Università di Padova – Dipartimento di Matematica Pura ed Applicata

Scuole di Dottorato in Matematica Pura e Matematica Computazionale

Seminario Dottorato 2009/10

Preface

This document offers a large overview of the nine months' schedule of Seminario Dottorato 2009/10. Our "Seminario Dottorato" (Graduate Seminar) is a double-aimed activity. At one hand, the speakers (usually Ph.D. students or post-docs, but sometimes also senior researchers) are invited to think how to communicate their own researches to a public of mathematically well-educated but not specialist people, by preserving both understandability and the flavour of a research report. At the same time, people in the audience enjoy a rare opportunity to get an accessible but also precise idea of what's going on in some mathematical research area that they might not know very well.
At the end of this year's activity, we are happy to remark once more that this philosophy has been generally understood by the speakers, who also nicely agreed to write down these notes to leave a concrete footstep of their participation: we thank them all warmly.

Padova, 26 June 2010

Corrado Marastoni, Tiziano Vargiolu, Matteo Dalla Riva

Abstracts (from Seminario Dottorato's web page)

Wednesday 14 October 2009

The $\bar{\partial}$-Neumann problem

TRAN Vu Khanh (Univ. Padova, Dip. Mat.)

The $\bar{\partial}$-Neumann problem is probably the most important and natural example of a non-elliptic boundary value problem, arising as it does from the Cauchy-Riemann system. The main tool to prove regularity of solutions in the study of this problem are L^2-estimates: subelliptic estimates, superlogarithmic estimates, compactness estimates. In the first part of the talk we give motivation and classical results on this problem. In the second part, we introduce general estimates for "gain of regularity" of solutions of this problem and relate it to the existence of weights with large Levi-form at the boundary. (Keywords: q-pseudoconvex/concave domains, subelliptic estimates, superlogarithmic estimates, compactness estimates, finite type, infinite type. MSC: 32D10, 32U05, 32V25.)

Wednesday 28 October 2009

Typicality and fluctuations: a different way to look at quantum statistical mechanics

Barbara FRESCH (Univ. Padova, Dep. of Chemical Sciences, Ph.D.)

Complex phenomena such as the characterization of the properties and the dynamics of many body systems can be approached from different perspectives, which lead to physical theories of completely different characters. A striking example of this is the duality, for a given physical system, between its thermodynamical characterization and the pure mechanical description. Finding a connection between these different approaches requires the introduction of suitable statistical tools. While classical statistical mechanics represents a conceptually clear framework, some problems arise if quantum mechanics is assumed as fundamental theory. In this talk, after a general introduction to the subject for non-experts, we shall discuss the emergence of thermodynamic properties from the underlying quantum dynamics.

Wednesday 18 November 2009

Injective modules and Star operations

Gabriele FUSACCHIA (Univ. Padova, Dip. Mat.)

The problem of classifying injective modules in terms of direct decompositions does not admit, in general, a solution. Exhaustive results have been obtained, however, when restricting to special classes of domains, such as Prufer domains, valuation domains and Noetherian domains. After recalling some basic notions on injective modules and direct decompositions, we provide examples of domains in which the classification is not possible, and we give the classical results on valuation and Noetherian domains. Next we introduce the notion of star operation over a domain, a special kind of closure operator defined over the fractional ideals. Thanks to this concept, we show how the classification on Noetherian domains can be generalized, allowing to completely classify special subclasses of injective modules over domains which are not Noetherian.

Wednesday 9 December 2009

Diffusion coefficient and the Speed of Propagation of traveling front solutions to KPP-type problems

Adrian VALDEZ (Ph.D., University of the Philippines)

In this talk, we shall concern ourselves with a general reaction-diffusion equation/system in a periodic setting concentrating on reaction terms of KPP-type. Our interest is focused on special solutions called traveling fronts. In particular, we look at how the minimal speed of propagation of such front solutions can be influenced by the different coefficients of the system. For this, an intensive discussion will be alloted specifically on the influence of the diffusion coefficient.

Wednesday 16 December 2009

On some aspects of McKay correspondence and its applications

Luca SCALA (University of Chicago)

When we quotient $\mathbb{C}^2$ by a finite subgroup G of $SL(2,\mathbb{C})$, and we take a minimal resolution Y of the kleinian singularity $\mathbb{C}^2/G$, then Y is a crepant resolution and the exceptional locus consists of a bunch of curves, whose dual graph is a Dynkin diagram of the kind A_n, D_n, E_6, E_7, E_8. In the eighties, McKay noticed that the Dynkin diagrams arising from resolutions of kleinian singularities are in tight connection with the representations of G. In the first and introductory part of the talk, we will explain the McKay correspondence and its key generalization by means of K-theory, due to Gonzalez-Sprinberg and Verdier. The latter point of view opens the way to the modern derived McKay correspondence, due to Bridgeland-King-Reid. We will then see some applications of the BKR theorem to the geometry of Hilbert schemes of points, due to Haiman, and some other consequences related to the cohomology of tautological bundles.

Wednesday 13 January 2010

Liouville-type results for linear elliptic operators

Luca ROSSI (Univ. Padova - Dip. Mat.)

This talk deals with some extensions of the classical Liouville theorem about bounded harmonic functions to solutions of more general partial differential equations. In the first part, I will introduce the only two technical tools needed to prove the Liouville-type result in the case of periodic elliptic operators: Schauder's a priori estimates and maximum principle. Next, I will discuss the role of the periodicity assumption, seeing what happens if one replaces it with almost periodicity.

Wednesday 10 February 2010

Examples of strategy designs in banking practice

Marco CORSI (Barclays Capital - London)

While academic theory of financial mathematics emphasizes the concept of no-arbitrage models, in common practice the presence of arbitrage opportunities in the market in some cases can explicitly be taken into account. In this talk we will see how different kind of strategies (implied volatility strategy, volatility arbitrage strategy, etc.) can be practically implemented.

Wednesday 24 February 2010

Moebius function and probabilistic zeta function associated to a group

Valentina COLOMBO (Univ. Padova, Dip. Mat.)

Many authors have studied the probabilistic zeta function associated to a finite group; in the last years the study has been extended to profinite groups. To understand how the probabilistic zeta function is defined, it is necessary to introduce another function associated to a group: the Moebius function. We will start considering finite groups: we will explain how these two functions are obtained and we will give some basic examples. Then we will define the profinite groups and proceed to investigate whether and how a probabilistic zeta function can be associated to them. This is not always possible: Mann has conjectured that for a particular class of profinite groups (PFG groups) the definition of this function makes sense. We will present some recent results which suggest that the conjecture is true.

Wednesday 10 March 2010

A pseudometric for unbounded linear operators, extension to operators defined on different open sets and an application to spectral stability estimates for eigenfunctions

Ermal FELEQI (Univ. Padova - Dip. Mat.)

A distance on closed linear subspaces/operators has long been known. It was introduced under the name of "gap" or "opening in a Hilbert space context by Krein and coworkers in the 1940s. The first part of the talk will be of an introductory character and the main properties of the gap between subspaces/operators will be illustrated with the focus laid on spectral stability results. Next it will be shown how the notion of gap between operators can be adapted to study the spectral stability problem of a certain class of (partial) differential operators upon perturbation of the open set where they are defined on. An extension of the gap for operators defined on different open sets will be proposed and it will be estimated in terms of the geometrical vicinity or proximity of the open sets. Then, this will permit to estimate the deviation of the eigenfunctions of certain second order elliptic operators with homogeneous Dirichlet boundary conditions upon perturbation of the open set where the said operators are defined on.

Wednesday 24 March 2010

Finite and countable mixtures

Cecilia PROSDOCIMI (Univ. Padova - Dip. Mat.)

The present talk deals with finite and countable mixtures of independent identically distributed (i.i.d.) sequences and of Markov chains. After an easy introduction on mixture models and their main properties, we focus on binary exchangeable sequences. These are mixtures of i.i.d. sequences by de Finetti theorem. We present a necessary and sufficient condition for an exchangeable binary sequence to be a mixture of a finite number of i.i.d. sequences. If this is the case, we provide an algorithm which completely solves the stochastic realization problem. In the second part of the talk we focus on partially exchangeable sequences, that are known to be mixtures of Markov chains after the work of Diaconis and Freedman, and Fortini et al. later. We present a characterization theorem for partially exchangeable sequences that are mixtures just of a finite or countable number of Markov chains, finding a connection with Hidden Markov Models. Our result extends an old theorem by Dharmadhikari on finite and countable mixtures of i.i.d. sequences.

Tuesday 13 April 2010

Analytic and algebraic varieties: the classical and the non archimedean case

Alice CICCIONI (Univ. Padova - Dip. Mat.)

The complex line, as a set of points, can be endowed with an analytic structure, as well as with an algebraic one. The choice of the topology and the related natural definition of functions on the space determine different geometric behaviors: in the example of the line, there are differential equations admitting solutions in both cases, and some that can be solved only in the analytic setting.

The first part of the talk will focus on the algebraic and analytic structures of a variety over the field of complex numbers, while in the second part we will give an overview of the analogous constructions for varieties defined over a non archimedean field, touching the theory of rigid analytic spaces and its relation to the study of varieties over a discrete valuation ring of mixed characteristic in the framework of syntomic cohomology.

Wednesday 28 April 2010

Interest rate derivatives pricing when the short rate is a continuous time finite state Markov process

VALENTINA PREZIOSO (Univ. Padova - Dip. Mat.)

The purpose of this presentation is to price financial products called "interest rate derivatives", namely financial instruments in which the owner of the contract has the right to pay or receive an amount of money at a fixed interest rate in a specific future date. The pricing of these products is here obtained by assuming that the spot rate (i.e. the interest rate at which a person or an institution can borrow money for an infinitesimally short period of time) is considered as a stochastic process characterized by "absence of memory" (i.e. a time-continuous Markov chain). We develop a pricing model inspired by work of Filipovic'-Zabczyk which assumes the spot rate to be a discrete-time Markov chain: we extend their structure by considering, instead of deterministic time points, the random time points given by the jump times of the spot rate as they occur in the market. We are able to price with the same approach several interest rate derivatives and we present some numerical results for the pricing of these products.

Wednesday 12 May 2010

The ℓ-primary torsion conjecture for abelian varieties and Mordell conjecture

ANNA CADORET (Universite' de Bordeaux 1)

Let k be a field. An abelian variety A over k is a proper group scheme over k. It can be thought of as a functor (with extra properties) from the category of k-schemes to the category of abelian groups. One nice result about such a functor is:

Theorem (Mordell-Weil): Assume that k is a finitely generated field of characteristic 0: then, for any finitely generated extension K of k, $A(K)$ is a finitely generated group. In particular, the torsion subgroup $A(K)_{\mathrm{tors}}$ of $A(K)$ is finite.

For a prime ℓ, the ℓ-primary torsion conjecture for abelian varieties asserts that the order of the ℓ-Sylow of $A(K)_{\mathrm{tors}}$ should be bounded uniformly only in terms of ℓ, K and the dimension g of A. For $g = 1$ (elliptic curves), this conjecture was proved by Y. Manin, in 1969. The main ingredient is a special version of Mordell conjecture for modular curves. The general Mordell conjecture was only proved in 1984, by G. Faltings. For $g = 2$, the ℓ-primary torsion conjecture remains entirely open. After reviewing the proof of Y. Manin, I would like to explain how the general version of Mordell conjecture can be used to prove —following basically Manin's argument— the ℓ-primary torsion conjecture for 1-dimensional families of abelian varieties (of arbitrary dimension). This result was obtained jointly with Akio Tamagawa (R.I.M.S.), in 2008.

Wednesday 26 May 2010

Holomorphic sectors and boundary behavior of holomorphic functions

Raffaele MARIGO (Univ. Padova - Dip. Mat.)

Forced extendibility of holomorphic functions is one of the most important problems in several complex variables: it is a well known fact that a function defined in an open set D of $\mathbb{C}^n$ extends across the boundary at a point where the Levi form of the boundary of D (i.e. the complex hessian of its defining function restricted to the complex tangent space) has at least one negative eigenvalue. A fundamental role in this result is played by analytic discs, i.e. holomorphic images of the standard disc.
After describing the construction of discs attached to a hypersurface by solving a functional equation - Bishop equation - in the spaces of differentiable functions with fractional regularity, we will show how they induce the phenomenon described above, as well as the propagation of holomorphic extendibility along a disc tangent to the boundary of the domain. Finally, we will introduce a new family of discs, nonsmooth along the boundary, that will allow us to establish analogous results under various geometric conditions on the boundary of the domain.

Wednesday 9 June 2010

Edge-connectivity augmentation

Roland GRAPPE (Univ. Padova - Dip. Mat.)

A graph is k-edge-connected if there exist k edge-disjoint paths between every pair of vertices. The problem of global edge-connectivity augmentation of a graph is as follows: given a graph and an integer k, add a minimum number of edges to the graph in order to make it k-edge-connected. We will comprehensively focus on this problem and the simple method of Frank that solves it. Then, we will see a few generalizations such as edge-connectivity augmentation of a graph with partition constraints (Bang-Jensen et al.), edge-connectivity augmentation of a hypergraph (Bang-Jensen and Jackson), and the unification of these two results (joint work with Bernáth and Szigeti). Eventually, these problems can be formulated in an abstract form, leading to further generalizations.

Wednesday 23 June 2010

Topology of Kähler and hyperkähler manifolds

Julien GRIVAUX (Université Paris 6 - Pierre et Marie Curie)

This talk is an introduction to complex geometry.
In the first part, we will introduce some of the basic objects in the field (such as complex manifolds, differential forms and cohomology groups) and see several examples. Then we will be able to state some fundamental results on the cohomology of compact Kähler manifolds, which are a special

class of complex manifolds. We will see how these results generate constraints on the topology of Kähler manifolds.
The last part will be devoted to the theory of hyperkähler manifolds, wich is an active area of current research in complex algebraic geometry.
The talk will be accessible for a general audience; but basic knowledge of differentiable manifolds will of course be helpful.

The $\bar{\partial}$-Neumann problem

Tran Vu Khanh (*)

Abstract. The $\bar{\partial}$-Neumann problem is probably the most important and natural example of a non-elliptic boundary value problem, arising as it does from the Cauchy-Riemann system. The main tool to prove regularity of solution in of study of this problem are L^2-estimates : subelliptic estimates, superlogarithmic estimates, compactness estimates...
In the first part of this note, we give motivation and classical results on this problem. In the second part, we introduce general estimates for "gain of regularity" of solutions of this problem and relate it to the existence of weights with large Levi-form at the boundary.
(MSC: 32D10, 32U05, 32V25. Keywords : q-pseudoconvex/concave domain, subelliptic estimate, superlogarithmic estimate, compactness estimate, finte type, infinite type.)

1 The $\bar{\partial}$-Neumann problem and classical results

1.1 The $\bar{\partial}$- Neumann problem

Let $z_1, ..., z_n$ be holomorphic coordinates in $\mathbb{C}^n$ with $x_j = \mathrm{Re}(z_j), y_j = \mathrm{Im}(z_j)$. Then the holomorphic and anti-holomorphic vector fields are

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right).$$

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary $b\Omega$. Given functions $\alpha_1, ..., \alpha_n$ on Ω, the problem of solving the equations

$$\frac{\partial v}{\partial \bar{z}_j} = \alpha_j, \qquad j = 1, ..., n \tag{1.1}$$

and studying the regularity of the solution is called *the inhomogeneous Cauchy-Riemann equations.*

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We must assume that the $(\alpha_j)_{j=1}^n$ satisfy **the compatibility condition**

$$\frac{\partial \alpha_i}{\partial \bar{z}_j} - \frac{\partial \alpha_j}{\partial \bar{z}_i} = 0 \qquad \text{for all } i, j = 1, ..., n \tag{1.2}$$

Let $\alpha = \sum \alpha_j d\bar{z}_j$ be (0,1)-form, we rewrite (1.1) and by $\bar{\partial} v = \alpha$ and (1.2) by $\bar{\partial}\alpha = 0$. The inhomogeneous Cauchy-Riemann equations are also called *the $\bar{\partial}$-problem.*

Question: Is there a solution $v \in C^\infty(U \cap \bar{\Omega})$ if the datum α_j belongs to $C^\infty(U \cap \bar{\Omega})$ (*local regularity*)?

The regular properties of v in the interior are well known (see next section). Regularity of v on the boundary is more delicate. Notice that not all solutions are smooth : in fact, not all holomorphic functions in Ω are smoothly extended to $\bar{\Omega}$. If h is such a function and v is a smooth solution, then $v + h$ is also *a solution* and not smooth on the closed domain since $\bar{\partial}(v + h) = \bar{\partial} v = \alpha$. So we do not look for *a solution* but for *the solution.* The optimal solution (the one of smallest in L_2-norm) is the solution orthogonal to the holomorphic functions; this is called the *canonical solution.* It is not known whether the canonical is smooth even if there is a smooth solution.

Moreover, the regularity of canonical solution of the $\bar{\partial}$-problem has some applications in SCV such as : Levi problem, Bergman projection, Holomorphic mappings, ...

The aim of the $\bar{\partial}$-Neumann problem is to study the canonical solution of $\bar{\partial}$-problem.

Before stating the $\bar{\partial}$-Neumann problem, we need definition of $\bar{\partial}^*$ the L_2-adjoint of $\bar{\partial}$. The $\bar{\partial}^*$ is defined as follows: Let

$$u = \sum u_j d\bar{z}_j \in \mathrm{Dom}(\bar{\partial}^*) \cap C^\infty(\Omega)^1,$$

and $\bar{\partial}^* u = g$ if

$$(w, g) = (\bar{\partial} w, u) \quad \text{for all} \quad w \in C^\infty(\bar{\Omega}).$$

We see that $\bar{\partial} w = \sum \frac{\partial w}{\partial \bar{z}_j} d\bar{z}_j$ and

$$(\bar{\partial} w, u) = \sum (\frac{\partial w}{\partial \bar{z}_j}, u_j) \quad \overset{\text{Stokes}}{=} \quad -\sum (w, \frac{\partial u_j}{\partial z_j}) + \int_{b\Omega} \sum w \overline{u_j \frac{\partial r}{\partial z_j}} dS.$$

Hence $\sum w \overline{u_j \frac{\partial r}{\partial z_j}} = 0$ on $b\Omega$ for all w. i.e. $u_j \frac{\partial r}{\partial z_j} = 0$ on $b\Omega$. The condition $u \in \mathrm{Dom}(\bar{\partial}^*)$ impies boundary condition on u.

We now state the $\bar{\partial}$-Neumann problem : given $\alpha \in L_2(\Omega)^1$, find $u \in L_2(\Omega)^1$ such that

$$\begin{cases} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) u = \alpha \\ u \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \\ \bar{\partial} u \in \mathrm{Dom}(\bar{\partial}^*), \bar{\partial}^* u \in \mathrm{Dom}(\bar{\partial}). \end{cases} \tag{1.3}$$

The $\bar\partial$-Neumann problem is a boundary value problem; the Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ equation is elliptic, but the boundary conditions (i.e. $u \in \text{Dom}(\bar\partial^*); \bar\partial u \in \text{Dom}(\bar\partial^*)$) make the problem not elliptic. If (1.3) has a solution for every α, then one defines the $\bar\partial$-Neumann operator $N := \Box^{-1}$, this commutes both with $\bar\partial$ and $\bar\partial^*$.

If $\bar\partial\alpha = 0$, we define $v = \bar\partial^* N\alpha$ then v is the canonical solution of the $\bar\partial$-problem. In fact,

$$\bar\partial v = \bar\partial\bar\partial^* N\alpha = \bar\partial\bar\partial^* N\alpha + \bar\partial^*\bar\partial N\alpha = \Box N\alpha = \alpha$$

and $(v, h) = (\bar\partial^* N\alpha, h) = (N\alpha, \bar\partial h) = 0$ for any h holomorphic. (i.e. $v \perp \text{Ker}\bar\partial$).

Question: What geometric conditions on $b\Omega$ guarantee the existence and regularity of solution of $\bar\partial$-Neumann problem?

1.2 Existence and regularity

Let us introduce the weighted L_2-norms. If $\phi \in C^\infty(\Omega)$ and $u \in L_2(\Omega)^1$ define

$$\|u\|^2_\phi = (u, u)_\phi = \|ue^{-\frac{\phi}{2}}\|^2 = \int_\Omega |u|^2 e^{-\phi} dV.$$

Denote $\bar\partial^*_\phi$ the adjoint of $\bar\partial$ in this weighted inner product. Using integration by part, we obtain the basic identity:

Theorem 1.1 [Morry-Kohn-Hormander]

$$\begin{aligned}\|\bar\partial u\|^2_\phi + \|\bar\partial^*_\phi u\|^2_\phi = &\sum_{ij} \|\frac{\partial u_i}{\partial \bar z_j}\|^2_\phi + \int_\Omega \sum \frac{\partial^2\phi}{\partial z_i \partial \bar z_j} u_i \bar u_j e^{-\phi} dV \\ &+ \int_{b\Omega} \sum \frac{\partial^2 r}{\partial z_i \partial \bar z_j} u_i \bar u_j e^{-\phi} dS,\end{aligned} \tag{1.4}$$

for any $u = \sum u_j d\bar z_j \in \text{Dom}(\bar\partial^*) \cap C^\infty(\bar\Omega)^1$.

Then if Ω is pseudoconvex, i.e., $\sum \frac{\partial^2 r}{\partial z_i \partial \bar z_j} u_i \bar u_j \geq 0$ on $b\Omega$, and $\phi = C|z|^2$ for some suitable constant C. we obtain

$$(\Box u, u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \gtrsim \|u\|^2, \tag{1.5}$$

and hence

$$\|\Box u\| \gtrsim \|u\|.$$

Thus, the $\bar\partial$-Neumann problem is solvable in L_2-norm on any smooth, bounded, pseudoconvex domain.

Denote $Q(u, u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$.

Regularity in the interior: For $u \in C_0^\infty(\Omega)$, then $\|\frac{\partial u}{\partial \bar{z}_j}\|^2 = \|\frac{\partial u}{\partial z_j}\|^2$. For $\phi = 0$, (1.4) implies that

$$Q(u,u) \gtrsim \|\frac{\partial u}{\partial \bar{z}_j}\|^2 = \frac{1}{2}\Big(\|\frac{\partial u}{\partial \bar{z}_j}\|^2 + \|\frac{\partial u}{\partial z_j}\|^2\Big)$$

Combining with (1.5), we get $Q(u,u) \geq \|u\|_1^2$ (*elliptic estimate*).

Then we get regularity property in the interior. So our interest is confined to boundary $b\Omega$.

Regularity at the boundary? The main methods used in investigating of the regularity at the boundary of the solution of $\bar{\partial}$-Neumann problem consist in non-elliptic estimates : subelliptic, superlogarithmic and compactness estimates.

Let $z_0 \in b\Omega$. Suppose U is a neighborhood of z_0. Consider the local boundary coordinates (defined on U) denote by $(t,r) = (t_1, ..., t_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ where r is defining function of Ω.

For $\varphi \in C^\infty(\bar{\Omega} \cap U)$, the tangential Fourier transform of φ, defined by

$$\mathcal{F}_t\varphi(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-i\langle t,x\rangle}\varphi(t,r)dt.$$

The standard tangential pseudo-differential operator is expressed by

$$\Lambda\varphi(t,r) = \mathcal{F}_t^{-1}\Big((1+|\xi|^2)^{1/2}\mathcal{F}_t\varphi(\xi,r)\Big).$$

Classes of non-elliptic estimates for the $\bar{\partial}$-Neum. prob. in a neighborhood U of $z_0 \in b\Omega$ are defined by

Definition 1.2

(i) Subelliptic estimate: there is a positive constant ϵ such that

$$||\Lambda^\epsilon u||^2 \lesssim Q(u,u);$$

(ii) Superlogarithmic estimate: for any $\eta > 0$ there is a positive constant C_η such that

$$||\log \Lambda u||^2 \leq \eta Q(u,u) + C_\eta\|u\|_0^2;$$

(iii) Compactness estimate: for any $\eta > 0$ there is a positive constant C_η such that

$$||u||^2 \lesssim \eta Q(u,u) + C_\eta\|u\|_{-1}^2;$$

for any $u \in C_c^\infty(\bar{\Omega} \cap U) \cap \mathrm{Dom}(\bar{\partial}^*)$.

Remark that subelliptic estimate $\Rightarrow$ Superlogarithmic estimate $\Rightarrow$ Compactness estimate.

Let us recall the following results.

Theorem 1.3

(i) [Folland-Kohn 72] *Subelliptic estimate implies local regularity. Moreover,* $\Box u \in H^s(V) \Rightarrow u \in H^{s+2\epsilon}(V')$ *for* $V' \subset V$*, where* ϵ *is order of subellipticity.*

(ii) [Kohn 02] *Superlogarithmic estimate implies local regularity. Moreover,* $\Box u \in H^s(V) \Rightarrow u \in H^s(V')$ *for* $V' \subset V$*.*

(iii) [Kohn-Nirenberg 65] *Compactness estimate over a covering* $\cup U_j \supset b\Omega$*, implies global regularity. Moreover* $\Box u \in H^s(\bar{\Omega}) \Rightarrow u \in H^s(\bar{\Omega})$*.*

Remark: Compactness estimate $\not\Rightarrow$ local regularity (see [Ch02]).

1.3 Geometric condition

When Ω is pseudoconvex, a great deal of work has been done about subelliptic estimates. The most general results have been obtained by Kohn and Catlin.

Theorem 1.4 [Kohn, Ann. of Math, 63-64] *Strongly pseudoconvex* $\Leftrightarrow$ $\frac{1}{2}$*-subelliptic estimate.*

Strongly pseudoconvex : $\partial\bar{\partial}r > 0$, on $b\Omega$ when $u \in \mathrm{Dom}(\bar{\partial}^*)$.

Definition 1.5 Finite type (D'Angelo finite type):

$$D(z_0) = \sup \frac{\mathrm{ord}_{z_0}(r(\phi))}{\mathrm{ord}_0 \phi}$$

where supremum is taken over all local holomorphic curves $\phi : \Delta \to \mathbb{C}^n$ with $\phi(0) = z_0$.

Examples of finite type :

(a) Strongly pseudoconvex $\Leftrightarrow D(z_0) = 2$.

(b) $r = \mathrm{Re}z_n + \sum |z_j|^{2m_j}$ in $\mathbb{C}^n$, then $D(z_0) = 2\max\{m_j\}$.

(c) $r = \mathrm{Re}z_3 + |z_2^b - z_1|^2$, then $D_(z_0) = \infty$, since $\mathcal{C} : \{z_2^b - z_1 = 0, z_3 = 0\} \subset b\Omega$.

(d) $r = \mathrm{Re}z_3 + |z_1|^{2a} + |z_2^b - z_1|^2$, then $D(z_0) = 2ab$.

(e) $r = \mathrm{Re}z_n + \exp(-\frac{1}{|z_j|^s})$ in $\mathbb{C}^n$, then $D(z_0) = \infty$.

Theorem 1.6 [Kohn 79, Acta Math.] *Let* Ω *be pseudoconvex+real analytic+finite type at* $z_0 \in \Omega$*. Then subelliptic estimate hold at* z_0*.*

Theorem 1.7 [Caltin 84-87, Ann. of Maths] *Let* Ω *be pseudoconve+finite type at* $z_0 \in \Omega$*. Then subelliptic estimate hold at* z_0*. Moreover*

$$\epsilon \leq \frac{1}{D(z_0)}.$$

Denote $S_\delta = \{z \in \mathbb{C}^n : -\delta < r < 0\}$.

One of main steps in Catlin's proof is the following reduction:

Theorem 1.8 *Suppose that $\Omega \subset\subset \mathbb{C}^n$ is a pseudoconvex domain defined by $\Omega = \{r < 0\}$, and that $z_o \in b\Omega$. Let U is a neighborhood of z_o. Suppose that there is a family smooth real-valued function $\{\Phi^\delta\}_{\delta>0}$ satisfying the properties:*

$$\begin{cases} |\Phi^\delta| \leq 1 \text{ on } U, \\ \partial\bar\partial\Phi^\delta \gtrsim 0 \text{ on } U, \\ \partial\bar\partial\Phi^\delta \gtrsim \delta^{-2\epsilon} \text{ on } U \cap S_\delta \end{cases}$$

Then there is a subelliptic estimate of order ϵ at z_o.

Remark 1.9 Superlogarithmic estimate for the $\bar\partial$-Neumann problem was first introduced by [Koh02]. Supperlogarithmic estimate might hold on some class of infinite type domains.

Remark 1.10 A great deal of work has been done on pseudoconvex domains, however, not much is known in the non-pseudoconvex case except from the results related to the celebrated $Z(k)$ condition [Hor65], [FK72] and the case of top degree $n-1$ of the forms due to Ho [Ho85].

2 f-estimates on q-pseudoconvex/concave domain

2.1 The q-pseudoconvex/concave domain

Let $\lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalue of $\partial\bar\partial r$ on $b\Omega$, for a pair of indices q_0, q $(q_0 \neq q)$ suppose that

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} r_{jj} \geq 0 \qquad \text{on } b\Omega. \tag{2.1}$$

Definition 2.1

(i) If $q > q_0$, we say that Ω is q-pseudoconvex at z_0.

(ii) If $q < q_0$, we say that Ω is q-pseudoconcave at z_0.

Special case:

(a) $q_0 = 0, q = 1 : \lambda_1 \geq 0$, this means $\partial\bar\partial r \geq 0$, that is, 1-pseudoconvex $\equiv$ pseudoconvex.

(b) $q_0 = n-1, q = n-2$, then (n-2)-pseudoconcave $\equiv$ pseudoconcave.

2.2 Main Theorem

Theorem 2.2 *Let Ω be q-pseudoconvex* (resp. *q-pseudoconcave*) *domain at z_0. Let U is a neighborhood of z_0. Assume that there exists a family function $\{\Phi^\delta\}_{\delta>0}$, satisfying properties*

$$\begin{cases} |\Phi^\delta| \le 1 \\ \sum_{j=0}^{q} \lambda_j^{\Phi^\delta} - \sum_{j=1}^{q_0} \Phi^\delta_{jj} \gtrsim f(\frac{1}{\delta})^2 \end{cases} \qquad \text{on } S_\delta \cap U$$

Then the estimate

$$||f(\Lambda)u||^2 \lesssim Q(u,u) \tag{2.2}$$

holds for any form with degree $k \ge q$ (resp. $k \le q$) where $f(\Lambda)$ is the operator with symbol $f((1+|\xi|^2)^{\frac{1}{2}})$. Moreover,

(i) *if* $\lim_{\delta\to 0} \frac{f(\frac{1}{\delta})}{(\frac{1}{\delta})^\epsilon} \ge C > 0$ *then (2.2) implies ϵ-subelliptic estimate;*

(ii) *if* $\lim_{\delta\to 0} \frac{f(\frac{1}{\delta})}{\log\frac{1}{\delta}} = +\infty$ *then (2.2) implies superlogarithmic estimate;*

(iii) *if* $\lim_{\delta\to 0} f(\frac{1}{\delta}) = +\infty$ *then (2.2) implies compactness estimates.*

Remark 2.3 Superlogarithmic estimates were never handled in this way, but by Kohn's subelliptic multipliers

Remark 2.4 Catlin only used his weight on *finite type* domains.

Remark 2.5 We get a generalization : Pseudoconvex $\rightsquigarrow$ q-Pseudoconvex/concave

2.3 Construction of the Catlin's weight

Strongly pseudoconvex domain: Let Ω be strongly pseudoconvex at z_0 then $\partial\bar\partial r > 0$ in a n.b.h. of z_0. Define

$$\Phi^\delta = -\log(-\frac{r}{\delta}+1)$$

Then

$$\partial\bar\partial\Phi^\delta \gtrsim \frac{1}{\delta}\partial\bar\partial r \gtrsim \delta^{-2\frac{1}{2}} \qquad z \in S_\delta.$$

We get $\frac{1}{2}$-subelliptic estimates on this class of domain.

Domain satisties $Z(k)$ condition:

Definition 2.6 [Z(k) condition] Ω satisfies $Z(k)$ condition if $\partial\bar{\partial}r$ has either at least $(n-k)$ positive eigenvalues or at least $(k+1)$ negative eigenvalues.

Theorem 2.7 *Let Ω be a domain of $\mathbb{C}^n$ which satisfies $Z(k)$ condition, then*

$$|||u|||_{1/2}^2 \lesssim Q(u,u)$$

holds for any u of degree k.

This is classical result of non-pseudoconvex domain. This theorem can be found in [FK72]. We give a new way to get $\frac{1}{2}$-subelliptic estimates by construction the family of Catlin's weight functions.

If Ω satisfies $Z(k)$ condition, then Ω is strongly k-pseudoconvex or strongly k-pseudoconcave. We define

$$\Phi^\delta = -\log(\frac{-r}{\delta}+1).$$

Similarly in the case strongly pseudoconvex, we get $\frac{1}{2}$-subelliptic estimates for any form of degree k.

Decoupled domain:

Theorem 2.8 *Let $\Omega \in \mathbb{C}^2$ be defined by*

$$r = Rew + P(z) < 0$$

where P is a subharmornic non-harmonic function. Further, suppose that there is an invertible function F with $\frac{F(|z|)}{|z|^2}$ increasing such that

$$\partial\bar{\partial}P(z) \gtrsim \frac{F(|z|)}{|z|^2}.$$

Then, f-estimate holds with $f(\delta^{-1}) = (F^{-1}(\delta))^{-1}$.

Example 2.1 If $P(z) = |z|^{2m}$,then $F(\delta) = \delta^{2m} \Rightarrow f(\delta^{-1}) = \delta^{-\frac{1}{2m}} \Rightarrow \frac{1}{2m}$-subelliptic estimate.

Example 2.2 If $P(z) = \exp(-\frac{1}{|z|^s})$, then $F(\delta) = \exp(-\frac{1}{\delta^s}) \Rightarrow f(\delta^{-1}) = \left(\log\frac{1}{\delta}\right)^{1/s} \Rightarrow f$-estimate holds for this f. So, if $0 < s < 1$ then $\lim_{\xi\to\infty}\frac{f(|\xi|)}{\log|\xi|} = \infty$, we get superlogarithmic estimate. Furthermore, we obtain compactness estimate for any $s > 0$.

Sketch of the proof of Theorem 2.8. Define

$$\Phi^\delta = -\frac{r}{\delta} + \log(|z|^2 + f(\delta^{-1})^{-2}).$$

Then on S_δ,

$$\partial\bar{\partial}\Phi^\delta \gtrsim \delta^{-1}\frac{F(|z|)}{|z|^2} + \frac{f(\delta^{-1})^{-2}}{(|z|^2 + f(\delta^{-1})^{-2})^2}.$$

If $|z| \geq f(\delta^{-1})^{-1}$,(e.i. $= f(\delta^{-1})^{-1} = F^{-1}(\delta)$) then

$$(I) \geq \delta^{-1}\frac{F(F^{-1}(\delta))}{f(\delta^{-1})^{-2}} = f(\delta^{-1})^2.$$

Otherwise, if $|z| \leq f(\delta^{-1})^{-1}$, then

$$(II) \gtrsim f(\delta^{-1})^2$$

So that $\partial\bar{\partial}\Phi^\delta \gtrsim f(\delta^{-1})^2$ on S_δ.

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Typicality and fluctuations: a different way to look at quantum statistical mechanics

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Abstract. Complex phenomena such as the characterization of the properties and the dynamics of many body systems can be approached from different perspectives, which lead to physical theories of completely different characters. A striking example of this is the duality, for a given physical system, between its thermodynamical characterization and the pure mechanical description. Finding a connection between these different approaches requires the introduction of suitable statistical tools. While classical statistical mechanics represents a conceptually clear framework, some problems arise if quantum mechanics is assumed as fundamental theory. In this note we shall discuss the emergence of thermodynamic properties from the underlying quantum dynamics.

The description of complex phenomena such as the characterization of the properties and the dynamics of many body systems can be approached from different perspective which leads to physical theories of completely different characters. A striking example of this is the duality, for a given physical system, between its thermodynamical characterization and the pure mechanical description. Thermodynamics has been initially formulated as a pure phenomenological science describing the behaviour of macroscopic systems. Indeed, it has been developed at a time when the atomistic nature of the matter was not well understood; nonetheless it is a fully self consistent physical theory whose validity is beyond any doubt today. At the end of the nineteenth century the increasing popularity of the atomic theory of matter stimulated the research of a microscopic foundation of thermodynamics, i.e. a connection between a pure mechanical description of a system and its thermodynamic properties. The natural tools to look for such a connection are of statistical nature. The birth of statistical mechanics due to the innovative work of Maxwell, Boltzmann and Gibbs [1] among others, introduced concepts from the theory of probability into the description of physical systems. The peculiarity of statistical mechanics is that it deals with probability distributions: on the one hand this is the reason of its success in connecting the microscopic mechanical description with other theories which account for macroscopic phenomena, on the other hand this is also the root of the difficulties one encounters when trying to rigorously justify its principles. In the framework of classical mechanics the statistical description of the equilibrium finds its conceptual justification

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in the ergodic theory [2]: the representative point of the system in the phase space moves according to non linear equations of motion generating a trajectory which is supposed to fill up the hyper surface of constant energy uniformly (microcanonical distribution), spending an equal amount of time in equal volumes of the phase space. Even if there is no rigorous proof of ergodicity for generic systems, within classical mechanics we could use our powerful computers to solve the Hamilton equations of motion for a given many body system, say a gas or a liquid in a small box. In practice this is done with the standard Molecular Dynamics simulations [3]. Beyond any technical issue, we need essentially a good model for the interparticle interactions and the specification of the initial conditions, i.e. the point of the phase space describing the initial state of the system. Then, by analyzing the trajectory of our system we could in principle reconstruct the phase space probability distribution typical of the classical statistical mechanics. Thus, we will very likely find a uniform distribution on the accessible region of the constant total energy surface of the phase space from the evolution of the whole system, and the Boltzmann canonical distribution for the states of a given molecule as long as the rest of the system acts as a thermal bath. Finally, once recognized these fundamental distributions, we could use them to determine equilibrium properties and macroscopic observables pertinent to our system, so establishing a clear connection between the microscopic description and the macroscopic description of our system.

