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Steiner's and Weyl's Tube Formulae in Sub-Riemannian Geometry

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Abstract

The objective of this thesis is to explore the geometric properties of submanifolds within sub-Riemannian structures through the study of the volume of their tubular neighborhoods. In particular, we aim at investigate the geometric information carried by the coefficients of the asymptotics of the latter volume as the size tends to zero.

First of all, we consider the case of smooth surfaces embedded in three-dimensional contact sub-Riemannian manifolds and that do not contain characteristic points. We derive a Steiner-like formula computing the asymptotics of the volume of the half-tubular neighborhoods up to the third order and with respect to any smooth measure.

Subsequently, we extend the Weyl's tube formula for non-characteristic submanifolds of class C^2 with arbitrary codimension, and the Steiner's formula for non-characteristic hypersurfaces of class C^2 in any sub-Riemannian manifold equipped with a smooth measure. The volume of the tubular and half-tubular neighborhoods is a smooth or real-analytic function of the size whenever the ambient sub-Riemannian structure and the assigned measure are smooth or real-analytic, respectively. Moreover, the coefficients of the Taylor expansion are written in terms of integrals of iterated divergences of the distance from the submanifold.

Finally, we present a result concerning the local integrability of the sub-Riemannian mean curvature of a surface in a 3D contact sub-Riemannian manifold in presence of isolated characteristic points. In particular, the main result focuses on the integrability of the mean curvature around the so-called mildly degenerate points and with respect to the Riemannian induced measure.

Sommario

Lo scopo di questa tesi è studiare le proprietà geometrice di sottovarietà nel contesto di strutture sub-Riemanniane attraverso lo studio del volume dei loro intorni tubolari. In particolare, l'obiettivo è di scoprire l'informazione geometrica contenuta nei coefficienti dell'espansione del volume del tubo quando il suo spessore tende a zero.

Per cominciare, si considera il caso di una superficie liscia in una varietà sub-Riemanniana di contatto tridimensionale e che non contiene punti caratteristici. In questo contesto si dimostra una formula di Steiner calcolando l'espansione del volume del mezzo intorno tubolare fino al terzo ordine e considerando una misura liscia arbitraria.

Successivamente, si generalizzano la formula di Weyl per sottovarietà C^2-regolari senza punti caratteristici e codimensione arbitraria e la formula di Steiner per ipersuperfici C^2-regolari e senza punti caratteristici in una qualsiasi struttura sub-Riemanniana con una misura liscia. Tali volumi sono lisci o analitici in funzione dello spessore a seconda che la varietà ambiente e la misura assegnata siano liscie o analitiche, rispettivamente. Inoltre, i coefficienti dell'espansione di Taylor sono espressi in termini di integrali delle divergenze iterate della funzione distanza dalla sottovarietà.

Infine, viene presentato un risultato di integrabilità locale della curvatura media sub-Riemanniana di una superficie in una varietà sub-Riemanniana di contatto tridimensionale in presenza di punti caratteristici isolati. In particolare, il risultato principale riguarda l'integrabilità della curvatura media intorno a punti finitamente degeneri e rispetto alla misura Riemanniana indotta sulla superficie.

Contents

Introduction

The exploration of the geometric properties of submanifolds within an ambient manifold, endowed with a prescribed geometric structure, constitutes a classical subject. A familiar example, whose study goes back to Gauss, is that of a surface embedded in the Euclidean three-dimensional space. In such conditions, the surface inherits a natural Riemannian structure by restricting the ambient metric tensor to the tangent space of the surface. The situation is less straightforward in the case the ambient space is a sub-Riemannian manifold. Indeed, the restriction of the geometric structure on the submanifold fails at producing a sub-Riemannian submanifold.

The aim of this thesis is to investigate geometrical properties of a submanifold embedded in a sub-Riemannian space trough the study of the volume of its tubular neighborhood. In particular, we extract geometrical information computing the asymptotics of the volume as the size of the tube tends to zero.

Let S be a compact submanifold of codimension m embedded in $\mathbb{R}^n$. The tube of size $r > 0$ around S, denoted with S_r, is defined as the set of points at euclidean distance at most r from S. Weyl's tube formula (cf. [**Wey39**]) states that the Lebesgue volume of S_r is expressed by the following polynomial of degree n in r:

$$\mathscr{L}^n(S_r) = \mathscr{L}^m(B(r,\mathbb{R}^m)) \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \frac{a_{2k}(S)}{(m+2)\dots(m+2k)} r^{2k}, \tag{0.1}$$

where $B(r,\mathbb{R}^m)$ is the Euclidean ball in $\mathbb{R}^m$ with radius r. The remarkable feature of these coefficients is that they are independent of the particular way the submanifold is (isometrically) embedded in the ambient space. To prove this fact Weyl finds that the coefficients are integrals of certain functions expressed in terms of the second fundamental form of the submanifold. The first term $a_m(S)$ is the volume of the S and $a_{m+2}(S)$ is the integral over S of the scalar curvature. In the case when S is an even-dimensional submanifold, the top coefficient is especially interesting: it is given by $(2\pi)^{\frac{m}{2}}\chi(S)$ where $\chi(S)$ is the Euler characteristic of S.

In the case S is a compact and oriented hypersurface, its tubular neighborhood can be decomposed in two disjoint *half-tubular neighborhoods* separated by the hypersurface itself. This is the case when S is the boundary of a regular domain Ω in $\mathbb{R}^n$. Therefore, the half-tube formula is naturally related with the Steiner's formula. Let $\Omega \subset M$, the r-tubular neighborhood of Ω is the set of points in M at distance at most r from Ω. The celebrated Steiner's formula states that the Lebesgue volume of the r-neighborhood $\Omega(r)$ of a bounded regular domain $\Omega \subset \mathbb{R}^n$ can be expressed as a polynomial of degree n in r

$$\mathscr{L}^n(\Omega(r)) = \mathscr{L}^n(\Omega) + \sum_{k=1}^{n} a_k(\Omega) r^k, \tag{0.2}$$

where the $a_k = a_k(\Omega)$ are coefficients known as Minkowski's quermassintegrals. In this context, we understand the external half-tube of S to be $\Omega(r) \setminus \Omega$.

While in (0.1) the coefficients a_{m+2k+1} are vanishing, actually this is not the case for (0.2). The reason lies in the fact that the volume of the half-tube depends on how the hypersurface is embedded in the ambient. In the case of a compact surface $S = \partial\Omega$ bounding a regular domain Ω in $\mathbb{R}^3$ the coefficients a_1, a_2, a_3 are computed explicitly as follows

$$\mathscr{L}^3(\Omega(r) \setminus \Omega) = r \int_S dA - r^2 \int_S H dA + \frac{r^3}{3} \int_S K dA, \tag{0.3}$$

where H is the mean curvature of S, computed with respect to the orientation of the surface given by the outward pointing normal of Ω, and K is the Gaussian curvature of S, while dA denotes the surface measure. In (0.3) the coefficient containin the mean curvature is the one that carries information about the embedding.

The study of tubular volumes has a long history and is closely tied to the development of convex and Riemannian geometry throughout the twentieth century. The original work of Steiner proves formula (0.2) for convex sets in the Euclidean spaces of dimension 2 and 3 [**Ste40**]. In [**Fed59**], Federer proved a localized version of the formula for a large class of non-smooth submanifolds, introducing the concept of sets of positive reach.

In the Riemannian setting the tube and half-tube formulas are not anymore polynomial. However, the coefficients arising in the asymptotics as the size of the tube tends to zero carry information about the geometric structure of the submanifold. In the 1970s and 1980s, Gray and Vanhecke made an extensive study of power series expansions for Riemannian tubular volumes (cf. [**GV82**, **GV83**]).

We refer the reader to the monograph of A. Gray [**Gra04**] for an exhaustive overview of this subject. Tubular neighborhoods arise in metric geometry in various ways. The Hausdorff distance between (closed, bounded) sets is defined using tubular neighborhoods. Moreover, Weyl's intrinsic formula for the volumes of tubular neighborhoods of Euclidean submanifolds was exploited to deduce higher-dimensional generalizations of the Gauss–Bonnet theorem by Allendoerfer and Weil in [**AW43**], and later by Chern in [**Che44**].

The study of sub-Riemannian geometry is motivated by applications in fields such as control theory and robotics, where systems have constraints on their movements. The geometry of sub-Riemannian manifolds plays a crucial role in understanding the optimal paths or trajectories within these constrained systems. Indeed, the distance is defined as the infimum of the lengths of all the curves connecting two points and satisfying some non-holonomic constraints. For instance, it appears also in the field of cognitive neuroscience to model the functional architecture of the area V1 of the primary visual cortex, as proposed by Petitot, Citti and Sarti (cf. [**Pet17**, **CS06**]).

A sub-Riemannian structure on a smooth manifold M is characterized by a pair (D, g), where g is a smoothly varying scalar product defined on a subset of preferred directions within the tangent bundle

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \dots, X_N(x)\} \subset T_x M, \qquad \forall x \in M.$$

Even though the rank of $\mathcal{D}$ may depend on the point, we refer to it as the *horizontal distribution*. In order to define the distance between two points we consider all the curves joining them and satisfying the property that the velocity is a combination of the directions in the distribution. Notice that, since the distribution may not cover the entire tangent space, a priori there might exist points not connected by any such curves. Therefore, it is crucial to require that the Lie algebra generated by the distribution has to fill the entire tangent bundle. In control theory, the latter is known as Hörmander's condition and it guarantees that for any pair of points in the same connected component there exists at least one horizontal curve joining them. Consequently, it becomes meaningful to define the sub-Riemannian distance between two points as the infimum over the lengths of admissible curves joining the assigned endpoints. The Chow-Rashevsky's Theorem encodes the controlability result and it states that the sub-Riemannian distance is finite. Moreover, the topology induced by the sub-Riemannian metric is equivalent to the original manifold's topology. In this context, Riemannian manifolds constitute a class of special sub-Riemannian structures (the ones in which the horizontal distribution coincides with the whole tangent bundle).

Proper sub-Riemannian spaces demonstrate remarkably different phenomena with respect to the Riemannian counterparts. Firstly, there exist points where it is not bi-Lipschitz equivalent to a Euclidean space, and as a consequence its Hausdorff dimension is strictly greater than the topological one in a neighborhood of those points. Moreover, sub-Riemannian manifolds are geodesic metric spaces, but not all the geodesics are characterized as in the Riemannian case. There are two classes of length minimizing curves: those which behave as in the Riemannian setting, called *normal*, and those that not, the *abnormal* ones. The abnormal curves are connected to several open problems in the field. Finally, since the scalar product associated to the metric is not defined on the whole tangent space, there is not an automatic way to define an intrinsic volume and a metric connection.

Since the sub-Riemannian distance between points is not smooth, in order to characterize the geometry of tubular neighborhoods, the analysis of the sub-Riemannian distance function from a subset plays a fundamental role. This topic is considered also in reference to the *isoperimetric inequalities*, that is the study of the relation between quantities representing the volume and the perimeter of regions in metric spaces (cf. [**Pan82**, **CDPT07**, **CY09**]).

In order to define geometric invariants in sub-Riemannian geometry a possible approach is to exploit the tool of Riemannian approximations. Indeed, a sub-Riemannian metric space is the limit, in the sense of the pointed Gromov-Hausdorff convergence, of suitable Riemannian metric structures defined on the same manifold. Hence, the Riemannian approximating technique consists in computing the well known Riemannian geometric objects in the approximating spaces and then define the corresponding sub-Riemannian ones taking the limit with respect to the approximation.

In the case that the distribution is of constant rank, the approximating scheme is naturally implemented. More precisely, one considers a maximal set of independent vector fields transversal to the distribution. Then, for $\varepsilon > 0$ the Riemannian metric g^ε on M is defined such that it is an extension of the sub-Riemannian metric g and setting the assigned transversal vector fields to be pairwise orthogonal, orthogonal to the distribution, and with norm $\frac{1}{\varepsilon}$. Indicating with d^ε the corresponding approximating Riemannian distace, it holds that $\mathsf{d}^\varepsilon \to \mathsf{d}$ uniformly on compacts in $M \times M$ as $\varepsilon \to 0$.

In [**BTV17**, **BTV20**] the Riemannian approximations technique is exploited in order to define a Gaussian curvature of surfaces in three-dimensional Heisenberg group with the goal to deduce a Gauss-Bonnet theorem. Then, in [**Vel20**, **WW20**] for three-dimensional sub-Riemannian manifolds

A powerful tool to investigate geometrical properties in Riemannian context is the asymptotic analysis. Hence, making use of this technique represents a classical approach in sub-Riemannian setting, especially when dealing with curvature-related objects, as it is for the present thesis. Indeed, the concept of curvature in Sub-Riemannian geometry is a bit different from the more familiar Riemannian curvature, which is typically described by a curvature tensor which involves the covariant derivatives of vector fields. In [**ABR18**] the authors propose a definition of the sub-Riemannian curvature analysing the asymptotic expansion of the square of the distance from a point along a geodesic (see also [**ABP19**]). Furthermore, the asymptotic procedure is exploited in the study of the heat kernel [**Bar13**, **RR21**, **ARR23**, **TW18**]. Here, we are interested in understanding how analytic properties of the tubular volume function reflects the intrinsic and extrinsic geometry of the submanifold.

Structure and contents of the thesis

In this thesis we prove several results about the Weyl's and Steiner's formulas in sub-Riemannian geometry contained in [**BB23**] and [**BRR24**].

First of all, we introduce some basic notions of sub-Riemannian geometry in Chapter 1. In particular, we present in detail the setting of a three-dimensional contact sub-Riemannian manifold. Moreover, we describe the Tanno connection, that is a metric connection representing a technical tool exploited in Chapter 2.

In Chapter 2 we investigate the validity of Steiner like (or half-tube like) formulas, considering the case of half-tubes built over smooth surfaces in three-dimensional sub-Riemannian contact manifolds and that do not contain characteristic points. We provide a geometrical interpretation of the coefficients appearing in the expansion, generalizing the results obtained in [**BFF$^+$15**, **Rit21**] for surfaces which are embedded in the Heisenberg group. Finally, we show some examples of application of the formula considering rotational surfaces in the Heisenberg group and some particular surfaces in the model spaces $SU(2)$ and $SL(2)$.

In Chapter 3 we present the generalization of the Weyl's tube formula for C^2 non-characteristic submanifolds with arbitrary codimension and of the Steiner's formula for C^2 non-characteristic hypersurfaces embedded in any sub-Riemannian manifold. The tubular-volume functions are smooth or real-analytic if the ambient sub-Riemannian structure and the assigned measure are smooth or real-analytic, respectively. Moreover, the coefficients of the expansion are written in terms of integrals of iterated divergences of the distance from the manifold.

Finally, in Chapter 4 we present a result about the (local) integrability of the sub-Riemannian mean curvature of a surface in a 3D contact sub-Riemannian manifold in presence of isolated characteristic points. In particular, the integrability is intended to be with respect to the Riemannian induced measure and it is analyzed in detail the case of degenerate points having the property of being mildly degenerate. The latter condition, as well as the related results, generalizes the one presented in [**Ros21**] in the setting of the three-dimensional Heisenberg group.

Introduction to Chapter 2

A three-dimensional contact sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$ where M is a three-dimensional manifold, $\mathcal{D}$ a bracket generating two-dimensional distribution in the tangent bundle, locally obtained as the kernel of a one-dimensional smooth form ω such that $\omega \wedge \mathrm{d}\omega \neq 0$, and g a smooth metric defined on the distribution such that the two-dimensional induced volume form vol_g coincides with the contact two-dimensional form $\mathrm{d}\omega|_{\mathcal{D}}$. The three-dimensional form $\omega \wedge \mathrm{d}\omega$ is then a canonical volume form associated with the contact structure and the volume measure ν is called the *Popp's volume* (for more details see [**BR13**]).

Let M be a three-dimensional contact sub-Riemannian manifold and let S be a surface that does not contain characteristic points (namely $\mathcal{D}_x \neq T_xS$ for every $x \in S$) and that bounds a closed regular domain $\Omega \subset M$. Moreover, let μ be a smooth measure on the manifold M, which in general can be written as $\mu = h\nu$ for a smooth positive function h. We study the measure μ of the sub-Riemannian half-tube built over S.

Given S a hypersurface with no characteristic points, the sub-Riemannian normal N to the surface S is a vector field that generates the unique direction in the distribution orthogonal to T_xS (which is well-defined thanks to the condition $\mathcal{D}_x \neq T_xS$). If the surface S is locally defined as $S = f^{-1}(0)$ where $f : M \to \mathbb{R}$ is a smooth function with $\mathrm{d}f \neq 0$, the vector field N on S is parallel to the horizontal gradient $\nabla_H f$ and of unit norm with respect the metric g. Therefore, N is unique up to a sign. Recall that the horizontal gradient $\nabla_H f$ is the unique vector field tangent to the distribution $\mathcal{D}$ that satisfies the identity

$$\mathrm{d}f(X) = g(\nabla_H f, X)$$

for every vector field X tangent to $\mathcal{D}$. Given an orthonormal basis X_1, X_2 on $\mathcal{D}$, it holds that $\nabla_H f = (X_1 f)\, X_1 + (X_2 f)\, X_2$. Also, we denote by X_0 the Reeb vector field associated with the contact form, whichis the unique, up to a sign, vector field transversal to $\mathcal{D}$ such that $\omega(X_0) = 1$).

Let us denote with dV_μ the 3-dimensional smooth volume form on M associated with μ. Namely, dV_μ is such that $\mu(B) = \int_B dV_\mu$ for every $B \subset M$ measurable. Furthermore, we can define a a smooth measure on the surface S associated with μ, that is the *induced sub-Riemannian area* σ_μ: for a measurable set $U \subset S$, we set

$$\sigma_\mu(U) = \int_U dA_\mu, \qquad \text{where} \qquad dA_\mu := \iota_N dV_\mu \tag{0.4}$$

is the 2-dimensional volume form on S obtained by restricting the volume dV_μ with respect to N.

The 3-dimensional volume form dV_μ and the sub-Riemannian area form induced on a hypersurface are related by the following sub-Riemannian version of the coarea formula, which plays a crucial role in the proof of our result. Given a measurable function $f : M \to \mathbb{R}$ and a smooth map $\Phi : M \to \mathbb{R}$ such that $\nabla_H \Phi \neq 0$ on M, it holds that

$$\int_M f \, \|\nabla_H \Phi\| \, dV_\mu = \int_{\mathbb{R}} \int_{\Phi^{-1}(t)} f|_{\Phi^{-1}(t)} \, dA^t_\mu \, dt,$$

where dA^t_μ is the induced sub-Riemannian area form on the surface $\Phi^{-1}(t)$. This formula, which involves only sub-Riemannian quantities, can be obtained by its classical Riemannian counterpart (cf. Proposition 39 and its proof).

Finally, the *sub-Riemannian mean curvature of the surface* S *associated with* μ is the smooth function $\mathcal{H}_\mu$ defined as

$$\mathcal{H}_\mu = -\mathrm{div}_\mu \left(\nabla_H \delta\right). \tag{0.5}$$

where δ denotes the sub-Riemannian signed distance function from the surface S, cf. Definition 23 for more details.

We stress that (0.5) defines $\mathcal{H}_\mu$ not only on S but in a neighborhood of the surface. In particular, the derivative $N(\mathcal{H}_\mu)$ has a meaning.

When $\mu = \nu$ the measure associated with the Popp's volume form $\omega \wedge \mathrm{d}\omega$, we omit the dependence on the measure in the notation. For $\mathcal{H} := \mathcal{H}_\nu$ we obtain the following expression in terms of a local frame (cf. Lemma 34)

$$\mathcal{H} = -X_1 X_1 \delta - X_2 X_2 \delta - c_{12}^2 \left(X_1 \delta\right) + c_{12}^1 \left(X_2 \delta\right).$$

The main result of this chapter is the following formula for the measure of the localized half-tubular neighborhood. Namely, for $U \subset S$ we consider the subset U_r of all points $x \in M \setminus \Omega$ for which the curve realizing the distance of the point x from the surface S has an end point in U.

Theorem 1. *Let $(M, \mathcal{D}, g)$ be a contact three-dimensional sub-Riemannian manifold equipped with a smooth measure μ. Let $S \subset M$ be an embedded smooth surface bounding a closed region Ω and let $U \subset S$ be an open and relatively compact set such that its closure $\overline{U}$ does not contain characteristic points. The volume of the localized half-tubular neighborhood U_r, is smooth with respect to r and satisfies for $r \to 0$:*

$$\mu\left(U_r\right) = \sum_{k=1}^{3} a_k(U, \mu) \frac{r^k}{k!} + o(r^3) \tag{0.6}$$

where the coefficients $a_k = a_k(U, \mu)$ have the following expressions:

$$a_1 = \int_U dA_\mu, \quad a_2 = -\int_U \mathcal{H}_\mu \, dA_\mu, \quad a_3 = \int_U \left(-N\left(\mathcal{H}_\mu\right) + \mathcal{H}_\mu^2\right) dA_\mu.$$

Here dA_μ is the sub-Riemannian area measure on S induced by μ, $\mathcal{H}_\mu$ is the mean sub-Riemannian curvature of S with respect to μ and N is the sub-Riemannian normal to the surface.

Remark 2. Since our result is local, the requirement on S to be the boundary of a regular closed region Ω can be relaxed. In fact, one can consider any smooth surface S and it is always possible to build a local half-tubular neighborhood around a subset $U \subset S$ open, oriented and relatively compact such that $\overline{U}$ does not contain any characteristic point. This can be done in a similar way as in [**Rit21**].

Remark 3. In contrast to the Euclidean case presented in (0.2), the formula (0.6) in Theorem 1 is not a polynomial in r. In the Euclidean case the polynomial expansion is related to a specific choice of the volume, the Lebesgue one. In the sub-Riemannian case, even for the choice of the natural volume this is not true. Indeed, as proved in [**BFF$^+$15**], in the first Heisenberg group $\mathbb{H}$ equipped with the Popp's measure ν the volume of U_r is analytic in r. Considering for instance as U an annulus around the origin in S the horizontal plane, one can compute all terms of the Taylor series and check that this is not a polynomial, see [**BFF$^+$15**, Example 5.1].

Remark 4. The sub-Riemannian mean curvature $\mathcal{H}$ can be equivalently defined as the limit of the Riemannian mean curvatures H^ε of S with respect to the Riemannian extension g^ε converging to the sub-Riemannian metric imposing that the Reeb vector field is orthogonal to $\mathcal{D}$ and has norm $1/\varepsilon$. The same conclusion holds also for $\mathcal{H}_\mu$ when the smooth measure μ is associated to a Riemannian extension which is not necessarily defined through the Reeb vector field. This result is proved in [**BTV17**, **BTV20**] for surfaces in the Heisenberg group equipped with the Popp's measure. See Section 1.3 for the analogue statement in the general case.

To compute the coefficients of expansion (0.6) one a priori needs the knowledge of the explicit expression of the sub-Riemannian distance, which in general is not possible. We provide a formula which permits to compute those coefficients only in terms of a function f locally defining S. We state the result in the case $\mu = \nu$ hence we omit the dependence on the measure in the notation.

Proposition 5. *Under the assumptions of Theorem 1, let us suppose that it is assigned the Popp's measure ν and that S is locally defined as the level set of a smooth function $f : M \to \mathbb{R}$ such that $\nabla_H f|_U \neq 0$ and $\langle \nabla_H f, \nabla_H \delta\rangle|_U > 0$. Then, we have the following formulas for a_2 and a_3 in expansion (0.6):*

$$a_2 = -\int_U \operatorname{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right) dA;$$

$$a_3 = \int_U \left[2X_S\left(\frac{X_0 f}{\|\nabla_H f\|}\right) - \left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2 - \kappa - \left\langle \operatorname{Tor}\left(X_0, X_S\right), \frac{\nabla_H f}{\|\nabla_H f\|}\right\rangle\right] dA$$

with $\kappa = \langle R(X_1, X_2)X_2, X_1\rangle$ where R and Tor *are the curvature and the torsion operators associated with the Tanno connection and the operative expression* *for the characteristic vector field is*

$$X_S = \frac{X_2 f}{\|\nabla_H f\|} X_1 - \frac{X_1 f}{\|\nabla_H f\|} X_2.$$

We stress that it is not a priori clear that it is possible to write such an expression since, looking for example at formula (0.7), the coefficient a_3 depends on a derivative of $\mathcal{H}$ along N, which is a vector field transversal to the surface. Hence, it might depend on the value of $\mathcal{H}$ outside S, as it happens for higher order coefficients (cf. Remark 50 for a comment in the Heisenberg group).

Remark 6. We notice that the second order coefficient a_2 appearing in (0.3) and the corresponding one a_2 of expansion (0.6) are both integral of a mean curvature[1]. One may then wonder if the same analogy holds for the third order coefficient a_3. Namely, if the coefficient a_3 of expansion (0.6) is the integral of a suitably defined sub-Riemannian Gaussian curvature of the surface appearing in (0.3).

In [**BTV17**, **BTV20**] the authors define the sub-Riemannian Gaussian curvature $\mathcal{K}_S$ of a surface S in the Heisenberg group $\mathbb{H}$ through Riemannian approximations. The expression of the limit, written in our notation is

$$\mathcal{K}_S = X_S(X_0\delta) - (X_0\delta)^2,$$

It can be checked that the integral of this quantity does not correspond to the third coefficient a_3, that thanks to Proposition 5 rewrites as follows

$$a_3 = \int_U 2X_S\left(X_0\delta\right) - \left(X_0\delta\right)^2\, dA. \tag{0.8}$$

being $\kappa = 0$ and Tor the null operator in the Heisenberg group. Our result agrees with the expression obtained by Ritoré in [**Rit21**].

We exploit Proposition 5 to provide some examples. One can specialize the result in the case of surfaces embedded in some particular three-dimensional contact manifolds, considering the metric invariants χ and κ defining the local geometry of the sub-Riemannian structure. This coefficients correspond to the torsion and the horizontal scalar curvature of a natural affine connection and appear in the coefficients. For a three-dimensional contact manifold with $\chi = 0$ and equipped with the Popp's measure ν, we have that

$$a_3 = \int_U 2X_S(X_0\delta) - (X_0\delta)^2 - \kappa\, dA.$$

Moreover, chosen in terms of a canonical basis X_1, X_2 for a left invariant sub-Riemannian contact structure on a Lie group with $\chi \neq 0$ (see for instance see [**ABB20**, Prop. 17.14]) and again equipped with the Popp's measure ν, it holds that

$$a_3 = \int_U 2X_S(X_0\delta) - (X_0\delta)^2 - \kappa + \chi\left((X_1\delta)^2 - (X_2\delta)^2\right)\, dA.$$

We stress that in the Heisenberg group $\kappa = \chi = 0$, recovering formula (0.8), while in general on contact manifolds the curvature and torsion operators associated with the Tanno connection are not necessarily vanishing.

[1]The extra factor 2 in the sub-Riemannian formula is due to the fact that in the Euclidean case one defines the mean curvature as one half of the sum of the two principal curvatures.

Finally, we compute expansion (0.6) in the case of a surface embedded in $\mathbb{H}$ with rotational symmetry with respect to the z-axis. In addition, we show that for "model" surfaces in the model spaces $SU(2)$ and $SL(2)$ (cf. [**BBCH21**], see also [**BH23**]), it holds $a_2 = a_3 = 0$ as for the horizontal plane in $\mathbb{H}$.

Introduction to Chapter 3

Let M be a sub-Riemannian manifold of dimension n equipped with a smooth measure μ. We say that a C^2 submanifold $S \subset M$ is non-characteristic if it holds that $\mathcal{D}_q + T_qS = T_qM$ for all $q \in S$. The goal of this chapter is to study the measure of the tubular neighborhood of S defined with respect to the sub-Riemannian distance d. Namely, we study the measure μ of

$$S_r = \{p \in M \colon 0 < \delta(p) = \mathsf{d}(p, S) < r\},$$

as $r \to 0$. In particular, when S is a C^2 hypersurface bounding a compact domain Ω, we derive the Steiner's formula for the measure of the half-tubular neighborhood $S_r^+ = \Omega \cap S_r$. (for the convenience of the reader we stress the fact that in Chapter 2 we refer to the external half-tube $S_r^+ = S_r \setminus \overline{\Omega}$).

Since we develop the theory for non smooth submanifolds, we state some regularity properties for the distance function δ. First of all, we show that if S is a C^2 non-characteristic submanifold, then δ, the sub-Riemannian distance function from S, satisfies the following properties

$$\delta \in C^2(S_{r_0}), \qquad \text{and} \qquad \delta^2 \in C^2(S_{r_0} \cup S).$$

Moreover, if the ambient sub-Riemannian structure and the assigned measure are smooth (or real-analytic), then any derivation of δ and δ^2 of order at most two is smooth (or real-analytic) along any *geodesic from* S and contained in S_{r_0}, i.e., along any length minimizing curve starting at S and whose length realizes δ from the end-point in S_{r_0} (see Lemma 62).

We recall that the *divergence* of a smooth vector field X on M is defined as the smooth function $\operatorname{div}(X)$ on M satisfying $\operatorname{div}(X)\,\mu = \mathcal{L}_X\mu$ (where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X). Hence, we consider the iterated divergences defined as

$$\operatorname{div}^0(X) = 1, \qquad \operatorname{div}^{k+1}(X) = \operatorname{div}(\operatorname{div}^k(X)X). \tag{0.9}$$

The iterated divergences of $\nabla_H\delta$ are defined as in (0.9), where the derivatives are meant in the sense of distributions.

Nevertheless, these quantities can be expressed in terms of derivatives of order at most two of δ, and hence they are well defined in a strong sense. More precisely, let $Y_1, \dots, Y_n$ be a local smooth (or real-analytic) frame for TM. Then, for all $k \in \mathbb{N}$ there exists a polynomial P^k, with smooth (or real-analytic) coefficients, homogeneous of degree k, in the variables $Y_\alpha\delta, Y_\alpha Y_\sigma\delta$ for $\alpha, \sigma = 1, \dots, n$ such that

$$\operatorname{div}^k(\nabla\delta)(p) = P^k\left(Y_\alpha\delta(p), Y_\alpha Y_\sigma\delta(p)\right).$$

The latter result (cf. Lemma 63) can be read as a generalization of [**BFF$^+$15**, Thm. 1.2]. The main difference is that here we cannot rely on a closed formula for the iterated divergences that holds in the Heisenberg group.

As a consequence, $\mathrm{div}^k(\nabla_H\delta) \in C(S_{r_0})$ for every $k \in \mathbb{N}$, it is smooth (or real-analytic) along any geodesic from S, and $\nabla_H\delta\left(\mathrm{div}^k(\nabla_H\delta)\right) \in C(S_{r_0})$ (cf. Proposition 66).

THEOREM 7. *Let M be a smooth (or real-analytic) sub-Riemannian manifold, equipped with a smooth (or real-analytic) measure μ, and let $S \subset M$ be a C^2 non-characteristic submanifold. Then, there exists $r_0 > 0$ such that*

$$[0, r_0) \ni r \mapsto \mu(S_r),$$

is smooth (or real-analytic) on $[0, r_0)$, where S_r is the tube around S.

