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# Lavrentiev Phenomenon and integral representation of the lower semicontinuous envelope for functionals with non convex and non continuous Lagrangians

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# Abstract

In this thesis, we investigate classical integral functionals in the Calculus of Variations and study conditions on the Lagrangian that guarantee the absence of the Lavrentiev Phenomenon. In particular, we focus on Lagrangians that are non-convex, possibly highly discontinuous, and unbounded with respect to the gradient variable.

Our first step to prove the non-occurrence of the Lavrentiev Phenomenon is to prove the representation of the lower semicontinuous envelope of a functional defined in $W^{1,\infty}(\Omega)$ as an integral functional whose Lagrangian is given by the bipolar of the original one.

Specifically, we adapt a technique presented in [40] for the bounded case, and a refinement of a method due to Cellina [29, 30], for the unbounded case. The integral representation also allows us to apply recent results [14, 17, 20] to the non-convex setting.

We establish the integral representation of the lower semicontinuous envelope and the corresponding absence of the Lavrentiev Phenomenon under very weak assumptions in the autonomous case and under suitable anti-jump conditions on the spatial variable in the non-autonomous case. Furthermore, using a technique presented in [14], we prove the strong convergence of the approximating sequence for minimizers or under specific growth assumptions on the Lagrangian.

These results provide new insights into the interplay between regularity assumptions, relaxation methods, and the avoidance of the Lavrentiev Phenomenon, thereby extending recent advances to a broader class of variational problems.

# Contents

# Chapter 1

# Motivations of the Research

## 1.1 The study of minima in the Calculus of Variations

We consider the following class of functionals

$$F_p(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx, \qquad u \in \varphi + W_0^{1,p}(\Omega),$$

where $1 \leq p \leq \infty$, $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $\varphi \in W^{1,\infty}(\Omega)$, and $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a suitable Borel function, classically named the Lagrangian, which is not necessarily convex with respect to its last variable.

Some of the most extensively studied problems in the Calculus of Variations are the analysis of existence, uniqueness, and regularity of minimizers in Sobolev spaces. More precisely, one aims to characterize the solutions of

$$\min \Big\{ F_p(u) \,\Big|\, u \in \varphi + W_0^{1,p}(\Omega) \Big\}.$$

There are two main approaches to address these problems. The first consists in studying the minimizer as a solution of the Euler-Lagrange equations associated with the functional $F_p$. The second is to apply the Direct Method in the Calculus of Variations, which allows one to investigate the existence and qualitative properties of minimizers without requiring an explicit solution.

The main limitation of the first approach is that, in many cases, certain assumptions on the regularity of the minimizers are required, and, in general, the analysis of the associated PDEs can be very challenging. Moreover, in some situations, the minimizer does not satisfy the Euler-Lagrange equations: an example, presented in [7], is the functional

$$\int_0^1 \left[ (x^2 - u^3)\xi^{14} + \varepsilon \xi^2 \right] dx$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = k,$$

for suitable choices of $\varepsilon$ and $k$.

The second approach is based on the Weierstrass Theorem by means of two fundamental properties of the functional, coercivity and sequential lower semicontinuity. In this method, it is essential to choose an appropriate topology for the underlying functional space, which has to be weak enough to guarantee the existence of suitable compact sets, yet strong enough to ensure the lower semicontinuity of the functional and it turns out that the weak topology in Sobolev spaces is well suited to this aim.

We recall that a functional $F$ is coercive if, for every $K > 0$, the set

$$\{u \in W^{1,p}(\Omega) \mid |F(u)| \leq K\}$$

is weakly precompact in $W^{1,p}(\Omega)$.

A functional $F : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, with $p \in [1, \infty)$, is sequentially lower semicontinuous with respect to the weak topology of $W^{1,p}(\Omega)$ if, for every sequence

$$u_k \rightharpoonup u,$$

we have

$$F(u) \leq \liminf_{k\to\infty} F(u_k).$$

Many classical problems are naturally formulated in the framework of 'regular' functions, such as, for example, the space of $C^1$ functions. However, in order to apply the Direct Method and actually obtain a minimizer, it is necessary to extend the problem to a larger space, typically a Sobolev space $W^{1,p}(\Omega)$ endowed with the weak topology.

In some cases, in fact, the actual minimizers belong to a space of less regular functions. One of the most well-known examples is the brachistochrone problem, where the solution of

$$\min \int_a^b \sqrt{\frac{1 + (u'(x))^2}{u(x)}}\, dx$$

is not a $C^1([a, b])$ function, but belongs only to $W^{1,1}([a, b])$.

Consequently, a substantial area of research in the Calculus of Variations and Partial Differential Equations is devoted to the study of the regularity properties of minimizers.

When the minimizer does not satisfy the regularity properties desired, one seeks to approximate it by a sequence of more regular functions converging, in an appropriate sense, to it. For instance, given $u \in \varphi + W_0^{1,p}(\Omega)$ with $p < q$, our goal is to construct a sequence $u_n \in \varphi + W_0^{1,q}(\Omega)$ such that

$$u_n \to u \quad \text{in} \quad W^{1,p}(\Omega)$$

and

$$F(u_n) \to F(u).$$

We note that, in order to apply the Direct Method in the Calculus of Variations, one only requires lower semicontinuity; hence, in general, it may happen that

$$F(u) < \liminf_{n\to\infty} F(u_n).$$

In 1926, Lavrentiev [53] provided an example in which it is not possible to approximate the value of the minimizer in $AC([a, b])$ by a sequence in $C^2([a, b])$, introducing the notion of what is now known as the *Lavrentiev Phenomenon* and then the problem of determining conditions guaranteeing good approximability properties emerged as a crucial problem in the Calculus of Variations.

## 1.2 The Lavrentiev Phenomenon

Given a topological space $X$, a dense subset $Y \subset X$, and a real-valued functional $E$, we say that the *Lavrentiev Phenomenon* occurs between $X$ and $Y$ for $E$ if

$$\inf_{u\in X} E(u) < \inf_{v\in Y} E(v).$$

More specifically, we focus on the case $X = W_\varphi^{1,p}(\Omega)$, $Y = W_\varphi^{1,q}(\Omega)$, and $E = F_p$, particularly when $p = 1$ and $q = \infty$. In 1934, Manià [54] provided an example with a polynomial Lagrangian in the one-dimensional scalar case:

$$\min_{\mathrm{id}+W_0^{1,1}([0,1])} \int_0^1 (x - u^3(x))^2 |u'(x)|^6\, dx < \inf_{\mathrm{id}+W_0^{1,\infty}([0,1])} \int_0^1 (x - u^3(x))^2 |u'(x)|^6\, dx\,.$$

Since this functional is not coercive, in 1995, Zhikov showed ([69]) that the functional

$$\int_0^1 (x - u^3(x))^2 |u'(x)|^6 + \varepsilon |u'(x)|^{\frac{5}{4}}\, dx$$

still presents the Lavrentiev Phenomenon. Furthermore, in the same paper he presented an example with a $p, q$-type Lagrangian depending only on $(x, \xi)$ with domain $\Omega = B_1(0) \subset \mathbb{R}^2$. He considered the following Lagrangian

$$f(x,\xi) = |\xi|^p + a(x)|\xi|^q, \quad a(x) := \begin{cases} \frac{x_1 x_2}{\sqrt{x_1^2+x_2^2}} & \text{if } x_1x_2 > 0, \\ 0 & \text{if } x_1x_2 \leq 0, \end{cases}$$

where $1 \leq p < 2 < 3 < q$ and proved that there exists a suitable boundary condition $\varphi$ such that

$$\inf_{\varphi + W_0^{1,p}(\Omega)} \int_\Omega f(x, \nabla u(x))\, dx < \inf_{\varphi + W_0^{1,q}(\Omega)} \int_\Omega f(x, \nabla u(x))\, dx.$$

We emphasize that in this example the choice of $a$ is crucial for the occurrence of the Lavrentiev Phenomenon. In [16] the authors proposed some continuity assumptions on the function $a$ to guarantee the non-occurrence of the Lavrentiev phenomenon for every double phase functional.

In both of these examples, the spatial variable plays a crucial role in the occurrence of the Lavrentiev Phenomenon. There exists an extensive literature of examples of Lavrentiev Phenomenon, particularly in the non-autonomous case; we cite [34] for the multidimensional scalar case, [5, 6, 43] for the applications of fractal sets in the multidimensional scalar case and [2, 4, 42] for the vectorial case. Other examples are proposed in [9, 50].

The starting point of this PhD thesis is the study of conditions ensuring the non-occurrence of the Lavrentiev Phenomenon, via the integral representation of the lower semicontinuous envelopes of integral functionals in the multidimensional scalar case.

Studying conditions that prevent the Lavrentiev Phenomenon is important in several areas of theoretical and applied analysis, including the regularity of solutions, the presence of microstructures in materials [45], and the numerical approximation of minimizers.

We observe that, if the functional $F_p$ admits a minimizer $u \in W^{1,q}(\Omega)$, then the Lavrentiev Phenomenon between $W_\varphi^{1,p}(\Omega)$ and $W_\varphi^{1,q}(\Omega)$ does not occur. This provides a link between the theory of regularity and the non-occurrence of the Lavrentiev Phenomenon. More generally, the problem reduces to the existence of a minimizing sequence $(u_n) \subset W_\varphi^{1,q}(\Omega)$ for $F_p$.

The one-dimensional case has been extensively studied and many authors have proved the non-occurrence of the Lavrentiev Phenomenon under weak assumptions on the Lagrangian; see, for example, [3, 52, 54, 58, 59, 60]. In [3], the authors proved that in the autonomous case, under certain boundedness hypotheses on the Lagrangian, for every $u \in W^{1,p}(]a,b[)$ there exists a sequence $(u_n)_n \subset W^{1,\infty}(]a,b[)$ such that

$$\|u_n - u\|_{W^{1,p}} \to 0 \quad \text{and} \quad \int_a^b f(u_n(x), u_n'(x))\, dx \to \int_a^b f(u(x), u'(x))\, dx.$$

We remark that the sequence $(u_n)$ constructed in the proof satisfies only

$$u_n(a) = u(a),$$

and not necessarily

$$u_n(b) = u(b).$$

In [58], the author extended this result to the non-autonomous case by assuming an anti-jump condition: for every $K$, there exist constants $k, \beta \geq 0$, a function $\gamma \in L^1(I)$, and $\varepsilon^* > 0$ such that for all $x_1, x_2 \in [x - \varepsilon^*, x + \varepsilon^*] \cap I$, $u \in B_K$, and $\xi \in \mathbb{R}^n$,

$$|f(x_2, u, \xi) - f(x_1, u, \xi)| \leq (kf(x, u, \xi) + \beta|\xi|^p + \gamma(x))|x_2 - x_1|.$$

In this case, the approximating sequence preserves both boundary values.

In the multidimensional scalar case, many authors have proved the absence of the Lavrentiev Phenomenon under the assumption that the Lagrangian is convex with respect to the gradient variable. As examples, we cite some works on the non-occurrence of the Lavrentiev Phenomenon in this setting: [8, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 28, 51, 56, 69] for the multidimensional scalar case and [8, 37, 38, 41, 42, 49] for the vectorial case.

In 1992, Buttazzo and Mizel [28] proposed to interpret the Lavrentiev Phenomenon as a relaxation problem for integral functionals, introducing the notion of the *Lavrentiev Gap*. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be sequentially lower semicontinuous on $X$, and let $Y \subset X$. They then define

$$F_Y(u) := \begin{cases} F(u), & \text{if } u \in Y, \\ +\infty, & \text{if } u \in X \setminus Y \end{cases}$$

and they consider the lower semicontinuous envelope of $F_Y$ on $X$:

$$\mathrm{sc}^-(F_Y) := \sup\{\, G \text{ l.s.c. on } X \mid G \leq F \text{ on } Y \,\}.$$

The Lavrentiev Gap at $u$ is defined as the difference between $\mathrm{sc}^-(F_Y)(u)$ and $F(u)$, that is,

$$\mathrm{sc}^-(F_Y)(u) = F(u) + L(u).$$

They consider in particular the case where $F = F_p$ is an integral functional with Lagrangian $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$. We recall that

$$F_p(u) = \int_\Omega f\big(x, u(x), \nabla u(x)\big)\, dx, \qquad u \in \varphi + W_0^{1,p}(\Omega),$$

and that, for every $u \in \varphi + W_0^{1,p}(\Omega)$,

$$\mathrm{sc}^-(F_{W^{1,q}})(u) = \inf\Big\{ \liminf_{n\to\infty} \int_\Omega f\big(x, u_n(x), \nabla u_n(x)\big)\, dx \,\Big|\, u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega)\Big\}.$$

We note that if the Lavrentiev Gap is identically zero, then the Lavrentiev Phenomenon does not occur. Many authors have established the non occurrence of the Lavrentiev Phenomenon by proving the absence of the Lavrentiev gap under the assumption that $f$ is convex with respect to its last variable. We report [16, 17, 18, 20, 21] for the multidimensional scalar case and [37, 38, 41] for the vectorial case.

## 1.3 Convexity and sequential lower semicontinuity

In the multidimensional scalar case, there is a strong connection between the weak sequential lower semicontinuity of $F_p$ in $W^{1,1}(\Omega)$ and the convexity of the Lagrangian with respect to its last variable. This relationship was first investigated by Tonelli [66] in the one-dimensional scalar case.

We recall Theorem 1.3 from [32]:

**Theorem 1.3.1.** *Let $n, N \in \mathbb{N}$, $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a non-negative continuous function. Define*

$$I(u) = \int_\Omega f\big(x, u(x), \nabla u(x)\big)\, dx.$$

1. *If the function $\xi \mapsto f(x, u, \xi)$ is convex, then $I$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega)$.*
2. *Conversely, if $n = 1$ or $N = 1$, and $I$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega)$, then $\xi \mapsto f(x, u, \xi)$ is convex.*

A more general version of this theorem, under weaker assumptions on the Lagrangian, can be found in [40].

For completeness, we recall that in the multidimensional vectorial case, convexity with respect to the last variable is replaced by the notion of quasiconvexity. A locally bounded Borel-measurable function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is quasiconvex if

$$\int_{B(0,1)} \big(f(A + \nabla\psi(x)) - f(A)\big)\, dx \geq 0$$

for all $A \in \mathbb{R}^{m\times d}$.

Acerbi and Fusco proved in [1] the following theorem.

**Theorem 1.3.2.** *Let* $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m} \to \mathbb{R}$*,* $(x, v, A) \mapsto f(x, v, A)$*, be a Carathéodory function such that*

$$0 \le f(x, v, A) \le a(x) + C\big(|v|^p + |A|^p\big),$$

*where* $C > 0$ *is a constant and* $a(x)$ *is an integrable function. Then the functional*

$$\mathcal{F}[u] = \int_\Omega f\big(x, u(x), \nabla u(x)\big)\, dx$$

*is sequentially weakly lower semicontinuous in the Sobolev space* $W^{1,p}(\Omega, \mathbb{R}^m)$*, with* $p > 1$*, if and only if* $f$ *is quasiconvex.*

## 1.4 Integral representation of the lower semicontinuous envelope

In the multidimensional scalar case, when the Lagrangian is no longer convex with respect to the gradient variable, one considers the weak lower semicontinuous envelope of $F_p$ in $\varphi + W_0^{1,p}(\Omega)$.

Given $u \in \varphi + W_0^{1,p}(\Omega)$, the objective is to construct a sequence $(u_n) \subset \varphi + W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))\, dx, \tag{1.4.1}$$

where $f^{**}$ denotes the bipolar (convex envelope) of $f$.

In terms of the integral representation of the lower semicontinuous envelope, the problem can be formulated as proving the following equality:

$$\inf\left\{ \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \;\middle|\; u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega) \right\} = \int_\Omega f^{**}(x, u(x), \nabla u(x))\, dx, \tag{1.4.2}$$

where $(u_n) \subset W_\varphi^{1,p}(\Omega)$.

The integral representation of the lower semicontinuous envelope has been extensively studied in the literature, starting with the works [63], [40], and [55]. Also in this case there is an extensive literature, we report only some paper focused on relaxing the continuity assumptions with respect to $u$ ([24], [25], [26], [27], [33]).

In [40], a constructive method is presented to solve the approximation problem for integral functionals with Carathéodory Lagrangians in the case $p = \infty$. The authors first consider a function

$$\tilde{f}(x, \xi) : \Omega \times B_K(0) \to \mathbb{R}$$

depending only on the spatial variable and the gradient, defined for gradients in a bounded set. For an affine function $u$, they show

$$f^{**}(x, \nabla u(x)) \le \sum_{i=1}^{n+1} f(x, \xi_i^x) + \varepsilon,$$

and use the Scorza-Dragoni Theorem to decompose the domain into small balls where the spatial variable can be treated as constant. They further subdivide these balls into (at most) $n+1$ regions where the approximating function is affine, with values $\xi_i^x$, and apply the McShane lemma to match the boundary data. Finally, techniques from numerical and functional analysis are used to extend the construction to the general case.

In [55], the authors proved the representation formula (1.4.2) in the case $p = +\infty$ for Lagrangians that are upper semicontinuous with respect to the gradient variable $\xi$ and continuous with respect to $u$, uniformly over bounded sets of $\xi$. The key idea is to show that the right-hand side of (1.4.2) defines an integral functional whose Lagrangian is exactly the bipolar $f^{**}$ of the original Lagrangian.

We report, for the sake of completeness, that other types of relaxation are also studied considering other definitions of convexity with respect to the last variable for the Lagrangian. We cite [61] and [48] for the relaxation in $L^\infty$ in the vectorial case. In these works supremal representations are presented in nonlocal and local setting, respectively.

# Chapter 2

# Description of the results

The aim of this thesis is to investigate the absence of the Lavrentiev Phenomenon via the integral representation of the lower semicontinuous envelopes for a broad class of integral functionals. The work addresses both convex and non-convex settings, including functionals with discontinuous or unbounded Lagrangians, and examines the existence of approximating sequences that preserve both energy and convergence properties.

The results of this thesis are presented in Chapters 3, 4 and 5 and each of them contains one of the following papers

Chapter 3: Tommaso Bertin, *Integral representations of lower semicontinuous envelopes and Lavrentiev Phenomenon for non continuous Lagrangians*, [10].

Chapter 4: Tommaso Bertin, Giulia Treu, *Integral representations of lower semicontinuous envelopes for non convex, possibly unbounded, Lagrangians*, [12].

Chapter 5: Tommaso Bertin, Paulin Huguet, *Relaxation of Non-Convex Integral Functionals in the Multidimensional Scalar Case*, [11].

Here we resume the main results obtained.

## 2.1 Absence of the Lavrentiev Phenomenon for Non-Continuous Lagrangians

The Chapter 3 addresses integral representations for relaxed functionals. We assume the following hypotheses:

**Hypothesis 2.1.1.** Let $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. We assume:

a) $f$ is Borel measurable in all its variables;

b) For every bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$, there exists $a \in L^1(\Omega)$ such that

$$|f(x, u, \xi)| \leq a(x) \quad \text{for all } (u, \xi) \in B;$$

c) For every $u \in W^{1,\infty}(\Omega)$, every bounded set $\tilde{B} \subset \mathbb{R}^N$, and every $\delta > 0$, there exists a compact set $T \subset \Omega$ with $|\Omega \setminus T| < \delta$ such that $f(x, u(x), \xi)$ is continuous in $x \in T$ uniformly with respect to $\xi \in \tilde{B}$;

d) For almost every $x \in \Omega$, the map $u \mapsto f(x, u, \xi)$ is continuous uniformly with respect to $\xi$ in bounded sets.

The main result of the chapter is the following theorem:

**Theorem 2.1.2.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz domain, and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Assumptions 2.1.1. Then, for every $\overline{u} \in W^{1,\infty}(\Omega)$, there exists a sequence $u_n \in \overline{u} + W_0^{1,\infty}(\Omega)$ such that*

$$\lim_{n\to\infty} \|u_n - \overline{u}\|_{L^\infty(\Omega)} = 0$$

*and*

$$\lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx = \int_\Omega f^{**}(x, \overline{u}(x), \nabla \overline{u}(x))\, dx.$$

*Moreover,*

$$\mathrm{sc}^-(F_\infty)(\overline{u}) = \int_\Omega f^{**}(x, \overline{u}(x), \nabla \overline{u}(x))\, dx,$$

*where $\mathrm{sc}^-(F_\infty)$ denotes the lower semicontinuous envelope of*

$$F_\infty(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

*with respect to the weak* topology of $W^{1,\infty}(\Omega)$.*

In particular, under Hypothesis 2.1.1 on the original Lagrangian, for every $u \in W^{1,\infty}(\Omega)$, we construct a sequence $u_n \in u + W_0^{1,\infty}(\Omega)$ such that

$$u_n \to u \quad \text{in } L^\infty(\Omega)$$

and formula (1.4.1) holds, that is,

$$\int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \to \int_\Omega f^{**}(x, u(x), \nabla u(x))\, dx.$$

Moreover, the lower semicontinuous envelope of an integral functional with a non-convex Lagrangian, with respect to the weak* topology of $W_\varphi^{1,\infty}(\Omega)$, can be represented as an integral functional whose Lagrangian is the bipolar of the original one. More precisely, the following representation formula holds:

$$\inf \left\{ \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx \;\middle|\; u_n \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega) \right\} = \int_\Omega f^{**}(x, u(x), \nabla u(x))\, dx.$$

The main novelty of the results presented in this chapter and in the whole thesis is that we do not assume continuity of the Lagrangian with respect to the gradient variable. The techniques used in the proofs are adapted from the classical reference [40]. To extend this approach to the non-continuous case, we assume a uniform version of the Lusin Theorem in the spatial variable, together with continuity with respect to $u$, uniform with respect to $\xi$ on bounded sets.

The main idea of the proof is to construct an approximating sequence for an affine function $u$ with the Lagrangian

$$\tilde{f}(x, \xi) = f_K(x, \overline{u}(x), \xi),$$

where

$$f_K(x, u, \xi) = \begin{cases} f(x, u, \xi), & \text{if } \|\xi\| \le K, \\ +\infty, & \text{if } \|\xi\| > K. \end{cases}$$

One then writes

$$f^{**}(x, \nabla u(x)) \le \sum_{i=1}^{n+1} f(x, \xi_i^x) + \varepsilon,$$

and uses the uniform Lusin Theorem to decompose the domain into small balls where the spatial variable can be treated as constant. Then, all small balls are further subdivided into at most $n+1$ regions where the approximating function is affine (with values $\xi_i^x$), and we use the McShane Lemma to match the boundary data. This last step requires the boundedness of the Lagrangian.

Next, we extend the construction to a general function $\overline{u} \in W^{1,\infty}(\Omega)$, approximating it using a technique similar to finite element methods. We also introduce dependence on the variable $u$, exploiting the uniform continuity with respect to $u$ for bounded $\xi$, to establish

$$\inf \left\{ \liminf \int_\Omega f(x, u_n, \nabla u_n)\, dx \;\middle|\; u_n \overset{*}{\rightharpoonup} \overline{u},\, \|\nabla u_n\|_\infty < K \right\} = \int_\Omega (f_K)^{**}(x, \overline{u}(x), \nabla \overline{u}(x))\, dx,$$

with $u_n \in \overline{u} + W_0^{1,\infty}(\Omega)$.

Passing to the limit as $K \to +\infty$, we obtain

$$\inf \left\{ \liminf \int_\Omega f(x, u_n, \nabla u_n)\, dx \;\middle|\; u_n \overset{*}{\rightharpoonup} \overline{u} \right\} = \int_\Omega f^{**}(x, \overline{u}(x), \nabla \overline{u}(x))\, dx,$$

with $u_n \in \overline{u} + W_0^{1,\infty}(\Omega)$. Using a diagonal argument, we can construct an approximating sequence $\overline{u}_n$ such that

$$\lim_{n\to\infty} \|\overline{u}_n - \overline{u}\|_\infty = 0,$$

and

$$\lim_{n\to\infty} \int_\Omega f(x, \overline{u}_n(x), \nabla \overline{u}_n(x))\, dx = \int_\Omega f^{**}(x, \overline{u}(x), \nabla \overline{u}(x))\, dx.$$

In the last part of the chapter, we exploit a recent result ([20]) to prove the absence of the Lavrentiev Phenomenon between the spaces $\varphi + W_0^{1,\infty}(\Omega)$ and $\varphi + W_0^{1,1}(\Omega)$ for autonomous functionals whose Lagrangian is non-convex and non-continuous with respect to the gradient variable. In the autonomous case, one can work under the following simpler set of hypotheses.

**Hypothesis 2.1.3.** The function $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies:

i) $f(u, \xi)$ is Borel measurable;

ii) $f(u, \xi)$ is continuous with respect to $u$, uniformly for $\xi$ in each bounded subset of $\mathbb{R}^N$;

iii) For every $u \in \mathbb{R}$, $f(u, \cdot)$ is bounded on bounded subsets of $\mathbb{R}^N$.

We now state the following theorem.

**Theorem 2.1.4.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, and let $\varphi \in W^{1,\infty}(\Omega)$. Assume that $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies Hypothesis 2.1.3 and is uniformly superlinear; that is, there exists a superlinear function $\theta : \mathbb{R}^N \to \mathbb{R}$ such that*

$$f(u, \xi) \geq \theta(\xi) \quad \text{for all } u \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N.$$

*Assume further that $f^{**}$ is continuous in both variables. Then, for every $u \in \varphi + W_0^{1,1}(\Omega)$,*

$$sc^-(\overline{F_1})(u) = \int_\Omega f^{**}(u(x), \nabla u(x))\, dx,$$

*where $sc^-(\overline{F_1})$ denotes the lower semicontinuous envelope of*

$$\overline{F_1}(u) = \begin{cases} \displaystyle\int_\Omega f(u(x), \nabla u(x))\, dx, & \text{if } u \in \varphi + W_0^{1,\infty}(\Omega), \\ +\infty, & \text{if } u \in \varphi + W_0^{1,1}(\Omega) \setminus W_0^{1,\infty}(\Omega), \end{cases}$$

*with respect to the weak topology of $W^{1,1}(\Omega)$.*

Thus, under these growth and regularity assumptions, for every $u \in W^{1,1}\varphi(\Omega)$ with Lipschitz boundary data, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,1}(\Omega)$$

and

$$\int_\Omega f(x, u_n(x), \nabla u_n(x))dx \to \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

In other words, the lower semicontinuous envelope on $W^{1,1}(\Omega)$ of an integral functional originally defined on $W^{1,\infty}(\Omega)$ admits an integral representation with Lagrangian given by the bipolar of the original Lagrangian.

The main idea of the proof is to apply [17, Theorem 1.1] to the relaxed functional with Lagrangian $f$. For every $u \in W^{1,1}_\varphi(\Omega)$, there exists an approximating sequence $(u_k) \subset W^{1,\infty}_\varphi(\Omega)$ such that

$$u_k \to u \quad \text{in } W^{1,1}(\Omega),$$

and

$$\lim_{k\to\infty} \int_\Omega f(x, u_k(x), \nabla u_k(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

For each $u_k$, we can construct a sequence $(u_k^n) \subset W^{1,\infty}_\varphi(\Omega)$ such that

$$u_k^n \to u_k \quad \text{in } L^\infty(\Omega),$$

and

$$\lim_{n\to\infty} \int_\Omega f(x, u_k^n(x), \nabla u_k^n(x))dx = \int_\Omega f^{**}(x, u_k(x), \nabla u_k(x))dx.$$

By a diagonal argument, we can then extract a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that

$$\lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx,$$

and, thanks to the uniform superlinearity of $f(x, u, \cdot)$,

$$u_n \rightharpoonup u \quad \text{in } W^{1,1}(\Omega).$$

## 2.2 Absence of the Lavrentiev Phenomenon for Unbounded Lagrangians

In Chapter 4, we extend the validity of (1.4.1) to the case where the Lagrangian depends only on the gradient and satisfies very weak assumptions, including situations in which it is finite only on a countable set. The assumptions are as follows.

(HF) (i) There exist $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$f(\zeta) \geq c_1|\zeta| + c_2 \qquad \text{for every } \zeta \in \mathbb{R}^N.$$

(ii) For every $\xi \in \mathbb{R}^N$, either $f^{**}(\xi) = f(\xi)$ or there exist $k \in \{1, \dots, N\}$, $\{\xi_i\}_{i=1}^{k+1} \subset \mathbb{R}^N$ and $\{\alpha_i\}_{i=1}^{k+1} \subset \mathbb{R}$ such that $\alpha_i > 0$ for every $i = 1, \dots, k+1$, $\sum_{i=1}^{k+1} \alpha_i = 1$, $\sum_{i=1}^{k+1} \alpha_i \xi_i = \xi$, and

$$\sum_{i=1}^{k+1} \alpha_i f(\xi_i) = f^{**}(\xi).$$

Moreover, we also assume that $\dim \operatorname{Span}(\xi_1, \dots, \xi_{k+1}) = k$.

(iii) For every $R > 0$ there exist $N + 1$ vectors $\zeta_j \in \mathbb{R}^N$ such that $\overline{B}(0, R) \subset \overline{\text{co}}\big(\cup_{j=1}^{N+1} \{\zeta_j\}\big)$ and

$$f(\zeta_j) = f^{**}(\zeta_j) < +\infty \qquad \text{for every } j = 1, \dots, N+1.$$

The main theorem is the following.

**Theorem 2.2.1.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, and let $f : \mathbb{R}^N \to \mathbb{R} \cup +\infty$ satisfy the assumption* (HF). *Let $u \in \varphi + W_0^{1,1}(\Omega)$ be differentiable almost everywhere.*

*Then, for every $\varepsilon > 0$, there exists a function $v \in \varphi + W_0^{1,1}(\Omega)$, differentiable almost everywhere, such that*

$$\int_\Omega f(\nabla v(x))dx \leq \int_\Omega f^{**}(\nabla u(x))dx + \varepsilon.$$

Thus, for every $u \in W_\varphi^{1,1}(\Omega)$ that is differentiable almost everywhere and for every $\varepsilon > 0$, we can construct an approximating sequence with the same boundary datum, converging to $u$, such that (1.4.1) holds.

The main novelty of this result is that the Lagrangian is assumed to be neither continuous nor bounded with respect to the gradient variable. Consequently, the approximating sequence must be constructed in such a way that its gradients take values only where the original Lagrangian is finite. Our approach is inspired by techniques developed in the study of non-convex problems in the Calculus of Variations and in non-convex Differential Inclusions, first introduced by Cellina in [30]. In particular, we rely on a recent refinement of this method presented in [31].

The main idea is to consider, for almost every $x \in \Omega$, a small neighborhood where $u$ is differentiable, and to approximate it with a pyramidal function whose differential takes values only in the region where the Lagrangian is finite. The main difficulty in applying this technique is that we do not assume, as in [30] and in subsequent works such as [31], that $(\nabla u(x_0), f^{**}(\nabla u(x_0)))$ lies in the interior of an $N$-dimensional face of the epigraph of $f^{**}$. In other words, we address the case where the value of $f^{**}$ is not necessarily representable as a convex combination of exactly $N+1$ points.

In this case, the approximating function belongs to $W_\varphi^{1,1}(\Omega)$. Moreover, if $u \in W_\varphi^{1,\infty}(\Omega)$, then the approximating function $v$ also lies in $W_\varphi^{1,\infty}(\Omega)$.

In the last part of the chapter, we apply the results of [17] to prove the following theorem.

**Theorem 2.2.2.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$, $\varphi \in W^{1,\infty}(\Omega)$, and assume that $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfies the assumption* (HF) *and is superlinear. Then, for every $u \in \varphi + W_0^{1,1}(\Omega)$, there exists a sequence $(u_n) \subset \varphi + W_0^{1,\infty}(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \textit{in } W^{1,1}(\Omega),$$

*and*

$$\lim_{n\to\infty} \int_\Omega f(\nabla u_n(x))\,dx = \int_\Omega f^{**}(\nabla u(x))\,dx.$$

This result shows that the Lavrentiev Phenomenon does not occur between $\varphi + W_0^{1,1}(\Omega)$ and $\varphi + W_0^{1,\infty}(\Omega)$, and it yields a representation of the lower semicontinuous envelope of an integral functional with an unbounded Lagrangian as an integral functional whose Lagrangian is the bipolar of the original one.

## 2.3 Convergence in $W^{1,p}$ and absence of the Lavrentiev Gap

In Chapter 5, we address questions arising from Chapter 3. The first problem is, given $u \in W^{1,\infty}(\Omega)$, to identify geometric conditions that guarantee the existence of a sequence $u_n \in u + W_0^{1,\infty}(\Omega)$ such that

$$u_n \overset{*}{\rightharpoonup} u$$

and formula (1.4.1) holds, that is,

$$\lim_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\,dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))\,dx.$$

Such a sequence attains the infimum in (1.4.2).

Given $u \in W_\varphi^{1,\infty}(\Omega)$, the convergence of the approximating sequence with respect to the weak-$*$ topology of $W_\varphi^{1,\infty}(\Omega)$ is closely related to the following condition: for every $K > 0$, there exists $K' > K$ such that

$$\left((f_{K'})^{**}\right)_K = (f^{**})_K \tag{2.3.1}$$

where $f_K$ is defined as in the Chapter 3.

If condition (2.3.1) is satisfied for every $K > 0$, then for every $u \in W^{1,\infty}_{\varphi}(\Omega)$ there exists an approximating sequence $u_n \in W^{1,\infty}_{\varphi}(\Omega)$ such that

$$u_n \xrightharpoonup{*} u \quad \text{in } W^{1,\infty}(\Omega).$$

Conversely, if such a sequence exists, then there exists $K' > \|u\|_{W^{1,\infty}}$ such that

$$(f_{K'})^{**}(x, u(x), \nabla u(x)) = f^{**}(x, u(x), \nabla u(x))$$

for almost every $x \in \Omega$.

We have identified two simple sufficient conditions, involving the so-called de-touching set and the growth of $f$ at $+\infty$.

