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Telescope Conjecture for t-Structures:

Quiver Representations over Commutative Noetherian Rings

Coordinator
Prof. Giovanni Colombo

Supervisor
Prof. Jorge Vitória

Co - Supervisor
Prof. Michal Hrbek (Czech Academy of Sciences)

PhD Candidate
Enrico Sabatini

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To my grandparents, Antoniana and Gilberto,
for everything you have done for our family.

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Abstract

This thesis establishes affirmative results on the *telescope conjecture* for t-structures in the derived category of quiver representations over commutative noetherian rings. After outlining the general context, we proceed by introducing certain restrictions either on the class of commutative rings or on the type of quivers considered. Specifically, we prove the telescope conjecture for t-structures in the derived category of representations of finite quivers over *commutative artinian rings*, and in the derived category of representations of *Dynkin quivers* over commutative noetherian rings. Finally, to address the problem in greater generality, we consider the derived category of representations of finite, acyclic quivers over commutative noetherian rings equipped with a *tensor triangulated* structure. In this setting, we prove that the telescope conjecture holds for tensor-compatible t-structures.

Contents

Introduction

The central topic of this thesis is the telescope conjecture for t-structures in the derived category of representations of quivers over commutative noetherian rings. Representations of quivers (i.e. directed graphs) over a ring R consist of a collection of R-modules, one for each vertex, and compatible R-linear maps, one for each arrow. It is well known that these objects (together with a suitable notion of homomorphism between them) form an abelian category. The main setup of this thesis is the derived category $\mathcal{D}(RQ)$ of representations of a finite quiver Q over a commutative noetherian ring R and we will study the so-called telescope conjecture in this triangulated category.

The telescope conjecture is a fundamental question in the study of triangulated categories, addressing the relationship between "large" and "small" objects. A triangulated (sub)category is called *compactly generated* if every object can be constructed from a set of compact objects using basic operations such as coproducts and extensions. Of particular interest from the viewpoint of localization theory are the *smashing subcategories*, namely those for which the associated localization functor preserves coproducts, and which encode structural decompositions of the ambient category. Classically, the *telescope conjecture* asks whether every smashing subcategory is compactly generated – that is, whether these well-behaved subcategories are always controlled by compact objects.

This problem was first formulated by Bousfield and Ravenel [Bou79, Rav84], who asked whether the smashing localizations of the stable homotopy category of spectra are precisely those given by localization away from a set of finite spectra, i.e. by the *telescopes* of finite spectra. This classical form of the telescope conjecture has a long and intricate history, but was recently resolved in the negative by Burklund, Hahn, Levy, and Schlank [BHLS23]. In view of this, it is now more natural to regard the telescope conjecture as a property of a triangulated category and to ask, for each case of interest, whether the property holds.

For algebraic triangulated categories, several positive results are known. For instance, the telescope conjecture holds for derived categories of noetherian schemes, as proved by Neeman [Nee92a] in the affine case, and Alonso, Jeremías, and Souto [AJS04] in general. Further positive results have been obtained for derived categories of rings, such as for truncated polynomial rings [DP08], for (one-sided) hereditary rings [KŠ10] and for absolutely flat rings [Ste14]. However, the conjecture fails even in this context, as shown by Keller's counterexample [Kel94]. Moreover, a criterion for rings of weak global dimension at most one was later established in [BŠ17]. Many of these (counter)examples are elegantly unified under the general framework developed in [Hrb25].

Affirmative answers to the telescope conjecture are often accompanied by a full classification of the relevant subcategories. For instance, in [Nee92a] it is proved that, for a commutative noetherian ring R, the smashing subcategories of $\mathcal{D}(R)$ are in bijection with the specialization-closed subsets of $\operatorname{Spec}(R)$, the prime spectrum of the ring; classification results for the proof of the telescope conjecture are also used in [AJS04, BŠ17]. In the context of representations of Dynkin quivers over commutative noetherian rings, the work of Antieau and Stevenson [AS16] extends Neeman's result. They established a bijection between the lattice of smashing subcategories of $\mathcal{D}(RQ)$ and the lattice of poset homomorphisms from $\operatorname{Spec}(R)$ to the lattice $\mathbf{Nc}(Q)$ of noncrossing partitions of the quiver, and proved that the telescope conjecture continues to hold in this broader setting.

More recently, a generalized version of the telescope conjecture has been proposed in the context of t-structures, involving also subcategories which are not necessarily triangulated, see [BH21, Question A] and [HN21, Question A.7]. A *t-structure* on a triangulated category provides a tool to decompose objects into orthogonal parts, in a way reminiscent of truncations in homological algebra and, to some extent, generalizing the decompositions associated to localizing pairs. Formally, a t-structure is given by a pair of (not necessarily triangulated) full subcategories, called the aisle and the coaisle respectively, and the intersection of these two subcategories forms the so-called heart of the t-structure, which is always an abelian category. In this way, t-structures provide a systematic method to recover abelian categories from triangulated ones and to study objects according to their homological properties.

Considering t-structures as the objects of study, the telescope conjecture asks whether every homotopically smashing t-structure (in the sense of [SŠV23]) is compactly generated. Conceptually, the notion of homotopically smashing characterizes those t-structures whose heart is Grothendieck; moreover, as shown in [SŠ23], the compactly generated ones are those with a locally finitely presented heart. For stable t-structures (i.e. the ones given by a pair of triangulated subcategories), the homotopically smashing condition is equivalent to requiring the aisle to be smashing. Hence this formulation of the telescope conjecture recovers the classical one. In what follows, we will refer to this case as the *stable telescope conjecture*, reserving the name *telescope conjecture* for the generalized version.

In the setting of derived categories of rings, this general version of the conjecture has been proved for commutative noetherian rings by Hrbek and Nakamura [HN21] and for (one-sided) hereditary rings by Angeleri Hügel and Hrbek [AH21]; while a criterion for rings of weak global dimension at most one is given in [BH21]. Just like for the stable telescope conjecture, the proof in the commutative noetherian case follows a classification result, this time by Alonso, Jeremías, and Saorín. In [AJS10] the authors prove that the compactly generated t-structures of $\mathcal{D}(R)$ are in bijection with filtrations (i.e. non-increasing sequences) of specialization closed subsets of $\mathrm{Spec}(R)$.

Another variation of the telescope conjecture can be found in the context of tensor triangulated categories, where the question is restricted to tensor-compatible subcategories. Indeed, one can ask whether every smashing tensor-ideal (resp. homotopically smashing tensor-t-structure) is compactly generated. For instance, Dubey and Sahoo [DS23a] found a classification of compactly generated t-structures on the derived category of a noetherian scheme and proved that the tensor version of the telescope conjecture is satisfied in the case of separated noetherian schemes.

As mentioned earlier, this thesis focuses on the telescope conjecture for t-structures in the context of representations of small categories over commutative noetherian rings. The project is also motivated by the desire to find a common framework to two previous approaches: on the one hand, we would like to extend the results of [AS16] from localizing subcategories to t-structures, and on the other hand, we would like to generalize the results of [AJS10, HN21] to representations of quivers over commutative noetherian rings.

In [AS16] the authors prove that there is a bijection between localizing subcategories of $\mathcal{D}(RC)$ and collections of localizing subcategories of $\mathcal{D}(\kappa(\mathfrak{p})C)$ indexed by prime ideals $\mathfrak{p}$ in $\mathrm{Spec}(R)$, where $\kappa(\mathfrak{p})$ is the residue field of R at $\mathfrak{p}$ (see Theorem 4.2.1). Moreover, for every $\mathfrak{p} \in \mathrm{Spec}(R)$ such that the localization $R_\mathfrak{p}$ is a regular local ring, this bijection restricts to a bijection between localizing subcategories of the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RC)$ and localizing subcategories of $\mathcal{D}(\kappa(\mathfrak{p})C)$. In this case, for any finite quiver Q, the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$ satisfies the stable telescope conjecture (see Theorem 4.2.3). We find the analogous results for t-structures, by proving that there is an injective assignment from homotopically smashing t-structures of $\mathcal{D}(RC)$ to collections of homotopically smashing t-structures of $\mathcal{D}(\kappa(\mathfrak{p})C)$ indexed over $\mathrm{Spec}(R)$ (see Theorem 6.1.6). Moreover, for any finite quiver Q, such assignment restricts to a bijection between homotopically smashing t-structures of the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$

and homotopically smashing t-structures of $\mathcal{D}(\kappa(\mathfrak{p})Q)$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ (see Theorem 6.2.4). This proves the telescope conjecture for all stalk subcategories $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ without assuming regularity of the local ring $R_{\mathfrak{p}}$. Therefore, by trivial gluing techniques, we prove the following:

Theorem A (Corollary 6.2.5)**.** *For any commutative artinian ring R and finite quiver Q, the derived category $\mathcal{D}(RQ)$ satisfies the telescope conjecture for t-structures.*

Let us stress that, under the regularity assumption of [AS16], the same gluing techniques would only recover the limit case of representations of quivers over fields, for which the result has already been established in [AH21]. As an application of this result, we exhibit examples of non-hereditary finite dimensional algebras satisfying the telescope conjecture (see Example 6.2.6). Moreover, the techniques used to prove Theorem A can be adapted to establish the telescope conjecture also for representations over certain non-noetherian rings, namely *perfect rings*.

