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Finite State N-player and Mean Field Games

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Riassunto

I giochi a campo medio rappresentano modelli limite per giochi dinamici, simmetrici ed a somma non zero, quando il numero N di giocatori tende all'infinito. In questa tesi consideriamo giochi a campo medio e ad N giocatori in cui la posizione di ogni giocatore appartiene ad un insieme degli stati finito. Il tempo è continuo e l'orizzonte temporale è finito. A differenza dei precedenti lavori sull'argomento, utilizziamo una rappresentazione probabilistica delle dinamiche in termini di equazioni differenziali stocastiche rispetto a misure aleatorie di Poisson. Per prima cosa dimostriamo l'esistenza di soluzioni del gioco a campo medio con controlli rilassati, sia open-loop che feedback, in ipotesi piuttosto generali. Basandoci sulla rappresentazione probabilistica e su un argomento di accoppiamento, mostriamo che le soluzioni del gioco a campo medio forniscono ε_N equilibri di Nash per il gioco ad N giocatori, in strategie sia open-loop che feedback (non rilassate), con $\varepsilon_N \leq \frac{\text{costante}}{\sqrt{N}}$. In ipotesi più forti troviamo anche soluzioni del gioco a campo medio con controlli feedback ordinari e dimostriamo l'unicità se l'orizzonte temporale è abbastanza piccolo oppure sotto ipotesi di monotonia.

Poi, assumendo che i giocatori controllino solamente il proprio tasso di transizione da stato a stato, mostriamo la convergenza, per N che tende all'infinito, del gioco ad N giocatori alla dinamica limite data dal sistema del gioco a campo medio costituito da due ODE accoppiate, una in avanti e l'altra all'indietro. Sfruttiamo la cosiddetta master equation che nel presente contesto finito dimensionale è una PDE del primo ordine nel simplesso delle misure di probabilità. Se la master equation possiede una soluzione classica, allora tale solutione può essere usata per provare la convergenza delle funzioni valore degli N giocatori e degli equilibri di Nash feedback, ed anche la proprietà di propagazine del chaos per le traiettorie ottimali associate. Una condizione sufficiente per la regolarità richiesta per la master equation è data dalle ipotesi di monotonia. Inoltre impieghiamo il risultato di convergenza per stabilire un Teorema Limite Centrale ed un Principio delle Grandi Deviazioni per l'evoluzione delle misure empiriche ottimali.

Infine analizziamo un'esempio in cui lo spazio degli stati è $\{-1, 1\}$ ed il costo è antimonotono, e mostriamo che il gioco a campo medio possiede esattamente tre soluzioni. L'equilibrio di Nash è sempre unico e proviamo che il gioco ad N giocatori ammette sempre un limite: seleziona una singola soluzione del gioco a campo medio, tranne in un caso critico, pertanto c'è propagazione del chaos. Anche le funzioni valore convergono ed il limite è dato dalla soluzione di entropia della master equation, la quale in questo caso può essere scritta come una legge di conservazione scalare. Inoltre, vedendo il sistema del gioco a campo medio come le condizioni necessarie di ottimalità di un problema di controllo deterministico, mostriamo che il gioco ad N giocatori seleziona esattamente l'ottimo di questo problema quando è unico.

Abstract

Mean field games represent limit models for symmetric non-zero sum dynamic games when the number N of players tends to infinity. In this thesis, we study mean field games and corresponding N-player games in continuous time over a finite time horizon where the position of each agent belongs to a finite state space. As opposed to previous works on finite state mean field games, we use a probabilistic representation of the system dynamics in terms of stochastic differential equations driven by Poisson random measures. Firstly, under mild assumptions, we prove existence of solutions to the mean field game in relaxed open-loop as well as relaxed feedback controls. Relying on the probabilistic representation and a coupling argument, we show that mean field game solutions provide symmetric ε_N-Nash equilibria for the N-player game, both in open-loop and in feedback strategies (not relaxed), with $\varepsilon_N \leq \frac{\text{constant}}{\sqrt{N}}$. Under stronger assumptions, we also find solutions of the mean field game in ordinary feedback controls and prove uniqueness either in case of a small time horizon or under monotonicity.

Then, assuming that players control just their transition rates from state to state, we show the convergence, as N tends to infinity, of the N-player game to a limiting dynamics given by a finite state mean field game system made of two coupled forward-backward ODEs. We exploit the so-called master equation, which in this finite-dimensional framework is a first order PDE in the simplex of probability measures. If the master equation possesses a unique regular solution, then such solution can be used to prove the convergence of the value functions of the N players and of the feedback Nash equilibria, and a propagation of chaos property for the associated optimal trajectories. A sufficient condition for the required regularity of the master equation is given by the monotonicity assumptions. Further, we employ the convergence results to establish a Central Limit Theorem and a Large Deviation Principle for the evolution of the N-player optimal empirical measures.

Finally, we analyze an example with $\{-1, 1\}$ as state space and anti-monotonous cost, and show that the mean field game has exactly three solutions. The Nash equilibrium is always unique and we prove that the N-player game always admits a limit: it selects one mean field game solution, except in one critical case, so there is propagation of chaos. The value functions also converge and the limit is the entropy solution to the master equation, which for two state models can be written as a scalar conservation law. Moreover, viewing the mean field game system as the necessary conditions for optimality of a deterministic control problem, we show that the N-player game selects the optimum of this problem when it is unique.

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Introduction

Mean field games (MFG), as independently introduced by [67] and by [54], represent limit models for symmetric non-zero-sum non-cooperative N-player dynamic games when the number N of players tends to infinity. In games with a large number of players it is very hard to find the Nash equilibria, even numerically, because of the curse of dimensionality. Letting the number of players be infinite, rather than very large, has provided great advantages in the tractability of such games. This passage to the limit, which in the original works was only heuristic, is possible assuming that the players are indistinguishable and the interaction among them is of *mean field* type, in the sense that the evolution of a single agent only depends on his position and on the average behaviour of all the others. The limit model thus describes the statistical distribution of a single representative player, in analogy with mean field models in statistical mechanics. As opposed to these classical models, in mean field games agents are *rational*, in the sense that they aim at minimizing a cost.

For an introduction to mean field games see [12], [47], [20] and [7]; the latter two works also deal with optimal control problems of McKean-Vlasov type. We refer to the recent books [18, 19] for a thorough treatment of the probabilistic side of the theory. Mean field games have seen a wide variety of applications, including models of oil production, volatility formation, population dynamics and economic growth; see [52, 63, 68] for some examples. There is by now a wealth of works dealing with different classes of mean field games; for a partial overview see [47] and the references therein. In this thesis, we restrict attention to a class of finite time horizon problems with continuous time dynamics and fully symmetric cost structure, where the position of each agent belongs to a finite state space.

The relation between the limit model (the mean field game) and the corresponding prelimit models (the N-player games) can be understood in two opposite directions: approximation and convergence. By approximation we mean that a solution of the mean field game allows to construct approximate Nash equilibria for the N-player games, where the approximation error is arbitrarily small for N big enough. By convergence we mean that Nash equilibria for the N-player games may be expected to converge to a solution of the mean field game as N tends to infinity.

Results in the approximation direction are more common and usually provide the justification for the definition of the mean field game. When the underlying dynamics is of Itô type without jumps, such results were established by [54] and, more recently, by for instance [17], [22] and [8]. When the dynamics is driven by generators of Lévy type, but with the control appearing only in the drift, an approximation result is found in [60]. Rigorous results on convergence to the mean field game limit in the non stationary case (finite time horizon) are even more recent. While the limits of N-player Nash equilibria in stochastic open-loop strategies can be completely characterized (see [65] and [42] for general systems of Itô type), the convergence problem is more difficult for Nash equilibria in Markov feedback strategies with global state information. A result in this direction is given by [46] in our finite state setting, via the infinitesimal generator, but only if the time horizon is small.

A breakthrough was achieved by Cardaliaguet, Delarue, Lasry and Lions in [14]. Their proof of convergence relies on having a regular solution to the so-called master equation. This is a kind of transport equation on the space of probability measures associated with the mean field game; its solution yields a solution to the mean field game for any initial time and initial distribution. If the mean field game is such that its master equation possesses a unique regular solution, then that solution can be used to prove convergence of the value functions of the N-player game to the solution to the master equation, as well as a form of convergence for the associated optimal feedback strategies and a propagation of chaos result for the corresponding optimal trajectories. An important ingredient in the proof is a coupling argument similar to the one employed in deriving the propagation of chaos property for uncontrolled mean field systems (c.f. [75]). This kind of coupling argument, in which independent copies of the limit process are compared to their prelimit counterparts, is useful also for obtaining approximation results; cf. for instance the above cited works by [54] and [17].

In this thesis, we focus on games where the position of each agent belongs to a given finite state space $\Sigma = \{1, \ldots, d\}$. In this setting, mean field games were first analyzed in discrete time by [45], and then in continuous time by [46] and [51], the latter with an application on graphs, and also by [4]. The approach of these works is based on PDE / ODE methods and the infinitesimal generator (Q matrix) of the system dynamics. Here, we adopt a different approach based on a probabilistic representation. We employ this formulation to prove, first, the existence of limiting solutions and the approximation result, under general assumptions. Then, under stronger assumptions for which the master equation possesses a classical solution, we show the convergence of the feedback Nash equilibria and a propagation of chaos property for the associated optimal trajectories. Moreover, we analyze a two state model for which the master equation has no smooth solution and show that the convergence results still hold.

We write the dynamics of the N-player game as a system of stochastic differential equations driven by independent stationary Poisson random measures with the same intensity measure ν, weakly coupled through the empirical measure of the system states:

$$X_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, X_i(s^-), \theta, \pi^i(s), m^N(s^-)) \mathcal{N}_i(ds, d\theta), \quad i = 1, \ldots, N, \tag{0.1}$$

where π^i is the control of player i (here in open loop form) with values in a compact set A and $m^N(s^-)$ is the empirical measure of the system immediately before time s. The dynamics for the one representative player of the mean field limit is analogously written as

$$X(t) = \xi + \int_0^t \int_\Theta \gamma(s, X(s^-), \theta, \pi(s), m(s)) \mathcal{N}(ds, d\theta), \tag{0.2}$$

where π is the control and $m : [0, T] \to \mathcal{P}(\Sigma)$ a deterministic flow of probability measures, which takes the place of m^N.

The function γ appearing in (0.1) and (0.2) can be chosen so that the corresponding state processes X_i^N, X have prescribed transition rates when the control and measure variable are held constant. Following an idea of [50], we choose $\Theta \subset \mathbb{R}^d$, let the intensity measure ν be given by d copies of Lebesgue measure on the line (cf. (1.9) below), and set

$$\gamma(t, x, \theta, a, m) := \sum_{y \in \Sigma} (y - x) \mathbb{1}_{]0, \Gamma_{x,y}(t,a,m)[}(\theta_y). \tag{0.3}$$

With this γ we have, as $h \downarrow 0$,

$$P\left[X(t+h) = y | X(t) = x\right] = \Gamma_{x,y}(t,a,m) \cdot h + o(h)$$

if $y \neq x$, for any constant control a and probability measure m. Thus, $\Gamma_{x,y}(t,a,m)$ is the transition rate from state x to state y.

Each player wants to optimize his cost functional over a finite time horizon T. The coefficients representing running and terminal costs may depend on the measure variable and are the same for all players, in light of the mean field structure of the problem. The solution of the mean field game can be seen as a fixed point. For a given flow of measures $m(\cdot)$, find a strategy π_m that is optimal and let $X^{\pi_m,m}$ be the corresponding solution of Eq. (0.2). Now find m such that $\text{Law}(X(t)) = m(t)$ for all $t \in [0,T]$.

Existence and approximation

The results on existence and approximation were presented in [25]. We will consider several types of controls: open-loop, feedback, relaxed open-loop and relaxed feedback. We first study the mean field game and show that it admits a solution in relaxed controls. Under mild hypotheses, we prove existence of solutions in relaxed open-loop controls using the Ky Fan fixed point theorem for point-to-set maps. This is analogous to the existence result obtained by [64] for general dynamics driven by Wiener processes. As there, we will characterize solutions to Eq. (0.2) through the associated controlled martingale problem. In order to write the dynamics when using a relaxed control, we need to work with relaxed Poisson measures in the sense of [62]; also see Appendix A below. The same assumptions that give existence in relaxed open-loop controls also yield existence of solutions in relaxed feedback controls. Relaxed controls are used only for the limit model.

Then we show that those relaxed mean field game solutions provide ε_N-Nash equilibria for the N-player game both in ordinary open-loop and ordinary feedback strategies. To this end, we approximate a limiting optimal relaxed control by an ordinary one, using a version of the chattering lemma that also works for feedback controls, at least in our finite setting. The approximating control is then shown to provide a symmetric ε_N-Nash equilibrium, with $\varepsilon_N \leq \frac{\text{constant}}{\sqrt{N}}$, decentralized when considering feedback strategies. As explained above, our proof relies on the probabilistic representation of the system and a coupling argument.

We also study the problem of finding solutions of the mean field game in ordinary feedback controls. There, we need stronger assumptions in order to guarantee the uniqueness of an optimal feedback control for any fixed m (existence always holds). Moreover, we prove that the feedback mean field game solution is unique either if the time horizon T is small enough or if the cost coefficients satisfy the monotonicity conditions of Lasry and Lions.

Roughly speaking, we need to assume only the continuity of the rates Γ in order to have relaxed or relaxed feedback mean field game solutions and to obtain ε_N-Nash equilibria for the N-player game, both open-loop and feedback. Under stronger assumptions, namely affine dependence of Γ on the control and strict convexity of the cost, we have uniqueness of the optimal feedback control for any m through the uniqueness of the minimizer of the associated Hamiltonian. Under assumptions similar to these latter, [4] study the problem in the framework of non-linear Markov processes and find $\frac{1}{N}$-Nash equilibria for the N-player game. That work considers ordinary feedback controls only, hence feedback solutions of the mean field game.

Let us mention several recent preprints which address the problems of existence and approximation in the finite state space scenario. In [36], continuous time mean field games with finite state space and finite action space are studied. The authors prove existence of solutions to the mean field game, corresponding to what we call solutions in relaxed feedback controls. Their prelimit models (the N-player games) are different and difficult to compare to ours since they are set in discrete time. In [23], finite state mean field games with major and minor players are considered. The authors provide a characterization of the mean field game in terms of viscosity solutions to a coupled system of integro-differential equations and establish existence of solutions. The connection with the underlying $N+1$-player game is made through the construction of approximate Nash equilibria. As opposed to the fully symmetric case considered here, in the presence of a major player, solutions to the limit system in open-loop and in feedback strategies are in general not equivalent. The third work we mention is [6]. There, the authors study a class of mean field games with jump diffusion dynamics. An existence result for the mean field game in the spirit of [64] is given. The authors also obtain a convergence result in a special situation where Nash equilibria for the N-player games can be found explicitly. In their model, the jump heights are directly (and linearly) controlled, not the jump intensities.

Convergence via the master equation

In order to establish the convergence results, which we presented in [27], we need to make stronger assumptions on the dynamics and on the cost. In the remaining part of this introduction, we consider only feedback Nash equilibria and so feedback mean field game solutions. Assume hence that the dynamics is given by (0.3) and that the transition rates Γ coincide with the control, in analogy with the original works of Lasry and Lions. More precisely, the control space is $[0,+\infty[^d$ and $\Gamma_{x,y} = a_y$, so that the feedback control $\alpha^i_y(t, x_i, \boldsymbol{x}^{N,i}_t)$ represents the rate at which player i decides to go from state x_i to state y, when $x_i \neq y$, $\boldsymbol{x}^{N,i}_t$ being the states of the other $N-1$ players at time t; c.f. (1.38) below. Player i aims at minimizing the cost

$$J_i(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T \Big(L(X_i(t), \alpha^i(t, \boldsymbol{X}_t)) + F^{N,i}(\boldsymbol{X}_t)\Big)\, dt + G^{N,i}(\boldsymbol{X}_T)\right],$$

where $\boldsymbol{X}_t := (X_1(t), \dots, X_N(t))$ is the vector of the N processes and $\boldsymbol{\alpha} = (\alpha^1, \dots, \alpha^N)$ is the vector of the feedback controls; for the moment, let $F^{N,i}$ and $G^{N,i}$ depend on i.

We follow the approach of [14], showing the convergence of the value functions of the N-player game to the (unique) classical solution of the master equation. The argument provides also the convergence of the feedback Nash equilibria and a propagation of chaos property for the associated optimal trajectories. The coupling technique necessary for the proof was the main motivation for writing the dynamics of the N players as stochastic differential equations driven by Poisson Random measures. Let us remark that, while Cardaliaguet et al. study the convergence problem also in the presence of a noise (Brownian motion) common to all the players, which makes things even more difficult, we do not consider here any common noise. In the discrete setting, this would result in considering dynamics with simultaneous jumps, which can be realized by adding another Poisson measure in (0.1), common to all the players; see for instance [2]. For a different treatment of the common noise in finite state mean field games, see [9].

In our framework, we show that there exists a unique feedback Nash equilibrium for the N-player game. It is provided by the Nash system of Nd^N coupled ODE's, indexed by

$\boldsymbol{x} = (x_1, \ldots, x_N) \in \Sigma^N$,

$$\begin{cases} -\frac{\partial v^{N,i}}{\partial t}(t,\boldsymbol{x}) - \sum_{j=1,\ j\neq i}^{N} a^*(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} + H(x_i, \Delta^i v^{N,i}) = F^{N,i}(\boldsymbol{x}), \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(\boldsymbol{x}). \end{cases}$$

In the above equation, H is the Hamiltonian corresponding to L and a^* its unique maximizer, and

$$\Delta^j g(\boldsymbol{x}) = (g(x_1, \ldots, y, \ldots, x_N) - g(x_1, \ldots, x_j, \ldots, x_N))_{y=1,\ldots,d} \in \mathbb{R}^d$$

denotes the finite difference of a function $g(\boldsymbol{x}) = g(x_1, \ldots, x_N)$ with respect to its j-th entry.

The study of convergence consists in finding a limit for the Nash system as N tends to infinity. To this end, the symmetry properties of the game we described above are required. Namely, the costs $F^{N,i}$ and $G^{N,i}$ must satisfy the mean field assumptions: there exist two functions F and G such that $F^{N,i}(\boldsymbol{x}) = F(x_i, m_{\boldsymbol{x}}^{N,i})$ and $G^{N,i}(\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i})$, where $m_{\boldsymbol{x}}^{N,i}$ denotes the empirical measure of all the players except for the i-th, which belongs to $\mathcal{P}(\Sigma)$. Thanks to these mean field assumptions, we shall say that the solution $v^{N,i}$ of the Nash system can be found in the form $v^{N,i}(t,\boldsymbol{x}) = V^N(t, x_i, m_{\boldsymbol{x}}^{N,i})$, for a suitable function V^N of time, space and measure; this makes the convergence problem more tractable. At a formal level, we can introduce the limiting equation assuming the existence of a function U such that $V^N(t, x_i, m_{\boldsymbol{x}}^{N,i}) \sim U(t, x_i, m_{\boldsymbol{x}}^{N,i})$ for large N. Then, let us analyze the different components of the Nash system and which should be their corresponding limits in terms of U. First, the i-th difference of $v^{N,i}$ should converge to

$$\begin{aligned} \Delta^i v^{N,i}(t,\boldsymbol{x}) &= \left(v^{N,i}\left(t, y, m_{\boldsymbol{x}}^{N,i}\right) - v^{N,i}\left(t, x_i, m_{\boldsymbol{x}}^{N,i}\right)\right)_{y=1,\ldots,d} \\ &\to (U(t,y,m) - U(t,x_i,m))_{y=1,\ldots,d} = \Delta^x U(t, x_i, m). \end{aligned}$$

For $j \neq i$ we should instead get

$$\begin{aligned} \Delta^j v^{N,i}(t,\boldsymbol{x}) &= \left(v^{N,i}\left(t, x_i, \frac{1}{N-1}\sum_{k\neq j,i}\delta_{x_k} + \frac{1}{N-1}\delta_y\right) - v^{N,i}\left(t, x_i, \frac{1}{N-1}\sum_{k\neq i}\delta_{x_k}\right)\right)_{y=1,\ldots,d} \\ &\sim \frac{1}{N-1} D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j), \end{aligned}$$

modulo terms of order $O(1/N^2)$, where a precise definition of $D^m U$, the derivative with respect to a probability measure, will be given in 1.1.3. Then, $H(x_i, \Delta^i v^{N,i}) \to H(x_i, \Delta^x U)$, and we should obtain

$$\begin{aligned} \sum_{j=1, j\neq i}^{N} a^*(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} &\sim \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} a^*(x_j, \Delta^x U(t, x_j, m_{\boldsymbol{x}}^{N,i})) \cdot D_m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j) \\ &\sim \int_\Sigma a^*(y, \Delta^y U(t, y, m_{\boldsymbol{x}}^{N,i})) \cdot D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, y) dm_{\boldsymbol{x}}^{N,i}(y) \\ &\to \int_\Sigma D^m U(t,x,m,y) \cdot a^*(y, \Delta^y U(t,y,m)) dm(y). \end{aligned}$$

Thus, we are able to introduce the master equation, that is the equation to which we would like to prove convergence

$$\begin{cases} -\frac{\partial U}{\partial t} + H(x, \Delta^x U) - \int_\Sigma D^m U(t,x,m,y) \cdot a^*(y, \Delta^y U(t,y,m)) dm(y) = F(x,m), \\ U(T,x,m) = G(x,m), \quad (x,m) \in \Sigma \times \mathcal{P}(\Sigma), \quad t \in [0,T]. \end{cases} \tag{0.4}$$

It is a first order PDE in $\mathcal{P}(\Sigma)$, the simplex of probability measures in $\mathbb{R}^d$. We solve it using the strategy developed in [14], which relies on the method of characteristics. Under the present assumptions, the fixed point argument yielding a solution to the mean field game, as described above, is captured by the so-called mean field game system which, in our finite space setting, consists of two coupled ODEs: a backward Hamilton-Jacobi-Bellman equation giving the value function of the limiting control problem and a forward Kolmogorov-Fokker-Planck equation describing the evolution of the limiting deterministic flow of probability measures. Indeed, the mean field game system can be seen as the characteristic curves of (0.4). We solve the mean field game system for any initial time and initial distribution: this defines a candidate solution to (0.4) and, in order to prove that it is differentiable with respect to the initial condition, we introduce and analyze a linearized mean field game system. To prove the well posedness of (0.4) for any time horizon, the sufficient hypotheses we make are the monotonicity assumptions of Lasry and Lions. However, we stress again that these assumptions play no role in the convergence argument, as it requires only the existence of a regular solution to (0.4).

Let us mention that the limit equations, in the finite state scenario, have been studied by several authors in the last years. An equation similar to the master equation (0.4), but holding in the whole space $\mathbb{R}^d$, was studied in [70], proving the well-posedness and regularity under stronger assumptions. The well posedness of the mean field game system was discussed in [45], in the discrete time framework, and then in [46] and [51], under monotonicity assumptions. The works [41] and [46] deal also with the problem of convergence, as T tends to infinity, to the stationary mean field game. The master equation was discussed, but only at a formal level, in [48], [49] and again in [46], in the first two with a particular focus on the two state problem.

Here, we also study the empirical measure process of the N-player optimal trajectories. Indeed, the convergence obtained allows to get a Central Limit Theorem and a Large Deviation Principle for the asymptotic behavior, as N tends to infinity, of such process. These results were the first of this type in mean field game theory and allow to better understand the convergence of the empirical distribution of the players to the limit measure. Both Central Limit Theorems and Large Deviation Principles have been investigated for classical finite state mean field interacting particle systems, i.e. where the prelimit jump rates are the same for any individual. The fluctuations are derived e.g. in [28], via the martingale problem, and in [30], by analyzing the infinitesimal generator. The large deviation properties are found e.g. in [69], [74], [73], and, more recently, in [39].

The key point for proving these results is to compare the prelimit optimal trajectories with the ones in which each player chooses the control induced by the master equation. The fluctuations are then found by analyzing the associated infinitesimal generator, while the Large Deviation properties are derived using a result in [39]. It is worth saying that the limit processes involve the solution to the master equation. Let us mention that such properties, enriched by a concentration of measure result, are now established in the diffusion case, via the master equation approach, by Delarue, Lacker and Ramanan in [33, 34].

We also mention the recent preprint [5], which appeared some days after submission of [27]. In that work, independently, the authors again use the master equation approach to find the same convergence results we prove here. They still rely on the idea developed in [14], but consider a probabilistic representation of the dynamics different from ours. Moreover, they also obtain a Central Limit Theorem for the fluctuations of the empirical measure processes. However, they prove it in a different way, that is, via a martingale Central Limit Theorem.

A two state model

We explained above that if there is uniqueness of mean field game solutions, which holds under monotonicity assumptions, then the master equation possesses a smooth solution which can be used to prove the convergence of the value functions of the N players and a propagation of chaos property for the associated optimal trajectories. In the last part of the thesis, we study the convergence problem in feedback strategies for an example exhibiting non-uniqueness of mean field game solutions, presented in [24]. The particular model we examine has dynamics in continuous time with players' states taking values in $\{-1, 1\}$ and players' feedback controls determining the rate of flipping their own state. Running costs only depend on the control actions, while terminal costs are anti-monotonous with respect to the state and measure variable. More precisely, the setup is the same as for the convergence argument described above, choosing for any $x = \pm 1$ and $m \in \mathcal{P}(\{-1, 1\})$

$$L(x, a) = \frac{|a_{-x}|^2}{2}, \qquad F(x, m) \equiv 0, \qquad G(x, m) = -x(m_1 - m_{-1}).$$

A similar example was first analyzed in [48, 49], where numerical evidence on the convergence behavior was presented. It should also be compared to analogous examples considered in the diffusion setting in Section 3.3 of [65] and in Section 3.3 of [3]. In the infinite time horizon and two state case, an example of non-uniqueness is studied in [31], via numerical simulations, where periodic orbits emerge as solutions to the mean field game.

For this two state example, the mean field game possesses exactly three solutions, given any initial distribution, as soon as the time horizon is large enough. Consequently, there is no regular solution to the master equation, while multiple weak (in the sense of distributions) solutions exist. For the N-player game, on the other hand, there is a unique symmetric Nash equilibrium in feedback strategies for each N, determined by the Nash system.

We show that the value functions associated with these Nash equilibria converge, as $N \to \infty$, to a particular solution of the master equation. In our case, the master equation can be written as a scalar conservation law in one space variable. The (weak) solution that is selected by the N-player Nash equilibria can then be characterized as the unique entropy solution of the conservation law. The entropy solution presents a discontinuity in the measure variable (at the distribution that assigns equal mass to both states). Convergence of the value functions is uniform outside any neighborhood of the discontinuity. We also prove propagation of chaos for the N-player state processes provided that their averaged initial distributions do not converge to the discontinuity. The proofs of convergence adapt the coupling arguments described above, based on the fact that the entropy solution is smooth away from its discontinuity, as well as a qualitative property of the N-player Nash equilibria, which prevents crossing of the discontinuity.

We also observe that the mean field game system of the two state example can be viewed as the necessary conditions for optimality, given by the Pontryagin maximum principle, of a deterministic control problem. Such characterization is shown to hold for general $\Sigma = \{1, \dots, d\}$, provided the costs admit a potential structure. This potential, or variational, formulation has been studied in several works in the continuous state setting, starting from [13, 15] in the deterministic case and then in [16] for degenerate diffusions. For our two state example, we prove that the mean field game solution which is selected by the limit of the N-player optimal trajectories, when propagation of chaos holds, is exactly the unique minimum of this deterministic control problem. In fact, neither the characterization of the right solution to the master equation as the entropy admissible one nor the potential structure of the problem are needed for the convergence proofs, but only the qualitative property of the Nash equilibrium: these two characterizations may permit to extend the convergence results to more general models.

Let us mention some works recently appeared as preprints that are related to our results. In [71], Nutz, San Martin, and Tan address the convergence problem for a class of mean field games of optimal stopping. The limit model there possesses multiple solutions, which are grouped into three classes according to a qualitative criterion characterizing the proportion of players that have stopped at any given time. Solutions in one of the three classes will always arise as limit points of N-player Nash equilibria, solutions in the second class may be selected in the limit, while solutions in the third class cannot be reached through N-player Nash equilibria. In [66], Lacker attacks the convergence problem in feedback strategies by probabilistic methods. For a class of games with non-degenerate Brownian dynamics that may exhibit non-uniqueness, the author shows that all limit points of the N-player feedback Nash equilibria are concentrated, as in the open-loop case, on weak solutions of the mean field game. These solutions are more general than randomizations of ordinary ("strong") solutions of the mean field game; their flows of measures, in particular, are allowed to be stochastic containing additional randomness. Still, uniqueness in ordinary solutions implies uniqueness in weak solutions, which permits to partially recover the results in [14]. The question of which weak solutions can appear as limits of feedback Nash equilibria in a situation of non-uniqueness seems to be mainly open. In this thesis, we give the definition of weak mean field game solutions for finite state mean field games, and then also find weak (feedback) solutions for the two state example that are or not supported on strong mean field game solutions.

In [32], Delarue and Foguen Tchuendom study a class of linear-quadratic mean field games with multiple solutions in the diffusion setting. They prove that by adding a common noise to the limit dynamics uniqueness of solutions is re-established. As a converse to this regularization by noise result, they identify the mean field game solutions that are selected when the common noise tends to zero as those induced by the (unique weak) entropy solution of the master equation of the original problem. The interpretation of the master equation as a scalar conservation law works in their case thanks to a one-dimensional parametrization of an a priori infinite dimensional problem. Limit points of N-player Nash equlibria are also considered in [32], but in stochastic open-loop strategies. Again, the mean field game solutions that are selected are those induced by the entropy solution of the master equation. Interestingly, they are not minimal cost solutions; the cost functionals of the game in fact select a different mean field game solution. In [32], the N-player limit and the vanishing common noise limit both select two solutions of the original mean field game with equal probability. This is due to the fact that in [32] the initial distribution for the state trajectories is chosen to sit at the discontinuity of the unique entropy solution of the master equation. In our case, we expect to see the same behavior if we started at the discontinuity.

It is worth mentioning that the opposite situation, with respect to the one treated here, is considered in the examples presented in [36] and in Section 7.2.5 of [18]. Namely, in those examples there is uniqueness of mean field game solutions, but there are multiple feedback Nash equilibria for the N-player game, for any N. This is due to the fact that in both cases the authors consider a finite action set (while for us it is continuous), so that in particular the Nash system is not well-posed. Thus they prove that there is a sequence of (feedback) Nash equilibria which converges to the mean field game limit, but also a sequence that does not converge.

Outline of the thesis

The rest of the thesis is organized as follows.

In Chapter 1, we state the definitions which will be used throughout the thesis. In Section 1.1, we first introduce the basic notation, the natural shape of γ in (0.3), and the definition of derivatives in the simplex. We present the various sets of assumptions to be used in the sequel. Then we verify all the assumptions for natural shapes of the dynamics and of the costs. In Section 1.2, we describe the N-player game and state the definition of Nash equilibrium both in open-loop and in feedback strategies. We introduce the Nash system in 1.2.1 and prove a Verification Theorem and a useful property of the equilibrium. Then, in Section 1.3, we describe the limiting dynamics and the definition of mean field game solution. Relaxed controls (open-loop and feedback) are introduced in 1.3.1, while a proper definition of relaxed Poisson measures is given in Appendix A. The mean field game system and the master equation are presented in 1.3.2. We conclude the chapter with Section 1.4 which states, for general finite state mean field games, the variational formulation as well as the definition of weak mean field game solutions.

In Chapter 2, we present the existence and approximation results. In Section 2.1, we establish existence of solutions to the mean field game in relaxed open-loop (Theorem 2.4) as well as relaxed feedback controls (Theorem 2.5). In Section 2.2, we find, under additional assumptions, mean field game solutions in non-relaxed feedback controls (Theorem 2.8), by proving the uniqueness of the optimal control for any flow of measures. In Section 2.3, we first establish a version of the chattering lemma that works also for feedback controls. Then we turn to the construction of approximate Nash equilibria coming from a solution of the mean field game, and derive the error bound mentioned above for feedback as well as open-loop strategies (Theorems 2.16 and 2.18).

In Chapter 3, we present the results, in particular about convergence, which hold when there is uniqueness of mean field game solutions. Section 3.1 contains the convergence results and their proofs: convergence of the value functions is in Theorem 3.1, while the propagation of chaos is in Theorem 3.2. In Section 3.2, we employ the convergence argument to derive the asymptotic behaviour of the empirical measure processes, that is, the Central Limit Theorem (Theorem 3.7) and the Large Deviation Principle (Theorem 3.8). Section 3.3 analyzes the well-posedness and regularity of the solution to the master equation. In Section 3.4, we collect the results about uniqueness: in Theorems 3.17 and 3.18 we prove uniqueness of feedback mean field game solutions, respectively for T small enough and under monotonicity, while in Theorem 3.19 we show that the strategy vector given by the Nash system is the unique feedback Nash equilibrium.

Finally, in Chapter 4, we deal with the two state example. Section 4.1 presents the model, starting from the limit, analyzed first in terms of the mean field game system (Subsection 4.1.1), then in terms of its master equation (Subsection 4.1.2), while (basic) general facts concerning scalar conservation laws are summarized in Appendix B. In Section 4.2, we show that the N-player Nash equilibria converge to the unique entropy solution of the master equation; cf. Theorems 4.6 and 4.7 for convergence of value functions and propagation of chaos, respectively. The property of the Nash equilibria used in the proofs of convergence is in Subsection 4.2.1. We show the numerical simulations in Subsection 4.2.4. In Section 4.3, we provide three possible extensions of this model. Subsection 4.3.1 gives an alternative characterization of the solution that is selected by the Nash equilibria in terms of a variational problem derived from the potential game structure of the two state example (Theorem 4.14), while Subsection 4.3.2 deals with weak mean field game solutions in this framework. We conclude with Subsection

4.3.3, by examining a modified example in which the transition rates are bounded below away from zero.

CHAPTER 1

N-player and mean field games

In this chapter, we firstly introduce the notation and the assumptions we will make use of in the sequel. Then we describe the N-player game and the mean field game, with the relative definitions of optimality, in the various setups which will be used in the next chapters. Finally, we state the variational formulation as well as the definition of weak mean field game solution.

1.1 Notation and assumptions

1.1.1 Basic notation

Here we clarify the notations used throughout the thesis. In Chapter 4 we make use of some other notation, which will be introduced there. We fix $\Sigma = \{1, \ldots, d\}$ to be the finite state space of any player. Let T be the finite time horizon and (A, d_A) be a compact metric space, the space of control values. Let Θ be a compact set in $\mathbb{R}^d$ and let ν be a Radon measure on Θ. Let

$$\mathcal{P}(\Sigma) := \{m \in \mathbb{R}^d : m_x \geq 0, \quad x = 1, \ldots, d; \quad m_1 + \ldots + m_d = 1\}$$

be the space of probability measures on Σ, which is the probability simplex in $\mathbb{R}^d$. Let $\gamma : [0, T] \times \Sigma \times \Theta \times A \times \mathcal{P}(\Sigma) \longrightarrow \{-d, \ldots, d\}$ be a measurable function (the one appearing in the dynamics (0.2) and (0.1)) such that $\gamma(t, x, \theta, a, m) \in \{1 - x, \ldots, d - x\}$. Let $c : [0, T] \times \Sigma \times A \times \mathcal{P}(\Sigma) \longrightarrow \mathbb{R}$, $G : \Sigma \times \mathcal{P}(\Sigma) \longrightarrow \mathbb{R}$ be measurable functions, representing the running and the terminal costs, respectively, which will be the same for all players, in light of the mean field assumptions.

We will denote by $\mathcal{N}$ any stationary Poisson random measure on $[0, T] \times \Theta$ with intensity measure ν on Θ, and by $\underline{\mathcal{N}} = (\mathcal{N}_1, \ldots, \mathcal{N}_N)$ a vector of N i.i.d. stationary Poisson random measures, each with the same law as $\mathcal{N}$. Also, denote by $\mathcal{M} = \mathcal{M}([0, T] \times \Theta \times A)$ the set of finite positive measures on $[0, T] \times \Theta \times A$ endowed with the topology of weak convergence.

The initial datum of the N-player game is represented by N i.i.d. random variables $\xi_1, \ldots, \xi_N$ with values in Σ and distributed as $m_0 \in \mathcal{P}(\Sigma)$. The vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ is in particular *exchangeable*, in the sense that the joint distribution is invariant under permutations, and is assumed to be independent of the noise. Similarly, the initial point of the limiting system will be represented by a random variable ξ with law m_0.

The state of player i at time t is denoted by $X_i(t)$, with $\boldsymbol{X}_t := (X_1(t), \ldots, X_N(t))$. The trajectories of each X_i are in $\mathcal{D} := D([0, T], \Sigma)$, the space of càdlàg functions from $[0, T]$ to

Σ endowed with the Skorokhod metric. For $\boldsymbol{x} = (x_1, \dots, x_N) \in \Sigma^N$, denote the empirical measures

$$m^N_{\boldsymbol{x}} := \frac{1}{N}\sum_{j=1}^N \delta_{x_j} \qquad m^{N,i}_{\boldsymbol{x}} := \frac{1}{N-1}\sum_{j=1, j\neq i}^N \delta_{x_j}.$$

Thus, $m^N_{\boldsymbol{X}}(t) := m^N_{\boldsymbol{X}_t}$ is the empirical measure of the N players and $m^{N,i}_{\boldsymbol{X}}(t) := m^{N,i}_{\boldsymbol{X}_t}$ is the empirical measure of all the players except the i-th. Clearly, they are $\mathcal{P}(\Sigma)$-valued stochastic processes. In the limiting dynamics, the empirical measure is replaced by a deterministic flow of probability measures $m : [0,T] \to \mathcal{P}(\Sigma)$.

The space of measures $\mathcal{P}(\Sigma)$ can be equipped with any norm in $\mathbb{R}^d$, as they are all equivalent, so we choose the Euclidean norm $|\cdot|$; when needed, we will use the 1-Wasserstein distance, denoted by d_1, and the sup-norm $|m|_\infty := \max_{x\in\Sigma} |m_x|$. We observe that $\mathcal{P}(\Sigma)$ is a compact and convex subset of $\mathbb{R}^d$. Denote by $\mathcal{C}([0,T], \mathcal{P}(\Sigma))$ the space of continuous functions from $[0,T]$ to $\mathcal{P}(\Sigma)$, endowed with the uniform norm. For future use, let us also recall the inequalities

$$\left| m^N_{\boldsymbol{x}} - m^N_{\boldsymbol{y}} \right| \le C_d \mathrm{d}_1(m^N_{\boldsymbol{x}}, m^N_{\boldsymbol{y}}) \le \frac{C_d}{N}\sum_{i=1}^N |x_i - y_i| \tag{1.1}$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \Sigma^N$, where the first inequality comes from the equivalence of all the metrics in $\mathcal{P}(\Sigma)$ and the second is well-known for the Wasserstein distance.

We restrict our attention to a subset of the space of flows of probability measures on $\mathcal{P}(\Sigma)$, denoted by $\mathcal{E} \subset \mathcal{C}([0,T], \mathcal{P}(\Sigma))$. We will show in Lemma 1.15 that all the flows considered belong to

$$\mathcal{E} := \{ m : [0,T] \longrightarrow \mathcal{P}(\Sigma) : |m(t) - m(s)| \le K_0 |t-s|, \quad m(0) = m_0 \}$$

where the constant is given by $K_0 := 2\nu(\Theta)\sqrt{d}$.

We will study several types of controls. Pathwise existence and uniqueness of solutions to the controlled dynamics (0.2), with trajectories that remain in Σ, is guaranteed by the following Lipschitz condition:

$$\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(t,y,\theta,a,m)| \nu(d\theta) \le K_1 |x-y| \tag{1.2}$$

for every $x, y \in \Sigma, s \in [0,T], a \in A$ and $m \in \mathcal{P}(\Sigma)$, where K_1 is a constant. The above condition is always satisfied in our model since $|x-y| \ge 1$ for each $x \neq y \in \Sigma$ and $\int_\Theta |\gamma(s,x,\theta,a,m)|\nu(d\theta) \le \nu(\Theta) d$; thus we may take $K_1 = 2\nu(\Theta)d$.

We identify the set of functions $g : \Sigma \longrightarrow \mathbb{R}$ with $\mathbb{R}^d$ and observe that any g is bounded and Lipschitz. For any $x \in \Sigma$, $0 \le t \le T$, $a \in A$, $m \in \mathcal{P}(\Sigma)$ and $g \in \mathbb{R}^d$ define the *generator*

$$\Lambda^{a,m}_t g(x) := \int_\Theta [g(x + \gamma(t,x,\theta,a,m)) - g(x)] \nu(d\theta) \tag{1.3}$$

and the *pre-Hamiltonian*

$$\mathfrak{f}(t,x,a,m,g) := \Lambda^{a,m}_t g(x) + c(t,x,a,m). \tag{1.4}$$

The *Hamiltonian* of the problem, in this general setting, is defined by

$$\mathfrak{H}(t,x,m,g) := \inf_{a\in A} \mathfrak{f}(t,x,a,m,g). \tag{1.5}$$

Given a function $g : \Sigma \to \mathbb{R}$ we denote its first finite difference $\Delta g(x) \in \mathbb{R}^d$ by

$$\Delta g(x) := \begin{pmatrix} g(1) - g(x) \\ \cdots \\ \cdots \\ g(d) - g(x) \end{pmatrix}.$$

When we have a function $g : \Sigma^N \to \mathbb{R}$, we denote with $\Delta^j g(\boldsymbol{x}) \in \mathbb{R}^d$ the first finite difference with respect to the j-th coordinate, namely

$$\Delta^j g(\boldsymbol{x}) := \begin{pmatrix} g(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_N) - g(\boldsymbol{x}) \\ \cdots \\ \cdots \\ g(x_1, \dots, x_{j-1}, d, x_{j+1}, \dots, x_N) - g(\boldsymbol{x}) \end{pmatrix}.$$

For future use, let us observe that, for $g : \Sigma \to \mathbb{R}$,

$$|\Delta g(x)| \le \sqrt{d} \max_y |\Delta g(x)|_y \le 2\sqrt{d}|g|_\infty \le 2\sqrt{d}|g|. \tag{1.6}$$

For a function $u : [t_0, T] \times \Sigma \longrightarrow \mathbb{R}$, we denote

$$||u|| := \sup_{t \in [t_0,T]} \max_{x \in \Sigma} |u(t,x)|. \tag{1.7}$$

We also use the notation $u(t) := (u_1(t), \dots, u_d(t)) = (u(t,1), \dots, u(t,d))$. When considering a function u with values in $\mathbb{R}^d$, its norm is defined as in (1.7), but where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$; while if the domain is Σ^N instead of Σ then the maximum in (1.7) is set in Σ^N.

1.1.2 Explicit rates

If not otherwise stated, the spaces A and Θ are, respectively, a generic compact metric space and a compact subset of $\mathbb{R}^d$, as described above. Here we shall consider a natural shape of the function γ, as well as particular structures for A and Θ, for which the transition rates of the Markov chain X solution of the dynamics (1.46) appear explicitly.

Consider then γ defined by

$$\gamma(t,x,\theta,a,m) := \sum_{y \in \Sigma} (y - x) \mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y). \tag{1.8}$$

with $\Gamma_{x,y} \ge 0$. The intensity measure ν on Θ is defined by

$$\nu(E) := \sum_{y=1}^{d} \ell(E \cap \Theta_y), \tag{1.9}$$

for any E in the Borel subsets $\mathcal{B}(\Theta)$, where $\Theta_y := \{\theta \in \Theta : \theta_z = 0 \quad \forall z \neq y\}$, which is viewed as a subset of $\mathbb{R}$, and ℓ is the Lebesgue measure on $\mathbb{R}$.

Thus we will consider the following structural assumption:

(H1) The function γ is defined by (1.8), the set Θ is $[0,K]^d$ if Γ is bounded by K, otherwise it is $[0,+\infty[^d$, and ν is given by (1.9).

The function Γ appearing in (1.8) yields the transition rates of the Markov chain X solution to (0.2); see (1.49) below. Note that here we allow Θ to be also unbounded. Again, if (H1) does not hold, then the γ, Θ and ν are generic.

Moreover, the measure ν defined in (1.9) has the property that

$$\int_\Theta \varphi(\theta_1,\ldots,\theta_d)\nu(d\theta) = \sum_{y=1}^d \int_{\Theta_y} \varphi(0,\ldots,\theta_y,\ldots,0)d\theta_y \tag{1.10}$$

for any bounded and measurable $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$. In particular,

$$\int_\Theta \sum_{y=1}^d \varphi_y(\theta_y)\nu(d\theta) = \sum_{y=1}^d \int_{\Theta_y} \varphi_y(\theta_y)d\theta_y \tag{1.11}$$

for any function $\varphi_y : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi_y(0) = 0$, $y \in \Sigma$. A proof of (1.10) can be found in [76], where the example (1.8) was treated.

These properties give the following easy, but important, consequence. Let $\Gamma(x)$ denote the row vector of Γ, so that $\Gamma_y(x) = \Gamma_{x,y}$, and $\cdot$ denote the scalar product between vectors.

Lemma 1.1. *If (H1) holds then*

$$\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(s,x,\theta,\widetilde{a},\widetilde{m})|\nu(d\theta) \le 2d \sum_{y\neq x} |\Gamma_{x,y}(t,a,m) - \Gamma_{x,y}(s,\widetilde{a},\widetilde{m})|, \tag{1.12}$$

$$\Lambda_t^{a,m} g(x) = \sum_{y=1}^d \Gamma_{x,y}(t,a,m)\,[g(y)-g(x)] = \Gamma(x)\cdot\Delta g(x), \tag{1.13}$$

for any $t,s \in [0,T]$, $a,\widetilde{a} \in A$, $m,\widetilde{m} \in \mathcal{P}(\Sigma)$, $g \in \mathbb{R}^d$ *and* $x \in \Sigma$.

Proof. We have

$$\begin{aligned}
&\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(s,x,\theta,\widetilde{a},\widetilde{m})|\nu(d\theta)\\
&= \int_\Theta \left| \sum_{y\in\Sigma} (y-x)\left[\mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y) - \mathbb{1}_{]0,\Gamma_{x,y}(s,\widetilde{a},\widetilde{m})[}(\theta_y)\right]\right| \nu(d\theta)\\
&\le \int_\Theta \sum_{y\neq x} |y-x| \left|\mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y) - \mathbb{1}_{]0,\Gamma_{x,y}(s,\widetilde{a},\widetilde{m})[}(\theta_y)\right| \nu(d\theta)\\
&\le 2d \int_\Theta \sum_{y\neq x} \left|\mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y) - \mathbb{1}_{]0,\Gamma_{x,y}(s,\widetilde{a},\widetilde{m})[}(\theta_y)\right| \nu(d\theta).
\end{aligned}$$

Applying (1.11), the latter expression is equal to

$$\begin{aligned}
&2d \sum_{y\neq x} \int_{\Theta_y} \left|\mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y) - \mathbb{1}_{]0,\Gamma_{x,y}(s,\widetilde{a},\widetilde{m})[}(\theta_y)\right| d\theta_y\\
&\qquad = 2d \sum_{y\neq x} |\Gamma_{x,y}(t,a,m) - \Gamma_{x,y}(s,\widetilde{a},\widetilde{m})|,
\end{aligned}$$

which gives (1.12).

Applying formula (1.10) we obtain

$$\begin{aligned}\Lambda_t^{a,m} g(x) &= \int_\Theta \left[g\left(x + \sum_{y\in\Sigma} (y-x) \mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y) \right) - g(x) \right] \nu(d\theta). \\ &= \sum_{y=1}^{d} \int_{\Theta_y} \left[g(x + (y-x)\mathbb{1}_{]0,\Gamma_{x,y}(t,a,m)[}(\theta_y)) - g(x) \right] d\theta_y \\ &= \sum_{y=1}^{d} \Gamma_{x,y}(t,a,m) \left[g(y) - g(x) \right].\end{aligned}$$

□

We will also consider the following assumption, for which the controls are exactly the transition rates:

(H2) Assumption (H1) holds, the control space is $A = [0, +\infty[^d$ and the transition rate is $\Gamma_{x,y}(t,x,a,m) = a_y$.

Occasionally, we will need the following, which ensures that the processes are ergodic:

(Erg) Assumption (H2) holds, but the control space is $A = [\kappa, +\infty[^d$, with $\kappa > 0$.

It is analogous to the uniform ellipticity assumption on the volatility, in the diffusion setting.

1.1.3 Derivatives in the simplex

We now introduce the concept of variation with respect to a probability measure m of a function $U : \mathcal{P}(\Sigma) \to \mathbb{R}$. Let us remark that the usual notion of gradient in $\mathbb{R}^d$ cannot be defined for such a function: since the domain is $\mathcal{P}(\Sigma)$ we are not allowed to define e.g. the directional derivative $\frac{\partial}{\partial m_1}$, as we would have to extend the definition of U outside the simplex; such a stronger differentiability in $\mathbb{R}^d$ will be assumed in Subsection 1.4.1 only.

Definition 1.2. *We say that a function $U : \mathcal{P}(\Sigma) \to \mathbb{R}$ is* differentiable *if there exists a function $D^m U : \mathcal{P}(\Sigma) \times \Sigma \to \mathbb{R}^d$ given by*

$$[D^m U(m,y)]_z := \lim_{h \downarrow 0} \frac{U(m + h(\delta_z - \delta_y)) - U(m)}{h}. \tag{1.14}$$

for $z = 1, \dots, d$. Moreover, we say that U is $\mathcal{C}^1$ if the function $D^m U$ is continuous in m.

Morally, we can think of $[D^m U(m,y)]_z$ as the (right) directional derivative of U with respect to m along the direction $\delta_z - \delta_y$. Observe that $m + h(\delta_z - \delta_y)$ might be outside the probability simplex when m is at the boundary, thus (1.14) is meaningful in the interionr only. However, for our purposes, this is not really a problem: in the limit $m(t)$ will be the distribution of the reference player and, as it will be clear in Chapter 3, we will evaluate the derivative (see (1.16) below for the precise definition) just against vectors of the form $m(t+s) - m(t)$, so that $m(t) + h(m(t+s) - m(t))$ belongs to the symplex.

Let us state an identity which will come useful in the following:

$$[D^m U(m,y)]_z = [D^m U(m,x)]_z + [D^m U(m,y)]_x, \tag{1.15}$$

for any $x, y, z \in \Sigma$. Its derivation is an immediate consequence of the linearity of the directional derivative. We can easily extend the above definition to the case of derivative with respect to a direction $\mu \in \mathcal{P}_0(\Sigma)$, with

$$\mathcal{P}_0(\Sigma) := \left\{ \mu \in \mathbb{R}^d \ : \ \mu_1 + \cdots + \mu_d = 0 \right\}.$$

Indeed, an element $\mu = (\mu_1, \ldots, \mu_d) = \sum_{z\in\Sigma} \mu_z \in \mathcal{P}_0(\Sigma)$ can be rewritten as a linear combination of $\delta_z - \delta_y$ as follows

$$\mu = \sum_{z\neq y} \mu_z(\delta_z - \delta_y),$$

for each $y \in \Sigma$, since $\sum_{z\neq y} \mu_z(\delta_z - \delta_y) = \sum_{z\neq y} \mu_z\delta_z - \left(\sum_{z\neq y} \mu_z\right)\delta_y$, and $\sum_{z\neq y} \mu_z = -\mu_y$.

This remark allows us to define the derivative of $U(m)$ along the direction $\mu \in \mathcal{P}_0(\Sigma)$ as a map $\frac{\partial}{\partial\mu}U : \mathcal{P}(\Sigma) \times \Sigma \to \mathbb{R}$, defined for each $y \in \Sigma$ by

$$\frac{\partial}{\partial\mu}U(m, y) := \sum_{z\neq y}^{d} \mu_z \left[D^m U(m, y)\right]_z = \mu \cdot D^m U(m, y), \tag{1.16}$$

where the last equality comes from the fact that $[D^m U(m, y)]_y = 0$. We also note that the definition of $\frac{\partial}{\partial\mu}U(m, y)$ does not actually depend on y, i.e.

$$\frac{\partial}{\partial\mu}U(m, y) = \frac{\partial}{\partial\mu}U(m, 1) \tag{1.17}$$

for every $y \in \Sigma$ and for this reason we will fix $y = 1$ when needed in the equations. Indeed, by means of identity (1.15) and the fact that $\mu \in \mathcal{P}_0(\Sigma)$, for each $y \in \Sigma$

$$\begin{aligned}\frac{\partial}{\partial\mu}U(m, 1) &= \sum_{z=1}^{d} \mu_z \left[D^m U(m, 1)\right]_z = [\text{identity (1.15)}] = \sum_{z=1}^{d} \left([D^m U(m, y)]_z + [D^m U(m, 1)]_y\right)\mu_z \\ &= \sum_{z=1}^{d} [D^m U(m, y)]_z \mu_z + [D^m U(m, 1)]_y \sum_{z=1}^{d} \mu_z = \sum_{z=1}^{d} [D^m U(m, y)]_z \mu_z = \frac{\partial}{\partial\mu}U(m, y).\end{aligned}$$

Let us remark that we could give another definition of derivative in the simplex: by fixing the origin in a vertex, we could consider the gradient in $\mathbb{R}^{d-1}$, as $\mathcal{P}(\Sigma)$ might be viewed as a subset of $\mathbb{R}^{d-1}$. Such definition is different from ours since the basis vectors $(\delta_y - \delta_x)_{y\in\Sigma}$ we consider are not orthogonal if embedded in $\mathbb{R}^{d-1}$; thus the gradient in $\mathbb{R}^{d-1}$ would give a multiplicative factor compared to Definition 1.2. Another important motivation for considering the derivative in the simplex as in Definition 1.2 is that it comes directly from the heuristic derivation of the master equation we showed in the Introduction.

For a function $U : \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R}$ we denote the variation with respect to the first coordinate in a point $(x, m) \in \Sigma \times \mathcal{P}(\Sigma)$ by $\Delta^x U(x, m)$. Also, denote by $\Gamma^\dagger$ the transpose of a matrix Γ.