**Lemma 2.3.1.** *Let $f : \mathbb{R}^N \to [0, \infty]$, and let $A \subset B \subset \mathbb{R}^N$ be such that for any $\xi_A \in A$ and $\xi_B \notin B$, there exists $\xi \in [\xi_A, \xi_B] \cap B$ satisfying*

$$f(\xi) = f^{**}(\xi).$$

*Then*

$$(\tilde{f}_B)^{**} = f^{**} \quad \text{on } A.$$

This condition is satisfied in many situations that are easy to verify. Some examples are given in the following corollary.

**Corollary 2.3.2.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$, and let $I$ be a compact interval of $\mathbb{R}$.*

- *If there exists $K > 0$ such that*

$$f(x, u, \xi) = f^{**}(x, u, \xi), \quad \text{for a.e. } x \in \Omega,\ \forall u \in I,\ \forall \xi \in \partial B_K, \tag{2.3.2}$$

  *then*

$$(\tilde{f}_K)^{**}(x, u, \xi) = f^{**}(x, u, \xi), \quad \text{for a.e. } x \in \Omega,\ \forall u \in I,\ \forall \xi \in B_K.$$

  *In particular, if (2.3.2) holds for a sequence $(K_n)$ with $K_n \to +\infty$ and for every compact interval $I$, then $f$ satisfies (2.3.1).*

- *Assume that the connected components of the sets*

$$\mathcal{D}(f, x, u) := \{\xi \in \mathbb{R}^N : f^{**}(x, u, \xi) < f(x, u, \xi)\}, \quad x \in \Omega,\ u \in I,$$

  *are uniformly bounded, i.e., there exists $M > 0$ such that for a.e. $x \in \Omega$ and all $u \in I$, every connected component of $\mathcal{D}(f, x, u)$ has diameter smaller than $M$. If this holds for all compact intervals $I$, then $f$ satisfies (2.3.1).*

The second sufficient condition is the uniform superlinearity of $f(x, u, \cdot)$ with respect to $(x, u)$.

**Theorem 2.3.3.** *Let $\Gamma$ be a set and $f : \Gamma \times \mathbb{R}^N \to [0, \infty]$ satisfy:*

- *there exists a superlinear function $\Phi : \mathbb{R}^N \to [0, \infty)$ such that*

$$f(s, \xi) \geq \Phi(\xi), \quad \forall s \in \Gamma,\ \forall \xi \in \mathbb{R}^N;$$

- *for any $\rho > 0$, there exists $\rho' \geq \rho$ such that $(\tilde{f}_{\rho'})^{**}$ is bounded on $\Gamma \times B_\rho$.*

*Then, for any $K > 0$, there exists $K' \geq K$ such that*

$$(\tilde{f}_{K'})^{**} = f^{**} \quad \text{on } \Gamma \times B_K.$$

The main idea in the proofs of these results is to write

$$f^{**}(x,u,\xi) = \sup\Big\{ l(\xi) \;\Big|\; l \text{ affine},\ l(\cdot) \leq f(x,u,\cdot) \Big\},$$

and to show that no affine function lies strictly above $f^{**}$ and below $f$.

We note in particular that, under our assumptions, this implies the continuity of $f^{**}$ with respect to $(u,\xi)$. This issue has been recently addressed, for example, in [55].

In the second part, we provide a generalization of the result in the Chapter 3 concerning the non-occurrence of the Lavrentiev phenomenon between $W^{1,1}_{\varphi}(\Omega)$ and $W^{1,\infty}_{\varphi}(\Omega)$ in the autonomous case.

**Theorem 2.3.4.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfy Hypothesis 2.1.3, and let $\varphi \in W^{1,\infty}(\Omega)$. Then, for any $u \in W^{1,1}_{\varphi}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that*

$$u_n \to u \quad \text{in } L^1(\Omega),$$

*and*

$$\int_\Omega f(u_n(x), \nabla u_n(x))\,dx \to \int_\Omega f^{**}(u(x), \nabla u(x))\,dx.$$

*Moreover:*

- *If there exists a superlinear function $\Phi : \mathbb{R}^N \to [0,\infty)$ such that*

  $$f(u,\xi) \geq \Phi(\xi), \quad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

  *then the sequence $(u_n)$ can be chosen so that*

  $$u_n \rightharpoonup u \quad \text{weakly in } W^{1,1}(\Omega);$$

- *If, for some $p \in (1,\infty)$, there exist constants $c_1, c_2 > 0$ such that*

  $$f(u,\xi) \geq c_1|\xi|^p - c_2, \quad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

  *and $u \in W^{1,p}(\Omega)$, then the sequence $(u_n)$ can be chosen so that*

  $$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega).$$

Furthermore, we propose conditions on an autonomous Lagrangian, which may be nonconvex and, in some cases, noncontinuous with respect to the last variable, such that for every function $u \in W^{1,p}_{\varphi}(\Omega)$ there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ satisfying

$$u_n \to u \quad \text{strongly in } W^{1,p}(\Omega),$$

and

$$\int_\Omega f(u_n(x), \nabla u_n(x))\,dx \to \int_\Omega f(u(x), \nabla u(x))\,dx.$$

In particular, we prove this result in the case where

$$f(u(x), \nabla u(x)) = f^{**}(u(x), \nabla u(x)).$$

The main novelty, even in this case, is that the Lagrangian $f$ is not assumed to be continuous with respect to the last variable. Another situation considered is when the Lagrangian is continuous and dominated by a convex function.

The key idea of the proofs is to apply the main theorem of [17] to the relaxed functional, or to an appropriate auxiliary functional, in order to construct a sequence with the same boundary datum that converges to $u$ both in $W^{1,p}(\Omega)$ and in energy. Then, using Fatou Lemma, we show that for each term of the auxiliary functional the sequence converges to $u$ in $W^{1,p}(\Omega)$ and in energy.

In the final part, we apply recent results ([14]) to extend our approach to the non-autonomous case, under a so-called "anti-jump" condition: for every $L = (L_1, L_2) \in (0, \infty)^2$, there exists a constant $C_L > 0$ such that for almost every $x \in \Omega$, for every $(u, \xi) \in [-L_1, L_1] \times \mathbb{R}^N$, and for every $\varepsilon > 0$, it holds

$$(g_\varepsilon^-)^{**}(x, u, \xi) \leq \frac{L_2}{\varepsilon^N} \quad \Rightarrow \quad g(x, u, \xi) \leq C_L \Big( 1 + (g_\varepsilon^-)^{**}(x, u, \xi) \Big), \tag{$\mathcal{H}_1$}$$

where

$$g_\varepsilon^-(x, u, \xi) = \operatorname*{ess\,inf}_{y \in \Omega \cap B_\varepsilon(x)} g(y, u, \xi),$$

and $(g_\varepsilon^-)^{**}(x, u, \xi)$ denotes its convex envelope.

In particular, we prove that, given $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,1}(\Omega),$$

and formula (1.4.1) holds also in the non-autonomous case. Moreover, all results of Section 2 can be extended, provided that the anti-jump condition is satisfied for every auxiliary function.

# Chapter 3

# The case of Lagrangians depending on $(x, u, \nabla u)$

In this chapter, we report the paper *Integral representations of lower semicontinuous envelopes and Lavrentiev Phenomenon for non continuous Lagrangians* ([10]).

## 3.1 Introduction

We consider the functional

$$F_q(u) = \int_\Omega f(x, u(x), \nabla u(x))dx \qquad u \in \varphi + W_0^{1,q}(\Omega)$$

where $1 \leq q \leq \infty$, $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $\varphi \in W^{1,q}(\Omega)$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a suitable Borel function that is not necessarily assumed to be convex in the last variable. Due to the lack of convexity of the Lagrangian, the functional is not sequentially lower semicontinuous with respect to the weak topology in the case $1 \leq q < \infty$, resp weak$^*$ topology in the case $q = \infty$. We address the problem of representing the sequential weak lower semicontinuous envelope of $F_q$.
To be more precise, for every $1 \leq p \leq q$, we define the functional

$$\overline{F_p}(u) = \begin{cases} F_q(u) & \text{if } u \in \varphi + W_0^{1,q}(\Omega) \\ +\infty & \text{if } u \in \varphi + W_0^{1,p}(\Omega) \setminus W_0^{1,q}(\Omega) \end{cases}$$

and we are interested in determining sufficient conditions for the following identity to hold

$$\mathrm{sc}^-(\overline{F_p})(u) = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx \qquad \forall u \in \varphi + W_0^{1,p}(\Omega) \tag{3.1.1}$$

where $\mathrm{sc}^-(\overline{F_p})(u)$ denotes the greatest sequentially weak (weak$^*$ in the case of $p = \infty$) lower semicontinuos functional, with respect to $W^{1,p}(\Omega)$, that is less or equal to $\overline{F_p}$ and $f^{**}$ is the convexified function of $f$ w.r.t. the last variable. In the literature this problem has been studied from two different points of view. From one hand if $f$ is not convex with respect to the last variable then $F_q$ is not weakly lower semicontinuous and it is interesting to consider $sc^-(F_q)$. On the other hand, even in the case where $f$ is convex w.r.t. the last variable, it is important to represent $sc^-(\overline{F_p})$ for $p < q$. In this chapter we put together the two approach. Despite the fact that the identity in (3.1.1) seems very natural, we know that it does not hold true, for the case $p < q$, even for 'very regular' functionals. In fact every functional exhibiting the so called *Lavrentiev phenomenon* (introduced for the first time by Lavrentiev in [53]) cannot satisfy the identity (3.1.1). In the classic example by Manià ([54]) it has been shown that

$$\min_{\mathrm{id}+W_0^{1,1}([0,1])} \int_0^1 (x - u^3(x))^2 |u'(x)|^6 \, dx < \inf_{\mathrm{id}+W_0^{1,\infty}([0,1])} \int_0^1 (x - u^3(x))^2 |u'(x)|^6 \, dx$$

and we notice that, in this case, the Lagrangian is not only smooth in the three variables, but it is also convex w.r.t. the derivative variable. Further examples in which the minimum, or the infimum, in $u \in \varphi + W_0^{1,p}(\Omega)$ is strictly less than the minimum, or the infimum, in $u \in \varphi + W_0^{1,q}(\Omega)$, with $p < q$, can be found for one-dimensional case in [7], [51] and for multidimensional case in [69]. In [39] the authors present some $p, q$ growth conditions to reach the regularity of the minimum for $sc^-(\overline{F_p})$. In the present chapter we focus our attention on the scalar multidimensional case but we have to mention, for the sake of completeness, that there are examples also in the vectorial case (see [9] for a wide list of examples). The problem of detecting conditions that prevent Lavrentiev phenomenon is important in particular for numerical approximations and engineering applications. The problem is well studied in dimension 1 with very weak hypotheses about the Lagrangian ([3], [58]). In higher dimension usually Lagrangians are assumed convex ([14], [17], [18], [20], [29], [30]).

The approach to the Lavrentiev phenomenon as a problem of representation of the relaxed functional has been considered for the first time, as far as we know, in [28] where the authors introduced the notion of Lavrentiev gap at a fixed $u \in \varphi + W_0^{1,p}(\Omega)$. Precisely they say that the *Lavrentiev gap* occurs at $u$, for convex Lagrangians, when the difference between $\mathrm{sc}^-(\overline{F_p})(u)$ and $\int_\Omega f(x, u(x), \nabla u(x))dx$ is strictly positive.

Starting from [63], [40] and [55] the problem of integral representation for $sc^-(\overline{F_p})$ was investigated by many authors ([23], [24], [25], [26], [27], [33], [45], [51]). Many of these papers are devoted to avoid the assumption of continuity of the Lagrangian with respect to $u$ and study the property of the functional $sc^-(\overline{F_p})$ over different subsets of $\Omega$. The main goal of our chapter is to prove the validity of (3.1.1) in the scalar multidimensional case without hypothesis of continuity with respect to $\xi$ for non autonomous Lagrangians in the case $p = q = \infty$ (cfr Theorem 3.3.3) and for autonomous Lagrangians in the case $p = 1, q = \infty$ (cfr Theorem 3.5.14).

In section 3.2 we start considering Lagrangians depending by $x$ and $\nabla u$. We take inspiration by the constructive method by Ekeland and Temam in [40] to find for every $u \in \varphi + W_0^{1,\infty}(\Omega)$ a function $v \in \varphi + W_0^{1,\infty}(\Omega)$ sufficiently near to $u$ with $F_\infty(v)$ sufficiently near to $\int_\Omega f^{**}(x, \nabla u(x))dx$. We modified the proof presented in [40, Proposition 3.2, page 330] to deal with the absence of continuity of $f$ w.r.t. $\nabla u$. We notice, in particular, the fact that our construction uses the Vitali covering Theorem. The main advantage of this argument is that it allows us to construct an explicit sequence $u_n$ converging to $u$ in $L^\infty(\Omega)$ and such that the value $F_\infty(u_n)$ converges to $\int_\Omega f^{**}(x, \nabla u(x))dx$. We deduce also a first relaxation result and the validity of (3.1.1) in the case $p = q = \infty$.

In section 3.3 we still consider the case $p = q = \infty$ and we extend the results of section 3.2 to the case of a general Lagrangian $f(x, u, \xi)$ satisfying a suitable set of assumption that we will denote by Hypothesis 3.3.1. In particular we use a truncation method for $f$ with respect to the variable $\xi$ considering the auxiliary function $f_K(x, u, \xi)$ equal to $f(x, u, \xi)$ if $\|\xi\| \le K$ and $+\infty$ otherwise. It is interesting to note that for $f_K$ it is possible to find a sequence $u_{n,K} \overset{*}{\rightharpoonup} u$ such that $\int_\Omega f_K(x, u_{n,K}(x), \nabla u_{n,K}(x))dx$ converges to $\int_\Omega f_K^{**}(x, u(x), \nabla u(x))dx$. Later on, for $f(x, u, \xi)$ we pass to the limit $K \to \infty$ and it can happen, in general, that there does not exist a sequence such that $u_n \overset{*}{\rightharpoonup} u$ and $F_\infty(u_n)$ converges to $\int_\Omega f^{**}(x, u(x), \nabla u(x))dx$.

In section 3.4 we observe that under our hypotheses, the non occurrence of Lavrentiev Phenomenon for the relaxed functional $\int_\Omega f^{**}(x, u(x), \nabla u(x))dx$ implies the non occurrence of Lavrentiev Phenomenon for the original functional.

In section 3.5 we firstly apply the general results of sections 3.3 and 3.4 to autonomous Lagrangians. This is a case of great interest in many applications of the Calculus of Variations and has the peculiarity that the assumptions in these case appear more natural. We focus on this family of Lagrangians to apply a recent result by Bousquet ([17]) which prove, in the case of autonomous, continuous and convex Lagrangian, for every $u \in \varphi + W_0^{1,1}(\Omega)$ the existence of a sequence $u_n \in \varphi + W_0^{1,\infty}(\Omega)$ such that $u_n \to u$ in $W^{1,1}$ and $F_\infty(u_n)$ converges to $F_1(u)$. This theorem allows us, under suitable hypothesis, to exclude the Lavrentiev phenomenon for autonomous non convex Lagrangians and to prove the validity of the (3.1.1) on the case where $q = \infty$ and $p = 1$. The validity of (3.1.1) with this special choice of $p$ and $q$ implies that the value of the functional

$$\int_\Omega f^{**}(x, u(x), \nabla u(x))dx$$

can be approximated evaluating

$$\int_\Omega f(x, u(x), \nabla u(x))dx$$

on a suitable sequence of Lipschitz functions. This fact has a great impact on numerical estimates of the functional since the Finite Element Method, for example, relies on the use of Lipschitz functions.

Future developments will be to find some general geometrical condition on Lagrangian $f$ to apply the result of ([17]) to $f^{**}$ and try to extend the relaxation results to the non autonomous cases. For the sake of completeness we cite an interesting paper ([14]), which has to appear, about an approximation result similar to ([17]) in the case of non autonomous Lagrangians continuous and convex.

## 3.2 An approximation result for Lagrangians depending only on $x$ and $\nabla u$

In this section we consider a Lagragian $\tilde{f}(x, \xi)$ which satisfies following Hypotheses.

**Hypothesis 3.2.1.** Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$, $B_K(0) \subset \mathbb{R}^n$ the open ball with center 0 and radius $K > 0$ and $\tilde{f} : \Omega \times B_K(0) \to \mathbb{R}$ be a function such that

a) $\tilde{f}(x, \xi) : \Omega \times B_K(0) \to \mathbb{R}$ coincides a.e. with a borelian function;

b) there exists $a \in L^1(\Omega)$ such that $0 \le \tilde{f}(x, \xi) \le a(x)$ for every $\xi \in B_K(0)$ ;

c) for all $\delta > 0$ then there exists $T \subset \Omega$ compact such that $|\Omega \setminus T| < \delta$ and $\tilde{f}(x, \xi)_{|T \times B_K(0)}$ is continuous with respect to $x$ uniformly as $\xi$ varies in $B_K(0)$ .

**Remark 3.2.2.** We do not assume any continuity for $\tilde{f}$ with respect to $\xi$. This is the main novelty with respect to [40], where the Lagrangian is assumed to be a Caratheodory function, and to [55], where the Lagrangian is upper semicontinuous with respect to $\xi$.

The next two lemmas state some properties directly implied by Hypothesis 3.2.1.

**Lemma 3.2.3.** *Let $\tilde{f} : \Omega \times B_K(0) \to \mathbb{R}$ satisfy Hypothesis 3.2.1 and let $T \subset \Omega$ be a compact set such that Hypothesis 1 c) is satisfied. Then $\tilde{f}(x, \xi)$ is uniformly continuous as $x$ varies in $T$ uniformly as $\xi$ varies in $B_K(0)$.*

*Proof.* Given $\varepsilon > 0$ for every $x \in T$ there exists a $\eta_x$ such that

$$|\tilde{f}(\tilde{x}, \xi) - f(x, \xi)| < \varepsilon$$

for every $\tilde{x} \in ]x - \eta_x, x + \eta_x[ \cap T$ and for every $\xi \in B_K(0)$ .

Since $\{]x - \eta_x, x + \eta_x[ \cap T\}_x$ is an open covering of $T$, which is compact, we can extract a finite subcovering and in particular there exists a $\eta > 0$ such that

$$|\tilde{f}(x_1, \xi) - \tilde{f}(x_2, \xi)| < \varepsilon$$

for every $x_1, x_2 \in T$ such that $|x_1 - x_2| < \eta$ and for every $\xi \in B_K(0)$.

□

Given a function $f(x, u, \xi)$ we indicate with $f^{**}(x, u, \xi)$ the bipolar of $f$ with respect to $\xi$. Actually $f^{**}(x, u, \xi)$ is the biggest function lower semicontinuous and convex with respect to $\xi$ lower or equal than $f(x, u, \xi)$ (cfr [40, Proposition 4.1, pag 18]).

In the next lemma we prove that Hypothesis 3.2.1 implies that $\tilde{f}^{**}$ is continuous on $T \times B_K(0)$ for every $T$ such that Hypothesis 3.2.1 $c$) holds and that $\tilde{f}^{**}$ is a Borel function on $\Omega \times B_K(0)$.

**Lemma 3.2.4.** *If $\tilde{f} : \Omega \times B_K(0) \to \mathbb{R}$ satisfies Hypothesis 3.2.1 then $\tilde{f}^{**}$ is continuous on $T \times B_K(0)$ for every $T$ as in Hypothesis 3.2.1 c). Furthermore $\tilde{f}^{**}$ is a borelian function on $\Omega \times B_K(0)$.*

*Proof.* First of all we prove that $\tilde{f}^{**}(x, \xi)$ is continuous w.r.t. $x$ in $T$ uniformly as $\xi$ varies in $B_K(0)$. We fix $x_0 \in T$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that if $x, x_0 \in T$ and $|x - x_0| < \eta$ then

$$|\tilde{f}(x, \xi) - \tilde{f}(x_0, \xi)| < \varepsilon \quad \forall \xi \in B_K(0)$$

and so we have

$$\tilde{f}^{**}(x,\xi)-\varepsilon\leq\tilde{f}(x,\xi)-\varepsilon\leq\tilde{f}(x_0,\xi)\,.$$

Now $\tilde{f}^{**}(x,\xi)-\varepsilon$ is convex in $\xi$ and and so

$$\tilde{f}^{**}(x,\xi)-\varepsilon\leq\tilde{f}^{**}(x_0,\xi)\,.$$

By Lemma 3.2.3 $\tilde{f}(\cdot,\xi)$ is uniformly continuous in $T$ uniformly as $\xi$ varies in $B_K(0)$, thus we can change the roles of $x$ and $x_0$ in the previous inequalities and we obtain

$$|\tilde{f}^{**}(x,\xi)-\tilde{f}^{**}(x_0,\xi)|<\varepsilon\quad\forall\xi\in B_K(0)\,.$$

Since $\tilde{f}^{**}(x,\cdot)$ is continuous in $B_K(0)$ for every $x$, for every sequence $(x_n,\xi_n)\in T\times B_K(0)$ converging to $(x_0,\xi_0)\in T\times B_K(0)$ we have that

$$\begin{aligned}&\lim_n|\tilde{f}^{**}(x_n,\xi_n)-\tilde{f}^{**}(x_0,\xi_0)|\\&\qquad\leq\lim_n|\tilde{f}^{**}(x_n,\xi_n)-\tilde{f}^{**}(x_0,\xi_n)|+\lim_n|\tilde{f}^{**}(x_0,\xi_n)-\tilde{f}^{**}(x_0,\xi_0)|=0\end{aligned}$$

i.e. $\tilde{f}^{**}$ is continuous on $T\times B_K(0)$.

Now, recalling that

$$f^{**}(x,\xi)=\lim_n(f_{|T_n\times B_K(0)})^{**}(x,\xi)$$

with $|\Omega\setminus T_n|\to 0$, the continuity of $(f_{|T_n\times B_K(0)})^{**}$ implies that $f^{**}$ is borelian on $\Omega\times\mathbb{R}^N$.

□

The next Theorem is inspired by the analogous one in [40, Chapter X, Proposition 3.2]. The main novelty here is that we do not assume continuity of the Lagrangian with respect to the variable $\xi$. Af far as we know, this is the first case in literature in which the identity (3.1.1) about integral representation of $sc^-(F_\infty)$ is proved without assuming continuity with respect to $\xi$.

**Theorem 3.2.5.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$, $K\in\mathbb{N}$ and $\tilde{f}:\Omega\times B_K(0)\to\mathbb{R}$ satisfiy Hypothesis 3.2.1.*
*Then for every $u\in W^{1,\infty}(\Omega)$ such that*

$$\|\nabla u\|_\infty<K$$

*there exists a sequence $v_n\in u+W_0^{1,\infty}(\Omega)$ such that*

$$\|\nabla v_n\|_\infty<K\,,\qquad\lim_n\|v_n-u\|_\infty\to 0$$

*and*

$$\lim_n\Big|\int_\Omega\tilde{f}(x,\nabla v_n(x))dx-\int_\Omega\tilde{f}^{**}(x,\nabla u(x))dx\Big|\to 0\,.$$

*Furthermore*

$$sc^-(F_\infty)(u)=\int_\Omega\tilde{f}^{**}(x,\nabla u(x))dx$$

*where with $sc^-(F_\infty)$ we denote the lower semicontinuous envelope of*

$$F_\infty(u)=\int_\Omega\tilde{f}(x,\nabla u(x))dx$$

*with respect to the weak$^*$ topology of $W^{1,\infty}(\Omega)$.*

*Proof. Step 1.* First of all we observe by Lemma 3.2.4 that $\tilde{f}^{**}(x,\xi)$ coincides a.e. with a borelian function on $\Omega\times B_K(0)$.

*Step 2.* For the first part is sufficient to show that for every $\varepsilon>0$ there exists $v_\varepsilon\in u+W_0^{1,\infty}(\Omega)$ with $\|\nabla v_\varepsilon\|<K$ such that

$$\|u-v_\varepsilon\|_\infty<\varepsilon,\qquad\Big|\int_\Omega\tilde{f}(x,\nabla v_\varepsilon(x))dx-\int_\Omega\tilde{f}^{**}(x,\nabla u(x))dx\Big|<\varepsilon\,.$$

Following the same approach as in [40] we start considering the case where $u$ is affine, i.e. $\nabla u = \overline{\xi}$ and we construct a Lipschitz function sufficiently close to $u$, preserving the boundary datum and such that its gradient is a.e. in a suitable set.

For every $x \in \Omega$ we can write $\tilde{f}^{**}(x, \overline{\xi})$ as [32, Remark 3.27]

$$\tilde{f}^{**}(x, \overline{\xi}) = \inf \left\{ \sum_{i=1}^{n+1} \alpha_i^x \tilde{f}(x, \xi_i^x) \Big| \, \alpha_i^x \geq 0, \xi_i^x \in B_K(0), \sum_{i=1}^{n+1} \alpha_i^x = 1, \sum_{i=1}^{n+1} \alpha_i^x \xi_i^x = \overline{\xi} \right\}$$

so that, for $\varepsilon > 0$, we can choose $\alpha_i^x \geq 0$, $\xi_i^x \in B_K(0)$ such that $\sum_{i=1}^{n+1} \alpha_i^x = 1$, $\sum_{i=1}^{n+1} \alpha_i^x \xi_i^x = \overline{\xi}$ and

$$\sum_{i=1}^{n+1} \alpha_i^x \tilde{f}(x, \xi_i^x) - \tilde{f}^{**}(x, \overline{\xi}) < \frac{\varepsilon}{9|\Omega|} \, .$$

We fix $0 < \delta < \varepsilon$ such that for every $\omega \subset \Omega$ with $|\omega| < \delta$ then

$$\int_\omega a(x) dx < \frac{\varepsilon}{24}$$

where $a \in L^1(\Omega)$ satisfies Hypothesis 3.2.1 $b$).

Hypothesis 3.2.1 $c$) implies there exists $T \subset \Omega$ compact such that $\tilde{f}^{**}_{|T \times B_K(0)}(\cdot, \overline{\xi})$ is uniformly continuous (since $T$ is compact), $\tilde{f}_{|T \times B_K(0)}(x, \xi)$ is uniformly continuous with respect to $x$ uniformly as $\xi$ varies in $B_K(0)$ and $|\Omega \setminus T| < \delta$ so that

$$\int_{\Omega \setminus T} a(x) dx < \frac{\varepsilon}{24} \, . \tag{3.2.1}$$

Moreover for every $x \in T$ there exists a neighbourhood of $x$, $U_x \subset \Omega$, such that

$$\forall y \in U_x \cap T \qquad |\tilde{f}(y, \xi) - \tilde{f}(x, \xi)| \leq \frac{\varepsilon}{9|\Omega|} \quad \forall \xi \in B_K(0) \tag{3.2.2}$$

and

$$\forall y \in U_x \cap T \qquad |\tilde{f}^{**}(y, \overline{\xi}) - \tilde{f}^{**}(x, \overline{\xi})| \leq \frac{\varepsilon}{9|\Omega|} \, .$$

In particular we have

$$\forall y \in U_x \cap T \qquad \left| \sum_{i=1}^{n+1} \alpha_i^x \tilde{f}(y, \xi_i^x) - \tilde{f}^{**}(y, \overline{\xi}) \right| \leq \frac{\varepsilon}{3|\Omega|} \, . \tag{3.2.3}$$

Now, for every $x \in T$, there exists a regular family of closed balls of center $x$ and radius $r$ denoted as $\overline{B}_r(x) \subset U_x$, $0 < r < r_x$, that covers $T$ in the sense of Vitali. Then we can apply Vitali Covering theorem (cfr [64, Chapter IV, 3, page 109]) to find a countable family of

$$\omega^j = \overline{B}_{r_j}(x_j) \tag{3.2.4}$$

such that

$$\big| T \setminus \bigcup_j \omega_j \big| = 0 \, , \qquad \omega^{j_1} \cap \omega^{j_2} = \emptyset \quad j_1 \neq j_2 \, .$$

We remark that in general

$$T \subsetneq \bigcup_j \omega^j \, .$$

By [40, Chapter X, Theorem 1.2, pag. 300] we deduce that for every $\omega^j$ and

$$0 < \delta_j < \frac{\delta}{2^j}$$

we can find $n+1$ subsets of $\omega^j$, $\omega_i^j$ and a locally Lipschitz function $v_j$ such that

$$\begin{gathered}\big||\omega_i^j| - \alpha_i^{x_j}|\omega^j|\big| \le \alpha_i^{x_j}\delta_j \quad \text{for} \quad 1 \le i \le n+1\,, \\ \nabla v_j = \xi_i^{x_j} \quad \text{on} \quad \omega_i^j\,, \\ \|\nabla v_j\|_\infty < K \quad \text{on} \quad \omega^j\,, \\ \|v_j - u\|_\infty \le \delta \quad \text{on} \quad \omega^j\,, \\ v_j = u \quad \text{on} \quad \partial\omega^j\,. \end{gathered} \tag{3.2.5}$$

In particular the first property implies

$$\big||\omega^j| - |\bigcup_i \omega_i^j|\big| \le \big||\omega^j| - \sum_i |\omega_i^j|\big| \le \frac{\delta}{2^j}\,. \tag{3.2.6}$$

We define the function

$$v_\varepsilon(x) := \begin{cases} v_j(x) & \text{if} \quad x \in \omega^j\,, \\ u(x) & \text{if} \quad x \in \Omega \setminus \bigcup\limits_{j=1}^{\infty} \omega^j \end{cases}$$

and it easily turns out that $v_\varepsilon \in u + W_0^{1,\infty}(\Omega)$, $\|\nabla v_\varepsilon\|_\infty < K$ and, since $\delta < \varepsilon$,

$$\|v_\varepsilon - u\|_\infty < \varepsilon\,.$$

*Step 3.* Our aim now is to evaluate

$$\Big|\int_\Omega \tilde f(x, \nabla v_\varepsilon(x))dx - \int_\Omega \tilde f^{**}(x, \nabla u(x))dx\Big|.$$

We start observing that

$$\begin{aligned}&\Big|\int_\Omega \tilde f(x, \nabla v_\varepsilon(x))dx - \int_\Omega \tilde f^{**}(x, \nabla u(x))dx\Big| \\ &\le \int_{\Omega\setminus \bigcup\limits_{j=1}^{\infty} \omega^j} |\tilde f(x, \nabla v_\varepsilon(x)) - \tilde f^{**}(x, \nabla u(x))|dx \\ &\quad + \sum_{j=1}^{\infty}\Big|\int_{\omega^j} \tilde f(x, \nabla v_\varepsilon(x))dx - \int_{\omega^j} \tilde f^{**}(x, \nabla u(x))dx\Big|\end{aligned} \tag{3.2.7}$$

Since $|T \setminus \cup_{j=1}^{\infty}\omega_j| = 0$, we can estimate the first term in the right hand side of (3.2.7) using (3.2.1) and recalling that, by assumption $b)$, $f^{**}$ is non negative. We then obtain

$$\int_{\Omega\setminus \bigcup\limits_{j=1}^{\infty} \omega^j} |\tilde f(x, \nabla v_\varepsilon(x)) - \tilde f^{**}(x, \nabla u(x))|dx \le \int_{\Omega\setminus T} a(x)dx < \frac{\varepsilon}{24}\,.$$

We consider now each term in the last sum of (3.2.7). We have that

$$\begin{aligned}&\Big|\int_{\omega^j} \tilde f(x, \nabla v_\varepsilon(x))dx - \int_{\omega^j} \tilde f^{**}(x, \nabla u(x))dx\Big| \\ &\le \Big|\int_{\omega^j} \tilde f(x, \nabla v_\varepsilon(x))dx - \sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j} \tilde f(x, \xi_i^{x_j})dx\Big| \\ &\quad + \Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j} \tilde f(x, \xi_i^{x_j})dx - \int_{\omega^j} \tilde f^{**}(x, \nabla u(x))dx\Big|\end{aligned} \tag{3.2.8}$$

and, recalling the definition of the functions $v_\varepsilon$ and $v_j$, we can estimate the right hand side in (3.2.8) as

$$\begin{aligned}
&\Big|\int_{\omega^j} \tilde f(x,\nabla v_j(x))dx - \sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\Big| \\
&\quad+\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx - \int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big| \\
\le&\Big|\sum_{i=1}^{n+1}\left[\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right]\Big| \\
&\quad+\int_{\omega^j\setminus\bigcup_i\omega_i^j}\tilde f(x,\nabla v_j(x))dx \\
&\quad+\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx - \int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big|.
\end{aligned} \tag{3.2.9}$$

We consider each term in the first sum of the right hand side of (3.2.9) and we obtain

$$\begin{aligned}
&\left|\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right| \\
&\le\left|\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j}) - \tilde f(x_j,\xi_i^{x_j})dx\right| \\
&+\left|\int_{\omega_i^j}\tilde f(x_j,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j}\tilde f(x_j,\xi_i^{x_j})dx\right| \\
&\quad+\alpha_i^{x_j}\left|\int_{\omega^j}\tilde f(x_j,\xi_i^{x_j}) - \tilde f(x,\xi_i^{x_j})dx\right|.
\end{aligned} \tag{3.2.10}$$

Since $\tilde f(x_j,\xi_i^{x_j})$ is constant, by the first property in (3.2.5), we can estimate the second term in the right hand side as follows

$$\left|\int_{\omega_i^j}\tilde f(x_j,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j}\tilde f(x_j,\xi_i^{x_j})dx\right| \le \alpha_i^{x_j}\delta_j|\tilde f(x_j,\xi_i^{x_j})|\,.$$

