\times \longrightarrow \Lambda_{\boldsymbol{fgh}}^\times$ by

$$\kappa_{\boldsymbol{f}}^* = (-\kappa_{\boldsymbol{f}} + \kappa_{\boldsymbol{g}} + \kappa_{\boldsymbol{h}} - 2)/2$$

Similarly, we define $\kappa_{\boldsymbol{g}}^*$ and $\kappa_{\boldsymbol{h}}^*$. We can see $\kappa_{\boldsymbol{fgh}}^* = \kappa_{\boldsymbol{f}}^* + \kappa_{\boldsymbol{g}}^* + \kappa_{\boldsymbol{h}}^*$. We have an Abel–Jacobi map

$$\mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}} : H^0_{\text{ét}}(Y_1(N,p), \boldsymbol{\mathcal{A}}(\boldsymbol{f})' \otimes \boldsymbol{\mathcal{A}}(\boldsymbol{g}) \otimes \boldsymbol{\mathcal{A}}(\boldsymbol{h})(-\kappa_{\boldsymbol{fgh}}^*)) \longrightarrow H^1(\mathbb{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})).$$

We refer to [23, Section 8.1] for more details.

We can find that $\det : \mathbb{Z}_p^2 \times \mathbb{Z}_p^2 \longrightarrow \mathbb{Z}_p$ maps $\mathsf{T}' \times \mathsf{T}$ to $\mathbb{Z}_p^\times$, we let

$$\mathbf{Det} = \mathbf{Det}_{N,p}^{\boldsymbol{fgh}} : \mathsf{T}' \times \mathsf{T} \times \mathsf{T} \longrightarrow \Lambda_{\boldsymbol{fgh}}$$

be the function which vanishes identically on $\mathsf{T}' \times (\mathsf{T} \times \mathsf{T})^0$ and on elements $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in $\mathsf{T}' \times (\mathsf{T} \times \mathsf{T})_0$ takes the value

$$\mathbf{Det}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \det(\boldsymbol{x}, \boldsymbol{y})^{\kappa_{\boldsymbol{h}}^*} \cdot \det(\boldsymbol{x}, \boldsymbol{z})^{\kappa_{\boldsymbol{g}}^*} \cdot \det(\boldsymbol{y}, \boldsymbol{z})^{\kappa_{\boldsymbol{f}}^*}.$$

We can check

$$\mathbf{Det}(\gamma \cdot \boldsymbol{x}, \gamma \cdot \boldsymbol{y}, \gamma \cdot \boldsymbol{z}) = \det(\gamma)^{\kappa_{\boldsymbol{fgh}}^*} \cdot \mathbf{Det}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$$

for every $\gamma \in \Gamma_0(p\mathbb{Z}_p)$. So we can view $\mathbf{Det}$ as an element of $\mathcal{A}'_{\boldsymbol{f}}\widehat{\otimes}\mathcal{A}'_{\boldsymbol{g}}\widehat{\otimes}\mathcal{A}'_{\boldsymbol{h}}(-\kappa_{\boldsymbol{fgh}}^*)$ which is invariant under the diagonal action of $\Gamma_0(p\mathbb{Z}_p)$. Since $\Gamma_0(p\mathbb{Z}_p)$-representation $\mathcal{A}'_{\boldsymbol{f}}\widehat{\otimes}\mathcal{A}_{\boldsymbol{g}}\widehat{\otimes}\mathcal{A}_{\boldsymbol{h}}$ corresponds to the sheaf $\boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\widehat{\otimes}\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\widehat{\otimes}\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}$ on $Y_1(N,p)$, we have

$$\mathbf{Det}_{N,p}^{\boldsymbol{fgh}} \in H^0_{\text{ét}}(Y_1(N,p), \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\widehat{\otimes}\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}}\widehat{\otimes}\boldsymbol{\mathcal{A}}'_{\boldsymbol{h}}(-\kappa_{\boldsymbol{fgh}}^*)).$$

We then define

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \frac{1}{a_p(\boldsymbol{k})} \cdot \mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}}(\mathbf{Det}_{N,p}^{\boldsymbol{fgh}}) \in H^1(\mathbb{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})),$$

Let (k,l,m) be a balanced triple of classical weights, let $\boldsymbol{r}=(r_1,r_2,r_3)=(k-2,l-2,m-2)$ and $r=(r_1+r_2+r_3)/2$. We can define the twisted diagonal class(cf. [23, Equation (157)])

$$\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)\in H^1_g(\mathbb{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)).$$

Then from [23, Theorem 8.1], we have the following result.

Theorem 7.1.3. *For each balanced triple $\omega=(k,l,m)$, we have*

$$(p-1)\alpha_{\boldsymbol{f}_k}\cdot\rho_\omega(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))=(1-\frac{\alpha_{\boldsymbol{f}_\alpha}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}})\cdot\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m).$$

We can show that the big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in H^1_{\mathrm{bal}}(\mathbb{Q}_p,V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ following the proof of [23, Corollary 8.2].

7.2 Reciprocity law

Now let $E'\longrightarrow X_1(N)$ be the universal elliptic curve over the modular curve defined over $\mathbb{Q}_p$. For any $n\geq 0$, we let W_n be the n-th Kuga–Sato variety over $X_1(N)$. See [19, Appendix] for the definition of Kuga–Sato variety. Now we fix a balanced weight (k,l,m). We write $(k,l,m)=(r_1+2,r_2+2,r_3+2)$, $r\colon=\frac{1}{2}(r_1+r_2+r_3)$, and we set

$$W\colon=W_{r_1}\times W_{r_2}\times W_{r_3}.$$

In [35, Section 3.1], Darmon and Rotger construct a homologically trivial cycle which is called the generalized diagonal cycle

$$\Delta_{k,l,m}\in \mathrm{CH}^{r+2}(W)_0\colon=\ker(\mathrm{CH}^{r+2}(W)\xrightarrow{\mathrm{cl}}H^{2r+4}_{\mathrm{dR}}(W/\mathbb{C}_p)).$$

We have the p-adic Abel–Jacobi map

$$\mathrm{AJ}_p\colon \mathrm{CH}^{r+2}(W)_0\longrightarrow[\mathrm{Fil}^{r+2}H^{2r+3}_{\mathrm{dR}}(W/\mathbb{C}_p)]^{\vee}.$$

For the construction of the p-adic Abel–Jacobi map, we refer to [25] and [64, (1.2)]. We consider $f\in S_k(N,\chi_f)$, $g\in S_l(N,\chi_g)$ and $h\in S_m(N,\chi_h)$. We assume that $\chi_f\cdot\chi_g\cdot\chi_h=1$. We denote $k=l+m-2t$ and $c=(k+l+m-2)/2$. In [35, Section 3], one can construct an element

$$\eta_f^{\alpha}\otimes\omega_g\otimes\omega_h\in\mathrm{Fil}^{r+2}H^{2r+3}_{\mathrm{dR}}(W/\mathbb{C}_p).$$

Hence we can consider the valuation of $\mathrm{AJ}_p(\Delta_{k,l,m})$ on $\eta_f^{\alpha}\otimes\omega_g\otimes\omega_h$. Using the finite polynomial cohomology in [45], Huang proves the following p-adic Gross–Zagier formula for a triple modular forms (f,g,h).