1.1.4 Assumptions

Let us summarize here the various sets of assumptions we will make use of:

(A) The function $\widehat{\gamma} : [0,T]\times\Sigma\times A\times\mathcal{P}(\Sigma) \longrightarrow L^1(\nu)$ defined by $\widehat{\gamma}(t,x,a,m) := \gamma(t,x,\cdot,a,m)$ is continuous in t,a,m (uniformly, and is bounded), that is, there exists a function w_γ such that $\lim_{h\downarrow 0} w_\gamma(h) = 0$ and

$$\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(s,x,\theta,\widetilde{a},\widetilde{m})|\nu(d\theta) \leq w_\gamma(|t-s| + \mathrm{d}_A(a,\widetilde{a}) + |m-\widetilde{m}|) \quad (1.18)$$

for every $t,s\in[0,T]$, $x\in\Sigma$, $a,\widetilde{a}\in A$, $m,\widetilde{m}\in\mathcal{P}(\Sigma)$;

(A') Assumption (A) holds and $\widehat{\gamma}$ is Lipschitz in $m\in\mathcal{P}(\Sigma)$:

$$\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(t,y,\theta,\widetilde{a},\widetilde{m})|\nu(d\theta) \leq K_1(|x-y| + |m-\widetilde{m}|); \quad (1.19)$$

(A") Assumption (A') holds and $\widehat{\gamma}$ is Lipschitz also in $a\in A$:

$$\int_\Theta |\gamma(t,x,\theta,a,m) - \gamma(t,y,\theta,\widetilde{a},\widetilde{m})|\nu(d\theta) \leq K_1(|x-y| + \mathrm{d}_A(a,\widetilde{a}) + |m-\widetilde{m}|); \quad (1.20)$$

(B) The running cost c is continuous (and bounded) in t,x,a,m and the terminal cost is continuous (and bounded) in x,m;

(B') Assumption (B) holds and the costs c and G are Lipschitz in m:

$$|c(t,x,a,m) - c(t,y,a,\widetilde{m})| + |G(x,m) - G(y,\widetilde{m})| \leq K_2(|x-y| + |m-\widetilde{m}|); \quad (1.21)$$

(B") Assumption (B') holds and the running cost c is Lipschitz also in a:

$$|c(t,x,a,m) - c(t,y,\widetilde{a},\widetilde{m})| + |G(x,m) - G(y,\widetilde{m})| \leq K_2[|x-y| + \mathrm{d}_A(a,\widetilde{a}) + |m-\widetilde{m}|]. \quad (1.22)$$

The above assumptions will be used in Sections 2.1 and 2.3 to find solutions of the mean field game and then approximate Nash equilibria for the N-player game, both in open-loop and in feedback form.

The next assumption will be more implicit. In order to obtain existence and uniqueness of feedback mean field game solutions we will make the additional hypothesis:

(C) For any t, x, m and g there exists a unique $\mathfrak{a}^* = \mathfrak{a}^*(t,x,m,g)$ minimizer of $\mathfrak{f}(t,x,a,m,g)$ in A;

We observe that for any fixed m and g the function $\mathfrak{a}^*(t,x)$ is measurable, thanks to Theorem D.5 in [53]. We remark also that the limiting dynamics (0.2) always admits a pathwise unique solution thanks to (1.2).

For the convergence argument in Chapter 3 we need stronger structural assumptions.

(H3) The running cost splits as $c(t,x,a,m) = L(x,a) + F(x,m)$;

(H4) Assumption (H2), (H3) hold and the *Lagrangian* L is $\mathcal{C}^1$ and uniformly convex in $a\in A = [0,+\infty[^d$, and F and G are Lipschitz continuous in m.

Under assumption (H2) and (H3), thanks to (1.13), the pre-Hamiltonian (1.4) and the Hamiltonian (1.5) can be written as

$$\mathfrak{f}(t,x,a,m,g) = -f(x,a,\Delta g(x)) + F(x,m), \tag{1.23}$$
$$\mathfrak{H}(t,x,m,g) = -H(x,\Delta g(x)) + F(x,m), \tag{1.24}$$

where

$$f(x,a,p) = -a\cdot p - L(x,a), \tag{1.25}$$
$$H(x,p) = \sup_{a\in A}\left\{-a\cdot p - L(x,a)\right\}, \tag{1.26}$$

for $p\in\mathbb{R}^d$ and $x\in\Sigma$.

Moreover, if (H4) holds, then the uniform convexity assumption implies that there exists a unique maximizer $a^*(x,p)$ of f, for every (x,p):

$$a^*(x,p) := \arg\min_{a\in A}\left\{a\cdot p + L(x,a)\right\} = \arg\max_{a\in A}\left\{-a\cdot p - L(x,a)\right\}. \tag{1.27}$$

In particular assumption (C) holds, with $\mathfrak{a}^*(t,x,m,g) = a^*(x,\Delta g(x))$.

We choose to work under assumption (H4) because it is common in mean field game theory, even if it is a bit stronger than what is needed; the extension to a time dependent cost c which does not split as $L+F$ is however straightforward. If (H4) holds, the optimum a^* is globally Lipschitz in p, and whenever H is differentiable it can be explicitly expressed as $a^*(x,p) = -D_pH(x,p)$; see Proposition 1 in [46] for the proof.

For the convergence argument we make assumptions on H, which are more implicit. We will work with two sets of assumptions on H. We first observe that it is enough to give hypotheses for $H(x,\cdot)$ on a sufficiently big compact subset of $\mathbb{R}^d$, i.e. for $|p|\le K$. Here the control set is not a priori compact; nevertheless, we will show that the optimal control belongs to a compact set, see next section for details (Lemma 1.7). This is true both for the N-player and the mean field game and it is a great advantage provided by the finiteness of the state space. Hence, when considering only the optimal controls for the convergence argument, we are allowed to set $\Theta = [0,K]^d$.

In what follows, the constant K is fixed:

(LipH) Assumption (H2) and (H3) hold and there exists a unique maximizex $a^*(x,p)$ of f. Moreover H and a^* are Lipschitz continuous in p, for $|p|\le K$.

We stress the fact that the above assumptions, together with the existence of a regular solution to (M), are alone sufficient for proving the convergence of the N-player game to the limiting mean field game dynamics. As observed above, Assumption (H4), which is more explicit, is a sufficient condition for (LipH). We also remark that (H4) implies (A”), (B”) and (C), except for the compactness of the control space.

In order to establish the well-posedness and the needed regularity for the master equation we make use of the following additional assumptions:

(RegH) Assumptions (H2) and (H3) hold. If $|p|\le K$, H is $\mathcal{C}^2$ with respect to p; H, D_pH and $D^2_{pp}H$ are Lipschitz in p and the second derivative is bounded away from 0, i.e. there exists a constant C_H such that

$$D^2_{pp}H(x,p) \ge C_H^{-1}; \tag{1.28}$$

(Mon) Assumption (H3) holds and the cost functions F and G are monotone in m in the Lasry-Lions sense, i.e., for every $m, \widetilde{m} \in \mathcal{P}(\Sigma)$,

$$\sum_{x\in\Sigma}(F(x,m) - F(x,\widetilde{m}))(m(x) - \widetilde{m}(x)) \geq 0, \tag{1.29}$$

and the same holds for G;

(RegFG) Assumption (H3) holds and the cost functions F and G are $\mathcal{C}^1$ with respect to m, with $D^m F$ and $D^m G$ bounded and Lipschitz continuous. In this case (1.29) is equivalent to say that

$$\sum_x \mu_x [D^m F(x,m,1) \cdot \mu] \geq 0 \tag{1.30}$$

for any $m \in \mathcal{P}(\Sigma)$ and $\mu \in \mathcal{P}_0(\Sigma)$.

Observe that the assumptions on H allow for quadratic Hamiltonian; and clearly (RegH) implies (LipH). As we will see, the above assumptions imply both the boundedness and Lipschitz continuity of $\Delta^x U$ and $D^m U$ with respect to m.

1.1.5 Examples

We conclude the section with some natural example for which the above assumptions are satisfied. As a general example for which Assumption (A) is satisfied, which is implicit, we can consider a dynamics where (H1) holds and the transition rates $\Gamma_{x,y}$ possess the desired regularity, in light of (1.12). Namely,

- if the rate Γ appearing in (1.8) is continuous in t, a and m, then (A) holds;
- if in addition Γ is Lipschitz in m, then (A') holds;
- if in addition Γ is Lipschitz also in a, then (A”) holds.

Note in particular that, under (H1), we allow the transition rate to depend on the control and also on m, which represents the distribution of the other players.

For assumption (C) to hold, it is enough to consider (H1) valid, with the rate Γ affine in a (e.g. if (H2) holds) and the cost c strictly convex in a, thanks to (1.13). Again, if (H4) holds, then (LipH) is satisfied, as well as (A”), (B”) and (C), except for the compactness of A. As an example of a suitable uniformly convex Lagrangian, we can take

$$L(a) = \frac{1}{\lambda}|a|^\lambda, \qquad 1 < \lambda \leq 2, \tag{1.31}$$

so that the corresponding Hamiltonian is

$$H(p) = \frac{1}{\lambda'}|p^-|^{\lambda'}, \qquad p \in \mathbb{R}^d, \tag{1.32}$$

where $\lambda' \geq 2$ is the conjugate exponent to λ, i.e. $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, and $p_x^- = -p_x$ if $p_x \leq 0$, and 0 otherwise, for each $x \in \Sigma$.

As far as the monotonicity assumption is concerned, the easiest example for the costs F and G is $F(x,m) = G(x,m) = m(x)$. Slightly more in general, one can consider $F(x,m) = \nabla\varphi(m)(x)$, φ being a real convex function on $\mathbb{R}^d$. Another important example is

$$F(x,m) = x\,\mathrm{Mean}(m) := x \sum_{y\in\Sigma} y m_y,$$

so that $\sum_{x\in\Sigma}(F(x,m)-F(x,\widetilde{m}))(m(x)-\widetilde{m}(x)) = [\mathrm{Mean}(m-\widetilde{m})]^2$: this kind of cost induces players to spread. Let us remark that in Chapter 4 we consider the opposite case, i.e. $F(x,m) = -x\,\mathrm{Mean}(m)$, so that players prefer to aggregate.

For the choice of a Lagrangian L for which (RegH) is satisfied, a bit of work is needed in order to recover the $\mathcal{C}^2$ regularity for H, since the maximization of f in (1.25) is performed only in the subset $A = [0,+\infty[^d$ of $\mathbb{R}^d$. Indeed, the standard Lagrangian in (1.31) is not suitable because the corresponding Hamiltoniam in (1.32) is $\mathcal{C}^1$ but not $\mathcal{C}^2$, in any neighborhood of 0.

Consider then F and G bounded respectively by K_F and K_G and the Lagrangian, not depending on x, defined by

$$L(a) := \left| a - \frac{K}{2}(1,\dots,1)^\dagger \right|^2, \tag{1.33}$$

with $K = 2(TK_F + K_G)$. The computations of a^* and H for such choice of L give

$$H(p) = \frac{|p|^2}{4} - \frac{K}{2}p\cdot(1,\dots,1)^\dagger, \qquad a^*(p) = -\frac{p}{2} + \frac{K}{2}(1,\dots,1)^\dagger \tag{1.34}$$

for $p \in \mathbb{R}^d$ with any component $p_x \le K$; while H has at least one linear component outside this interval.

It is trivial to verify that H is in $\mathcal{C}^1(\mathbb{R}^d)$, and thus (LipH) is satisfied, while H is not in $\mathcal{C}^2(\mathbb{R}^d)$ because of the linear components. Nevertheless, (1.28) is satisfied whenever $|p|_\infty \le K$. It is easy to verify that the value functions are bounded by $TK_F + K_G$, and thus their gradients (the p argument) are bounded by K in the sup-norm, thanks to (1.6); see also Lemma 1.7 below. Then (1.34) implies that $|a^*|_\infty \le K$ if $|p|_\infty \le K$. Moreover, the Lipschitz continuity of D_pH and $D^2_{pp}H$ is trivially holding because of expression (1.34) for $|p|_\infty \le K$, thus (RegH) follows.

1.2 N-player game

In the prelimit, we consider a system of N symmetric players governed by the dynamics

$$X_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, X_i(s^-), \theta, \pi^i(s), m^N(s^-))\mathcal{N}_i(ds, d\theta) \qquad i = 1,\dots,N, \tag{1.35}$$

where $\boldsymbol{X} = (X_1,\dots,X_N)$ and $m^N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}$. Here, the controls π^i are in open-loop form. Let us specify the controls to be used in the N-player game.

Definition 1.3. *Define the set of* strategy vectors *as*

$$\mathcal{S}^N := \{((\Omega,\mathcal{F},P;\mathbb{F}),\boldsymbol{\pi},\boldsymbol{\xi},\underline{\mathcal{N}})\}$$

where $(\Omega,\mathcal{F},P;\mathbb{F})$ is a filtered probability space, $\boldsymbol{\xi} := (\xi_1,\dots,\xi_N)$ is a vector of N i.i.d. $\mathcal{F}_0$-measurable random variables with law m_0, the initial points, $\underline{\mathcal{N}} = (\mathcal{N}_1,\dots,\mathcal{N}_N)$ is a vector of N i.i.d. stationary Poisson random measures with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with intensity measure ν on Θ, $\mathcal{F}_T = \mathcal{F}$, and $\boldsymbol{\pi} = (\pi^1,\dots,\pi^N)$ is a vector of A-valued $\mathbb{F}$-predictable processes π^i. We will often write $\boldsymbol{\pi}\in\mathcal{S}^N$ to indicate the process $\boldsymbol{\pi}$.

Define the set of feedback strategy vectors *as*

$$\mathcal{A}^N := \{((\Omega,\mathcal{F},P;\mathbb{F}),\boldsymbol{\alpha},\boldsymbol{\xi},\underline{\mathcal{N}})\}$$

where $\boldsymbol{\alpha} = (\alpha^1,\dots,\alpha^N) : [0,T]\times\Sigma^N \to A^N$ is measurable and the filtered probability space and the $\boldsymbol{\xi}$ and $\underline{\mathcal{N}}$ are as above. We will often write $\boldsymbol{\alpha}\in\mathcal{A}^N$ to indicate the function $\boldsymbol{\alpha}$.

We observe that the above definition of feedback strategy vector is not standard, as it is given together with the probability space and the noise. We give such a definition because in this way any strategy gives a unique pathwise solution to dynamics (1.35). Indeed, provided that $\widehat{\gamma}$ is Lipschitz in m, we have pathwise existence and uniqueness of solutions to the system (1.35), for any $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^N) \in \mathcal{S}^N$.

Given a feedback strategy vector $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^N) \in \mathcal{A}^N$, equation (1.35) is written as

$$X_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, X_i(s^-), \theta, \alpha^i(s, \boldsymbol{X}(s^-)), m^N(s^-))\mathcal{N}_i(ds, d\theta) \tag{1.36}$$

for each $i = 1, \ldots, N$. The same assumption as above provides existence and uniqueness of solutions X_i to this equation, so we can define the related open-loop control $\boldsymbol{\pi}_{\boldsymbol{\alpha}}$ by

$$\boldsymbol{\pi}^i_{\boldsymbol{\alpha}}(t) := \alpha^i(t, \boldsymbol{X}(t^-)).$$

In view of Definition 1.3, the open-loop control $\boldsymbol{\pi}_{\boldsymbol{\alpha}}$ has to be given together with a filtered probability space, a vector of initial conditions and a vector of Poisson random measures, which we impose to be the same as those given with the feedback control $\boldsymbol{\alpha}$.

Next, we define the object of the minimization. Let $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^N) \in \mathcal{S}^N$ be a strategy vector and $\boldsymbol{X} = (X_i, \ldots, X_N)$ be the solution to dynamics (1.35). For $i = 1, \ldots, N$ set

$$J_i(\boldsymbol{\pi}) := \mathbb{E}\left[\int_0^T c(t, X_i(t), \pi^i(t), m^N(t))dt + G(X_i(T), m^N(T))\right]. \tag{1.37}$$

Define also $J_i(\boldsymbol{\alpha}) := J_i(\boldsymbol{\pi}_{\boldsymbol{\alpha}})$ for any $\boldsymbol{\alpha} \in \mathcal{A}^N$.

We look for *approximate* Nash equilibria for the N-player game. So let us define what are the perturbed strategy vectors we consider.

Notation 1.4. *Let $\widetilde{\pi}$ be an A-valued $\mathbb{F}$-predictable process. For a strategy vector $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^N)$ in $\mathcal{S}^N$ denote by $[\boldsymbol{\pi}^{-i}; \widetilde{\pi}]$ the strategy vector such that*

$$[\boldsymbol{\pi}^{-i}; \widetilde{\pi}]^j = \begin{cases} \pi^j & j \neq i \\ \widetilde{\pi} & j = i. \end{cases}$$

For a feedback strategy vector $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^N) \in \mathcal{A}^N$, let $\widetilde{\boldsymbol{X}}$ be the solution to

$$\begin{cases} \widetilde{X}_i(t) = & \xi_i + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_i(s^-), \theta, \widetilde{\pi}(s), \widetilde{m}^N(s^-))\mathcal{N}_i(ds, d\theta) \\ \widetilde{X}_j(t) = & \xi_j + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_j(s^-), \theta, \alpha^j(s, \widetilde{\boldsymbol{X}}(s^-)), \widetilde{m}^N(s^-))\mathcal{N}_j(ds, d\theta) \\ & \text{if } j \neq i. \end{cases}$$

Denote then by $[\boldsymbol{\alpha}^{-i}; \widetilde{\pi}] \in \mathcal{S}^N$ the strategy vector such that

$$[\boldsymbol{\alpha}^{-i}; \widetilde{\pi}]^j(t) = \begin{cases} \alpha^j(t, \widetilde{\boldsymbol{X}}(t^-)) & j \neq i \\ \widetilde{\pi}(t) & j = i. \end{cases}$$

Definition 1.5. *Let $\varepsilon > 0$. A strategy vector $\boldsymbol{\pi}$ is said to be an ε-*Nash equilibrium *if for each $i = 1, \ldots, N$*

$$J_i(\boldsymbol{\pi}) \leq J_i([\boldsymbol{\pi}^{-i}; \widetilde{\pi}]^j) + \varepsilon$$

for every $\widetilde{\pi}$ such that $[\boldsymbol{\pi}^{-i}; \widetilde{\pi}]$ is a strategy vector.

A vector $\boldsymbol{\alpha} \in \mathcal{A}^N$ *is called a* feedback ε-Nash equilibrium *if*

$$J_i(\boldsymbol{\alpha}) \le J_i([\boldsymbol{\alpha}^{-i}; \widetilde{\pi}]) + \varepsilon$$

for every $\widetilde{\pi}$ *such that* $[\boldsymbol{\alpha}^{-i}; \widetilde{\pi}]$ *is a strategy vector.*

A strategy vector is called a Nash equilibrium *if* $\varepsilon = 0$.

We remark that the above definition of feedback ε-Nash equilibrium is not standard. Indeed, the perturbed strategy vector $[\boldsymbol{\alpha}^{-i}; \widetilde{\pi}]$ is usually required to be in feedback form. In our definition, a slightly more restrictive (or stronger) condition is used since the perturbing strategy $\widetilde{\pi}$ is allowed to be in open-loop form. As a consequence, the approximation result of Section 2.3 will be slightly stronger than with the standard definition.

1.2.1 Nash system

In this section, we specialize the above setting assuming (H4); in particular we have that players control exactly their transition rate. Consider then a feedback strategy vector $\boldsymbol{\alpha}$ and the dynamics (1.36), where γ and ν are given respectively by (1.8) and (1.9). Assuming (H2), we can suppose that the probability space is given and the filtration $\mathbb{F}$ is the natural filtration provided by the Poisson random measures. Thus for any $\boldsymbol{\alpha} \in \mathcal{A}$ there exists a pathwise unique solution $\boldsymbol{X}$ to (1.36), which is such that

$$P\left[X_i(t+h) = y \middle| \boldsymbol{X}_t = \boldsymbol{x}\right] = \alpha^i_y(t, \boldsymbol{x})h + o(h) \tag{1.38}$$

as $h \downarrow 0$, for each $y \neq x_i$ and $\boldsymbol{x} \in \Sigma^N$. A proof of (1.38) can be found in [76]. Since α is the vector of the transition rates of the Markov chain, we set $\alpha^i_x(x) := -\sum_{y \neq x} \alpha^i_y(x)$.

If (H3) holds, the cost (1.37) is written as

$$J_i(\boldsymbol{\alpha}) := \mathbb{E}\left[\int_0^T \Big(L(X_i(t), \alpha^i(t, \boldsymbol{X}_t)) + F(X_i(t), m^{N,i}_{\boldsymbol{X}}(t))\Big)\, dt + G(X_i(T), m^N_{\boldsymbol{X}}(T))\right], \tag{1.39}$$

for any $\boldsymbol{\alpha} \in \mathcal{A}$, whereas $\boldsymbol{X}$ is the corresponding solution to (1.36). For technical reasons, in this cost we write $m^{N,i}$ instead of m^N, which does not really make a difference when N is large. Let us now introduce the functional

$$J_i(t,\boldsymbol{x},\boldsymbol{\alpha}) := \mathbb{E}\left[\int_t^T [L(X^{t,\boldsymbol{x}}_i(s), \alpha^i(s, \boldsymbol{X}^{t,\boldsymbol{x}}_s)) + F(X^{t,\boldsymbol{x}}_i(s), m^{N,i}_{\boldsymbol{X}^{t,\boldsymbol{x}}}(s))]ds + G(X^{t,\boldsymbol{x}}_i(T), m^{N,i}_{\boldsymbol{X}^{t,\boldsymbol{x}}}(T))\right] \tag{1.40}$$

where

$$X^{t,\boldsymbol{x}}_i(s) = x_i + \int_t^s \int_\Theta \gamma(X^{t,\boldsymbol{x}}_i(r^-), \theta, \alpha^i(r, \boldsymbol{X}^{t,\boldsymbol{x}}_{r^-}))\mathcal{N}_i(dr, d\theta) \qquad s \in [t,T].$$

We work under hypotheses that guarantee the existence of a unique maximizer $a^*(x,p)$ defined in (1.27). With this notation, the Nash system associated to the above differential game is given by the system presented in the Introduction:

$$\begin{cases} -\frac{d}{dt} v^{N,i} - \sum_{j=1,\ j\neq i}^N a^*(x_j, \Delta^j v^{N,j}) \cdot \Delta^j v^{N,i} + H(x_i, \Delta^i v^{N,i}) = F(x_i, m^{N,i}_{\boldsymbol{x}}), \\ v^{N,i}(T, \boldsymbol{x}) = G(x_i, m^{N,i}_{\boldsymbol{x}}). \end{cases} \tag{Nash}$$

This is a system of Nd^N coupled ODEs, whose well-posedness for all $T > 0$ can be proved through standard ODEs techniques, because of the Lipschitz continuity of the vector fields involved in the equations (Assumption (LipH)): there exists a unique solution in the set of bounded and continuous functions.

We are now able to relate system (Nash) to the Nash equilibria for the N-player game through the following

Proposition 1.6 (Verification Theorem)**.** *Assume (LipH) and let $v^{N,i}$, $i = 1, \dots, N$ be a classical solution to system* (Nash)*. Then the feedback strategy vector $\boldsymbol{\alpha} = (\alpha^1, \dots, \alpha^N)$ defined by*

$$\alpha^i(t, \boldsymbol{x}) := a^*(x_i, \Delta^i v^{N,i}(t, \boldsymbol{x})) \qquad i = 1, \dots, N, \tag{1.41}$$

is the unique Nash equilibrium for the N-player game and the $v^{N,i}$'s are the value functions *of the game, i.e.*

$$v^{N,i}(t, \boldsymbol{x}) = J_i(t, \boldsymbol{x}, \boldsymbol{\alpha}) = \inf_{\pi \in \mathcal{S}} J_i(t, \boldsymbol{x}, [\boldsymbol{\alpha}^{-i}; \pi]). \tag{1.42}$$

Proof. Let $\pi \in \mathcal{S}$ be any open-loop control, so that $\pi(t) \in [0, +\infty[^d$ for any t thanks to (H2), and let $\boldsymbol{X}^{t,\boldsymbol{x}}$ be the dynamics related to the strategy vector $[\boldsymbol{\alpha}^{-i}; \pi]$; denote for simplicity $\boldsymbol{X} = \boldsymbol{X}^{t,\boldsymbol{x}}$. Fixing $i \in \{1, \dots, N\}$, because of the uniqueness of the maximizer in (1.27), we have

$$\frac{\partial v}{\partial t}^{N,i} + \sum_{j \neq i} \sum_{y=1}^{d} a_y^*(x_j, \Delta^j v^{N,j})[\Delta^j v^{N,i}(t, \boldsymbol{x})]_y + \pi(t) \cdot \Delta^i v^{N,i}(t, \boldsymbol{x}) + L(x_i, \pi(t)) + F(x_i, m_{\boldsymbol{x}}^{N,i}) \geq 0$$

for any $t, \boldsymbol{x}, \omega$. Applying first Itô formula (Theorem II.5.1 in [55], p. 66) and then (1.13) and the above inequality, we obtain

$$\begin{aligned}
v^{N,i}(t, \boldsymbol{x}) &= \mathbb{E}\left[v^{N,i}(T, \boldsymbol{X}_T) - \int_t^T \frac{\partial v}{\partial t}^{N,i}(s, \boldsymbol{X}_s) ds\right] \\
&\quad - \sum_{j=1}^{N} \mathbb{E}\left[\int_t^T \int_\Theta \left[v^{N,i}(X_1(s), \dots, X_j(s) + \gamma(X_j(s), \theta, [\boldsymbol{\alpha}^{-i}; \pi](s, \boldsymbol{X}_s)), \dots, X_N(s))\right.\right. \\
&\qquad \left.\left. - v^{N,i}(\boldsymbol{X}_s)\right] \nu(d\theta) ds\right] \\
&= \mathbb{E}\left[v^{N,i}(T, \boldsymbol{X}_T)\right. \\
&\quad \left. - \int_t^T \left(\frac{\partial v}{\partial t}^{N,i}(s, \boldsymbol{X}_s) + \sum_{j \neq i} \alpha^{j,*}(s, \boldsymbol{X}_s) \cdot \Delta^j v^{N,i}(s, \boldsymbol{X}_s) + \pi(t) \cdot \Delta^i v^{N,i}(t, \boldsymbol{X}_s)\right) ds\right] \\
&\leq \mathbb{E}\left[G(X_i(T), m_{\boldsymbol{X}}^{N,i}(T)) + \int_t^T \Big(L(X_i(s), \pi(s)) + F(X_i(s), m_{\boldsymbol{X}}^{N,i}(s))\Big) ds\right] \\
&=: J_i(t, \boldsymbol{x}, [\boldsymbol{\alpha}^{-i}; \pi]).
\end{aligned}$$

Replacing π by α^i the inequalities become equalities. □

The next result shows that the value functions, as well as their gradients, are always bounded. Although the proof, which employs the form of the cost (1.39), is very simple, it is very important. In particular, the uniform bound on the gradients, and thus on the Nash equilibrium, is a particular feature of the finite state space models, which usually does not hold for diffusion-based models. Note that this also implies the uniform boundedness of the Nash equilibria (1.41). We will also show that the Nash equilibrium is unique in this class in Theorem 3.19. It is for this reason that the only local regularity (assumptions

(LipH) and (RegH)) for $H(x,p)$ with respect to p is enough for getting the convergence and the well-posedness results. Further, the following result allows to set $\Theta = [0,K]^d$ when considering only the optimal controls.

Lemma 1.7. *If (H4) holds then there exists a constant K for which the classical solution to* (Nash) *is such that, for any N,i,j,*

$$\max_{t\in[0,T],\boldsymbol{x}\in\Sigma^N} |v^{N,i}(t,\boldsymbol{x})| \le K \tag{1.43}$$

$$\max_{t\in[0,T],\boldsymbol{x}\in\Sigma^N} |\Delta^j v^{N,i}(t,\boldsymbol{x})| \le K. \tag{1.44}$$

Proof. Under (H4), the costs F and G are bounded: denote by K_F and K_G their bounds. Moreover, the uniform convexity of $L(x,\cdot)$ implies that it has a unique minimum $a_0(x)$, for any x; set $L(x,a_0(x)) = L_0(x)$. Therefore, for any $N,i,\boldsymbol{x}$ and $\boldsymbol{\alpha}$

$$J_i(t,\boldsymbol{x},\boldsymbol{\alpha}) \ge T\left(\min_{x\in\Sigma} L_0(x) - K_F\right) - K_G.$$

For the converse inequality, let $\alpha^{i,0}(t,\boldsymbol{x}) = a_0(x_i)$ for any t and fix the controls $\boldsymbol{\alpha}^{-i}$ of the other players.

$$\inf_{\pi\in\mathcal{S}} J_i(t,\boldsymbol{x},[\boldsymbol{\alpha}^{-i};\pi]) \le J_i(t,\boldsymbol{x},[\boldsymbol{\alpha}^{-i};\alpha^{i,0}]) \le T\left(\max_{x\in\Sigma} L_0(x) + K_F\right) + K_G.$$

Comparing these two inequalities we get (1.43), and so (1.6) yields (1.44). □

We are interested in studying the limit of the (Nash) system as $N\to\infty$. An easy but crucial consequence of the mean field assumptions on the costs and the uniqueness of solution to system (Nash) is that the solution $v^{N,i}$ of such system enjoys symmetric properties.

Lemma 1.8. *There exists $v^N : [0,T]\times\Sigma^N \to \mathbb{R}^d$ such that the solutions $v^{N,i}$ to system* (Nash) *satisfy, for $i=1,\dots,N$,*

$$v^{N,i}(t,\boldsymbol{x}) = v^N(t,x_i,(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)), \tag{1.45}$$

for any $(t,x)\in[0,T]\times\Sigma$, and the function

$$\Sigma^{N-1} \ni (y_1,\dots,y_{N-1}) \to v^N(t,x,(y_1,\dots,y_{N-1}))$$

is invariant under permutations of $(y_1,\dots,y_{N-1})$.

Proof. Let $\widetilde{\boldsymbol{x}}$ be defined from $\boldsymbol{x}$ after exchanging x_k with x_j, for $j\ne k\ne i$. Because of the mean field assumptions, we have

$$F^{N,i}(\boldsymbol{x}) := F(x_i, m^{N,i}_{\boldsymbol{x}}) = F(x_i, m^{N,i}_{\widetilde{\boldsymbol{x}}}) =: F^{N,i}(\widetilde{\boldsymbol{x}})$$

and the same for G. Thus, by the uniqueness of solution to (Nash), we conclude that $v^{N,i}(t,\boldsymbol{x}) = v^{N,i}(t,\widetilde{\boldsymbol{x}})$. □

The above proposition motivates the study of a possible convergence of system (Nash) to a limiting system, by analyzing directly the limit of the functions v^N.

1.3 Mean field game

The *mean field* limiting system consists of a single player whose state evolves according to the dynamics

$$X(t) = \xi + \int_0^t \int_\Theta \gamma(s, X(s^-), \theta, \pi(s), m(s)) \mathcal{N}(ds, d\theta), \quad t \in [0, T]. \tag{1.46}$$

Here the empirical measure appearing in (1.35) is replaced by a deterministic *flow* of probability measures $m : [0, T] \longrightarrow \mathcal{P}(\Sigma)$.

Definition 1.9. *The set of* open-loop controls *is the set*

$$\mathcal{S} := \{((\Omega, \mathcal{F}, P; \mathbb{F}), \pi, \xi, \mathcal{N})\}$$

where $(\Omega, \mathcal{F}, P; \mathbb{F})$ *is a filtered probability space,* ξ *is an* $\mathcal{F}_0$*-measurable random variable with law* m_0*, the initial condition,* $\mathcal{N}$ *is a stationary Poisson random measure with respect to the filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ *with intensity measure* ν *on* Θ*,* $\mathcal{F}_T = \mathcal{F}$*, and* π *is an* A*-valued* $\mathbb{F}$*-predictable process. We will often write* $\pi \in \mathcal{S}$ *to indicate the process* π*.*

Define the set of feedback controls *as*

$$\mathcal{A} := \{((\Omega, \mathcal{F}, P; \mathbb{F}), \alpha, \xi, \mathcal{N})\}$$

where $\alpha : [0, T] \times \Sigma \to A$ *is measurable and the filtered probability space, the initial condition and the Poisson random measure* $\mathcal{N}$ *are as above. We will often write* $\alpha \in \mathcal{A}$ *to indicate the function* α*.*

We remark that the feedback control is given with the probability space and the noise, in analogy with Definition 1.3 for the prelimit system.

Thanks to the Lipschitz condition (1.2), the limiting dynamics is well defined. More precisely, given any open-loop control $((\Omega, \mathcal{F}, P; \mathbb{F}), \pi, \xi, \mathcal{N}) \in \mathcal{S}$ and flow of measures $m \in \mathcal{E}$, there exists a pathwise unique solution X of Eq. (1.46), which we will denote by $X_{\pi,m}$. Similarly, given any feedback control $((\Omega, \mathcal{F}, P; \mathbb{F}), \alpha, \xi, \mathcal{N}) \in \mathcal{A}$ and flow of measures $m \in \mathcal{E}$, there exists a pathwise unique process $X = X_{\alpha,m}$ solving

$$X(t) = \xi + \int_0^t \int_\Theta \gamma(s, X(s^-), \theta, \alpha(s, X(s^-)), m(s)) \mathcal{N}(ds, d\theta), \quad t \in [0, T]. \tag{1.47}$$

The corresponding open-loop control is then defined as

$$\pi^\alpha(t) := \alpha(t, X_{\alpha,m}(t^-)). \tag{1.48}$$

In view of Definition 1.9, the open-loop control π^α has to be given together with a filtered probability space, an initial condition and a Poisson random measure, which we impose to be the same as those given with the feedback control α.

If Assumption (H1) holds then the transition rates of the Markov chain solution to (1.46) are explicit. In fact, for each $y \neq x$, as $h \downarrow 0$, we have

$$P\left[X(t+h) = y | X(t) = x\right] = \mathbb{E}_{t,x}[\Gamma_{x,y}(t, \pi(t), m(t))] \cdot h + o(h), \tag{1.49}$$

where $X = X_{\pi,m}$ is the solution of (1.46) under the control $\pi \in \mathcal{S}$ and flow of measures $m \in \mathcal{E}$ and $\mathbb{E}_{t,x}$ denotes expectation with respect to the conditional probability $P[\cdot | X(t) = x]$. In particular, if π corresponds to a feedback control α, i.e. (1.48) holds, then the transition rate becomes $\Gamma_{x,y}(t, \alpha(t,x), m(t))$. For a proof of (1.49), see again [76].

We define the object of the minimization for the mean field game. For any $\pi \in \mathcal{S}$ and $m \in \mathcal{E}$ set

$$J(\pi, m) := \mathbb{E}\left[\int_0^T c(s, X_{\pi,m}(s), \pi(s), m(s))ds + G(X_{\pi,m}(T), m(T))\right]. \tag{1.50}$$

Define also $J(\alpha, m) := J(\pi^\alpha, m)$ for any $\alpha \in \mathcal{A}$.

The notion of solution for the limiting mean field game is the following.

Definition 1.10. *An* open-loop solution of the mean field game *is a triple*

$$(((\Omega, \mathcal{F}, P; \mathbb{F}), \pi, \xi, \mathcal{N}), m, X)$$

such that

1. $((\Omega, \mathcal{F}, P; \mathbb{F}), \pi, \xi, \mathcal{N}) \in \mathcal{S}$, $m \in \mathcal{E}$, $(X(t))_{t\in[0,T]}$ *is adapted to the filtration* $\mathbb{F}$ *and* $X = X_{\pi,m}$*;*
2. Optimality*:* $J(\pi, m) \leq J(\widetilde{\pi}, m)$ *for every* $\widetilde{\pi} \in \mathcal{S}$*;*
3. Mean Field Condition*:* $\mathrm{Law}(X(t)) = m(t)$ *for every* $t \in [0,T]$*.*

We say that $(((\Omega, \mathcal{F}, P; \mathbb{F}), \alpha, \xi, \mathcal{N}), m, X)$ *is a* feedback solution of the mean field game *if* $\alpha \in \mathcal{A}$ *and* $(((\Omega, \mathcal{F}, P; \mathbb{F}), \pi^\alpha, \xi, \mathcal{N}), m, X)$ *is an open-loop solution of the mean field game, where* π^α *is defined in (1.48).*

In our writing, we will often drop the filtered probability space and the Poisson random measure from the notation.

In condition (3) of the above definition, $\mathrm{Law}(X(t)) := P \circ X(t)^{-1}$ as usual. Let us denote by $\mathrm{Flow}(X) : [0,T] \longrightarrow \mathcal{P}(\Sigma)$ the *flow* of the process X, that is, $\mathrm{Flow}(X).t := \mathrm{Law}(X(t))$. Then the mean field condition can be written as $\mathrm{Flow}(X) = m$.

1.3.1 Relaxed controls

The space $\mathcal{S}$ is not itself compact. In order to always have convergence along subsequences, we need to enlarge the space of controls, considering relaxed controls and related relaxed Poisson measures. They are used only for the limiting system.

Definition 1.11. *A* deterministic relaxed control *is a measure* ρ *on the Borel sets* $\mathcal{B}([0,T]\times A)$ *such that*

$$\rho([0,t[\times A) = \rho([0,t] \times A) = t \quad \forall t \in [0,T]. \tag{1.51}$$

The space of deterministic relaxed controls will be denoted by $\mathcal{Q}$*.*

Given $\rho \in \mathcal{Q}$, the time derivative exists for Lebesgue-almost every $t \in (0,T]$; it is the probability measure ρ_t on A given by

$$\rho_t(E) := \lim_{h\to 0} \frac{\rho([t-h,t] \times E)}{h}, \quad E \in \mathcal{B}(A). \tag{1.52}$$

As a consequence, ρ can be factorized according to

$$\rho(dt, da) = \rho_t(da)dt. \tag{1.53}$$

The space $\mathcal{Q}$ is endowed with the topology of weak convergence of measures, i.e. $\rho_n \to \rho$ if and only if

$$\int_0^T \int_A \varphi(s,a)\rho_n(ds,da) \quad \longrightarrow \quad \int_0^T \int_A \varphi(s,a)\rho(ds,da) \tag{1.54}$$

for every continuous φ on $[0,T]\times A$. Moreover there exists a metric which makes $\mathcal{Q}$ a compact metric space (for instance, [62]).

Definition 1.12. *The space of* (stochastic) relaxed controls *is*

$$\mathcal{R} = \{((\Omega,\mathcal{F},P;\mathbb{F}),\rho,\xi,\mathcal{N})\}$$

where $(\Omega,\mathcal{F},P;\mathbb{F})$ is a filtered probability space, ρ is a $\mathcal{Q}$-valued random variable such that $\rho([0,\cdot]\times E)$ is $\mathbb{F}$-adapted for every $E\in\mathcal{B}(A)$, and $\mathcal{N}$ is a stationary Poisson random measure with respect to the filtration $\mathbb{F}$ with intensity measure ν on Θ. We will often write $\rho\in\mathcal{R}$ to denote the process ρ.

The space of relaxed feedback controls *is the set*

$$\mathcal{Y} := \{((\Omega,\mathcal{F},P;\mathbb{F}),\Upsilon,\xi,\mathcal{N})\}$$

where $\Upsilon : [0,T]\times\Sigma \longrightarrow \mathcal{P}(A)$ is measurable, $\mathcal{P}(A)$ is endowed with the topology of weak convergence, and the filtered probability space, the initial condition and the Poisson random measure are as above. We will often write $\Upsilon\in\mathcal{Y}$ to denote the process Υ.

The relaxed feedback control is given with the probability space and the noise, in analogy with Definition 1.9. Because of (1.52), the derivative $(\rho_t(E))_{0\le t\le T}$ is an $\mathbb{F}$-predictable process for any $E\in\mathcal{B}(A)$. An ordinary open-loop control $\pi\in\mathcal{S}$ can be viewed as a relaxed control $\rho^\pi\in\mathcal{R}$ in which the derivative in time is a Dirac measure:

$$\rho^\pi([0,t]\times E) = \int_0^t \rho_s^\pi(E)ds = \int_0^t \delta_{\pi(s)}(E)ds.$$

We also have to introduce the corresponding *relaxed Poisson measure* in order to have well-defined dynamics. This will be done properly in Appendix A. Given any $\rho\in\mathcal{R}$, Borel sets $\Theta_0\subseteq\Theta$, $A_0\subseteq A$, the relaxed Poisson measure $\mathcal{N}_\rho$ related to the relaxed control ρ has the property that the processes

$$\mathcal{N}_\rho(t,\Theta_0,A_0) - \nu(\Theta_0)\rho([0,t]\times A_0) \tag{1.55}$$

are $\mathbb{F}$-martingales, and are orthogonal for disjoint $\Theta_0\times A_0$. This martingale property and the fact that $\mathcal{N}_\rho$ is a counting measure valued process define the distribution of $\mathcal{N}_\rho$ and the joint law of $(\mathcal{N}_\rho,\rho,\xi,\mathcal{N})$ uniquely (see Appendix A). The martingale property (1.55) also implies that the process

$$\int_0^t \int_\Theta \int_A \varphi(s,\theta,a)\mathcal{N}_\rho(ds,d\theta,da) - \int_0^t \int_\Theta \int_A \varphi(s,\theta,a)\nu(d\theta)\rho_s(da)ds \tag{1.56}$$

is an $\mathbb{F}$-martingale, for any bounded and measurable φ. For an ordinary control $\pi\in\mathcal{S}$ (or the relaxed control it induces), the corresponding relaxed Poisson measure is explicitly given by

$$\mathcal{N}_\pi(t,\Theta_0,A_0) := \int_0^t \int_{\Theta_0} \mathbb{1}_{A_0}(\pi(s))\mathcal{N}(ds,d\theta). \tag{1.57}$$

The stochastic differential equation (1.46) in this more general framework with a relaxed Poisson measure is written as

$$X(t) = \xi + \int_0^t \int_\Theta \int_A \gamma(s, X(s^-), \theta, a, m(s)) \mathcal{N}_\rho(ds, d\theta, da) \tag{1.58}$$

for any relaxed control $\rho \in \mathcal{R}$ and $m \in \mathcal{E}$.

Given a relaxed feedback control $\Upsilon \in \mathcal{Y}$ and a process X, define the corresponding relaxed open-loop control through

$$\rho^{\Upsilon,X}(dt, da) := [\Upsilon(t, X(t^-))](da)dt. \tag{1.59}$$

Let $\mathcal{N}_{\rho^{\Upsilon,X}}$ be the relaxed Poisson measure corresponding to $\rho^{\Upsilon,X}$. Equation (1.58) then becomes

$$X(t) = \xi + \int_0^t \int_\Theta \int_A \gamma(s, X(s^-), \theta, a, m(s)) \mathcal{N}_{\rho^{\Upsilon,X}}(ds, d\theta, da), \tag{1.60}$$

where the solution process X appears also in the relaxed Poisson measure.

The proof of the following lemma is given in Appendix A.1.

Lemma 1.13. *For any $m \in \mathcal{E}$ and $\rho \in \mathcal{R}$, respectively $\Upsilon \in \mathcal{Y}$, there exists a pathwise unique solution to the stochastic differential equation (1.58), respectively (1.60).*

The solutions to (1.58) and (1.60) will be denoted by $X_{\rho,m}$ and $X_{\Upsilon,m}$ respectively. For $\Upsilon \in \mathcal{Y}$, let ρ^Υ denote the corresponding relaxed control defined by (1.59), that is, ρ^Υ is the relaxed open-loop control such that

$$\rho_t^\Upsilon(da) := [\Upsilon(t, X_{\Upsilon,m}(t^-))](da). \tag{1.61}$$

In view of Definition 1.12, the relaxed open-loop control ρ^Υ has to be given together with a filtered probability space, an initial condition and a Poisson random measure, which we impose to be the same as those coming with the relaxed feedback control Υ.

Let $\rho \in \mathcal{R}$ and $m \in \mathcal{E}$. Let $X = X_{\rho,m}$. Thanks to the martingale property (1.56), we obtain that the process

$$\begin{aligned} M_g^X(t) &= g(X(t)) - g(X(0)) \\ &\quad - \int_0^t \int_\Theta \int_A \left[g(X(s) + \gamma(s, X(s), \theta, a, m(s))) - g(X(s)) \right] \nu(d\theta) \rho_s(da) ds \end{aligned} \tag{1.62}$$

is an $\mathbb{F}$-martingale, for any $g \in \mathbb{R}^d$. This yields the Dynkin formula

$$\begin{aligned} &\mathbb{E}[g(X(t))] - \mathbb{E}[g(\xi)] \\ &= \mathbb{E} \int_0^t \int_\Theta \int_A \left[g(X(s) + \gamma(s, X(s), \theta, a, m(s))) - g(X(s)) \right] \nu(d\theta) \rho_s(da) ds. \end{aligned} \tag{1.63}$$

The cost to be minimized is

$$J(\rho, m) := \mathbb{E} \left[\int_0^T \int_A c(s, X_{\rho,m}(s), a, m(s)) \rho_s(da) ds + G(X_{\rho,m}(T), m(T)) \right]. \tag{1.64}$$

Define also $J(\Upsilon, m) := J(\rho^\Upsilon, m)$ for $\Upsilon \in \mathcal{Y}$. The definitions of *relaxed solution* of the mean field game (1.58) and *relaxed feedback solution* are analogous to Definition 1.10, where ordinary controls are replaced by relaxed controls.

Definition 1.14. *A* relaxed solution of the mean field game *is a triple*

$$(((\Omega, \mathcal{F}, P; \mathbb{F}), \rho, \xi, \mathcal{N}), m, X)$$

such that

1. $((\Omega, \mathcal{F}, P; \mathbb{F}), \rho, \xi, \mathcal{N}) \in \mathcal{R}$, $m \in \mathcal{E}$, $(X(t))_{t\in[0,T]}$ *is adapted to the filtration* $\mathbb{F}$ *and* $X = X_{\rho,m}$*;*
2. Optimality*:* $J(\rho, m) \leq J(\sigma, m)$ *for every* $\sigma \in \mathcal{R}$*;*
3. Mean Field Condition*:* $\text{Law}(X(t)) = m(t)$ *for every* $t \in [0, T]$.

We say that $(((\Omega, \mathcal{F}, P; \mathbb{F}), \Upsilon, \xi, \mathcal{N}), m, X)$ *is a* relaxed feedback solution of the mean field game *if* $\Upsilon \in \mathcal{Y}$ *and* $\left(\left((\Omega, \mathcal{F}, P; \mathbb{F}), \rho^{\Upsilon}, \xi, \mathcal{N}\right), m, X\right)$ *is a relaxed solution of the mean field game, where* ρ^{Υ} *is defined in (1.61).*

In our writing, we will often drop the filtered probability space and the Poisson random measure from the notation.

In Section 2.1 we will show both the existence of relaxed MFG solutions, via a fixed point argument, and the existence of relaxed feedback MFG solutions. In order to apply a fixed point theorem, we want to find a suitable space where all the flows of probability measures lie. As before, set $K_0 := 2\nu(\Theta)\sqrt{d}$ and denote by

$$\mathcal{E} := \{m : [0, T] \longrightarrow \mathcal{P}(\Sigma) : |m(t) - m(s)| \leq K|t - s|, \quad m(0) = m_0\} \tag{1.65}$$

the space of Lipschitz continuous flows of probability measures, with the same Lipschitz constant K_0 and initial point m_0. This space is easily seen to be convex and compact with respect to the uniform norm, thanks to the Ascoli-Arzelà theorem. The following lemma allows to restrict attention to flows of probability measures in $\mathcal{E}$.

Lemma 1.15. *Let* $\pi \in \mathcal{S}$*, or* $\rho \in \mathcal{R}$*, and let* $m : [0, T] \longrightarrow \mathcal{P}(\Sigma)$ *be any measurable deterministic flow of probability measures. Then the flow of the solution process* $\text{Flow}(X_{\pi,m})$*, or* $\text{Flow}(X_{\rho,m})$*, is in* $\mathcal{E}$*.*

Proof. We prove the claim for relaxed controls, so the conclusion follows also when considering the subset of ordinary controls. Let $g : \Sigma \longrightarrow \mathbb{R}$ be a function, which is then Lipschitz and bounded and can be viewed as a vector in $\mathbb{R}^d$. Let $\rho \in \mathcal{R}$ and m be fixed, and set $X = X_{\rho,m}$. The function $m : [0, T] \to \mathcal{P}(\Sigma)$ has a priori no regularity, except for being measurable. By the Dynkin formula (1.63) we have, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned}
&\mathbb{E}[g(X(t))] - \mathbb{E}[g(X(s))] \\
&= \int_s^t \int_\Theta \int_A \mathbb{E}[g(X(r) + \gamma(r, X(r), \theta, a, m(r))) - g(X(r))]\rho_r(da)\nu(d\theta)dr.
\end{aligned}$$

Hence

$$\begin{aligned}
&|\mathbb{E}[g(X(t))] - \mathbb{E}[g(X(s))]| \\
&\leq \int_s^t \int_\Theta \int_A \mathbb{E}|g(X(r) + \gamma(r, X(r), \theta, a, m(r))) - g(X(r))|\rho_r(da)\nu(d\theta)dr \\
&\leq \int_s^t \int_\Theta \int_A 2|g|_\infty \rho_r(da)\nu(d\theta)dr = 2\nu(\Theta)|g|_\infty(t - s)
\end{aligned}$$

thanks to the fact that ρ_r is a probability measure on A for any r. Clearly, $\mathbb{E}[g(X(t))] = g \cdot \mathrm{Law}(X(t))$. Thus, for any t and s,

$$\begin{aligned}|\,\mathrm{Law}(X(t)) - \mathrm{Law}(X(s))| &= \sqrt{\sum_{x=1}^{d} [e_x \cdot (\mathrm{Law}(X(t)) - \mathrm{Law}(X(s)))]^2} \\ &\le \sqrt{\sum_{x=1}^{d} |t-s|^2 4|e_x|_\infty^2 \nu(\Theta)^2} = 2\nu(\Theta)\sqrt{d}|t-s|,\end{aligned}$$

which gives the claim. □

We will use the characterization of solutions to (1.58) via the controlled martingale problem. The proof of the following lemma is omitted; it can be derived by mimicking the one of Theorem 2.8.1 in [61, p. 42].

Lemma 1.16. *Let $((\Omega', \mathcal{F}', P'; \mathbb{F}'), \rho, \xi, \mathcal{N}) \in \mathcal{R}$ and $m \in \mathcal{E}$. Then X solves equation (1.58) in distribution if and only if the process $M_g^X(t)$ defined in (1.62) is an $\mathbb{F}'$-martingale for any $g \in \mathbb{R}^d$. The underlying filtered probability space can always be assumed to be $D([0,T],\Sigma) \times \Omega$, where Ω is the canonical space for $(\mathcal{N}_\rho, \rho, \xi, \mathcal{N})$ defined in Appendix A, $\mathbb{F}$ the canonical filtration, and X is the canonical process.*

The martingale property holds if and only if

$$\mathbb{E}\left[\psi(X(t_i); i \le j)(M_g^X(t+s) - M_g^X(t))\right] = 0 \tag{1.66}$$

for every $\psi : \Sigma^j \to \mathbb{R}$ and every choice of j, t, s, t_i, $i = 1, \dots, j$ such that $0 \le t_i \le t \le t+s$.

1.3.2 Mean field game system and master equation

The solution of the mean field game can be seen as a fixed point: for a given flow of measures m, find a strategy ρ_m that is optimal and let $X^{\pi_m,m}$ be the corresponding solution to (1.58), then find m such that $\mathrm{Flow}(X) = m$. If Assumption (C) holds, we will show that the optimal control is unique (even over relaxed controls), for any fixed m, and so there exists a feedback MFG solution (not relaxed); see Theorem 2.8. Moreover, feedback solutions are shown in Section 3.4 to be unique either if the time horizon is small or if the Lasry-Lions monotonicity assumptions apply.

This fixed point argument is the basis for the analytic formulation of mean field games. In order to study the convergence of the Nash system, let us assume here (LipH), so that in particular (H2) holds and hence the limiting dynamics simplifies: we have, for any $y \neq x$, as $h \downarrow 0$,

$$P\left[X_\alpha(t+h) = y | X_\alpha(t) = x\right] = \alpha_y(t,x)h + o(h), \tag{1.67}$$

where $X = X_\alpha$ is the solution to (1.47) under the feedback control α (there is no dependence on m in the dynamics). The cost (1.50) in this setting becomes

$$J(\alpha, m) := \mathbb{E}\left[\int_0^T (L(X_\alpha(t), \alpha(t, X_\alpha(t))) + F(X_\alpha(t), m(t)))\, dt + G(X_\alpha(T), m(T))\right]. \tag{1.68}$$

Given m, the optimal control is unique and provided by the value function V_m of the problem, which satisfies the backward Hamilton-Jacobi-Bellman (HJB) equation that clearly depends on m. The unique optimal control is then given by

$$\alpha_m(t,x) := a^*(x, \Delta^x V_m(t,x)); \tag{1.69}$$

see Section 2.2 for the details under the weaker Assumption (C) and Section 3.4 for the proper definition of uniqueness. Then we impose that the flow of the corresponding optimal process is exactly the m we started with, which thus has to solve a forward Kolmogorov-Fokker-Planck (KFP) equation. The coupling between these two equations yields the celebrated mean field game system, whose unknowns are two functions $(u, m) : [0, T] \times \Sigma \to \mathbb{R}$:

$$\begin{cases} -\frac{d}{dt}u(t,x) + H(x, \Delta^x u(t,x)) = F(x, m(t)), \\ \frac{d}{dt}m_x(t) = \sum_y m_y(t) a^*_x(y, \Delta^y u(t,y)), \\ u(T,x) = G(x, m(T)), \\ m_x(t_0) = m_{x,0}. \end{cases} \tag{MFG}$$

In our discrete space setting, it is a strongly coupled forward-backward system of ODEs, labeled by $x \in \Sigma$. Again, existence of solutions follows from the fixed point argument (Theorem 2.8), while uniqueness does not hold in general, due to the forward-backward structure. Solutions are unique either if the time horizon is small, so that the fixed point becomes a contraction, or under monotonicity assumptions; see 3.4.1 and 3.4.2 below. In particular, a solution (u, m) of the MFG system yields a feedback MFG solution (in the sense of Definition 1.10), by means of (1.69) whereas $V_m = u$. The limiting optimal control (1.69) is bounded independently of m, by mimicking the proof of Lemma 1.7, and thus we are allowed to consider controls in a compact set only and let $\Theta = [0, K]^d$ when dealing with the optimal control and the Nash equilibria.

As already mentioned, recently in [14] a new technique involving the use of the so-called master equation was introduced to get the exact relation between symmetric N-player games and mean field games. Moreover, in the Introduction we already motivated heuristically the convergence result of system (Nash) to the master equation. As it will be clear from the convergence argument, all that is needed is the existence of a regular solution to the master equation. To be specific on the needed regularity, we give the following

Definition 1.17. *A function $U : [0, T] \times \Sigma \times \mathcal{P}(\Sigma) \to \mathbb{R}$ is said to be a* classical solution *to* (M) *if it is continuous in all its arguments, $\mathcal{C}^1$ in t and $\mathcal{C}^1$ in m and, for any $(t, x, m) \in [0, T] \times \Sigma \times \mathcal{P}(\Sigma)$ we have*

$$\begin{cases} -\frac{\partial U}{\partial t} + H(x, \Delta^x U) - \int_\Sigma D^m U(t,x,m,y) \cdot a^*(y, \Delta^y U(t,y,m)) dm(y) = F(x,m), \\ U(T,x,m) = G(x,m), \quad (x,m) \in \Sigma \times \mathcal{P}(\Sigma). \end{cases} \tag{M}$$

In particular then $\Delta^x U(t, x, \cdot) : \mathcal{P}(\Sigma) \to \mathbb{R}^d$ is bounded and Lipschitz continuous and $D^m U(t, x, \cdot) : \mathcal{P}(\Sigma) \to \mathbb{R}^{d \times d}$ is bounded.