However quantum mechanics is the theory of the microscopic world which is believed to be more fundamental than classical mechanics. Thus, the following question arises: how statistical mechanics, and thus thermodynamics, emerges from the underling quantum mechanical description? There is no obvious answer to this question. Even if we could perform a simulation of a large many body quantum system in analogy to the ideal molecular dynamics experiment invoked above, there is not a straightforward relation between the results of such a calculation, i.e. the time dependent wavefunction $\psi(t)$ of the isolated system, and the standard quantum statistical description based on the statistical density matrix. Of course, one could identify the latter quantity with the time average of the instantaneous density matrix determined by the wavefunction, but in such a case it is not clear why an evolving isolated system should necessarily leads to the microcanonical statistical density matrix. Indeed, the simplistic idea of a classic ergodic system whose trajectory fills up uniformly the constant energy surface is never applicable to the quantum evolution due to the linearity of the Schroedinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{1}$$

which implies the existence of many additional constants of the motion. More specifically one can always write the wavefunction at the time t in the energy representation in terms of N time independent populations and N time dependent phases (with N denoting the dimension of the corresponding Hilbert space)

$$\psi(t) = \sum_{k=1}^{N} P_k e^{\alpha_k(t)} \tag{2}$$

This implies that, given an initial pure state at $t = 0$, the motion of its representative point in the phase space is confined to the torus containing that initial state, as it is pictorially represented in Figure 1.

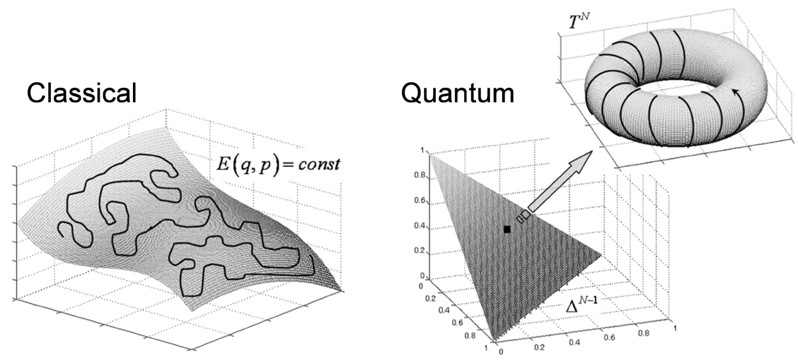

Figure 1: Pictorial representation of the motion of a point in the phase space: (left panel) according to the ergodic foundations of classical statistical mechanics the representative point is supposed to fill up uniformily the constant energy surface. (Right panel) Due to the linearity of the Schroedinger equation the motion of a representative point is confined on a N dimensional torus.

As a direct consequence one finds that the equilibrium value of any property, defined as asymptotic time average, always depends on the detail of the initial state. Nonetheless, if the concept of thermal equilibrium is meaningful for a quantum system, then we would expect that the equilibrium average of at least some functions of interest depends on the total energy of the system, but it is independent on all other aspects of the initial state.

Recently, interesting results toward the resolution of this puzzle has been proposed by several authors [4] by considering quantum statistical mechanics from a different standpoint. One of the key ingredients of this new perspective consists of shifting the focus from the averages of the traditional quantum statistics back to the role and predictability of one single realization of a system and its environment described by a quantum mechanical pure state associated to a wavefunction. The central argument to connect the quantum mechanical description to the statistical and thermodynamical characterization relies on the idea that typical behaviors can emerge from different quantum pure states.

In this presentation we shall illustrate how we have applied this idea to explain the emergence of thermodynamic properties from the pure quantum mechanical description of a composite system. By considering different ensembles of pure states defined on the basis of arbitrary contstraints which the member of the particular ensemble has to satisfy we show that typicality may be a crucial step in establishing a firm connection between macroscopic thermodynamics and microscopic quantum dynamics.

To this aim we consider the parameterization of the wave function in terms of populations and phases as in equation (2) and derive the corresponding ensemble distributions from the inherent geometry of the Hilbert space. Such distributions can be investigated by employing Monte Carlo sampling techniques. From the numerically generated statistical sample one can in principle study the ensemble distribution of any function of the quantum state. To illustrate this we shall consider the distribution of the Shannon entropy

$$S = \sum_{k=1}^{N} P_k ln P_k \tag{3}$$

which characterizes the pure states in the energy representation. In order to investigate how typical values of any function of interest behaves in the asymptoyic limit of very large system $n \to \infty$, we shell present an approximation of the ensemble distributions obtained by means of the minimization of the informational functional. Its validity can be assessed through the comparison with the numerical evidences [5]. Finally the application of such methodologies to the problem of deriving a meaningful thermodynamical characterization of a simple spin system is discussed [6].

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Injective modules and star operations

Gabriele Fusacchia (*)

Abstract. The problem of classifying injective modules does not admit a general solution. Exhaustive results have been obtained, however, when restricting to modules over special classes of domains, such as Prufer domains, valuation domains and Noetherian domains.
After recalling some basic notions on injective modules and direct decompositions, we provide examples of domains in which the classification is not possible, and we give the classical results on valuation and Noetherian domains.
Next we introduce the notion of star operation over a domain, a special kind of closure operator defined over the fractional ideals. Thanks to this concept, we show how the classification on Noetherian domains can be generalized, allowing to completely classify some special subclasses of injective modules over domains which are not Noetherian.

1 Classifying injective modules

From now on, we will assume R to be an integral domain, with K its quotient field. Given an R-module H, we recall that H is said to be **injective** if for every choice of modules A, B and morphisms f, g the following diagram can always be completed by a third morphism h which makes it commute:

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\ H & \end{tikzcd}$$

In the category of R-modules, the injective modules occupy a key position thanks to their special properties, which have been studied thoroughly: among them, we focus on the fact that every R-module M can always be embedded in an injective module. Moreover, this embedding can be chosen in a minimal way, as the following definition and theorem show:

Definition 1.1 Let M, H be R-modules, with $M \leq H$:

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- H is an **essential extension** of M if whenever $C \leq H$ is a submodule of H, then $C \cap M \neq 0$.
- If H is also injective, H is called an **injective hull** of M.

Theorem 1.2 *The injective hull of a module M always exists and is unique up to isomorphism over M. It is denoted by $E(M)$.*

Moreover,

- *$E(M)$ is the smallest injective module containing M.*
- *$E(M)$ is the largest essential extension of M.*

Our aim is to study the direct decompositions of injective modules, that is, their writings as direct sums of submodules; as such, we are interested in understanding the structure of those modules which only admit trivial decompositions. We start with the following definition:

Definition 1.3 An R-module A is said to be **indecomposable** if whenever A can be written as $A = B \oplus C$, then either $B = 0$ or $C = 0$.

When dealing with modules which are both indecomposable and injective, we are able to give a first description of their structure:

Theorem 1.4 *H is injective and indecomposable if and only if*

$$H \cong E(R/I),$$

with I irreducible ideal of R.

We are now ready to state the main problem we want to address.

Problem (Classification of injective modules). *Given an injective module H, is it possible to write it as:*

(a) a direct sum of injective indecomposable modules?

$$H \cong \bigoplus_{\lambda \in \Lambda} E(R/I_\lambda)$$

(b) The injective hull of such a direct sum?

$$H \cong E\left[\bigoplus_{\lambda \in \Lambda} E(R/I_\lambda)\right]$$

Moreover, given either of the two above writings, are the indecomposable summands uniquely determined? (Up to isomorphism)

Remark 1.5 The second, weaker option, (b), is motivated by the fact that an arbitrary direct sum of injective modules is not necessarily injective, so that we might need to "enlarge" it enough (that is, consider its injective hull) to make it so.

As a first step, we can give a positive answer to the last question asked by the problem, provided by Azumaya's Theorem:

Theorem 1.6 [Azumaya, 1950] *If one of the two writings listed in the problem exists, then it is unique up to isomorphism.*

In light of the above result, we can focus on the first part of the problem, which is strongly related to the following questions:

- What is the structure of an indecomposable injective module?
- When is an arbitrary sum of injective modules still injective?

We will now give the answers to these questions in two special cases: the classification on valuation domains and on Noetherian domains. The latter, moreover, will provide a framework which we will then generalize to the non-Noetherian case.

Recall that a valuation domain is characterized by having the set of its ideals totally ordered. In particular, every ideal is irreducible.

Theorem 1.7 [The classification on valuation domains] *If R is a valuation domain then the injective indecomposable modules are exactly those isomorphic to $E(R/I)$, for each ideal I of R.*

Moreover, every injective module H has a unique writing as the injective hull of a direct sum of indecomposable modules:

$$H \cong E\left[\bigoplus_{\lambda\in\Lambda} E(R/I_\lambda)\right]$$

The above tells us that the problem of classifying injective modules on a valuation domain is completely solved; observe that in general the description of an injective module will be of the type (b), as previously denoted in the Classification Problem. In the case of Noetherian domains, characterized by satisfying the Ascending Chain Condition (ACC) on the set of all ideals, we can actually obtain the description of type (a):

Theorem 1.8 [The classification on Noetherian domains]

If R is a Noetherian domain then the injective indecomposable modules are exactly those isomorphic to $E(R/P)$, for each prime ideal P of R.

Moreover, every injective module H has a unique writing as a direct sum of indecomposable modules:

$$H \cong \bigoplus_{\lambda\in\Lambda} E(R/P_\lambda)$$

Now, before introducing the tools needed to generalize the Noetherian case, we want to give more details about the general situation (that is, without assuming special conditions on our domain), in order to better understand why the Classification Problem is not generally solvable. The following Theorem provides the answer:

Theorem 1.9 [Dauns, 1987] *An injective module H can always be written as a direct sum of two injective submodules,*

$$H = C \oplus D,$$

such that:

- *C is the injective hull of a direct sum of indecomposable injective modules.*
- *D has no indecomposable direct summands.*

C and D are uniquely determined by H, up to isomorphism.

We call a module such as D a **super-decomposable** module. It is clear that such a module cannot be described as a direct sum of indecomposable modules. Thus, for Noetherian and valuation domains, D is always 0.

This ceases to be true, however, as soon as we start dropping these powerful hypotheses. As a matter of fact, even a "good" class of domains such that the Bézout domains (those in which every finitely generated ideal is principal) possesses super-decomposable injective modules. The following is the sketch of a construction which allows to obtain such a module:

(a) We start from an atomless boolean lattice $\mathcal{B}$ and obtain from $\mathcal{B}$ a lattice-ordered group Γ.

  (i) We consider $\mathcal{B} = \{I \subset \mathbb{R} \,|\, I = I_1 \cup \ldots \cup I_n\}$, finite unions of:

  $$\emptyset \quad , \quad \mathbb{R} \quad , \quad [a, b) \quad , \quad (-\infty, c) \quad , \quad [d, +\infty)$$

  With the usual set operations of union and intersection, $\mathcal{B}$ is an **atomless boolean lattice**.

  (ii) In the lattice-ordered abelian group $\mathbb{Z}^{\mathbb{R}}$ (pointwise ordering), we consider the **lattice-ordered subgroup** generated by the characteristic functions of the elements of $\mathcal{B}$:

  $$\Gamma = < \{\mathbb{1}_I\}_{I \in \mathcal{B}} >, \quad \mathbb{1}_I \text{ characteristic function of } I$$

  where $\mathbb{1}_I$ is the characteristic function of $I \in \mathcal{B}$.

(b) Find a Bézout domain R with divisibility group Γ.

  (i) We use the following:

  Theorem 1.10 [Kaplansky-Jaffard-Ohm] *Given a **lattice-ordered** abelian group Γ there exists a **Bézout** domain R with quotient field K whose **group of divisibility** $\Gamma(R) = K^*/U(R)$ is order-isomorphic to Γ.*

(ii) We apply (KJO) to $\Gamma =< \{\mathbb{1}_I\}_{I\in\mathcal{B}} >$ and find R Bézout domain such that $\Gamma(R) = \Gamma$.

(c) Find an R-module M such that M and all its cyclic submodules are super-decomposable. Then $E(M)$ is super-decomposable.

(i) Consider $J \neq \emptyset, \mathbb{R}$ in $\mathcal{B}$. Then $\mathbb{1}_{\mathbb{R}} = \mathbb{1}_J + \mathbb{1}_{\mathbb{R}\setminus J}$

(ii) Choose $a, b, c \in R$ whose classes in Γ are $\mathbb{1}_{\mathbb{R}}, \mathbb{1}_J$, and $\mathbb{1}_{\mathbb{R}\setminus J}$. Then $M = R/aR$ decomposes as $R/bR \oplus R/cR$.

(iii) Applying again the above argument, it can be seen that not only M is super-decomposable, its cyclic submodules are as well.

(iv) Thanks to this stronger property, the injective hull $E(M)$ is super-decomposable.

2 Star operations

Recall that a *fractional* ideal I of R is a submodule of the quotient field K such that $dI \subseteq R$ for some $0 \neq d \in R$. For example, the module $\frac{1}{2}\mathbb{Z})$ is a fractional ideal of $\mathbb{Z}$. We denote by $F(R)$ set of nonzero fractional ideals of R.

Definition 2.1 A **star operation** on R is a map in the set $F(R)$ of fractional ideals

$$\star : F(R) \to F(R) \quad , \quad I \mapsto I^\star$$

such that:

($\star_1$) $R^\star = R$ and $(xI)^\star = xI^\star$, for all $0 \neq x \in K$

($\star_2$) $I \subseteq J \Rightarrow I^\star \subseteq J^\star$

($\star_3$) $I \subseteq I^\star$, and $I^{\star\star} = I^\star$

Thus, a star operation $\star$ can be seen as a closure operator on the ideals of a domain; in particular, if I is an ideal such that $I = I^\star$ (that is, I is "closed" with respect to $\star$), we say I is a **$\star$-ideal**.

Finally, as we will see later we are mostly interested in those star operations which distribute over finite intersections of ideals, that is, such that $(I \cap J)^\star = I^\star \cap J^\star$. If a star operation satisfies this property, we say it is **stable**.

Note that it is always possible to define a star operation on a domain, since the identity map (usually denoted by d) is, trivially, a star operation; we also list some of the main nontrivial examples, which can always be defined:

$$\begin{array}{rrlcll}
 & \mathbf{v}: & I \mapsto I^v & = & (I^{-1})^{-1} = (R:(R:I)) & \\
 & \mathbf{t}: & I \mapsto I^t & = & \bigcup J^v & J \subseteq I \quad f.g. \\
\text{(always stable)} & \mathbf{w}: & I \mapsto I^w & = & \bigcup (I:J), \quad J^v = R \quad \text{f.g.} & \\
\text{(always stable)} & \mathbf{d}: & I \mapsto I^d & = & I &
\end{array}$$

where $(I : J)$ is the ideal $\{x \in K \mid xJ \subseteq I\}$.

While the above examples are defined independently from the particular domain, other star operations can be obtained in special cases: one of the most studied constructions is made possible when a family $\{R_\alpha\}$ exists, composed by overrings of R such that $R = \bigcap R_\alpha$. We can then define the star operation:

$$\star_A : \quad I \mapsto \quad I^{\star_A} \quad = \quad \bigcap IR_\alpha$$

A special case, which turns out to be always stable, is obtained if we have$\{R_\alpha\} = \{R_P\}_{P\in\Delta}, \Delta \subseteq \mathrm{Spec}(R)$; in this case we write:

$$\star_\Delta : \quad I \mapsto I^{\star_\Delta} \quad = \quad \bigcap\nolimits_\Delta IR_P$$

We are now ready to introduce the key property needed to generalize the Noetherian case.

Definition 2.2 Given a domain R with a star operation $\star$, we say that R is **$\star$-Noetherian**, if the *Ascending Chain Condition* holds on the set of all the $\star$-ideals.

The most studied example of $\star$-Noetherian domain is provided by the class of v-Noetherian domains, which are called **Mori** domains.

We will see, however, that this condition alone is not enough to give a satisfactory generalization. The following actually provides the proper framework:

Definition 2.3 We say that R is **Strong $\star$-Noetherian**, if R is $\star$-Noetherian and $\star$ is *stable.*

Here the main example is that of w-Noetherian domains, or **Strong Mori** domains. Moreover, as a trivial but significant example, choosing $\star = d$ (the identity) gives back the classical definition of Noetherian domain.

We conclude this section by focusing on some special subsets of the prime spectrum of a domain over which a star operation $\star$ is defined:

- If P is a prime ideal which is also a $\star$-ideal, it is called a **$\star$-prime** ideal.
- A $\star$-ideal which is maximal in the set of $\star$-ideals is necessarily a $\star$-prime ideal, and is said to be a **$\star$-maximal** ideal.

While the above definitions clearly generalize the classical notions of prime and maximal ideals (which are simply obtained by choosing $\star = d$), the main difference is that we are not in general granted the existence of $\star$-prime or $\star$-maximal ideals. This is not a problem, however, once we assume $\star$-Noetherianity:

Theorem 2.4 *If R is a $\star$-Noetherian domain, then $\star$-prime and $\star$-maximal ideals exist and every $\star$-ideal is contained in a $\star$-maximal ideal.*

3 Classes of modules related to star operations

As we previously observed, dropping the hypothesis of Noetherianity makes the Classification Problem, in general, unsolvable. Our aim, therefore, is to restrict ourselves to a special subclass of injective modules, determined by a fixed star operation $\star$. Then, if the domain we are working on is $\star$-Noetherian, we can hope to give a complete classification of this subclass.

Our first step to find these special injective modules is finding those we are *not* interested in: in a sense, we want to understand what it means for a module to behave as the zero module under "the point of view" of a fixed star operation.

Recall that for every R-module M and for every element x in M, the set

$$\mathrm{Ann}_R\, x = \{r \in R \,|\, rx = 0\}$$

is an ideal of R, called the **annihilator** of x. In particular, $x = 0$ if and only if $\mathrm{Ann}_R\, x = R$.

Definition 3.1 Suppose a star operation $\star$ is defined on R.

- For x in M, we say that x is **$\star$-null** if $(\mathrm{Ann}_R\, x)^\star = R$.
- M is $\star$-null if all its elements are $\star$-null.

For any module M, the subset of M composed by its $\star$-null elements is in fact a $\star$-null submodule, called the **$\star$-null part** of M, and denoted by $\tau_\star M$.

Next we define, in a similar way, those modules which we plan to classify:

Definition 3.2 Suppose a star operation $\star$ is defined on R.

- For x in M, we say that x is **co-$\star$** if $(\mathrm{Ann}_R\, x)^\star = \mathrm{Ann}_R\, x$.
- M is co-$\star$ if all its elements are co-$\star$.

With the next theorem, we can see the role played by co-$\star$ and $\star$-null modules in describing injective modules; observe that we do not require $\star$-Noetherianity.

Theorem 3.3 *Given an injective module H, every star operation $\star$ induces a direct decomposition of H, unique up to isomorphism:*

$$H = E(A) \oplus B \oplus E(\tau_\star H)$$

where:

- *A is a maximal co-$\star$ module.*
- *B contains no $\star$-null elements and no co-$\star$ elements.*

The above can be greatly improved by choosing a *stable* star operation:

Theorem 3.4 *Given an injective module H, every stable star operation $\star$ induces a direct decomposition of H, unique up to isomorphism:*

$$H = A \oplus E(\tau_\star H)$$

where A is a maximal co-$\star$ module.

4 Injective modules over $\star$-Noetherian domains

The last two theorems of the previous section tell us which classes of modules we can completely classify: this will depend on whether our domain is $\star$-Noetherian or Strong $\star$-Noetherian. In the first case, the classification will concern the injective hulls of co-$\star$ modules, while in the second case it will directly focus on injective co-$\star$ modules.

Theorem 4.1

Suppose R is a $\star$-Noetherian domain with $\star$-finite character.

Then the ***injective hulls of co-$\star$ modules*** *are given by the direct sums $D \oplus C$, where:*

$$D \cong \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$$

with P_λ ***$\star$-prime*** *ideal such that R_{P_λ} is Noetherian, and:*

$$C \cong E\left[\bigoplus_{\delta \in \Delta} E(R/Q_\delta)\right]$$

with Q_δ ***$\star$-prime*** *ideal such that R_{Q_δ} is* ***not*** *Noetherian.This decomposition is uniquely determined, up to isomorphism.*

Assuming R to be Strong $\star$-Noetherian makes things much clearer: indeed, Strong $\star$-Noetherianity behaves much better than simple $\star$-Noetherianity as a generalization of classical Noetherianity. To see this, we first recall two fundamental results concerning Noetherian domains:

Theorem 4.2 [Matlis, 1958] *The injective modules over a Noetherian domain R are given by the direct sums*

$$H \cong \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$$

with P_λ prime ideal.

Theorem 4.3 *R is a Noetherian domain if and only if every direct sum of injective modules is injective.*

In 2008 Kim, Kim and Park gave a analogous results for Strong Mori domains:

Theorem 4.4 [Kim, Kim, Park, 2008] *The injective **co-w** modules over a **Strong Mori** domain R are given by the direct sums*

$$H \cong \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$$

*with P_λ **w-prime** ideal.*

Theorem 4.5 *R is a **Strong Mori** domain if and only if every direct sum of injective **co-**w modules is injective.*

Finally, we are able to give a complete generalization, which reduces to the previous cases once we choose $\star = d$ or $\star = w$:

Theorem 4.6 *The injective **co-$\star$** modules over a **Strong $\star$-Noetherian** domain R are given by the direct sums*

$$H \cong \bigoplus_{\lambda \in \Lambda} E(R/P_\lambda)$$

*with P_λ **$\star$-prime** ideal.*

Theorem 4.7 *R is a **Strong $\star$-Noetherian** domain if and only if every direct sum of injective **co-$\star$** modules is injective.*

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Diffusion coefficient and the speed of propagation of traveling front solutions to KPP-type problems

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Abstract. We are concerned with a general reaction-diffusion equation/system in a periodic setting concentrating on reaction terms of KPP-type. Our interest is focused on special solutions called traveling fronts. In particular, we look at how the minimal speed of propagation of such front solutions can be influenced by the different coefficients of the system. For this, an intensive discussion is alloted specifically on the influence of the diffusion coefficient.

1 Introduction

Front propagation is a phenomenon which occur in many scientific areas. In spite of the different applications, this basic phenomenon can all be modelled using nonlinear parabolic partial differential equations or systems of such equations. A simple example is the following general homogeneous reaction-diffusion equation:

$$u_t = \Delta u + f(u) \tag{1}$$

where $u = u(t, x)$ is a scalar function which may accordingly stand for the concentration of a chemical reactant, population density of a biological species, or temperature of a reacting mixture; $\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ describes the diffusion; and $f(u)$ is the term representing the reaction.

The equation (1) above was introduced in the pioneering works of Fisher in 1937 [F] to describe the spreading phenomenon in population genetics with the logistics law for the reaction term $f(u) = u(1-u)$. Almost simultaneously, this equation was the object of the fundamental article of Kolmogorov, Petrovsky and Piskunov [KPP] which laid the ground for the study of nonlinear parabolic equations and introduced some of the essential tools in this field.

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1.1 Some definitions

Let us now consider a simple example of reaction-diffusion equation in dimension 1 with boundary condition:

$$
\left\{\begin{array}{ll} u_t = u_{xx} + f(u) & \text{in } \mathbb{R}\times\mathbb{R} \\ u(+\infty, x) = 1 & \text{in } \mathbb{R} \\ u(-\infty, x) = 0 & \text{in } \mathbb{R}. \end{array}\right. \tag{2}
$$

The simplest non-trivial solution of (2) is the **travelling front** solution of the form $u = \phi(x + ct) \equiv \phi(\xi)$, where c is the wave speed and ϕ is the wave profile that connects 0 and 1. Substituting this form into (2), we obtain

$$
\phi_{\xi\xi} - c\phi_\xi + f(\phi) = 0, \tag{3}
$$

with boundary conditions $\lim_{\xi\longrightarrow-\infty}\phi(\xi) = 0$ and $\lim_{\xi\longrightarrow\infty}\phi(\xi) = 1$. We also impose the condition that $\phi(\xi) \geq 0$. The above problem can be thought of as a nonlinear eigenvalue problem with eigenvalue c and eigenfunction ϕ. Moreover, note that as soon as a travelling front solution ϕ is known, we get another solution moving in the opposite direction at the same speed by transforming ξ to $-\xi$ and c to $-c$. This new solution ϕ will take values 1 at $\xi = -\infty$ and 0 at $\xi = \infty$. We also get other front solutions by translation, *i.e.*, changing ξ to ξ + constant.

We can easily extend this concept of travelling front solutions to higher dimensions. Travelling front solutions for the homogeneous reaction-diffusion equation (1) are of the form $u(t, x) = \phi(x \cdot e + ct)$. They are planar, they propagate in the unit direction $e = (e_1, \ldots, e_n) \in \mathbb{S}^{n-1}$

$$
\left(\text{where } \mathbb{S}^{n-1} := \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i^2 = 1 \right\}\right)
$$

with speed c and they connect two uniform stationary solutions, namely the two zeros of f, 0 and 1. Note that using a similar change of variable $\xi = x \cdot e + ct$, we obtain the same form as (3).

In the KPP case, one can easily show that traveling front solutions exist for any speed c greater than some critical value c^*. Such a value we call the **critical speed**.

1.2 Heterogeneities

Front propagation in heterogeneous media has been studied only recently since the trail-blzing work of Kolmogorov, Petrovsky, and Pishkunov [KPP], and Fisher during the late 1930's on the traveling fronts in reaction-diffusion equations. One reason for this is the many mathematical difficulties brought about by the heterogeneity of the problem.

When studying the propagation of fronts in heterogeneous media, one usually considers either periodic media or random media. We shall limit our discussion here to the case of periodic media. The survey paper of J. Xin [X] gives a detailed exposition in the case of random media.

Aptly, our first source of heterogeneity comes from the geometry of our domain. An example of such a problem is when we consider the homogeneous equation set in a domain $\Omega \subset \mathbb{R}^n$ which is the whole space with periodic array of holes, imposing a Neumann boundary condition, as follows:

$$\left\{ \begin{array}{rcll} u_t - \Delta u & = & f(u) & \text{in } \mathbb{R} \times \Omega \\ \dfrac{\partial u}{\partial \nu} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4}$$

Obviously, there are no travelling front solutions to the Neumann problem (4). Thus one is led to extend the notion of travelling fronts and define what is called **pulsating travelling fronts**:

Definition 1 A **pulsating travelling front**(PTF) solution propagating in the direction $-e \in \mathbb{S}^{n-1}$ is a globally (in time) defined solution $u(t,x)$ with the following properties:

(a) The solution $u(t,x)$ satisfies

$$\begin{array}{lll} u(t,x) \to 0 & \text{as} & x \cdot e \to -\infty \\ u(t,x) \to 1 & \text{as} & x \cdot e \to +\infty \end{array}$$

with limits uniform with respect to $y = x - (x \cdot e)e$ for each $t \in \mathbb{R}$.

(b) There exists $c > 0$ called the average speed of the front such that

$$u(t + \ell_i \frac{e \cdot e_i}{c}, x) = u(t, x + \ell_i e_i), \quad \text{for all } i = 1, ..., n.$$

Heterogeneities inherent in the equation may be due to any of the following factors. One, an underlying flow $q(x) = (q_1(x), \ \dots, \ q_n(x))$ which gives rise to a transport of the scalar u. One is thus led to consider the following reaction-diffusion-advection equation:

$$u_t \ - \ \Delta u \ + \ q(x) \cdot \nabla u \ = \ f(u) \tag{5}$$

where $\nabla u = \left[\dfrac{\partial u}{\partial x_i} \right]_{1 \leq i \leq n}$ is the gradient of u with respect to x.

Another source of heterogeneity is when diffusion is not anymore isotrophic(*i.e.*, the diffusion coefficient is not the identity matrix I_n) but is anisotrophic, or in a more general case, position-dependent. This will lead us to a diffusion term of the form

$$\nabla \cdot (A(x) \nabla u) = \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \tag{6}$$

or

$$\text{Trace}(A(x) D^2 u) = \sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{7}$$

where $A(x) = [a_{ij}(x)]_{1 \leq i,j \leq n}$, an $n \times n$ matrix, is the diffusion coefficient. We refer to a reaction-diffusion equation with diffusion term like in (6) to be in divergence form, and with a diffusion term like in (7) to be in non-divergence form.

Finally, heterogeneity may be due to the nonlinearity f. This happens when the reaction term itself becomes directly dependent on the space Ω, *i.e.*, of the form $f(x,u)$.

2 Setting of the Problem

In this paper, we try to address how the heterogeneity caused by the diffusion affect the minimal speed of propagation of a pulsating front solution traveling in a certain direction in $\mathbb{R}^n$.

To be specific, we shall be looking at the reaction-diffusion-advection equation of the following form:

$$\left\{\begin{array}{rcll} u_t &=& \mathrm{Trace}(A(x,y)D^2u) + q(x,y)\cdot\nabla u + \ f(x,y,u), & t\in\mathbb{R},\ (x,y)\in\Omega \\ & & \nu A\nabla u(x,y) \ = \ 0, & t\in\mathbb{R},\ (x,y)\in\partial\Omega, \end{array}\right. \tag{8}$$

where $A(x)=[a_{ij}(x)]$ is the diffusion matirx, Ω an open, connected subset of $\mathbb{R}^n$, ν denotes the outward unit normal on the boundary $\partial\Omega$ of Ω. We sahll assume throughout the paper that the diffusion matrix $A(x)$, the advection vector q and the reaction term $f(x,s)$ as well as the geometry Ω are periodic. On the whole, we will follow the assumptions on the coefficients a_{ij}, q, and f and on the general periodic framework as in the paper [BHN].

2.1 Assumptions

We now make precise the setting of the paper.

Let $n\geq 1$ be the space dimension and let d be an integer such that $1\leq d\leq n$. Call $x=(x_1,\ldots,x_d)$ and $y=(x_{d+1},\ldots,x_n)$. Let $\ell_1,\ldots,\ell_d$ be d postive numbers and let Ω be a nonempty connected open subset of $\mathbb{R}^n$ with C^3 boundary such that

$$\left\{\begin{array}{cc} \exists R\geq 0,\qquad \forall (x,y)\in\Omega, & |y|\leq R, \\ \forall (k_1,\ldots,k_d)\in \ell_1\mathbb{Z}\times\ldots\times\ell_d\mathbb{Z}, & \Omega=\Omega+\sum_{i=1}^n k_i\vec{e}_i, \end{array}\right. \tag{9}$$

where $(\vec{e}_i)_{1\leq i\leq n}$ is the canonical basis in $\mathbb{R}^n$. Let $\mathcal{C}$ be the set defined by

$$\mathcal{C}=\{(x,y)\in\Omega | x\in(0,\ell_1)\times\ldots\times(0,\ell_d)\}.$$

A function g is said to be L-periodic with respect to $x\in\Omega$ if

$$g(x+k,y)=g(x,y) \tag{10}$$

almost everywhere in Ω for all $k\in\ell_1\mathbb{Z}\times\ldots\times\ell_d\mathbb{Z}$. By pulsating traveling front solutions, we mean special solutions which are classical time-global soltions u of (8) satisfying $0\leq u\leq 1$ and

$$\left\{\begin{array}{l} \forall k\in\prod_{i=1}^d \ell_i\mathbb{Z},\ \forall (t,x,y)\in\mathbb{R}\times\overline{\Omega},\quad u\left(t-\dfrac{k\cdot e}{c},x,y\right)=u(t,x+k,y), \\ \qquad u(t,x,y)\longrightarrow 0, \qquad \text{as } x\cdot e\longrightarrow +\infty \\ \qquad u(t,x,y)\longrightarrow 1, \qquad \text{as } x\cdot e\longrightarrow -\infty \end{array}\right. \tag{11}$$

where the above limits hold locally in t and uniformly in y and in the direction of $\mathbb{R}^d$ which are orthogonal to e. Here, $e = (e^1, \cdots, e^d)$ is a given unit vector in $\mathbb{R}^d$. Such a solution satisfying (11) is then called a *pulsating travelling front* propagating in the direction of e. We say that c is the *effective* unknown speed $c \neq 0$.

Under the assumptions above, it was proved in [BHN] and [BH] that there exists $c^*(\vec{e}) > 0$ called the **minimal speed** such that pulsating traveling fronts u in the direction $\vec{e}$ with the speed c exists if and only if $c \geq c^*(\vec{e})$. Moreover, all such pulsating fronts are increasing in time t.

2.2 Main Problem

This study is done to further extend the result in [BH], and [BHN] by looking at one specific cause of heterogeneity: **diffusion**. Particularly, we now want to generalize the result obtained in [BHN] concerning the effect of the diffusion coefficient to the minimal speed of propagation when it is no longer the identity matrix I_n but a general matrix field which is a function of the domain.

Specifically, suppose we have two reaction-diffusion-advection equations with diffusion coefficients $A(x)$ and $B(x)$ satisfying the assumptions in subsection 2.1 and having the property

$$0 < A(x) \leq B(x) \tag{12}$$

in the sense that for any $\xi \in \mathbb{R}^n$

$$0 < \langle A(x)\xi, \xi\rangle \leq \langle B(x)\xi, \xi\rangle \qquad \text{for all } x \in \mathbb{R}^n, \tag{13}$$

can we compare their corresponding minimum speed of propagation?

3 The Result

We are now ready to state our result. Before we do that, we introduce the concept of rational directions with respect to our lattice $\ell_1\mathbb{Z} \times \ldots \times \ell_n\mathbb{Z}$ where the ℓ_is were defined in subsection 2.1. Define $\mathbb{T}^{n-1}$ the set of all rational directions with respect to $\ell_1\mathbb{Z} \times \ldots \times \ell_n\mathbb{Z}$ to be

$$\mathbb{T}^{n-1} := \{x \in \mathbb{S}^{n-1} : \ \exists r \in \mathbb{R} \setminus \{0\} \text{ such that } rx \in \ell_1\mathbb{Z} \times \ldots \ell_n\mathbb{Z}\}. \tag{14}$$

Observe that $\mathbb{T}^{n-1}$ is dense in $\mathbb{S}^{n-1}$. Indeed, take $e_0 \in \mathbb{S}^{n-1} \setminus \mathbb{T}^{n-1}$. For any $\epsilon > 0$, define a cone $\mathcal{C}_\epsilon(e_0)$ centered at e_0 to be

$$\mathcal{C}_\epsilon(e_0) := \{ke \mid \forall k \in \mathbb{R} \setminus \{0\}, \ \forall e \in \mathbb{S}^{n-1} \text{ such that } \|e - e_0\| < \epsilon\}$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$. Then

$$\mathcal{C}_\epsilon(e_0) \cap (\ell_1\mathbb{Z} \times \ldots \times \ell_n\mathbb{Z}) \neq \emptyset, \qquad \text{for any } \epsilon > 0.$$

Our main theorem is as follows:

Theorem 2 *Consider the reaction-diffusion equation in (8) with coefficients satisfying the assumptions in subsection 2.1. Suppose* $q \equiv 0$, $f(x, s) = f(s)$ *and let* $A(x) = [a_{ij}(x)]$ *and* $B(x) = [b_{ij(x)}]$ *be symmetric positive definite* $n \times n$ *matrices for every* $x \in \mathbb{R}^n$*. If there exists a* $\gamma \in \mathbb{R}^n$ *such that*

$$A(x) \leq B(x + \gamma), \qquad \forall x \in \mathbb{R}^n \tag{15}$$

in the sense of (13), then

$$c^*_A(e) \leq c^*_B(e), \qquad \forall e \in \mathbb{T}^{n-1}. \tag{16}$$

The proof follows closely the work in [RV]. Although the said work is done in the one-dimensional case, one can connect our multidimensional problem to such by "cutting" the eigenfunctions through a rational direction e. Thus, the result stated here is limited to rational directions.

A sufficient condition is posed in [V] so that (16) holds true for any direction $e \in \mathbb{R}^n$. It would be interesting to know if such a condition can be dropped.