In the same way that the classification of the smashing subcategories and the stable telescope conjecture have been generalized from the derived category of a commutative noetherian ring [Nee92a] to the derived category of representations of Dynkin quivers over such rings by [AS16], we generalize the classification of the compactly generated t-structures [AJS10] and the telescope conjecture for t-structures [HN21] to this broader context. In particular, for representations of Dynkin quivers, the classification of localizing subcategories in [AS16] restricts to a classification of smashing subcategories. This shows that the latter are in bijection with poset homomorphisms between the spectrum of the ring $\operatorname{Spec}(R)$ and the lattice of noncrossing partitions of the quiver $\mathbf{Nc}(Q)$ (see Theorem 4.2.8) and extends Neeman's classification for commutative noetherian rings (see Theorem 2.1.4).

As mentioned before, [AJS10] establishes that compactly generated t-structures in the derived category of a commutative noetherian ring are classified by filtrations (i.e. non-increasing sequences) of specialization closed subsets of the spectrum $\operatorname{Spec}(R)$ (see Theorem 2.2.2). Later, [HN21] shows that homotopically smashing t-structures are cogenerated by shifts of indecomposable injective modules, in a way compatible with this classification, i.e. in such a way that the aisle is determined by a filtration of specialization closed subsets (see Lemma 2.2.4). This fact is fundamental in proving the telescope conjecture in this context (see Theorem 2.2.5).

In this thesis we will show that, in the context of representations of Dynkin quivers over commutative noetherian rings, compactly generated t-structures are in bijection with poset homomorphisms between $\operatorname{Spec}(R)$ and filtrations of noncrossing partitions $\operatorname{Filt}(\mathbf{Nc}(Q))$ (see 7.2.2 (1)). Moreover, just as in the commutative case, we show that homotopically smashing t-structures are cogenerated by shifts of vertexwise indecomposable injective representations, and that they are compactly generated (see Theorem 7.1.12). Thereby we have the following:

Theorem B (Theorem 7.2.2 (2))**.** *For any commutative noetherian ring R and Dynkin quiver Q, the derived category $\mathcal{D}(RQ)$ satisfies the telescope conjecture for t-structures.*

Moreover, within this context, we also prove that compactly generated aisles are determined on cohomology (see Corollary 7.2.5), obtaining a classification of wide subcategories of $\operatorname{mod}(RQ)$ (see Theorem 7.2.7). This generalizes both Takahashi's classification over commutative noetherian rings [Tak08] and the work of Ingalls and Thomas over Dynkin algebras [IT09]. This also completes the picture outlined in [IK24], where similar results are obtained for torsion classes and Serre subcategories of $\operatorname{mod}(RQ)$.

We also consider the derived category of representations of a finite quiver over a commutative noetherian ring endowed with the (derived) vertexwise tensor product. This turns out to be a tensor triangulated category (*tt-category*). In this framework, we compute the *Balmer spectrum* of its compact objects and derive several *tt-classifications.* This computation is noteworthy because, although the category is compactly generated, its compact objects do not form a rigid tt-category. Explicit computations of Balmer spectra in such non-rigid situations are rare – see

[LS13, Xu14, MT17, BCS19] – and, among these, only the first two consider compact objects of a non-rigidly compactly generated tt-category.

The relevance of our example also lies in the fact that, despite the lack of rigidity, it is still the subcategory of compact objects that captures the relevant information about the ambient category. Usually, this is the expected behavior in rigidly-compactly generated tt-categories, where the subcategory of compact objects $\mathcal{T}^c$ coincides with the subcategory of rigid objects $\mathcal{T}^d$. However we are not in such situation. In fact, when $\mathcal{T}^c$ fails to be rigid – or even to form a tt-subcategory, as it may happen when $\mathcal{T}$ is not a compactly generated tt-category – it is generally regarded as more appropriate, as suggested by [BCHS23], to consider the spectrum of rigid objects $\mathrm{Spc}(\mathcal{T}^d)$ rather than the one of the compacts. For example, [Ken25] exhibits big tt-categories where either $\mathcal{T}^c \subsetneq \mathcal{T}^d$ or $\mathcal{T}^d \subsetneq \mathcal{T}^c$, and in both instances the meaningful spectrum is $\mathrm{Spc}(\mathcal{T}^d)$. Our example suggests that the recourse to rigid objects may be relevant only when $\mathcal{T}$ is not a compactly generated tt-category (see Definition 3.2.1).

In particular, we show that the Balmer spectrum $\mathrm{Spc}(\mathcal{D}^c(RQ))$ is a disjoint union of copies of the prime spectrum $\mathrm{Spec}(R)$ (see Theorem 8.1.3), extending the result of [LS13], and that its specialization-closed subsets classify all thick tensor-ideals of $\mathcal{D}^c(RQ)$ (see Corollary 8.1.6). This computation is also significant from the viewpoint of abstract tensor triangular geometry. Indeed, in the rigid case, if the spectrum decomposes as a disjoint union of subspaces, then the category itself splits as a direct sum of subcategories indexed by those components, but our category is indecomposable.

Moving from the compact objects to the ambient big tt-category, we explain how the absence of rigidity obstructs the construction of the Balmer-Favi support, and we introduce an *ad hoc* support which we prove to be *stratifying* (see Theorem 8.1.13). Next, we turn to the recently introduced notion of tensor-t-structures, defined with respect to a fixed tensor-closed suspended subcategory $\mathcal{T}^{\leq 0}$ containing the tensor unit. In this framework, one can formulate the *tensor telescope conjecture*, which asks whether every homotopically smashing tensor-t-structure is compactly generated. We provide a classification of compactly generated tensor-t-structures with respect to the standard aisle $\mathcal{D}^{\leq 0}(RQ)$ in terms of filtrations of specialization-closed subsets of the Balmer spectrum (see Corollary 8.2.6), and we establish the following result:

Theorem C (Theorems 8.3.4 and 8.3.10)**.** *For any commutative noetherian ring R and finite acyclic quiver Q, the derived category $\mathcal{D}(RQ)$ satisfies the telescope conjecture for tensor-t-structures with respect to $\mathcal{D}^{\leq 0}(RQ)$. More generally, the same holds for tensor-t-structures with respect to any tensor-closed suspended subcategory generated by a filtration system with Dynkin support (see Definition 8.3.5).*

This result not only generalizes Theorem B – which plays a crucial role in its proof – but also marks a significant step toward establishing the telescope conjecture for representations of finite acyclic simply-laced quivers.

Structure of the thesis. The present thesis is divided into two parts: the first one contains the *preliminaries*, and the second the author's *original contributions* to the research field. Each part is divided into four chapters. In the first part, we begin with the basics of *triangulated categories* and their application to the *derived category of a commutative noetherian ring*, see Chapters 1 and 2 respectively. We then introduce *tensor triangulated categories* and their actions on compactly generated triangulated categories in Chapter 3, and conclude by applying this theory to the derived category of *representations of small categories over commutative noetherian rings*, see Chapter 4. The second part opens with Chapter 5, containing new results on representations of small categories, including semi-stable versions (i.e. versions for t-structures) of the *local-to-global principle* and *minimality of stalk subcategories*, as well as an analysis of the *tensor triangulated structure* of the derived category. The former will play a central role in Chapters 6 and 7, while the latter provides the framework for Chapter 8. These final three chapters contain the results announced in the introduction, presented in the same order.

Part I

Preliminaries

Chapter 1

Triangulated Categories

We now present some preliminaries on *triangulated categories*. For a modern introduction to this topic we refer the reader to Neeman [Nee01, Chapter 1].

Definition 1.0.1. Let $\mathcal{T}$ be an additive category, and $\Sigma : \mathcal{T} \to \mathcal{T}$ an autoequivalence. A *triangle* in $\mathcal{T}$ is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the compositions $v \circ u$, $w \circ v$, and $(\Sigma u) \circ w$ vanish.

A *morphism of triangles* in $\mathcal{T}$ is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

whose rows are triangles. It is called *isomorphism* if a, b, and c are isomorphisms.