Moreover, we say that U is a regular solution *to* (M) *if it is a classical solution and $D^m U(t, x, \cdot)$ is also Lipschitz continuous in m, uniformly in (t, x).*

Let us observe that in the master equation, thanks to property (1.15) of the derivative, we could replace $D^m U(t, x, m, y)$ by $D^m U(t, x, m, 1)$. Under sufficient conditions, we will prove in Section 3.3 the existence and uniqueness of a regular solution to (M). Generally speaking, the master equation summarizes all the information needed to get solutions to the mean field game: namely, the system (MFG) provides the characteristic curves for (M).

Remark 1.18. *If U is a classical solution to* (M) *then*

$$\begin{cases} \frac{d}{dt}m_x(t) = \sum_y m_y(t) a^*_x(y, \Delta^y U(t,y,m(t))) \\ m(t_0) = m_0 \end{cases} \tag{1.70}$$

admits a unique solution m, since U is Lipschitz in m and thus also the HJB equation has a unique solution u, given by

$$u(t,x) = U(t,x,m(t)). \tag{1.71}$$

Therefore U induces a unique solution (u,m) of the mean field game system, and a unique optimal control, given by

$$\alpha(t,x) := a^*(x, \Delta^x U(t,x,m(t))). \tag{1.72}$$

In order to show the existence of a regular solution to (M), in Theorem 3.9, we have to show the opposite, that is, a solution to the MFG system, when it is unique, provides a unique regular solution to the master equation. We will apply the method of characteristics, which consists in showing that the function defined by $U(t_0,x,m_0) = u(t_0,x)$ solves the master equation, where (u,m) is the solution to the mean field game system starting at time t_0 up to time T, with $m(t_0) = m_0$; see Section 3.3. Note in particular that uniqueness of MFG solution is required in order to define a regular solution to (M).

1.4 Extensions

In this section we provide other two approaches to the limit model, the mean field game. These may allow in the future to prove the convergence results of Chapters 3 and 4 for more general models. The first approach deals with a particular, but large, class of mean field games, which are called potential since the costs are potentials of a functional of the measure. The goal is to show that the MFG system can be viewed as the necessary conditions for optimality of a deterministic control problem. While the second analysis deals with weak mean field game solutions, which are called weak as the flow of measures m might be random, as opposed to the definitions given above in which m is always deterministic. Both approaches will be treated more in details in Section 4.3 for the two state model considered there.

1.4.1 Potential mean field games

We show that the mean field game system, in some case, can be viewed as the necessary condition for optimality, given by the Pontryagin maximum principle of a *deterministic* optimal control problem in $\mathbb{R}^d$. This analysis can be conducted only for *potential* mean field games, that is, if the costs have a potential structure: in this case the mean field game system (MFG) is indeed an Hamiltonian system. Thus we assume here (H4) and that there exist functions $\mathcal{F}, \mathcal{G} : \mathcal{P}(\Sigma) \to \mathbb{R}$ such that

$$\nabla^m \mathcal{F}(m) = F(x,m), \qquad \nabla^m \mathcal{G}(m) = G(x,m), \tag{1.73}$$

for any $m \in \mathcal{P}(\Sigma)$ and $x \in \Sigma$, where ∇^m is the standard gradient in $\mathbb{R}^d$. Note that here $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}^1$ in the classical sense, as we will always consider in this section.

Let us now specify the control problem. The action set is the set of matrices $\mathbb{A} := [0,+\infty[^{d\times d}$; for $\Gamma \in \mathbb{A}$, denote by $\Gamma(x) \in A = [0,+\infty[^d$ the row vectors of Γ, for any $x \in \Sigma$, i.e. $\Gamma_y(x) = \Gamma_{x,y}$. The set of controls, denoted by $\mathscr{A}$, is then the set of measurable functions $\Gamma : [0,T] \to \mathbb{A}$; let us remark that the admissible controls considered here are deterministic and open-loop. The controlled dynamics is given by the Kolmogorov-Fokker-Planck ODE

$$\dot{m}_x = \sum_{y \neq x} \left(m_y \Gamma_x(t,y) - m_x \Gamma_y(t,x) \right); \tag{1.74}$$

namely, the control $\Gamma \in \mathscr{A}$ represents the transition rates of a Markov chain and the state variable of the problem is its law. Note that the diagonal arguments $\Gamma_{x,x}$ are never considered and that the dynamics remains in $\mathcal{P}(\Sigma)$ for any time, so we do not need to impose any state constraint.

The cost to be minimized, for $\Gamma \in \mathscr{A}$, is

$$\mathcal{J}(\Gamma) := \int_0^T \left(\mathcal{L}(m(t), \Gamma(t)) + \mathcal{F}(m(t))\right) dt + \mathcal{G}(m(T)), \tag{1.75}$$

where the Lagrangian $\mathcal{L} : [0,T] \times \mathbb{A} \to \mathbb{R}$ is given by

$$\mathcal{L}(m, \Gamma) := \sum_x m_x L(x, \Gamma(x)). \tag{1.76}$$

The pre-Hamiltonian and Hamiltonian of the problem are respectively

$$\mathscr{H}(m, g, \Gamma) := -b(m, \Gamma) \cdot g - \mathcal{L}(m, \Gamma) - \mathcal{F}(m), \tag{1.77}$$

$$\mathcal{H}(m, g) := \sup_{\Gamma \in \mathbb{A}} \mathscr{H}(m, g, \Gamma), \tag{1.78}$$

for any $g \in \mathbb{R}^d$, where $b_x : [0,T] \times \mathbb{A} \to \mathbb{R}$ given by

$$b_x(m, \Gamma) := \sum_{y \neq x} \left(m_y \Gamma_x(y) - m_x \Gamma_y(x)\right) \tag{1.79}$$

is the vector field of the dynamics (1.74). A straightforward computation leads to

$$\mathcal{H}(m, g) = \sum_{x \in \Sigma} m_x H(x, \Delta g(x)) - \mathcal{F}(m) \tag{1.80}$$

and the unique argmax of $\mathscr{H}$ is given by

$$\Gamma^*(x, g) = a^*(x, \Delta g(x)). \tag{1.81}$$

Moreover we obtain

$$\frac{\partial \mathcal{H}}{\partial m_x}(m, g) = H(x, \Delta g(x)) - F(x, m) \tag{1.82}$$

$$\frac{\partial \mathcal{H}}{\partial g_x}(m, g) = -b_x(m, \Gamma^*(\cdot, g)) \tag{1.83}$$

Therefore the corresponding Hamiltonian system, arising form the Pontryagin maximum principle, reads

$$\begin{cases} \dot{u} = \frac{\partial \mathcal{H}}{\partial m}(m, u), \qquad t \in [0,T] \\ \dot{m} = -\frac{\partial \mathcal{H}}{\partial u}(m, u), \\ m(0) = m_0, \qquad u(T) = \nabla^m \mathcal{G}(m(T)), \end{cases} \tag{1.84}$$

which is exactly the mean field game system (MFG), setting $a_x^*(x) := -\sum_{y \neq x} a_y^*(x)$. Further, the Hamilton-Jacobi-Bellman equation of the problem is given by

$$\begin{cases} -\frac{\partial \mathcal{U}}{\partial t} + \mathcal{H}(m, \nabla^m \mathcal{U}) = 0, \\ \mathcal{U}(T, m) = \mathcal{G}(m), \end{cases} \tag{1.85}$$

for a function $\mathcal{U} : [0,T] \times \mathcal{P}(\Sigma) \to \mathbb{R}$.

Lemma 1.19. *Under (H4) and* (1.73),

1. *There exists an optimum of the control problem* (1.74)-(1.75)*;*
2. *The MFG system* (1.84) *represents the necessary conditions for optimality, given by the Pontryagin maximum principle;*
3. *The value function of the problem is the unique viscosity solution to Equation* (1.85)*;*
4. *If (Mon) also holds then the value function is in $\mathcal{C}^1(\mathcal{P}(\Sigma))$ and the optimal control is unique and given by the feedback*

$$\Gamma(t,x,m) = \Gamma^*(x, \nabla^m \mathcal{U}(t,m)) = a^*(x, \Delta^x \nabla^m \mathcal{U}(t,m)). \tag{1.86}$$

Proof. The first claim is given by Theorem 5.2.1 p. 94 in [10], which can be applied since the dynamics is linear in Γ and the running cost is convex in Γ. Conclusions (2) and (3) are standard. For (4), if the costs are monotone then $\mathcal{F}$ and $\mathcal{G}$ are convex and thus the claims follow. □

Next, we show shat, at least at a heuristic level, the derivative of Equation (1.85) turns out to be exactly the master equation (M). Indeed, assuming that the solution U to the master equation is differentiable in the classical sense with respect to m, so that $[D^m U(t,x,m,y)]_z = \frac{\partial U}{\partial m_z} - \frac{\partial U}{\partial m_y}$, we have (omitting t, m)

$$\int_\Sigma D^m U(x,m,y)\cdot a^*(y, \Delta U(y)) m(dy) = \sum_y \sum_{z\neq y} \left(\frac{\partial U(x)}{\partial m_z} - \frac{\partial U(x)}{\partial m_y} \right) a^*_z(y) m_y$$

$$= \sum_y \sum_{z\neq y} \frac{\partial U(x)}{\partial m_z} \Big(a^*_z(y, \Delta U(y)) m_y - a^*_y(z, \Delta U(z)) m_z \Big) = \nabla^m U(x)\cdot b(m, \Gamma^*),$$

where $\Gamma^*(y) = a^*(y, \Delta U(y))$ and b is the vector field given by (1.79). Thus Equation (M) becomes

$$-\frac{\partial U_x}{\partial t} + H(x, \Delta^x U) - b(m, \Gamma^*)\cdot \nabla^m U = F(x,m). \tag{1.87}$$

Let now $\mathcal{U}$ be a classical solution to (1.85) and set $U(t,m) = \nabla^m \mathcal{U}(t,m)$. Assuming $\mathcal{U} \in \mathcal{C}^2$, we have $\frac{\partial U_x}{\partial m_y} = \frac{\partial U_y}{\partial m_x}$. This fact, together with (1.73) and (1.84) allows to conclude that U solves (1.87): indeed, deriving (1.85) by m_x we obtain

$$0 = -\frac{\partial U_x}{\partial t} + \frac{\partial \mathcal{H}}{\partial m_x}(m,U) + \frac{\partial \mathcal{H}}{\partial u}(m,U)\cdot \frac{\partial U}{\partial m_x} = -\frac{\partial U_x}{\partial t} + H(x, \Delta^x U) - F(x,m) - b(m,\Gamma^*)\cdot \nabla^m U \tag{1.88}$$

and the terminal condition is clearly satisfied.

Therefore we have showed that (at least formally) the master equation, if the costs are potential, can be written as

$$\begin{cases} -\frac{\partial U}{\partial t} + \frac{\partial}{\partial m}\left[\mathcal{H}(m,U)\right], & t\in[0,T[, \quad m\in\mathcal{P}(\Sigma) \\ U(T,x,m) = G(x,m) \end{cases} \tag{1.89}$$

which reads as a system of d conservation laws in space dimension d. Such system of PDEs is known to be ill-posed in general, while the theory of *entropy solutions* is developed for $d=1$, providing well-posedness of the equation; we will examine this notion for the two state model of Chapter 4. Notably, the Hamiltonian $\mathcal{H}$ is space-dependent and the equation is set in the symplex, which is a bounded subset of $\mathbb{R}^d$. Nevertheless, boundary conditions are not needed since the domain is invariant under the action of the characteristics.

1.4.2 Weak mean field game solutions

The notion of weak MFG solution has to be introduced in order to handle the case of a random flow of measures, which may arise as the limit of the empirical measures of the N-players, when (strong) MFG are not unique. This notion, in the diffusion setting, was first examined in [21] to study mean field games with a common noise, in which the limiting measure is actually a conditional law, thus always random, even when solutions are unique. Then this concept was specialized for mean field games without common noise in [65], where also examples were treated. We give now the definition for finite state mean field games.

Let $\mathbb{M}$ be the set of integer valued random measures on $[0,T]\times\Theta$ with the topology of weak convergence, and recall $\mathcal{D}=D([0,T],\Sigma)$. For $\mu\in\mathcal{P}(\mathcal{D})$ let $\mu(t)\in\mathcal{P}(\Sigma)$ denote the image measure under the map $x\mapsto x(t)$. Let

$$\mathcal{Z}:=\mathbb{M}\times\mathcal{Q}\times\mathcal{D} \tag{1.90}$$

and $\mathcal{F}^{\mathcal{Z}}_t$ denotes its canonical filtration. For $\eta\in\mathcal{P}(\mathcal{Z})$ let $\eta^x:=\eta(\mathcal{M}\times\mathcal{A}\times\cdot)$ denote the $\mathcal{D}$ marginal. Given any $\eta\in\mathcal{P}(\mathcal{Z})$, dynamics (1.58) is still well-posed, whereas $m=\eta^x$: denote its pathwise unique solution by X_{ρ,η^x}. The cost (1.64) is still well-defined, with $m=\eta^x$.

Definition 1.20. *A* weak relaxed mean field game solution *is a triple*

$$\left(\left((\Omega,\mathcal{F},P;\mathbb{F}),\rho,\xi,\mathcal{N}\right),\eta,X\right)$$

such that

1. $((\Omega,\mathcal{F},P;\mathbb{F}),\rho,\xi,\mathcal{N})\in\mathcal{R}$, *$\eta$ is a random element of $\mathcal{P}(\mathcal{Z})$ such that $\eta(E)$ is $\mathcal{F}_t$-measurable for each $E\in\mathcal{F}^{\mathcal{Z}}_t$ and $t\in[0,T]$, X is $\mathbb{F}$-adapted and $X=X_{\rho,\eta^x}$;*
2. *ξ, $\mathcal{N}$ and η are independent, $m_0=\mathrm{Law}(\xi)$ and $\eta(0)=\delta_0\otimes\delta_0\otimes m_0$ is deterministic;*
3. Compatibility condition*: $\sigma(\rho_s:s\le t)$ is conditionally independent of $\mathcal{F}^{\xi,\mathcal{N},\eta}_T$ given $\mathcal{F}^{\xi,\mathcal{N},\eta}_t$, for every $t\in[0,T]$, where*
$$\mathcal{F}^{\xi,\mathcal{N},\eta}_t:=\sigma(\xi,\mathcal{N}_s,\eta(E):s\le t,E\in\mathcal{F}^{\mathcal{Z}}_t);$$
4. Optimality*: $J(\rho,\eta^x)\le J(\widetilde{\rho},\eta^x)$ for any $\widetilde{\rho}\in\mathcal{R}$;*
5. Consistency condition*: η is a version of the conditional law of $(\mathcal{N},\rho,X)$ given η, i.e.*
$$\eta(E)=P\left[(\mathcal{N},\rho,X)\in E|\eta\right]\quad\forall E\in\mathcal{B}(\mathcal{Z}),a.s. \tag{1.91}$$

Moreover (omitting the filtered probability space, the initial condition and the Poisson random measure) in the notation, we say that a triple (π,η,X) is a *weak open-loop MFG solution* if $\pi\in\mathcal{S}$ and (ρ^π,η,X) is a weak relaxed MFG solution, with $\rho^\pi_t=\delta_{\pi(t)}$. For a *measurable* function $\beta:[0,T]\times\Sigma\times\mathcal{P}(\Sigma)$, we say that (β,η,X) is a *weak feedback MFG solution* if (π^β,η,X) is a weak open-loop MFG solution, where

$$\pi^\beta(t)=\beta(t,X(t^-),\eta^x(t))\qquad \ell\otimes P\text{ - a.e. }(t,\omega). \tag{1.92}$$

Since the flow of measures is random, its joint distribution with the control (which includes the noise and the initial condition) has to be specified: for this reason η is defined as a random measure on $\mathcal{P}(\mathcal{Z})$. If η is deterministic then we recover the notions of MFG solutions given above, which might be called *strong* MFG solutions. Let us observe that, indeed, when η is deterministic, the consistency condition given here implies a stronger mean field condition than in the previous definitions, as it considers the law of the processes in the space of trajectories and not just their marginals laws (or flows). This is due to the fact that here we need to handle also the law of the (open-loop) control, whose filtration might be larger that the one of η. Very recently, a slightly different notion of weak feedback MFG solution has been given in [66]. We remark that (1.92) defines feedback controls which differ from the ones considered before (Definition 1.9), since β also depends on the measure, while before the dependence on m was included in the time argument of α, as m was deterministic. Note also that the value of the control in the initial time is meaningless, as it always appears inside the integral in time, hence we could impose condition (2) above.

Weak MFG solutions exist under (A) and (B), thanks to Theorem 2.4, as strong MFG solutions are also weak MFG solutions. Moreover, if (Mon) holds, then weak MFG solutions are unique (as strong solutions are), since the proof of Theorem 3.18 still works in this case. When strong MFG solutions are not unique, a weak MFG solution is for instance a random measure supported in the set of strong MFG solutions. However there might be weak MFG solutions that are not supported on strong MFG solutions; see 4.3.2 for an example in the two state model.

Appendix A: Relaxed Poisson measures

In order to state the definition of the relaxed Poisson random measure we first need to define the canonical space of integer valued random measures on a metric space E. Following [57], the setting is:

- $\overline{\Omega}$ is the set of sequences $(t_n, y_n) \subset [0, +\infty] \times E$ such that (t_n) is increasing and $t_n < t_{n+1}$ if $t_n < +\infty$; set $t_0 := 0$ and $t_\infty := \lim_n t_n$;
- if $\overline{\omega} = (t_n, y_n)_{n\in\mathbb{N}}$ write $T_n(\overline{\omega}) := t_n$ and $Y_n(\overline{\omega}) := y_n$;
- the *canonical random measure* is

$$\overline{\mathcal{N}}(\overline{\omega}, B) := \sum_{n\in\mathbb{N}} \mathbb{1}_{\{T_n(\overline{\omega})<\infty\}} \delta_{(T_n(\overline{\omega}), Y_n(\overline{\omega}))}(B)$$

  for any $B \in \mathcal{B}([0, +\infty[\times E)$;

- $\overline{\mathcal{G}}_t := \sigma\left(\overline{\mathcal{N}}(\cdot, B) : B \in \mathcal{B}([0, t] \times E)\right)$, $\overline{\mathcal{F}}_0$ is given, $\overline{\mathcal{F}}_t = \overline{\mathcal{F}_0} \vee \left(\cap_{s<t}\overline{\mathcal{G}}_s\right)$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}_\infty$ and $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$.

The filtered space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}})$ is then the *canonical space of integer valued random measures on E*. A probability measure on it is the law of an integer valued random measure on E, given an initial condition on $\overline{\mathcal{F}_0}$. Note that the canonical measure $\overline{\mathcal{N}}$ is not the identity: for this reason we can work with $\mathcal{M} = \mathcal{M}([0, +\infty[\times E)$ as the state space of a random measure. Moreover, the set of integer valued random measures is vaguely closed in $\mathcal{M}$: see Theorem 15.7.4 in [58] and the references therein.

Let now Ξ be any integer valued random measure defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. It is determined by a sequence of stopping times T_n and random variables X_n which are $\mathcal{F}_{T_n}$-measurable. To any Ξ is associated its *compensator*, that is, a positive random measure η on E such that

1. $\eta([0,t] \times B)_{t \geq 0}$ is predictable for any $B \in \mathcal{B}(E)$;

2. $(\Xi([0,t \wedge T_n] \times B) - \eta([0,t \wedge T_n] \times B))_{t \geq 0}$ is an $\mathbb{F}$-martingale for each n and B;

3. $\eta(\{t\} \times E) \leq 1$ for each t and $\eta([T_\infty, \infty[\times E) = 0$.

The compensator exists and is unique (up to a modification on a P-null set) for any Θ. The proof can be found in [56], where the author also shows that a process with the above properties uniquely determines an integer valued random measure.

Consider then an arbitrary measurable space $(\Omega', \mathcal{F}')$ and define $\Omega := \overline{\Omega} \times \Omega'$. Set $\overline{\mathcal{F}_0} := \left\{\varnothing, \overline{\Omega}\right\}$ and $\mathcal{F}_0 := \overline{\mathcal{F}_0} \otimes \mathcal{F}'$. The canonical random measure $\overline{\mathcal{N}}$ on $\overline{\Omega}$ is extended to Ω via $(T_n, Y_n)(\overline{\omega}, \omega') := (T_n, Y_n)(\overline{\omega})$. Set $\mathcal{F}_t := \overline{\mathcal{F}}_t \vee \mathcal{F}_0$.

Theorem 1.21 ([56]). *Let P_0 be a probability measure on $(\Omega, \mathcal{F}_0)$ and η a predictable random measure satisfying (1) and (3). Then there exists a unique probability measure P on $(\Omega, \mathcal{F}_\infty)$ whose restriction to $\mathcal{F}_0$ is P_0 and for which η is the compensator of $\overline{\mathcal{N}}$.*

By means of this theorem, we are able to define properly a relaxed Poisson measure. Consider a relaxed control $((\Omega'', \mathcal{F}'', P''; \mathbb{F}''), \rho, \xi, \mathcal{N}) \in \mathcal{R}$ and let $\Omega' = \mathcal{Q} \times \Sigma \times \overline{\Omega}$ be the state space of the process ρ, the initial distribution ξ and the Poisson random measure $\mathcal{N}$. The σ-algebra $\mathcal{F}'$ is generated by the processes and P_0 is the joint law of $(\rho, \xi, \mathcal{N})$. So a relaxed Poisson measure $\mathcal{N}_\rho$, related to the relaxed control ρ, is an integer valued random measure on $[0,T] \times \Theta \times A$ whose compensator η, calculated on $[0,t]$, Θ_0, A_0, is $\nu(\Theta_0)\rho([0,t] \times A_0)$. Its law is uniquely determined on $\overline{\Omega}$ and thus has the martingale properties (1.55) and (1.56). Moreover, the joint law of $(\mathcal{N}_\rho, \rho, \xi, \mathcal{N})$ is uniquely determined.

We could also give an explicit construction of $\mathcal{N}_\rho$. Let $\rho \in \mathcal{R}$ and (π_n) be a sequence in $\mathcal{S}$ which tends to ρ in the sense of Lemma 2.12, the chattering lemma. Denote by ρ^{π_n} the relaxed control representation of π_n and construct $\mathcal{N}_{\pi_n}$ as in (1.57): $\mathcal{N}_{\pi_n}(t, \Theta_0, A_0) := \int_0^t \int_{\Theta_0} \mathbb{1}_{A_0}(\pi_n(s))\mathcal{N}(ds, d\theta)$. Then, by Theorem 2.1, the sequence $(X_{\pi_n}, \rho^{\pi_n}, \mathcal{N}_{\pi_n})$ is tight and any subsequence converges in distribution to $(X_\rho, \rho, \mathcal{N}_\rho)$. The marginals are uniquely defined in this way, while to show that the joint law of $(\rho, \mathcal{N}_\rho)$ is unique we need to invoke the above Theorem 1.21.

A.1: Proof of Lemma 1.13

Let $m \in \mathcal{E}$ be fixed, which we shall omit. Let $\mathcal{X}$ be the space of stochastic processes with paths in $\mathcal{D}$ and equip it with the norm $||X||_\mathcal{X} = \mathbb{E}\left[\sup_{0 \leq t \leq T} |X(t)|\right]$. Let $\rho \in \mathcal{R}$ and define the map $\Psi : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$\Psi_t(X) := \xi + \int_0^t \int_\Theta \int_A \gamma(s, X(s^-), \theta, a)\mathcal{N}_\rho(ds, d\theta, da)$$

for any $X \in \mathcal{X}$. If we prove that this map is a contraction in the norm $|| \cdot ||_\mathcal{X}$, then pathwise existence and uniqueness of solutions to equation (1.58) follow. We have, for any $X, Y \in \mathcal{X}$,

$$|\Psi_t(X) - \Psi_t(Y)| \leq \int_0^t \int_\Theta \int_A |\gamma(s, X(s^-), \theta, a) - \gamma(s, Y(s^-), \theta, a)|\mathcal{N}_\rho(ds, d\theta, da),$$

hence

$$\mathbb{E}\left[\sup_{0\le t\le T}|\Psi_t(X)-\Psi_t(Y)|\right]$$
$$\le \mathbb{E}\int_0^T\int_\Theta\int_A|\gamma(s,X(s),\theta,a)-\gamma(s,Y(s),\theta,a)|\rho_s(da)\nu(d\theta)ds$$
$$\le K_1\mathbb{E}\int_0^T\int_A|X(s)-Y(s)|\rho_s(da)ds\le K_1T\mathbb{E}\left[\sup_{0\le t\le T}|X(s)-Y(s)|\right]$$

thanks to (1.2) and the fact that ρ_s is a probability measure. Therefore Ψ is a contraction if $T<\frac{1}{K_1}$, and so uniqueness is proved for small time horizon; but then iterating the same argument, we have uniqueness for any T.

Consider now $\Upsilon\in\mathcal{Y}$ and define $\widehat{\Psi}:\mathcal{X}\longrightarrow\mathcal{X}$ by

$$\widehat{\Psi}_t(X):=\xi+\int_0^t\int_\Theta\int_A\gamma(s,X(s^-),\theta,a)\mathcal{N}_{\rho^\Upsilon,X}(ds,d\theta,da)$$

for any process $X\in\mathcal{X}$. Then for any X and Y we have $||\widehat{\Psi}(X)-\widehat{\Psi}(Y)||_{\mathcal{X}}\le||Z_1||_{\mathcal{X}}+||Z_2||_{\mathcal{X}}$ where

$$Z_1(t):=\int_0^t\int_\Theta\int_A|\gamma(s,X(s^-),\theta,a)-\gamma(s,Y(s^-),\theta,a)|\mathcal{N}_{\rho^\Upsilon,Y}(ds,d\theta,da)$$

and

$$Z_2(t):=\int_0^t\int_\Theta\int_A|\gamma(s,X(s^-),\theta,a)|\left|\mathcal{N}_{\rho^\Upsilon,X}-\mathcal{N}_{\rho^\Upsilon,Y}\right|(ds,d\theta,da),$$

where $|\Xi|$ denotes the total variation of the signed measure Ξ defined for any $B\in\mathcal{B}([0,T]\times\Theta\times A)$ by $|\Xi|(B):=\sup_{E\subseteq B}|\Xi(E)|$; while the total variation norm is $||\Xi||_{TV}=|\Xi|([0,T]\times\Theta\times A)$. The first term Z_1 is bounded as above yielding $||Z_1||_{\mathcal{X}}\le K_1T||X-Y||_{\mathcal{X}}$. For the second term, we use $|\gamma|\le d$ to obtain

$$\sup_{0\le t\le T}Z_2(t)\le d||\mathcal{N}_{\rho^\Upsilon,X}-\mathcal{N}_{\rho^\Upsilon,Y}||_{TV}=d\sup_{E\subset[0,T]\times\Theta\times A}\left|\mathcal{N}_{\rho^\Upsilon,X}(E)-\mathcal{N}_{\rho^\Upsilon,Y}(E)\right|.$$

Thanks to (1.55) and (1.51), we have $\mathbb{E}||\mathcal{N}_{\rho^\Upsilon,X}-\mathcal{N}_{\rho^\Upsilon,Y}||_{TV}\le 2T\nu(\Theta)$, saying that the right-hand side above is finite P-a.s. Since the measure $\mathcal{N}_{\rho^\Upsilon,X}-\mathcal{N}_{\rho^\Upsilon,Y}$ is integer valued, we can assume that the above supremum is attained on a set $B(\omega)$ for P-a.e. ω, giving thus a random set B. Moreover, we may assume that on such a set the random measure considered is positive. The martingale property (1.56) now gives

$$||Z_2||_{\mathcal{X}}\le d\mathbb{E}\left[\mathcal{N}_{\rho^\Upsilon,X}(B)-\mathcal{N}_{\rho^\Upsilon,Y}(B)\right]$$
$$=d\left|\mathbb{E}\int_0^T\int_\Theta\int_A\mathbb{1}_B(t,\theta,a)[\Upsilon(t,X(t))-\Upsilon(t,Y(t))](da)\nu(d\theta)dt\right|$$
$$\le d\mathbb{E}\int_0^T|\Upsilon(t,X(t))-\Upsilon(t,Y(t))|(A)\nu(\Theta)dt$$
$$\le 2\nu(\Theta)d\mathbb{E}\int_0^T|X(t)-Y(t)|dt\le K_1T||X-Y||_{\mathcal{X}},$$

where in the last line above we have used the fact that Υ is a probability measure and $|x-y|\ge1$ for each $x\ne y\in\Sigma$. Therefore, for $T<\frac{1}{2K_1}$, the map $\widehat{\Psi}$ is a contraction; the claim follows iterating the above procedure.

CHAPTER 2

Existence and approximation

In this chapter, we state the existence and approximation results, presented in [25]. Firstly, we prove existence of solutions to the mean field game in relaxed open-loop as well as relaxed feedback controls. Then we find, under additional assumptions, mean field game solutions in non-relaxed feedback controls. In the last section, we first establish a version of the chattering lemma that works also for feedback controls. Then we turn to the construction of approximate Nash equilibria coming from a solution of the mean field game, and derive the error bound for feedback as well as open-loop strategies.

2.1 Relaxed mean field game solutions

2.1.1 Tightness and continuity for m fixed

Consider a sequence of random variables

$$(X^n, \rho^n, \mathcal{N}_{\rho^n}) \tag{2.1}$$

where ρ^n is a relaxed control, $\mathcal{N}_{\rho^n}$ is the related relaxed Poisson measure and $X^n = X_{\rho^n,m}$, $m \in \mathcal{E}$ is fixed. The state space of these random variables is $D([0,T],\Sigma) \times \mathcal{Q} \times \mathcal{M}$, where $\mathcal{M} = \mathcal{M}([0,T] \times \Theta \times A)$ denotes the set of finite positive measures on $[0,T] \times \Theta \times A$ endowed with the topology of weak convergence.

The following is of fundamental importance, and is similar to Theorem 13.2.1 in [62, p. 363].

Theorem 2.1. *Assume (A) and (B). Then*

1. *any sequence of the form (2.1) is tight;*
2. *the limit in distribution $(X, \rho, \widetilde{\mathcal{N}})$ of any converging subsequence is such that $\widetilde{\mathcal{N}}$ is the relaxed Poisson measure related to the relaxed control ρ and $X = X_{\rho,m}$ in distribution;*
3. *$J(\rho, m)$ is continuous in ρ.*

Proof. (1) The sequence of relaxed controls is tight as $\mathcal{Q}$ is compact. For any $\varepsilon > 0$, the set

$$\mathcal{K}_\varepsilon := \left\{ \Xi \in \mathcal{M} : \Xi([0,T] \times \Theta \times A) \leq \frac{T\nu(\Theta)}{\varepsilon} \right\}$$

is compact in $\mathcal{M}$, since $[0,T]\times\Theta\times A$ is compact. From (1.51) and the martingale property (1.55), it follows that $N_{\rho^n}(t,\Theta,A)-t\nu(\Theta)$ is a martingale for any n and so $\mathbb{E}[\mathcal{N}_{\rho^n}(T,\Theta,A)]=T\nu(\Theta)$. Therefore, by Chebychev's inequality,

$$P\left(\mathcal{N}_{\rho^n}\notin\mathcal{K}_\varepsilon\right)=P\left(\mathcal{N}_{\rho^n}(T,\Theta,A)>\frac{T\nu(\Theta)}{\varepsilon}\right)\le\mathbb{E}[\mathcal{N}_{\rho^n}(T,\Theta,A)]\cdot\frac{\varepsilon}{T\nu(\Theta)}=\varepsilon$$

for any n, saying that the sequence of relaxed Poisson measures is tight. The properties of the stochastic integral give

$$\mathbb{E}\left[|X^n(\tau+h)-X^n(\tau)|^2\Big|\mathcal{F}_\tau\right]=O(h)$$

for any $\mathbb{F}$-stopping time τ, uniformly in n, which yields the tightness of the processes in $D([0,T],\Sigma)$ by Aldous's criterion [1].

(2) By abuse of notations, denote by $(X^n,\rho^n,\mathcal{N}_{\rho^n})$ the subsequence which converges in distribution to $(X,\rho,\widetilde{\mathcal{N}})$. From the martingale property (1.55), it follows that $\widetilde{\mathcal{N}}(t,\Theta_0,A_0)-\nu(\Theta_0)\rho(t,A_0)$ is a martingale for any Borel sets $A_0\subset A$ and $\Theta_0\subset\Theta$, where the limiting measure is defined on the canonical space and the filtration is the canonical filtration (both defined in Appendix A). The limit random measure $\widetilde{\mathcal{N}}$ is integer valued (Theorem 15.7.4 in [58]), so the uniqueness property says that $\widetilde{\mathcal{N}}=\mathcal{N}_\rho$ in distribution. The claim $X=X_{\rho,m}$ in distribution will be shown also in the proof of Theorem 2.4, where m is not fixed, using the controlled martingale problem, so we do not repeat the argument here.

(3) $\lim_{n\to\infty}J(\rho_n,m)=J(\rho,m)$ since c and G are bounded and continuous by assumption (B). □

By the chattering lemma, which we will present later as Lemma 2.12, we have

$$\min_{\rho\in\mathcal{R}}J(\rho,m)=\inf_{\pi\in\mathcal{S}}J(\pi,m).$$

The minimum on the left hand side exists by the above Theorem 2.1. The infimum on the right hand side is actually a minimum, too; see Theorem 2.6 below, where the existence of optimal feedback controls will be shown. However, there might exist more optima among relaxed open-loop controls than among ordinary feedback controls.

2.1.2 Fixed point argument

Let $2^{\mathcal{E}}$ be the set of subsets of $\mathcal{E}$ and define the point-to-set map $\Phi:\mathcal{E}\longrightarrow 2^{\mathcal{E}}$ by

$$\Phi(m):=\{\mathrm{Flow}(X_{\rho,m}):J(\rho,m)\le J(\sigma,m)\quad\forall\sigma\in\mathcal{R}\},\qquad m\in\mathcal{E}.\tag{2.2}$$

A flow $m\in\mathcal{E}$ is called a *fixed point* of this point-to-set map if $m\in\Phi(m)$. We need this map since the optimal control is not necessarily unique.

By construction, Φ has a fixed point if and only if there exists a relaxed solution to the mean field game, in the sense of Definition 1.14. In order to prove the existence of a fixed point, we are going to apply Theorem 1 in [40], which requires the following definition.

Definition 2.2. *Let $\mathcal{E}$ be a metric space. A map $\Phi:\mathcal{E}\longrightarrow 2^{\mathcal{E}}$ is said to have* closed graph *if $m_n\in\mathcal{E}$, $y_n\in\mathcal{E}$, $y_n\in\Phi(m_n)$ for any $n\in\mathbb{N}$ and $m_n\to m$, $y_n\to y$ in $\mathcal{E}$ implies $y\in\Phi(m)$.*

Proposition 2.3 (Ky Fan)**.** *Let $\mathcal{E}$ be a non empty, compact and convex subset of a locally convex metric topological vector space. Let $\Phi:\mathcal{E}\longrightarrow 2^{\mathcal{E}}$ have closed graph and assume that $\Phi(m)$ is non empty and convex for any $m\in\mathcal{E}$. Then the set of fixed points of Φ is non empty and compact.*

By means of this proposition we are now able to state and prove the following main theorem concerning existence of relaxed solutions, while uniqueness is not guaranteed.

Theorem 2.4. *Under assumptions (A) and (B) there exists at least one relaxed solution of the mean field game.*

Proof. We want to show the existence of a fixed point for the map $\Phi : \mathcal{E} \longrightarrow 2^{\mathcal{E}}$ defined in (2.2), applying Proposition 2.3. Recall that any element of $\Phi(m)$ is in $\mathcal{E}$ by Lemma 1.15, and the set $\mathcal{E}$ defined in (1.65) is a compact and convex subset of $\mathcal{C}([0,T],\mathcal{P}(\Sigma))$ endowed with the uniform norm. By Theorem 2.1, $\Phi(m)$ is non empty for any m. It remains to prove that $\Phi(m)$ is convex and Φ has closed graph.

$\Phi(m)$ **is convex**. Let m be fixed and let $\rho_1, \rho_2 \in \mathcal{R}$ be such that $\mathrm{Flow}(X_{\rho_1,m})$ and $\mathrm{Flow}(X_{\rho_2,m})$ belong to $\Phi(m)$, i.e. ρ_1 and ρ_2 are optimal controls for m, and take $\lambda \in [0,1]$. Let χ be a Bernoulli random variable with parameter λ, $\mathcal{F}_0$ measurable and independent of ρ_1 and ρ_2. Define $\rho_3 \in \mathcal{R}$ by

$$\rho_3([0,t[\times E) := \rho_1([0,t[\times E)\mathbb{1}_{\{\chi=1\}} + \rho_2([0,t[\times E)\mathbb{1}_{\{\chi=0\}}$$

for any $E \in \mathcal{B}(A)$ and $t \in [0,T]$. We have

$$\begin{aligned}\mathbb{E}[\varphi(X_{\rho_3,m})] &= \mathbb{E}\left[\varphi(X_{\rho_3,m})|\,\chi=1\right]P(\chi=1) + \mathbb{E}\left[\varphi(X_{\rho_3,m})|\,\chi=0\right]P(\chi=0) \\ &= \lambda\mathbb{E}[\varphi(X_{\rho_1,m})] + (1-\lambda)\mathbb{E}[\varphi(X_{\rho_2,m})]\end{aligned}$$

for every $\varphi \in \mathcal{C}_b(D([0,T],\Sigma),\mathbb{R})$. This implies that

$$\mathrm{Law}(X_{\rho_3,m}) = \lambda\,\mathrm{Law}(X_{\rho_1,m}) + (1-\lambda)\,\mathrm{Law}(X_{\rho_2,m}) \tag{2.3}$$

and then in particular

$$\mathrm{Flow}(X_{\rho_3,m}) = \lambda\,\mathrm{Flow}(X_{\rho_1,m}) + (1-\lambda)\,\mathrm{Flow}(X_{\rho_2,m}). \tag{2.4}$$

Since ρ_1 and ρ_2 are optimal for m we have, thanks to (2.3),

$$\begin{aligned}J(\rho_3,m) &= J(\rho_1,m)P(\chi=1) + J(\rho_2,m)P(\chi=0) \\ &\le \lambda J(\sigma,m) + (1-\lambda)J(\sigma,m) = J(\sigma,m)\end{aligned}$$

for any $\sigma \in \mathcal{R}$, which means that also ρ_3 is optimal for m and hence (2.4) says that $\Phi(m)$ is convex.

Φ **has closed graph**. Let $m_n, y_n, m, y \in \mathcal{E}$ be such that $m_n \to m$, $y_n \to y$ in $\mathcal{E}$ and $y_n \in \Phi(m_n)$ for every $n \in \mathbb{N}$. We have to prove that $y \in \Phi(m)$. Let $\rho_n \in \mathcal{R}$ be optimal for m_n and such that $y_n = \mathrm{Flow}(X_{\rho_n,m_n})$. Set $X_n := X_{\rho_n,m_n}$ and let $\mathcal{N}_n := \mathcal{N}_{\rho_n}$ be the relaxed Poisson measure related to ρ_n.

The tightness of the sequence $(X_n,\rho_n,\mathcal{N}_n)$ is proved as in Theorem 2.1. Let $(X_{n_k},\rho_{n_k},\mathcal{N}_{n_k})$ be a subsequence which converges in distribution to $(X,\rho,\widetilde{\mathcal{N}})$. We have $\widetilde{\mathcal{N}} = \mathcal{N}_\rho$ in distribution, i.e. it is the relaxed Poisson measure related to ρ. In order to prove that $X = X_{\rho,m}$ in distribution, we use the controlled martingale problem formulation stated in Lemma 1.16, and hence let us assume that the processes are defined in the canonical space.

Property (1.66) holds for X_{n_k}, ρ_{n_k} and m_{n_k}, any $k \in \mathbb{N}$. Let $M_g^{n_k}$ denote the process defined by

$$\begin{aligned}&M_g^{n_k}(t) = g(X_{n_k}(t)) - g(X_{n_k}(0)) \\ &- \int_0^t \int_\Theta \int_A \left[g(X_{n_k}(s) + \gamma(s,X_{n_k}(s),\theta,a,m_{n_k}(s))) - g(X_{n_k}(s))\right]\nu(d\theta)\rho_s^{n_k}(da)ds,\end{aligned}$$

for any $g \in \mathbb{R}^d$. Property (1.66) and the convergence in distribution of the sequence $(X_{n_k}, \rho_{n_k}, \mathcal{N}_{n_k})$ imply that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mathbb{E}\left[\psi(X_{n_k}(t_i); i \le j)(M_g^{n_k}(t+s) - M_g^{n_k}(t))\right] \\ &= \mathbb{E}\left[\psi(X(t_i); i \le j)(M_g(t+s) - M_g(t))\right] \end{aligned}$$

thanks to continuity assumption (A), uniform convergence of m_n and (1.54). Therefore we have proved that $X = X_{\rho,m}$ in distribution.

Thus we obtain

$$\lim_{k\to\infty} \mathrm{Law}(X_{n_k}) = \mathrm{Law}(X_{\rho,m}),$$

which implies the convergence

$$\lim_{k\to\infty} \sup_{t\in[0,T]} |\,\mathrm{Law}(X_k(t)) - \mathrm{Law}(X(t))| = 0,$$

that is, $\mathrm{Flow}(X_{n_k}) \to \mathrm{Flow}(X)$ uniformly. The convergence is then proved along a subsequence, but by hypothesis the limit $\mathrm{Flow}(X_n) \to y$ exists in $\mathcal{E}$, hence $y = \mathrm{Flow}(X) = \mathrm{Flow}(X_{\rho,m})$.

It remains to prove that ρ is optimal for m. Again the convergence in distribution of the sequence $(X_{n_k}, \rho_{n_k}, \mathcal{N}_{n_k})$ implies that $\lim_k J(\rho_{n_k}, m_{n_k}) = J(\rho, m)$ thanks to continuity assumption (B), uniform convergence of m_n and (1.54). Then from the optimality of ρ_n for m_n, i.e. $J(\rho_{n_k}, m_{n_k}) \le J(\sigma, m_{n_k})$ for every $\sigma \in \mathcal{R}$, taking the limit as $k \to \infty$ we get $J(\rho, m) \le J(\sigma, m)$ for every $\sigma \in \mathcal{R}$, which means that ρ is optimal for m and thus $y = \mathrm{Flow}(X_{\rho,m}) \in \Phi(m)$ as required. □

2.1.3 Relaxed feedback mean field game solutions

Theorem 2.4 provides a relaxed (open-loop) solution of the mean field game. Under the same assumptions we obtain here a relaxed feedback mean field game solution which has the same cost and flow of the open-loop one. This result is similar to Theorem 3.7 in [64] and will provide approximate feedback Nash equilibria for the N-player game.

Theorem 2.5. *Assume (A) and (B) and let $(((\Omega, \mathcal{F}, P; \mathbb{F}), \rho, \xi, \mathcal{N}), m, X_{\rho,m})$ be a relaxed mean field game solution. Then there exists a relaxed feedback control $\Upsilon \in \mathcal{Y}$ such that the tuple $(((\Omega, \mathcal{F}, P; \mathbb{F}), \Upsilon, \xi, \mathcal{N}), m, X_{\Upsilon,m})$ is a relaxed feedback mean field game solution; namely*

$$\mathrm{Flow}(X_{\Upsilon,m}) = \mathrm{Flow}(X_{\rho,m}) = m, \tag{2.5}$$

$$J(\Upsilon, m) = J(\rho, m). \tag{2.6}$$

Proof. The flow $m \in \mathcal{E}$ is fixed and set $X = X_{\rho,m}$. We claim that there exists a measurable function $\Upsilon : [0,T] \times \Sigma \longrightarrow \mathcal{P}(A)$ such that

$$\Upsilon(t, X(t)) = \mathbb{E}[\rho_t | X(t)] \qquad \ell \otimes P\text{-almost every } (t,\omega) \in [0,T] \times \Omega.$$

This holds if and only if

$$\int_A \varphi(t, X(t), a)[\Upsilon(t, X(t))](da) = \mathbb{E}\left[\int_A \varphi(t, X(t), a)\rho_t(da) \middle| X(t)\right] \tag{2.7}$$

for any bounded and measurable $\varphi : [0,T] \times \Sigma \times A \longrightarrow \mathbb{R}$. In order to construct Υ, define the probability measure R on $[0,T] \times \Sigma \times A$ by

$$R(B) := \frac{1}{T}\mathbb{E}\left[\int_0^T \int_A \mathbb{1}_B(t, X(t), a)\rho_t(da)dt\right], \quad B \in \mathcal{B}([0,T] \times \Sigma \times A).$$

Then build Υ by disintegration of R:

$$R(dt, ds, da) = R_1(dt, dx)[\Upsilon(t,x)](da)$$

where R_1 denotes the $[0,T] \times \Sigma$ marginal of R and $\Upsilon : [0,T] \times \Sigma \longrightarrow \mathcal{P}(A)$ is measurable. Following [64], we show that such Υ satisfies (2.7): for every bounded and measurable $\psi : [0,T] \times \Sigma \longrightarrow \mathbb{R}$ we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \psi(t, X(t)) \int_A \varphi(t, X(t), a)[\Upsilon(t, X(t))](da)dt\right] \\
&= T \int_{[0,T]\times\Sigma} \psi(t,x) \int_A \varphi(t,x,a)[\Upsilon(t,x)](da)R_1(dt,dx) \\
&= T \int_{[0,T]\times\Sigma\times A} \psi(t,x)\varphi(t,x,a)R(dt,dx,da) \\
&= \mathbb{E}\left[\int_0^T \psi(t, X(t)) \int_A \varphi(t, X(t), a)\rho_t(da)dt\right],
\end{aligned}$$

which provides (2.7) thanks to Lemma 5.2 in [11]. Having Υ, (2.7) yields

$$\begin{aligned}
&\int_\Theta \int_A \gamma(t, X(t), \theta, a, m(t))[\Upsilon(t, X(t))](da)\nu(d\theta) \\
&\qquad = \mathbb{E}\left[\int_\Theta \int_A \gamma(t, X(t), \theta, a, m(t))\rho_t(da)\nu(d\theta)\middle| X(t)\right]
\end{aligned}$$

$\ell \otimes P$-almost everywhere.

Then we solve equation (1.60) in the same probability space of X, under the relaxed feedback control Υ, and denote by $Y = X_{\Upsilon,m}$ its solution. By the Dynkin formula (1.63), we have for any $g \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}\left[g(X(t))\right] &= \mathbb{E}\left[g(\xi)\right] + \mathbb{E}\left[\int_0^t \int_A \Lambda_s^a g(X(s))\rho_s(da)ds\right] \\
&= \mathbb{E}\left[g(\xi)\right] + \mathbb{E}\left[\mathbb{E}\left[\int_0^t \int_A \Lambda_s^a g(X(s))\rho_s(da)\middle| X(s)\right] ds\right]
\end{aligned}$$

and then thanks to (2.7)

$$\mathbb{E}\left[g(X(t))\right] = \mathbb{E}\left[g(\xi)\right] + \mathbb{E}\left[\int_0^t \int_A \Lambda_s^a g(X(s))[\Upsilon(s, X(s))](da)ds\right], \tag{2.8}$$

while Dynkin's formula for Y yields

$$\mathbb{E}\left[g(Y(t))\right] = \mathbb{E}\left[g(\xi)\right] + \mathbb{E}\left[\int_0^t \int_A \Lambda_s^a g(Y(s))[\Upsilon(s, Y(s))](da)ds\right]. \tag{2.9}$$

Comparing (2.8) and (2.9) we obtain that Law$(X(t))$ and Law$(Y(t))$, which are vectors in $\mathcal{P}(\Sigma) \subset \mathbb{R}^d$, satisfy the same ODE in integral form, namely

$$g \cdot \zeta(t) = g \cdot \mathrm{Law}(\xi) + \int_0^t \int_\Sigma \int_A \Lambda_s^a g(x)[\Upsilon(s,x)](da)[\zeta(t)](dx)ds, \quad t \in [0,T],$$

for any $g \in \mathbb{R}^d$, the unknown being denoted by $\zeta : [0,T] \to \mathcal{P}(\Sigma)$. Taking $g = e_j$, $j = 1, \dots, d$, the corresponding system of ODEs, which is clearly linear in π, has a unique absolutely continuous solution $\zeta \in \mathcal{E}$, hence (2.5) is proved.

Similarly, (2.7) gives

$$\begin{aligned} J(\rho, m) &= \mathbb{E}\left[\int_0^T \int_A c(t, X(t), a, m(t))\rho_t(da)dt + G(X(T), m(T))\right] \\ &= \mathbb{E}\left[\int_0^T \int_A c(t, X(t), a, m(t))[\Upsilon(s, X(s))](da)dt + G(X(T), m(T))\right] \end{aligned}$$

and then we use (2.5) to conclude that

$$\begin{aligned} J(\rho, m) &= \mathbb{E}\left[\int_0^T \int_A c(t, Y(t), a, m(t))[\Upsilon(s, Y(s))](da)dt + G(Y(T), m(T))\right] \\ &= J(\Upsilon, m). \end{aligned}$$

□

2.2 Feedback mean field game solutions

2.2.1 Feedback optimal control for m fixed

We show the existence of an optimal non-relaxed feedback control α_m for $J(\pi, m)$ for any m, using the verification theorem for the related *Hamilton-Jacobi-Bellman* equation. Let $m \in \mathcal{E}$ be fixed.

For any $t \in [0,T]$, $x \in \Sigma$ and $\pi \in \mathcal{S}$ let $X_\pi^{t,x}$ be the solution to

$$X_\pi^{t,x}(s) = x + \int_t^s \int_\Theta \gamma(r^-, X_\pi^{t,x}(r^-), \theta, \pi(r), m(r))\mathcal{N}(dr, d\theta) \tag{2.10}$$

and set

$$J(t, x, \pi, m) := \mathbb{E}\left[\int_t^T c(s, X_\pi^{t,x}(s), \pi(s), m(s))ds + G(X_\pi^{t,x}(T), m(T))\right].$$

Next, define the *value function* by

$$V_m(t, x) := \inf_{\pi \in \mathcal{S}} J(t, x, \pi, m). \tag{2.11}$$

Recall that the generator was defined in (1.3) by

$$\Lambda_t^{a,m} g(x) := \int_\Theta [g(x + \gamma(t, x, \theta, a, m)) - g(x)]\nu(d\theta)$$

for any t, x, a, m and $g \in \mathbb{R}^d$. For a function $v = v(t, x)$ the generator will be applied to the space variable, i.e. denote $\Lambda_t^{a,m} v(t, x) = \Lambda_t^{a,m} v(t, \cdot)(x)$.

Thanks to Theorem D.5 in [53] on measurable selectors, there exists a feedback control $\alpha_m \in \mathcal{A}$ (i.e. measurable) such that

$$\alpha_m(t, x) \in \operatorname{argmin}_{a \in A} \left\{\Lambda_t^{a,m(t)} V_m(t, x) + c(t, x, a, m(t))\right\}, \tag{2.12}$$

where V_m is the value function (2.11). Let us remark that the above minimum exists for any t and x if (A) and (B) hold, as the right hand side turns out to be a continuous function of the variable a, since the value function is trivially Lipschitz continuous in x.

Theorem 2.6. *Assume (A) and (B). Let $m \in \mathcal{E}$. Then any feedback control α_m defined by (2.12) is optimal, that is, $J(\alpha_m, m) \leq J(\pi, m)$ for any $\pi \in \mathcal{S}$.*

In order to prove Theorem 2.6, we use the *Hamilton-Jacobi-Bellman* equation of the problem (see, for instance, Chapter 3 in [43]):

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) + \inf_{a \in A} \left\{ \Lambda_t^{a,m(t)} v(t,x) + c(t,x,a,m(t)) \right\} = 0 & \text{in } [0,T[\times\Sigma \\ v(T,x) = G(x, m(T)) & \text{in } \Sigma \end{cases} \tag{2.13}$$

for a function $v \colon [0,T] \times \Sigma \to \mathbb{R}$. Using the definition of the Hamiltonian $\mathfrak{H}$ in (1.5), (2.13) can be written as

$$\begin{cases} \frac{d}{dt} v(t,x) + \mathfrak{H}(t,x,v(t)) = 0, & \text{in } [0,T[\times\Sigma \\ v(T,x) = G(x), & \text{in } \Sigma \end{cases} \tag{2.14}$$

which is in fact an ODE indexed by $x \in \Sigma$, where $v(t)$ denotes the vector $(v(t,1), \ldots, v(t,d))$.

Define a *classical solution* to (2.14) as an absolutely continuous function v from $[0,T]$ to $\mathbb{R}^d$ such that, in vector form, $v(t) = G + \int_t^T \mathfrak{H}(s, v(s))ds$ for every $t \in [0,T]$. We apply to our problem the following verification theorem, which is a version of Theorem 3.8.1 in [43, p. 135]:

Proposition 2.7 (Verification)**.** *Let v be a classical solution to (2.14), and let α_m be any feedback control such that (2.12) holds for Lebesgue almost every t. Then*

$$v(t,x) = J(t,x,\alpha_m,m) = V_m(t,x)$$

for any $t \in [0,T]$ and $x \in \Sigma$, where V_m is the value function (2.11).

We are now in the position to prove Theorem 2.6.