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On some aspects of the McKay correspondence

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"Les objets concernés par cet article
sont essentiellement les solides platoniciens"
G. Gonzalez-Sprinberg and J. L. Verdier

1 Introduction

When we quotient $\mathbb{C}^2$ by a finite subgroup G of $SL(2,\mathbb{C})$, and we take a minimal resolution Y of the kleinian singularity $\mathbb{C}^2/G$, then Y is a crepant resolution and the exceptional locus consists of a bunch of curves, whose dual graph is a Dynkin diagram of the kind A_n, D_n, E_6, E_7, E_8. In the eighties, McKay noticed that the Dynkin diagrams arising from resolutions of kleinian singularities are in tight connection with the representations of G. In the first part we will explain the McKay correspondence and its key generalization by means of K-theory, due to Gonzalez-Sprinberg and Verdier. The latter point of view opens the way to the modern derived McKay correspondence, due to Bridgeland-King-Reid. We will then see some applications of the BKR theorem to the geometry of Hilbert schemes of points, due to Haiman, and some other consequences related to the cohomology of tautological bundles. In all the exposition, we will always work with algebraic varieties over $\mathbb{C}$; moreover, we will always suppose that all singular varieties are normal.

2 Rational double points

The objects of interest in this section are certain classes of singularities of surfaces and their resolutions. If X is an algebraic variety, we denote with X_{reg} and with X_{sing} the open set of regular points and the closed set of singular points, respectively. We recall the definition of resolution of singularities.

Definition 2.1 Let X an algebraic variety. A resolution of singularities of X is a smooth variety Y, equipped with a proper birational morphism $\mu : Y \longrightarrow X$, such that μ induces an isomorphism between $Y \setminus \mu^{-1}(X_{\mathrm{sing}})$ and X_{reg}. The set $Exc(\mu) := \mu^{-1}(X_{\mathrm{sing}})$, where μ fails to be an isomorphism, is called the exceptional locus.

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We say that an algebraic variety X has *rational singularities* if there is a resolution $\mu : Y \longrightarrow X$ such that $\mathbf{R}\mu_* \mathcal{O}_X \simeq \mathcal{O}_Y$, or equivalently, such that $\mu_* \mathcal{O}_Y \simeq \mathcal{O}_X$, and the higher direct images of the structural sheaf of Y vanish: $R^i \mu_* \mathcal{O}_X \simeq 0$. We are interested in (germs of) isolated singularities of surfaces, that is, we consider a small neighbourhood of a surface X around the only singular point x. We denote the germ with (X, x). For such a singularity consider the maximal ideal $\mathfrak{m}_x$ in the local ring $\mathcal{O}_{X,x}$: the Zariski cotangent space of X at x is then $\mathfrak{m}_x/\mathfrak{m}_x^2$. For an isolated singularity (X, x), being rational can be rephrased in terms of complex analytic geometry as follows:

Definition 2.2 [27], [6]. A germ of n-dimensional isolated singularity (X, x) is rational if and only if for all regular holomorphic n-form $\sigma \in H^0(X \setminus \{x\}, \Omega^n_X)$ on $X \setminus \{x\}$, the pull-back $\mu^*\sigma \in H^0(Y \setminus Exc(\mu), \Omega^n_Y)$ extends to a regular holomorphic form on the whole Y. This is equivalent to saying that any holomorphic n-form σ defined in a deleted neighbourhood $U \setminus \{x\}$ of x is square integrable around x, that is

$$\int_{U'} \sigma \wedge \bar{\sigma} < +\infty$$

for U' sufficiently small relatively compact.

Germs of isolated rational *surface singularities* have been extensively studied by Artin in [1]: among other results, he proves that an isolated rational surface singularity has multiplicity exactly $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 - 1$. Since one can always embed any germ (X, x) in its tangent space at the point x, one has consequently that a rational double point is always embeddable in $\mathbb{C}^3$. Hence the isolated surface singularities that we are interested in are all of the form $(X, x) = (V(f), 0)$, where $f \in \mathbb{C}[x, y, z]$ is a polynomial in 3 variables, with $\nabla f(0) = 0$.

Remark 2.3 In this case, if Y is a resolution of singularities, then the exceptional set with the reduced structure $E = \mu^{-1}(0)_{\text{red}}$ is a divisor (necessarily a curve in Y) and it is always connected (by Zariski main theorem, since X is normal). However E can be reducible. We will write $E = \cup_i C_i$, where C_i are the irreducible components.

Definition 2.4 Let X an n-dimensional algebraic variety. Consider the open immersion $j : X_{\text{reg}} \hookrightarrow X$. Consider the sheaf $\omega_X := j_*\Omega^n_{X_{\text{reg}}}$ and suppose that it is a line bundle. A resolution of singularities $\mu : Y \longrightarrow X$ is called *crepant*$^{(*)}$ if $\mu^*\omega_X \simeq \omega_Y$, where $\omega_Y := \Omega^n_Y$ is the canonical line bundle of Y. For an isolated singularity (X, x) this means that for any holomorphic n-form σ defined on a heighbourhood of x, the form $\mu^*\sigma$ is a holomorphic n-form on $\mu^{-1}(U) \setminus Exc(\mu)$ which can be extended to a holomorphic n-form to $\mu^{-1}(U)$ without zeros on $\mu^{-1}(U)$.

Remark 2.5 We will say that the resolution $\mu : Y \longrightarrow X$ is *minimal* if it does not factorize through another resolution $\mu' : Y' \longrightarrow X$. It follows that if the germ of surface singularity (X, x) is rational and the resolution $\mu : Y \longrightarrow X$ is minimal, then it is crepant.

$^{(*)}$This means that there is no *discrepancy* between ω_Y and $\mu^*\omega_X$

Example 2.6 Consider the polynomial $f = x^2 + y^2 - z^2$ (figure 1). The surface $X = V(f)$ has an isolated singularity at the origin. The singularity is rational, as it can be seen as follows. Since $df = 2xdx + 2ydy - 2zdz = 0$ on X, the differential form on X: $\sigma = -dx \wedge dy/2z = -dy \wedge dz/2y = dx \wedge dz/2y$ is a well defined rational form on X; moreover it is regular and nondegenerate at every nonsingular point of X, hence it is a volume form on $X \setminus \{0\}$. A resolution of X can be obtained considering the blow-up $h : \mathrm{Bl}_0\mathbb{C}^3 \longrightarrow \mathbb{C}^3$ of the origin in $\mathbb{C}^3$, and taking Y as the strict transform of X, that is, the Zariski closure of $h^{-1}(X \setminus \{0\})$. On can easily prove that the form $h^*\sigma$, defined on $Y \setminus E$, can be extended to the whole Y as a volume form. The blow-up has indeed 3-charts; one of them (the others are analogous) has coordinates λ, μ, z, with $x = \lambda z$, $y = \mu z$, and Y is defined on this chart by $\lambda^2 + \mu^2 - 1 = 0$. The exceptional divisor is the circle $E = Y \cap \{z = 0\}$. The differential form $\tau = -d\mu \wedge dz/2\lambda = d\lambda \wedge dz/2\mu$ is a rational volume form on Y, coinciding with $h^*\sigma$ on $Y \setminus E$.

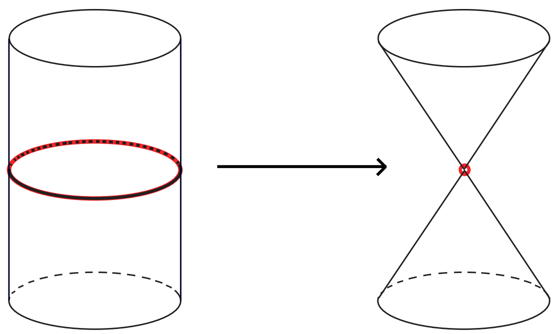


Figure 1: Minimal resolution of the A_1-singularity $x^2 + y^2 - z^2 = 0$.

Example 2.7 Consider the polynomial $f = x^2 - y^2z - z^3$ (figure 2). The surface $V(f)$ has an isolated rational singularity at the origin. In order to solve it, we need two blow-ups. After the first one, there will be three distinct singular points in the exceptional divisor. Blowing-up the three of them at once, we get the resolution Y. The exceptional divisor E is a union of four rational curves C_i.

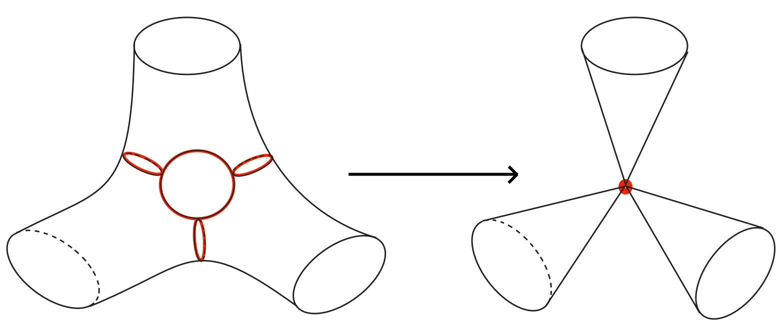
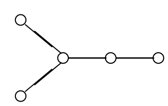
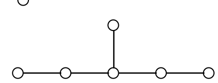
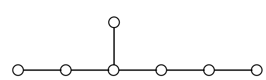
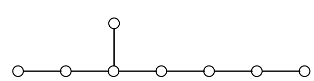


Figure 2: Minimal resolution of the D_4-singularity $x^2 - y^2z - z^3 = 0$.

Definition 2.8 If (X, x) is germ of an isolated rational surface singlarity and $\mu : Y \longrightarrow X$ a minimal resolution with exceptional divisor $E = \sum_i C_i$, the cycle $W = \sum_i r_i C_i$ given by the nonreduced scheme $W := \mu^{-1}(x)$ is called the *fundamental cycle*.

Remark 2.9 If (X, x) is a germ of rational surface singularities and $\mu : Y \longrightarrow X$ is a minimal resolution, we can understand completely the kind of curves C_i appearing as irreducible components of E and the structure of their intersections [2, chapter 3, §2,3]. Indeed

- The autointersection C_i^2 of each curve is -2. This is equivalent to the fact that all curve are rational, and actually isomophic to $\mathbb{P}_1$.
- If we draw a point for each curve C_i and a line between points if the two corresponding curve intersect, the diagram we obtain are all and only the following Dynkin diagrams. It is clear that isomorphic germ singularities will generate the same diagrams, so the following is a classification of isomophism classes of rational double points. The matrix $(C_i \cdot C_j)_{ij}$, whose information is equivalent to the information given by the diagrams, is called the *intersection matrix*.

A_n	$x^2 + y^2 + z^{n+1}$	
D_n	$x^2 + y^2 z + z^{n-1}$	
E_6	$x^2 + y^3 + z^4$	
E_7	$x^2 + y^3 + yz^3$	
E_8	$x^2 + y^3 + z^5$	

Remark 2.10 Rational double points have many other beautiful and interesting characterizations and connections, not only in terms of algebraic geometry, but also of complex analysis, Lie groups, differential topology, algebraic topology and fundamental groups, Morse theory, catastrophe theory and many others. See for example [14] and [38].

3 Finite subgroups of $SU(2)$

Interesting isolated surface singularities come from quotients $\mathbb{C}^2/G$, with G a finite group of $SL(2, \mathbb{C})$. Any finite subgroup of $SL(2, \mathbb{C})$ is conjugated to a subgroup of $SU(2)$, the reason being that, by averaging, one can build a G-invariant hermitian metric in $\mathbb{C}^2$. In this section we will say some words on the classification of finite subgroups of $SU(2)$.

As an immediate consequence of its definition, the group $SU(2)$ is diffeomeorphic to the sphere S^3 and hence simply connected. Moreover there is a $2:1$ covering map

$\pi : SU(2) \longrightarrow SO(3)$, that realizes it as the universal cover of $SO(3)$, or, in other terms, as $Spin(3)$.

Remark 3.1 Since the only element of order 2 in $SU(2)$ is -1, we have that if G is a finite subgroup of $SU(2)$, then, up to conjugation, G is a cyclic group of finite order, or $G = \pi^{-1}(G')$ with G' a finite group of $SO(3)$, that is, a *binary polyhedral group*. Indeed if $|G|$ is odd, then $G \cap \ker \pi = \{1\}$, hence $G \simeq \pi(G)$, and hence it has to be cyclic. Otherwise, if $|G|$ is even, then, by Sylow theorem, it contains a subgroup of order a power of 2, and hence an element of order 2, that is, it has to contain the kernel. Hence, $G = \pi^{-1}\pi(G)$.

Remark 3.2 After the previous remark, to classify, up to conjugation, finite subgroups of $SU(2)$, we just have to classify finite subrgroups of $SO(3)$. Let G a finite subgroup of $SO(3)$. Let p a point of $\mathbb{R}^3$, $p \neq 0$. Then the orbit Gp can be planar or not. If Gp is planar, then G is *cyclic of order* n, or a *dihedral group* (of order $2n$), that is, the symmetry group of a polygon with n sides. On the other hand, if the orbit is not planar, then it is the set of vertices of a regular polyhedron, and G is its symmetry group. Regular polyhedra, or platonic solids, have been classified first by Theaetetus (417 B.C. – 369 B.C.), a Greek mathematician contemporary to Plato, and the classification has been reported by Plato himself in [34] and by Euclid in the Elements [18]. Since esahedron and octahedron, dodecahedron and icosaheron are dual couples of platonic solids, they have the same symmetry group. Hence, all possible symmetry groups of platonic solids are: the *tetrahedral group*, isomorphic to the alternating group $\mathfrak{A}_4$, with 12 elements, the *octrahedral group*, isomorphic to the symmetric group $\mathfrak{S}_4$, with 24 elements, the *icosahedral group*, isomorphic to $\mathfrak{S}_5$, with 60 elements.

As a consequence of the previous two remarks, we can write down the list of all possible finite subroups G of $SU(2)$ (and of $SL(2,\mathbb{C})$) up to conjugation.

C_n	cyclic	n
BD_{4n}	binary dihedral	$4n$, $n \geq 2$
BT_{24}	binary tetrahedral	24
BO_{48}	binary octahedral	48
BI_{120}	binary icosahedral	120

4 Quotient singularities

Consider now the quotient $\mathbb{C}^2/G$, with G a finite subgroup of $SL(2,\mathbb{C})$. We have that 0 is the unique point with nontrivial stabilizer. Since the action of G on $\mathbb{C}^2 \setminus \{0\}$ is free, the quotient $(\mathbb{C}^2 \setminus \{0\})/G$ is a smooth variety. The topological[(*)] space $\mathbb{C}^2/G$ can be given the structure of an affine algebraic surface with an isolated singularity in $[0]$.

Remark 4.1 If X is an affine variety with ring of regular functions $A(X)$, then X can be recovered from $A(X)$ taking the spectrum $\operatorname{Spec} A(X)$ of $A(X)$, that is, considering all the prime ideals of $A(X)$, if we want the whole scheme structure, and just by taking the

[(*)]Here we take the Zariski topology on $\mathbb{C}^2$ and the quotient topology on $\mathbb{C}^2/G$.

maximal spectrum Max $A(X)$, if we just want to recover the closed points, that is, the structure of algebraic variety.

As a consequence of the previous remark, in order to put on the quotient $\mathbb{C}^2/G$ a structure of affine variety, it is just necessary to assign the algebra of regular functions $A(\mathbb{C}^2/G)$. We remark that the projection $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$ has to induce by pull-back a morphism between the algebras of regular functions (as it does for continuous functions)

$$\pi^* : A(\mathbb{C}^2/G) \longrightarrow A(\mathbb{C}^2)^G \ , \tag{4.1}$$

since we want that the pull-back on $\mathbb{C}^2$ of any regular function on $\mathbb{C}^2/G$ has to be automatically G-invariant. We require that the pull-back (4.1) is actually an *isomorphism.* Hence, as an algebraic scheme,

$$\mathbb{C}^2/G := \operatorname{Spec} A(\mathbb{C}^2)^G \simeq \operatorname{Spec} \mathbb{C}[x,y]^G \ .$$

The reassuring thing is that, topologically, the variety underlying Spec $A(\mathbb{C}^2)^G$ is homeomorphic to the original topological quotient $\mathbb{C}^2/G$. The last thing we need is the understanding of the invariants $\mathbb{C}[x,y]^G$. This is provided by the following theorem.

Theorem 4.2 [Klein, 1884, [26]] *Let G a finite subgroup of $SL(2,\mathbb{C})$. Then the ring of G-invariants $\mathbb{C}[x,y]^G$ is generated by three invariants polynomial $P,Q,R \in \mathbb{C}[x,y]^G$, with a unique relation $S(P,Q,R) = 0$.*

As a consequence of Klein theorem, we have an epimorphism $\mathbb{C}[u,v,w] \twoheadrightarrow \mathbb{C}[x,y]^G$, sending u on P, v on Q and w on R. The kernel is generated by the principal ideal $(S(u,v,w))$. Hence passing to the quotient we get an isomorphism:

$$\mathbb{C}[u,v,w]/(S) \simeq \mathbb{C}[x,y]^G \ .$$

As a consequence we get the immersion:

$$\mathbb{C}^2/G \simeq \operatorname{Spec} \mathbb{C}[x,y]^G \simeq \operatorname{Spec} \mathbb{C}[u,v,w]/(S) \hookrightarrow \operatorname{Spec} \mathbb{C}[u,v,w] \simeq \mathbb{C}^3 \ ,$$

where $\mathbb{C}^2/G$ is embedded in $\mathbb{C}^3$ as the hypersurface of equation $S(u,v,w) = 0$. Hence $(\mathbb{C}^2/G, [0]) \simeq (V(S), 0)$ is an isolated surface singularity. The quotient singularities of the form $\mathbb{C}^2/G$, with G a finite group of $SL(2,\mathbb{C})$ are called *kleinian singularities.*

Example 4.3 Let $G = \mathbb{Z}_m$, the cyclic group with m elements, acting on $\mathbb{C}^2$ in the following way. If ϵ is a primitive m-root of unity, then it acts on (x,y) by sending it to $(\epsilon x, \epsilon^{-1} y)$. The invariant polynomials $\mathbb{C}[x,y]^G$ are generated by $P = x^m$, $Q = y^m$, $R = xy$, with the relation $R^m = PQ$. Hence $\mathbb{C}^2/G$ can be embedded in $\mathbb{C}^3$ as the hypersurface of equation $w^m = uv$, that, with a change of coordinates, becomes $u^2 + v^2 + w^m = 0$. We remark that it is one of the rational double point listed at page 41, as a A_n-singularity.

Example 4.4 Consider the binary dihedral group BD_{4n}. It is generated by two elements $G = \langle \alpha, \beta \rangle$,

$$\alpha = \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon} \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where ϵ is a primitive $2n$-root of unity; here $\alpha^n = \beta^2 = -1$, $\alpha\beta = \beta\alpha^{-1}$. The invariant polynomials are: $P = x^n + y^n$, $Q = x^2y^2$, $R = xy(x^{2n} - y^{2n})$. The relation S is $S(P, Q, R) = R^2 - P^2Q + 4Q^{n+1}$. Hence $\mathbb{C}^2/G$ can be embedded in $\mathbb{C}^3$ as the hypersurface of equation $u^2 - v^2w + 4w^{n+1} = 0$, or, after a change of variable, of equation $u^2 + vw^2 + w^{n+1} = 0$. This is also listed on page 41 as a D_{n+2} singularity.

It is not by chance that the singularities obtained by the previous two examples are rational double points. In general one has:

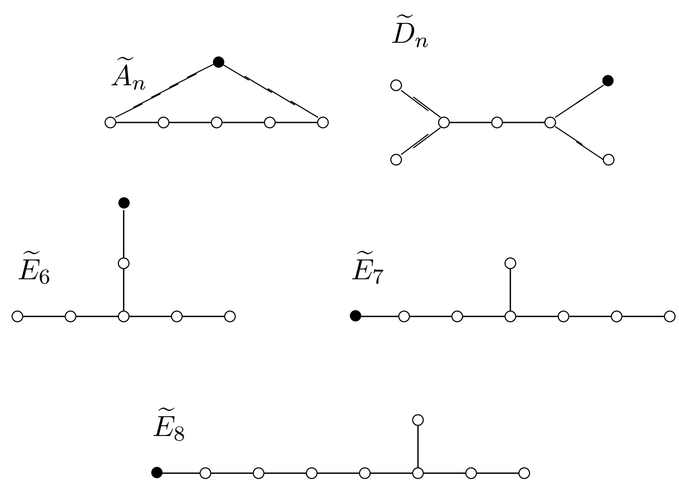
Theorem 4.5 [Du Val, 1934, [11], [12], [13]] *If G is a finite subgroup of $SL(2, \mathbb{C})$, the kleinian singularity $\mathbb{C}^2/G$ is a rational double point. For each finite subgroup of G, up to conjugation, we have exactly one isomorphism class of singularities. They correspond to each other in the following way.*

A_n	$x^2 + y^2 + z^{n+1}$	C_n	*cyclic*
D_n	$x^2 + y^2z + z^{n+1}$	$BD_{4(n-2)}$	*binary dihedral*
E_6	$x^2 + y^3 + z^4$	BT_{24}	*binary tetrahedral*
E_7	$x^2 + y^3 + yz^3$	BO_{48}	*binary octahedral*
E_8	$x^2 + y^3 + z^5$	BI_{120}	*binary icosahedral*

See also [37, Chapter IV, §4.3].

5 The McKay correspondence

In the eighties [32, 33] John McKay had the idea to relate, in a purely combinatorial but completely unexpected way, the geometry of a minimal resolution of a kleinian singularity $\mathbb{C}^2/G$, and in particular the intersection graph of the irreducible components of the exceptional divisor, with the irreducible representations of G. In order to be able to explain such a correspondence, we have to introduce the *extended Dynkin diagrams*, obtained by the ADE diagrams, by adding to each of them a point, in the following way.



Remark 5.1 The newly added point is motivated by the following. Consider a minimal resolution $\mu : Y \longrightarrow X = \mathbb{C}^2/G$ of a Kleinian singularity. Let now C_0 the strict transform of a general hyperplane section in X. If $C_1, \dots, C_n$ are the irreducible components of the exceptional divisor E, whose intersection graph is the old Dynkin diagram, the intersection matrix $\tilde{A}$ of the set of curves $C_0, C_1, \dots, C_n$ corresponds to the extended Dynkin diagram, with C_0 corresponding to $\bullet$.

Consider now the set $\operatorname{Irr} G = \{\rho_0, \dots, \rho_n\}$ of irreducible representations of G. Here ρ_0 is the trivial representation. Associate to $\operatorname{Irr} G$ the matrix $A = (a_{ij})$ whose terms a_{ij} are the coefficient of $\mathbb{C}^2 \otimes \rho_j$ in terms of ρ_i:

$$\mathbb{C}^2 \otimes \rho_j = \bigoplus_i \rho_i^{\oplus a_{ij}} .$$

McKay proved:

Theorem 5.2 [McKay, 1980] *There is a bijection*

$$\operatorname{Irr} G \longleftrightarrow \{C_0, \dots, C_n\} \tag{5.1}$$

such that $\rho_i \mapsto C_i$ for all i, and

(i) $(C_i \cdot C_j) = a_{ij} - 2\delta_{ij}$ *(or $(C_i \cdot C_j)_{ij} = A - 2\mathrm{id} = \tilde{A}$);*

(ii) $\dim \rho_i = r_i$*, where $W = \sum_i r_i C_i = \mu^{-1}(0)$ is the fundamental cycle.*

Remark 5.3 The cohomology classes $[C_i]$ of curves $\{C_0, \dots, C_n\}$ form a basis of the second cohomology group $H^2(Y, \mathbb{Z})$.

6 Geometric McKay correspondence

A few years after McKay result, Gonzalez-Sprinberg and Verdier [20] succeeded in giving a geometric construction of McKay correspondence. They actually prove a more general correspondence at the K-theory level, which induces McKay's one. We recall that, for a smooth algebraic variety V, the K-theory $K(V)$ is the ring generated by locally free sheaves (vector bundles) on V with a relation $E = E_1 + E_2$ whenever E is an extension of E_1 and E_2, that is, whenever we have a short exact sequence: $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$; the multiplication is given by the tensor product. The use of K-theory allows to to reinterpret the terms of the correspondence. Indeed

- the set of curves $\{C_0, \dots, C_n\}$ (which actually give information on the second cohomology $H^2(Y, \mathbb{Z})$ of the minimal resolution) is replaced with the larger K-theory ring $K(Y)$. It is not difficult to prove that

  $$K(Y) \simeq \mathbb{Z} \oplus \operatorname{Pic}(Y) \simeq \mathbb{Z} \oplus H^2(Y, \mathbb{Z})$$

  via the map that associate to a vector bundle E the couple $(\operatorname{rk} E, c_1(E))$ given by its rank $\operatorname{rk} E \in \mathbb{Z}$ and its first Chern class $c_1(E) \in H^2(Y, \mathbb{Z})$. Hence the ring $K(Y)$

allows to recover the information given by the second cohomology $H^2(Y, \mathbb{Z})$ and its basis $\{C_0, \dots, C_n\}$.

- Even if $K(\mathbb{C}^2)$ does not provide much information, since it is trivial—$K(\mathbb{C}^2) \simeq \mathbb{Z}$—the G-equivariant K-theory of $\mathbb{C}^2$, that is ring generated by G-equivariant vector bundles (with analogous relations given by extensions), gives the needed informations. Indeed one can prove that the map $K_G(\mathbb{C}^2) \longrightarrow R(G)$, associating to a G-equivariant vector bundle E its 0-fiber $E(0)$, is a ring isomorphism, with inverse $\rho \mapsto \mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} \rho$.

The geometric construction of the correspondece is now built as follows. Consider the *reduced fibered product* $\mathcal{Z} := (Y \times_X \mathbb{C}^2)_{\mathrm{red}}$. Then one has a (non cartesian) diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\ Y \arrow[r, "\mu"] & X = \mathbb{C}^2/G \end{tikzcd}$$

Remark 6.1 One can prove easily that p and q are birational, while it is more difficult (and it is a key point, as we will see later) to prove that q is flat and finite of degree $|G|$.

Gonzalez-Sprinberg and Verdier define a morphism of groups $\lambda : R(G) \longrightarrow K(Y)$ as a composition:

$$\lambda : R(G) \simeq K_G(\mathbb{C}^2) \xrightarrow{p^*} K_G(\mathcal{Z}) \xrightarrow{q_*^G} K(Y) \ ,$$

that is, for any $\rho \in R(G)$

$$\lambda(\rho) := q_*^G(p^*\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} \rho) = q_*^G(\mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}} \rho) \ ;$$

where q_*^G is the G-invariant push forward, that is, the push-forward followed by the functor of G-fixed points $[-]^G$. The morphism λ is a K-theoretical integral transform of kernel $\mathcal{Z}$. We have the following result, stating the geometric realization of the McKay correspondence.

Theorem 6.2 [Gonzalez-Sprinberg, Verdier, 1983, [20]] *The morphism λ is an isomorphism of abelian groups such that:*

(i) *If $\rho_i \in \operatorname{Irr} G$, then $c_1(\lambda(\rho_i)) = [C_i] \in H^2(Y, \mathbb{Z})$;*

(ii) *$c_1(\lambda(\rho_i)) \cdot c_1(\lambda(\rho_j)) = a_{ij}$ for $i \neq j$;*

(iii) *$[W] = \sum_i (\dim \rho_i) c_1(\lambda(\rho_i)) \in H^2(Y, \mathbb{Z})$.*

Remark 6.3 Remark that the composition: $\operatorname{Irr} G \hookrightarrow R(G) \xrightarrow{\lambda} K(Y) \longrightarrow H^2(Y, \mathbb{Z})$ realizes the classical McKay correspondence (5.1).

Definition 6.4 The sheaves $\mathcal{F}_{\rho_i} := \lambda(\rho_i) = q_*^G(\mathcal{O}_{\mathcal{Z}} \otimes_{\mathbb{C}} \rho_i)$, where ρ_i is an irreducible representation of G, are called *Gonzalez-Sprinberg-Verdier sheaves.*

7 Derived McKay correspondence

The geometric McKay correspondence of Gonzalez-Sprinberg and Verdier is the turning point for more recent developments. Notably, after the work [20], some of the questions that could be raised were how to generalize the result in higher dimensions, to general smooth varieties (instead of $\mathbb{C}^n$) and how to lift it to the derived category level. One of the key difficulties in order to answer these questions is how to replace Y in all generality: the existence of a crepant resolution of singularities is indeed not at all guaranteed in dimension 3 or more [35].

The fundamental point is to consider Y as a *moduli space*, that is a variety parametrizing some kind of objects on $\mathbb{C}^2$. A closer look at the Gonzalez-Sprinberg-Verdier construction allows to guess what are the objects that Y could parametrize. In the diagram (6) the reduce fiber product $\mathcal{Z}$ inherits a G-action (through the factor $\mathbb{C}^2$); moreover, the morphism $q : \mathcal{Z} \longrightarrow Y$ is *flat* and finite of degree $|G|$, as we remarked; finally q is G-invariant. Consequently $\mathcal{Z} \subseteq Y \times \mathbb{C}^2$ can be seen as a *flat family over Y of G-equivariant subschemes of $\mathbb{C}^2$ of length $|G|$.*

The precise construction was built by Ito and Nakamura in 1996 [24], [23] for a general smooth quasi-projective variety M, equipped with the action of a finite group G, and goes under the name of Nakamura G-Hilbert scheme $\mathrm{Hilb}^G(M)$.

Definition 7.1 The G-Hilbert scheme $\mathrm{GHilb}(M)$ of G-clusters on M is the scheme representing the functor:

$$\underline{\mathrm{GHilb}(M)} : \mathrm{Sch}/\mathbb{C} \longrightarrow \mathrm{Sets}$$

associating to a scheme S the set

$$\begin{aligned} \underline{\mathrm{GHilb}(M)}(S) := \{ & Z \subset S \times M, Z \text{ closed } G\text{-invariant subscheme}, \\ & \text{flat and finite over } S \text{ such that } H^0(\mathcal{O}_{Z_s}) \simeq \mathbb{C}[G] \text{ for all } s \in S\} \,. \end{aligned}$$

The irreducible component of $\mathrm{GHilb}(M)$ containing free G-orbits is called the *Nakamura G-Hilbert scheme*, and it is indicated with $\mathrm{Hilb}^G(M)$.

Remark 7.2 The necessity of taking the irreducible component containing free orbits comes from the fact that, in general, the scheme $\mathrm{GHilb}(M)$ is very bad: it is not irreducible and not even equidimensional. One has a natural morphism $\mu : \mathrm{Hilb}^G(M) \longrightarrow M/G$, called the G-Hilbert-Chow morphism, which sends a free orbit Gx over the class $[x]$; it is birational and dominant.

Remark 7.3 Being a the scheme representing the functor $\underline{\mathrm{GHilb}(M)}$, the scheme $\mathrm{GHilb}(M)$ is a fine moduli space of *G-clusters*, that is, G-invariant subschemes ξ of M of length $|G|$, such that $H^0(\mathcal{O}_\xi)$ is isomorphic to the regular representation $\mathbb{C}[G]$ of G. Hence there is a universal family $\mathcal{Z}$ of G-clusters, $\mathcal{Z} \subseteq \mathrm{GHilb}(M) \times M$. The restriction of $\mathcal{Z}$ to $\mathrm{Hilb}^G(M)$ provides a flat and finite family of G-clusters over $\mathrm{Hilb}^G(M)$.

Denote from now one with Y the Nakamura G-Hilbert scheme $\mathrm{Hilb}^G(M)$. We have the diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"'] & M \arrow[d, "\pi"] \\ Y \arrow[r, "\mu"] & M/G = X \end{tikzcd}$$

The morphisms p and μ are birational, the morphisms q and π are finite of generic degree $|G|$, q is flat. Remark that p is G-equivariant.

Remark 7.4 An algebraic variety X is said to be Cohen-Macauley if all local rings $\mathcal{O}_{X,x}$, for all points $x \in X$, are Cohen-Macauley. In this case (see [22]) there exists a dualizing sheaf ω_X°, that allows Serre duality. The variety X is said to be Gorenstein (or to have Gorenstein singularities) if it is Cohen-Macauley and the dualizing sheaf is actually a line bundle; in that case $\omega_X^\circ \simeq j_*\omega_{X_{\mathrm{reg}}}$, where j is the open immersion $X_{\mathrm{reg}} \hookrightarrow X$. For the quotient M/G of a smooth variety by a finite group to be Gorenstein, it suffices (and is actually equivalent) that the stabilizer G_x of any points acts on the tangent space T_xM as a subgroup of $SL(T_xM)$. Indeed in this case the canonical line bundle ω_M is preserved by G, and hence it is locally trivial as a G-line bundle; therefore it descends to a line bundle $\omega_{M/G}$ on M/G, which coincide with the canonical line bundle on the smooth points of M/G; it is isomorphic to the dualizing sheaf $\omega_{M/G}^\circ$ of M/G.

With these premises Bridgeland, King and Reid proved in 2001, under some reasonable hypothesis, a general derived category version of the geometric McKay correspondence.

Theorem 7.5 [Bridgeland-King-Reid, 2001, [5]] *Suppose that M is a smooth quasi-projective variety, $G \subseteq \mathrm{Aut}(M)$ is a finite group of automorphism of M and that:*

(i) *M/G is Gorenstein*

(ii) $\dim Y \times_{M/G} Y \leq \dim Y + 1$.

Then $Y = \mathrm{Hilb}^G(M)$ is a crepant resolution of M/G and the Fourier-Mukai functor:

$$\mathbf{\Phi} := \mathbf{R}p_* \circ q^* = \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M) \tag{7.1}$$

is an equivalence between the bounded derived category of coherent sheaves on Y and the bounded derived category of G-equivariant coherent sheaves on M.

Remark 7.6 The derived equivalence (7.1) was first proved by Kapranov and Vasserot [25] in the classical case of McKay correspondence, $M = \mathbb{C}^2$, $G \subseteq SL(2, \mathbb{C})$. The key point in the proof is in any case the use of Nakamura G-Hilbert scheme.

Remark 7.7 The theorem 7.5 implies that the geometric McKay correspondence holds for three dimensional quotient singularities $\mathbb{C}^3/G$, with $G \subseteq SL(3, \mathbb{C})$. Already in dimension 4, it can be seen that the hypothesis of theorem 7.5 are not always verified. Some quotients $\mathbb{C}^4/G$ do not admit any crepant resolution [35]. In general, the conjectural equivalence $\mathbf{D}^b(Y) \longrightarrow \mathbf{D}_G(\mathbb{C}^n)$ if Y is a crepant resolution of $\mathbb{C}^n/G$, $G \subseteq SL(n, \mathbb{C})$, is called the *derived McKay correspondence conjecture.*

8 Applications and new directions

8.1 Haiman's work

In order to study the $n!$ conjecture, Haiman worked out the situation of the action of the symmetric group on the n-cartesian product of a smooth surface. Let X a smooth quasi-projective surface and consider the cartesian product X^n. The symmetric variety S^nX is the quotient $X^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group. Consider the Hilbert scheme $X^{[n]}$, parametrizing length n-subschemes of X. The Hilbert-Chow morphism $\mu : X^{[n]} \longrightarrow S^nX$, is defined as $\mu(\xi) = \sum_{x \in X} \mathrm{length}(\mathcal{O}_{\xi,x})x$. The following facts are well known.

(i) The symmetric variety S^nX has rational singularities (see [6], [3]).

(ii) By a theorem of Fogarty [19], the Hilbert scheme $X^{[n]}$ is smooth of dimension $2n$ and the Hilbert-Chow morphism provides a crepant resolution of singularities of S^nX.

(iii) S^nX is Gorenstein, since the stabilizer of any point x is a subgroup of $SL(T_xX^n)$ and hence the canonical bundle ω_{X^n} is locally trivial as $\mathfrak{S}_n$-sheaf.

(iv) μ is a semismall resolution. This follows from works of Briançon [4], Ellingsrud-Stromme [17] or Ellingsrud-Lehn [16] and the stratification of S^nX in terms of partitions of n.

We remark that we are in a situation very similar to the classical McKay correspondence. Moreover almost all the hypothesis of Bridgeland-King-Reid theorem are verified, since S^nX is Gorenstein and the Hilbert-Chow morphism is semismall, and hence $\dim X^{[n]} \times_{S^nX} X^{[n]} \leq \dim X^{[n]} + 1$. It remains to compare the Nakamura $\mathfrak{S}_n$-Hilbert scheme $Y = \mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ with the Hilbert scheme of points $X^{[n]}$ and to understand what is the universal $\mathfrak{S}_n$-cluster $\mathcal{Z}$. It turns out that the right universal family of $\mathfrak{S}_n$-clusters is provided by Haiman's *isospectral Hilbert scheme.*

Definition 8.1 Let X a smooth quasi-projective algebraic surface. The *isospectral Hilbert scheme* B^n of n points on the surface X is the *reduced* fibered product

$$B^n := (X^{[n]} \times_{S^nX} X^n)_{\mathrm{red}} \, .$$

As a consequence we are given a noncartesian diagram:

$$\begin{tikzcd} B^n \arrow[r, "p"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^n X \end{tikzcd}$$

with p birational and q finite. Haiman proves the following

Theorem 8.2 [Haiman, 2001, [21]]

(1) *The isospectral Hilbert scheme is irreducible of dimension* $2n$ *and can be identified with the blow-up of the union of the pairwise diagonals* $\cup_{i<j}\Delta_{ij}$ *in* X^n*:*

$$B^n \simeq \mathrm{Bl}_{\cup_{i<j}\Delta_{ij}} X^n.$$

(2) *The isospectral Hilbert scheme* B^n *is normal, Cohen-Macauley and Gorenstein.*

Remark 8.3 The Cohen-Macauley property implies the flatness of the morphism q, since a finite surjective morphism between a Cohen-Macauley variety and a smooth one is necessarily flat ([15], chapter 18). Consequently the morphism $q : B^n \longrightarrow X^{[n]}$ is flat and finite of degree $n!$. Moreover, the isospectral Hilbert scheme B^n inherits a $\mathfrak{S}_n$-action, since it is the blow-up of X^n along a closed $\mathfrak{S}_n$-invariant subscheme. This fact implies that B^n is a flat family of $\mathfrak{S}_n$-cluster and gives origin to a map: $\varphi : X^{[n]} \longrightarrow \mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$, which allows to compare the two Hilbert schemes. It is now easy to prove that φ is an isomorphism, that is, *the Hilbert scheme* $X^{[n]}$ *can be identified with the Nakamura G-Hilbert scheme* $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ and B^n can be identified with the universal $\mathfrak{S}_n$-cluster $\mathcal{Z}$.