Definition 1.0.2. A *triangulated category* is an additive category $\mathcal{T}$ with an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ (called the suspension functor) and a family of *distinguished triangles*, i.e. triangles satisfying the following axioms:

(TR0) Triangles isomorphic to distinguished triangles are distinguished. Triangles of the form

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

are distinguished.

(TR1) Every morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

(TR2) A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished if and only if the shifted triangle

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is distinguished.

(TR3) Every commutative diagram with solid arrows

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

whose rows are distinguished triangles can be completed to a commutative diagram, i.e. to a morphism of triangles.

(TR4) Given composable morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z$, the commutative diagram with solid arrows

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & A & \longrightarrow & \Sigma X \\ \| & & \downarrow{\scriptstyle v} & & \vdots & & \| \\ X & \xrightarrow{v \circ u} & Z & \longrightarrow & B & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle u} & & \| & & \vdots & & \downarrow{\scriptstyle \Sigma u} \\ Y & \xrightarrow{v} & Z & \longrightarrow & C & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & \| & & \downarrow \\ A & \dashrightarrow & B & \dashrightarrow & C & \dashrightarrow & \Sigma A \end{array}$$

whose rows are distinguished triangles can be completed to a commutative diagram whose fourth row is a distinguished triangle.

In the following, we will denote distinguished triangles by $X \to Y \to Z \xrightarrow{+}$.

Notation 1.0.3. Let $\mathcal{T}$ be a triangulated category. Given subcategories $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the full subcategory of objects $A \in \mathcal{T}$ for which there is a distinguished triangle

$$X \longrightarrow A \longrightarrow Y \xrightarrow{+}$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

We also write:

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) := \{f \in \mathrm{Hom}_{\mathcal{T}}(X, Y) \mid X \in \mathcal{X},\ Y \in \mathcal{Y}\}$$

$$\mathcal{X}^{\perp} := \{Y \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, Y) = 0\} \text{ and } {}^{\perp}\mathcal{Y} := \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{Y}) = 0\}$$

Definition 1.0.4. Let $(\mathcal{T}, \Sigma)$ and $(\mathcal{T}', \Sigma')$ be triangulated categories. A *triangulated functor* is an additive functor $F : \mathcal{T} \to \mathcal{T}'$ satisfying the following axioms:

(TF1) There is a natural isomorphism $\phi : F \circ \Sigma \Rightarrow \Sigma' \circ F$.

(TF2) For every distinguished triangle of $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the following is a distinguished triangle of $\mathcal{T}'$

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\phi_X \circ Fw} \Sigma' \circ FX$$

A central role in the study of triangulated categories is played by those that are *compactly generated*. Roughly speaking, these are triangulated categories in which the entire structure is determined by a small collection of compact objects. This provides access to powerful tools, such as *Brown representability* and stronger *localization* techniques.

Definition 1.0.5. Let $\mathcal{T}$ be a triangulated category. An object $C \in \mathcal{T}$ is called *compact* if for every family $(X_i)_{i\in I}$ with coproduct $\coprod X_i$ existing in $\mathcal{T}$, the canonical map

$$\coprod \mathrm{Hom}_{\mathcal{T}}(C, X_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}\left(C, \coprod X_i\right)$$

is an isomorphism. The subcategory of compact objects is denoted $\mathcal{T}^c$.

Definition 1.0.6. A triangulated category $\mathcal{T}$ with coproducts is called *compactly generated* if $\mathcal{T}^c$ is skeletally small and $\mathcal{T}$ is the smallest triangulated subcategory containing $\mathcal{T}^c$ and closed under coproducts.

Theorem 1.0.7 (Brown representability, [Nee96, Theorems 4.1, 5.1])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category, $\mathcal{T}'$ any triangulated category and $F : \mathcal{T} \to \mathcal{T}'$ a triangulated functor. Then, F preserves coproducts if and only if it has a right adjoint $G : \mathcal{T}' \to \mathcal{T}$. Moreover, in this case, F preserves compact objects if and only if G preserves coproducts.*

1.1 Localization theory

Localization theory provides a systematic way to study triangulated categories by decomposing them into smaller, more manageable pieces. The central idea is to "complement" a given subcategory by constructing a new triangulated category in which the objects of that subcategory are formally forced to become zero – namely by forming the quotient by that subcategory. It often happens that this quotient category admits suitable triangulated functors back to the original category, producing the desired decomposition. In this section, we recall the basic definitions and constructions of *Verdier localization*, examine its fundamental properties, and discuss how it interacts with compact generation. Our presentation follows Neeman [Nee01, Section 2.1]; see also Krause [Kra10].

Definition 1.1.1. Let $\mathcal{T}$ be a triangulated category. A strictly full additive subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *triangulated subcategory* if satisfies the following axioms:

(TS1) An object X of $\mathcal{T}$ is in $\mathcal{S}$ if and only if ΣX is in $\mathcal{S}$;

(TS2) For every distinguished triangle of $\mathcal{T}$

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+}$$

such that two out of three among X, Y, Z is in $\mathcal{S}$, so is the third.

Note that, the two axioms are equivalent to say that the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is a triangulated functor. A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick* if in addition:

(TS3) It is closed under taking direct summands of its objects.

If $\mathcal{T}$ has coproducts (resp. products). A thick subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *localizing* (resp. *colocalizing*) if in addition:

(TS4) It is closed under coproducts (resp. products).

Notation 1.1.2. Given a set of objects $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\mathsf{thick}_{\mathcal{T}}\langle\mathcal{X}\rangle$ (resp. $\mathsf{loc}_{\mathcal{T}}\langle\mathcal{X}\rangle$) the smallest thick (resp. localizing) subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and we call it the thick (resp. localizing) subcategory *generated by* $\mathcal{X}$. Moreover, we denote by $\mathrm{Thick}(\mathcal{T})$ (resp. $\mathrm{Loc}(\mathcal{T})$) the lattice of thick (resp. localizing) subcategories of $\mathcal{T}$ ordered by inclusion.

Given a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$, the *kernel* of F is defined to be the full subcategory of $\mathcal{T}$ whose objects map to objects of $\mathcal{T}'$ isomorphic to 0, i.e.

$$\mathrm{Ob}(\ker F) = \{X \in \mathcal{T} \mid FX \cong 0\}$$

Thick subcategories are kernels of triangulated functors.

Proposition 1.1.3. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. Then there exists a triangulated category $\mathcal{T}/\mathcal{S}$ (called the* Verdier quotient*) and an essentially surjective triangulated functor $Q_\mathcal{S} : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ (called the* Verdier localization*) such that:*

1. *$\mathcal{S} = \ker Q_\mathcal{S}$;*
2. *For every other triangulated category $\mathcal{D}$ and triangulated functor $F : \mathcal{T} \to \mathcal{D}$, if $\mathcal{S} \subseteq \ker F$, then F factors uniquely through $Q_\mathcal{S}$.*

Remark 1.1.4. The triangulated category $\mathcal{T}/\mathcal{S}$ is not guaranteed to be locally small, in general; i.e. the morphisms between two given objects may form a proper class. One way to make sure this does not happen is by showing that the localization functor $Q_\mathcal{S}$ admits an adjoint, on either side. Indeed, assume for example that $Q_\mathcal{S}$ has a right adjoint $\rho_\mathcal{S}$, then for every $Y = Q_\mathcal{S}X, Y' = Q_\mathcal{S}X'$ in $\mathcal{T}/\mathcal{S}$, we have

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(Q_\mathcal{S}X, Q_\mathcal{S}X') \cong \mathrm{Hom}_\mathcal{T}(X, \rho_\mathcal{S}Q_\mathcal{S}X')$$

In certain cases, the Verdier quotient functor admits an adjoint back to the original triangulated category. When this occurs, the category admits a canonical decomposition into two hom-orthogonal subcategories.

Proposition 1.1.5 ([Kra10, Proposition 4.9.1])**.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a thick subcategory of $\mathcal{T}$. Then the following are equivalent:*

1. *The inclusion functor $i_\mathcal{S} : \mathcal{S} \hookrightarrow \mathcal{T}$ admits a right (resp. left) adjoint;*
2. *The Verdier localization functor $Q_\mathcal{S} : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a right (resp. left) adjoint;*
3. *The Verdier quotient $\mathcal{T}/\mathcal{S}$ is triangulated equivalent to $\mathcal{S}^\perp$ (resp. ${}^\perp\mathcal{S}$) via the composite $Q_\mathcal{S} \circ i_{\mathcal{S}^\perp}$ (resp. $Q_\mathcal{S} \circ i_{{}^\perp\mathcal{S}}$);*
4. *$\mathcal{S} * \mathcal{S}^\perp = \mathcal{T}$ (resp. ${}^\perp\mathcal{S} * \mathcal{S} = \mathcal{T}$).*

Definition 1.1.6. Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *Bousfield localizing* (resp. *Bousfield colocalizing*) if it satisfies the equivalent properties of Proposition 1.1.5.