Lemma 7.2.1. *We let*

$$\mathcal{E}_1(f)=(1-\beta_f^2\chi_f^{-1}(p)p^{-k}),\quad \mathcal{E}_0(f)=(1-\beta_f^2\chi_f^{-1}(p)p^{1-k})$$

and

$$\mathcal{E}=(1-\beta_f\alpha_g\alpha_hp^{-c})(1-\beta_f\alpha_g\beta_hp^{-c})(1-\beta_f\beta_g\alpha_hp^{-c})(1-\beta_f\beta_g\beta_hp^{-c}).$$

Then we have

$$\mathscr{L}_p(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{g}},\widetilde{\boldsymbol{h}})(k,l,m)=(-1)^{t-1}\frac{\mathcal{E}(f,g,h)}{(t-1)!\mathcal{E}_0(f)\mathcal{E}_1(f)}\times AJ_p(\Delta_{k,l,m})(\eta_f^{\alpha}\otimes\omega_g\otimes\omega_h)$$

Proof. See [44, Theorem 3.9]. □

We know the class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in H^1_{\mathrm{bal}}(\mathbb{Q}_p,V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$. Using the p-adic Gross–Zagier formula above, we can prove the following reciprocity law.

Theorem 7.2.2.

$$\mathscr{L}_{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})))=\mathscr{L}_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}).\tag{7.2.1}$$

Proof. See [44, Theorem 4.23]. □

7.3 Some discussions in exceptional case

We do not prove any results in exceptional case, but we would like to show when the exceptional zero phenomenon occurs following [23, Section 1.2]. It would be interesting to deal with the exception case in the future work.

Let $\mathcal{H}_{\boldsymbol{g}}$ be the g-improving plane in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ defined by the equation

$$\boldsymbol{k} - \boldsymbol{l} + \boldsymbol{m} = k - l + m.$$

Let $\mathcal{O}_{\boldsymbol{gh}} = \mathcal{O}_{U_{\boldsymbol{g}}} \widehat{\otimes}_L \mathcal{O}_{U_{\boldsymbol{h}}}$ and $\nu_{\boldsymbol{g}} \colon \mathcal{O}_{\boldsymbol{fgh}} \longrightarrow \mathcal{O}_{\boldsymbol{gh}}$ be the map sending $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to its restriction $F(\boldsymbol{l} - \boldsymbol{m} + k + m - l, \boldsymbol{l}, \boldsymbol{m})$ to $\mathcal{H}_{\boldsymbol{g}}$. Set $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} := V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_{\nu_{\boldsymbol{g}}} \mathcal{O}_{\boldsymbol{gh}}$ and denote by

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} \in H^1(\mathbb{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$$

the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under the morphism induced in cohomology by $\nu_{\boldsymbol{g}}$. Define the Euler factor

$$\mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = 1 - \frac{\bar{\chi}_{\boldsymbol{g}} \cdot b_p(\boldsymbol{l})}{c_p(\boldsymbol{m}) \cdot a_p(\boldsymbol{l} - \boldsymbol{m} + k + m - l)} \cdot p^{(k-l+m-2)/2} \in \mathcal{O}_{\boldsymbol{gh}}.$$

In [23, Section 9.3], we can show that there is a canonical g-improved balanced diagonal class

$$\kappa_g^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$$

and we have the following factorisation

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \cdot \kappa_g^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}).$$

If $\mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = 0$, this implies the specialisation of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at ω_0 vanishes independently of whether the complex L-function $L(f_k \otimes g_l \otimes h_m, s)$ vanishes at $s = (k + l + m - 2)/2$. This is the first source of exceptional zeros. The second source of the exceptional zeros is the vanishing of the Euler factor in Equation 6.2.1.

7.4 Non exceptional case

When ω_0 is not exceptional, we prove the result following the proof in [23, Section 9.1].

Theorem 7.4.1. *When ω_0 is not exceptional, $L(f_k \otimes g_l \otimes h_m, s)$ vanishes at $s = (k + l + m - 2)/2$, we have $\kappa(\widetilde{\boldsymbol{f}}_k, \widetilde{\boldsymbol{g}}_l, \widetilde{\boldsymbol{h}}_m)$ is crystalline at p.*

Proof. From the interpolation formula Equation 6.2.1, since ω_0 is not exceptional, we can see that

$$L(f_k \otimes g_l \otimes h_m, (k + l + m - 2)/2) = 0 \text{ if and only if } \mathscr{L}_p(\widetilde{\boldsymbol{f}}_k, \widetilde{\boldsymbol{g}}_l, \widetilde{\boldsymbol{h}}_m) = 0$$

for each level N test vector $(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})$ for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. Combining Theorem 7.2.1, Theorem 5.2.1 and the fact that ω_0 is not exceptional, we have

$$\mathscr{L}_p(\widetilde{\boldsymbol{f}}_k \otimes \widetilde{\boldsymbol{g}}_l \otimes \widetilde{\boldsymbol{h}}_m) = \mathcal{E}_{\omega_0} \cdot \langle \exp_p^*(\kappa(\widetilde{\boldsymbol{f}}_k, \widetilde{\boldsymbol{g}}_l, \widetilde{\boldsymbol{h}}_m)_f), \eta_{\widetilde{\boldsymbol{f}}_k} \omega_{\widetilde{\boldsymbol{g}}_l} \omega_{\widetilde{\boldsymbol{h}}_m} \rangle_{\widetilde{\boldsymbol{f}}_k \widetilde{\boldsymbol{g}}_l \widetilde{\boldsymbol{h}}_m}$$

for a non-zero algebraic number $\mathcal{E}_{\omega_0}$. Hence the Garrett L-function $L(f_k \otimes g_l \otimes h_m, s)$ vanishes at $s = (k + l + m - 2)/2$ if and only if

$$\langle \exp_p^*(\kappa(\widetilde{\boldsymbol{f}}_k, \widetilde{\boldsymbol{g}}_l, \widetilde{\boldsymbol{h}}_m)_f), \eta_{\widetilde{\boldsymbol{f}}_k} \omega_{\widetilde{\boldsymbol{g}}_l} \omega_{\widetilde{\boldsymbol{h}}_m} \rangle_{\widetilde{\boldsymbol{f}}_k \widetilde{\boldsymbol{g}}_l \widetilde{\boldsymbol{h}}_m} = 0,$$