The important consequence is that the Bridgeland-King-Reid theorem works in the situation of diagram (8.1):

Corollary 8.4 *The Fourier-Mukai functor:*

$$\mathbf{\Phi} = \mathbf{R}p_* \circ q^* : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

is an equivalence of derived categories.

8.2 Cohomology of representations of tautological bundles

Let X a smooth quasi-projective algebraic surface and let L a line bundle on X. Let $\Xi \subseteq X^{[n]} \times X$ the universal subscheme. It is flat and finite over $X^{[n]}$ of degree n.

Definition 8.5 The tautological bundle over $X^{[n]}$ associated to the line bundle L is the rank n vector bundle:

$$L^{[n]} := (p_{X^{[n]}})_* p_X^* L$$

where $p_{X^{[n]}}$ and p_X are the projections of Ξ over $X^{[n]}$ and over X respectively.

Tautological bundles on Hilbert schemes are interesting for many reasons; they play an important role in the topology of $X^{[n]}$, since their Chern classes are important for understanding the structure of the cohomology ring $H^*(X^{[n]}, \mathbb{Q})$ [28]; moreover in many occasions cohomology computations on moduli spaces of sheaves on surfaces can be reduced to cohomology computations on Hilbert schemes of points [9], [10], [29], [31], [30] where the knowledge of the behaviour of tautological bundles can be necessary.

Notation 8.6 Let $\emptyset \neq I \subseteq \{1, \ldots, n\}$ a multi-index; we denote with $p_I : X^n \longrightarrow X^I$ the projection onto the factors in I; let $i_I : X \hookrightarrow X^I$ the diagonal immersion. We denote with L_I the sheaf on X^n defined by: $L_I = p_I^*(i_I)_* L$: it is supported on the partial diagonal Δ_I. Denote with $\mathcal{C}^\bullet_L$ the complex:

$$0 \longrightarrow \oplus_{i=1}^n L_i \longrightarrow \oplus_{|I|=2} L_I \longrightarrow \ldots \longrightarrow L_{\{1,\ldots,n\}} \longrightarrow 0 \; ,$$

where $\oplus_{|I|=p+1} L_I$ is placed in degree p and where the arrows are given by Čech-like restrictions. It is exact in degree $\neq 0$. The group $\mathfrak{S}_n$ acts naturally on each factor $\mathcal{C}^p_L = \oplus_{|I|=p+1} L_I$, making the complex $\mathfrak{S}_n$-equivariant.

As a consequence of corollary 8.4, the cohomology of the Hilbert scheme $H^*(X^{[n]}, F)$ with values in any coherent sheaf F can be obtained as the $\mathfrak{S}_n$-equivariant hypercohomology $\mathbb{H}^*_{\mathfrak{S}_n}(X, \mathbf{\Phi}(F))$ on X^n with values in the image $\mathbf{\Phi}(F)$ of F for the Bridgeland-King-Reid equivalence. We proved

Theorem 8.7 [Scala, 2009, [36]] *The image of the tautological bundle $L^{[n]}$ via the BKR equivalence is*

$$\mathbf{\Phi}(L^{[n]}) \simeq \mathcal{C}^\bullet_L$$

in the $\mathfrak{S}_n$-equivariant derived category $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$. Moreover there is a natural morphism

$$\underbrace{\mathcal{C}_L{}^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_L{}^\bullet}_{l\text{-times}} \longrightarrow \mathbf{\Phi}(L^{[n]^{\otimes l}})$$

whose mapping cone is acyclic in degree > 0. This means that $R^q p_* q^*(L^{[n]^{\otimes l}}) = 0$ for all $q > 0$ and in degree zero the morphism: $p_* q^*(L^{[n]})^{\otimes l} \longrightarrow p_* q^*(L^{[n]^{\otimes l}})$ is surjective, the kernel being the torsion subsheaf.

As a consequence, the sheaf $\mathbf{\Phi}(L^{[n]^{\otimes l}}) \simeq p_* q^*(L^{[n]^{\otimes l}})$ can be identified with the $E^{0,0}_\infty$ term of the hyperderived spectral sequence $E^{p,q}_1 = \oplus_{i_1+\cdots+i_l=p} \mathrm{Tor}_{-q}(\mathcal{C}^{i_1}_L, \ldots, \mathcal{C}^{i_l}_L)$, associated to the l-fold derived tensor product $\mathcal{C}_L{}^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_L{}^\bullet$. Working out the term $E^{0,0}_\infty$ of the spectral sequence in all generality is hard, due to evident technical difficulties. Nonetheless, for applications to computations of equivariant cohomology, all we really need is the knowledge of the $\mathfrak{S}_n$-*invariants* $\mathbf{\Phi}((E^{[n]})^{\otimes l})^{\mathfrak{S}_n}$ of the image $\mathbf{\Phi}((E^{[n]})^{\otimes l})$, which can be obtained as the term $\mathcal{E}^{0,0}_\infty$ of the spectral sequence $\mathcal{E}^{p,q}_1 = (E^{p,q}_1)^{\mathfrak{S}_n}$ of invariants of the original spectral sequence $E^{p,q}_1$. In some lucky cases the new spectral sequence $\mathcal{E}^{p,q}_1$ degenerate at level E_2, and provides the following results.

Theorem 8.8 [Scala, 2009, [36]]

(i) *Let $a \in X$ and let $\mathcal{J}$ the kernel of the morphism $S^{n-1}H^*(\mathcal{O}_X) \longrightarrow S^{n-2}H^*(\mathcal{O}_X)$ induced by the morphism $S^{n-2}X \longrightarrow S^{n-1}X$ sending x to $a+x$. The cohomology of the double tensor power $L^{[n]} \otimes L^{[n]}$ of a tautological bundle is isomorphic to*

$$H^*(X^{[n]}, L^{[n]} \otimes L^{[n]}) \simeq H^*(L^{\otimes 2}) \otimes \mathcal{J} \oplus H^*(L)^{\otimes 2} \otimes S^{n-2}H^*(\mathcal{O}_X)$$

as $\mathbb{Z}$-graded modules and $\mathfrak{S}_2$-representations.

(ii) *The cohomology of the general exterior power $\Lambda^k L^{[n]}$ is isomorphic to*

$$H^*(X^{[n]}, \Lambda^k L^{[n]}) \simeq \Lambda^k H^*(L) \otimes S^{n-k}H^*(\mathcal{O}_X)\ .$$

8.3 Conclusions

There are many other aspects of McKay correspondence that we could not touch, in connection with valuation theory, string theory, motivic integration, noncommutative geometry, perverse sheaves, Gromov-Witten invariants and quantum cohomology, Donaldson-Thomas invariants, orbifolds, mirror symmetry, for example. See [35].

Research in McKay correspondence is still extremely active: we just mention the crepant resolution conjecture of Chen-Ruan [7], and the derived McKay correspondence conjecture. For the latter it seems that an encouraging direction is the use of moduli spaces of representations of the McKay quiver. See [8].

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Liouville-type results for linear elliptic operators

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Abstract. This talk deals with some extensions of the classical Liouville theorem about bounded harmonic functions to solutions of more general elliptic partial differential equations. In the first part, we will introduce the only two technical tools needed to prove the Liouville-type result in the case of periodic elliptic operators: the maximum principle and Schauder's a priori estimates. Next, we will discuss the role of the periodicity assumption, seeing what happens if one replaces it with almost periodicity.

1 Introduction

Partial differential equations (PDE's) are equations of the form

$$Lu = f \quad \text{in } \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^N$, N integer, $f : \Omega \to \mathbb{R}$ is a given function and L is an operator involving an unknown function $u : \Omega \to \mathbb{R}$ and its partial derivatives. The order of a PDE coincides with the highest order of derivative appearing in the equation. Thus, a general second order *linear* equation (that is, with L linear) can be written as

$$a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u = f(x), \quad x \in \Omega, \tag{2}$$

(the convention is adopted for summation from 1 to N on repeated indices) where $(a_{ij})_{i,j}$, $(b_i)_i$ and c are respectively a given matrix, vector and scalar field on Ω. Such equation is said to be *uniformly elliptic* if the matrix field $(a_{ij})_{i,j}$ is symmetric and there exists a positive constant λ such that

$$\forall\, x \in \Omega,\ \xi = (\xi_1, \dots \xi_N) \in \mathbb{R}^N, \quad a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2. \tag{3}$$

Throughout the paper, we always assume that L is a second order linear operator satisfying (3). The basic example to have in mind is $L = \Delta$.

PDE's have been introduced in the 18th century to model physical phenomena. Starting from the work of Euler, d'Alembert, Lagrange and Laplace, the theory of PDE's has

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known a great expansion and represents nowadays one of the most prolific subjects of research in mathematics. Their interest is due to the applications not only to the description of the real world, but also to different fields of pure mathematics, such as differential and algebraic geometry. The complete and rigorous treatment of elliptic PDE's is essentially based on two tools: the maximum principle, first proved by Hopf in the 1920s, and the apriori estimates, initiated by Schauder and Caccioppoli in the 1930s. Though almost everything is known today about second order linear elliptic equations in bounded domains, several very natural questions are still open in the unbounded case.

A very classical result due to Liouville is that if u is a bounded solution of $\Delta u = 0$ in $\mathbb{R}^N$ then u has to be constant. In other words, the only bounded harmonic functions in the whole space are the constants. This fact immediately follows from the fact that harmonic functions satisfy the mean value property. Here, we address the following question: does the same result hold if one replaces Δ with a general second order uniformly elliptic linear operator L ? The answer is no, if one does not impose some conditions on L, as it is shown by the equation $u'' - u$ in $\mathbb{R}$, whose space of solutions is generated by $u_1 = \sin x$ and $u_2 = \cos x$. We will show that the answer is yes if the coefficients of L are periodic in all the variables, with the same periods $l_1, \dots l_N$, and $c \leq 0$. We next show, with an explicit counterexample, that the periodicity assumption cannot be relaxed.

The main difficulty with respect to the classical Liouville theorem is that, for general L, we do not have an analogue of the mean value property. Thus, a completely different approach is required to treat the equation. This will be based on two of the most important tools in the theory of elliptic PDE's: the maximum principle and the apriori estimates.

In Section 2, we prove the maximum principle and some of its consequences. Apriori estimates are discussed in Section 3. In Section 4, we derive the Liouville-type result for periodic operators. In the last section, we exhibit the counter-example in the case of almost-periodic operators. Two exhaustive references for the material contained in the first two sections are [3] (for Section 2) and [2] (Chapter 3 for Section 2 and Chapter 6 for Section 3. For the last two sections, we refer to [4], where the results are derived in the more general framework of parabolic equations and strong solutions in the sense of Sobolev spaces. Though some technical difficulties arise due to the larger generality considered in [4], the basic ideas of some of the proofs contained there can be found in these notes.

2 Maximum principle

Theorem 2.1 [Weak maximum principle] *Let Ω be a bounded domain and L be a uniformly elliptic operator with bounded coefficients and $c \leq 0$. Then, any function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $Lu \geq 0$ in Ω satisfies*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

Proof. The proof is divided into two parts.
Step 1: the conclusion holds if $Lu > 0$ in Ω.

If the above statement does not hold then there exists a point $x_0 \in \Omega$ where u achieves its nonnegative maximum. Then, we get

$$0 < Lu(x_0) = a_{ij}(x_0)\partial_{ij}u(x_0) + c(x_0)u(x_0) \leq a_{ij}(x_0)\partial_{ij}u(x_0).$$

This is impossible, because the latter term is the trace of the product between $(a_{ij}(x_0))_{i,j}$, which is nonnegative definite, and the hessian of u at x_0, which is nonpositive definite.
Step 2: the general case.
For $\beta > 0$, define the following function: $v(x) := e^{\beta x_1}$. Direct computation yields

$$\forall\, x \in \Omega, \quad Lv(x) = a_{11}(x)\beta^2 + b_1(x)\beta + c(x) \geq \lambda\beta^2 + b_1(x)\beta + c(x).$$

Hence, we can chose β large enough in such a way that $Lv > 0$ in Ω. As a consequence, for any $\varepsilon > 0$, the function $u_\varepsilon := u + \varepsilon v$ satisfies $Lu_\varepsilon > 0$ in Ω. Applying the step 1 we then derive

$$\sup_\Omega u \leq \sup_\Omega u_\varepsilon \leq \sup_{\partial\Omega} u_\varepsilon^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon \sup_{\partial\Omega} v.$$

The result follows by letting ε go to 0. □

Let us point out that the uniform ellipticity of L in the above result can be slightly weakened (cf. [2], [3]).

An immediate consequence of the weak maximum principle is the uniqueness result for the Dirichlet problem.

Corollary 2.2 *Under the assumptions of Theorem 2.1, the problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

admits at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

The weak maximum principle does not prevent u from having a nonnegative interior maximum. However, if it is the case then u is necessarily constant. This can be proved by use of the Hopf lemma (see Lemma 3.4 in [2]) which, in turns, is a consequence of the weak maximum principle.

Theorem 2.3 [Strong maximum principle] *Let Ω be a general domain and L be a uniformly elliptic operator with bounded coefficients and $c \leq 0$. Then, any function $u \in C^2(\Omega)$ such that $Lu \geq 0$ in Ω cannot achieve a nonnegative maximum in Ω unless it is constant.*

3 A priori estimates

From now on, we assume that the coefficients of L and the function f are uniformly Hölder continuous:

$$a_{ij}, b_i, c, f \in C^{0,\alpha}(\overline{\Omega}),$$

for some $\alpha \in (0, 1)$.

Theorem 3.1 [Schauder's interior estimates] *Let Ω be an open set and $u \in C^2(\Omega)$ be a bounded solution of (1). Then, for any Ω' compactly contained in Ω, there exists a positive constant C, only depending on the coefficients of L and the distance between Ω^c and Ω', such that*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)}).$$

The proof of the above result can be found in [2], Theorem 6.2 and Lemma 6.16.

Corollary 3.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of solutions of (1) uniformly bounded in $L^\infty(\Omega)$. Then, $(u_n)_{n\in\mathbb{N}}$ converges, up to subsequences, in $C^2_{loc}(\Omega)$ to a solution u^* of (1).*

Proof. Let K be a compact subset of Ω. Theorem 3.1 implies that the u_n are uniformly bounded in $C^{2,\alpha}(K)$. Thus, their derivatives up to the second order are equicontinuous in K. Applying the Arzela-Ascoli theorem we then infer that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging in $C^2(K)$ to a function depending on K. Since this holds for any compact $K \subset \Omega$, considering an increasing sequence of compact sets invading Ω and then using a diagonal extraction, we can find a common subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to a function u^* in $C^2_{loc}(\Omega)$. Eventually, u^* satisfies (1). □

4 Periodic operators

We consider now the homogeneous equation

$$Lu = 0 \quad \text{in } \mathbb{R}^N. \tag{4}$$

The coefficients a_{ij}, b_i, c of L are always assumed to be Hölder continuous. We say that L is periodic if there exist N positive constants $l_1, \dots, l_N$ such that

$$\forall\, k \in \{1, \dots, N\},\ x \in \mathbb{R}^N, \quad a_{ij}(x + l_k e_k) = a_{ij}(x),$$

$$b_i(x + l_k e_k) = b_i(x), \quad c(x + l_k e_k) = c(x),$$

where $\{e_1, \cdots, e_N\}$ is the canonical basis of $\mathbb{R}^N$.

Theorem 4.1 *Let L be a periodic operator with $c \leq 0$. Then, any bounded solution of (4) is necessarily constant.*

Proof. Let u be a bounded solution of (4). We proceed in two steps.
Step 1: u is periodic.
Fix $k \in \{1, \dots, N\}$ and define the following function: $\psi(x) := u(x + l_k e_k) - u(x)$. Assume by way of contradiction that $M := \sup_{\mathbb{R}^N} \psi > 0$. Let $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ be such that $\lim_{n\to\infty} \psi(x_n) = M$. For $n \in \mathbb{N}$, let $z_n \in \prod_{i=1}^N l_i\mathbb{Z}$ satisfy $y_n := x_n - z_n \in [0, l_1) \times \cdots \times [0, l_N)$. We have that, up to subsequences, $(y_n)_{n\in\mathbb{N}}$ converges to a point $\overline{y} \in [0, l_1] \times \cdots \times [0, l_N]$. Define the sequence of functions $(u_n)_{n\in\mathbb{N}}$ by setting $u_n(x) := u(x + z_n)$. Clearly, the u_n satisfy (4). Thus, Corollary 3.2 implies that they converge in $C^2_{loc}(\mathbb{R}^N)$ to a bounded

solution u^* of (4). The function $\psi^*(x) := u^*(x + l_k e_k) - u^*(x)$ is a solution of (4) too. Moreover, it is bounded above by M and satisfies

$$\psi^*(\overline{y}) = \lim_{n\to\infty}(u_n(y_n + l_k e_k) - u_n(y_n)) = \lim_{n\to\infty}\psi(x_n) = M.$$

That is, ψ^* achieves a positive maximum in $\mathbb{R}^N$. Theorem 2.3 then yields $\psi^* \equiv M$. As a consequence,

$$\lim_{n\to\infty} u^*(nl_k e_k) = \lim_{n\to\infty}(u^*(0) + nM) = +\infty,$$

which is impossible. This shows that $u(x+l_k e_k) \le u(x)$ for $x \in \mathbb{R}^N$. The reverse inequality, and then the periodicity of u, is obtained by replacing u with $-u$.
Step 2: u is constant.
Being periodic, u achieves its maximum and minimum on $\mathbb{R}^N$. Up to replace u with $-u$ if need be, we can assume that the maximum of u is nonnegative. Applying once again Theorem 2.3 we infer that u is constant. □

We remark that if $L = \Delta$ then the Liouville theorem holds for solutions which are only bounded from one side. Namely, an harmonic function in the whole space which is bounded above or below is necessarily constant. Instead, the two-sides boundedness is required for general L, as it is shown for example by the function $u(x) = e^x$ which satisfies $u'' - u = 0$ in $\mathbb{R}$.

5 Almost periodic operators

Concerning the sharpness of the hypotheses of Theorem 4.1, we have seen in the introduction that the condition $c \le 0$ cannot be dropped. Let us mention that it can be relaxed by requiring that the *periodic principal eigenvalue* of $-L$ is nonnegative (see [4]). We now discuss the sharpness of the periodicity assumption.

A natural generalization of periodic functions of a single real variable are *almost periodic* functions, introduced by Bohr in 1925. This notion can be readily extended to functions of several variables through a characterization of **continuous** almost periodic functions due to Bochner. A comprensive treatment of almost periodic functions can be found in the book of Fink [1].

Definition 5.1 We say that a function $f \in C^0(\mathbb{R}^N)$ is almost periodic (a. p.) if from any arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ can be extracted a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(f(x + x_{n_k}))_{k\in\mathbb{N}}$ converges uniformly in $x \in \mathbb{R}^N$.

It is straightforward to check that continuous periodic functions are a. p. (this is no longer true if we drop the continuity assumption). We say that a linear operator is a. p. if its coefficients are a. p.

By explicitly constructing a counter-example, we show below that the Liouville type result of Theorem 4.1 does not hold in general if we require the operator to be only a. p.

Counter-example 1 There exists an a. p. function $b : \mathbb{R} \to \mathbb{R}$ such that the space of bounded solutions to

$$u'' + b(x)u' = 0 \quad \text{in } \mathbb{R} \tag{5}$$

has dimension 2, and it is generated by the function $u_1 \equiv 1$ and a function u_2 which is not a. p..

The reason why the Liouville type result fails in the a. p. case is that bounded solutions of a. p. equations with nonpositive zero order term may not be a. p. Indeed, it is not hard to prove that, for an a. p. operator L with $c \leq 0$, the unique a. p. solutions of (4) are the constants.

Actually, the function b in Counter-example 1 is *limit periodic*, in the sense of the following definition.

Definition 5.2 We say that a function $f \in C^0(\mathbb{R}^N)$ is limit periodic if there exists a sequence of continuous periodic functions converging uniformly to f in $\mathbb{R}^N$.

Limit periodic functions are a subset of a. p. functions because, as it is easily seen from Definition 5.1, the space of a. p. functions is closed with respect to the L^∞ norm.

Let us prove Counter-example 1. To do this, we first construct a discontinuous function σ, then we modify it to obtain a Lipschitz continuous limit periodic function b.

We start defining σ on the interval $(-1, 1]$ by setting

$$\sigma(x) = \begin{cases} -1 & \text{if } -1 < x \leq 0, \\ 1 & \text{if } 0 < x \leq 1. \end{cases}$$

Then, in $(-3, 3]$:

$$\forall\, x \in (-3, -1], \qquad \sigma(x) = \sigma(x+2) - 1,$$

$$\forall\, x \in (1, 3], \qquad \sigma(x) = \sigma(x-2) + 1,$$

and, by iteration,

$$\forall\, x \in (-3^{n+1}, -3^n], \qquad \sigma(x) = \sigma(x + 2 \cdot 3^n) - \frac{1}{(n+1)^2}, \tag{6}$$

$$\forall\, x \in (3^n, 3^{n+1}], \qquad \sigma(x) = \sigma(x - 2{\cdot}3^n) + \frac{1}{(n+1)^2}. \tag{7}$$

By construction, the function σ satisfies $\|\sigma\|_{L^\infty(\mathbb{R})} = 1 + \sum_{n=1}^{\infty} n^{-2}$, and it is odd except for the set $\mathbb{Z}$, in the sense that $\sigma(-x) = -\sigma(x)$ for $x \in \mathbb{R} \backslash \mathbb{Z}$.

Proposition 5.3 *There exists a sequence of bounded periodic functions $(\phi_n)_{n \in \mathbb{N}}$ converging uniformly to σ in $\mathbb{R}$ and such that*

$$\forall\, n \in \mathbb{N}, \qquad \phi_n \in C(\mathbb{R} \backslash \mathbb{Z}), \qquad \phi_n \textit{ has period } 2{\cdot}3^n.$$

Proof. Fix $n \in \mathbb{N}$. For $x \in (-3^n, 3^n]$ set $\phi_n(x) := \sigma(x)$, then extend ϕ_n to the whole real line by periodicity, with period $2{\cdot}3^n$. We claim that

$$\|\sigma - \phi_n\|_{L^\infty(\mathbb{R})} \le \sum_{k=n+1}^{\infty} \frac{1}{k^2},$$

which would conclude the proof. We prove our claim by a recursive argument, showing that the property

$$(\mathcal{P}_i) \qquad \forall\, x \in (-3^{n+i}, 3^{n+i}], \qquad |\sigma(x) - \phi_n(x)| \le \sum_{k=n+1}^{n+i} \frac{1}{k^2}$$

holds for every $i \in \mathbb{N}$. Let us check $(\mathcal{P}_1)$. By (6) and (7) we get

$$\sigma(x) = \begin{cases} \sigma(x + 2{\cdot}3^n) - \dfrac{1}{(n+1)^2} & \text{if } -3^{n+1} < x \le -3^n \\ \phi_n(x) & \text{if } -3^n < x \le 3^n \\ \sigma(x - 2{\cdot}3^n) + \dfrac{1}{(n+1)^2} & \text{if } 3^n < x \le 3^{n+1}\ . \end{cases}$$

Property $(\mathcal{P}_1)$ then follows from the periodicity of ϕ_n.

Assume now that $(\mathcal{P}_i)$ holds for some $i \in \mathbb{N}$. Let $x \in (-3^{n+i+1}, 3^{n+i+1}]$. If $x \in (-3^{n+i}, 3^{n+i}]$ then

$$|\sigma(x) - \phi_n(x)| \le \sum_{k=n+1}^{n+i} \frac{1}{k^2} \le \sum_{k=n+1}^{n+i+1} \frac{1}{k^2}.$$

Otherwise, set

$$y := \begin{cases} x + 2{\cdot}3^{n+i} & \text{if } x < 0 \\ x - 2{\cdot}3^{n+i} & \text{if } x > 0\ . \end{cases}$$

Note that $y \in (-3^{n+i}, 3^{n+i}]$ and $|x - y| = 2{\cdot}3^{n+i}$. Thus, (6), (7), $(\mathcal{P}_i)$ and the periodicity of ϕ_n yield

$$\begin{aligned} |\sigma(x) - \phi_n(x)| &\le |\sigma(x) - \sigma(y)| + |\sigma(y) - \phi_n(y)| \\ &\le \frac{1}{(n+i+1)^2} + \sum_{k=n+1}^{n+i} \frac{1}{k^2} \\ &= \sum_{k=n+1}^{n+i+1} \frac{1}{k^2}. \end{aligned}$$

This means that $(\mathcal{P}_{i+1})$ holds and then the proof is concluded. $\square$

Note that σ is not limit periodic because it is discontinuous on $\mathbb{Z}$.

Proposition 5.4 *The function σ satisfies*

$$(8) \qquad \forall\, x \ge 1, \qquad \int_0^x \sigma(t)dt \ge \frac{x}{2(\log_3 x + 1)^2}.$$

Proof. For $y \in \mathbb{R}$, define $F(y) := \int_0^y \sigma(t)dt$. Let us preliminarily show that, for every $n \in \mathbb{N}$, the following formula holds:

$$\forall\, y \in [0, 3^n], \qquad F(y) \geq \frac{y}{2n^2}. \tag{9}$$

We shall do it by iteration on n. It is immediately seen that (9) holds for $n = 1$. Assume that (9) holds for some $n \in \mathbb{N}$. We want to prove that (9) holds with n replaced by $n+1$. If $y \in [0, 3^n]$ then

$$F(y) \geq \frac{y}{2n^2} \geq \frac{y}{2(n+1)^2}.$$

If $y \in (3^n, 2{\cdot}3^n]$ then, by computation,

$$F(y) = F(2{\cdot}3^n - y) + \int_{2\cdot 3^n - y}^{y} \sigma(t)dt \geq \frac{2{\cdot}3^n - y}{2n^2} + \int_{-(y-3^n)}^{y-3^n} \sigma(\tau + 3^n)d\tau.$$

Using property (7), one sees that

$$\begin{aligned}\int_{-(y-3^n)}^{y-3^n} \sigma(\tau + 3^n)d\tau &= \int_{-(y-3^n)}^{0} \sigma(\tau + 3^n)d\tau + \int_0^{y-3^n} \sigma(\tau - 3^n)d\tau + \frac{y - 3^n}{(n+1)^2} \\ &= \frac{y - 3^n}{(n+1)^2}\ ,\end{aligned}$$

where the last equality holds because σ is odd except in the set $\mathbb{Z}$. Hence,

$$F(y) \geq \frac{2{\cdot}3^n - y}{2n^2} + \frac{y - 3^n}{(n+1)^2} \geq \frac{y}{2(n+1)^2}.$$

Let now $y \in (2{\cdot}3^n, 3^{n+1}]$. Since $F(2{\cdot}3^n) \geq 3^n(n+1)^{-2}$, as we have seen before, and (7) holds, it follows that

$$F(y) = F(2{\cdot}3^n) + \int_{2\cdot 3^n}^{y} \sigma(t)dt \geq \frac{3^n}{(n+1)^2} + F(y - 2{\cdot}3^n) + \frac{y - 2{\cdot}3^n}{(n+1)^2}.$$

Using the hypothesis (9) we then get

$$F(y) \geq \frac{y - 3^n}{(n+1)^2} + \frac{y - 2{\cdot}3^n}{2n^2} \geq \frac{y}{2(n+1)^2}.$$

We have proved that (9) holds for any $n \in \mathbb{N}$. Consider now $x \geq 1$. We can find an integer $n = n(x)$ such that $x \in [3^{n-1}, 3^n)$. Applying (9) we get $F(x) \geq x(2n^2)^{-1}$. Therefore, since $n \leq \log_3 x + 1$, we infer that

$$F(x) \geq \frac{x}{2(\log_3 x + 1)^2}.$$

□

In order to define the function b, we introduce the following auxiliary function $z \in C(\mathbb{R})$ vanishing on $\mathbb{Z}$: $z(x) := 2|x|$ if $x \in [-1/2, 1/2]$, and it is extended by periodicity with period 1 outside $[-1/2, 1/2]$. Then we set

$$b(x) := \sigma(x)z(x).$$

The definition of b is easier to understand by its graph (see Figure 1).

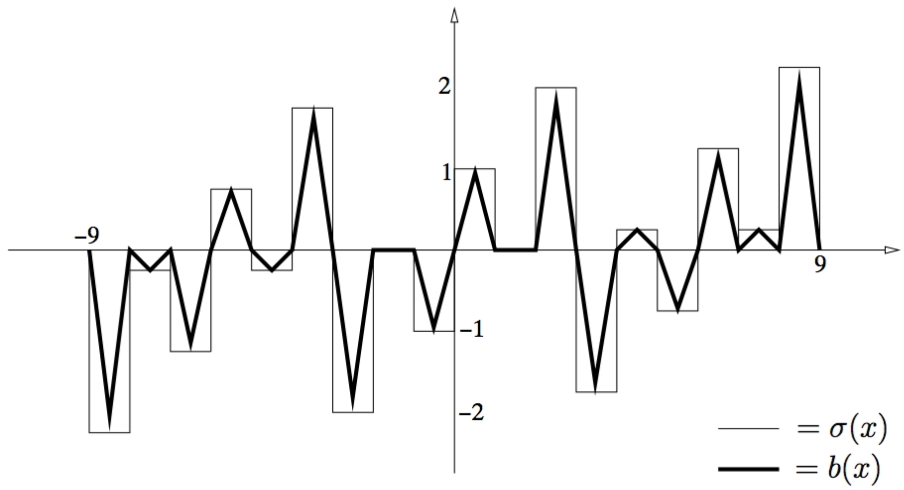

Figure 1: graphs of σ and b.

Proposition 5.5 *The function b is odd and limit periodic.*

Proof. Let us check that b is odd. For $x \in \mathbb{Z}$ we find $b(-x) = 0 = -b(x)$, while, for $x \in \mathbb{R} \backslash \mathbb{Z}$,

$$b(-x) = \sigma(-x)z(-x) = -\sigma(x)z(x) = -b(x).$$

In order to prove that b is limit periodic, consider the sequence of periodic functions $(\phi_n)_{n \in \mathbb{N}}$ given by Proposition 5.3. Then define

$$\psi_n(x) := \phi_n(x)z(x).$$

Clearly, the functions ψ_n are continuous (because z vanishes on $\mathbb{Z}$) and periodic, with period $2{\cdot}3^n$ (because z has period 1). Also, for $n \in \mathbb{N}$,

$$|b - \psi_n| = |\sigma - \phi_n| z \leq |\sigma - \phi_n|.$$

Therefore, ψ_n converges uniformly to b as n goes to infinity. $\square$

Proposition 5.6 *The solutions of (5) are generated by $u_1 \equiv 1$ and a non-a. p. bounded function u_2.*

Proof. The two generators of the space of solutions of (5) are $u_1 \equiv 1$ and

$$u_2(x) := \int_0^x \exp\left(-\int_0^y b(t)dt\right) dy.$$

Since u_2 is strictly increasing, it cannot be a. p. So, to prove the statement it only remains to show that u_2 is bounded. By construction, it is clear that, for $m \in \mathbb{Z}$,

$$\int_0^m b(t)dt = \frac{1}{2}\int_0^m \sigma(t)dt.$$

Consequently, by (8), we get for $x \geq 1$

$$\int_0^x b(t)dt = \frac{1}{2}\int_0^{[x]} \sigma(t)dt + \int_{[x]}^x b(t)dt \geq \frac{x-1}{4(\log_3 x + 1)^2} - \|b\|_{L^\infty(\mathbb{R})}$$

and then

$$\begin{aligned} 0 \leq u_2(x) &\leq e^{\|b\|_{L^\infty(\mathbb{R})}} \int_0^x \exp\left(-\frac{y-1}{4(\log_3 y + 1)^2}\right) dy \\ &\leq e^{\|b\|_{L^\infty(\mathbb{R})}} \int_0^{+\infty} \exp\left(-\frac{y-1}{4(\log_3 y + 1)^2}\right) dy. \end{aligned}$$

Since b is odd, it follows that u_2 is odd too and then it is bounded on $\mathbb{R}$. □

Remark 1 The function $b = \sigma z$ constructed above is uniformly Lipschitz continuous, with Lipschitz constant equal to $2\|\sigma\|_{L^\infty(\mathbb{R})}$. Actually, one could use a suitable C^∞ function instead of z in order to obtain a function $b \in C^\infty(\mathbb{R})$.

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Möbius function and probabilistic zeta function associated to a group

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Abstract. The study of the probabilistic zeta function is known as non commutative number theory of subgroups growth. To understand how the probabilistic zeta function is defined, it is necessary to introduce another function associated to a group: the Möbius function. We will start considering finite groups: we will explain how these two functions are obtained. Then we will define the profinite groups and proceed to investigate whether and how a probabilistic zeta function can be associated to them. We will present some recent results contained in [1].

1 Preliminaries

We start giving some preliminary definitions and properties; all the following statements on topological and profinite groups can be found in [7].

Let G be a group; we denote by $\mathcal{L}_G$ the set of all the subgroups of G. We consider on $\mathcal{L}_G$ the following relation:

$$H \leq K \text{ if and only if } H \subseteq K.$$

This is an *order relation* (i.e. it is reflexive, antisymmetric and transitive); then $\mathcal{L}_G$ is an ordered set.

Definition 1.1 A subgroup N of G is said *normal* if and only if $g^{-1}Ng = N$, for any $g \in G$. If G and $\langle 1 \rangle$ are the only normal subgroups of G then G is called *simple*.

Definition 1.2 A group G is *abelian* if and only if $gh = hg$, for any $h, g \in G$.

Example The Symmetric group $\mathrm{Sym}(n)$ is the group of all the permutations of the elements $\{1, \ldots, n\}$. The Alternating group $\mathrm{Alt}(n)$ is the set of all the permutations of $\mathrm{Sym}(n)$ which can be written as a product of an even number of cycles of length 2. $\mathrm{Alt}(n)$ is a normal subgroup of $\mathrm{Sym}(n)$, and

$$|\,\mathrm{Alt}(n)| = |\,\mathrm{Sym}(n)|/2.$$

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1.1 Topological groups

Definition 1.3 A *topological space* is a set X together with a family of subsets, called open sets, satisfying the following conditions:

i) the empty set and X are both open sets;

ii) the intersection of any two open sets is an open set;

iii) the union of any collection of open sets is an open set.

The set of open sets is called a *topology* on X. A subset of X is called closed if its complement is open.

Example Any set X may be regarded as a topological space with respect to the topology in which each subset is open (the *discrete* topology on X).

Definition 1.4 Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if for each open set U of Y the set $f^{-1}(U) = \{x \in X | \, f(x) \in U\}$ is open in X. A map $f : X \to Y$ is said to be a *homeomorphism* if it is bijective and if both f and f^{-1} are continuous.

Consider now a family $(X_\lambda | \, \lambda \in \Lambda)$ of topological spaces.

Definition 1.5 The *Cartesian product* of the family $(X_\lambda | \, \lambda \in \Lambda)$ is the topological space $C = \mathrm{Cr}(X_\lambda | \, \lambda \in \Lambda)$ whose elements are all the vectors (x_λ) with entries indexed by the elements of Λ. The projection map π_λ is the map which takes an element of C to its value at λ. The *product topology* on C has as its open sets all unions of sets of the form

$$\pi_{\lambda_1}^{-1}(U_1) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_n)$$

with n finite, each λ_i in Λ and U_i open in X_{λ_i}.

We are now able to define a topological group.

Definition 1.6 A *topological group* is a set G which is both a group and a topological space and for which the map $(x, y) \mapsto xy^{-1}$ from $G \times G$ (with the product topology) to G is continuous.

Example Any group G with the discrete topology may be regarded as a topological group.

Definition 1.7 Let G_1 and G_2 be topological groups. A map $f : G_1 \to G_2$ is an *isomorphism* of topological groups if it is an isomorphism of groups and a homeomorphism.

From the definition of the product topology, it follows immediately the following result.

Lemma 1.8 *Let $(G_\lambda | \, \lambda \in \Lambda)$ be a family of topological groups, and write*

$$C = \mathrm{Cr}(G_\lambda | \, \lambda \in \Lambda).$$

Define multiplication in C pointwise (so that $(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda)$ for all $(x_\lambda), (y_\lambda)$ in C). With respect to this multiplication and the product topology, C becomes a topological group.

1.2 Profinite groups: some definitions

Now we introduce a particular class of infinite groups: the profinite groups. These groups will be very useful in our argumentations.