In order to evaluate the other two terms in (3.2.10) we notice that, by the definition of $\omega^j$ in (3.2.4) and the property (3.2.2)

$$\forall x\in\omega_j\cap T\quad |\tilde f(x,\xi_i^{x_j}) - \tilde f(x_j,\xi_i^{x_j})| \le \frac{\varepsilon}{9|\Omega|}$$

and so

$$\left|\int_{\omega_i^j\cap T}\tilde f(x,\xi_i^{x_j})dx - \int_{\omega_i^j\cap T}\tilde f(x_j,\xi_i^{x_j})dx\right| \le \frac{\varepsilon|\omega_i^j|}{9|\Omega|}\,,$$

$$\left|\alpha_i^{x_j}\int_{\omega^j\cap T}\tilde f(x_j,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j\cap T}\tilde f(x,\xi_i^{x_j})dx\right| \le \frac{\varepsilon\alpha_i^{x_j}|\omega^j|}{9|\Omega|}\,.$$

It follows that

$$\left|\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx - \alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right| \le$$

$$\frac{\varepsilon|\omega_i^j|}{9|\Omega|} + \int_{\omega_i^j\setminus T}a(x) + \alpha_i^{x_j}\delta_j|\tilde f(x_j,\xi_i^{x_j})| + \frac{\varepsilon\alpha_i^{x_j}|\omega^j|}{9|\Omega|} + \alpha_i^{x_j}\int_{\omega^j\setminus T}a(x)\,.$$

Coming back to the sum on $i$ in (3.2.9) and recalling the first property of (3.2.5) we have

$$\Big|\sum_{i=1}^{n+1}\left[\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx-\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right]\Big|\le \delta_j\max_i\left\{|\tilde f(x_j,\xi_i^{x_j})|\right\}+\frac{\varepsilon(2|\omega^j|)}{9|\Omega|}+\frac{\varepsilon\delta_j}{9|\Omega|}+\int_{\omega^j\setminus T}a(x)+\int_{\bigcup_i\omega_i^j\setminus T}a(x)\,. \quad (3.2.11)$$

Since $|\tilde f(x,\xi)|\le a(x)$ and $\tilde f_{|T\times B_K(0)}(x,\xi)$ is uniformly continuous in $x$ uniformly as $\xi$ varies in $B_K(0)$, then $\tilde f_{|T\times B_K(0)}(x,\xi)$ is bounded. In fact fixed $\overline{x}$ then $\tilde f(\overline{x},\xi)$ is bounded for every $\xi\in B_K(0)$ and it exists a modulus of continuity such that

$$|\tilde f(x,\xi)-\tilde f(\overline{x},\xi)|\le\gamma(|x-\overline{x}|)$$

for every $x\in T$ and $\xi\in B_K(0)$. Thus there exists $C>0$ which does not depend by $x_j$ or $\xi_i^{x_j}$ such that

$$\max_i\left\{|\tilde f(x_j,\xi_i^{x_j})|\right\}\le C\,.$$

Now we note that $C$ depends only by the choose of the compact $T$. So we can take $\delta_j$ sufficiently small such that

$$\frac{2\varepsilon|\omega^j|}{9|\Omega|}+\delta_jC+\frac{\varepsilon\delta_j}{9|\Omega|}\le\frac{\varepsilon|\omega^j|}{3|\Omega|}\,.$$

Now (3.2.11) becomes

$$\Big|\sum_{i=1}^{n+1}\left[\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx-\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right]\Big|\le \frac{\varepsilon|\omega^j|}{3|\Omega|}+\int_{\omega^j\setminus T}a(x)+\int_{\bigcup_i\omega_i^j\setminus T}a(x)$$

and we have completed the estimate of the first term in the right hand of (3.2.9).
Now we turn to evaluate the second term. By assumption $b)$

$$\int_{\omega^j\setminus\bigcup_i\omega_i^j}\tilde f(x,\nabla v_j(x))dx\le\int_{\omega^j\setminus\bigcup_i\omega_i^j}a(x)dx$$

where we recall by (3.2.6) that

$$\big||\omega^j|-|\bigcup_i\omega_i^j|\big|\le\frac{\delta}{2^j}\,.$$

We split the third term in the right hand side of (3.2.9)

$$\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx-\int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big| \le\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j\cap T}\tilde f(x,\xi_i^{x_j})dx-\int_{\omega^j\cap T}\tilde f^{**}(x,\nabla u(x))dx\Big|+\int_{\omega^j\setminus T}a(x)dx\,.$$

By (3.2.3) we have

$$\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j\cap T}\tilde f(x,\xi_i^{x_j})dx-\int_{\omega^j\cap T}\tilde f^{**}(x,\nabla u(x))dx\Big|\le\frac{\varepsilon|\omega^j|}{3|\Omega|}\,.$$

Considering together these tree evaluations we can rewrite (3.2.8) as

$$\begin{aligned}
&\Big|\int_{\omega^j}\tilde f(x,\nabla v_\varepsilon(x))dx-\int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big|\\
&\quad\le\Big|\int_{\omega^j}\tilde f(x,\nabla v_\varepsilon(x))dx-\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\Big|\\
&\qquad+\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx-\int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big|\\
&\quad\le\Big|\sum_{i=1}^{n+1}\left[\int_{\omega_i^j}\tilde f(x,\xi_i^{x_j})dx-\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx\right]\Big|\\
&\qquad+\int_{\omega^j\setminus\bigcup_i\omega_i^j}\tilde f(x,\nabla v_j(x))dx\\
&\qquad+\Big|\sum_{i=1}^{n+1}\alpha_i^{x_j}\int_{\omega^j}\tilde f(x,\xi_i^{x_j})dx-\int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big|\\
&\quad\le\frac{\varepsilon|\omega^j|}{3|\Omega|}+2\int_{\omega^j\setminus T}a(x)+\int_{\bigcup_i\omega_i^j\setminus T}a(x)+\int_{\omega^j\setminus\bigcup_i\omega_i^j}a(x)dx+\frac{\varepsilon|\omega^j|}{3|\Omega|}\,.
\end{aligned}$$

We recall by (3.2.1)

$$\int_{\Omega\setminus T}a(x)dx<\frac{\varepsilon}{24}$$

and furthermore by (3.2.6)

$$\big|\cup_j(\omega^j\setminus\cup_i\omega_i^j)\big|=\sum_j(|\omega^j|-\sum_i|\omega_i^j|)\le\delta$$

so that also in this case

$$\int_{\bigcup_j(\omega^j\setminus\bigcup_i\omega_i^j)}a(x)dx<\frac{\varepsilon}{24}\,.$$

Since by definition $\omega^j$ are mutually disjoints we can conclude recalling the estimate (3.2.7) which become

$$\begin{aligned}
&\Big|\int_\Omega\tilde f(x,\nabla v_\varepsilon(x))dx-\int_\Omega\tilde f^{**}(x,\nabla u(x))dx\Big|\\
&\quad\le\int_{\Omega\setminus T}a(x)dx+\sum_{j=1}^{\infty}\Big|\int_{\omega^j}\tilde f(x,\nabla v_\varepsilon(x))dx-\int_{\omega^j}\tilde f^{**}(x,\nabla u(x))dx\Big|\\
&\quad<\frac{\varepsilon}{24}+\frac{2|\Omega|\varepsilon}{3|\Omega|}+\frac{\varepsilon}{6}\,.
\end{aligned}$$

i.e.

$$\Big|\int_\Omega\tilde f(x,\nabla v_\varepsilon(x))dx-\int_\Omega\tilde f^{**}(x,\nabla u(x))dx\Big|<\varepsilon\,.$$

*Step 4.* Now we consider the case where $u$ is piecewise affine, so that we can split $\Omega$, minus a negligible set $N'$, in an union of disjoint Lipschitz open sets $\{\Omega_d\}_d$, with $1\le d\le M$, where $u$ is affine. Thanks to the previous steps, for every $d\in\{1,\dots,M\}$, we can find a function $v^d$ such that

$$v^d\in u+W_0^{1,\infty}(\Omega_d)\,,\quad\|v^d-u\|_\infty<\varepsilon\,,\quad\|\nabla v^d\|_\infty<K$$

and

$$\Big|\int_{\Omega_d}\tilde f(x,\nabla v^d(x))dx-\int_{\Omega_d}\tilde f^{**}(x,\nabla u(x))dx\Big|<\varepsilon\frac{|\Omega_d|}{|\Omega|}\,.$$

So we define

$$v_\varepsilon(x) := \begin{cases} v^d(x) & \text{in} \quad \Omega_d \\ u(x) & \text{in} \quad N' \end{cases}$$

and we have

$$v_\varepsilon \in u + W_0^{1,\infty}(\Omega)\,, \quad \|v_\varepsilon - u\|_\infty < \varepsilon\,, \quad \|\nabla v_\varepsilon\|_\infty < K$$

and

$$\Big| \int_\Omega \tilde{f}(x, \nabla v_\varepsilon(x))dx - \int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx \Big| < \varepsilon\,.$$

*Step 5.* In the general case if $u \in W^{1,\infty}(\Omega)$ by [40, Chapter X, Proposition 2.9, page 317] then there exists a sequence of Lipschitz open sets $\Omega_l \subset \Omega$ and a sequence $u_l \in u + W_0^{1,\infty}(\Omega)$ such that $u_l$ is piecewise affine over $\Omega_l$ and

$$|\Omega \setminus \Omega_l| \to 0\,, \qquad \|\nabla u_l\|_\infty \le \|\nabla u\|_\infty + c(l) \quad \text{where } c(l) \to 0\,,$$

$$\|u_l - u\|_\infty \to 0\,, \qquad \nabla u_l \to \nabla u \quad \text{a.e. in } \Omega\,.$$

Since $\tilde{f}^{**}(x, \cdot)$ is continuous for every $x$, then $\tilde{f}^{**}(x, \nabla u_l(x))$ converges a.e. to $\tilde{f}^{**}(x, \nabla u(x))$. So, by assumption $b)$ and Lebesgue dominated convergence Theorem, we obtain

$$\lim_{l \to +\infty} \int_\Omega |\tilde{f}^{**}(x, \nabla u_l(x)) - \tilde{f}^{**}(x, \nabla u(x))|dx = 0\,.$$

So there exists $\tilde{u} \in u + W_0^{1,\infty}(\Omega)$ and $\tilde{\Omega} \subseteq \Omega$ such that $\tilde{u}$ is piecewise affine over $\tilde{\Omega}$,

$$\int_{\Omega \setminus \tilde{\Omega}} a(x) < \frac{\varepsilon}{4}\,, \quad \|\nabla \tilde{u}\|_\infty < K\,, \quad \|\tilde{u} - u\|_\infty < \frac{\varepsilon}{2}\,,$$

and

$$\int_\Omega |\tilde{f}^{**}(x, \nabla \tilde{u}(x)) - \tilde{f}^{**}(x, \nabla u(x))|dx < \frac{\varepsilon}{4}\,.$$

Now, using the results of the previous steps, we can find $\tilde{v} \in \tilde{u} + W_0^{1,\infty}(\tilde{\Omega})$ such that

$$\|\nabla \tilde{v}\|_\infty < K\,, \quad \|\tilde{v} - \tilde{u}\|_\infty < \frac{\varepsilon}{2}$$

and

$$\Big| \int_{\tilde{\Omega}} \tilde{f}(x, \nabla \tilde{v}(x)) - \tilde{f}^{**}(x, \nabla \tilde{u}(x))dx \Big| < \frac{\varepsilon}{4}\,.$$

So taking

$$v_\varepsilon(x) := \begin{cases} \tilde{v}(x) & \text{in} \quad \tilde{\Omega} \\ \tilde{u}(x) & \text{in} \quad \Omega \setminus \tilde{\Omega} \end{cases}$$

then we have

$$v_\varepsilon \in u + W^{1,\infty}(\Omega)\,, \quad \|v_\varepsilon - u\|_\infty < \varepsilon\,, \quad \|\nabla v_\varepsilon\|_\infty < K$$

and

$$\Big| \int_\Omega \tilde{f}(x, \nabla v_\varepsilon(x))dx - \int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx \Big| < \varepsilon\,.$$

*Step 6.* Now we want to prove the second part of the statement about the integral representation of $sc^-(F_\infty)$. By previous steps for every $u \in W^{1,\infty}(\Omega)$ there exists a sequence $u_n \in u + W_0^{1,\infty}(\Omega)$ such that

$$\|\nabla u_n\|_\infty \le K, \quad \|u_n - u\|_\infty \to 0$$

and

$$\lim_{n \to \infty} \int_\Omega \tilde{f}(x, \nabla u_n(x))dx = \int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx\,.$$

Up to a subsequence, we can suppose that $u_n \overset{*}{\rightharpoonup} u$, i.e. $u_n$ weakly$^*$ converges to $u$ in $W^{1,\infty}$ and we know, by a generalization of Tonelli theorem about lower semicontinuity of integral functionals with convex Lagrangian (cfr [40, Chapter VIII, Theorem 2.1, page 243]), that

$$\int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx \le \liminf_{n\to\infty} \int_\Omega \tilde{f}(x, \nabla v_n(x))dx$$

for every $v_n \in u + W_0^{1,\infty}(\Omega)$.
Thus we have that

$$\inf\left\{ \liminf \int_\Omega \tilde{f}(x, \nabla v_n)dx \middle| v_n \overset{*}{\rightharpoonup} u, v_n \in u + W_0^{1,\infty}(\Omega) \right\} = \int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx\,.$$

i.e.

$$sc^-(F_\infty)(u) = \int_\Omega \tilde{f}^{**}(x, \nabla u(x))dx$$

where $sc^-(F_\infty)$ is the lower semicontinuous envelope respect to the weak$^*$ topology of $W^{1,\infty}(\Omega)$.

□

**Remark 3.2.6.** We want to discuss assumption $c)$ of Hypothesis 3.2.1. First of all we notice that assumption $a)$ and Lusin's Theorem would imply that for every $\xi$ and for every $\delta > 0$ there exists $T_\xi \subset \Omega$ compact such that $|\Omega \setminus T_\xi| < \delta$ and $\tilde{f}(\cdot, \xi)_{|T_\xi}$ is continuous. In other words in our hypothesis c) we state a property that requires that Lusin's Theorem holds uniformly as $\xi$ varies in $B_K(0)$.

In [40][Theorem 3.3, page 332], the authors assumed that $f(x, \xi)$ is a function Carathéodory, then Scorza-Dragoni Theorem implies that our assumption $c)$ holds true.

Moreover, we remark that the function $\tilde{f}(x, \xi) := g(x)h(\xi)$, with $g(x) \in L^1(\Omega)$ and $h(\xi)$ borelian and bounded on bounded sets, satisfies our assumption, but is not Carathéodory.

## 3.3 An approximation result for the general case

In this section we want to generalize the validity of the integral representation formula (3.1.1) with $p = q = \infty$ proved in Theorem 3.2.5 to a general Lagrangian $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfies following hypotheses.

**Hypothesis 3.3.1.** The function $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies

a) $f(x, u, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a borelian function;

b) for every bounded set of $B \subset \mathbb{R} \times \mathbb{R}^N$ there exists $a \in L^1(\Omega)$ such that $|f(x, u, \xi)| \le a(x)$ for all $(u, \xi) \in B$,

c) for every $u \in W^{1,\infty}(\Omega)$, for every $\tilde{B} \subset \mathbb{R}^N$ bounded set and for every $\delta > 0$ there exists $T \subset \Omega$ compact such that $|\Omega \setminus T| < \delta$ and $f(x, u(x), \xi)$ is continuous with respect to $x \in T$ uniformly as $\xi$ varies in $\tilde{B}$,

d) for almost every $x$ the function $f(x, u, \xi)$ is continuous with respect to $u$ uniformly as $\xi$ varies in bounded sets.

We introduce the following auxiliary functions that will be useful in the next proposition and in the main theorem of this section. For every function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and for every $K$ in $\mathbb{N}$ we define $f_K$ as

$$f_K(x, u, \xi) = \begin{cases} f(x, u, \xi) & \text{if } \ \|\xi\| \le K\,, \\ +\infty & \text{if } \ \|\xi\| > K\,. \end{cases} \tag{3.3.1}$$

We remark that $(f_K)^{**}(x, u, \xi) \ge (f^{**})_K(x, u, \xi)$ for every $(x, u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and in general the strict inequality may hold. Moreover it is straightforward that

$$(f_K)^{**}(x, u, \xi) \ge (f_{K+1})^{**}(x, u, \xi)$$

for every $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and in particular $(f_K)^{**}$ is a monotone decreasing sequence pointwise convergent to $f^{**}$.

**Proposition 3.3.2.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Hypotheses 3.3.1.*
*For every $\overline{u} \in W^{1,\infty}(\Omega)$ and for every $K \in \mathbb{N}$ such that*

$$\|\overline{u}\|_\infty + \|\nabla \overline{u}\|_\infty < K$$

*there exists a sequence $u_{K,n} \in \overline{u} + W_0^{1,\infty}(\Omega)$ such that*

$$\|\nabla u_{K,n}\|_\infty < K \,, \qquad u_{K,n} \overset{*}{\rightharpoonup} \overline{u}$$

*and*

$$\lim_{n\to\infty} \left| \int_\Omega f(x, u_{K,n}(x), \nabla u_{K,n}(x))dx - \int_\Omega (f_K)^{**}(x, \overline{u}(x), \nabla \overline{u}(x))dx \right| = 0 \,.$$

*Proof.* By Hypothesis 3.3.1 $b)$ we fix $\overline{u} \in W^{1,\infty}(\Omega)$ and we can choose $K \in \mathbb{N}$ and $a_K \in L^1(\Omega)$ such that

$$\|\overline{u}\|_\infty + \|\nabla \overline{u}\|_\infty < K \tag{3.3.2}$$

and

$$|f(x, \overline{u}(x), \xi)| \le a_K(x)$$

for every $\xi \in B_K(0)$.
We define $\tilde{f}_K : \Omega \times B_K \to \mathbb{R}$ as

$$\tilde{f}_K(x,\xi) := f(x, \overline{u}(x), \xi) + a_K(x) \,.$$

We observe that $\tilde{f}_K$ satisfies Hypothesis 3.2.1, in fact

- $\tilde{f}_K(x,\xi) : \Omega \times B_K \to \mathbb{R}$ is borelian,
- $0 \le \tilde{f}_K(x,\xi) \le 2a_K(x)$ for all $(x,\xi) \in \Omega \times B_K(0)$,
- for every $\delta > 0$ then there exists a compact set $T \subset \Omega$ such that $|\Omega \setminus T| < \delta$ and $\tilde{f}_K(x,\xi)$ is continuous with respect to $x \in T$ uniformly as $\xi$ varies in $B_K(0)$ .

We can apply Theorem 3.2.5 to $\tilde{f}_K$ and thus for every $n \in \mathbb{N}$ we can find $u_{K,n}$ in $\overline{u} + W_0^{1,\infty}(\Omega)$ such that

$$\|\nabla u_{K,n}\|_\infty < K \,, \qquad \|u_{K,n} - \overline{u}\|_\infty < \frac{1}{n}$$

and

$$\Big| \int_\Omega \tilde{f}_K(x, \nabla u_{K,n}(x))dx - \int_\Omega (\tilde{f}_K)^{**}(x, \nabla \overline{u}(x))dx \Big| < \frac{1}{n} \tag{3.3.3}$$

i.e.

$$\Big| \int_\Omega f(x, \overline{u}(x), \nabla u_{K,n}(x))dx - \int_\Omega (f_K)^{**}(x, \overline{u}(x), \nabla \overline{u}(x))dx \Big| < \frac{1}{n} \,.$$

Now, in order to prove our propositions, it is sufficient to show that

$$\lim_{n\to+\infty} \Big| \int_\Omega f(x, u_{K,n}(x), \nabla u_{K,n}(x))dx - \int_\Omega f(x, \overline{u}(x), \nabla u_{K,n}(x))dx \Big| = 0 \,.$$

We recall that, by (3.3.2) and (3.3.3), there exists $\tilde{K}$ in $\mathbb{N}$ such that

$$\|u_{K,n}\|_\infty < \tilde{K}$$

and so

$$(u_{K,n}(\cdot), \nabla u_{K,n}(\cdot)) \in [-\tilde{K}, \tilde{K}] \times B_K(0) \quad \text{a.e. in} \quad \Omega \,.$$

We recall that Hypothesis 3.3.1 d) implies that $f(x,u,\xi)$ is uniformly continuous on $u$ in $[-\tilde{K},\tilde{K}]$ uniformly as $\xi$ varies in $B_K(0)$ for almost every $x$ and we observe that, from this fact and the uniform convergence of $u_{K,n}$ to $\overline{u}$ in $\Omega$, it follows

$$f(x,u_{K,n}(x),\nabla u_{K,n}(x)) - f(x,\overline{u}(x),\nabla u_{K,n}(x)) \to 0$$

for a.e. $x \in \Omega$.
By Hypotheses 3.3.1 $b)$ there exists $a_{\tilde{K}} \in L^1(\Omega)$ that

$$|f(x,u,\xi)| \le a_{\tilde{K}}(x)$$

in $\Omega \times [-\tilde{K},\tilde{K}] \times B_K(0)$ and so by Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty} \Big| \int_\Omega f(x,u_{K,n}(x),\nabla u_{K,n}(x))dx - \int_\Omega f(x,\overline{u}(x),\nabla u_{K,n}(x))dx \Big| = 0\,.$$

Therefore, passing to a suitable subsequence, we get the conclusion.

□

Now we are ready to prove the main result of this section.

**Theorem 3.3.3.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Hypotheses 3.3.1.*
*For every $\overline{u} \in W^{1,\infty}(\Omega)$ exists a sequence $u_n \in \overline{u} + W_0^{1,\infty}(\Omega)$ such that*

$$\lim_{n\to\infty} \|u_n - \overline{u}\|_\infty = 0$$

*and*

$$\lim_{n\to\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

*Furthermore*

$$sc^-(F_\infty)(\overline{u}) = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx$$

*where with $sc^-(F_\infty)$ we denote the lower semicontinuous envelope of*

$$F_\infty(u) = \int_\Omega f(x,u,\nabla u(x))dx$$

*with respect to the weak* topology of $W^{1,\infty}(\Omega)$.*

*Proof.* For every $K$ in $\mathbb{N}$ we consider the function $f_K(x,u,\xi)$ defined in (3.3.1) and its lower semi-continuous convex envelope w.r.t. $\xi$ that, as usual, we denote by $(f_K)^{**}(x,u,\xi)$. We remark that

$$(f_K)^{**}(x,u,\xi) \ge (f_{K+1})^{**}(x,u,\xi)$$

for every $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$ and in particular $(f_K)^{**}$ is a monotone decreasing sequence pointwise convergent to $f^{**}$.
Fixed $\overline{u} \in W^{1,\infty}(\Omega)$ we have $(f_K)^{**}(\cdot,\overline{u}(\cdot),\nabla\overline{u}(\cdot))$ is a decreasing sequence pointwise a.e. convergent to $f^{**}(\cdot,\overline{u}(\cdot),\nabla\overline{u}(\cdot))$.
Let $\overline{K}$ such that

$$\|\overline{u}\|_\infty + \|\nabla\overline{u}\|_\infty < \overline{K}\,.$$

By Hypothesis 3.3.1 b), there exists $a_{\overline{K}} \in L^1(\Omega)$ such that for every $K \ge \overline{K}$ and for a.e. $x \in \Omega$

$$(f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x)) \le (f_K)(x,\overline{u}(x),\nabla\overline{u}(x)) = f(x,\overline{u}(x),\nabla\overline{u}(x)) \le a_{\overline{K}}(x)\,.$$

Monotone convergence theorem implies that

$$\lim_{K\to+\infty} \int_\Omega a_{\overline{K}}(x) - (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx = \int_\Omega a_{\overline{K}}(x) - f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx$$

and also

$$\lim_{K\to+\infty}\int_\Omega (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

By Proposition 3.3.2, for every $\overline{u}\in W^{1,\infty}(\Omega)$ and for every $K\in\mathbb{N}$ such that

$$\|\overline{u}\|_\infty + \|\nabla\overline{u}\|_\infty < K$$

there exists a sequence $u_{K,n}\in\overline{u}+W_0^{1,\infty}(\Omega)$ such that

$$\|\nabla u_{K,n}\| < K\,,\quad u_{K,n}\overset{*}{\rightharpoonup}\overline{u}$$

and

$$\lim_{n\to\infty}\Big|\int_\Omega f(x,u_{K,n}(x),\nabla u_{K,n}(x))dx - \int_\Omega (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\Big| = 0\,.$$

Taking the double limit

$$\lim_{K\to+\infty}\left(\lim_{n\to\infty}\int_\Omega f(x,u_{K,n}(x),\nabla u_{K,n}(x))dx\right) = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

This implies the existence of a sequence $\overline{u}_n$ in $\overline{u}+W_0^{1,\infty}(\Omega)$ such that

$$\lim_{n\to\infty}\|\overline{u}_n-\overline{u}\|_\infty = 0\,,$$

$$\lim_{n\to\infty}\int_\Omega f(x,\overline{u}_n(x),\nabla\overline{u}_n(x))dx = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

Furthermore for every $K$

$$\inf\left\{\liminf\int_\Omega f(x,u_n(x),\nabla u_n(x))dx\,\middle|\,u_n\overset{*}{\rightharpoonup}\overline{u},\ \|\nabla u_n\|_\infty < K\right\} \le \int_\Omega (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,. \tag{3.3.4}$$

Since $(f_K)^{**}(x,u,\xi)$ is convex and l.s.c. with respect to $\xi$ we have (for example by [40, Chapter VIII, Theorem 2.1, page 243]) that the functional

$$\int_\Omega (f_K)^{**}(x,u(x),\nabla u(x))dx$$

is sequentially lower semicontinuous with respect to the weak topology of $W^{1,1}(\Omega)$. So a fortiori

$$\inf\left\{\liminf\int_\Omega f(x,u_n(x),\nabla u_n(x))dx\,\middle|\,u_n\overset{*}{\rightharpoonup}\overline{u},\ \|\nabla u_n\|_\infty < K\right\} \ge \int_\Omega (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx \tag{3.3.5}$$

and then (3.3.4) and (3.3.5) imply that the equality holds

$$\inf\left\{\liminf\int_\Omega f(x,u_n(x),\nabla u_n(x))dx\,\middle|\,u_n\overset{*}{\rightharpoonup}\overline{u},\ \|\nabla u_n\|_\infty < K\right\} = \int_\Omega (f_K)^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

Finally we have that, for $K\to+\infty$,

$$\inf\left\{\liminf\int_\Omega f(x,u_n(x),\nabla u_n(x))dx\,\middle|\,u_n\overset{*}{\rightharpoonup}\overline{u}\right\} = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

i.e.

$$sc^-(F_\infty(\overline{u})) = \int_\Omega f^{**}(x,\overline{u}(x),\nabla\overline{u}(x))dx\,.$$

□

**Remark 3.3.4.** We notice that Hypothesis 3.3.1 d) on the continuity of the Lagrangian w.r.t. $u$ is strongly used in the proof of our result. The same assumption is also present in previous papers [40] and [55].

**Remark 3.3.5.** We point out the fact that, as a corollary, we can replace Hypotheses 3.3.1 c) and d) with the more restrictive request that for every bounded set $\tilde{B} \subset \mathbb{R}^N$ and for every $\delta > 0$ there exists a compact set $T \subset \Omega$ such that $|\Omega \setminus T| < \delta$ and $f(x, u, \xi)$ is continuous with respect to $(x, u) \in T \times \mathbb{R}$ uniformly as $\xi$ varies in $\tilde{B}$.

**Remark 3.3.6.** Proposition 3.3.2 can be seen as a generalization of Theorem 3.7 in [40, Chapter X], at least for the case of Lipschitz functions. In particular we underline that in [40] the authors assume that the Lagrangian is a Carathéodory function, continuous w.r.t. $(u, \xi)$.

**Remark 3.3.7.** In [55] is presented a relaxation result under the assumption that $f(x, u, \xi)$ is continuous on $u$ uniformly as $\xi$ varies in a bounded set of $\mathbb{R}^N$ and $f$ is also assumed to be upper semicontinuous with respect to $\xi$. Thus $f(x, u, \xi)$ is upper semicontinuous with respect to $(u, \xi)$. In the next examples we show that our Hypotheses 3.3.1 c) and d) are neither more general nor more restrictive than the one presented in [55].

**Example 3.3.8.** We consider the set $\Omega := ]-\frac{\pi}{2}, \frac{\pi}{2}[$ and the function $f : ]-\frac{\pi}{2}, \frac{\pi}{2}[ \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, \xi) := \begin{cases} 0 & \text{if} \quad \xi > \tan(x) \\ 1 & \text{if} \quad \xi \leq \tan(x) \,. \end{cases}$$

It is easy to check that $f(x, \xi)$ is upper semicontinuous on $\Omega \times \mathbb{R}$. On the other hand, given $M \in \mathbb{R}$ and $\delta < 2 \arctan M$, we have that, for every $T \subset ]-\frac{\pi}{2}, \frac{\pi}{2}[$ such that $|]-\frac{\pi}{2}, \frac{\pi}{2}[ \setminus T| < \delta$, there exists $\bar{x} \in T$ such that $f$ is not continuous at $(\bar{x}, \tan \bar{x})$. This shows that the assumptions in [55] do not imply our hypotheses.

**Example 3.3.9.** On the other side if $f(x, u, \xi) := g(x, u)h(\xi)$ with

$$g(x, u) := \|x\| + |u|$$

and

$$h(\xi) := \begin{cases} 1 & \text{if} \quad \xi \in \mathbb{Q} \\ -1 & \text{if} \quad \xi \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

then $f(x, u, \xi)$ satisfies Hypothesis 3.3.1 but it is not upper semicontinuous with respect to $\xi$.

## 3.4 Lavrentiev Phenomenon

In this section we apply the results of previous sections to show that whenever the Lavrentiev phenomenon does not occur for the functional

$$\int_\Omega f^{**}(x, u(x), \nabla u(x))dx$$

then it does not occur also for the functional

$$\int_\Omega f(x, u(x), \nabla u(x))dx.$$

**Theorem 3.4.1.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$, let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Hypotheses 3.3.1, $\varphi \in W^{1,\infty}(\Omega)$ and $1 \leq p < +\infty$.*

*If*

$$\inf_{\varphi + W_0^{1,p}(\Omega)} \int_\Omega f^{**}(x, u(x), \nabla u(x))dx = \inf_{\varphi + W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(x, u(x), \nabla u(x))dx \tag{3.4.1}$$

*then*

$$\inf_{\varphi + W_0^{1,p}(\Omega)} \int_\Omega f(x, u(x), \nabla u(x))dx = \inf_{\varphi + W_0^{1,\infty}(\Omega)} \int_\Omega f(x, u(x), \nabla u(x))dx \,.$$

*Proof.* The properties of $f$ and $f^{**}$ imply that, for every $\varphi \in W^{1,\infty}(\Omega)$,

$$\inf_{\varphi+W_0^{1,q}(\Omega)} \int_\Omega f^{**}(x,u(x),\nabla u(x))dx \leq \inf_{\varphi+W_0^{1,q}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx \tag{3.4.2}$$

for every $1 \leq q \leq \infty$. We have also that

$$\inf_{\varphi+W_0^{1,p}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx \leq \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx$$

and, using both hypothesis (3.4.1) and (3.4.2) we get

$$\inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(x,u(x),\nabla u(x))dx \leq \inf_{\varphi+W_0^{1,p}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx\,.$$

Furthermore Theorem 3.3.3 states that

$$\inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx = \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(x,u(x),\nabla u(x))dx$$

and so

$$\begin{aligned}\inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(x,u(x),\nabla u(x))dx &\leq \inf_{\varphi+W_0^{1,p}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx \\ &\leq \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f(x,u(x),\nabla u(x))dx = \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(x,u(x),\nabla u(x))dx\,. \end{aligned} \tag{3.4.3}$$

It follows that all the inequalities in (3.4.3) are equalities, proving the thesis.

□

## 3.5 Approximation results for the autonomous case

Now we focus on autonomous Lagrangians, in order to apply some recent results about this case. As far as we know, also in the autonomous case our results were never proved before without assuming the continuity of Lagrangian respect to $\xi$. So we prefer report explicitly them also for Lagrangians which satisfy the following hypotheses which imply Hypothesis 3.3.1 and that are quit natural.

**Hypothesis 3.5.1.** The function $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies

i) $f(u,\xi) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is borelian,

ii) $f(u,\xi)$ is continuous with respect to $u$ uniformly as $\xi$ varies on each bounded set of $\mathbb{R}^N$,

iii) $f(u,\cdot)$ is bounded on bounded sets of $\mathbb{R}^N$ for every $u \in \mathbb{R}$.

Now we show that Hypotheses 3.5.1 imply Hypotheses 3.3.1.

**Lemma 3.5.2.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy Hypothesis 3.5.1. Then $f$ satisfies Hypotheses 3.3.1.*

*Proof.* Hypotheses 3.3.1 $a$) and $d$) are immediately verified and to prove b) we can observe, arguing as in Lemma 3.2.3, that $f(u,\xi)$ is uniformly continuous with respect to $u$ in bounded sets of $\mathbb{R}$ uniformly as $\xi$ varies in bounded sets of $\mathbb{R}^N$.
Thus for every $u \in W^{1,\infty}(\Omega)$ and $B_K(0) \subset \mathbb{R}^N$ there exists a non decreasing modulus of continuity $\omega(\cdot)$ such that

$$|f(\overline{u}(x),\xi) - f(\overline{u}(y),\xi)| \leq \omega(|\overline{u}(x) - \overline{u}(y)|) \leq \omega(M|x-y|)$$

for every $\xi$ in $B_K(0)$. So $f(\overline{u}(x),\xi)$ is uniformly continuous for a.e. $x \in \Omega$ uniformly as $\xi$ varies in $B_K(0)$.
Finally to prove the validity of Hypothesis 3.3.1 $c$) it is sufficient to show that $f$ is bounded on bounded

sets. Given $(u,\xi) \in [-M,M] \times B_K(0)$, the function $f(u,\xi)$ is uniformly continuous with respect to $u \in [-M,M]$ uniformly as $\xi$ varies on $B_K(0)$. Furthermore for every $u \in \mathbb{R}$ then $f(u,\cdot)$ is bounded on $B_K(0)$ by Hypothesis 3.5.1.
So there exists a finite set $\{u_i | i = 1,\ldots,N\} \subset [-M,M]$ such that

$$|f(u,\xi)| \leq 1 + \max_{1\leq i\leq N} \left\{|f(u_i,\xi)|\right\} \leq 1 + \max_{1\leq i\leq N} \left\{ \sup_{\xi\in B_K(0)} \left\{|f(u_i,\xi)|\right\} \right\} \in \mathbb{R}$$

for every $u \in [-M,M]$ and for every $\xi \in B_K(0)$.