Let S be a non-characteristic hypersurface, the sub-Riemannian normal to S is defined as the unique, up to a sign, unitary vector field $N \in \mathcal{D}$ such that

$$\int_U i_N d\mu = \max_{X\in\mathcal{D}:\ \|X\|=1} \int_U i_X d\mu \qquad \forall\, U \subset S.$$

where i is the restriction operator. If the hypersurface is locally defined as the level set of f, a C^2 locally defining function (that is with $\mathrm{d}f|_S \neq 0$), then the horizontal normal on S is parallel to the horizontal gradient of f. Recall that the horizontal gradient of f, written in terms of a generating family $X_1, \dots, X_N$ for $\mathcal{D}$ is $\nabla_H f = \sum_{i=1}^N (X_i f) X_i$. Moreover, from the very definition, N defines a natural sub-Riemannian induced perimeter measure on S, setting that, for every measurable $U \subset S$

$$\sigma(U) = \int_U d\sigma \qquad \text{where} \qquad d\sigma = i_N d\mu.$$

The key tool in order compute the asymptotic expansion of the tubular-volume function is to apply recursively a sub-Riemannian version of the Mean Value Lemma to the function $v = \mathrm{div}^k(\nabla_H\delta)$. Let us consider $v \in C(S_{r_0})$ such that $\nabla_H\delta(v) \in C(S_{r_0})$ and that $v \circ \gamma \in C^1(0,1)$ for every γ geodesic from S. We define $F : [0, r_0) \to \mathbb{R}$ as

$$F(r) := \int_{S_r} v\, d\mu, \qquad \forall\, r \in [0, r_0).$$

The Mean Value Lemma, in Theorem 72, states that $F \in C^2((0, r_0))$ and $\forall\, r \in (0, r_0)$

$$F'(r) = \int_{\delta^{-1}(r)} v\, d\sigma_r \qquad F''(r) = \int_{\delta^{-1}(r)} \left(v\, \mathrm{div}(\nabla_H\delta) + \nabla_H\delta(v)\right) d\sigma_r,$$

where σ_r is the induced sub-Riemannian perimeter measure on $\delta^{-1}(r)$.

THEOREM 8. *Let M be a sub-Riemannian manifold, equipped with a smooth measure μ, and let S be a C^2 non-characteristic submanifold. Then, there exists $r_0 > 0$ such that the function $[0, r_0) \ni r \mapsto \mu(S_r)$ is smooth and, for every $K \in \mathbb{N}$,*

$$\mu(S_r) = \sum_{k=1}^{K} a_k r^k + o(r^N), \qquad \textit{as } r \to 0^+,$$

with for all $k \in \{1, \dots, K\}$

$$a_k = \frac{1}{k!} \lim_{r\to 0} \int_{\delta^{-1}(r)} \mathrm{div}^{k-1}(\nabla_H\delta)\, d\sigma_r, \tag{0.10}$$

where σ_r is the sub-Riemannian perimeter measure induced by μ on $\delta^{-1}(r)$.

The iterated divergences are not smooth on S, nonetheless, we are able to precisely characterize their singularities. For $k \in \mathbb{N} \setminus \{0\}$, we define the following functions:

$$\Theta^0 := 1, \qquad \Theta^k := \left(\frac{\operatorname{div}(\nabla_H \delta^2)}{2} - k \right) \Theta^{k-1} + \frac{\nabla_H \delta^2 \left(\Theta^{k-1} \right)}{2},$$

where the derivatives are meant in the distributional sense. The functions Θ^k are actually well-defined on $S_{r_0} \cup S$ in the strong sense, and represent the principal part of the iterated divergences. Indeed, on S_{r_0} we have that $\operatorname{div}^k(\nabla_H \delta) = \frac{1}{\delta^k}\Theta^k$. for all $k \in \mathbb{N}$.

Finally, we recall that in the case of a smooth submanifold of codimension m, [**Ros22**, Lemma 4.1] states the existence of a unique intrinsic probability measure on the submanifold induced by μ. Following the same construction, we define μ_S as the unique $(n-m)$-measure of class C^1 on S such that for every $f \in C_c(M)$

$$\lim_{r \to 0} \frac{1}{\mathscr{L}^m(B(r, \mathbb{R}^m))} \int_{S_r} f \, d\mu = \int_S f \, d\mu_S,$$

where $\mathscr{L}^m(B(r, \mathbb{R}^m))$ is the standard Lebesgue measure of the ball of radius r in $\mathbb{R}^m$ and $d\mu$, $d\mu_S$ are the differential forms associated to the measures μ and μ_S respectively.

Proposition 9. *Under the assumptions and notations of Theorem 8, it holds that*

$$\mu(S_r) = \frac{2\pi^{\frac{m}{2}}}{m\,\Gamma\left(\frac{m}{2}\right)} \mu_S(S) r^m + o(r^m), \qquad \text{as } r \to 0.$$

If S is a C^2 non-characteristic hypersurface in M bounding a compact region $\Omega \subset M$, then the above regularity properties for δ and its derivatives (of at most the second order) are valid up to S. Therefore, we specialize Theorem 8 computing the limits in (0.10) and we obtain the measure of the half-tubular neighborhood $S_r^+ = \Omega \cap S_r$.

Theorem 10 (Steiner's formula). *Let M be a smooth (or real-analytic) sub-Riemannian manifold, equipped with a smooth (or real-analytic) measure μ. Moreover, let $S \subset M$ be a C^2 non-characteristic hypersurface bounding a relatively compact domain in M. Then, there exists $r_0 > 0$ such that $[0, r_0) \ni r \mapsto \mu(S_r^+)$, is smooth (or real-analytic) on $[0, r_0)$, where S_r^+ is the half-tube of S. Moreover, for every $K \in \mathbb{N}$,*

$$\mu(S_r^+) = \sum_{k=1}^{K} a_k(S, \mu) r^k + o(r^N), \qquad \text{as } r \to 0^+,$$

where for every $k \in \{1, \dots, K\}$,

$$a_k = \frac{1}{k!} \int_S \operatorname{div}^{k-1}(\nabla \delta) \, d\sigma,$$

with σ the induced sub-Riemannian perimeter measure on S.

Introduction to Chapter 4

In conclusion of this thesis, we address the question of the integrability of the sub-Riemannian mean curvature around isolated characteristic points of a smooth surface embedded in a three-dimensional sub-Riemannian contact manifold. In particular, we prove the local integrability in presence of mildly degenerate characteristic points. The aim of the present Chapter is to extend the definition of such class of points introduced in [**Ros21**] for the Heisenberg group as well as the results of local integrability.

Let $(M, \mathcal{D}, g)$ be a three-dimensional sub-Riemannian contact manifold equipped with a smooth measure μ. Let us consider a smooth surface S embedded in M. A point $p \in S$ is called characteristic if the horizontal distribution at p coincides with the tangent space to S. Let $f \in C^\infty(M)$ be a function defining the surface as its zero level set and with $\mathrm{d}f \neq 0$. The sub-Riemannian mean curvature is the smooth function defined on non-characteristic points as

$$\mathcal{H}_\mu = -\mathrm{div}_\mu \left(\frac{\nabla_H f}{\|\nabla_H f\|} \right),$$

where $\nabla_H f$ is the sub-Riemannian gradient of f. The latter definition is independent of the locally defining function, and thus is equivalent to (0.5), where the defining function is the sub-Riemannian distance from the surface.

The integrability problem becomes meaningful at characteristic points, since $\mathcal{H}_\mu$ blows up at such points, being $\nabla_H f = 0$. The sub-Riemannian mean curvature is a geometrical invariant which naturally emerges in different areas of geometric analysis, such as in the theory of minimal surfaces [**CHMY05**, **DGN07**, **HP08**, **Mon13**, **Pau04**]. Moreover, as showed in the previous chapters, it is involved in the Steiner-like formulas for the volume of half-tubes over hypersurfaces, and it appears also in the study of the heat content asymptotics in sub-Riemannian manifolds.

Since characteristic points are the horizontal critical points of the defining function f, we can further characterize them considering a horizontal second order differential condition. Namely, we take into account the *horizontal Hessian* of f, that is a well defined bilinear form on the distribution at characteristic points, and it is obtained by computing the derivatives of $\nabla_H f$ along the directions tangent to the distribution. Therefore, we individuate the class of *degenerate characteristic points*, that are points in which the horizontal hessian has not full rank as an endomorphism of the distribution. This condition, as well as the characteristic one, does not depend on f and it is intrinsic of the embedding of the surface.

As it is shown in [**Ros21**, Thm. 1.1], if the surface contains only isolated non-degenerate characteristic points, then the sub-Riemannian mean curvature is locally integrable. Things are less understood for degenerate characteristic points. In [**Ros21**] the author individuates a property for such points that guarantees the local integrability of the mean curvature in the Heisenberg group. Specifically, it is introduced a finite-order condition on the regular parametrization of an intrinsic curve lying on the surface and passing trough the point. An isolated characteristic point satisfying such a condition is called *mildly degenerate*. In this chapter we extend the latter concept when dealing with general three-dimensional sub-Riemannian contact manifolds.

We prove the local integrability of the mean curvature with respect to the Riemannian induced measure by μ on the surface. Namely, we consider the usual Riemannian induced measure obtained from the Riemannian structure defined as a suitable extension of the assigned sub-Riemannian metric. The main result of the chapter is the following extension of [**Ros21**, Thm. 1.2].

THEOREM 11. *Let S be a smooth surface embedded in a three-dimensional contact manifold $(M, \mathcal{D}, g)$ equipped with a smooth measure μ. Let uss suppose that S contains only isolated characteristic points that are non-degenerate or mildly degenerate. Then, it holds that*

$$\mathcal{H}_\mu \in L^1_{loc}(S, \sigma_R),$$

where σ_R is the induced Riemannian measure on S by μ.

We stress the fact that in [**Ros21**] the sub-Riemannian mean curvature and the induced measure are defined in terms of the canonical Popp's volume, that coincides with the Lebesgue one. On the contrary, here we consider any smooth measure μ.

REMARK 12. Let M be a constant rank sub-Riemannian structure. The sub-Riemannian mean curvature of a hypersurface is locally integrable with respect to the sub-Riemannian induced measure independently of any feature that a characteristic point may have (see [**Ros21**, Lemma 3.1] and [**DGN12**, Prop. 3.5] for Carnot groups of arbitrary step).

Since in a real-analytic context a degenerate characteristic point satisfies the mild degeneration condition automatically, we extend [**Ros21**, Thm. 3.1].

COROLLARY 13. *Let S be an embedded real-analytic surface in a three dimensional sub-Riemannian and real-analytic contact manifold $(M, \mathcal{D}, g)$ equipped with a smooth measure μ. Assume that all the characteristic points are isolated, then*

$$\mathcal{H}_\mu \in L^1_{loc}(S, \sigma_R)$$

where σ_R is the induced Riemannian measure on S by μ.

REMARK 14. Actually, we prove the local integrability of $\|\nabla_H f\|^{-1}$ for which the mild degeneration assumption is sharp and cannot be improved, as shown in [**Ros21**, Remark 5.6] with an example taken from [**DGN12**]. This stronger result implies the local integrability of the sub-Riemannian mean curvature. Nevertheless, the same example shows that the mean curvature may be still locally integrable at non mildly degenerate points.

List of publications

The research presented in this work appears in the following papers:

[**BB23**] Davide Barilari and Tania Bossio, *Steiner and tube formulae in 3D contact sub-Riemannian geometry*, Communications in Contemporary Mathematics, 2023;

[**BRR24**] Tania Bossio, Luca Rizzi and Tommaso Rossi, *Sub-Riemannian Steiner's and Weyl's tube formulae*, in preparation.

CHAPTER 1

Preliminaries

In this chapter we recall the basic definitions and some results concerning sub-Riemannian geometry. Moreover we specialize the subject to the three-dimensional contact sub-Riemannian manifolds, presenting, in addition, the canonical Tanno connection. For a more detailed and general presentation see [**ABB20**].

1. Sub-Riemannian geometry

Let M be a smooth, connected n-dimensional smooth manifold. A smooth sub-Riemannian structure on M is defined by a set of $N \in \mathbb{N}$ global smooth vector fields $X_1, \dots, X_N$, called a *generating family*. We indicate with $\mathcal{D}$ the *sub-Riemannian distribution*. Namely, the collection of vector subspaces in the tangent bundle

$$\mathcal{D}_p = \operatorname{span}\{X_1(p), \dots, X_N(p)\} \subseteq T_pM, \qquad \forall\, p \in M. \tag{1.1}$$

We assume that the generating family is *bracket-generating*, i.e.,that the Lie algebra generated by the distribution,

$$\mathcal{L}\mathrm{ie}(\mathcal{D}) = \{[X_1, [\dots, [X_{k-1}, X_k] \dots] \mid X_1, \dots, X_k \in \mathcal{D}, k \in \mathbb{N}\},$$

evaluated at the point p, coincides with T_pM for all $p \in M$. The generating family induces a norm on the distribution at p:

$$\|v\|_p = \inf \left\{ \sum_{i=1}^N u_i^2 : \sum_{i=1}^N u_i X_i(p) = v \right\}, \qquad \forall\, v \in \mathcal{D}_p,$$

which, in turn, defines an inner product g_p on $\mathcal{D}_p$ by polarization.

The manifold M, equipped with the above structure, is said to be a *smooth sub-Riemannian manifold*. We say that M is a *real-analytic sub-Riemannian manifold* if M is a real-anaytic manifold and the generating family $X_1, \dots, X_N$ consists of real-analytic vector fields.

A curve $\gamma : [0, 1] \to M$ is *horizontal*, if it is absolutely continuous and

$$\dot\gamma_t \in \mathcal{D}_{\gamma_t}, \qquad \text{for a.e. } t \in [0, 1].$$

This implies that there exists a *control* $u : [0, 1] \to \mathbb{R}^N$, such that

$$\dot\gamma_t = \sum_{i=1}^N u_i(t) X_i(\gamma_t), \qquad \text{for a.e. } t \in [0, 1].$$

Moreover, we require that $u \in L^2([0, 1], \mathbb{R}^N)$. We define the *length* of a horizontal curve as:

$$\ell(\gamma) = \int_0^1 \|\dot\gamma_t\|_{\gamma_t} dt.$$

Finally, the *sub-Riemannian distance* is defined, for any $p, q \in M$, by

$$\mathsf{d}(p,q) = \inf\{\ell(\gamma) : \gamma \text{ horizontal curve joining } p \text{ and } q\}.$$

By the Chow-Rashevskii Theorem, the bracket-generating assumption ensures that the distance d is finite, continuous and it induces the same topology as the manifold one.

REMARK 15. The above definition includes all classical constant-rank sub-Riemannian structures as in [**Mon02**, **Rif14**] (where $\mathcal{D} \subset TM$ is a sub-bundle), but also general rank-varying sub-Riemannian structures.

2. Geodesics and Hamiltonian flow

Let M be a smooth (or real-analytic) sub-Riemannian manifold. A *geodesic* is a horizontal curve $\gamma : [0,1] \to M$, parametrized with constant speed, such that it is locally length-minimizing. The *sub-Riemannian Hamiltonian* is the smooth (real-analytic respectively) function $H : T^*M \to \mathbb{R}$, given by

$$H(\lambda) := \frac{1}{2}\sum_{i=1}^{N} \langle \lambda, X_i \rangle^2, \qquad \lambda \in T^*M,$$

where $X_1, \dots, X_N$ is a generating family for the sub-Riemannian structure, and $\langle \lambda, \cdot \rangle$ denotes the duality pairing, i.e., the action of covectors on vectors. The *Hamiltonian vector field* $\vec{H}$ on T^*M is then defined by

$$\varsigma(\cdot, \vec{H}) = dH,$$

where $\varsigma \in \Lambda^2(T^*M)$ is the canonical symplectic form.

Solutions $\lambda : [0,1] \to T^*M$ of the *Hamilton equations*

$$\dot{\lambda}_t = \vec{H}(\lambda_t), \tag{1.2}$$

are called *normal extremals*. Their projections $\gamma_t = \pi(\lambda_t)$ on M, where $\pi : T^*M \to M$ is the bundle projection, are locally length-minimizing horizontal curves parametrized with constant speed, and are called *normal geodesics*. If γ is a normal geodesic with normal extremal λ, then its speed is given by $\|\dot{\gamma}\| = \sqrt{2H(\lambda)}$. In particular

$$\ell(\gamma|_{[0,t]}) = t\sqrt{2H(\lambda_0)}, \qquad \forall\, t \in [0,1].$$

The *exponential map* $\exp_p : T_p^*M \to M$, with base point $p \in M$ is

$$\exp_p(\lambda) = \pi \circ e^{\vec{H}}(\lambda), \qquad \lambda \in T_p^*M,$$

where $e^{\vec{H}}$ denotes the flow of $\vec{H}$, which we assume to be well-defined up to time 1. This is the case, for example, when (M, d) is a complete metric space.

There is another class of length-minimizing curves in sub-Riemannian geometry, called *abnormal geodesics*. As for the normal case, to these curves it corresponds an extremal lift, which however may not follow the Hamiltonian dynamics (1.2). Here we only observe that an abnormal extremal lift $(\lambda_t)_{t\in[0,1]}$ satisfies

$$\langle \lambda_t, \mathcal{D}_{\pi(\lambda_t)} \rangle = 0 \quad \text{and} \quad \lambda_t \neq 0, \qquad \forall\, t \in [0,1],$$

that is $H(\lambda_t) \equiv 0$. Note that a geodesic may be abnormal and normal at the same time.

3. Horizontal gradient and divergence

Let M be a sub-Riemannian manifold. Let μ be a smooth measure on M, defined by a positive tensor density. The *divergence* of a smooth vector field is defined by

$$\mathrm{div}(X)\,\mu = \mathcal{L}_X \mu, \qquad \forall\, X \in \Gamma(TM),$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X. The *horizontal gradient* of a function $f \in C^1(M)$, denoted by $\nabla_H f$, is defined as the horizontal vector field, such that

$$g_p(\nabla_H f(p), v) = \mathrm{d}_p f(v), \qquad \forall\, v \in \mathcal{D}_p,$$

where v acts as a derivation on f. In terms of a generating family as in (1.1), one has

$$\nabla_H f = \sum_{i=1}^N X_i(f) X_i \qquad \text{and} \qquad \|\nabla_H f\|^2 = \sum_{i=1}^N X_i(f)^2 = 2H(\mathrm{d}f),$$

for every $f \in C^1(M)$, cf. [**RS23**, Cor. A.2]. Finally, the *sub-Laplacian* is the operator $\Delta = \mathrm{div} \circ \nabla$, acting on $C^2(M)$. Again, we may write its expression with respect to a generating family (1.1), obtaining

$$\Delta f = \sum_{i=1}^N X_i^2(f) + X_i(f)\mathrm{div}(X_i), \qquad \forall\, f \in C^2(M).$$

4. Three-dimensional contact sub-Riemannian manifolds

Definition 16. A *three-dimensional contact sub-Riemannian manifold* is a triple $(M, \mathcal{D}, g)$ constitued by a smooth manifold M of dimension three equipped with a two dimensional vector distribution $\mathcal{D}$ that is the kernel of a one-dimensional form ω such that $\omega \wedge \mathrm{d}\omega \neq 0$. We endow $\mathcal{D}$ with a smooth metric g.

Up to a multiplication by a never vanishing function, one can always normalize the contact form with respect to g by requiring that $\mathrm{d}\omega|_{\mathcal{D}}$ coincides with the volume form defined by the metric g on $\mathcal{D}$, that is $\mathrm{d}\omega(X_1, X_2) = 1$ for every positively oriented orthonormal frame of $\mathcal{D}$. In the sequel we always assume this normalization.

The conditions $\ker \omega = \mathcal{D}$ and $\omega \wedge \mathrm{d}\omega \neq 0$ are equivalent to saying that for any pair of locally defined and linearly independent vector fields X and Y in $\mathcal{D}$, the Lie bracket $[X, Y]$ is not contained in $\mathcal{D}$. The latter fact can be deduced by the Cartan formula applied to the contact one-form. More precisely, let τ be a smooth one-form and U, V smooth vector fields on M. The Cartan formula states that $\tau([U, V]) + \mathrm{d}\tau(U, V) = U(\tau(V)) - V(\tau(U))$. Hence,

$$0 \neq \mathrm{d}\omega(X, Y) = \omega([Y, X]) \qquad \forall X, Y \in \mathcal{D}.$$

In other words $\mathcal{D}$ is Lie bracket generating.

The *Reeb vector field* X_0 is the unique vector field such that

$$i_{X_0}\omega = 1, \qquad i_{X_0}\mathrm{d}\omega = 0, \tag{1.3}$$

where i_{X_0} denotes the interior product. For every positively oriented local orthonormal frame X_1, X_2 of $\mathcal{D}$ it holds that X_1, X_2, X_0 is a local frame for TM satisfying the identity $[X_2, X_1] = X_0 \mod \mathcal{D}$.

In a three-dimensional contact sub-Riemannian manifold all length-minimizers are normal curves (cf. [**ABB20**, Prop. 4.8]). That is, they are all obtained projecting on M the solutions of the Hamiltonian equation (1.2), where the Hamiltonian in this setting is

$$H(\lambda) = H(p, x) = \frac{1}{2}\left[\langle p, X_1(x)\rangle^2 + \langle p, X_2(x)\rangle^2\right].$$

To obtain arclength parametrized length minimizers one has to consider solutions of (1.2) with $\lambda_0 \in H^{-1}(1/2)$.

Finally, we can introduce on M a Riemannian metric such that X_1, X_2, X_0 is a local orthohormal frame for TM, i.e., extending the metric g defined on $\mathcal{D}$ by declaring that X_0 has unit norm and is orthogonal to $\mathcal{D}$. The Riemannian metric will be denoted with $\langle\cdot,\cdot\rangle$ and the corresponding norm with $\|\cdot\|$.

Definition 17. Given a smooth function $f : M \to \mathbb{R}$, we denote with $\nabla_H f$ its horizontal gradient, which is the unique horizontal vector field satisfying the identity $\mathrm{d}f(v) = g(\nabla_H f, v)$ for every $v \in \mathcal{D}$.

Denoting by ∇f the Riemannian gradient of f with respect to the ambient Riemannian metric, in terms of a local orthonormal frame X_1, X_2, X_0 chosen as above, the expressions for the gradients are the following:

$$\nabla f = (X_1 f)X_1 + (X_2 f)X_2 + (X_0 f)X_0,$$
$$\nabla_H f = (X_1 f)X_1 + (X_2 f)X_2.$$

From the properties defining the Reeb vector field (1.3) and the normalization of the contact form we get that for every positively oriented orthonormal frame of $\mathcal{D}$ we have the Lie brackets can be written as follows

$$\begin{aligned}[X_2, X_1] &= c^1_{12}X_1 + c^2_{12}X_2 + X_0,\\ [X_1, X_0] &= c^1_{01}X_1 + c^2_{01}X_2,\\ [X_2, X_0] &= c^1_{02}X_1 + c^2_{02}X_2.\end{aligned} \tag{1.4}$$

for suitable smooth functions c^k_{ij} defined on M. We notice that these functions are constant if the sub-Riemannian structure is left-invariant on a Lie group.

Definition 18. Let $J : TM \to TM$ be the linear morphism of vector bundles defined by the identities

$$J(X_1) = X_2, \quad J(X_2) = -X_1, \quad J(X_0) = 0.$$

Notice that $g(X, JY) = -\mathrm{d}\omega(X, Y)$, for every X, Y in TM, hence $g(X, JX) = 0$, and that $J|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ preserves the sub-Riemannian metric.

The following lemma follows from a direct computation.

Lemma 19. *Let $Y_1, Y_2 = JY_1$ and $X_1, X_2 = JX_1$ be two pairs of horizontal vector fields satisfying*

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

for some smooth function $\theta : M \to \mathbb{R}$. Then we have

$$[Y_2, Y_1] = [X_2, X_1] - X_1(\theta)\, X_1 - X_2(\theta)\, X_2.$$

5. Tanno connection

Let us now introduce the Tanno connection, which is a canonical connection on contact manifold, see for instance [**Tan89**, **ABR17**].

DEFINITION 20. The Tanno connection ∇ is the unique linear connection on TM satisfying

(i) $\nabla g = 0$, $\nabla X_0 = 0$;
(ii) $\mathrm{Tor}(X,Y) = -\langle X, JY\rangle X_0 = \mathrm{d}\omega(X,Y)X_0$ for all $X, Y \in \mathcal{D}$;
(iii) $\mathrm{Tor}(X_0, JX) = -J\,\mathrm{Tor}(X_0, X)$ for any vector field X on M.

where Tor denotes the Torsion of the connection ∇.

REMARK 21. The Tanno connection ∇ commutes with the operator J, i.e.,

$$\nabla_X JY = J\nabla_X Y.$$

Indeed, $J\nabla_X Y$ is horizontal by definition of J, while $\nabla_X JY$ is horizontal since

$$0 = X\langle JY, X_0\rangle = \langle \nabla_X JY, X_0\rangle + \langle JY, \nabla_X X_0\rangle = \langle \nabla_X JY, X_0\rangle.$$

Moreover, from the fact that J preserves the metric, we deduce that

$$\langle \nabla_X JY, JY\rangle = \frac{1}{2}X\langle JY, JY\rangle = \frac{1}{2}X\langle Y, Y\rangle = \langle \nabla_X Y, Y\rangle = \langle J\nabla_X Y, JY\rangle.$$

Finally, since $\langle Y, JY\rangle = 0$ one obtains that

$$0 = X\langle Y, JY\rangle = \langle \nabla_X Y, JY\rangle + \langle Y, \nabla_X JY\rangle = \langle -J\nabla_X Y, Y\rangle + \langle \nabla_X JY, Y\rangle.$$

LEMMA 22. *For every vector field X on M, it holds*

$$\langle \mathrm{Tor}\,(X_0, X)\,, JX\rangle = \frac{1}{2}\langle [JX, X_0]\,, X\rangle + \frac{1}{2}\langle [X, X_0]\,, JX\rangle\,.$$

PROOF. Using (iii) of Definition 20, we have the equality

$$\langle \mathrm{Tor}\,(X_0, X)\,, JX\rangle = \langle \mathrm{Tor}\,(X_0, JX)\,, X\rangle\,.$$

On the other hand, by the definition of the torsion, we have that

$$\langle \mathrm{Tor}\,(X_0, X)\,, JX\rangle = \langle \nabla_{X_0} X + [X_0, X]\,, JX\rangle\,,$$
$$\langle \mathrm{Tor}\,(X_0, JX)\,, X\rangle = \langle \nabla_{X_0} JX + [X_0, JX]\,, X\rangle\,.$$

Adding the two equations one obtains the statement recalling in addition that $\langle \nabla_{X_0} JX, X\rangle = -\langle \nabla_{X_0} X, JX\rangle$, being X and JX orthogonal. □

Let $\tau : \mathcal{D} \to \mathcal{D}$ be the linear operator $\tau(X) = \mathrm{Tor}(X_0, X)$ for X a vector field in $\mathcal{D}$. The symmetric matrix representing the operator τ with respect to the orthonormal and positively oriented frame X_1, X_2 for $\mathcal{D}$ satisfying (1.4) is

$$\begin{pmatrix} c^1_{01} & \dfrac{c^1_{02} + c^2_{01}}{2} \\ \dfrac{c^1_{02} + c^2_{01}}{2} & c^2_{02} \end{pmatrix}. \tag{1.6}$$

Notice that, for a smooth function f, the notation ∇f is compatible with its Riemannian gradient, if one interprets ∇ as the Tanno connection.

CHAPTER 2

Steiner's formula in 3D contact sub-Riemannian geometry

In this Chapter we show the results contained in [**BB23**]. We prove a Steiner formula for regular surfaces with no characteristic points in 3D contact sub-Riemannian manifolds endowed with an arbitrary smooth volume. The formula we obtain, which is equivalent to a half-tube formula, is of local nature. It can thus be applied to any surface in a region not containing characteristic points. Moreover, we provide a geometrical interpretation of the coefficients appearing in the expansion, and compute them on some relevant examples in three-dimensional sub-Riemannian model spaces. These results generalize those obtained in [**BFF$^+$15**] and [**Rit21**] for the Heisenberg group.

Structure of the Chapter

Let $(M, \mathcal{D}, g)$ be a three-dimensional contact manifold equipped with a smooth volume measure μ. Let S be a smooth surface embedded in M bounding a closed region Ω and let us consider $U \subset S$ open, relatively compact and such that $\overline{U}$ does not contain characteristic points.

In Section 1 we state the regularity properties of the sub-Riemannian distance function from S and define the local half-tubular neighborhood U_r as the union of the length-minimizing curves contained in S_r realizing the distance from the surface S with an endpoint on U. Secondly, exploiting the transversality conditions, we characterize the latter length-minimizing curves computing their initial covector in T^*M. Moreover, we define the sub-Riemannian mean curvature with respect to μ and compute the expression in coordinates of the mean sub-Riemannian curvature with respect to the Popp volume. Furthermore, we define an orthogonal moving frame with respect to the surface and prove a sub-Riemannian version of the coarea formula in order to derive a formula for the measure of U_r.

Section 2 is devoted to the study of the Jacobian of the exponential map at a fixed time with the goal to derive a handy expression for the computations.

The proof of Theorem 42 is contained in Section 3. The technical tool that we exploit in the computations is the Tanno connection. It is useful in order to obtain a geometrical interpretation in terms of curvature objects.

In Section 4, we present an operative way to compute the coefficients in expansion (2.27) without having explicit access to the expression of distance function (cf. Proposition 47). Moreover, we specialize our result in the case of three-dimensional contact manifolds with particular values of the geometric invariants χ and κ.

In conclusion, in section 5 we provide some applications of the results to rotational surfaces in the Heisenberg group and model surfaces in the Lie groups $SU(2)$ and $SL(2)$.

1. Half-tubular neighborhoods and coarea formula

Let $(M, \mathcal{D}, g)$ be a three-dimensional contact manifold equipped with a smooth volume measure μ and let S be a smooth surface embedded in M bounding a closed region Ω. Let us consider a positively oriented local orthonormal frame on $\mathcal{D}$ given by the vector fields X_1, X_2, and let us extend the sub-Riemannian metric g to the Riemannian one for which the Reeb vector field X_0 is orthogonal to the distribution and with unit norm. Moreover, let us denote with $\omega^1, \omega^2, \omega^0$ on T^*M the dual 1-forms to X_1, X_2, X_0. Notice that $\omega^0 = \omega$ is the normalized contact form. Recall that $\omega \wedge \omega^1 \wedge \omega^2 = \omega \wedge d\omega$ is the Popp's volume form of the three-dimensional contact sub-Riemannian manifold (see for instance [**ABB20**]).

1.1. Regularity of the distance and tubular neighborhoods. We recall that a point $x \in S$ is said to be characteristic if $T_xS = \mathcal{D}_x$ and that the set $\Gamma(S)$ of characteristic points in S is closed and has zero measure on the surface S, see [**Bal03**]. As a consequence, we have a well-defined characteristic foliation on $S \setminus \Gamma(S)$ defined by the intersection $\mathcal{D}_x \cap T_xS$.