where $\eta_{\widetilde{\boldsymbol{f}}_k} \omega_{\widetilde{\boldsymbol{g}}_l} \omega_{\widetilde{\boldsymbol{h}}_m}$ is the specialisation of $\eta_{\widetilde{\boldsymbol{f}}} \omega_{\widetilde{\boldsymbol{g}}} \omega_{\widetilde{\boldsymbol{h}}}$ at $\omega = (k, l, m)$. Since $\eta_{\widetilde{\boldsymbol{f}}} \omega_{\widetilde{\boldsymbol{g}}} \omega_{\widetilde{\boldsymbol{h}}}$ is the basis of 1 dimesional space $(\mathbf{D}(\widetilde{\boldsymbol{f}})^-)^{\Gamma=1} \otimes (\mathbf{D}(\widetilde{\boldsymbol{g}})^+(-1 - \boldsymbol{\kappa}_{U_{\widetilde{\boldsymbol{g}}}} - \chi_{\widetilde{\boldsymbol{g}}}(p)))^{\Gamma=1} \otimes (\mathbf{D}(\widetilde{\boldsymbol{h}})^+(-1 - \boldsymbol{\kappa}_{U_{\widetilde{\boldsymbol{h}}}} - \chi_{\widetilde{\boldsymbol{h}}}(p)))^{\Gamma=1}$, we have that the Garrett L-function $L(f_k \otimes g_l \otimes h_m, s)$ vanishes at $s = (k + l + m - 2)/2$ if and only if $\exp_p^*(\kappa(\widetilde{\boldsymbol{f}}_k, \widetilde{\boldsymbol{g}}_l, \widetilde{\boldsymbol{h}}_m)_f) = 0$, which proves the result. □

CHAPTER 8

Proof of the main theorem

8.1 Logarithms

Set $V_{\mathrm{dR}}(f,g,h) := \mathbf{D}_{\mathrm{dR}}(V(f,g,h))$. From the discussion in [21, Section 3.1.2], we have a perfect duality

$$\langle\cdot,\cdot\rangle_{fgh} \colon V_{\mathrm{dR}}(f,g,h) \otimes_L V_{\mathrm{dR}}(f,g,h) \longrightarrow L, \tag{8.1.1}$$

which identifies $V_{\mathrm{dR}}(f,g,h)/\operatorname{Fil}^0$ with the L-linear dual of $\operatorname{Fil}^0 V_{\mathrm{dR}}(f,g,h)$. We have the Bloch–Kato logarithm

$$\log_p \colon H^1_f(\mathbb{Q}_p, V(f,g,h)) \longrightarrow V_{\mathrm{dR}}(f,g,h)/\operatorname{Fil}^0.$$

We then define the ++-logarithm

$$\log^{++} = \langle \log_p(\cdot), \eta_f^{\alpha} \otimes \omega_g \otimes \omega_h \rangle_{fgh} \colon H^1_f(\mathbb{Q}_p, V(f,g,h)) \longrightarrow L.$$

to be the composition of the Bloch–Kato p-adic logarithm with evaluation on the class

$$\eta_f^{\alpha} \otimes \omega_g \otimes \omega_h \in \operatorname{Fil}^0 V_{\mathrm{dR}}(f,g,h)$$

under the duality $\langle\cdot,\cdot\rangle_{fgh}$.

8.2 The main theorem

The main theorem is the following

Theorem 8.2.1. *Assume that the complex Garrett L-function $L(f \otimes g \otimes h, (k+l+m-2)/2) = 0$ and ω_0 is not exceptional. Then*

$$\frac{1 - \dfrac{\alpha_g \alpha_h}{\alpha_f}}{1 - \dfrac{\alpha_f}{p\alpha_g\alpha_h}} \langle\!\langle \kappa(f,g,h), y \rangle\!\rangle_{\boldsymbol{fgh}} = \log^{++}(\mathrm{res}_p(y)) \cdot \mathscr{L}_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \mod \mathscr{I}^2$$

for $y \in \mathrm{Sel}(\mathbb{Q}, V(f,g,h))$.

8.3 Proof of the main theorem

The following proof is identical to the proof of [22, Theorem 3.2], which is given in [22, Section 3.4]. We reproduce it here for the reader's convenience with the same notation.

We fix a 1-cocyle

$$\widetilde{z} = (z, z^+, a) \in C^1(V(f,g,h), D(f,g,h)^+)$$

which represents the diagonal class $\kappa(f,g,h)$ in $H^1(V(f,g,h),D(f,g,h)^+)$. Here

$$z \in C^1(G_{Np}, V(f,g,h)), \quad z^+ \in C^1_{\varphi,\gamma}(D(f,g,h)^+)$$

and $a = (a_\ell)_{\ell|Np} \in \bigoplus_{\ell|Np} K^0_\ell(V(f,g,h))$ which satisfy $dz = 0$, $dz^+ = 0$, $\mathrm{res}_{Np}(z) = i^+(z^+) - da$.

We then fix another 1-cocycle $Z \in C^1(G_{Np}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ representing $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. We have $dZ = 0$. In addition, from

$$\rho_{w_0} \colon C^1(G_{Np}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow C^1_{\mathrm{cont}}(G_{Np}, V(f,g,h)),$$

we have $\rho_{w_0}(Z) = z$. On the other hand, using the morphism of complexes

$$\rho_{w_0} \colon C^\bullet(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+) \longrightarrow C^\bullet(V(f,g,h), D(f,g,h)^+),$$

we have that the 1-cocyle $\widetilde{z}$ is then lifted by a 1-cochain of the form

$$\widetilde{Z} = (Z, Z^+, A) \in C^1(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+),$$

where the cochain $Z^+ \in C^1_{\varphi,\gamma}(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+)$ is a lift of z^+ and $A = (A_\ell)_{\ell|Np} \in \bigoplus_{\ell|Np} K^0_\ell(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ is a lift of a.

Since $\widetilde{z}$ is a 1-cocyle, the differential $\mathrm{d}\widetilde{Z}$ in $C^2(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+)$ can be written as

$$d\widetilde{Z} = (\boldsymbol{k} - k) \cdot \widetilde{Z}_{\boldsymbol{k}} + (\boldsymbol{l} - l) \cdot \widetilde{Z}_{\boldsymbol{l}} + (\boldsymbol{m} - m) \cdot \widetilde{Z}_{\boldsymbol{m}}$$

with 2-cochains $\widetilde{Z}_\cdot$, for $\cdot = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$. These 2-cochains are in $C^2(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ and then we write $\widetilde{Z}_\cdot = (Z_\cdot, Z^+_\cdot, W_\cdot)$ for $Z_\cdot \in C^2(G_{Np}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$, $Z^+_\cdot \in C^2_{\varphi,\gamma}(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+)$, and the 1-cochains $W_\cdot = (W_{\cdot,\ell})_{\ell|Np} \in \bigoplus_{\ell|Np} K^1_\ell(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ satisfying

$$(\boldsymbol{k} - k) \cdot W_{\boldsymbol{k}} + (\boldsymbol{l} - l) \cdot W_{\boldsymbol{l}} + (\boldsymbol{m} - m) \cdot W_{\boldsymbol{m}} = i^+(Z^+) - \mathrm{res}_{Np}(Z) - dA. \tag{8.3.1}$$

Lemma 8.3.1. *Let $\widetilde{z} \in C^1(V(f,g,h), D(f,g,h)^+)$ be a 1-cocycle and let $\widetilde{Z} \in C^1(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+)$ and $\widetilde{Z}_{\boldsymbol{k}}$, $\widetilde{Z}_{\boldsymbol{l}}$ and $\widetilde{Z}_{\boldsymbol{m}} \in C^2(V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+)$ satisfy*

1. $\rho_{\omega_0}(\widetilde{Z}) = \widetilde{z}$
2. $d(\widetilde{Z}) = (\boldsymbol{k} - k) \cdot \widetilde{Z}_{\boldsymbol{k}} + (\boldsymbol{l} - l) \cdot \widetilde{Z}_{\boldsymbol{l}} + (\boldsymbol{m} - m) \cdot \widetilde{Z}_{\boldsymbol{m}}$.