In particular, if $\mathcal{T}$ has coproducts (resp. products), then Bousfield localizing (resp. Bousfield colocalizing) subcategories are localizing (resp. colocalizing).

Corollary 1.1.7. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a thick subcategory of $\mathcal{T}$. Then $\mathcal{S}$ is Bousfield localizing if and only if $\mathcal{S}^\perp$ is Bousfield colocalizing and $\mathcal{S} = {}^\perp(\mathcal{S}^\perp)$. Moreover, in this case, we have the following* localization sequence*, i.e. a diagram*

$$\left(\mathcal{T}/\mathcal{S}^\perp \overset{\psi}{\cong}\right) \mathcal{S} \underset{\psi\circ Q_{\mathcal{S}^\perp}}{\overset{i_\mathcal{S}}{\underset{\perp}{\rightleftarrows}}} \mathcal{T} \underset{i_{\mathcal{S}^\perp}\circ\varphi}{\overset{Q_\mathcal{S}}{\underset{\perp}{\rightleftarrows}}} \mathcal{T}/\mathcal{S} \left(\overset{\varphi}{\cong} \mathcal{S}^\perp\right)$$

where the isomorphisms $\varphi : \mathcal{T}/\mathcal{S} \to \mathcal{S}^\perp$ and $\psi : \mathcal{T}/\mathcal{S}^\perp \to \mathcal{S}$ are the inverses of those in Proposition 1.1.5 (3), and respectively identify the right adjoint of $Q_\mathcal{S}$ with the inclusion of $\mathcal{S}^\perp$ and the right adjoint of $i_\mathcal{S}$ with the Verdier localization by $\mathcal{S}^\perp$.

Bousfield localizing subcategories are kernels (resp. images) of localization (resp. colocalization) functors – and also the dual holds.

Definition 1.1.8. Let $\mathcal{T}$ be a triangulated category. A triangulated functor $\mathrm{L} : \mathcal{T} \to \mathcal{T}$ (resp. $\Gamma : \mathcal{T} \to \mathcal{T}$) is a *localization functor* (resp. *colocalization functor*) if there exists a natural transformation $\eta : \mathrm{id}_{\mathcal{T}} \to \mathrm{L}$ (resp. $\varepsilon : \Gamma \to \mathrm{id}_{\mathcal{T}}$) such that the natural transformations $L\eta_{\_}, \eta_{\mathrm{L}\_} : \mathrm{L} \to \mathrm{L}^2$ (resp. $\Gamma\varepsilon_{\_}, \varepsilon_{\Gamma\_} : \Gamma^2 \to \Gamma$) are the same and are natural isomorphisms of functors.

Proposition 1.1.9. *Let $\mathcal{T}$ be a triangulated category. The we have that:*

1. *Any localization functor $\mathrm{L} : \mathcal{T} \to \mathcal{T}$ gives rise to a Bousfield (co)localizing subcategory $\mathcal{S} = \ker \mathrm{L}$ (resp. $\mathcal{C} = \operatorname{im} \mathrm{L}$).*

2. *Any colocalization functor $\Gamma : \mathcal{T} \to \mathcal{T}$ gives rise to a Bousfield (co)localizing subcategory $\mathcal{S} = \operatorname{im} \Gamma$ (resp. $\mathcal{C} = \ker \Gamma$).*

3. *Any Bousfield localizing subcategory $\mathcal{S}$, give rise to a localization functor $\mathrm{L}_{\mathcal{S}} = \rho_{\mathcal{S}} \circ Q_{\mathcal{S}}$ and a colocalization functor $\Gamma_{\mathcal{S}} = \lambda_{\mathcal{S}^{\perp}} \circ Q_{\mathcal{S}^{\perp}}$ such that*
$$\ker \mathrm{L}_{\mathcal{S}} = \operatorname{im} \Gamma_{\mathcal{S}} = \mathcal{S} \text{ and } \operatorname{im} \mathrm{L}_{\mathcal{S}} = \ker \Gamma_{\mathcal{S}} = \mathcal{S}^{\perp}$$
and for any $X \in \mathcal{T}$ the localization triangle of Proposition 1.1.5 (4) is
$$\Delta_{\mathcal{S}}(X) := \ \mathrm{L}_{\mathcal{S}}X \longrightarrow X \longrightarrow \Gamma_{\mathcal{S}}X \xrightarrow{+}$$

When the localization functor preserves coproducts, the Verdier quotient by its image admits both a left and a right adjoint. In this case, the resulting decomposition of the triangulated category is known as a *recollement*.

Proposition 1.1.10 ([Kra10, Proposition 5.5.1])**.** *Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{S}$ a Bousfield localizing subcategory of $\mathcal{T}$. Then the following are equivalent:*

1. *The right adjoint of the inclusion functor $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{T}$ preserves coproducts;*

2. *The right adjoint of the Verdier localization functor $Q_{\mathcal{S}} : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ preserves coproducts;*

3. *The localization functor $\mathrm{L}_{\mathcal{S}} : \mathcal{T} \to \mathcal{T}$ preserves small coproducts.*

Definition 1.1.11. Let $\mathcal{T}$ be a triangulated category with coproducts. A subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *smashing* if it satisfies the equivalent properties of Proposition 1.1.10.

Notation 1.1.12. We denote by $\mathrm{Smash}(\mathcal{T})$ the poset of smashing subcategories of $\mathcal{T}$ ordered by inclusion. Recall that it forms a lattice when $\mathcal{T}$ is compactly generated (see [Kra00, Corollary 4.12]).

Corollary 1.1.13. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{S}$ a Bousfield localizing subcategory of $\mathcal{T}$. Then $\mathcal{S}$ is smashing if and only if $\mathcal{S}^{\perp}$ is Bousfield localizing and (Bousfield) colocalizing. Moreover, in this case, we have the following* recollement*, i.e. a diagram*

$$\mathcal{S}^{\perp} \;\underset{\overset{\longleftarrow}{\psi' \circ Q_{(\mathcal{S}^{\perp})^{\perp}}}}{\overset{\overset{\varphi \circ Q_{\mathcal{S}}}{\longleftarrow}}{\xrightarrow{\ i_{\mathcal{S}^{\perp}}\ }}}\; \mathcal{T} \;\underset{\overset{\longleftarrow}{i_{(\mathcal{S}^{\perp})^{\perp}} \circ \varphi'}}{\overset{\overset{i_{\mathcal{S}} \circ \psi}{\longleftarrow}}{\xrightarrow{\ Q_{\mathcal{S}^{\perp}}\ }}}\; \mathcal{T}$$

where the isomorphisms φ, ψ and φ', ψ' are those arising from the localization sequences of $\mathcal{S}$ and $\mathcal{S}^{\perp}$ respectively, as in Corollary 1.1.7. Moreover, this recollement also give rise to two pairs of localization and co-localization functors $(\mathrm{L}_{\mathcal{S}}, \Gamma_{\mathcal{S}})$ *and* $(\mathrm{L}_{\mathcal{S}^{\perp}}, \Gamma_{\mathcal{S}^{\perp}})$.

Proposition 1.1.14 ([Kra10, Theorem 5.6.1]). *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{X} \subseteq \mathcal{T}^c$ a set of compact objects. Then the smallest localizing subcategory $\mathcal{S} = \mathsf{loc}_{\mathcal{T}}\langle \mathcal{X} \rangle$ containing $\mathcal{X}$ is a smashing subcategory.*

Definition 1.1.15. A compactly generated triangulated category $\mathcal{T}$ is said to satisfy the *stable telescope conjecture* if the converse of Proposition 1.1.14 holds, namely if any smashing subcategory is a compactly generated localizing subcategory.

In the following chapters, we will examine some well-known examples of triangulated categories satisfying the *stable telescope conjecture*, such as the derived category $\mathcal{D}(R)$ of a commutative noetherian ring R (see Section 2.1) and the derived category $\mathcal{D}(RQ)$ of representations of a Dynkin quiver Q over a commutative noetherian ring R (see Section 4.2). Moreover, from the results in Chapter 6 we can derive new examples, namely the derived category $\mathcal{D}(RQ)$ of representations of a finite quiver Q over a commutative artinian ring R.

1.2 Derived categories

We now recall some standard material on *derived categories* and *derived functors*; for reference see Kashiwara and Schapira [KS06, Sections 1.3-7] and Krause [Kra21, Chapter 4].