Definition 23. Let us consider on M the sub-Riemannian signed distance function from the surface S:

$$\delta_S(p) = \begin{cases} \inf \{\mathsf{d}(p, x) \mid x \in S\}, & p \notin \Omega, \\ -\inf \{\mathsf{d}(p, x) \mid x \in S\}, & p \in \Omega. \end{cases}$$

Our aim is to obtain results for the volume of the external sub-Riemannian r-half-tubular neighborhood of the surface S, that is

$$S_r = \{x \in M \setminus \Omega \mid 0 < \delta_S(x) < r\}.$$

More precisely, we are going to study local results considering a r-half-tubular neighborhood of open and relatively compact subsets of S that do not contain characteristic points. In order to do that, we report a key result about the smoothness of the sub-Riemannian distance function from surfaces.

Theorem 24. *Let S be a smooth surface bounding a closed domain Ω in a three-dimensional contact sub-Riemannian manifold M. Let us consider $U \subset S$ open, relatively compact and such that $\overline{U}$ does not contain characteristic points. Then*

(1) there exist $r > 0$ and a smooth map $G : (-r, r) \times U \to M$ that is a diffeomorphism on the image and such that for all $(t, x) \in (-r, r) \times U$

$$\delta_S(G(t, x)) = t \quad \textit{and} \quad dG(\partial_t) = \nabla_H \delta_S;$$

(2) δ_S is smooth on $G((-r, r) \times U)$, with $\|\nabla_H \delta_S\| = 1$.

Theorem 24 can be found in [**FPR20**, Prop. 3.1] and in [**Ros22**, Thm. 3.3] for compact hypersurfaces and for submanifolds of arbitrary codimension, respectively, without characteristic points, embedded in sub-Riemannian manifolds. The version presented here refers to [**Ros22**, Thm. 3.7], that, in turn, is a refinement of the previously mentioned results for the signed distance from non characteristic hypersurfaces. Our goal is to study the measure of the following subsets of S_r.

Definition 25. Let $U \subset S$ be open, relatively compact and such that the closure does not contain characteristic points. For $r > 0$ we define the (external) local half-tubular neighborhood of S relatively to U as

$$U_r = G((0, r) \times U) \subset S_r, \tag{2.3}$$

where $G : (-r, r) \times U \to M$ is the diffeomorphism of Theorem 24.

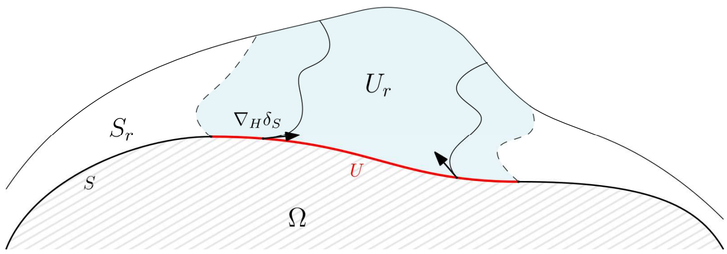

Figure 1. The external local half-tubular neighborhood U_r.

Remark 26. Notice that the set U_r is well-defined thanks to Proposition 31. In fact, the map G is unique and explicitly characterized in (2.12). The localized half-tubular neighborhood U_r can be described as the union of the length-minimizing curves contained in S_r realizing the distance from the surface S with an endpoint on U. The description through the diffeomorphism G guarantees that U_r is open and smooth in M, hence measurable.

Proof of Theorem 24. We report only a sketch of the proof because it follows verbatim from that of [**Ros22**, Thm. 3.7] with only minor adjustments to adapt it to our setting. The set $\Gamma(S)$ of characteristic points in S, is closed, thus we can consider $\widetilde{U}$ an open neighborhood of $\overline{U}$, the compact closure of U in S, that does not contain characteristic points. Let us define the annihilating bundle

$$\mathcal{A}\widetilde{U} = \{(x, \lambda) \in T^*M \mid x \in \widetilde{U}, \langle \lambda, T_xS \rangle = 0\}.$$

The exponential map $E : \mathcal{A}\widetilde{U} \to M$ defined by $E(\lambda) = \pi \circ e^{\vec{H}}(\lambda)$ is a local diffeomorphism at every $(x, 0) \in \overline{U}$. Since $\overline{U}$ is compact, there exists $r > 0$ such that the restriction of E to the open set

$$\{(x, \lambda) \mid x \in U, 0 < \sqrt{2H(\lambda)} < r\} \subset \mathcal{A}\widetilde{U},$$

is a diffeomorphism onto its image. Finally, being U oriented, there exists a smooth outward pointing non-vanishing section of $\mathcal{A}\widetilde{U}$, i.e., $\lambda : \widetilde{U} \to \mathcal{A}\widetilde{U}$ such that $E(\lambda(x)) \notin \Omega$ for every $x \in U$. Without loss of generality, we can suppose that $2H(\lambda(x)) = 1$, then the map we are looking for is defined as

$$G(t, x) = E(t\lambda(x)). \qquad \square$$

Remark 27. The requirement on U to have compact closure and without characteristic points is essential in order to define the diffeomorphism in Theorem 24. The absence of characteristic points guarantees the smoothness of the map, while the hypothesis of compactness of the closure of U guarantees the existence of a minimum $r > 0$ for which we can define the map G.

1.2. Transversality conditions. Let us fix $U \subset S$ open, relatively compact and such that its closure does not contain any characteristic point. Thanks to Theorem 24, for a sufficiently small $r > 0$ the set $G((-r, r) \times U)$ is a open submanifold in M. For the sake of simplicity, from now on we refer to $G((-r, r) \times U) \subset M$ as ambient space, and we denote with δ the sub-Riemannian signed distance function from U. In this way δ is a smooth function and since $\nabla_H \delta \neq 0$, we can treat δ as a defining function for U, i.e., $U = \delta^{-1}(0)$ and $U_r = \delta^{-1}(]0, r[)$.

In coordinates, identity $\|\nabla_H \delta\|^2 = 1$ of Theorem 24 is written as

$$(X_1\delta)^2 + (X_2\delta)^2 = 1, \tag{2.4}$$

where we recall that $\nabla_H \delta = (X_1\delta)X_1 + (X_2\delta)X_2$.

One can also introduce on U a characteristic vector field as follows

$$X_S = (X_2\delta)X_1 - (X_1\delta)X_2, \tag{2.5}$$

and consider the orthogonal basis on TU composed by X_S and

$$Y_S = (X_0\delta)\,\nabla_H \delta - X_0. \tag{2.6}$$

Remark 28. Notice that X_S is a smooth vector field of unit norm, and it spans $\mathcal{D}_x \cap T_x S$ for every $x \in U$. Moreover, we recall that if the surface contains characteristic points then X_S is still well-defined through (2.5) replacing the sub-Riemannian distance function δ with a smooth function $f : M \to \mathbb{R}$ that locally defines the surface as its zero level set, i.e., $X_S = (X_2 f)X_1 - (X_1 f)X_2$. In this case, the resulting vector field is not of unit norm and it vanishes at characteristic points.

For $x \in U_r$, the distance from the surface $\delta(x)$ can be rewritten as

$$\delta(x) = \inf \left\{ \ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\| dt \colon \gamma : x_0 \to x \text{ horizontal}, \, x_0 \in U \right\}. \tag{2.7}$$

Let us consider $\gamma : [0, \delta(x)] \to M$ an arclength parametrized curve that realizes the distance (2.7) from $x = \gamma(\delta(x))$ to the surface. A lift $\lambda : [0, \delta(x)] \to T^*M$ of γ that is a solution of (1.2) has to satisfy the transversality conditions given by the Pontryagin Maximum Principle (PMP) for optimal control problems with constraints on initial and terminal points (see [**AS04**, Thm. 12.13]). More precisely, the condition on $\lambda(0) = \lambda_0$ is

$$\langle \lambda_0, v \rangle = 0, \qquad \forall v \in T_{\gamma(0)}U. \tag{2.8}$$

Exploiting this fact, one can compute the initial covector that identifies the arclength parametrized curve that realizes the distance of a fixed point in U_r from the surface.

Proposition 29. *Let $\gamma : [0, T] \to M$ be an arclength parametrized curve with $\gamma(0) \in U$ that minimizes the distance from $\gamma(T) \in U_r$ to U, and let $\lambda : [0, T] \to T^*M$ be the corresponding lift solving (1.2). Then the initial covector is*

$$\lambda_0 = (X_1\delta)\,\omega^1 + (X_2\delta)\,\omega^2 + (X_0\delta)\,\omega \in T^*_{\gamma(0)}M. \tag{2.9}$$

Moreover, $\gamma(t) = \pi \circ e^{t\vec{H}}(\lambda_0)$ and $\dot{\gamma}(0) = \nabla_H \delta$.

PROOF. Let us consider the local ortogonal basis on U composed by X_S, Y_S defined in (2.5) and (2.6). The transversality condition (2.8) applied to the vector field X_S is written as follows

$$0 = \langle \lambda_0, X_S \rangle = (X_2\delta) \langle \lambda_0, X_1 \rangle - (X_1\delta) \langle \lambda_0, X_2 \rangle$$

from which one gets $(\langle \lambda_0, X_1 \rangle, \langle \lambda_0, X_2 \rangle) = c\,(X_1\delta, X_2\delta)$ with $c \in \mathbb{R}$. Turning the attention to Y_S,

$$0 = \langle \lambda_0, Y_S \rangle = (X_0\delta)\,(X_1\delta)\,c\,(X_1\delta) + (X_0\delta)\,(X_2\delta)\,c\,(X_2\delta) - \langle \lambda_0, X_0 \rangle .$$

Hence, from (2.4) we have that $\langle \lambda_0, X_0 \rangle = c(X_0\delta)$. That means that the covector λ_0 is a multiple of

$$(X_1\delta)\,\omega^1 + (X_2\delta)\,\omega^2 + (X_0\delta)\,\omega \in T^*_{\gamma(0)}M.$$

Finally, writing in coordinates the Hamiltonian system (1.2), we have that

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

Then, we deduce that

$$\dot{\gamma}(0) = \langle \lambda_0, X_1 \rangle X_1 + \langle \lambda_0, X_2 \rangle X_2 = c\,(X_1\delta)\,X_1 + c\,(X_2\delta)\,X_2.$$

Requiring γ to have unit speed, (2.4) implies $c = 1$ and we conclude that

$$\dot{\gamma}(0) = \nabla_H \delta(x), \quad \lambda_0 = (X_1\delta)\,\omega^1 + (X_2\delta)\,\omega^2 + (X_0\delta)\,\omega. \qquad \square$$

REMARK 30. Given $x \in U$, there is a unique arclength parametrized curve $\gamma_x : [0, T] \to M$, such that $\gamma(0) = x$ and that realizes the distance of $\gamma(T)$ from U. Namely, $\gamma_x(t) = \pi \circ e^{t\vec{H}}(\lambda_0)$ with $\lambda_0 \in T^*_x M$ prescribed by (2.9). Therefore, Theorem 24 together with Proposition 29 provides a description of U_r both as the disjoint union of copies of U and as the disjoint union of arclength parametrized minimizing curves escaping from U.

In conclusion, let us introduce the smooth map $\lambda_0 : U \to T^*M$ as follows:

$$\lambda_0(x) = (X_1\delta)\,\omega^1 + (X_2\delta)\,\omega^2 + (X_0\delta)\,\omega \in T^*_x M. \tag{2.11}$$

We have proved the following

PROPOSITION 31. *The map G of Theorem 24 is uniquely characterized as*

$$G(t, x) = \gamma_x(t) = \pi \circ e^{t\vec{H}}(\lambda_0(x)) \tag{2.12}$$

for every $t \in (-r, r)$ and $x \in U$, with λ_0 defined by (2.11).

1.3. The mean sub-Riemannian curvature. Given a smooth volume form μ on M, we recall that the divergence of a vector field X with respect to a measure μ is the function denoted by $\mathrm{div}_\mu X$ such that $\mathcal{L}_X \mu = (\mathrm{div}_\mu X)\mu$ (where $\mathcal{L}_X$ is the Lie derivative along the vector field X).

DEFINITION 32. The *sub-Riemannian mean curvature with respect to a smooth measure* μ of a surface U embedded in a three-dimensional contact manifold and that does not contain characteristic points, is the smooth function $\mathcal{H}_\mu : U_r \to \mathbb{R}$

$$\mathcal{H}_\mu = -\mathrm{div}_\mu \left(\nabla_H \delta\right).$$

Moreover, the *sub-Riemannian mean curvature* is the smooth function $\mathcal{H} : U_r \to \mathbb{R}$ defined as $\mathcal{H} = \mathcal{H}_\nu$, where ν is the canonical measure on M associated with the Popp's volume $\omega \wedge \mathrm{d}\omega = \omega \wedge \omega^1 \wedge \omega^2$.

REMARK 33. Let us consider $f : U_r \to \mathbb{R}$ a smooth defining function for U, i.e., such that $U = f^{-1}(0)$ and $\mathrm{d}f \neq 0$ on U. Then $\|\nabla_H f\| \neq 0$ on U (since U does not contain characteristic points), and it holds that

$$\mathcal{H}|_U = -\mathrm{div}_\nu \left(\frac{\nabla_H f}{\|\nabla_H f\|} \right). \tag{2.13}$$

Actually, identity (2.13) is valid if the orientation of the vector field $\nabla_H f$ coincides with that of $\nabla_H \delta$, in other words if $\langle \nabla_H f, \nabla_H \delta \rangle > 0$. Otherwise the equality holds with the opposite sign. Notice that in general the right hand side of (2.13) is well defined on S but outside the surface it still depends on the choice of the function f. Moreover, we recall that since every smooth measure μ can be interpreted as a multiple of the Popp's measure by a smooth density, i.e., $\mu = h\nu$ with $h : M \to \mathbb{R}$ non vanishing. From the properties of the divergence operator we have that

$$\mathrm{div}_{h\nu} X = \mathrm{div}_\nu X + \frac{Xh}{h},$$

and we obtain the formula

$$\mathcal{H}_\mu = -\mathrm{div}_{h\nu}(\nabla_H \delta) = \mathcal{H} - \frac{\nabla_H \delta(h)}{h}. \tag{2.14}$$

LEMMA 34. *The expression in coordinates of the sub-Riemannian mean curvature $\mathcal{H} : U_r \to \mathbb{R}$ with respect to the Popp measure ν is:*

$$\mathcal{H} = -X_1 X_1 \delta - X_2 X_2 \delta - c_{12}^2 (X_1 \delta) + c_{12}^1 (X_2 \delta). \tag{2.15}$$

In particular, in terms of (1.4),

$$\mathrm{div}_\nu (X_1) = c_{12}^2 \quad \text{and} \quad \mathrm{div}_\nu (X_2) = -c_{12}^1. \tag{2.16}$$

PROOF. Applying the linearity of the divergence and the Leibnitz rule for which $\mathrm{div}(aX) = X(a) + a\mathrm{div}(X)$ for a a smooth function and X a smooth vector field,

$$\begin{aligned}\mathcal{H} &= -\mathrm{div}_\nu ((X_1\delta) X_1 + (X_2\delta) X_2) \\ &= -X_1 X_1 \delta - (X_1\delta)\, \mathrm{div}_\nu (X_1) - X_2 X_2 \delta - (X_2\delta)\, \mathrm{div}_\nu (X_2)\end{aligned}$$

Now, considering that for every $i = 0, 1, 2$

$$\mathcal{L}_X \omega^i = \sum_{j=0}^{2} \left\langle \mathcal{L}_X \omega^i, X_j \right\rangle \omega^j = \sum_{j=0}^{2} \left\langle \omega^i, \mathcal{L}_{-X} X_j \right\rangle \omega^j = \sum_{j=0}^{2} \left\langle \omega^i, [-X, X_j] \right\rangle \omega^j,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X, we obtain that

$$\begin{aligned}\mathrm{div}_\nu (X)\, \omega \wedge \omega^1 \wedge \omega^2 &= \mathcal{L}_X \omega \wedge \omega^1 \wedge \omega^2 + \omega \wedge \mathcal{L}_X \omega^1 \wedge \omega^2 + \omega \wedge \omega^1 \wedge \mathcal{L}_X \omega^2 \\ &= \sum_{i=0}^{2} \left\langle \omega^i, [X_i, X] \right\rangle \omega \wedge \omega^1 \wedge \omega^2.\end{aligned}$$

Hence, $\mathrm{div}_\nu (X_1) = c_{12}^2$ and $\mathrm{div}_\nu (X_2) = -c_{12}^1$ which proves the statement. □

REMARK 35. Following the proof of Lemma 34, one can obtain an explicit expression of $\mathcal{H}_\mu$ with $\mu = h\nu$ a smooth measure where h is a positive smooth density. More precisely, let X_θ be a vector field transverse to the distribution $\mathcal{D}$ such that $\mu(X_1, X_2, X_\theta) = 1$. Let us denote by $\{\omega^1, \omega^2, \omega^\theta\}$ the dual basis to $\{X_1, X_2, X_\theta\}$, then $\mu = \omega^\theta \wedge \omega^1 \wedge \omega^2$. Furthermore, let us consider the structure functions associated to X_1, X_2, X_θ defined as in (1.4) (notice that X_θ here is transverse but not necessarily the Reeb vector field). Namely, we set

$$c_{ij}^k = \langle \omega^k, [X_j, X_i] \rangle, \qquad i, j, k = 1, 2, \theta.$$

Replacing X_0 and ω with X_θ and ω^θ in the proof of Lemma 34, one obtains

$$\operatorname{div}_\mu (X_1) = c_{12}^2 - c_{\theta 1}^\theta, \qquad \operatorname{div}_\mu (X_2) = -c_{12}^1 - c_{\theta 2}^\theta.$$

and consequently

$$\mathcal{H}_\mu = -X_1 X_1 \delta - X_2 X_2 \delta - (c_{12}^2 - c_{\theta 1}^\theta)(X_1 \delta) + (c_{12}^1 + c_{\theta 2}^\theta)(X_2 \delta). \tag{2.17}$$

Let us consider a smooth measure μ on M which is the Riemannian volume with respect to a Riemannian metric g that extends the sub-Riemannian metric. Therefore, an orthonormal frame for g is of the form $\{X_1, X_2, X_\theta\}$ as in Remark 35.

One can define a corresponding Riemannian approximation of the contact sub-Riemannian manifold as (M, g^ε), where g^ε is the Riemannian metric for which $\{X_1, X_2, \varepsilon X_\theta\}$ is an orthonormal frame.

The next proposition states that the sub-Riemannian mean curvature with respect to μ is the limit of the corresponding weighted Riemannian mean curvature.

PROPOSITION 36. *Let μ be a smooth measure which is the Riemannian volume with respect to a metric g that extends the sub-Riemannian metric. The mean sub-Riemannian curvature $\mathcal{H}_\mu$ of a regular surface S in M is the limit of the mean curvatures H_μ^ε of S in the corresponding Riemannian approximations*

$$\mathcal{H}_\mu = \lim_{\varepsilon \to 0} H_\mu^\varepsilon.$$

For the reader's convenience, the proof is postponed in Appendix A.

1.4. Coarea formula. In order to compute the volume of U_r we need a sub-Riemannian version of the coarea formula. We decompose the space $U_r = \bigcup_{t \in (0,r)} U^t$, where $U^t = \delta^{-1}(t)$ and we choose an adapted frame to such a decomposition.

DEFINITION 37. We define the orthogonal moving frame on TU_r :

$$\begin{aligned} F_1 &= (X_2 \delta) X_1 - (X_1 \delta) X_2, \\ F_2 &= (X_0 \delta) \nabla_H \delta - X_0, \\ N &= \nabla_H \delta. \end{aligned}$$

We notice that, for fixed $t \in (0, r)$, the vectors F_1, F_2 define a basis of TU^t, while N is the sub-Riemannian normal. Moreover, F_1, F_2 extend X_S and Y_S

defined on U given in (2.5) and (2.6) respectively. The corresponding dual basis on T^*U_r is:

$$\begin{aligned}\phi^1 &= (X_2\delta)\,\omega^1 - (X_1\delta)\,\omega^2,\\ \phi^2 &= -\omega,\\ \eta &= (X_1\delta)\,\omega^1 + (X_2\delta)\,\omega^2 + (X_0\delta)\,\omega.\end{aligned}$$

Let us now recall the classical Riemannian coarea formula for smooth functions (see for instance [**Cha06**]).

Proposition 38 (Coarea formula). *Let (M, g) be a Riemannian manifold equipped with a smooth measure μ. Let $\Phi : M \to \mathbb{R}$ be a smooth function such that $\nabla\Phi \neq 0$ and $f : M \to \mathbb{R}$ be a measurable non negative function, then*

$$\int_M f\,\|\nabla\Phi\|\,dV_\mu = \int_{\mathbb{R}}\int_{\Phi^{-1}(t)} f\,dA_t^R dt,$$

where dV_μ and dA_t^R are respectively the volume element on M induced by μ, and the induced Riemannian area form on $\Phi^{-1}(t)$.

We recall that the induced area form on $\Phi^{-1}(t)$, for $t \in \mathbb{R}$, is given by the interior product of the volume form dV_μ by $\frac{\nabla\Phi}{|\nabla\Phi|}$, the Riemannian normal to the surface, i.e.,

$$dA_t^R = \iota\left(\frac{\nabla\Phi}{\|\nabla\Phi\|}\right) dV_\mu\,|_{T\Phi^{-1}(t)}\,.$$

Now, let us consider $(M, \mathcal{D}, g)$ a three-dimensional contact manifold equipped with a smooth measure μ. Consider $\Phi : M \to \mathbb{R}$ a smooth function such that $\nabla_H\Phi \neq 0$, we define the induced sub-Riemannian area form on the surfaces $\Phi^{-1}(t)$

$$dA_t = \iota\left(\frac{\nabla_H\Phi}{\|\nabla_H\Phi\|}\right) dV_{\mu|T\Phi^{-1}(t)}.$$

This is the restriction of the volume form dV_μ on M, induced by the measure μ, by the sub-Riemannian normal $\frac{\nabla_H\Phi}{\|\nabla_H\Phi\|}$ to the surface $\Phi^{-1}(t)$.

We are ready to reformulate the coarea formula as follows.

Proposition 39 (Sub-Riemannian Coarea). *Let $(M, \mathcal{D}, g)$ be a three-dimensional contact manifold equipped with a smooth measure μ. Given $\Phi : M \to \mathbb{R}$ a smooth function such that $\|\nabla_H\Phi\| \neq 0$ and $f : M \to \mathbb{R}$ measurable non negative*

$$\int_M f\,\|\nabla_H\Phi\|\,d\mu = \int_{\mathbb{R}}\int_{\Phi^{-1}(t)} f_{|\Phi^{-1}(t)} dA_t dt,$$

where dA_t is the induced sub-Riemannian area form on $\Phi^{-1}(t)$.

Turning the attention to our problem and taking in account the bases previously defined on U_r (Definition 37), we have that the volume form dV_μ coincides with $\eta \wedge \phi^1 \wedge \phi^2$ up to a smooth density $h : M \to \mathbb{R}$. Then, the sub-Riemannian area form induced by μ on U^t is

$$dA_t = \iota\,(N)\,dV_\mu = h \cdot \iota\,(N)\,\eta \wedge \phi^1 \wedge \phi^2 = h \cdot \phi^1 \wedge \phi^2. \tag{2.18}$$

Finally, applying the sub-Riemannian coarea formula, and from (2.4),

$$\mu(U_r) = \int_{U_r} \frac{\|\nabla_H \delta\|}{\|\nabla_H \delta\|} dV_\mu = \int_0^r \int_{U^t} dA_t dt. \tag{2.19}$$

Proof. To obtain the result we have to clarify the relation between the induced Riemannian area form and the sub-Riemannian one on $\Phi^{-1}(t)$. As noticed before, the measure μ is proportional to the measure associated with Popp's volume up to a smooth density, and the same relation is true for the associated measures, volumes and area forms. Therefore, without lack of generality, we suppose that dV_μ is the Popp's volume $\omega \wedge \omega^1 \wedge \omega^2$. First of all, let us consider the following moving frame of vector fields that coincides with that of Definition 37 when $\Phi = \delta$:

$$\begin{aligned} F_1 &= \|\nabla_H \Phi\|^{-2} \left[(X_2\Phi)\, X_1 - (X_1\Phi)\, X_2 \right], \\ F_2 &= (X_0\Phi)\, \nabla_H \Phi - \|\nabla_H \Phi\|^2 X_0, \\ N &= \|\nabla_H \Phi\|^{-2} \nabla_H \Phi. \end{aligned}$$

The corresponding dual basis of 1-forms is written as follows:

$$\begin{aligned} \phi^1 &= (X_2\Phi)\, \omega^1 - (X_1\Phi)\, \omega^2, \\ \phi^2 &= -\|\nabla_H \Phi\|^{-2} \omega, \\ \eta &= (X_1\Phi)\, \omega^1 + (X_2\Phi)\, \omega^2 + (X_0\Phi)\, \omega. \end{aligned}$$

For a fixed $t \in \mathbb{R}$, the vector fields F_1 and F_2 are tangent to the level set $\Phi^{-1}(t)$, while N is transversal. The 1-form η annihilates the tangent space, while $\phi^1 \wedge \phi^2$ is an area form on the surface. Moreover, we have that

$$\begin{aligned} \eta \wedge \phi^1 \wedge \phi^2 (X_0, X_1, X_2) &= \eta(X_1)\, \phi^1(X_2)\, \phi^2(X_0) - \eta(X_2)\, \phi^1(X_1)\, \phi^2(X_0) \\ &= \|\nabla_H \Phi\|^{-2} \left[(X_1\Phi)^2 + (X_2\Phi)^2 \right] = 1, \end{aligned}$$

and this means that the volume form $\eta \wedge \phi^1 \wedge \phi^2$ coincides with the volume form $\omega \wedge \mathrm{d}\omega$ associated with the ν measure. Now, let us consider the Riemannian and sub-Riemannian induced area forms on $\Phi^{-1}(t)$:

$$dA_t^R = \iota\left(\frac{\nabla \Phi}{\|\nabla \Phi\|} \right) d\mu, \quad \text{and} \quad dA_t = \iota\left(\frac{\nabla_H \Phi}{\|\nabla_H \Phi\|} \right) d\mu.$$

To compute those area forms, we need to write the Riemannian outer normal in our new basis

$$\frac{\nabla \Phi}{\|\nabla \Phi\|} = \frac{\nabla_H \Phi + (X_0\Phi) X_0}{\|\nabla \Phi\|} = \rho N + A F_1 + B F_2.$$

We obtain that $A = 0$ because F_1 is orthogonal to $\nabla\Phi$. Moreover, projecting on X_0, it holds that

$$B = -\frac{1}{\|\nabla_H \Phi\|^2} \frac{X_0 \Phi}{\|\nabla \Phi\|}.$$

Then, substituting and computing the scalar product with N,

$$\frac{1}{\|\nabla \Phi\|} = \frac{\rho}{\|\nabla_H \Phi\|^2} - \frac{(X_0\Phi)^2}{\|\nabla_H \Phi\|^2 \|\nabla \Phi\|}$$

and we conclude that

$$\rho = \frac{\|\nabla_H \Phi\|^2}{\|\nabla \Phi\|} + \frac{(X_0\Phi)^2}{\|\nabla \Phi\|} = \|\nabla \Phi\| .$$

Since we consider 2-forms on $T\{\Phi^{-1}(t)\}$, we deduce that

$$dA_t^R = \iota\left(\frac{\nabla\Phi}{\|\nabla\Phi\|}\right) dV_\mu = \iota\,(\rho N)\, dV_\mu = \|\nabla\Phi\|\, \phi^1 \wedge \phi^2,$$

$$dA_t = \iota\left(\frac{\nabla_H\Phi}{\|\nabla_H\Phi\|}\right) dV_\mu = \iota\,(\|\nabla_H\Phi\|\, N)\, dV_\mu = \|\nabla_H\Phi\|\, \phi^1 \wedge \phi^2.$$

Finally, applying the coarea Riemannian formula in Proposition 38,

$$\int_M f\,\|\nabla_H\Phi\|\, dV_\mu = \int_{\mathbb{R}}\int_{\{\Phi=t\}} f\frac{\|\nabla_H\Phi\|}{\|\nabla\Phi\|} dA_t^R dt = \int_{\mathbb{R}}\int_{\{\Phi=t\}} f dA_t dt. \qquad \square$$

2. A localized formula for the volume of the half-tube

The measure of the localized half-tubular neighborhood U_r is expressed in (2.19) as

$$\mu\,(U_r) = \int_0^r \int_{U^t} dA_t dt,$$

where, for $t \in (0, r)$, dA_t is the sub-Riemannian area form (2.18) induced by the measure μ on the surface $U^t = G(t, U)$, with $G : (0, r) \times U \to U_r$ the diffeomorphism of Theorem 24. One can further transform the previous formula considering the diffeomorphism G at fixed time. Namely, the sub-Riemannian area form is

$$dA_t \mid_{(t,p)} = h(t,p) \cdot \phi^1 \wedge \phi^2 \mid_{(t,p)} = h(t,p) \cdot G_*\left(\phi^1 \wedge \phi^2 \mid_{(0,p)}\right),$$

and the expression for the measure of U_r becomes the following

$$\mu\,(U_r) = \int_0^r \int_U \left|\det\left(\mathrm{d}_p G|_{(t,p)}\right)\right| \frac{h(t,p)}{h(0,p)} dA_\mu dt,$$

where with dA_μ we denote dA_0, that is the induced sub-Riemannian area form on U by the measure μ in (2.18). We deduce that $\mu(U_r)$ is smooth at $r = 0$ and, in order to obtain its asymptotics stated in Theorem 42, we need to compute the Taylor expansion near $t = 0$ of the function

$$t \mapsto \left|\det\left(\mathrm{d}_p G|_{(t,p)}\right)\right| h(t,p),$$

for a fixed $p \in U$. For this purpose, we express the matrix representing $\mathrm{d}_p G|_{(t,p)} : T_pU \to T_{G(t,p)}U^t$ with respect to some bases as follows: on $T_{G(t,p)}U^t$, we choose $\{F_1, F_2\}|_{U^t}$ of the moving frame of Definition 37; while on T_pU we consider $\{X_S(p), Y_S(p)\}$ introduced in (2.5) and (2.6). Moreover, we define following vector fields on U_r:

$$V_1(t,p) = \mathrm{d}G\,(X_S(p))\,, \quad V_2(t,p) = \mathrm{d}G\,(Y_S(p))\,,$$

where in the left hand side (t, p) is a shorthand for computing the vector field at $G(t, p) \in U_r$.

PROPOSITION 40. *Let $(t,p) \in (0,r) \times U$, the matrix representing the linear operator $d_pG|_{(t,p)} : T_pU \to T_{G(t,p)}U^t$ with respect to the bases X_S, Y_S for T_pU and F_1, F_2 for $T_{G(t,p)}U^t$, is*

$$\mathrm{d}_pG|_{(t,p)} = \begin{pmatrix} -\langle V_1, JN\rangle & -\langle V_2, JN\rangle \\ -\langle V_1, X_0\rangle & -\langle V_2, X_0\rangle \end{pmatrix}(t,p), \tag{2.22}$$

where $N = \nabla_H\delta$, J is the linear map of Definition 18 and X_0 is the Reeb vector field of the contact manifold. In particular, it holds that

$$\mu(U_r) = \int_U\int_0^r \Big| \langle V_1, JN\rangle \langle V_2, X_0\rangle - \langle V_2, JN\rangle \langle V_1, X_0\rangle \Big|(t,p)\frac{h(t,p)}{h(0,t)}dtdA_\mu. \tag{2.23}$$

Before giving the proof we compare the vector fields V_1, V_2, F_1, F_2, and present a property that is useful also for the incoming computations.