Then we have $\widetilde{z}_\cdot := \rho_{\omega_0}(\widetilde{Z}_\cdot)$ are 2-cocycles of $C^2(V(f,g,h), D(f,g,h)^+)$ and

$$-\widetilde{\beta}_{\boldsymbol{fgh}}(\kappa(f,g,h)) = (\boldsymbol{k} - k) \cdot cl(\widetilde{z}_{\boldsymbol{k}}) + (\boldsymbol{l} - l) \cdot cl(\widetilde{z}_{\boldsymbol{l}}) + (\boldsymbol{m} - m) \cdot cl(\widetilde{z}_{\boldsymbol{m}}). \tag{8.3.2}$$

Here $cl(\cdot)$ means that the class represented by $\cdot$.

Proof. This follows from the proof of [71, Lemma 5.5] using the Koszul complex. □

For $\mathbf{D} = D(f,g,h)$ or $\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, we have $p^- \colon C^\bullet_{\varphi,\gamma}(\mathbf{D}) \longrightarrow C^\bullet_{\varphi,\gamma}(\mathbf{D}^-)$. We let

$$X_\cdot = p^-(W_{\cdot,p}) \in C^1_{\varphi,\gamma}(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^-)$$

and

$$x_\cdot = \rho_{w_0}(X_\cdot) \in C^1_{\varphi,\gamma}(D(f,g,h)^-).$$

We also let $A^-_p = p^-(A_p)$.

Applying p^- to Equation 8.3.1, we get

$$(\boldsymbol{k} - k) \cdot X_{\boldsymbol{k}} + (\boldsymbol{l} - l) \cdot X_{\boldsymbol{l}} + (\boldsymbol{m} - m) \cdot X_{\boldsymbol{m}} = -p^-(\mathrm{res}_{Np}(Z)) - dA^-_p. \tag{8.3.3}$$

As Z is a 1-cocyle, this implies that the 1-cochain $x_\cdot$ are 1-cocyles. We set

$$\mathfrak{x}_\cdot = \mathrm{cl}(x_\cdot) \in H^1(D(f,g,h)^-).$$

On the other hand, Z is a 1-cocycle, then we have $\rho_{w_0}(Z_\cdot) = 0$. Hence

$$\widetilde{z}_\cdot = (0, \rho_{w_0}(Z_\cdot^+), \rho_{w_0}(W_\cdot)).$$

We set $\imath_{\mathrm{ur}}\colon \mathrm{Sel}(\mathbb{Q}, V(f,g,h)) \hookrightarrow H^1(V(f,g,h), D(f,g,h)^+)$,

$$\cdot^+\colon H^1(V(f,g,h), D(f,g,h)^+) \longrightarrow H^1(D(f,g,h)^+).$$

and

$$\jmath\colon H^1(D(f,g,h)^-) \longrightarrow H^2(V(f,g,h), D(f,g,h)^+).$$

Since $C^\bullet_{\mathrm{cont}}(\mathbb{Q}_\ell, V(f,g,h))$ is acyclic for $\ell \neq p$, this implies

$$\mathrm{cl}(\widetilde{z}_\cdot) = \jmath(\mathfrak{x}_\cdot) \tag{8.3.4}$$

From Equation 8.3.2 and 8.3.4, we have

$$\begin{aligned}\langle\!\langle \kappa(f,g,h), \cdot \rangle\!\rangle &= \langle \widetilde{\beta}_{\boldsymbol{fgh}}(\kappa(f,g,h)), \cdot \rangle_{\mathrm{Nek}} \otimes \mathscr{I}/\mathscr{I}^2 \\ &= -\sum \langle \jmath(\mathfrak{x}_{\boldsymbol{u}}), \cdot \rangle_{\mathrm{Nek}} \cdot (\boldsymbol{u} - u_0) \\ &= -\sum \langle \mathfrak{x}_{\boldsymbol{u}}, \cdot^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_0).\end{aligned} \tag{8.3.5}$$

Here $\langle \cdot, \cdot \rangle_{\mathrm{Tate}}\colon H^1(D(f,g,h)^+) \otimes_L H^1(D(f,g,h)^-) \longrightarrow L$ is the local Tate duality induced by the perfect pairing. The last equality follows from the adjointness of the maps $\jmath$ and $\cdot^+$ with respect to the pairings $\langle \cdot, \cdot \rangle_{\mathrm{Nek}}$ and $\langle \cdot, \cdot \rangle_{\mathrm{Tate}}$(See [71, Lemma 5.7] and [13, Corollary 4.16]).

As in Section 5.1, we have $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ and $F^2\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. The projection

$$p^-\colon \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^-$$

maps $F^2\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ onto $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$, hence it induces a morphism

$$P_f\colon H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f).$$

Lemma 8.3.2. *There exist $\mathfrak{Y}_{\boldsymbol{k}}$, $\mathfrak{Y}_{\boldsymbol{l}}$, $\mathfrak{Y}_{\boldsymbol{m}} \in H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f)$ such that*

$$P_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = (\boldsymbol{k} - k) \cdot \mathfrak{Y}_{\boldsymbol{k}} + (\boldsymbol{l} - l) \cdot \mathfrak{Y}_{\boldsymbol{l}} + (\boldsymbol{m} - m) \cdot \mathfrak{Y}_{\boldsymbol{m}}.$$

Furthermore, we have $\mathfrak{x}_{\boldsymbol{u}} = -\rho_{w_0}(\mathfrak{Y}_{\boldsymbol{u}})$.