Let $\mathcal{A}$ be an additive category. We denote by $\mathcal{C}(\mathcal{A})$ the category of cochain complexes over $\mathcal{A}$ and cochain maps

$$X^\bullet = (X^i, d_X^i) = (\ldots \to X^i \xrightarrow{d_X^i} X^{i+1} \to \ldots) \text{ and } f^\bullet = (f^i : X^i \to Y^i) \textit{ s.t. } d_Y^i \circ f^i = f^{i+1} \circ d_X^i$$

where $X^i \in \mathcal{A}$ and $f^i \in \operatorname{Hom}_{\mathcal{A}}(X^i, Y^i)$. We will often omit the superscript $^\bullet$ when it is unnecessary.

Definition 1.2.1. Let $\mathcal{A}$ be an additive category. Then:

- The *shift* functor $[1] : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ is the autoequivalence defined by

$$\left(X[1]^i, d^i_{X[1]}\right) := \left(X^{i+1}, -d_X^{i+1}\right) \text{ and } f[1]^i := f^{i+1}$$

  we denote by $[n] = [1]^n$ for any $n \in \mathbb{Z}$.

- For a morphism $f : X \to Y$, the *cone* of f is the complex defined as

$$\operatorname{cone} f := \left(X[1]^i \oplus Y^i, \begin{bmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{bmatrix} \right)$$

Given an additive category, it is often useful to study complexes of its objects and the morphisms between such complexes up to *homotopy*. This leads naturally to the *homotopy category*, which provides a first example of a triangulated category.

Definition 1.2.2. Given morphisms of complexes $f, g : X \to Y$, we say that f and g are *homotopic*, and write $f \sim g$, if there is a collection of maps $h = \left(h^i : X^i \to Y^{i-1}\right)_{i \in \mathbb{Z}}$

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & X^{i-1} & \longrightarrow & X^i & \xrightarrow{d_X^i} & X^{i+1} & \longrightarrow & \ldots \\ & & & \swarrow^{h^i} & & \swarrow^{h^{i+1}} & & & \\ \ldots & \longrightarrow & Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \longrightarrow & Y^{i+1} & \longrightarrow & \ldots \end{array}$$

such that $f^i - g^i = d_Y^{i-1} \circ h^i + h^{i+1} \circ d_X^i$ for every $i \in \mathbb{Z}$. In this case, h is called *homotopy*. A morphism $f : X \to Y$ is *null-homotopic* if $f \sim 0$, and a complex X is *contractible* if id_X is null-homotopic.

Note that, for any two complexes X and Y, being homotopic defines an equivalence relation on the hom-set $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)$.

Definition 1.2.3. Let $\mathcal{A}$ be an additive category. The *homotopy category* $\mathcal{K}(\mathcal{A})$ is the category with same objects as $\mathcal{C}(\mathcal{A})$ and morphisms defined by $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) := \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y)/\sim$.

Remark 1.2.4. Let $\mathcal{A}$ be an additive category, then any additive functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}$ to an additive category $\mathcal{C}$ such that $F(f) = 0$ for every null-homotopic morphism f, or equivalently, $F(X) = 0$ for every contractible complex X, induces an additive functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{C}$.

In particular, for any two additive categories $\mathcal{A}$ and $\mathcal{B}$, an additive functor $F : \mathcal{A} \to \mathcal{B}$ induces additive functors $\mathcal{C}(F) : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ and $\mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$.

Similarly, the shift functor $[1] : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ induces a shift functor $[1] : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$.

Proposition 1.2.5 ([KS06, Theorem 11.2.6])**.** *Let $\mathcal{A}$ be an additive category, then the category $\mathcal{K}(\mathcal{A})$ together with the shift functor $[1] : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ and the class of distinguished triangles given by triangles of the form*

$$X \xrightarrow{f} Y \longrightarrow \operatorname{cone} f \longrightarrow X[1]$$

is a triangulated category and functors of the form $\mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ are triangulated.

Definition 1.2.6. Let $\mathcal{A}$ be an abelian category. Then:

- For a complex $X = \left(X^i, d_X^i\right)$, we call the *i-th cohomology* of X the object of $\mathcal{A}$ defined by
$$H^i X = \ker d_X^i / \operatorname{im} d_X^{i-1}$$
- A complex X is called *acyclic* if $H^i(X) = 0$ for all $i \in \mathbb{Z}$.
- A morphism of complexes $f : X \to Y$ is called *quasi-isomorphism* if it induces isomorphisms on cohomology, or equivalently, if $\operatorname{cone} f$ is acyclic.

Definition 1.2.7. Let $\mathcal{A}$ be a (co)complete abelian category. Consider the category $\mathcal{C}(\mathcal{C}(\mathcal{A}))$ of bicomplexes. Given a bicomplex $(X^{\bullet,\bullet}, d_1, d_2)$, its product (resp. coproduct) *totalization* is the complex

$$\mathrm{Tot}^n_\Pi(X^{\bullet,\bullet}) := \prod_{i+j=n} X^{i,j} \text{ and } d^n := d_1 + (-1)^n d_2$$

where d_1 and d_2 are the horizontal and vertical differentials (resp. $\mathrm{Tot}_\amalg(X^{\bullet,\bullet})$ defined analogously using coproducts).

Given two complexes $X, Y \in \mathcal{C}(\mathcal{A})$, define the Hom-complex as

$$\mathrm{Hom}_\mathcal{A}(X, Y) := \mathrm{Tot}_\Pi\left(\mathrm{Hom}_\mathcal{A}\left(X^{-i}, Y^j\right)\right)$$

Note that the Hom-complex gives an additive bifunctor $\mathcal{C}(\mathcal{A})^{\mathrm{op}} \times \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathbb{Z})$, and hence a (triangulated) bifunctor $\mathcal{K}(\mathcal{A})^{\mathrm{op}} \times \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathbb{Z})$.

Lemma 1.2.8 ([Kra21, Example 4.1.5])**.** *Let $\mathcal{A}$ be a complete abelian category. Then for every two complexes $X, Y \in \mathcal{C}(\mathcal{A})$ and $i \in \mathbb{Z}$*

$$H^i\left(\mathrm{Hom}_\mathcal{A}(X, Y)\right) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y[i])$$

While the homotopy category captures chain complexes up to homotopy, it does not identify as zero those complexes whose cohomology vanishes in every degree. To remedy this, one formally quotients out all acyclic complexes, obtaining the *derived category*. This construction refines the homotopy category by focusing on the homological information of complexes.

Definition 1.2.9. The thick subcategory of acyclic complexes in $\mathcal{K}(\mathcal{A})$ is denoted by $\mathcal{K}_{\mathrm{ac}}(\mathcal{A})$. The *derived category* of $\mathcal{A}$ is defined as the Verdier quotient

$$\mathcal{D}(\mathcal{A}) := \mathcal{K}(\mathcal{A})/\mathcal{K}_{\mathrm{ac}}(\mathcal{A})$$

The localization functor will be denoted by $Q_{\mathcal{A}} : \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$.

Recall that, by Remark 1.1.4, $\mathcal{D}(\mathcal{A})$ may not be a locally small category and that to solve this problem it is enough to provide an adjoint to the localization functor $Q_{\mathcal{A}}$.

Definition 1.2.10. Let $\mathcal{A}$ be an abelian category. A complex X is *K-injective* if for every acyclic complex C the Hom-complex $\mathrm{Hom}_{\mathcal{A}}(C, X)$ is acyclic, or equivalently, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(C, X) = 0$. We denote by $\mathcal{K}_{\mathrm{inj}}(\mathcal{A})$ the full subcategory of K-injective objects.

Dually, a complex X is *K-projective* if for every acyclic complex C the Hom-complex $\mathrm{Hom}_{\mathcal{A}}(X, C)$ is acyclic, or equivalently, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, C) = 0$. We denote by $\mathcal{K}_{\mathrm{proj}}(\mathcal{A})$ the full subcategory of K-projective objects.

Lemma 1.2.11. *The subcategories of K-injectives and of K-projectives satisfy the following:*

1. $\mathcal{K}^+(\mathrm{Inj}\,\mathcal{A}) \subseteq \mathcal{K}_{\mathrm{inj}}(\mathcal{A})$ *and* $\mathcal{K}^-(\mathrm{Proj}\,\mathcal{A}) \subseteq \mathcal{K}_{\mathrm{proj}}(\mathcal{A})$.
2. *Both are thick subcategories of* $\mathcal{K}(\mathcal{A})$. *Moreover,* $\mathcal{K}_{\mathrm{inj}}(\mathcal{A})$ *is closed under existing products and* $\mathcal{K}_{\mathrm{proj}}(\mathcal{A})$ *is closed under existing coproducts.*

Given a complex X, a *K-injective resolution* of X is a quasi-isomorphism $X \to I$, with I K-injective. Dually, a *K-projective resolution* of X is a quasi-isomorphism $P \to X$, with P K-projective. If every complex admits a K-injective (resp. K-projective) resolution, we say that $\mathcal{K}(\mathcal{A})$ has *enough K-injectives* (resp. *K-projectives*).