By construction, for every $(t,p) \in U_r$, the vector fields V_1, V_2 are tangent to U^t. As already mentioned, the same is true for F_1, F_2, however they differ from the couple V_1, V_2 just introduced, as the following lemma states.

LEMMA 41. *Let us consider the vector fields F_1, N defined in Definition 37, and V_1, V_2 introduced in (2.21). For every $(t,p) \in U_r$, we have $[F_1, N] \neq 0$ and*

$$[V_1, N] = [V_2, N] = 0. \tag{2.24}$$

PROOF. Since F_1 and N are independent horizontal vector fields in a three-dimensional contact sub-Riemannian manifold, the Hörmander condition implies that the Lie bracket $[F_1, N]$ has to be everywhere nonvanishing.

Now, let us choose in U some coordinates (u_1, u_2) with corresponding coordinate fields $\{\partial_{u_1}, \partial_{u_2}\}$. The vector fields X_S, Y_S are tangent to U and can be written as

$$X_S = a_X^1\partial_{u_1} + a_X^2\partial_{u_2}, \quad Y_S = a_Y^1\partial_{u_1} + a_Y^2\partial_{u_2}.$$

with $a_X^1, a_X^2, a_Y^1, a_Y^2$ smooth functions on U. Moreover, since $G : (0,r)\times U \to U_r$ a diffeomorphism, we have that, for $i = 1, 2$,

$$[\mathrm{d}G(\partial_t), \mathrm{d}G(\partial_{u_1})] = \mathrm{d}G[\partial_t, \partial_{u_1}] = 0.$$

Finally, by Theorem 24, the horizontal gradient of the sub-Riemannian distance $N = \nabla_H\delta$ is $\mathrm{d}G(\partial_t)$. Thus, it holds that

$$[V_1, N] = \mathrm{d}G\left[a_X^1\partial_{u_1} + a_X^2\partial_{u_2}, \partial_t\right] = -\partial_t(a_X^1)\partial_{u_1} - \partial_t(a_X^2)\partial_{u_2} = 0.$$

An analogous computation proves that $[V_2, N] = 0$. □

PROOF OF PROPOSITION 40. To write the matrix representing $\mathrm{d}_pG|_{(t,p)}$, we need to project the vector fields $V_1(t,p)$ and $V_2(t,p)$ along the orthonormal basis $\frac{F_1}{\|F_1\|}, \frac{F_2}{\|F_2\|}$ in $T_{G(t,p)}U^t$. Then the linear combinations we are looking for are the following

$$V_i = \frac{\langle V_i, F_1\rangle}{\|F_1\|^2}F_1 + \frac{\langle V_i, F_2\rangle}{\|F_2\|^2}F_2, \qquad i = 1, 2,$$

and the matrix of our interest is

$$d_pG|_{(t,p)} = \begin{pmatrix} \dfrac{\langle V_1, F_1\rangle}{\|F_1\|^2} & \dfrac{\langle V_2, F_1\rangle}{\|F_1\|^2} \\ \dfrac{\langle V_1, F_2\rangle}{\|F_2\|^2} & \dfrac{\langle V_2, F_2\rangle}{\|F_2\|^2} \end{pmatrix} (t,p).$$

Now, recall that N is horizontal and unitary, so that, considering in addition JN and X_0 we have an orthonormal frame for TU_r. Therefore, looking at the formula in Definition 37, we have that in the new basis

$$F_1 = -JN \quad \text{and} \quad F_2 = (X_0\delta)N - X_0.$$

Hence $\|F_1\|^2 = 1$ and $\|F_2\|^2 = 1 + (X_0)^2$. Moreover, due to the fact that the Riemannian gradient of δ, i.e., $\nabla\delta = N + (X_0\delta)X_0$, is orthogonal to every surface $U^t = \delta^{-1}(t)$, we have that, for $i = 1, 2$, $\langle V_i, N\rangle = -(X_0\delta)\langle V_i, X_0\rangle$. Therefore,

$$\langle V_i, F_2\rangle = (X_0\delta)\langle V_i, N\rangle - \langle V_i, X_0\rangle = -\left(1 + (X_0\delta)^2\right)\langle V_i, X_0\rangle.$$

Finally, substituting we conclude the proof. □

3. The Taylor expansion of the volume

We are ready to prove the main result of this chapter.

THEOREM 42. *Let $(M, \mathcal{D}, g)$ be a contact three-dimensional sub-Riemannian manifold equipped with a smooth measure μ. Let $S \subset M$ be an embedded smooth surface bounding a closed region Ω and let $U \subset S$ be an open and relatively compact set such that its closure $\overline{U}$ does not contain characteristic points. The volume of the localized half-tubular neighborhood U_r, is smooth with respect to r and satisfies for $r \to 0$:*

$$\mu(U_r) = \sum_{k=1}^{3} a_k(U, \mu)\frac{r^k}{k!} + o(r^3) \tag{2.27}$$

where the coefficients $a_k = a_k(U, \mu)$ have the following expressions:

$$a_1 = \int_U dA_\mu, \quad a_2 = -\int_U \mathcal{H}_\mu \, dA_\mu, \quad a_3 = \int_U \left(-N(\mathcal{H}_\mu) + \mathcal{H}_\mu^2\right) dA_\mu.$$

Here dA_μ is the sub-Riemannian area measure on S induced by μ, $\mathcal{H}_\mu$ is the mean sub-Riemannian curvature of S with respect to μ and N is the sub-Riemannian normal to the surface.

We recall that (2.23), in Proposition 40, expresses the volume of the localized half-tubular neighborhood U_r with respect to μ as

$$\mu(U_r) = \int_U \int_0^r \Big| \langle V_1, JN\rangle \langle V_2, X_0\rangle - \langle V_2, JN\rangle \langle V_1, X_0\rangle \Big| (t,p)\frac{h(t,p)}{h(0,t)} dt dA_\mu,$$

where $h : U_r \to \mathbb{R}$ is the smooth density of μ with respect the Popp's measure ν (i.e., $\mu = h\nu$), dA_μ is the induced sub-Riemannian area form on U by μ (i.e., dA_0 in (2.18)), V_1, V_2 are the vector fields defined in (2.21), and $N = \nabla_H\delta$ is the sub-Riemannian gradient of the sub-Riemannian distance from the surface.

In order to prove the statement in Theorem 42, we fix $p \in U$, and we proceed in calculating the Taylor expansion centered in $t=0$ of the smooth function

$$C(t,p) = |\langle V_1, JN\rangle \langle V_2, X_0\rangle - \langle V_2, JN\rangle \langle V_1, X_0\rangle|(t,p), \tag{2.29}$$

Therefore, for $k=1,2,3$ the coefficients of the expansion (2.27) are given by

$$a_k = \int_U \sum_{\substack{i,j=0,\dots,k-1\\ i+j=k-1}} \binom{k-1}{i} \partial_t^i C \cdot \frac{\partial_t^j h}{h}(0,p) dA_\mu,$$

from which we obtain the expressions in (2.28) thanks to (2.14).

We start the proof observing a useful relation between the terms appearing in (2.29). Recall that $N = \partial_t$ thanks to Theorem 24.

Lemma 43. *Let us consider the vector fields V_i defined in (2.21), for $i = 1,2$. On the set U_r it holds that*

$$\partial_t \langle V_i, X_0\rangle = N\langle V_i, X_0\rangle = \langle V_i, JN\rangle .$$

Proof. We employ the properties of the Tanno connection introduced in Definition 20. Recalling property (2.24) and that $\nabla_X N$ is parallel to JN for every vector field X since N is unitary and horizontal, we deduce that

$$\begin{aligned} N\langle V_i, X_0\rangle &= \langle \nabla_N V_i, X_0\rangle \\ &= \langle \nabla_{V_i} N + [N, V_i] + \mathrm{Tor}(N, V_i), X_0\rangle = \langle \mathrm{Tor}(N, V_i), X_0\rangle . \end{aligned}$$

We split V_i in its the horizontal and Reeb component to exploit the properties of the torsion:

$$\begin{aligned} N\langle V_i, X_0\rangle &= \langle \mathrm{Tor}(N, V_i - \langle V_i, X_0\rangle X_0) + \mathrm{Tor}(N, \langle V_i, X_0\rangle X_0), X_0\rangle \\ &= -\langle N, JV_i - \langle V_i, X_0\rangle JX_0\rangle \end{aligned}$$

and we conclude since $JX_0 = 0$. □

Set $c_i(t,p) = \langle V_i, X_0\rangle$. Thanks to Lemma 43, formula (2.29) can be written as

$$C(t,p) = |\dot{c}_1 c_2 - c_1 \dot{c}_2|(t,p),$$

We recall that $C(t,p)$ comes from the determinant of matrix (2.22), that for every $p \in U$ at $t=0$ has the following expression:

$$\begin{pmatrix} -\dot{c}_1(0) & -\dot{c}_2(0) \\ -c_1(0) & -c_2(0) \end{pmatrix} = \begin{pmatrix} -\langle V_1, JN\rangle & -\langle V_2, JN\rangle \\ -\langle V_1, X_0\rangle & -\langle V_2, X_0\rangle \end{pmatrix}(0,p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.30}$$

Since $C(0,p) = |\dot{c}_1 c_2 - c_1 \dot{c}_2|(0,p) = 1$, we deduce the first term a_1 of the expansion (2.27) in Theorem 42. Namely

$$a_1 = \int_U C(0,p) dA_\mu = \int_U dA_\mu.$$

Again, taking into account the identities in (2.30), we obtain the following expressions for the derivatives of C in $t=0$:

$$\partial_t C(0,p) = (\ddot{c}_1 c_2 - c_1 \ddot{c}_2)(0,p) = -\ddot{c}_1(0,p); \tag{2.31}$$

$$\partial_t^2 C(0,p) = \left(c_1^{(3)} c_2 + \ddot{c}_1 \dot{c}_2 - \dot{c}_1 \ddot{c}_2 - c_1 c_2^{(3)}\right)(0,p) = -c_1^{(3)}(0,p) + \ddot{c}_2(0,p). \tag{2.32}$$

The work now is about to explicit these formulae with respect the sub-Riemannian curvature operators. In order to do that, we show the relation between the Lie brackets and the canonical Tanno connection when considering the key vector fields for the surface N and JN.

LEMMA 44. *We have that at every point of U_r*

$$[JN, N] = -\mathcal{H}JN - (X_0\delta)\,N + X_0,$$

where $\mathcal{H} : U_r \to \mathbb{R}$ is the sub-Riemannian mean curvature of the surface with respect to the Popp measure ν in Definition 32. Moreover,

$$\nabla_{JN}N = -\mathcal{H}JN \quad \text{and} \quad \nabla_N N = -(X_0\delta)JN.$$

PROOF. Exploiting (2.4), we can express the vector fields N and JN as a rotation of X_1, X_2:

$$\begin{pmatrix} N \\ JN \end{pmatrix} = \begin{pmatrix} X_1\delta & X_2\delta \\ -X_2\delta & X_1\delta \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Therefore, by Lemma 19,

$$\begin{aligned} \langle [JN, N], JN\rangle &= \langle [X_2, X_1] - X_1(\theta)\,X_1 - X_2(\theta)\,X_2, \sin\theta X_1 + \cos\theta X_2\rangle \\ &= \sin\theta c_{12}^1 + \cos\theta c_{12}^2 - \sin\theta X_1(\theta) - \cos\theta X_2(\theta) \\ &= -(X_2\delta)\,c_{12}^1 + (X_1\delta)\,c_{12}^2 + X_1X_1\delta + X_2X_2\delta, \end{aligned}$$

that is exactly the expression of $-\mathcal{H}$ in coordinates, see Lemma 34. With similar computations we obtain that

$$\begin{aligned} \langle [JN, N], N\rangle &= \langle [X_2, X_1] - X_1(\theta)\,X_1 - X_2(\theta)\,X_2, \cos\theta X_1 - \sin\theta X_2\rangle \\ &= (X_1\delta)\,c_{12}^1 + (X_2\delta)\,c_{12}^2 + X_1X_2\delta - X_2X_1\delta \\ &= (X_1\delta)\,c_{12}^1 + (X_2\delta)\,c_{12}^2 - [X_2, X_1]\delta = -(X_0\delta). \end{aligned}$$

Finally, since $\mathrm{Tor}\,(JN, N) = -X_0$, we have that

$$[JN, N] = \nabla_{JN}N - \nabla_N JN + X_0,$$

and thus we conclude from the fact that N and JN are unitary. In particular,

$$\nabla_{JN}N = -\mathcal{H}JN, \qquad -\nabla_N JN = -J\nabla_N N = -(X_0\delta)N. \qquad \square$$

The next two propositions concern the computations of the elements to obtain the second and third term in the Taylor expansion of Theorem 42.

PROPOSITION 45. *For every $p \in U$, it holds that*

$$\partial_t C(0, p) = \langle [JN, N], JN\rangle\,(0, p) = -\mathcal{H}(p)$$

where $\mathcal{H} : U_r \to \mathbb{R}$ is the mean sub-Riemannian curvature of the surface with respect to the Popp measure ν defined in Definition 32.

PROOF. Considering equation (2.31) and recalling that the Tanno connection is metric, we have

$$\partial_t C(t, p) = -\partial_t \langle V_1, JN\rangle = -\langle \nabla_N V_1, JN\rangle - \langle V_1, \nabla_N JN\rangle\,.$$

Evaluating at $t = 0$ the second addendum vanishes being $V_1(0, p) = -JN(0, p)$. For what concerns the first addendum, the idea is to get rid of V_1 applying the definition of the torsion operator and property (2.24). Hence we have that

$$\begin{aligned} -\langle \nabla_N V_1, JN \rangle (0, p) &= -\langle \nabla_{V_1} N + [N, V_1] + \mathrm{Tor}(N, V_1), JN \rangle (0, p) \\ &= \langle \nabla_{JN} N + \mathrm{Tor}(N, JN), JN \rangle (0, p). \end{aligned}$$

We conclude thanks to Lemma 44 and the fact that $\mathrm{Tor}(N, JN) = X_0$. □

PROPOSITION 46. *Given $p \in U$, it holds that*

$$\partial_t^2 C(0, p) = \left[X_S(X_0\delta) - (X_0\delta)^2 - \langle R(JN, N)N, JN \rangle - \langle \nabla_{X_0} N, JN \rangle \right] (0, p). \tag{2.34}$$

Moreover, considering the mean curvature $\mathcal{H} : U_r \to \mathbb{R}$ of Definition 32, the previous formula becomes

$$\partial_t^2 C(0, p) = \mathcal{H}^2 - N(\mathcal{H}). \tag{2.35}$$

PROOF. To obtain (2.34) we compute $c_1^{(3)}(0, p)$ and $\ddot{c}_2(0, p)$ separately, being $\ddot{C}(0, p)$ expressed as their difference in (2.32).

Step 1. We compute $c_1^{(3)}(0, p)$. Recall that $\dot{c}_1(t, p) = \langle V_1, JN \rangle (t, p)$, consequently:

$$\begin{aligned} c_1^{(3)}(t, p) &= N^2 \langle V_1, JN \rangle \\ &= \langle \nabla_N \nabla_N V_1, JN \rangle + 2 \langle \nabla_N V_1, \nabla_N JN \rangle + \langle V_1, \nabla_N \nabla_N JN \rangle. \end{aligned}$$

Computing the second and the third addendum in $t = 0$, we have respectively:

$$\begin{aligned} \langle \nabla_N V_1, \nabla_N JN \rangle &= \langle \nabla_{V_1} N + [N, V_1] + \mathrm{Tor}(N, V_1), \nabla_N JN \rangle \\ &= \langle -\nabla_{JN} N + \mathrm{Tor}(N, -JN), \nabla_N JN \rangle \\ &= \langle -\nabla_{JN} N - X_0, \nabla_N JN \rangle = 0, \end{aligned}$$

because $\nabla_{JN} N, \nabla_N JN$ are orthogonal to each other, and then

$$\begin{aligned} \langle V_1, \nabla_N \nabla_N JN \rangle &= -\langle JN, \nabla_N \nabla_N JN \rangle \\ &= -N \langle JN, \nabla_N JN \rangle + \langle \nabla_N JN, \nabla_N JN \rangle \\ &= \langle \nabla_N N, \nabla_N N \rangle \\ &= (X_0\delta)^2. \end{aligned}$$

Therefore, $c_1^{(3)}(0, p) = \left[\langle \nabla_N \nabla_N V_1, JN \rangle + (X_0\delta)^2 \right] (0, p)$.

Step 2. We compute $\ddot{c}_2(t, p)$. Since $\dot{c}_2(t, p) = \langle V_2, JN \rangle (t, p)$, it holds that

$$\ddot{c}_2(t, p) = N \langle V_2, JN \rangle = \langle \nabla_N V_2, JN \rangle + \langle V_2, \nabla_N JN \rangle.$$

Evaluating each addendum in $t = 0$, where $V_2(0, p) = (X_0\delta) N - X_0$, applying (2.24) and Lemma 44, we deduce that

$$\begin{aligned} \langle \nabla_N V_2, JN \rangle &= \langle \nabla_{V_2} N + [N, V_2] + \mathrm{Tor}(N, V_2), JN \rangle \\ &= \left\langle \nabla_{(X_0\delta)N - X_0} N + \mathrm{Tor}(N, (X_0\delta) N - X_0), JN \right\rangle \\ &= (X_0\delta) \langle \nabla_N N, JN \rangle - \langle \nabla_{X_0} N + \mathrm{Tor}(N, X_0), JN \rangle \\ &= -(X_0\delta)^2 + \langle [N, X_0], JN \rangle \end{aligned}$$

and

$$\langle V_2, \nabla_N JN\rangle = \langle (X_0\delta)\, N - X_0, \nabla_N JN\rangle = (X_0\delta)^2 .$$

Putting together, we have that $\ddot{c}_2(0,p) = \langle [N, X_0]\, , JN\rangle\, (0,p)$.

Step 3. We have the following intermediate expression for $\partial_t^2 C(0,p)$:

$$\partial_t^2 C(0,p) = \Big[- \langle \nabla_N\nabla_N V_1, JN\rangle - (X_0\delta)^2 + \langle [N, X_0]\, , JN\rangle\Big]\, (0,p).$$

Let us focus on the term still depending on the Jacobi field V_1. In order to get rid of V_1 we need to use the curvature operator associated with the connection ∇, i.e., $R(X;Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. Then, applying (2.24) and Lemma 44, we have:

$$\begin{aligned} - \langle \nabla_N\nabla_N V_1, JN\rangle &= - \langle \nabla_N\nabla_{V_1} N + \nabla_N \operatorname{Tor}(N, V_1)\, , JN\rangle \\ &= - \Big\langle \nabla_{V_1}\nabla_N N + R(N, V_1)N + \nabla_{[N,V_1]}N, JN\Big\rangle \\ &\quad - \langle \nabla_N \operatorname{Tor}(N, V_1)\, , JN\rangle \\ &= \langle V_1(X_0\delta)JN + R(V_1, N)N, JN\rangle \\ &\quad - \langle \nabla_N \operatorname{Tor}(N, V_1)\, , JN\rangle . \end{aligned}$$

Since the first scalar product is tensorial in the arguments in which V_1 appears, evaluating in $t = 0$ we have:

$$- \langle \nabla_N\nabla_N V_1, JN\rangle = -JN(X_0\delta) - \langle R(JN, N)N, JN\rangle - \langle \nabla_N \operatorname{Tor}(N, V_1)\, , JN\rangle .$$

At this point, looking at the scalar product involving the torsion term, we decompose V_1 along the orthonormal frame N, JN, X_0, i.e.,

$$\begin{aligned} \operatorname{Tor}(N, V_1) &= \operatorname{Tor}(N, \langle V_1, N\rangle N + \langle V_1, JN\rangle JN + \langle V_1, X_0\rangle X_0) \\ &= \langle V_1, JN\rangle X_0 + \langle V_1, X_0\rangle \operatorname{Tor}(N, X_0). \end{aligned}$$

Evaluating at $t = 0$, we get $\langle V_1, X_0\rangle = 0$ and we obtain that

$$\begin{aligned} - \langle \nabla_N \operatorname{Tor}(N, V_1)\, , JN\rangle &= - \langle N\, \langle V_1, JN\rangle\, X_0 + N\, \langle V_1, X_0\rangle \operatorname{Tor}(N, X_0)\, , JN\rangle \\ &= - \langle (V_1, JN) \operatorname{Tor}(N, X_0)\, , JN\rangle \\ &= \langle \operatorname{Tor}(N, X_0)\, , JN\rangle . \end{aligned}$$

Since on U we have that $X_S = -JN$, we get that on $(0,p) \in U$

$$- \langle \nabla_N\nabla_N V_1, JN\rangle\, (0,p) = X_S\, (X_0\delta) + \langle -R(JN, N)N + \operatorname{Tor}(N, X_0)\, , JN\rangle .$$

Finally:

$$\begin{aligned} \partial_t^2 C(0,p) &= - \langle \nabla_N\nabla_N V_1, JN\rangle - (X_0\delta)^2 + \langle [N, X_0]\, , JN\rangle \\ &= X_S(X_0\delta) - (X_0\delta)^2 \\ &\quad + \langle -R(JN, N)N + \operatorname{Tor}(N, X_0) + [N, X_0]\, , JN\rangle \\ &= X_S(X_0\delta) - (X_0\delta)^2 - \langle R(JN, N)N, JN\rangle - \langle \nabla_{X_0} N, JN\rangle . \end{aligned}$$

Step 4. The last part of the proof is to obtain (2.35). Exploiting Lemma 44, we deduce that

$$
\begin{aligned}
-\langle R(JN,N)N,JN\rangle &= \left\langle -\nabla_{JN}\nabla_N N + \nabla_N\nabla_{JN}N + \nabla_{[JN,N]}N, JN\right\rangle \\
&= JN(X_0\delta) - N(\mathcal{H}) + \left\langle \nabla_{-\mathcal{H}JN-(X_0\delta)N+X_0}N, JN\right\rangle \\
&= -X_S(X_0\delta) - N(\mathcal{H}) + \mathcal{H}^2 + (X_0\delta)^2 + \langle \nabla_{X_0}N, JN\rangle .
\end{aligned}
$$

Substituting in (2.34), we obtain the statement. □

4. An operative formula for the coefficients in the asymptotics

In this Section we prove that one can compute the coefficients in expansion (2.27) of Theorem 42 only in terms of a smooth function f locally defining the surface $S = f^{-1}(0)$, without the explicit knowledge of the distance function δ.

PROPOSITION 47. *Under the assumptions of Theorem 42, let us suppose that it is assigned the Popp's measure ν and that S is locally defined as the level set of a smooth function $f : M \to \mathbb{R}$ such that $\nabla_H f|_U \neq 0$ and $\langle \nabla_H f, \nabla_H \delta\rangle|_U > 0$. Then, we have the following formulas for a_2 and a_3 in expansion* (2.27)*:*

$$
a_2 = -\int_U \operatorname{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right) dA;
$$

$$
a_3 = \int_U \left[2X_S\left(\frac{X_0 f}{\|\nabla_H f\|}\right) - \left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2 - \kappa - \left\langle \mathrm{Tor}\left(X_0, X_S\right), \frac{\nabla_H f}{\|\nabla_H f\|}\right\rangle\right] dA
$$

with $\kappa = \langle R(X_1,X_2)X_2,X_1\rangle$ where R and Tor *are the curvature and the torsion operators associated with the Tanno connection and the operative expression for the characteristic vector field is*

$$
X_S = \frac{X_2 f}{\|\nabla_H f\|}X_1 - \frac{X_1 f}{\|\nabla_H f\|}X_2.
$$

REMARK 48. This property is true for the coefficients a_1 and a_2 appearing in the in expansion (2.27), as it is evident from their explicit expression. Indeed, as noticed in Remark 33, the mean sub-Riemannian curvature $\mathcal{H}$, is well defined on the surface by (2.13). This is not a priori clear for the coefficient a_3, since it depends on a derivative of $\mathcal{H}$ along N, which is a vector field transversal to the surface. Thus, it might depend on the value of $\mathcal{H}$ outside S.

Before proving Proposition 47, we recall the two local metric invariants χ and κ, that are smooth functions on M characterizing a three-dimensional contact manifold. These quantities are strictly related to the curvature operator associated with the Tanno connection ∇.

The first metric functional invariant κ of a three-dimensional sub-Riemannian contact manifold is the Tanno sectional curvature of the distribution, i.e., the smooth function defined as

$$
\kappa = \langle R(X_1,X_2)X_2,X_1\rangle,
$$

where X_1, X_2 is a local orthonormal frame for $\mathcal{D}$. Denoting by $\tau : \mathcal{D} \to \mathcal{D}$ the linear map $\tau(X) = \mathrm{Tor}(X_0, X)$, we define the second metric invariant χ as the function

$$\chi = \sqrt{-\det \tau}.$$

In terms of (1.4) the metric invariants χ and κ we have the following expression:

$$\chi^2 = -c_{01}^1 c_{02}^2 + \frac{\left(c_{01}^2 + c_{02}^1\right)^2}{4},$$
$$\kappa = X_2\left(c_{12}^1\right) - X_1\left(c_{12}^2\right) - \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2 + \frac{c_{01}^2 - c_{02}^1}{2}.$$

For more details see [**ABB20**, Sections 17.2, 17.6] (see also [**Bar13**]).

Proof of Proposition 47. As explained in Remark 48, we need to prove the result only in the case of a_3. Since both f and δ locally define the surface, as their zero level set and $\langle \nabla_H f, \nabla_H \delta\rangle|_U > 0$, there exists a smooth function h such that $f = e^h \delta$. Exploiting (2.4), we have that for $i = 0, 1, 2$

$$\left.\left(\frac{X_i f}{\|\nabla_H f\|}\right)\right|_U = \left.\left(\frac{X_i \delta + \delta X_i h}{\|\nabla_H \delta + \delta \nabla_H h\|}\right)\right|_U = X_i \delta|_U. \tag{2.36}$$

Therefore, the equivalence of the definition of X_S in the current statement with that of (2.5) is proved.

Recalling (2.34) we have the following expression for the coefficient a_3:

$$a_3 = \int_U \left[X_S(X_0\delta) - (X_0\delta)^2 - \langle R(JN,N)N, JN\rangle - \langle \nabla_{X_0} N, JN\rangle\right] dA.$$

Let us focus on the first term in the integral. We observe that

$$\begin{aligned}\nabla_H\left(\frac{X_0 f}{\|\nabla_H f\|}\right) &= \nabla_H\left(\frac{X_0\delta + \delta X_0 h}{\|\nabla_H \delta + \delta\nabla_H h\|}\right)\\ &= \frac{\nabla_H(X_0\delta) + (X_0 h)\nabla_H \delta}{\|\nabla_H\delta + \delta\nabla_H h\|} + (X_0\delta)\nabla_H\left(\frac{1}{\|\nabla_H\delta + \delta\nabla_H h\|}\right).\end{aligned}$$

And evaluating at $U \subset S$

$$\begin{aligned}\nabla_H\left(\frac{X_0 f}{\|\nabla_H f\|}\right) &= \nabla_H(X_0\delta) + (X_0 h)N - (X_0\delta)\frac{1}{2}\nabla_H(1 + 2\delta N h + \delta^2 \|\nabla_H h\|^2)\\ &= \nabla_H(X_0\delta) + (X_0 h - (X_0\delta)Nh)N.\end{aligned}$$

Since X_S is orthogonal to N, we deduce that on U

$$X_S(X_0\delta) = \langle -JN, \nabla_H(X_0\delta)\rangle = X_S\left(\frac{X_0 f}{\|\nabla_H f\|}\right).$$

Moreover, since N, JN is a local orthonormal frame for the sub-Riemannian structure, as well as X_1, X_2, a direct computation shows that

$$\langle R(JN,N)N, JN\rangle = \langle R(X_1,X_2)X_2, X_1\rangle = \kappa.$$

Hence, from (2.36), we deduce that on U

$$-(X_0\delta)^2 - \langle R(JN,N)N, JN\rangle = -\left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2 - \kappa.$$

Finally, we observe that

$$-\langle \nabla_{X_0} N, JN\rangle = \langle \mathrm{Tor}(N, X_0) + [N, X_0], JN\rangle .$$

Then, making the computations for the second addendum we obtain that

$$\begin{aligned}\langle [N, X_0], JN\rangle &= \langle [(X_1\delta)X_1 + (X_2\delta)X_2, X_0], -(X_2\delta)X_1 + (X_1\delta)X_2\rangle \\ &= (X_2\delta)(X_0X_1\delta) - (X_1\delta)(X_0X_2\delta) \\ &\quad + \langle (X_1\delta)[X_1, X_0] + (X_2\delta)[X_2, X_0], -(X_2\delta)X_1 + (X_1\delta)X_2\rangle \\ &= (X_2\delta)(X_1X_0\delta) - (X_1\delta)(X_2X_0\delta) - c_{01}^2(X_2\delta)^2 + c_{02}^1(X_1\delta)^2 \\ &\quad + 2(c_{02}^2 - c_{01}^1)(X_1\delta)(X_2\delta).\end{aligned}$$

Recalling (1.6), we deduce $\langle [N, X_0], JN\rangle = -JN(X_0\delta) + 2\,\langle \mathrm{Tor}(X_0, N), JN\rangle$, thus

$$-\langle \nabla_{X_0} N, JN\rangle = X_S(X_0\delta) - \langle \mathrm{Tor}(X_0, N), X_S\rangle .$$

The torsion operator is tensorial in its arguments, we conclude thanks to (2.36). □

Remark 49. If we specify Theorem 42 to the three-dimensional Heisenberg group $\mathbb{H}$, we obtain a result that is consistent with [**Rit21**]. Indeed, chosen an orthonormal frame X_1, X_2 for the distribution, we have the following relations:

$$[X_2, X_1] = X_0 \quad \text{and} \quad [X_1, X_0] = [X_2, X_0] = 0.$$

The geometric invariant κ vanishes and thus, exploiting Proposition 47, we compute the expansion (2.27) with respect to ν:

$$\nu(U_r) = r\int_U dA - \frac{r^2}{2}\int_U \mathcal{H}dA + \frac{r^3}{6}\int_U 2X_S\left(X_0\delta\right) - \left(X_0\delta\right)^2 dA + o(r^3).$$

We warn the reader that in [**Rit21**] the author chooses an orthonormal basis X, Y, T for the Riemannian extension, that in our notations reads $X = X_1$, $Y = X_2$ and $X_0 = 2T$. Moreover, the vector field $-e_1$ corresponds to X_S in our notations, as well as dP to dA, and $\frac{N}{\|N_h\|}$ to the Riemannian gradient $\nabla\delta = \nabla_H\delta + (X_0\delta)X_0$. In particular, $2\frac{\langle N,T\rangle}{\|N_h\|} = X_0\delta$. The third term in the expansion presented in [**Rit21**]

$$-\frac{2r^3}{3}\int_U e_1\frac{\langle N, T\rangle}{\|N_h\|} + \frac{\langle N, T\rangle^2}{\|N_h\|^2}dP,$$

coincides with the one obtained here.