Definition 1.9 Let G be a topological group. G is a *profinite* group if and only if it is isomorphic (as a topological group) to a closed subgroup of a Cartesian product of finite groups.

Example The group $G \cong \mathrm{Cr}(\mathrm{Alt}(n) | \, n \in \mathbb{N})$ is a profinite group (any Alternating group $\mathrm{Alt}(n)$ is considered with the discrete topology).

Proposition 1.10 *If G is a profinite group, then $G \cong \varprojlim G/N$, where N ranges over all normal subgroups of G of finite index.*

Remark 1.11 We observe that, using the previous proposition, the study of a profinite group G can be completely reduced to the study of its finite epimorphic images, that is finite groups.

Let G be a profinite group. When we talk about *generators* of a profinite group, we mean generators as a topological group. Then a set X generates *topologically* G if and only if G is the minimal closed subgroup of G containing X, that is G coincides with the closure of the abstract subgroup generated by X.

Definition 1.12 The closure in G of a subset X is $\overline{X} = \cap XN$, with N ranging over all open normal subgroups of G.

Then it is easily proved the following:

Proposition 1.13 *X generates G if and only if each finite factor group G/N is generated by XN/N.*

It follows:

Proposition 1.14 *G is finitely generated, by d elements, if and only if each finite factor group G/N can be generated by d elements.*

2 The Möbius function on a poset

Let $(P, \leq)$ be a partially ordered set (*poset*), with $\leq$ an order relation in P; moreover we require that P is *locally finite*, that is each interval in P is finite.

Definition 2.1 The *Möbius function* μ on $P \times P$ is defined as follows:

$$\mu(x,y) = \begin{cases} 0 & \text{if } x \not\leq y \\ 1 & \text{if } x = y \\ -\sum\limits_{x<z\leq y} \mu(z,y) & \text{if } x < y \end{cases}$$

Example The set of natural numbers $\mathbb{N}$ with the divisibility relation ($a \leq b$ if and only if $a|b$) is a locally finite poset. Fix $y = 8$:

$$\mu(x,8) = \begin{cases} 0 & \text{if } x \not| \, 8 \\ 1 & \text{if } x = 8 \\ -\sum\limits_{\substack{z \neq x \\ x|z \text{ and } z|8}} \mu(z,8) & \text{if } x|8 \text{ and } x \neq 8 \end{cases}$$

Then: $\mu(4,8) = -\mu(8,8) = -1$,
$\mu(2,8) = -(\mu(4,8) + \mu(8,8)) = 0$,
$\mu(1,8) = -(\mu(2,8) + \mu(4,8) + \mu(8,8)) = 0$.

Remark 2.2 In particular $\mu(1,n)$, for any $n \in \mathbb{N}$, coincides exactly with the value $\mu(n)$ of the Möbius function, as defined in Number Theory.

3 Finite groups

Let G be a *finite* group. Then the poset $\mathcal{L}_G$ is finite, and we may define the Möbius function on $\mathcal{L}_G$ as follows (see Definition 2.1): for any $H, K \leq G$

$$\mu(H,K) = \begin{cases} 0 & \text{if } H \not\leq K \\ 1 & \text{if } H = K \\ -\sum\limits_{H<Z\leq K} \mu(Z,K) & \text{if } H < K \end{cases}$$

In particular if $K = G$, for any $H \leq G$, we obtain

$$\mu(H,G) = \begin{cases} 1 & \text{if } H = G \\ -\sum\limits_{H<Z\leq G} \mu(Z,G) & \text{if } H < G \end{cases}$$

We call $\mu(H,G)$ the *Möbius number* of H in G.

Example Consider $G = \mathrm{Sym}(3) = \{\mathbb{1}, (12), (13), (23), (123), (132)\}$ and $\mathcal{L}_G$:

$$\begin{tikzcd}[arrows=dash]
& \mathrm{Sym}(3) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\langle(123)\rangle \arrow[dr] & \langle(12)\rangle \arrow[d] & \langle(13)\rangle \arrow[dl] & \langle(23)\rangle \arrow[dll] \\
& \langle\mathbb{1}\rangle & &
\end{tikzcd}$$

Using the above formula, we are able to calculate the Möbius number of each subgroup in G; obviously $\mu(G, G) = 1$. The Möbius number of each maximal subgroup of G is -1, then we obtain: $\mu(\langle(123)\rangle, G) = \mu(\langle(12)\rangle, G) = \mu(\langle(13)\rangle, G) = \mu(\langle(23)\rangle, G) = -1$. Finally we have $\mu(\langle\mathbb{1}\rangle, G) = 3$.

3.1 The probability $P_G(t)$

We denote by $P_G(t)$ the *probability* that t randomly chosen elements of G generate G itself. To establish this probability, in the finite case, we have simply to calculate the number of systems of generators of G.

Definition 3.1 Define the function

$$\begin{array}{rcl} \phi_G : \mathbb{N} & \to & \mathbb{N} \cup \{0\} \\ t & \mapsto & \phi_G(t) \end{array}$$

where $\phi_G(t)$ is the number of the ordered t-tuples $(g_1, \ldots, g_t)$ of elements of G which generate G. This function is called the *Eulerian function* associate to G.

Then probability $P_G(t)$ can be regarded as the quotient between $\phi_G(t)$ and the number of all the ordered t-tuples of elements of G:

$$P_G(t) = \frac{\phi_G(t)}{|G|^t} \qquad \forall\, t \in \mathbb{N}.$$

Proposition 3.3 [Hall, [2]] *Let G be a finite group. Then, for any $t \in \mathbb{N}$,*

$$\phi_G(t) = \sum_{H \leq G} \mu(H, G)|H|^t.$$

By this result, we obtain:

$$P_G(t) = \sum_{H \leq G} \frac{\mu(H, G)}{|G : H|^t}.$$

3.2 The probabilistic zeta function

We may interpolate the integer function P_G and define $P_G(s)$ for any $s \in \mathbb{C}$:

$$P_G(s) = \sum_{H \leq G} \frac{\mu(H,G)}{|G:H|^s}.$$

By rearranging the addends in the above sum, we obtain a Dirichlet polynomial as follows:

$$P_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} \tag{1}$$

where

$$a_n := \sum_{|G:H|=n} \mu(H,G).$$

Definition 3.3 The multiplicative inverse of the complex function $P_G(s)$, so defined, is called the *probabilistic zeta function* of G.

Example Consider $G = \mathrm{Sym}(3)$. To obtain the formula of the probabilistic zeta function of G, we have to calculate the function $P_G(s)$. By (1), we proceed to compute the coefficients a_n, with n the index of a subgroup of G. Note that $|G| = 6$; it is known that the index of any subgroup of G divides the order of G, then we may conclude that $n \in \{1, 2, 3, 6\}$. For each possible value of n, we know all the subgroups of G with index n; moreover we have already calculated the Möbius number of each subgroup of G. Hence:

$$a_1 = \sum_{|G:H|=1} \mu(H,G) = \mu(G,G) = 1,$$

$$a_2 = \sum_{|G:H|=2} \mu(H,G) = \mu(\langle(123)\rangle, G) = -1,$$

$$a_3 = \sum_{|G:H|=3} \mu(H,G) = \mu(\langle(12)\rangle, G) + \mu(\langle(13)\rangle, G) + \mu(\langle(23)\rangle, G) = -3,$$

$$a_6 = \sum_{|G:H|=6} \mu(H,G) = \mu(\langle\mathbb{1}\rangle, G) = 3.$$

Then, by (1), we have:

$$P_{\mathrm{Sym}(3)}(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} = 1 - \frac{1}{2^s} - \frac{3}{3^s} + \frac{3}{6^s}.$$

4 Profinite groups

Suppose now G an *infinite* group. What is the probability that a random finite subset generates G? To answer this question we need a probability measure defined on G; so we have to restrict our argumentation to profinite groups. In fact these groups are compact topological groups, and so they have a finite *Haar measure*; normalizing this measure, we may consider G as a probability space: this means that the measure of a subset X of G is construed as the probability that a random element of G lies in X.
Let G be a *profinite* group. We observe that the subgroup lattice $\mathcal{L}_G$ is not in general a locally finite poset. Hence we consider the set of subgroups of G with finite index in G (this is a locally finite poset), and we define on this set the Möbius function, as in Definition 2.1: for each finite index subgroup H of G,

$$\mu(H,G) = \begin{cases} 1 & \text{if } H = G \\ -\displaystyle\sum_{H<Z\leq G} \mu(Z,G) & \text{otherwise} \end{cases}$$

4.1 The probability $P_G(t)$

As in the finite case, we denote by $P_G(t)$ the probability that t randomly chosen elements of G generate (topologically) G itself. Using Proposition 1.14, it is easy to prove the following:

Proposition 4.1 *Let G be a profinite group. Then*

$$P_G(t) = \inf_N P_{G/N}(t)$$

where N ranges over all open normal subgroups of G.

We have already seen that, if G is a finite group, the function $P_G(t)$ can be interpolated in the complex plane by a Dirichlet polynomial. So it is natural to ask: if G is a profinite group, can the integer function $P_G(t)$ be interpolated by a complex function with "good" analytic properties?
We present in the following section a particular case for which we have a positive answer to this question.

4.2 Example: $\hat{\mathbb{Z}}$

To explain this example, we have to introduce the profinite completion.
Let G be an arbitrary group and let $\mathcal{I}$ be the family of all normal subgroups of finite index. We can construct a topology $\mathcal{T}$ on G in the following way: a subset of G is open if it is a union of cosets Kg of subgroups $K \in \mathcal{I}$. With respect to $\mathcal{T}$, G is a topological group (for more details see [7]). Then, applying the construction "inverse limit", we can associate uniquely to the finite quotients of G a profinite group $\hat{G}$, called the *profinite completion* of G:

$$\hat{G} = \varprojlim G/N$$

where N ranges over all normal subgroups of G of finite index.

Remark 4.2 Note that $\hat{G}$ has the same finite quotients of G and, by Proposition 4.1, it follows

$$P_{\hat{G}}(t) = \inf_N P_{G/N}(t).$$

If $G = \mathbb{Z}$ we obtain the *profinite completion* $\hat{\mathbb{Z}}$. It is known (see for example [4]) that the integer function $P_{\hat{\mathbb{Z}}}(t)$ can be interpolated by a complex function $P_{\hat{\mathbb{Z}}}(s)$ defined as follows: for any complex variable s,

$$P_{\hat{\mathbb{Z}}}(s) = \frac{1}{\zeta(s)}$$

where ζ is the *Riemann zeta function*. Recall the definition of ζ:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right)^{-1} \tag{2}$$

Remark 4.3 The Riemann zeta function is a very well known mathematical object; using its properties, we can deduce relevant informations on the probability to generate $\hat{\mathbb{Z}}$.

- $P_{\hat{\mathbb{Z}}}(1) = 0$, since ζ diverges if $s = 1$: this corresponds to the probability to take a number relatively prime with all the primes.
- $P_{\hat{\mathbb{Z}}}(2) = 1/\zeta(2) = 6/\pi^2$: two integers generate $\mathbb{Z}$ if and only if they are relatively prime; then this is the probability that two integers are relatively prime.

Remark 4.4 Using (2), we can write the complex function $P_{\hat{\mathbb{Z}}}(s)$ as a Dirichlet series, in this way:

$$P_{\hat{\mathbb{Z}}}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Then good analytic properties can be deduced for $P_{\hat{\mathbb{Z}}}(s)$.

4.3 The probabilistic zeta function

In the previous section we have analyzed the example of $\hat{\mathbb{Z}}$: the integer function $P_{\hat{\mathbb{Z}}}(t)$ is interpolated, in a "good" analytic way, by a complex function $P_{\hat{\mathbb{Z}}}(s)$. Can we obtain the same for any profinite group G?

Let G be a profinite group. Suppose that the function $P_G(t)$ can be interpolated, in some half plane of the complex plane, by a function of the form:

$$P_G(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s},$$

that is expressed as a Dirichlet series. We call *probabilistic zeta function* of G the multiplicative inverse of the complex function $P_G(s)$ with these properties.

Remark 4.5 Then the probabilistic zeta function plays, for a profinite group G, the same role as the Riemann zeta function for $\hat{\mathbb{Z}}$. The Riemann zeta function was introduced to codify the properties of prime numbers. Note that prime numbers correspond to the indexes of the maximal subgroups of any group; so, in some sense, the probabilistic zeta function codifies the behavior of the maximal subgroups of G.

We want now to establish for which profinite groups this probabilistic zeta function can be defined. We use the following:

Proposition 4.6 [Mann, [5]] *Let G be a profinite group and let $\{N_i\}_{i\in\mathbb{N}}$ be a family of open normal subgroups such that each open subgroup of G contains one of them. Then*

$$P_G(t) = \inf_{i\in\mathbb{N}} P_{G/N_i}(t).$$

We have to restrict our investigation to *finitely generated* profinite groups. In fact if G is a finitely generated profinite group the family $\{N_i\}_{i\in\mathbb{N}}$ can be chosen such that it is a descending chain. Then

$$P_G(t) = \lim_{i\to\infty} P_{G/N_i}(t) = \lim_{i\to\infty}\left(\sum_{N_i\leq H\leq G}\frac{\mu(H,G)}{|G:H|^t}\right).$$

Consider the series

$$\sum_{H\leq_o G}\frac{\mu(H,G)}{|G:H|^s} \qquad \text{(S)}$$

for any complex variable s. We observe that the series (S), with s replaced by a positive integer t, and with the above insertion of parentheses, converges to $P_G(t)$. Thus the series (S) with the above insertion of parentheses, if it converges, is a candidate for the interpolating function $P_G(s)$. It remains to verify:

- if (S) converges in some half plane of the complex plane;
- if the addends in (S) can be ordered in a "nice" way: for example such that the Dirichlet series $\sum_{n\in\mathbb{N}} a_n/n^s$, with $a_n := \sum_{|G:H|=n}\mu(H,G)$, converges and interpolates the probability function $P_G(t)$.

Remark 4.7 We note that if the series (S) converges *absolutely* in some half plane, then the integer function P_G can be interpolated, in the domain of convergence, by a complex function $P_G(s)$ which coincides with the series (S), and it can be expressed in the following way:

$$P_G(s) := \sum_{n\in\mathbb{N}}\frac{a_n}{n^s}$$

where

$$a_n := \sum_{|G:H|=n} \mu(H,G).$$

In this case the definition of the probabilistic zeta function makes sense.

Then our aim becomes to investigate whether the series (S) converges absolutely in some half complex plane.

4.4 Problem

Given a finitely generated profinite group G, we want to establish whether the series (S), associated to it, converges absolutely in some half complex plane.

Definition 4.8 For any $m \in \mathbb{N}$, denote by $b_m(G)$ the number of all the subgroups H with $|G : H| = m$ and $\mu(H,G) \neq 0$. We say:

- $b_m(G)$ *grows polynomially* if there exists α such that $b_m(G) \leq m^{\alpha}$, for each $m \in \mathbb{N}$.
- $|\mu(H,G)|$ *grows polynomially* if there exists β such that, for each finite index subgroup H of G, $|\mu(H,G)| \leq |G : H|^{\beta}$.

In 2005 Mann proved the following:

Theorem 4.9 [Mann, [6]] *The series (S) converges absolutely in some half plane of the complex plane if and only if both $|\mu(H,G)|$ and $b_m(G)$ grow polynomially.*

Definition 4.10 A profinite group G is *positively finitely generated* (PFG), if for some t, the probability $P_G(t)$ is positive.

Remark 4.11 Note that a PFG group is a finitely generated profinite group.

In the same paper (see [6]), Mann has formulated the following conjecture:

Conjecture 4.12 [Mann] *Let G a* PFG *group. Then $|\mu(H,G)|$ grows polynomially in the index of H and $b_m(G)$ grows polynomially in m.*

Remark 4.13 By Theorem 4.9, it is equivalent to conjecture that the series (S) associated to a PFG group G converges absolutely in some half complex plane, and that the probability function $P_G(t)$ can be interpolated, in the domain of convergence, by a complex function expressed as a Dirichlet series (see Remark 4.7).

Recently Lucchini (see [3]) has proved that in order to decide whether a PFG group satisfies Mann's conjecture, it suffices to investigate the behavior of the Möbius function of the finite almost simple groups related to G: in fact to any PFG group can be associated, in a suitable way, some almost simple groups. Recall that a group X is called *almost simple* if and only if $S \leq X \leq \mathrm{Aut}(S)$, with S a non abelian finite simple group.
In the same paper, Lucchini proved that Mann's conjecture holds if the following is true:

Conjecture 4.14 [Lucchini, [3]] *There exists a constant c such that if X is a finite almost simple group, then $b_m(X) \leq m^c$ and $|\mu(Y, X)| \leq |X : Y|^c$ for each $m \in \mathbb{N}$ and each $Y \leq X$.*

This conjecture has been proved for some particular almost simple groups, as shown in the next section.

4.5 Results

It is well known that, for any $n \geq 5$, the Alternating group $\mathrm{Alt}(n)$ is a non abelian simple group, and it is normal in $\mathrm{Sym}(n)$; then $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ are almost simple groups. In [1] we have proved that Lucchini's conjecture holds for all the Alternating and Symmetric groups:

Theorem 4.15 [A. Lucchini, V.C.] *There exists a constant c such that $\forall n \in \mathbb{N}$, if $X \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$, then $b_m(X) \leq m^c$ and $|\mu(Y, X)| \leq |X : Y|^c$, for each $m \in \mathbb{N}$ and each $Y \leq X$.*

Using this result, we are able to establish the validity of Mann's conjecture for some particular PFG groups; in fact the above theorem allows us to formulate the following:

Theorem 4.16 [A. Lucchini, V.C.] *If G is a* PFG *group, and for each open normal subgroup N of G all the composition factors of G/N are either abelian or Alternating groups, then $|\mu(H, G)|$ and $b_m(G)$ grow polynomially. This means that the series (S) associated to G converges absolutely in some half complex plane.*

Example The group $G = \mathrm{Cr}(\mathrm{Alt}(n) | \, n \in \mathbb{N})$ satisfies the hypotheses of Theorem 4.16; then the series (S) of G converges absolutely to the probability $P_G(t)$, for any integer t in the domain of convergence of (S).

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Gaps between linear operators and spectral stability estimates

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Introduction

A distance on closed linear subspaces/operators has long been known. It was introduced under the name of "gap" or "opening" in a Hilbert space context by M. G. Krein and coworkers in the 1940s. The first part of the talk will be of an introductory character and the main properties of the gap between subspaces/operators will be illustrated with the focus laid on spectral stability results.

Next, it will be shown how the notion of gap between operators can be adapted to study the spectral stability problem of a certain class of (partial) differential operators upon perturbation of the open set where they are defined on. An extension of the gap for operators defined on different open sets will be proposed and it will be estimated in terms of the geometrical vicinity or proximity of the open sets. Then, this will permit to estimate the deviation of the eigenfunctions of certain second order elliptic operators with homogeneous Dirichlet boundary conditions upon perturbation of the open set where the said operators are defined on.

1 Gap between linear subspaces

Definition 1 The *gap* between two linear subspaces M and N of a normed space Z is defined by the following formula:

$$\delta(M,N) = \sup_{\substack{u \in M \\ \|u\|=1}} \operatorname{dist}(u,N),$$

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where

$$\mathrm{dist}(u, N) = \inf_{v \in N} \|u - v\|.$$

Moreover, one defines

$$\delta(0, N) = 0$$

for any linear subspace N.

Definition 2 If M and N are linear subspaces of a normed space Z, the *symmetric gap* between M and N is defined by

$$\hat{\delta}(M,\, N) = \max\{\delta(M, N),\ \delta(N, M)\}.$$

Theorem 1 *If M and N are linear subspaces of a Hilbert space Z, then*

$$\hat{\delta}(M, N) = \|P_{\overline{M}} - P_{\overline{N}}\|,$$

where $P_{\overline{M}}$, $P_{\overline{N}}$ are the orthogonal projectors onto $\overline{M}$, $\overline{N}$ respectively.

2 Gap between operators

The Cartesian product $X \times Y$ of two normed spaces X, Y is made into a normed space with the usual definition of addition, multiplication by scalars and the norm defined by

$$\|(u, v)\|_{X \times Y} = (\|u\|_X^2 + \|v\|_Y^2)^{\frac{1}{2}} \quad \text{for all} \quad u \in X,\ v \in Y.$$

If the norms of X and Y derive, respectively, from certain inner products $(\cdot, \cdot)_X$, $(\cdot, \cdot)_Y$, then $X \times Y$ can also be equipped with an inner product defined by

$$\big((u, v), (u', v')\big)_{X \times Y} = (u, u')_X + (v, v')_Y$$

for all $u, u' \in X$, $v, v' \in Y$, from which the norm $\| \cdot \|_{X \times Y}$ is derived. Thus, if X, Y are Banach (Hilbert) spaces, then $X \times Y$ is also a Banach (Hilbert) space. As usual, the subscripts from the norms and inner products will be dropped whenever no possibility for ambiguity arises.

Definition 3 Let X, Y be normed spaces and let

$$S : D(S) \subset X \to Y,$$

$$T : D(T) \subset X \to Y$$

be linear operators acting from X to Y. Then the *gap between S and T* is defined by

$$\delta(S, T) = \delta(G(S), G(T)),$$

$$\hat{\delta}(S, T) = \hat{\delta}(G(S), G(T)),$$

where $G(S)$, $G(T)$ are the graphs of the operators S, T respectively.
More explicitly,

$$\delta(S,T) = \sup_{\substack{u \in \mathrm{D}(S) \\ \|u\|_X^2 + \|Su\|_Y^2 = 1}} \inf_{v \in \mathrm{D}(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y\right)^{\frac{1}{2}}.$$

If S, T are self-adjoint linear operators in a Hilbert space X,

$$\delta(S,T) = \delta(T,S).$$

The gap is invariant under inversion, that is, if S, T are invertible operators, then

$$\delta(S^{-1}, T^{-1}) = \delta(S,T).$$

Theorem 2 (Kato [4]) *Let T be an invertible operator with bounded inverse $T^{-1} \in \mathcal{B}(Y,X)$. If S is a closed operator such that*

$$\hat{\delta}(S,T) < (1 + \|T^{-1}\|^2)^{-1/2},$$

then S is invertible, $S^{-1} \in \mathcal{B}(Y,X)$ and

$$\|S^{-1} - T^{-1}\| \leq \frac{1 + \|T^{-1}\|^2}{1 - (1 + \|T^{-1}\|^2)^{1/2} \delta(S,T)} \delta(S,T).$$

3 Some spectral stability results

Let $T \in \mathcal{C}(X)$. Assume that $\sigma(T)$ is separated into two parts σ_1 and σ_2 by a rectifiable simple closed curve (or, more generally, a finite number of such curves with no point of any of these curves contained in the interior of any other) in such a way that it encloses an open set containing σ_1 in its interior and σ_2 in its exterior. Let

$$P[T] = -\frac{1}{2\pi i} \int_\Gamma (T - \xi)^{-1} d\xi.$$

Let

$$N[T] = P[T]X.$$

Theorem 3 *Let X be a Hilbert space and T a self-adjoint operator in X. Let λ_0 be an eigenvalue of T and assume that it is an isolated point of $\sigma(T)$. Precisely, assume that it is the center of a circle of radius $r > 0$ that we denote by $C_r^+(\lambda_0)$ and that does not enclose* *any other point of $\sigma(T)$ except λ_0. Assume also that any other point of $\sigma(T)$ different from λ_0 has distance from λ_0 not less than $2r$. Set*

$$\delta = \frac{r}{[2 + (|\lambda_0| + r)^2]\,(1 + r^2)^{1/2}}.$$

Any self-adjoint operator S with $\delta(S,T) < \delta$ has spectrum $\sigma(S)$ separated by $C_r^+(\lambda_0)$ into two parts $\sigma_0(S)$ and $\sigma'(S)$ ($C_r^+(\lambda_0)$ running in $P(S)$) and

$$\hat{\delta}(N[S], N[T]) \leq \frac{r}{\delta - \delta(S,T)}\delta(S,T).$$

The following facts are needed to prove this theorem. If $A \in \mathcal{B}(X,Y)$, then

$$\delta(S + A, T + A) \leq (2 + \|A\|^2)\delta(S,T).$$

If $\xi \in \mathbb{C}$ is in the resolvent set of T,

$$\|(T - \xi)^{-1}\| = \frac{1}{\mathrm{dist}(\xi,\, \sigma(T))}.$$

We are going to apply the estimates of the previous theorem to obtain estimates for the eigenfunctions.

Lemma 1 (Barbatis, Burenkov and Lamberti [1]) *Let M and N be finite dimensional subspaces of a Hilbert space X, $\dim M = \dim N = m$, and let $u_1, \ldots, u_m$ be an orthonormal basis for M. Then there exists an orthonormal basis $v_1, \ldots, v_m$ for N such that*

$$\|u_k - v_k\| \leq 5^k \hat{\delta}(M,N), \qquad k = 1, \ldots, m.$$

Combining the above results we obtain the following theorem.

Theorem 4 *Let X be a Hilbert space and T, S self-adjoint operators. Let λ_0, r, δ be as in the assumptions of Theorem 3. Moreover, assume that the eigenvalue λ_0 of T has multiplicity m. Then, if $\hat{\delta}(S,T) < \delta$, there are at most m distinct eigenvalues of S in the disc $D_r(\lambda_0)$ such that the sum of the relative multiplicities is exactly m. Moreover, if $\varphi_1[S], \ldots, \varphi_m[S]$ is a an orthonormal set of eigenfunctions of S corresponding to the aforementioned set of eigenvalues of S, then there exists an orthonormal set of eigenfunctions $\varphi_1[T], \ldots, \varphi_m[T]$ of T corresponding to the eigenvalue λ_0 such that*

$$\|\varphi_k[S] - \varphi_k[T]\| \leq 5^k \frac{r}{\delta - \delta(S,T)}\delta(S,T)$$

for each $k = 1, \ldots, m$.

4 Direct sum operator

Let X, X' be normed spaces: any subspace $M \subset X$ can be seen as a subspace of $X \times X'$ by identifying it with

$$\{(u, 0) \ : \ u \in M\};$$

analogously can be done for any subspace $M' \subset X'$ by identifying it with the subspace

$$\{(0, u') \ : \ u' \in M'\}$$

of $X \times X'$.

Hence, in this notation we have $X \times X' = X \dotplus X'$.

If $u \in X$ and $v \in X'$, as we said, we can see u and v as elements of $X \dotplus X'$ by identifying them with $(u, 0)$ and $(0, v)$ respectively; in this context we have

$$(u, v) = u + v.$$

Definition 4 Let X, X', Y, Y' be normed spaces and let

$$S : D(S) \subset X \to Y,$$

$$S' : D(S') \subset X' \to Y'$$

be linear operators. Then the *direct sum operator* $S \dotplus S'$ is defined in the following way:

$$\mathrm{D}(S \dotplus S') = \mathrm{D}(S) \dotplus \mathrm{D}(S'),$$

and

$$(S \dotplus S')(u + u') = Su + S'u'$$

for all $u \in \mathrm{D}(S)$, $u' \in \mathrm{D}(S')$.

If X, X', Y, Y' are Hilbert spaces we write $S \oplus S'$ instead of $S \dotplus S'$.

Properties.

(i)

$$\mathrm{N}(S \dotplus S') = \mathrm{N}(S) \dotplus \mathrm{N}(S'), \qquad R(S \dotplus S') = R(S) \dotplus R(S').$$

(ii)

$$G(S \dotplus S') = G(S) \dotplus G(S').$$

(iii) $S \dotplus S'$ is invertible if, and only if, S, S' are both invertible and, in this case,

$$(S \dotplus S')^{-1} = S^{-1} \dotplus S'^{-1}.$$

(iv) $S \dotplus S'$ is a closable operator if, and only if, S, S' are closable operators and,

$$\overline{S \dotplus S'} = \bar{S} \dotplus \bar{S}';$$

in particular, if S, S' are closed operators, then so is $S \dotplus S'$.

(v) $S \dotplus S'$ is a densely defined operator if, and only if, S, S' are densely defined operators and in this case

$$(S \dotplus S')^* = S^* \dotplus S'^*.$$

(vi) $S \dotplus S'$ is a bounded (respectively, compact) operator if, and only if, both S and S' are bounded (respectively, compact) operators.

In the following is assumed that $X = Y$ and $X' = Y'$.

(vii)

$$\sigma(S \dotplus S') = \sigma(S) \cup \sigma(S') \qquad \rho(S \dotplus S') = \rho(S) \cap \rho(S').$$

(viii)

$$\sigma_p(S \dotplus S') = \sigma_p(S) \cup \sigma_p(S'),$$

and for any $\lambda \in \mathbb{C}$,

$$\mathrm{N}(S \dotplus S' - \lambda I_{X \dotplus X'}) = \mathrm{N}(S - \lambda I_X) \dotplus \mathrm{N}(S' - \lambda I_{X'}),$$

hence, in particular, the geometrical multiplicity of λ as an eigenvalue of $S \dotplus S'$ is equal to the sum of the geometrical multiplicities of λ as an eigenvalue of S and S' (this assertion is valid with the understanding that if λ is not an actual eigenvalue of an operator, then λ is looked at as an eigenvalue of geometric multiplicity zero of the said operator).

(ix) $S \dotplus S'$ is an operator with compact resolvent if, and only if, S, S' are operators with compact resolvent.

Now let $X = Y$, $X' = Y'$ be Hilbert spaces.

(x) $S \oplus S'$ is self-adjoint (essentially self-adjoint) if, and only if, both S, S' are self-adjoint (essentially self-adjoint) operators.

(xi) If S, S' are non-negative symmetric densely defined linear operators, then about the Friedreich extensions subsists the result

$$(S \oplus S')^F = S^F \oplus S'^F.$$

5 The Dirichlet Laplacian operator

For an open set $\Omega \in \mathbb{R}^n$, the Dirichlet Laplacian is usually defined via the Friedrich's extension procedure.

Let Ω be an open set in $\mathbb{R}^n$, $(n \in \mathbb{N})$. Then there exists a non-negative self-adjoint operator

$$-\Delta_{\Omega,\mathrm{D}} : \mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) \subset L^2(\Omega) \to L^2(\Omega)$$

called *the (generalized) Dirichlet Laplacian relative to the domain* Ω*,* and defined in the following way:

$u \in \mathrm{D}(-\Delta_{\Omega,\mathrm{D}})$ if, and only if $u \in H_0^1(\Omega)$ and there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} dx = \int_\Omega f \overline{v} dx$$

for all $v \in H_0^1(\Omega)$; in this case one defines

$$-\Delta_{\Omega,\mathrm{D}} u = f.$$

We have

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) = H_0^1(\Omega) \cap H^\Delta(\Omega)$$

and

$$(-\Delta_{\Omega,\mathrm{D}})u = -\Delta_w u$$

for all $u \in \mathrm{D}(-\Delta_{\Omega,\mathrm{D}})$.

Moreover, $\lambda \in \mathbb{C}$ is an eigenvalue of $-\Delta_{\Omega,\mathrm{D}}$ and $u \in \mathrm{D}(-\Delta_{\Omega,\mathrm{D}})$, $u \neq 0$, a corresponding eigenfunction if, and only if,

$$\int_\Omega \nabla u \cdot \overline{\nabla v} dx = \lambda \int_\Omega u \overline{v} dx$$

for all $v \in H_0^1(\Omega)$. Thus, $\lambda \in \mathbb{R}$ and $\lambda > 0$.

If the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact (this is always the case when Ω is a bounded set), then the operator $-\Delta_{\Omega,\mathrm{D}}$ has compact resolvent, hence its spectrum is discrete, that is to say, it consists only of isolated eigenvalues of finite multiplicity. If we arrange these eigenvalues in non-decreasing order repeating them as many times as their multiplicities

$$0 < \lambda_1[\Omega] \leq \lambda_2[\Omega] \cdots \leq \lambda_2[\Omega] \leq \cdots ,$$

then

$$\lim_{n\to\infty} \lambda_n[\Omega] = \infty.$$

Furthermore, $L^2(\Omega)$ admits an orthonormal basis $\{\varphi_n[\Omega]\}_{n\in\mathbb{N}}$ consisting of eigenfunctions of $-\Delta_{\Omega_1,\mathrm{D}}$ (in these notations, the eigenfunction $\varphi_n[\Omega]$ is chosen to correspond to the eigenvalue $\lambda_n[\Omega]$, $n \in \mathbb{N}$). For reference, see Davies, E. B. [2].

Theorem 5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and assume that Ω has a $C^{1,r}$ boundary for some $\frac{1}{2} < r \leq 1$. or, in alternative, that Ω is a convex set or a polyhedron. Then*

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) = H_0^1(\Omega) \cap H^2(\Omega).$$

If Ω is a convex set this classical result is due to Kadlec. The result in the case of an open set with a C^2 boundary or of a polyhedron is also classical and can be found in Ladyzhenskaya and Uralt'zeva's book on elliptic equations. The result in the case of an open set with a $C^{1,r}$ boundary $(1/2 < r < 1)$ seems to be recent and is mentioned in Gesztesy, F. and Mitrea, M. [3].

Theorem 6 *Let $n = 1, 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $\Omega \in C^{1,r}$ with $\frac{1}{2} < r \leq 1$, or that Ω is an open convex set or an open polyhedron. Then*

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) = \overline{\{u \in C^2(\Omega) \cap C(\overline{\Omega}) \cap H^2(\Omega) \ : \ u|_{\partial\Omega} = 0\}},$$

where the closure is in $H^2(\Omega)$.

6 Gap between operators defined on different open sets

Let Ω_1, Ω_2 be open sets in $\mathbb{R}^n$. Then we have the natural identifications:

$$L^2(\Omega_1 \cup \Omega_2) = L^2(\Omega_1) \oplus L^2(\Omega_2 \setminus \Omega_1),$$

$$L^2(\Omega_1 \cup \Omega_2) = L^2(\Omega_2) \oplus L^2(\Omega_1 \setminus \Omega_2).$$

Definition 5

$$\delta(-\Delta_{\Omega_1,\mathrm{D}},\, -\Delta_{\Omega_2,\mathrm{D}}) \ = \\ \delta((-\Delta_{\Omega_1,\mathrm{D}}) \oplus (-\Delta_{\Omega_2\setminus\overline{\Omega_1},\mathrm{D}}),\, (-\Delta_{\Omega_2,\mathrm{D}}) \oplus (-\Delta_{\Omega_1\setminus\overline{\Omega_2},\mathrm{D}}))$$

Let Ω_1, Ω_2 be bounded open sets in $\mathbb{R}^n$ such that $\overline{\Omega_2} \subset \Omega_1$, then

$$\delta(-\Delta_{\Omega_1,\mathrm{D}}, -\Delta_{\Omega_2,\mathrm{D}}) = \\ \sup_{\substack{u \in C^2(\Omega_1)\cap C(\overline{\Omega_1}) \\ u|_{\partial\Omega_1} = 0 \\ \mathcal{K}(u) = 1}} \quad \inf_{\substack{v \in C^2(\Omega_2)\cap C^2(\Omega_1\setminus\overline{\Omega_2}) \\ \cap C(\overline{\Omega_1}) \\ \Delta v \in L^2(\Omega_2)\cap L^2(\Omega_1\setminus\overline{\Omega_2}) \\ v|_{\partial\Omega_1} = 0,\ v|_{\partial\Omega_2} = 0}} \mathcal{K}(u - v),$$

where

$$\mathcal{K}(w) = \Big(\|w\|^2_{L^2(\Omega_2)} + \|\Delta w\|^2_{L^2(\Omega_1)} + \|\Delta w\|^2_{L^2(\Omega_2\setminus\overline{\Omega_1})}\Big)^{\frac{1}{2}}$$

for any $w \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \cap C(\overline{\Omega_2})$.

Theorem 7 *Let $n \leq 3$, $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$ and $\gamma = 1$ if $n = 1$. Let Ω_1 be an open bounded set in $\mathbb{R}^n$ with a $C^{1,r}$ boundary for some $\frac{1}{2} < r \leq 1$, or an open convex set, or an open polyhedron that satisfies an exterior right cone condition. Then there exists $M > 0$ such that*

$$\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) \leq M\varepsilon^\gamma,$$

for all $\varepsilon > 0$ and for all open sets Ω_2 that satisfy an exterior and an interior right cone condition and for which $(\Omega_1)_\varepsilon \subset \Omega_2 \subset \overline{\Omega_2} \subset \Omega_1$.