Proposition 1.2.12 ([KS06, Corollary 14.1.8],[Kra21, Proposition 4.3.4])**.** *Let $\mathcal{A}$ be an abelian category. Then, we have that:*

1. *If $\mathcal{A}$ is Grothendieck, $\mathcal{K}(\mathcal{A})$ has enough K-injectives.*
2. *If $\mathcal{A}$ is complete, has enough injectives and products are exact, $\mathcal{K}(\mathcal{A})$ has enough K-injectives and $\mathcal{K}_{\mathrm{inj}}(\mathcal{A})$ is the smallest thick subcategory of $\mathcal{K}(\mathcal{A})$ which is closed under products and contains all injective objects of $\mathcal{A}$. In particular, $\mathcal{K}_{\mathrm{inj}}(\mathcal{A}) \subseteq \mathcal{K}(\mathrm{Inj}\,\mathcal{A})$.*
3. *If $\mathcal{A}$ is cocomplete, has enough projectives and coproducts are exact, $\mathcal{K}(\mathcal{A})$ has enough K-projectives and $\mathcal{K}_{\mathrm{proj}}(\mathcal{A})$ is the smallest thick subcategory of $\mathcal{K}(\mathcal{A})$ which is closed under coproducts and contains all projective objects of $\mathcal{A}$. In particular, $\mathcal{K}_{\mathrm{proj}}(\mathcal{A}) \subseteq \mathcal{K}(\mathrm{Proj}\,\mathcal{A})$.*

Proposition 1.2.13 ([Kra21, Proposition 4.3.1])**.** *The following are equivalent:*

1. *$\mathcal{K}(\mathcal{A})$ has enough K-injectives (resp. K-projectives).*
2. *The localization functor $Q_{\mathcal{A}} : \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ admits a right (resp. left) adjoint.*

Moreover, in this case:

- *The K-injective resolutions $X \mapsto \mathbf{i}X$ induce a left adjoint to the inclusion $\mathcal{K}_{\mathrm{inj}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A})$ (resp. the K-projective resolutions $\mathbf{p}X \mapsto X$ induce a right adjoint to $\mathcal{K}_{\mathrm{proj}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A})$);*
- *The composition $\mathcal{K}_{\mathrm{inj}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A}) \xrightarrow{Q_{\mathcal{A}}} \mathcal{D}(\mathcal{A})$ (resp. $\mathcal{K}_{\mathrm{proj}}(\mathcal{A}) \hookrightarrow \mathcal{K}(\mathcal{A}) \xrightarrow{Q_{\mathcal{A}}} \mathcal{D}(\mathcal{A})$) is an equivalence of triangulated categories;*
- *For every two complexes X and Y it holds that*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, \mathbf{i}Y) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\mathbf{p}X, Y)$$

Remark 1.2.14. Let $\mathcal{A}$ be an abelian category, then any triangulated functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{T}$ to a triangulated category $\mathcal{T}$ such that $F(f)$ is an isomorphism for every quasi-isomorphism f, or equivalently, $F(X) = 0$ for every acyclic complex X, induces a triangulated functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$.

In particular, an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories such that $Q_{\mathcal{B}} \circ \mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ sends quasi-isomorphisms to isomorphisms, or equivalently, acyclic complexes to zero, induces a triangulated functor $\mathcal{D}(F) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$. This is the case for exact functors.

Similarly, the injective (resp. projective) resolution functor $\mathbf{i} : \mathcal{K}(\mathcal{A}) \to \mathcal{K}_{\text{inj}}(\mathcal{A})$ induces a functor $\mathbf{i} : \mathcal{D}(\mathcal{A}) \to \mathcal{K}_{\text{inj}}(\mathcal{A})$ (resp. $\mathbf{p} : \mathcal{K}(\mathcal{A}) \to \mathcal{K}_{\text{proj}}(\mathcal{A})$ induces $\mathbf{p} : \mathcal{D}(\mathcal{A}) \to \mathcal{K}_{\text{proj}}(\mathcal{A})$).

However, in general, triangulated functors $\mathcal{K}(\mathcal{A}) \to \mathcal{T}$ do not send quasi-isomorphisms to isomorphisms (similarly not every additive functor $\mathcal{A} \to \mathcal{B}$ is exact). Thus, it is not always possible to construct such "derived" functors from $\mathcal{D}(\mathcal{A})$ to $\mathcal{T}$ (resp. from $\mathcal{D}(\mathcal{A})$ to $\mathcal{D}(\mathcal{B})$). It is possible, though, to "approximate" them.

Definition 1.2.15. Let $\mathcal{A}$ be an abelian category and $F : \mathcal{K}(\mathcal{A}) \to \mathcal{T}$ be a triangulated functor to a triangulated category $\mathcal{T}$.

- A *left derived functor* of F is a triangulated functor $\mathbf{L}F : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ together with a natural transformation $\eta : \mathbf{L}F \circ Q_{\mathcal{A}} \Rightarrow F$, such that for every other triangulated functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ and natural transformation $\gamma : G \circ Q_{\mathcal{A}} \Rightarrow F$ there exists a unique natural transformation $\mu : G \Rightarrow \mathbf{L}F$ such that $\gamma = \eta \circ \mu_{Q_{\mathcal{A}}-}$.
- A *right derived functor* of F is a triangulated functor $\mathbf{R}F : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ together with a natural transformation $\epsilon : F \Rightarrow \mathbf{R}F \circ Q_{\mathcal{A}}$, such that for every other triangulated functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ and natural transformation $\gamma : F \Rightarrow G \circ Q_{\mathcal{A}}$ there exists a unique natural transformation $\mu : \mathbf{R}F \Rightarrow G$ such that $\gamma = \mu_{Q_{\mathcal{A}}-} \circ \varepsilon$.
- For an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we call left (resp. right) derived functor of F the left (resp. right) derived functor of $Q_{\mathcal{B}} \circ \mathcal{K}(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$

Remark 1.2.16. If $F : \mathcal{A} \to \mathcal{B}$ is exact, then the triangulated functor $\mathcal{D}(F) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is both a left and right derived functor of F.

Proposition 1.2.17 ([Kra21, Proposition 4.3.11])**.** *Let $\mathcal{A}$ be an abelian category and $F : \mathcal{K}(\mathcal{A}) \to \mathcal{T}$ be a triangulated functor to a triangulated category $\mathcal{T}$.*

1. *If $\mathcal{K}(\mathcal{A})$ has enough K-projectives, denoting by $X \mapsto \mathbf{p}X$ the projective resolutions, then $\mathbf{L}F := F \circ \mathbf{p} : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ is a left derived functor of F.*
2. *If $\mathcal{K}(\mathcal{A})$ has enough K-injectives, denoting by $X \mapsto \mathbf{i}X$ the injective resolutions, then $\mathbf{R}F := F \circ \mathbf{i} : \mathcal{D}(\mathcal{A}) \to \mathcal{T}$ is a right derived functor of F.*

Example 1.2.18. The most significant examples of derived functors are the following:

- Let $(\mathcal{A}, \otimes_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}})$ be a closed symmetric monoidal abelian category with coproducts. For any two complexes X and Y in $\mathcal{C}(\mathcal{A})$, we define

$$\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X, Y) := \mathrm{Tot}_{\Pi}\left(\mathrm{Hom}_{\mathcal{A}}(X^{-i}, Y^{j})\right) \text{ and } X \otimes^{\bullet}_{\mathcal{A}} Y := \mathrm{Tot}_{\amalg}\left(X^{i} \otimes_{\mathcal{A}} Y^{j}\right)$$

  Then, if $\mathcal{K}(\mathcal{A})$ has enough K-injectives and K-projectives, these induce two derived functors

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\_, \_) : \mathcal{D}(\mathcal{A})^{\mathrm{op}} \times \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathbb{Z}) \text{ and } \_ \otimes^{\mathbf{L}}_{\mathcal{A}} \_ : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$$

- Let $\mathcal{G}$ be a Grothendieck abelian category and I a small category. Then $\mathcal{G}^I$ denotes the Grothendieck category of diagrams of shape I in $\mathcal{G}$. Let $\Delta_I : \mathcal{G} \to \mathcal{G}^I$ be the *diagonal functor* (of shape I), i.e. the functor which maps each object $G \in \mathcal{G}$ to the constant diagram