Remark 50. The explicit knowledge of the sub-Riemannian distance function δ from the surface S, is in general necessary in order to compute expansion (2.27) with order greater or equal than four. Indeed, this is true already for surfaces S in the Heisenberg group $\mathbb{H}$, thanks to the results in [**BFF⁺15**]. Let $U \subset S$ relatively compact and such that the closure does not contain any characteristic point. We have that $\nu(U_r) = \sum_{k=1}^4 \frac{r^k}{k!}a_k + o(r^4)$, with

$$a_4 = \int_U (X_0\delta)^2\mathcal{H} + 2X_0X_0\delta\, dA, \tag{2.38}$$

where $\mathcal{H} = -X_1X_1\delta - X_2X_2\delta$ is the mean sub-Riemannian curvature of Definition 32 specified in the case of $\mathbb{H}$.

Let us consider a smooth local defining function $f : \mathbb{H} \to \mathbb{R}$ such that $U = f^{-1}(0)$ and $f = e^h \delta$ on U_r. The first term in (2.38) can be equivalently written as:

$$(X_0\delta)^2 \mathcal{H}|_U = -\left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2 \operatorname{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right).$$

Moreover, since

$$X_0 X_0 f = e^h \left(\delta X_0 X_0 h + \delta (X_0 h)^2 + 2(X_0 h)(X_0 \delta) + X_0 X_0 \delta\right),$$

and $\|\nabla_H f\|\Big|_U = e^h \|\nabla_H \delta + \delta \nabla_H h\|\Big|_U = e^h$, we obtain that on the surface

$$X_0 X_0 \delta = \frac{X_0 X_0 f}{\|\nabla_H f\|} - 2(X_0 h)\frac{X_0 f}{\|\nabla_H f\|}.$$

The last identity shows that, in order to compute a_4, the explicit knowledge of the distance function δ is needed (i.e., one cannot get rid of h).

We are ready to specialize our result for some particular three-dimensional sub-Riemannian contact manifolds. Firstly, we consider the class of three-dimensional contact manifold with $\chi = 0$. We recall that for this type of spaces there exists a local orthonormal frame such that

$$[X_1, X_0] = \kappa X_2, \quad [X_2, X_0] = -\kappa X_1, \quad [X_2, X_1] = X_0. \tag{2.39}$$

In addition, when the structure of the contact manifold is left invariant on a Lie group with $\chi \neq 0$, we can choose a canonical frame X_1, X_2 of the distribution that is unique up to a sign and such that:

$$[X_1, X_0] = c_{01}^2 X_2, \quad [X_2, X_0] = c_{02}^1 X_1, \quad [X_2, X_1] = X_0 + c_{12}^1 X_1 + c_{12}^2 X_2, \tag{2.40}$$

and with (cf. [**ABB20**, Prop. 17.14])

$$\chi = \frac{c_{01}^2 + c_{02}^1}{2},$$

Taking into account these spaces and exploiting Proposition 47 we have the following expression for the term a_3 in Theorem 42.

COROLLARY 51. *Given a three-dimensional contact manifold with $\chi = 0$,*

$$a_3 = \int_U 2X_S(X_0\delta) - (X_0\delta)^2 - \kappa \, dA. \tag{2.41}$$

In the case of a left invariant sub-Riemannian contact structure on a three-dimensional Lie group with $\chi \neq 0$, we have that

$$a_3 = \int_U 2X_S(X_0\delta) - (X_0\delta)^2 - \kappa + \chi\left[(X_1\delta)^2 - (X_2\delta)^2\right] dA$$

with X_1, X_2 the unique (up to a sign) canonical basis chosen such that (2.40) *holds.*

REMARK 52. The coefficients a_1, a_2, a_3 of expansion (2.27) in Theorem 42 are integrals of iterated horizontal divergence of δ. More precisely,

$$a_k = \int_U \operatorname{div}_\mu^{k-1}(\nabla_H \delta)\, dA_\mu, \qquad k = 1, 2, 3. \tag{2.42}$$

The iterated divergences of a vector field X, with respect to a smooth measure μ, on M are defined as

$$\operatorname{div}_\mu^0(X) = 1, \qquad \operatorname{div}_\mu^{k+1}(X) = \operatorname{div}_\mu(\operatorname{div}_\mu^k(X)X).$$

In particular, identity (2.42) holds tautologically for a_1 and by Definition 32 for a_2. For what concerns the coefficient a_3, we have that

$$\operatorname{div}_\mu^2(\nabla_H\delta) = \operatorname{div}_\mu(\operatorname{div}_\mu(\nabla_H\delta)\nabla_H\delta) = -\operatorname{div}(\mathcal{H}_\mu N) = -N(\mathcal{H}_\mu) + \mathcal{H}_\mu^2,$$

where $N = \nabla_H\delta$ by Theorem 24 and thanks to the property of the divergence for which $\operatorname{div}(aX) = X(a) + a\operatorname{div}(X)$ for every smooth function a. This fact extends the result in [**BFF$^+$15**] for a surface in the Heisenberg group endowed with the Popp measure. In fact, $\nu(U_r)$ is analytic in r

$$\nu(U_r) = \sum_{k=1}^{\infty} \frac{r^k}{k!} \int_{\partial\Omega} \operatorname{div}^{k-1}(\nabla_H\delta)\, d\mathcal{H}^3_{d_{cc}},$$

where $d\mathcal{H}^3_{d_{cc}}$ is the three-dimensional Hausdorff measure of the metric of the Carnot-Carathéodory distance d_{cc} on the surface. Moreover, the authors prove that the iterated divergences are polynomials of certain second order derivatives of the distance function for which it is given a precise recursive formula.

5. Surfaces in model spaces

We compute expansion (2.27) in Theorem 42 for a class of surfaces in model sub-Riemannian spaces, with μ equal to the Popp measure ν. We exploit Proposition 47, avoiding the explicit computation of the sub-Riemannian distance from the surface.

First, we consider the class of rotational surfaces in the Heisenberg group $\mathbb{H}$. Afterwards, following the construction presented in [**BBCH21**, Section 5], we take into account two specific surfaces in the model spaces $SU(2)$ and $SL(2)$. We show that for these surfaces the coefficients a_2 and a_3 in expansion (2.27) are always vanishing. This generalizes the case of the horizontal plane in $\mathbb{H}$ shown in [**BFF$^+$15**].

5.1. Rotational surfaces in $\mathbb{H}$. Let us consider the Heisenberg group $\mathbb{H} = (\mathbb{R}^3, \omega)$, where the contact form is

$$\omega = dz - \frac{y}{2}dx + \frac{x}{2}dy,$$

the distribution $\mathcal{D} = \ker\omega$ is spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z} \quad \text{and} \quad X_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z},$$

and the Reeb vector field is $X_0 = \frac{\partial}{\partial z}$. Moreover, let $f : \mathbb{H} \to \mathbb{R}$ be defined by

$$f(x, y, z) = z - g(\rho)$$

where $\rho = \sqrt{x^2 + y^2}$ and $g : [0, +\infty) \to \mathbb{R}$ is a smooth function such that $g'(0) = 0$. The level set $S = f^{-1}(0)$ is a smooth surface with a unique characteristic point in $(0, 0, g(0))$. Indeed, if one considers other points of the surface, the vector field $-y\partial_x + x\partial_y$ is tangent to the surface at such points, but it is not contained in the distribution $\mathcal{D}$, while the distribution coincides with the tangent plane to S at $(0, 0, g(0))$ because $\partial_x f = \partial_y f = 0$ at such a point.

Let us consider $U \subset S$ an open and relatively compact set such that its closure does not contain the characteristic point $(0,0,g(0))$. In order to obtain the coefficient a_2, we proceed to compute the mean curvature $\mathcal{H}$ restricted on the surface exploiting (2.13). From (2.16) in Lemma 34, the divergence with respect to the Popp's measure ν of both X_1 and X_2 in $\mathbb{H}$ is zero, therefore the quantity we need to is the following

$$-\mathcal{H}|_S = \frac{X_1X_1f + X_2X_2f}{\|\nabla_H f\|} + \nabla_H f\left(\frac{1}{\|\nabla_H f\|}\right). \tag{2.44}$$

We compute separately the elements composing the previous formula:

$$X_1 f = \frac{y}{2} - g'(\rho)\frac{x}{\rho}, \qquad X_2 f = -\frac{x}{2} - g'(\rho)\frac{y}{\rho};$$

$$X_1X_1f + X_2X_2f = -g''(\rho) - \frac{g'(\rho)}{\rho};$$

$$\frac{1}{\|\nabla_H f\|} = \frac{1}{\sqrt{(X_1f)^2 + (X_2f)^2}} = \frac{2}{\sqrt{\rho^2 + 4g'(\rho)^2}}.$$

Hence we can compute

$$\nabla_H f = \left(\frac{y}{2} - g'(\rho)\frac{x}{\rho}\right)\partial_x - \left(\frac{x}{2} + g'(\rho)\frac{y}{\rho}\right)\partial_y + \frac{\rho^2}{4}\partial_z;$$

$$\nabla_H f\left(\frac{1}{\|\nabla_H f\|}\right) = -2\frac{\rho + 4g'(\rho)g''(\rho)}{(\rho^2 + 4g'(\rho)^2)^{\frac{3}{2}}}\nabla_H f(\rho) = 2g'(\rho)\frac{\rho + 4g'(\rho)g''(\rho)}{(\rho^2 + 4g'(\rho)^2)^{\frac{3}{2}}}.$$

Plugging into (2.44), we obtain that

$$-\mathcal{H} = -2\frac{g''(\rho) + \frac{g'(\rho)}{\rho}}{\sqrt{\rho^2 + 4g'(\rho)^2}} + 2g'(\rho)\frac{\rho + 4g'(\rho)g''(\rho)}{(\rho^2 + 4g'(\rho)^2)^{\frac{3}{2}}} = -2\frac{4g'(\rho)^3 + \rho^3 g''(\rho)}{\rho(\rho^2 + 4g'(\rho)^2)^{\frac{3}{2}}}.$$

Finally, we compute the elements needed for a_3 exploiting Proposition 47:

$$\begin{aligned} X_S &= \frac{X_2 f}{\|\nabla_H f\|}X_1 - \frac{X_1 f}{\|\nabla_H f\|}X_2 \\ &= \frac{1}{\|\nabla_H f\|}\left[-\left(\frac{x}{2} + g'(\rho)\frac{y}{\rho}\right)\partial_x - \left(\frac{y}{2} - g'(\rho)\frac{x}{\rho}\right)\partial_y - \frac{\rho}{2}g'(\rho)\partial_z\right] \end{aligned}$$

and

$$\begin{aligned} 2X_S\left(\frac{X_0 f}{\|\nabla_H f\|}\right) - \left(\frac{X_0 f}{\|\nabla_H f\|}\right)^2 &= -\frac{2}{\|\nabla_H f\|^3}\left(\rho + 4g'(\rho)g''(\rho)\right)X_S(\rho) \\ &\quad - \frac{1}{\|\nabla_H f\|^2} \\ &= 4g'(\rho)\frac{\rho g''(\rho) - g'(\rho)}{\|\nabla_H f\|^4}. \end{aligned}$$

We conclude that

$$\begin{aligned} \nu(U_r) = r\int_U dA - r^2\int_U \frac{4g'(\rho)^3 + \rho^3 g''(\rho)}{\rho(\rho^2 + 4g'(\rho)^2)^{\frac{3}{2}}}dA \\ + \frac{r^3}{6}\int_U 4g'(\rho)\frac{rg''(\rho) - g'(\rho)}{(\rho^2 + 4g'(\rho)^2)^2}dA + o(r^3). \end{aligned}$$

5.2. Examples in model spaces. Let us consider in the Heisenberg group $\mathbb{H}$ the plane $S = f^{-1}(0)$ with $f(x,y,z) = z$. The surface S bounds $\Omega = \{z \leq 0\}$ and has an unique characteristic point in the origin. Let $U \subset S$ be open and relatively compact such that $(0,0,0) \notin \overline{U}$. Considering the same setting as in the previous subsection, one can compute expansion (2.27) of Theorem 42 obtaining that

$$\nu(U_r) = r\int_U dA + o(r^3). \tag{2.45}$$

This result agrees with the example presented in [**BFF$^+$15**]. In what follows the spaces of interest are the Lie groups $SU(2)$ and $SL(2)$ equipped with standard sub-Riemannian structures. Together with the Heisenberg group, these spaces play the role of model spaces for the three-dimensional contact sub-Riemannian manifolds.

5.2.1. *The Special Unitary Group* $SU(2)$. The Special Unitary group is the Lie group of the 2×2 unitary matrices of determinant 1 with the matrix multiplication,

$$SU(2) = \left\{ \begin{pmatrix} z+iw & y+ix \\ -y+ix & z-iw \end{pmatrix} : x,y,z,w \in \mathbb{R},\, x^2+y^2+z^2+w^2 = 1 \right\}.$$

The Lie algebra $su(2)$ is the algebra of the 2×2 skew-Hermitian matrices with vanishing trace. Let us identify $SU(2)$ with the unitary sphere $S^3 \subset \mathbb{R}^4$, and given $k > 0$, we define the left-invariant vector fields on the Lie group:

$$\begin{aligned} X_1 &= k\left(z\frac{\partial}{\partial x} - w\frac{\partial}{\partial y} - x\frac{\partial}{\partial z} + y\frac{\partial}{\partial w}\right), \\ X_2 &= k\left(w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} - x\frac{\partial}{\partial w}\right), \\ X_0 &= 2k^2\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}\right). \end{aligned}$$

The relations given by the Lie brackets between these vector fields correspond to that in (2.39) with $\kappa = 4k^2$. Now, let us consider the smooth function $f(x,y,z,w) = w$ that defines the surface $S = \{w = 0\} \simeq S^2$. We have that

$$X_1 f = ky, \quad X_2 f = -kx, \quad X_0 f = 2k^2 z.$$

There are two characteristic points $(0,0,\pm 1,0)$, that are two opposite poles in the sphere, and correspond to $\pm\mathbb{I}$, where $\mathbb{I}$ is the identity 2×2 matrix. Let us choose $U \subset S$ relatively compact such that the closure does not contain any characteristic point. In coordinates,

$$\|\nabla_H f\| = k\sqrt{x^2+y^2}, \qquad \nabla_H f|_U = k^2 yz\frac{\partial}{\partial x} - k^2 xz\frac{\partial}{\partial y} + k^2(x^2+y^2)\frac{\partial}{\partial w}.$$

From (2.16) in Lemma 34, we have that $\operatorname{div}X_1 = \operatorname{div}X_2 = 0$, and so we compute the mean sub-Riemannian curvature as previously with (2.44).

$$-\mathcal{H}|_U = \frac{X_1(y) - X_2(x)}{\sqrt{x^2+y^2}} - \frac{x\nabla_H f(x) + y\nabla_H f(y)}{k(x^2+y^2)^{\frac{3}{2}}} = 0.$$

For a_3 we exploit Proposition 47 (more precisely (2.41) in Corollary 51):

$$\frac{X_0 f}{\|\nabla_H f\|} = \frac{2kz}{\sqrt{x^2+y^2}}, \qquad X_S = \frac{k}{\sqrt{x^2+y^2}}\left(-xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y} + (x^2+y^2)\frac{\partial}{\partial z}\right).$$

And thus, we have the following formula, which proves (2.45)

$$\begin{aligned} a_3 &= \int_U \left[\frac{4k}{\sqrt{x^2+y^2}} X_S(z) - \frac{4kz}{(x^2+y^2)^{\frac{3}{2}}}(xX_S(x) + yX_S(y)) - \frac{4k^2z^2}{x^2+y^2} - 4k^2\right] dA \\ &= \int_U \left[4k^2 + \frac{4k^2z^2}{x^2+y^2} - \frac{4k^2z^2}{x^2+y^2} - 4k^2\right] dA = 0. \end{aligned}$$

5.2.2. *The Special Linear Group* $SL(2,\mathbb{R})$. The special linear group is the Lie group of 2×2 matrices with determinant 1 with the matrix multiplication,

$$SL(2,\mathbb{R}) = \left\{(x,y,z,w) \in \mathbb{R}^4 : xw - yz = 1\right\}.$$

The corresponding Lie algebra $sl(2,\mathbb{R})$ is the vector space of the 2×2 real matrices with vanishing trace. For $k > 0$, we define the left invariant vector fields

$$\begin{aligned} X_1 &= k\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}\right), \\ X_2 &= k\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}\right), \\ X_0 &= 2k^2\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}\right). \end{aligned}$$

The commutation relations for these vector fields, are of type in (2.44) with $\kappa = -4k^2$. Let us consider $f(x,y,z,w) = y - z$ defining the plane $S = f^{-1}(0)$. We obtain that

$$X_1 f = k(x-w), \qquad X_2 f = -k(y+z), \qquad X_0 f = 2k^2(x+w).$$

The characteristic points are $(\pm 1, 0, 0, \pm 1)$ that correspond to $\pm\mathbb{I}$, where $\mathbb{I}$ is the identity 2×2 matrix. Let us choose $U \subset S$ relatively compact such that the closure does not contain any characteristic point. We have that

$$\|\nabla_H f\| = k\sqrt{(x-w)^2 + (y+z)^2}$$

and, considering that on S it holds the relation $xw = 1 + y^2$,

$$\nabla_H f_{|S} = k^2 y(w+x)\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial x}\right) + k^2(x^2+y^2-1)\frac{\partial}{\partial y} + k^2(1-y^2-w^2)\frac{\partial}{\partial z}.$$

From (2.16) in Lemma 34, we have that $\operatorname{div} X_1 = \operatorname{div} X_2 = 0$, and so we compute the mean sub-Riemannian curvature as previously with (2.44).

$$\begin{aligned}
-\mathcal{H}|_U &= \frac{X_1(x-w) - X_2(y+z)}{\sqrt{(x-w)^2+(y+z)^2}} - \frac{(x-w)\nabla_H f(x-w) + (y+z)\nabla_H f(y+z)}{k((x-w)^2+(y+z)^2)^{\frac{3}{2}}} \\
&= 2k\frac{y-z}{\sqrt{(x-w)^2+(y+z)^2}} - k\frac{-(x-w)2y(x+w)+(y+z)(x^2-w^2)}{(x^2+2y^2+w^2-2)^{\frac{3}{2}}} \\
&= 0.
\end{aligned}$$

For the coefficient a_3, we compute the characteristic vector field X_S on U

$$X_S = \frac{-k^2}{\|\nabla_H f\|}\left[(x^2+y^2-1)\frac{\partial}{\partial x} + y(w+x)\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) + (y^2+w^2-1)\frac{\partial}{\partial w}\right].$$

And thus, from Proposition 47 (more precisely, from (2.41) in Corollary 51),

$$\begin{aligned}
a_3 &= \int_U \left[2X_S\left(\frac{2k(x+w)}{\sqrt{(x-w)^2+(y+z)^2}}\right) - \left(\frac{2k(x+w)}{\sqrt{(x-w)^2+(y+z)^2}}\right)^2 + 4k^2\right] dA \\
&= \int_U \left[\frac{4kX_S(x+w)}{\sqrt{(x-w)^2+(y+z)^2}} - 4k(x+w)\frac{(x-w)X_S(x-w)+2yX_S(y+z)}{((x-w)^2+(y+z)^2)^{\frac{3}{2}}}\right. \\
&\qquad \left. - \frac{4k^2(x+w)^2}{(x-w)^2+(y+z)^2} + 4k^2\right] dA \\
&= \int_U \left[-4k^2\frac{x^2+2y^2+w^2-2}{(x-w)^2+(y+z)^2} - 4k^2(x+w)\frac{(x-w)(x^2-w^2)+4y^2(x+w)}{((x-w)^2+(y+z)^2)^2}\right. \\
&\qquad \left. + \frac{4k^2(x+w)^2}{(x-w)^2+(y+z)^2} + 4k^2\right] dA = 0
\end{aligned}$$

after some simplifications. We conclude, again, that (2.45) holds.

CHAPTER 3

Steiner's and Weyl's formulae in sub-Riemannian geometry

The present Chapter contains some preliminary results that are part of [**BRR24**] (paper in preparation). We prove the Steiner's and Weyl's tube formulae for non-characteristic submanifold of arbitrary codimension within a sub-Riemannian structure equipped with a smooth volume. As we develop the theory for C^2 submanifolds, we investigate some regularity properties of the distance function necessary to deduce the regularity of the tubular volume. Furthermore, we compute the coefficients of the asymptotic expansion as the size of the tube approaches to zero. These coefficients are associated with the integrals of the so called *iterated divergences* of the horizontal gradient of the distance.

Structure of the Chapter

Let us consider S, a submanifold of a sub-Riemannian manifold M equipped with a smooth measure μ. The goal of this Chapter is to study the measure of S_r, the tubular neighborhood of S defined with respect to the sub-Riemannian distance. In the case S bounds a regular domain Ω, we consider the (internal) half-tubular neighborhood $S_r^+ = S_r \cap \Omega$.

In Section 1 we recall some basic definitions and facts about δ, the sub-Riemannian distance from S, and we define the tubular and half-tubular neighborhoods trough the sub-Riemannian exponential map.

In Section 2 we proceed to investigate the regularity of δ, deducing that it has the same global regularity as S. In addition, it is smooth (or real-analytic) along the geodesics from S whenever M and μ are smooth (or real-analytic). This properties are inherited also by the derivatives of δ in all directions up to the second order.

In Section 3 we take into account the iterated divergences of the horizontal gradient of δ. Due to the lack of regularity of δ, these quantities a priori are defined in the distributional sense. Actually, we show that they can be expressed in terms of at most second order derivatives of δ, and hence they are well-defined in the strong sense.

Since the distance is not regular enough at S when the codimension is ≥ 1, we consider δ^2 instead. In Section 4 we introduce the *auxiliary iterated divergences* and we prove some regularity properties.

As a consequence of all the previous considerations on the regularity of δ, in Section 5 we deduce the smoothness (or real-analyticity) of the tubular-volume and we state an improved version of the sub-Riemannian Mean Value Lemma.

Finally, in Section 6 we obtain the sub-Riemannian Steiner's and Weyl's tube formulae and we characterize more precisely the coefficients of the interested asymptotics.

1. Preliminaries

Let M be a smooth (or real-analytic) sub-Riemannian manifold. In the present chapter we indicate the horizontal gradient of a function $f \in C^1(M)$ with ∇f, omitting the subscript $_H$ in contrast with the notation of the other chapters.

1.1. Length-minimizers to the boundary. Let us consider a closed embedded submanifold $S \subset M$ of class C^1. The *distance from* S is defined as

$$\delta : M \to [0, \infty); \qquad \delta(p) := \min\{\mathsf{d}(p, q) : q \in S\}.$$

We say that a horizontal curve $\gamma : [0, 1] \to M$ is a *geodesic from* S if it is a constant speed (globally length-minimizing) geodesic, such that

$$\gamma_0 \in S, \qquad \gamma_1 \in M \setminus S \qquad \text{and} \qquad \ell(\gamma) = \delta(p).$$

If $\gamma : [0, 1] \to M$ is a geodesic from S, any corresponding extremal lift, $\lambda : [0, 1] \to T^*M$, must satisfy the transversality conditions of [**AS04**, Thm. 12.4]:

$$\langle \lambda_0, v\rangle = 0, \qquad \forall\, v \in T_{\gamma_0}S. \tag{3.1}$$

In other words, the initial covector λ_0 must belong to the *annihilator bundle*

$$\mathcal{A}S = \{\lambda \in T^*M : \langle \lambda, T_{\pi(\lambda)}S\rangle = 0\}.$$

Moreover, the *normal exponential map* is the restriction of the sub-Riemannian exponential map to the annihilator bundle, namely

$$E : \mathcal{A}S \to M; \qquad E(\lambda) := \exp_{\pi(\lambda)}(\lambda).$$

Definition 53 (Non-characteristic submanifold). Let M be a sub-Riemannian manifold and let $S \subset M$ be a closed embedded submanifold of class C^1. A point $q \in S$ is said to be *non-characteristic* if

$$\mathcal{D}_q + T_qS = T_qM.$$

We denote by $\mathrm{Char}(S)$ the set of characteristic points of S. We say that S is a *non-characteristic submanifold* if $\mathrm{Char}(S) = \emptyset$. Moreover, if S is a non-characteristic hypersurface of class C^ℓ ($\ell \geq 1$) and bounds a relatively compact open set Ω, we say that Ω is a C^ℓ *non-characteristic domain.*

When S is a non-characteristic submanifold, (3.1) implies that the curves that realizes the distance from S are all normal, cf. [**Ros22**, Lemma 3.4].

Proposition 54. *Let M be a sub-Riemannian manifold and let $S \subset M$ be a closed embedded submanifold of class C^1. Let $\gamma\colon [0, 1] \to M$ be a geodesic from S. If γ is an abnormal geodesic, then γ_0 is a characteristic point of S.*

1.2. Regularity of the distance from a submanifold. We report a regularity result for the distance δ from a non-characteristic submanifold S, see [**FPR20**, Prop. 3.1] for the case of codimension 1 and [**Ros22**, Thm. 3.3] for the general case. Here and below, we set

$$S_r := \{p \in M : 0 < \delta < r\}. \tag{3.2}$$

THEOREM 55. *Let M be a sub-Riemannian manifold and let $S \subset M$ be a compact non-characteristic submanifold of class C^ℓ, with $\ell \geq 2$. Then, the following statements hold:*

i) δ is Lipschitz with respect to d and $\|\nabla\delta\| \leq 1$ a.e.;
ii) there exists $r_0 > 0$ such that $\delta \in C^{\ell-1}(S_{r_0})$ and

$$\|\nabla\delta\|_g = \sqrt{2H(\mathrm{d}\delta)} = 1, \qquad \text{on } S_{r_0}. \tag{3.3}$$

In addition, $\delta^2 \in C^{\ell-1}(S_{r_0} \cup S)$;
iii) the normal sub-Riemannian exponential map

$$E : \{\lambda \in \mathcal{A}S : \sqrt{2H(\lambda)} < r_0\} \to S_{r_0} \cup S$$

defines a $C^{\ell-1}$-diffeomorphism such that $\delta(E(\lambda)) = \sqrt{2H(\lambda)}$.

The proof of the previous theorem is analogous to the one found in [**Ros22**], taking into account the suitable regularity. Without adding further hypotheses, the assumption $\ell \geq 2$ is necessary, see [**Foo84**].

REMARK 56. In Section 2, we show that if S is a non-characteristic submanifold of class C^ℓ, then δ is actually of class C^ℓ on S_{r_0}. Additionally, we prove that δ and its derivatives up to order ℓ have the same regularity as the ambient structure when restricted to geodesics from S.

1.3. The tube and the half-tube. Let us define the *unit annihilator bundle* as the sub-bundle of $\mathcal{A}S$ given by:

$$\mathcal{A}^1S := \{\lambda \in \mathcal{A}S : 2H(\lambda) = 1\} \subset \mathcal{A}S, \tag{3.4}$$

and let us consider the map

$$E^1 : \mathbb{R} \times \mathcal{A}^1S \to M, \qquad E^1(t, \lambda) := E(t\lambda). \tag{3.5}$$

Recall that, from item iii) of Theorem 55, E is a diffeomorphism onto its image when restricted to the set $\{\lambda \in \mathcal{A}S : 2H(\lambda) < r_0^2\}$. As a consequence (the restriction of) the map $E^1 : (0, r_0) \times \mathcal{A}^1S \to M$ defines a diffeomorphism onto its image S_{r_0}, and it holds that

$$\mathrm{d}E^1(\partial_r) = \nabla\delta. \tag{3.6}$$

When $S = \partial\Omega$ is the boundary of a C^k non-characteristic domain $\Omega \subset M$ (thus S is a two-sided hypersurface), we adopt the following conventions. First of all, $\delta : M \to \mathbb{R}$ is the *signed distance from* S, with positive sign in Ω, namely

$$\delta(p) := \begin{cases} \inf\{\mathsf{d}(p,q) : q \in S\}, & \text{if } p \in \Omega \\ -\inf\{\mathsf{d}(p,q) : q \in S\}, & \text{if } p \in M \setminus \Omega. \end{cases} \tag{3.7}$$

Secondly, $\mathcal{A}^1_+ S$ is the connected component of the unit annihilator bundle[1] such that the image the map E is contained in Ω. Namely,

$$\mathcal{A}^1_+ S = \{\lambda \in \mathcal{A}^1 S : E^1(t,\lambda) \in \Omega \text{ for } t > 0 \text{ small}\}.$$

In this case, we regard the map E^1 as defined on $\mathbb{R} \times S$, and it holds the following improvement of Theorem 55 (cf. [**Ros22**, Thm. 3.7]).

PROPOSITION 57. *Let M be a sub-Riemannian manifold and let $\Omega \subset M$ be a non-characteristic domain of class C^ℓ, with $\ell \geq 2$. Let δ be the signed distance function defined in (3.7), then*

$$\delta \in C^{\ell-1}\left(S_{r_0} \cup S\right).$$

Moreover, $\mathrm{d}\delta$ *defines a non-trivial section of* $\mathcal{A}^1_+ S$ *and the map*

$$G = E^1 \circ \mathrm{d}\delta : [0, r_0) \times S \to M \qquad (r,q) \mapsto E(r\mathrm{d}_q\delta) = E^1(r,,\mathrm{d}_q\delta),$$

is a $C^{\ell-1}$-diffeomorphism onto its image such that

$$\delta(E^1(r,q)) = r \qquad \textit{and} \qquad \mathrm{d}G(\partial_r) = \mathrm{d}E^1(\partial_r) = \nabla\delta.$$

We are in position to provide the definition of the tubular and half-tubular neighborhoods. The following definitions are of local nature and the definition of the half-tube is equivalent to (2.3) presented in Chapter 2.

DEFINITION 58 (Tube and half-tube). Let M be a sub-Riemannian manifold, S be a closed non-characteristic submanifold of class C^ℓ, with $\ell \geq 2$, and $U \subset S$ be an open set. Then, the *tube* over U of size $r \in (0, r_0)$ is the open set

$$U_r := \{E^1(t,\lambda) : \lambda \in \mathcal{A}^1 S,\ \pi(\lambda) \in U \text{ and } 0 < t < r\}. \tag{3.8}$$

Moreover, if $S = \partial\Omega$ is the boundary of a non-characteristic domain $\Omega \subset M$, then it is defined the *half-tube* over $U \subset S$ of size r, as

$$U_r^+ := \{E^1(t,q) : q \in U \text{ and } 0 < t < r\}. \tag{3.9}$$

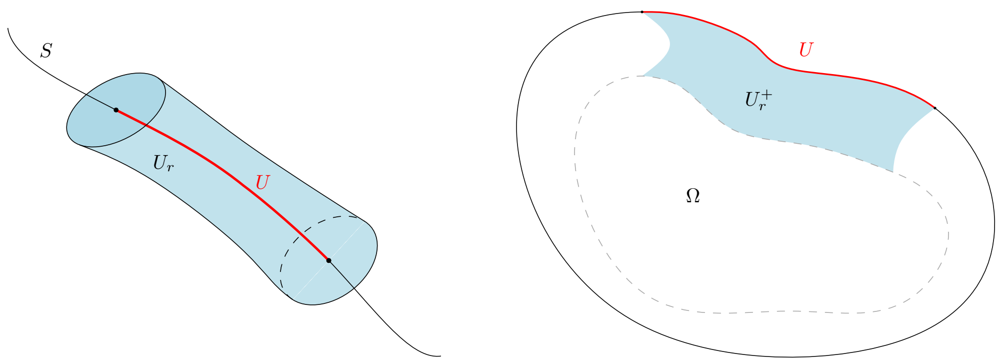

FIGURE 1. The tube U_r and the half-tube U_r^+, respectively.