7 Spectral stability estimates

Let us consider again an orthonormal basis $\{\varphi_k[\Omega]\}_{k\in\mathbb{N}}$ of $L^2(\Omega)$, consisting of eigenfunctions of $-\Delta_{\Omega,\mathrm{D}}$ where for each $k \in \mathbb{N}$, $\varphi_k[\Omega]$ is an eigenfunction for the eigenvalue $\lambda_k[\Omega]$; let $k, m \in \mathbb{N}$ be such that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega] \le \cdots \le \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega],$$

(the first inequality on the left is not present if $k = 1$) then we set

$$N_{k,m}[\Omega] = \operatorname{span}\{\varphi_k[\Omega], \ldots, \varphi_{k+m-1}[\Omega]\}.$$

Theorem 8 *Let $n \le 3$, $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$ and $\gamma = 1$ if $n = 1$. Let Ω_1 be a bounded open set of $\mathbb{R}^n$. Assume that $\Omega_1 \in C^{1,r}$ ($\frac{1}{2} < r \le 1$), or that Ω_1 is convex, or that Ω_1 is a polyhedron that satisfies an exterior right cone condition. Let $\lambda[\Omega_1]$ be an eigenvalue of multiplicity m ($m \in \mathbb{N}$) of $-\Delta_{\Omega_1,\mathrm{D}}$, that is, there exists $k \in \mathbb{N}$ such that $\lambda[\Omega_1] = \lambda_k[\Omega_1] = \cdots = \lambda_{k+m-1}[\Omega_1]$. Then there exist $M_0, \varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon < \varepsilon_0$ and Ω_2 is such that $(\Omega_1)_\varepsilon \subset \Omega_2 \subset \overline{\Omega_2} \subset \Omega_1$, and Ω_2 satisfies an interior and an exterior right cone condition, then*

$$\hat{\delta}(N_{k,m}[\Omega_2],\ N_{k,m}[\Omega_1]) \le M_0\varepsilon^\gamma.$$

Theorem 9 *Let $n \le 3$, $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$, and $\gamma = 1$ if $n = 1$. Let Ω_1 be an open set of $\mathbb{R}^n$. Assume that $\Omega_1 \in C^{1,r}$ for some $\frac{1}{2} < r \le 1$, or that Ω_1 is convex, or that Ω_1 is a polyhedron and satisfies an exterior right cone condition. Let $\lambda[\Omega_1]$ be an eigenvalue of multiplicity m ($m \in \mathbb{N}$) of $-\Delta_{\Omega_1,\mathrm{D}}$, that is, there exists $k \in \mathbb{N}$ such that $\lambda[\Omega_1] = \lambda_k[\Omega_1] = \cdots = \lambda_{k+m-1}[\Omega_1]$. Then there exist $c_0, \varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon < \varepsilon_0$ and Ω_2 is an open set that satisfies an exterior and an interior right cone condition and such that $(\Omega_1)_\varepsilon \subset \Omega_2 \subset \overline{\Omega_2} \subset \Omega_1$, and $\varphi_{k+1}[\Omega_2], \ldots, \varphi_{k+m}[\Omega_2]$ is an orthonormal set of eigenfunctions of $-\Delta_{\Omega_2,\mathrm{D}}$ corresponding to the eigenvalues $\lambda_k[\Omega_2] \le \cdots \le \lambda_{k+m-1}[\Omega_2]$, then there exists an orthonormal set of eigenfunctions $\varphi_k[\Omega_1], \ldots, \varphi_{k+m-1}[\Omega_1]$ of $-\Delta_{\Omega_1,\mathrm{D}}$ corresponding to the eigenvalue $\lambda[\Omega_1]$ such that*

$$\|\varphi_{k+i}[\Omega_1] - \varphi_{k+i}[\Omega_2]\|_{L^2(\Omega_2)} \le c_0\varepsilon^\gamma$$

for all $i = 0, \ldots, m-1$.

8 Proof of Theorem 7

Let $u \in C^2(\Omega_1) \cap C(\overline{\Omega_1})$, $\Delta u \in L^2(\Omega_1)$ and $u|_{\partial\Omega_1} = 0$

First we prove that

$$I(u) := \inf_{\substack{v \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \cap C(\overline{\Omega_1}) \\ v|_{\partial\Omega_1} = 0,\ v|_{\partial\Omega_2} = 0}} \mathcal{K}(u - v) \le (\operatorname{meas}\Omega_1)^{\frac{1}{2}} \|u\|_{C(\partial\Omega_2)}.$$

We have to estimate

$$I(u) = \inf_{\substack{w \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \cap C(\overline{\Omega_1}) \\ w|_{\partial\Omega_2} = u|_{\partial\Omega_2},\ w|_{\partial\Omega_1} = 0}} \mathcal{K}(w).$$

Let $w_1 \in C^2(\Omega_2) \cap C(\overline{\Omega_2})$ be such that

$$\begin{cases} \Delta w_1 & = & 0 & \text{in } \Omega_2 \\ w_1|_{\partial\Omega_2} & = & u|_{\partial\Omega_2} \end{cases}$$

and $w_2 \in C^2(\Omega_1 \setminus \overline{\Omega_2}) \cap C(\overline{\Omega_1} \setminus \Omega_2)$ be such that

$$\begin{cases} \Delta w_2 & = & 0 & \text{in } \Omega_1 \setminus \overline{\Omega_2} \\ w_2|_{\partial\Omega_2} & = & u|_{\partial\Omega_2} \\ w_2|_{\partial\Omega_1} & = & 0 \end{cases} .$$

Such functions w_1 and w_2 exist due to the assumption that the open sets Ω_2, $\Omega_1 \setminus \overline{\Omega_2}$ satisfy an exterior right cone condition.

We put

$$w_0 = \begin{cases} w_1 & \text{in } \Omega_2 \\ w_2 & \text{in } \overline{\Omega_1} \setminus \Omega_2 \end{cases} .$$

Therefore

$$\begin{aligned} I(u) &\leq \mathcal{K}(w_0) \\ &= \left(\int_{\Omega_1} w_0^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega_2} {w_1}^2 dx + \int_{\Omega_2 \setminus \Omega_1} {w_2}^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

By the maximum principle

$$\begin{aligned} I(u) &\leq (\text{meas } \Omega_2 \|w_1\|^2_{C(\overline{\Omega_2})} + \text{meas } (\Omega_1 \setminus \Omega_2) \|w_2\|^2_{C(\overline{\Omega_1} \setminus \Omega_2)})^{\frac{1}{2}} \\ &= (\text{meas } \Omega_2 \|u\|^2_{C(\partial\Omega_2)} + \text{meas } (\Omega_1 \setminus \Omega_2) \|u\|^2_{C(\partial\Omega_2)})^{\frac{1}{2}} \\ &= (\text{meas } \Omega_1)^{\frac{1}{2}} \|u\|_{C(\partial\Omega_2)}. \end{aligned}$$

So far we have proved that

$$(*) \qquad \delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) \leq (\text{meas } \Omega_1)^{\frac{1}{2}} \sup_{\substack{u \in C^2(\Omega_1) \cap C(\overline{\Omega_1}) \\ u|_{\partial\Omega_1} = 0 \\ \|u\|^2_{L^2(\Omega_1)} + \|\Delta u\|^2_{L^2(\Omega_1)} = 1}} \|u\|_{C(\partial\Omega_2)}.$$

In the following estimates we use this Sobolev's embedding theorem

$$H^2(\Omega_1) \subset C^{\gamma}(\overline{\Omega_1}),$$

which holds since

$$\frac{n}{2} + \gamma < 2.$$

We also use an a priori L^2-estimate for solutions to elliptic equations.

Now let $u \in C^2(\Omega_1) \cap C(\overline{\Omega_1})$ be such that $\mathcal{K}(u) = 1$, $u|_{\partial\Omega_1} = 0$ and $x \in \partial\Omega_2$. Let also $y \in \partial\Omega_1$ be such that $\mathrm{dist}(x, \partial\Omega_1) = |x - y|$. Then, since $x \notin \partial\Omega_1$, we have

$$\begin{aligned} |u(x)| &= |x-y|^\gamma \frac{|u(x)|}{|x-y|^\gamma} \\ &\le \varepsilon^\gamma \sup_{\substack{x,y \in \overline{\Omega_1} \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} \\ &\le M_1 \varepsilon^\gamma ||u||_{H^2(\Omega_1)} \\ &\le M_2 \varepsilon^\gamma \left(\int_{\Omega_1} \left(|u|^2 + |\Delta u|^2\right) dx \right)^{1/2} \\ &= M_2 \varepsilon^\gamma, \end{aligned}$$

where M_1, M_2 are positive constants that do not depend on the function u but only on the open set Ω_1; M_1 derives from the aforementioned Sobolev embedding theorem, while M_2 derives from a global regularity $H^2(\Omega_1)$-estimate for solutions of elliptic equations (this explains the regularity assumptions on Ω_1). From this estimate and $(*)$ follows the estimate of Theorem 7 with $M = M_2(\mathrm{meas}\ \Omega_1)^{1/2}$. $\square$

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Finite and countable mixtures

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1 Introduction

The present note deals with finite and countable mixtures of independent identically distributed (i.i.d.) sequences and of Markov chains. Mixture models are well known in the Bayesian-statistic community. *Finite* mixtures of Markov chains (resp. of i.i.d. sequences) are an appropriate statistical model when the population is naturally divided into clusters, and the time evolution of a sample is Markovian (resp. an i.i.d. sequence), but with distribution dependent on which cluster the sample belongs to. These models have been applied in different contexts. See for example [2] for an application to navigation patterns on a web site or [9] for an application to bond rating migration.

In the first part of the note we focus on Mixtures of i.i.d. sequences. These are exchangeable processes by de Finetti's theorem. We present a verifiable criterion to establish whether an exchangeable binary process is a *finite* mixture of i.i.d. sequences. If this is the case, we provide an algorithm which completely identifies the mixing measure of the mixture. In the second part of the talk we focus on mixtures of Markov chains, which are known to be partially exchangeable processes after the work of Diaconis and Freedman. We present a characterization theorem for partially exchangeable processes which are mixtures just of a finite or countable number of Markov chains, finding a connection with Hidden Markov Models. Our result extends an old theorem by Dharmadhikari on finite and countable Mixtures of i.i.d. sequences. Moreover we partially extend the results we give for exchangeable processes, providing a criterion to establish whether a partially exchangeable binary process is a *finite* mixture of Markov chains.

2 Mixtures of i.i.d. sequences

This section deals with Mixtures of i.i.d. sequences. We give the definition of mixtures of i.i.d. sequences and of exchangeable processes, recalling the wellknown de Finetti's theorem which links the two notions. After that, we report a characterization of countable and finite Mixtures of i.i.d. sequences due to Dharmahikari, which involves Hidden Markov Models.

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In our main result give a characterization of finite mixtures of i.i.d. sequences, looking at the rank of an appropriate family of Hankel matrices. Differently from Dharmahikari's result, our characterization can be easily tested. We also give an algorithm to identify the unique mixing measure associated with a finite mixture of i.i.d. sequences.

2.1 Preliminary notions

For a finite binary string $y_1^n = y_1 \dots y_n$ let $n_1 = \sum_{t=1}^n y_t$ be the number of 1-s in the string, and $p = \mathbb{P}\{Y_1 = 1\}$. Then for an i.i.d. process (Y_n)

$$\mathbb{P}\{Y_1^n = y_1^n\} = \prod_{t=1}^{n} \mathbb{P}\{Y_1 = y_t\} = p^{n_1} (1-p)^{n-n_1}. \tag{1}$$

We give the following

Definition 1 A binary process (Y_n) is a *mixture of i.i.d. sequences* if there exists a measure μ on the interval $[0,1]$ such that

$$\mathbb{P}\{Y_1^n = y_1^n\} = \int_0^1 p^{n_1} (1-p)^{n-n_1} d\mu(p). \tag{2}$$

As stated by de Finetti's theorem below, Mixtures of i.i.d. sequences can be identified with exchangeable processes. To define them, let $\sigma = \sigma_1^n = \sigma_1 \dots \sigma_n \in \{0,1\}^n$ be a binary string of length n. Denoting by $S(n)$ the group of permutations of $I_n := \{1, 2, \dots, n\}$ we say that the binary string τ of length n is a *permutation* of σ if there exists a permutation $\pi \in S(n)$ such that

$$\tau_1^n = \tau_1 \tau_2 \dots \tau_n = \sigma_{\pi(1)} \sigma_{\pi(2)} \dots \sigma_{\pi(n)}.$$

Definition 2 The binary stochastic process (Y_n), $n = 1, 2, \dots$, taking values into $\{0,1\}^\infty$, is *exchangeable* if for all n, for all $\sigma = \sigma_1^n$ and all permutations τ of σ

$$\mathbb{P}\big\{Y_1^n = \sigma_1^n\big\} = \mathbb{P}\big\{Y_1^n = \tau_1^n\big\}.$$

The simplest example of exchangeable process is an i.i.d process. The random variables of an exchangeable process are identically distributed, but not necessarily independent.

Mixtures of i.i.d. sequences and exchangeable processes are linked by the classic de Finetti theorem below (see [4]).

Theorem 1 (de Finetti 1937) *The stochastic process (Y_n) is exchangeable if and only if (Y_n) is a mixture of i.i.d. sequences. When this is the case the measure μ in (2) is unique.*

2.1.1 Countable mixtures of Markov chains

Imposing constraints on the de Finetti measure μ one narrows the class of exchangeable processes. In this note attention will be restricted to the subclasses defined below.

Definition 3 (Y_n) is a *finite* mixture of i.i.d. sequences if the de Finetti measure μ in equation (2) is concentrated on a finite set, i.e. there exist $N < \infty$ points $p_1, p_2, \ldots, p_N$ in $[0,1]$, with $\pi_k := \mu(p_k)$ such that $\sum_{k=1}^N \pi_k = 1$. Therefore it holds

$$\mu(\cdot) = \sum_{k=1}^{N} \pi_k \delta_{p_k}(\cdot), \tag{3}$$

where δ_p indicates the Dirac measure concentrated in p.
(Y_n) is a *countable* mixture of i.i.d. sequences if μ is concentrated on a countable set K.

For a finite binary mixture of i.i.d. sequences equation (2), which provides the probability of a trajectory, takes the explicit form

$$\mathbb{P}\{Y_1^n = y_1^n\} = \sum_{k=1}^{N} \pi_k p_k^{n_1} (1-p_k)^{n-n_1}. \tag{4}$$

The measure μ associated with a finite binary mixture of i.i.d. sequences and therefore all the finite joint distributions of the mixture itself are determined by the parameters $\mathbf{p} := (p_1, \ldots, p_N)$ and $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_N)$.

Next to the widely known de Finetti characterization of general exchangeable processes there is a specialized result characterizing the subclasses of finite and countable mixtures of i.i.d. processes in terms of hidden Markov models (HMM).

Definition 4 The process (Y_n), taking values in a countable set I, is a finite (countable) HHM if there exists a Markov chain (X_n) with values in a finite (countable) state space $\mathcal{X}$ and a function $f : \mathcal{X} \to I$ such that $Y_n = f(X_n)$ for all n.

The characterization of finite and countable mixtures of i.i.d. binary sequences, due to Dharmadhikari [5] is as follows.

Theorem 2 (Dharmadhikari 1964) *The binary exchangeable process (Y_n) is a finite (countable) mixture of i.i.d. sequences if and only if (Y_n) is a finite (countable) HMM.*

2.2 Characterization of finite mixtures

In this section we propose an alternative to Dharmadhikar's Theorem 2 to characterize exchangeable processes which are finite Mixtures of i.i.d. sequences. The fringe benefit of our approach is that it lends itself naturally to the identification of the mixing measure μ, when it is concentrated on a finite set, as we will see in the next section. More precisely we pose and solve the following

Problem 1 *Given the joint distributions $p_Y(y_1^n) := \mathbb{P}\{Y_1^n = y_1^n\}$ of a binary exchangeable process (Y_n), establish whether (Y_n) is a finite mixture of i.i.d. sequences. If it is, find the mixing measure μ.*

Note that by Theorem 1 we know that (Y_n) is a mixture of i.i.d. sequences.

In the following Theorem 3 we will give a characterization of finite mixtures of i.i.d. processes looking at the rank of a suitable class of Hankel matrices, constructed in terms of the $p_Y(1^m)$, where 1^m indicates the string of m consecutive 1s. It is not surprising that we can restrict our attention to this subclass of joint distributions. In fact the $p_Y(1^m)$, $m = 1, 2, \ldots$, completely specify the general joint distributions of a binary exchangeable process (Y_n) (see [7] Chapter VII.4).

Given any binary process (Y_n), not necessarily exchangeable, one can associate to it a Hankel matrix whose elements are the probabilities of the strings 1^m. Formally, for any $n \in \mathbb{N}$, let $H_n = (h_{ij})_{0 \le i,j \le n}$ be the $(n+1) \times (n+1)$ Hankel matrix, with entries $h_{ij} := h_{i+j} = p_Y(1^i 1^j) = p_Y(1^{i+j})$, with the convention that $p_Y(1^0) = 1$:

$$\mathbf{H_n} := \begin{pmatrix} 1 & p_Y(1) & \ldots & p_Y(1^n) \\ p_Y(1) & p_Y(11) & \ldots & p_Y(1^{n+1}) \\ p_Y(11) & p_Y(111) & \ldots & p_Y(1^{n+2}) \\ \vdots & \vdots & \vdots & \vdots \\ p_Y(1^n) & p_Y(1^{n+1}) & \ldots & p_Y(1^{2n}) \end{pmatrix}. \tag{5}$$

The semi-infinite matrix H_∞ is defined in the obvious way.

The theorem below, styled as a theorem of the alternative, characterizes the binary exchangeable processes which are finite Mixtures of i.i.d. sequences.

Theorem 3 *Let (H_n) be the Hankel matrices of the binary exchangeable process (Y_n) defined in equation (5), then exactly one of the following two statements holds.*

- *There exists a finite N such that*

$$\operatorname{rank}(H_n) = \begin{cases} n+1 & \text{for } n = 0, \ldots, N-1 \\ N & \text{for } n \ge N \end{cases} \tag{6}$$

- $\operatorname{rank}(H_n) = n + 1$ *for all* $n \in \mathbf{N}$.

(Y_n) is a mixture of N i.i.d. sequences if and only if (6) holds.

Sketch of the proof of Theorem 3

In order to characterize the exchangeable processes which are finite mixtures of i.i.d. sequences one has to be able to characterize the de Finetti measures μ which are concentrated on a finite set of points, (see Theorem 1 and the ensuing Definition 3). As it will be explained below this is directly related to some properties of the moments of the measure μ.

Let μ be a probability measure on $[0, 1]$. The m-th moment α_m of μ is defined as

$$\alpha_m := \int_0^1 x^m \mu(dx). \tag{7}$$

Assume that the moments α_m are all finite and define the moments matrices and their determinants as

$$\mathbf{M_n} := \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_n & \dots & \dots & \alpha_{2n} \end{pmatrix} \quad \text{and } d_n := \det(M_n). \tag{8}$$

In particular, for the de Finetti measure μ of an exchangeable process the moments are given by

$$\alpha_m = \int_0^1 p^m d\mu(p) = p_Y(1^m). \tag{9}$$

We thus get immediately that the moments matrices of μ and the Hankel matrices defined in equation (5) coincide for all $n \in \mathbb{N}$, i.e.

$$M_n = H_n.$$

Once we have the previous relation, to go further, recall the definition of point of increase of a real valued function.

Definition 5 Given a real valued function f, we say that x_0 is a *point of increase* for f if $f(x_0 + h) > f(x_0 - h)$ for all $h > 0$.

To determine whether the de Finetti measure μ is concentrated it is enough to study the set of points of increase of its distribution function,

$$F^\mu(x) := \mu([0, x]) \text{ for any } x \in [0, 1]. \tag{10}$$

It is easy to prove that F^μ has exactly N points of increase $p_1, \dots, p_N$ if and only if μ is concentrated on $p_1, \dots, p_N$.

To finish the characterization, adapting an argument in Cramer [3], Chapter 12.6, one can prove the following Lemma.

Lemma 1 *If F^μ has N points of increase, then $d_n \neq 0$ for $n = 0, \dots, N-1$ and $d_n = 0$ for $n \geq N$.*
If F^μ has infinitely many points of increase, then $d_n \neq 0$ for any n.

Piecing together these considerations the proof of Theorem 3 is completed.

2.3 Computation of the mixing measure

Theorem 3 in the previous section gives a criterion to check whether an exchangeable binary process (Y_n) is a finite mixture of i.i.d. processes. In this section we propose an algorithm to compute the parameters $(\mathbf{p}, \boldsymbol{\pi})$ that identify the mixing measure μ. The algorithm is a by-product of Theorem 5 below, which follows by a well known theorem on Hankel matrices reported below, in a simplified version sufficient for our purposes:

Theorem 4 *Let $H_\infty = (h_{i+j})_{i,j\geq 0}$ be an infinite Hankel matrix of rank N. Then the entries of H_∞ satisfy an N-term recurrence equation of the form:*

$$h_m = a_{N-1}h_{m-1} + a_{N-2}h_{m-2} + \cdots + a_0 h_{m-N} \tag{11}$$

for suitable $(a_0, a_1, \ldots, a_{N-1})$ and for all $m \geq N$.
Define the characteristic polynomial

$$q(x) := x^N - a_{N-1}x^{N-1} - a_{N-2}x^{N-2} - \cdots - a_0, \tag{12}$$

and assume that its roots $p_1, \ldots, p_N$ are all simple. Then there exist linear combinators $\pi_1, \ldots, \pi_N$, such that the entries h_m of the Hankel matrix H_∞ can be written as

$$h_m = \sum_{k=1}^{N} \pi_k p_k^m. \tag{13}$$

This result, adapted to the Hankel matrix of a binary exchangeable process which is a finite mixture of i.i.d. sequences, is the basis for the identification of μ. One can prove the following

Theorem 5 *Let (Y_n) be a mixture of N i.i.d. binary sequences and μ the associated de Finetti measure. Then the rank of the matrix H_∞ associated with (Y_n) is N, the roots of the polynomial $q(x)$ defined in equation (12) are all distinct, and the measure μ is concentrated on the roots $p_1, \ldots, p_N$ of $q(x)$. Moreover the linear combinators $\pi_1, \ldots, \pi_N$ in equation (13) are all in $]0,1[$ and add up to 1. We have $\mu(p_k) = \pi_k$ for $k = 1; \ldots, N$.*

Theorems 4 and 5 allow us to construct an algorithm to identify the parameters $(\mathbf{p}, \boldsymbol{\pi})$ which identify the mixing measure μ associated with a mixture (Y_n) of N i.i.d. sequences, given the Hankel matrix H_N. The algorithm is presented below.

2.3.1 The algorithm

(Y_n) is a mixture of N i.i.d. sequences, thus by Theorem 5 the measure μ is concentrated on the N distinct roots of the polynomial $q(x)$ defined in equation (12). To identify μ we have to find $\mathbf{p}$ and $\boldsymbol{\pi}$. We find $p_1, \ldots, p_N$ as the roots of $q(x)$, we thus first need the coefficients $(a_0, \ldots, a_{N-1})$ of the polynomial $q(x)$, (with obvious meaning of notation $a_N = 1$). To get them it is enough to construct the matrix H_N as defined in equation (5). The matrix H_N in fact is a submatrix of H_∞, thus its entries satisfy the recurrence equation 11 in Theorem 4. This gives

$$\begin{aligned} h_N &= a_{N-1}h_{N-1} + a_{N-2}h_{N-2} + \cdots + a_0 h_0 \\ h_{N+1} &= a_{N-1}h_N + a_{N-2}h_{N-1} + \cdots + a_0 h_1 \\ &\cdots\cdots \\ h_{2N-1} &= a_{N-1}h_{2N-2} + a_{N-2}h_{2N-3} + \cdots + a_0 h_{N-1} \end{aligned}$$

Denoting by $H^{(N)} = (h_N, h_{N+1}, \dots, h_{2N-1})^\top$ the N-dimension vector of the first N elements of the $(N+1)$-th column of the matrix H_N and by $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^\top$, we can write the previous set of equations in matrix form as follows

$$H_{N-1}\mathbf{a} = H^{(N)}. \tag{14}$$

Compute the coefficients $(a_0, a_1, \dots, a_{N-1})$ of the polynomial $q(x)$ solving the linear system in equation (14) in the unknown $\mathbf{a}$. Once $\mathbf{a}$ is known, construct the polynomial $q(x)$ and find its roots, getting the points $p_1, \dots, p_N$ where μ is concentrated.

To find the weights $\pi_1, \dots, \pi_N$ recall that, by equations (9) and (4) for any m we have

$$h_m = p_Y(1^m) = \sum_{k=1}^{N} \pi_k p_k^m. \tag{15}$$

Defining

$$V := \begin{pmatrix} 1 & p_1 & p_1^2 & \dots & p_1^{N-1} \\ 1 & p_2 & p_2^2 & \dots & p_2^{N-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & p_N & p_N^2 & \dots & p_N^{N-1} \end{pmatrix} \tag{16}$$

$$W := \begin{pmatrix} \pi_1 & 0 & 0 & \dots & 0 \\ 0 & \pi_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \pi_N \end{pmatrix}, \tag{17}$$

the previous equation (15) implies

$$H_{N-1} = V^\top W V. \tag{18}$$

The points $p_1, \dots, p_N$ and the entries $h_0 = 1, h_1 = p_Y(1), \dots, h_{(N-1)} = p_Y(1^{(N-1)})$ are known, thus we can construct the matrices V and H_{N-1}, and find the matrix W inverting the previous relationship (note that V is always invertible), getting

$$W = (V^\top)^{-1} H_{N-1} V^{-1}. \tag{19}$$

The weights $\pi_1, \dots, \pi_N$ are the diagonal elements of W. This completes the construction of the measure μ.

Here is a summary of the algorithm proposed above:

- Find $\mathbf{a}$ solving the linear system
$$H_{N-1}\mathbf{a} = H^{(N)}$$
- Find the roots $p_1, p_2, \dots, p_N$ of the polynomial
$$q(x) := x^N - a_{N-1}x^{N-1} - a_{N-2}x^{N-2} - \dots - a_0.$$
- Construct the matrix V defined in equation (16)
- Compute the matrix W of the weights using the expression
$$W = (V^{-1})^\top H_{N-1} V^{-1}.$$

3 Mixtures of Markov chains

This section deals with mixtures of Markov chains. We have seen in the previous section de Finetti's theorem, which connects Mixtures of i.i.d. sequences to exchangeable processes. We will start the section recalling the definitions of mixture of Markov chains and of partial exchangeability, and the strict connection between them. This connection, i.e. the generalization of de Finetti's theorem, was proved first by Diaconis and Freedman, but pointed out by de Finetti himself many years earlier.[(*)] Then we restrict our attention to countable mixtures of Markov chains, extending the characterization of countable Mixtures of i.i.d. sequences due to Dharmadhikari to mixtures of Markov chains. At the end of the section we provide a characterization of finite mixtures of Markov chains, which is easily verifiable looking at the rank of a suitable class of Hankel matrices. This result partially extends the results we gave in the last section for finite Mixtures of i.i.d. sequences, but, differently from the previous section, we can not explicitly identify the mixing measure.

3.1 Preliminary notions

Let $\mathcal{P}$ the set of the transition matrices for Markov chains (Y_n) with values in $\{0,1\}$. Let $P \in \mathcal{P}$

$$P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix},$$

with $P_{ij} := \mathbb{P}\{Y_{n+1} = j \mid Y_n = i\}$ for $i,j = 0,1$. It holds $P_{00} + P_{01} = 1$ and $P_{10} + P_{11} = 1$. For a Markov chain starting at y_1 and with transition matrix P we have

$$\mathbb{P}\{Y_1^n = y_1^n\} = \prod_{t=1}^{n-1} P_{y_t y_{t+1}}.$$

We give the following

Definition 6 Y is a *mixture of homogeneous Markov chains* if there exists a probability μ on $\{0,1\} \times \mathcal{P}$ such that for any finite string y_1^n

$$\mathbb{P}\{Y_1^n = y_1^n\} = \int_{\mathcal{P}} \prod_{t=1}^{n-1} P_{y_t y_{t+1}} \mu(y_1, dP).$$

We recall below the definition of partially exchangeable processes, which are strictly connected with mixtures of Markov chains. To state it, we need the following

Definition 7 Two strings $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ and $\tau = \tau_1 \tau_2 \ldots \tau_n$ are *transition equivalent*, if they start with the same letter and they exhibit the same number of transitions from letter i to letter j, for $i,j \in \{0,1\}$.

[(*)] In the literature there are many different notions which go under the name of *partial exchangeability*. The conjecture of de Finetti about mixtures of Markov chains actually refers to a slightly different notion of partial exchangeability than the one used by Diaconis and Freedman and by us in this note. For a discussion on the relationship between these different definitions of partial exchangeability and for a complete proof of de Finetti's conjecture see [8].

Write $\sigma \sim \tau$ for transition equivalent strings. For example $\sigma = 1100101$ and $\tau = 1101001$ are transition equivalent, $\rho = 1110001$ is a permutation of σ, but σ and ρ are not transition equivalent. We can now give the following

Definition 8 The process (Y_n) is *partially exchangeable* if for transition equivalent strings $\sigma \sim \tau$

$$\mathbb{P}\big\{Y_1^n = \sigma_1^n\big\} = \mathbb{P}\big\{Y_1^n = \tau_1^n\big\}.$$

The prime example of partially exchangeable process is an exchangeable process, since the finite-dimensional joint distributions of an exchangeable process must be invariant under a larger class of permutations. A time homogeneous Markov chain starting at a fixed state is partially exchangeable.

The connection between partially exchangeable processes and mixture of Markov chains is stated in the following

Theorem 6 (Diaconis and Freedman 1980) *Let (Y_n) be a recurrent process.*
(Y_n) is partially exchangeable if and only if (Y_n) is a mixture of Markov chains.

3.2 Countable mixtures of Markov chains

We now restrict our attention to a special class of mixtures of Markov chains. To be precise

Definition 9 A binary mixture of Markov chains is *countable* (finite), if for any $y_1 \in \{0,1\}$, there are a *countable* (finite) set of indices K_{y_1}, and matrices $P^1, P^2, \ldots, P^k, \ldots$, with $k \in K_{y_1}$, such that letting $\pi_{y_1}^k := \mu(y_1, P^k)$, we get $\sum_k \pi_{y_1}^k = 1$.

In this case we get

$$\mathbb{P}\{Y_1^n = y_1^n\} = \sum_{k \in K} \pi_{y_1}^k \prod_{t=1}^{n-1} P_{y_t y_{t+1}}^k.$$

We have seen above that de Finetti's theorem for Mixtures of i.i.d. sequences extends to mixtures of Markov chains. Moreover in the previous section we have recalled a characterization of countable Mixtures of i.i.d. sequences. We can thus close the circle of theorems 1, 6 and 2 proving the following

Theorem 7 *Let (Y_n) be partially exchangeable.*
(Y_n) is a countable *mixture of Markov chains if and only if (Y_n) is a* countable *HHM.*

Proof. See [10]. □

3.3 Finite mixtures of Markov chains

Recall the notation $p_Y(1^n) := \mathbb{P}\{Y_1^n = 1^n\}$. In this section we pose and solve the following problem

Problem 2 *Given $p_Y(0), p_Y(0^2), \dots$ and $p_Y(1), p_Y(1^2), \dots$ for a partially exchangeable process (Y_n) decide whether (Y_n) is a finite mixture of Markov chains. If it is, find a lower and an upper bound for the number of Markov chains in the mixture.*

We assume the following technical condition

Condition 1 *μ factorizes as*

$$\mu(y_1, P) = \tilde{\mu}(y_1)\bar{\mu}(P),$$

where $\tilde{\mu}(\cdot)$ is a measure on $\{0,1\}$ and $\bar{\mu}(\cdot)$ is a measure on $\mathcal{P}$.

We are thus assuming that the random choice of the transition matrices in the mixture is independent from the initial condition.

For any element A of the Borel σ-field on $[0,1]$, let $\nu^{(0)}$ be the measure on $[0,1]$ defined as

$$\nu^{(0)}(A) := \bar{\mu}(S_A^{(0)}),$$

where

$$S_A^{(0)} := \{P \in \mathcal{P} \mid P_{00} \in A\}.$$

Define in the analogous way ν^1 and $S_A^{(1)}$. For a mixtures of Markov chains (Y_n) we have

$$\begin{aligned} p_Y(0^m) &= \int_{\mathcal{P}} P_{00}^{m-1}\mu(0, dP) = \int_{\mathcal{P}} P_{00}^{m-1}\tilde{\mu}(0)\bar{\mu}(dP) \\ &= \tilde{\mu}(0)\int_{\mathcal{P}} P_{00}^{m-1}\bar{\mu}(dP) = \tilde{\mu}(0)\int_{[0,1]} q^{m-1}\nu^{(0)}(dq). \end{aligned}$$

Thus $p_Y(0^m)$ is $(m-1)$-th moment of $\nu^{(0)}$, up to the multiplying constant $\tilde{\mu}(0)$.

Define for any $n \in \mathbb{N}$

$$\mathbf{H_n}^{(0)} := \begin{pmatrix} p_Y(0) & p_Y(00) & \dots & p_Y(0^{n+1}) \\ p_Y(00) & p_Y(000) & \dots & p_Y(0^{n+2}) \\ p_Y(000) & p_Y(0000) & \dots & p_Y(0^{n+3}) \\ \vdots & \vdots & \vdots & \vdots \\ p_Y(0^{n+1}) & p_Y(0^{n+2}) & \dots & p_Y(0^{2n+1}) \end{pmatrix},$$

and in the same way $\mathbf{H_n}^{(1)}$.

Let r_0 be the first integer n such that $det(\mathbf{H_n}^{(0)}) = 0$. If $det(\mathbf{H_n}^{(0)}) \neq 0$ for any n, put $r_0 = +\infty$. Define r_1 in the same way.

The following theorem gives the desired characterization of finite mixtures of Markov chains and the bounds on the number of Markov chains in a finite mixture.

Theorem 8 *(Y_n) is a finite mixture of Markov chains if and only if $\max\{r_0, r_1\} < +\infty$. If (Y_n) is a mixture of N Markov chains, then*

$$\max\{r_0, r_1\} \leq N \leq r_0\, r_1.$$

Proof. The proof of the theorem is carried out similarly to the proof of Theorem 3, for the details see [10]. □

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Analytic and algebraic varieties: the classical and the non archimedean case

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Abstract. The complex line, as a set of points, can be endowed with an analytic structure, as well as with an algebraic one. The choice of the topology and the related natural definition of functions on the space determine different geometric behaviors: in the example of the line, there are differential equations admitting solutions in both cases, and some that can be solved only in the analytic setting. The first part of the talk will focus on the algebraic and analytic structures of a variety over the field of complex numbers, while in the second part we will give an overview of the analogous constructions for varieties defined over a non archimedean field, touching the theory of rigid analytic spaces and its relation to the study of varieties over a discrete valuation ring of mixed characteristic in the framework of syntomic cohomology.

1 Introduction: sheaf theory and cohomology

In order to define a **variety**, we must specify

- a set of points X
- a topology on X
- a *sheaf of functions* over X.

Even if there exists a more general notion of *sheaf* (as functor), for our purposes it is enough to give the definition of sheaf of functions on a topological space.

Let X be a topological space. A **sheaf of functions** over X is the data of the set

$$\{\text{functions on } U\}$$

for every open $U \subset X$. This assigment is characterized by two main properties:
i) For each inclusion of open sets $V \subset U$ there is a restriction map

$$\rho_{V,U} : \{\text{functions on } U\} \to \{\text{functions on } V\}, \quad f \mapsto f_{|V}$$

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such that $\rho_{U,U} = \mathrm{id}$ and the restriction maps are compatible, i.e. if we have three open sets $W \subset V \subset U$, then $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$.
ii) (Crucial property) Consider a covering $\{U_i\}_{i \in I}$ of an open $U \subset X$, and for any i a function f_i on U_i. If for any $U_i \cap U_j \neq \emptyset$ we have $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$, then there must exist a unique function f on U such that $f_{|U_i} = f_i$.

Now that we have an idea of what a sheaf is, we can make some historical remarks concerning the development of the definition of cohomology.

Roughly speaking, we can consider **cohomology** as a method of assigning algebraic invariants to a geometric object. This object may either be simply a topological space or it may be provided with the structure of a variety; in another direction, its *natural setting* may be the field of complex numbers as well as some less familiar algebraic structure (this concept will be clarified with the example below)... There exists many cohomological theories, depending on the space we want to study.

In 1945 Eilenberg and Steenrod considered a cohomological theory with fixed coefficients to be a functor

$$\mathbf{Top} \to \mathbf{Ab}.$$

An important change of point of view came up in 1957, when Grothendieck defined the cohomological theory of a fixed topological space X as a functor

$$\mathbf{Sh}_{/X} \to \mathbf{Ab}.$$

The last definition is more general, in the sense that any sheaf (not necessarily constant!) can be taken as *coefficients* for cohomology. Furthermore, the category $\mathbf{Sh}_{/X}$ is an **abelian category** (i.e. a category as **Ab**, endowed with the notions of kernel, cokernel, exact sequence), hence it is provided with a good algebraic structure, while the category of topological spaces is not.

2 An example over $\mathbb{C}$

We want to give an idea of the construction of the theory of algebraic and analytic varieties over non archimedean fields, analyzing for first a simple example in the familiar *archimedean* setting of the complex numbers. In order to endow the complex line with a structure of algebraic variety, obtaining $\mathbb{C}^{alg}$, we must specify which are the topology and the sheaf of functions we want to consider on $\mathbb{C}$:

- points: elements of $\mathbb{C}$
- Zariski topology: in the general definition, closed sets are the algebraic sets (i.e. zeros of polynomials), but in the particular case of case of the line Zariski topology coincides with cofinite topology
- the sheaf of **algebraic functions**: $\mathcal{O}_{\mathbb{C}^{an}}$ defined by

$$U \mapsto \{\text{locally rational functions (with complex coefficients) on } U\}.$$

Now we are going to define the analytic structure of the complex line: the set of points does not change, but we choose another topology on it and consequently a different sheaf of functions:

- points: elements of $\mathbb{C}$
- Euclidean topology (generated by the set of open balls)
- the sheaf of **analytic functions**: $\mathcal{O}_{\mathbb{C}^{an}}$ defined by
$$U \mapsto \left\{ \text{functions locally } f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \right\}.$$

2.1 Serre's GAGA

The GAGA theorem, due to Serre in 1950, is a strong result of comparison between the analytic and the algebraic structure of a variety defined over the field of complex numbers. Before giving the general form of GAGA theorem, we will compare the algebraic and analytic settings in the example of the line, and then we will see how this naive approach can be generalized.