Proof of Theorem 2.6. In view of Proposition 2.7, we have just to show that there exists a classical solution to (2.14). Hence it is enough to prove that $\mathfrak{H} = \mathfrak{H}(t,g)$ is globally Lipschitz continuous in $g \in \mathbb{R}^d$, uniformly in $t \in [0,T]$. So let t be fixed and take $g, \widetilde{g} \in \mathbb{R}^d$ and $x \in \Sigma$. Recall that

$$\mathfrak{H}(t,x,g) := \min_{a \in A} \left\{ \int_\Theta [g(x + \gamma(t,x,\theta,a,m(t))) - g(x)] \nu(d\theta) + c(t,x,a,m(t)) \right\},$$

and let $\widetilde{a}$ be a minimizer for $\mathfrak{H}(t,x,\widetilde{g})$. Then

$$\begin{aligned} \mathfrak{H}(t,x,g) - \mathfrak{H}(t,x,\widetilde{g}) &= \min_{a \in A} \left\{ \int_\Theta [g(x + \gamma(t,x,\theta,a,m(t))) - g(x)] \nu(d\theta) + c(t,x,a,m(t)) \right\} \\ &\quad - \int_\Theta [\widetilde{g}(x + \gamma(t,x,\theta,\widetilde{a},m(t))) - \widetilde{g}(x)] \nu(d\theta) - c(t,x,\widetilde{a},m(t)) \\ &\leq \int_\Theta [g(x + \gamma(t,x,\theta,\widetilde{a},m(t))) - g(x)] \nu(d\theta) - \int_\Theta [\widetilde{g}(x + \gamma(t,x,\theta,\widetilde{a},m(t))) - \widetilde{g}(x)] \nu(d\theta) \\ &\leq \int_\Theta |g(x + \gamma(t,x,\theta,\widetilde{a},m(t))) - g(x) - \widetilde{g}(x + \gamma(t,x,\theta,\widetilde{a},m(t))) + \widetilde{g}(x)| \, \nu(d\theta) \\ &= \int_\Theta |(g - \widetilde{g})(x + \gamma(t,x,\theta,\widetilde{a},m(t))) - (g - \widetilde{g})(x)| \, \nu(d\theta) \leq 2\nu(\Theta) \max_{y \in \Sigma} |(g - \widetilde{g})_y| \,. \end{aligned}$$

Changing the role of g and $\widetilde{g}$ we obtain the converse, hence

$$|\mathfrak{H}(t,x,g) - \mathfrak{H}(t,x,\widetilde{g})| \leq 2\nu(\Theta) \max_{y \in \Sigma} |g(y) - \widetilde{g}(y)|$$

for any x, which implies

$$\max_{x\in\Sigma} |\mathfrak{H}(t,x,g) - \mathfrak{H}(t,x,\widetilde{g})| \leq 2\nu(\Theta) \max_{y\in\Sigma} |g(y) - \widetilde{g}(y)| .$$

Therefore $\mathfrak{H}$ is Lipschitz continuous in g in the norm $|\cdot|_\infty$, which is equivalent to the Euclidean norm in $\mathbb{R}^d$. □

2.2.2 Uniqueness of the feedback control for m fixed

Consider the *pre-Hamiltonian*, as defined in (1.4),

$$\mathfrak{f}(t,x,a,m,g) := \int_\Theta [g(x+\gamma(t,x,\theta,a,m)) - g(x)]\,\nu(d\theta) + c(t,x,a,m)$$

for $(t,x,a,m) \in [0,T]\times\Sigma\times A\times\mathcal{P}(\Sigma)$ and $g\in\mathbb{R}^d$. We make the additional assumption (C); so let us recall that $\mathfrak{a}^*(t,x,m,g)$ is the unique minimizer of $\mathfrak{f}(t,x,a,m,g)$ in $a\in A$. Define for $m\in\mathcal{E}$ the feedback control

$$\alpha_m(t,x) := \mathfrak{a}^*(t,x,m(t),V_m(t,\cdot)) \tag{2.15}$$

where V_m is the value function (2.11).

Theorem 2.8. *Assume (A), (B) and (C). Given $m\in\mathcal{E}$, let $\sigma\in\mathcal{R}$ be any optimal relaxed control for m and let $X_{\sigma,m}$ be the corresponding solution to (1.58). Then $\sigma_t = \delta_{\alpha_m(t,X_{\sigma,m}(t))}$ for $\ell\otimes P$-almost every (t,ω), that is, σ corresponds to the feedback control α_m.*

This result and the proof of Theorem 2.4 imply that any relaxed solution of the mean field game must correspond to a feedback solution:

Corollary 2.9. *Assume (A), (B) and (C). Then there exists a feedback solution (α,m,X) of the mean field game, and any solution is such that its control coincides with α_m.*

Let $Q\in\mathcal{P}(A)$, and define

$$\widehat{\mathfrak{f}}(t,x,Q,m,g) := \int_A \mathfrak{f}(t,x,a,m,g)Q(da).$$

Lemma 2.10. *If $\mathfrak{f}$ is continuous in a, then*

$$\min_{Q\in\mathcal{P}(A)} \widehat{\mathfrak{f}}(t,x,Q,m,g) = \min_{a\in A}\mathfrak{f}(t,x,a,m,g) \tag{2.16}$$

for any t,x,m and g. Moreover, if (C) holds, then there exists a unique $Q^*\in\mathcal{P}(A)$ such that

$$\widehat{\mathfrak{f}}(t,x,Q^*,m,g) = \min_{Q\in\mathcal{P}(A)}\widehat{\mathfrak{f}}(t,x,Q,m,g) = \min_{a\in A}\mathfrak{f}(t,x,a,m,g) = \mathfrak{f}(t,x,\mathfrak{a}^*,m,g)$$

and $Q^* = \delta_{\mathfrak{a}^*}$, where $\mathfrak{a}^* = \mathfrak{a}^*(t,x,m,g)$.

Proof. If $\mathfrak{f}$ is continuous in a, then $\widehat{\mathfrak{f}}$ is continuous in $Q\in\mathcal{P}(A)$ in the weak topology. Since $\mathcal{P}(A)$ is compact, there exists a minimum: let Q^* be a minimizer. For fixed t,x,m and g we have

$$\min_{Q\in\mathcal{P}(A)}\widehat{\mathfrak{f}}(t,x,Q,m,g) \leq \min_{Q=\delta_a, a\in A}\int_A \mathfrak{f}(t,x,a,m,g)Q(da) = \min_{a\in A}\mathfrak{f}(t,x,a,m,g)$$

and

$$\min_{Q\in\mathcal{P}(A)}\int_A \mathfrak{f}(t,x,a,m,g)Q(da) \geq \min_{Q\in\mathcal{P}(A)}\int_A \mathfrak{f}(t,x,\mathfrak{a}^*,m,g)Q(da) = \mathfrak{f}(t,x,\mathfrak{a}^*,m,g),$$

which means that $\widehat{\mathfrak{f}}(t,x,Q^*,m,g) = \mathfrak{f}(t,x,\mathfrak{a}^*,m,g)$.

Consider $\mathfrak{f}(t,x,a,m,g) - \mathfrak{f}(t,x,\mathfrak{a}^*,m,g)$ as a function of a: it is non-negative and, if (C) holds, it equals zero if and only if $a = \mathfrak{a}^*$. Therefore,

$$0 = \widehat{\mathfrak{f}}(t,x,Q^*,m,g) - \mathfrak{f}(t,x,\mathfrak{a}^*,m,g) = \int_A \left[\mathfrak{f}(t,x,a,m,g) - \mathfrak{f}(t,x,\mathfrak{a}^*,m,g)\right] Q^*(da),$$

which implies the claim, namely that $Q^*(\{\mathfrak{a}^*\}) = 1$. □

Remark 2.11. *Note that if (C) does not hold, then Q^* is supported on the set of all minimizers of $\mathfrak{f}$. Thus it might not be a Dirac measure. This implies that there may exist an optimal relaxed control which is not an ordinary control (not even open-loop).*

Proof of Theorem 2.8. Fix $m \in \mathcal{E}$. Let $\sigma \in \mathcal{R}$ be an optimal relaxed control and denote by $X_\sigma = X_{\sigma,m}$ the corresponding optimal trajectory. By the chattering lemma, which we will state later as Lemma 2.12[1],

$$\begin{aligned}\mathbb{E}\left[V(0,X_\sigma(0))\right] &= \min_{\rho\in\mathcal{R}} J(\rho,m) = J(\sigma,m)\\ &= \mathbb{E}\left[\int_0^T\int_A c(t,X_\sigma(t),a,m(t))\sigma_t(da)dt + G(X_\sigma(T),m(T))\right],\end{aligned}$$

where $V = V_m$ is the value function defined in (2.11). Thanks to (2.13), the Hamilton-Jacobi-Bellman equation, and (2.16), we have

$$\frac{\partial}{\partial t}V(t,x) + \widehat{\mathfrak{f}}(t,x,\sigma_t,V(t,\cdot)) \geq 0 \quad \text{for all } t,x,\omega. \tag{2.17}$$

By the Dynkin formula (1.63) and the terminal condition for V,

$$\begin{aligned}\mathbb{E}\left[V(0,X_\sigma(0))\right] &= \mathbb{E}\left[V(T,X_\sigma(T))\right]\\ &\quad - \mathbb{E}\left[\int_0^T\left(\frac{\partial}{\partial t}V(t,X_\sigma(t)) + \int_A \Lambda_t^{a,m(t)}V(t,X_\sigma(t))\sigma_t(da)\right)dt\right]\\ &= \mathbb{E}\left[\int_0^T\int_A c(t,X_\sigma(t),a,m(t))\sigma_t(da)dt + G(X_\sigma(T),m(T))\right]\\ &\quad - \mathbb{E}\left[\int_0^T \frac{\partial}{\partial t}V(t,X_\sigma(t)) + \widehat{\mathfrak{f}}(t,X_\sigma(t),\sigma_t,V(t,\cdot))dt\right].\end{aligned}$$

It follows that

$$\mathbb{E}\left[\int_0^T \frac{\partial}{\partial t}V(t,X_\sigma(t)) + \widehat{\mathfrak{f}}(t,X_\sigma(t),\sigma_t,V(t,\cdot))dt\right] = 0,$$

hence, in view of (2.17),

$$\frac{\partial}{\partial t}V(t,X_\sigma(t)) + \widehat{\mathfrak{f}}(t,X_\sigma(t),\sigma_t,V(t,\cdot)) = 0$$

[1] Here only the open-loop part of the chattering lemma is needed, which is well known, and so we postpone the proof of the lemma to Section 5, where we also give the feedback part.

for $\ell \otimes P$-almost every (t,ω), which means that

$$\sigma_t \in \operatorname{argmin}_{Q\in\mathcal{P}(A)} \widehat{\mathfrak{f}}(t, X_{\sigma,m}(t), \sigma_t, V(t,\cdot))$$

for $\ell \otimes P$-almost every (t,ω). If (C) holds, then, by Lemma 2.10, the unique minimizer of $Q \mapsto \widehat{\mathfrak{f}}(t,x,Q,m(t),V(t,\cdot))$ is the measure $Q^* = \delta_{\mathfrak{a}^*} \in \mathcal{P}(A)$ with $\mathfrak{a}^* = \mathfrak{a}^*(t,x,m(t),V(t,\cdot))$. It follows that $\sigma_t = \delta_{\alpha_m(t,X_\sigma(t))}$ for $\ell \otimes P$-almost every (t,ω).

□

2.3 Approximation of N-player game

2.3.1 Approximation of relaxed controls

In order to get an ε-Nash equilibrium for the N-player game in open-loop strategies, respectively in feedback strategies, we have first to find an approximation of the optimal relaxed control, respectively relaxed feedback control, for the mean field game. To this end, we will make use of the following version of the chattering lemma.

Lemma 2.12 (Chattering). *For any relaxed control $\rho \in \mathcal{R}$, there exists a sequence of stochastic open-loop controls $\pi_n \in \mathcal{S}$ such that, denoting by $\rho^{\pi_n}(dt,da) = \delta_{\pi_n(t)}(da)dt$ their relaxed control representation,*

$$\lim_{n\to\infty} \rho^{\pi_n} = \rho \quad P\text{-a.s.},$$

where the limit is in the weak topology in $\mathcal{M}([0,T]\times A)$. Moreover, any π_n takes values in a finite subset of A.

For any relaxed feedback control $\Upsilon \in \mathcal{Y}$, there exists a sequence of feedback controls $\alpha_n \in \mathcal{A}$ such that

$$\lim_{n\to\infty} \delta_{\alpha_n(t,x)}(da)dt = [\Upsilon(t,x)](da)dt \tag{2.18}$$

uniformly in $x \in \Sigma$ and

$$\lim_{n\to\infty} \rho^{\alpha_n} = \rho^{\Upsilon} \quad \text{in distribution}, \tag{2.19}$$

where ρ^{α_n} denotes the relaxed control representation of the open-loop control π^{α_n} corresponding to α^n, as in (1.48), and ρ^{Υ} is defined in (1.61); i.e. $\rho^{\alpha_n}_t(da) = \delta_{\alpha_n(t,X_{\alpha_n}(t^-))}(da)$ and $\rho^{\Upsilon}_t(da) = [\Upsilon(t,X_\Upsilon(t^-))](da)$.

Proof. The first part is proved as Theorem 3.5.2 in [61, p. 59], and the construction of the approximating sequence in the proof gives the α_n for the second part; let us show how to build them. Let $\Upsilon \in \mathcal{Y}$, cover A by M_r disjoint sets C^r_i which contain a point a^r_i and set $A^r := \{a^r_i : i \le M_r\}$, a finite subset of A. For any $\Delta > 0$ and i,j define the function

$$\tau^{\Delta r}_{ij}(x) := \int_{i\Delta}^{(i+1)\Delta} [\Upsilon(s,x)](C^r_j)ds.$$

Divide any interval $[(i+1)\Delta,(i+2)\Delta[$ into M^r subintervals $I^{\Delta r}_{ij}(x)$ of length $\tau^{\Delta r}_{ij}(x)$ and define the feedback control $\alpha^{\Delta r}$, which is piecewise constant, by

$$\alpha^{\Delta r}(t,x) := \begin{cases} a_0 & t\in[0,\Delta[\\ a^r_j & t \in I^{\Delta r}_{ij}(x) \end{cases}$$

where a_0 is an arbitrary value in A. The proof in [61] shows that

$$\lim_{\substack{r\to 0\\ \Delta\to 0}} \delta_{\alpha^{\Delta^r}(t,x)}(da)dt = [\Upsilon(t,x)](da)dt$$

weakly, for any $x \in \Sigma$. Since Σ is finite we obtain that there exists a sequence of ordinary feedback controls (α_n) such that (2.18) holds uniformly in x. Let $m \in \mathcal{E}$ be fixed and X_n be the solution to (1.47) corresponding to the feedback control α_n. By Theorem 2.1, the sequence X_n is tight and there are a subsequence, which we still denote as (X_n), and a process X such that $\lim_{n\to\infty} X_n = X$ in distribution. Possibly applying the Skorokhod representation (Theorem 4.30 in [59], p.79), we may assume that this convergence is with probability one in the space of càdlàg functions $D([0,T],\Sigma)$ equipped with the Skorokhod metric. This implies in particular that

$$P\left(\lim_{n\to\infty} X_n(t) = X(t) \text{ for any } t \notin E\right) = 1, \tag{2.20}$$

where E is the finite random set of discontinuity points (the jumps) of X.

Let now $\varphi \in \mathcal{C}([0,T]\times A)$ be any continuous function, which is also bounded as A is compact. We have to show the convergence to zero, almost surely, of

$$\int_0^T \int_A \varphi(t,a)[\delta_{\alpha_n(t,X_n(t^-))} - \Upsilon(t,X(t^-))](da)dt = Y_n + Z_n,$$

where

$$Y_n = \int_0^T \left[\varphi(t,\alpha_n(t,X_n(t^-))) - \varphi(t,\alpha_n(t,X(t^-)))\right] dt$$
$$Z_n = \int_0^T \int_A \varphi(t,a)[\delta_{\alpha_n(t,X(t^-))} - \Upsilon(t,X(t^-))](da)dt.$$

Any feedback control is Lipschitz in x, i.e. $\mathrm{d}_A(\alpha_n(t,x),\alpha_n(t,y)) \le \mathrm{Diam}(A)|x-y|$, and so Y_n tends to zero thanks to (2.20), the continuity of φ and dominated convergence. As to Z_n, write $Z_n = \sum_{x\in\Sigma} Z_n^x$ where

$$Z_n^x := \int_0^T \int_A \mathbb{1}_{B_x}(t)\varphi(t,a)[\delta_{\alpha_n(t,x)} - \Upsilon(t,x)](da)dt$$

and B_x is the random set in $[0,T]$ where $X(t) = x$. For each x, the random set D_x of discontinuity points of the function $\mathbb{1}_{B_x}(t)\varphi(t,a)$ is a subset of $E_x \times A$ for some finite random set $E_x \subset [0,T]$. Thus D_x has null measure with respect to the limiting control $\Upsilon(t,x)(da)dt$ with probability one, for each x, thanks to Definition 1.11. Hence by (2.18) we get that Z_n^x tends to zero for each x and so does Z_n since Σ is finite.

Let $\pi_n(t) = \alpha_n(t,X_n(t^-))$ be the open-loop control corresponding to α_n and ρ^n its relaxed control representation. We have just proved that $\lim_{n\to\infty}\rho^n = [\Upsilon(t,X(t^-))](da)dt$ P-almost surely and thus Theorem 2.1 says that X must have the same law as the solution to (1.60) under the relaxed feedback control Υ. That solution is unique by Lemma 1.13, meaning that $X = X_\Upsilon$ in distribution. Therefore (2.19) follows since $\rho_t^\Upsilon = \Upsilon(t,X_\Upsilon(t^-))$ by (1.61). □

Remark 2.13. *In the above proof we strongly used the finiteness of Σ to get the approximation in feedback controls. While the result in the open-loop setting holds for general state space Σ, when considering feedback controls it is not clear whether the above lemma can be generalized to uncountably infinite state spaces.*

We are now able to state the approximation result:

Proposition 2.14. *Let $m \in \mathcal{E}$, $\rho \in \mathcal{R}$ and $\Upsilon \in \mathcal{Y}$. Then for every $\varepsilon > 0$ there exist $\pi \in \mathcal{S}$ and $\alpha \in \mathcal{A}$ such that*

$$\mathbb{E}\left[\sup_{t\geq 0} |X_{\pi,m}(t) - X_{\rho,m}(t)|\right] \leq \varepsilon \tag{2.21}$$

$$\mathbb{E}\left[\sup_{t\geq 0} |X_{\alpha,m}(t) - X_{\Upsilon,m}(t)|\right] \leq \varepsilon \tag{2.22}$$

$$|J(\pi,m) - J(\rho,m)| \leq \varepsilon. \tag{2.23}$$

$$|J(\alpha,m) - J(\Upsilon,m)| \leq \varepsilon. \tag{2.24}$$

Proof. Let (π_n) be a sequence in $\mathcal{S}$ that approximates ρ as in Lemma 2.12. Then we apply Theorem 2.1 to the sequence $(X_{\pi_n,m}, \pi_n, m)$: it is tight, a subsequence $(X_{\pi_{n_k},m}, \pi_{n_k}, m)$ converges in distribution to $(X_{\rho,m}, \rho, m)$ and $\lim_{k\to\infty} J(\pi_{n_k}, m) = J(\rho, m)$. Thus there exist $\pi_{n_k} =: \pi$ for which (2.21) and (2.23) hold. In a similar way, one proves (2.22) and (2.24) for feedback controls. □

2.3.2 ε_N-Nash equilibria

We can now define the approximate Nash equilibrium for the N-player game, first in open-loop form.

Notation 2.15. *Let $\left(((\Omega,\mathcal{F},P;\mathbb{F}),\pi,\xi,\mathcal{N}), m, X_{\rho,m}\right)$ be a relaxed solution of the mean field game, which exists assuming (A) and (B) by Theorem 2.4. Fix $N \in \mathbb{N}$ and let $\pi \in \mathcal{S}$ be as in Proposition 2.14, satisfying (2.21) and (2.23) with $\varepsilon = \frac{1}{\sqrt{N}}$. Then $((\Omega',\mathcal{F}',P';\mathbb{F}'),\boldsymbol{\pi},\boldsymbol{\xi},\underline{\mathcal{N}})$ denotes the strategy vector where $\boldsymbol{\pi} = (\pi^1,\ldots,\pi^N)$, $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_N)$, $\underline{\mathcal{N}} = (\mathcal{N}_1,\ldots,\mathcal{N}_N)$, such that*

$$\mathrm{Law}\left((\pi^1,\xi_1,\mathcal{N}_1),\ldots,(\pi^N,\xi_N,\mathcal{N}_N)\right) = (\mathrm{Law}(\pi,\xi,\mathcal{N}))^{\otimes N}. \tag{2.25}$$

Such a control exists by considering, for instance, Ω' and P' to be the product space and the product measure, respectively, with $(\pi^i,\xi_i,\mathcal{N}_i)(\omega') := (\pi,\xi,\mathcal{N})(\omega_i)$ for $\omega' = (\omega_1,\ldots,\omega_N)$.

Equation (2.25) says that this control is symmetric. The following is our main result, whose proof is carried out in the next subsection. In addition to (A) and (B), we make the Lipschitz assumptions (A') and (B').

Theorem 2.16. *Assume (A') and (B'). Then the strategy vector defined in Notation 2.15 is an ε_N-Nash equilibrium for the N-player game for any N where $\varepsilon_N \leq \frac{C}{\sqrt{N}}$ and $C = C(T,d,\nu(\Theta),K_1,K_2)$ is a constant.*

An analogous result holds when considering feedback strategies, but we state it separately.

Notation 2.17. *Let $\left(((\Omega,\mathcal{F},P;\mathbb{F}),\Upsilon,\xi,\mathcal{N}), m, X_{\Upsilon,m}\right)$ be a relaxed feedback solution of the mean field game, which exists assuming (A) and (B) by Theorem 2.5. Fix $N \in \mathbb{N}$ and let $\alpha \in \mathcal{A}$ be as in Proposition 2.14, satisfying (2.22) and (2.24) with $\varepsilon = \frac{1}{\sqrt{N}}$. Then the tuple $((\Omega',\mathcal{F}',P';\mathbb{F}'),\boldsymbol{\alpha},\boldsymbol{\xi},\underline{\mathcal{N}})$ denotes the feedback strategy vector where $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_N)$, $\underline{\mathcal{N}} = (\mathcal{N}_1,\ldots,\mathcal{N}_N)$, $\boldsymbol{\alpha} = (\alpha^1,\ldots,\alpha^N)$ such that*

$$\alpha^i(t,x^N) := \alpha(t,x_i) \tag{2.26}$$

for any t, i and $\boldsymbol{x} = (x_1,\ldots,x_N) \in \Sigma^N$, and the $(\xi_i,\mathcal{N}_i)$ are N i.i.d copies of $(\xi,\mathcal{N})$.

Equation (2.26) says that this feedback strategy vector is symmetric and decentralized. In order to obtain feedback ε-Nash equilibria from a mean field game solution, we need the Lipschitz assumptions (A") and (B").

Theorem 2.18. *Assume (A"), (B"). Then the feedback strategy vector defined in Notation 2.17 is a feedback ε_N-Nash equilibrium for the N-player game for any N where $\varepsilon_N \leq \frac{C}{\sqrt{N}}$ and $C = C(T, d, \nu(\Theta), K_1, K_2)$ is a constant.*

2.3.3 Proofs of Theorems 2.16 and 2.18

In the following C will denote any constant which depends on T, d, $\nu(\Theta)$ and the Lipschitz constants K_1 and K_2, but not on N, and is allowed to change from line to line. We focus first on open-loop controls. Fix $N \in \mathbb{N}$ and let the strategy vector $\boldsymbol{\pi}$ be as in Notation 2.15. We play this strategy in the N-player game:

$$X_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, X_i(s^-), \theta, \pi^i(s), m^N(s^-))\mathcal{N}_i(ds, d\theta) \qquad i = 1, \dots, N. \tag{2.27}$$

This will be coupled with $\boldsymbol{Y}$ defined by

$$Y_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, Y_i(s^-), \theta, \pi^i(s), m(s))\mathcal{N}_i(ds, d\theta) \qquad i = 1, \dots, N. \tag{2.28}$$

Let $m^N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}$ be the empirical measure of the system (2.27) and $\overline{m}^N$ be the empirical measure of (2.28). Denote $\overline{m}(t) := \text{Law}(X_{\pi,m}(t))$. By (2.21) we have

$$|\overline{m}(t) - m(t)| \leq \frac{1}{\sqrt{N}} \tag{2.29}$$

for any $t \geq 0$, since $\text{Flow}(X_{\rho,m}) = m$. From (2.25) it follows that

$$\text{Law}(Y_i, \pi^i, \xi_i, \mathcal{N}_i) = \text{Law}(X_{\pi,m}, \pi, \xi, \mathcal{N}), \quad i \in \{1, \dots, N\}.$$

This implies, thanks to Theorem 1 in [44], that

$$\mathbb{E}|\overline{m}^N(t) - \overline{m}(t)| \leq \frac{C}{\sqrt{N}} \tag{2.30}$$

for any $t \in [0, T]$ and $N \in \mathbb{N}$, where C is a constant. This upper bound in $N^{-\frac{1}{2}}$ cannot be improved, since for these discrete measures a lower bound still in $N^{-\frac{1}{2}}$ can be found, see again [44].

Lemma 2.19. *Under assumption (A'), for every $t \geq 0$ and $i = 1, \dots, N$*

$$\mathbb{E}|m^N(t) - m(t)| \leq \frac{C}{\sqrt{N}} \tag{2.31}$$

$$\mathbb{E}|X_i(t) - Y_i(t)| \leq \frac{C}{\sqrt{N}}. \tag{2.32}$$

Proof. From (2.29) and (2.30) it follows that

$$\mathbb{E}|\overline{m}^N(t) - m(t)| \leq \frac{C}{\sqrt{N}}. \tag{2.33}$$

We estimate $|m^N(t) - \overline{m}^N(t)|$ using the (1.1), which follows from the 1-Wasserstein metric, (2.27), (2.28) and the Lipschitz assumption (1.19):

$$\mathbb{E}|m^N(t) - \overline{m}^N(t)| \le \frac{C}{N}\sum_{i=1}^{N}\mathbb{E}|X_i(t) - Y_i(t)|$$
$$\le \frac{C}{N}\sum_{i=1}^{N}\mathbb{E}|X_i(0) - Y_i(0)| + \frac{C}{N}\sum_{i=1}^{N}\int_0^t\int_\Theta \mathbb{E}|\gamma(s, X_i(s), \theta, \pi^i(s), m^N(s^-))$$
$$- \gamma(s, Y_i(s), \theta, \pi^i(s), m(s))|\nu(d\theta)ds$$
$$\le \frac{C}{N}\sum_{i=1}^{N}K_1\int_0^t \Big[\mathbb{E}|X_i(s) - Y_i(s)| + \mathbb{E}|m^N(s) - m(s)|\Big]\,ds.$$

Hence applying (2.33)

$$\mathbb{E}|m^N(t) - m(t)| + \frac{C}{N}\sum_{i=1}^{N}\mathbb{E}|X_i(t) - Y_i(t)|$$
$$\le \mathbb{E}|\overline{m}^N(t) - m(t)| + \frac{2C}{N}\sum_{i=1}^{N}\mathbb{E}|X_i(t) - Y_i(t)|$$
$$\le \frac{C}{\sqrt{N}} + 2K_1\frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big[\mathbb{E}|X_i(s) - Y_i(s)| + \mathbb{E}|m^N(s) - m(s)|\Big]\,ds.$$

Then we obtain, by Gronwall's lemma,

$$\mathbb{E}|m^N(t) - m(t)| + \frac{C}{N}\sum_{i=1}^{N}\mathbb{E}|X_i(t) - Y_i(t)| \le \frac{C}{\sqrt{N}} + 2K_1\int_0^t e^{2K(t-s)}\frac{C}{\sqrt{N}}ds \le \frac{C}{\sqrt{N}}.$$

Similarly we show (2.32): using (2.27), (2.28) and (2.31) we get, for any i,

$$\mathbb{E}|X_i(t) - Y_i(t)|$$
$$\le \int_0^t\int_\Theta \mathbb{E}\left|\gamma(s, X_i(s), \theta, \pi^i(s), m^N(s)) - \gamma(s, Y_i(s), \theta, \pi^i(s), m(s))\right|\nu(d\theta)ds$$
$$\le K_1\int_0^t \Big[\mathbb{E}|X_i(s) - Y_i(s)| + \mathbb{E}|m^N(s) - m(s)|\Big]\,ds$$
$$\le K_1\int_0^t \left[\mathbb{E}|X_i(s) - Y_i(s)| + \frac{C}{\sqrt{N}}\right]ds$$

and hence $\mathbb{E}|X_i(t) - Y_i(t)| \le \frac{C}{\sqrt{N}}$ by Gronwall's lemma. □

We are now in the position to state the result about the costs. Because of the symmetry of the problem, for the prelimit we shall consider only player one ($i = 1$).

Lemma 2.20. *Under assumptions (A') and (B')*

$$|J_1(\boldsymbol{\pi}) - J(\rho, m)| \le \frac{C}{\sqrt{N}}. \tag{2.34}$$

Proof. Inequality (2.23), together with notation 2.15, yields

$$|J(\pi, m) - J(\rho, m)| \leq \frac{C}{\sqrt{N}}. \tag{2.35}$$

While from (1.21), (2.31) and (2.32) we have

$$\begin{aligned} |J_1(\boldsymbol{\pi}) - J(\pi, m)| &\leq \mathbb{E}|G(X_1(T), m^N(T)) - G(Y_1(T), m(T))| \\ &\quad + \mathbb{E}\int_0^T |c(t, X_1(t), \pi^1(t), m^N(t)) - c(t, Y_1(t), \pi^1(t), m(t))| dt \\ &\leq K_2 \int_0^T \left[\mathbb{E}|X_1(t) - Y_1(t)| + \mathbb{E}|m^N(t) - m(t)|\right] dt \\ &\quad + K_2 \left[\mathbb{E}|X_1(T) - Y_1(T)| + \mathbb{E}|m^N(T) - m(T)|\right] \\ &\leq K_2 T \frac{C}{\sqrt{N}} + K_2 \frac{C}{\sqrt{N}} \leq \frac{C}{\sqrt{N}}, \end{aligned}$$

which, combined with (2.35), gives the claim. □

We consider then any $\widetilde{\pi} \in \mathcal{S}$ and the perturbed strategy vector $[\pi^{N,-1}, \widetilde{\pi}]$. We denote by $\widetilde{\boldsymbol{X}}$ the solution to

$$\widetilde{X}_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_i(s^-), \theta, [\pi^{N,-1}, \widetilde{\pi}]_i(s), \widetilde{m}^N(s^-))\mathcal{N}_i(ds, d\theta) \tag{2.36}$$

for each $i = 1, \ldots, N$. Set also $\widetilde{Y}_1 := X_{\widetilde{\pi},m}$ and $\widetilde{m}^N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{X}_i(t)}$.

Lemma 2.21. *Under assumption (A'), for any $t \geq 0$ and $\widetilde{\pi} \in \mathcal{S}$*

$$\mathbb{E}|m^N(t) - \widetilde{m}^N(t)| \leq \frac{C}{N} \tag{2.37}$$

$$\mathbb{E}|\widetilde{m}^N(t) - m(t)| \leq \frac{C}{\sqrt{N}} \tag{2.38}$$

$$\mathbb{E}|\widetilde{X}_1(t) - \widetilde{Y}_1(t)| \leq \frac{C}{\sqrt{N}}. \tag{2.39}$$

Proof. We make the rough estimate

$$\begin{aligned} \mathbb{E}|m^N(t) - \widetilde{m}^N(t)| &\leq \frac{1}{N}\mathbb{E}|X_1(t) - \widetilde{X}_1(t)| + \frac{1}{N}\sum_{i=2}^N \mathbb{E}|X_i(t) - \widetilde{X}_i(t)| \\ &\leq \frac{d}{N} + \frac{1}{N}\sum_{i=2}^N \int_0^t \int_\Theta \mathbb{E}|\gamma(s, X_i(s), \theta, \pi^i(s), m^N(s)) \\ &\qquad - \gamma(s, \widetilde{X}_i(s), \theta, \pi^i(s), \widetilde{m}^N(s))|\nu(d\theta)ds \\ &\leq \frac{d}{N} + \frac{1}{N}\sum_{i=2}^N K_1 \int_0^t \left[\mathbb{E}|X_i(s) - \widetilde{X}_i(s)| + \mathbb{E}|m^N(s) - \widetilde{m}^N(s)|\right] ds. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E}|m^N(t) - \widetilde{m}^N(t)| + \frac{1}{N}\sum_{i=2}^N \mathbb{E}|X_i(t) - \widetilde{X}_i(t)| \\ &\qquad \leq \frac{d}{N} + 2K_1 \frac{1}{N}\sum_{i=2}^N \int_0^t \left[\mathbb{E}|X_i(s) - \widetilde{X}_i(s)| + \mathbb{E}|m^N(s) - \widetilde{m}^N(s)|\right] ds \end{aligned}$$

and then, by Gronwall's lemma,

$$\mathbb{E}|m^N(t) - \widetilde{m}^N(t)| + \frac{1}{N}\sum_{i=2}^{N}\mathbb{E}|X_i(t) - \widetilde{X}_i(t)| \le \frac{d}{N}e^{2K_1T} \le \frac{C}{N}.$$

Therefore (2.37) is proved. Estimate (2.38) follows from (2.37) and (2.31) and the fact that $\frac{1}{N} \le \frac{1}{\sqrt{N}}$ for any $N \in \mathbb{N}$. While (2.39) is a consequence of (2.38):

$$\begin{aligned}
&\mathbb{E}|\widetilde{X}_1(t) - \widetilde{Y}_1(t)| \\
&\le \int_0^t \int_\Theta \mathbb{E}\left|\gamma(s, \widetilde{X}_1(s), \theta, \widetilde{\pi}(s), \widetilde{m}^N(s)) - \gamma(s, \widetilde{Y}_1(s), \theta, \widetilde{\pi}(s), m(s))\right| \nu(d\theta)ds \\
&\le K_1 \int_0^t \left[\mathbb{E}|\widetilde{X}_1(s) - \widetilde{Y}_1(s)| + \mathbb{E}|m^N(s) - m(s)|\right] ds \\
&\le K_1 \int_0^t \left[\mathbb{E}|\widetilde{X}_1(s) - \widetilde{Y}_1(s)| + \frac{C}{\sqrt{N}}\right] ds
\end{aligned}$$

and we conclude by Gronwall's lemma. □

Lemma 2.22. *Under assumptions (A') and (B')*

$$|J_1([\boldsymbol{\pi}^{-1}, \widetilde{\pi}]) - J(\widetilde{\pi}, m)| \le \frac{C}{\sqrt{N}}. \tag{2.40}$$

Proof. Inequalities (1.21), (2.38) and (2.39) give

$$\begin{aligned}
|J_1([\boldsymbol{\pi}^{-1}, \widetilde{\pi}]) - J(\widetilde{\pi}, m)| \le& \ \mathbb{E}|G(\widetilde{X}_1(T), \widetilde{m}^N(T)) - G(\widetilde{Y}_1(T), m(T))| \\
&+ \mathbb{E}\int_0^T |c(t, \widetilde{X}_1(t), \widetilde{\pi}(t), \widetilde{m}^N(t)) - c(t, \widetilde{Y}_1(t), \widetilde{\pi}(t), m(t))| dt \\
\le& K_2 \int_0^T \left[\mathbb{E}|\widetilde{X}_1(t) - \widetilde{Y}_1(t)| + \mathbb{E}|\widetilde{m}^N(t) - m(t)|\right] dt \\
&+ K_2\left[\mathbb{E}|\widetilde{X}_1(T) - \widetilde{Y}_1(T)| + \mathbb{E}|\widetilde{m}^N(T) - m(T)|\right] \\
\le& K_2 T\frac{C}{\sqrt{N}} + K_2\frac{C}{\sqrt{N}} \le \frac{C}{\sqrt{N}}.
\end{aligned}$$

□

Theorem 2.16 is now a consequence of Lemmata 2.20 and 2.22:

Proof of Theorem 2.16. Inequalities (2.34), (2.40), and the optimality of ρ yield

$$J_1(\boldsymbol{\pi}) \le J(\rho, m) + \frac{C}{\sqrt{N}} \le J(\widetilde{\pi}, m) + \frac{C}{\sqrt{N}} \le J_1([\boldsymbol{\pi}^{-1}, \widetilde{\pi}]) + \frac{C}{\sqrt{N}}.$$

□

Remark 2.23. *We observe that $\boldsymbol{\pi}$ is still an ε_N-Nash equilibrium if we assume only (B) instead of (B'), but without the estimate of the order of convergence $\varepsilon_N \le \frac{C}{\sqrt{N}}$. Namely, there exists a sequence (ε_N) such that $\lim_{N\to\infty} \varepsilon_N = 0$.*

Proof of Theorem 2.18. The argument is the same as in the proof of Theorem 2.16. The difference is that equations (2.27), (2.28) and (2.36) become respectively, for each $i = 1, \dots, N$,

$$X_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, X_i(s^-), \theta, \alpha(s, X_i(s^-)), m^N(s^-)) \mathcal{N}_i(ds, d\theta),$$
$$Y_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, Y_i(s^-), \theta, \alpha(s, Y_i(s^-)), m(s)) \mathcal{N}_i(ds, d\theta)$$

and

$$\widetilde{X}_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_i(s^-), \theta, [\boldsymbol{\alpha}^{-1}, \widetilde{\pi}]_i(s), \widetilde{m}^N(s^-)) \mathcal{N}_i(ds, d\theta),$$

where the latter means that

$$\widetilde{X}_1(t) = \xi_1 + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_1(s^-), \theta, \widetilde{\pi}(s), \widetilde{m}^N(s^-)) \mathcal{N}_1(ds, d\theta)$$

and

$$\widetilde{X}_i(t) = \xi_i + \int_0^t \int_\Theta \gamma(s, \widetilde{X}_i(s^-), \theta, \alpha(s, \widetilde{X}_i(s^-)), \widetilde{m}^N(s^-)) \mathcal{N}_i(ds, d\theta)$$

for $i = 2, \dots, N$, thanks to Notation 1.4. The estimates we need to apply Gronwall's lemma, in particular in the proof of Lemma 2.21, are found using also (1.20) and the fact that $\mathrm{d}_A(\alpha(s,x), \alpha(s,y)) \leq \mathrm{Diam}(A)|x - y|$ for every s and each x and y in the finite Σ. □

CHAPTER 3

Convergence under uniqueness: the master equation

We present here the results which hold when there is uniqueness of mean field game solutions. We begin by showing the convergence of the value functions and the propagation of chaos property for the optimal trajectories. Then we employ the convergence argument to derive the asymptotic behaviour of the empirical measure processes, that is, the Central Limit Theorem and the Large Deviation Principle. After that, we analyze the well-posedness and regularity of the solution to the master equation, which is required for the above results. These results were presented in [27], but under the more restrictive condition that the transition rates are bounded below away from zero (Assumption (Erg)). Here we manage to prove, in particular, the convergence results without requiring this assumption. In the last section, we prove uniqueness of feedback mean field game solutions under slightly more general assumptions, either for T small enough or under monotonicity, and show that the strategy vector given by the Nash system is the unique feedback Nash equilibrium.

3.1 The convergence argument

In this section we take for granted the well-posedness of the master equation (M) and focus on the study of the convergence. As we consider only the feedback Nash equilibrium and the optimal processes, we can fix here and in the rest of the chapter the filtered probability space and the Poisson random measures. We give the precise statement of the convergence in terms of two theorems: the first one describes the convergence of the value functions, while the second one is a propagation of chaos for the optimal trajectories.

For any $i \in \{1, \dots, N\}$ and $x \in \Sigma$, set

$$w^{N,i}(t_0, x, m_0) := \sum_{x_1=1}^{d} \cdots \sum_{x_{i-1}=1}^{d} \sum_{x_{i+1}=1}^{d} \cdots \sum_{x_N=1}^{d} v^{N,i}(t_0, \boldsymbol{x}) \prod_{j \neq i} m_0(x_j),$$

where $\boldsymbol{x} = (x_1, \dots, x_N)$, and

$$||w^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0)||_{L^1(m_0)} := \sum_{x=1}^{d} |w^{N,i}(t_0, x, m_0) - U(t_0, x, m_0)| m_0(x).$$

The main result is given by the following

Theorem 3.1. *Assume (LipH) and that (M) admits a unique regular solution U in the sense of Definition 1.17. Fix $N \geq 1$, $(t_0, m_0) \in [0,T] \times \mathcal{P}(\Sigma)$, $\boldsymbol{x} \in \Sigma^N$ and let $(v^{N,i})$ be the solution to (Nash). Then for any $i \in \{1, \dots, N\}$*

$$|v^{N,i}(t,\boldsymbol{x}) - U(t,x_i,m^N_{\boldsymbol{x}})| \leq \frac{C}{N} \tag{3.1}$$

$$|\Delta^i v^{N,i}(t,\boldsymbol{x}) - \Delta^i U(t,x_i,m^N_{\boldsymbol{x}})| \leq \frac{C}{N} \tag{3.2}$$

$$||w^{N,i}(t,\cdot,m_0) - U(t,\cdot,m_0)||_{L^1(m_0)} \leq \frac{C}{\sqrt{N}}. \tag{3.3}$$

In (3.1) *and* (3.3)*, the constant C does not depend on i, t, m_0, $\boldsymbol{x}$ nor N.*

As stated above, the convergence can be studied also in terms of the optimal trajectories. Consider the optimal process $\boldsymbol{Y_t} = (Y_1(t), \dots, Y_N(t))_{t\in[0,T]}$ for the N-player game,

$$Y_i(t) = \xi_i + \int_0^t \int_\Theta \sum_{y\in\Sigma} (y - Y_i(s^-)) \mathbb{1}_{]0,\alpha^i_y(s,\boldsymbol{Y}_{s^-})[}(\theta_y) \mathcal{N}_i(ds,d\theta), \quad t \in [0,T] \tag{3.4}$$

where $\alpha^i_y(t,\boldsymbol{x})$ is the optimal feedback, i.e. $\alpha^i_y(t,\boldsymbol{x}) := [a^*(x_i, \Delta^i v^{N,i}(t,\boldsymbol{x}))]_y$: it solves (1.36) with γ given by (1.8). Moreover, let $\widetilde{\boldsymbol{X}}_t = (\widetilde{X}_1(t), \dots, \widetilde{X}_N(t))_{t\in[0,T]}$ be the i.i.d. process solution to

$$\widetilde{X}_{i,t} = \xi_i + \int_0^t \int_\Theta \sum_{y\in\Sigma} (y - \widetilde{X}_i(s^-)) \mathbb{1}_{]0,\widetilde{\alpha}^i_y(s,\widetilde{\boldsymbol{X}}_{s^-})[}(\theta_y) \mathcal{N}_i(ds,d\theta), \quad t \in [0,T] \tag{3.5}$$

with $\widetilde{\alpha}^i_y(t,\boldsymbol{x}) := [a^*(x_i, \Delta^x U(t,x_i,m(t)))]_y$, m being the unique solution to the mean field game, which is induced by U; see Remark 1.18. It follows that $\mathrm{Law}(\widetilde{X}_i(t)) = m(t)$.

Theorem 3.2. *Under the same assumptions of Theorem 3.1, for any $N \geq 1$ and any $i \in \{1, \dots, N\}$, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_i(t) - \widetilde{X}_i(t)\right|\right] \leq \frac{C}{\sqrt{N}} \tag{3.6}$$

for some constant $C > 0$ independent of m_0 and N. In particular we obtain the Law of Large Numbers

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|m^N_{\boldsymbol{Y}}(t) - m(t)\right|\right] \leq CN^{-\frac{1}{9}}. \tag{3.7}$$

Notice that the supremum is taken inside the mean, giving the convergence in the space of trajectories. For this reason, we have a slow convergence of order $N^{-1/9}$ in (3.7), coming from a result in [72] about the convergence of the empirical measures of a decoupled system (c.f. Lemma 3.6 below). Instead, if the supremum were taken outside the mean, the convergence would be of order $N^{-1/2}$ as in (3.6), thanks to the result in [44] we recalled also in (2.30).

3.1.1 Convergence of the value functions

The first step in the proof of Theorem 3.1 is to show that the projection of U onto empirical measures

$$u^{N,i}(t,\boldsymbol{x}) := U(t,x_i,m^{N,i}_{\boldsymbol{x}}) \tag{3.8}$$

satisfies the system (Nash) up to a term of order $O\left(\frac{1}{N}\right)$. We recall that the value functions, as well as their discrete gradients and the Nash equilibria, are bounded uniformly in N thanks to Lemma 1.7; the $u^{N,i}$-s are also bounded uniformly in N because of the regularity of U.

The following proposition makes rigorous the intuition we already used in the heuristic derivation of the master equation (M). In what follows, C will denote any constant independent of $i, N, m_0, \boldsymbol{x}$, which is allowed to change from line to line.

Proposition 3.3. *Let U be a regular solution to the master equation and $u^{N,i}(t,\boldsymbol{x})$ be defined as in (3.8). Then, for $j \neq i$,*

$$\Delta^j u^{N,i}(t,\boldsymbol{x}) = \frac{1}{N-1} D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j) + \tau^{N,i,j}(t,\boldsymbol{x}), \tag{3.9}$$

where $\tau^{N,i,j} \in \mathcal{C}([0,T] \times \Sigma^N; \mathbb{R}^d)$, $||\tau^{N,i,j}|| \leq \frac{C}{(N-1)^2}$.

Proof. Observe first that $[\Delta^j u^{N,i}(t,\boldsymbol{x})]_{x_j} = 0 = [D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)]_{x_j}$ by definition, so we set $[\tau^{N,i,j}(t,\boldsymbol{x})]_{x_j} := 0$. Consider then $y \neq x_j$: $[\Delta^j u^{N,i}(t,\boldsymbol{x})]_y = U(t, x_i, \frac{1}{N-1}\sum_{k\neq i,j} \delta_{x_k} + \frac{1}{N-1}\delta_y) - U(t, x_i, m_{\boldsymbol{x}}^{N,i})$ by definition. By standard computations we get

$$\begin{aligned}
&U\left(t, x_i, \frac{1}{N-1}\sum_{k\neq i,j} \delta_{x_k} + \frac{1}{N-1}\delta_y\right) - U(t, x_i, m_{\boldsymbol{x}}^{N,i}) \\
&= U\left(t, x_i, m_{\boldsymbol{x}}^{N,i} + \frac{1}{N-1}(\delta_y - \delta_{x_j})\right) - U(t, x_i, m_{\boldsymbol{x}}^{N,i}) \\
&= \int_0^{\frac{1}{N-1}} \left[D^m U(m_{\boldsymbol{x}}^{N,i} + s(\delta_y - \delta_{x_j}), x_j)\right]_y ds \\
&= \int_0^{\frac{1}{N-1}} \left(\left[D^m U(m_{\boldsymbol{x}}^{N,i} + s(\delta_y - \delta_{x_j}), x_j)\right]_y + \left[D^m U(m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y - \left[D^m U(m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y\right) ds \\
&= \frac{1}{N-1}\left[D^m U(m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y + \int_0^{\frac{1}{N-1}}\left(\left[D^m U(m_{\boldsymbol{x}}^{N,i} + s(\delta_y - \delta_{x_j}), x_j)\right]_y - \left[D^m U(m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y\right) ds \\
&= \frac{1}{N-1}\left[D^m U(t, x_i, m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y + O\left(\frac{1}{(N-1)^2}\right),
\end{aligned}$$

where the last equality is derived by exploiting the Lipschitz continuity in m of $D^m U$

$$\begin{aligned}
&\left|\int_0^{\frac{1}{N-1}} \left(\left[D^m U(m_{\boldsymbol{x}}^{N,i} + s(\delta_y - \delta_{x_j}), x_j)\right]_y - \left[D^m U(m_{\boldsymbol{x}}^{N,i}, x_j)\right]_y\right) ds\right| \\
&\qquad \leq C \int_0^{\frac{1}{N-1}} \left|s(\delta_y - \delta_{x_j})\right| ds = O\left(\frac{1}{(N-1)^2}\right).
\end{aligned}$$

For every component y of $D^m U$ we have found the thesis, and thus the same holds for the whole vector. □

In the next proposition we show that the $u^{N,i}$-s almost solve the system (Nash):

Proposition 3.4. *Under the assumptions of Theorem 3.1, the functions $(u^{N,i})_{i=1,\dots,N}$ solve*

$$\begin{cases} -\frac{\partial u^{N,i}}{\partial t}(t,\boldsymbol{x}) - \sum_{j=1,\ j\neq i}^N a^*(x_j, \Delta^j u^{N,j}) \cdot \Delta^j u^{N,i} + H(x_i, \Delta^i u^{N,i}) = F(x_i, m_{\boldsymbol{x}}^{N,i}) + r^{N,i}(t,\boldsymbol{x}) \\ u^{N,i}(T,\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}), \end{cases} \tag{3.10}$$

with $r^{N,i} \in \mathcal{C}([0,T] \times \Sigma^N)$, $||r^{N,i}|| \leq \frac{C}{N}$.

Proof. We know that U solves

$$-\partial_t U + H(x,\Delta^x U) - \int_\Sigma D^m U(t,x,m,y)\cdot a^*(y,\Delta^y U(t,y,m))dm(y) = F(x,m),$$

and $U(T,x,m)=G(x,m)$.

Computing the equation in $(t,x_i,m_{\boldsymbol{x}}^{N,i})$ we get

$$\begin{aligned}-\partial_t U(t,x_i,m_{\boldsymbol{x}}^{N,i}) &+ H(x_i,\Delta^x U(t,x_i,m_{\boldsymbol{x}}^{N,i}))\\ &- \int_\Sigma D^m U(t,x_i,m_{\boldsymbol{x}}^{N,i},y)\cdot a^*(y,\Delta^x U(t,y,m_{\boldsymbol{x}}^{N,i}))dm_{\boldsymbol{x}}^{N,i}(y) = F(x_i,m_{\boldsymbol{x}}^{N,i}),\end{aligned}$$

with the correct final condition $u^{N,i}(t,\boldsymbol{x}) = U(T,x_i,m_{\boldsymbol{x}}^{N,i}) = G(x_i,m_{\boldsymbol{x}}^{N,i})$. By definition of empirical measure we can rewrite

$$\begin{aligned}-\partial_t U(t,x_i,m_{\boldsymbol{x}}^{N,i}) &+ H(x_i,\Delta^x U(t,x_i,m_{\boldsymbol{x}}^{N,i}))\\ &- \frac{1}{N-1}\sum_{j=1,j\neq i}^N D^m U(t,x_i,m_{\boldsymbol{x}}^{N,i},x_j)\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i})) = F(x_i,m_{\boldsymbol{x}}^{N,i}).\end{aligned}$$

Thanks to Proposition 3.3, we have

$$\begin{aligned}\frac{1}{N-1}\sum_{j=1,j\neq i}^N & D^m U(t,x_i,m_{\boldsymbol{x}}^{N,i},x_j)\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i}))\\ &= \sum_{j=1,j\neq i}^N \Delta^j u^{N,i}(t,\boldsymbol{x})\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i}))\\ &- \sum_{j=1,j\neq i}^N \tau^{N,i,j}(t,\boldsymbol{x})\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i}))\\ &=: 1) \ + \ 2).\end{aligned}$$

For the first term we add and subtract the quantity $a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,j}))$:

$$\begin{aligned}1) &= \sum_{j\neq i}\Delta^j u^{N,i}(t,\boldsymbol{x})\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i})) - a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,j}))\\ &+ \sum_{j\neq i}\Delta^j u^{N,i}(t,\boldsymbol{x})\cdot a^*(x_j,\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,j}))\\ &= (A)+(B).\end{aligned}$$

For (A) we have, using first the Lipschitz continuity of a^* with respect to the second variable and then the Lipschitz continuity of $\Delta^x U$ with respect to m:

$$\begin{aligned}(A) &\leq \sum_{j\neq i}\Delta^j u^{N,i}(t,\boldsymbol{x})\cdot(\Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,i}) - \Delta^x U(t,x_j,m_{\boldsymbol{x}}^{N,j}))\\ &\leq C\sum_{j\neq i}||\Delta^j u^{N,i}||\cdot|m_{\boldsymbol{x}}^{N,i}-m_{\boldsymbol{x}}^{N,j}|\\ &\leq \frac{C}{N-1}\sum_{j\neq i}||\Delta^j u^{N,i}|| \leq \frac{C}{N},\end{aligned}$$

where the last inequality is a consequence of (3.9) and the uniform bound on $||D^m U||$ for the solution to (M). Part (B) of 1) is instead what we want to obtain in the equation for $u^{N,i}$, so we leave it as it is. For the term 2), we simply note that U is bounded from above, and thus the whole term 2) is also of order $O\left(\frac{1}{N}\right)$. □

We are now in the position to prove the convergence of the value functions to the regular solution to the master equation.