[1]If $S = \partial\Omega$ then $\{\lambda \in \mathcal{A}S : 2H(\lambda) = 1\} \cong S \times \{-1, 1\} \cong S \sqcup S$, thus it has two connected components isomorphic to S.

According to Theorem 55, for every $p \in S_{r_0}$, there exists a unique $\lambda \in \mathcal{A}S$ such that $\sqrt{2H(\lambda)} = \delta(p) < r_0$, $E(\lambda) = p$ and the curve

$$[0,1] \ni t \mapsto E^1\left(t\delta(p), \frac{\lambda}{\sqrt{2H(\lambda)}}\right) = E(t\lambda) \tag{3.10}$$

is the unique geodesic from S starting at $\pi(\lambda) \in S$. In particular, according to Proposition 57, if $S = \partial\Omega$ is the boundary of a non-characteristic domain, we deduce that the curve $[0,1] \ni t \mapsto E^1(tr_0, q)$ is the unique geodesic from S starting at $q \in S$ and lying in Ω.

Therefore, if U is the whole submanifold S in (3.8), we obtain an equivalent definition for S_r in (3.2). In addition, the tubular and half-tubular neighborhoods can be regarded as the collection of all the geodesics from S (and lying in Ω in the half-tube case).

2. Regularity of the distance from a submanifold

In this section, we prove a regularity result for the distance δ from a submanifold. We recall some basic regularity properties of the Hamiltonian flow.

Let M be a complete sub-Riemannian structure and let $\Phi : \mathbb{R} \times T^*M \to M$ be the *extended Hamiltonian flow* on T^*M, namely

$$\Phi(t, \lambda) := e^{t\vec{H}}(\lambda), \qquad \forall\, (t, \lambda) \in \mathbb{R} \times T^*M. \tag{3.11}$$

In what follows, (M, d) is always assumed to be complete.

LEMMA 59. *Let M be a smooth (or real-analytic) sub-Riemannian manifold. Then, the extended Hamiltonian flow $\Phi : \mathbb{R} \times T^*M \to T^*M$ is smooth (or real-analytic).*

PROOF. By definition of extended flow, the curve $t \mapsto \Phi(t, \lambda)$ is the solution to the following ODE on the cotangent bundle:

$$\begin{cases} \dot{\lambda}(t) = \vec{H}(\lambda(t)), \\ \lambda(0) = \lambda, \end{cases}$$

or, equivalently,

$$\begin{cases} \partial_t \Phi(t, \lambda) = \vec{H}(\Phi(t, \lambda)) \\ \Phi(0, \cdot) = \mathrm{id}. \end{cases}$$

On the one hand, if the sub-Riemannian structure on M is smooth, then the Hamiltonian vector field $\vec{H}$ is smooth on the cotangent bundle. Thus, Φ is smooth as well. On the other hand, if the sub-Riemannian structure is real-analytic, then $\vec{H}$ a real-analytic vector field. Therefore, applying Cauchy–Kovaleskaya theorem, cf. [**Hör03**, Thm. 9.4.5], we deduce that Φ is real-analytic. □

Furthermore, for the sake of clarity, we report the following corollary.

COROLLARY 60. *Let M be a smooth (or real-analytic) sub-Riemannian structure. Then, the linearized extended Hamiltonian flow, namely*

$$\Psi : \mathbb{R} \times T(T^*M) \to T(T^*M), \qquad \Psi(t, \xi) := \mathrm{d}_{\pi(\xi)}\Phi(t, \cdot)(\xi) = e^{t\vec{H}}_{*,\pi(\xi)}(\xi), \tag{3.12}$$

is smooth (or real-analytic).

LEMMA 61. *Let M be a smooth sub-Riemannian manifold and $S \subset M$ be a non-characteristic submanifold of class C^2. Let $p \in S_{r_0}$ and let $\gamma : [0,1] \to M$ be the unique geodesic from S with and end point at p. Then,*

$$\mathrm{d}_{\gamma_t}\delta = \Phi((t-s)\delta(p), \mathrm{d}_{\gamma_s}\delta) \qquad \forall\, t,s \in (0,1]. \tag{3.13}$$

where Φ is the extended Hamiltonian flow in (3.11). If $S = \partial\Omega$ for a non-characteristic domain $\Omega \subset M$, (3.13) holds for every $t,s \in [0,1]$.

PROOF. Let $\lambda \in \mathcal{A}S$ be the unique covector such that $E(\lambda) = p$. The curve γ is characterized as

$$\gamma_t = \pi \circ \Phi(t,\lambda) \qquad \forall\, t \in [0,1],$$

where $\pi : T^*M \to M$ is the canonical projection. We may apply [**Rif14**, Lemma 2.15] to show that $\Phi(1,\lambda) = \delta(p)\mathrm{d}_p\delta$. Indeed, let $q := \pi(\lambda)$ and consider the function $\phi := \delta^2$. Then, ϕ is in $C^1(S_{r_0} \cup S)$ by Theorem 55 and, in addition it satisfies

$$\phi(p) = \mathsf{d}^2(q,p) \qquad \text{and} \qquad \phi(z) \le \mathsf{d}^2(q,z), \quad \forall\, z \in M.$$

Therefore, the unique minimizing curve γ is the projection of a normal extremal with final covector $\frac{1}{2}d_p\phi = \delta(p)\mathrm{d}_p\delta \in T_p^*M$. Moreover, since S is non-characteristic, all geodesics from S are strictly normal, meaning that the final covector of γ is unique and thus it holds $\Phi(1,\lambda) = \frac{1}{2}\mathrm{d}_p\phi$, as claimed. Analogously, for every $t \in (0,1]$, we obtain that

$$\Phi(1,t\lambda) = \delta(\gamma_t)\mathrm{d}_{\gamma_t}\delta = t\delta(p)\mathrm{d}_{\gamma_t}\delta. \tag{3.14}$$

In addition, since H is homogeneous of degree 2, $\Phi(1,t\lambda) = t\Phi(t,\lambda)$, thus (3.14) becomes:

$$\Phi(t,\lambda) = \delta(p)\mathrm{d}_{\gamma_t}\delta, \qquad \forall\, t \in (0,1]. \tag{3.15}$$

Now fix $t,s \in (0,1]$. From identity (3.15), evaluated at time t and s, and using the group property of $\Phi(\cdot,\lambda)$, we deduce that

$$\lambda = \delta(p)\Phi(-t\delta(p), \mathrm{d}_{\gamma_t}\delta) = \delta(p)\Phi(-s\delta(p), \mathrm{d}_{\gamma_s}\delta),$$

from which (3.13) follows.

Finally, if $S = \partial\Omega$, we can take the limit as $t \to 0$ (or $s \to 0$) in (3.13), since the signed distance is C^1 up to S, cf. Proposition 57. □

LEMMA 62. *Let M be a smooth (or real-analytic) sub-Riemannian manifold and let S be a C^ℓ non-characteristic submanifold, with $\ell \ge 2$. Then, there exists $r_0 > 0$ such that*

$$\delta \in C^\ell(S_{r_0}), \qquad \textit{and} \qquad \delta^2 \in C^\ell(S_{r_0} \cup S). \tag{3.16}$$

Moreover, let X, Y be smooth (or real-analytic) vector fields locally defined on $S_{r_0} \cup S$. Then, the functions δ, $X\delta$, $YX\delta$ are smooth (or real-analytic) along any geodesic from S and contained in S_{r_0}.

If $S = \partial\Omega$ for a non-characteristic domain $\Omega \subset M$, then the above regularity properties are valid up to S (when δ is in this case the signed distance).

PROOF. By Theorem 55, $E : \{\lambda \in \mathcal{A}S : \sqrt{2H(\lambda)} < r_0\} \to S_{r_0} \cup S$ is a $C^{\ell-1}$ diffeomorphism. Thus, exploiting (3.15), for all $p \in S_{r_0}$ it holds

$$\mathrm{d}_p\delta = \Phi\left(\delta(p), \frac{E^{-1}(p)}{\delta(p)}\right). \tag{3.17}$$

It follows that $\delta \in C^\ell(S_{r_0})$. For the squared distance, observe that (3.17) implies that

$$\mathrm{d}_p\delta^2 = 2\delta(p)\mathrm{d}_p\delta = 2\Phi(1, E^{-1}(p)).$$

This shows that $\delta^2 \in C^\ell(S_{r_0} \cup S)$ and concludes the proof of (3.16).

We turn our attention to the regularity along the geodesics from S. Let $\gamma : [0,1] \to M$ be such a curve, with $\gamma_1 = p \in S_{r_0}$. First of all, $\delta(\gamma_t) = \delta(p)t$, which is indeed analytic. Second of all, $(0,1] \ni t \mapsto \mathrm{d}_{\gamma_t}\delta$ is smooth (or real-analytic) as a consequence of Lemma 61, and in particular of (3.13), with $s = 1$. Finally, for every vector field X, let us consider the evaluation map $h_X : T^*M \to \mathbb{R}$ defined as $h_X(\lambda) = \langle \lambda, X\rangle$. Then, we write

$$YX\delta|_p = \mathrm{d}_p(X\delta)Y = h_Y(\mathrm{d}_p(h_X(\mathrm{d}\delta))) = h_Y \circ \mathrm{d}_{\mathrm{d}_p\delta}h_X \circ \mathrm{d}_p\mathrm{d}\delta.$$

where $\mathrm{d}\delta : S_{r_0} \to T^*M$ is of class $C^{\ell-1}$. Since X, Y are smooth (or real-analytic) vector fields, so are h_Y and $\mathrm{d}h_X$. We are left to show that $\mathrm{d}(\mathrm{d}\delta) : TS_{r_0} \to T(T^*M)$ is smooth (or real-analytic) along γ. Let us reformulate (3.13) using the map E^1 defined in (3.5): if $\delta(p) = r$, then $\gamma_t = E^1(tr, \lambda)$ for some $\lambda \in \mathcal{A}^1S$ and

$$\mathrm{d}_{E^1(tr,\lambda)}\delta = \Phi((t-s)r, \mathrm{d}_{E^1(sr,\lambda)}\delta) \qquad \forall\, t, s \in (0,1]. \tag{3.18}$$

Fix a frame $\{v_1, \dots, v_{n-1}\}$ for $\mathcal{A}^1_{+,\lambda}(S)$ and define $Y_i(t) := (\mathrm{d}_\lambda E^1(tr, \cdot))(v_i)$. Then, for $t \in (0,1]$, setting $Y_n(t) := \dot\gamma_t$, the family $\{Y_1(t), \dots, Y_n(t)\}$ is a smooth (or real-analytic) moving frame for $T_{\gamma_t}M$ along γ. On the one hand, for every $i = 1, \dots, n-1$, differentiating (3.18) with respect to λ, and evaluating along γ_t, we obtain that for all $t, s \in (0,1]$

$$\mathrm{d}_{\gamma_t}\mathrm{d}\delta(Y_i(t)) = \mathrm{d}_\lambda(\mathrm{d}\delta \circ E^1(tr, \cdot))(v_i) = \Psi((t-s)r, \mathrm{d}_{\gamma_s}\mathrm{d}\delta(Y_i(s))), \tag{3.19}$$

where Ψ is the linearization of Φ, see (3.12). On the other hand, a direct computation using Lemma 61 shows that

$$\mathrm{d}_{\gamma_t}\mathrm{d}\delta(Y_n(t)) = \partial_t\Phi((t-s)r, \mathrm{d}_{\gamma_s}\delta), \qquad \forall\, t, s \in (0,1]. \tag{3.20}$$

Therefore, we conclude that (the matrix representation of) the map $(0,1] \ni t \mapsto \mathrm{d}_{\gamma_t}\mathrm{d}\delta$ is smooth (or real-analytic) thanks to Corollary 60 and the analogous regularity of $\mathrm{d}\delta$. This concludes the proof for non-characteristic submanifolds.

If $S = \partial\Omega$, in order to conclude, it is enough to show that $\delta \in C^\ell(S_{r_0} \cup S)$. Indeed, if this is the case, as (3.18) now holds for $t \in [0,1]$, the identities (3.19) and (3.20) can now be extended up to $t = 0$, implying the claimed regularity. Since S is of class C^ℓ, there exists a (global, without loss of generality) C^ℓ defining function for S, that is there exists $f \in C^\ell(M)$ such that

$$\partial\Omega = \{f = 0\}, \qquad \mathrm{d}f|_{\partial\Omega} \neq 0.$$

We can also assume that $\sqrt{2H(\mathrm{d}f)} = 1$ on $S = \partial\Omega$. As a consequence, since now $\mathcal{A}S$ has one-dimensional fiber, up to changing sign, $\mathrm{d}\delta = \mathrm{d}f$ on S and, applying Lemma 61 for $p \in S_{r_0} \cup S$, $t = 1$ and $s = 0$, we obtain

$$\mathrm{d}_p\delta = \Phi(\delta(p), \mathrm{d}_{\pi\circ E^{-1}(p)} f),$$

Then, as E is a $C^{\ell-1}$ diffeomorphism by item iii) of Theorem 55, we deduce that $\mathrm{d}\delta$ is of class $C^{\ell-1}$ on $S_{r_0} \cup S$. □

3. Regularity of the iterated divergences of $\nabla\delta$

In this section, we study the regularity of the so-called *iterated divergences*. We recall that for a smooth vector field $X \in \Gamma(TM)$ and $k \in \mathbb{N}$, we set

$$\operatorname{div}^0(X) := 1, \qquad \operatorname{div}^{k+1}(X) := \operatorname{div}(\operatorname{div}^k(X)X), \tag{3.21}$$

We apply this definition to the vector field $X = \nabla\delta$, which by Lemma 62 is not smooth in general. Hence, a priori, definition (3.21) must be understood in the sense of distributions. However, we are going to prove that the iterated divergences are well-defined in the strong sense.

Lemma 63. *Let M be a smooth (or real-analytic) sub-Riemannian manifold, $S \subset M$ be a C^2 non-characteristic submanifold. Let $Y_1, \ldots, Y_n$ be a local smooth (or real-analytic) frame for TM defined on a neighborhood $\mathcal{O} \subset M$. Then, for all $k \in \mathbb{N}$ there exists a polynomial P^k, with smooth (or real-analytic) coefficients on $\mathcal{O}$, homogeneous of degree k, in the variables $Y_\alpha\delta, Y_\alpha Y_\sigma\delta$ for $\alpha, \sigma = 1, \ldots, n$ such that,*

$$\operatorname{div}^k(\nabla\delta)(p) = P^k\left(Y_\alpha\delta(p), Y_\alpha Y_\sigma\delta(p)\right), \qquad \forall\, p \in \mathcal{O} \cap S_{r_0}.$$

If $S = \partial\Omega$ for a non-characteristic domain $\Omega \subset M$, then p can be chosen in $\mathcal{O} \cap (S_{r_0} \cup S)$.

Remark 64. In case M is a Lie group equipped with a left-invariant sub-Riemannian structure and a left-invariant (or right-invariant) measure, one can choose $Y_1, \ldots, Y_n$ to be a left-invariant global frame, adapted to the sub-Riemannian distribution. In this case the P^k's are polynomials with constant coefficients, canonically associated with the sub-Riemannian structure once a left-invariant frame is fixed. This is the case, for example, for all Carnot groups.

Remark 65. Lemma 63 can be read as a generalization of [**BFF$^+$15**, Thm. 1.2]. The main difference with the latter result is that we can not rely on a closed formula for the iterated divergences that holds in the Heisenberg group.

Proof of Lemma 63. We proceed by induction on k. Let $X_1, \ldots, X_N$ be a smooth (or real-analytic) generating family for the sub-Riemannian structure. In the following, we consider the indexes $\alpha, \sigma, \varrho \in \{1, \ldots, n\}$ and $h \in \{1, \ldots, N\}$ and we adopt the convention in which repeated indexes are summed. We have that $X_h = a_h^\alpha Y_\alpha$ for some smooth (or real-analytic) functions a_h^α defined on $\mathcal{O}$. Hence,

$$\nabla\delta = (X_h\delta)X_h = a_h^\alpha a_h^\sigma (Y_\alpha\delta)Y_\sigma.$$

The case $k=0$ is trivial since $\mathrm{div}^0(\nabla\delta)=1$ by definition. For the case $k=1$ we have that

$$\begin{aligned}\mathrm{div}(\nabla\delta) &= (X_hX_h\delta)+(X_h\delta)\mathrm{div}(X_h)\\ &= a_h^\alpha a_h^\sigma\left[(Y_\alpha Y_\sigma\delta)+(Y_\alpha\delta)\mathrm{div}(Y_\sigma)\right]\\ &\quad + a_h^\alpha(Y_\alpha a_h^\sigma)(Y_\sigma\delta)+a_h^\alpha(Y_\sigma a_h^\sigma)(Y_\alpha\delta)\end{aligned} \tag{3.22}$$

is a homogeneous polynomial of degree 1 with smooth (or real-analytic) coefficients in the variables $Y_\alpha\delta, Y_\alpha Y_\sigma\delta$.

Then, let us suppose that the statement is true for some $k\geq 1$, i.e. that

$$\mathrm{div}^k(\nabla\delta)=P^k\left(Y_\alpha\delta, Y_\alpha Y_\sigma\delta\right) \qquad \text{for } \alpha,\sigma=1,\dots,n.$$

We prove that the statement holds also for $\mathrm{div}^{k+1}(\nabla\delta)$. Exploiting (3.22) and the inductive hypothesis in (3.21), we obtain that

$$\mathrm{div}^{k+1}(\nabla\delta)=\mathrm{div}(\nabla\delta)\,\mathrm{div}^k(\nabla\delta)+\nabla\delta(\mathrm{div}^k(\nabla\delta))=P^1P^k+\nabla\delta(P^k), \tag{3.23}$$

where $P^1=\mathrm{div}(\nabla\delta)$, and we omit the explicit dependence on the variables. We claim that (3.23) is homogeneous of degree $k+1$. By Leibniz, it is sufficient to show that for $i,j=1,\dots,n$ fixed

$$\nabla\delta(Y_i\delta) \qquad \text{and} \qquad \nabla\delta(Y_iY_j\delta)$$

are homogeneous polynomials of degree 2 in the variables $Y_\alpha\delta, Y_\alpha Y_\sigma\delta$, where $\nabla\delta(Y_iY_j\delta)$ has to be intended in the distributional sense.

On the one hand, the claim follows easily for $\nabla\delta(Y_i\delta)$ by the definition of $\nabla\delta$. Indeed,

$$\nabla\delta(Y_i\delta)=a_h^\alpha a_h^\sigma(Y_\alpha\delta)(Y_\sigma Y_i\delta).$$

On the other hand, considering the bracket relations $[Y_\alpha,Y_\sigma]=c_{\alpha\sigma}^\varrho Y_\varrho$ where $c_{\alpha\sigma}^\varrho:\mathcal{O}\to\mathbb{R}$ are smooth (or real-analytic), we obtain the following distributional identity:

$$\begin{aligned}\nabla\delta(Y_iY_j\delta) &= a_h^\alpha a_h^\sigma(Y_\alpha\delta)(Y_\sigma Y_iY_j\delta)\\ &= a_h^\alpha a_h^\sigma(Y_\alpha\delta)\left[(Y_iY_\sigma Y_j\delta)+c_{\sigma i}^\varrho(Y_\varrho Y_j\delta)\right]\\ &= a_h^\alpha a_h^\sigma(Y_\alpha\delta)\left[(Y_iY_jY_\sigma\delta)+Y_i\left(c_{\sigma j}^\varrho(Y_\varrho\delta)\right)+c_{\sigma i}^\varrho(Y_\varrho Y_j\delta)\right].\end{aligned} \tag{3.24}$$

Finally, to get rid of the term containing the third order derivative of δ, we exploit the following relation obtained differentiating twice (3.3) along the vector fields Y_i, Y_j. Namely, differentiating in the sense of distributions the Eikonal equation, we deduce that

$$Y_iY_j(\|\nabla\delta\|^2)=0 \qquad \Rightarrow \qquad (X_h\delta)(Y_iY_jX_h\delta)+(Y_iX_h\delta)(Y_jX_h\delta)=0. \tag{3.25}$$

Hence, by a direct computation, from (3.25) we deduce that

$$0=a_h^\alpha a_h^\sigma(Y_\alpha\delta)(Y_iY_jY_\sigma\delta)+\tilde{P},$$

where $\tilde{P}$ is a suitable homogeneous polynomial of degree 2 in the variables $Y_\alpha\delta$, $Y_\alpha Y_\sigma\delta$. Therefore, the distribution $a_h^\alpha a_h^\sigma(Y_\alpha\delta)(Y_iY_jY_\sigma\delta)$ is represented by the function $\tilde{P}$, which is continuous. Then, substituting the latter expression in (3.24), we conclude the proof. □

We can now prove a fine regularity result for the iterated divergences and their derivative along geodesics needed in Theorem 71.

PROPOSITION 66. *Let M be a smooth (or real-analytic) sub-Riemannian manifold and $S \subset M$ be a C^2 non-characteristic submanifold. Then, for every $k \in \mathbb{N}$, the following holds true*

(1) $\mathrm{div}^k(\nabla\delta) \in C(S_{r_0})$;
(2) $\mathrm{div}^k(\nabla\delta) \circ E^1(\cdot,\lambda)$ *is smooth (or real-analytic) on* $(0,r_0)$, *for every* $\lambda \in \mathcal{A}^1 S$;
(3) $\partial_t^j\Big(\mathrm{div}^k(\nabla\delta)\circ E^1\Big) \in C((0,r_0)\times\mathcal{A}^1S)$, *for every* $j \in \mathbb{N}$.

Where $E^1 : (0,r_0)\times\mathcal{A}^1S \to S_{r_0}$ *is the map defined in* (3.5).

Moreover, if $S = \partial\Omega$ *for a non-characteristic domain* $\Omega \subset M$, *then, with* δ *the signed distance, the regularity properties above can be extended up to* $r = 0$.

PROOF. Items (1) and (2) are a direct consequence of Lemma 63 and Lemma 62. We are left to show (3). Namely, that,

$$\partial_t^j\Big(\mathrm{div}^k(\nabla\delta)\circ E^1\Big) \in C((0,r_0)\times\mathcal{A}^1S), \qquad \forall\, k,j\in\mathbb{N}.$$

For $j = 1$, this follows from the definition of iterated divergences (3.21), indeed

$$\begin{aligned}\partial_t\Big(\mathrm{div}^k(\nabla\delta)\circ E^1\Big) &= \nabla\delta\Big(\mathrm{div}^k(\nabla\delta)\Big)\circ E^1 \\ &= \Big(\mathrm{div}^{k+1}(\nabla\delta) - \mathrm{div}(\nabla\delta)\mathrm{div}^k(\nabla\delta)\Big)\circ E^1,\end{aligned} \tag{3.26}$$

which is continuous on $(0,r_0)\times\mathcal{A}^1S$, by item (1). Differentiating (3.26) with respect to t, we deduce continuity for $j \geq 2$ by a simple induction argument. In the case of $S = \partial\Omega$, the argument above implies the claimed regularity up to $r = 0$. □

REMARK 67. If the submanifold S is of class C^ℓ (with $\ell \geq 2$), then $\mathrm{div}^k(\nabla\delta) \in C^{\ell-2}(S_{r_0})$ and $\partial_t^j\Big(\mathrm{div}^k(\nabla\delta)\circ E^1\Big) \in C^{\ell-2}((0,r_0)\times\mathcal{A}^1S)$, for every $j \in \mathbb{N}$.

4. Auxiliary iterated divergences

The iterated divergences are not smooth on S, nonetheless, we are able to precisely characterize the on S through the squared distance δ^2.

DEFINITION 68 (Auxiliary iterated divergences). Let M be a sub-Riemannian manifold, equipped with a smooth (or real-analytic) measure μ, and let S be a C^2 non-characteristic submanifold. Let $r_0 > 0$ be the parameter identified by Theorem 55. We define the auxiliary iterated divergences Θ^k iteratively as

$$\Theta^0 := 1, \qquad \Theta^k := \left(\frac{\mathrm{div}(\nabla\delta^2)}{2} - k\right)\Theta^{k-1} + \frac{\nabla\delta^2\left(\Theta^{k-1}\right)}{2}, \tag{3.27}$$

where the derivative are meant in the distributional sense.

We show that the functions Θ^k are actually well-defined in the strong sense on $S_{r_0} \cup S$, and represent the principal part of the iterated divergences.

LEMMA 69. *Let M be a smooth (or real-analytic) sub-Riemannian manifold, $S \subset M$ be a C^2 non-characteristic submanifold. For every $k \in \mathbb{N}$,*

$\Theta^k \in C(S_{r_0} \cup S)$ *and it is smooth (or real-analytic) along geodesics from* S *contained in* S_{r_0}. *Moreover,*

$$\operatorname{div}^k(\nabla\delta) = \frac{1}{\delta^k}\Theta^k, \qquad \text{on } S_{r_0}. \tag{3.28}$$

Proof. To prove the first part of the statement we proceed in the same spirit as in the proof of Lemma 63, but this time exploiting the fact that $\delta^2 \in C^2(S_{r_0} \cup S)$ (while $\delta \in C^2(S_{r_0})$). More precisely, let $Y_1, \dots, Y_n$ be a local frame in a neighborhood of $p \in S_{r_0} \cup S$. We prove by induction that Θ^k, for $k \in \mathbb{N}$, is a well-defined function and it can be written as a polynomial (with smooth or real-analytic coefficients) in the variables $Y_i\delta^2, Y_iY_j\delta^2$ for $i, j = 1, \dots, n$.

With the notation of the proof of Lemma 63, let $\{X_1, \dots, X_N\}$ be a smooth (or real-analytic) generating family for the sub-Riemannian structure. For $k = 1$, by definition we have

$$\operatorname{div}(\nabla\delta^2) = X_hX_h\delta^2 + X_h\delta^2\operatorname{div}(X_h).$$

Since $X_h = a_h^\alpha Y_\alpha$ for some smooth (or real-analytic) functions a_h^α defined on $\mathcal{O}$, then we conclude.

Assume now that the claim is valid for Θ^k and we prove it for Θ^{k+1}. Taking into account (3.27), the only term to discuss is $\nabla\delta^2\left(\Theta^k\right)$. By the induction step and the Leibniz rule, we are left to show that, for $i, j = 1, \dots, n$ fixed,

$$\nabla\delta^2\left(Y_i\delta^2\right) \qquad \text{and} \qquad \nabla\delta^2\left(Y_iY_j\delta^2\right)$$

can be expressed as polynomials in the variables $Y_\alpha\delta^2, Y_\alpha Y_\sigma\delta^2$. The first term writes as

$$\nabla\delta^2\left(Y_i\delta^2\right) = X_h\delta^2X_hY_i\delta^2$$

where the right-hand side is a continuous polynomial. For the second term, we use the bracket relations $[Y_\alpha, Y_\sigma] = c_{\alpha\sigma}^\varrho Y_\varrho$ where $c_{\alpha\sigma}^\varrho : \mathcal{O} \to \mathbb{R}$ are smooth (or real-analytic) functions, and we obtain the following distributional identity:

$$\begin{aligned}\nabla\delta^2\left(Y_iY_j\delta^2\right) &= \left(X_h\delta^2\right)\left(X_hY_iY_j\delta^2\right)\\ &= \left(X_h\delta^2\right)a_h^\alpha\left[Y_iY_jY_\alpha\delta^2 + c_{\alpha i}^\varrho Y_\varrho Y_j\delta^2 + Y_i\left(c_{\alpha j}^\varrho Y_\varrho\delta^2\right)\right].\end{aligned} \tag{3.29}$$

To get rid of the term containing derivations of the third order of δ^2, we observe that $\|\nabla\delta^2\|^2 = 4\delta^2$. Hence differentiating in the sense of the distributions the latter relation we deduce that

$$\left(X_h\delta^2\right)\left(Y_iY_jX_h\delta^2\right) + \left(Y_iX_h\delta^2\right)\left(Y_jX_h\delta^2\right) = 4Y_iY_j\delta^2,$$

from which we obtain a distributional expression for $(X_h\delta^2)(Y_iY_jX_h\delta^2)$ and especially for $(X_h\delta^2)\,a_h^\alpha\,(Y_iY_jY_\alpha\delta^2)$. Therefore, from (3.29) we obtain the following distributional identity:

$$\nabla\delta^2\left(Y_iY_j\delta^2\right) = 4Y_iY_j\delta^2 + \tilde{P},$$

where $\tilde{P}$ is a homogeneous polynomial of degree two, with smooth (or real-analytic) coefficients, in the variables $Y_\alpha\delta^2, Y_\alpha Y_\sigma\delta^2$. We conclude that Θ^{k+1} is continuous and can be written as a polyonomial of degree $k+1$ in the same variables. The proof of the first part of the statement is concluded.

Finally, we prove (3.28) on S_{r_0}. For $k=0$ the identity is trivial. Now observe that

$$\mathrm{div}(\nabla\delta) = \mathrm{div}\left(\frac{1}{\delta}\frac{\nabla\delta^2}{2}\right) = \frac{1}{\delta}\left(\frac{\mathrm{div}(\nabla\delta^2)}{2} - 1\right).$$

We conclude by induction on k and exploiting the definition of the iterated divergence in (3.21):

$$\begin{aligned}\mathrm{div}^{k+1}(\nabla\delta) &= \mathrm{div}(\nabla\delta)\frac{\Theta^k}{\delta^k} + \nabla\delta\left(\frac{\Theta^k}{\delta^k}\right)\\ &= \left(\frac{\mathrm{div}(\nabla\delta^2)}{2} - 1 - k\right)\frac{\Theta^k}{\delta^{k+1}} + \frac{\nabla\delta^2}{2\delta}\left(\frac{\Theta^k}{\delta^k}\right) = \frac{1}{\delta^{k+1}}\Theta^{k+1},\end{aligned}$$

where in the last line we exploited the identity $\|\nabla\delta\| = 1$, valid on S_{r_0}. □

Lemma 70. *Let M be a smooth sub-Riemannian manifold, and let $S \subset M$ be a C^2 non-characteristic submanifold of codimension m. Then,*

$$\mathrm{div}(\nabla\delta^2)(p) = 2m \qquad \forall\, p \in S.$$

Proof. For any $f \in C^2(M)$ with a critical point at $p \in M$ (i.e. $\mathrm{d}_p f = 0$), the Hessian at p can be defined as the quadratic form $D_p^2 f : T_pM \to \mathbb{R}$ such that

$$D_p^2 f(v) = \left.\frac{d^2}{dt^2}\right|_{t=0} f(\gamma_t), \qquad \forall\, v \in T_pM,$$

where $\gamma : [0,\varepsilon) \to M$ is any C^2 curve such that $\dot\gamma_0 = v$. The restriction of $D_p^2 f$ to the sub-Riemannian distribution $\mathcal{D}_p$ (equipped with the metric g_p) yields a symmetric linear operator whose trace coincides with the sub-Laplacian of f at p (see [**ABR18**, Sec. 4.4]):

$$\mathrm{div}(\nabla f)(p) = \mathrm{Trace}(D_p^2 f : \mathcal{D}_p \to \mathcal{D}_p).$$

The above result is of easy deduction in the case of a constant-rank distribution, but holds true also in the general rank-varying case, using the formulas in [**RS23**, Appendix A]. For the case of the squared distance from S, for which all points $p \in S$ are critical, we obtain

$$\mathrm{div}(\nabla\delta^2)(p) = \sum_{i=1}^{N} \left.\frac{d^2}{dt^2}\right|_{t=0} f \circ \gamma_i(t), \qquad p \in S,$$

where $N = N(p) = \dim \mathcal{D}_p$, and $\gamma_1, \dots, \gamma_N$ are C^2 curves such that their tangent vectors at $t=0$ are an orthonormal basis of $(\mathcal{D}_p, g_p)$. Thanks to the non-characteristic assumption, we can choose $\gamma_1, \dots, \gamma_m$ to be unit-speed geodesics from S, and $\gamma_{m+1}, \dots, \gamma_{N-m}$ to be C^2 unit-speed curves in S. The result follows, since $\delta^2(\gamma_i(t)) = t^2$ for $i = 1, \dots, m$ while $\delta^2(\gamma_j(t)) \equiv 0$ for $j = m+1, \dots, N-m$. □

5. Regularity of the tube-volume and the Mean Value Lemma

This section is devoted to the proof of the regularity of the volume of the tube, that ensures the existence of a Taylor expansion as the radius tends to zero. Moreover, we present an improved version of the sub-Riemannian Mean

Value Lemma that in turn is the technical tool that permits to characterize the coefficients of the asymptotics of the volume of the tube.