Proof. We set $D(f,g,h)_f := D(f)^- \otimes_{\mathscr{R}_L} D(g)^+ \otimes_{\mathscr{R}_L} D(h)^+((4-k-l-m)/2)$. It is a direct summand of $D(f,g,h)^-$ and the specialisation map ρ_{ω_0} induces an isomorphism

$$\rho_{\omega_0}\colon \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{\omega_0} \mathcal{O}_{\boldsymbol{fgh}} \cong D(f,g,h)_f.$$

Since the kernel of evaluation at ω_0 on $\mathcal{O}_{\boldsymbol{fgh}}$ is generated by a regular sequence and

$$H^2(D(f,g,h)_f) = 0,$$

the specialisation isomorphism ρ_{ω_0} induces an isomorphism

$$\rho_{\omega_0}\colon H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \otimes_{\omega_0} L \cong H^1(D(f,g,h)_f). \tag{8.3.6}$$

From Theorem 7.4.1, $\kappa(f,g,h) = \rho_{\omega_0}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ is crystalline at p. And we know that $H^1_f(\mathbb{Q}_p, V(f,g,h))$ is $H^1(D(f,g,h)^+)$ from the proof of Lemma 4.8.1, hence

$$\boldsymbol{\kappa} := P_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})))$$

belongs to the kernel of Equation 8.3.6, thus we prove the first statement.

Let $\mathfrak{Y}_{\boldsymbol{u}}$ in $H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f)$ be local classes satisfying

$$\boldsymbol{\kappa}_f = \sum \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_0).$$

We only prove the case that $\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})$ is equal to $-\mathfrak{x}_{\boldsymbol{u}}$ for $\boldsymbol{u} = \boldsymbol{k}$. The other proofs are similar.

From Equation 8.3.3 and the construction $\mathrm{cl}(Z) = \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, we have

$$\mathrm{cl}\left(\sum X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_0)\right) = -\sum i_f(\mathfrak{Y}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_0) \in H^1(\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^-), \tag{8.3.7}$$

where i_f denotes the morphism in cohomology induced by the inclusion $\mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \longrightarrow \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^-$. Let $\nu \colon \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{f}}$ be the surjective morphism of rings sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to $F(\boldsymbol{k}, l, m)$ and set

$$\mathbf{D}(\boldsymbol{f}, g, h)^- = \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^- \otimes_\nu \mathscr{O}_{\boldsymbol{f}} \quad \textit{and} \quad \mathbf{D}(\boldsymbol{f})^{-++} = \mathbf{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_\nu \mathscr{O}_{\boldsymbol{f}}.$$

Then from Equation 8.3.3, we have $\nu(X_{\boldsymbol{k}})$ is a 1-cocyle in $C^\bullet_{\varphi,\gamma}(\mathbf{D}(\boldsymbol{f}, g, h)^-)$ and Equation 8.3.7 gives

$$(\boldsymbol{k} - k) \cdot (\mathrm{cl}(\nu(X_{\boldsymbol{k}})) + \nu(\mathfrak{Y}_{\boldsymbol{k}})) = 0.$$

We know that the $(\boldsymbol{k} - k)$-torsion of $H^1(\mathbf{D}(\boldsymbol{f}, g, h)^-)$ is a quotient of $H^0(D(f, g, h)^-)$, which is zero since we assume that (f, g, h) is not exceptional. Then $\nu(\mathfrak{Y}_{\boldsymbol{k}}) = -\mathrm{cl}(\nu(X_{\boldsymbol{k}}))$, hence we have $\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}}) = -\mathfrak{x}_{\boldsymbol{k}}$. ☐

Let $\widetilde{y}$ be an element of $H^1(V(f, g, h), D(f, g, h)^+)$. Equation 8.3.5 and Lemma 8.3.2 give the identity

$$\langle\!\langle \kappa(f, g, h), \widetilde{y} \rangle\!\rangle_{\boldsymbol{fgh}} = \sum \langle \rho_{w_0}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_0). \tag{8.3.8}$$

If $\widetilde{y} = \imath_{\mathrm{ur}}(y)$ corresponds to $y \in \mathrm{Sel}(\mathbb{Q}, V(f, g, h))$, then the image of $\widetilde{y}^+$ under the map induced in cohomology by the inclusion $i^+ \colon D(f, g, h)^+ \longrightarrow D(f, g, h)$ is equal to the restriction of y at p.

Lemma 8.3.3. *We have that*

$$\langle \rho_{w_0}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+ \rangle_{Tate} = \log^{++}(\mathrm{res}_p(y)) \cdot \langle \exp_p^*(\rho_{w_0}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_g \otimes \omega_h \rangle_{fgh}. \tag{8.3.9}$$

where $\exp_p^* \colon H^1(D(f, g, h)) \longrightarrow \mathscr{D}_{dR}(D(f, g, h))$ *is the Bloch–Kato dual exponential map in Theorem 4.3.2.*

Proof. The Bloch–Kato dual exponential map in Theorem 4.3.2 gives an morphism

$$\exp_p^* \colon H^1(D(f, g, h)) \longrightarrow \mathscr{D}_{\mathrm{dR}}(D(f, g, h)).$$

As $i^+(\widetilde{y}^+) = \mathrm{res}_p(y)$, it follows that

$$\langle \rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}}), \widetilde{y}^+ \rangle_{\mathrm{Tate}} = \langle \exp_p^*(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})), \log_p(\mathrm{res}_p(y)) \rangle_{fgh}. \tag{8.3.10}$$

Here $\langle \cdot, \cdot \rangle_{fgh}$ is the pairing in Equation 8.1.1. Since the $\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})$ belongs to $H^1(D(f, g, h)_f)$, and for the linear form

$$\langle \exp_p^*(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})), \rangle_{fgh} \colon V_{\mathrm{dR}}(f, g, h)/\,\mathrm{Fil}^0 \longrightarrow L,$$

there is a commutative diagram

$$\begin{array}{ccc} V_{\mathrm{dR}}(f, g, h)/\,\mathrm{Fil}^0 & \longrightarrow & L \\ \downarrow \mathrm{pr}^{++} & \nearrow & \\ \mathscr{D}_{\mathrm{dR}}(D(f) \otimes_{\mathscr{R}_L} D(g)^+ \otimes_{\mathscr{R}_L} D(h)^+)/\,\mathrm{Fil}^0 & & \end{array}$$

By definition, we have

$$\mathrm{pr}^{++}(\log_p(\mathrm{res}_p(y))) = \log^{++}(\mathrm{res}_p(y)) \cdot \eta_f \otimes \omega_g \otimes \omega_h,$$

then the result follows from Equation 8.3.10. ☐

We set

$$\exp_p^{++*}(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})) \colon = \langle \exp_p^*(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_g \otimes \omega_h \rangle_{fgh},$$

Then Equation 8.3.8 and 8.3.9 give the equality

$$\langle\!\langle \kappa(f, g, h), y \rangle\!\rangle_{\boldsymbol{fgh}} = \log^{++}(\mathrm{res}_p(y)) \cdot \sum \exp^{++*}(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})) \cdot (\boldsymbol{u} - u_0) \tag{8.3.11}$$

for $y \in \mathrm{Sel}(\mathbb{Q}, V(f,g,h))$.