Proof of Theorem 3.1. In order to prove (3.1), we apply Itô Formula to the function $\Psi(t, \boldsymbol{Y}_t) = (u^{N,i}(t, \boldsymbol{Y}_t) - v^{N,i}(t, \boldsymbol{Y}_t))^2$,

$$d\Psi(t, \boldsymbol{Y}_t) = \frac{\partial \Psi(t, \boldsymbol{Y}_t)}{\partial t} + \sum_{j=1}^{N} \int_{\Theta} [\Psi(t, \widetilde{\boldsymbol{Y}}^j_{t-}) - \Psi(t, \boldsymbol{Y}_{t-})] \mathcal{N}_j(dt, d\theta),$$

$$\widetilde{\boldsymbol{Y}}^j_t = \left(Y_{1,t}, \ldots, Y_{j-1,t}, Y_{j,t} + \sum_{y \in \Sigma} (y - Y_{j,t}) \mathbb{1}_{]0, \alpha^j_y[}(\theta_y), Y_{j+1,t}, \ldots, Y_{N,t} \right),$$

and, as above,

$$\alpha^j_y(t, \boldsymbol{Y}_t) = \left[a^*(Y_{j,t}, \Delta^j v^{N,j}(t, \boldsymbol{Y}_t)) \right]_y.$$

It follows that

$$\begin{aligned} d\Psi(t, \boldsymbol{Y}_t) &= 2(u^{N,i}(t, \boldsymbol{Y}_t) - v^{N,i}(t, \boldsymbol{Y}_t))(\partial_t u^{N,i} - \partial_t v^{N,i}) \\ &+ \sum_{j=1}^{N} \int_{\Theta} [(u^{N,i}(t, \widetilde{\boldsymbol{Y}}^j_{t-}) - v^{N,i}(t, \widetilde{\boldsymbol{Y}}^j_{t-}))^2 - (u^{N,i}(t, \boldsymbol{Y}_{t-}) - v^{N,i}(t, \boldsymbol{Y}_{t-}))^2] \mathcal{N}_j(dt, d\theta). \end{aligned}$$

Now, let us fix a deterministic initial condition $\boldsymbol{Y}_t = \boldsymbol{\xi}$. Integrating on the time interval $[t, T]$, we get

$$\begin{aligned} &[u^{N,i}(T, \boldsymbol{Y}_T) - v^{N,i}(T, \boldsymbol{Y}_T)]^2 = \\ &= [u^{N,i}(t, \boldsymbol{Y}_t) - v^{N,i}(t, \boldsymbol{Y}_t)]^2 + 2 \int_t^T (u^{N,i}(s, \boldsymbol{Y}_s) - v^{N,i}(s, \boldsymbol{Y}_s))(\partial_t u^{N,i}(s, \boldsymbol{Y}_s) - \partial_t v^{N,i}(s, \boldsymbol{Y}_s)) ds \\ &+ \sum_{j=1}^{N} \int_t^T \int_{\Theta} [(u^{N,i}(s, \widetilde{\boldsymbol{Y}}^j_{s-}) - v^{N,i}(s, \widetilde{\boldsymbol{Y}}^j_{s-}))^2 - (u^{N,i}(s, \boldsymbol{Y}_{s-}) - v^{N,i}(s, \boldsymbol{Y}_{s-}))^2] \mathcal{N}_j(ds, d\theta). \end{aligned}$$

For brevity, in the remaining part of the proof we set $u^i_s := u^{N,i}(s, \boldsymbol{Y}_s)$, $v^i_s := v^{N,i}(s, \boldsymbol{Y}_s)$, $\alpha^j_s := \alpha^j(s, \boldsymbol{Y}_s)$ and $\overline{\alpha}^j_s := \overline{\alpha}^j(s, \boldsymbol{Y}_s)$. Next, taking the expectation and applying (1.13), we obtain

$$\begin{aligned} \mathbb{E}[(u^i_T - v^i_T)^2] &= \mathbb{E}[(u^i_t - v^i_t)^2] + 2\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s) ds \right] \\ &+ \sum_{j=1}^{N} \mathbb{E}\left[\int_t^T \alpha^j(s, \boldsymbol{Y}_s) \cdot \Delta^j [(u^i_s - v^i_s)^2] ds \right]. \end{aligned}$$

Let us first study the term $\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s) ds\right]$. Applying equations (3.10) and (Nash) we get

$$\begin{aligned} &\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s) ds \right] \\ &= \mathbb{E}\Bigg[\int_t^T (u^i_s - v^i_s) \Bigg\{ \sum_{j=1, j \neq i}^{N} \left(-\overline{\alpha}^j_s \cdot \Delta^j u^i_s + \alpha^j_s \cdot \Delta^j v^i_s + \alpha^j_s \cdot \Delta^j u^i_s - \alpha^j_s \cdot \Delta^j u^i_s \right) \\ &- H(Y_{i,s}, \Delta^i v^i_s) + H(Y_{i,s}, \Delta^i u^i_s) - r^{N,i}(s, \boldsymbol{Y}_s) \Bigg\} ds \Bigg]. \end{aligned}$$

Note that we also added and subtracted $\alpha^j \cdot \Delta^j u^i_s$ in the last line so that we can use the Lipschitz properties of H, a^* and the bound on $r^{N,i}$ to get the correct estimates. Specifically, we can rewrite

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s) ds\right] \\
&= \mathbb{E}\Bigg[\int_t^T (u^i_s - v^i_s)\Bigg\{\sum_{j=1,j\neq i}^N \Big((\alpha^j_s - \overline{\alpha}^j_s)\cdot \Delta^j u^i_s - \alpha^j_s \cdot (\Delta^j u^i_s - \Delta^j v^i_s)\Big) \\
&- H(Y_{i,s}, \Delta^i v^i_s) + H(Y_{i,s}, \Delta^i u^i_s) - r^{N,i}(s, \boldsymbol{Y}_s)\Bigg\} ds\Bigg].
\end{aligned}$$

Putting things together,

$$\begin{aligned}
&\mathbb{E}[(u^i_T - v^i_T)^2] \\
&= \mathbb{E}[(u^i_t - v^i_t)^2] + 2\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)(\partial_t u^i_s - \partial_t v^i_s) ds\right] + \sum_{j=1}^N \mathbb{E}\left[\int_t^T \alpha^j_s \cdot \Delta^j [(u^i_s - v^i_s)^2] ds\right] \\
&= \mathbb{E}[(u^i_t - v^i_t)^2] + 2\mathbb{E}\Bigg[\int_t^T (u^i_s - v^i_s)\Bigg\{\sum_{j\neq i}^N \Big((\alpha^j_s - \overline{\alpha}^j_s)\cdot \Delta^j u^i_s - \alpha^j_s \cdot (\Delta^j u^i_s - \Delta^j v^i_s)\Big) \\
&- H(Y_{i,s}, \Delta^i v^i_s) + H(Y_{i,s}, \Delta^i u^i_s) - r^{N,i}(s, \boldsymbol{Y}_s)\Bigg\} ds\Bigg] + \mathbb{E}\left[\int_t^T \alpha^i_s \cdot \Delta^i [(u^i_s - v^i_s)^2] ds\right] \\
&+ \sum_{j\neq i}^N \mathbb{E}\left[\int_t^T \alpha^j_s \cdot \Delta^j [(u^i_s - v^i_s)^2] ds\right] \\
&= \mathbb{E}[(u^i_t - v^i_t)^2] + 2\mathbb{E}\Bigg[\int_t^T (u^i_s - v^i_s)\left\{\sum_{j\neq i}^N \Big((\alpha^j_s - \overline{\alpha}^j_s)\cdot \Delta^j u^i_s - \alpha^j_s \cdot (\Delta^j u^i_s - \Delta^j v^i_s)\Big)\right\} ds \\
&+ \int_t^T \sum_{j\neq i}^N \frac{1}{2}\alpha^j_s \cdot \Delta^j [(u^i_s - v^i_s)^2] ds + \int_t^T (u^i_s - v^i_s)(-H(Y_{i,s}, \Delta^i v^i_s) + H(Y_{i,s}, \Delta^i u^i_s) - r^{N,i}(s, \boldsymbol{Y}_s)) ds\Bigg] \\
&+ \mathbb{E}\left[\int_t^T \alpha^i_s \cdot \Delta^i [(u^i_s - v^i_s)^2] ds\right].
\end{aligned}$$

On the other hand, observing that $\Delta^j[(u^i - v^i)^2] = \Delta^j(u^i - v^i) \times (\Delta^j(u^i - v^i) + 2(\mathbf{1}(u^i - v^i)))$, $\times$ being the element by element product between vectors and $\mathbf{1} = (1, \dots, 1)^\dagger$, the expression

$$\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)\left\{\sum_{j=1,j\neq i}^N \Big(-2\alpha^j_s \cdot (\Delta^j u^i_s - \Delta^j v^i_s)\Big)\right\} ds + \int_t^T \sum_{j=1,j\neq i}^N \Big(\alpha^j_s \cdot \Delta^j [(u^i_s - v^i_s)^2]\Big) ds\right]$$

can be simplified as follows

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T (u^i_s - v^i_s)\left\{\sum_{j=1,j\neq i}^N \Big(-2\alpha^j_s \cdot (\Delta^j u^i_s - \Delta^j v^i_s)\Big)\right\} ds + \int_t^T \sum_{j=1,j\neq i}^N \Big(\alpha^j_s \cdot \Delta^j [(u^i_s - v^i_s)^2]\Big) ds\right] \\
&= \sum_{j=1,j\neq i}^N \mathbb{E}\Bigg[\int_t^T \Big\{-2\alpha^j_s \cdot (u^i_s - v^i_s)(\Delta^j u^i_s - \Delta^j v^i_s)
\end{aligned}$$

$$+\alpha_s^j \cdot (\Delta^j(u_s^i - v_s^i) \times (\Delta^j(u_s^i - v_s^i) + 2(\mathbf{1}(u_s^i - v_s^i)))\Big\} ds\Bigg]$$

$$= \sum_{j=1, j\neq i}^{N} \mathbb{E}\left[\int_t^T \alpha_s^j \cdot (\Delta^j(u_s^i - v_s^i))^2 ds\right].$$

Thus, we have found

$$\begin{aligned}0 =& \mathbb{E}[(u_T^i - v_T^i)^2] \\ =& \mathbb{E}[(u_t^i - v_t^i)^2] + 2\mathbb{E}\Bigg[\int_t^T (u_s^i - v_s^i)\Bigg\{ \sum_{j=1,j\neq i}^{N} \Big((\alpha_s^j - \overline{\alpha}_s^j)\cdot \Delta^j u_s^i\Big) \\ &- H(Y_{i,s}, \Delta^i v_s^i) + H(Y_{i,s}, \Delta^i u_s^i) - r^{N,i}(s, \mathbf{Y}_s)\Bigg\} ds\Bigg] + \mathbb{E}\left[\int_t^T \alpha_s^i \cdot \Delta^i[(u_s^i - v_s^i)^2] ds\right] \\ &+ \sum_{j=1,j\neq i}^{N} \mathbb{E}\left[\int_t^T \alpha_s^j \cdot (\Delta^j(u_s^i - v_s^i))^2 ds\right].\end{aligned}$$

Now, using again the expression for $\Delta^i((u_s^i - v_s^i)^2)$,

$$\begin{aligned}&\mathbb{E}\left[\int_t^T \alpha_s^i \cdot \Delta^i[(u_s^i - v_s^i)^2] ds\right] \\ &\quad = \mathbb{E}\left[\int_t^T \alpha_s^i \cdot (\Delta^i(u_s^i - v_s^i))^2 ds\right] + \mathbb{E}\left[\int_t^T \alpha_s^i \cdot (\Delta^i(u_s^i - v_s^i) \times 2(\mathbf{1}(u_s^i - v_s^i) ds\right],\end{aligned}$$

so that we can rewrite the previous as

$$\begin{aligned}&\mathbb{E}[(u_t^i - u_t^i)^2] + \sum_{j=1}^{N} \mathbb{E}\left[\int_t^T \alpha_s^j \cdot (\Delta^j(u_s^i - v_s^i))^2 ds\right] \qquad (3.11) \\ &= -2\mathbb{E}\Bigg[\int_t^T (u_s^i - v_s^i)\Bigg\{\sum_{j=1,j\neq i}^{N}\Big((\alpha_s^j - \overline{\alpha}_s^j)\cdot \Delta^j u_s^i\Big) - H(Y_{i,s}, \Delta^i v_s^i) + H(Y_{i,s}, \Delta^i u_s^i) - r^{N,i}(s, \mathbf{Y}_s)\Bigg\} ds\Bigg] \\ &- \mathbb{E}\left[\int_t^T \alpha_s^i \cdot (\Delta^i(u_s^i - v_s^i) \times 2(\mathbf{1}(u_s^i - v_s^i)))ds\right].\end{aligned}$$

Recalling that $\alpha_s^j \geq 0$ (since it is a vector of transition rates), we can estimate, erasing the sum in the left hand side,

$$\begin{aligned}&\mathbb{E}[(u_t^i - v_t^i)^2] \\ &\leq 2\mathbb{E}\Bigg[\int_t^T \Big|u_s^i - v_s^i\Big| \Bigg\{\sum_{j\neq i}^{N} \Big|(\alpha_s^j - \overline{\alpha}_s^j)\cdot \Delta^j u_s^i\Big| + \Big|H(Y_{i,s}, \Delta^i v_s^i) - H(Y_{i,s}, \Delta^i u_s^i)\Big| + \Big|r^{N,i}(s, \mathbf{Y}_s)\Big|\Bigg\} ds\Bigg] \\ &+ 2\mathbb{E}\left[\int_t^T \Big|u_s^i - v_s^i\Big| \Big|\alpha_s^i \cdot \Delta^i(u_s^i - v_s^i)\Big| ds\right].\end{aligned}$$

We now use the Lipschitz continuity of H and a^* (assumption (LipH), the bound on α and the bounds on $||r^{N,i}|| \leq \frac{C}{N}$ and $||\Delta^j u^i|| \leq \frac{1}{N}||D^m U|| \leq \frac{C}{N}$ proved in Propositions 3.3 and 3.4 to obtain

$$\mathbb{E}[(u_t^i - v_t^i)^2]$$

$$\leq 2\mathbb{E}\left[\int_t^T \left|u_s^i - v_s^i\right| \left\{\frac{C}{N}\sum_{j=1,j\neq i}^{N}\left|\Delta^j u_s^j - \Delta^j v_s^j\right| + C\left|\Delta^i(v_s^i - u_s^i)\right| + \frac{C}{N}\right\} ds\right]$$
$$+ 2C\mathbb{E}\left[\int_t^T \left|u_s^i - v_s^i\right| \left|\Delta^i(u_s^i - v_s^i)\right| ds\right]$$
$$\leq \frac{C}{N}\mathbb{E}\left[\int_t^T |u_s^i - v_s^i| ds\right] + \frac{C}{N}\sum_{j\neq i}\mathbb{E}\left[\int_t^T |u_s^i - v_s^i| \left|\Delta^j(u_s^j - v_s^j)\right| ds\right]$$
$$+ C\mathbb{E}\left[\int_t^T |u_s^i - v_s^i| \left|\Delta^i(u_s^i - v_s^i)\right| ds\right].$$

By the convexity inequality $AB \leq \varepsilon A^2 + \frac{B^2}{4\varepsilon}$ we can further estimate the right hand side to get

$$\mathbb{E}[(u_t^i - v_t^i)^2] \leq \frac{C}{N^2} + C\mathbb{E}\left[\int_t^T |u_s^i - v_s^i|^2 ds\right] + \frac{C}{N}\sum_{j=1}^{N}\mathbb{E}\left[\int_t^T \left|\Delta^j(u_s^j - v_s^j)\right|^2 ds\right]. \quad (3.12)$$

Since all the functions are evaluated on the optimal trajectories, we can use the bounds

$$|u^{N,i}(s, \boldsymbol{Y}_s) - v^{N,i}(s, \boldsymbol{Y}_s)| \leq \max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|$$
$$|\Delta^i u^{N,i}(s, \boldsymbol{Y}_s) - \Delta^i v^{N,i}(s, \boldsymbol{Y}_s)| \leq C\max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|$$

for any $s \in [t, T]$, where the latter is a consequence of (1.6). Recalling that the initial condition is deterministic, (3.12) gives

$$|u^{N,i}(t, \boldsymbol{\xi}) - v^{N,i}(t, \boldsymbol{\xi})| \leq \frac{C}{N^2} + \frac{C}{N}\sum_{j=1}^{N}\int_t^T \max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|^2 ds \quad (3.13)$$

for any $\boldsymbol{\xi} \in \Sigma^N$. Hence we can take the max in Σ^N and then average $\left(\frac{1}{N}\sum_{i=1}^N\right)$ to obtain

$$\frac{1}{N}\sum_{i=1}^{N}\max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|^2 \leq \frac{C}{N^2} + C\int_t^T \frac{1}{N}\sum_{i=1}^{N}\max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|^2 ds.$$

Therefore Gronwall's Lemma applied to the quantity $\frac{1}{N}\sum_{i=1}^N \max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(s, \boldsymbol{x}) - v^{N,i}(s, \boldsymbol{x})|^2$ yields

$$\frac{1}{N}\sum_{i=1}^{N}\max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(t, \boldsymbol{x}) - v^{N,i}(t, \boldsymbol{x})|^2 \leq \frac{C}{N^2}$$

which, by the arbitrary of t, holds for any $t \in [0, T]$. Thus this inequality, applied to the right hand side of (3.13), gives

$$\max_{\boldsymbol{x}\in\Sigma^N} |u^{N,i}(t, \boldsymbol{x}) - v^{N,i}(t, \boldsymbol{x})|^2 \leq \frac{C}{N^2}$$

for any i, which immediately implies (3.1). Inequality (3.2) is a consequence of (3.1) and (1.6).

For (3.3), consider now the initial data $\boldsymbol{\xi} := (\xi_1 \ldots, \xi_N)$, the ξ_i's are i.i.d. and m_0-distributed: we have

$$||w^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0)||_{L^1(m_0)} =$$

$$
\begin{aligned}
&= \sum_{x_i=1}^{d} |w^{N,i}(t_0, x_i, m_0) - U(t_0, x_i, m_0)| m_0(x_i) \\
&= \sum_{x_i=1}^{d} \left| \sum_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_N=1}^{d} v^{N,i}(t, \boldsymbol{x}) \prod_{j\neq i} m_0(x_j) - U(t, x_i, m_0) \right| m_0(x_i) \\
&= \sum_{x_i=1}^{d} \left| \sum_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_N=1}^{d} \left\{ v^{N,i}(t, \boldsymbol{x}) \prod_{j\neq i} m_0(x_j) - u^{N,i}(t, \boldsymbol{x}) \prod_{j\neq i} m_0(x_j) \right.\right. \\
&\left.\left. + u^{N,i}(t, \boldsymbol{x}) \prod_{j\neq i} m_0(x_j) \right\} - U(t, x_i, m_0) \right| m_0(x_i) \\
&\leq \mathbb{E}|v^{N,i}(t, \boldsymbol{\xi}) - u^{N,i}(t, \boldsymbol{\xi})| + \sum_{x_1,\dots,x_N=1}^{d} |u^{N,i}(t, \boldsymbol{x}) - U(t, x_i, m_0)| \prod_{j=1}^{N} m_0(x_j), \qquad (3.14)
\end{aligned}
$$

By (3.1), the first term in (3.14) is of order $1/N$. For the second term we further estimate, using again the Lipschitz continuity of U with respect to m,

$$
\begin{aligned}
&\sum_{x_1,\dots,x_N=1}^{d} |u^{N,i}(t, \boldsymbol{x}) - U(t, x_i, m_0)| \prod_{j=1}^{N} m_0(x_j) \\
&\qquad = \sum_{x_1,\dots,x_N=1}^{d} |U(t, x_i, m_{\boldsymbol{x}}^{N,i}) - U(t, x_i, m_0)| \prod_{j=1}^{N} m_0(x_j) \\
&\qquad \leq C\mathbb{E}\left[\mathrm{d}_1(m_{\boldsymbol{\xi}}^{N,i}, m_0) \right] \leq \frac{C}{\sqrt{N}},
\end{aligned}
$$

where in the last inequality we used that $\mathbb{E}\left[\mathrm{d}_1(m_{\boldsymbol{\xi}}^N, m_0)\right] \leq \frac{C}{\sqrt{N}}$, thanks to Theorem 1 of [44], where d_1 is the 1-Wasserstein distance and $m_{\boldsymbol{\xi}}^N$ is the corresponding empirical measure. Overall, we have bounded (3.14) by a term of order $1/\sqrt{N}$, and thus (3.3) is also proved. □

3.1.2 Propagation of chaos

The central part of the proof of the propagation of chaos is based on comparing the optimal trajectories associated to $v^{N,i}$ with the ones associated to $u^{N,i}$. Hence, consider the processes

$$
X_i(t) = \xi_i + \int_0^t \int_\Theta \sum_{y\in\Sigma} (y - X_i(s^-)) \mathbf{1}_{]0, \overline{\alpha}_y^i(s, \boldsymbol{X}_{s^-})[}(\theta_y) \mathcal{N}_i(ds, d\theta), \quad t \in [0, T] \qquad (3.15)
$$

where $\overline{\alpha}_y^i(t, \boldsymbol{x}) := [a^*(x_i, \Delta^i u^{N,i}(t, \boldsymbol{x}))]_y$. Observe that the processes $\boldsymbol{X}$ and $\boldsymbol{Y}$ are exchangeable, and that both the Nash equilibrium α and $\overline{\alpha}$ are uniformly bounded.

Theorem 3.5. *With the notation introduced above, under the assumptions of Theorem 3.1, we have*

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |Y_i(t) - X_i(t)| \right] \leq \frac{C}{N}, \qquad (3.16)
$$

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |m_{\boldsymbol{Y}}^N(t) - m_{\boldsymbol{X}}^N(t)| \right] \leq \frac{C}{N}, \qquad (3.17)
$$

Proof. We first exploit the representation (3.15) and (3.4) of the dynamics to get, for any t,

$$\sup_{s\in[0,t]}|X_{i,s}-Y_{i,s}|$$
$$\leq \int_0^t \Big|\gamma(X_{i,s^-},\theta,a^*(X_{i,s^-},\Delta^i u^{N,i}(\boldsymbol{X}_{s^-})))-\gamma(Y_{i,s^-},\theta,a^*(Y_{i,s^-},\Delta^i v^{N,i}(\boldsymbol{Y}_{s^-})))\Big|\mathcal{N}_i(ds,d\theta).$$

Then, taking the expectation and using (1.12) and the Lipschitz continuity of a^*, we obtain

$$\begin{aligned}\varphi(t):=&\ \mathbb{E}\Bigg[\sup_{s\in[0,t]}|X_{i,s}-Y_{i,s}|\Bigg]\\ \leq& C\mathbb{E}\Bigg[\int_0^t\Big|a^*(X_{i,s},\Delta^i u^{N,i}(\boldsymbol{X}_s))-a^*(Y_{i,s},\Delta^i v^{N,i}(\boldsymbol{Y}_s))\Big|\,ds\Bigg]+C\mathbb{E}\Bigg[\int_0^t|X_{i,s}-Y_{i,s}|\,ds\Bigg]\\ \leq&\ C\mathbb{E}\left[\int_0^t|X_{i,s}-Y_{i,s}|ds\right]+C\mathbb{E}\left[\int_0^T\Big|\Delta^i u^{N,i}(\boldsymbol{Y}_s)-\Delta^i v^{N,i}(\boldsymbol{Y}_s)\Big|\,ds\right]\\ &+C\mathbb{E}\left[\int_0^t\Big|\Delta^x U\left(s,X_{i,s},m^{N,i}_{\boldsymbol{X}_s}\right)-\Delta^x U\left(s,Y_{i,s},m^{N,i}_{\boldsymbol{Y}_s}\right)\Big|\,ds\right]\\ \leq&\ \frac{C}{N}+C\mathbb{E}\left[\int_0^t\sup_{r\in[0,s]}|X_{i,r}-Y_{i,r}|ds\right]+C\mathbb{E}\left[\int_0^t\sup_{r\in[0,s]}\Big|m^{N,i}_{\boldsymbol{X}_r}-m^{N,i}_{\boldsymbol{Y}_r}\Big|\,ds\right],\end{aligned}\tag{3.18}$$

where we applied (3.2) and the Lipschitz continuity in m of $\Delta^x U$ in the last inequality. Hence inequality (1.1) and the exchangeability of $(\boldsymbol{X},\boldsymbol{Y})$ yield

$$\begin{aligned}\varphi(t)\leq&\ \frac{C}{N}+C\int_0^t\left(\mathbb{E}\left[\sup_{r\in[0,s]}|X_{i,r}-Y_{i,r}|\right]+\frac{1}{N-1}\sum_{j\neq i}\mathbb{E}\left[\sup_{r\in[0,s]}|X_{j,r}-Y_{j,r}|\right]\right)ds\\ \leq&\ \frac{C}{N}+C\int_0^t\varphi(s)ds\end{aligned}$$

and by Gronwall's inequality we get (3.16). Finally (3.16), applying again (1.1), gives (3.17). □

Finally, we get to the proof of the propagation of chaos (Theorem 3.2). Recall that the $Y_{i,t}$'s are the optimal processes, i.e. the solutions to system (3.4), the $X_{i,t}$'s are the processes associated to the functions $u^{N,i}$, i.e. they solve system (3.15), while the $\widetilde{X}_{i,t}$'s - to which we would like to prove convergence - are the decoupled limit processes (they solve system (3.5)). First, we need the following lemma

Lemma 3.6. *Let* $\widetilde{\boldsymbol{X}}_t=(\widetilde{X}_{i,t})_{i\in1,\ldots,N}$ *be* N *i.i.d. processes with values in* $\mathbb{R}$*, with* $\mathrm{Law}(\widetilde{X}_{i,t})=m(t)$*. Then*

$$\mathbb{E}\left|m^{N,i}_{\widetilde{\boldsymbol{X}}_t}-m_t\right|\leq\frac{C}{\sqrt{N}}\qquad\forall t\in[0,T]\tag{3.19}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|m^{N,i}_{\widetilde{\boldsymbol{X}}_t}-m_t\right|\right]\leq CN^{-1/9},\tag{3.20}$$

where C *does not depend on* t *nor* N*.*

Inequality (3.19) follows again from Theorem 1 in [44], while a proof of (3.20) can be found for example in [72].

Proof of Theorem 3.2. The assertion of the theorem is proved if we show that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_{i,t}-\widetilde{X}_{i,t}|\right]\leq\frac{C}{\sqrt{N}}. \tag{3.21}$$

Indeed, by the triangle inequality and (3.16) in Theorem 3.5 we can estimate

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_{i,t}-\widetilde{X}_{i,t}|\right]&\leq\mathbb{E}\left[\sup_{t\in[0,T]}|Y_{i,t}-X_{i,t}|\right]+\mathbb{E}\left[\sup_{t\in[0,T]}|X_{i,t}-\widetilde{X}_{i,t}|\right]\\&\leq C\left(\frac{1}{N}+\frac{1}{\sqrt{N}}\right).\end{aligned}$$

We are then left to prove (3.21). As in the proof of (3.16), we have

$$\begin{aligned}\varphi(t)&:=\mathbb{E}\left[\sup_{s\in[0,t]}|X_{i,s}-\widetilde{X}_{i,s}|\right]\\&\leq\mathbb{E}\left[\int_0^t\left|a^*(X_{i,s},\Delta^i u^{N,i}(\boldsymbol{X}_s))-a^*(\widetilde{X}_{i,s},\Delta^x U(s,\widetilde{X}_{i,s},m(s)))\right|ds+\int_0^t\left|X_{i,s}-\widetilde{X}_{i,s}\right|ds\right]\\&\leq\mathbb{E}\left[\int_0^t\left|a^*(X_{i,s},\Delta^x U(r,X_{i,s},m^{N,i}_{\boldsymbol{X}_s}))-a^*(X_{i,s},\Delta^x U(s,\widetilde{X}_{i,s},m^{N,i}_{\widetilde{\boldsymbol{X}}_s}))\right|ds+\int_0^t\left|X_{i,s}-\widetilde{X}_{i,s}\right|ds\right.\\&\left.+\int_0^t\left|a^*(X_{i,s},\Delta^x U(s,\widetilde{X}_{i,s},m^{N,i}_{\widetilde{\boldsymbol{X}}_s}))-a^*(\widetilde{X}_{i,s},\Delta^x U(s,\widetilde{X}_{i,s},m(s)))\right|ds\right].\end{aligned}$$

By the Lipschitz continuity of the optimal controls, and of $\Delta^x U$, we can write

$$\begin{aligned}\varphi(t)&\leq C\int_0^t\mathbb{E}\left[|X_{i,s}-\widetilde{X}_{i,s}|+\left|m^{N,i}_{\widetilde{\boldsymbol{X}}_s}-m^{N,i}_{\boldsymbol{X}_s}\right|+\left|m^{N,i}_{\widetilde{\boldsymbol{X}}_s}-m(s)\right|\right]ds\\&\leq C\int_0^t\mathbb{E}\left[|X_{i,s}-\widetilde{X}_{i,s}|+\frac{1}{N-1}\sum_{j\neq i}|X_{j,s}-\widetilde{X}_{j,s}|+\left|m^{N,i}_{\widetilde{\boldsymbol{X}}_s}-m(s)\right|\right]ds.\end{aligned}$$

Using (3.19) of Lemma 3.6 and the exchangeability of the processes, we obtain

$$\begin{aligned}\varphi(t)&\leq C\int_0^t\left(\mathbb{E}\left[\sup_{r\in[0,s]}|X_{i,r}-\widetilde{X}_{i,r}|\right]+\frac{1}{N-1}\sum_{j\neq i}\mathbb{E}\left[\sup_{r\in[0,s]}|X_{j,r}-\widetilde{X}_{j,r}|\right]\right)ds+T\frac{C}{\sqrt{N}}\\&\leq C\int_0^t\varphi(s)ds+\frac{C}{\sqrt{N}},\end{aligned}$$

which, by Gronwall's Lemma, concludes the proof of (3.6). Finally (3.7) follows from (3.6) and (3.20), using also (1.1). □

3.2 Fluctuations and large deviations

The convergence results, Theorem 3.1 and 3.2, allow to derive a Central Limit Theorem and a Large Deviation Principle for the asymptotic behaviour of the empirical measure process of the N-player game optimal trajectories. First of all, we recall from Proposition 1.8 that, for any i, the value function $v^{N,i}$ of player i in the N-player game is invariant under permutations of $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$. This is equivalent to say that the value functions can be viewed as functions of the empirical measure of the system, i.e. there exists a map $V^N:[0,T]\times\Sigma\times\mathcal{P}(\Sigma)$ such that

$$v^{N,i}(t,\boldsymbol{x})=V^N(t,x_i,m^{N,i}_{\boldsymbol{x}}) \tag{3.22}$$

for any $i=1,\ldots,N$, $t\in[0,T]$ and $\boldsymbol{x}\in\Sigma^N$.

3.2.1 Dynamics of the empirical measure process

We consider the empirical measure process of the optimal evolution $\boldsymbol{Y}$ - defined in (3.4) - of the N-player game. If the system is in $\boldsymbol{x}$ at time t, then the rate at which player i goes from x_i to y is given, via the optimal control, by

$$a^*_y(x_i, \Delta^i V^N(t, x_i, m^{N,i}_{\boldsymbol{x}})) =: \Gamma^N_{x_i,y}(t, m^N_{\boldsymbol{x}}), \tag{3.23}$$

i.e. by a function Γ^N which depends only on the empirical measure $m^N_{\boldsymbol{x}}$ and on the number of players N.

Thus the empirical measure of the system $(m^N_t)_{t\in[0,T]}$, $m^N_t := m^N_{\boldsymbol{Y}}(t) = \frac{1}{N}\sum_{i=1}^N \delta_{Y_{i,t}}$, evolves as a (time-inhomogeneous) Markov process on $[0,T]$, with values in $S_N := \mathcal{P}(\Sigma)\cap\frac{1}{N}\mathbb{Z}^d$. The number of players in state x, when the empirical measure is m, is Nm_x. Hence the jump rate of m^N in the direction $\frac{1}{N}(\delta_y - \delta_x)$ at time t is $Nm_x\Gamma^N_{x,y}(t,m)$. Therefore the generator of the time-inhomogeneous Markov process m^N is given, at time t, by

$$\mathfrak{L}^N_t g(m) := N \sum_{x,y\in\Sigma} m_x \Gamma^N_{x,y}(t,m)\left[g\left(m + \frac{1}{N}(\delta_y - \delta_x)\right) - g(m)\right], \tag{3.24}$$

for any $g : S_N \longrightarrow \mathbb{R}$.

Theorem 3.2 implies that the empirical measures converge in L^1 - on the space of trajectories $D([0,T], \mathcal{P}(\Sigma))$ - to the deterministic flow of measures m which is the unique solution of the mean field game system, whose dynamics is given by the KFP ODE

$$\begin{cases} \frac{d}{dt}m(t) = \Gamma(t, m(t))^\dagger m(t) \\ m(0) = m_0, \end{cases} \tag{3.25}$$

where Γ is the matrix defined by

$$\Gamma_{x,y}(t,m) := a^*_y(x, \Delta^x U(t,x,m)) \tag{3.26}$$

and U is the solution to the master equation. Viewing $m(t)$ as a Markov process - and so we will write m_t in this section -, its infinitesimal generator is given, at time t, by

$$\mathfrak{L}_t g(m) := \sum_{x,y\in\Sigma} m_x \Gamma_{x,y}(t,m)[D^m g(m,x)]_y \tag{3.27}$$

for any $g : \mathcal{P}(\Sigma) \longrightarrow \mathbb{R}$. Thanks to (1.15), the generator can be equivalently written as

$$\mathfrak{L}_t g(m) := \sum_{x,y\in\Sigma} m_x \Gamma_{x,y}(t,m)[D^m g(m,1)]_y = m^\dagger \Gamma(t,m) D^m g(m,1). \tag{3.28}$$

In order to prove the asymptotic results, we will also consider the empirical measure of the process $\boldsymbol{X}$ defined in (3.15), in which each player chooses the same control $\Gamma_{x,y}$ independent of N. We denote by $\overline{m}^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}$ the empirical measure process of $\boldsymbol{X}$, whose generator is given, for any $g : \mathcal{P}(\Sigma) \longrightarrow \mathbb{R}$, by

$$\overline{\mathfrak{L}}^N_t g(m) := N \sum_{x,y\in\Sigma} m_x \Gamma_{x,y}(t,m)\left[g\left(m + \frac{1}{N}(\delta_y - \delta_x)\right) - g(m)\right]. \tag{3.29}$$

3.2.2 Central Limit Theorem

A natural refinement of the Law of Large Numbers (3.7) consists in studying the fluctuations around the limit, that is the asymptotic distribution of $m_t^N - m_t$.

This can be done through a functional Central Limit Theorem: we define the fluctuation flow

$$\eta_t^N := \sqrt{N}(m_t^N - m_t) \quad t \in [0,T] \tag{3.30}$$

and study its asymptotic behavior as N tends to infinity. We follow a classical weak convergence approach based on uniform convergence of the generator of the fluctuation flow (3.30) to a limiting generator of a diffusion process to be determined; see e.g. [30] for reference. Before stating the theorem we observe that the process (3.30) has values in $\mathcal{P}_0(\Sigma)$, which in the following we treat as a subset of $\mathbb{R}^d$.

Theorem 3.7 (Central Limit Theorem)**.** *Let U be a regular solution to the master equation and assume (LipH). Then the fluctuation flow η_t^N in* (3.30) *converges, as $N \to \infty$, in the sense of weak convergence of stochastic processes, to a limiting Gaussian process η_t which is the solution of the linear SDE*

$$\begin{cases} d\eta_t = \Big(\Gamma(t,m_t)^\dagger \eta_t + b(t,m_t,\eta_t)\Big)\, dt + \sigma(t,m_t) dB_t, \\ \eta_0 = \bar{\eta}, \end{cases} \tag{3.31}$$

where $\bar{\eta}$ is the limit of η_0^N in distribution, B is a standard d-dimensional Brownian motion, Γ is the transition rate matrix in (3.26)*, $b \in \mathbb{R}^d$ is linear in μ and defined, for any $y \in \Sigma$ and $\mu \in \mathcal{P}_0(\Sigma)$, by*

$$b(t,m,\mu)_y := \sum_{x\in\Sigma} m_x \left[D^m \Gamma_{x,y}(t,m,1)\cdot \mu\right], \tag{3.32}$$

and $\sigma \in \mathbb{R}^{d\times d}$ is given by the relations

$$(\sigma^2)_{x,y}(t,m) = -(m_x\Gamma_{x,y}(t,m) + m_y\Gamma_{y,x}(t,m)), \ \text{for } x \neq y, \tag{3.33}$$

$$(\sigma^2)_{x,x}(t,m) = \sum_{y\neq x}(m_y\Gamma_{y,x}(t,m) + m_x\Gamma_{x,y}(t,m)). \tag{3.34}$$

In particular the matrix σ^2 is the opposite of the generator of a Markov chain, is symmetric and positive semidefinite with one null eigenvalue, and the same properties hold for σ, meaning that $\eta_t \in \mathcal{P}_0(\Sigma)$ for any t.

Proof. The key observation is that we can reduce ourselves to study the asymptotics of the fluctuation flow

$$\mu_t^N := \sqrt{N}(\overline{m}_t^N - m_t), \tag{3.35}$$

which is more standard since $\overline{m}_t^N$, whose generator $\overline{\mathfrak{L}}$ is defined in (3.29), is the empirical measure of an uncontrolled system of N mean-field interacting particles. Indeed, by (3.17) we have that $\sqrt{N}(m^N - \overline{m}^N)$ tends to 0 almost surely as N goes to infinity.

Thus, it remains to prove the convergence in law of (3.35) to the solution to (3.31). The convergence of μ_0^N (and η_0^N) to the initial condition $\bar{\eta}$ follows from the Central Limit Theorem for the i.i.d. sequence of initial conditions ξ_i in systems (22) and (28). Then, we compute the generator of (3.35) for $t \geq 0$. We note that μ_t^N is obtained from $\overline{m}_t^N$ through a time dependent, linear invertible transformation $\Phi_t : S_N \to \mathcal{P}_0(\Sigma) \subset \mathbb{R}^d$, defined by

$$\Phi_t(m) := \sqrt{N}(m - m_t),$$

with inverse $\Phi_t^{-1}(\mu) := m_t + \frac{\mu}{\sqrt{N}}$. Thus, the generator $\mathfrak{M}_t^N$ of (3.35) can be written as

$$\mathfrak{M}_t^N g(\mu) = \overline{\mathfrak{L}}_t^N [g \circ \Phi_t](\Phi_t^{-1}(\mu)) + \frac{\partial}{\partial t}[g \circ \Phi_t](\Phi_t^{-1}(\mu)), \tag{3.36}$$

for any $g : \mathcal{P}_0(\Sigma) \to \mathbb{R}$ regular and with compact support (we can extend the definition of g to be a smooth function in the whole space $\mathbb{R}^d$, so that the usual derivatives are well defined). We have

$$\begin{aligned}\frac{\partial}{\partial t}[g \circ \Phi_t](\Phi_t^{-1}(\mu)) &= -\sqrt{N}\nabla_\mu g(\mu) \cdot \frac{d}{dt} m_t = -\sqrt{N}\nabla_\mu g(\mu) \cdot \left(\Gamma\,(t, m_t)^\dagger\, m_t\right) \\ &= -\sqrt{N} \sum_{x,y\in\Sigma} \frac{\partial}{\partial \mu_y} g(\mu)\Gamma_{x,y}(t, m_t)(m_t)_x.\end{aligned}$$

where the second equality follows from the KFP equation for m_t. For the remaining part in (3.36), we have

$$\begin{aligned}\overline{\mathfrak{L}}_t^N [g \circ \Phi_t](\Phi_t^{-1}(\mu)) &= N \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \times \\ &\quad \times \left\{[g \circ \Phi_t]\left(m_t + \frac{\mu}{\sqrt{N}} + \frac{1}{N}(\delta_y - \delta_x)\right) - [g \circ \Phi_t]\left(m_t + \frac{\mu}{\sqrt{N}}\right)\right\} \\ &= N \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \left\{g\left(\mu + \frac{1}{\sqrt{N}}(\delta_y - \delta_x)\right) - g(\mu)\right\}.\end{aligned}$$

Thus, we have found

$$\begin{aligned}\mathfrak{M}_t^N g(\mu) &= N \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \left\{g\left(\mu + \frac{1}{\sqrt{N}}(\delta_y - \delta_x)\right) - g(\mu)\right\} \\ &\quad - \sqrt{N} \sum_{x,y\in\Sigma} \frac{\partial}{\partial \mu_y} g(\mu)\Gamma_{x,y}(t, m_t)(m_t)_x.\end{aligned}$$

In order to perform a Taylor expansion of the generator, we first develop the term

$$\begin{aligned}& g\left(\mu + \frac{1}{\sqrt{N}}(\delta_y - \delta_x)\right) - g(\mu) \\ &= \frac{1}{\sqrt{N}} \nabla_\mu g(\mu) \cdot (\delta_y - \delta_x) + \frac{1}{2N}(\delta_y - \delta_x)^\dagger D^2_{\mu\mu} g(\mu)(\delta_y - \delta_x) + O\left(\frac{1}{N^{3/2}}\right).\end{aligned}$$

Substituting, we get

$$\begin{aligned}\mathfrak{M}_t^N g(\mu) &= \sqrt{N} \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \nabla_\mu g(\mu) \cdot (\delta_y - \delta_x) \\ &\quad + \frac{1}{2} \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) (\delta_y - \delta_x)^\dagger D^2_{\mu\mu} g(\mu)(\delta_y - \delta_x) \\ &\quad - \sqrt{N} \sum_{x,y\in\Sigma} \frac{\partial}{\partial \mu_y} g(\mu)\Gamma_{x,y}(t, m_t)(m_t)_x + O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

Now, we note that

$$\begin{aligned}& \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \nabla_\mu g(\mu) \cdot (\delta_y - \delta_x) \\ &\quad = \sum_{x,y\in\Sigma} \left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) \frac{\partial}{\partial \mu_y} g(\mu),\end{aligned}$$

since $\sum_y \Gamma_{x,y} = 0$. This property allows us to rewrite

$$\begin{aligned}\mathfrak{M}_t^N g(\mu) = &\sum_{x,y\in\Sigma} \mu_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right)\frac{\partial}{\partial \mu_y} g(\mu)\\ &+ \sqrt{N}\sum_{x,y\in\Sigma}(m_t)_x \frac{\partial}{\partial\mu_y} g(\mu)\left[\Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) - \Gamma_{x,y}(t,m_t)\right]\\ &+ \frac{1}{2}\sum_{x,y\in\Sigma}\left(m_t + \frac{\mu}{\sqrt{N}}\right)_x \Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right)(\delta_y - \delta_x)^\dagger D^2_{\mu\mu} g(\mu)(\delta_y - \delta_x) + O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

Then, using the Lipschitz continuity of Γ as we did in Proposition 3, we linearize the term

$$\Gamma_{x,y}\left(t, m_t + \frac{\mu}{\sqrt{N}}\right) - \Gamma_{x,y}(t,m_t) = \frac{1}{\sqrt{N}} D^m\Gamma_{x,y}(t,m_t,1)\cdot\mu + O\left(\frac{1}{N}\right).$$

We thus deduce that

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\sup_{\mu\in\mathcal{P}_0(\Sigma)} |\mathfrak{M}_t^N g(\mu) - \mathfrak{M}_t g(\mu)| = 0$$

for any g, the convergence being of order $\frac{1}{\sqrt{N}}$, where

$$\begin{aligned}\mathfrak{M}_t g(\mu) := &\sum_{x,y\in\Sigma}\mu_x\Gamma_{x,y}(t,m_t)\frac{\partial}{\partial\mu_y} g(\mu) + \sum_{x,y\in\Sigma}(m_t)_x\left[D^m\Gamma_{x,y}(t,m_t,1)\cdot\mu\right]\frac{\partial}{\partial\mu_y}g(\mu)\\ &+ \frac{1}{2}\sum_{x,y\in\Sigma}(m_t)_x\Gamma_{x,y}(t,m_t)(\delta_y-\delta_x)^\dagger D^2_{\mu\mu}g(\mu)(\delta_y-\delta_x).\end{aligned} \tag{3.37}$$

The proof is then completed if we show that the generator (3.37) is associated to the SDE (3.31).

The drift component can be immediately identified, since

$$\sum_{x,y\in\Sigma}\mu_x\Gamma_{x,y}(t,m_t)\frac{\partial}{\partial\mu_y}g(\mu) = \left(\Gamma(t,m_t)^\dagger\mu\right)\cdot\nabla_\mu g(\mu),$$

and

$$\sum_{x,y\in\Sigma}(m_t)_x\left[D^m\Gamma_{x,y}(t,m_t,1)\cdot\mu\right]\frac{\partial}{\partial\mu_y}g(\mu) = b(t,\mu)\cdot\nabla_\mu g(\mu).$$

For the diffusion component, we first note that, for each $x, y \in \Sigma$,

$$(\delta_y-\delta_x)^\dagger D^2_{\mu\mu}g(\mu)(\delta_y-\delta_x) = \frac{\partial^2}{\partial\mu_y\mu_y}g(\mu) + \frac{\partial^2}{\partial\mu_x\mu_x}g(\mu) - \frac{\partial^2}{\partial\mu_x\mu_y}g(\mu) - \frac{\partial^2}{\partial\mu_y\mu_x}g(\mu),$$

so that

$$\begin{aligned}&\frac{1}{2}\sum_{x,y\in\Sigma}(\delta_y-\delta_x)^\dagger D^2_{\mu\mu}g(\mu)(\delta_y-\delta_x)(m_t)_x\Gamma_{x,y}(t,m_t)\\ &\quad= \frac{1}{2}\sum_{x,y\in\Sigma}\left[\frac{\partial^2}{\partial\mu_y\mu_y}g(\mu) + \frac{\partial^2}{\partial\mu_x\mu_x}g(\mu) - \frac{\partial^2}{\partial\mu_x\mu_y}g(\mu) - \frac{\partial^2}{\partial\mu_y\mu_x}g(\mu)\right](m_t)_x\Gamma_{x,y}(t,m_t),\end{aligned}$$

which is equal to

$$\frac{1}{2}\mathrm{Tr}(\sigma^2(t,m_t)D^2_{\mu\mu}g(\mu)) = \frac{1}{2}\sum_{x,y\in\Sigma}(\sigma^2(t,m_t))_{x,y}\frac{\partial^2}{\partial\mu_x\mu_y}g(\mu),$$

if we define $(\sigma^2)_{x,y\in\Sigma}$ by the relations (3.33) and (3.34).

Finally, we observe that the limiting process η_t defined in (3.31) takes values in $\mathcal{P}_0(\Sigma)$, as required. Indeed, by diagonalizing σ^2 - which is symmetric and such that its rows sum to 0 - we get that all the eigenvectors, besides the constant one relative to the null eigenvalue, have components which sum to 0 (by orthogonality). The same properties hold for the square root matrix σ, so that equation (3.31) preserves the space $\mathcal{P}_0(\Sigma)$. □

3.2.3 Large Deviation Principle

We state the large deviation result, which is a sample path Large Deviation Principle on $D([0,T];\mathcal{P}(\Sigma))$. To define the rate function, we first introduce the local rate function $\chi : \mathbb{R} \longrightarrow [0,+\infty]$,

$$\chi(r) := \begin{cases} r\log r - r + 1 & r > 0, \\ 1 & r = 0, \\ +\infty & r < 0. \end{cases} \tag{3.38}$$

For $t \in [0,T]$, $m \in \mathcal{P}(\Sigma)$ and $\mu \in \mathcal{P}_0(\Sigma)$, define

$$\Lambda(t,m,\mu) := \inf \left\{ \sum_{x,y\in\Sigma} m_x \Gamma_{x,y}(t,m) \chi\left(\frac{q_{x,y}}{\Gamma_{x,y}(t,m)}\right) : q_{x,y} \geq 0, \sum_{x,y\in\Sigma} q_{x,y}(\delta_y - \delta_x) = \mu \quad \forall x,y \right\} \tag{3.39}$$

and set, for $\lambda : [0,T] \longrightarrow \mathcal{P}(\Sigma)$,

$$\mathcal{I}(\lambda) := \begin{cases} \int_0^T \Lambda(t,\lambda(t),\dot{\lambda}(t))dt & \text{if } \lambda \text{ is absolutely continuous and } \lambda(0) = m_0 \\ +\infty & \text{otherwise.} \end{cases} \tag{3.40}$$

We need the additional property, common in large deviations theory, that the processes are ergodic, which is guaranteed for instance Assumption (Erg). We are now able to state the Large Deviation Principle. We equip $D([0,T];\mathcal{P}(\Sigma))$ with the Skorokhod J_1-topology and denote by $\mathcal{B}(D([0,T];\mathcal{P}(\Sigma)))$ the associated Borel σ-algebra.

Theorem 3.8 (Large Deviation Principle)**.** *Let U be a regular solution to the master equation and assume (LipH) and (Erg). Also, assume that the initial conditions $(m_0^N)_{N\in\mathbb{N}}$ are deterministic and $\lim_N m_0^N = m_0$. Then the sequence of empirical measure processes $(m^N)_{N\in\mathbb{N}}$ satisfies the sample path Large Deviation Principle on $D([0,T];\mathcal{P}(\Sigma))$ with the (good) rate function $\mathcal{I}$. Specifically,*

(i) if $E \in \mathcal{B}(D([0,T];\mathcal{P}(\Sigma)))$ is closed then

$$\limsup_N \frac{1}{N} \log P(m^N \in E) \leq - \inf_{\lambda\in E} \{\mathcal{I}(\lambda)\} \tag{3.41}$$

(ii) if $E \in \mathcal{B}(D([0,T];\mathcal{P}(\Sigma)))$ is open then

$$\liminf_N \frac{1}{N} \log P(m^N \in E) \geq - \inf_{\lambda\in E} \{\mathcal{I}(\lambda)\} \tag{3.42}$$

(iii) For any $M < +\infty$ the set

$$\{\lambda \in D([0,T];\mathcal{P}(\Sigma)) : \mathcal{I}(\lambda) \leq M\} \tag{3.43}$$

is compact.

We remark that the initial conditions are assumed to be deterministic only for simplicity, otherwise there would be another term in the rate function $\mathcal{I}$. Before proving Theorem 3.8, let us give another characterization of $\mathcal{I}$. For $m \in \mathcal{P}(\Sigma)$ and $p \in \mathbb{R}^d$, define

$$\Psi(t,m,p) := \sum_{x,y} m_x \Gamma_{x,y}(t,m) \left[e^{p\cdot(\delta_y - \delta_x)} - 1 \right] \tag{3.44}$$

and let Λ^0 be the Legendre transform of Ψ:

$$\Lambda^0(t,m,\mu) = \sup_{p\in\mathbb{R}^d} \left[p\cdot\mu - \Psi(t,m,p) \right]. \tag{3.45}$$

Define $\mathcal{I}^0$ as in (3.40) but with Λ replaced by Λ^0. Via a standard result in convex analysis, Proposition 6.2 in [39] shows that $\Lambda = \Lambda^0$ and then $\mathcal{I} = \mathcal{I}^0$.

Several authors studied large deviation properties of mean field interacting processes similar to ours. However, most of them deal with the case in which the prelimit jump rates, $m_x^N \Gamma^N$, are constant and equal to the limiting rates $m_x \Gamma$; see e.g. [69], [74] and [73]. We mention that in this latter paper, as in many others, it is also assumed that the jump rates of the prelimit process are bounded from below and away from 0; this does not apply to our case, since the number of agents in a state x could be zero, implying that $m_x^N \Gamma_{x,y}^N$ might also be zero.

To prove the claim, we apply the results in [39]: to our knowledge, it is the first paper which proves a Large Deviation Principle considering the jump rates of any player depending on N (and deals also with systems with simultaneous jumps). Theorem 3.4.1 in [77] shows, however, the exponential equivalence of the processes m^N and the processes $\overline{m}^N$ given by (3.29) in which the jump rates of the prelimit system $m_x^N \Gamma^N$ are replaced by $m_x \Gamma$, which does not depend on N; the proof uses a coupling of the two Markov chains. The results in [39] and [77] are derived assuming the following properties:

1. the dynamics of any agent is ergodic and the jump rates are uniformly bounded;
2. for each $x, y \in \Sigma$, the limiting jump rates $\Gamma_{x,y}$ are Lipschitz continuous in m;
3. for each $x, y \in \Sigma$, given any sequence $m^N \in S_N$ such that $\lim_N m^N = m$,

$$\lim_N \sup_{0\le t\le T} |\Gamma_{x,y}^N(t,m^N) - \Gamma_{x,y}(t,m)| = 0. \tag{3.46}$$

We are now in the position to prove the large deviations result.

Proof of Theorem 3.8. The fact that $\mathcal{I}$ is a good rate function, i.e condition (iii), is proved for instance in Theorem 1.1 of [38]. Due to Theorem 3.9 in [39], in order to prove the claims (i) and (ii), it is enough to show the three properties above. Actually [39] studies time homogeneous Markov processes, but their results still apply in the non-homogeneous case if one proves the uniform in time convergence given by (3.46).

Property (1) holds in our model since the jump rates of any player are bounded below away from 0 by Assumption (Erg), and are bounded above by Lemma 1.7. Property (2) is true because of the regularity of the solution U to the master equation. Let $x, y \in \Sigma$, $m^N = m_{\boldsymbol{x}}^N \in S_N$, $\boldsymbol{x} = (x_1, \dots, x_N) \in \Sigma^N$ and $m_{\boldsymbol{x}}^N \to m$. Then

$$\begin{aligned} |\Gamma_{x,y}^N(t,m_{\boldsymbol{x}}^N) - \Gamma_{x,y}(t,m)| &\le |\Gamma_{x,y}^N(t,m_{\boldsymbol{x}}^N) - \Gamma_{x,y}(t,m_{\boldsymbol{x}}^N)| \\ &+ |[\Gamma_{x,y}(t,m_{\boldsymbol{x}}^N) - \Gamma_{x,y}(t,m)| =: A + B. \end{aligned}$$

The first term goes to zero, uniformly over time, thanks to (3.1) and the Lipschitz continuity of a^*. While B converges to 0, uniformly over t, for the regularity of U. □

3.3 The master equation: well-posedness and regularity

In this section we study the well-posedness of Equation (M) under the assumptions of monotonicity and regularity for F, G, H we already introduced (Mon), (RegFG), (RegH). A preliminary remark is that, thanks to Proposition 1 in [46], if H is differentiable (and this is indeed the case of our assumptions) then

$$a_x^*(y,p) = -\frac{\partial}{\partial p_x} H(y,p). \tag{3.47}$$

For this reason, we will in the following use a^* interchangeably with $-D_p H$.

Theorem 3.9. *Assume (Mon), (RegFG) and (RegH). Then there exists a unique classical solution to* (M) *in the sense of Definition 1.17. Moreover it is regular.*

The proof exploits the renowned method of characteristics, which consists in proving that

$$U(t_0, x, m_0) := u(t_0, x) \tag{3.48}$$

solves (M), u being the solution of the mean field game system (MFG) with initial time t_0 and initial distribution m_0. In order to perform the computations, we have to prove the regularity in m of the function $U(t_0, x, m)$ defined above. In particular, we have to show that $D^m U$ exists and is bounded. For this, we follow the strategy shown in [14] - which is developed in infinite dimension - adapting it to our discrete setting. The idea consists in studying the well-posedness and regularity properties of the linearized version of the system (MFG), whose solution will end up coinciding with $D^m U \cdot \mu_0$, for all possible directions $\mu_0 \in \mathcal{P}_0(\Sigma)$. In the remaining part of this section, C will denote any constant which does not depend on t_0, m_0, and is allowed to change from line to line.

3.3.1 Estimates on the mean field game system

We start by proving the well-posedness of the system (MFG)

$$\begin{cases} -\frac{d}{dt} u(t,x) + H(x, \Delta^x u(t,x)) = F(x, m(t)), \\ \frac{d}{dt} m_x(t) = \sum_y m_y(t) a_x^*(y, \Delta^y u(t,y)), \\ u(T,x) = G(x, m(T)), \\ m_x(t_0) = m_{x,0}, \end{cases}$$

and a useful a priori estimate on its solution (u, m). The existence of solutions follows from Theorem 2.8; see also Proposition 4 of [46]. On the other hand the uniqueness of solution, under our assumptions, is a consequence of the following a priori estimates. Before stating the proposition, we recall the notation $||u|| := \sup_{t \in [t_0, T]} \max_{x \in \Sigma} |u(t,x)|$.

Proposition 3.10. *Assume (Mon), (RegFG) and (RegH). Let* (u_1, m_1) *and* (u_2, m_2) *be two solutions to* (MFG) *with initial conditions* $m_1(t_0) = m_0^1$ *and* $m_2(t_0) = m_0^2$*. Then*

$$||u_1 - u_2|| \le C|m_0^1 - m_0^2|, \tag{3.49}$$

$$||m_1 - m_2|| \le C|m_0^1 - m_0^2|. \tag{3.50}$$

Proof. Without loss of generality, let us set $t_0 = 0$. Let $u := u_1 - u_2$ and $m := m_1 - m_2$. The proof is carried out in three steps.