$\Delta_I G(i) = G$ and $\Delta_I G(\alpha) = \mathrm{id}_G$ for each object i and morphism α in I. Then the *limit* and *colimit* functors (of shape I) are defined as the right and left adjoint of Δ_I (which exists by bicompleteness of $\mathcal{G}$):

$$\mathcal{G} \;\begin{array}{c} \xleftarrow{\;\mathrm{colim}_I\;} \\ \xrightarrow{\;\Delta_I\;} \\ \xleftarrow[\;\lim_I\;]{} \end{array}\; \mathcal{G}^I$$

In particular, if we further assume that $\mathcal{K}(\mathcal{G})$ has enough K-projectives – this hypothesis is actually superfluous by [Gro13, Proposition 1.30], but to avoid it we should have used the theory of model categories to define derived functors – since the diagram functor Δ_I is exact, that adjoint triple lifts to:

$$\mathcal{D}(\mathcal{G}) \;\begin{array}{c} \xleftarrow{\;\mathbf{L}\mathrm{colim}_I\;} \\ \xrightarrow{\;\mathcal{D}(\Delta_I)\;} \\ \xleftarrow[\;\mathbf{R}\lim_I\;]{} \end{array}\; \mathcal{D}\left(\mathcal{G}^I\right)$$

We call *homotopy limit* and *homotopy colimit* functors (of shape I) the derived functors $\mathrm{holim}_I := \mathbf{R}\lim_I$ and $\mathrm{hocolim}_I := \mathbf{L}\mathrm{colim}_I$, respectively.

Definition 1.2.19. Let $\mathcal{G}$ be a Grothendieck category and I a small directed category. We call *direct homotopy colimit* functor, the left derived functor of the (exact) direct colimit of shape I:

$$\underrightarrow{\mathrm{hocolim}}_I := \mathbf{L}\underrightarrow{\mathrm{colim}}_I : \mathcal{D}\left(\mathcal{G}^I\right) \longrightarrow \mathcal{D}(\mathcal{G})$$

Note that in this case it always exists, indeed by Remark 1.2.16 we can take $\mathbf{L}\underrightarrow{\mathrm{colim}}_I = \mathcal{D}\left(\underrightarrow{\mathrm{colim}}_I\right)$. Denoting by $\mathrm{dia}_I : \mathcal{D}\left(\mathcal{G}^I\right) \longrightarrow \mathcal{D}(\mathcal{G})^I$ the *diagram functor*, which sends objects of $\mathcal{D}\left(\mathcal{G}^I\right)$ to ordinary I-shaped diagrams on $\mathcal{D}(\mathcal{G})$, we say that a subcategory $\mathcal{X} \subseteq \mathcal{D}(\mathcal{G})$ is *closed under directed homotopy colimit* if, for any directed small category I and any object $\mathbb{X} \in \mathcal{D}\left(\mathcal{G}^I\right)$ such that all components $(\mathrm{dia}_I\mathbb{X})(i) \in \mathcal{X}$, we have that $\underrightarrow{\mathrm{hocolim}}_I\, \mathbb{X} \in \mathcal{X}$ – analogous definitions hold for closure under general homotopy limits and homotopy colimits.

1.2.1 Algebraic triangulated categories

Although, for most of the results in this thesis, the generality of a derived category $\mathcal{D}(\mathcal{G})$ of a Grothendieck category $\mathcal{G}$ is sufficient to cover our setup and define *directed homotopy colimits*, we will need – for Section 6.2.2 – to define homotopy colimits in compactly generated triangulated categories that are not necessarily derived categories of Grothendieck categories. For this reason, we recall the notion of *algebraic triangulated category*. We refer to Keller [Kel06] and Hovey [Hov99] for details and unexplained terminology on dg-categories and model categories respectively.

Definition 1.2.20 ([Kra07, Lemma 7.5])**.** A triangulated category $\mathcal{T}$ is called *algebraic* if there exists a fully faithful exact functor $\mathcal{T} \longrightarrow \mathcal{K}(\mathcal{A})$ for some additive category $\mathcal{A}$.

Note that derived categories of Grothendieck categories are algebraic. Indeed, by the existence of K-injective resolutions, the derived category $\mathcal{D}(\mathcal{G})$ is equivalent to the full triangulated subcategory $\mathcal{K}_{\mathrm{Inj}}(\mathcal{G})$ of K-injective objects in $\mathcal{K}(\mathcal{G})$.

Remark 1.2.21 ([Kel06, Theorem 3.8]). Any compactly generated algebraic triangulated category is equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of a small dg-category $\mathcal{A}$, and by [Bec14, Proposition 1.3.5], $\mathcal{D}(\mathcal{A})$ is equivalent to the homotopy category $\mathrm{Ho}(\mathrm{dgMod}(\mathcal{A}))$ of the category of dg-modules over $\mathcal{A}$ endowed with a model structure. Thus, for any directed small category I, we define the direct homotopy colimit functor as the left derived functor (in the sense of model categories) of the direct colimit functor:

$$\underrightarrow{\mathrm{hocolim}}_I := \mathbf{L}\underrightarrow{\mathrm{colim}}_I : \mathrm{Ho}\left(\mathrm{dgMod}(\mathcal{A})^I\right) \longrightarrow \mathrm{Ho}(\mathrm{dgMod}(\mathcal{A})) \cong \mathcal{D}(\mathcal{A})$$

and the closure under directed homotopy colimits analogously to the one of Definition 1.2.19.

Definition 1.2.22. Given a compactly generated algebraic triangulated category $\mathcal{T}$, we say that a subcategory $\mathcal{X}$ is *closed under directed homotopy colimits* if there exists a small dg-category $\mathcal{A}$ and an equivalence $F : \mathcal{T} \longrightarrow \mathcal{D}(\mathcal{A})$ such that $F(\mathcal{X})$ is closed under homotopy colimits in $\mathcal{D}(\mathcal{A})$.

This definition recovers the one given in Definition 1.2.19 when $\mathcal{T} = \mathcal{D}(\mathcal{G})$ is compactly generated. Moreover, by [CS18], in this case the choice of the dg-category $\mathcal{A}$ is unique. However, in general, it is not clear whether this definition is independent of such a choice; nevertheless, we will see in Remark 6.1.3 that it is, in the cases relevant to us.

1.3 t-Structures

Let $\mathcal{T}$ be a triangulated category. A subcategory of $\mathcal{T}$ is called *suspended* (resp. *cosuspended*) if it is closed under direct summands and positive (resp. negative) shifts and extensions; it is called *cocomplete* (resp. *complete*) if it is closed under coproducts (resp. products) and, when $\mathcal{T}$ is compactly generated algebraic, *homotopically smashing* if it is closed under directed homotopy colimits.

Notation 1.3.1. Given a set of objects $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\mathsf{susp}_{\mathcal{T}}\langle\mathcal{X}\rangle$ (resp. $\mathsf{cosusp}_{\mathcal{T}}\langle\mathcal{S}\rangle$) the smallest suspended (resp. cosuspended) subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and we call it the suspended (resp. cosuspended) subcategory *generated by* $\mathcal{X}$. We will use the decorations $\amalg, \Pi, \mathbf{hs}$ in the apex to mean that the subcategory is the smallest among the cocomplete, complete and (when $\mathcal{T}$ is compactly generated algebraic) homotopically smashing, respectively. Moreover, we denote by $\mathrm{Susp}(\mathcal{T})$ the lattice of suspended subcategories of $\mathcal{T}$ ordered by inclusion.

In a triangulated category, *t-Structures* provide a tool to decompose objects into orthogonal parts, generalizing the decompositions arising from localizing subcategories to ones that are not necessarily triangulated.

Definition 1.3.2. A *t-structure* in $\mathcal{T}$ consists of a pair of subcategories $(\mathcal{U}, \mathcal{V})$ satisfying:

(tS1) $\Sigma\mathcal{U} \subseteq \mathcal{U}$;

(tS2) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$;

(tS3) $\mathcal{U} * \mathcal{V} = \mathcal{T}$.

We call $\mathcal{U}$ the *aisle* and $\mathcal{V}$ the *coaisle* of the t-structure and the triangles of Axiom (tS3) the *truncation triangles* with respect to the t-structure $(\mathcal{U}, \mathcal{V})$.