From Proposition 5.2.1, we have a morphism

$$\mathscr{L}_{\boldsymbol{f}} \colon H^1(\mathbf{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow \mathcal{O}_{\boldsymbol{fgh}}.$$

For each local class $\mathcal{Z}$ in $H^1_{\mathrm{bal}}(\mathbb{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$, we have

$$\mathscr{L}_{\boldsymbol{f}}(\mathcal{Z})(\omega_0) = (p-1)\alpha_f \cdot \frac{1 - \dfrac{\alpha_g\alpha_h}{\alpha_f}}{1 - \dfrac{\alpha_f}{p\alpha_g\alpha_h}} \cdot \exp_p^{++*}(\rho_{\omega_0}(\mathcal{Z})).$$

We apply $\mathscr{L}_{\boldsymbol{f}}$ to both sides of the equation in Lemma 8.3.2 and use the explicit reciprocity law in Theorem 7.2.1 to obtain

$$\mathscr{L}_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \mod \mathscr{I}^2 = \dot{\alpha_f}\frac{1 - \dfrac{\alpha_g\alpha_h}{\alpha_f}}{1 - \dfrac{\alpha_f}{p\alpha_g\alpha_h}} \cdot \sum \exp^{++*}(\rho_{\omega_0}(\mathfrak{Y}_{\boldsymbol{u}})) \cdot (\boldsymbol{u} - u_0).$$

Then the theorem follows from the previous equation and Equation 8.3.11.

CHAPTER 9

p-adic BSD conjecture for AI p-adic L-function

In this part, we formulate a p-adic BSD conjecture for AI p-adic L-function. We let E be an elliptic curve defined over the field $\mathbb{Q}$ and its conductor is N_E. We also let ϱ_1, ϱ_2 be two-dimensional odd Artin representations of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$, i.e.,

$$\varrho_i \colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(V_{\varrho_i}).$$

Set $\varrho \colon = \varrho_1 \otimes_{\mathbb{Q}} \varrho_2$ and $V_\varrho \colon = V_{\varrho_1} \otimes_{\mathbb{Q}(\varrho)} V_{\varrho_2}$. We denote by K_ϱ the extension of $\mathbb{Q}$ cut out by ϱ. Assume that the $\det(\varrho_1) \cdot \det(\varrho_2) = 1$. From the modularity theorem, we can associate E with a cuspidal newform $f = \sum_{n \geq 1} a_n(f) \cdot q^n \in S_2(N_E)$. From the work of Khare and Wintenberger [52, Corollary 10.2], we can associate ϱ_1 and ϱ_2 to normalised Hecke eigenforms

$$g = \sum_{n \geq 1} a_n(g) \cdot q^n \in M_1(N_g, \chi_g)$$

and

$$h = \sum_{n \geq 1} a_n(h) \cdot q^n \in M_1(N_h, \chi_h)$$

of levels N_g and N_h that are equal to the conductors of ϱ_1 and ϱ_2 respectively and also the characters χ_g and χ_h which satisfy $\chi_g \cdot \chi_h = 1$. Assume that E has good ordinary reduction at p. We consider the p-th Hecke polynomial $T_p(X) = X^2 - a_p(f) \cdot X + 1_{N_E}(p) \cdot p$ of f, here 1_{N_E} is the trivial character modulo N_E. Then we denote by α_f the root of $T_p(X)$ with positive p-adic valuation and by β_f the other root, which is a p-adic unit. In addition, we suppose that E does not have complex multiplication and that f is non-critical. Then the critical p-stabilisation

$$f_{\alpha_f}(q) = f(q) - \beta_f \cdot f(q^p) \in S_2(N_E p)$$

is the specialisation at weight two of a unique cuspidal Coleman family $\boldsymbol{f}$ for suitable connected open disc $U_{\boldsymbol{f}}$ centred at 2 in the weight space $\mathcal{W}$. Here $\mathcal{O}_{U_{\boldsymbol{f}}}$ is the ring of bounded analytic functions on $U_{\boldsymbol{f}}$ with respect to Gauss norm. Let ξ be either g or h, and let α_ξ and β_ξ be the roots of p-th Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$. We assume that p does not divide N_ξ and ξ satisfies Assumption 3.5.3 (2) (b). The p-stabilisation

$$\xi_{\alpha_\xi}(q) = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_1(N_\xi p, \chi_\xi)_L$$

is the weight one specialisation of a unique cuspidal Coleman family $\boldsymbol{\xi}$ for some $U_{\boldsymbol{\xi}}$ which is an affinoid disc centered in 1 in $\mathcal{W}_L$.

Let Σ^{cl} denote the set of classical triples, the intersection of $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ with $\mathbb{Z}^3_{\geq 1}$. Under the assumption $\chi_g \cdot \chi_h = 1$, for each $(k, l, m) \in \Sigma^{\mathrm{cl}}$, the complex Garrett L-function $L(f_k \otimes g_l \otimes h_m, s)$ admits an analytic continuation and satisfies a functional equation with sign ± 1 relating its values at s and $k + l + m - 2 - s$. The assumption $(N_g, N_h, N_E) = 1$ guarantees that the signs in the functional equations are equal to $+1$ for all classical triples (k, l, m) such that $k \geq l + m$. See [35, Section 4.1].

In this case, the complex Garrett L-function $L(E,\varrho,s) := L(f\otimes g\otimes h, s)$ vanishes to even order at the central critical point $s=1$. We denote by $\mathscr{L}_p(E,\varrho)$ the AI p-adic L-function $\mathscr{L}_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and we know that $L_p(E,\varrho) := \mathscr{L}_p(E,\varrho)^2$ interpolates the central critical values $L\left(f_k\otimes g_l\otimes h_m, 1\right)$ from Equation 6.2.1. Now we define by

$$E(K_\varrho)^\varrho := (E(K_\varrho)\otimes_{\mathbb{Z}} V_\varrho)^{\mathrm{Gal}(K_\varrho/\mathbb{Q})}$$

the Mordell–Weil group of E twisted by ϱ. Enlarging L if necessary, we fix isomorphisms

$$\gamma_g\colon V_{\varrho_1}\otimes_{\mathbb{Q}(\varrho)} L\cong V(g), \quad \textit{and} \quad \gamma_h\colon V_{\varrho_2}\otimes_{\mathbb{Q}(\varrho)} L\cong V(h)$$

of $L[G_{\mathbb{Q}}]$-modules. Using the Kummer map and the Shapiro isomorphism yields an injective morphism

$$(E(K_\varrho)\otimes_{\mathbb{Z}} V_\varrho)^{\mathrm{Gal}(K_\varrho/\mathbb{Q})}\otimes_{\mathbb{Q}(\varrho)} L\longrightarrow \mathrm{Sel}(\mathbb{Q}, V(f)\otimes_{\mathbb{Q}(\varrho)} V_\varrho).$$