□

We are ready to state the results in the autonomous case.

**Proposition 3.5.3.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Hypothesis 3.5.1.*
*For every $\overline{u} \in W^{1,\infty}(\Omega)$ and for every $K \in \mathbb{N}$ such that*

$$\|\overline{u}\|_\infty + \|\nabla\overline{u}\|_\infty < K$$

*there exists a sequence $u_{K,n} \in \overline{u} + W_0^{1,\infty}(\Omega)$ such that*

$$\|\nabla u_{K,n}\|_\infty < K\,, \qquad u_{K,n} \overset{*}{\rightharpoonup} \overline{u} \quad in \quad W^{1,\infty}(\Omega)$$

*and*

$$\lim_{n\to\infty} \left| \int_\Omega f(u_{K,n}(x), \nabla u_{K,n}(x))dx - \int_\Omega (f_K)^{**}(\overline{u}(x), \nabla\overline{u}(x))dx \right| = 0\,.$$

*Proof.* In view of Lemma 3.5.2 it is a immediate consequence of Proposition 3.3.2.

□

**Theorem 3.5.4.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy Hypothesis 3.5.1.*
*For every $\overline{u} \in W^{1,\infty}(\Omega)$ there exists a sequence $u_n \in \overline{u} + W_0^{1,\infty}(\Omega)$ such that*

$$\lim_{n\to\infty} \|u_n - \overline{u}\|_\infty = 0$$

*and*

$$\lim_{n\to\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(\overline{u}(x), \nabla\overline{u}(x))dx\,.$$

*Furthermore*

$$sc^-(F_\infty)(\overline{u}) = \int_\Omega f^{**}(\overline{u}(x), \nabla\overline{u}(x))dx$$

*where with $sc^-(F_\infty)$ we denote the lower semicontinuous envelope of*

$$F_\infty(u) = \int_\Omega f(u(x), \nabla u(x))dx$$

*with respect to the weak* topology of $W^{1,\infty}(\Omega)$.*

*Proof.* As for the previous proposition we have only to notice that it is an immediate application of Theorem 3.3.3.

□

**Remark 3.5.5.** The assumption that $f$ is bounded on bounded sets in particular implies that, for every $K > 0$, $(f_K)^{**}$ is bounded from below.

**Remark 3.5.6.** We notice that in our setting it could happen that $f^{**} \equiv -\infty$. In this case Theorem 3.5.4 implies that there exists a sequence in $u_n \in W^{1,\infty}(\Omega)$ such that $u_n$ converges to $\bar{u}$ in $L^\infty$ and

$$\lim_{n\to\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = -\infty\,.$$

**Remark 3.5.7.** If $f(x, u, \xi) := g(x, u) + h(u, \xi)$ where $g$ is Carathéodory and $h$ satisfies Hypothesis 3.5.1 then $f(x, u, \xi)$ satisfies Hypotheses 3.3.1 but does not necessarily satisfy the hypotheses assumed on [40] or on [55].

Now we can apply the previous results about Lavrentiev Phenomenon for the autonomous case.

**Theorem 3.5.8.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$, let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy Hypothesis 3.5.1, $\varphi \in W^{1,\infty}(\Omega)$ and $1 \le p < \infty$.*

*If*

$$\inf_{\varphi+W_0^{1,p}(\Omega)} \int_\Omega f^{**}(u(x), \nabla u(x))dx = \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(u(x), \nabla u(x))dx\,.$$

*then*

$$\inf_{\varphi+W_0^{1,p}(\Omega)} \int_\Omega f(u(x), \nabla u(x))dx = \inf_{\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f(u(x), \nabla u(x))dx\,.$$

*Proof.* In view of Lemma 3.5.2 it is an immediate consequence of Theorem 3.4.1.

□

Now we recall a recent result by Pierre Bousquet ([17, Theorem 1.1]) and then we will use it to prove Theorem 3.5.14.

**Theorem 3.5.9.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and $\varphi \in W^{1,\infty}(\Omega)$. Assume $g : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^+$ continuous in both variable and convex with respect to the last variable. For every $u \in W_\varphi^{1,1}(\Omega)$ there exists a sequence $(u_n)_n \in W_\varphi^{1,\infty}(\Omega)$ such that $u_n$ strongly converges to $u$ in $W^{1,1}(\Omega)$ and*

$$\int_\Omega g(u_n(x), \nabla u_n(x))dx \to \int_\Omega g(u(x), \nabla u(x))dx\,.$$

**Theorem 3.5.10.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$. Let $\varphi \in W^{1,\infty}(\Omega)$ and assume $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies Hypothesis 3.5.1 and $f^{**}$ continuous in both variables, then*

$$\inf_{u\in\varphi+W_0^{1,1}(\Omega)} \int_\Omega f(u(x), \nabla u(x))dx = \inf_{u\in\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f(u(x), \nabla u(x))dx\,.$$

*Proof.* We can apply Theorem 3.5.9 to $f^{**}$ and so we note that

$$\inf_{u\in\varphi+W_0^{1,1}(\Omega)} \int_\Omega f^{**}(u(x), \nabla u(x))dx = \inf_{u\in\varphi+W_0^{1,\infty}(\Omega)} \int_\Omega f^{**}(u(x), \nabla u(x))dx\,.$$

Now the assumptions of Theorem 3.5.8 are satisfied.

□

**Remark 3.5.11.** In general, under Hypothesis 3.5.1, the function $(u, \xi) \to f^{**}(u, \xi)$ could be not continuous (cfr [55, Example 3.9]). In [55, Corollary 3.12] the authors proved that if

i) $f(u, \xi)$ is continuous in $u$ uniformly with respect to $\xi \in \mathbb{R}^N$

or

ii) $f(u, \xi) \ge \lambda_1 \|\xi\|^\alpha + \lambda_2$ with $\alpha > 1$, $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}$

then $f^{**}(u, \xi)$ is continuous in both variables.

In the next lemma we prove that Hypothesis 3.5.1 implies that $(f_K)^{**}$ is continuous on $\mathbb{R} \times B_K(0)$ for every $K \in \mathbb{N}$ and that $f^{**}$ is a Borel function on $\mathbb{R} \times \mathbb{R}^N$.

**Lemma 3.5.12.** *If $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies Hypothesis 3.5.1 then $(f_K)^{**}$ is continuous on $\mathbb{R} \times B_K(0)$ for every $K \in \mathbb{N}$. Furthermore $f^{**}$ is a borelian function on $\mathbb{R} \times \mathbb{R}^N$.*

*Proof.* First of all we prove that $(f_K)^{**}(u, \xi)$ is continuous w.r.t. $u$ uniformly as $\xi$ varies in $B_K(0)$. We fix $u_0 \in \mathbb{R}$ and Hypothesis 3.5.1, ii) guarantees that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|u - u_0| < \delta$ then

$$|f(u, \xi) - f(u_0, \xi)| < \varepsilon \quad \forall \xi \in B_K(0)$$

and so we have

$$(f_K)^{**}(u, \xi) - \varepsilon \leq f(u, \xi) - \varepsilon \leq f(u_0, \xi) .$$

Now $(f_K)^{**}(u, \xi) - \varepsilon$ is convex in $\xi$ and and so

$$(f_K)^{**}(u, \xi) - \varepsilon \leq (f_K)^{**}(u_0, \xi) .$$

Since $f(\cdot, \xi)$ is uniformly continuous in $[-M, M]$ uniformly as $\xi$ varies in $B_K(0)$, thus we can change the roles of $u$ and $u_0$ in the previous inequalities and we obtain

$$|(f_K)^{**}(u, \xi) - (f_K)^{**}(u_0, \xi)| < \varepsilon \quad \forall \xi \in B_K(0) .$$

Since $(f_K)^{**}(u, \cdot)$ is continuous in $B_K(0)$ for every $u$, for every sequence $(u_n, \xi_n) \in \mathbb{R} \times B_K(0)$ converging to $(u_0, \xi_0) \in \mathbb{R} \times B_K(0)$ such that

$$\begin{aligned} \lim_n \, &|(f_K)^{**}(u_n, \xi_n) - (f_K)^{**}(u_0, \xi_0)| \\ &\leq \lim_n |(f_K)^{**}(u_n, \xi_n) - (f_K)^{**}(u_0, \xi_n)| \\ &\quad + \lim_n |(f_K)^{**}(u_0, \xi_n) - (f_K)^{**}(u_0, \xi_0)| \\ &= 0 \end{aligned}$$

i.e. $(f_K)^{**}$ is continuous on $\mathbb{R} \times B_K(0)$.

Now, recalling that

$$f^{**}(u, \xi) = \inf_K (f_K)^{**}(u, \xi) = \lim_K (f_K)^{**}(u, \xi),$$

the continuity of $(f_K)^{**}$ implies that $f^{**}$ is borelian on $\mathbb{R} \times \mathbb{R}^N$.

□

**Corollary 3.5.13.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies Hypothesis 3.5.1. If for every $K > 0$ there exist $K' \geq K$ such that*

$$(f_{K'})^{**}_{|\mathbb{R} \times B_K} = (f^{**})_{|\mathbb{R} \times B_K}$$

*then $f^{**}$ is continuous in both variables.*

*Proof.* By Lemma 3.5.12 we know that $(f_{K'})^{**}_{|\mathbb{R} \times B_K}$ is continuous in both variables. Thus every restriction of $f^{**}$ is continuous and so $f^{**}$ is globally continuous in both variables.

□

Now we state a relaxation result that holds under the assumption of uniform superlinearity of the Lagrangian.

**Theorem 3.5.14.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$. Let $\varphi \in W^{1,\infty}(\Omega)$ and assume $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies Hypothesis 3.5.1, be uniformly superlinear, i.e. there exists a superlinear function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(u, \xi) \geq \theta(\xi) \quad \forall u \in \mathbb{R}$$

*and $f^{**}$ be continuous in both variables.*
*Then for every $u \in \varphi + W_0^{1,1}(\Omega)$*

$$sc^-(\overline{F_1})(u) = \int_\Omega f^{**}(u(x), \nabla u(x))dx$$

*where $sc^-(\overline{F_1})$ is the lower semicontinuous envelope of*

$$\overline{F_1}(u) = \begin{cases} \int_\Omega f(u(x), \nabla u(x))dx & \text{if} \quad u \in \varphi + W_0^{1,\infty}(\Omega) \\ +\infty & \text{if} \quad u \in \varphi + W_0^{1,1}(\Omega) \setminus W_0^{1,\infty}(\Omega) \end{cases}$$

*with respect to the weak topology of $W^{1,1}(\Omega)$.*

*Proof.* By theorem 3.5.9 for every $u \in \varphi + W_0^{1,1}(\Omega)$ there exists a sequence $(u_k)_k \subset \varphi + W_0^{1,\infty}(\Omega)$ such that

$$u_k \to u \quad \text{strongly in } W^{1,1}(\Omega)$$

and

$$\lim_k \int_\Omega f^{**}(u_k(x), \nabla u_k(x))dx = \int_\Omega f^{**}(u(x), \nabla u(x))dx\,.$$

By Theorem 3.5.4, for every $u_k$ there exist a sequence $(u_k^{(n)})_n \subset \varphi + W_0^{1,\infty}(\Omega)$ such that

$$u_k^{(n)} \to u_k \quad \text{in} \quad L^\infty(\Omega)$$

and

$$\lim_n \int_\Omega f(u_k^{(n)}(x), \nabla u_k^{(n)}(x))dx = \int_\Omega f^{**}(u_k(x), \nabla u_k(x))dx\,.$$

Thus there exists a sequence $u_n \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$u_n \to_{L^1} u$$

and

$$\lim_n \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(u(x), \nabla u(x))dx\,.$$

Since $f(u, \xi)$ is uniformly superlinear there exists $\theta$ superlinear such that

$$\sup_n \int_\Omega \theta(\nabla u_n(x))dx \le \sup_n \int_\Omega f(u_n(x), \nabla u_n(x))dx < +\infty$$

thus, unless considering a subsequence, $\nabla u_n$ converges weakly in $L^1$ to some $v \in L^1$. It is easy to see (using the integration by parts) that actually $v = \nabla u$.
So we have

$$u_n \rightharpoonup_{W^{1,1}} u\,.$$

Since by [40, Chapter VIII, Theorem 2.1, page 243]

$$\int_\Omega f^{**}(u(x), \nabla u(x))dx \le \int_\Omega f(v_n(x), \nabla v_n(x))dx$$

for every $v_n \rightharpoonup_{W^{1,1}} u$ then

$$sc^-(\overline{F_1})(u) = \int_\Omega f^{**}(u(x), \nabla u(x))dx\,.$$

□

**Remark 3.5.15.** An interesting question regards which conditions guarantee the continuity of $f^{**}$ in both variables. A first result in this direction will be present in the Chapter 5.

**Remark 3.5.16.** In general the result of Theorem 3.5.14 for non autonomous Lagrangians is false and it is possible to find counterexamples in [34], in [14] and in [16].

# Chapter 4

# The case of unbounded Lagrangians depending on $\nabla u$

In this chapter, we report the paper *Integral representations of lower semicontinuous envelopes for non convex, possibly unbounded, Lagrangians* ([12]).

## 4.1 Introduction

The Lavrentiev phenomenon, first discovered by M. Lavrentiev in 1926 [53], represents a fundamental problem in the Calculus of Variations. It occurs when the infimum of a variational functional taken over a smaller, smoother class of admissible functions, for instance Lipschitz or $\mathcal{C}^1$ functions, is strictly greater than the infimum taken over a larger, weaker class, such as Sobolev functions. In other words, the expected approximation of minimizers by smooth functions fails, revealing a gap between the two infima and highlighting situations where regularity and approximation properties break down. Several examples of its occurrence have been exhibited in various cases of the Calculus of Variations also for very regular Lagrangians [2, 5, 6, 34, 42, 43, 50, 54, 69].

The problem of determining sufficient conditions for the non occurrence of the Lavrentiev gap has been studied in various settings, including problems with non-standard growth conditions, constraints, or degeneracies in the integrand. We cite here some of the results in this direction, without claiming to be exhaustive [3, 8, 10, 16, 14, 17, 20, 21, 22, 28, 37, 38, 41, 42, 49, 51, 52, 54, 58, 69]

Understanding the mechanisms that give rise to or prevent the Lavrentiev phenomenon not only deepens the theoretical framework of the calculus of variations but also has implications for numerical analysis, materials science, and the theory of partial differential equations. Recently, its implications in AI have been highlighted. We underline that, while most of the results cited above deal with Lagrangians that are convex in the gradient variable, in the framework of AI non convex problems play a crucial role.

In this work, we consider the functional

$$F(u) = \int_\Omega f(\nabla u)\, dx \qquad u \in \varphi + W_0^{1,1}(\Omega) \tag{4.1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz domain, $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a Borel function, $\varphi$ is assumed to be Lipschitz.

In Theorem 4.4.3, we prove that the functional (4.1.1) does not exhibit the Lavrentiev phenomenon as soon as we assume $f$ is superlinear and satisfies a very weak assumption, see (HF') (ii) and (iii). We remark that $f$ is not necessarily continuous, convex, or bounded, as it has been underlined in Example 4.2.1.

The main difficulty in the proof of this result consists in the fact that we cannot use approximations of the function $u$ by means of convolutions. Instead, we must explicitly construct a suitable approximating sequence that has the property that each function in the sequence has a gradient taking values where $f$ and its lower semicontinuous convex envelope $f^{**}$ coincide. This is achieved in Theorem 4.3.1 where, refining a construction by Cellina [30] (see also [45]) later developed in [67, 65, 31], we show that, for any almost everywhere differentiable Sobolev function $u$ and any $\varepsilon > 0$, we can define a suitable function $v$, coinciding with $u$ on $\partial\Omega$, such that

$$F(v) \leq F^{**}(u) + \varepsilon$$

where

$$F^{**}(v) = \int_{\Omega} f^{**}(\nabla v)\, dx.$$

The proof of the non occurrence of the Lavrentiev phenomenon for the functional $F$ then works as follows. First of all, we exploit some recent results [17, 20] that state the non occurrence of the gap for autonomous convex problems, applying them to the functional $F^{**}$. Then, denoting by $u_n$ a sequence of Lipschitz functions that is minimizing for $F^{**}$, we apply Theorem 4.3.1 to each element $u_n$, obtaining a new sequence of Lipschitz $v_n$ that is minimizing for the functional $F$.

## 4.2 Notation, preliminaries and assumptions

We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^N$ and by $\|\cdot\|_\infty$ the norm in $L^\infty$; by $e_i$ the $i$-th vector of the canonical basis in $\mathbb{R}^N$ and by $a \cdot b$ the scalar product between two vectors $a, b \in \mathbb{R}^N$. In the whole chapter $f$ will denote a Borel function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$.
We will use the lower semicontinuous convex envelope of $f$ i.e., the function $f^{**} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Recalling ([32],[40]) can be represented as

$$f^{**}(\xi) = \inf\left\{ \sum_{i=1}^{N+1} \alpha_i f(\xi_i) : \sum_{i=1}^{N+1} \alpha_i \xi_i = \xi,\, \alpha_i \geq 0,\, \sum_{i=1}^{N+1} \alpha_i = 1 \right\}$$

or also as

$$f^{**}(\xi) = \sup\left\{ l(\xi) : l \text{ affine}, l \leq f \right\}.$$

We assume that $f$ satisfy the following properties.

(HF) (i) There exist $c_1 > 0$, $c_2 \in \mathbb{R}$ such that

$$f(\zeta) \geq c_1|\zeta| + c_2 \qquad \text{for every } \zeta \in \mathbb{R}^N$$

(ii) For every $\xi \in \mathbb{R}^N$ either $f^{**}(\xi) = f(\xi)$ or there exist $k \in \{1, \ldots, N\}$, $\{\xi_i\}_{i=1\ldots,k+1} \subset \mathbb{R}^N$ and $\{\alpha_i\}_{i=1\ldots,k+1} \subset \mathbb{R}$ such that $\alpha_i$ is strictly positive for every $i = 1, \ldots, k+1$, $\sum_{i=1}^{k+1} \alpha_i = 1$, $\sum_{i=1}^{k+1} \alpha_i \xi_i = \xi$ and

$$\sum_{i=1}^{k+1} \alpha_i f(\xi_i) = f^{**}(\xi).$$

Moreover we also assume that $\dim \mathrm{Span}(\xi_1, ..., \xi_{k+1}) = k$.

(iii) For every $R > 0$ there exist $N+1$ vectors $\zeta_j \in \mathbb{R}^N$ such that $\overline{B}(0, R) \subset \overline{\mathrm{co}}(\cup_{j=1}^{N+1}\{\zeta_j\})$ and

$$f(\zeta_j) = f^{**}(\zeta_j) < +\infty \qquad \text{for every } j = 1, ..., N+1.$$

The first of the following examples will show that the assumption (HF) is very general and includes functions that assume the value $+\infty$ on very large sets, possibly a.e. on $\mathbb{R}^N$. The second one will exhibit a function that does not satisfy (HF) (ii).

**Example 4.2.1.** Condition (HF) (ii) is satisfied by any function $f : \mathbb{R}^N \to \mathbb{R}$ such that for any face $\mathcal{F}$ of the epigraph of $f^{**}$ it holds

$$\mathcal{F} = \mathrm{co}\{(\eta, f^{**}(\eta)) : \, f^{**}(\eta) = f(\eta) \text{ and } (\eta, f^{**}(\eta)) \in \mathcal{F}\}$$

These class includes, for instance, functions that could be equal to $+\infty$ almost everywhere. The function defined by

$$f(\eta) = \begin{cases} |\eta|^2, & \text{if } \eta \in \mathbb{Z}^N, \\ |\eta|^2 + g(\eta), & \text{otherwise.} \end{cases} \tag{4.2.1}$$

where $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $g(\eta) \geq 0$ , satisfies assumption (HF) (ii).

**Remark 4.2.2.** We underline the fact that, in the case $N = 2$, the function defined in (4.2.1) assumes the value $+\infty$ on every line passing through the origin with an irrational slope.

**Example 4.2.3.** We consider the function defined by

$$f(\eta) = g(|\eta|),$$

where

$$g(t) = \begin{cases} |t^2 - 1|, & \text{if } t \neq 1, \\ 1, & \text{if } t = 1, \end{cases}$$

then $f$ does not satisfy condition (HF) (ii). In fact $f^{**}(\eta) = 0$ for every $\eta \in \overline{B(0,1)}$ while $f(\eta) > 0$ for every $\eta \in \mathbb{R}^N$.

In the proofs of our results we will use some consequences of assumption (HF).

**Lemma 4.2.4.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfy assumption* (HF). *Then the following properties hold.*

*(a) The effective domain of $f^{**}$ is $\mathbb{R}^N$, i.e., $f^{**} : \mathbb{R}^N \to \mathbb{R}$;*

*(b) If $\xi$, $\xi_i$, $\alpha_i$ are as in* (HF) (ii), *then $f(\xi_i) = f^{**}(\xi)$ and there exist $a \in \mathbb{R}^n$ such that $a \in \partial f^{**}(\xi) \cap (\cap_{i=1,\dots,k+1} \partial f^{**}(\xi_i))$.*

To conclude this section, we list some further notations that we will use in the chapter. Since we will need to use Lebesgue measures in spaces of different dimensions, when dealing with the measure in $\mathbb{R}^N$, we write $\lambda$; whereas when dealing with the measure in a subspace of dimension $k$, we use $\lambda_k$. We use the standard notations for the Sobolev spaces $W^{1,1}(\Omega)$, $W^{1,\infty}(\Omega)$, $W_0^{1,1}(\Omega)$ and $W_0^{1,\infty}(\Omega)$. In particular, we denote by $\rightharpoonup$ the weak convergence in $W^{1,1}(\Omega)$, by $\overset{*}{\rightharpoonup}$ the weak* convergence in $W^{1,\infty}(\Omega)$, and by $\|\cdot\|_{W^{1,1}}$ and $\|\cdot\|_{W^{1,\infty}}$ respectively the norms in $W^{1,1}(\Omega)$ and in $W^{1,\infty}(\Omega)$.

## 4.3 An approximation result

In this section, we prove a theorem that constitutes the first step in identifying sufficient conditions to prevent the Lavrentiev phenomenon for integral functionals with non-convex and discontinuous Lagrangians. Given a function $u \in W_\varphi^{1,1}$, assumed also to be differentiable almost everywhere in $\Omega$, we will construct a function $v$ whose gradients take values where the functions $f$ and $f^{**}$ coincide. The construction is carried out so that the value of $F(v)$ is not far from $F^{**}(u)$.

**Theorem 4.3.1.** *Let be $\Omega$ a open bounded Lipschitz subset of $\mathbb{R}^N$ and assume that $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfies assumption* (HF). *Let $u \in \varphi + W_0^{1,1}(\Omega)$ be differentiable almost everywhere.*
*Then for every $\varepsilon > 0$ there exists $v \in \varphi + W_0^{1,1}(\Omega)$, differentiable almost everywhere, such that*

$$\int_\Omega f(\nabla v)\, dx \le \int_\Omega f^{**}(\nabla u)\, dx + \varepsilon\,.$$

*Proof.* The construction of the function $v$ proceeds in several steps. We start by defining a piecewise affine function that will be used to modify the function $u$ in a suitable neighborhood of a fixed point $x_0$ where $u$ is differentiable. We also provide an estimate of the difference between $F^{**}(u)$ and the value $F^{**}(\tilde{u})$ in the modified function. In the last step of the proof we use the Vitali Covering Theorem to obtain the desired function $v$. The proof is inspired by by a technique due to Cellina [30] in the framework of non-convex problems. The main difficulty we have to face here is that we do not assume, as in [30] and subsequent papers [65, 31, 67], that $(\nabla u(x_0), f^{**}(\nabla u(x_0)))$ is in the interior of a $N$-dimensional face of the epigraph of $f^{**}$.
In the whole proof we assume, without restriction, that $\int_\Omega f^{**}(\nabla u)\, dx < +\infty$.

*Step 1. Local construction in the case $k = N$.*

We remark that in this case we can argue as in [31] and take advantage of the construction performed there. We repeat the construction here for the sake of clarity and, moreover, to later underline the difficulties of the case $k < N$.
We fix a point $x_0 \in \Omega$ such that $u$ is differentiable at $x_0$ and we denote $\xi = \nabla u(x_0)$. Let $\xi_i$, $\alpha_i$ $1 \leq k \leq N$, the vectors and the coefficients involved in (HF) (ii). We define

$$\frac{1}{\alpha} = \max\{|x| : \max_{1,\ldots,N+1}(\xi_i - \xi) \cdot x \leq 1\} \qquad \text{and} \qquad \beta = \max_{i=1,\ldots N+1} |\xi_i - \xi|.$$

and we notice that assumption (HF) (ii) implies $\alpha > 0$ and that

$$\alpha|x - x_0| \leq \max_{i=1,\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) \leq \beta|x - x_0| \quad \forall x \in \mathbb{R}^N.$$

Now, for every $s \in \mathbb{R}$, $s > 0$, we define

$$\begin{aligned}\tilde{\omega}_{x_0,s}(x) := u(x_0) + \xi \cdot (x - x_0)& \\ + \max_{i=1\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) - \frac{1}{2}s&\end{aligned}$$

and

$$\omega_{x_0,s}(x) := \tilde{\omega}_{x_0,s}(x)\, I_{B(x_0,s)}(x)$$

where

$$I_{B_s(x_0)}(x) = \begin{cases} 1 & \text{if } x \in \overline{B(x_0,s)} \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that there exists $\overline{s}$ such that for every $0 < s < \overline{s}_{x_0} = \min\{\overline{s}, 2\alpha\overline{s}\}$, we have

$$u(x) \leq \omega_{x_0,s}(x) \qquad \text{on} \quad \partial B\left(x_0, \frac{s}{\alpha}\right). \tag{4.3.1}$$

and

$$\omega_{x_0,s}(x) \leq u(x) \qquad \text{on} \quad B\left(x_0, \frac{s}{(2\beta + \alpha)}\right). \tag{4.3.2}$$

To prove (4.3.1) and (4.3.2) we first recall that the differentiability property of $u$ in $x_0$ implies that there exists $\overline{s} > 0$ such that $B(x_0, \overline{s}) \subset \Omega$ and, for every $x \in B(x_0, \overline{s})$, we have

$$\begin{aligned}u(x_0)+&\xi \cdot (x - x_0) - \frac{\alpha}{2}|x - x_0| \\ &< u(x) < u(x_0) + \xi \cdot (x - x_0) + \frac{\alpha}{2}|x - x_0|.\end{aligned} \tag{4.3.3}$$

It follows, for every $x \in \partial B\big(x_0, \frac{s}{\alpha}\big)$,

$$\begin{aligned}u(x) &\leq u(x_0) + \xi \cdot (x - x_0) + \frac{\alpha}{2}|x - x_0| \\ &\leq u(x_0) + \xi \cdot (x - x_0) + \frac{1}{2}\max_{1,\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) \\ &= u(x_0) + \xi \cdot (x - x_0) + \max_{1,\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) \\ &\quad - \frac{1}{2}\max_{1,\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) \\ &\leq u(x_0) + \xi \cdot (x - x_0) + \max_{1,\ldots,N+1}(\xi_i - \xi) \cdot (x - x_0) \\ &\quad - \frac{\alpha}{2}|x - x_0|.\end{aligned}$$

and the proof of (4.3.1) is completed observing that if $x \in \partial B\left(x_0, \frac{s}{\alpha}\right)$ the last expression coincides with $\omega_{x_0,,s}(x)$.

To prove (4.3.2) we consider $x \in B\left(x_0, \frac{s}{2\beta+\alpha}\right)$ and we compute

$$\begin{aligned}
&u(x_0) + \xi \cdot (x - x_0) + \max_{1,\dots,N+1} (\xi_i - \xi) \cdot (x - x_0) - \frac{s}{2} \\
&\leq u(x_0) + \xi \cdot (x - x_0) + \beta|x - x_0| - \frac{s}{2} \\
&\leq u(x_0) + \xi \cdot (x - x_0) + \beta|x - x_0| - \frac{1}{2}\left(2\beta + \alpha\right)|x - x_0| \\
&= u(x_0) + \xi \cdot (x - x_0) - \frac{\alpha}{2}|x - x_0| \leq u(x).
\end{aligned}$$

Inequality (4.3.1) implies that the function

$$\tilde{u}_{x_0,s} := \min\{u, \omega_{x_0,s}(x)\}$$

coincides with $u$ in $\Omega \setminus B\left(x_0, \frac{s}{\alpha}\right)$, hence also on $\partial\Omega$.
We denote by $E_{x_0,s}$ the set $\{x \in B\left(x_0 \frac{s}{\alpha}\right) : \tilde{u}_{x_0,s} = \omega_{x_0,s}(x)\}$ and we notice that, (4.3.1) and (4.3.2)

$$B\left(x_0, \frac{s}{2\beta+\alpha}\right) \subset E_{x_0,s} \subset B\left(x_0, \frac{s}{\alpha}\right). \tag{4.3.4}$$

Now we want to estimate $F^{**}(\tilde{u}_{x_0,s})$. A key point is the property $(b)$ in Lemma 4.2.4. We observe that, since $\tilde{u}_{x_0,s} = u$, and hence also $\nabla\tilde{u}_{x_0,s} = \nabla u$, on $\Omega \setminus E_{x_0,s}$ it is sufficient to compare

$$\int_{E_{x_0,s}} f^{**}(\nabla u)\,dx \qquad \text{and} \qquad \int_{E_{x_0,s}} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx.$$

The convexity of $f^{**}$, assumption (HF) (ii) and Lemma 4.2.4 imply that we can choose a selection $p(\cdot) \in \partial f^{**}(\cdot)$ such that $p(\nabla\tilde{u}_{x_0,s}(x)) = a$ for a.e. $x \in \Omega$. It follows

$$\begin{aligned}
&\int_{E_{x_0,s}} f^{**}(\nabla u)\,dx \\
&\qquad \geq \int_{E_{x_0,s}} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx + \int_{E_{x_0,s}} a(\nabla u - \nabla\tilde{u}_{x_0,s})\,dx \\
&\qquad = \int_{E_{x_0,s}} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx + \int_{B(x_0,\frac{s}{\alpha})} a(\nabla u - \nabla\tilde{u}_{x_0,s})\,dx \\
&\qquad = \int_{E_{x_0,s}} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx.
\end{aligned}$$

We notice that the last two equalities follow from the fact that $\nabla u = \nabla\tilde{u}_{x_0,s}$ a.e on $B(x_0, \frac{s}{\alpha}) \setminus E_{x_0,s}$ and observing that $\mathrm{div} a = 0$.
Finally we also underline that assumption (HF) (ii) and (iii) imply

$$\int_{E_{x_0,s}} f^{**}(\nabla u)\,dx \geq \int_{E_{x_0,s}} f(\nabla\tilde{u}_{x_0,s})\,dx.$$

*Step 2. Construction of a suitable piecewise affine function in the case $k < N$.*
As in the previous step, we fix a point $x_0 \in \Omega$ such that $u$ is differentiable at $x_0$, we denote $\xi = \nabla u(x_0)$ and we fix $R$ such that $R > |\xi| + 1$. Let $\xi_i$, $\alpha_i$, $\zeta_j$ be the vectors and the coefficients that satisfy assumption (HF) (ii) and (HF) (iii) for $\xi$ and $R$.
To begin with we need to fix some parameters that are needed in the construction. As we will see, some of them will depend on the choice of $x_0$. In order to keep the notation light, we will not always make explicit the dependence on $x_0$. We fix $\varepsilon > 0$. We assumed $f^{**}(\nabla u) \in L^1(\Omega)$ and so there exists $\delta > 0$ such that if $\Omega' \subset \Omega$ and $\lambda(\Omega') < \delta$ implies

$$\int_{\Omega'} |f^{**}(\nabla u)|\,dx < \frac{\varepsilon}{3\lambda(\Omega)}. \tag{4.3.5}$$

We still denote by $a$ the vector in $\mathbb{R}^N$ given by Lemma 4.2.4 (b) such that $a \in \partial f^{**}(\xi_i)$ for every $i = 1, \ldots, k+1$ and we choose $\gamma\, \eta \in \mathbb{R}$ such that

$$0 < \gamma < \min \left\{ \frac{\varepsilon}{3^{N+2}(N-k)|a|\lambda(\Omega)}, 1 \right\} \tag{4.3.6}$$

$$\eta = \frac{\delta}{\lambda(\Omega)}. \tag{4.3.7}$$

We set

$$M_{x_0} := \max_{i,j} \{|f(\xi_i)|, |f(\zeta_j)|, 1\} < +\infty \tag{4.3.8}$$

and we fix the real number $S_\eta$ such that

$$S_\eta \geq \max \left\{ \frac{1}{1 - \left(1 - \frac{\varepsilon}{M_{x_0}\lambda(\Omega)3^{N+1}}\right)^{\frac{1}{N-k}}}, \frac{1}{1 - (1 - \frac{\eta}{3^N})^{\frac{1}{N-k}}}, \frac{3}{2} \right\}. \tag{4.3.9}$$

Without losing generality, we assume that

$$\xi_i \in \text{Span}\{e_1, \ldots, e_k\} =: V \qquad \forall i = 1, \ldots, k+1$$

and we denote

$$V^\perp := \text{Span}\{e_{k+1}, \ldots, e_N\}.$$

Given a point $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$ we denote by $x' = (x_1, \ldots, x_k)$ its projection on $V$ and by $x'' = (x_{k+1}, \ldots, x_N)$ its projection on $V^\perp$. We consider the function $w_k : V \to \mathbb{R}$ defined by

$$w_k(x') := \max_{1,\ldots,k+1} (\xi_i - \xi) \cdot x'.$$

and we will also use

$$|x''|_{\infty, V^\perp} = \max_{j=k+1,\ldots,N} |x_j|$$

$$B := \{x' \in V : w_k(x') < \gamma\}$$

$$Q := \{x = (x', x'') \in \mathbb{R}^N : w_k(x') < \gamma \text{ and } |x''|_{\infty, V^\perp} < S_\eta\},$$

$$\tilde{Q} := \{x = (x', x'') \in \mathbb{R}^N : w_k(x') < \gamma \text{ and } |x''|_{\infty, V^\perp} < S_\eta - 1\}.$$

In the following, we will use that

$$\lambda(Q) = \lambda_k(B) S_\eta^{N-k} \tag{4.3.10}$$

$$\lambda(\tilde{Q}) = \lambda_k(B)(S_\eta - 1)^{N-k} \tag{4.3.11}$$

$$\lambda_{N-1}(\partial\tilde{Q} \setminus \partial Q) = 2(N-k)\lambda_k(B) S_\eta^{N-k-1} \tag{4.3.12}$$

We introduce the function $w : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w(x) := \max\{w_k(x'), \frac{\gamma}{S_\eta}|x''|_{\infty, V^\perp}\}$$

Using the positive $1-$homogeneity of $w_k$ and $|\cdot|_{\infty, V^\perp}$ and the definition of $Q$, it follows that the function $w$ is 1-positively homogeneous, and satisfies the following properties:

$$\{x \in \mathbb{R}^N : w(x) < \gamma\} = Q;$$

$$\text{there exist} \quad 0 < \alpha < \beta \quad \text{such that} \quad \alpha|x| \leq w(x) \leq \beta|x| \quad \forall x \in \mathbb{R}^N \tag{4.3.13}$$

where the last property can be obtained in a similar way to the analogous one of Step 1.