Let M be a sub-Riemannian manifold equipped with a smooth measure μ and let S be a non-characteristic submanifold of class C^2. Taking into account Theorem 55, there exists $r_0 > 0$ such that we can define the $(n-1)$-form on the tube S_{r_0} as

$$d\sigma = i_{\nabla\delta} d\mu,$$

where $d\mu$ is the n-form associated to μ. Note that $\delta^{-1}(r)$ is a non characteristic hypersurface for every $r \in (0, r_0)$. Thus, $d\sigma$ defines σ_r, the C^1 sub-Riemannian perimeter measure on $\delta^{-1}(r)$ induced by μ.

For the computations that will follow we assign a never vanishing measure α of class C^1 on $\mathcal{A}^1 S$, the unit annihilator bundle (cf. (3.4)). Moreover, taking into account the C^1 diffeomorphism $E^1 : (0, r_0) \times \mathcal{A}^1 S \to S_{r_0}$ defined in (3.5), we can relate α and μ defining $\mathrm{f} : (0, \mathrm{r}_0) \times \mathcal{A}^1\mathrm{S} \to \mathbb{R}$ such that

$$E^{1^*} d\mu\Big|_{(r,\lambda)} = \mathrm{f}(r, \lambda)\, dr \wedge d\alpha|_\lambda \qquad \forall\, (r, \lambda) \in (0, r_0) \times \mathcal{A}^1 S, \tag{3.30}$$

with $E^{1^*} d\mu$ the pull-back of $d\mu$ with respect to E^1. Moreover, notice that since $\mathrm{d}E^1(\partial_r) = \nabla\delta$, then

$$E^{1^*} d\sigma\Big|_{(r,\lambda)} = \mathrm{f}(r, \lambda) d\alpha|_\lambda \qquad \forall\, (r, \lambda) \in (0, r_0) \times \mathcal{A}^1 S. \tag{3.31}$$

From the analysis of the regularity of the function f follows the regularity result for the volume of the tube.

Theorem 71. *Let M be a smooth (or real-analytic) sub-Riemannian manifold, equipped with a smooth (or real-analytic) measure μ, and let $S \subset M$ be a C^2 non-characteristic submanifold. Then, there exists $r_0 > 0$ such that*

$$[0, r_0) \ni r \mapsto \mu(S_r),$$

is smooth (or real-analytic) on $[0, r_0)$, where S_r is the tube around S.

Proof. Since S is closed and with codimension ≥ 1, then for every $r \in [0, r_0)$ it holds that $\mu(S_r \cup S) = \mu(S_r)$. In addition, recalling that $S_r = E^1((0, r) \times \mathcal{A}^1 S)$, after a change of variables, the former rewrites as follows:

$$\mu(S_r) = \int_{E^1((0,r)\times\mathcal{A}^1 S)} d\mu = \int_0^r \int_{\mathcal{A}^1 S} E^{1^*} d\mu, \qquad \forall\, r \in (0, r_0).$$

First of all, as a consequence of the observation in (3.10), the map E^1 can be expressed in terms of the extended Hamiltonian flow as follows:

$$E^1(t, \lambda) = (\pi \circ \Phi)(t, \lambda)$$

Therefore, recalling that Ψ is the linearized Hamiltonian flow, cf. (3.12), the differential of the map $\lambda \mapsto E^1(t, \lambda)$ is given by

$$\mathrm{d}_\lambda E^1(t, \cdot) = \mathrm{d}_\lambda (\pi \circ \Phi)(t, \cdot) = \mathrm{d}_\lambda \pi \circ \Psi(t, \cdot).$$

Second of all, let $\alpha \in \bigwedge^{n-1}(\mathcal{A}^1 S)$ be the fixed tensor density on $\mathcal{A}^1 S$ and $V_1, \ldots, V_{n-1}$ be a local frame for $T(\mathcal{A}^1 S)$ around some $\lambda_0 \in \mathcal{A}^1 S$ such that

$d\alpha(V_1,\dots,V_{n-1})=1$. Then, taking into account f as in (3.30), we have that

$$\begin{aligned}\mathrm{f}(\mathrm{t},\lambda)&=E^{1*}\mu\Big|_{(t,\lambda)}(\partial_t,V_1,\dots,V_{n-1})\\&=\mu|_{E^1(t,\lambda)}\left(\mathrm{d}_{(t,\lambda)}E^1(\partial_t),\dots,\mathrm{d}_{(t,\lambda)}E^1(V_i),\dots\right)\\&=\mu|_{E^1(t,\lambda)}\left(\partial_t\left(\pi\circ\Phi(t,\lambda)\right),\dots,\mathrm{d}_\lambda\pi\circ\Psi(t,V_i),\dots\right).\end{aligned}\tag{3.32}$$

On the one hand, if μ and the sub-Riemannian structure are smooth, then function $(0,r_0)\ni t\mapsto \mathrm{f}(t,\lambda)$ is smooth for every $\lambda\in\mathcal{A}^1S$. On the other hand, if μ and and the sub-Riemannian structure are a real-analytic, by Lemma 59 and Corollary 60, all the entries in (3.32) are real-analytic functions of $t\in[0,r_0)$, uniformly with respect to $\lambda\in\mathcal{A}^1S$ (being $\mathcal{A}^1S$ a compact submanifold of T^*M). Therefore, for every $\lambda\in\mathcal{A}^1S$, the function $t\mapsto\mathrm{f}(\mathrm{t},\lambda)$ is also real-analytic on $(0,r_0)$, uniformly with respect to λ, meaning that there exists a constant $C>0$ such that

$$\sup_{\lambda\in\mathcal{A}^1S}\ \sup_{t\in(0,r_0)}\left|\partial_t^k\mathrm{f}(t,\lambda)\right|\le C^{k+1}k!,\qquad\forall\,k\in\mathbb{N}.\tag{3.33}$$

More precisely, the constant C only depends on the analyticity constants of the functions $E^1=\pi\circ\Phi$, $\mathrm{d}_\lambda E^1(t,\cdot)=\mathrm{d}_\lambda\pi\circ\Psi(t,\cdot)$ and of the measure μ. The desired conclusion now follows by the estimate (3.33), which allows to extend the map $\mathrm{f}(\cdot,\lambda)$ to a real-analytic function on $[0,r_0)$, uniformly with respect to $\lambda\in\mathcal{A}^1S$. □

In order to state the Mean Value Lemma we consider the following function $F:(0,r_0)\to\mathbb{R}$ defined as follows. Given $v\in C(S_{r_0})$, define:

$$F(r):=\int_{S_r}v(p)\,d\mu(p),\qquad\forall\,r\in[0,r_0).\tag{3.34}$$

Reasoning as in the proof of Theorem 71, we may use the map (3.5) to change variables in the integral, and, exploiting (3.30), we obtain:

$$\int_{S_r}v\,d\mu=\int_0^r\int_{\mathcal{A}^1S}(v\circ E^1)(t,\lambda)\mathrm{f}(t,\lambda)\,dt\,d\alpha(\lambda),\qquad\forall\,r\in(0,r_0).$$

Hence, we immediately deduce that $F\in C^1([0,r_0))$ and

$$F'(r)=\int_{\mathcal{A}^1S}(v\circ E^1)(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda)=\int_{\delta^{-1}(r)}v\,d\sigma.\tag{3.35}$$

For the second derivative, we show an improved version of the Mean Value Lemma for non-characteristic submanifolds, see [**RR21**, **ARR23**, **Sav01**].

THEOREM 72. *Let M be a smooth sub-Riemannian manifold, equipped with a smooth measure μ, and let $S\subset M$ be a C^2 non-characteristic submanifold. Assume that $v\in C(S_{r_0})$ and, for every $\lambda\in\mathcal{A}^1S$, $v\circ E^1(\cdot,\lambda)\in C^1((0,r_0))$ and $\partial_t v\circ E^1\in C((0,r_0)\times\mathcal{A}^1S)$. Then, the function F, defined as in (3.34), is $C^2((0,r_0))$ and, for all $r\in(0,r_0)$,*

$$\begin{aligned}F''(r)&=\int_{\mathcal{A}^1S}\left(v\,\mathrm{div}(\nabla\delta)+g(\nabla v,\nabla\delta)\right)\circ E^1(r,\lambda)\,\mathrm{f}(r,\lambda)d\alpha(\lambda),\\&=\int_{\delta^{-1}(r)}\left(v\,\mathrm{div}(\nabla\delta)+g(\nabla v,\nabla\delta)\right)\,d\sigma,\end{aligned}\tag{3.36}$$

where α is a fixed non-vanishing measure on $\mathcal{A}^1S$ and σ the sub-Riemannian induced measure by μ.

Proof. For ease of notation, denote by $f_r^1 := (f \circ E^1)(r, \cdot) : \mathcal{A}^1 S \to \mathbb{R}$, for every $f \in C(S_{r_0})$ and $r \in (0, r_0)$. Firstly, note that the assumptions on the function v guarantee that $F \in C^2((0, r_0))$. Indeed, from Theorem 71 and the hypothesis on v, the function $g(r, \lambda) := v_r^1(\lambda)\mathrm{f}(r, \lambda)$ is such that

$$g(\cdot, \lambda) \in C^1((0, r_0)), \quad \forall\, \lambda \in \mathcal{A}^1 S \qquad \text{and} \qquad \partial_r g \in C((0, r_0) \times \mathcal{A}^1 S). \tag{3.37}$$

Thus, from (3.35), we conclude that $F' \in C^1((0, r_0))$, showing the claimed regularity. Secondly, for a fixed $r \in (0, r_0)$, we compute the second derivative at r starting from (3.35). Differentiating under the integral sign (which is admissible thanks to (3.37)), we obtain:

$$\begin{aligned} F''(r) &= \int_{\mathcal{A}^1 S} \partial_r \Big(v_r^1\, \mathrm{f}(r, \cdot) \Big)\, d\alpha \\ &= \int_{\mathcal{A}^1 S} \Big(\partial_r v_r^1\, \mathrm{f}(r, \cdot) + v_r^1\, \partial_r (\log \mathrm{f})(r, \cdot) \Big) \mathrm{f}(r, \cdot)\, d\alpha \end{aligned}$$

On the one hand, since $\mathrm{d}E^1(\partial_r) = \nabla \delta$, we deduce that

$$\partial_r v_r^1(r, \lambda) = g(\nabla v, \nabla \delta)|_{E^1(r, \lambda)}. \tag{3.38}$$

On the other hand, in order to describe the logarithmic derivative of f, we observe that by the property of the divergence operator,

$$\mathrm{div}_{\mathrm{f}\, dr \wedge \alpha}(\partial_r) = \mathrm{div}_{\, dr \wedge \alpha}(\partial_r) + \partial_r (\log \mathrm{f}) = \partial_r (\log \mathrm{f}).$$

Then again, recalling that by definition ${E^1}^* \mu = \mathrm{f}\, dr \wedge \alpha$, we may compute

$$\begin{aligned} \mathrm{div}_{\mathrm{f}\, dr \wedge \alpha}(\partial_r) \mathrm{f}\, dr \wedge \alpha &= \mathrm{div}_{{E^1}^* \mu}(\partial_r) {E^1}^* \mu = \mathcal{L}_{\partial_r}({E^1}^* \mu) = \mathrm{d}(i_{\partial_r}({E^1}^* \mu)) \\ &= {E^1}^*(\mathrm{d}(i_{\nabla \delta} \mu))) = (\mathrm{div}(\nabla \delta) \circ E^1) {E^1}^* \mu. \end{aligned}$$

Therefore, we deduce that

$$\partial_r (\log \mathrm{f}) = \mathrm{div}(\nabla \delta) \circ E^1. \tag{3.39}$$

Plugging equalities (3.38) and (3.39) in the expression for the second derivative of F, we obtain (3.36) and conclude the proof. □

6. Steiner's formula and Weyl's tube formula

In this section, we deduce Weyl's tube formula and Steiner's formula for non-characteristic submanifolds of class C^2 in a sub-Riemannian manifold. Thanks to Therorem 71, a Taylor expansion at any order always exists. And, we consider the iterated divergences of Section 3 to compute its coefficients.

Theorem 73. *Let M be a sub-Riemannian manifold, equipped with a smooth measure μ, and let S be a C^2 non-characteristic submanifold. Then, there exists $r_0 > 0$ such that the function $[0, r_0) \ni r \mapsto \mu(S_r)$ is smooth and, for every $K \in \mathbb{N}$,*

$$\mu(S_r) = \sum_{k=1}^{K} a_k r^k + o(r^N), \qquad \text{as } r \to 0^+,$$

with for all $k \in \{1, \dots, K\}$

$$a_k = \frac{1}{k!} \lim_{r \to 0} \int_{\delta^{-1}(r)} \mathrm{div}^{k-1}(\nabla \delta)\, d\sigma, \tag{3.40}$$

where σ is the sub-Riemannian perimeter measure induced by μ.

PROOF. Theorem 71 implies that $[0,r_0)\ni r\mapsto \mu(S_r)$ is smooth. For the sake of notation, denote by $F(r):=\mu(S_r)$, for every $r\in[0,r_0)$. Thus, Theorem 72 applies for the function F (which is as in (3.34) with $v\equiv 1$ on S_{r_0}) and we obtain

$$F''(r)=\int_{\mathcal{A}^1S}\Big(\mathrm{div}(\nabla\delta)\circ E^1\Big)(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda),\qquad \forall\, r\in(0,r_0), \tag{3.41}$$

where we have used that $\nabla 1=0$ and α is a fixed measure on $\mathcal{A}^1S$.

We claim that, for every $k\geq 2$,

$$F^{(k)}(r)=\int_{\mathcal{A}^1S}\Big(\mathrm{div}^{k-1}(\nabla\delta)\circ E^1\Big)(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda),\qquad \forall\, r\in(0,r_0). \tag{3.42}$$

The equality in (3.41) is precisely (3.42) for $k=2$ as $\mathrm{div}^1(\nabla\delta)=\mathrm{div}(\nabla\delta)$.

Proceeding by induction on k, assume (3.42) is true for $k\geq 2$, then we compute $F^{(k+1)}$:

$$\begin{aligned}F^{(k+1)}(r)&=\frac{d}{dr}F^{(k)}(r)=\frac{d}{dr}\left(\int_{\mathcal{A}^1S}\Big(\mathrm{div}^{k-1}(\nabla\delta)\circ E^1\Big)(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda)\right)\\&=\frac{d^2}{dr^2}\left(\int_0^r\int_{\mathcal{A}^1S}\Big(\mathrm{div}^{k-1}(\nabla\delta)\circ E^1\Big)(t,\lambda)\mathrm{f}(t,\lambda)\,d\alpha(\lambda)\,dt\right),\\&=\frac{d^2}{dr^2}\left(\int_{S_r}\mathrm{div}^{k-1}(\nabla\delta)\,d\mu\right).\end{aligned}$$

Thus, we may apply Theorem 72 with $v=\mathrm{div}^{k-1}(\nabla\delta)$, which satisfies its regularity assumptions, by Proposition 66. Thus, we obtain:

$$\begin{aligned}F^{(k+1)}(r)&=\int_{\mathcal{A}^1S}[v\,\mathrm{div}(\nabla\delta)+g\,(\nabla v,\nabla\delta)]\circ E^1(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda)\\&=\int_{\mathcal{A}^1S}\mathrm{div}^k(\nabla\delta)\circ E^1(r,\lambda)\mathrm{f}(r,\lambda)\,d\alpha(\lambda),\end{aligned}$$

where in the last equality we used the definition of iterated divergence in (3.21). This shows the validity of the claim (3.42). The proof is then concluded applying (3.31). □

We specialize Theorem 73 to the case where $S=\partial\Omega$ is the boundary of a non-characteristic domain to obtain Steiner's formula. Recall that, according to our conventions (cf. (3.7)), δ denotes the signed distance from $\partial\Omega$ and S_r^+ is the half-tube of size r around $\partial\Omega$ in Ω (cf. (3.9)).

THEOREM 74 (Steiner's formula). *Let M be a smooth (or real-analytic) sub-Riemannian manifold equipped with a smooth (or real-analytic) measure μ. Let us consider $\Omega\subset M$ be a non-characteristic domain of class C^2. Then, there exists $r_0>0$ such that the function $[0,r_0)\ni r\mapsto\mu(S_r^+)$ is smooth (or real-analytic), and, for every $K\in\mathbb{N}$,*

$$\mu(S_r^+)=\sum_{k=1}^{K}a_kr^k+o(r^N),\qquad \text{as } r\to 0^+, \tag{3.43}$$

where S_r^+ is half-tube over $S=\partial\Omega$ in Ω of radius r and

$$a_k=\frac{1}{k!}\int_{\partial\Omega}\mathrm{div}^{k-1}(\nabla\delta)\,d\sigma,\qquad \forall\, k\in\{1,\dots,K\}. \tag{3.44}$$

PROOF. The proof of the regularity of the volume function follows verbatim Theorem 71 reading S_r^+ and $\mathcal{A}_+^1 S$ in place of S_r and $\mathcal{A}^1 S$ respectively. The proof of (3.44) follows verbatim the proof of Theorem 73 and in turn Theorem 72 with the same substitutions as before. Moreover, from Proposition 66, the iterated divergences of $\nabla\delta$ are continuous up to $S = \partial\Omega$. Therefore, the classical dominated convergence Theorem allows to compute the limits in (3.40), where this time the $\delta^{-1}(r)$ are defined with respect to the signed distance and thus contained in Ω. □

Theorem 74 allows to compute the asymptotic expansion of $r \mapsto \mu(S_r)$ as $r \to 0^+$, where $S_r = S_r^+ \sqcup S \sqcup S_r^-$ is the *whole* tube of size r, where

$$S_r^- := \{E^1(t,\lambda) : \lambda \in \mathcal{A}_+^1 S \text{ and } -r < t < 0\}$$

is the exterior half-tube with respect to Ω. Indeed,

$$\mu(S_r) = \mu(S_r^+) + \mu(S_r^-),$$

and S_r^- can be regarded as the half-tube of $S = \partial\Omega$ when we consider the complement $M \setminus \bar{\Omega}$ as the non-characteristic domain of reference. Hence, (3.43) still holds with the coefficients in (3.44) defined with $-\delta$, in place of δ. This proves the following corollary, recalling that for every vector field

$$\mathrm{div}^k(-X) = (-1)^k \mathrm{div}^k(X) \qquad \forall\, k \in \mathbb{N}.$$

COROLLARY 75. *Let M be a smooth sub-Riemannian manifold and let $\Omega \subset M$ be a non-characteristic domain of class C^2. Then, there exists $r_0 > 0$ such that the function $[0, r_0) \ni r \mapsto \mu(S_r)$ is smooth and, for every $N \in \mathbb{N}$,*

$$\mu(S_r) = 2\sum_{k=1}^{N} a_{2k-1} r^{2k-1} + o(r^N), \qquad \text{as } r \to 0^+,$$

where $S_r = S_r^+ \sqcup S_r^-$ is the tube over $S = \partial\Omega$ of radius r and the coefficients a_{2k-1} are defined in (3.44).

6.1. The coefficients of the Weyl's tube formula. We provide a more precise description of some of the coefficients in (3.40) of Theorem 73, in terms of the auxiliary iterated divergences (cf. Definition 68).

First of all, we define the induced measure by μ on the C^2 non-characteristic submanifold S. Recall that in the case of a smooth submanifold, [**Ros22**, Lemma 4.1] states the existence of a unique intrinsic probability measure on the submanifold induced by μ. Following the same construction, we define a measure on S.

DEFINITION 76. Let S be a non-characteristic surface in M a sub-Riemannian manifold equipped with a smooth volume μ. Define μ_S the unique C^1 measure induced by μ on S such that for every $f \in C_c(M)$

$$\lim_{r\to 0} \frac{1}{\mathscr{L}^m(B(r,\mathbb{R}^m))} \int_{S_r} f\, d\mu = \int_S f(s)\, d\mu_S(s),$$

with $\mathscr{L}^m(B(r,\mathbb{R}^m))$ the Lebesgue measure of the ball of radius r in $\mathbb{R}^m$.

Secondly, up to a local trivialization of $\mathcal{A}S$ and $\mathcal{A}^1 S$, as a consequence of Theorem 55 we can consider the following identifications through the map E^1

$$S_{r_0} \cup S \sim_{loc} S \times B(r_0, \mathbb{R}^m) \qquad \text{and} \qquad \delta^{-1}(r) \sim_{loc} S \times \mathbb{S}^{m-1},$$

where, $\mathbb{S}^{m-1}$ is the sphere in $\mathbb{R}^m$. Then, there exists a smooth density $f \in C^\infty(S_{r_0} \cup S)$ with $f|_S = 1$ and such that, if we consider consider polar coordinates (r, θ), we write on S_{r_0}

$$d\mu = r^{m-1} f dr \wedge d\theta \wedge d\mu_S,$$
$$d\sigma = r^{m-1} f d\theta \wedge d\mu_S,$$

where we recall that $d\sigma = i_{\partial_r} d\mu$ thanks to (3.6). Therefore, for every measurable function $h : S_{r_0} \to \mathbb{R}$ and $r \in (0, r_0)$, we write

$$\int_{\delta^{-1}(r)} h\, d\sigma = r^{m-1} \int_{S\times\mathbb{S}^{m-1}} h(s, r, \theta) f(s, r, \theta)\, d\mu_S(s)\, d\theta. \tag{3.45}$$

Remark 77. Taking into account (3.30) and (3.31) we can consider α a non vanishing measure on $\mathcal{A}^1 S$ such that f $= r^{m-1} E^{1*}(f)$.

From now on, with a slight abuse of notation, we denote by Θ^k also its expression in coordinates (s, r, θ).

Lemma 78. *Under the assumptions and notations of Theorem 73, it holds the following identity*

$$a_k = \frac{1}{k!} \lim_{r\to 0} \int_{\delta^{-1}(r)} \operatorname{div}^{k-1}(\nabla\delta)\, d\sigma = 0, \qquad \forall\, k < m.$$

Proof. Putting together (3.45) and (3.28), we have that, in polar coordinates, (3.40) rewrites as follows

$$a_k = \frac{1}{k!} \lim_{r\to 0} r^{m-k} \int_{S\times\mathbb{S}^{m-1}} \Theta^{k-1} f(s, r, \theta)\, d\mu_S(s)\, d\theta.$$

Since Θ^{k-1} and f are continuous functions on the relatively compact set $S_{r_0} \cup S$, the statement is trivially true as $k < m$. ☐

Lemma 79. *Let us consider the auxiliary iterated divergences defined in (3.27), it holds that*

$$\Theta^k|_S = \begin{cases} 1 & k = 0 \\ \prod_{j=1}^k (m-j) & 1 \le k \le m-1 \\ 0 & k \ge m \end{cases}.$$

Proof. Since Θ^k is a continuous function, we consider $\lim_{r\to 0} \partial_r \Theta^k(s, r, \theta)$ that exists and is finite for every $(s, \theta) \in S \times \mathbb{S}^{m-1}$ fixed. Furthermore, taking into account the identification $\partial_r = \nabla\delta$ given by Theorem 55, we write

$$\partial_r \Theta^k = g\left(\nabla\delta, \nabla\Theta^k\right) = \frac{1}{r} g\left(\frac{\nabla\delta^2}{2}, \nabla\Theta^k\right).$$

Hence, it holds necessarily that

$$0 = \lim_{r\to 0} g\left(\frac{\nabla\delta^2}{2}, \nabla\Theta^k\right) = \lim_{r\to 0} \Theta^{k+1} - \Theta^k \left(\frac{\operatorname{div}(\nabla\delta^2)}{2} - k - 1\right),$$

where in the last equality we applied the definition of Θ^k in (3.27). Consequently, for every $k \in \mathbb{N}$ we have that

$$\lim_{r\to 0} \Theta^k(s, r, \theta) = \lim_{r\to 0} \prod_{j=1}^k \left(\frac{\operatorname{div}(\nabla\delta^2)}{2} - j\right).$$

We conclude since $\operatorname{div}(\nabla\delta^2)|_S = 2m$ thanks to Lemma 70. □

Finally, from the previous results we deduce the expansion for the tubular volume up to order m, the codimension of the submanifold.

PROPOSITION 80. *Under the assumptions and notations of Theorem 73, it holds that*

$$\mu(S_r) = \frac{2\pi^{\frac{m}{2}}}{m\Gamma\left(\frac{m}{2}\right)}\mu_S(S)r^m + o(r^m), \qquad as\ r \to 0.$$

CHAPTER 4

Integrability of the mean curvature of surfaces in 3D contact sub-Riemannian geometry

In this final chapter, we address the question of the integrability of the sub-Riemannian mean curvature around isolated characteristic points of a smooth surface embedded in a three-dimensional sub-Riemannian contact manifold. In particular, we prove the local integrability in presence of mildly degenerate characteristic points. The aim of the present Chapter is to extend the definition of such class of points introduced in [**Ros21**] for the Heisenberg group as well as the results of local integrability.

Structure of the Chapter

Let $(M, \mathcal{D}, g)$ be a three-dimensional contact manifold equipped with a smooth measure μ. The goal of this final chapter is to study whenever the sub-Riemannian mean curvature of a smooth surface is locally integrable in presence of isolated characteristic points and with respect to any Riemannian induced measure.

In section 1 we introduce the horizontal hessian with respect to an assigned affine connection in order to individuate the class of degenerate characteristic points. Then, the condition of mild degeneration is defined trough the construction of a critical curve passing through the isolated characteristic point of interest and tangent to the kernel of the horizontal hessian at such point.

In Section 2 we proceed by providing an estimate for the sub-Riemannian mean curvature with the aim to reduce the study of its integrability to the study of the integrability of the reciprocal of the norm of the horizontal gradient of a locally defining function for the surface. We conclude the section with a discussion about the Riemannian and sub-Riemannian induced measures.

Finally, in Section 3, we prove the (local) integrability result in presence of mildly degenerate characteristic points deducing Theorem 11 and Corollary 13, the main results of this chapter.

1. The mild degeneration condition

Let us consider $(M, \mathcal{D}, g)$ a three-dimensional contact manifold equipped with a smooth measure μ. Let us suppose that the sub-Riemannian distribution is locally generated by the orthonormal vector fields X_1 and X_2 and let us indicate with X_0 the Reeb vector field.

Moreover, let S be a smooth surface embedded in M. We say that f is a *locally defining function for* S if there exists an open $V \subset M$ such that

$$f \in C^\infty(V), \qquad U = S \cap V = \{f = 0\}, \qquad \mathrm{d}f|_U \neq 0.$$

1.1. The Horizontal Hessian. In order to classify the characteristic points, we introduce the Horizontal Hessian of a function with respect to an affine connection (cf. also [**BBCH21**]).

DEFINITION 81. Let be assigned $\widetilde{\nabla}$ an affine connection on a three-dimensional sub-Riemannian contact manifold $(M, \mathcal{D}, g)$. The horizontal Hessian of $f \in C^2(M)$ with respect to $\widetilde{\nabla}$ is the $(0,2)$-tensor defined as

$$\mathrm{Hess}^{\widetilde{\nabla}}(f)(V,W) = g\left(\widetilde{\nabla}_V \nabla_H f, W\right) \qquad \forall\, V, W \in \mathcal{D}.$$

As observed in [**Ros21**, Lemma 2.1], the horizontal Hessian does not depend on the choice of the affine connection when evaluated at characteristic points of the surface.

DEFINITION 82. Let S be a smooth surface embedded in a three-dimensional sub-Riemannian contact manifold and let f be a locally defining function for S. Moreover, let $p \in S$ be a characteristic point of S. The *horizontal Hessian at a characteristic point* $\nabla^2_H f(p)$ is represented by the following matrix written with respect to the horizontal frame

$$\nabla^2_H f(p) = \begin{pmatrix} X_1X_1f(p) & X_2X_1f(p) \\ X_1X_2f(p) & X_2X_2f(p) \end{pmatrix}. \tag{4.1}$$

A characteristic point $p \in S$ is called *degenerate* if $\det\left(\nabla^2_H f(p)\right) = 0$.

REMARK 83. Up to a constant, the horizontal hessian at a characteristic point does not depend on the choice of the defining function for the surface S. The degenerate condition is intrinsic of how S is embedded in the sub-Riemannian structure. Indeed, let f and g be two locally defining function for S, then there exists h a smooth function such that $g = \pm e^h f$. The claim follows from the fact that for any X, Y smooth vector fields in $\mathcal{D}$ it holds that

$$XYg = \pm e^h \left(f\, XYh + (Xh)(Yf) + (Yh)(Xf) + XYf\right).$$

Hence, at a characteristic point $p \in S$ we have that $\nabla^2_H g(p) = \pm e^h(p)\nabla^2_H f(p)$.

LEMMA 84. *Let $p \in S$ be a degenerate characteristic point, then $\nabla^2_H f(p)$ has rank exactly one as an endomorphism of $\mathcal{D}_p$. Moreover,*

$$J\nabla_H(Xf)(p) \in \ker \nabla^2_H f(p) \qquad \forall\, X \in \mathcal{D},$$

where $J : \mathcal{D} \to \mathcal{D}$ is the rotation map that sends (X_1, X_2) onto $(X_2, -X_1)$.

PROOF. From (1.4), the fact that p is characteristic and $\mathrm{d}f \neq 0$, we deduce that $[X_2, X_1]f(p) = X_0 f(p) \neq 0$. Hence, $[X_2, X_1]f = X_2X_1f - X_1X_2f \neq 0$, meaning that the matrix in (4.1) is not null.