Step 1. Use of Monotonicity. The couple (u, m) solves

$$\begin{cases} -\frac{d}{dt}u(t,x) + H(x, \Delta^x u_1(t,x)) - H(x, \Delta^x u_2(t,x)) = F(x, m_1(t)) - F(x, m_2(t)) \\ \frac{d}{dt}m(t,x) = \sum_y \left[m_1(t,y)a_x^*(y, \Delta^y u_1(t,y)) - m_2(t,y)a_x^*(y, \Delta^y u_2(t,y))\right] \\ u(T,x) = G(x, m_1(T)) - G(x, m_2(T)) \\ m(0,x) = m_0^1 - m_0^2. \end{cases} \tag{3.51}$$

Since $\frac{d}{dt}\sum_x m(x)u(x) = \sum_x m(x)\frac{du}{dt}(x) + \sum_x \frac{dm}{dt}(x)u(x)$, integrating over $[0, T]$ we have

$$\begin{aligned} &\sum_x \left[m(T,x)u(T,x) - m(0,x)u(0,x)\right] \\ &= \int_0^T \sum_x \left[H(x, \Delta^x u_1) - H(x, \Delta^x u_2) - F(x, m_1) + F(x, m_2)\right](m_1(x) - m_2(x))dt \\ &+ \int_0^T \sum_x \sum_y \left[m_1(y)a_x^*(y, \Delta^y u_1) - m_2(y)a_x^*(y, \Delta^y u_2)\right](u_1(x) - u_2(x))dt. \end{aligned}$$

Using the fact that $\sum_x a_x^*(y) = 0$ and the initial-final data, we can rewrite

$$\begin{aligned} &\sum_x [G(x, m_1) - G(x, m_2)](m_1(x) - m_2(x)) + \int_0^T \sum_x \left[F(x, m_1) - F(x, m_2)\right](m_1(x) - m_2(x))dt \\ &= \sum_x (m_0^1(x) - m_0^2(x))(u_1(0,x) - u_2(0,x)) \\ &\quad + \int_0^T \sum_x \{[H(x, \Delta^x u_1) - H(x, \Delta^x u_2)](m_1(x) - m_2(x)) \\ &\quad + \Delta^x u \cdot [m_1(x)a^*(x, \Delta^x u_1) - m_2(x)a^*(x, \Delta^x u_2)]\}\, dt. \end{aligned}$$

We now apply the monotonicity of F and G in the first line and the uniform convexity of H in the last two lines. In fact, recalling that $a_y^*(x,p) = -\frac{\partial}{\partial p_y}H(x,p)$, by (RegH) we have that, for each x,

$$\begin{aligned} H(x, \Delta^x u_1) - H(x, \Delta^x u_2) - \Delta^x u \cdot \frac{\partial}{\partial p}H(x, \Delta^x u_1) &\le -C^{-1}|\Delta^x u|^2 \\ H(x, \Delta^x u_2) - H(x, \Delta^x u_1) + \Delta^x u \cdot \frac{\partial}{\partial p}H(x, \Delta^x u_2) &\le -C^{-1}|\Delta^x u|^2. \end{aligned}$$

Hence we obtain

$$\int_0^T \sum_x |\Delta^x u(x)|^2(m_1(x) + m_2(x))dt \le C(m_0^1 - m_0^2) \cdot (u_1(0) - u_2(0)) \tag{3.52}$$

Step 2. Estimate on Kolmogorov-Fokker-Planck equation. Integrating the second equation in (3.51) over $[0, t]$, we get

$$m(t,x) = m(0,x) + \int_0^t \sum_y \left[m_1(s,y)a_x^*(y, \Delta^y u_1(s,y)) - m_2(s,y)a_x^*(y, \Delta^y u_2(s,y))\right] ds.$$

The boundedness and Lipschitz continuity of the rates give

$$\max_x |m(t,x)| \le C|m_0^1 - m_0^2| + C\int_0^t \max_x |m(s,x)|ds + C\int_0^t \sum_x |\Delta^x u(s,x)|m_1(s,x)ds$$

and hence, by Gronwall's Lemma,

$$||m|| \leq C|m_0^1 - m_0^2| + C\int_0^T \sqrt{\sum_x |\Delta^x u(t,x)|^2 m_1(x)}dt. \tag{3.53}$$

This, together with inequality (3.52), yields

$$||m|| \leq C(|m_0^1 - m_0^2| + |m_0^1 - m_0^2|^{1/2}||u||^{1/2}). \tag{3.54}$$

Step 3. Estimate on Hamilton-Jacobi-Bellman equation. Integrating the first equation in (3.51) over $[t,T]$, we get

$$u(t,x) = G(x,m_1(T)) - G(x,m_2(T)) + \int_t^T [F(x,m_1) - F(x,m_2) + H(x,\Delta^x u_2) - H(x,\Delta^x u_1)]\, ds.$$

Using the Lipschitz continuity of F, G, H and the bound $\max_x |\Delta^x u(x)| \leq C \max_x |u(x)|$ we obtain

$$\max_x |u(t,x)| \leq C|m_1(T) - m_2(T)| + C\int_t^T |m_1(s) - m_2(s)|ds + C\int_t^T \max_x |u(s,x)|ds$$

and then Gronwall's Lemma gives

$$||u|| \leq C||m||. \tag{3.55}$$

This bound (3.55) and estimate (3.54) yield claim (3.50), using the convexity inequality $AB \leq \varepsilon A^2 + \frac{1}{4\varepsilon}B^2$ for $A, B > 0$. Again (3.55) finally proves claim (3.49). □

3.3.2 Linearized mean field game system

For proving Theorem 3.9, we introduce the linearized version of system (MFG) around its solutions and then prove that it provides the derivative of $u(t_0,x)$ with respect to the initial condition m_0.

As a preliminary step, we study a related linear system of ODE's, which will come useful several times.

$$\begin{cases} -\frac{d}{dt}w(t,x) - a^*(x,\Delta^x u)\cdot\Delta^x w(t,x) = D^m F(x,m(t),1)\cdot\eta(t) + b(t,x) \\ \frac{d}{dt}\eta(t,x) = \sum_y \eta_y a_x^*(y,\Delta^y u) + \sum_y m_y(t) D_p a_x^*(y,\Delta^x u)\cdot\Delta^y w + c(t,x) \\ w(T,x) = D^m G(x,m(T),1)\cdot\eta(T) + w_T(x) \\ \eta(t_0,\cdot) = \eta_0, \end{cases} \tag{3.56}$$

The unknowns are w and η, while b, c, w_T, η_0 are given measurable functions, with $c(t) \in \mathcal{P}_0(\Sigma)$, and (u,m) is the solution to (MFG). We state an immediate but useful estimate regarding the first of the two equations in (3.56).

Lemma 3.11. *If (RegFG) holds then the equation*

$$\begin{cases} -\frac{d}{dt}w(t,x) - a^*(x,\Delta^x u)\cdot\Delta^x w(t,x) = D^m F(x,m(t),1)\cdot\eta(t) + b(t,x) \\ w(T,x) = D^m G(x,m(T),1)\cdot\eta(T) + w_T(x) \end{cases} \tag{3.57}$$

has a unique solution for each final condition $w_T(x)$ and satisfies

$$||w|| \leq C\left[\max_x |w_T(x)| + ||\eta|| + ||b||\right]. \tag{3.58}$$

Proof. The well-posedness of the equation is immediate from classical ODE's theory. Integrating over the time interval $[t,T]$ and using that

$$a^*(x,\Delta^x u)\cdot\Delta^x w(t,x)=\sum_y a^*_y(x,\Delta^x u)w_y(t),$$

we find

$$w(t,x)-w(T,x)-\int_t^T\sum_y a^*_y(x,\Delta^x u)w_y(s)ds=\int_t^T D^mF\cdot\eta(s)ds+\int_t^T b(s,x)ds.$$

Substituting the expression for $w(T,x)$, and using the bound on the control and on the derivatives of F and G we can estimate

$$\begin{aligned}\max_x|w(t,x)|&\le\max_x|w_T(x)|+C\max_x|\eta(T,x)|\\&+C\int_t^T\max_x|w(s,x)|ds+C\int_t^T\max_x|\eta(s,x)|ds+\int_t^T\max_x|b(s,x)|ds\end{aligned}$$

and thus, applying Gronwall's Lemma and taking the supremum on t, we get (3.58). □

In the next result we prove the well-posedness of system (3.56) together with useful a priori estimates on its solution.

Proposition 3.12. *Assume (RegH), (Mon) and (RegFG). Then for any (measurable)* b,c,w_T, *the linear system* (3.56) *has a unique solution* $(w,\eta)\in\mathcal{C}^1([0,T];\mathbb{R}^d\times\mathcal{P}_0(\Sigma))$. *Moreover it satisfies*

$$||w||\le C(|w_T|+||b||+||c||+|\eta_0|)\tag{3.59}$$
$$||\eta||\le C(|w_T|+||b||+||c||+|\eta_0|).\tag{3.60}$$

Proof. Without loss of generality we assume $t_0=0$. We use a fixed-point argument to prove the existence of a solution to (3.56). Uniqueness will be then implied by estimates (3.59) and (3.60), thanks to the linearity of the system.

We define the map $\Phi:\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))\to\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$ as follows: for a fixed $\eta\in\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$ we consider the solution $w=w(\eta)$ to equation (3.57), and define $\Phi(\eta)$ to be the solution of the second equation in (3.56) with $w=w(\eta)$. In order to prove the existence of a fixed point of Φ, which is clearly a solution to (3.56), we apply Leray-Schauder Fixed Point Theorem. We remark the fact that more standard fixed point theorems are not applicable to this situation since we cannot assume that η belongs to a compact subspace of $\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$, since $\mathcal{P}_0(\Sigma)$ is not compact. First of all, we note that $\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$ is convex and that the map Φ is trivially continuous, because of the linearity of the system. Moreover, using the equation for η in system (3.56), it is easy to see that Φ is a compact map, i.e. it sends bounded sets of $\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$ into bounded sets of $\mathcal{C}^1([0,T];\mathcal{P}_0(\Sigma))$. Thus, to apply Leray-Schauder Theorem it remains to prove that the set $\{\eta\ :\ \eta=\lambda\Phi(\eta)$ for some $\lambda\in[0,1]\}$ is bounded in $\mathcal{C}([0,T];\mathcal{P}_0(\Sigma))$.

Let us fix a η such that $\eta=\lambda\Phi(\eta)$. Then the couple (w,η) solves

$$\begin{cases}-\frac{d}{dt}w(t,x)-a^*(x,\Delta^x u)\cdot\Delta^x w(t,x)=\lambda\left(D^mF(x,m(t),1)\cdot\eta(t)+b(t,x)\right)\\ \frac{d}{dt}\eta(t,x)=\sum_y\eta_y a^*_x(y,\Delta^y u)+\lambda\left(\sum_y m_y(t)D_pa^*_x(y,\Delta^x u)\cdot\Delta^y w+c(t,x)\right)\\ w(T,x)=\lambda\left(D^mG(x,m(T),1)\cdot\eta(T)+w_T(x)\right)\\ \eta(t_0,\cdot)=\lambda\eta_0.\end{cases}$$

First, we note that we can restrict to $\lambda > 0$, since otherwise $\eta = 0$. Therefore, we can use the equations (for brevity we omit the dependence of a^* on the second variable) to get

$$\begin{aligned}\frac{d}{dt}\sum_x w(t,x)\eta_x(t) =& -\lambda\sum_x \eta(t,x)[D^m F(x,m(t),1)\cdot\eta(t) + b(t,x)]\\ &-\sum_{x,y}\eta_x(t)a^*_y(x)[w(t,y)-w(t,x)] + \sum_{x,y}\eta_y(t)a^*_x(y)w(t,x)\\ &+\lambda\sum_{x,y} m_y w(t,x)D_p\alpha^*_x(y)\cdot\Delta^y w + \lambda\sum_x c(t,x)w(t,x).\end{aligned}$$

The second line is 0, using the fact that $\sum_x \eta_x(t) = 0$ and changing x and y in the second double sum. Integrating over $[0,T]$ and using the expression for $w(T,x)$ we obtain

$$\begin{aligned}&\lambda\sum_x \eta_x(T)\,[D^m G(x,m(T),1)\cdot\eta(T) + w_T(x)] - \lambda w(0)\cdot\eta_0\\ &\quad= -\lambda\int_0^T\sum_x \eta_x(t)[D^m F(x,m(t),1)\cdot\eta(t) + b(t,x)]dt\\ &\quad\quad+\lambda\int_0^T\sum_{x,y} m_y D_p\alpha^*_x(y)\cdot\Delta^y w(w(t,x)-w(t,y))dt\\ &\quad\quad+\lambda\int_0^T\sum_x c(t,x)w(t,x)dt - \lambda\int_0^T \eta(t,x)D^m G(x,m(T),1)\cdot\eta(T)dt,\end{aligned}$$

where in the second term of the sum we have also used that $\sum_{x,y}[m_y D_p a^*_x(y)\cdot\Delta^y w]w(t,y) = 0$.

Dividing by $\lambda > 0$ and bringing the terms with F and G on the left hand side, together with the term in m and $D_p a^*$, we can rewrite

$$\begin{aligned}&-\int_0^T\sum_{x,y} m_y\Delta^y w D_p a^*_x(y)\cdot\Delta^y w dt + \int_0^T\sum_x \eta(t,x)[D^m F(x,m(t),1)\cdot\eta(t)]dt\\ &+\sum_x \eta(T,x)D^m G(x,m(T),1)\cdot\eta(T)\\ &\quad= -\sum_x w_T(x)\eta(T,x) + \sum_x w(0,x)\eta_0(x) - \int_0^T\sum_x \eta(t,x)b(t,x)dt + \int_0^T\sum_x c(t,x)w(t,x)dt.\end{aligned}$$

We observe that, by (Mon) and (RegFG), we have

$$\sum_x \eta(t,x)[D^m F(x,m(t),1)\cdot\eta(t)] \geq 0, \tag{3.61}$$

$$\sum_x \eta(T,x)[D^m G(x,m(T),1)\cdot\eta(T)] \geq 0. \tag{3.62}$$

Furthermore assumption (1.28) yields

$$-\int_0^T\sum_{x,y} m_y\Delta^y w D_p a^*_x(y)\cdot\Delta^y w dt \geq C^{-1}\int_0^T\sum_x m_x|\Delta^x w|^2 dt,$$

so that we can estimate the previous equality by

$$\begin{aligned}C^{-1}\int_0^T\sum_x m_x|\Delta^x w|^2 dt &\leq |w_T\cdot\eta(T)| + |w(0)\cdot\eta_0| + \int_0^T|c(t)\cdot w(t)|\,dt + \int_0^T|\eta(t)\cdot b(t)|\,dt\\ &\leq |w_T||\eta(T)| + |w(0)||\eta_0| + \int_0^T|c(t)||w(t)|dt + \int_0^T|\eta(t)||b(t)|dt\end{aligned} \tag{3.63}$$

On the other hand, by the equation for η we have

$$\eta(t,x) = \eta_0(x) + \int_0^t \sum_y \eta(s,y)\alpha_x^*(y)ds + \int_0^t \left[\sum_y m_y D_p a_x^*(y) \cdot \Delta^y w + c(x)\right] ds,$$

and thus

$$|\eta(t,x)| \le |\eta_0(x)| + M\int_0^t \sum_y |\eta_y| ds + C\int_0^t \left[\sum_y m_y|\Delta^y w| + |c(x)|\right] ds,$$

so that, by Gronwall's Lemma and taking the sum for $x \in \Sigma$ and the sup over $t \in [0,T]$,

$$\begin{aligned}
||\eta|| &\le C|\eta_0| + C\int_0^T \sum_x \sqrt{m_x}\sqrt{m_x}|\Delta^x w| dt + C||c|| \\
&\le C|\eta_0| + C\int_0^T \sqrt{\sum_x (\sqrt{m_x})^2}\sqrt{\sum_x m_x|\Delta^x w|^2} dt + C||c|| \\
&= C|\eta_0| + C\int_0^T \sqrt{\sum_x m_x|\Delta^x w|^2} dt + C||c|| \\
&\le C|\eta_0| + C\sqrt{\int_0^T \sum_x m_x|\Delta^x w|^2 dt} + C||c||.
\end{aligned}$$

Now, we use estimate (3.63) on $\int_0^T \sum_x m_x|\Delta^x w|^2$ that we found above to get

$$\begin{aligned}
||\eta|| &\le C||c|| + C|\eta_0| + C\left(|\eta_0||w(0)| + |w_T||\eta(T)| + \int_0^T |c(t)||w(t)| + \int_0^T |\eta(t)||b(t)|\right)^{\frac{1}{2}} \\
&\le C||c|| + C|\eta_0| + C\left(|w(0)|^{1/2}|\eta_0|^{1/2} + |w_T|^{1/2}|\eta(T)|^{1/2}| + ||c||^{1/2}||w||^{1/2} + ||\eta||^{1/2}||b||^{1/2}\right).
\end{aligned}$$

We further estimate the right hand side using bound (3.58):

$$\begin{aligned}
||\eta|| &\le C(||c|| + |\eta_0|) \\
&\quad + C\left[|w_T|^{1/2}|\eta(T)|^{1/2} + (||c||^{1/2} + |\eta_0|^{1/2})(|w_T|^{1/2} + ||\eta||^{1/2} + ||b||^{1/2}) + ||\eta||^{1/2}||b||^{1/2}\right].
\end{aligned}$$

Using the inequality $AB \le \varepsilon A^2 + \frac{1}{4\varepsilon}B^2$ for $A, B > 0$, we obtain

$$||\eta|| \le C(||c|| + |w_T| + ||b|| + |\eta_0|) + \frac{1}{2}||\eta||,$$

which implies (3.60). Then (3.59) follows from (3.58). □

Given the solution (u,m) to system (MFG), with initial condition m_0 for m and final condition G for u, we introduce the linearized system:

$$\begin{cases}
-\frac{d}{dt}v(t,x) - a^*(x, \Delta^x u(t,x)) \cdot \Delta^x v(t,x) = D^m F(x, m(t), 1) \cdot \mu(t) \\
\frac{d}{dt}\mu_x(t) = \sum_y \mu_y(t) a_x^*(y, \Delta^y u(t,y)) + \sum_y m_y D_p a_x^*(y, \Delta^y u) \cdot \Delta^y v(t,x) \\
v(T,x) = D^m G(x, m(T), 1) \cdot \mu(T) \\
\mu(t_0) = \mu_0 \in \mathcal{P}_0(\Sigma).
\end{cases} \tag{LIN}$$

Note that in the right hand side of the first equation $D^m F(x, m(t), 1)\cdot\mu(t) = D^m F(x, m(t), y)\cdot \mu(t)$ for every $y \in \Sigma$, using identity (1.15) and the fact that $\mu(t) \in \mathcal{P}_0(\Sigma)$ for every t (i.e. identity (1.17)). For this reason we just fixed the choice to $D^m F(x, m(t), 1)$ and $D^m G(x, m(T), 1)$ in system (LIN).

The existence and uniqueness of a solution $(v, \mu) \in \mathcal{C}^1([0, T]; \mathbb{R}^d \times \mathcal{P}_0(\Sigma))$ is ensured by Proposition 3.12. The aim is to show that the solution (v, μ) to system (LIN) satisfies

$$v(t_0, x) = D^m U(t_0, x, m_0, 1) \cdot \mu_0. \tag{3.64}$$

This proves that the solution U defined via (3.48) is differentiable with respect to m_0 in any direction μ_0, with derivative given by (3.64), and also that $D^m U$ is continuous in m. Equality (3.64) is implied by the following

Theorem 3.13. *Assume (RegH), (Mon) and (RegFG). Let (u, m) and $(\widetilde{u}, \widetilde{m})$ be the solutions to* (MFG) *respectively starting from (t_0, m_0) and $(t_0, \widetilde{m}_0)$. Let (v, μ) be the solution to* (LIN) *starting from (t_0, μ_0), with $\mu_0 := \widetilde{m}_0 - m_0$. Then*

$$||\widetilde{u} - u - v|| + ||\widetilde{m} - m - \mu|| \leq C|m_0 - \widetilde{m}_0|^2. \tag{3.65}$$

Proof. Set $w := \widetilde{u} - u - v$ and $\eta := \widetilde{m} - m - \mu$, they solve (3.56)

$$\begin{cases} -\frac{d}{dt}w(t,x) - a^*(x, \Delta^x u)\cdot \Delta^x w(t,x) = D^m F(x, m(t), 1)\cdot \eta(t) + b(t,x) \\ \frac{d}{dt}\eta(t,x) = \sum_y \eta_y a^*_x(y, \Delta^y u) + \sum_y m_y(t) D_p a^*_x(y, \Delta^x u)\cdot \Delta^y w + c(t,x) \\ w(T,x) = D^m G(x, m(T), 1)\cdot \eta(T) + w_T(x) \\ \eta(t_0, \cdot) = 0, \end{cases}$$

with

$$\begin{aligned} b(t,x) &:= A(t,x) + B(t,x) \\ A(t,x) &:= -\int_0^1 [D_p H(x, \Delta^x u + s(\Delta^x \widetilde{u} - \Delta^x u)) - D_p H(x, \Delta^x u)]\cdot (\Delta^x \widetilde{u} - \Delta^x u) ds \\ B(t,x) &:= \int_0^1 [D^m F(x, m + s(\widetilde{m} - m), 1) - D^m F(x, m, 1)]\cdot (\widetilde{m} - m) ds \\ c(t,x) &:= \sum_y (\widetilde{m}_y - m_y) D_p a^*_x(y, \Delta^y u)\cdot (\Delta^y \widetilde{u} - \Delta^y u) \\ &\quad + \sum_y \widetilde{m}_y \int_0^1 [D_p a^*_x(y, \Delta^x u + s(\Delta^x \widetilde{u} - \Delta^x u)) - D_p a^*_x(y, \Delta^y u)]\cdot (\Delta^y \widetilde{u} - \Delta^y u) ds \\ w_T(x) &:= \int_0^1 [D^m G(x, m(T) + s(\widetilde{m}(T) - m(T)), 1) - D^m G(x, m(t), 1)]\cdot (\widetilde{m}(T) - m(T)) ds. \end{aligned}$$

Using the assumptions, namely the Lipschitz continuity of $D_p H$, $D^2_{pp} H$, $D^m F$ and $D^m G$, and the bound $\max_x |\Delta^x u| \leq C|u|$, we estimate

$$\begin{aligned} ||b|| &\leq ||A|| + ||B|| \\ ||A|| &\leq C||\widetilde{u} - u||^2 \\ ||B|| &\leq C||\widetilde{m} - m||^2 \\ |w_T| &\leq C|\widetilde{m}(T) - m(T)|^2 \\ ||c|| &\leq C||\widetilde{m} - m||\cdot ||\widetilde{u} - u|| + C||\widetilde{u} - u||^2. \end{aligned}$$

Applying (3.59) and (3.60) to the above system and then (3.49) and (3.50), we obtain

$$\begin{aligned}||w|| + ||\eta|| &\leq C(|w_T| + ||b|| + ||c||)\\ &\leq C\left(||\widetilde{u} - u||^2 + ||\widetilde{m} - m||^2 + ||\widetilde{m} - m|| \cdot ||\widetilde{u} - u||\right)\\ &\leq C|m_0 - \widetilde{m}_0|^2.\end{aligned}$$

□

3.3.3 Proof of Theorem 3.9

We are finally in the position to prove the main theorem of this section.

3.3.3.1 Existence

Let U be the function defined by (3.48), i.e. $U(t_0, x, m_0) := u(t_0, m_0)$. We have shown in the above Theorem 3.13 that U is $\mathcal{C}^1$ in m, while the fact that it is $\mathcal{C}^1$ in t is clear. We compute the limit, as h tends to 0, of

$$\begin{aligned}&\frac{U(t_0+h, x, m_0) - U(t_0, x, m_0)}{h}\\ &= \frac{U(t_0+h, x, m_0) - U(t_0+h, x, m(t_0+h))}{h} + \frac{U(t_0+h, x, m(t_0+h)) - U(t_0, x, m_0)}{h}.\end{aligned} \tag{3.66}$$

For the first term, we have, for any $y \in \Sigma$,

$$\begin{aligned}&U(t_0+h, x, m(t_0+h)) - U(t_0+h, x, m(t_0))\\ &= [m_s := m(t_0) + s(m(t_0+h) - m(t_0))]\\ &= \int_0^1 \frac{\partial}{\partial(m(t_0+h) - m(t_0))} U(t_0+h, x, m_s, y)ds\\ &= \int_0^1 D^m U(t_0+h, x, m_s, y) \cdot (m(t_0+h) - m(t_0))ds\\ &= \int_0^1 ds \int_{t_0}^{t_0+h} D^m U(t_0+h, x, m_s, y) \cdot \left(\sum_{k=1}^d m_k(t) a^*(k, \Delta^k u(t))\right) dt\\ &= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{z=1}^d \sum_{k=1}^d m_k(t) \left[D^m U(t_0+h, x, m_s, y)\right]_z a_z^*(k, \Delta^k u(t))dt.\end{aligned}$$

Using identity (1.15), we obtain

$$\begin{aligned}&U(t_0+h, x, m(t_0+h)) - U(t_0+h, x, m(t_0))\\ &= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{z=1}^d \sum_{k=1}^d m_k(t) \left[D^m U(t_0+h, x, m_s, k)\right]_z a_z^*(k, \Delta^k u(t))dt\\ &+ \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{z=1}^d \sum_{k=1}^d m_k(t) \left[D^m U(t_0+h, x, m_s, y)\right]_k a_z^*(k, \Delta^k u(t))dt\\ &= \int_0^1 ds \int_{t_0}^{t_0+h} \sum_{z=1}^d \sum_{k=1}^d m_k(t) \left[D^m U(t_0+h, x, m_s, k)\right]_z a_z^*(k, \Delta^k u(t))dt,\end{aligned}$$

where the last equality follows from

$$\begin{aligned}\sum_{z=1}^{d}\sum_{k=1}^{d} m_k(t)\left[D^m U(t_0+h,x,m_s,y)\right]_k a_z^*(k,\Delta^k u(t))\\ =\sum_{k=1}^{d} m_k(t)\left[D^m U(t_0+h,x,m_s,y)\right]_k \sum_{z=1}^{d} a_z^*(k,\Delta^k u(t))=0,\end{aligned}$$

since $\sum_{z=1}^{d} a_z^*=0$, as $a_k^*(k)=-\sum_{z\neq k} a_z^*(k)$.

Summarizing, we have found that,

$$\begin{aligned}&U(t_0+h,x,m(t_0+h))-U(t_0+h,x,m(t_0))\\ &\qquad=\int_0^1 ds\int_{t_0}^{t_0+h} dt\int_\Sigma D^m U(t_0+h,x,m_s,y)\cdot a^*(y,\Delta^y u(t))m(t)(dy).\end{aligned}$$

Dividing by h and letting $h\downarrow 0$, we get

$$\begin{aligned}\lim_{h\to 0}&\frac{U(t_0+h,x,m(t_0+h))-U(t_0+h,x,m(t_0))}{h}\\ &=\int_\Sigma D^m U(t_0,x,m_0,y)\cdot a^*(y,\Delta^y u(t_0))dm_0(y)\\ &=\int_\Sigma D^m U(t_0,x,m_0,y)\cdot a^*(y,\Delta^x U(t_0,y,m_0))dm_0(y),\end{aligned}$$

using the continuity of $D^m U$ in time and dominate convergence to take the limit inside the integral in ds.

The second term in (3.66), for $h>0$, is instead

$$U(t_0+h,x,m(t_0+h))-U(t_0,x,m_0)=u_x(t_0+h)-u_x(t_0)=h\frac{d}{dt}u_x(t_0)+o(h),$$

and thus

$$\lim_{h\downarrow 0}\frac{U(t_0+h,x,m(t_0+h))-U(t_0,x,m_0)}{h}=\frac{d}{dt}u_x(t_0).$$

Finally, we can rewrite (3.66), after taking the limit $h\downarrow 0$, to obtain

$$\begin{aligned}\partial_t U(t_0,x,m_0)&=-\int_\Sigma D^m U(t_0,x,m_0,y)\cdot a^*(y,\Delta^x U(t_0,y,m_0))dm_0(y)\\ &\quad+\frac{d}{dt}u_x(t_0)=[\text{using the equation for } u]\\ &=-\int_\Sigma D^m U(t_0,x,m_0,y)\cdot a^*(y,\Delta^x U(t_0,y,m_0))dm_0(y)\\ &\quad+H(x,\Delta^x U(t_0,x,m_0))-F(x,m_0),\end{aligned}$$

and thus

$$-\partial_t U(t_0,x,m_0)+H(x,\Delta^x U(t_0,x,m_0))-\int_\Sigma D^m U(t_0,x,m_0,y)\cdot a^*(y,\Delta^y U)dm_0(y)=F(x,m_0),$$

which is exactly (M) computed in (t_0,m_0).

3.3.3.2 Uniqueness

Let us consider another solution V of (M). Since $||D^mV|| \leq C$, we know that V is Lipschitz with respect to m, and so is $\Delta^x V$. From this remark and the Lipschitz continuity of a^* with respect to p , it follows that the equation

$$\begin{cases} \frac{d}{dt}\widetilde{m}(t) = \sum_y \widetilde{m}_y(t) a^*(y, \Delta^y V(t, y, \widetilde{m}(t))) \\ \widetilde{m}(t_0) = m_0 \end{cases}$$

admits a unique solution in $[t_0, T]$.

If we now set $\widetilde{u}(t,x) := V(t,x,\widetilde{m}(t))$, we can compute (using for e.g. $D^mV(\cdot,\cdot,\cdot,1)$)

$$\begin{aligned}
\frac{d}{dt}\widetilde{u}(t,x) &= \partial_t V(t,x,\widetilde{m}(t)) + D^mV(t,x,\widetilde{m}(t),1)\cdot\frac{d}{dt}\widetilde{m}(t)\\
&= [\text{using the equation for } \widetilde{m}]\\
&= \partial_t V(t,x,\widetilde{m}(t)) + D^mV(t,x,\widetilde{m}(t),1)\cdot\left(\sum_y \widetilde{m}_y(t) a^*(y,\Delta^y V(t,y,\widetilde{m}(t)))\right)\\
&= [\text{using identity (1.15) on } D^mV(\cdot,\cdot,\cdot,1)]\\
&= \partial_t V(t,x,\widetilde{m}(t)) + \int_\Sigma D^mV(t,x,\widetilde{m}(t),y)\cdot a^*(y,\Delta^y V(t,y,\widetilde{m}(t)))\widetilde{m}(t)(dy)\\
&= [\text{using the equation for } V]\\
&= H(x,\Delta^x V(t,x,\widetilde{m}(t))) - F(x,\widetilde{m}) = H(x,\Delta^x\widetilde{u}(t,x)) - F(x,\widetilde{m}(t)),
\end{aligned}$$

and thus the pair $(\widetilde{u}(t),\widetilde{m}(t))$ satisfies

$$\begin{cases} -\frac{d}{dt}\widetilde{u}(t,x) + H(x,\Delta^x\widetilde{u}(t,x)) = F(x,\widetilde{m}(t)),\\ \frac{d}{dt}\widetilde{m}_x(t) = \sum_j \widetilde{m}_y(t) a^*_x(y,\Delta^y\widetilde{u}(t,y)),\\ \widetilde{u}(T,x) = V(T,x,\widetilde{m}(T)) = G(x,\widetilde{m}(T)),\\ \widetilde{m}(t_0) = m_0. \end{cases}$$

Namely, $(\widetilde{u},\widetilde{m})$ solves the system (MFG), whose solution is unique thanks to Proposition 3.10, so that we can conclude $V(t_0,x,m_0) = U(t_0,x,m_0)$ for each (t_0,x,m_0), and thus the uniqueness of solutions to (M) follows.

3.3.3.3 Regularity

It remains to prove that the unique classical solution defined via (3.48) is regular, in the sense of Definition 1.17, i.e. that D^mU is Lipschitz continuous with respect to m, uniformly in t, x.

So let (u_1, m_1) and (u_2, m_2) be two solution to (MFG) with initial conditions $m_1(t_0) = m_0^1$ and $m_2(t_0) = m_0^2$, respectively. Let also (v_1, μ_1) and (v_2, μ_2) be the associated solutions to (LIN) with $\mu_1(t_0) = \mu_2(t_0) = \mu_0$. Recall from equation (3.64) that $v_1(t_0, x) = D^mU(t_0, x, m_0^1, 1)\cdot\mu_0$ and $v_2(t_0, x) = D^mU(t_0, x, m_0^2, 1)\cdot\mu_0$, thus we have to estimate the norm $||v_1 - v_2||$.

Set $w := v_1 - v_2$ and $\eta := \mu_1 - \mu_2$. They solve the linear system (3.56) with $\eta_0 = 0$ and

$$\begin{aligned}
b(t,x) &:= [D^mF(x,m_1,1) - D^mF(x,m_2,1)]\cdot\mu_2 + [a^*(x,\Delta^x u_1) - a^*(x,\Delta^x u_2)]\cdot\Delta^x v_2\\
c(t,x) &:= \sum_y \mu_{2,y}\left[a^*_x(y,\Delta^y u_1) - a^*_x(y,\Delta^y u_2)\right]
\end{aligned}$$

$$+ \sum_y [m_{1,y} D_p a_x^*(y, \Delta^y u_1) - m_{2,y} D_p a_x^*(y, \Delta^y u_2)] \cdot \Delta^x v_2$$
$$w_T(x) := [D^m G(x, m_1(T), 1) - D^m G(x, m_2(t), 1)] \cdot \mu_2.$$

Using the Lipschitz continuity of $D_p H$, $D^2_{pp} H$, $D^m F$ and $D^m G$, applying the bounds (3.59) to v_2 and (3.60) to μ_2 and also (3.49) and (3.50), we estimate

$$||b|| \leq C||m_1 - m_2|| \cdot ||\mu_2|| + C||u_1 - u_2|| \cdot ||v_2|| \leq C|m_0^1 - m_0^2| \cdot |\mu_0|$$
$$||c|| \leq C||u_1 - u_2|| \cdot ||\mu_2|| + C||m_1 - m_2|| \cdot ||v_2|| + C||u_1 - u_2|| \cdot ||v_2|| \leq C|m_0^1 - m_0^2| \cdot |\mu_0|$$
$$|w_T| \leq C||m_1 - m_2|| \cdot ||\mu_2|| \leq C|m_0^1 - m_0^2| \cdot |\mu_0|.$$

Then (3.59) gives

$$||w|| \leq C(||b|| + ||c|| + |w_T|) \leq C|m_0^1 - m_0^2| \cdot |\mu_0|,$$

which, since $w(t_0, x) = \big(D^m U(t_0, x, m_0^1, 1) - D^m U(t_0, x, m_0^2, 1)\big) \cdot \mu_0$, yields

$$\begin{aligned} \max_x |D^m U(t_0, x, m_0^1, 1) &- D^m U(t_0, x, m_0^2, 1)| \\ &\leq C \max_x \sup_{\mu_0 \in \mathcal{P}_0(\Sigma)} \frac{\big|\big(D^m U(t_0, x, m_0^1, 1) - D^m U(t_0, x, m_0^2, 1)\big) \cdot \mu_0\big|}{|\mu_0|} \\ &\leq C|m_0^1 - m_0^2|. \end{aligned}$$

3.4 Uniqueness results

We conclude this chapter by giving the results about uniqueness of both the feedback MFG solution and the Nash equilibrium, and we try to provide minimal assumptions. We start by giving the proper definition of uniqueness for the mean field game. When considering only feedback controls, we can assume that the stochastic basis, the noise and the initial condition are fixed.

Definition 3.14. *Two feedback mean field game solutions* (α, m, X) *and* $(\widetilde{\alpha}, \widetilde{m}, \widetilde{X})$ *are called* equivalent *if* $m(t) = \widetilde{m}(t)$ *for any* t*. We say that* (α, m, X) *is the* unique *feedback MFG solution (for* m_0 *given) if there are no non-equivalent MFG solutions and any other solution* $(\widetilde{\alpha}, m, \widetilde{X})$ *is such that*

$$\alpha(t, X(t^-)) = \widetilde{\alpha}(t, \widetilde{X}(t^-)) \qquad \ell \otimes P \text{ - a.e. } (t, \omega); \tag{3.67}$$

or equivalently

$$\alpha(t, x) = \widetilde{\alpha}(t, x) \qquad \ell \text{ - a.e. } t, \quad m(t) \text{ - every } x. \tag{3.68}$$

We remark that the above definition does not imply that the feedback functions α and $\widetilde{\alpha}$ of two equivalent MFG solutions, when uniqueness holds, are equal for any t and x: they are equal when evaluated on the optimal process, which is in fact uniquely determined almost surely. However note that this possible non-uniqueness of the feedback controls is not really a problem, as the controls might be different in stated that are not visited by the optimal process. Instead, we will see below that the Nash equilibrium is uniquely determined for any t and $\boldsymbol{x}$. Concerning MFG solutions, we can conclude that such α and $\widetilde{\alpha}$ are equal (for each x, ℓ-a.e. x) if the optimal process visits any state at any time with a positive probability. This holds for instance either if assumption (Erg) holds or if the initial distribution m_0, which is part of the solution, is supported everywhere and the transition rates given by the optimal control are bounded. So we provide sufficient conditions for which this latter fact is a priori true, which can be of interest itself.

Lemma 3.15. *If assumption (H4) holds then any open-loop MFG solution* (π, m, X) *is such that there exists* $K > 0$ *for which* $|\pi(t,\omega)| \le K$ *for* $\ell \otimes P$*-a.e.* (t,ω).

Proof. By contradiction, suppose π is such that $[\ell \otimes P](|\pi(t)| \ge K) > 0$ for any K, and let $X = X_{\pi,m}$. We show that π can not be optimal. Let B_K be he set of (t,ω) where $|\pi(t)| \ge K$ and let π_K be the truncated (bounded) control

$$\pi_K(t) = \begin{cases} \pi(t) & \text{in } B_K^C \\ a_0(X(t)) & \text{in } B_K, \end{cases}$$

where $a_0(x)$ is the unique minimizer in a of $L(x,a)$, whose value is $L(x, a_0(x)) = L_0(x)$. Denoting by X_K the process corresponding to π_K, we get that there exists a constant C such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X(t) - X_K(t)|\right] \le C\mathbb{E}\left[\int_0^T |\pi(t) - \pi_K(t)|dt\right],$$

thanks to (1.12), by proceeding as in the proof of Theorem 3.5. By using also the Lipschitz continuity of L, F and G in x, we find (by possibly changing C)

$$\begin{aligned} &J(\pi,m) - J(\pi_K,m) \\ &\ge \mathbb{E}\left[\int_0^T (L(X(t),\pi(t)) - L(X_K(t), a_0(X(t))))\,\mathbb{1}_{B_K} dt\right] - C\mathbb{E}\left[\sup_{t\in[0,T]} |X(t) - X_K(t)|\right] \\ &\ge C\mathbb{E}\left[\int_0^T (L(X(t),\pi(t)) - L(X(t), a_0(X(t))))\,\mathbb{1}_{B_K} dt\right] - C\mathbb{E}\left[\int_0^T |\pi(t) - a_0(X(t))|\mathbb{1}_{B_K} dt\right]. \end{aligned}$$

The uniform convexity of L implies that there exists a constant c_0 such that

$$L(x,a) - L(x,a_0(x)) \ge c_0|a - a_0(x)|^2$$

for any a. Therefore, for $K > |a_0|_\infty$, we obtain

$$\begin{aligned} J(\pi,m) &- J(\pi_K,m) \\ &\ge c_0C\mathbb{E}\left[\int_0^T |\pi(t) - a_0(X(t))|^2\mathbb{1}_{B_K} dt\right] - C\mathbb{E}\left[\int_0^T |\pi(t) - a_0(X(t))|\mathbb{1}_{B_K} dt\right] \\ &\ge C(c_0(K - |a_0|_\infty) - 1)\mathbb{E}\left[\int_0^T |\pi(t) - a_0(X(t))|\mathbb{1}_{B_K} dt\right] \\ &\ge C(c_0(K - |a_0|_\infty) - 1)(K - |a_0|_\infty)[\ell \otimes P](|\pi(t)| \ge K) \end{aligned}$$

which is positive if K is large enough, implying that π is not optimal. □

We observe that, by Theorem 2.8, if e.g. (H4) holds, then the feedback mean field game solution is unique, in the sense of Definition 3.14, if and only if there are no non-equivalent MFG solutions. In order to have uniqueness, we must require the feedback control to be unique for m fixed, which holds e.g. under (H4), and so it can be uniquely determined via

(1.69), even if it is not determined for any t, x, but only in the sense of (3.68). Hence we now look for solutions only in the form (α_m, m), where α_m has to be uniquely defined and given by (1.69). In fact, we look for uniqueness of solutions of the mean field game system (MFG). Note that the optimal process can be omitted in the notation and that, in order to have uniqueness when α_m is well defined, we have just to show that there are no MFG solutions with different m, i.e. non-equivalent.

3.4.1 Uniqueness of the mean field game solution for small time

In this subsection, we consider a slightly more general setting: namely, the cost does not split into $c = L + F$ (Assumption H3) and the transition rate is not exactly the control. Moreover, we provide quantitative estimates for the maximal time for which there is uniqueness of feedback MFG solutions.

We assume (H1), that is, we focus only on the dynamics for γ in (1.8), and ν defined in (1.9) with $\Theta := [0, K]^d$. Moreover, we assume that $A = \Theta$ (see Lemma 1.7) and, for $x \neq y$,

$$\Gamma_{x,y}(t, a, m) = a_y + \zeta(m),$$

where $\zeta : \mathcal{P}(\Sigma) \to \mathbb{R}$ is some Lipschitz continuous function with Lipschitz constant K_ζ. Since Γ determines the transition rates, we set $\Gamma_{x,x}(t, a, m) := -\sum_{y \neq x} \Gamma_{x,y}(t, a, m)$, $x \in \Sigma$. We assume that the cost c in the variable a is in $\mathcal{C}^1(A)$, $\nabla_a c$ is Lipschitz continuous in the variable m with Lipschitz constant K_a and c is uniformly convex, that is, there exists $c_0 > 0$ such that

$$c(t, x, \widetilde{a}, m) - c(t, x, a, m) \geq \nabla_a c(t, x, a, m) \cdot (\widetilde{a} - a) + c_0 |\widetilde{a} - a|^2$$

for all $t, x, a, \widetilde{a}, m$.

This setup is analogous to the one considered in [46], except for the additional ζ in the rate, and to our Assumption (H4), except for ζ again and for the fact that c does not split. For any $g \in \mathbb{R}^d$ there exist a unique minimizer $\mathfrak{a}^*(t, x, m, g)$ of $\mathfrak{f}(t, x, a, m, g)$, which in this setting becomes

$$\begin{aligned} \mathfrak{f}(t, x, a, m, g) &= \sum_{y=1}^{d} \Gamma_{x,y}(t, a, m)[g(y) - g(x)] + c(t, x, a, m) \\ &= \sum_{y=1}^{d} (a_y + \zeta(m))[g(y) - g(x)] + c(t, x, a, m). \end{aligned} \tag{3.69}$$

We need $\mathfrak{a}^*$ to be Lipschitz continuous in m and g; this fact is proved in Proposition 1 in [46]. We state the result in the following

Lemma 3.16. *Under the above assumptions (in this subsection), the function $\mathfrak{a}^*$ is Lipschitz continuous in m and g:*

$$|\mathfrak{a}^*(t, x, m, g) - \mathfrak{a}^*(t, x, \widetilde{m}, g)| \leq \frac{K_a}{c_0} |m - \widetilde{m}| \tag{3.70}$$

$$|\mathfrak{a}^*(t, x, m, g) - \mathfrak{a}^*(t, x, m, \widetilde{g})| \leq \frac{1}{c_0} |g - \widetilde{g}| \tag{3.71}$$

for any $t, x, m, \widetilde{m}, g, \widetilde{g}$.

Define $\alpha_m(t,x) = \mathfrak{a}^*(t,x,m(t),V_m(t,\cdot))$ as in (2.15): it is the unique feedback control for given flow of measures $m \in \mathcal{E}$, where $V_m(t,x)$ is the value function defined in (2.11) with respect to m. The cost functions c and G are uniformly bounded and so is the value function: let us denote by K_V the maximum of its absolute value. Denote by K_ζ the maximum of ζ and fix the constants

$$
\begin{aligned}
C_1 &:= 2Kd^2 + 2d\sqrt{d}K^d,\\
C_2 &:= 2d\sqrt{d}\frac{K_a}{c_0} + 2d^2K_\zeta,\\
C_3 &:= \frac{2d^2}{c_0},\\
C_4 &:= K_2 + 2dK_VK_\zeta + 2\sqrt{d}K_V\frac{K_a}{c_0} + K_2\frac{K_a}{c_0},\\
C_5 &:= 2K_V\frac{\sqrt{d}}{c_0} + \frac{K_2}{c_0} + \sqrt{d}(K_\zeta + K).
\end{aligned}
$$

Let $T^* > 0$ be such that

$$2T^*\sqrt{d}e^{T^*C_1}\left[C_2 + C_3(K_2 + T^*C_4)e^{T^*C_5}\right] = 1. \tag{3.72}$$

Theorem 3.17. *Under the assumptions of this subsection, for any* $0 < T < T^*$ *there exists a unique feedback MFG solution* (α_m, m) *of the mean field game.*

Proof. In the notation of Theorem 2.4, the map $\Phi : \mathcal{E} \to \mathcal{E}$ is defined by $\Phi(m) = \{\mathrm{Flow}(X_{\alpha_m,m})\}$, a singleton. If we prove that this map is a contraction for small time horizon T, then the assertion follows by the Banach-Cacciopoli Theorem. So let $m, \widetilde{m} \in \mathcal{E}$ and set $X := X_{\alpha_m,m}$ and $\widetilde{X} := X_{\alpha_{\widetilde{m}},\widetilde{m}}$.

First we prove that the value function V_m is Lipschitz continuous with respect to m. Thanks to the HJB equation (2.13) we have

$$V_m(t,x) = V_m(T,x) + \int_t^T \mathfrak{f}(s,x,\mathfrak{a}^*(s,x,m(s),V_m(s,\cdot)),m(s),V_m(s,\cdot))ds.$$

The pre-Hamiltonian $\mathfrak{f}$ is Lipschitz in (a,m,g); in fact, by (1.22) and (3.69) we have

$$
\begin{aligned}
&|\mathfrak{f}(t,x,a,m,g) - \mathfrak{f}(t,x,\widetilde{a},\widetilde{m},\widetilde{g})|\\
&\le 2|g|_\infty\left(\sqrt{d}|a-\widetilde{a}| + dK_\zeta|m-\widetilde{m}|\right) + K_2(|a-\widetilde{a}| + |m-\widetilde{m}|) + \sqrt{d}(K_\zeta+K)|g-\widetilde{g}|.
\end{aligned}
$$

Then using (3.70) and (3.71) we obtain

$$
\begin{aligned}
&|V_m(t,x) - V_{\widetilde{m}}(t,x)|\\
&\qquad\le K_2|m(T)-\widetilde{m}(T)| + \int_t^T \left[C_4|m(s)-\widetilde{m}(s)| + C_5|V_m(s)-V_{\widetilde{m}}(s)|\right] ds\\
&\qquad\le K_2|m(T)-\widetilde{m}(T)| + C_4(T-t)||m-\widetilde{m}||_\infty + \int_t^T C_5|V_m(s)-V_{\widetilde{m}}(s)|ds
\end{aligned}
$$

for any x, hence Gronwall's lemma implies that

$$|V_m(t) - V_{\widetilde{m}}(t)| \le \sqrt{d}(K_2 + TC_4)e^{TC_5}||m-\widetilde{m}||_\infty$$

for any $0 \le t \le T$.

Therefore, by applying again (3.70) and (3.71), we obtain

$$\begin{aligned}
&\mathbb{E}|X(t) - \widetilde{X}(t)| \\
&\leq \int_0^t \int_\Theta \mathbb{E}|\gamma(s, X(s), \theta, \alpha(s, X(s), m(s), V_m(t,\cdot)), m(s)) \\
&\quad - \gamma(s, \widetilde{X}(s), \theta, \alpha(s, \widetilde{X}(s), \widetilde{m}(s), V_{\widetilde{m}}(t,\cdot)), \widetilde{m}(s))|\nu(d\theta)ds \\
&\leq \int_0^t \left[C_1\mathbb{E}|X(s) - \widetilde{X}(s)| + C_2|m(s) - \widetilde{m}(s)| + \frac{2d\sqrt{d}}{c_0}|V_m(t,\cdot) - V_{\widetilde{m}}(t,\cdot)|\right] ds \\
&\leq \int_0^t \left[C_1\mathbb{E}|X(s) - \widetilde{X}(s)| + C_2|m(s) - \widetilde{m}(s)| + C_3(K_2 + C_4T)e^{TC_5}||m - \widetilde{m}||_\infty\right] ds
\end{aligned}$$

and thus, again by Gronwall's lemma,

$$\mathbb{E}|X(t) - \widetilde{X}(t)| \leq T^* e^{T^*C_1}\left[C_2 + C_3(K_2 + T^*C_4)e^{T^*C_5}\right]||m - \widetilde{m}||_\infty$$

for any $0 \leq t \leq T$. Since $|\operatorname{Law}(X(t)) - \operatorname{Law}(\widetilde{X}(t))| \leq 2\sqrt{d}\mathbb{E}|X(t) - \widetilde{X}(t)|$ we have

$$\begin{aligned}
\sup_{0\leq t\leq T} |\operatorname{Law}(X(t)) - \operatorname{Law}(\widetilde{X}(t))| &=: ||\operatorname{Flow}(X) - \operatorname{Flow}(\widetilde{X})||_\infty \\
&\leq 2T^*\sqrt{d}e^{T^*C_1}\left[C_2 + C_3(K_2 + T^*C_4)e^{T^*C_5}\right]||m - \widetilde{m}||_\infty,
\end{aligned}$$

and then the claim holds for T^* satisfying (3.72). □

3.4.2 Uniqueness of the mean field game solution under monotonicity

Uniqueness of mean field game solutions was shown in Theorem 2 in [47] for arbitrary time horizon under the Lasry-Lions monotonicity assumptions. Here, we give a different proof of this result, which relies on the probabilistic representation of the mean field game, and allows for less restrictive assumptions on the data. This idea of this proof was first developed in [21] in the diffusion setting. Specifically, we assume (Mon) and that the function γ in the dynamics (1.46) does not depend on $m \in \mathcal{P}(\Sigma)$; we do not need to assume (H2), i.e. that the control is exactly the rate.

Theorem 3.18. *Suppose that (A), (B), (C) and (Mon) hold and that γ does not depend on m. Then there exists a unique feedback MFG solution (α, m, X) of the mean field game.*

Proof. Let (α_m, m, X) and $(\widetilde{\alpha}, \widetilde{m}, \widetilde{X})$ be two different feedback mean field game solutions. Since the dynamics does not depend on $m \in \mathcal{P}(\Sigma)$, we have $X = X_\alpha = X_{\alpha,m} = X_{\alpha,\widetilde{m}}$ and $\widetilde{X} = X_{\widetilde{\alpha}} = X_{\widetilde{\alpha},m} = X_{\widetilde{\alpha},\widetilde{m}}$. As the solutions are different, they are non-equivalet: there exists t^* such that $m(t^*) \neq \widetilde{m}(t^*)$. Thus $m \neq \widetilde{m}$ in a neighborhood of t^*, as they are continuous. Recalling that we have $\alpha = \alpha_m$ and $\widetilde{\alpha} = \alpha_{\widetilde{m}}$, we obtain $\alpha(t, X(t)) \neq \widetilde{\alpha}(t, X(t))$ in a set of positive measure, implying that $\widetilde{\alpha}$ is not optimal for m, in lights of Theorem 2.8.

Therefore the optimality of α yields $J(\alpha, m) < J(\widetilde{\alpha}, m)$, and similarly $J(\widetilde{\alpha}, \widetilde{m}) < J(\alpha, \widetilde{m})$, hence

$$\begin{aligned}
0 < J(\widetilde{\alpha}, m) - J(\alpha, m) &= \mathbb{E}\left[G(\widetilde{X}(T), m(T)) - G(X(T), m(T))\right] \\
&+ \mathbb{E}\left[\int_0^T [L(\widetilde{X}(t), \widetilde{\alpha}(t, \widetilde{X}(t))) + F(\widetilde{X}(t), m(t))\right. \\
&\qquad \left. - L(X(t), \alpha(t, X(t))) - F(X(t), m(t))]dt\right],
\end{aligned}$$

$$
\begin{aligned}
0 \le J(\alpha,\widetilde{m}) - J(\widetilde{\alpha},\widetilde{m}) &= \mathbb{E}\left[G(X(T),\widetilde{m}(T)) - G(\widetilde{X}(T),\widetilde{m}(T))\right] \\
&+ \mathbb{E}\left[\int_0^T [L(X(t),\alpha(t,X(t))) + F(X(t),\widetilde{m}(t)) \right. \\
&\left. -L(\widetilde{X}(t),\widetilde{\alpha}(t,\widetilde{X}(t))) - F(\widetilde{X}(t),\widetilde{m}(t))]dt\right].
\end{aligned}
$$

Summing these two inequalities and using the fact that $\text{Law}(X(t)) = m(t)$ for any t, we obtain

$$
\begin{aligned}
0 &< \mathbb{E}\left[G(\widetilde{X}(T),m(T)) - G(X(T),m(T)) + G(X(T),\widetilde{m}(T)) - G(\widetilde{X}(T),\widetilde{m}(T))\right] \\
&+ \mathbb{E}\left[\int_0^T [F(\widetilde{X}(t),m(t)) - F(X(t),m(t)) \right. \\
&\left. +F(X(t),\widetilde{m}(t)) - F(\widetilde{X}(t),\widetilde{m}(t))]dt\right] \\
&= \sum_{x\in\Sigma}(G(x,m(T)) - G(x,\widetilde{m}(T)))(\widetilde{m}_x(T) - m_x(T)) \\
&+ \int_0^T \left[\sum_{x\in\Sigma}(F(x,m(t)) - F(x,\widetilde{m}(t)))(\widetilde{m}_x(t) - m_x(t))\right] dt. \le 0,
\end{aligned}
$$

where in the latter we used (1.29); a contradiction. □

3.4.3 Uniqueness of the Nash equilibrium

Let the number of players N be fixed, as well as the stochastic basis and the Poisson random measures. We say that the feedback Nash equilibrium $\boldsymbol{\alpha} = (\alpha^1,\cdots,\alpha^N) : [0,T]\times\Sigma^N \to A^N$ is *unique* in a class $\mathcal{B}^N$ of feedback strategies if, given another feedback Nash equilibrium $\boldsymbol{\alpha}^* \in \mathcal{B}^N$, we have $\alpha^i(t,\boldsymbol{x}) = \alpha^{*,i}(t,\boldsymbol{x})$ for each i and $\boldsymbol{x}\in\Sigma^N$ and ℓ-almost every $t\in[0,T]$. Note that this definition involves only the strategies, as functions of time and space: in particular the initial condition is not considered. The uniqueness of Nash equilibria, in the diffusion setting, is proved in Proposition 6.27 of [19].

Theorem 3.19. *Under (LipH), the Nash equilibrium $\boldsymbol{\alpha}$ provided by the* (Nash) *system, i.e.*

$$
\alpha^i(t,\boldsymbol{x}) := a^*(x_i,\Delta^i v^{N,i}(t,\boldsymbol{x})) \qquad i = 1,\dots,N,
$$

is the unique feedback Nash equilibrium in the class of bounded feedback strategy vectors.

Proof. We remark that the $v^{N,i}$-s, as well as their gradients, are uniformly bounded thanks to Lemma (1.7). Let $\boldsymbol{\alpha}^*$ be any Nash equilibrium. We consider the best response problem for player i: having fixed the $\alpha^{*,j}$ for $j\neq i$, it consists in finding the optimal control for the functional

$$
J_i(\pi^i) := \mathbb{E}\left[\int_0^T \Big(L(X_i(t),\pi^i(t)) + F(X_i(t),m^{N,i}_{\boldsymbol{X}}(t))\Big)\, dt + G(X_i(t),m^{N,i}_{\boldsymbol{X}}(t))\right],
$$

over open-loop controls π^i, where $\boldsymbol{X}$ solves

$$
\begin{cases}
X_i(t) = & \xi_i + \int_0^t\int_\Theta \gamma(X_i(s^-),\theta,\pi^i(s))\mathcal{N}_i(ds,d\theta) \\
X_j(t) = & \xi_j + \int_0^t\int_\Theta \gamma(X_j(s^-),\theta,\alpha^{*,j}(s,\boldsymbol{X}(s^-)))\mathcal{N}_j(ds,d\theta) \qquad \text{if } j\neq i.
\end{cases}
$$

The HJB equation related to this problem reads

$$\begin{cases} -\frac{\partial w}{\partial t}^{N,i}(t,\boldsymbol{x}) - \sum_{j=1,\ j\neq i}^{N} \alpha^{*,j}(t,\boldsymbol{x}) \cdot \Delta^j w^{N,i} + H(x_i, \Delta^i w^{N,i}) = F(x_i, m_{\boldsymbol{x}}^{N,i}), \\ w^{N,i}(T,\boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}). \end{cases} \tag{3.73}$$

There exists a unique bounded and continuous solution $(w^{N,i})_i$ since the $\alpha^{*,j}$-s are bounded and (LipH) holds.