We define the function $\tilde{w} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\tilde{w}(x) := \max\{w_k(x'), \gamma(|x''|_{\infty,V^\perp} - (S_\eta - 1))\}$$

and we prove that

$$\tilde{w}(x) \leq w(x) \quad \text{in} \quad Q\,, \tag{4.3.14}$$
$$\tilde{w}(x) = w(x) \quad \text{on} \quad \partial Q \tag{4.3.15}$$
$$\tilde{w}(x) = w_k(x) \quad \text{in} \quad \tilde{Q}\,. \tag{4.3.16}$$

In fact, since $w_k \geq 0$ and $|x''|_{\infty,V^\perp} \leq S_\eta - 1$ in $\tilde{Q}$, then (4.3.16) follows. To prove (4.3.14) and (4.3.15), it is sufficient to note that if $|x''| \leq S_\eta$

$$\gamma(|x''|_{\infty,V^\perp} - (S_\eta - 1)) \leq \frac{\gamma}{S_\eta}|x''|_{\infty,V^\perp}$$

where the equality holds if (and only if) $|x''|_{\infty,V^\perp} = S_\eta$.
Now let $A$ be the set

$$A := \{x \in Q : \tilde{w}(x) = \gamma(|x''|_{\infty,V^\perp} - (S_\eta - 1))\}$$

and notice that, by the definition of $\tilde{w}$ and (4.3.16), it follows that

$$A \subset Q \setminus \tilde{Q} \qquad \text{and} \qquad |\nabla \tilde{w}(x)| = \gamma < 1 \text{ a.e. in } A.$$

Denoting by $\zeta_j$ the vectors chosen at the beginning of this step, since $B(\xi, 1) \subset B(0, R{+}1) \subset \overline{\mathrm{co}}\,(\cup_{j=1}^{N+1}\{\zeta_j\})$, it follows that $\nabla \tilde{w}(x) \in B(0, 1) \subset (\cup_{j=1}^{N+1}\{\zeta_j - \xi\})$ and hence we can use the construction performed by Cellina in [30] and then subsequently refined by many authors (see [31] and references therein) to obtain a function $\overline{w}_A : A \to \mathbb{R}$ such that

$$\tilde{w}(x) - \frac{1}{2} \leq \overline{w}_A(x) \leq \tilde{w}(x) \quad \text{ for every } x \in A, \tag{4.3.17}$$

$$\overline{w}_A(x) = \tilde{w}(x) \quad \text{ for every } x \in \partial A$$

and, observing that $0 \in \operatorname{int} \overline{co}\{\zeta_j - \xi\}$

$$\nabla \overline{w}_A(x) = \zeta_j - \nabla u(x_0) \quad \text{ a.e. in } A.$$

Now we consider the function $\overline{w} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\overline{w}(x) := \begin{cases} \tilde{w}(x) & \text{if } x \in \mathbb{R}^N \setminus A \\ \overline{w}_A(x) & \text{if } x \in A\,. \end{cases}$$

*Step 3. Local construction in the case $K < N$.*
In this step we use the function $\overline{w}$ defined in Step 2 to obtain the desired local modification of the function $u$ in a suitable neighborhood of the point $x_0$ fixed at the beginning of Step 2. First of all, we recall that in Step 1 we already remarked that, by the differentiability of $u$ at $x_0$, there exist $\overline{r} \in \mathbb{R}$ such that $B(x_0, \overline{r}) \subset \Omega$ and inequalities (4.3.3) holds for every $x \in B(x_0, \overline{r})$.
We consider the set

$$Q_{x_0,s} = \{x \in \mathbb{R}^n : w(x - x_0) < s\gamma\} = x_0 + sQ$$

and we have that there exists $\overline{s}_{x_0}$ such that, for every $0 < s < \overline{s}_{x_0}$, $Q_{x_0,s} \subset B(x_0, \overline{r})$. For every $s \in ]0, \overline{s}_{x_0}[$ it also holds:

$$u(x) \leq u(x_0) + \xi \cdot (x - x_0) + w(x - x_0) - \frac{s}{2}\gamma \quad \forall x \in \partial Q_{x_0,s} \tag{4.3.18}$$

and

$$u(x_0) + \xi \cdot (x - x_0) + w(x - x_0) - \frac{s}{2}\gamma \leq u(x) \quad \forall x \in Q_{x_0,\frac{s}{3}}\,. \tag{4.3.19}$$

In fact, using the second inequality in (4.3.3) and (4.3.13), we get

$$
\begin{aligned}
u(x) &\le u(x_0) + \xi \cdot (x - x_0) + \frac{\alpha}{2}|x - x_0| \\
&\le u(x_0) + \xi \cdot (x - x_0) + \frac{1}{2} w(x - x_0) \\
&= u(x_0) + \xi \cdot (x - x_0) + w(x - x_0) - \frac{1}{2} w(x - x_0)
\end{aligned}
$$

and recalling that on $\partial Q_{x_0,s}$ we have $w(x - x_0) = s\gamma$ we obtain (4.3.18). To deduce (4.3.19) we notice that in $Q_{x_0,\frac{s}{3}}$ it holds $3w(x - x_0) \le s\gamma$; we use the first inequality in (4.3.3) and (4.3.13), so that

$$
\begin{aligned}
& u(x_0) + \xi \cdot (x - x_0) + w(x - x_0) - \frac{s}{2}\gamma \\
&\le u(x_0) + \xi \cdot (x - x_0) + w(x - x_0) - \frac{3}{2} w(x - x_0) \\
&\le u(x_0) + \xi \cdot (x - x_0) - \frac{\alpha}{2}|x - x_0| \le u(x).
\end{aligned}
$$

Now we define

$$\overline{w}_s(x) = s\overline{w}\Big(\frac{x}{s}\Big)$$

and we note that $\overline{w}_s(x) = s\gamma$ for every $x \in s\partial Q$, and $\overline{w}_s \le w(x)$ for every $x \in sQ$. Hence, from (4.3.14) and (4.3.15), we also have that

$$u(x) \le u(x_0) + \xi \cdot (x - x_0) + \overline{w}_s(x - x_0) - \frac{s}{2}\gamma \quad \forall x \in \partial Q_{x_0,s} \tag{4.3.20}$$

and

$$u(x_0) + \xi \cdot (x - x_0) + \overline{w}_s(x - x_0) - \frac{s}{2}\gamma \le u(x) \quad \forall x \in Q_{x_0,\frac{s}{3}}. \tag{4.3.21}$$

We define

$$\tilde{\omega}_{x_0,s}(x) = u(x_0) + \xi \cdot (x - x_0) + \overline{w}_s(x - x_0) - \frac{s}{2}\gamma$$

and

$$\omega_{x_0,s} = \tilde{\omega}_{x_0,s} I_{Q_{x_0,s}}$$

where

$$I_{Q_{x_0,s}}(x) = \begin{cases} 1 & \text{in} \quad Q_{x_0,s}\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Analogously to Step 1, we consider the function $\tilde{u}_{x_0,s} : \Omega \to \mathbb{R}$ defined by

$$\tilde{u}_{x_0,s}(x) := \min\{u(x),\, \omega_{x_0,s}\}$$

and we denote by $E_{x_0,s}$ the set $\{x \in Q_{x_0,s} :\, \tilde{u}_{x_0,s} = \omega_{x_0,s}(x)\}$. By (4.3.20) and (4.3.21), it follows that

$$Q_{x_0,\frac{s}{3}} \subset E_{x_0,s} \subset Q_{x_0,s} \tag{4.3.22}$$

We observe that

$$\nabla \tilde{u}_{x_0,s}(x) = \nabla u(x) \quad \text{for a.e. } x \in Q_{x_0,s} \setminus E_{x_0,s}, \tag{4.3.23}$$

$$\nabla \tilde{u}_{x_0,s}(x) \in \{\xi_i,\, i = 1, \dots, k+1\} \cup \{\zeta_j,\, j = 1, \dots, N+1\} \quad \text{for a.e. } x \in \tilde{Q}_{x_0,s} \cap E_{x_0,s}, \tag{4.3.24}$$

$$f(\nabla \tilde{u}_{x_0,s}(x)) = f^{**}(\nabla \tilde{u}_{x_0,s}(x)) \ \text{ a.e. } x \in E_{x_0,s}. \tag{4.3.25}$$

Analogously to Step 1, we compare

$$\int_{E_{x_0,s}} f^{**}(\nabla u)\, dx \qquad \text{and} \qquad \int_{E_{x_0,s}} f(\nabla \tilde{u}_{x_0,s})\, dx.$$

We start by considering

$$\begin{aligned}
&\int_{E_{x_0,s}} f^{**}(\nabla u)\,dx \qquad (4.3.26)\\
=&\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla u)\,dx - \int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla \tilde{u}_{x_0,s})\,dx\\
&+\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla \tilde{u}_{x_0,s})\,dx + \int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} f^{**}(\nabla u)\,dx
\end{aligned}$$

By the convexity of $f^{**}$ and Lemma 4.2.4 (b), we can choose a selection $p(\cdot)\in\partial f^{**}(\cdot)$ such that $p(\xi_i)=a$ for every $i=1,\dots,k+1$, so that also using (4.3.24) we get

$$\begin{aligned}
&\int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} f^{**}(\nabla u)\,dx \qquad (4.3.27)\\
&\geq \int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} f^{**}(\nabla \tilde{u}_{x_0,s})\,dx + \int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} a\cdot(\nabla u-\nabla\tilde{u}_{x_0,s})\,dx\\
&= \int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} f(\nabla \tilde{u}_{x_0,s})\,dx + \int_{\tilde{Q}_{x_0,s}} a\cdot(\nabla u-\nabla\tilde{u}_{x_0,s})\,dx
\end{aligned}$$

where, in the last equality, we applied (4.3.23). The fact that $a$ is constant on $\tilde{Q}_{x_0,s}$ implies that

$$\int_{\tilde{Q}_{x_0,s}} a\cdot(\nabla u-\nabla\tilde{u}_{x_0,s})\,dx = \int_{\partial\tilde{Q}_{x_0,s}} (u-\tilde{u}_{x_0,s})a\cdot\nu_x dH^{N-1} \qquad (4.3.28)$$

where $\nu_x$ denotes the external normal to $\tilde{Q}_{x_0,s}$. By construction and (4.3.20), recalling that $B=\{x'\in V: w_k(x')<\gamma\}$, it follows that

$$\begin{aligned}
\{x\in\partial\tilde{Q}_{x_0,s}:u(x)\neq\tilde{u}_{x_0,s}(x)\}&\subset\partial\tilde{Q}_{x_0,s}\setminus\partial Q_{x_0,s}\\
&=\{x\in\mathbb{R}^n:\,x'\in sB \text{ and } |x''|_{\infty,V^\perp}=s(S_\eta-1)\}
\end{aligned}$$

thus, recalling (4.3.12),

$$\begin{aligned}
&\lambda_{N-1}(\{x\in\partial\tilde{Q}_{x_0,s}:u(x)\neq\tilde{u}_{x_0,s}(x)\})\\
&\quad\leq\lambda_{N-1}(s(\partial\tilde{\Omega}\setminus\partial Q))\\
&\quad=2(N-k)s^{N-1}\lambda_k(B)(S_\eta-1)^{N-k-1}.
\end{aligned}$$

Furthermore, using (4.3.13), (4.3.17) and (4.3.3), we have, on $\partial\tilde{Q}_{x_0,s}$,

$$|\tilde{u}_{x_0,s}(x)-u(x)|\leq\frac{3}{2}s\gamma$$

so that, returning to (4.3.28), we have the following

$$\begin{aligned}
&\left|\int_{\tilde{Q}_{x_0,s}} a\cdot(\nabla u-\nabla\tilde{u}_{x_0,s})dx\right|\\
&\quad\leq 3\gamma(N-k)|a|s^N\lambda_k(B)(S_\eta-1)^{N-k-1}\\
&\quad=3^{N+1}\gamma(N-k)|a|\frac{s^N\lambda_k(B)(S_\eta-1)^{N-k-1}}{3^N}\\
&\quad=3^{N+1}\gamma(N-k)|a|\lambda(Q_{x_0,\frac{s}{3}})\\
&\quad\leq 3^{N+1}\gamma(N-k)|a|\lambda(E_{x_0,s}).
\end{aligned}$$

Recalling (4.3.6) we get

$$\left|\int_{E_{x_0,s}\cap\tilde{Q}_{x_0,s}} a\cdot(\nabla u-\nabla\tilde{u}_{x_0,s})\,dx\right|\leq\frac{\varepsilon}{3}\frac{\lambda(E_{x_0,s})}{\lambda(\Omega)}\,. \qquad (4.3.29)$$

Now we want to estimate the measure of the set of integration in the expression

$$\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla u)\,dx\,.$$

By the choice of $S_\eta$ in (4.3.9) and by (4.3.10) and (4.3.11), we obtain

$$\begin{aligned}\lambda(E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})) &\le \lambda(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s}) \\ &= s^N\lambda(Q\setminus\tilde{Q}) \\ &= s^N\lambda_k(B)\left(S_\eta^{N-k}-(S_\eta-1)^{N-k}\right) \\ &\le 3^N\left(\frac{s}{3}\right)^N\lambda_K(B)\left(S_\eta^{N-k}-(S_\eta-1)^{N-k}\right) \\ &\le \eta\lambda(Q_{x_0,\frac{s}{3}})\le\delta\frac{\lambda(E_{x_0,s})}{\lambda(\Omega)}.\end{aligned} \tag{4.3.30}$$

It remains to estimate

$$\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx.$$

Using (4.3.8) and (4.3.25) we obtain

$$|f^{**}(\nabla\tilde{u}_{x_0,s}(x))| = |f(\nabla\tilde{u}_{x_0,s}(x))|\le M_{x_0}$$

for a.e. $x\in E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})$, so that, using once again (4.3.9), (4.3.10) and (4.3.11), we obtain

$$\begin{aligned}&\left|\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx\right| \\ &\le M_{x_0}\lambda(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s}) = M_{x_0}s^N\lambda_K(B)\left(S_\eta^{N-k}-(S_\eta-1)^{N-k}\right) \\ &\le\frac{\varepsilon}{3}\frac{\lambda(E_{x_0,s})}{\lambda(\Omega)}\end{aligned} \tag{4.3.31}$$

Collecting (4.3.26), (4.3.27), (4.3.29), (4.3.30), (4.3.31) and recalling that $f^{**}(\nabla\tilde{u}_{x_0,s}(x)) = f((\nabla\tilde{u}_{x_0,s}(x))$ for a.e. $x\in E_{x_0,s}$ we obtain that

$$\begin{aligned}\int_{E_{x_0,s}} f(\nabla\tilde{u}_{x_0,s})\,dx &= \int_{E_{x_0,s}} f^{**}(\nabla\tilde{u}_{x_0,s})\,dx \\ &\le\int_{E_{x_0,s}} f^{**}(\nabla u)\,dx \\ &\quad+\left|\int_{E_{x_0,s}\cap(Q_{x_0,s}\setminus\tilde{Q}_{x_0,s})} f^{**}(\nabla u)\,dx\right| + 2\frac{\varepsilon}{3}\frac{\lambda(E_{x_0,s})}{\lambda(\Omega)}.\end{aligned} \tag{4.3.32}$$

*Step 4. Construction of the function $v$.*
We consider the bounded measurable set

$$\tilde{\Omega} := \{x\in\Omega : u \text{ is differentiable at } x \text{ and } f(\nabla u(x))\ne f^{**}(\nabla u(x))\}\,.$$

For every $x\in\tilde{\Omega}$ we can consider the family of sets $E_{x,s}$, $0<s<\overline{s}_x$ determined in Step 1 for the case where $\nabla u(x)$ satisfies assumption (HF) (ii) with $k=N$ or in Step 3 for the case $k<N$.
We prove, as in Claim 2.5 of [31], that for every $E_{x,s}$ there exists a closed set $F_{x,s}$ such that

$$\lambda(E_{x,s}) = \lambda(F_{x,s}) \tag{4.3.33}$$

and

$$E_{x,s} \subseteq F_{x,s} \subseteq B\left(x, \frac{s}{\alpha}\right) \qquad \text{if} \qquad k = N \tag{4.3.34}$$
$$E_{x,s} \subseteq F_{x,s} \subseteq Q_{x,s} \qquad \text{if} \qquad k < N.$$

To this aim, we consider

$$G_{x,s} = \begin{cases} \{x \in B(x,s) : u(x) > \omega_{x,s}(x)\} & \text{if } k = N \\ \{x \in Q_{x,s} : u(x) > \omega_{x,s}(x)\} & \text{if } k < N \end{cases}$$

The assumption that $u$ is a.e. differentiable implies that, for a.e. $y \in G_{x,s}$, there exists $B(y, r_y)$ such that $B(y,s) \subseteq G_{x,s}$. Defining

$$F_{x,s} = \begin{cases} B(x,s) \setminus \cup_{y \in G_{x,s}} B(y,s) & \text{if } k = N \\ Q_{x,s} \setminus \cup_{x \in G_{x,s}} B(y,s) & \text{if } k < N \end{cases}$$

we completed the proof of (4.3.33) and (4.3.34) and recalling also (4.3.4) and (4.3.22) we have that the family $F_{x,s}$, for $x$ varying in $\tilde{Q}$, is a Vitali covering of $\tilde{Q}$, (see Theorem 3.1 in [62]). Then there exists an at most countable family $(F_{x_n,s_n}) = (F_n)_{n \in \mathbb{N}}$ of mutually disjoint elements of $\mathcal{F}$ such that

$$\lambda\left(\tilde{\Omega} \setminus \cup_{n \in \mathbb{N}} F_n\right) = \lambda\left(\tilde{\Omega} \setminus \cup_{n \in \mathbb{N}} E_n\right) = 0$$

where we denoted $E_n := E_{x_n,s_n}$.
Finally we can define

$$v(x) = \begin{cases} \tilde{u}_{x_n}(x) & \text{if } x \in F_n \\ u(x) & \text{if } x \notin \cup_n F_n\,. \end{cases}$$

Exploiting (4.3.32) we obtain

$$\begin{aligned} \int_\Omega f(\nabla v)dx &= \int_{\Omega \setminus \cup_n F_n} f^{**}(\nabla u)dx + \int_{\cup_n F_n} f^{**}(\nabla v)dx \\ &= \int_{\Omega \setminus \cup_n F_n} f^{**}(\nabla u)dx + \sum_n \int_{F_n} f^{**}(\nabla v)dx \\ &\leq \int_{\Omega \setminus \cup_n F_n} f^{**}(\nabla u)dx + \sum_n \int_{F_n} f^{**}(\nabla u)dx \\ &\quad + \sum_n \left| \int_{E_n \cap (Q_{x_n,s_n} \setminus \tilde{Q}_{x_n,s_n})} f^{**}(\nabla v)\, dx \right| \\ &\quad + \frac{2}{3}\varepsilon \sum_n \frac{\lambda(E_{x_n,s_n})}{\lambda(\Omega)} \\ &\leq \int_\Omega f^{**}(\nabla u(x)dx + \varepsilon. \end{aligned}$$

We notice that in the last inequality we used the estimate (4.3.30) on the measure of the sets $E_n \cap (Q_{x_n,s_n} \setminus \tilde{Q}_{x_n,s_n})$ and the choices of $\eta$ and $\delta$ in (4.3.7) and (4.3.5). To complete the proof it is sufficient to notice that, by construction, we have $v = \varphi$ on $\partial\Omega$ and that the last inequality, together with assumption (HF) (i) and Poincaré inequality, implies that $v \in \varphi + W_0^{1,1}(\Omega)$.

□

## 4.4 Non-occurrence of the Lavrentiev gap

The aim of this section is to apply the construction of Section 4.3 to obtain a result on the non-occurrence of the Lavrentiev gap. First of all, the next theorem is a consequence of Theorem 4.3.1 that shows that if $u \in \varphi + W_0^{1,\infty}(\Omega)$ then there exists a sequence $v_n \in \varphi + W_0^{1,\infty}(\Omega)$ such that approximates $u$ in energy, Here we need to slightly modify the assumption (HF) requiring that the growth of the Lagrangian is superlinear. To be more precise, we formulate the following

(HF') (i) There exist $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty$ and

$$f(\zeta) \geq \phi(|\zeta|) \qquad \text{for every } \zeta \in \mathbb{R}^N$$

(ii) For every $\xi \in \mathbb{R}^N$ either $f^{**}(\xi) = f(\xi)$ or there exist $k \in \{1, \dots, N\}$, $\{\xi_i\}_{i=1\dots,k+1} \subset \mathbb{R}^N$ and $\{\alpha_i\}_{i=1\dots,k+1} \subset \mathbb{R}$ such that $\alpha_i$ is strictly positive for every $i = 1, \dots, k+1$, $\sum_{i=1}^{k+1} \alpha_i = 1$, $\sum_{i=1}^{k+1} \alpha_i \xi_i = \xi$ and

$$\sum_{i=1}^{k+1} \alpha_i f(\xi_i) = f^{**}(\xi).$$

Moreover we also assume that $\dim \mathrm{Span}(\xi_1, ..., \xi_{k+1}) = k$.

(iii) For every $R > 0$ there exist $N+1$ vectors $\zeta_j \in \mathbb{R}^N$ such that $\overline{B}(0,R) \subset \overline{\text{co}}(\cup_{j=1}^{N+1}\{\zeta_j\})$ and

$$f(\zeta_j) = f^{**}(\zeta_j) < +\infty \qquad \text{for every } j = 1, ..., N+1.$$

**Remark 4.4.1.** By [44][Theorem 4.98], if $f$ is superlinear and lower semicontinuous, then it satisfies (HF') (ii).

Now we can prove the following theorem.

**Theorem 4.4.2.** *Let $\Omega$ be an open bounded Lipschitz subset of $\mathbb{R}^N$ and assume that $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfies assumption (HF'). Then for every $u \in W^{1,\infty}(\Omega)$ there exists $u_n \in u + W_0^{1,\infty}(\Omega)$ such that*

$$u_n \overset{*}{\rightharpoonup} u \quad \textit{in} \quad W^{1,\infty}(\Omega)$$

*and*

$$\lim_n \int_\Omega f(\nabla u_n) dx = \int_\Omega f^{**}(\nabla u) dx.$$

*Proof.* First of all we notice that, by Lemma 4.2.4 $(a)$, it follows that $\int_\Omega f^{**}(\nabla u)\, dx$ is finite. Moreover $u$ is a.e. differentiable in $\Omega$, so that we can repeat the same construction that we made in the proof of Theorem 4.3.1.
We fix $u \in W^{1,\infty}(\Omega)$ and $R = \|\nabla u\|_\infty + 1$. We can consider the set

$$\Omega' = \{x \in \Omega : \ f^{**}(\nabla u(x)) \neq f(\nabla u(x)) \text{ and } u \text{ is differentiable at } x\}$$

For every $x \in \Omega'$ there exist $k(x) \in \{1, \dots, N\}$, $\xi_i(x)$, $\alpha_i(x)$ satisfying (HF) (ii). Moreover (HF) (iii) states that there exist $\zeta_j$, $j = 1, \dots, N+1$, such that

$$\overline{B}(0,R) \subset \overline{\text{co}}(\cup_{j=1}^{N+1}\{\zeta_j\})$$

and we notice that the vectors $\zeta_j$ do not depend on the choice of $x$.
Using Theorem 4.3.1 we deduce that, for every $n \in \mathbb{N}$, there exists $v_n \in u + W_0^{1,1}(\Omega)$ such that

$$\int_\Omega f(\nabla v_n) dx \leq \int_\Omega f^{**}(\nabla u) dx + \frac{1}{n}$$

We claim that $v_n \in u + W_0^{1,\infty}(\Omega)$ and that there exists $K \in \mathbb{R}$ such that $\|\nabla v_n\|_\infty \leq K$ for every $n \in \mathbb{N}$. To prove this claim it is sufficient to recall that in the proof of Theorem 4.3.1 we obtain that $\nabla v_n(x) \in \{\cup_{i=1,\dots,k(x)+1}\xi_i(x), \cup_{j=1,\dots,N+1}\zeta_j\}$, for a.e. $x \in \Omega'$, and then it is sufficient to show that there exists $M > 0$ such that for every $x \in \Omega'$

$$|\xi_i(x)| \leq M \qquad \text{for every } i = 1, \dots, k(x)+1.$$

To this aim we fix $x \in \Omega'$ and, in order to keep the notation light, we denote $\xi = \nabla u(x)$ and we drop the dependence on $x$ in the $\xi_i$ and $k$. By Lemma 4.2.4 (b) it follows that there exists $a \in \partial f^{**}(\xi)$ such that

$$f^{**}(\xi_i) = f^{**}(\xi) + a \cdot (\xi_i - \xi) \qquad \text{for every } i = 1, \dots k+1. \tag{4.4.1}$$

We define

$$g(\zeta) := \max_{b \in \partial f^{**}(\eta), \eta \in \overline{B(0, \|\nabla u\|_\infty)}} \quad f^{**}(\eta) + b \cdot (\zeta - \eta)$$

and we observe that, by (4.4.1) and the convexity of $f^{**}$, it turns out that,

$$g(\xi_i) = f^{**}(\xi_i) \qquad \text{for every } i = 1, \dots k+1. \tag{4.4.2}$$

The inequality

$$g(\zeta) \le \max_{\eta \in \overline{B(0, \|\nabla u\|_\infty)}} \quad f^{**}(\eta) + \max_{b \in \partial f^{**}(\eta), \eta \in \overline{B(0, \|\nabla u\|_\infty)}} |b|(|\zeta| + \|\nabla u\|_\infty)$$

shows that the function $g$ grows at most linearly and hence, by the superlinearity of $f$ and $f^{**}$ there exists $M > 0$ such that if $|\zeta| > M$ then $g(\zeta) < f^{**}(\zeta)$. Thus, by (4.4.2), the claim is proved and we have that $\|\nabla v_n\|_\infty \le K := \max\{M, |\zeta_j|, \, j = 1, \dots, N+1\}$.
A standard argument implies the existence of a subsequence that we still denote by $v_n$ such that $v_n \overset{*}{\rightharpoonup} v$ and, by the weak$^*$-lower semicontinuity of $F^{**}$, we obtain

$$\begin{aligned}\int_\Omega f^{**}(\nabla u)dx \le \liminf_n \int_\Omega f^{**}(\nabla v_n)dx = \liminf_n \int_\Omega f(\nabla v_n)dx \\ \le \limsup_n \int_\Omega f(\nabla v_n)dx \le \int_\Omega f^{**}(\nabla u)dx\end{aligned}$$

□

Now we can apply the previous result to prove the non-occurrence of the so-called Lavrentiev gap.

**Theorem 4.4.3.** *Let $\Omega$ be an open, bounded, Lipschitz subset of $\mathbb{R}^N$. Let $\varphi$ be in $W^{1,\infty}(\Omega)$ and assume that $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfies assumption* (HF'). *Then for every $u \in \varphi + W_0^{1,1}(\Omega)$ there exists $u_n \in \varphi + W_0^{1,\infty}(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \textit{in} \quad W^{1,1}(\Omega)$$

*and*

$$\lim_n \int_\Omega f(\nabla u_n)dx = \int_\Omega f^{**}(\nabla u)dx.$$

*Proof.* Is sufficient to consider the case where $\int_\Omega f^{**}(\nabla u)\,dx$. Theorem 1.1 in [17] and Theorem 5 in [20] state that for every $u \in \varphi + W_0^{1,1}(\Omega)$ there exists a sequence in $u_n \in \varphi + W^{1,\infty}(\Omega)$ such that $u_n$ strongly converge in $W^{1,1}(\Omega)$ to $u$ and

$$\lim_n \int_\Omega f^{**}(\nabla v_n)dx = \int_\Omega f^{**}(\nabla u)dx.$$

By Theorem 4.4.3, for every $v_n$ we can construct a sequence $\{v_n^h\}_{h\in\mathbb{N}}$ such that $v_n^h \in \varphi + W_0^{1,\infty}(\Omega)$ for every $h$, and

$$v_n^h \overset{*}{\rightharpoonup} v_n \quad \text{in} \quad W^{1,\infty}(\Omega),$$

$$\lim_n \int_\Omega f(\nabla v_n^h)dx = \int_\Omega f^{**}(\nabla v_n)dx.$$

Thus, via a diagonal argument, we can determine a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\lim_n \int_\Omega f(\nabla u_n)dx = \int_\Omega f^{**}(\nabla u)dx$$

and then we also have that

$$\int_\Omega \phi(|\nabla u_n|)\, dx \leq \int_\Omega f(\nabla u_n) dx \leq \int_\Omega f^{**}(\nabla u) dx + C$$

for a suitable $C \in \mathbb{R}$. By de la Vallée Poussin Theorem we can conclude that, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,1}(\Omega)$. □

As a consequence, we can also prove the non-occurrence of the Lavrentiev phenomenon for the functional $F$. We will argue as in Theorem 30 in [10].

**Theorem 4.4.4.** *Let $\Omega$ be an open, bounded and Lipschitz subset of $\mathbb{R}^N$, let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfy* (HF') *and $\varphi$ be in $W^{1,\infty}(\Omega)$. Then*

$$\inf_{\varphi + W_0^{1,1}(\Omega)} \int_\Omega f(\nabla u) dx = \inf_{\varphi + W_0^{1,\infty}(\Omega)} \int_\Omega f(\nabla u) dx\,.$$

*Proof.* It is sufficient to observe that

$$\begin{aligned} \inf_{\varphi + W_0^{1,\infty}(\Omega)} \int_\Omega f(\nabla u) dx &= \inf_{\varphi + W_0^{1,1}(\Omega)} \int_\Omega f^{**}(\nabla u) dx \\ &\leq \inf_{\varphi + W_0^{1,1}(\Omega)} \int_\Omega f(\nabla u) dx \leq \inf_{\varphi + W_0^{1,\infty}(\Omega)} \int_\Omega f(\nabla u) dx \end{aligned}$$

where the first equality has been proved in Theorem 4.4.3. □

# Chapter 5

# Sufficient conditions of the existence of a convergent approximating sequence

In this chapter, we report the paper *Relaxation of Non-Convex Integral Functionals in the Multidimensional Scalar Case* ([11]).

## 5.1 Introduction

We consider the following integral functional

$$E[f](u) = \int_{\Omega} f(x, u(x), \nabla u(x))dx,$$

where $\Omega$ is an open, bounded, Lipschitz subset of $\mathbb{R}^N$, $u \in W^{1,p}(\Omega, \mathbb{R})$ with $1 \leq p \leq +\infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is a Lagrangian which satisfies suitable hypotheses. Given $\varphi \in W^{1,\infty}(\Omega)$ and $1 \leq p < q \leq +\infty$, we say that the Lavrentiev Phenomenon occurs between $W^{1,p}_{\varphi}(\Omega)$ and $W^{1,q}_{\varphi}(\Omega)$ if

$$\inf_{W^{1,p}_{\varphi}(\Omega)} E[f] < \inf_{W^{1,q}_{\varphi}(\Omega)} E[f],$$

where $W^{1,p}_{\varphi}(\Omega)$ denotes the space of $W^{1,p}$ functions on $\Omega$ agreeing with $\varphi$ on $\partial\Omega$. Lavrentiev [53] and Manià [54] proposed the first examples of Lavrentiev Phenomenon. In particular, the example of Manià is a polynomial Lagrangian in the one-dimensional scalar case. Further examples, in particular in the non autonomous case, can be found in [5], [6], [14], [21], [34], [42], [43] and [69] .