Finally, we observe that

$$J\nabla_H(X_1f)(p) = -X_2X_1f(p)X_1 + X_1X_1f(p)X_2) \in \ker \nabla^2_H f(p),$$
$$J\nabla_H(X_2f)(p) = -X_2X_2f(p)X_1 + X_1X_2f(p)X_2) \in \ker \nabla^2_H f(p).$$

Thus, given $X \in \mathcal{D}$, then $X = aX_1 + bX_2$ for some $a, b \in C^\infty(M)$, and

$$J\nabla_H(Xf)(p) = a(p)J\nabla_H(X_1f)(p) + b(p)J\nabla_H(X_2f)(p) \in \ker \nabla^2_H f(p). \qquad \square$$

1.2. The critical curve. Let $p \in S$ be a degenerate isolated characteristic point. Let be assigned N_p the unique (up to a sign) unitary vector in $\ker \nabla_H^2 f(p) \subset \mathcal{D}_p$. We consider $N \in \mathcal{D}$ any vector field obtained as a unitary smooth horizontal extension of N_p in a neighborhood of p. Its expression with respect to the local frame is

$$N = \cos\theta X_1 + \sin\theta X_2, \tag{4.2}$$

where θ is a real and smooth function defined on the domain of N that values $\theta_0 \in \mathbb{R}$ in p. Moreover, we define the orthogonal and unitary vector field

$$T = JN = -\sin\theta X_1 + \cos\theta X_2. \tag{4.3}$$

DEFINITION 85. Let p be a degenerate characteristic point on a surface S and let f be a locally defining function for S around p. Moreover, N be an extension of a unitary vector $N_p \in \ker \nabla_H^2 f(p)$ as in (4.2). The *critical curve of p with respect to N* is defined as the set of points

$$\mathcal{C} = \{Nf = 0\} \cap S.$$

REMARK 86. The definition of the critical curve does not depend on the choice of the locally defining function for S. Indeed, let f and g be two locally defining function for S, then there exists h a smooth function such that $g = \pm e^h f$. From a direct computation, it holds that

$$\begin{cases} g = \pm e^h f = 0 \\ Ng = \pm e^h (fNh + Nf) = 0 \end{cases} \iff \begin{cases} f = 0 \\ Nf = 0 \end{cases}.$$

LEMMA 87. *The definition of $\mathcal{C}$, the critical curve, is well posed. Moreover, let $\gamma : (-\varepsilon, \varepsilon) \to S$ be a regular parametrization of $\mathcal{C}$ such that $\gamma(0) = p$. Then, $\dot\gamma(0)$ is a parallel to N_p.*

PROOF. Since $\mathcal{C} = \{\phi = 0\}$ with $\phi = (f, Nf)$ a smooth map, we prove that the definition is well posed having ϕ maximum rank at p.

More precisely, since $\nabla_H f(p) = 0$ and $\mathrm{d}f \neq 0$, from (1.4) we deduce that

$$[N, T]\, f(p) = X_0 f(p) \neq 0. \tag{4.4}$$

To conclude, we have to find a vector in $\mathcal{D}_p$ such that the derivation of Nf in its direction does not vanish at p. From (4.4), we deduce that $NTf(p)$ and $TNf(p)$ are not both vanishing. The following computation shows that

$$\begin{aligned} NTf(p) &= (\cos\theta_0 X_1(p) + \sin\theta_0 X_2(p)) \left[(-\sin\theta X_1 + \cos\theta X_2)\, f\right] \\ &= \cos\theta_0 \sin\theta_0 \left(-X_1X_1 f(p) + X_2X_2 f(p)\right) \\ &\quad + \cos^2\theta_0 X_1X_2 f(p) - \sin^2\theta_0 X_2X_1 f(p) \\ &\quad - \left(\cos\theta_0 X_1(\sin\theta)(p) + \sin\theta_0 X_2(\sin\theta)(p)\right) X_1 f(p) \\ &\quad + \left(\cos\theta_0 X_1(\cos\theta)(p) + \sin\theta_0 X_2(\cos\theta)(p)\right) X_2 f(p) \\ &= \begin{pmatrix} -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} X_1X_1 f(p) & X_2X_1 f(p) \\ X_1X_2 f(p) & X_2X_2 f(p) \end{pmatrix} \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} \\ &= g\left(T_p, \nabla_H^2 f(p) N_p\right) = 0. \end{aligned}$$

Therefore, necessarily $TNf\,(p) \neq 0$.

Analogously, it is easy to show that $NNf(p) = 0$. Indeed,

$$\begin{aligned} NNf(p) &= (\cos\theta_0 X_1(p) + \sin\theta_0 X_2(p))\left[(\cos\theta X_1 + \sin\theta X_2)\, f\right] \\ &= \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \end{pmatrix} \begin{pmatrix} X_1X_1f(p) & X_2X_1f(p) \\ X_1X_2f(p) & X_2X_2f(p) \end{pmatrix} \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} \\ &= g\left(N_p, \nabla^2_H f(p) N_p\right) = 0. \end{aligned}$$

□

Finally we define the mild degeneration condition for a characteristic point.

Definition 88. Let p be a degenerate characteristic point of a surface S embedded in $(M, \mathcal{D}, g)$ and let f be a locally defining function for S around p. The point p is mildly degenerate if there exist two vector fields $N, T \in \mathcal{D}$ defined as in (4.2) and (4.3) such that

$$s \mapsto Tf\,(\gamma(s)) \text{ has finite order as } s \to 0,$$

where $\gamma : (-\varepsilon, \varepsilon) \to S$ is a regular parametrization of $\mathcal{C}$, the critical curve of p with respect to N.

Remark 89. The critical curve $\mathcal{C}$ is not necessarily horozontal and thus there is no relation with the characteristic foliation out of the characteristic point. Moreover, the mild degeneration condition does not depend on the choice of the regular parametrization of $\mathcal{C}$. However, the fact that the condition is independent of the extension N of $N_p \in \ker \nabla^2_H f$ is still an open question.

2. The sub-Riemannian mean curvature

Let us consider S a smooth surface embedded in a three-dimensional contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ equipped with a smooth measure μ. We recall that $p \in S$ is a characteristic point if $\nabla_H f = 0$ where f is a smooth locally defining function for S. Let us denote with $Char(S)$ the set of all the characteristic points of the surface. $Char(S)$ is closed and of zero measure in S (cf. [**Bal03**]).

We recall that sub-Riemannian mean curvature with respect to the Popp's measure ν is the smooth function $\mathcal{H} : U \setminus Char(S) \to \mathbb{R}$ defined as

$$\mathcal{H} = -\mathrm{div}\left(\frac{\nabla_H f}{\|\nabla_H f\|}\right),$$

where $\nabla_H f = (X_1 f)X_1 + (X_2 f)X_2$ is the sub-Riemannian gradient of f.

Remark 90. In this section we consider the sub-Riemannian mean curvature defined with respect to the Popp volume ν. However, all the results are valid also for the mean sub-Riemannian curvature defined with respect to any smooth measure μ. Indeed, we can write $\mu = h\nu$ for a suitable $h \in C^\infty(M)$ and, by the properties of the divergence operator

$$\mathrm{div}_{h\nu}(X) = \mathrm{div}_\nu(X) + \frac{Xh}{h}. \tag{4.5}$$

Moreover, let ω be the contact form defining the sub-Riemannian structure, we recall that the three-dimensional form associated to the Popp measure ν

is $dV_\nu = \omega \wedge \mathrm{d}\omega$, that is never vanishing. Then, $dV_\mu = h\,\omega \wedge \mathrm{d}\omega$ and for any vector field Y on M and every measurable $U \subset S$,

$$\int_U \frac{Xh}{h}\, i_Y dV_\mu = \int_U (Xh)\, i_Y dV_\nu$$

concluding that the second addendum in (4.5) is locally integrable with respect to any smooth measure induced by μ on S.

The following result permits to reduce the study of the integrability of the mean sub-Riemannian curvature to that of the integrability of $\|\nabla_H f\|^{-1}$.

LEMMA 91. *Let f be a locally defining function for a relatively compact subset $U \subset S$. There exists $C \geq 0$ such that the sub-Riemannian mean curvature satisfies the folloqing inequality on U:*

$$|\mathcal{H}| \leq \frac{C}{\|\nabla_H f\|}.$$

PROOF. From the properties of the divergence operator we deduce that

$$\begin{aligned} |\mathcal{H}| &= \left| -\frac{\mathrm{div}(\nabla_H f)}{\|\nabla_H f\|} - \nabla_H f \left(\frac{1}{\|\nabla_H f\|} \right) \right| \\ &\leq \frac{|\mathrm{div}(\nabla_H f)|}{\|\nabla_H f\|} + \frac{|\nabla_H f(\|\nabla_H f\|^2)|}{2\|\nabla_H f\|^3}. \end{aligned} \tag{4.6}$$

On the one hand, recalling (1.4) and (2.16), we have that

$$|\mathrm{div}(\nabla_H f)| \leq |X_1 X_1 f + X_2 X_2 f - c_{12}^1 X_2 f + c_{12}^2 X_1 f|$$

and so the frist addendum in (4.6) is bounded. On the other hand,

$$|\nabla_H f(\|\nabla_H f\|^2)| \leq \sum_{i,j=1,2} |(X_i f)(X_j f)(X_i X_j f)| \leq \sum_{i,j=1,2} \frac{\|\nabla_H f\|^2}{2} |(X_i X_j f)|.$$

The conclusion holds from the smoothness of f. □

2.1. The induced area measures. Let us define the Riemannian extension of the sub-Riemannian metric setting the Reeb vector field X_0 as of unitary norm and orthogonal to the distribution $\mathcal{D}$. The classical induced Riemannian area measure on S is defined contracting the three-dimensional volume form associated to μ with respect to the normalized Riemannian gradient of f. Namely, let for every measurable $U \subset S$, we define the Riemannian measure σ_R on S setting

$$\sigma_R(U) = \int_U dA^R, \qquad \text{where} \qquad dA^R = i_{\frac{\nabla f}{\|\nabla f\|}} dV_\mu$$

is the two-dimensional form on S obtained as a restriction of the three-dimensional volume form dV_μ associated with μ with respect to the (never vanishing) vector field

$$\frac{\nabla f}{\|\nabla f\|} = \frac{(X_1 f)X_1 + (X_2 f)X_2 + (X_0 f)X_0}{\sqrt{(X_1 f)^2 + (X_2 f)^2 + (X_0 f)^2}}.$$

Furthermore, we consider the natural sub-Riemannian area measure on S induced by μ, setting for all measurable sets $U \subset S$

$$\sigma_{SR}(U) = \max_{X \in \mathcal{D}:\, \|X\|=1} \int_U i_X dV_\mu. \tag{4.7}$$

REMARK 92. The definition for submanifolds with characteristic points is well posed but does not produce a positive density measure even when μ is the Popp volume. Moreover, this definition at non-characteristic points is equivalent to (0.4). Indeed, the horizontal normal to the surface realizes the maximum in (4.7).

We recall that the horizontal normal is defined at non-characteristic points of the surface as

$$\frac{\nabla_H f}{\|\nabla_H f\|} = \frac{(X_1 f)X_1 + (X_2 f)X_2}{\sqrt{(X_1 f)^2 + (X_2 f)^2}}.$$

From the computations contained in the proof of Proposition 39, we deduce that

$$\sigma_{SR} = \frac{\|\nabla_H f\|}{\|\nabla f\|}\sigma_R,$$

from which holds the following result.

LEMMA 93. *Let S be a surface embedded in $(M, \mathcal{D}, g)$ equipped with a smooth measure μ. Moreover, let f be a locally defining function for S. It holds that*

$$\frac{1}{\|\nabla_H f\|} \in L^1_{loc}\left(S, \sigma_{SR}\right),$$

where σ_{SR} is the induced sub-Riemannian measure on S by μ. In particular, the sub-Riemannian mean curvature $\mathcal{H}_\mu$ is locally integrable with respect to the induced sub-Riemannian area measure.

PROOF. Let U be a compact and measurable subset of the domain of f in S. Since $\mathrm{d}f \neq 0$, there exist constants C_1, $C_2 > 0$ such that

$$C_1 \leq \|\nabla f\| \leq C_2.$$

Hence, the integrability with respect to σ_{SR} is granted by Lemma 91. Indeed,

$$\int_U |\mathcal{H}| dA^{SR} \leq \int_U \frac{C}{\|\nabla_H f\|}\frac{\|\nabla_H f\|}{\|\nabla f\|} dA^R \leq \frac{C}{C_1}\sigma_R(U),$$

for every $U \subset S$ measurable. □

REMARK 94. Actually, we would have considered any Riemannian extension of the sub-Riemannian metric to define the Riemannian induced measure by μ without affecting the integrability results. Indeed, we need only that $\|\nabla f\|$ is locally not vanishing, fact granted by the condition $\mathrm{d}f \neq 0$. More precisely, if we consider two different Riemannian extensions g_R and $g_{R'}$, then we obtain the relation

$$\sigma_{R'} = \frac{\|\nabla f\|_{g_{R'}}}{\|\nabla f\|_{g_R}}\sigma_R.$$

3. Local integrability results

The main result of this Chapter is the following Proposition, from which, together with Lemma 91, descends Theorem 11.

PROPOSITION 95. *Let S be a surface embedded in a three-dimensional contact manifold $(M, \mathcal{D}, g)$ equipped with a smooth measure and let f be a locally defining function for S. Let $p \in S$ be a isolated mildly degenerate characteristic point. It holds that*

$$\frac{1}{\|\nabla_H f\|} \in L^1_{loc}(S, \sigma_R),$$

where σ_R is the induced Riemannian measure on S.

PROOF. Let us consider N, T the vector fields defined in (4.2) and (4.3) realizing the mild degeneration condition for p. For the following computations, it is useful to consider local coordinates $(x, y, z) \in \mathbb{R}^3$ centered in p such that $f = z - g(x, y)$ for a suitable smooth function g and

$$(N, T)(p) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Moreover, let us indicate with h the smooth density function of σ_R in coordinates Our goal is to obtain a bound for

$$\int_U \frac{1}{\|\nabla_H f\|} h(x, y) dx dy,$$

where U is an open relatively compact set in $\mathbb{R}^2$ containing 0.

We obtain the following equivalent expression for the horizontal gradient of f:

$$\|\nabla_H f\| = \sqrt{(Nf)^2 + (Tf)^2}.$$

Indeed, N, T constitute an orthonormal frame for $\mathcal{D}$ as well as X_1, X_2. And, taking into account (4.2) and (4.3), it holds that

$$\begin{aligned}(Nf)^2 + (Tf)^2 &= (\cos\theta(X_1 f) + \sin\theta(X_2 f))^2 + (-\sin\theta(X_1 f) + \cos\theta(X_2 f))^2 \\ &= (X_1 f)^2 + (X_2 f)^2.\end{aligned}$$

Operating the following change of variables $\phi : U \to \widetilde{U} \subset \mathbb{R}^2$

$$\begin{cases} x = x \\ t = Nf(x, y) \end{cases},$$

it holds that $\|\nabla_H f\|^2 = t^2 + (Tf(\phi^{-1}(x, t)))^2$. Moreover, the Jacobian is

$$\frac{1}{|J(x, y)|} = \frac{1}{|\partial y N f(x, y)|} = \frac{1}{|TNf(x, y)|},$$

that does not vanish at $p = (0, 0)$ as proved in Lemma 87. We deduce that there exist C_2, $C_2 > 0$ such that

$$C_1 < \frac{h(x, y)}{|J(x, y)|} < C_2 \qquad \forall (x, y) \in \widetilde{U}.$$

(eventually restricting U). Hence, we reduce to study the following quantity

$$\int_{\widetilde{U}} \frac{1}{\|\nabla_H f\|} dx dt = \int_{\phi(U)} \frac{1}{\sqrt{t^2 + (Tf(\phi^{-1}(x, t)))^2}} dx dt.$$

The Taylor expansion $Tf\left(\phi^{-1}(x,t)\right)$ with respect to $t \to 0$ is

$$Tf\left(\phi^{-1}(x,t)\right) = \xi(x) + t\left(\alpha + R(x,t)\right), \qquad \text{with } \lim_{(x,t)\to 0} R(x,t) = 0$$

and $\alpha \in \mathbb{R}$. Note that $\mathcal{C} = \{Nf(x,y) = t = 0\}$, so $\{(x,0)\}$ is the support of γ, the critical curve of p with respect to N. Then, the hypothesis of p being mildly degenerate explixates as follows:

$$\xi(x) = Tf\left(\phi^{-1}(x,0)\right) = Tf\left(\gamma(x)\right) = cx^k\left(1 + r(x)\right)$$

with $\lim_{x\to 0} r(x) = 0$ and the natural parametrizazion $\gamma(x) = (x,0)$. We deduce that

$$\begin{aligned}\|\nabla_H f\|^2(x,t) &= t^2 + \left(cx^k\left(1+r(x)\right) + t\left(\alpha + R(x,t)\right)\right)^2\\ &= \left(1+\alpha^2\right)t^2 + c^2x^{2k} + cx^{2k}\left(r^2(x) + 2r(x)\right)\\ &\quad + t^2\left(R^2(x,y) + 2\alpha R(x,t)\right) + 2cx^k t\left(\alpha + R(x,t)\right)\left(1+r(x)\right).\end{aligned}$$

Finally, we consider the weighted polar coordinates

$$\begin{cases} cx^k = \rho\cos\theta \\ \sqrt{1+\alpha^2}\,t = \rho\sin\theta \end{cases}$$

so that $(\rho,\theta) = \psi(x,t)$. Substituting, we obtain

$$\|\nabla_H f\|^2 = \rho^2\left(1 + \frac{\alpha}{\sqrt{1+\alpha^2}}\sin 2\theta + R_{pol}(\rho,\theta)\right)$$

with

$$\begin{aligned}R_{pol}(\rho,\theta) =& \cos^2\theta\left(r^2(x) + 2r(x)\right) + \frac{\sin^2\theta}{1+\alpha^2}\left(R^2(x,y) + 2\alpha R(x,t)\right)\\ &+ \frac{\sin 2\theta}{\sqrt{1+\alpha^2}}\left(R(x,t) + \alpha r(x) + R(x,t)r(x)\right)\end{aligned}$$

that tends to 0 as $(x,t) \to 0$, or better we say that $\lim_{\rho\to 0} R_{pol}(\rho,\theta) = 0$. Furthermore,

$$1 + \frac{\alpha}{\sqrt{1+\alpha^2}}\sin 2\theta \geq 1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}} = m^2 > 0.$$

Thus, eventually restricting the domain of integration, $\|\nabla_H f\| \geq \frac{|m|}{2}\rho$, and so

$$\int_U \frac{1}{\|\nabla_H f\|}\, dA^R \leq \frac{2}{\sqrt{m^2\left(1+\alpha^2\right)}k\sqrt[k]{c}} \int_{\phi(U)} \left(\rho\,|\sin\theta|\right)^{\frac{1}{k}-1} d\rho d\theta$$

that is integrable if $k \geq 1$. The last requirement on k is satisfied thanks to the hypothesis of the characteristic point being isolated. Indeed, the set of the characteristic points is described by the condition $\{t = 0,\ \xi(x) = 0\}$ and so ξ is non identically vanishing in a neighborhood of p. ☐

In conclusion, we prove Corollary 13 that is a local integrability result in the case when the considered objects are real-analytic. We stress the fact that the assigned measure μ does not need to be analytic. Indeed, the mild degeneration condition depends only on the sub-Riemannian structure and on the locally defining function for the surface.

Proof. On the one hand, the statement in presence of isolated non-degenerate characteristic points holds directly from [**Ros21**, Thm. 1.1]. On the other hand, let us consider an isolated degenerate characteristic point $p \in S$, we show that the hypotheses of Theorem 11 are satisfied. Since the sub-Riemannian structure and the locally defining function for S are real-analytic, then, we can consider any real-analytic extension N of $N_p \in \ker \nabla_H^2 f$ as in (4.2). As a consequence, the critical curve of p with respect to N is a real-analytic submanifold. and a degenerate characteristic point is necessarily mildly degenerate. □

APPENDIX A

Proof of Proposition 36

We denote by $\langle \cdot, \cdot \rangle_\varepsilon$ the scalar product induced by g^ε. Moreover, let ∇^ε be the Levi-Civita connection associated to g^ε. In order to compute the mean curvature H^ε of S with respect to the g^ε, we consider the orthonormal frame on TS

$$X_S = (X_2\delta)X_1 - (X_1\delta)X_2 \qquad Y^\varepsilon = \frac{\varepsilon(X_\theta\delta)(X_1\delta)X_1 + \varepsilon(X_\theta\delta)(X_2\delta)X_2 - \varepsilon X_\theta}{\sqrt{1+\varepsilon^2(X_\theta\delta)^2}}.$$

Notice that X_S is the characteristic vector field in (2.5) which is horizontal and does not depend on ε. Moreover, if X_θ is the Reeb vector field X_0, then Y^ε is Y_S in (2.6) normalized with respect to g^ε. The mean curvature H^ε is the trace of the second fundamental form computed with respect to the frame X_S, Y. More precisely,

$$H^\varepsilon = \langle \nabla^\varepsilon_{X_S} X_S, N^\varepsilon \rangle_\varepsilon + \langle \nabla^\varepsilon_{Y^\varepsilon} Y^\varepsilon, N^\varepsilon \rangle_\varepsilon \tag{A.1}$$

where N^ε is the Riemannian gradient with respect to g^ε of S

$$N^\varepsilon = \frac{(X_1\delta)X_1 + (X_2\delta)X_2 + \varepsilon^2(X_\theta\delta)X_\theta}{\sqrt{1+\varepsilon^2(X_\theta\delta)^2}}.$$

The key tool for the computations is the Koszul formula, i.e., for U, V, Z vector fields, we have that

$$\langle \nabla^\varepsilon_U V, Z \rangle_\varepsilon = \frac{1}{2}\left(\langle [U,V], Z \rangle_\varepsilon - \langle [V,Z], U \rangle + \langle [Z,U], V \rangle_\varepsilon \right). \tag{A.2}$$

Furthermore recall that

$$[X_j, X_i] = c^1_{ij} X_1 + c^2_{ij} X_2 + c^\theta_{ij} X_\theta \quad \text{for } i,j = 1,2,\theta. \tag{A.3}$$

Notice that c^k_{ij} do not depend on ε. Let us compute the first term in (A.1)

$$\begin{aligned}
\nabla^\varepsilon_{X_S} X_S &= \nabla^\varepsilon_{(X_2\delta)X_1 - (X_1\delta)X_2} (X_2\delta)X_1 - (X_1\delta)X_2 \\
&= \left((X_2\delta)X_1X_2\delta - (X_1\delta)X_2X_2\delta \right) X_1 \\
&\quad + \left(-(X_2\delta)X_1X_1\delta + (X_1\delta)X_2X_1\delta \right) X_2 \\
&\quad + (X_2\delta)^2 \nabla^\varepsilon_{X_1} X_1 + (X_1\delta)^2 \nabla^\varepsilon_{X_2} X_2 \\
&\quad - (X_1\delta)(X_2\delta)(\nabla^\varepsilon_{X_1} X_2 + \nabla^\varepsilon_{X_2} X_1).
\end{aligned}$$

Differentiating (2.4) with respect to X_1 and X_2 and X_θ we obtain that

$$\begin{aligned}
&(X_1\delta)X_1X_1\delta + (X_2\delta)X_1X_2\delta = 0, \\
&(X_1\delta)X_2X_1\delta + (X_2\delta)X_2X_2\delta = 0, \\
&(X_1\delta)X_\theta X_1\delta + (X_2\delta)X_\theta X_2\delta = 0.
\end{aligned} \tag{A.4}$$

Taking into account (A.3), we have that

$$\begin{aligned}
\nabla^{\varepsilon}_{X_1} X_1 &= c^1_{12} X_2 - \varepsilon^2 c^1_{\theta 1} X_\theta;\\
\nabla^{\varepsilon}_{X_1} X_2 &= -c^1_{12} X_1 + \frac{1}{2}(-c^\theta_{12} - \varepsilon^2 c^1_{\theta 2} - \varepsilon^2 c^2_{\theta 1}) X_\theta\\
\nabla^{\varepsilon}_{X_2} X_1 &= c^2_{12} X_2 + \frac{1}{2}(c^\theta_{12} - \varepsilon^2 c^1_{\theta 2} - \varepsilon^2 c^2_{\theta 1}) X_\theta\\
\nabla^{\varepsilon}_{X_2} X_2 &= -c^2_{12} X_2 - \varepsilon^2 c^2_{\theta 2} X_\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla^{\varepsilon}_{X_S} X_S = & \left(-(X_1\delta) X_1 X_1 \delta - (X_1\delta) X_2 X_2 \delta\right) X_1\\
&+ \left(-(X_2\delta) X_1 X_1 \delta - (X_2\delta) X_2 X_2 \delta\right) X_2\\
&+ (X_2\delta)^2 (c^1_{12} X_2 - \varepsilon^2 c^1_{\theta 1} X_\theta) + (X_1\delta)^2 (-c^2_{12} X_2 - \varepsilon^2 c^2_{\theta 2} X_\theta)\\
&- (X_1\delta)(X_2\delta)\left(-c^1_{12} X_1 + c^2_{12} X_2 + -(c^1_{\theta 2} + c^2_{\theta 1}) X_\theta\right)\\
= & -\left(X_1 X_1 \delta + X_2 X_2 \delta - c^1_{12}(X_2\delta) + c^2_{12}(X_1\delta)\right)((X_1\delta) X_1 + (X_2\delta) X_2)\\
&- \varepsilon^2 ((X_2\delta)^2 c^1_{\theta 1} + (X_1\delta)^2 c^2_{\theta 2} - (X_1\delta)(X_2\delta)(c^1_{\theta 2} + c^2_{\theta 1})) X_\theta.
\end{aligned}$$

Simplifying the computations we obtain that

$$\begin{aligned}
\langle \nabla^{\varepsilon}_{X_S} X_S, N^\varepsilon \rangle_\varepsilon = & \frac{-X_1 X_1 \delta - X_2 X_2 \delta + c^1_{12}(X_2\delta) - c^2_{12}(X_1\delta)}{\sqrt{1 + \varepsilon^2 (X_\theta \delta)^2}}\\
&- \varepsilon^2 (X_\theta\delta) \frac{(X_2\delta)^2 c^1_{\theta 1} + (X_1\delta)^2 c^2_{\theta 2} - (X_1\delta)(X_2\delta)(c^1_{\theta 2} + c^2_{\theta 1})}{\sqrt{1 + \varepsilon^2 (X_\theta \delta)^2}}.
\end{aligned}$$

Finally, recalling (2.15), we conclude that

$$\langle \nabla^{\varepsilon}_{X_S} X_S, N^\varepsilon \rangle_\varepsilon = \frac{\mathcal{H}}{\sqrt{1 + \varepsilon^2 (X_\theta \delta)^2}} + O(\varepsilon^2). \tag{A.5}$$

Next, we compute the second term in (A.1). Let us set $N = (X_1\delta) X_1 + (X_2\delta) X_2$ as in Definition 37, so that we can rewrite Y^ε and N^ε in the following way

$$Y^\varepsilon = \frac{\varepsilon (X_\theta\delta) N - \varepsilon X_\theta}{\sqrt{1 + \varepsilon^2 (X_\theta \delta)^2}}, \qquad N^\varepsilon = \frac{N + \varepsilon^2 (X_\theta\delta) X_\theta}{\sqrt{1 + \varepsilon^2 (X_\theta \delta)^2}}.$$

Since Y^ε and N^ε are orthogonal,

$$\langle \nabla^{\varepsilon}_{Y^\varepsilon} Y^\varepsilon, N^\varepsilon \rangle_\varepsilon = \frac{\varepsilon^2}{(1 + \varepsilon^2 (X_\theta\delta)^2)^{\frac{3}{2}}} \langle \nabla^{\varepsilon}_{(X_\theta\delta)N - X_\theta}((X_\theta\delta)N - X_\theta), N + \varepsilon^2 (X_\theta\delta) X_\theta \rangle_\varepsilon.$$

In particular,

$$\begin{aligned}
\nabla^{\varepsilon}_{(X_\theta\delta)N - X_\theta}((X_\theta\delta)N - X_\theta) = & (X_\theta\delta)^2 \nabla^{\varepsilon}_N N - (X_\theta\delta) \nabla^{\varepsilon}_N X_\theta + \nabla^{\varepsilon}_{X_\theta} X_\theta\\
&+ ((X_\theta\delta) N X_\theta \delta - X_\theta X_\theta \delta) N.
\end{aligned}$$

Since ∇^ε is a metric connection, then $\nabla^\varepsilon_U V$ is orthogonal to V for every pair of vector fields U, V. Hence

$$\begin{aligned}\langle \nabla^\varepsilon_{Y^\varepsilon} Y^\varepsilon, N^\varepsilon\rangle_\varepsilon = &-\varepsilon^4 \frac{(X_\theta\delta)^3 \langle \nabla^\varepsilon_N N, X_\theta\rangle_\varepsilon}{(1+\varepsilon^2(X_\theta\delta)^2)^{\frac{3}{2}}} \\ &+ \varepsilon^2 \frac{\langle (X_\theta\delta)\nabla^\varepsilon_N X_\theta - \nabla^\varepsilon_{X_\theta} X_\theta, N\rangle_\varepsilon}{(1+\varepsilon^2(X_\theta\delta)^2)^{\frac{3}{2}}} \\ &- \varepsilon^2 \frac{(X_\theta\delta) N X_\theta\delta - X_\theta X_\theta \delta}{(1+\varepsilon^2(X_\theta\delta)^2)^{\frac{3}{2}}}.\end{aligned} \tag{A.6}$$

Using again the Koszul formula in (A.2), and (A.4), we obtain that

$$\begin{aligned}\langle \nabla^\varepsilon_N N, X_\theta\rangle_\varepsilon &= -\langle \nabla^\varepsilon_N X_\theta, N\rangle_\varepsilon \\ &= -\langle [N, X_\theta], N\rangle_\varepsilon \\ &= -(X_1\delta)^2 c^1_{\theta 1} - (X_1\delta)(X_2\delta)(c^1_{\theta 2} + c^2_{\theta 1}) - (X_2\delta)^2 c^2_{\theta 2},\end{aligned} \tag{A.7}$$

that does not depend on ε. On the other hand,

$$\langle \nabla^\varepsilon_{X_\theta} X_\theta, N\rangle_\varepsilon = \langle [N, X_\theta], X_\theta\rangle_\varepsilon = \frac{(X_1\delta)c^\theta_{\theta 1} + (X_2\delta)c^\theta_{\theta 2}}{\varepsilon^2}. \tag{A.8}$$

Substituting (A.7) and (A.8) into (A.6), we have

$$\langle \nabla^\varepsilon_{Y^\varepsilon} Y^\varepsilon, N^\varepsilon\rangle_\varepsilon = O(\varepsilon^2) - \frac{(X_1\delta)c^\theta_{\theta 1} + (X_2\delta)c^\theta_{\theta 2}}{(1+\varepsilon^2(X_\theta\delta)^2)^{\frac{3}{2}}} \tag{A.9}$$

Taking the limit for $\varepsilon \to 0$ in (A.5) and (A.9)

$$\lim_{\varepsilon\to 0} H^\varepsilon = \lim_{\varepsilon\to 0} \frac{\mathcal{H} + (X_1\delta)c^\theta_{\theta 1} + (X_2\delta)c^\theta_{\theta 2}}{(1+\varepsilon^2(X_\theta\delta)^2)^{\frac{3}{2}}} + O(\varepsilon^2)$$

one recovers (2.17).

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