Let $\boldsymbol{X}^*$ be the optimal process related to the Nash equilibrium $\boldsymbol{\alpha}^*$ and let $\pi^{*,i}(t) = \alpha^{*,i}(t, \boldsymbol{X}^*_{t^-})$, for $\ell \otimes P$-almost every (t,ω), be the corresponding open-loop control for player i. By the definition of Nash equilibrium, the feedback strategy $\alpha^{*,i}$ is optimal for the best response problem, and so is its open-loop counterpart $\pi^{*,i}$. By mimicking the proof of Theorem 2.8, we find that any optimal control $\pi^{*,i}$, with corresponding optimal process $\boldsymbol{X}^*$, is such that $\pi^{*,i}(t) = a^*(X_i^*(t^-), \Delta w^{N,i}(t, \boldsymbol{X}^*_{t^-}))$ for $\ell \otimes P$-almost every (t,ω). Therefore

$$\alpha^{*,i}(t, \boldsymbol{X}^*_{t^-}) = a^*(X_i^*(t^-), \Delta^i w^{N,i}(t, \boldsymbol{X}^*_{t^-})) \qquad \ell \otimes P \text{ - a.e. } (t,\omega) \tag{3.74}$$

where $\boldsymbol{X}^*$ is the optimal process. Since the time horizon is finite and the transition rates, given by the Nash equilibrium $\boldsymbol{\alpha}^*$, of the Markov chain $\boldsymbol{X}^*$ are bounded, we obtain that $P(\boldsymbol{X}^*_t = \boldsymbol{x}) > 0$ for ℓ-a.e. t and any $\boldsymbol{x}$, if we take an initial distribution which is supported everywhere. Hence (3.74) implies that

$$\alpha^{*,i}(t,\boldsymbol{x}) = a^*(x_i, \Delta^i w^{N,i}(t,\boldsymbol{x})) \qquad \ell \text{ - a.e. } t, \quad \forall \boldsymbol{x}.$$

By inserting the above identity into (3.73) we find that the $w^{N,i}$ solve exactly the Nash system, thus $w^{N,i} = v^{N,i}$ by uniqueness of solutions to (Nash), which implies that $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$ for almost every t and each $\boldsymbol{x}$. □

CHAPTER 4

Convergence without uniqueness: a two state model

We examine here the model, presented in [24], in which the position of each agent belongs to $\{-1,1\}$. We saw in Chapter 3 that, if there is uniqueness of mean field game solutions, which holds under monotonicity assumptions on the costs, then the master equation possesses a smooth solution which can be used to prove the convergence of the value functions of the N players and a propagation of chaos property for the associated optimal trajectories. In the example considered here the terminal cost is anti-monotonous, so that we show that the mean field game admits exactly three solutions. We prove that the N-player game always admits a limit: it selects one mean field game solution, except in one critical case, so there is propagation of chaos. The value functions also converge and the limit is the entropy solution to the master equation, which can be written as a scalar conservation law. Moreover, viewing the mean field game system as the necessary conditions for optimality of a deterministic control problem as in 1.4.1, we show that the N-player game selects the optimum of this problem when it is unique.

4.1 The two state model

We let $\Sigma = \{-1,1\}$ be the state space. An element $m \in \mathcal{P}(\Sigma)$ can be determined by its mean, denoted $\mu = m(1) - m(-1) \in [-1,1]$; compared to the notations used before, we have $(\mu, -\mu) \in \mathcal{P}_0(\Sigma)$. We assume (H2), (H3) and choose

$$L(x,a) := \frac{a_{-x}^2}{2}, \qquad F(x,\mu) \equiv 0, \qquad G(x,\mu) := -x\mu.$$

The final cost favors alignment with the majority, while the running cost is a simple quadratic cost which forces players to not choose a high transition rate. Compared to condition (1.29), note that the final cost is *anti-monotonous*, as

$$\sum_{x\in\Sigma} (G(x,m) - G(x,\widetilde{m}))(m(x) - \widetilde{m}(x)) = -(\mu - \widetilde{\mu})^2 \leq 0.$$

We write $\alpha^i(t,\boldsymbol{x}) \in [0,+\infty[$ for $\alpha^i_{-x_i}(t,\boldsymbol{x})$, i.e. the rate at which player i *flips* its state from x_i to $-x_i$, and also α for α_{-x}, p for p_{-x} and identify $\Delta^x u(x)$ with its non-zero component

$u(-x) - u(x)$. The limiting cost reads

$$J(\alpha, m) = \mathbb{E}\left[\int_0^T \frac{|\alpha(t, X(t))|^2}{2} dt - X(T)m(T)\right] \tag{4.1}$$

The associated Hamiltonian is given, for $p \in \mathbb{R}$, by

$$H(x,p) = \sum_{a \geq 0} \left\{ ap - \frac{a^2}{2} \right\} = \frac{(p^-)^2}{2},$$

with $a^*(x,p) = p_x^- = p^- = -H'(p)$, where p^- denotes here the negative part of p. Note that this Hamiltonian is in $\mathcal{C}^1$, with Lipschitz derivative, but not in $\mathcal{C}^2$. Moreover, we denote by ζ the first marginal of m, so that $m = (m_1, m_{-1}) = \left(\frac{1+\mu}{2}, \frac{1-\mu}{2}\right) = (\zeta, 1-\zeta)$.

4.1.1 The mean field game system

The first equation in (MFG), i.e the HJB equation for the value function $u(t,x)$, reads

$$-\frac{d}{dt}u(t,x) + \left[(\Delta^x u(t,x))^-\right]^2 = 0.$$

Setting $z(t) := u(t,-1) - u(t,1)$, we have that $z(t)$ solves

$$\begin{cases} \dot{z} = \frac{z|z|}{2} \\ z(T) = 2\mu(T). \end{cases} \tag{4.2}$$

and the corresponding optimal control is given by $\alpha(1) = z^-$, which is the transition rate from 1 to -1, and $\alpha(-1) = z^+$. This equation must be coupled with the forward Kolmogorov equation for the mean, which reads $\dot{\mu} = -\mu|z| + z$. The mean field game system takes therefore the form

$$\begin{cases} \dot{z} = \frac{z|z|}{2} \\ \dot{\mu} = -\mu|z| + z \\ z(T) = 2\mu(T) \\ \mu(0) = \mu_0. \end{cases} \tag{4.3}$$

Proposition 4.1. *Let $T(\mu_0)$ be the unique solution in $\left[\frac{1}{2}, 2\right]$ of the equation:*

$$|\mu_0| = \frac{(2T-1)^2(T+4)}{27T}. \tag{4.4}$$

Then, for every $\mu_0 \in [-1,1]$ (4.3) admits

(i) a unique solution for $T < T(\mu_0)$;

(ii) two distinct solutions for $T = T(\mu_0)$, if $\mu_0 \neq 0$;

(iii) three distinct solutions for $T > T(\mu_0)$.

Proof. Note that (4.2) can be solved as a final value problem, giving

$$z(t) = \frac{2\mu(T)}{|\mu(T)|(T-t)+1}. \tag{4.5}$$

This can then be inserted in the forward Kolmogorov equation, giving

$$\mu(t) = (\mu_0 - \operatorname{sgn}(\mu(T)))\left(\frac{|\mu(T)|(T-t)+1}{|\mu(T)|T+1}\right)^2 + \operatorname{sgn}(\mu(T)). \tag{4.6}$$

These are actually solutions of (4.3) if and only if the consistency relation obtained by setting $t = T$ in (4.6) holds, i.e. $\mu(T) = M$ solves

$$T^2M^3 + T(2-T)M|M| + (1-2T)M - \mu_0 = 0. \tag{4.7}$$

Moreover, distinct solutions of (4.7) correspond to distinct solutions of (4.3). We first look for nonnegative solutions of (4.7). Set

$$f(M) := T^2M^3 + T(2-T)M^2 + (1-2T)M - \mu_0.$$

Note that

$$f'(M) < 0 \quad \Longleftrightarrow \quad M \in \left]-\frac{1}{T}, \frac{2T-1}{3T}\right[.$$

If $T \leq \frac{1}{2}$ then f is strictly increasing in $(0, +\infty)$, so the equation $f(M) = 0$ admits a unique nonnegative solution if $\mu_0 \geq 0$, otherwise there is no nonnegative solution. If $T > \frac{1}{2}$, then f restricted to $(0, +\infty)$ has a global minimum at $M^* = \frac{2T-1}{3T}$. If $\mu_0 > 0$ then there is still a unique nonnegative solution, while for $\mu_0 = 0$ there are two nonnegative solutions, one of which is zero. If, instead, $\mu_0 < 0$, so that $f(0) > 0$, the equation $f(M) = 0$ has zero, one or two nonnegative solutions, depending on whether $f(M^*) > 0$, $f(M^*) = 0$ or $f(M^*) < 0$ respectively. Observing that

$$f(M^*) = -\mu_0 - \frac{(2T-1)^2(T+4)}{27T},$$

we see that those three alternatives occur if $T < T(\mu_0)$, $T = T(\mu_0)$ and $T > T(\mu_0)$ respectively. The case $M \leq 0$ is treated similarly. □

Note that $T(0) = \frac{1}{2}$ and $T(\pm 1) = 2$. It will be useful to characterize the behaviour when $\mu_0 = 0$.

Corollary 4.2. *If* $\mu_0 = 0$ *then* (4.3) *admits*

(i) a unique solution for $T \leq \frac{1}{2}$*, which is* $z \equiv \mu \equiv 0$*;*

(ii) three distinct solutions for $T > \frac{1}{2}$*: the solution constantly 0 and other two solutions* (z_+, μ_+) *and* (z_-, μ_-) *such that*

$$z_+(t) = -z_-(t) > 0, \qquad \mu_+(t) = -\mu_-(t) > 0, \qquad \forall t \in]0, T] \tag{4.8}$$

$$\mu_+(T) = \mu_-(T) = \frac{T(T-2) + \sqrt{T^3(T+4)}}{2T^2}. \tag{4.9}$$

4.1.1.1 Uniqueness under monotonicity

The non-uniqueness for large T derives from the terminal cost which we take anti-monotonous. We known from the previous chapter that there is uniqueness of MFG solutions if the coupling is monotonous. Let us verify it for this model, i.e. let us briefly examine here the opposite case, where the terminal condition in (4.2) is $u(T,x) = x\mu(T)$.

So the MFG system (4.3) becomes

$$\begin{cases} \dot{z} = \frac{z|z|}{2} \\ \dot{\mu} = -\mu|z| + z \\ z(T) = -2\mu(T) \\ \mu(0) = \mu_0 \end{cases} \tag{4.10}$$

Given $\mu(T) = M$, the solution is then

$$\begin{aligned} z(t) &= -\frac{2M}{|M|(T-t)+1} \\ \mu(t) &= (\mu_0 + \operatorname{sgn}(M))\left(\frac{|M|(T-t)+1}{|M|T+1}\right)^2 - \operatorname{sgn}(M). \end{aligned}$$

Hence there is a solution of the mean field game if and only M solves

$$T^2M^3 + T(T+2)M|M| + (1+2T)M - \mu_0 = 0. \tag{4.11}$$

Thanks to Descartes's rule of signs, we see that there is a unique solution $M > 0$ if and only if $\mu_0 > 0$, otherwise there are no solutions. Viceversa, there is a unique solution $M < 0$ if and only if $\mu_0 < 0$, otherwise there are no solutions. While if $\mu_0 = 0$ there is the unique solution $M = 0$. This holds for any time horizon T.

4.1.2 The master equation

For this model the master equation (M) takes the form

$$\begin{aligned} -\frac{\partial U}{\partial t}(t,x,m) + \frac{1}{2}\left[(\Delta^x U(t,x,m))^-\right]^2 \qquad\qquad & (x,m) \in \{-1,1\} \times \mathcal{P}(\{-1,1\}) \\ - m_1 \frac{\partial U}{\partial(m_{-1}-m_1)}(t,x,m)\,(\Delta^x U(t,1,m))^- & \\ - m_{-1} \frac{\partial U}{\partial(m_1-m_{-1})}(t,x,m)\,(\Delta^x U(t,-1,m))^- = 0, & \\ U(T,x,m) = -x(m_{-1}-m_1),. & \end{aligned} \tag{4.12}$$

As for the MFG system, we can rewrite the above as an equation for the difference $U(-1)-U(1)$ in the variable $\mu = m_{-1} - m_1$: setting

$$Z(t,\mu) := U\left(t,-1,\frac{1+\mu}{2},\frac{1-\mu}{2}\right) - U\left(t,1,\frac{1+\mu}{2},\frac{1-\mu}{2}\right),$$

we easily derive a closed equation for Z, now written as a forward equation,

$$\begin{cases} \frac{\partial Z}{\partial t} + \frac{\partial}{\partial \mu}\left(\mu\frac{Z|Z|}{2} - \frac{Z^2}{2}\right) = 0, & \mu \in [-1,1] \\ Z(0,\mu) = 2\mu. \end{cases} \tag{4.13}$$

We observe that this equation has the form of a scalar *conservation law* with space-dependent flow $g(\mu, z) := \mu\frac{z|z|}{2} - \frac{z^2}{2}$. Again, system (4.3) represents the characteristic curves to Equation (4.13). This equation is not standard since the flow g is space-dependent and the equation is stated in the bounded domain $[-1, 1]$; note that g is $\mathcal{C}^1$ and concave in z, for $\mu \in [-1, 1]$. Basic facts concerning conservation laws will be summarized in the Appendix B below. These equations typically posses unique smooth solutions for small time, but develop singularities in finite time: weak solutions exist but uniqueness may fail. If the initial datum were decreasing, then there would exist a classical solution for any time horizon, since the flow is concave. This is exactly the case discussed above in which monotonicity holds, which is the opposite to the one considered here. To recover uniqueness (for any T) the notion of *entropy solution* is introduced. The complete definition of entropy solution, as well as its specialization to piecewise smooth functions, is postponed to the Appendix B.

For equation (4.13), the entropy solution can be explicitly found. Let

$$f(M, t, \mu) := t^2M^3 + t(2-t)M|M| + (1-2t)M - \mu \tag{4.14}$$

and $M^*(t, \mu)$ denote the unique solution to $f(M, t, \mu) = 0$ with the same sign of μ, if $\mu \neq 0$; M^* is defined for any time and let $M^*(t, 0) \equiv 0$. Define

$$Z^*(t, \mu) := \frac{2M^*(t, \mu)}{t|M^*(t, \mu)| + 1}. \tag{4.15}$$

Theorem 4.3. *The function Z^* defined in* (4.15) *is the unique entropy admissible weak solution to* (4.13).

Proof. The function Z^* is piecewise C^1 with the discontinuity in $\mu = 0$, for any $t > 1/2$. So it is enough to check whether the conditions of Proposition 4.22 are satisfied. The master equation holds in the classical sense for any $\mu \neq 0$. We have $g(0, z) = -\frac{z^2}{2}$, hence condition (RH) becomes $|Z_l^*(t)| = |Z_r^*(t)|$, while the (Lax) condition in this case reads $Z_l^*(t) \leq Z_r^*(t)$. These properties hold by Corollary 4.2. Uniqueness is given by Theorem 4.23. We remark that the conservation law is set in the domain $[-1, 1]$ without any boundary condition, but this is not a problem as we have invariance of the domain under the action of the characteristics. □

Remark 4.4. *We observe that to this entropy solution of* (4.13)*, inverting the time, there corresponds a unique solution of* (4.12)*. Indeed, once Z is known,* (4.12) *becomes linear and it is well posed.*

We know from Remark 1.18, if there were a regular solution to the master equation (4.13), i.e. Lipschitz in μ, then this solutions provides a unique solution to the mean field game system (4.3), since the KPF equation would be well posed for any initial condition, when using the control $z(t) = Z(T - t, \mu(t))$ induced by the solution to the master equation:

$$\begin{cases} \dot{\mu} = -\mu|Z(T-t, \mu)| + Z(T-t, \mu) \\ \mu(0) = \mu_0. \end{cases} \tag{4.16}$$

In our example there are no regular solutions to the master equation; however the entropy solution still induces a unique mean field game solution, if $\mu_0 \neq 0$.

Proposition 4.5. *Let Z^* be the entropy solution defined in* (4.15)*. Then* (4.16) *admits a unique solution μ^*, if $\mu_0 \neq 0$: it is the unique solution which does not change sign, for any time.*

Proof. If $\mu_0 > 0$ then $Z^*(T-t,\mu)$ remains positive if t and $|\mu-\mu_0|$ are small. So we have a unique solution, for small time, and it is such that $\mu(t) > 0$ and the induced control is positive. We iterate this procedure starting from $\mu(t_0) > 0$, and noting that $\mu(t) > \mu_0$ for any t. Therefore we end up with the required solution, which is positive and such that $\mu(t) > \mu_0$. This solution is unique since $Z^*(t,\mu)$ is Lipschitz for $\mu \in [\mu_0, 1]$. In fact the other two solutions described in Proposition 4.1 would require the optimal control to be negative for any time, and this is not possible when considering the entropy solution Z^*. The same argument gives the claim when $\mu_0 < 0$. □

4.1.3 The $N+1$-player game

First of all, we want to write the (Nash) system in a simpler way, as it is done in [46], in order to reduce the number of equations which is now $N2^N$. This will also allow to perform numerical simulations. It is convenient to consider now the game played by $N+1$ players, labeled by the integers $\{0, 1, \dots, N\}$. In what follows, we use N rather than $N+1$ as apex in all objects related to the $N+1$-player game. We make the *ansatz* that there exists a function $V^N : [0,T] \times \Sigma \times \mathcal{P}(\Sigma)$ such that the value functions of the game can be written as

$$v^{N,i}(t,\boldsymbol{x}) = V^N(t, x_i, m_{\boldsymbol{x}}^{N,i}), \qquad i = 0, \dots, N. \tag{4.17}$$

Under this condition, the (Nash) system is equivalent to the following equation for V_N:

$$\begin{aligned} &-\frac{d}{dt}V^N(x,m) + \frac{[(\Delta^x V^N(x,m))^-]^2}{2} \qquad (4.18)\\ &= Nm_1 \left[V^N\left(-1, m+\frac{\delta_x-\delta_1}{N}\right) - V^N\left(1, m+\frac{\delta_x-\delta_1}{N}\right)\right]^- \left[V^N\left(x,m+\frac{\delta_{-1}-\delta_1}{N}\right) - V^N(x,m)\right]\\ &+Nm_{-1}\left[V^N\left(1,m+\frac{\delta_x-\delta_{-1}}{N}\right) - V^N\left(-1,m+\frac{\delta_x-\delta_{-1}}{N}\right)\right]^- \left[V^N\left(x,m+\frac{\delta_1-\delta_{-1}}{N}\right) - V^N(x,m)\right] \end{aligned}$$

which admits a unique bounded solution, for any N.

Heuristically, the above Equation converges to the master equation (4.12). Indeed, if $m^N \to m$ deterministic, as $N \to \infty$, then

$$\begin{aligned} &\lim_N \left[V^N\left(-1, m+\frac{\delta_x-\delta_1}{N}\right) - V^N\left(1, m+\frac{\delta_x-\delta_1}{N}\right)\right]^- = [\Delta^x U(1,m)]^-\\ &\lim_N \frac{\left[V^N\left(x, m+\frac{\delta_{-1}-\delta_1}{N}\right) - V^N(x,m)\right]}{\frac{1}{N}} = \frac{\partial U(x,m)}{\partial(m_{-1}-m_1)} \end{aligned}$$

and similarly for the other terms. In fact, system (4.18) is a particular space discretization of (4.12).

Any measure $m \in \mathcal{P}(\Sigma)$ is determined by its first component $\zeta \equiv m_1$. Let

$$\zeta_{\boldsymbol{x}}^{N,i} := \frac{1}{N}\sum_{j=0, j\neq i}^{N} \delta_{\{x_j=1\}} \in \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1\right\} =: S_N$$

be the fraction of the "other" players having state 1, and $\zeta_{\boldsymbol{x}}^N = \frac{1}{N+1}\sum_{i=0}^N \mathbb{1}_{\{Y_i(t)=1\}}$. Comparing with the notations used before, note that $\zeta_{\boldsymbol{x}}^{N,i} = \frac{1+\mu_{\boldsymbol{x}}^{N+1,i}}{2}$. Thus we now view V^N as a function of ζ and note that, by symmetry of system (4.18) and of the terminal condition, we have

$$V^N(t,1,\zeta) = V^N(t,-1,1-\zeta). \tag{4.19}$$

We can therefore redefine $V^N(t,\zeta) := V^N(t,1,\zeta)$ and then derive the following closed equation for V^N:

$$
\begin{aligned}
-\frac{d}{dt}&V^N(t,\zeta) + H(V^N(1-\zeta) - V^N(\zeta)) \\
&= N\zeta\left[V^N(1-\zeta) - V^N(\zeta)\right]^-\left[V^N\left(\zeta - \frac{1}{N}\right) - V^N(\zeta)\right] \\
&\quad + N(1-\zeta)\left[V^N\left(\zeta + \frac{1}{N}\right) - V^N\left(1-\zeta-\frac{1}{N}\right)\right]^-\left[V^N\left(\zeta+\frac{1}{N}\right) - V^N(\zeta)\right] \\
V^N&(T,\zeta) = -(2\zeta - 1)
\end{aligned}
\tag{4.20}
$$

with $H(p) = \frac{(p^-)^2}{2}$. Clearly, if $\zeta = 0$ (resp. $\zeta = 1$) then there is only the second term (resp. the first) in the right hand side of the equation. This is a system of $N+1$ ODEs indexed by $\zeta \in S_N$ whose well posedness is ensured by the Lipschitz continuity of the Hamiltoniam and its argmax $p^- = -H'(p)$: indeed, the right hand side is easily seen to be Lipschitz with respect to V^N for fixed N, but the Lipschitz constant is proportional to N.

The solution to system (4.20) gives the unique Nash equilibrium of the N+1-player game: by symmetry, each player chooses the same feedback control, i.e. the same function, denoted by β^N, of his private state and the fraction of the other player in state 1; the Nash equilibrium $\boldsymbol{\alpha}$ is given by

$$\alpha^i(t,\boldsymbol{x}) = \beta^N(t,x,\zeta^{N,i}_{\boldsymbol{x}}) := [V^N(t,-x,\zeta^{N,i}_{\boldsymbol{x}}) - V^N(t,x,\zeta^{N,i}_{\boldsymbol{x}})]^-. \tag{4.21}$$

We recall that such control represents the transition rate for any player to go from x to $-x$ at time t, when the fraction of the other players in state 1 is $\zeta^{N,i}$. Setting

$$Z^N(t,\zeta) := V^N(t,1-\zeta) - V^N(t,\zeta),$$

we have

$$\beta^N(t,1,\zeta) = Z^N(t,\zeta)^-, \qquad \beta^N(t,-1,\zeta) = Z^N(t,\zeta)^+. \tag{4.22}$$

We observe that system (4.20) can not be closed as an equation for Z^N, while this is possible for the macroscopic limit, i.e. the master equation. Let us also point out some trivial, but useful, estimates which can be seen as consequences of Lemma 1.7:

$$\left|V^N(t,\zeta)\right| \le 1, \qquad \left|Z^N(t,\zeta)\right| \le 2 \tag{4.23}$$

for any N, $t \in [0,T]$ and $\zeta \in S_N$.

4.1.3.1 Best response map

We observe here that Equation (4.18) can be also derived in a more game-theoretic framework, by proceeding as in Section 3 in [46], i.e examining the best response map for the N+1 player problem. The difference from what we did in Section 1.2 is that here we assume a priori the symmetry of all the feedback strategies. We introduce here some notation which will not be used in the following sections, as here we just make a remark.

We consider a reference player, labeled by 0, and assume that the other N players are symmetric and use the same feedback strategy $\beta(t,x,m^N)$, which is a deterministic function of time, state of the player and empirical measure of the other N players, included the reference player:

$$P(X_i(t+h) = -x | X_i(t) = x, m^{N,i}(t) = m) = \beta(t,x,m)h + o(h)$$

for $i = 0, \dots, N$, where $m^{N,i}(t)$ denotes the empirical measures of the N players without player i and included the reference player. The reference player chooses his control $\widehat{\beta}$ at time t, i.e.

$$P(X_0(t+h) = -x | X_0(t) = x, m^{N,0}(t) = m) = \widehat{\beta}(t,x,m)h + o(h),$$

in order to minimize

$$J(t,x,m,\beta,\widehat{\beta}) := \mathbb{E}^{t,y,m}\left[\int_t^T \frac{|\widehat{\beta}(s)|^2}{2} ds - X_0(T)\mu^{N,0}(T)\right], \tag{4.24}$$

where the apex means that we are conditioning on $X_0(t) = x$ and $m^N(t) = m \in S^N$, and $\mu^{N,0}(T) = \text{Mean}[m^{N,0}(T)] = \frac{1}{N}\sum_{i=0}^N X_i(T)$ is the empirical mean of the N symmetric player at the terminal time.

There exists a unique optimal control $\widehat{\beta} =: \widehat{\Phi}(\beta)$, which depends on β, and is called the best response. It follows that β is a Nash equilibrium if $\widehat{\Phi}(\beta) = \beta$. The value function $V^N(t,y,m)$ of the game is then the minimum in (4.24) and, by Theorem 6 in [46], it solves system (4.18).

4.2 Convergence results

We state the main convergence results. The first deals with the convergence of the value function V^N, the unique solution solutions to system (4.20), and study its limit as $N \to +\infty$. We show that its limit corresponds to the entropy solution of the master equation (4.12). More precisely, let U be the solution of (4.12) corresponding to the entropy solution Z of (4.13). Define, for $\zeta \in [0,1]$

$$U^*(t,\zeta) := U(t,1,2\zeta - 1).$$

Note that, for $T - t > \frac{1}{2}$, if $T > \frac{1}{2}$, $U^*(t,\cdot)$ is discontinuous at $\zeta = \frac{1}{2}$, but it is smooth elsewhere. The next result establishes that V^N converges to U^* uniformly outside any neighborhood of $\zeta = \frac{1}{2}$.

Theorem 4.6 (Convergence of the value functions). *For any $N \geq 1$, $\varepsilon > 0$, $t \in [0,T]$ and $\zeta \in S^N \setminus \left]\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right[$, we have*

$$|V^N(t,\zeta) - U^*(t,\zeta)| \leq \frac{C_\varepsilon}{N}, \tag{4.25}$$

where C_ε does not depend on N nor on t, ζ, but $\lim_{\varepsilon \downarrow 0} C_\varepsilon = +\infty$.

It immediately follows that the convergence is pointwise and thus also in L^p, for any $p \geq 1$; however it can not be uniform. The result clearly holds also for $V^N(t,-1,\zeta)$ because of (4.19).

The second result gives the propagation of chaos property for the optimal trajectories. Let us recall that by propagation of chaos it is meant that the limit of the empirical measures is deterministic, or equivalently that the limit of their laws is a Dirac's delta. Consider then the initial datum $\boldsymbol{\xi}$ i.i.d with $P(\xi_i = 1) = \zeta_0$ and $\mathbb{E}[\xi_i] = \mu_0 = 2\zeta_0 - 1$, and denote by $\boldsymbol{Y}_t = (Y_0(t), \dots, Y_N(t))$ the optimal trajectories of the $N+1$-player game, i.e. when agents play the Nash equilibrium given by (4.21). Also, denote by $\widetilde{\boldsymbol{X}}$ the i.i.d process in which players choose the local control $\widetilde{\alpha}^i(t,\pm 1) := [Z^*(t,\mu^*(t))]^{\mp}$, where Z^* is the entropy solution to (4.13) and μ^* is the unique mean field game solution induced by Z^*, if $\mu_0 \neq 0$ ($\zeta_0 \neq \frac{1}{2}$), that is the one which does not change sign (and $\zeta^* = \frac{1+\mu^*}{2}$); in fact, thanks to Proposition 4.5, we also have $\mathbb{E}[\widetilde{X}_i(t)] = \mu^*(t)$.

Theorem 4.7 (Propagation of chaos)**.** *If $\zeta_0 \neq \frac{1}{2}$ then, for any N and $i = 0, \dots, N$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_i(t) - \widetilde{X}_i(t)|\right] \leq \frac{C_{\zeta_0}}{\sqrt{N}}, \tag{4.26}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\zeta^N_{\boldsymbol{Y}}(t) - \zeta^*(t)|\right] \leq \frac{C_{\zeta_0}}{N^{\frac{1}{9}}}, \tag{4.27}$$

where C_{ζ_0} does not depend on N, and $\lim_{\zeta_0 \to \frac{1}{2}} C_{\zeta_0} = +\infty$.

Let us give the idea of the proof of both the results. We do not use the characterization of the solution to the master equation as the entropy admissible one, because it is very hard to obtain compactness estimates on system (4.20), even in the space of functions with bounded variation. An approach in this direction was instead followed in [46], where the authors proved compactness estimates on the Nash system, but only if T is small enough. This lack of estimates was in fact the main reason for considering the master equation approach we developed in Section 3.1 for finite state models. The entropy solution Z^* we consider here is regular outside the discontinuity: if we show that the optimal trajectories of the $N+1$-player game do non cross the discontinuity then we are allowed to use the master equation approach in order to prove the convergence results. The key point for proving the convergence in then to show a qualitative property of the Nash equilibrium, i.e. that it does not cross the discontinuity.

Remark 4.8. *Theorem 4.7 states that the N-player game selects one MFG solution, if $\mu_0 \neq 0$. We recall that the other two solution always have a physical meaning: indeed, thanks to Theorem 2.18, the decentralized feedback strategy vector $\overline{\boldsymbol{\alpha}} = (\overline{\alpha}^0, \dots, \overline{\alpha}^N)$ given by $\overline{\alpha}^i(t, \boldsymbol{x}) = \overline{\alpha}(t, x_i)$, where $\overline{\alpha}(t, \pm 1) = z(t)^{\mp}$ is the limiting control induced by any MFG solution (z, m), is a $\frac{1}{\sqrt{N}}$-Nash equilibrium; however they induce a completely different behaviour in the N players.*

Actually in Theorem 2.18 the controls were a priori assumed to belong to a compact set, but the result can be still applied in this model since the any MFG solution z is bounded by 2.

What is left to prove for this model is a propagation of chaos result when $\mu_0 = 0$. Let μ_+, resp. μ_-, be the mean field game solution always positive, resp. always negative, in light of Corollary (4.2). What is evident from the simulations (see Section 4.2.4) is the following

Conjecture 4.9. *Let $\mu_0 \neq 0$ and μ^N be the empirical mean related to the optimal trajectories of the N-player game, viewed as a random variable in $D([0,T], [-1,1])$. Then*

$$\lim_N \mathrm{Law}(\mu^N) = \frac{1}{2}\delta_{\mu_+} + \frac{1}{2}\delta_{\mu_-}. \tag{4.28}$$

The limit of the empirical measures is not deterministic: in this sense there is no propagation of chaos when $\mu_0 = 0$, i.e. the initial point is exactly in the discontinuity. Unfortunately we did not manage to prove this result for our model, since it is difficult to tract the Nash system in a neighborhood of the discontinuity. We remark that a similar result, but when dealing with open-loop controls, was very recently obtained in [32] for a linear quadratic mean field game in dimension 1.

4.2.1 Characterization of the Nash equilibrium

Recall that the Nash equilibrium is given by (4.21): $\beta^N(t,1,\zeta) = Z^N(t,\zeta)^-$ and $\beta^N(t,-1,\zeta) = Z^N(t,\zeta)^+$, where $Z^N(t,\zeta) := V^N(t,1-\zeta) - V^N(t,\zeta)$. We show that if a player agrees with the majority, i.e. $x_i = 1$ and $\zeta_{\boldsymbol{x}}^{N,i} \geq \frac{1}{2}$, or $x_i = -1$ and $\zeta_{\boldsymbol{x}}^N \leq \frac{1}{2}$, then he keeps his state by applying the control zero.

Theorem 4.10. *For any N and $\zeta \in S_N = \left\{0, \frac{1}{N}, \dots, 1\right\}$, we have*

$$Z^N(t,\zeta) \geq 0 \qquad (\beta^N(t,1,\zeta) = 0) \qquad \text{if } \zeta \geq \frac{1}{2}, \tag{4.29}$$

$$Z^N(t,\zeta) \leq 0 \qquad (\beta^N(t,-1,\zeta) = 0) \qquad \text{if } \zeta \leq \frac{1}{2}. \tag{4.30}$$

Proof. The proof is done via time discretization. Fix N and consider the explicit backward Euler scheme for Equation (4.20):

$$\begin{aligned} V_h^N(t-h,\zeta) &= V_h^N(t,\zeta) - h\frac{\left[\left(V_h^N(t,1-\zeta) - V_h^N(t,\zeta)\right)^-\right]^2}{2} \\ &\quad + hN\zeta\left(V_h^N(t,1-\zeta) - V_h^N(t,\zeta)\right)^-\left(V_h^N\left(t,\zeta-\frac{1}{N}\right) - V_h^N(t,\zeta)\right) \\ &\quad + hN(1-\zeta)\left(V_h^N\left(t,\zeta+\frac{1}{N}\right) - V_h^N\left(t,1-\zeta-\frac{1}{N}\right)\right)^-\left(V_h^N\left(t,\zeta+\frac{1}{N}\right) - V_h^N(t,\zeta)\right) \end{aligned}$$

for $\zeta \in S_N$, where h is the time step and $t \in \mathcal{T}_h = \{0, h, \dots, T-h, T\}$, with final condition $V(T,\zeta) = -(2\zeta - 1)$. Since the Euler scheme converges, i.e. $\lim_{h\to 0} V_h^N(t,\zeta) = V^N(t,\zeta)$, claim (4.29) is proved if we show that, for h small,

$$Z_h^N(t,\zeta) \geq 0 \qquad t \in \mathcal{T}_h, \quad \zeta \geq \frac{1}{2}, \tag{4.31}$$

as we have $Z^N(t,\zeta) = -Z^N(t,1-\zeta)$ by (4.19). We will prove more, i.e. also that for h small

$$V_h^N(t,\zeta) - V_h^N\left(t,\zeta+\frac{1}{N}\right) \geq 0 \qquad t \in \mathcal{T}_h, \quad \zeta \geq \frac{1}{2}, \tag{4.32}$$

meaning that the value function (of a player in state 1) is decreasing on the right of the discontinuity.

We prove (4.31) and (4.32) by backward induction on t; as N and h are fixed, let us now set $V_h^N = V$. The claims are clearly true in T, and assuming they are true in t, we have

$$\begin{aligned} Z(t-h,\zeta) &= Z(t,\zeta) - h\frac{1}{2}Z(t,\zeta)^2 \\ &\quad + hN(1-\zeta)Z(t,\zeta)\left(V\left(t,1-\zeta-\frac{1}{N}\right) - V(t,1-\zeta)\right) \\ &\quad - hN(1-\zeta)Z\left(t,\zeta+\frac{1}{N}\right)\left(V\left(t,\zeta+\frac{1}{N}\right) - V(t,\zeta)\right) \\ &= Z(t,\zeta)\left[1 - h\left(\frac{1}{2}Z(t,\zeta) + N(1-\zeta)\left(V(t,1-\zeta) - V\left(t,1-\zeta-\frac{1}{N}\right)\right)\right)\right] \\ &\quad + hN(1-\zeta)Z\left(t,\zeta+\frac{1}{N}\right)\left(V(t,\zeta) - V\left(t,\zeta+\frac{1}{N}\right)\right), \end{aligned}$$

which is non-negative by the induction hypothesis and by (4.23), if $h \leq \frac{1}{N+1}$. Concerning (4.32), it holds

$$
\begin{aligned}
V(t-h,\zeta) - V\left(t-h,\zeta+\frac{1}{N}\right)
&= V(t,\zeta) - V\left(t,\zeta+\frac{1}{N}\right) \\
&+ hN(1-\zeta)Z\left(t,\zeta+\frac{1}{N}\right)\left(V\left(t,\zeta+\frac{1}{N}\right) - V(t,\zeta)\right) \\
&- hN\left(1-\zeta-\frac{1}{N}\right)Z\left(t,\zeta+\frac{2}{N}\right)\left(V\left(t,\zeta+\frac{2}{N}\right) - V\left(t,\zeta+\frac{1}{N}\right)\right)
\end{aligned}
$$

and the latter term is non-negative by induction hypothesis, so that

$$
V(t-h,\zeta) - V\left(t-h,\zeta+\frac{1}{N}\right) \geq \left(V(t,\zeta) - V\left(t,\zeta+\frac{1}{N}\right)\right)\left[1 - h\left(N(1-\zeta)Z\left(t,\zeta+\frac{1}{N}\right)\right)\right]
$$

which is non-negative if $h \leq \frac{1}{N}$. We have proved claim (4.29), indeed, for $\zeta \neq 0, \frac{1}{2}$, but the proof can be easily adapted to these extreme points. Then (4.30) follows from (4.29) and (4.19). □

4.2.2 Convergence of the value functions

We exploit the convergence argument developed in Section 3.1. It is necessary to reintroduce the notation

$$
v^{N,i}(t,\boldsymbol{x}) = V^N(t,x_i,\zeta_{\boldsymbol{x}}^{N,i}), \qquad u^{N,i}(t,\boldsymbol{x}) = U^*(t,x_i,\zeta_{\boldsymbol{x}}^{N,i})
$$

for $i = 0,\dots,N$, where $\zeta^{N,i} = \frac{1}{N}\sum_{j=0 j\neq i}^{N} \delta_{\{x_i=1\}}$ is the fraction of the other players in 1. Denote also $S_N^\varepsilon := S_N \setminus (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$ and $\zeta^N(t) = \frac{1}{N+1}\sum_{i=0}^{N} \mathbb{1}_{\{Y_i(t)=1\}}$. The following is the adaptation to the present model of Propositions 3.3 and 3.4.

Proposition 4.11. *For any $t \in [0,T]$, $\varepsilon > 0$ and any $\boldsymbol{x}$ such that $\zeta_{\boldsymbol{x}}^{N,i} \in S_N^\varepsilon$, if $N \geq \frac{\varepsilon}{2}$, we have*

$$
\Delta^j u^{N,i}(t,\boldsymbol{x}) = -\frac{1}{N+1}\frac{\partial}{\partial\zeta}U^*(t,x,\zeta) + \tau^{N,i,j}(t,\boldsymbol{x}), \tag{4.33}
$$

for any $j \neq i$, with $|\tau^{N,i,j}(t,\boldsymbol{x})| \leq \frac{C_\varepsilon}{N^2}$, and the function $U^*(t,\zeta)$ solves

$$
\begin{aligned}
-\frac{d}{dt}U^*(t,\zeta) &+ H(U^*(1-\zeta) - U^*(\zeta)) \qquad (4.34)\\
&= N\zeta\left[U^*(1-\zeta) - U^*(\zeta)\right]^-\left[U^*\left(\zeta-\frac{1}{N}\right) - U^*(\zeta)\right] + r^N(t,\zeta) \\
&+ N(1-\zeta) + \left[U^*\left(\zeta+\frac{1}{N}\right) - U^*\left(1-\zeta-\frac{1}{N}\right)\right]^-\left[U^*\left(\zeta+\frac{1}{N}\right) - U^*(\zeta)\right],
\end{aligned}
$$

with $|r^N(t,\zeta)| \leq \frac{C_\varepsilon}{N}$. The constant C_ε is proportional to the Lipschitz constant of the master equation outside the discontinuity, which behaves like $\varepsilon^{-\frac{2}{3}}$.

Thanks to the characterization of the Nash equilibrium (Theorem 4.10), if the initial condition $\boldsymbol{Y}_{t_0}$ is deterministic and such that

$$
\zeta^N(t_0) \geq \varepsilon_N := \frac{N}{N+1}\left(\frac{1}{2}+\varepsilon\right) + \frac{1}{N+1}, \tag{4.35}
$$

then $\zeta^N(t) \geq \varepsilon_N$ for any $t \geq t_0$, almost surely. This implies that if $Y_i(t) = 1$ then the fraction of the other players in state 1 is $\zeta^{N,i}(t) = \frac{1}{N}\sum_{j\neq i} \mathbb{1}_{\{Y_j(t)=1\}} \geq \frac{1}{2} + \varepsilon$ for any t, and the same clearly holds if $Y_i(t) = -1$. We can argue symmetrically if $\zeta^N(t_0) \leq 1 - \varepsilon_N$; so we define

$$\Sigma_\varepsilon^N = \left\{\boldsymbol{x} \in \{-1,1\}^{N+1} : \zeta_{\boldsymbol{x}}^N \geq \varepsilon_N \text{ or } \zeta_{\boldsymbol{x}}^N \leq 1 - \varepsilon_N\right\}. \tag{4.36}$$

Therefore, computing V^N (or U^*) in the optimal trajectories $\boldsymbol{Y}_t$ when starting from $\boldsymbol{Y}_{t_0} \in \Sigma_\varepsilon^N$, we have

$$v^{N,i}(t,\boldsymbol{Y}_t) = V^N(t, Y_i, \zeta^{N,i}(t)) = \begin{cases} V^N(t,1,\zeta^{N,i}(t)) & Y_i(t) = 1 \\ V^N(t,1,1-\zeta^{N,i}(t)) & Y_i(t) = -1 \end{cases} \tag{4.37}$$

using the identity $V^N(t,-1,\zeta) = V^N(t,1,1-\zeta)$. Thus we are either or the right or on the left of the strip centered in the discontinuity and so

$$v^{N,i}(t,\boldsymbol{Y}_t) \leq \max_{\zeta^N \in S_N^\varepsilon} V^N(t,\zeta^N); \tag{4.38}$$

hence

$$|v^{N,i}(t,\boldsymbol{Y}_t) - u^{N,i}(t,\boldsymbol{Y}_t)| \leq \max_{\zeta^N \in S_N^\varepsilon} |V^N(t,\zeta^N) - U^*(t,\zeta^N)| \tag{4.39}$$

for any t, almost surely. We observe also that

$$\max_{\boldsymbol{x}\in\Sigma_\varepsilon^N} |v^{N,i}(t,\boldsymbol{x}) - u^{N,i}(t,\boldsymbol{x})| = \max_{\zeta^N \in S_N^\varepsilon} |V^N(t,\zeta^N) - U^*(t,\zeta^N)| \tag{4.40}$$

and further

$$\begin{aligned}
&|\Delta^i v^{N,i}(t,\boldsymbol{Y}_t) - \Delta^i v^{N,i}(t,\boldsymbol{Y}_t)| \\
&\qquad = |V^N(t,-Y_i(t),\zeta^{N,i}(t)) - U^*(t,-Y_i(t),\zeta^{N,i}(t)) \\
&\qquad\qquad - V^N(t,Y_i(t),\zeta^{N,i}(t)) + U^*(t,Y_i(t),\zeta^{N,i}(t))| \\
&\qquad \leq 2 \max_{\zeta^N \in S_N^\varepsilon} |V^N(t,\zeta^N) - U^*(t,\zeta^N)|.
\end{aligned} \tag{4.41}$$

Proof of Theorem 4.6. We choose a deterministic initial condition $\boldsymbol{Y}_t \in \Sigma_\varepsilon^N$; proceeding as in the proof of Theorem 3.1, we find

$$\begin{aligned}
&\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] + \sum_{j=0}^N \mathbb{E}\left[\int_t^T \alpha^j(s,\mathbf{Y}_s)\Big(\Delta^j[u_s^{N,i} - v_s^{N,i}]\Big)^2 ds\right] \\
&= -2\mathbb{E}\Bigg[\int_t^T (u_s^{N,i} - v_s^{N,i})\Bigg\{ - r^N(s,\mathbf{Y}_s) + H(\Delta^i u_s^{N,i}) - H(\Delta^i v_s^{N,i}) \\
&+ \sum_{j=0,j\neq i}^N (\alpha_s^j - \overline{\alpha}_s^j)\Delta^j u_s^{N,i} + \alpha_s^i(\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i})\Bigg\} ds\Bigg],
\end{aligned} \tag{4.42}$$

which is the analogous to (3.11), where α_s^i is the Nash equilibrium played by player i, $\overline{\alpha}_s^i = [U^*(s,-Y_i(s),\zeta_{\mathbf{Y}}^{N,i}(s)]^-$ is the control induced by U^* and all the functions are evaluated on the optimal trajectories, e.g. $v_s^{N,i} := v^{N,i}(s,\boldsymbol{Y}_s)$. We raise all the positive sum on the lhs and estimate the rhs using the Lipschitz properties of H, the bounds on $r^{N,i}$ and $\Delta^j u^i$ given by Proposition 4.11, and the bound on α^j given by (4.23), to get, if $N \geq \frac{2}{\varepsilon}$,

$$\begin{aligned}
&\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] \\
&\leq \frac{C}{N}\mathbb{E}\left[\int_t^T |u_s^{N,i} - v_s^{N,i}|ds\right] + C\mathbb{E}\left[\int_t^T |u_s^{N,i} - v_s^{N,i}||\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}|ds\right] \\
&+ \frac{C}{N}\sum_{j=0,j\neq i}^{N} \mathbb{E}\left[\int_t^T |u_s^{N,i} - v_s^{N,i}||\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|ds\right],
\end{aligned}$$

which can be further estimated via the convexity inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ yielding

$$\begin{aligned}
\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] \leq \frac{C}{N^2} &+ C\mathbb{E}\left[\int_t^T \left|u_s^{N,i} - v_s^{N,i}\right|^2 ds\right] + C\mathbb{E}\left[\int_t^T \left|\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}\right|^2 ds\right] \\
&+ \frac{C}{(N+1)}\sum_{j=0}^{N} \mathbb{E}\left[\int_t^T |\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|^2 ds\right].
\end{aligned}$$

Here C denotes any constant which may depend on ε. Since all the functions are evaluated on the optimal trajectories, we apply (4.39) and (4.41) to obtain

$$|u^{N,i}(t,\boldsymbol{Y}_t) - v^{N,i}(t,\boldsymbol{Y}_t)|^2 \leq \frac{C}{N^2} + C\int_t^T \max_{\zeta\in S_N^\varepsilon} |U^*(s,\zeta) - V^N(s,\zeta)|^2 ds$$

for any deterministic initial condition $\boldsymbol{Y}_t \in \Sigma_\varepsilon^N$. Therefore (4.40) gives

$$\max_{\zeta\in S_N^\varepsilon} |U^*(t,\zeta) - V^N(t,\zeta)|^2 \leq \frac{C}{N^2} + C\int_t^T \max_{\zeta\in S_N^\varepsilon} |U^*(s,\zeta) - V^N(s,\zeta)|^2 ds \tag{4.43}$$

and thus Gronwall's lemma applied to the quantity $\max_{\zeta\in S_N^\varepsilon} |U^*(s,\zeta) - V^N(s,\zeta)|^2$ allows to conclude that

$$\max_{\zeta\in S_N^\varepsilon} |U^*(t,\zeta) - V^N(t,\zeta)|^2 \leq \frac{C}{N^2}e^{C(T-t)} \leq \frac{C}{N^2}, \tag{4.44}$$

which immediately implies (4.25), but only if $N \geq \frac{\varepsilon}{2}$. By changing the value of $C = C_\varepsilon$, the thesis follows for any N. □

4.2.3 Propagation of chaos

Here we deal with the proof of Theorem 4.7. Denote by $X_i(t)$ the dynamics of the i-th player when choosing the control

$$\bar{\alpha}^i(t,\boldsymbol{x}) = [\Delta^i U^*(t,x_i,\zeta_{\boldsymbol{x}}^{N,i})]^- \tag{4.45}$$

induced by the master equation.

Theorem 4.12. *If $\zeta_0 \neq \frac{1}{2}$ then, for any N and $i = 0,\dots,N$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_i(t) - X_i(t)|\right] \leq \frac{C_{\zeta_0}}{N}, \tag{4.46}$$

where C does not depend on N, and $\lim_{\zeta_0\to\frac{1}{2}} C_{\zeta_0} = +\infty$.

Proof. Let $\zeta_0 = \frac{1}{2} + 2\varepsilon$ and consider the set A_ε where both $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ belong to Σ_ε^N, for any time. We have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |X_i(s) - Y_i(s)|\right] \leq C\mathbb{E}\left[\int_0^t |X_i(s) - Y_i(s)| + |\Delta^i u^{N,i}(s, \boldsymbol{X}_s) - \Delta^i v^{N,i}(s, \boldsymbol{Y}_s)|ds\right] \\
&\leq C\mathbb{E}\left[\int_0^t |X_i(s) - Y_i(s)|ds\right] + C\mathbb{E}\left[\mathbb{1}_{A_\varepsilon}\int_0^t |\Delta^i u^{N,i}(s, \boldsymbol{Y}_s) - \Delta^i v^{N,i}(s, \boldsymbol{Y}_s)|ds\right] \\
&+ C\mathbb{E}\left[\mathbb{1}_{A_\varepsilon}\int_0^t |\Delta^i u^{N,i}(s, \boldsymbol{X}_s) - \Delta^i u^{N,i}(s, \boldsymbol{Y}_s)|ds\right] + CP(A_\varepsilon^C)
\end{aligned}$$

and now we apply (4.25), the Lipschitz continuity of U^* in Σ_ε^N and the exchangeability of the processes to get, if $N \geq \frac{2}{\varepsilon}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |X_i(s) - Y_i(s)|\right] \leq \frac{C}{N} + C\int_0^t \mathbb{E}|X_i(s) - Y_i(s)|ds + CP(A_\varepsilon^C) \\
&+ C\mathbb{E}\Big[\mathbb{1}_{A_\varepsilon}\int_0^t |U^*(s, X_i(s), \zeta_{\boldsymbol{X}}^{N,i}(s)) - U^*(s, X_i(s), \zeta_{\boldsymbol{Y}}^{N,i}(s))| \\
&\qquad + |U^*(s, -X_i(s), \zeta_{\boldsymbol{X}}^{N,i}(s)) - U^*(s, -X_i(s), \zeta_{\boldsymbol{Y}}^{N,i}(s))|ds\Big] \\
&\leq \frac{C}{N} + C\int_0^t \mathbb{E}|X_i(s) - Y_i(s)|ds + CP(A_\varepsilon^C) + C\mathbb{E}\left[\mathbb{1}_{A_\varepsilon}\int_0^t \frac{1}{N}\sum_{j\neq i}|X_j(s) - Y_j(s)|ds\right] \\
&\leq \frac{C}{N} + C\int_0^t \mathbb{E}|X_i(s) - Y_i(s)|ds + CP(A_\varepsilon^C). \qquad (4.47)
\end{aligned}$$

We can bound the probability of A_ε^C considering the process in which the rates are equal to 0, for any time, i.e. the constant process equal to the initial condition. Thanks to the shape of the Nash equilibrium and of the control induced by the solution to the master equation, we have

$$P(A_\varepsilon^C) = P(\exists t: \text{ either } \zeta_{\boldsymbol{X}}^N(t) \text{ or } \zeta_{\boldsymbol{Y}}^N(t) \notin \Sigma_\varepsilon^N) \leq 2P(\zeta_{\boldsymbol{\xi}}^N \notin \Sigma_\varepsilon^N). \qquad (4.48)$$

Observe that $(N+1)\zeta_{\boldsymbol{\xi}}^N \sim \text{Bin}(N+1, \frac{1}{2} + 2\varepsilon)$; by standard Markov inequality, we bound

$$\begin{aligned}
P\left(\zeta_{\boldsymbol{\xi}}^N < \varepsilon_N\right) &\leq P\left(\left|\zeta_{\boldsymbol{\xi}}^N - \frac{1}{2} - 2\varepsilon\right| > \frac{1}{2} + 2\varepsilon - \varepsilon_N\right) \leq \frac{\text{Var}\left[\zeta_{\boldsymbol{\xi}}^N\right]}{\left(\frac{1}{2} + 2\varepsilon - \varepsilon_N\right)^2} \\
&= \frac{1}{N+1}\frac{\left(\frac{1}{2} + 2\varepsilon\right)\left(\frac{1}{2} - 2\varepsilon\right)}{\left(\frac{1}{2} + 2\varepsilon - \frac{N}{N+1}\left(\frac{1}{2} + \varepsilon\right) - \frac{1}{N+1}\right)^2} \leq \frac{C}{N\varepsilon} \qquad (4.49)
\end{aligned}$$

if $N \geq \frac{2}{\varepsilon}$, so that $\frac{1}{2} + 2\varepsilon - \varepsilon_N \geq \frac{\varepsilon}{4}$. Putting this latter estimate (4.49) into (4.47), and denoting $\varphi(t) := \mathbb{E}\left[\sup_{s\in[0,t]} |X_i(s) - Y_i(s)|\right]$, we obtain

$$\varphi(t) \leq \frac{C}{N\varepsilon} + C\int_0^t \varphi(s)ds \qquad (4.50)$$

which, by Gronwall's lemma, gives (4.46), but only if $N \geq \frac{2}{\varepsilon}$. By changing the value of C, the claim follows for any N. □

We are now in the position to prove Theorem 4.7.