The study of the non-occurrence of the Lavrentiev Phenomenon, in particular for $p = 1$, $q = \infty$, is important for the application of several numerical approximation techniques, for example the finite elements method. More generally, given $u \in W^{1,p}(\Omega)$, we seek a more regular sequence $(u_n) \subset W^{1,q}(\Omega)$ that converges in some sense to $u$ and such that

$$E[f](u_n) \to E[f](u).$$

This problem has been studied in the one-dimensional case with weak assumptions on the Lagrangian ([3],[58]); usually in the multidimensional scalar case the Lagrangian is assumed to be convex with respect to the gradient variable ([14],[17], [18], [20], [29], [30]). The relation between relaxation of functionals and the Lavrentiev Phenomenon was proposed, as far as we know, for the first time in [28]. This problem has been well studied in the literature, in particular when $f$ is convex with respect to the last variable ( [17], [18], [20], [21], [37], [38], [41]). Roughly speaking, the weak sequential lower semicontinuity of $E[f]$ in $W^{1,1}(\Omega)$ is equivalent to the convexity of the Lagrangian with respect to the last variable. If the Lagrangian is no more convex with respect to the gradient variable, it is interesting to study the weak-$*$ lower semicontinuous envelope of $E[f]$ in $W^{1,\infty}_{\varphi}(\Omega)$. Assuming some continuity assumptions with respect to $(u, \xi)$, in [40] and [55] the following integral representation formula holds: for every $u \in W^{1,\infty}(\Omega)$,

$$\inf\left\{\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx \;\middle|\; (u_n)\subset W^{1,\infty}_u(\Omega),\ u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega)\right\}$$
$$= \int_\Omega f^{**}(x,u(x),\nabla u(x))dx, \quad (5.1.1)$$

where $f^{**}$ is the bipolar of the Lagrangian with respect to the last variable.

The integral representation of the lower semicontinuous envelope was studied by many authors, we cite for example [23], [24], [25], [26], [27], [33], [35], [45], [51], starting from [63], [40] and [55]. In [10] the formula (5.1.1) is proved without any continuity assumptions with respect to the last variable, but under some conditions on the state variable $u$. Furthermore, given $u \in W^{1,\infty}(\Omega)$, there exists $(u_n) \subset u + W^{1,\infty}_0(\Omega)$ such that

$$\|u_n - u\|_{L^\infty} \to 0, \quad \text{and} \quad E[f](u_n) \to E[f^{**}](u). \quad (5.1.2)$$

At this point, the natural question is whether, given $u \in W^{1,\infty}(\Omega)$, there exists $(u_n) \subset u + W^{1,\infty}_0$ such that

$$u_n \rightharpoonup^* u, \quad \text{and} \quad E[f](u_n) \to E[f^{**}](u), \quad (5.1.3)$$

which would be stronger than (5.1.2). To be more precise, along all this chapter, we will use that a sequence $(u_n) \subset W^{1,\infty}(\Omega)$ goes weakly-$*$ to some $u$, denoted by $u_n \rightharpoonup^* u$ if

$$\|u_n - u\|_{L^\infty} \to 0 \quad \text{and} \quad \sup_n \|\nabla u_n\|_{L^\infty} < +\infty. \quad (5.1.4)$$

In Section 5.2, we expose a condition that implies the existence of an approximating sequence as in (5.1.3). The main novelty compared to other similar results, such as in [40] or [55], is that the assumed regularity of the Lagrangian is weaker than usual (especially with respect to the last variable) and that the convergence considered here is with respect to the weak-$*$ topology in $W^{1,\infty}(\Omega)$ (compared to uniform convergence in already existing literature). It is referenced as Theorem 5.2.23 in the following and is stated as such:

**Theorem 5.2.23.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfy Hypothesis 5.2.4. Assume that $f$ also satisfies condition ($\mathcal{K}$). Then for every $u \in W^{1,\infty}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_u(\Omega)$ such that*

$$u_n \rightharpoonup^* u \quad \textit{in} \quad W^{1,\infty}(\Omega),$$

*and*

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx = \int_\Omega f^{**}(x,u(x),\nabla u(x))dx.$$

We do not detail conditions 5.2.4 and ($\mathcal{K}$) here. Let us only mention that Hypothesis 5.2.4 refers to regularity assumptions on $f$, while condition ($\mathcal{K}$) is a geometrical condition introduced in Section 5.2.3. The main appeal of this new result compared to the already existing literature is condition ($\mathcal{K}$), which as stated before, ensure the existence of a converging sequence in $W^{1,\infty}(\Omega)$ weak-$*$. We find in Corollaries 5.2.27 and 5.2.31 two distinct conditions which are sufficient for this condition ($\mathcal{K}$) to hold. The first condition is the uniform boundedness of the connected components of the detachment set (i.e., the set where $f^{**} < f$); the second condition is the superlinearity in the variable $\xi$ uniformly in the other two variables. The main idea of the proofs is to show that, under one of these conditions, one has

$$\inf\left\{\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx \;\middle|\; u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega)\right\}$$
$$= \inf\left\{\liminf_{n\to+\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))dx \;\middle|\; u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega), \quad \|\nabla u_n\|_{L^\infty} < K\right\},$$

where $u_n = u$ on $\partial\Omega$ in the trace sense and $K$ is chosen large enough so that $\|\nabla u\|_{L^\infty} < K$.

In Section 5.3, we focus on the autonomous case (that is, no dependency of the Lagrangian on the $x$ variable). In particular, we apply the existence of an approximating sequence in $W^{1,\infty}(\Omega)$ that satisfies (5.1.2) to the nonconvex case. Firstly, we use Theorem 3.5.4 to avoid the Lavrentiev phenomenon for a large class of Lagrangian (see Theorem 5.3.4). Then we prove the following

**Theorem 5.3.5.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.3.3, and let $\varphi \in W^{1,\infty}(\Omega)$. Then for any $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{in} \quad L^1(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(u(x), \nabla u(x))dx.$$

*Moreover,*

- *if there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ superlinear such that*

$$f(u, \xi) \geq \Phi(\xi), \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

  *then the sequence $(u_n)$ can be chosen so that $u_n \rightharpoonup u$ weakly in $W^{1,1}(\Omega)$;*

- *if for some $p \in (1, \infty)$ it holds that*

$$f(u, \xi) \geq c_1|\xi|^p - c_2, \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

  *for some $c_1, c_2 > 0$, and $u \in W^{1,p}(\Omega)$ then the sequence $(u_n)$ can be taken so that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

The main idea is to apply the result [17, Theorem 1.1] to $E[f^{**}]$ to find an approximating sequence $(u_k) \subset W^{1,\infty}_\varphi(\Omega)$ such that $u_k \to u$ in $W^{1,1}(\Omega)$ and $E[f^{**}](u_k) \to E[f^{**}](u)$. For every $u_k$, we can construct an approximating sequence as in (5.1.2), then use a diagonal extraction argument. Furthermore, in Theorem 5.3.9, if $E[f](u) = E[f^{**}](u)$ we recover the strong convergence in $W^{1,p}(\Omega)$ of the approximating sequence:

**Theorem 5.3.9.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfies Hypothesis 5.3.3, as well as $\varphi \in W^{1,\infty}(\Omega)$. Assume $u \in W^{1,p}_\varphi(\Omega)$ for some $p \in [1, \infty)$, and satisfies*

$$\int_\Omega f^{**}(u(x), \nabla u(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

*Then there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

We remark that, in all the results presented above, the Lagrangian is not necessarily assumed to be continuous with respect to the variable $\xi$. If the Lagrangian is continuous and is dominated by a convex function $g$ such that $E[g](u) < +\infty$, (Theorem 5.3.14 and corollaries) we can find a sequence $(u_n) \subset W^{1,\infty}(\Omega)$ such that $u_n \to u$ strongly in $W^{1,p}(\Omega)$ and $E[f](u_n) \to E[f](u)$. In fact, all of these results also ensure the conservation of the boundary condition all along the approximating sequence, assuming that $u_{|\partial\Omega} \in \mathrm{Lip}(\partial\Omega)$. The main idea is to use [17, Theorem 1.1] to the dominating function and use Fatou Lemma. We can also see these results as integral representations of the lower semicontinuous envelope with respect to the strong topology of $W^{1,p}(\Omega)$, this is the content of Remark 5.3.7.

Then in Section 5.4, we apply a result in [14] to extend the results of Section 5.3 to the non-autonomous case, assuming an anti-jump condition with respect to the variable $x$ (called condition $(\mathcal{H}_1)$ in this chapter). The main novelty is to prove that condition $(\mathcal{H}_1)$ for a non convex Lagrangian implies that the condition holds also for its bipolar. At this point, most of the proofs follow similar patterns to the ones in Section 5.3. The main additional difficulty is to make sure that condition $(\mathcal{H}_1)$ stays true for the various auxiliary Lagrangians we consider for calculation purpose.

## 5.2 Geometrical conditions for the weak-$*$ approximation in $W^{1,\infty}(\Omega)$

This section is devoted to finding geometric conditions on the Lagrangian $f$ such that for every $u \in W^{1,\infty}(\Omega)$ there exists a sequence $(u_n) \subset u + W_0^{1,\infty}(\Omega)$ such that

$$u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega), \tag{5.2.1}$$

and

$$\int_\Omega f(x, u_n(x), \nabla u_n(x))dx \to \int_\Omega f^{**}(x, u(x), \nabla u(x))dx. \tag{5.2.2}$$

Our main result and contribution for this section will be Theorem 5.2.23, stating that, under a geometric condition on the Lagrangian (see condition $(\mathcal{K})$), a sequence such that (5.2.1) and (5.2.2) hold does exist. We will then, in Section 5.2.4 work to state a few sufficient properties on the Lagrangian $f$ ensuring that it satisfies $(\mathcal{K})$.

**Notation 5.2.1.** In the whole chapter, we will use the following conventions and notations:

- $\Omega$ is an open, bounded, Lipschitz domain of $\mathbb{R}^N$, for some $N \geq 1$.
- Given $K > 0$, $B_K$ will denote the closed ball of radius $K$ and center 0 in $\mathbb{R}^N$.
- Given $\varphi \in W^{1,\infty}(\Omega)$, we will write

  $$W_\varphi^{1,p}(\Omega) = \varphi + W_0^{1,p}(\Omega) = \Big\{u \in W^{1,p}(\Omega) \;:\; u = \varphi \quad \text{on} \quad \partial\Omega\Big\},$$

  for every $p \in [1, \infty]$. Hence, $W_\varphi^{1,p}(\Omega)$ is the set of functions in $W^{1,p}(\Omega)$ which agree with $\varphi$ (in the sense of the trace) on $\partial\Omega$.

### 5.2.1 On measurability and basic assumptions

**Definition 5.2.2.** Given a metric space $X$, we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of $X$. Moreover, if $X$ is a subset of an euclidean space, then $\mathcal{L}(X)$ will denote its Lebesgue $\sigma$-algebra. We will say that a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$ is Lebesgue-Borel measurable if it is measurable for the $\sigma$-algebras

$$\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^N) \longrightarrow \mathcal{B}([0, \infty]).$$

This notion of measurability for Lagrangians is made in such a way that, if $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}^N$ are Lebesgue measurable, then

$$x \mapsto f(x, u(x), v(x)),$$

is measurable on $\Omega$, and thus the quantity $\int_\Omega f(x, u(x), v(x))dx$ makes sense. This will of course be used in the case $u \in W^{1,1}(\Omega)$ and $v = \nabla u$. In a large part of the literature on the subject, the usual assumptions made on the Lagrangian is the following Carathéodory property:

**Definition 5.2.3.** We say that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$ is a Carathéodory function if it is Lebesgue-Borel measurable and for a.e. $x \in \Omega$, the mapping $(u, \xi) \mapsto f(x, u, \xi)$ is continuous on $\mathbb{R}^N$.

In this Thesis, we will usually have a less restrictive assumption on the Lagrangian. In particular, the continuity in $\xi$ will be withdrawn from the assumptions for most of Section 5.2. We assume the following for the Lagrangian $f$:

**Hypothesis 5.2.4.** The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfies

a) $f$ is Lebesgue-Borel measurable;

b) for a.e. $x \in \Omega$, the function $u \mapsto f(x, u, \xi)$ is continuous with respect to $u$ uniformly as $\xi$ varies in bounded sets. That is, for every bounded set $B \subset \mathbb{R}^N$, for every $u_0 \in \mathbb{R}$,

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u \in \mathbb{R}, \forall \xi \in B, \\ &\qquad\qquad |u - u_0| < \delta \quad \Rightarrow \quad \big| f(x, u, \xi) - f(x, u_0, \xi) \big| < \varepsilon; \end{aligned}$$

c) for every bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$, there exists $a \in L^1(\Omega)$ such that $f(x, u, \xi) \leq a(x)$ for a.e. $x \in \Omega$ and all $(u, \xi) \in B$;

d) for every $u \in W^{1,\infty}(\Omega)$, for every bounded set $B \subset \mathbb{R}^N$ and for every $\eta > 0$ there exists $T \subset \Omega$ compact such that $|\Omega \setminus T| < \eta$ and $x \mapsto f(x, u(x), \xi)$ is continuous on $T$ uniformly as $\xi$ varies in $B$, that is, for every $x_0 \in T$,

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in T, \forall \xi \in B, \\ &\qquad\qquad |x - x_0| < \delta \quad \Rightarrow \quad \big| f(x, u(x), \xi) - f(x_0, u(x_0), \xi) \big| < \varepsilon. \end{aligned}$$

**Remark 5.2.5.** We point out the fact that we can replace Hypotheses 5.2.4-b) and -d) with the more restrictive request that for every bounded set $B \subset \mathbb{R}^N$ and for every $\eta > 0$ there exists a compact set $T \subset \Omega$ such that $|\Omega \setminus T| < \eta$ and $f$ is continuous with respect to $(x, u) \in T \times \mathbb{R}$ uniformly as $\xi$ varies in $B$.

A typical example of a Lagrangian $f$ satisfying Hypothesis 5.2.4, is if the dependency in $\xi$ is bounded and separated from the other variables. That is, if

$$f(x, u, \xi) = g(x, u)h(\xi),$$

for some $g : \Omega \times \mathbb{R} \to [0, \infty)$ Carathéodory satisfying that for every compact interval $I \subset \mathbb{R}$, there exists $a \in L^1(\Omega)$ such that $g(x, u) \leq a(x)$, for a.e. $x \in \Omega$ and all $u \in I$; and $h : \mathbb{R}^N \to [0, \infty)$ Borel bounded on bounded set. Another interesting case, which is very useful in application is for Carathéodory Lagrangians:

**Proposition 5.2.6.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a Carathéodory function. Then $f$ satisfies assumptions 5.2.4-a), -b) and -d).*

*Proof.* 5.2.4-a) is obvious by definition. To prove 5.2.4-b), it is enough to use the fact that, for a.e. $x \in \Omega$, the map $(u, \xi) \mapsto f(x, u, \xi)$ is continuous. The rest follows from a simple compactness argument on the bounded set $B$.

We now prove 5.2.4-d). Let $u \in W^{1,\infty}(\Omega)$, $B \subset \mathbb{R}^N$ a bounded set and $\eta > 0$. Let $g(x, \xi) := f(x, u(x), \xi)$. Then $g$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N)$-measurable and continuous with respect to $\xi$. By the Scorza-Dragoni Theorem (see [40, Chapter VIII, Section 1.3]), there exists $T \subset \Omega$ compact such that $|\Omega \setminus T| < \eta$ and $g$ is continuous on $T \times \mathbb{R}^N$. Now since $B$ is bounded, another compactness argument shows that $g$ is continuous with respect to $x \in T$ uniformly as $\xi$ varies in $B$. $\square$

**Remark 5.2.7.** In this chapter, we will work only with non-negative Lagrangians for simplicity. However, it might be interesting to keep in mind that the various approximation results in Section 5.2.3 (Proposition 5.2.18, and Theorems 5.2.20, 5.2.23) would still hold if $f$ were only assumed to be real-valued, and with assumption 5.2.4-c) replaced by

c') for every bounded set $B \subset \mathbb{R} \times \mathbb{R}^N$, there exists $a \in L^1(\Omega)$ such that $|f(x, u, \xi)| \leq a(x)$ for a.e. $x \in \Omega$ and all $(u, \xi) \in B$.

Indeed, if b') were to hold, considering the (non-negative) auxiliary Lagrangian $g(x, u, \xi) = f(x, u, \xi) + a(x)$, it would satisfy Hypothesis 5.2.4. All of these approximation results would thus hold for $g$, and thus for $f$ as an immediate calculation would show.

### 5.2.2 Basic tools, notations and some reminder about convexification

We introduce the following technical definition:

**Definition 5.2.8.** For every $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$ and for every $T \subset \mathbb{R}^N$ we define

$$\tilde{f}_T(x, u, \xi) := \begin{cases} f(x, u, \xi) & \text{if} \quad \xi \in T \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2.3}$$

In the special case $T = B_K$ for some $K > 0$, we let

$$\tilde{f}_K := \tilde{f}_{B_K}.$$

As convexification is a crucial tool whenever mentioning weak or weak-$*$ convergence for integral functional, we define it and mention some standard facts on the subject.

**Definition 5.2.9.** Let $f : \mathbb{R}^N \to [0, \infty]$. We define the function $f^{**} : \mathbb{R}^N \to [0, \infty]$ as the convexification of $f$. That is, $f^{**}$ is the greatest lower semicontinuous convex function on $\mathbb{R}^N$ which stays under $f$.

The following facts are classical, the reader can have a look at [40, Chapter I] for a more complete overview of convex analysis.

**Proposition 5.2.10.** *Let* $f : \mathbb{R}^N \to [0, \infty]$.

- *The function $f^{**}$ can be written as the supremum of all affine maps on $\mathbb{R}^N$ which bound $f$ from below.*
- *Assume that $f$ takes finite values in a neighborhood of some $\xi \in \mathbb{R}^N$. Then*

$$f^{**}(\xi) = \inf \left\{ \sum_i \alpha_i f(\xi_i) \right\}, \tag{5.2.4}$$

*where the infimum is taken over all convex combinations $(\alpha_i, \xi_i) \subset [0, 1] \times \mathbb{R}^N$ such that $\sum_i \alpha_i = 1$ and $\xi = \sum_i \alpha_i \xi_i$.*

*Proof.* The first point is a consequence of the Hahn-Banach theorem (see [40, Chapter I, Proposition 3.1] for a detailed proof). We prove the second point. Let $g : \mathbb{R}^N \to [0, \infty]$ be the functional on the right hand side of (5.2.4). We claim that $g$ is convex. Indeed, take $\xi, \zeta \in \mathbb{R}^N$ as well as $\lambda \in (0, 1)$. Let $\sum_i \alpha_i \xi_i$ and $\sum_j \beta_j \zeta_j$ be convex combinations of $\xi$ and $\zeta$ respectively. Then $\sum_i (1 - \lambda) \alpha_i \xi_i + \sum_j \lambda \beta_j \zeta_j$ is a convex combination of $(1 - \lambda)\xi + \lambda\zeta$, and by definition of $g$,

$$g((1 - \lambda)\xi + \lambda\zeta) \leq (1 - \lambda) \sum_i \alpha_i f(\xi_i) + \lambda \sum_j \beta_j f(\zeta_j).$$

Now taking the infimum over all convex combinations of $\xi$ and $\zeta$ respectively, we finally obtain $g((1 - \lambda)\xi + \lambda\zeta) \leq (1 - \lambda) g(\xi) + \lambda g(\zeta)$. Now using the fact that $f^{**}$ is convex and $f^{**} \leq f$, for every $\xi$,

$$f^{**}(\xi) = \inf \left\{ \sum_i \alpha_i f^{**}(\xi_i) \right\} \leq \inf \left\{ \sum_i \alpha_i f(\xi_i) \right\} = g(\xi),$$

where the infimum is again taken over all convex combinations of $\xi$. Therefore $f^{**} \leq g \leq f$. Now assume that $f$ is finite in a neighborhood of some $\xi \in \mathbb{R}^N$. By [40, Chapter I, Corollary 2.3], $g$ is continuous at point $\xi$ and by [40, Chapter I, Proposition 5.2], there exists an affine map $\ell$ on $\mathbb{R}^N$ such that

$$\ell \leq g \leq f, \quad \text{and} \quad \ell(\xi) = g(\xi).$$

By the first point of the proposition, one has $f^{**}(\xi) \geq \ell(\xi) = g(\xi)$, which achieves the proof. □

**Remark 5.2.11.** Using the notation introduced in (5.2.3), $(\tilde{f}_T)^{**}$ is the greatest function which is convex, lower semicontinuous with respect to the last variable and stays under $f$ on $T$. In particular, $(\tilde{f}_K)^{**} \geq f^{**}$ on $B_K$ and in general the strict inequality may hold. Indeed, consider for instance $f(\xi) = \exp(-|\xi|)$. Then $f^{**} \equiv 0$ on $\mathbb{R}^N$, but for every $K > 0$,

$$(\tilde{f}_K)^{**} = e^{-K} \quad \text{on} \quad B_K.$$

The previous remark allows to formulate the following lemma:

**Lemma 5.2.12.** *Let $f : \mathbb{R}^N \to [0, \infty)$. Then the family of mapping $((\tilde{f}_K)^{**})_{K>0}$ is non-increasing as $K \to +\infty$ and converges pointwise to $f^{**}$.*

*Proof.* The fact that the family is non-increasing as $K \to +\infty$ easily follows from the definition of $(\tilde{f}_K)^{**}$. Now define $g := \lim_{K\to+\infty}(\tilde{f}_K)^{**} = \inf_{K>0}(\tilde{f}_K)^{**}$. By definition of $f^{**}$, it is clear that $f^{**} \leq g$ (see Remark 5.2.11). Moreover, since the family $((\tilde{f}_K)^{**})_{K>0}$ is totally ordered, it holds that $g$ is convex and $g \leq f$ on all of $\mathbb{R}^N$ (given $\xi \in \mathbb{R}^N$, one can choose $K > |\xi|$). Since $f$ is finite valued, so is $g$ and thus $g$ is lower semicontinuous on $\mathbb{R}^N$. Therefore $g \leq f^{**}$. □

In the case of an integral functional which may depend on $x$ and $u$ and not just on $\nabla u$, it is relevant to consider the convexification only with respect to the variable $\xi$:

**Definition 5.2.13.** Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$. We define the function $f^{**} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ as the convexification of $f$ with respect to the last variable. More specifically, for every $(x, u) \in \Omega \times \mathbb{R}$, $f^{**}(x, u, \cdot)$ is the greatest lower semicontinuous convex function on $\mathbb{R}^N$ which stays under $f(x, u, \cdot)$.

Since the Lagrangian $f$ might be non-convex with respect to the last variable, we prove that $f^{**}$ is at least measurable.

**Lemma 5.2.14.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.2.4. Then for every $K > 0$, $(\tilde{f}_K)^{**}$ is a Carathéodory function on $\Omega \times \mathbb{R} \times (B_K)^\circ$. Furthermore $f^{**}$ is Lebesgue-Borel measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^N$.*

Here $(B_K)^\circ$ denote the centered open ball of radius $K$.

*Proof.* Firstly, we prove that $u \mapsto (\tilde{f}_K)^{**}(x, u, \xi)$ is continuous uniformly as $\xi$ varies in $B_K$, for a.e. $x \in \Omega$ and for every $K > 0$. We fix $u_0 \in \mathbb{R}$ and Hypothesis 5.2.4-b) guarantees that for a.e. $x \in \Omega$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|u - u_0| < \delta$ then

$$|f(x, u, \xi) - f(x, u_0, \xi)| < \varepsilon, \qquad \forall \xi \in B_K, \tag{5.2.5}$$

and so we have

$$(\tilde{f}_K)^{**}(x, u, \xi) - \varepsilon \leq f(x, u, \xi) - \varepsilon \leq f(x, u_0, \xi), \qquad \forall \xi \in B_K. \tag{5.2.6}$$

Now $(\tilde{f}_K)^{**}(x, u, \xi) - \varepsilon$ is convex and lower semicontinuous in $\xi$ and so

$$(\tilde{f}_K)^{**}(x, u, \xi) - \varepsilon \leq (\tilde{f}_K)^{**}(x, u_0, \xi), \qquad \forall \xi \in B_K.$$

Reversing the roles of $u$ and $u_0$ in (5.2.6), one gets

$$|(\tilde{f}_K)^{**}(x, u, \xi) - (\tilde{f}_K)^{**}(x, u_0, \xi)| < \varepsilon, \qquad \forall \xi \in B_K. \tag{5.2.7}$$

Now for every $u \in \mathbb{R}$, since the function $f(x, u, \cdot)$ is finite-valued, it holds that (see [40, Chapter I, Corollary 2.3])

$$(\tilde{f}_K)^{**}(x, u, \cdot) \quad \text{is continuous on} \quad (B_K)^\circ. \tag{5.2.8}$$

Therefore, if $(u_n, \xi_n) \subset \mathbb{R} \times (B_K)^\circ$ is a sequence converging to $(u_0, \xi_0) \in \mathbb{R} \times (B_K)^\circ$, then

$$\begin{aligned}
\limsup_{n\to+\infty} & |(\tilde{f}_K)^{**}(x, u_n, \xi_n) - (\tilde{f}_K)^{**}(x, u_0, \xi_0)| \\
&\leq \limsup_{n\to+\infty} |(\tilde{f}_K)^{**}(x, u_n, \xi_n) - (\tilde{f}_K)^{**}(x, u_0, \xi_n)| \\
&+ \limsup_{n\to+\infty} |(\tilde{f}_K)^{**}(x, u_0, \xi_n) - (\tilde{f}_K)^{**}(x, u_0, \xi_0)| \\
&= 0,
\end{aligned}$$

by (5.2.7) and (5.2.8). That is, for a.e. $x \in \Omega$, $(\tilde{f}_K)^{**}(x,\cdot,\cdot)$ is continuous in $\mathbb{R} \times (B_K)^\circ$. By [40, Chapter VIII, Proposition 1.1], to prove that $(\tilde{f}_K)^{**}$ is Carathéodory, it is therefore enough to show that $x \mapsto (\tilde{f}_K)^{**}(x,u,\xi)$ is Lebesgue measurable for every $(u,\xi) \in \mathbb{R} \times (B_K)^\circ$. We fix $(u_0,\xi_0) \in \mathbb{R} \times (B_K)^\circ$. Using 5.2.4-d), there exists an increasing sequence of compact sets $(T_n)$ of $\Omega$ such that $|\Omega \setminus T_n| \to 0$ and for every $n$, the mapping $x \mapsto f(x,u_0,\xi)$ is continuous on $T_n$ uniformly as $\xi$ varies in $B_K$. Now by a similar argument to the one developed above (equations (5.2.5)-(5.2.8)), $x \mapsto (\tilde{f}_K)^{**}(x,u_0,\xi_0)$ is continuous on $T_n$. Taking the limit as $n \to +\infty$, this same mapping is an almost everywhere limit of continuous maps, and is thus Lebesgue measurable on $\Omega$.

Now, for every $K \in \mathbb{N}$, we let $g_K$ be defined as such:

$$g_K(x,u,\xi) = \begin{cases} (\tilde{f}_K)^{**}(x,u,\xi) & \text{if} \quad |\xi| \le K/2 \\ +\infty & \text{if} \quad |\xi| > K/2. \end{cases}$$

Since $(\tilde{f}_K)^{**}$ is Carathéodory on $\Omega \times \mathbb{R} \times (B_K)^\circ$, then $g_K$ is Lebesgue-Borel measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. Recalling Lemma 5.2.12, we have

$$f^{**}(x,u,\xi) = \inf_K g_K(x,u,\xi) = \lim_{K \to +\infty} g_K(x,u,\xi),$$

hence, $f^{**}$ is Lebesgue-Borel measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. □

**Remark 5.2.15.** It might be interesting to point out that $f^{**}$ may not be a Carathéodory function in full generality. The issue is that the joint continuity in $(u,\xi)$ may not be satisfied. We give here an example, which was initially presented in [55, Example 3.11]. Consider the following mapping $f(u,\xi) := (|\xi|+1)^{|u|}$ for $(u,\xi) \in \mathbb{R} \times \mathbb{R}^1$. It is clear that it is continuous, however

$$f^{**}(u,\xi) = \begin{cases} f(u,\xi) & \text{if} \quad |u| \ge 1 \\ 1 & \text{if} \quad |u| < 1. \end{cases}$$

which is not continuous (not even lower semicontinuous).

However, we will now study some conditions on the Lagrangian $f$ to ensure that $f^{**}$ is Carathéodory. The problem has already been studied in [55, Corollary 3.12]. Assuming a $p-$growth or continuity in $u$ uniformly in $\xi$, the authors proved that $f^{**}$ is a Carathéodory function. Theorem 5.2.17 below, with the help of a different kind of argument, extends the $p$-growth case to superlinear Lagrangians.

**Proposition 5.2.16.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfies Hypothesis 5.2.4. If for every compact interval $I \subset \mathbb{R}$ and $K > 0$, there exists $K' \ge K$ such that*

$$(\tilde{f}_{K'})^{**} = f^{**} \quad \text{on} \quad I \times B_K, \tag{5.2.9}$$

*then $f^{**}$ is continuous on $\mathbb{R} \times \mathbb{R}^N$.*

*Proof.* It is enough to prove that $f^{**}$ is continuous on $I \times B_K$ for every compact interval $I \subset \mathbb{R}$ and $K > 0$. Fix such a choice of $I$ and $K$. Let $K' \ge K$ such that (5.2.9) apply. If $K' > K$, by Lemma 5.2.14, $(\tilde{f}_{K'})^{**}$ is continuous on $I \times B_K$, and (5.2.9) gives our conclusion. If $K = K'$, we cannot apply immediately Lemma 5.2.14 because $(\tilde{f}_{K'})^{**}$ may not be continuous on $\{\xi \in \partial B_K\}$. However, choosing $K'' = K' + 1 > K$, then

$$(\tilde{f}_{K'})^{**} \ge (\tilde{f}_{K''})^{**} \ge f^{**} = (\tilde{f}_{K'})^{**} \quad \text{on} \quad I \times B_K.$$

Thus $(\tilde{f}_{K''})^{**} = f^{**}$ on $I \times B_K$, and since $(\tilde{f}_{K''})^{**}$ is continuous on $I \times B_K$ by Lemma 5.2.14, this proves the result in the case $K' = K$. □

The condition (5.2.9) used in the statement of the previous proposition is quite similar to condition ($\mathcal{K}$) presented below in Section 5.2.3, which will play a major role in our discussion. Actually, a similar argument given in Remark 5.2.22 was used in the proof of Proposition 5.2.16.

This result leads to the following, less abstract statement:

**Theorem 5.2.17.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfy Hypothesis 5.2.4. Furthermore, assume that for a.e. $x \in \Omega$ and every compact interval $I \subset \mathbb{R}$, there exists $\Phi_{x,I} : \mathbb{R}^N \to [0,\infty)$ superlinear such that*

$$f(x,u,\xi) \geq \Phi_{x,I}(\xi), \qquad \forall u \in I,\ \forall \xi \in \mathbb{R}^N. \tag{5.2.10}$$

*Then $f^{**}$ is a Carathéodory function.*

Before proving it rigorously, we will need some other tools, in particular regarding superlinearity (Theorem 5.2.28). We refer the reader to the end of Section 5.2.4 for a detailed proof of Theorem 5.2.17.

### 5.2.3 Known results of weak-$*$ relaxation on $W^{1,\infty}(\Omega)$ and condition $(\mathcal{K})$.

We now present multiple results of relaxation on $W^{1,\infty}(\Omega)$ with respect to the weak-$*$ topology. As mentioned above, the main contribution of this section is Theorem 5.2.23. Before that, we need some intermediary results, already proved (for instance by P. Marcellini and C. Sbordone in [55]). The following proposition, proved in [10], is the most recent formulation,with the weakest initial assumptions that we are aware of.

**Proposition 5.2.18.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfy Hypothesis 5.2.4. For every $u \in W^{1,\infty}(\Omega)$ and for every $K \in \mathbb{N}$ such that*

$$\|\nabla u\|_{L^\infty} < K, \tag{5.2.11}$$

*there exists a sequence $(u_{K,n}) \subset W^{1,\infty}_u(\Omega)$ such that*

$$\|\nabla u_{K,n}\|_{L^\infty} < K\,, \qquad u_{K,n} \rightharpoonup^* u \quad \textit{in} \quad W^{1,\infty}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(x, u_{K,n}(x), \nabla u_{K,n}(x))dx = \int_\Omega (\tilde{f}_K)^{**}(x,u(x),\nabla u(x))dx.$$

Here, the weak-$*$ convergence denoted by $\rightharpoonup^*$ refers to the one introduced in (5.1.4). We recall that, consistent with Notation 5.2.1, $W^{1,\infty}_u(\Omega)$ denotes the space of functions in $W^{1,\infty}(\Omega)$ which agree with $u$ on $\partial\Omega$.