See [22, Section 2.3] for more details. Using the isomorphism of $L[G_{\mathbb{Q}}]$-modules

$$\gamma_g\otimes\gamma_h\colon V_\varrho\otimes_{\mathbb{Q}(\varrho)} L\cong V(g)\otimes_L V(h),$$

we have

$$E(K_\varrho)^\varrho\longrightarrow \mathrm{Sel}(\mathbb{Q}, V(f,g,h)).$$

The p-adic height pairing

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{fgh}}\colon E(K_\varrho)^\varrho\otimes_{\mathbb{Q}(\varrho)} E(K_\varrho)^\varrho\longrightarrow \mathscr{I}/\mathscr{I}^2$$

is defined to be restriction of the canonical p-adic height pairing (see Equation 4.7.3) to the Mordell–Weil group $E(K_\varrho)^\varrho$. Here $\mathscr{I}$ is the ideal of $\mathcal{O}_{\boldsymbol{fgh}}$ vanishing at $\omega_0=(2,1,1)$. Using the height pairing, we define the regulator $R_p(E,\varrho)$ to be

$$R_p(E,\varrho) := \det(\langle\!\langle P_i,P_j\rangle\!\rangle_{\boldsymbol{fgh}})_{1\le i,j\le r(E,\varrho)}, \tag{9.0.1}$$

where P_i is a $\mathbb{Q}(\varrho)$-basis of Mordell–Weil group $E(K_\varrho)^\varrho$ and $r(E,\varrho) := \dim_{\mathbb{Q}(\varrho)} E(K_\varrho)^\varrho$. We have

$$R_p(E,\varrho)\in(\mathscr{I}^{r(E,\varrho)}/\mathscr{I}^{r(E,\varrho)+1})/(\mathbb{Q}(\varrho)^2)^\times.$$

We say that a non zero element F of $\mathcal{O}_{\boldsymbol{fgh}}$ has order of vanishing $r\in\mathbb{Z}_{\ge 0}$ at $\omega_0=(2,1,1)$ if it belongs to $\mathscr{I}^r-\mathscr{I}^{r+1}$ and we denote by F^* the image of F in the quotient $\mathscr{I}^r/\mathscr{I}^{r+1}$.

Conjecture 9.0.1. *The p-adic L-function $L_p(E,\varrho)$ has order of vanishing $r(E,\varrho)$ at $\omega_0=(2,1,1)$, and the following equality holds up to $(\mathbb{Q}(\varrho)^2)^\times$.*

$$L_p(E,\varrho)^* = R_p(E,\varrho) \mod \mathscr{I}^{r(E,\varrho)}/\mathscr{I}^{r(E,\varrho)+1}$$

In particular, the p-adic height pairing $\langle\!\langle\cdot,\cdot,\rangle\!\rangle_{\boldsymbol{fgh}}$ is non-degenerate.

Bibliography

[1] Fabrizio Andreatta and Adrian Iovita, *Triple product p-adic L-functions associated to finite slope p-adic families of modular forms*, Duke Mathematical Journal **170** (2021), no. 9, 1989 – 2083. i, 2, 3, 34, 35

[2] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens, *Overconvergent Eichler–Shimura isomorphisms*, Journal of the Institute of Mathematics of Jussieu **14** (2015), no. 2, 221–274. 14, 15, 16

[3] Avner Ash and Glenn Stevens, *Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues.*, Journal für die reine und angewandte Mathematik **365** (1986), 192–220. 18

[4] ———, *p-adic deformations of arithmetic cohomology*, preprint (2008). 15, 16

[5] Joël Bellaïche, *Critical p-adic L-functions*, Inventiones mathematicae **189** (2012), no. 1, 1–60. 13, 14, 16, 17, 18

[6] Joël Bellaıche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque **324** (2009), 1–314. 14

[7] Joël Bellaïche and Mladen Dimitrov, *On the eigencurve at classical weight* 1 *points*, Duke Mathematical Journal **165** (2016), no. 2. 18

[8] Denis Benois, *A generalization of Greenberg's L-invariant*, American journal of mathematics **133** (2011), no. 6, 1573–1632. 25

[9] ———, *On extra zeros of p-adic L-functions: the crystalline case*, Iwasawa Theory 2012: State of the Art and Recent Advances, Springer, 2014, pp. 65–133. 27

[10] ———, *Selmer complexes and the p-adic hodge theory*, Arithmetic and Geometry. London Mathematical Society, Lecture Note Series **420** (2015), 36–88. i, 2, 26, 29

[11] ———, *p-adic heights and p-adic Hodge theory*, Mém. Soc. Math. Fr. (N.S.) (2021), no. 167, vi + 135. MR 4219074 i, 2, 3, 26, 28, 29

[12] Denis Benois and Kâzım Büyükboduk, *Interpolation of Beilinson–Kato elements and p-adic L-functions*, Annales mathématiques du Québec **46** (2022), no. 2, 231–287. 16, 17, 19

[13] ———, *On the exceptional zeros of p-non-ordinary p-adic L-functions and a conjecture of Perrin-Riou*, Transactions of the American Mathematical Society **376** (2023), no. 01, 231–284. i, 2, 28, 29, 45

[14] Laurent Berger, *Représentations p-adiques et équations différentielles*, Inventiones mathematicae **148** (2002), no. 2, 219–284. 5, 8, 9, 10

[15] ———, *Construction de (φ, Γ)-modules: représentations p-adiques et B-paires*, Algebra & Number Theory **2** (2008), no. 1, 91–120.

[16] ———, *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Astérisque **319** (2008), 13–38. 8, 9

[17] ———, *Trianguline representations*, Bulletin of the London Mathematical Society **43** (2011), no. 4, 619–635. 8, 20, 21

[18] Laurent Berger and Pierre Colmez, *Familles de représentations de de Rham et monodromie p-adique*, Astérisque **319** (2008), 303–337. 10, 11

[19] M. Bertolini, H. Darmon, and K. Prasanna, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148. 41

[20] Massimo Bertolini, Henri Darmon, and Rodolfo Venerucci, *Heegner points and Beilinson–Kato elements: a conjecture of Perrin-Riou*, Advances in Mathematics **398** (2022), 108172. 3, 19, 21

[21] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, *Diagonal classes and the Bloch–Kato conjecture*, (2020). 3, 32, 43

[22] ———, *On p-adic analogues of the Birch and Swinnerton-Dyer conjecture for Garrett L-functions*, Preprint **1** (2021), no. 3, 7. i, 1, 2, 3, 43, 50

[23] ———, *Reciprocity laws for balanced diagonal classes*, Heegner points, Stark–Heegner points and diagonal classes (2022). 1, 2, 3, 15, 16, 18, 19, 20, 22, 27, 31, 32, 37, 38, 39, 40, 41, 42

[24] ———, *Errata to «reciprocity laws for balanced diagonal classes»*, (2024). 16, 18

[25] Amnon Besser, *A generalization of Colemans p-adic integration theory*, Inventiones mathematicae **142** (2000), no. 2, 397–434. 41

[26] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400. 38

[27] Kevin Buzzard, *Eigenvarieties*, L-functions and Galois representations (2007), 59–120. 16