At level of topological spaces there is a continuous map

$$i : \mathbb{C}^{an} \to \mathbb{C}^{alg}$$

due to the fact that Euclidean topology is finer than Zariski topology.

On the other hand, at level of sheaves of functions there is an inclusion in the opposite direction

$$\mathcal{O}_{\mathbb{C}^{alg}} \hookrightarrow i_* \mathcal{O}_{\mathbb{C}^{an}},$$

because every rational function defined on a Zariski open U with poles outside U can be considered in particular as an analytic function, if we regard U as an analytic open by the map i (this is the meaning of the *direct image* i_* in this case).

In order to have an equality at level of global functions we must compactify: in our example, functions globally defined on $\mathbb{P}_1^{alg}$ are constant functions, exactly as the functions globally defined on $\mathbb{P}_1^{an}$ (by Liouville theorem).

General GAGA result

Let $(X, \mathcal{O}_X)$ be a scheme of finite type over $\mathbb{C}$.

- There is a topological space X^{an}, which as set consists of the closed points of X, with an inclusion
$$i : X^{an} \hookrightarrow X.$$
- There is a sheaf $\mathcal{O}_X^{an}$ on X^{an} such that $(X^{an}, \mathcal{O}_X^{an})$ is an analytic space. The association
$$(X, \mathcal{O}_X) \mapsto (X^{an}, \mathcal{O}_X^{an})$$
is functorial.

- For every sheaf $\mathcal{F}$ on X there is a sheaf $\mathcal{F}^{an}$ on X^{an} (functorial association) and a map of sheaves of O_X-modules
$$\mathcal{F} \to i_*\mathcal{F}^{an}$$

- If X is a projective scheme over $\mathbb{C}$ and $\mathcal{F}$ is a coherent sheaf on X, then
$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an}).$$

2.2 Differential equations

We have just seen how, compactifying the space, the functions globally defined in the analytic and algebraic setting coincide. On the other hand, when we deal with differential equations, compactifying is not enough to have similar behaviours between the algebraic structure and the analytic one; let's investigate what happens by mean of an example. Consider the following differential equations:

- $L_1(f) = z\frac{df}{dz} - 3f = 0$
- $L_2(f) = \frac{df}{dz} - f = 0$

The first one has z^3 as solution, that means it admits solutions in both settings (algebraic and analytic). The second one has e^z as solution (only analytic). If we want algebraic and analytic solutions to coincide, we need to consider differential equations having a *regular singular point* at ∞:

Definition 1 (regular singular point) Consider a linear differential equation of n-th order $\sum_{i=0}^{n} p_i(z)f^{(i)}(z)$ with $p_i(z)$ meromorphic functions (we can assume $p_n(z) = 1$). A point a is said to be a regular singular point if $p_{n-i}(z)$ has a pole of order at most i at a.

Let's return to our example. By the change of variable
$$z = \frac{1}{z_\infty}, \qquad \frac{d}{dz} = -z_\infty^2 \frac{d}{dz_\infty}$$
applied to the previous differential equations, we get

- $L_1(f) = \frac{df}{dz_\infty} + \frac{3}{z_\infty}f$
- $L_2(f) = \frac{df}{dz_\infty} + \frac{1}{z_\infty^2}f$

so the first one has ∞ as regular singular point while for the second one ∞ is not a regular singular point.

3 Algebraic and analytic over $\mathbb{Q}_p$

Before passing to the study of a non archimedean example, we want to recall some definitions about the notion of absolute value over a field.

Definition 2 An *absolute value* over a field K is a map $|\quad|: K \to \mathbb{R}_{\geq 0}$ s.t.
(i)$| a |= 0 \Leftrightarrow a = 0$
(ii)$| ab |=| a || b |$
(iii)$| a + b |\leq| a | + | b |$ (triangle inequality)
for all $a, b \in K$. If furthermore $| a + b |\leq \max\{| a |, | b |\}$ the absolute value is called *non Archimedean*.

Fix a prime number p. Let $x \in \mathbb{Q}^{\times}$. The p-adic valuation of x, $v_p(x)$, is the only integer such that

$$x = p^{v_p(x)} \frac{a}{b} \qquad p \nmid a, \quad p \nmid b.$$

We define the *p-adic absolute value* of x by $| x |_p= p^{-v_p(x)}$ if $x \neq 0$, and we set $| 0 |_p= 0$. This is a classical example on non archimedean absolute value.

We recall that, if $(K, |\quad|)$ is a field provided with an absolute value, then we define $\hat{K}$ as the smallest complete field containing K. Here are two fundamental examples:

- the completion of $\mathbb{Q}$ with respect to the usual absolute value is $\mathbb{R}$;
- the completion of $\mathbb{Q}$ with respect to the p-adic absolute value (non archimedean) is the field $\mathbb{Q}_p$.

3.1 An example over $\mathbb{Q}_p$: an idea of rigid analytic spaces

Concerning the non archimedean algebraic setting, we will only say that it is possible to define the algebraic line over $\mathbb{Q}_p$ using a tool that generalizes the construction of the algebraic complex line: the *spectrum* of a ring.

We are interested now in the problem of constructing the analitic structure over the field $\mathbb{Q}_p$: how to define functions?

In the archimedean framework of the complex numbers, there exist two main different concepts of *good* functions: holomorphic functions (defined via complex differentiability) and analytic functions (i.e. convergent power series). We can prove the equivalence of these definitions by using a formula involving the definition of integral along a path (the Cauchy formula).

The first idea to approach the non archimedean setting could be to give the definition of path over $\mathbb{Q}_p$. Unfortunately, the topology of $\mathbb{Q}_p$ is **totally disconnected**, i.e. any subset of $\mathbb{Q}_p$ consisting of more than just one point is not connected.
Hence the concept of path in $\mathbb{Q}_p$ is not useful, because any continuous map

$$\sigma : [0, 1] \to \mathbb{Q}_p$$

must be constant.

We could then try to define $f : U \to \mathbb{Q}_p$ to be locally analytic if it admits a convergent power series around each $x \in U$.
Now, if $U = D^-(0, 1)$ and $0 < r < 1$ then

$$U = \bigcup_{a \in U} D^-(a, r)$$

is a partition of U into disjoint discs D_i. Consider f_i a different constant function for every i, then $f : U \to \mathbb{Q}_p$ defined by $f_{|U_i} = f_i$ is locally analytic but does not glue to a globally convergent power series expansion as expected.

A way to solve this problem is to consider as open subsets only some *admissible open subsets* obtaining a **Grothendieck topology**, that is a generalization of the concept of topology for which we still have a good notion of glueing of functions. *Rigid analytic spaces* will be spaces equipped with a Grothendieck topology, hence they will be **Grothendieck sites**.

We want to remark that a GAGA result also holds in the framework of non archimedean fields: it relates K-schemes of finite type (and sheaves on them) with rigid analytic spaces (and sheaves).

See Bosch [3] for a complete treatement of the theory of rigid analytic spaces.

4 Rigid analytic spaces in the framework of syntomic cohomology

Syntomic cohomology is a cohomological theory for varieties defined over a discrete valuation ring of mixed characteristic.

Definition 3 A discrete valuation ring (DVR) is a principal ideal domain $\mathcal{V}$ with a unique non-zero prime ideal $\mathbf{m}$. We will call $k = \frac{\mathcal{V}}{\mathbf{m}}$ the residue field and $K = \mathrm{frac}(\mathcal{V})$ the fraction field of $\mathcal{V}$. A DVR $\mathcal{V}$ is said to be of mixed characteristic $(0, p)$ if $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p$.

The main example for us is the ring of p-adic numbers $\mathbb{Z}_p$, that is the ring of integers of the field $\mathbb{Q}_p$. In this case $\mathbf{m} = (p)$, $K = \mathbb{Q}_p$ and $k = \mathbb{F}_p$.

4.1 Syntomic cohomology

Let $\mathcal{V}$ be a DVR of mixed characteristic, and let X be a smooth variety over $\mathcal{V}$. There are two varieties *related* to X by which we can define a cohomology for X:

- the *special fiber* X_k, i.e. a variety over k: we can compute its rigid cohomology by mean of the **rigid analytic space** associated to the formal scheme $\hat{\overline{X}}_{\mathcal{V}}$ (formal completion along the special fiber), where $\overline{X}_{\mathcal{V}}$ is a compactification of $X_{\mathcal{V}}$ (see Berthelot [1])
- the *generic fiber* X_K, i.e. a variety over K: we can compute its de Rham cohomology.

There is a map, called *specialization*, that links the rigid cohomology of X_k with the de Rham cohomology of X_K. By the specialization morphism it is possible to give the definition of *syntomic cohomology* of the variety X (see Besser [2]).

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Interest rate derivatives pricing when the short rate is a continuous time finite state Markov process

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Abstract. The purpose of this presentation is to price financial products called "interest rate derivatives", namely financial instruments in which the owner of the contract has the right to pay or receive an amount of money at a fixed interest rate in a specific future date. The pricing of these products is here obtained by assuming that the spot rate (i.e. the interest rate at which a person or an institution can borrow money for an infinitesimally short period of time) is considered as a stochastic process characterized by "absence of memory" (i.e. a time-continuous Markov chain). We develop a pricing model inspired by work of Filipovic'-Zabczyk which assumes the spot rate to be a discrete-time Markov chain: we extend their structure by considering, instead of deterministic time points, the random time points given by the jump times of the spot rate as they occur in the market. We are able to price with the same approach several interest rate derivatives and we present some numerical results for the pricing of these products.

The most popular financial contract is the *zero-coupon bond*, a contract which guarantees its holder the payment of a unit monetary amount at time T, with no intermediate payments: its value at time $t < T$ is denoted by $p(t,T)$ and $p(T,T) = 1$ for all T. Let a filtered probability space be given by $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \mathbb{P})$ where $\mathbb{P}$ is the physical measure. Under the hypothesis of *Absence of Arbitrage Opportunity* (for which it is not possible to make profit without risk), the bond price can be written in terms of the *"short (spot) rate"*, the interest rate prevailing in the market at which an investment accrues continuously to every instant t:

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t]$$

with $\widetilde{\mathbb{P}}$ a martingale measure equivalent to $\mathbb{P}$.

The short rate r can be modeled by several possible stochastic processes as diffusion processes, Lévy processes and also by deterministic functions. For instance Filipovic' and Zabczyk [5] consider the spot rate $r(t)$ a Markov chain (MC) with a finite state space. Since the short rate is a MC in discrete time, the number of jumps in a fixed time interval is deterministic.

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However, in real markets the spot rate does generally not change at deterministic times but it rather " jumps" at random times.

This suggests to model the spot rate as a continuous time Markov chain (CTMC) with a finite state space $E = \{r^1, r^2, \dots, r^N\}$, $N \in \mathbb{N}$, $r^i \in \mathbb{R}$, $i = 1, \dots, N$. We assume that under the above martingale measure $\tilde{\mathbb{P}}$ the transition intensity matrix of the chain is given by $Q = \{q_{i,j}\}_{i,j=1,\cdots,N}$. For a maturity T and an evaluation time t, the number of jumps of the spot rate between t and T (denoted by $\nu_{t,T}$), namely the number of transition of the MC, is random and can take arbitrarily large values.

The purpose of this study is to obtain, in a setup where the short rate evolves as a continuous time Markov process, explicit formulae for prices of bond and *interest rate derivatives*, contracts where the holder has the right to pay or receive interest at a fixed rate instead of a floating rate. The pricing of bonds and interest derivatives as caps and swaptions will be shown to be particular cases of the pricing of a fictitious financial product, namely the " Prototype product" which thus represents a unified approach to the pricing of interest rate related products.

We call **Prototype product** a financial product which guarantees to deliver at maturity T a certain payoff $\vartheta_0(r(T))$ which depends on the value taken by the spot rate at the date of maturity T:

$$\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}} \text{ with } r^i \in E \text{ and } w_i \in \{0\} \cup \mathbb{R}_+.$$

Recall that for a given current state $r^i \in E$ $(1 \leq i \leq N)$ which below we generically denote by v:

- the time to the next jump is an exponential random variable with parameter $q(v) \triangleq q_i = \sum_{j=1, j\neq i}^{N} q_{i,j}$;
- the transition probability to the next state $u = r^j$ is
  $p_{v,u} \triangleq p_{i,j} = \begin{cases} p_{i,j} = \frac{q_{i,j}}{q_i} & \text{if } i \neq j \\ p_{i,j} = 0 & \text{if } i = j. \end{cases}$

Furthermore, let ν_t denote the number of jumps of the CTMC up to time t and $\mathfrak{M} \triangleq \{\vartheta: E \to \{0\} \cup \mathbb{R}_+ \mid \vartheta(v) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{v = r^i\}},\ w_i \in \{0\} \cup \mathbb{R}_+,\ \forall i = 1, \dots, N\}$ endowed with the sup norm. We have now the following results:

Proposition 1 *For a martingale measure $\tilde{\mathbb{P}}$, the price of the Prototype product at time $t < T$ can be represented as*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t}) \tag{1}$$

where the functions ϑ_k are obtained recursively, after k steps, by iterating the operator $\mathcal{T}$:

$$\mathcal{T}\vartheta(v) \triangleq \int_{\mathbb{R}} q(v) e^{-(v+q(v))s} \Big(\sum_{u \in E} p_{v,u} \vartheta(u) \Big) ds, \quad \vartheta \in \mathfrak{M}.$$

Proposition 2 *The operator $\mathcal{T} : \mathfrak{M} \to \mathfrak{M}$ is a contraction operator with fixed point $\vartheta^* = 0$ and contraction constant $\gamma \triangleq \sup_{v \in E} \frac{q(v)}{v+q(v)}$.*

For the distribution of $\nu_{t,T}$ we have:

Proposition 3 *The following holds*

$$
(2) \qquad \begin{cases} \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^m) = \displaystyle\sum_{\substack{i_1,\dots,i_k=1 \\ i_1 \neq m, i_2 \neq i_1, \dots, i_k \neq i_{k-1}}}^{N} e^{q_m t - q_{i_k} T} \varphi_k(Q) \cdot \Psi_k(t,T,Q) \\ \tilde{\mathbb{P}}(\nu_{t,T} = 0 | r_{\nu_t} = r^m) = e^{-q_m(T-t)} \end{cases}
$$

where m is a fixed index in $\{1, \dots, N\}$, Ψ_k is the multiple integral

$$
(3) \qquad \Psi_k(t,T,Q) \triangleq \int_t^T e^{(q_{i_1} - q_m)t_1} \int_{t_1}^T e^{(q_{i_2} - q_{i_1})t_2} \cdots \int_{t_{k-1}}^T e^{(q_{i_k} - q_{i_{k-1}})t_k} \mathrm{dt_k} \dots \mathrm{dt_2} \mathrm{dt_1}
$$

and

$$
\varphi_k(Q) \triangleq q_{m,i_1} \cdot \dots \cdot q_{i_{k-1},i_k}.
$$

Formula (1) involves an infinite number of terms to compute; however, since the ϑ_ks are obtained by applying k times a contraction operator on ϑ_0, there exists -for an arbitrarily small ϵ- a natural number n_ϵ such that $V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) \triangleq \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t}) \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t})$ approximates arbitrarily well the real price $V_{\vartheta_0,t,T}$ in the sense that

$$
(4) \qquad |V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) - V_{\vartheta_0,t,T}(r_{\nu_t})| < \epsilon \text{ uniformily in } (t, T, r_{\nu_t}).
$$

The pricing of bond and other interest rate derivatives can now be obtained as follows

P1 : a bond which matures at time T can be viewed as a Prototype product with payoff $\varphi_0(\cdot) = \sum_{i=1}^N w_i \mathbf{I}_{\{\cdot = r^i\}}$ where $w_i \equiv 1$: the price of a T-bond evaluated at time t is equal to $V_{\varphi_0,t,T}$;

P2 : the prices of both caps and swaptions can be represented as linear combinations of the prices of N Prototype products $V_{\psi_0^n,t,T}$ with payoffs ψ_0^n defined as follows for each $n \in \{1, \dots, N\}$:

$$
\begin{cases} \psi_0^n(\cdot) = \sum_{i_0=1}^N w_{i_0}(n) \mathbf{I}_{\{\cdot = r^{i_0}\}} \\ w_{i_0}(n) = \begin{cases} 0, & i_0 \neq n \\ 1, & i_0 = n \end{cases} \end{cases}
$$

Moreover, on the basis of the result in (4), we are able to obtain computable expressions for the financial products already mentioned above:

P1$_\epsilon$: a " good" approximation of the bond price is obtained by considering $V^\epsilon_{\varphi_0,t,T}$ instead of $V_{\varphi_0,t,T}$ in the expression for **P1**;

P2$_\epsilon$: " good" approximations of the prices of caps and swaptions are obtained by considering $V^{\epsilon}_{\psi_0^n,t,T}$ instead of $V_{\psi_0^n,t,T}$ for each $n \in \{1,\dots,N\}$ in the expression for **P2**.

We have finally written a code in `C++` to illustrate numerically the theoretical results.

In view of the expressions for **P1**$_\epsilon$ and **P2**$_\epsilon$, the pricing of bonds and other interest rate derivatives can always be reduced to the pricing of a generic Prototype product $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t})\tilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$ (where n_ϵ is determined a priori in accordance with the value of ϵ) and we call *"Prototype Product Method"* (***PPM***) this approach to price bonds, caps and swaptions. We adopt two different approaches:

PPM(EF) : the explicit formula as in accordance with **P1**$_\epsilon$ and **P2**$_\epsilon$;

PPM(MC) : a full simulation approach based on the MonteCarlo technique for which

$$\frac{1}{M}\sum_{l=1}^{M}[\widetilde{Q}^{\nu^l_{t,T}} \cdot \theta_0(\underline{r})]_i \overset{M\to\infty}{\longrightarrow} V_{\vartheta_0,t,T}(r^i) \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

with M the number of steps ***"stepsMC"*** and $\nu^l_{t,T}$ the l-th simulation outcome of the random variable $\nu_{t,T}$.

One possible test of the validity of the pricing approach proposed in this study is as follows: starting from a continuous-time affine model for the short rate, for which an exact analytic bond pricing formula (***CF***) is available, first approximate the diffusion by a CTMC using "Kushner's approximation" (***K-A***) (see Di Masi-Runggaldier [3]) and then apply the Prototype Product Method (***PPM(EF)*** and ***PPM(MC)***). Moreover we compute the prices with a method largely used in finance called Recombining Binomial Tree (***RBT***) method by using the algorithm suggested in Costabile-Leccadito-Massabò [2]. Since ***PPM*** requires the intermediate spatial discretization to obtain a CTMC model to which then to apply our method, the numerical results feel the effects of the error of this approximation. Let the trajectories of the spot rate be given by the following stochastic differential equation (CIR affine term structure model)

$$\begin{cases} dr(t) = \theta(k - r(t))dt + \sigma\sqrt{r(t)}dW_t \\ r(0) = \widetilde{r}. \end{cases}$$

In the following the numerical results for the prices of T-maturity zero coupon bonds:

Bond prices with **CF**, **RBT** and **PPM(MC)+K-A**
(***stepsMC***=***stepsRBT***=500)

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}(=r^i)$	0.01	0.01	0.01	0.02	0.02	0.02
k	0.01	0.01	0.01	0.02	0.02	0.02
θ	0.8	0.8	0.8	0.5	0.5	0.5
σ	0.1	0.1	0.1	0.05	0.05	0.05
CF	0.995014	0.980245	0.951463	0.990051	0.960822	0.905047
RBT	0.995042	0.980302	0.951556	0.99007	0.960898	0.905226
PPM(MC)+K-A	**0.995024**	**0.980276**	0.951621	0.990143	0.960734	0.905318

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}(=r^i)$	0.03	0.03	0.03	0.02	0.02	0.02
k	0.03	0.03	0.03	0.02	0.02	0.02
θ	1.1	1.1	1.1	1.2	1.2	1.2
σ	0.1	0.1	0.1	0.1	0.1	0.1
CF	0.985116	0.941861	0.861095	0.990053	0.960849	0.905072
RBT	0.985146	0.941974	0.86135	0.990072	0.960926	0.905251
PPM(MC)+K-A	**0.985128**	**0.941968**	**0.861319**	**0.990059**	0.95647	0.90193

Bond prices with **CF**, **RBT**,**PPM(MC)+K-A** and **PPM(EF)+K-A** (***stepsMC***=***stepsRBT***=500)

T(years)	0.5	0.5	0.5	0.5
$\widetilde{r}(=r^i)$	0.1	0.1	0.2	0.3
k	0.1	0.1	0.2	0.3
θ	0.1	0.4	0.2	0.3
σ	0.1	0.05	0.2	0.3
CF	0.951249	0.951234	0.904977	0.86114
RBT	0.951343	0.951329	0.905157	0.861394
PPM(MC)+K-A	0.951022	0.950859	0.905229	0.861104
PPM(EF)+K-A	**0.951324**	0.951723	**0.905012**	0.861756

The Prototype Product Method, by using either the explicit formulae or the Monte Carlo simulations, is competitive with the **RBT** method which is widely used to compute the price of zero-coupon bonds (see results in **bold**). Moreover we are able to obtain numerical results for prices of caps and swaptions with the same complexity required for the computation of bond prices (considered as a particular case of Prototype Product) because all the prices of these interest rate derivatives can be viewed as linear combinations of Prototype Product prices. On the other hand, better than other existing approaches, our method applies straightforwardly to CTMC models for the short rate. Above all it can be easily extended to the more realistic multifactor case contrary to Recombining Binomial Tree method for which the dimensionality of the solution approach grows considerably with the number of the factors. We can in fact easily generalize the pricing based on the Prototype product when the short rate is expressed as a function of several correlated CTMCs.

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Holomorphic sectors and boundary behavior of holomorphic functions

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Abstract. Forced extendibility of holomorphic functions is one of the most important problems in several complex variables: it is a well known fact that a function defined in an open set Ω of $\mathbb{C}^n$ extends across the boundary at a point where the Levi form of ∂D (i.e. the complex hessian of its defining function restricted to the complex tangent space) has at least one negative eigenvalue. A fundamental role in this result is played by analytic discs, i.e. holomorphic images of the standard disc. After describing the construction of discs attached to a hypersurface, we will show how they induce the phenomenon described above. Finally, we will introduce a new family of discs, nonsmooth along the boundary, that will allow us to establish analogous results under various geometric conditions on the boundary of the domain.

1 Forced extendibility of holomorphic functions

The forced extendibility is one major peculiarity of holomorphic functions of several complex variables: one can find open sets Ω of $\mathbb{C}^n$ $(n \geq 2)$ and an extension map

$$hol(\Omega) \longrightarrow hol(\widetilde{\Omega}),$$

where $\widetilde{\Omega}$ is a bigger open set in $\mathbb{C}^n$. This is in striking contrast with the situation that can be found in the real setting, where given an open set $\Omega \in \mathbb{R}^n$ and a boundary point x_0, there always exist functions that do not extend to any full neighborhood of x_0 (e.g. $\frac{1}{|x-x_0|}$).

One important tool in determining sets of extendibility of holomorphic functions is given by the study of analytic discs.

Definition 1 An *analytic disc* is a holomorphic map

$$A : \Delta \longrightarrow \mathbb{C}^n,$$

continuous up to the boundary of the standard disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$; given a subset $M \subset \mathbb{C}^n$, a disc A is said to be *attached* to M if $A(\partial\Delta) \subset M$.

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Let Ω be an open set of $\mathbb{C}^n$ and f a holomorphic function defined in Ω; by the celebrated theorem of Baouendi and Treves of [2], we can find a sequence of polynomials f_λ that converges locally on $\partial\Omega$ to f. Given a sufficiently small disc A attached to $\partial\Omega$, the sequence $f_\lambda \circ A$ converges on $\partial\Delta$; but then, for the maximum principle, it must converge inside Δ, i.e. the sequence f_λ converges on the set $A(\bar{\Delta})$. Suppose now that $\widetilde{\Omega}$ is an open set filled up by small analytic discs attached to $\partial\Omega$: then the sequence of polynomials f_λ converges in $\widetilde{\Omega}$ to a holomorphic function.

The aim of the next sections will be to attach discs to the boundary of open sets in $\mathbb{C}^n$, and to give examples of geometric conditions under which there actually is holomorphic extension (i.e. the open union of discs attached to $\partial\Omega$ lies outside Ω).

2 Attaching discs to a real hypersurface in $\mathbb{C}^n$ ([1], [5], [10])

Let $M = \partial\Omega$ be a smooth real hypersurface in $\mathbb{C}^n$, locally given by a real equation $r = 0$, and $p \in M$. After a complex affine change of coordinates, we can suppose that $p = 0$ and that, in a neighborhood of 0, M is the set

$$\{(x + iy, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : y = h(x, z')\}, \tag{1}$$

where h is a real function such that $h(0) = 0$ and $dh(0) = 0$. Given an analytic disc $w : \bar{\Delta} \to \mathbb{C}^{n-1}$ with $w(1) = 0$ and a point $z = (x + iy, z') \in M$, we want to find $u + iv : \bar{\Delta} \to \mathbb{C}$ such that $A(\cdot) = (u(\cdot) + iv(\cdot), z' + w(\cdot))$ is a disc attached to M with $A(1) = z$. Suppose the hypersurface M is rigid, that is, the function h is independent of the real variable x; then, setting $v(\cdot) = h(z' + w(\cdot))$ and $u(\cdot) = -T_1(v(\cdot)) + x$, we can find a function satisfying our requirements. Here T_1 denotes the Hilbert transform on the unit circle $\partial\Delta$ normalized at 1, defined by

$$T_1 : u|_{\partial\Delta} \mapsto v|_{\partial\Delta},$$

where v is the unique real function on $\partial\Delta$ such that $v(1) = 0$ and $u + iv$ extends holomorphically inside Δ. By Privalov's classical theorem T_1 is a continuous functional between the Lipschitz spaces $C^{k,\alpha}(\partial\Delta)$ ($k \geq 0$, $0 < \alpha < 1$).

The previous result holds even in the more general setting, where the function h depends on x. In fact we have:

Theorem 1 *Let M be a smooth real hypersurface given by (1) around $p = 0$. Then, for any disc $w \in C^{k,\alpha}(\bar{\Delta}, \mathbb{C}^{n-1})$, small in $C^{k,\alpha}$-norm and with $w(1) = 0$, and for any $z = (x+iy, z') \in M$ close to 0, there exists a unique analytic disc $A(\cdot) = (u(\cdot)+iv(\cdot), z'+w(\cdot))$ attached to M with $A(1) = z$.*

3 Extension along Levi directions

We will review Lewy's extension theorem in the form of Boggess-Polking [6], closely following the approach of [10]. Let M be a smooth real hypersurface given by $r = 0$.

Definition 2 The *Levi form* of $M = \partial\Omega$ at a point p is the hermitian form

$$L_M(p)(X, \overline{Y}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) X_j \overline{Y_k},$$

for $X, Y \in T_p^{\mathbb{C}} M = T_p M \cap i T_p M$ (the *complex tangent space*).

The Levi form is invariant under holomorphic change of coordinates, and its rank and signature are well defined (that is, independent of the defining function r), up to the choice of an orientation: as a convention, we will suppose that the open set Ω is given by $\{r < 0\}$.

When the Levi form of M has at least one negative eigenvalue, holomorphic functions defined in Ω extend across the boundary:

Theorem 2 *Let $M = \partial\Omega$ as above. Suppose that*

$$\partial_{w_0} \partial_{\bar{w}_0} h(p) < 0$$

for a complex tangential vector w_0. Then there is a full neighborhood U of p in $\mathbb{C}^n$ with an extension map

$$hol(U \cap \Omega) \longrightarrow hol(U).$$

Proof. We will construct a family of discs $\{A\}$ attached to M, with $A(1) = z$ describing a neighborhood of p, and prove they are transversal to M at 1 with a uniform bound for the angle they form with TM; then the rays $A([0, 1])$ will fill up the desired neighborhood of p, forcing the extension of the holomorphic functions defined on Ω.

Let M be defined by (1) in coordinates $(x + iy, z')$, and define the z'-component of a disc $A_{z,\eta}$ (for $z = (x + iy, z')$ close to p and η small) as $w_\eta(\tau) = \eta w_0(1 - \tau)$. By Theorem 1, we can find a disc $A_{z,\eta}(\cdot) = (u_\eta(\cdot) + iv_\eta(\cdot), z' + w_\eta(\cdot))$ attached to M and such that $A_{z,\eta}(1) = z$. Fix $z = p = 0$; it is easy to see (by the normal form of the hypersurface) that the Taylor development of $\partial_t v_\eta$ (for $\tau = te^{i\vartheta} \in \Delta$) with respect to η reduces to

$$\partial_t v_\eta = \partial_t \partial_\eta^2 v_\eta|_{\eta=0} \frac{\eta^2}{2} + o^2.$$

Recalling that $v_\eta = h$ on $\partial\Delta$, and applying a further change of holomorphic coordinates, we can prove that

$$\partial_\eta^2 v_\eta = 2\partial_{w_0} \partial_{\bar{w}_0} h |1 - \tau|^2 \text{ on } \partial\Delta.$$

Since $|1 - \tau|^2|_{\partial\Delta} = 2\mathrm{Re}\,(1 - \tau)|_{\partial\Delta}$, we have

$$\partial_t \partial_\eta^2 v_\eta|_{\eta=0} = -4\partial_{w_0} \partial_{\bar{w}_0} h > 0,$$

that is, the ray of the disc A_η is transversal to $\partial\Omega$ and points outside Ω. The final step of the proof consists in moving z near 0 for a fixed small η_0, obtaining the desired family of discs. $\square$

4 Higher type hypersurfaces and α-discs

We will now extend the previous result to a more general case. Suppose the hypersurface M is of type k at a point p, for $k \geq 3$, that is

$$\begin{cases} \partial^{\alpha}_{z_j}\partial^{\beta}_{\bar{z}_k} h = 0 \text{ for } |\alpha| + |\beta| < k \\ \partial^{\alpha}_{w_0}\partial^{\beta}_{\bar{w}_0} h < 0 \text{ for some } |\alpha| + |\beta| = k. \end{cases} \tag{2}$$

Applying the same construction used in Theorem 2, one realizes that, in order to control the sign of the lower order term of the Taylor development at $\eta = 0$, we need to restrict to a sector of complex angle π/k, and hence to attach discs which are not C^1 along the boundary as in the previous situation. A new class of discs with Lipschitz boundary is then needed to overcome the technical difficulty of determine the directions of the discs at their singular points.

Definition 3 Let $0 < \alpha < 1$, $d = d(\alpha)$ the unique positive integer such that $d\alpha < 1 \leq (d+1)\alpha$, and fix β satisfying

$$\begin{cases} 0 < \beta \leq (d+1)\alpha - 1 \text{ if } (d+1)\alpha > 1 \\ 0 < \beta \leq (d+2)\alpha - 1 \text{ if } (d+1)\alpha = 1. \end{cases}$$

The class $\mathcal{P}^{\alpha}$ is defined as

$$\mathcal{P}^{\alpha}(\partial\Delta) = \mathbb{C}_d[(1-\tau)^{\alpha}] + C^{1,\beta}(\partial\Delta) \subset C^{\alpha}(\partial\Delta),$$

where $\mathbb{C}_d[(1-\tau)^{\alpha}]$ is the space of complex polynomials of degree at most d in the variable $(1-\tau)^{\alpha}$.

This is the smaller subspace of C^{α}, closed under the Hilbert transform, containing our model disc $(1-\tau)^{\alpha}$, hence it is immediate to prove an analogous of Theorem 1 in such a class. The peculiarity of this construction is given by the fact that the normal component of $\mathcal{P}^{\alpha}$-discs is of class C^1 (see [8]). It is now possible to prove the following

Theorem 3 *Let $M = \partial\Omega$ be a smooth hypersurface given by (1) in a neighborhood of $p = 0$, and suppose that (2) holds at p for a $w_0 \in T^{\mathbb{C}}_p M$. Then there is a full neighborhood U of p in $\mathbb{C}^n$ with an extension map*

$$hol(U \cap \Omega) \longrightarrow hol(U).$$

As a final remark, we point out that the result still holds if one just consider boundary values of holomorphic functions, defined in a sector with complex angle π/k. This technique can then be used to prove generalizations of classical results about extension of CR functions, and propagation of extendibility for CR functions (see [3], [8], [9]).

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Edge-connectivity augmentation

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Abstract. Edge-connectivity augmentation of a graph is as follows: given a graph and an integer k, find a minimum set of edges of which addition makes the graph k-edge-connected. The following pages are an brief introduction to edge-connectivity augmentation problems. In particular, we focus on the partition constrained version of edge-connectivity augmentation of a graph [1], and overview the ingredients of its generalization to hypergraphs by Bernáth, Grappe and Szigeti [6].

Introduction

Graphs are often used to model real-world communication networks: a graph is a set of points called vertices with connexions between them, called edges. The robustness of a network is the minimum number of failure of connexions that disconnects it, and its study is of practical interest. Here, we are interested in increasing the robustness of a network by adding new connexions. Motivated by practical questions, we require the addition of a minimum number of connexions. In terms of graphs, the notion of edge-connectivity captures the robustness of a network, and increasing it leads to the problems of edge-connectivity augmentation.

The following few pages are intended to introduce the subject of edge-connectivity augmentation and we overview various positive results in the field. By positive, we mean solvable in polynomial time. We try to give an intuition of what matters in such problems, and as proofs may be found in the list of references, we skip all of them. For a survey, we refer to [7].

Let us begin with a few definitions. Let $G = (V, E)$ be a graph, V denotes the set of vertices and E the set of edges. E is a set of couples of vertices where repetitions are allowed, that is there may be several copies of the same edge. For a set $X \subset V$, let $\delta_G(X)$ be the set of edges in E containing one vertex of X and one of $V - X$, and let $d_G(X) = |\delta_G(X)|$. Let k be an integer, the graph G is called ***k*-edge-connected** if there exists k edge-disjoint paths between every pair of vertices, or equivalently by Menger's theorem, if $d_G(X) \geq k$ for every non empty $X \subset V$. When $k = 1$, we simply say that the graph is **connected**. A **component** of G subset X of vertices such that $d_G(X) = 0$ that

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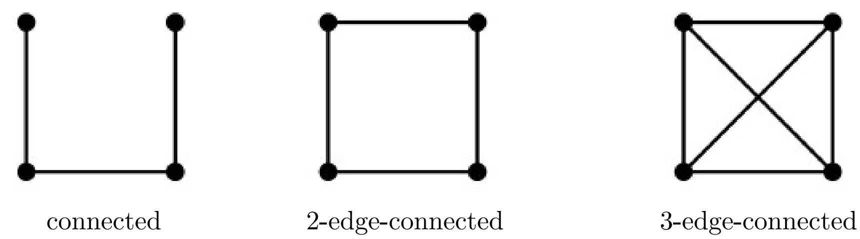
forms a connected graph with the edges in E having both ends in X. We will denote the set of all subpartitions of V by $\mathcal{S}(V)$.

Here are a few examples.



The outline of these pages follows. We first introduce edge-connectivity augmentation of a graph and the simple method of Frank [5] that solves it. Then the generalization obtained by adding partition constraints is overviewed. In Section 2, we transpose these problems in the framework of hypergraphs. Finally, we mention an abstract generalization of edge-connectivity augmentation problems in terms of covering a function by a graph.

1 Graphs

1.1 Edge-connectivity augmentation of a graph

In this section, we deal with the problem of edge-connectivity augmentation of a graph, which is as follows. Given a graph and an integer k, find a minimum set of edges of which addition makes the graph k-edge-connected. A strategy to approach this kind of problem consists in determining a suitable lower, and then try to prove that this bound is achieved. For edge-connectivity augmentation of a graph, we may indeed find a lower bound that is always achieved, and the method of Frank [5], described below, is a very simple and efficient approach to prove that.

The case $k = 1$ is an easy exercise: $\#component(G) - 1$ is the correct number of edges to be added to a graph G to make it connected. For $k \geq 2$, a natural lower bound for the number of edges needed to make a graph $G = (V, E)$ k-edge-connected is obtained by considering the deficiencies of subsets of vertices:

$$\alpha := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d_G(X)) \rceil : \mathcal{X} \in \mathcal{S}(V)\}.$$

This value α is indeed a lower bound because at least $k - d_G(X)$ new edges must enter a non empty set $X \subset V$ with $d_G(X) < k$, and adding a new edge to the graph may decrease the sum that appears above by at most 2. Watanabe and Nakamura [8] proved that it is always achieved.