Proof of Theorem 4.7. Thanks to (4.46), it is enough to show that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_i(t) - \widetilde{X}_i(t)|\right] \leq \frac{C_{\zeta_0}}{\sqrt{N}}. \tag{4.51}$$

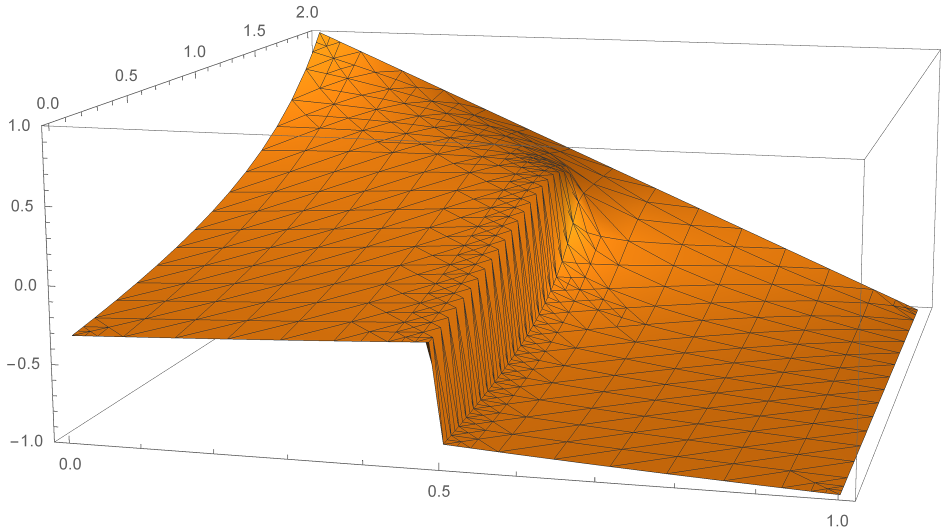
Let $\zeta_0 = \frac{1}{2} + 2\varepsilon$ and consider the set A_ε where both $\boldsymbol{X}_t$ and $\widetilde{\boldsymbol{X}}_t$ belong to Σ_ε^N, for any time. Arguing as in the proof of Theorem 4.12, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |X_i(s) - \widetilde{X}_i(s)|\right] \leq C\int_0^t \mathbb{E}|X_i(s) - \widetilde{X}_i(s)|ds + CP(A_\varepsilon^C)\\
&+ C\mathbb{E}\Big[\mathbb{1}_{A_\varepsilon}\int_0^t |U^*(s, X_i(s), \zeta_{\boldsymbol{X}}^{N,i}(s)) - U^*(s, X_i(s), \zeta_{\widetilde{\boldsymbol{X}}}^{N,i}(s))|\\
&\qquad + |U^*(s, -X_i(s), \zeta_{\widetilde{\boldsymbol{X}}}^{N,i}(s)) - U^*(s, -X_i(s), \zeta(s))|ds\Big]\\
&\leq C\int_0^t \mathbb{E}|X_i(s) - \widetilde{X}_i(s)|ds + CP(A_\varepsilon^C) + C\mathbb{E}\left[\int_0^t \frac{1}{N}\sum_{j\neq i}|X_j(s) - \widetilde{X}_j(s)|ds\right] + C\sup_t \mathbb{E}\left|\zeta_{\widetilde{\boldsymbol{X}}_t}^N - \zeta(t)\right|\\
&\leq \frac{C}{\sqrt{N}} + C\int_0^t \mathbb{E}|X_i(s) - \widetilde{X}_i(s)|ds + CP(A_\varepsilon^C),
\end{aligned}$$

where in the latter estimate we used (3.19). We can bound the probability of A_ε^C as before and thus Gronwall's Lemma allows to obtain (4.51). Finally (4.26), applying again (1.1) and (3.20), gives (4.27). □

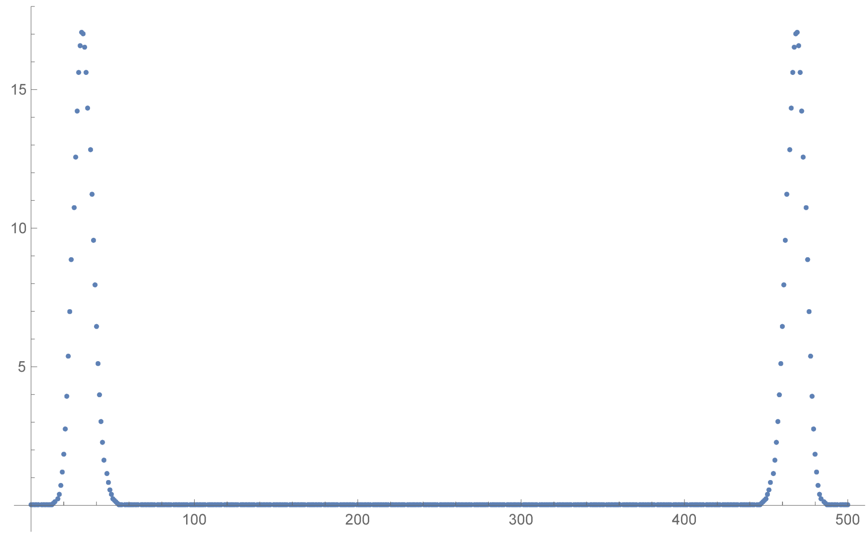
4.2.4 Numerical simulations

We provide here the numerical study which led to the convergence results. The first is the simulation of the value function $V^N(t,\zeta)$, for $N = 1000$ and $T = 2$. Let us remark that a numerical study of an example similar to ours was performed in [49].



The equation is backward with time going from 0 to 2, and in the abscissa there is $\zeta \in [0,1]$. We see that there is a shock at $\zeta = \frac{1}{2}$, for $0 \leq t \leq \frac{3}{2} = T - \frac{1}{2}$, which confirms Theorem 4.6.

The second simulation deals with the propagation of chaos for $\mu_0 = 0$. Here there is the (rescaled) plot of Law$[\zeta^N(T)]$, for $N = 500$ and $T = 2$.



We see that the law of the empirical measure is concentrated, at the terminal time, in two deltas, leading to Conjecture 4.9. On the converse, the simulation of Law$[\zeta^N(t)]$ for $\mu_0 \neq 0$ shows that it remains a delta for any time, as Theorem 4.7 states.

4.3 Extensions of the model

In this Section, we examine more in detail, on this model, the properties of the potential structure and of the weak mean field game solutions introduced in 1.4. Moreover we treat a modified example where the rates are bounded below away from zero: we show that the mean field game solutions have the same behaviour and the convergence results still hold if the Nash equilibrium possess the qualitative property described above.

4.3.1 Potential mean field game

In Subsection 1.4.1 we showed that the MFG system (4.3) can be viewed as the necessary conditions for optimality, given by the Pontryagin maximum principle, of a deterministic optimal control problem in $\mathbb{R}^2$. We specialize such setup to the model considered here and show that the N-player game selects exactly the minimum of this problem when it is unique, i.e. when $m_0 \neq 0$.

Consider the controlled dynamics, representing the KFP equation,

$$\begin{cases} \dot{m}_1 = m_{-1}\Gamma_{-1} - m_1\Gamma_1 \\ \dot{m}_{-1} = m_1\Gamma_1 - m_{-1}\Gamma_{-1} \\ m(0) = m_0. \end{cases} \tag{4.52}$$

The state variable is $m(t) = (m_1(t), m_{-1}(t))$, which represents the law of a Markov chain in $\{-1, 1\}$. Here the control is $\Gamma(t) = (\Gamma_1(t), \Gamma_{-1}(t)) \in \mathbb{A} := [0, +\infty[^2$, deterministic and open-loop, whereas Γ_x represents the transition rates of the Markov chain from state $-x$ to state x. The cost to be minimized is

$$\mathcal{J}(\Gamma) := \int_0^T \left(m_1(t)\frac{|\Gamma_1(t)|^2}{2} + m_{-1}(t)\frac{|\Gamma_{-1}(t)|^2}{2} \right) dt + \mathcal{G}(m(T)), \tag{4.53}$$

where $\mathcal{G}(m_1, m_{-1}) := -\frac{(m_1 - m_{-1})^2}{2}$ is such that

$$\frac{\partial}{\partial m_1}\mathcal{G}(m) = -(m_1 - m_{-1}) =: G(1, m)$$
$$\frac{\partial}{\partial m_{-1}}\mathcal{G}(m) = m_1 - m_{-1} =: G(-1, m),$$

meaning that the mean field game is potential, as $\nabla^m \mathcal{G}(m) = G(\cdot, m)$.

The Hamiltonian of this problem, for $g \in \mathbb{R}^2$, is

$$\mathcal{H}(m, g) = \sup_{\Gamma \in \mathbb{A}} \left\{ -b(m, \Gamma) \cdot g - m_1 \frac{\Gamma_1^2}{2} - m_{-1} \frac{\Gamma_{-1}^2}{2} \right\} = m_1 \frac{[(g_{-1} - g_1)^-]^2}{2} + m_{-1} \frac{[(g_1 - g_{-1})^-]^2}{2},$$

where $b_x(m, \Gamma) = m_{-x} a_{-x} - m_x a_x$, for $x = \pm 1$, is the vector field in (4.52), and the argmax of the Hamiltonian is

$$\Gamma_1^*(g) = (g_{-1} - g_1)^-$$
$$\Gamma_{-1}^*(g) = (g_1 - g_{-1})^-.$$

Thus the HJB equation of the control problem reads

$$\begin{cases} -\frac{\partial \mathcal{U}}{\partial t} + \mathcal{H}(m, \nabla^m \mathcal{U}) = 0 & m \in \mathcal{P}(\{-1, 1\}) \\ \mathcal{U}(T, m) = \mathcal{G}(m) \end{cases} \tag{4.54}$$

and its characteristics curves are given by the MFG system

$$\begin{cases} -\dot{u}_1 + \frac{[(u_{-1} - u_1)^-]^2}{2} = 0 \\ -\dot{u}_{-1} + \frac{[(u_1 - u_{-1})^-]^2}{2} = 0 \\ \dot{m}_1 = m_{-1}\Gamma_{-1} - m_1 \Gamma_1 \\ \dot{m}_{-1} = m_1 \Gamma_1 - m_{-1}\Gamma_{-1} \\ u_{\pm 1}(T) = G(\pm 1, m(T)), \quad m(0) = m_0. \end{cases} \tag{4.55}$$

The conclusions of Lemma 1.19 hold in this model; moreover, (4.54) can be reduce to dimension 1, as it is stated in $\mathcal{P}(\Sigma)$, and thus the derivative of the value function is known to be the unique entropy solution to (4.12), which can also be reduced to dimension 1. We observed in 1.4.1 that the potential structure is equivalent to the formulation of the master equation as a system of conservation laws, in any dimension. If $d = 2$ the master equation can always be reduced to dimension 1, thus it always admits a potential formulation; see also [48]. However the reduced master equation (4.13) can not be interpreted as the derivative of a HJB equation of a control problem, since its flow is not convex.

We know that, if T is large enough, there are three solutions to the MFG system. The control problem (4.52)-(4.53) has a minimum, so we wonder which of these solutions is indeed an optimum.

Firstly, we need to investigate some properties of the roots of (4.7). Let $T > \frac{1}{2}$ be fixed. Let $M_1(\mu_0) < M_2(\mu_0) < M_3(\mu_0)$ be the three solutions to (4.7). If $\mu_0 = 0$ denote $M_- = M_1(0) < 0$, $M_+ = M_3(0) > 0$; we have $M_2(0) = 0$ and $M_+ = -M_-$. If $\mu_0 > 0$ then, by Proposition 4.1, $M_3(\mu_0) > 0$ and $M_1(\mu_0), M_2(\mu_0) < 0$; if $\mu_0 < 0$ then $M_3(\mu_0) < 0$ and $M_1(\mu_0), M_2(\mu_0) > 0$.

Lemma 4.13. *Let $\mu_0 > 0$ and $T > T(\mu_0)$ be fixed. Then*

1. *The function* $[0,\mu_0] \ni \mu \mapsto M_3(\mu) \in [0,1]$ *is increasing,* $M_2(\mu)$ *is decreasing and* $M_1(\mu)$ *is increasing. In particular for any* $\mu \in [0,\mu_0]$

$$M_3(\mu) > M_+ = |M_-| > |M_1(\mu)| > |M_2(\mu)| > M_2(0) = 0 \tag{4.56}$$

2. *We have* $M_1(\mu) < -\frac{2T-1}{3T} < M_2(\mu) < 0$ *and for any* $\mu \in [0,\mu_0]$

$$\left|M_2(\mu) + \frac{2T-1}{3T}\right| > \left|M_1(\mu) + \frac{2T-1}{3T}\right|. \tag{4.57}$$

The case $\mu_0 < 0$ *is symmetric.*

Proof. Claim (1) derives from the proof of Proposition 4.1. For claim (2), $M_1(\mu)$ and $M_2(\mu)$ are the two negative roots of $f(M) = T^2M^3 - T(2-T)M^2 + (1-2T)M - \mu = 0$. The roots of $f'(M)$ are $q := -\frac{2T-1}{3T}$ and $\frac{1}{T}$. Hence $M_1 < q < M_2 < 0$, $f(q) > 0$ and we have, by Taylor's formula (which here is actually a change of variable),

$$f(q+\varepsilon) = f(q) + f'(q)\varepsilon + \frac{f''(q)}{2}\varepsilon^2 + \frac{f'''(q)}{6}\varepsilon^3 = f(q) + \frac{f''(q)}{2}\varepsilon^2 + T^2\varepsilon^3$$
$$f(q-\varepsilon) = f(q) - f'(q)\varepsilon + \frac{f''(q)}{2}\varepsilon^2 - \frac{f'''(q)}{6}\varepsilon^3 = f(q) + \frac{f''(q)}{2}\varepsilon^2 - T^2\varepsilon^3$$

for any $\varepsilon > 0$. Thus $f(q+\varepsilon) - f(q-\varepsilon) = 2T^2\varepsilon^3 > 0$ for any $\varepsilon > 0$, which implies (4.57). □

For $i = 1,2,3$, denote by m_i, u_i the solution to the MFG system (4.55) corresponding to M_i, and let also $\Gamma_i := \Gamma^*(u_i)$ be the optimal control.

Theorem 4.14. *Let* $\mu_0 > 0$ *and* $T > T(\mu_0)$ *be fixed. Then for any* $\mu \in [0,\mu_0]$ *and* $i = 1,2,3$ *we have* $\mathcal{J}(\Gamma_i) = \varphi(M_i(\mu))$*, where* $\varphi : [-1,1] \to [-1,1]$,

$$\varphi(M) := M^2\left(T - \frac{1}{2} - T|M|\right). \tag{4.58}$$

Moreover for any $\mu \in]0,\mu_0]$

$$\varphi(M_+) = \varphi(M_-) < \varphi(0) = 0, \tag{4.59}$$
$$\varphi(M_3(\mu)) < \varphi(M_+) < \varphi(M_1(\mu)), \tag{4.60}$$
$$\varphi(M_1(\mu)) < \varphi(M_2(\mu)) > 0,. \tag{4.61}$$

Proof. The first claim and (4.58) follow directly from (4.53) and (4.6).

We start proving (4.60). The roots of φ' are 0 and $\pm q$, with $q := -\frac{2T-1}{3T}$. The function φ is then increasing if either $M < q$ or $0 < M < -q$. Thus (4.60) follows from (4.56) and the fact that $\varphi(M_+) = \varphi(M_-)$, as $\varphi(M)$ only depends on $|M|$.

Next, we show that $\varphi(M_+) < 0 = \varphi(0)$. Since M_+ solves $T^2M^2 + T(2-T)M + 1 - 2T = 0$, we obtain for $M = M_+$

$$\varphi(M) = \frac{M^2}{2}(2T - 1 - 2TM) = \frac{M^2}{2}(T^2M^2 - T^2M) = \frac{T^2M^3}{2}(M-1) < 0$$

because $M_+ < 1$.

To prove (4.61), we first note that we have just showed that it holds in $\mu = 0$: $\varphi(M_1(0)) = \varphi(M-) = \varphi(M_+) < 0 = \varphi(0) = \varphi(M_2(0))$. We also know that $\varphi(M_1(\mu)) > \varphi(M_1(0))$ and $\varphi(M_2(\mu)) > \varphi(M_2(0))$, thanks to the monotonicity behavior of φ and Lemma 4.13. Hence suppose by contradiction that there exists $\mu \in]0, \mu_0]$ such that $\varphi(M_1(\mu)) = \varphi(M_2(\mu)) = c$, for some $c > 0$. This implies that both $M_1(\mu)$ and $M_2(\mu)$ are negative roots of $\varphi(M) - c$. Thus they are also negative roots of

$$\psi(M) := T\varphi(M) - Tc - f(M) = \frac{3}{2}TM^2 - (1-2T)M + \mu - Tc = 0$$

and $\psi'(q) = 0$, where $q = -\frac{2T-1}{3T}$ as above. Since ψ has degree 2, it follows that $|M_2(\mu) - q| = |M_1(\mu) - q|$, but this contradicts (4.57). Therefore there is no μ for which $\varphi(M_1(\mu)) = \varphi(M_2(\mu))$, and then if (4.61) holds for $\mu = 0$ (which is (4.59)) then it is true for any $\mu \in [0, \mu_0]$. □

We have thus proved that the N-player game selects the optimum of this deterministic control problem, when it is unique, i.e. $m_0 \neq 0$. Denoting, as before, by (μ^*, z^*) the unique MFG solution with $\mathrm{sgn}(\mu^*(T)) = \mathrm{sgn}(\mu^*(0))$, and by Γ_* the corresponding optimal control, we have for $\mu_0 > 0$

$$\mathcal{J}(\Gamma_*) < \mathcal{J}(\Gamma_1) < \mathcal{J}(\Gamma_2). \tag{4.62}$$

Instead, if $\mu_0 = 0$ then Γ_+ and Γ_- are both optimal, while $\Gamma \equiv 0$ is not a minimum. This could explain why the limit of the optimal empirical means is random and supported in μ_+ and μ_- with probability $\frac{1}{2}$; see Conjecture 4.9.

4.3.2 Weak mean field game solutions

Here we examine the weak mean field game solutions, as defined in Subsection 1.4.2, of our two state model. We show that they can be a simple randomization of the strong MFG solutions, but there are also weak (open-loop) MFG solutions that are not supported in the set of strong MFG solutions. This holds also for weak feedback MFG solution, in analogy with the "illuninating example" of [65] and [66].

For a (random) measure $m \in \mathcal{P}(\{-1, 1\})$, we denote its mean by $\bar{m} := m(1) - m(-1)$; and similarly for $\eta \in \mathcal{P}(\mathcal{Z})$ we write $\bar{\eta}^x(t) := \eta^x(t, 1) - \eta^x(t, -1)$. Hence we have $G(x, m) = -x\bar{m}$. To fix the ideas, let $T = 2$, so that there always exist 3 strong MFG solutions, in light of Proposition 4.1; and let also $\mu_0 = 0$ Let us firs study the optimization problem arising in the definition of weak MFG solution. So suppose $(\pi, \eta, X) \in \mathcal{S} \times \mathcal{P}(\mathcal{Z}) \times \mathcal{D}$ satisfies (1-3) of Definition 1.20; π is an open-loop control. Since $X(t)$ is a measurable function of π and $\mathcal{N}$ we have (as $E[X_0] = 0$),

$$\begin{aligned}
J(\pi, \eta^x) &= \mathbb{E}\left[\int_0^2 \frac{|\pi(t)|^2}{2} dt - X(2)\bar{\eta}^x(2)\right] \\
&= \int_0^2 \mathbb{E}\left[\frac{|\pi(t)|^2}{2} - \left(\int_\Theta \gamma(X(t^-), \pi(t), \theta)\nu(d\theta)\right)\bar{\eta}^x(2)\right] dt \\
&= \int_0^2 \mathbb{E}\left[\frac{|\pi(t)|^2}{2} - \left(\int_\Theta \gamma(X(t^-), \pi(t), \theta)\nu(d\theta)\right)\mathbb{E}\left[\bar{\eta}^x(2)|\mathcal{F}_t^{\pi,\mathcal{N}}\right]\right] dt
\end{aligned}$$

Now we use that $\mathcal{F}_t^\pi$ is conditionally independent of $\mathcal{F}_T^{X_0,\mathcal{N},\eta}$ given $\mathcal{F}_t^{X_0,\mathcal{N},\eta}$ and the independence of $X_0, \mathcal{N}$ and η to obtain

$$\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^{\pi,\mathcal{N}}] = \mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^{\pi}] = \mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^{X_0,\mathcal{N},\eta}] = \mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^{\eta}],$$

where $\mathcal{F}_t^\eta := \sigma(\eta(E) : E \in \mathcal{F}_t^{\mathcal{Z}})$. Thus

$$J(\pi, \eta^x) = \mathbb{E}\left[\int_0^2 \frac{|\pi(t)|^2}{2} - \left(\int_\Theta \gamma(X(t^-), \pi(t), \theta)\nu(d\theta)\right) \mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]dt\right] \tag{4.63}$$

If $M = \bar{\eta}^x(2)$ was deterministic, the optimal (feedback) control was given by $z(t) = \frac{2M}{|M|(2-t)+1}$. Therefore if M is replaced by $\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]$ in (4.63), a feedback control $\widehat{\beta}(t, x, \eta^x)$ is optimal for η if $\widehat{\beta}(t, 1, \eta^x) = \mathfrak{z}^-(t, \eta^x)$ and $\widehat{\beta}(t, -1, \eta^x) = \mathfrak{z}^+(t, \eta^x)$ for $\ell \otimes P$-a.e. (t, ω), with

$$\mathfrak{z}(t, \eta^x) = \frac{2\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]}{|\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]|(2-t)+1}. \tag{4.64}$$

Note that, a priori, the function $\mathfrak{z}$, and so also $\widehat{\beta}$, depend on the whole trajectory of η^x, not only on $\eta^x(t)$: it is not immediately a weak feedback MFG solution.

If $m_0 = 0$ and $T = 2$ there are three strong MFG solutions, in light of Corollary 4.2, corresponding to $M = \mu(2) \in \{0, \sqrt{3}/2, -\sqrt{3}/2\}$. The optimal controls are respectively provided by $z_0 \equiv 0$ and

$$z_\pm(t) = \pm\frac{\sqrt{3}}{\frac{\sqrt{3}}{2}(2-t)+1}$$

We now exhibit two different weak feedback MFG solutions: in the first the terminal average $\bar{\eta}^x(2)$ is supported on $S = \left\{0, \sqrt{3}/2, -\sqrt{3}/2\right\}$, while in the second $\bar{\eta}^x(2) \in \left\{\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}\right\}$, so that $P(\bar{\eta}^x(2) \in S) = 0$.

Lemma 4.15. *Any randomization of strong MFG solutions is a weak feedback MFG solution (β, η, X), with $P(\eta^x(2) \in S) = 1$.*

Proof. Let χ be a random variable valued on $\{0, -1, 1\}$, independent of X_0 and $\mathcal{N}$. Let

$$\mathfrak{z}(t) := z_+(t)\mathbb{1}_{\{\chi=1\}} + z_-(t)\mathbb{1}_{\{\chi=-1\}}, \tag{4.65}$$

i.e. we choose one of the three controls at $t = 0$ depending on the value of χ. Let $X^{\mathfrak{z}}$ be the corresponding dynamics and define

$$\eta := P\left[(\mathcal{N}, \mathfrak{z}, X^{\mathfrak{z}}) \in \cdot|\chi\right]. \tag{4.66}$$

Of course η is χ-measurable and, on the other hand,

$$\bar{\eta}_2^x = \mathbb{E}[X^{\mathfrak{z}}(2)|\chi] = \frac{\sqrt{3}}{2}\chi \in S. \tag{4.67}$$

Then χ is η measurable, and hence we obtain that $\eta := P\left[(\mathcal{N}, \mathfrak{z}, X^{\mathfrak{z}}) \in \cdot|\eta\right]$, that is, condition (1.91) holds.

So it remains to check that (4.64) is satisfied. Clearly $\mathcal{F}_t^\eta = \sigma(\chi)$ for any t and so

$$\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta] = \mathbb{E}[\bar{\eta}^x(2)|\chi] = \bar{\eta}^x(2) = \frac{\sqrt{3}}{2}\chi.$$

Therefore for any $t > 0$ (recall that the value of the control at $t = 0$ is meaningless) and ω

$$\mathfrak{z}(t) = \frac{\sqrt{3}\chi}{\frac{\sqrt{3}}{2}(2-t)+1} = \frac{2\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]}{|\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]|(2-t)+1},$$

which means that the control $\mathfrak{z}$ defined in (4.65) is optimal, as it satisfies (4.64).

Moreover $\eta^x(t)$ is χ-measurable for any t and then $\mathcal{F}_t^\eta = \sigma(\eta^x(t))$. This implies that

$$\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta] = \mathbb{E}[\bar{\eta}^x(2)|\eta^x(t)] = \frac{\sqrt{3}}{2}\chi$$

and thus there exist a function $Z : [0,2] \times [-1,1] \to [-2,2]$ such that the optimal control is $\mathfrak{z}(t,\eta) = \mathfrak{z}(t) = Z(t, \bar{\eta}^x(t))$. This means that we found a weak feedback MFG solution, with feedback control β given by $\beta(t, \pm 1, \mu) = Z(t,\mu)^{\mp}$. More precisely, Z is given by

$$Z(t,\mu) = \begin{cases} \frac{\sqrt{3}\,\mathrm{sgn}(\mu)}{\frac{\sqrt{3}}{2}(2-t)+1} & \text{if } \mu \neq 0 \\ 0 & \text{if } \mu = 0, \end{cases}$$

which in particular says that Z is measurable. □

Lemma 4.16. *There exists a weak feedback mean field game solution* (β, η, X) *such that* $P(\eta^x(2) \in S) = 0$.

Proof. Let χ be a random variable valued on $\{-1,1\}$ with $\mathbb{E}[\chi] = 0$, independent of X_0 and $\mathcal{N}$ and let

$$\mathfrak{z}(t) := \frac{\chi(\sqrt{5}-1)}{\frac{\sqrt{5}-1}{2}(2-t)} \mathbb{1}_{]1,2]}(t), \tag{4.68}$$

i.e. we choose the control zero until time $t=1$ and then one of the other two controls $z_\pm$, depending on the value of χ. Note that the constant $M_T = \mu(T)$ given by (4.9) is precisely $\frac{\sqrt{5}-1}{2}$ for $T = 1$. In fact we go from $t=1$ to $t=2$ choosing one of the controls different from 0, starting from 0 at $t=1$, which is the same as going from $t=0$ to $t=1$ starting from 0, and thus the constant we have to use in the definition (4.68) is M_T.

As above, let $X^{\mathfrak{z}}$ be the corresponding dynamics and define $\eta := P\left[(\mathcal{N}, \mathfrak{z}, X^{\mathfrak{z}}) \in \cdot|\chi\right]$. Of course η is χ-measurable and, on the other hand,

$$\bar{\eta}_2^x = \mathbb{E}[X^{\mathfrak{z}}(2)|\chi] = \frac{\sqrt{5}-1}{2}\chi \notin S. \tag{4.69}$$

Then χ is η measurable, and hence we obtain that $\eta := P\left[(\mathcal{N}, \mathfrak{z}, X^{\mathfrak{z}}) \in \cdot|\eta\right]$, that is, condition (1.91) holds. So it remains to check that (4.64) is satisfied. Now

$$\mathcal{F}_t^\eta = \begin{cases} \{\varnothing, \Omega\} & t \leq 1 \\ \sigma(\chi) & 1 < t \leq 2 \end{cases}$$

and so

$$\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta] = \begin{cases} \frac{\sqrt{5}-1}{2}\mathbb{E}[\chi] = 0 & t \leq 1 \\ \frac{\sqrt{5}-1}{2}\chi & 1 < t \leq 2. \end{cases}$$

Therefore for any t and ω it holds

$$\mathfrak{z}(t) = \frac{2\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]}{|\mathbb{E}[\bar{\eta}^x(2)|\mathcal{F}_t^\eta]|(2-t)+1},$$

which means that the control $\mathfrak{Z}$ defined in (4.68) is optimal, as it satisfies (4.64). Moreover, as before, $\eta^x(t)$ is χ-measurable for any t and then $\mathcal{F}_t^\eta = \sigma(\eta^x(t))$. This implies that $\mathbb{E}[\overline{\eta}^x(2)|\mathcal{F}_t^\eta] = \mathbb{E}[\overline{\eta}^x(2)|\eta^x(t)]$ and thus, defining $Z : [0,2] \times [-1,1] \to [-2,2]$ by

$$Z(t,\mu) = \begin{cases} 0 & \text{if } 0 < t \leq 1, \quad \forall \mu \\ \frac{(\sqrt{5}-1)\,\mathrm{sgn}(\mu)}{\frac{\sqrt{5}-1}{2}(2-t)+1} & \text{if } 1 < t \leq 2, \quad \mu \neq 0, \end{cases}$$

we have $\mathfrak{Z}(t,\eta) = Z(t,\overline{\eta}^x(t))$ for $\ell \otimes P$ - a.e. (t,ω). In particular Z is measurable and the induced open-loop control is predictable: we found a weak feedback MFG solution. Note that the value of Z in $\mu = 0$ does not need to be defined, as $\overline{\eta}^x(t) = 0$ with null probability, if $1 < t \leq 2$. □

Remark 4.17. *We choose $T = 2$ and $m_0 = 0$ just for simplicity: with different horizon and initial condition the solutions are similar, but in the second example we would have to take χ with $\mathbb{E}[\chi] = -m_0$.*

In the above proof the time $t = 1$ can be replaced by any $t_0 \in]0, 3/2[$, choosing the constant M_{T-t_0} given by (4.9) instead of M_1.

Among the large set of weak (feedback) MFG solutions, the N-player game selects only one of them in the limit: either one strong solution, if $\mu_0 \neq 0$ (Theorem 4.7), or a randomization of two strong solutions if $\mu_0 = 0$ (Conjecture 4.9). Therefore in this example we can conclude that the set of weak (feedback) MFG solutions, as defined in 1.20, is too wide to capture the limits of the feedback Nash equilibria of the N-player game. Recently in [66] in the diffusion setting, limit points of closed-loop Nash equilibria are characterized to be weak feedback MFG solutions, but it is not clear whether they are just randomization of strong solutions or not. This is true, instead, when dealing with open loop controls: in [65] it is shown in an example that there exists a sequence of open-loop Nash equilibria converging to a weak MFG solution which is not a randomization of strong solutions.

4.3.3 A modified example

Here we consider a modified framework allowing only controls bounded from below (Assumption (Erg)), i.e. $\alpha(t,x) \geq \kappa > 0$. Most of the results are analogous to the previous setting, so we just sketch them, but the convergence proof is different: it involves a large deviation principle, and so it could be of interest itself. It still relies on a characterization of the Nash equilibrium as in Theorem 4.10, but unfortunately we have not managed to prove it.

Consider the Lagrangian $L_\kappa(a) = \frac{|a-\kappa|^2}{2}$, so that the running cost is still zero if a player chooses the control equal to the minimum; the final cost is te same, anti-monotonous. The Hamiltonian of the problem is

$$H_\kappa(p) := \sup_{a \geq \kappa} \left\{ -ap - \frac{(a-\kappa)^2}{2} \right\} = -\kappa p + \frac{(p^-)^2}{2}, \tag{4.70}$$

whose argmax is given by $a^*_\kappa(p) := \kappa + p^-$. The mean field game system becomes

$$\begin{cases} \dot{z} = z\left(\frac{|z|}{2} + 2\kappa\right) \\ \dot{\mu} = -\mu(|z| + 2\kappa) + z \\ z(T) = 2\mu(T) \\ \mu(0) = \mu_0. \end{cases} \tag{4.71}$$

In order to solve system (4.71), we again suppose $\mu(T) = M$ is given so that we can find $z(t)$. As one can check via computation,

$$z(t) := \frac{4\kappa M}{(2\kappa + |M|)e^{(T-t)2\kappa} - |M|} \tag{4.72}$$

and substituting this expression in the KFP equation, we find

$$\mu(t) = \frac{e^{2\kappa t}(|M| - e^{2\kappa(T-t)}(2\kappa + |M|))^2 \Big(\mu_0 + \frac{(-1+e^{2\kappa t})M(2e^{2\kappa T}(1+e^{2\kappa t})\kappa + (-2e^{2\kappa t} + e^{2\kappa T} + e^{2\kappa(t+T)})|M|)}{(e^{2\kappa t}|M| - e^{2\kappa T}(2\kappa + |M|))^2}\Big)}{(|M| - e^{2\kappa T}(2\kappa + |M|))^2}. \tag{4.73}$$

By imposing the mean field condition $\mu(T) = M$ we can characterize the MFG solutions via the solutions M to

$$-M + \frac{4e^{2\kappa T}\kappa^2 \left[\mu_0 + \frac{(-1+e^{2\kappa T})M(2e^{2\kappa T}(1+e^{2\kappa T})\kappa + (-e^{2\kappa T} + e^{4\kappa T})|M|)}{(e^{2\kappa T}|M| - e^{2\kappa T}(2\kappa + |M|))^2}\right]}{(|M| - e^{2\kappa T}(2\kappa + |M|))^2} = 0. \tag{4.74}$$

Note that this is a generalization of the case $\kappa = 0$: indeed, for $\kappa \to 0$ we recover the previous mean field condition, given by (4.7). The above Equation can be rewritten as

$$M^3(e^{2\kappa T} - 1)^2 - M|M|(e^{2\kappa T} - 1)[(1 - 4\kappa)e^{2\kappa T} - 1] + 2\kappa M[e^{4\kappa T}(2\kappa - 1) + 1] - 4e^{2\kappa T}\kappa^2\mu_0 = 0. \tag{4.75}$$

We can now state the analogous of Proposition 4.1.

Proposition 4.18. *If $\kappa \geq \frac{1}{2}$ the MFG solution is unique for any T and μ_0; while if $\kappa > \frac{1}{2}$ and $T \leq T_\kappa := \frac{-\log(1-2\kappa)}{4\kappa}$ the MFG solution is unique for any μ_0. Let $T_\kappa(\mu_0)$ be the unique solution in $[T_\kappa, +\infty[$ of*

$$\begin{aligned}
|\mu_0| = \frac{1}{4e^{2\kappa T}\kappa^2}\Bigg\{ & \frac{1}{3(-1+e^{2\kappa T})^2} 2\kappa \left(1 + e^{4\kappa T}(-1 + 2\kappa)\right) \Big[-1 + e^{2\kappa T}(2 - 4\kappa) + e^{4\kappa T}(-1 + 4\kappa) \\
& - \sqrt{(-1 + e^{2\kappa T})^2(1 - 6\kappa + e^{2\kappa T}(-2 + 8\kappa) + e^{4\kappa T}(1 - 2\kappa + 4\kappa^2)}\Big] \\
& + \frac{1}{9(-1+e^{2\kappa T})^3}(1 + e^{2\kappa T}(-1 + 4\kappa)) \Big[1 + e^{4\kappa T}(1 - 4\kappa) + e^{2\kappa T}(-2 + 4\kappa) \\
& + \sqrt{(-1 + e^{2\kappa T})^2(1 - 6\kappa + e^{2\kappa T}(-2 + 8\kappa) + e^{4\kappa T}(1 - 2\kappa + 4\kappa^2)}\Big]^2 \\
& + \frac{1}{27(-1+e^{2\kappa T})^4} \Big[1 + e^{4\kappa T}(1 - 4\kappa) + e^{2\kappa T}(-2 + 4\kappa) \\
& + \sqrt{(-1 + e^{2\kappa T})^2(1 - 6\kappa + e^{2\kappa T}(-2 + 8\kappa) + e^{4\kappa T}(1 - 2\kappa + 4\kappa^2))}\Big]^3 \Bigg\}.
\end{aligned} \tag{4.76}$$

Then for any $\mu_0 \in [-1, 1]$, the MFG system (4.71) admits

(i) a unique solution for $T < T_\kappa(\mu_0)$;

(ii) two solutions if $T = T_\kappa(\mu_0)$;

(iii) three distinct solutions for $T > T_\kappa(\mu_0)$.

Note that $\lim_{\kappa\downarrow 0} T_\kappa = \frac{1}{2}$, as in Corollary 4.2, and $\lim_{\kappa\uparrow\frac{1}{2}} T_\kappa = +\infty$. In fact, the introduction of a lower bound κ increases the time for which there is uniqueness of solutions. Moreover the three distinct solutions, when they exist, possess the same properties as for $\kappa = 0$. Namely, there is a unique solution, denoted by $(z^*_\kappa, \mu^*_\kappa)$, which does not change sign, if $\mu_0 \neq 0$, and is the one that exists for any T. While if $\mu_0 = 0$ the three solutions are the constant 0, the one always positive and the one always negative, if $T > T_\kappa$.

The master equation and the Nash system have the same shape as in (4.12) and (4.20), where the Hamiltonian is replaced by H_κ and p^- by a^*_κ. The master equation can still be written as a scalar conservation law, whose entropy solution, denoted by $Z^*_\kappa(t,\mu)$, has the same properties as before: is has a shock at $\mu = 0$, for $t > T_k$, and is smooth elsewhere. If we show that the solution to the Nash system enjoys the same properties, then we are able to prove the convergence of the value functions as well as a propagation of chaos for $\mu_0 \neq 0$. Further, the claim of Conjecture 4.9 should remain true, due to the numerical simulations which show the same behaviour as in 4.2.4.

Thus we now consider $0 < \kappa < 1/2$ and $T > T_\kappa$. Denote $V^N_\kappa(t,\zeta) = V^N_\kappa(t,1,\mu)$ and $Z^N_\kappa(t,\zeta) = V^N_\kappa(t,1-\zeta) - V^N_\kappa(t,\zeta)$, so that the Nash equilibrium is given by

$$\beta^N_\kappa(t,\pm 1,\zeta) = \kappa + Z^N_\kappa(t,\zeta)^\mp.$$

Let $U_\kappa(t,x,\mu)$ be the solution to the master equation corresponding to the the entropy solution $Z^*_\kappa(t,\mu)$ and define $U^*_\kappa(t,\zeta) = U_\kappa(t,1,2\zeta-1)$. Let also $\boldsymbol{Y}^\kappa$, $\boldsymbol{X}^\kappa$ and $\widetilde{\boldsymbol{X}}^\kappa$ be as in Section 4.2.

Theorem 4.19. *Fix $N \geq 1$ and $0 < \kappa < \frac{1}{2}$. Assume that for any $\zeta \in S_N = \left\{0, \frac{1}{N}, \dots, 1\right\}$*

$$Z^N_\kappa(t,\zeta) \geq 0 \qquad \text{if } \zeta \geq \frac{1}{2}, \tag{4.77}$$

$$Z^N_\kappa(t,\zeta) \leq 0 \qquad \text{if } \zeta \leq \frac{1}{2}. \tag{4.78}$$

Then for any $t \in [0,T]$, $\varepsilon > 0$ and $\zeta \in S^N \setminus \left]\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right[$, we have

$$|V^N_\kappa(t,\zeta) - U^*_\kappa(t,\zeta)| \leq \frac{C_{\varepsilon,\kappa}}{N}, \tag{4.79}$$

where $C_{\varepsilon,\kappa}$ does not depend on N nor on t,ζ. Moreover if $\zeta_0 \neq \frac{1}{2}$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y^\kappa_i(t) - \widetilde{X}^\kappa_i(t)|\right] \leq \frac{C_{\zeta_0,\kappa}}{\sqrt{N}}, \tag{4.80}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y^\kappa_i(t) - X^\kappa_i(t)|\right] \leq \frac{C_{\zeta_0,\kappa}}{N}. \tag{4.81}$$

Proof. We start by proving (4.79), and omit the κ from the notation. Let $\varepsilon > 0$ be fixed and consider a deterministic initial condition $\boldsymbol{\xi} = \boldsymbol{Y}_t$ at time t such that $\zeta^N_{\boldsymbol{\xi}} \in \Sigma^N_\varepsilon$, where Σ^N_ε is defined by (4.36) and ε_N by (4.35). Let $\bar{\varepsilon} = \bar{\varepsilon}(T,\kappa,\varepsilon) := \frac{\varepsilon}{2}e^{-2\kappa T}$, fix $N \geq \frac{2}{\varepsilon}$ and consider the set

$$A_\varepsilon := \left\{\boldsymbol{Y}_s \in \Sigma^N_{\bar{\varepsilon}} \quad \forall s \in [t,T]\right\}.$$

Firstly, we bound the probability of A^C_ε. So consider the process $\widetilde{\boldsymbol{Y}}$ in which the transition rates of each $\widetilde{Y}_i$ are all constant and equal to the minimum κ, with the same initial condition $\boldsymbol{Y}_t$.

Thanks to the properties of the Nash equilibrium (4.77) and (4.78), we have $P(A_\varepsilon^C) \le P(\widetilde{A}_\varepsilon^C)$, where $\widetilde{A}_\varepsilon$ is the set where $\widetilde{\boldsymbol{Y}}_s \in \Sigma_{\bar\varepsilon}^N$ for any $s \in [t,T]$. The fraction or players in state 1, denoted by $\widetilde{\zeta}^N(s)$ for $t \le s \le T$, of this process has a non-zero probability of crossing the discontinuity, due to $\kappa > 0$, thus we can not argue as for $\kappa = 0$.

We are allowed to consider a sequence of deterministic initial conditions such that

$$\lim_{N\to\infty} \zeta_{\boldsymbol{\xi}}^N =: \zeta_t^* \in [0,1] \setminus \left]1/2-\varepsilon, 1/2+\varepsilon\right[=: S^\varepsilon; \tag{4.82}$$

in particular the limit exists. We have that the $\widetilde{Y}_i$-s are independent processes (even if not identically distributed), and the sequence of processes $\left(\widetilde{\zeta}^N(s)\right)_{s\in[t,T]}$ satisfies a sample path large deviation principle on $D([0,T],[0,1])$, thanks to a version of Sanov's Theorem; see e.g. [35] and [37]. We actually need only the upper bound:

$$\limsup_{N\to\infty} \frac{1}{N} \log P(\widetilde{A}_{\varepsilon,N}^C) \le -\mathcal{I}_{T,\kappa,\varepsilon}, \tag{4.83}$$

where $\mathcal{I}$ is a good rate functional, $\mathcal{I}_{T,\kappa,\varepsilon} = \inf_{\lambda\in\, \overline{B_{T,\kappa,\varepsilon}}} \mathcal{I}(\lambda)$ and

$$B_{T,\kappa,\varepsilon} := \left\{ \lambda \in D([0,T],[0,1]) : \lambda(s) \notin \left]\frac{1}{2}-\bar\varepsilon, \frac{1}{2}-\bar\varepsilon\right[\quad \forall s \in [t,T] \right\}.$$

Thanks to (4.82), the sequence of processes $\left(\widetilde{\zeta}^N(s)\right)_{s\in[t,T]}$ satisfies a propagation of chaos property with the limit given by $\zeta^*(s) = \frac{1}{2} + \left(\zeta_t^* - \frac{1}{2}\right) e^{-2\kappa(s-t)}$ for $t \le s \le T$: it is provided by the solution to the KFP equation when $z=0$. It is well known that the rate functional is always positive and, if the propagation of chaos holds, $\mathcal{I}(\lambda) = 0$ if and only if $\lambda = \zeta^*$. Therefore we can conclude that $\mathcal{I}_{T,\kappa,\varepsilon} > 0$, because of the choice of $\bar\varepsilon$: we have $|\zeta^*(s) - 1/2| \ge 2\bar\varepsilon$ for all $s \in [t,T]$ and for any choice of t and $\zeta_t^* \in S^\varepsilon$, thus ζ^* does not belong to the closure of $B_{T,\kappa,\varepsilon}$. This implies that

$$P(A_\varepsilon^C) \sim e^{-N\mathcal{I}_{T,\kappa,\varepsilon}}. \tag{4.84}$$

Moreover, the solution U^* to the master equation is smooth outside $[1/2-\bar\varepsilon, 1/2+\bar\varepsilon]$ and so the conclusions of Proposition 4.11 follow in the same way for $N \ge 2/\bar\varepsilon$. We obtain Equation (4.42) as above:

$$\begin{aligned}
&\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] + \sum_{j=0}^N \mathbb{E}\left[\int_t^T \alpha^j(s,\mathbf{Y}_s)\Big(\Delta^j[u_s^{N,i} - v_s^{N,i}]\Big)^2 ds\right] \\
&= -2\mathbb{E}\left[\int_t^T (u_s^{N,i} - v_s^{N,i})\bigg\{ -r^N(s,\mathbf{Y}_s) + H(\Delta^i u_s^{N,i}) - H(\Delta^i v_s^{N,i}) \right. \\
&\left. + \sum_{j=0, j\neq i}^N (\alpha_s^j - \overline{\alpha}_s^j)\Delta^j u_s^{N,i} + \alpha_s^i(\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}) \bigg\} ds \right]
\end{aligned}$$

with the same notation. Now we split the mean in $\mathbb{E}[\mathbb{1}_{A_\varepsilon} \ldots] + \mathbb{E}[\mathbb{1}_{A_\varepsilon^C} \ldots]$. The second term is bounded by

$$\mathbb{E}[\mathbb{1}_{A_\varepsilon^C} \ldots] \le CNP(A_\varepsilon^C) \sim CNe^{-CN} \le \frac{C}{N^2}$$

for $N \geq N_\varepsilon$ large enough. While in the event A_ε we can use Lipschitz properties of H_κ and a^*_κ and the bounds on $r^{N,i}$ and $\Delta^j u^{N,i}$. On the left hand side, we raise the positive sum $\sum_{j\neq i}$ and estimate $\alpha^i \geq \kappa$, to get

$$\begin{aligned}
&\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] + \kappa \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T \left|\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}\right|^2 ds\right] \leq \\
&\leq \frac{C}{N}\mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |u_s^{N,i} - v_s^{N,i}| ds\right] + C\mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |u_s^{N,i} - v_s^{N,i}||\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}| ds\right] \\
&+ \frac{C}{N+1} \sum_{j=0, j\neq i}^{N} \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |u_s^{N,i} - v_s^{N,i}||\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}| ds\right] + CNP(A_\varepsilon^C).
\end{aligned}$$

The right hand side can be further bounded using the inequality $ab \leq \delta a^2 + \frac{b^2}{4\delta}$, so that we can write

$$\begin{aligned}
&\mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] + \kappa \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T \left|\Delta^i u_s^{N,i} - \Delta^i v_s^{N,i}\right|^2 ds\right] \\
&\leq \frac{C}{N^2} + C\mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T \left|u_s^{N,i} - v_s^{N,i}\right|^2 ds\right] + \frac{\kappa}{2(N+1)} \sum_{j=0}^{N} \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|^2 ds\right] \\
&\leq \frac{C}{N^2} + C\mathbb{E}\left[\int_t^T \left|u_s^{N,i} - v_s^{N,i}\right|^2 ds\right] + \frac{\kappa}{2(N+1)} \sum_{j=0}^{N} \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|^2 ds\right].
\end{aligned} \tag{4.85}$$

Averaging $(\frac{1}{N+1}\sum_{i=0}^N)$ we obtain

$$\begin{aligned}
\frac{1}{N+1}\sum_{i=0}^{N} \mathbb{E}[(u_t^{N,i} - v_t^{N,i})^2] + \frac{\kappa}{2(N+1)} \sum_{j=0}^{N} \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|^2 ds\right] \\
\leq \frac{C}{N^2} + C\int_t^T \frac{1}{N+1} \sum_{i=0}^{N} \mathbb{E}\left[|u_s^{N,i} - v_s^{N,i}|^2\right] ds
\end{aligned}$$

and thus Gronwall's Lemma, applied to the quantity $\frac{1}{N+1}\sum_{i=0}^N \mathbb{E}\left[|u_s^{N,i} - v_s^{N,i}|^2\right]$ yields, raising the positive term of the lhs,

$$\sup_{t\leq s\leq T}\left\{\frac{1}{N+1}\sum_{i=0}^{N}\mathbb{E}\left[\left|u^{N,i}(s,\boldsymbol{Y}_s) - v^{N,i}(s,\boldsymbol{Y}_s)\right|^2\right]\right\} \leq \frac{C}{N^2},$$

which also implies

$$\frac{\kappa}{2(N+1)} \sum_{j=0}^{N} \mathbb{E}\left[\mathbb{1}_{A_\varepsilon} \int_t^T |\Delta^j u_s^{N,j} - \Delta^j v_s^{N,j}|^2 ds\right] \leq \frac{C}{N^2}. \tag{4.86}$$

Applying (4.86) to the rhs of (4.85) and using Gronwall's Lemma again, we have

$$|u^{N,i}(t,\boldsymbol{\xi}) - v^{N,i}(t,\boldsymbol{\xi})|^2 \leq \frac{C}{N^2} \tag{4.87}$$

for any deterministic $\boldsymbol{\xi} \in \Sigma_\varepsilon^N$, which immediately gives (4.79), in light of (4.40).

To prove (4.80), we first observe that (4.86) can be derived in the same way for more general non-deterministic initial condition. Indeed, assuming now that the initial time is 0 and the initial condition $\boldsymbol{\xi}$ is i.i.d with $P(\xi_i = 1) = \frac{1}{2} + 2\varepsilon$, the same argument we used above yields $P(A_\varepsilon^C) \leq CN^{-2}$ and thus, by summing on both sides of (4.86) the same quantity appearing on the lhs, but with A_ε replaced by A_ε^C, and then using the exchangeability of the process $\boldsymbol{Y}$, we deduce

$$\mathbb{E}\left[\int_0^T \left|\Delta^i v^{N,i}(s, \boldsymbol{Y}_s) - \Delta^i u^{N,i}(s, \boldsymbol{Y}_s)\right| ds\right] \leq \frac{C}{N}. \tag{4.88}$$

Consider now the set E_ε where both $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ belong to $\Sigma_{\tilde{\varepsilon}}^N$, for any time. We can bound

$$P(E_\varepsilon^C) = P\left(\exists t: \text{ either } \zeta_{\boldsymbol{X}}^N(t) \text{ or } \zeta_{\boldsymbol{Y}}^N(t) \notin \Sigma_{\tilde{\varepsilon}}^N\right) \leq 2P\left(\exists t: \zeta_{\widetilde{\boldsymbol{Y}}}^N(t) \notin \Sigma_{\tilde{\varepsilon}}^N\right) \leq 2P(\widetilde{A}_\varepsilon^C) \leq \frac{C}{N}.$$

Proceeding as in the proof of (4.46), applying (4.88), the Lipschitz continuity of U^* in E_ε and the exchangeability of the processes, we find

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |X_i(s) - Y_i(s)|\right] \leq C\mathbb{E}\left[\int_0^t |X_i(s) - Y_i(s)| + |\Delta^i u^{N,i}(s, \boldsymbol{X}_s) - \Delta^i v^{N,i}(s, \boldsymbol{Y}_s)| ds\right] \\
&\leq C\mathbb{E}\left[\int_0^t |X_i(s) - Y_i(s)| ds\right] + C\mathbb{E}\left[\mathbb{1}_{E_\varepsilon} \int_0^t |\Delta^i u^{N,i}(s, \boldsymbol{Y}_s) - \Delta^i v^{N,i}(s, \boldsymbol{Y}_s)| ds\right] \\
&+ C\mathbb{E}\left[\mathbb{1}_{E_\varepsilon} \int_0^t |\Delta^i u^{N,i}(s, \boldsymbol{X}_s) - \Delta^i u^{N,i}(s, \boldsymbol{Y}_s)| ds\right] + CP(E_\varepsilon^C) \\
&\leq C\int_0^t \mathbb{E}|X_i(s) - Y_i(s)| ds + \frac{C}{N} + C\mathbb{E}\left[\mathbb{1}_{E_\varepsilon} \int_0^t \frac{1}{N}\sum_{j\neq i} |X_j(s) - Y_j(s)| ds\right] + \frac{C}{N} \\
&\leq \frac{C}{N} + C\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s} |X_i(r) - Y_i(r)|\right] ds
\end{aligned}$$

and thus Gronwall's inequality gives (4.80).

Finally, (4.81) derives from (4.80) as in Theorem 4.7. Indeed, we obtained the claims only for N large enough, but changing the value of $C = C_\varepsilon$ the thesis follows for any N. □

Appendix B: Entropy solutions to scalar conservation laws

We recall here some fact about entropy solutions to scalar conservation laws. A standard reference for the followinf results is [29]. We consider the Cauchy problem, for $\mu \in \mathbb{R}$, $t \in [0, T]$

$$\begin{cases} \partial_t z + \partial_\mu [g(\mu, z)] = 0, \\ z(0, \mu) = z_0(\mu). \end{cases} \tag{4.89}$$

The function g, called the flow, is not standard as it is space-dependent. We always assume that $g \in \mathcal{C}^1(\mathbb{R}^2)$.

Definition 4.20. *A function $z \in L^1_{loc}([0, T] \times \mathbb{R}) \cap \mathcal{C}([0, T[, L^1_{loc}(\mathbb{R}))$ is called an* entropy solution *to* (4.89) *if*

$$\lim_{t\downarrow 0} z(t) = z_0 \qquad \text{in } L^1_{loc}(\mathbb{R}) \tag{4.90}$$

and one of the following two equivalent conditions holds:

1. *for any entropy-entropy flux pair* (η, q), *that is, for any* $\eta \in \mathcal{C}^2(\mathbb{R})$ *convex and* $q = q(\mu, z)$ *such that* $\partial_z q(\mu, z) = \partial_z g(\mu, z)\eta'(z)$,

$$\partial_t \eta(z) + \partial_\mu [q(\mu, z)] + \eta'(z) g_\mu(\mu, z) - q_\mu(\mu, z) \leq 0, \tag{4.91}$$

in distribution, i.e for any $\varphi \in \mathcal{C}_C^\infty(]0, T[\times \mathbb{R})$, $\varphi \geq 0$,

$$\int_0^T \int_{\mathbb{R}} \{\eta(z)\varphi_t + q(\mu, z)\varphi_\mu + [q_\mu(\mu, z) - \eta'(z) g_\mu(\mu, z)]\varphi\} \, d\mu dt \geq 0; \tag{4.92}$$

2. *for any* $c \in \mathbb{R}$

$$\partial_t |z - c| + \partial_\mu [\mathrm{sgn}(z - c)(g(\mu, z) - g(\mu, c))] + \mathrm{sgn}(z - c) g_\mu(\mu, c) \leq 0, \tag{4.93}$$

in distribution, that is, for any $\varphi \in \mathcal{C}_C^\infty(]0, T[\times \mathbb{R})$, $\varphi \geq 0$,

$$\int_0^T \int_{\mathbb{R}} \{|z - c|\varphi_t + \mathrm{sgn}(z - c)(g(\mu, z) - g(\mu, c))\varphi_\mu - \mathrm{sgn}(z - c) g_\mu(\mu, c)\varphi\} \, d\mu dt \geq 0. \tag{4.94}$$

Lemma 4.21. *The two conditions in the above definition are equivalent and imply that* z *is a weak solution to* (4.89) *in the sense of distributions.*

The entropy condition can be specialized when z is a piecewise smooth function.

Proposition 4.22. *Let* z *be a function piecewice* $\mathcal{C}^1$ *whose discontinuity points belong to the smooth curve* $\mu = \lambda(t)$. *Then* z *is an entropy solution to* (4.89) *if and only if*

1. z *solves* (4.89) *in the classical sense where it is smooth;*

2. *the initial condition* $z(0, \mu) = z_0(\mu)$ *holds in the classical sense;*

3. *denoting the limits* $z_r(t) := \lim_{\mu \downarrow \lambda(t)} z(t, \mu)$, $z_l(t) := \lim_{\mu \uparrow \lambda(t)} z(t, \mu)$, *the* Rankine-Hugoniot *condition holds: for* ℓ*-a.e.* t

$$\dot{\lambda}(t) = \frac{g(\lambda(t), z_r(t)) - g(\lambda(t), z_l(t))}{z_r(t) - z_l(t)}; \tag{RH}$$

4. *the* Lax stability condition *holds:*

$$\frac{g(\lambda(t), c) - g(\lambda(t), z_r(t))}{c - z_r(t)} < \dot{\lambda}(t) < \frac{g(\lambda(t), c) - g(\lambda(t), z_l(t))}{c - z_l(t)} \tag{Lax}$$

for ℓ*-a.e.* t *and any* c *strictly between* z_l *and* z_r.

The Rankine-Hugoniot condition is equivalent to state that z is a weak solution to the scalar conservation law. The Lax condition can be reformulated saying that the graph of $g(\lambda(t), \cdot)$ stays above the chord joining z_r and z_l, if $z_r < z_l$, while the graph stays below the chord when $z_l < z_r$.

The main result about the theory of conservation laws is the following

Theorem 4.23. *If* $z_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *then there exists a unique entropy admissible weak solution* $z \in \mathcal{C}([0, T], L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *to* (4.89).

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