[28] Frédéric Cherbonnier and Pierre Colmez, *Représentations p-adiques surconvergentes*, Inventiones mathematicae **133** (1998), no. 3, 581–611. 8

[29] ———, *Théorie d'Iwasawa des représentations p-adiques d'un corps local*, Journal of the American Mathematical Society **12** (1999), no. 1, 241–268. 7

[30] Robert Coleman and Bas Edixhoven, *On the semi-simplicity of the U_p-operator on modular forms*, arXiv preprint alg-geom/9611013 (1996). 13

[31] Robert Coleman and Barry Mazur, *The eigencurve*, London Mathematical Society Lecture Note Series (1998), 1–114. 16

[32] Robert F Coleman, *p-adic Banach spaces and families of modular forms*, Inventiones mathematicae **127** (1997), no. 3, 417–479. 16

[33] Pierre Colmez, *Espaces vectoriels de dimension finie et représentations de de rham*, Astérisque **319** (2008), 117–186. 5, 6

[34] ———, *Représentations triangulines de dimension* 2, Astérisque **319** (2008), no. 213-258, 83. 20

[35] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula*, Ann. Sci. Éc. Norm. Supér.(4) **47** (2014), no. 4, 779–832. 1, 33, 34, 35, 41, 49

[36] Gerd Faltings, *p-adic Hodge theory*, Journal of the American Mathematical Society **1** (1988), no. 1, 255–299. 20

[37] Jean-Marc Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111. 5

[38] ———, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184. 7

[39] ———, *Représentations p-adiques des corps locaux (1 ère partie)*, The Grothendieck Festschrift, Springer, 2007, pp. 249–309. 7, 8

[40] Jean-Marc Fontaine and Yi Ouyang, *Theory of p-adic Galois representations*, preprint (2008). 5, 6, 20

[41] Michael Harris and Stephen S Kudla, *The central critical value of a triple product L-function*, Annals of Mathematics (1991), 605–672. 34

[42] Laurent Herr, *Sur la cohomologie galoisienne des corps p-adiques*, Bulletin de la Société mathématique de France **126** (1998), no. 4, 563–600. 24, 25

[43] ______, *Une approche nouvelle de la dualité locale de Tate*, Mathematische Annalen **320** (2001), 307–337. 24, 25

[44] Ting-Han Huang, *Triple product p-adic L-functions for finite slope families and a p-adic Gross-Zagier formula*, arXiv preprint arXiv:2403.18620 (2024). i, 2, 3, 31, 32, 41

[45] Ting-Han Huang and Ju-Feng Wu, *Finite polynomial cohomology with coefficients*, arXiv preprint arXiv:2209.10289 (2022). 41

[46] ATSUSHI ICHINO, *Trilinear forms and the central values of triple product L-functions*, DUKE MATHEMATICAL JOURNAL **145** (2008), no. 2. 34

[47] Ananyo Kazi, *Overconvergent modular and de Rham sheaves and p-adic interation of the Gauss–Manin connection*, (2022). 35

[48] Kiran Kedlaya and Ruochuan Liu, *On families of φ, Γ-modules*, Algebra & Number Theory **4** (2011), no. 7, 943–967. 10, 11

[49] Kiran Kedlaya, Jonathan Pottharst, and Liang Xiao, *Cohomology of arithmetic families of (φ, Γ)-modules*, Journal of the American Mathematical Society **27** (2014), no. 4, 1043–1115. 21, 26

[50] Kiran S Kedlaya, *A p-adic local monodromy theorem*, Annals of mathematics (2004), 93–184. 8

[51] ______, *Slope filtrations revisited*, Documenta Mathematica **10** (2005), 447–525. 8

[52] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serres modularity conjecture (I)*, Inventiones mathematicae **178** (2009), no. 3, 485–504. 49

[53] Ruochuan Liu, *Cohomology and duality for (φ, Γ)-modules over the Robba ring*, International Mathematics Research Notices **2007** (2007), no. 9, rnm150–rnm150. 24

[54] ______, *Triangulation of refined families*, Commentarii Mathematici Helvetici **90** (2015), no. 4, 831–904. 20

[55] David Loeffler and Sarah Livia Zerbes, *Rankin–Eisenstein classes in Coleman families*, Research in the Mathematical Sciences **3** (2016), no. 1, 1–53. 15, 18, 21, 32

[56] James S Milne, *Lectures on étale cohomology*, 2012. 38

[57] ______, *Étale cohomology (pms-33), volume 33*, Étale Cohomology (PMS-33), Volume 33, Princeton university press, 2016. 19

[58] T. Miyake, *Modular forms*, Springer-Verlag, Berlin, 1989. 39

[59] Santiago Molina, *Finite slope triple product p-adic L-functions over totally real number fields*, Journal of Number Theory **223** (2021), 267–306. 35

[60] Kentaro Nakamura, *Classification of two-dimensional split trianguline representations of p-adic fields*, Compositio Mathematica **145** (2009), no. 4, 865–914.

[61] ______, *Iwasawa theory of de Rham-modules over the Robba ring*, Journal of the Institute of Mathematics of Jussieu **13** (2014), no. 1, 65–118. 9, 24, 25

[62] ______, *A generalization of Katos local ε-conjecture for (φ, Γ)-modules over the Robba ring*, Algebra & Number Theory **11** (2017), no. 2, 319–404.

[63] J. Nekovář, *Selmer complexes*, Astérisque (2006), no. 310, viii+559. i, 1, 2, 28

[64] Jan Nekovár, *p-adic Abel-Jacobi maps and p-adic heights*, The Arithmetic and Geometry of Algebraic Cycles (Banff, Canada, 1998), CRM Proc. Lect. Notes **24** (2000), 367–379. 41

[65] Jan Nekovář and Wiesława Nizioł, *Syntomic cohomology and p-adic regulators for varieties over p-adic fields*, Algebra & Number Theory **10** (2016), no. 8, 1695–1790. 38

[66] Ilya Piatetski-Shapiro and Stephen Rallis, *Rankin triple L functions*, Compositio Mathematica **64** (1987), no. 1, 31–115. 33

[67] Jonathan Pottharst, *Analytic families of finite-slope Selmer groups*, Algebra & Number Theory **7** (2013), no. 7, 1571–1612. 26

[68] Jonathan Pottharst and Liang Xiao, *On the parity conjecture in finite-slope families*, arXiv preprint arXiv:1410.5050 (2014). 29

[69] Dipendra Prasad, *Trilinear forms for representations of* GL(2) *and local ϵ-factors*, Compositio Mathematica **75** (1990), no. 1, 1–46. 33

[70] Takeshi Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Inventiones mathematicae **137** (1999), no. 2, 233–411. 20

[71] Rodolfo Venerucci, *Exceptional zero formulae and a conjecture of Perrin-Riou*, Inventiones mathematicae **203** (2016), 923–972. i, 2, 44, 45

[72] Thomas Crawford Watson, *Rankin triple products and quantum chaos*, Princeton University, 2002. 34