Theorem 1 (Watanabe and Nakamura [8]) *Let $G = (V, E)$ be a graph and $k \geq 2$. Then the minimum number of graph edges of which addition to G results in a k-edge-connected graph is α.*

Frank developed a both simple and efficient method that proves Theorem 1, based on the technique of splitting off. It consists of two steps. First, add a special vertex s to the starting graph, and a minimum set of edges between s and the graph in order to satisfy the desired connectivity property. Second, apply the technique of splitting off, that is replace two edges incident to s by an edge between the corresponding vertices of the original graph if the desired connectivity property remains valid. Repeat this operation in order to get rid of the edges incident to s and finally delete the isolated vertex s. The set of new edges obtained provides an optimal solution of the problem, and is indeed the method used to prove every theorem mentioned below. We will now focus on a generalization involving partition constraints.

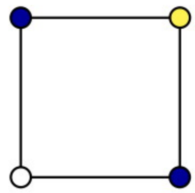
1.2 Edge-connectivity augmentation of a graph with partition constraints

In a real-world network, given two points, it is not always possible to add a connexion between them. A way to model this is to forbid the addition of connexions between some points, for example by partitioning them and allowing new connexions only between points that belong to different sets of the partition. In terms of graph, this problem is solved in [1], where the authors are given not only a graph and an integer k, but also a partition $\mathcal{P}$ of the vertex set and they ask for the new edges to connect distinct members of this partition. Taking each vertex of the graph as a set of the partition gives edge-connectivity augmentation of a graph as a special case.

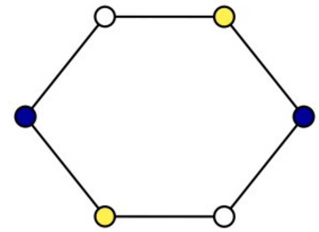
Note that α is again a lower bound for this problem. Moreover, the value β below is a lower bound that arises directly from the fact that one can not add an edge between vertices that lie in the same set of the partition $\mathcal{P}$, and hence a lower bound for the problem of edge-connectivity augmentation of a graph with partition constraints is $\max\{\alpha, \beta\}$, where

$$\beta := \max\{\max\{\sum_{Y \in \mathcal{Y}} (k - d_G(Y)) : \mathcal{Y} \in \mathcal{S}(P)\} : P \in \mathcal{P}\}.$$

This problem is polynomially solved in [1], where they show that the natural lower bound is almost always the correct answer. In the proof of their theorem, special graphs called configurations appear, they are the ones failing the lower bound. There are two families of such graphs, the C_4- and the C_6-configurations, each of them generalizing the fact that the lower bound is not achieved when one wants to make the following graphs 3-edge-connected, respecting the partition constraints.



C_4



C_6

The sets of the partition are represented above by different colors of vertices. For instance, the cycle on four vertices C_4 partitioned as below requires the addition of three edges to be made 3-edge-connected, respecting the partition constraints, while the lower bound is 2. It comes from the properties of the optimal solutions for the unconstrained problem: for the C_4, the unique one contains both diagonal edges; for the C_6, every one of them contains a diagonal edge. Such properties do not hold for longer cycles.

The result solving the problem of edge-connectivity augmentation of a graph with partition constraints is the following.

Theorem 2 (Bang-Jensen, Gabow, Jordán, Szigeti [1]) *Let $G = (V, E)$ be a graph, $\mathcal{P}$ a partition of V and $k \geq 2$. Then the minimum number of edges between distinct members of $\mathcal{P}$ of which addition to G results in a k-edge-connected graph is* $\max\{\alpha, \beta\}$ *if G contains no C_4- and no C_6-configuration,* $\max\{\alpha, \beta\} + 1$ *otherwise.*

We refer to [1] for a precise definition of configurations in graphs. An important fact is that one can check in polynomial time whether a given graph contains a configuration or not.

2 Hypergraphs

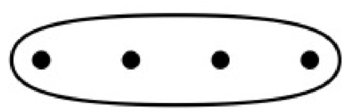
In this section, we transpose the above problems in the frameworks of hypergraphs. We first mention the result of Bang-Jensen and Jackson [2] that solves the problem of edge-connectivity augmentation of a hypergraph, and then the theorem of Bernáth, Grappe and Szigeti [6] that handles the addition of partition constraints.

Hypergraphs are objects generalizing graphs, they are composed of vertices and hyperedges, and a hyperedge is allowed to connect more than two vertices at once. That is a hyperedge is a subset of vertices. A graph is a hypergraph where every hyperedge contains exactly two vertices. The definition of edge-connectivity for hypergraphs is similar to graphs. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{E}$ being a set of subsets of V, and let k be an integer. For a set $X \subset V$, let $\delta_{\mathcal{H}}(X)$ be the set of hyperedges in $\mathcal{E}$ containing at least one vertex of X and at least one of $V - X$, and let $d_{\mathcal{H}}(X) = |\delta_{\mathcal{H}}(X)|$. The hypergraph $\mathcal{H}$ is called ***k*-edge-connected** if $d_{\mathcal{H}}(X) \geq k$ for every non empty $X \subset V$.

2.1 Edge-connectivity augmentation of a hypergraph

Now, given a hypergraph and an integer k, we would like to make the hypergraph k-edge-connected by adding a minimum number of hyperedges. Without further condition, it is not a relevant problem as it is sufficient to add the hyperedge V a minimum number of times. As graphs are hypergraphs, it would be pleasant to generalize Theorem 1 in this framework, hence we require for the new hyperedges to be of size two, that is they are graph edges. Hence, the problem of edge-connectivity augmentation of a hypergraph is as follows: given a hypergraph $\mathcal{H}$ and an integer k, find a minimum set of edges to be added to $\mathcal{H}$ to make it k-edge-connected.

First, note that the lower bound α defined in Section 1 directly generalizes to hypergraphs. An important difference with edge-connectivity augmentation of graphs is shown by the example below: making the following hypergraph 2-edge-connected is equivalent to making the edgeless graph on four vertices connected.



A hypergraph with four vertices and one hyperedge.

Hence, the problem of edge-connectivity augmentation of a hypergraph also contains the problem of making a graph connected. A new lower bound ω arises for this fact, because we need #component$(G) - 1$ edges to make a graph G connected:

$$\omega := \max\{\#\text{component}(\mathcal{H} - \mathcal{F}) - 1 : \mathcal{F} \subseteq \mathcal{E}, |\mathcal{F}| = k - 1\}.$$

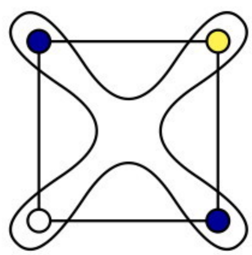
Bang-Jensen and Jackson [2] solved the problem of making a hypergraph k-edge-connected by adding a minimum number of graph edges, they proved that the lower bound is always achieved, and their proof yields a polynomial algorithm.

Theorem 3 (Bang-Jensen, Jackson [2]) *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and k an integer. Then the minimum number of graph edges of which addition to $\mathcal{H}$ results in a k-edge-connected hypergraph is* $\max\{\alpha, \omega\}$.

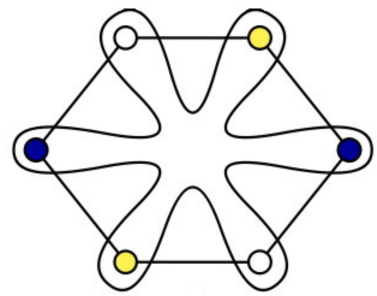
2.2 Edge-connectivity augmentation of a hypergraph with partition constraints

A common generalization of the above mentioned problems is edge-connectivity augmentation of a hypergraph with partition constraints: given a hypergraph $\mathcal{H}$ a partition $\mathcal{P}$ of the vertex set and an integer k, find a minimum set of edges between distinct members of $\mathcal{P}$ to be added to $\mathcal{H}$ to make it k-edge-connected.

As this problem contains the three previous ones, we know that $\max\{\alpha, \beta, \omega\}$ is a lower bound. Moreover, it is not always achieved because of the C_4- and C_6-configurations in graphs. These two families may be generalized in the framework of hypergraphs, and here are the simplest examples of configurations for hypergraphs.



a $\mathcal{C}_4$-configuration



a $\mathcal{C}_6$-configuration

They are obtained from configurations for graphs in a manner that should recall why edge-connectivity augmentation of a hypergraph contains the problem of making a graph connected: adding a hyperedge containing every vertex to a configuration for graphs gives a configuration for hypergraphs. The converse is not true, but it is a rough intuition of what is going on. See [6] for the precise definition of $\mathcal{C}_4$- and $\mathcal{C}_6$-configurations.

It turns out that, beside $\mathcal{C}_4$- and $\mathcal{C}_6$-configurations, the lower bound is always achieved, see [6]. As one more edge is needed for a configuration, the following theorem holds.

Theorem 4 (Bernáth, Grappe, Szigeti [6]) *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Then the minimum number of graph edges between different members of $\mathcal{P}$ of which addition to $\mathcal{H}$ results in a k-edge-connected hypergraph is* $\max\{\alpha, \beta, \omega\}$ *if $\mathcal{H}$ contains no $\mathcal{C}_4$- and no $\mathcal{C}_6$-configuration,* $\max\{\alpha, \beta, \omega\} + 1$ *otherwise.*

This theorem implies Theorem 3 and Theorem 2. We emphasize that the proof yields a polynomial algorithm to find the desired set of edges.

3 Conclusion

To conclude, we simply mention that edge-connectivity augmentation problems may be formulated in an abstract form, in terms of covering a function by a graph. For instance, making a hypergraph $\mathcal{H} = (V, \mathcal{E})$ k-edge-connected by adding graph edges is equivalent to finding a graph $K = (V, F)$ such that $d_K(X) \geq k - d_{\mathcal{H}}(X)$ for every non empty $X \subset V$. The function p defined by $p(X) := k - d_{\mathcal{H}}(X)$ is symmetric crossing supermodular, see [3], hence given such a function p the problem of finding a graph K having a minimum number of edges such that $d_K(X) \geq p(X)$ for every non empty $X \subset V$ contains the problem of edge-connectivity augmentation of a hypergraph. This problem is the covering of p by a graph. Without further details, we mention that the abstract generalizations of the results overviewed in these pages may be solved efficiently: the problem of covering a symmetric crossing supermodular function by a graph was solved by Benczúr and Frank [3] and its partition constrained generalization was solved by Bernáth, Grappe and Szigeti [4].

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Topological properties of Kähler and hyperkähler manifolds

Julien Grivaux (*)

1 Introduction

Complex geometry can be defined as the study of complex manifolds. A complex manifold is a topological space which is locally similar to an open set of $\mathbb{C}^n$ via the choice of local coordinates in a neighborhood of each point, change of variables between different system of coordinates having to be holomorphic. The simplest example of complex manifold appearing in classical complex analysis is the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: z and w are holomorphic coordinates on $\widehat{\mathbb{C}} \setminus \{\infty\}$ and $\widehat{\mathbb{C}} \setminus \{0\}$ respectively, related on $\widehat{\mathbb{C}} \setminus \{0, \infty\}$ by the formula $z = 1/w$.

We will mainly deal here with *Kähler manifolds*, which are complex manifolds endowed with some extra structure (given by a closed differential form of degree 2). The first aim of this note is to present some fundamental results of the theory, such as the Hodge decomposition theorem and the hard Lefschetz theorem, and to explain how the Kähler hypothesis yields topological restrictions on the manifold. The second part is devoted to hyperkähler manifolds, which is an active area of current research. Hyperkähler manifolds, also called *quaternionic-Kähler manifolds*, are compact Kähler manifolds admitting a nondegenerate holomorphic 2-form. We will explain Bogomolov's approach to Verbitsky's results on the de Rham cohomology ring of hyperkähler manifolds.

2 Complex manifolds

We begin by the definition of a complex manifold. Exactly as in the theory of differentiable manifolds, we start by defining suitable atlases of local coordinates:

Definition 2.1 Let M be a separated topological space.

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(i) A *holomorphic atlas* $\mathcal{A}$ on M is a family of homeomorphisms $\{\phi_i : U_i \xrightarrow{\sim} \phi_i(U_i)\}_{i\in I}$ (called *local holomorphic coordinates*, or *holomorphic charts*) such that:

  - the U_i's form an open covering of M,
  - there exists an integer n such that for every i in I, $\phi_i(U_i)$ is an open subset of $\mathbb{C}^n$,
  - for every i, j in I such that $U_i \cap U_j \neq \emptyset$, the transition functions

$$\phi_{ij} = \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \xrightarrow{\sim} \phi_j(U_i \cap U_j)$$

  are holomorphic.

(ii) Two holomorphic atlases (U_i, ϕ_i) and (U'_i, ϕ'_i) on M are *equivalent* if all functions $\phi'_j \circ \phi_i^{-1}$ are holomorphic.

Then the definition of a complex manifold runs as follows:

Definition 2.2

(i) A *complex manifold* is a separated topological space endowed with an equivalence class of holomorphic atlases.

(ii) If M is a differentiable manifold, a complex structure on M is an equivalence class of holomorphic atlases on M such that all the holomorphic coordinates are smooth (for the differentiable structure of M).

Therefore we have inclusions

$$\{\text{complex manifolds}\} \subset \{\text{differentiable manifolds}\} \subset \{\text{topological spaces}\}$$

Let us give examples of complex manifolds:

• **Open sets in** $\mathbb{C}^n$. Any open set U of $\mathbb{C}^n$ is naturally a complex manifold, an atlas being given by the chart $\mathrm{id} : U \xrightarrow{\sim} U$.

• **Complex tori.** Let d be a positive integer, Γ be a lattice in $\mathbb{C}^d$, i.e. $\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \ldots \oplus \mathbb{Z}e_{2d}$, where $(e_1, \ldots, e_{2d})$ is a basis of $\mathbb{C}^d$ as a real vector space, $M = \mathbb{C}^d/_{\Gamma}$ and let $\pi : \mathbb{C}^d \longrightarrow M$ be the projection. The map π is a local homeomorphism (it is in fact a covering map). For any z in $\mathbb{C}^d$ let U_z be a small complex ball around z such that $\pi : U_z \xrightarrow{\sim} \pi(U_z)$ is a homeomorphism. We take $\mathcal{A} = \left\{\pi^{-1} : \pi(U_z) \xrightarrow{\sim} U_z,\ z \in \mathbb{C}^d\right\}$. It can be checked that $\mathcal{A}$ is a holomorphic atlas, the transition functions being of the type $z \mapsto z + \gamma$, where $\gamma \in \Gamma$. The complex manifold $M = \mathbb{C}^d/_{\Gamma}$ is called a *complex torus of dimension d*. It is diffeomorphic, as a differentiable manifold, to the manifold $(\mathbb{S}^1)^{2d}$.

• **Projective spaces.** Let $n \in \mathbb{N}^*$. The *complex projective space* $\mathbb{P}^n(\mathbb{C})$ of dimension n is the set of all lines of $\mathbb{C}^{n+1}$. A structure of complex manifold on $\mathbb{P}^n(\mathbb{C})$ is defined as follows:

for $1 \leq i \leq n$, we define U_i as the set of all l in $\mathbb{P}^n(\mathbb{C})$ such that l is not contained in the hyperplane $\{z_i = 0\}$. Then each U_i is homeomorphic to $\mathbb{C}^n$, the homeomorphism being given by $\phi_i(l) = l \cap \{z_i = 1\}$. The transition functions for the charts $\{\phi_i\}$ are readily seen to be holomorphic. For instance, if $n = 1$, $\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and $\phi_{12}(z) = 1/z$. The projective spaces $\mathbb{P}^n(\mathbb{C})$ are compact.

• **Hopf manifolds.** For $n \in \mathbb{N}^*$, let $\mathcal{S}_n$ be the quotient of $\mathbb{C}^n \setminus \{0\}$ obtained by identifying for any z in $\mathbb{C}^n \setminus \{0\}$ the points z and $2z$. Since the natural map from $\mathbb{C}^n \setminus \{0\}$ to $\mathcal{S}_n$ is a covering map, the argument already used for complex tori provides a holomorphic atlas on $\mathcal{S}_n$. The associated complex manifolds are called *Hopf manifolds*. They are diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.

• **The Iwasawa manifold.** Let G be the *complex Heisenberg group*, i.e. the set of all matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, $(a, b, c) \in \mathbb{C}^3$; and $\Gamma = G \cap \mathrm{Gl}(3, \mathbb{Z} \oplus i\mathbb{Z})$. The quotient $M = {}_{\Gamma}\backslash G$ is a complex manifold of dimension three, called the *Iwasawa manifold*.

Given a orientable differentiable manifold M of even dimension, it is natural to ask the following question: does there exist a complex structure on M? Little is known on this question when $\dim(M) \geq 6$, except a topological obstruction to the existence of almost-complex structures (a notion weaker than the notion of complex structure). In dimension two, every compact orientable manifold admits a complex structure. For results in dimension four, the interested reader is referred to [1, Chap. IV §9].

To conclude this part, let us mention the following famous problem, which is still open:

Question 2.3 (Chern) *Does there exist a complex structure on the sphere $\mathbb{S}^6$?*

3 Differential forms on a complex manifold

Let Ω be an open set of $\mathbb{C}^n$ and $f : \Omega \longrightarrow \mathbb{C}$ be a smooth function. Its differential df is a differential form of degree one on Ω, which can be decomposed as the sum of a $\mathbb{C}$-linear part and a $\mathbb{C}$-antilinear part

$$df = \partial f + \overline{\partial} f = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i \right) + \left(\sum_{i=1}^{n} \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i \right)$$

where

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right).$$

In terms of smooth differentiable forms, if $\mathcal{A}^k_{\mathbb{C}}(\Omega)$ denotes the set of complex-valued smooth differentiable forms of degree k on Ω , then $\mathcal{A}^1_{\mathbb{C}}(\Omega) = \mathcal{A}^{1,0}_{\mathbb{C}}(\Omega) \oplus \mathcal{A}^{0,1}_{\mathbb{C}}(\Omega)$ and the differential $d : \mathcal{A}^0_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^1_{\mathbb{C}}(\Omega)$ can be decomposed as $\partial + \overline{\partial}$, where

$$\partial : \mathcal{A}^0_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^{1,0}_{\mathbb{C}}(\Omega) \quad \text{and} \quad \overline{\partial} : \mathcal{A}^0_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^{0,1}_{\mathbb{C}}(\Omega).$$

This decomposition holds in fact in every degree: if $1 \leq k \leq 2n$, $\mathcal{A}^k_{\mathbb{C}}(\Omega) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{\mathbb{C}}(\Omega)$, where

$$\mathcal{A}^{p,q}_{\mathbb{C}}(\Omega) = \Big\{ \sum_{|I|=p,\,|J|=q} \phi_{IJ}\; dz^I \wedge d\bar{z}^J,\ \phi_{IJ} : \Omega \longrightarrow \mathbb{C} \text{ is smooth} \Big\};$$

and the de Rham derivative $d : \mathcal{A}^k_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^{k+1}_{\mathbb{C}}(\Omega)$ can be decomposed as $\partial + \overline{\partial}$, where

$$\partial : \mathcal{A}^{p,q}_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^{p+1,q}_{\mathbb{C}}(\Omega) \quad \text{and} \quad \overline{\partial} : \mathcal{A}^{p,q}_{\mathbb{C}}(\Omega) \longrightarrow \mathcal{A}^{p,q+1}_{\mathbb{C}}(\Omega).$$

Forms in $\mathcal{A}^{p,q}_{\mathbb{C}}(\Omega)$ are called $(p\text{–}q)$ *forms*. Since $d^2 = 0$, $\partial^2 = 0$, $\overline{\partial}^{\,2} = 0$ and $\overline{\partial}\partial + \partial\overline{\partial} = 0$.

Let η be a differential form of type $(p\text{–}q)$ in an open set Ω of $\mathbb{C}^n$. If we apply a change of variables given by a biholomorphism between Ω and another open subset Ω' of $\mathbb{C}^n$, the form η expressed in these new holomorphic coordinates is a $(p\text{–}q)$ form on Ω'. Therefore, the notion of a $(p\text{–}q)$ form on a complex manifold M is intrinsically defined: a differential form on M is of type $(p\text{–}q)$ if it is of type $(p\text{–}q)$ in any holomorphic coordinate system. For any integer k, $\mathcal{A}^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{\mathbb{C}}(M)$ and the de Rham differential d can be decomposed as $\partial + \bar{\partial}$, where

$$\partial : \mathcal{A}^{p,q}_{\mathbb{C}}(M) \longrightarrow \mathcal{A}^{p+1,q}_{\mathbb{C}}(M) \quad \text{and} \quad \overline{\partial} : \mathcal{A}^{p,q}_{\mathbb{C}}(M) \longrightarrow \mathcal{A}^{p,q+1}_{\mathbb{C}}(M).$$

The relations $d^2 = 0$ and $\bar{\partial}^2 = 0$ allow to define de Rham (resp. Dolbeault) cohomology groups on complex manifolds. These vector spaces measure the defect between the kernel of d (resp. $\bar{\partial}$) and the image of d (resp. $\bar{\partial}$).

Definition 3.1 Let M be a differential manifold.

(i) The *de Rham cohomology groups* $H^k(M, \mathbb{R})$ and $H^k(M, \mathbb{C})$ are defined by

$$H^k(M, \mathbb{R}) = \{\alpha \in \mathcal{A}^k_{\mathbb{R}}(M) \text{ such that } d\alpha = 0\} \Big/ \{d\eta,\ \eta \in \mathcal{A}^{k-1}_{\mathbb{R}}(M)\}.$$

and

$$H^k(M, \mathbb{C}) = \{\alpha \in \mathcal{A}^k_{\mathbb{C}}(M) \text{ such that } d\alpha = 0\} \Big/ \{d\eta,\ \eta \in \mathcal{A}^{k-1}_{\mathbb{C}}(M)\}.$$

(ii) Suppose that M is endowed with a complex structure. If $p, q \in \mathbb{N}$, $H^{p,q}(M)$ is the image of the map

$$\{\alpha \in \mathcal{A}^{p,q}_{\mathbb{C}}(M) \text{ such that } d\alpha = 0\} \longrightarrow H^{p+q}(M, \mathbb{C}).$$

It is a subspace of $H^{p+q}(M, \mathbb{C})$.

(iii) Under the same hypothesis as in (ii), if p q are two integers, the *Dolbeault cohomology groups* $H^{p,q}_{\overline{\partial}}(M)$ are defined by

$$H^{p,q}_{\overline{\partial}}(M) = \{\alpha \in \mathcal{A}^{p,q}_{\mathbb{C}}(M) \text{ such that } \overline{\partial}\alpha = 0\} \Big/ \{\overline{\partial}\eta,\ \eta \in \mathcal{A}^{p,q-1}_{\mathbb{C}}(M)\}.$$

Remark 3.2

(i) It is clear that $H^k(M,\mathbb{C}) = H^k(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

(ii) For all integers p and q, $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

(iii) If M is compact, the de Rham and Dolbeault cohomology groups are finite dimensional complex vector spaces (the proof of these facts uses methods of functional analysis, such as Green operators and L^2 theory).

In the compact case, we can therefore get some information about the cohomology by considering the dimensions of these vector spaces.

Definition 3.3 Let M be a compact differentiable manifold of dimension d.

– The *Betti numbers* of M are defined for $0 \leq k \leq d$ by

$$b_k(M) = \dim_{\mathbb{R}} H^k(M,\mathbb{R}) = \dim_{\mathbb{C}} H^k(M,\mathbb{C}).$$

– If M is endowed with a complex structure, the *Hodge numbers* of M are defined for $0 \leq p, q \leq \frac{d}{2}$ by $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(M)$.

Let us finish this part on differential forms and cohomology with some identities and inequalities between Hodge and Betti numbers. First there are some symmetry relations coming from duality. To explain this, let M be a real manifold of dimension d. For every integer k such that $0 \leq k \leq d$, we introduce the following pairing:

$$\mathcal{A}^k_{\mathbb{C}}(M) \times \mathcal{A}^{d-k}_{\mathbb{C}}(M) \longrightarrow \mathbb{C}, \qquad (\alpha,\beta) \longmapsto \int_M \alpha \wedge \beta \tag{1}$$

Theorem 3.4 *Let M be a compact differentiable manifold of dimension d.*

– ***Poincaré Duality***

For any integer k with $0 \leq k \leq d$, the pairing (1) induces a perfect pairing

$$\langle\ ,\ \rangle : H^k(M,\mathbb{R}) \times H^{d-k}(M,\mathbb{R}) \longrightarrow \mathbb{R}.$$

In particular, $b_k(M) = b_{d-k}(M)$.

– ***Kodaira-Serre Duality***

Suppose that M is endowed with a complex structure, and put $d = 2n$. Then for every integers k, p and q such that $p+q = k$ and $0 \leq p, q \leq n$, the pairing (1) induces a perfect pairing

$$\langle\ ,\ \rangle : H^{p,q}_{\overline{\partial}}(M) \times H^{n-p,n-q}_{\overline{\partial}}(M) \longrightarrow \mathbb{C}.$$

In particular, $h^{p,q}(M) = h^{n-p,n-q}(M)$.

There are also some other relations between Hodge and Betti numbers (called the Fröhlicher relations) coming from the study of the Hodge-de Rham spectral sequence. Here is the result:

Proposition 3.5 *Let M be a complex compact manifold. Then*

(i) $\displaystyle\sum_{p,q\in\mathbb{N}}(-1)^{p+q}h^{p,q}(M)=\sum_{r\in\mathbb{N}}(-1)^r b_r(M)$,

(ii) $\forall r\in\mathbb{N}$, $\displaystyle\sum_{p+q=r}h^{p,q}(M)\geq b_r(M)$.

4 Kähler manifolds

To define Kähler manifolds, which form an extremely important class of complex manifolds, we need to introduce a notion of positivity for real differential forms of type (1–1) on a complex manifold. As usual, we begin with an open subset of $\mathbb{C}^n$.

Definition 4.1

(i) Let Ω be an open set of $\mathbb{C}^n$ and $\omega=\frac{i}{2}\sum_{1\leq\alpha,\beta\leq n}\omega_{\alpha\overline{\beta}}\,dz^\alpha\wedge d\overline{z}^{\,\beta}$ be in $\mathcal{A}^{1,1}_{\mathbb{R}}(\Omega)$. We say that ω is *positive* if the matrix $\omega_{\alpha\overline{\beta}}$ is hermitian positive definite.

(ii) Let M be a complex manifold. A form ω in $\mathcal{A}^{1,1}_{\mathbb{R}}(M)$ is *positive* if it is positive (as defined in (i)) in any system holomorphic coordinates.

Remark that if ω is a positive (1–1) form on a complex manifold of dimension n, the form ω^n is a volume form on M. Indeed if $\omega=\frac{i}{2}\sum_{1\leq\alpha,\beta\leq n}\omega_{\alpha\overline{\beta}}\,dz^\alpha\wedge d\overline{z}^{\,\beta}$ in local holomorphic coordinates, then $\omega^n=\det\big(\omega_{\alpha\overline{\beta}}\big)\,dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n$.

Definition 4.2 A complex manifold M is a *Kähler manifold* if there exists a positive (1–1) form ω on M such that $d\omega=0$. Such a form is called a *Kähler form.*

Examples.

- For any integer n, the form $\omega=\frac{i}{2}\sum_{1\leq i\leq n}dz_i\wedge d\bar{z}_i$ is a Kähler form on $\mathbb{C}^n$.
- If Γ is a lattice of $\mathbb{C}^n$, the Kähler form on $\mathbb{C}^n$ is invariant by the action of Γ, so it induces a Kähler form on the complex torus $\mathbb{C}/_{\Gamma}$.
- For any integer n, the form $\omega=\frac{i}{2}\,\partial\bar{\partial}\,\log\|z\|^2$ is invariant by the natural $\mathbb{C}^\times$ action on $\mathbb{C}^{n+1}\setminus\{0\}$, it induces a Kähler form on $\mathbb{P}^n(\mathbb{C})$ which is called the *Fubini-Study form.*
- Any complex submanifold of a Kähler manifold is Kähler. In particular, any *projective variety* (i.e. any complex submanifold of a projective space) is Kähler.

The existence of a Kähler form on a complex manifold generates strong constraints on its cohomology rings. From now on we will deal only with *compact* complex manifolds.

Proposition 4.3 *If M is a Kähler manifold, the even Betti numbers of M are nonzero. More precisely, if ω is the Kähler form and n is the complex dimension of M, the powers ω^i $(1 \leq i \leq n)$ of ω are nonzero in the associated de Rham cohomology groups $H^{2i}(X, \mathbb{R})$.*

Proof. If $1 \leq i \leq n$, then $d\omega^i = 0$. To prove that ω^i is nonzero in $H^{2i}(M, \mathbb{R})$, we pair it with the cohomology class ω^{n-i}. We get $\langle \omega^i, \omega^{n-i} \rangle = \int_M \omega^n > 0$, since ω^n is a volume form. □

This proposition proves that the Hopf manifolds do not admit complex Kähler structures, since their second Betti numbers vanish.

One of the central results in the theory of Kähler manifolds is the Hodge decomposition theorem:

Theorem 4.4 (Hodge) *Let M be a compact Kähler manifold. Then*

(i) *For every integer k, $H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$.*

(ii) *For all integers p and q, $H^{p,q}(M) \simeq H^{p,q}_{\overline{\partial}}(M)$.*

This result is not obvious at all. Indeed, if $\alpha \in \mathcal{A}^k_{\mathbb{C}}(M)$ satisfies $d\alpha = 0$, it is no longer the case for its $(p$–$q)$ components.

Corollary 4.5 *If M is a compact Kähler manifold, then*

(i) *For every integer k, $b_k(M) = \sum_{p+q=k} h^{p,q}(M)$.*

(ii) ***Hodge symmetry**: for all integers p and q, $h^{p,q}(M) = h^{q,p}(M)$.*

(iii) *For every integer p, if η is a holomorphic $(p$–$0)$ form on M, then $d\eta = 0$.*

Let us mention a topological corollary of the Hodge decomposition theorem:

Corollary 4.6 *If M is a compact Kähler manifold, its odd Betti numbers are even.*

Another cornerstone of the theory of Kähler manifolds is the so called *hard Lefschetz theorem*:

Theorem 4.7 *Let (M, ω) be a compact Kähler manifold of complex dimension n and let $L : H^*(M, \mathbb{R}) \longrightarrow H^{*+2}(M, \mathbb{R})$ be the Lefschetz operator defined by $L\alpha = \omega \wedge \alpha$. Then for every integer i such that $0 \leq i \leq n$, $L^i : H^{n-i}(M, \mathbb{R}) \longrightarrow H^{n+i}(M, \mathbb{R})$ is an isomorphism.*

Corollary 4.8 *Let M be a compact Kähler manifold of complex dimension n. Then*

(i) *the sequence of the even Betti numbers* $\{b_{2i}(M),\ 0 \leq 2i \leq n\}$ *increases,*

(ii) *the sequence of the odd Betti numbers* $\{b_{2i+1}(M),\ 0 \leq 2i+1 \leq n\}$ *increases.*

Let us look again at the Iwasawa manifold M defined by

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},\ (a,b,c) \in \mathbb{C}^3 \right\}, \qquad M = {}_{G \cap \mathrm{Gl}(3, \mathbb{Z} \oplus i\mathbb{Z})} \backslash G$$

The vector space of left-invariant 1-forms on G is isomorphic to the dual Lie algebra of G. A specific basis is given by the three 1-forms da, db and $dc - a\,db$. Since these forms are left-invariant by G, hence by $G \cap \mathrm{Gl}(3, \mathbb{Z} \oplus i\mathbb{Z})$, they induce 1-forms α, β and γ on M. It is possible to verify that

- α, $\overline{\alpha}$, β, $\beta \wedge \overline{\beta}$ form a basis of $H^1(M, \mathbb{C})$.
- $\alpha \wedge \overline{\alpha}$, $\beta \wedge \overline{\beta}$, $\alpha \wedge \overline{\beta}$, $\overline{\alpha} \wedge \beta$, $\overline{\alpha} \wedge \gamma \wedge \overline{\alpha} \wedge \overline{\gamma}$, $\beta \wedge \gamma$ and $\overline{\beta} \wedge \overline{\gamma}$ form a basis of $H^2(M, \mathbb{C})$.

Therefore, the Betti numbers of M are $b_0 = b_6 = 1,\ b_1 = b_5 = 4,\ b_2 = b_4 = 8,\ b_3 = 10$.

It appears that the restrictions we gave on the Betti numbers of a compact Kähler manifold are not strong enough to show that the Iwasawa manifold M is not Kähler. However, it is fairly easy to prove that M (endowed with its usual complex structure) does not admit a Kähler form. Indeed, γ is a holomorphic 1-form on M and $d\gamma \neq 0$. It could nevertheless happen that M be Kähler for a different complex structure, but it turns out that it is not the case. This can be proved with the help of the following result:

Theorem 4.9 [6] *If M is a compact Kähler manifold and $\mathcal{A}(M) = \bigoplus_k \mathcal{A}^k(M)$ is the de Rham algebra of differential forms on M, then $\mathcal{A}(M)$ is formal as a dg-algebra, i.e. is quasi-isomorphic to the direct sum of its cohomology objects.*

5 Hyperkähler manifolds

Hyperkähler manifolds are Kähler manifolds carrying a holomorphic symplectic form. The original definition comes from differential geometry and corresponds to a quaternionic structure on the tangent bundle, satisfying some integrability conditions.

Definition 5.1 A compact Kähler manifold M of complex dimension $2n$ is *hyperkähler* if

- there exists a holomorphic 2-form σ on M such that σ^n is everywhere nonzero,
- $h^{2,0}(M) = 1$ and $h^{1,0}(M) = 0$.

Example 5.2 A $K3$-surface, e.g. a smooth quartic in $\mathbb{P}^3(\mathbb{C})$, is hyperkähler.

Remark that if M is hyperkähler, the form σ is holomorphic, so that $d\sigma = 0$. Furthermore, the form σ^n is a holomorphic form of maximal degree (i.e. a holomorphic volume form) which is everywhere nonzero, which means that M is a Calabi-Yau manifold.

In this theory, the problem is no longer to prove that a given manifold cannot be hyperkähler, but to find some examples of hyperkähler manifolds. Here is a list of known examples:

- Beauville's examples [2]:
  - punctual Hilbert schemes of $K3$-surfaces,
  - generalized Kummer manifolds (which are constructed from punctual Hilbert schemes of 2-dimensional complex tori).
- O'Grady's examples [10], [11]:
  - one family of examples in dimension 6,
  - one family of examples in dimension 10.

These examples are obtained by desingularising moduli spaces of semistable sheaves on $K3$ and abelian surfaces.

Conjecture 5.3 *If M is hyperkähler, it can be deformed to one of the examples listed above.*

This conjecture means that very few examples (up to deformation) are expected. Even if the theory of deformations of hyperkähler manifolds is well understood, explicit deformations of hyperkähler manifolds are not easy to construct (see for instance [3], [5], [9], [12]).

We are now going to see that the hyperkähler condition yields huge restrictions on the de Rham cohomology ring. Let M be a hyperkähler manifold of dimension $2n$. By the Hodge decomposition theorem,

$$H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M) = \mathbb{C}\sigma \oplus H^{1,1}(M) \oplus \mathbb{C}\,\overline{\sigma}.$$

It can be proved, using deformation theory (see [4]) that the submanifold Q of M defined by

$$Q = \{x \in H^2(M,\mathbb{C}) \text{ such that } x^{n+1} = 0\}$$

is a smooth quadric hypersurface of $H^2(M,\mathbb{C})$.

Definition 5.4 The *Beauville-Bogomolov* quadratic form q on $H^2(M,\mathbb{C})$ is defined by

$$q(a\sigma + \omega + b\,\overline{\sigma}\,) = ab + \frac{n}{2}\int_M \omega^2 \wedge \sigma^{n-1} \wedge \overline{\sigma}^{\,n-1},$$

where σ is normalized by the condition $\displaystyle\int_M \sigma^n \wedge \overline{\sigma}^{\,n-1} = 1$.

The form q is a nondegenerate quadratic form on $H^2(M, \mathbb{C})$ which can also be written in a more simple way as $q(x) = x^{n+1} \wedge \overline{\sigma}^{n-1}$. Then $Q = \{x \in M \text{ such that } q(x) = 0\}$.

The following result was initially proved by Verbitsky [13]. Another proof, using deformation theory was given by Bogomolov [4].

Theorem 5.5 *Let M be a hyperkähler manifold of complex dimension $2n$. Then the subring generated in $H^*(M, \mathbb{C})$ by $H^2(M, \mathbb{C})$ is equal to Sym$\big(H^2(M, \mathbb{C})\big)/_{\mathcal{J}}$, where $\mathcal{J}$ is the ideal of Sym$\big(H^2(M, \mathbb{C})\big)$ generated by the relations*

$$\langle\, x^{n+1} = q(x) \text{ if } x \in H^2(M, \mathbb{C}) \,\rangle \quad \text{and} \quad \langle\, x = 0 \text{ if } \deg x > 4n \,\rangle.$$

Since there is only one isomorphism class of complex nondegenerate quadratic form on a finite dimensional complex vector space, this theorem implies that there exist universal graded commutative algebras $\big(R_{n,d}\big)_{n,d \in \mathbb{N}}$ satisfying the following remarkable property:

If M is any hyperkähler manifold of complex dimension $2n$ such that $b_2(M) = d$, the subring generated in $H^*(M, \mathbb{C})$ by $H^2(M, \mathbb{C})$ is isomorphic to $R_{n,d}$.

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