By the previous result we have that for every $K \in \mathbb{N}$, $u \in W^{1,\infty}(\Omega)$ satisfying (5.2.11),

$$\inf \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_u(\Omega) \\ u_n \rightharpoonup^* u \\ \|\nabla u_n\|_{L^\infty} < K \end{array} \right\} \leq \int_\Omega (\tilde{f}_K)^{**}(x,u(x),\nabla u(x))dx. \tag{5.2.12}$$

Since $(\tilde{f}_K)^{**}$ is Carathéodory and is convex, lower semicontinuous with respect to $\xi$ we have the following (see for instance [47, Chapter 4, Theorem 4.5]):

**Lemma 5.2.19** (Tonelli)**.** *The integral functional*

$$u \mapsto \int_\Omega (\tilde{f}_K)^{**}(x,u(x),\nabla u(x))dx,$$

*is sequentially lower semicontinuous with respect to the weak topology of $W^{1,1}(\Omega)$.*

So a fortiori, since any sequence converging weakly-$*$ in $W^{1,\infty}(\Omega)$ also converges weakly in $W^{1,1}(\Omega)$, we have that

$$\inf \left\{ \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_u(\Omega) \\ u_n \rightharpoonup^* u \\ \|\nabla u_n\|_{L^\infty} < K \end{array} \right\} \geq \int_\Omega (\tilde{f}_K)^{**}(x,u(x),\nabla u(x))dx. \tag{5.2.13}$$

(Here we used the fact that $f \geq (\tilde{f}_K)^{**}$ on $\Omega \times \mathbb{R} \times B_K$). Thus, combining (5.2.12) and (5.2.13),

$$\min \left\{ \liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_u(\Omega) \\ u_n \rightharpoonup^* u \\ \|\nabla u_n\|_{L^\infty} < K \end{array} \right\} = \int_\Omega (\tilde{f}_K)^{**}(x, u(x), \nabla u(x))dx. \tag{5.2.14}$$

The fact that the infimum is in fact a minimum in (5.2.14) comes from the existence of a minimizing sequence (see Proposition 5.2.18). Now $x \mapsto (\tilde{f}_K)^{**}(x, u, \nabla u)$ is a non-increasing sequence bounded from above and converging pointwise to $x \mapsto f^{**}(x, u, \nabla u)$ (by Lemma 5.2.12). Thus taking the limit as $K \to +\infty$ and using a diagonal argument, we obtain the following result, stated in [55] in the continuous case, or in [10] for the general case:

**Theorem 5.2.20.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.2.4. For every $u \in W^{1,\infty}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_u(\Omega)$ such that*

$$u_n \to u \quad \textit{in} \quad L^\infty(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

*Furthermore,*

$$\inf \left\{ \liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_u(\Omega) \\ u_n \rightharpoonup^* u \end{array} \right\} = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx. \tag{5.2.15}$$

Notice that (5.2.15) is obtained by taking the limit as $K \to +\infty$ in (5.2.14). Now our aim is to find some conditions on the Lagrangian $f$ in order to reach the infimum in (5.2.15), that is, our question is whether there exists a sequence $(u_n) \subset W^{1,\infty}_u(\Omega)$ such that

$$u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega), \tag{5.2.16}$$

and

$$\int_\Omega f(x, u_n(x), \nabla u_n(x))dx \to \int_\Omega f^{**}(x, u(x), \nabla u(x))dx. \tag{5.2.17}$$

We will see that in general, such a sequence does not exist. Before moving on, we study the situation in more details. Let $u \in W^{1,\infty}(\Omega)$ and $(u_n) \subset W^{1,\infty}(\Omega)$ be such that $u_n \rightharpoonup^* u$. Let $K' > \|\nabla u\|_{L^\infty}$ such that

$$\|\nabla u_n\|_{L^\infty} < K', \qquad \forall n \in \mathbb{N}.$$

Consider for instance the Lagrangian $f(\xi) = \exp(-|\xi|)$ presented in Remark 5.2.11. Then because $f$ is bounded from below by $\exp(-K')$ on $B_{K'}$, necessarily,

$$\liminf_{n\to+\infty} \int_\Omega f(\nabla u_n(x))dx \geq |\Omega| e^{-K'} > 0 = E[f^{**}](u).$$

In particular, (5.2.17) cannot hold. More generally, if $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is any Lagrangian satisfying Hypothesis 5.2.4, by (5.2.14),

$$\int_\Omega (\tilde{f}_{K'})^{**}(x, u(x), \nabla u(x))dx \leq \liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx.$$

Thus, recalling that $(\tilde{f}_{K'})^{**} \geq f^{**}$, the existence of a $K' \geq \|\nabla u\|_{L^\infty}$ such that

$$(\tilde{f}_{K'})^{**}(x, u(x), \nabla u(x)) = f^{**}(x, u(x), \nabla u(x)), \qquad \text{for a.e. } x \in \Omega,$$

is a necessary condition in order to reach the infimum in (5.2.15). Actually this is also a sufficient condition, as it is stated in the following Theorem 5.2.23.

The previous discussion allows us to characterize the property (named condition $\mathcal{K}$ in the sequel) which is needed on the Lagrangian to ensure the existence of a sequence satisfying (5.2.16) and (5.2.17).

**Definition 5.2.21.** We say that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$ satisfies condition ($\mathcal{K}$) if for every compact interval $I \subset \mathbb{R}$ and $K > 0$, there exists $K' \geq K$ and $\tilde{\Omega} \subset \Omega$ such that $|\Omega \setminus \tilde{\Omega}| = 0$ and

$$(\tilde{f}_{K'})^{**} = f^{**} \quad \text{on} \quad \tilde{\Omega} \times I \times B_K. \tag{$\mathcal{K}$}$$

**Remark 5.2.22.** A simple observation is that, if $g : \mathbb{R}^N \to [0, \infty]$, then

$$(\tilde{g}_{K'})^{**} \geq (\tilde{g}_{K''})^{**} \geq g^{**} \quad \text{on} \quad B_K,$$

for every $K'' \geq K' \geq K > 0$ (by Remark 5.2.11). Therefore, if condition ($\mathcal{K}$) applies for $f$ and some $K' \geq K$, then it will still apply for any $K'' \geq K'$.

**Theorem 5.2.23.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.2.4. Assume that f also satisfies condition ($\mathcal{K}$). Then for every $u \in W^{1,\infty}(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_u(\Omega)$ such that*

$$u_n \rightharpoonup^* u \quad \textit{in} \quad W^{1,\infty}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

*In particular, the sequence $(u_n)$ reaches the minimum for* (5.2.15).

*Proof.* Let $u$ be in $W^{1,\infty}(\Omega)$, then there exists a compact interval $I \subset \mathbb{R}$ such that

$$u(x) \in I,$$

for almost every $x$ in $\Omega$ and there exists $K > 0$ such that

$$\|\nabla u\|_{L^\infty} < K.$$

By assumption we can take $K' \geq K$ such that condition ($\mathcal{K}$) applies, and so in particular

$$(\tilde{f}_{K'})^{**}(x, u(x), \nabla u(x)) = f^{**}(x, u(x), \nabla u(x)), \qquad \text{for a.e. } x \in \Omega.$$

By Proposition 5.2.18 there exists a sequence $(u_n) \subset W^{1,\infty}_u(\Omega)$ such that

$$u_n \rightharpoonup^* u \quad \text{in} \quad W^{1,\infty}(\Omega),$$

and

$$\int_\Omega f(x, u_n(x), \nabla u_n(x))dx \to \int_\Omega (\tilde{f}_{K'})^{**}(x, u(x), \nabla u(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

Since by Theorem 5.2.20,

$$\inf \left\{ \liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_u(\Omega) \\ u_n \rightharpoonup^* u \end{array} \right\} = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

then $(u_n)$ reaches the minimum in (5.2.15). □

### 5.2.4 Sufficient geometric and analytic conditions for ($\mathcal{K}$).

Now we seek conditions on $f$ so that the Lagrangian satisfies condition ($\mathcal{K}$). We begin by stating some results when $f$ depends only on the third variable $\xi$.

**Lemma 5.2.24.** *Let $f : \mathbb{R}^N \to [0, \infty]$. Assume that $A \subset B \subset \mathbb{R}^N$ satisfy that for any $\xi_A \in A$ and $\xi_B \notin B$, there exists $\xi \in [\xi_A, \xi_B] \cap B$ such that $f(\xi) = f^{**}(\xi)$. Then*

$$(\tilde{f}_B)^{**} = f^{**} \quad \textit{on} \quad A.$$

*Proof.* Firstly, one has $\tilde{f}_B \geq f$, hence $(\tilde{f}_B)^{**} \geq f^{**}$. We turn to the proof of the converse inequality: assume by contradiction that it is false, then without loss of generality, one may assume that $0 \in A$ and that $(\tilde{f}_B)^{**}(0) > f^{**}(0)$. In particular, thank to Proposition 5.2.10, there must exist an affine map $\ell$ on $\mathbb{R}^N$ such that

$$\ell \leq \tilde{f}_B \quad \text{and} \quad \ell(0) > f^{**}(0). \tag{5.2.18}$$

Define $g := f^{**} - \ell$. Then $g$ is a convex function satisfying $g(0) < 0$ and

$$g(\zeta) \geq 0, \qquad \forall \zeta \in \mathbb{R}^N \setminus B. \tag{5.2.19}$$

Indeed, for $\zeta \notin B$, there exists by assumption $\xi \in [0, \zeta] \cap B$ such that $f(\xi) = f^{**}(\xi)$. If $f(\xi) = +\infty$, then $g(\xi) = +\infty$. Otherwise,

$$g(\xi) = f^{**}(\xi) - \ell(\xi) \geq f^{**}(\xi) - \tilde{f}_B(\xi) = f^{**}(\xi) - f(\xi) = 0.$$

Therefore, in any case $g(\xi) \geq 0$ and because $g(0) < 0$, (5.2.19) holds. We derive from (5.2.19) that $\ell \leq f$ on $\mathbb{R}^N$, indeed:

- on $B$, one has $f = \tilde{f}_B$ and the conclusion follows from (5.2.18) in this case;
- on $\mathbb{R}^N \setminus B$, this comes from (5.2.19) and the fact that $f^{**} \leq f$.

Finally, one gets that $\ell$ is an affine map which bounds $f$ from below, thus $\ell \leq f^{**}$. But this is in clear contradiction with (5.2.18), therefore the proof is complete. □

Lemma 5.2.24 has a few consequences:

**Corollary 5.2.25.** *Let $f : \mathbb{R}^N \to [0, \infty]$.*

- *If $A \subset \mathbb{R}^N$ is closed and $f = f^{**}$ on $\partial A$, then $(\tilde{f}_A)^{**} = f^{**}$ on $A$.*
- *Assume that the detachment set of $f$ defined by*

$$\mathcal{D}(f) := \{\xi \in \mathbb{R}^N \ : \ f^{**}(\xi) < f(\xi)\}$$

*has uniformly bounded connected components (i.e. there exists $M > 0$ such that $\operatorname{diam}(C) \leq M$ for any such component). Then for any $K > 0$, there exists $K' \geq K$ such that*

$$(\tilde{f}_{K'})^{**} = f^{**} \quad \textit{on} \quad B_K.$$

*Proof.*

- Taking $B = A$ in Lemma 5.2.24 and noticing that by assumption $\partial A \subset A$, it gives the result.
- Take $K > 0$ and let $K' > K + M$. Then for any $\xi_1 \in B_K$, $\xi_2 \notin B_{K'}$, one has $|\xi_1 - \xi_2| > M$ and as such, there must exist some $\xi \in [\xi_1, \xi_2]$ with $|\xi| \leq K'$ such that $f(\xi) = f^{**}(\xi)$. The result follows from Lemma 5.2.24.

□

**Remark 5.2.26.** The assumption that $A$ is closed in the first assertion of the previous corollary is crucial. Take for instance $f : \mathbb{R} \to [0, \infty)$ defined by

$$f(\xi) = \begin{cases} |\xi|^2 & \text{if} \quad \xi < 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can verify easily (using the second point of Proposition 5.2.10 for instance) that

$$f^{**}(\xi) = \begin{cases} |\xi|^2 & \text{if} \quad \xi \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Taking $A = (-\infty, 1)$, then $f = f^{**}$ on $\partial A = \{1\}$. However, $f$ is convex on $A$ and therefore

$$(\tilde{f}_A)^{**} = f \neq f^{**} \quad \text{on} \quad A.$$

We now generalize Corollary 5.2.25 to the case of $x$ and $u$ dependency, whose proof is immediate:

**Corollary 5.2.27.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$. Let $I$ be a compact interval of $\mathbb{R}$.*

- *If there exists $K > 0$ such that*

$$f(x, u, \xi) = f^{**}(x, u, \xi), \qquad \text{for a.e. } x \in \Omega,\ \forall u \in I,\ \forall \xi \in \partial B_K, \tag{5.2.20}$$

  *Then*

$$(\tilde{f}_K)^{**}(x, u, \xi) = f^{**}(x, u, \xi), \qquad \text{for a.e. } x \in \Omega,\ \forall u \in I,\ \forall \xi \in B_K.$$

  *In particular, if* (5.2.20) *holds for a sequence $(K_n)$ of positive numbers such that $K_n \to +\infty$, and for every such $I$, then $f$ satisfies* ($\mathcal{K}$).

- *Assume that the connected components of the sets*

$$\mathcal{D}(f, x, u) := \{\xi \in \mathbb{R}^N \ :\ f^{**}(x, u, \xi) < f(x, u, \xi)\}, \qquad x \in \Omega,\ u \in I,$$

  *are uniformly bounded. That is, there exists $M > 0$ such that for a.e. $x \in \Omega$, for all $u \in I$, every connected component of $\mathcal{D}(f, x, u)$ has diameter smaller than $M$. If this holds for all compact interval $I$, then $f$ satisfies* ($\mathcal{K}$).

The second condition that we present here, independent of the geometry of the detachment set, is the superlinearity of the Lagrangian with respect to the variable $\xi$. In the following result, we will consider the dependency in $(x, u)$ as auxiliary, taken into account in the parametric set $\Gamma$. It is an abstract statement and one may identify $\Gamma$ as $\Omega \times I$, for $I \subset \mathbb{R}$ a compact interval (see Corollary 5.2.31 below which gives a more explicit statement).

**Theorem 5.2.28.** *Let $\Gamma$ be a set and $f : \Gamma \times \mathbb{R}^N \to [0, \infty]$ be such that:*

- *there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ superlinear such that*

$$f(s, \xi) \geq \Phi(\xi), \qquad \forall s \in \Gamma,\ \forall \xi \in \mathbb{R}^N; \tag{5.2.21}$$

- *for any $\rho > 0$, there exists $\rho' \geq \rho$ such that $(\tilde{f}_{\rho'})^{**}$ is bounded on $\Gamma \times B_\rho$.*

*Then for any $K > 0$, there exists $K' \geq K$ such that*

$$(\tilde{f}_{K'})^{**} = f^{**} \quad \text{on} \quad \Gamma \times B_K.$$

**Remark 5.2.29.** Following the same convention introduced in Definition 5.2.13, the convexification in this Theorem is to be understood with respect to the variable $\xi \in \mathbb{R}^N$.

*Proof.* Let $\rho' > 0$ satisfy the second assumption for $\rho = K + 1$, and define

$$M = \sup_{\Gamma \times B_{K+1}} (\tilde{f}_{\rho'})^{**} < +\infty.$$

By (5.2.21), there exists $K' \geq \rho'$ such that

$$f(s, \xi) \geq (M + 1)|\xi| + M, \qquad \forall s \in \Gamma,\ \forall \xi \in \mathbb{R}^N \setminus B_{K'}. \tag{5.2.22}$$

Fix now $\varepsilon \in (0, 1]$ and consider $(s_0, \xi_0) \in \Gamma \times B_K$, then by Proposition 5.2.10, there exists an affine map $\ell$ of slope $\zeta \in \mathbb{R}^N$, (i.e. $\nabla \ell \equiv \zeta$), such that

$$\ell \leq \tilde{f}_{K'}(s_0, \cdot) \quad \text{and} \quad \ell(\xi_0) \geq (\tilde{f}_{K'})^{**}(s_0, \xi_0) - \varepsilon. \tag{5.2.23}$$

Owing to (5.2.23), one has

$$-1 \leq (\tilde{f}_{K'})^{**}(s_0, \xi_0) - \varepsilon \leq \ell(\xi_0) \leq \ell\left(K \frac{\zeta}{|\zeta|}\right) = \ell\left((K+1)\frac{\zeta}{|\zeta|}\right) - |\zeta| \leq M - |\zeta|,$$

because $\ell \leq (\tilde{f}_{K'})^{**}(s_0, \cdot)$ and $\sup_{B_{K+1}}(\tilde{f}_{K'})^{**}(s_0, \cdot) \leq M$. Hence $|\zeta| \leq M+1$. Our goal is to show that $\ell$ is an affine minorant of $f(s_0, \cdot)$. The fact that $\ell \leq f(s_0, \cdot)$ on $B_{K'}$ is an immediate consequence of (5.2.23). Now let us consider $\xi$ which is not in $B_{K'}$, then by (5.2.22):

$$\ell(\xi) = \langle \xi, \zeta \rangle + \ell(0) \leq (M+1)|\xi| + M \leq f(s_0, \xi),$$

which proves that $\ell \leq f(s_0, \cdot)$ on $\mathbb{R}^N$ and thus that $\ell \leq f^{**}(s_0, \cdot)$. In particular, one has by (5.2.23) :

$$(\tilde{f}_{K'})^{**}(s_0, \xi_0) \leq f^{**}(s_0, \xi_0) + \varepsilon,$$

letting $\varepsilon \to 0$, it gives $(f_{K'})^{**} \leq f^{**}$ on $\Gamma \times B_K$. The converse inequality being obvious, we finally obtain that

$$(f_{K'})^{**} = f^{**} \quad \text{on} \quad \Gamma \times B_K.$$

□

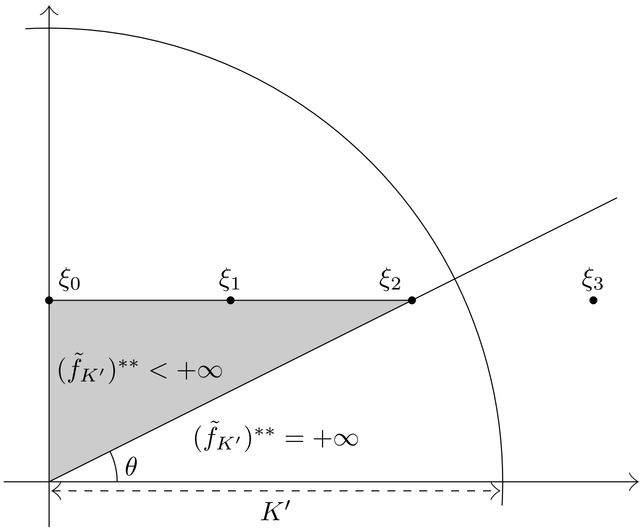


Figure 5.1: $(\tilde{f}_{K'})^{**}$ is infinite on the angular sector guided by $\theta$. Yet, it is not the case for $f^{**}$.

**Remark 5.2.30.** The second assumption of Theorem 5.2.28 is crucial in this result. So much that it may fail without it: consider for instance the function (depending only on $\xi$), $f : \mathbb{R}^2 \to [0, \infty]$ defined by:

$$\begin{cases} f(0,0) & = 0 \\ f(\xi_n) & = |\xi_n|^2 \qquad \forall n \in \mathbb{N} \\ f(\xi) & = +\infty \qquad \text{otherwise,} \end{cases}$$

where $\xi_n := (n, 1)$. Then $f$ is superlinear (taking for instance $\Phi(\xi) = |\xi|^2$ is enough), but we claim that it does not satisfy $(\mathcal{K})$. Indeed, $f$ is finite on the set

$$A := \{(0,0)\} \cup \{\xi_0, \xi_1, \dots\},$$

therefore $f^{**}$ is finite on $\operatorname{conv} A \supset [0, \infty[\times]0, 1]$. But for any $K' > 0$, $B_{K'}$ contains only a finite number of the $\xi_n$ and thus $(\tilde{f}_{K'})^{**}$ is infinite on an angular sector (see Figure 5.1).

For the sake of comprehensibility, we now state a version of Theorem 5.2.28 in the case where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$:

**Corollary 5.2.31.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$. Assume that for every compact interval $I \subset \mathbb{R}$,*

- *there exists $\Phi_I : \mathbb{R}^N \to [0, \infty)$ superlinear such that*

$$f(x, u, \xi) \geq \Phi_I(\xi), \qquad \text{for a.e. } x \in \Omega,\ \forall u \in I,\ \forall \xi \in \mathbb{R}^N; \tag{5.2.24}$$

- *there exists $\tilde{\Omega} \subset \Omega$ such that $|\Omega \setminus \tilde{\Omega}| = 0$ and for any $\rho > 0$, $f$ is bounded on $\tilde{\Omega} \times I \times B_\rho$.*

*Then $f$ satisfies condition ($\mathcal{K}$).*

*Proof.* Let $\Omega_0 \subset \Omega$ a measurable set with full measure such that (5.2.24) holds for every $x \in \Omega_0$. Let $\Gamma = (\Omega_0 \cap \tilde{\Omega}) \times I$. Then the first condition of Theorem 5.2.28 holds by construction, and the second comes from the fact that

$$(\tilde{f}_\rho)^{**} \leq f, \quad \text{on} \quad \Omega \times I \times B_\rho,$$

thus the choice $\rho' = \rho$ is enough. □

On another note, Theorem 5.2.28 allows us to prove Theorem 5.2.17, which gave a sufficient condition for $f^{**}$ to be Carathéodory..

*Proof of Theorem 5.2.17.* By Lemma 5.2.14, we know that $f^{**}$ is Lebesgue-Borel measurable. It is therefore enough to show that it is continuous with respect to $(u, \xi)$ for a.e. $x \in \Omega$. Let $A_0$ be a measurable subset of $\Omega$ with full measure such that (5.2.10) holds for every $x \in A_0$ and every compact interval $I$. According to Hypothesis 5.2.4-c), for any integer $n \geq 1$, there exists $a_n \in L^1(\Omega, [0, \infty))$ and a measurable set $A_n \subset \Omega$ with full measure such that

$$f(x, u, \xi) \leq a_n(x) < +\infty, \qquad \forall x \in A_n,\ \forall (u, \xi) \in [-n, n] \times B_n. \tag{5.2.25}$$

For $x \in A := \cap_{n \geq 0} A_n$, define $g : (u, \xi) \mapsto f(x, u, \xi)$. We will show that $g$ satisfies the assumption of Theorem 5.2.28 with $\Gamma = I$. Fix $I \subset \mathbb{R}$ a compact interval. Firstly by (5.2.25), $g$ is bounded on bounded sets and thus the second assumption of Theorem 5.2.28 is satisfied with $\rho' = \rho$. For the first assumption, notice that by (5.2.10), it is satisfied with $\Phi := \Phi_{x,I}$. We can therefore finally apply Theorem 5.2.28, which tells us that $g$ satisfies (5.2.9) of Proposition 5.2.16 and thus that $g^{**}$ is continuous. The conclusion follows from the fact that $A$ has full measure in $\Omega$, and hence we have proved that $(u, \xi) \mapsto f^{**}(x, u, \xi)$ is continuous for a.e. $x \in \Omega$: $f^{**}$ is a Carathéodory function. □

**Remark 5.2.32.** In this Section 5.2.4, we have given two distinct sets of assumptions ensuring that the Lagrangian satisfies ($\mathcal{K}$). On the one hand, a boundedness condition on the detachment set (Corollaries 5.2.25 and 5.2.27). On the other hand, the superlinearity of the Lagrangian (Theorem 5.2.28 and Corollary 5.2.31). We wish to show that these sets of assumptions are independent and none implies the other.

- Let $f : (\xi_1, \xi_2) \mapsto (1 + \sin(\xi_1))(1 + \sin(\xi_2))$. Then $f$ is clearly not superlinear, however $f^{**} \equiv 0$ and thus $f$ satisfies the detachment set condition, (second point of Corollary 5.2.25) because $f = f^{**}$ on a square grid of side-length $2\pi$.

- Conversely, let $f : \mathbb{R}^2 \to [0, \infty)$ defined by

  $$f(\xi_1, \xi_2) := \big(|\xi_1| - 1\big)^2 + |\xi_2|^2.$$

  It is clearly superlinear. However, we show that the line $\{\xi_1 = 0\}$ is contained in the detachment set of $f$, thus contradicting the boundedness condition. Indeed, notice that by Proposition 5.2.10, for every $\xi_2 \in \mathbb{R}$,

  $$\begin{aligned} f^{**}(0, \xi_2) &\leq \frac{1}{2}(f(-1, \xi_2) + f(1, \xi_2)) \\ &= |\xi_2|^2 \\ &< |\xi_2|^2 + 1 \\ &= f(0, \xi_2). \end{aligned}$$

## 5.3 Approximation in $W^{1,p}(\Omega)$ for the autonomous case

In this paragraph, we state an important result which answers the question of the Lavrentiev gap when the Lagrangian is *autonomous*, that is $f = f(u,\xi)$ does not depend on $x$. Some regularity is still needed (in particular, the convexity with respect to $\xi$). The detailed proof may be found in [17].

**Theorem 5.3.1.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a continuous Lagrangian, which is convex in $\xi$ (the last variable), and let $\varphi \in W^{1,\infty}(\Omega)$. Then for any $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,1}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

*Moreover, if $u \in W^{1,p}(\Omega)$ for some $p > 1$, then the sequence $(u_n)$ can be chosen so that $u_n \to u$ strongly in $W^{1,p}(\Omega)$.*

**Remark 5.3.2.** We can see Theorem 5.3.1 through another perspective by defining another functional:

$$E_{rel}[f](u) = \inf \left\{ \liminf_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx \;\middle|\; \begin{array}{c} (u_n) \subset W^{1,\infty}_\varphi(\Omega) \\ u_n \to_{W^{1,1}} u \end{array} \right\}.$$

By the Fatou Lemma we have that if $f(u,\xi)$ is continuous then for every $u \in W^{1,1}_\varphi(\Omega)$

$$E[f](u) \leq E_{rel}[f](u),$$

with an equality for every $u \in W^{1,\infty}_\varphi(\Omega)$. Now Theorem 5.3.1 implies that if $g(u,\xi)$ is continuous and convex with respect to the last variable then

$$E[g] = E_{rel}[g] \quad \text{on} \quad W^{1,1}_\varphi(\Omega).$$

Now a natural question arising from this statement is to wonder whether the assumption concerning convexity is needed for this result to hold. In this Section, we will extend this result to a larger class of Lagrangians. In the autonomous case we have a simpler set of assumptions.

**Hypothesis 5.3.3.** The function $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfies

a) $f$ is Borel,

b) $f$ is continuous with respect to $u$ uniformly as $\xi$ varies in bounded sets of $\mathbb{R}^N$,

c) $f$ is bounded on bounded sets,

d) $f^{**}$, the bipolar of $f$ with respect to the second variable, is continuous in $(u,\xi)$.

In the autonomous case Hypothesis 5.3.3 implies Hypothesis 5.2.4 (this is pretty clear but a detailed proof is given in [10, Lemma 18]). We recall that Theorem 5.2.17 gives a sufficient condition for 5.3.3-d) to hold: it is enough to assume that $f$ is superlinear with respect to $\xi$ uniformly as $u$ varies in bounded sets of $\mathbb{R}$. It might be of interest to note that again, Hypothesis 5.3.3 does not request that $f$ is continuous with respect to $(u,\xi)$.

Of course, Theorem 5.3.1 implies the non occurrence of the Lavrentiev phenomenon for continuous and autonomous Lagrangians which are convex in $\xi$. Actually, more is true, as stated in the following result:

**Theorem 5.3.4.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfy Hypothesis 5.3.3. Then no Lavrentiev phenomenon occurs for the integral functional $E[f]$, that is:*

$$\inf \left\{ E[f](u) \;:\; u \in W^{1,1}_\varphi(\Omega) \right\} = \inf \left\{ E[f](u) \;:\; u \in W^{1,\infty}_\varphi(\Omega) \right\}.$$

*Proof.* For the sake of simplicity, write $X = W^{1,1}_\varphi(\Omega)$ and $Y = W^{1,\infty}_\varphi(\Omega)$. Then

$$\inf_X E[f] \geq \inf_X E[f^{**}] = \inf_Y E[f^{**}] = \inf_Y E[f] \geq \inf_X E[f].$$

Indeed, the first and last inequalities are clear, the second comes from Theorem 5.3.1 and the third from Theorem 5.2.20. □

As a byproduct of this proof, we deduce that (if $f$ satisfies Hypothesis 5.3.3), for any $p \in [1, \infty)$,

$$\inf \Big\{ E[f](u) \ : \ u \in W^{1,p}_\varphi(\Omega) \Big\} = \inf \Big\{ E[f^{**}](u) \ : \ u \in W^{1,p}_\varphi(\Omega) \Big\}. \tag{5.3.1}$$

### 5.3.1 Weak relaxation on $W^{1,1}(\Omega)$.

Now we extend Theorem 5.3.1 to the non convex case. The following theorem is a generalization of [10, Theorem 30].

**Theorem 5.3.5.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.3.3, and let $\varphi \in W^{1,\infty}(\Omega)$. Then for any $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{in} \quad L^1(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(u(x), \nabla u(x))dx.$$

*Moreover,*

- *if there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ superlinear such that*

$$f(u, \xi) \geq \Phi(\xi), \qquad \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N, \tag{5.3.2}$$

  *then the sequence $(u_n)$ can be chosen so that $u_n \rightharpoonup u$ weakly in $W^{1,1}(\Omega)$;*

- *if for some $p \in (1, \infty)$ it holds that*

$$f(u, \xi) \geq c_1|\xi|^p - c_2, \qquad \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N, \tag{5.3.3}$$

  *for some $c_1, c_2 > 0$, and $u \in W^{1,p}(\Omega)$ then the sequence $(u_n)$ can be taken so that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

**Remark 5.3.6.** Notice that, compared to Theorem 5.3.1, the assumptions on $f$ are relaxed: indeed in particular, $f$ is not assumed to be convex in $\xi$. However, the conclusion is slightly weaker in the sense that the strong convergence of Theorem 5.3.1 was replaced by an $L^1$ (or $W^{1,1}$-weak) convergence. Apart from that, both results have similar conclusions, up to noticing that $f = f^{**}$ in the assumptions of Theorem 5.3.1.

*Proof of Theorem 5.3.5.* Fix $u \in W^{1,1}_\varphi(\Omega)$. If $\int_\Omega f^{**}(u(x), \nabla u(x))dx = +\infty$, the result follows from Fatou Lemma. Indeed, take any sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ converging to $u$ strongly in $W^{1,1}(\Omega)$ (such a sequence always exists). Then up to a subsequence, $(u_n(x), \nabla u_n(x)) \to (u(x), \nabla u(x))$ for a.e. $x \in \Omega$, and by continuity of $f^{**}$, it holds that $f^{**}(u_n(x), \nabla u_n(x)) \to f^{**}(u(x), \nabla u(x))$ for a.e. $x \in \Omega$. Thus

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx &\geq \liminf_{n\to+\infty} \int_\Omega f^{**}(u_n(x), \nabla u_n(x))dx \\ &\geq \int_\Omega f^{**}(u(x), \nabla u(x))dx \\ &= +\infty.\end{aligned}$$

We assume from now on that $E[f^{**}](u) < +\infty$. By Theorem 5.3.1, there exists a sequence $(v_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that $v_n \to u$ in $W^{1,1}(\Omega)$ and $E[f^{**}](v_n) \to E[f^{**}](u)$. Now for each $n$, one may use Theorem 5.2.20 to find a sequence $(v_n^k) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that $\|v_n^k - v_n\|_{L^\infty} \to 0$ (in particular, $v_n^k \to v_n$ in $L^1$) and

$$\liminf_{k\to+\infty} E[f](v_n^k) \leq E[f^{**}](v_n) + 1/n.$$

Using a diagonal argument, one obtains a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that $u_n \to u$ in $L^1$ and $E[f](u_n) \to E[f^{**}](u)$. To conclude, notice that conditions (5.3.2) and (5.3.3) respectively gives weak compactness for the approximating sequence in $W^{1,1}$ (by the Dunford-Pettis Theorem) and $W^{1,p}$. Therefore it is an immediate corollary that the approximating sequence weakly converges to $u$ in that case. □

**Remark 5.3.7.** We can see Theorem 5.3.5 from a functional point of view.
We define

$$\tilde{E}_1[f](u) := \inf\left\{ \liminf_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx \;\middle|\; \begin{matrix} (u_n) \subset W^{1,\infty}_{\varphi}(\Omega) \\ u_n \rightharpoonup_{W^{1,1}} u \end{matrix} \right\}. \tag{5.3.4}$$

If $f^{**}$ is convex in $\xi$ and continuous in $(u, \xi)$, by [47, Chapter 4, Theorem 4.5] the integral functional

$$u \mapsto \int_\Omega f^{**}(u(x), \nabla u(x))dx,$$

is weakly lower semicontinuous on $W^{1,1}(\Omega)$. So we have

$$\int_\Omega f^{**}(u(x), \nabla u(x))dx \leq \liminf_{n\to+\infty} \int_\Omega f(v_n(x), \nabla v_n(x))dx,$$

for every $(v_n) \subset W^{1,1}(\Omega)$ such that $v_n \rightharpoonup_{W^{1,1}} u$, that is

$$\int_\Omega f^{**}(u(x), \nabla u(x))dx \leq \tilde{E}_1[f](u).$$

Now we know by Theorem 5.3.5 that if $f$ satisfies Hypothesis 5.3.3 and is uniformly superlinear then for every $u \in W^{1,1}_{\varphi}(\Omega)$,

$$E[f^{**}](u) = \tilde{E}_1[f](u),$$

and furthermore the infimum in (5.3.4) is actually a minimum.

### 5.3.2 A geometric condition to recover strong convergence.

In Theorem 5.3.5 we do not have the strong convergence of the approximating sequence. Now we study other cases in which the approximating sequence converges strongly in $W^{1,1}(\Omega)$. Before stating the main result, we will need a useful lemma.

**Lemma 5.3.8.** *Let $v \in L^1(\Omega, \mathbb{R}^N)$. Then there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ convex and superlinear such that*

$$\int_\Omega \Phi(v(x))dx < +\infty.$$

*Proof.* The proof is rather straightforward and uses a similar argument to the one used in the proof of the Dunford-Pettis Theorem. Let $(M_n)$ be an increasing sequence of positive real numbers such that $M_n \to +\infty$ and

$$\int_{\{|v|\geq M_n\}} |v|dx \leq 2^{-n}, \qquad \forall n \in \mathbb{N}. \tag{5.3.5}$$

Then define

$$\Phi(\xi) := \sum_{n\in\mathbb{N}} \left(|\xi| - M_n\right)^+, \qquad \forall \xi \in \mathbb{R}^N,$$

where we used the standard notation $a^+ := \max(a, 0)$. The assumption that $M_n \to +\infty$ ensures that the sum in the definition of $\Phi$ is effectively finite at each point, and therefore $\Phi$ takes values into $[0, \infty)$. Each of the functions $\xi \mapsto (|\xi| - M_n)^+$ is convex, thus so is $\Phi$ as a supremum of convex functions. Also, the identity $\Phi(\xi)/|\xi| = \sum_n (1 - M_n/|\xi|)^+$ ensures that $\Phi$ is superlinear. To achieve the proof, it is finally enough to use (5.3.5) and notice the following:

$$\int_\Omega \Phi(v(x))dx = \sum_{n=0}^{+\infty} \int_{\{|v| \geq M_n\}} \big(|v| - M_n\big)dx \leq \sum_{n=0}^{+\infty} 2^{-n} < +\infty.$$

□

**Theorem 5.3.9.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfies Hypothesis 5.3.3, and let $\varphi \in W^{1,\infty}(\Omega)$. Assume $u \in W^{1,p}_\varphi(\Omega)$ for some $p \in [1, \infty)$, and satisfies*

$$\int_\Omega f^{**}(u(x), \nabla u(x))dx = \int_\Omega f(u(x), \nabla u(x))dx. \tag{5.3.6}$$

*Then there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

In fact, equality (5.3.6) happens to be also a necessary for the existence of such an approximating sequence (see Remark 5.3.13).

*Proof.* Assume first that $p = 1$. In the case where $E[f](u) = +\infty$, this follows from the Fatou lemma just as in the proof of Theorem 5.3.5. Otherwise, we assume $E[f](u) < +\infty$. By Lemma 5.3.8, there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ convex and superlinear such that

$$\int_\Omega \Phi(\nabla u(x))dx < +\infty.$$

We define

$$g(u, \xi) := f(u, \xi) + \Phi(\xi) + \sqrt{1 + |\xi|^2}.$$

Because $g(u, \xi) \geq \Phi(\xi)$, Theorem 5.2.17 ensures that $g^{**}$ is Carathéodory and thus $g$ satisfies Hypothesis 5.3.3. By Theorem 5.3.5, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,1}(\Omega)$ and

$$\begin{aligned}\int_\Omega g(u_n(x), \nabla u_n(x))dx &\to \int_\Omega g^{**}(u(x), \nabla u(x))dx \\ &\leq E[f](u) + \int_\Omega \Phi(\nabla u(x))dx + \int_\Omega \sqrt{1 + |\nabla u(x)|^2}dx.\end{aligned}$$

(The inequality is a simple consequence of $g^{**} \leq g$.) Therefore,

$$\begin{aligned}&\limsup_{n\to+\infty} \left( E[f](u_n) + \int_\Omega \Phi(\nabla u_n(x))dx + \int_\Omega \sqrt{1 + |\nabla u_n(x)|^2}dx \right) \\ &\leq E[f](u) + \int_\Omega \Phi(\nabla u(x))dx + \int_\Omega \sqrt{1 + |\nabla u(x)|^2}dx < +\infty\end{aligned} \tag{5.3.7}$$

By convexity with respect to the last variable and continuity, we have again by Tonelli's result (see Lemma 5.2.19) the following estimates

$$\begin{aligned}E[f](u) = \int_\Omega f^{**}(u(x), \nabla u(x))dx &\leq \liminf_{n\to+\infty} \int_\Omega f^{**}(u_n(x), \nabla u_n(x))dx \\ &\leq \liminf_{n\to+\infty} E[f](u_n),\end{aligned}$$

as well as

$$\int_\Omega \Phi(\nabla u(x))dx \le \liminf_{n\to+\infty} \int_\Omega \Phi(\nabla u_n(x))dx,$$

$$\int_\Omega \sqrt{1+|\nabla u(x)|^2}dx \le \liminf_{n\to+\infty} \int_\Omega \sqrt{1+|\nabla u_n(x)|^2}dx,$$

and thus, combining this with (5.3.7), one see that we have in fact convergence of each of these terms. In particular:

$$E[f](u_n) \to E[f](u) \quad \text{and} \quad \int_\Omega \sqrt{1+|\nabla u_n|^2}dx \to \int_\Omega \sqrt{1+|\nabla u|^2}dx.$$

From the second point, we deduce that $u_n \to u$ strongly in $W^{1,1}(\Omega)$ (see Lemma 5.3.10 below), which achieves the proof. When $p > 1$, the proof is similar, as one only needs to take $g(u,\xi) = f(u,\xi) + |\xi|^p$ in this case.

□

**Lemma 5.3.10.** *Let $p \in [1,\infty)$ and $(v_n) \subset L^p$. Assume that $v_n \rightharpoonup v$ weakly in $L^p$.*

- *If $p = 1$ and $\int \sqrt{1+|v_n|^2} \to \int \sqrt{1+|v|^2}$, then $v_n \to v$ strongly in $L^1$.*
- *If $p > 1$ and $\int |v_n|^p \to \int |v|^p$, then $v_n \to v$ strongly in $L^p$.*

*Proof.* For the case $p = 1$, see [46, Section 1.3.4, Proposition 1]. For $p > 1$, this is a consequence of the uniform convexity of the space $L^p$. □

**Remark 5.3.11.** It might be relevant to emphasize the fact that, using the inequality $f^{**} \le f$, condition (5.3.6) in Theorem 5.3.9 is satisfied if and only if

$$f(u(x), \nabla u(x)) = f^{**}(u(x), \nabla u(x)), \qquad \text{for a.e. } x \in \Omega.$$

This Theorem 5.3.9 has as a consequence the following interesting corollary, which prevent the Lavrentiev gap for functions $\overline{u}$ which are already known to be minimizers of the energy.

**Corollary 5.3.12.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ satisfies Hypothesis 5.3.3, and let $\varphi \in W^{1,\infty}(\Omega)$. If $\overline{u}$ is a minimizer of $E[f]$ on $W^{1,p}_\varphi(\Omega)$, for some $p \in [1,\infty)$, then there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to \overline{u} \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(\overline{u}(x), \nabla \overline{u}(x))dx.$$

*Proof.* By Theorem 5.3.9, it is enough to prove that $\overline{u}$ satisfies (5.3.6). Thank to equality (5.3.1), we obtain

$$\begin{aligned} E[f^{**}](\overline{u}) &\ge \inf\left\{E[f^{**}](u) \ : \ u \in W^{1,p}_\varphi(\Omega)\right\} \\ &= \inf\left\{E[f](u) \ : \ u \in W^{1,p}_\varphi(\Omega)\right\} \\ &= E[f](\overline{u}). \end{aligned}$$

The reverse inequality comes from $f^{**} \le f$. □

**Remark 5.3.13.** We can see Theorem 5.3.9 as a generalization of Theorem 5.3.1 to the case of non convex Lagrangians. Indeed, if $f = f^{**}$, then (5.3.6) is trivially satisfied for every $u$ and we recover the original statement of Theorem 5.3.1. In fact more can be said on this matter: Theorem 5.3.9 states that (5.3.6) is a sufficient condition for the existence of a sequence $(u_n)$ strongly converging to $u$ satisfying $E[f](u_n) \to E[f^{**}](u)$. We claim that under the assumption that $f$ is continuous, the converse is also true. Indeed, in that case, by the Fatou Lemma,

$$E[f](u) \le \liminf_{n\to+\infty} E[f](u_n) = E[f^{**}](u),$$

and the fact that $f^{**} \le f$ gives the reverse inequality.

### 5.3.3 Convergence results for convex-dominated Lagrangians.

In the following theorem we show another case in which $E[f](u) = E_{rel}[f](u)$ (the notation $E_{rel}[f]$ was introduced in Remark 5.3.2), with some interesting implications.

**Theorem 5.3.14.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ continuous, and let $\varphi \in W^{1,\infty}(\Omega)$. Let $u \in W^{1,p}_{\varphi}(\Omega)$ for some $p \in [1, \infty)$ and assume that there exists $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ globally continuous and convex with respect to the last variable, such that $f \leq g$ and*

$$\int_{\Omega} g(u(x), \nabla u(x))dx < +\infty. \tag{5.3.8}$$

*Then there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_{\Omega} f(u_n(x), \nabla u_n(x))dx = \int_{\Omega} f(u(x), \nabla u(x))dx.$$

In fact, the previous theorem is an immediate consequence of the more general following one:

**Theorem 5.3.15.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ continuous, and let $\varphi \in W^{1,\infty}(\Omega)$. Let $u \in W^{1,p}_{\varphi}(\Omega)$ for some $p \in [1, \infty)$ and assume that there exists a continuous $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfying Hypothesis 5.3.3, such that $f \leq g$,* (5.3.8) *is satisfied, as well as*

$$\int_{\Omega} g^{**}(u(x), \nabla u(x))dx = \int_{\Omega} g(u(x), \nabla u(x))dx.$$

*Then there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_{\Omega} f(u_n(x), \nabla u_n(x))dx = \int_{\Omega} f(u(x), \nabla u(x))dx.$$

*Proof.* We begin by noticing that $g$ satisfies the assumptions of Theorem 5.3.9. Therefore, there exists $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that

$$u_n \to u \quad \text{strongly in} \quad W^{1,p}(\Omega),$$

and

$$\lim_{n\to+\infty} \int_{\Omega} g(u_n(x), \nabla u_n(x))dx = \int_{\Omega} g(u(x), \nabla u(x))dx. \tag{5.3.9}$$

Since $f$ and $g - f$ are continuous and non-negative, by the Fatou Lemma,

$$\liminf_{n\to+\infty} E[f](u_n) \geq E[f](u), \tag{5.3.10}$$

and

$$\liminf_{n\to+\infty} E[g - f](u_n) \geq E[g - f](u). \tag{5.3.11}$$

Thus, using (5.3.9) and (5.3.11),

$$\begin{aligned}
\limsup_{n\to+\infty} E[f](u_n) &= \limsup_{n\to+\infty} \Big[E[g](u_n) - E[g - f](u_n)\Big] \\
&\leq E[g](u) - \liminf_{n\to+\infty} E[g - f](u_n) \\
&\leq E[g](u) - E[g - f](u) \\
&= E[f](u).
\end{aligned}$$

Which, with the help of (5.3.10), concludes the proof. □

Here is a consequence of Theorem 5.3.14:

**Proposition 5.3.16.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ continuous, $p, q \in [1, \infty)$ such that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and let $\varphi \in W^{1,\infty}(\Omega)$. Assume that there exists $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ globally continuous and convex with respect to the last variable, such that*

$$f(u, \xi) \leq g(u, \xi) \leq c(f(u, \xi) + |\xi|^p + |u|^q + 1), \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

*for some $c > 0$. Then for any $u \in W^{1,p}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n \to +\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

*Proof.* In the case where $E[f](u) = +\infty$, this follows from the continuity of $f$ and the Fatou Lemma. Otherwise, we can apply Theorem 5.3.14. By assumption, $g$ is convex with respect to $\xi$ and furthermore

$$\begin{aligned}\int_\Omega g(u(x), \nabla u(x))dx &\leq c \left( \int_\Omega f(u(x), \nabla u(x))dx + \|\nabla u\|^p_{L^p} + \|u\|^q_{L^q} + |\Omega| \right) \\ &< +\infty,\end{aligned}$$

which is what we needed. □

Proposition 5.3.16 has a few interesting consequences:

**Corollary 5.3.17.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ continuous such that $f^{**}$ is continuous, $p, q \in [1, \infty)$ such that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and let $\varphi \in W^{1,\infty}(\Omega)$. Assume that*

$$f(u, \xi) \leq c(f^{**}(u, \xi) + |\xi|^p + |u|^q + 1), \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

*for some $c > 0$. Then for any $u \in W^{1,p}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n \to +\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

*Proof.* It is sufficient to take $g := c(f^{**} + |\xi|^p + |u|^q + 1)$ in Proposition 5.3.16. □

**Corollary 5.3.18.** *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a continuous function, $p, q \in [1, \infty)$ such that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and let $\varphi \in W^{1,\infty}(\Omega)$. Assume that $f$ is "convex at infinity"; that is, there exists $\tilde{f} : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ convex in $\xi$ and continuous as well as $K > 0$ such that*

$$f(u, \xi) = \tilde{f}(u, \xi), \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N \setminus B_K.$$

*Assume furthermore that*

$$\sup \left\{ f(u, \xi) - \tilde{f}(u, \xi) \ :\ \xi \in B_K \right\} = O(|u|^q), \quad \textit{as } |u| \to \infty. \tag{5.3.12}$$

*Then for any $u \in W^{1,p}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n \to +\infty} \int_\Omega f(u_n(x), \nabla u_n(x))dx = \int_\Omega f(u(x), \nabla u(x))dx.$$

*Proof.* By (5.3.12) and continuity in $u$ of the term on its left-hand side, there exists $M > 0$ such that

$$f(u,\xi) \le \tilde{f}(u,\xi) + M(|u|^q + 1), \qquad \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

Now let $g(u,\xi) := \tilde{f}(u,\xi) + M(|u|^q + 1)$ and apply Proposition 5.3.16. □

The following result consists in the case where $f = f(\xi)$ is $C^2$ and not far from being convex (more specifically, the eigenvalues of the Hessian of $f$ are not too largely negative). Before stating it, here are some notations: given $f : \mathbb{R}^N \to [0,\infty)$ a $C^2$ function, we define the function $\lambda[f] : \mathbb{R}^N \to \mathbb{R}$ by

$$\lambda[f](\xi) := \inf_{|\zeta|=1} \langle \nabla^2 f(\xi)\cdot\zeta,\, \zeta\rangle, \qquad \forall \xi \in \mathbb{R}^N.$$

Hence $\lambda[f](\xi)$ is the smallest eigenvalues of the Hessian of $f$ at point $\xi$. Notice that $f$ being $C^2$, $\lambda[f]$ is continuous on $\mathbb{R}^N$. We can now state the result:

**Corollary 5.3.19.** *Let $f : \mathbb{R}^N \to [0,\infty)$ be a function of class $C^2$ and $p \in [1,\infty)$. Denote*

$$\theta(r) := \sup_{|\xi|=r} \Big(\lambda[f](\xi)\Big)^-, \qquad \forall r \ge 0. \tag{5.3.13}$$

*(Here we used the notation $a^- := \max(-a, 0)$). We assume that $\theta(r) = O(r^{p-2})$, as $r \to +\infty$. Moreover, if $p = 1$, we assume that $\int_0^{+\infty} \theta(r)dr < +\infty$. Then for any $u \in W^{1,p}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_\Omega f(\nabla u_n(x))dx = \int_\Omega f(\nabla u(x))dx.$$

*Proof.* As always, we can assume that $\int_\Omega f(\nabla u)dx < +\infty$, otherwise the result follows by the Fatou Lemma. Notice first that if $\theta \equiv 0$, then $f$ is convex and Theorem 5.3.1 can immediately be applied. We assume in the following that $\theta$ is not identically 0 on $[0,\infty)$. Let $\gamma : [0,\infty) \to [0,\infty)$ be $C^2$ such that $\gamma(0) = \gamma'(0) = 0$ to be fixed later on, as well as $g_1 : \xi \mapsto \gamma(|\xi|)$ defined on $\mathbb{R}^N$. As such, $g_1$ is a $C^2$ radial function on $\mathbb{R}^N$. Let $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\mathcal{B}$ an orthonormal basis of $\mathbb{R}^N$ having $\xi/|\xi|$ as its first vector, then a straightforward calculation using the radial property of $g_1$ gives that its Hessian quadratic form is represented in the basis $\mathcal{B}$ as the following diagonal matrix:

$$\operatorname{diag}\Big(\gamma''(|\xi|),\ |\xi|^{-1}\gamma'(|\xi|),\ \dots,\ |\xi|^{-1}\gamma'(|\xi|)\Big),$$

and so,

$$\operatorname{spectrum}\big[\nabla^2 g_1(\xi)\big] = \{\gamma''(|\xi|),\ |\xi|^{-1}\gamma'(|\xi|)\}, \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.3.14}$$

- If $p > 1$, choose $\gamma$ to be a $C^2$ function on $[0,\infty)$ such that $\gamma(0) = \gamma'(0) = 0$ and such that $\gamma(r) = r^p$ if $r \ge 1$. Then (5.3.14) gives the existence of some $c > 0$ such that it holds

  $$\lambda[g_1](\xi) \ge c|\xi|^{p-2}, \qquad \forall \xi \in \mathbb{R}^N \setminus B_2.$$

  To be clear, here $B_2$ refers to the centered ball of radius 2 in $\mathbb{R}^N$. Now using the assumption we made on $\theta$, one can find $R \ge 2$ and $M \ge 1$ such that,

  $$\lambda[g_1](\xi) \ge M^{-1}\theta(|\xi|), \qquad \forall \xi \in \mathbb{R}^N \setminus B_R. \tag{5.3.15}$$

  If $f + Mg_1$ is convex on $\mathbb{R}^N$, then we can take $g := f + Mg_1$ and apply Theorem 5.3.14. Indeed, recall that $u \in W^{1,p}(\Omega)$ and thus that $\int_\Omega g_1(\nabla u)dx < +\infty$. Otherwise, because $f$ and $g_1$ are $C^2$ on $\mathbb{R}^N$, it holds that

  $$0 > \inf_{\xi \in B_R} \lambda[f + Mg_1](\xi) =: -c > -\infty. \tag{5.3.16}$$

  Let $g_2 : \mathbb{R}^N \to [0,\infty)$ be any $C^2$ convex function with linear growth satisfying that $\lambda[g_2] \ge c$, on $B_R$. Then, by (5.3.13), (5.3.15) and (5.3.16), $f + Mg_1 + g_2$ is convex on $\mathbb{R}^N$. Taking $g = f + Mg_1 + g_2$ in Theorem 5.3.14 gives the result.

- If $p = 1$, the proof is slightly different. Firstly, instead of taking $\gamma(r) = r$, we choose $\gamma$ such that $\gamma'' = \theta$ (as well as $\gamma(0) = \gamma'(0) = 0$). We claim that $\gamma''(r) = O(r^{-1}\gamma'(r))$, as $r \to +\infty$. Indeed, $\gamma'$ is a non decreasing function, converging to $\tilde{c} := \int_0^{+\infty} \theta > 0$ . Thus $r^{-1}\gamma'(r) \sim \tilde{c}r^{-1}$. However, $\gamma''(r) = \theta(r) = O(r^{-1})$ by assumption. Let $R > 0$ and $M \geq 1$ be such that for every $r > R$, it holds that
$$\gamma''(r) \leq Mr^{-1}\gamma'(r). \tag{5.3.17}$$
Using the fact that $M \geq 1$, as well as (5.3.14) and (5.3.17), we obtain once again (5.3.15). Moreover, we have that $g_1$ has sublinear growth (because $\gamma'$ is bounded), thus again $\int_\Omega g_1(\nabla u)dx < +\infty$. From this point on, the proof follows just as in the case $p > 1$.

□

## 5.4 Generalization to the non-autonomous case

In this section we want to extend Theorem 5.3.5 to the non autonomous case, applying the results from [14]. Firstly we report [14, Theorem 3], the main conditions used in the paper are the following : we say that a measurable function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfies condition ($\mathcal{H}_1$) if for every $L = (L_1, L_2) \in (0, \infty)^2$, there exists a constant $C_L > 0$ such that for a.e. $x \in \Omega$ and every $(u, \xi) \in [-L_1, L_1] \times \mathbb{R}^N$, for every $\varepsilon > 0$, it holds
$$(g_\varepsilon^-)^{**}(x, u, \xi) \leq \frac{L_2}{\varepsilon^N} \quad \Rightarrow \quad g(x, u, \xi) \leq C_L(1 + (g_\varepsilon^-)^{**}(x, u, \xi)), \tag{$\mathcal{H}_1$}$$
where
$$g_\varepsilon^-(x, u, \xi) = \operatorname*{ess\,inf}_{y \in \Omega \cap B_\varepsilon(x)} g(y, u, \xi). \tag{5.4.1}$$

We also say that a measurable function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfies condition ($\mathcal{H}_2$) if there exists $\theta \in [1, \infty]$, $a \in L^\theta(\Omega)$ and a real number $u_0 > 0$ such that
$$g(x, u, 0) \leq a(x)|u|^{\frac{p^*}{\theta'}}, \qquad \text{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R} \setminus [-u_0, u_0]. \tag{$\mathcal{H}_2$}$$
Here $p^*$ and $\theta'$ are respectively the Sobolev and Hölder conjugate exponents of $p$ and $\theta$. We now state the result proved in [14]:

**Theorem 5.4.1.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a Carathéodory function which is convex with respect to the last variable, and let $\varphi \in W^{1,\infty}(\Omega)$. Also assume that $f$ satisfies ($\mathcal{H}_1$). If $N \geq 2$, we assume furthermore that $f$ satisfy ($\mathcal{H}_2$). Then, for every $u \in W^{1,p}_\varphi(\Omega)$ such that $E[f](u) < +\infty$ there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that $u_n \to u$ strongly in $W^{1,p}(\Omega)$ and*
$$\lim_{n \to +\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f(x, u(x), \nabla u(x))dx.$$

**Hypothesis 5.4.2.** We will work with the following set of assumptions:

a) $f$ satisfies Hypothesis 5.2.4,

b) $f$ satisfies condition ($\mathcal{H}_1$),

c) if the dimension $N$ of $\Omega$ is strictly greater than 1, $f$ satisfies condition ($\mathcal{H}_2$),

d) $f^{**}$ is a Carathéodory function.

We introduce a technical lemma about an anti-jump condition to impose on the original Lagrangian $f$ to obtain the condition ($\mathcal{H}_1$) for $f^{**}$.

**Lemma 5.4.3.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a measurable function. Then for any $\varepsilon > 0$,*
$$((f^{**})_\varepsilon^-)^{**} = (f_\varepsilon^-)^{**}.$$

The notation in this lemma is as introduced in (5.4.1)

*Proof.* Since $f^{**} \le f$, then
$$((f^{**})^-_\varepsilon)^{**} \le (f^-_\varepsilon)^{**} \tag{5.4.2}$$
To show the other inequality, we first recall that, using Proposition 5.2.10,
$$(f^-_\varepsilon)^{**}(x,u,\xi) = \inf\Big\{ \sum_i \alpha_i \Big( \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} f(y,u,\xi_i)\Big) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\},$$
and
$$(f^{**})^-_\varepsilon(x,u,\xi) = \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} \Big( \inf\Big\{ \sum_i \alpha_i f(y,u,\xi_i) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\}\Big).$$
For every convex combination we have that
$$\sum_i \alpha_i \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} f(y,u,\xi_i) \le \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} \sum_i \alpha_i f(y,u,\xi_i),$$
and
$$\begin{aligned} &\inf\Big( \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} \Big\{ \sum_i \alpha_i f(y,u,\xi_i) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\}\Big) \\ &= \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} \Big( \inf\Big\{ \sum_i \alpha_i f(y,u,\xi_i) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\}\Big), \end{aligned}$$
and so,
$$\begin{aligned} &\inf\Big\{ \sum_i \alpha_i \Big( \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} f(y,u,\xi_i)\Big) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\} \\ &\le \operatorname*{ess\,inf}_{y\in\Omega\cap B_\varepsilon(x)} \Big( \inf\Big\{ \sum_i \alpha_i f(y,u,\xi_i) \,\Big|\, \sum_i \alpha_i\xi_i = \xi \Big\}\Big), \end{aligned}$$
that is, $(f^-_\varepsilon)^{**} \le (f^{**})^-_\varepsilon$. Since $(f^-_\varepsilon)^{**}$ is a convex function with respect to $\xi$, we have
$$(f^-_\varepsilon)^{**} \le ((f^{**})^-_\varepsilon)^{**},$$
thus with (5.4.2),
$$(f^-_\varepsilon)^{**} = ((f^{**})^-_\varepsilon)^{**}.$$
□

**Lemma 5.4.4.** *Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty)$ be a measurable function. Then if $f$ satisfies condition $(\mathcal{H}_1)$, $f^{**}$ satisfies it as well.*

*Proof.* By the previous lemma we can rewrite condition $(\mathcal{H}_1)$ for $f$ as
$$((f^{**})^-_\varepsilon)^{**}(x,u,\xi) \le \frac{L_2}{\varepsilon^n} \Rightarrow f(x,u,\xi) \le C_L(1 + ((f^{**})^-_\varepsilon)^{**}(x,u,\xi))$$
and since $f^{**} \le f$ we have
$$((f^{**})^-_\varepsilon)^{**}(x,u,\xi) \le \frac{L_2}{\varepsilon^n} \Rightarrow f^{**}(x,u,\xi) \le C_L(1 + ((f^{**})^-_\varepsilon)^{**}(x,u,\xi)).$$
□

Now we can extend Theorem 5.3.5 to the non-autonomous case.

**Theorem 5.4.5.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ satisfy Hypothesis 5.4.2, and let $\varphi \in W^{1,\infty}(\Omega)$. Then for any $u \in W^{1,1}_\varphi(\Omega)$, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

*Moreover,*

- *if there exists $\Phi : \mathbb{R}^N \to [0, \infty)$ superlinear such that*

$$f(x, u, \xi) \geq \Phi(\xi), \qquad \text{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

*then the sequence $(u_n)$ can be chosen so that $u_n \rightharpoonup u$ weakly in $W^{1,1}(\Omega)$;*

- *if for some $p \in (1, \infty)$ it holds that*

$$f(x, u, \xi) \geq c_1|\xi|^p - c_2, \qquad \text{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

*for some $c_1, c_2 > 0$, and $u \in W^{1,p}(\Omega)$ then the sequence $(u_n)$ can be taken so that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

*Proof.* Just as in the proof of Theorem 5.3.5, we can assume without loss of generality that $E[f^{**}](u) < +\infty$. By assumptions $f^{**}$ is a Carathéodory function and if $N \geq 2$, for $|u| \geq u_0$ and a.e. $x \in \Omega$,

$$f^{**}(x, u, 0) \leq f(x, u, 0) \leq a(x)|u|^{\frac{p^*}{\theta r}},$$

thus $f^{**}$ satisfies $(\mathcal{H}_2)$ in this case. By Lemma 5.4.4 and the fact that $f$ satisfies $(\mathcal{H}_1)$, for every $L = (L_1, L_2) \in (0, \infty)^2$, there exists a constant $C_L > 0$ such that for a.e. $x \in \Omega$ and every $(u, \xi) \in [-L_1, L_1] \times \mathbb{R}^N$, for every $\varepsilon > 0$ it holds

$$((f^{**})^-_\varepsilon)^{**}(x, u, \xi) \leq \frac{L_2}{\varepsilon^N} \Longrightarrow f^{**}(x, u, \xi) \leq C_L(1 + ((f^{**})^-_\varepsilon)^{**}(x, u, \xi)).$$

Thus we can apply Theorem 5.4.1 to $f^{**}$ (which is a Carathéodory function by 5.4.2-d)) and so, for every $u \in W^{1,1}_\varphi(\Omega)$ there exists a sequence $(v_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that $v_n \to u$ in $W^{1,1}(\Omega)$ and

$$\lim_{n\to+\infty} \int_\Omega f^{**}(x, v_n(x), \nabla v_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

By Theorem 5.2.20, for every $n$ there exists a sequence $(v_n^k) \subset W^{1,\infty}_\varphi(\Omega)$ such that $v_n^k \to u$ in $L^\infty(\Omega)$ as $k \to \infty$, and

$$\lim_{k\to+\infty} \int_\Omega f(x, v_n^k(x), \nabla v_n^k(x))dx = \int_\Omega f^{**}(x, u_n(x), \nabla u_n(x))dx.$$

Thus, using a diagonal argument, there exists a sequence $(u_n) \subset W^{1,\infty}_\varphi(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ and

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))dx = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx.$$

If there exists $\Phi$ superlinear such that

$$\sup_n \int_\Omega \Phi(\nabla u_n(x))dx \leq \sup_n \int_\Omega f(x, u_n(x), \nabla u_n(x))dx < +\infty,$$

then, up to considering a subsequence, $(\nabla u_n)$ converges weakly in $L^1$ to some $v \in L^1$. It is easy to see (using an integration by parts argument) that actually $v = \nabla u$. So we have

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,1}(\Omega).$$

If $p > 1$ the proof is the same taking $\Phi(\xi) = c_1|\xi|^p - c_2$. □

At this point, we finally generalize Theorem 5.3.9 to the non-autonomous case.

**Theorem 5.4.6.** *Let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ *satisfy Hypothesis 5.4.2, and let* $\varphi \in W^{1,\infty}(\Omega)$*. Assume* $u \in W^{1,p}_{\varphi}(\Omega)$ *for some* $p \in [1, \infty)$*, and satisfies*

$$\int_{\Omega} f^{**}(x, u(x), \nabla u(x))dx = \int_{\Omega} f(x, u(x), \nabla u(x))dx.$$

*Then there exists a sequence* $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ *such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x))dx = \int_{\Omega} f(x, u(x), \nabla u(x))dx.$$

*Proof.* We work as in the proof of Theorem 5.3.9. For the case $p = 1$, we make use again of Lemma 5.3.8 to get the existence of some $\Phi : \mathbb{R}^N \to [0, \infty)$ superlinear and convex such that $\int_{\Omega} \Phi(\nabla u(x))dx < +\infty$. Then we define

$$g(x, u, \xi) := f(x, u, \xi) + \Phi(\xi) + \sqrt{1 + |\xi|^2}.$$

The only thing we have to show is that, under the assumption that $f$ satisfies $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, $g$ satisfies them as well. The remainder of the proof will follows then just as in Theorem 5.3.9. Notice that, by definition of $g^-_\varepsilon$ and the fact that $g - f$ does not depend on $x$ and $u$, it holds

$$g^-_\varepsilon(x, u, \xi) = f^-_\varepsilon(x, u, \xi) + \Phi(\xi) + \sqrt{1 + |\xi|^2}.$$

Therefore we have, from the fact that $\Phi$ was assumed to be convex, that

$$(g^-_\varepsilon)^{**}(x, u, \xi) \geq (f^-_\varepsilon)^{**}(x, u, \xi) + \Phi(\xi) + \sqrt{1 + |\xi|^2}, \tag{5.4.3}$$

by definition of $(g^-_\varepsilon)^{**}$. By Hypothesis 5.4.2, given $L = (L_1, L_2) \in (0, \infty)^2$, there exists a constant $C_L > 0$ such that for a.e. $x \in \Omega$ and every $(u, \xi) \in [-L_1, L_1] \times \mathbb{R}^N$, for every $\varepsilon > 0$, it holds

$$(f^-_\varepsilon)^{**}(x, u, \xi) \leq \frac{L_2}{\varepsilon^N} \quad \Rightarrow \quad f(x, u, \xi) \leq C_L(1 + (f^-_\varepsilon)^{**}(x, u, \xi)).$$

If for a.e. $x \in \Omega$ and every $(u, \xi) \in [-L_1, L_1] \times \mathbb{R}^N$ for every $\varepsilon > 0$,

$$(g^-_\varepsilon)^{**}(x, u, \xi) \leq \frac{L_2}{\varepsilon^N},$$

then $(f^-_\varepsilon)^{**}(x, u, \xi) \leq L_2/\varepsilon^N$ by (5.4.3), and thus

$$f(x, u, \xi) \leq C_L(1 + (f^-_\varepsilon)^{**}(x, u, \xi)).$$

So, taking $\tilde{C} = \max(1, C_L)$, we have by (5.4.3),

$$\begin{aligned} g(x, u, \xi) &\leq C_L(1 + (f^-_\varepsilon)^{**}(x, u, \xi)) + \Phi(\xi) + \sqrt{1 + |\xi|^2} \\ &\leq \tilde{C}(1 + (g^-_\varepsilon)^{**}(x, u, \xi)), \end{aligned}$$

and thus $g$ satisfies $(\mathcal{H}_1)$. Now in the case $N \geq 2$, since $f$ satisfies condition $(\mathcal{H}_2)$ there exists $C' > 0$ such that

$$f(x, u, 0) + 1 + \Phi(0) \leq C' a(x)|u|^{\frac{p^*}{\theta'}}, \qquad \text{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R} \setminus [-u_0, u_0].$$

Here, up to modify $a$ by taking $\tilde{a} := \max(a, 1)$, we assumed that the function $a$ in condition $(\mathcal{H}_2)$ satisfies $a(x) \geq 1$ on $\Omega$.By Theorem 5.2.17 we have that $g^{**}$ is a Carathéodory function since $g$ is superlinear with respect to the last variable. So $g$ satisfies Hypothesis 5.4.2, thus we can apply Theorem 5.4.5 to $g$ and argue as in the proof of Theorem 5.3.9. If $p > 1$ the proof is the same replacing $\Phi(\xi) + \sqrt{1 + |\xi|^2}$ by $|\xi|^p$.

□

In [14, Appendix A] there are some applications of Theorem 5.4.1 to the case $f(x, u, \cdot)$ nonconvex but dominated by a convex function $g$. The following result follows this idea:

**Theorem 5.4.7.** *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a Carathéodory function, and let $\varphi \in W^{1,\infty}(\Omega)$. Let $u \in W^{1,p}_{\varphi}(\Omega)$ for some $p \in [1, \infty)$ and assume that there exists $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ a Carathéodory function satisfying the assumptions of Theorem 5.4.1 and $a \in L^1(\Omega, [0, \infty))$ such that*

$$f(x, u, \xi) \leq g(x, u, \xi) + a(x), \qquad \textit{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

*and*

$$\int_{\Omega} g(x, u(x), \nabla u(x))dx < +\infty.$$

*Then there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in} \quad W^{1,p}(\Omega),$$

*and*

$$\lim_{n\to+\infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x))dx = \int_{\Omega} f(x, u(x), \nabla u(x))dx.$$

*Proof.* Applying Theorem 5.4.1 to $g$, there exists a sequence $(u_n) \subset W^{1,\infty}_{\varphi}(\Omega)$ converging to $u$ in $W^{1,p}(\Omega)$ such that

$$\lim_{n\to+\infty} \int_{\Omega} g(x, u_n(x), \nabla u_n(x)) + a(x)dx = \int_{\Omega} g(x, u(x), \nabla u(x)) + a(x)dx < +\infty.$$

The reminder of the proof follows just as in Theorem 5.3.14, using the fact that $f$ is Carathéodory and the Fatou Lemma. □

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