Proposition 1.3.3 ([BBD82, Proposition 1.3.3],[KV88, Proposition 1],[SŠV23, Proposition 5.2])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{T}$. Then it holds that:*

1. *$\mathcal{U} = {}^{\perp}\mathcal{V}$ and $\mathcal{V} = \mathcal{U}^{\perp}$.*

2. *The axiom (tS1) is equivalent to $\mathcal{V} \subseteq \Sigma\mathcal{V}$.*

3. *$\mathcal{U}$ (resp. $\mathcal{V}$) is a cocomplete suspended (resp. complete cosuspended) subcategory; when $\mathcal{T}$ is compactly generated algebraic, it is closed under homotopy colimits (resp. homotopy limits).*

4. *A (co)suspended subcategory is a (co)aisle if and only if the inclusion functor has a right (resp. left) adjoint.*

5. *Denoting by* $\tau_{\mathcal{U}}^{<}$ *and* $\tau_{\mathcal{V}}^{>}$ *the right adjoint of* $i_{\mathcal{U}}$ *and the left adjoint of* $i_{\mathcal{V}}$*, respectively, for any* $X \in \mathcal{T}$ *the truncation triangle of* X *with respect to* $(\mathcal{U}, \mathcal{V})$ *is given by*

$$\tau_{\mathcal{U}}^{<}X \longrightarrow X \longrightarrow \tau_{\mathcal{V}}^{>}X \stackrel{+}{\longrightarrow}$$

Definition 1.3.4. The functors $\tau_{\mathcal{U}}^{<}$ and $\tau_{\mathcal{V}}^{>}$ are called the *left* and *right truncation functors* of the t-structure.

Example 1.3.5. Let $\mathcal{T} = \mathcal{D}(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. The *standard t-structure* in $\mathcal{D}(\mathcal{A})$ is the pair $(\mathcal{D}^{\leq 0}(\mathcal{A}), \mathcal{D}^{\geq 1}(\mathcal{A}))$, defined by

$$\mathcal{D}^{\leq 0}(\mathcal{A}) := \{X \in \mathcal{D}(\mathcal{A}) \mid H^n(X) = 0 \; for \; any \; n > 0\}$$

$$\mathcal{D}^{\geq 1}(\mathcal{A}) := \{X \in \mathcal{D}(\mathcal{A}) \mid H^n(X) = 0 \; for \; any \; n < 1\}$$

The truncation functors with respect to $(\mathcal{D}^{\leq 0}(\mathcal{A}), \mathcal{D}^{\geq 1}(\mathcal{A}))$ are induced by the *smart truncations* of complexes, namely

$$\tau^{\leq 0}X := (\ldots \to X_{-1} \xrightarrow{d^{-1}} \ker d^0 \to 0 \to \ldots) \text{ and } \tau^{\geq 1}X := (\ldots \to 0 \to \operatorname{coker} d^0 \xrightarrow{d^1} X_2 \to \ldots)$$

It is often possible to generate a t-structure from a set of objects $\mathcal{X} \subseteq \mathcal{T}$ and, in these cases, we have an explicit description of the objects in the aisle (see [AJS03, Theorem 3.4] for (1) in the case where $\mathcal{T}$ is the derived category of a Grothendieck category, and [Nee21, Proposition 2.5] for (2) in the case where $\mathcal{X}$ is any set of objects). We will call this the aisle *generated by* $\mathcal{X}$ and denote it by $\mathsf{aisle}_{\mathcal{T}}\langle \mathcal{X} \rangle$.

Theorem 1.3.6 ([Nee21, Theorem 2.3],[KN13, Theorem A.9])**.** *Let* $\mathcal{T}$ *be a compactly generated triangulated category, and* $\mathcal{X}$ *be a set of objects of* $\mathcal{T}$*. Then:*

1. *The smallest cocomplete suspended subcategory* $\mathsf{susp}_{\mathcal{T}}^{\amalg}\langle \mathcal{X} \rangle$ *containing* $\mathcal{X}$ *is an aisle. In particular, it is equal to the full subcategory* ${}^{\perp}(\mathcal{X}[\geq 0]^{\perp})$*.*

2. *If* $\mathcal{X} \subseteq \mathcal{T}^c$ *is a set of compact objects, any object* $X \in \mathsf{susp}_{\mathcal{T}}^{\amalg}\langle \mathcal{X} \rangle$ *fits in a triangle*

$$\textstyle\coprod_{i \geq 0} X_i \longrightarrow X \longrightarrow \coprod_{i \geq 0} X_i[1] \stackrel{+}{\longrightarrow}$$

where X_i *is an* i*-fold extension of coproducts of non-negative shifts of objects in* $\mathcal{X}$*.*

Remark 1.3.7. In general, the analogue of Theorem 1.3.6 (1) for coaisles does not hold. However, it can be achieved in the following, more restricted, context – we refer to *loc. cit.* for undefined terminologies. Let $\mathcal{T} = \mathcal{D}(\mathbb{K}Q)$ be the derived category of an hereditary algebra of finite representation type. By [Bel00, Theorem 12.20], $\mathcal{D}(\mathbb{K}Q)$ is a pure-semisimple triangulated category, i.e. any object is pure-injective. Then, by [LV20, Proposition 5.10, Corollary 5.11], for any set of objects $\mathcal{E} \subseteq \mathcal{D}(\mathbb{K}Q)$, the subcategory $\mathsf{cosusp}_{\mathbb{K}Q}^{\Pi}(\mathcal{E})$, is equal to the coaisle $({}^{\perp}\mathcal{E}[\leq 0])^{\perp}$.

Definition 1.3.8. A t-structure $(\mathcal{U}, \mathcal{V})$ in a compactly generated algebraic triangulated category $\mathcal{T}$ is called:

- *Compactly generated* if $\mathcal{U} = \mathsf{aisle}_{\mathcal{T}}\langle \mathcal{X} \rangle$, for some $\mathcal{X} \subseteq \mathcal{T}^c$.
- *Homotopically smashing* if $\mathcal{V}$ is closed under directed homotopy colimit.
- *Stable* if $\mathcal{U} = \Sigma\mathcal{U}$, or equivalently, $\mathcal{V} = \Sigma\mathcal{V}$.

Notation 1.3.9. Since being compactly generated is mainly a property of the aisle and homotopically smashing of the coaisle, we denote by $\mathrm{Aisle}_{\mathrm{cg}}(\mathcal{T})$ (resp. $\mathrm{Coaisle}_{\mathrm{hs}}(\mathcal{T})$) the lattice of compactly generated aisles (resp. homotopically smashing coaisles) of $\mathcal{T}$ ordered by inclusion.

Proposition 1.3.10 ([Kra00, Theorem A], [SŠV23, Proposition 7.2])**.** *The following holds:*

1. *A t-structure $(\mathcal{U}, \mathcal{V})$ is a stable if and only if $\mathcal{U}$ (resp. $\mathcal{V}$) is a Bousfield (co)localizing subcategory.*

2. *A stable t-structure $(\mathcal{U}, \mathcal{V})$ is homotopically smashing if and only if $\mathcal{U}$ is a smashing subcategory.*

3. *Any compactly generated t-structure is homotopically smashing.*

Definition 1.3.11. A compactly generated algebraic triangulated category $\mathcal{T}$ is said to satisfy the (*semi-stable*) *telescope conjecture* if the converse of Proposition 1.3.10 (3) holds, namely if any homotopically smashing t-structure is compactly generated.

In the next chapter, we will show that the derived category $\mathcal{D}(R)$ of a commutative noetherian ring R satisfies the *telescope conjecture* (see Section 2.2). Moreover, in the second part of this thesis we present new cases in which this conjecture holds—namely, the derived category $\mathcal{D}(RQ)$ of representations of a finite quiver Q over a commutative artinian ring R (see Chapter 6) and the derived category $\mathcal{D}(RQ)$ of representations of a Dynkin quiver Q over a commutative noetherian ring R (see Chapter 7).

Chapter 2

Commutative Noetherian Rings

In this section, R denotes a commutative noetherian ring and $\mathrm{Mod}(R)$ the category of R-modules. The set of prime ideals of R, will be called the *spectrum of* R and denoted by $\mathrm{Spec}(R)$. For an ideal $\mathfrak{a} \subseteq R$, we write

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

The *Zariski topology* on $\mathrm{Spec}(R)$ is the topology whose closed subsets are the sets of the form $V(\mathfrak{a})$ for all ideals $\mathfrak{a} \subseteq R$. In particular, a basis of open subsets is given by

$$\{\mathcal{O}(r) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid r \notin \mathfrak{p}\} \mid r \in R\}$$

Definition 2.0.1. A subset V of $\mathrm{Spec}(R)$ is said to be *specialization closed* if for any $\mathfrak{p} \in V$ we have that $V(\mathfrak{p}) \subseteq V$, or equivalently, if it is a (possibly infinite) union of Zariski closed subsets and we denote by $\mathbf{V}(\mathrm{Spec}(R))$ the set of specialization closed subsets of $\mathrm{Spec}(R)$.

The spectrum of R is deeply related to the structure of the category $\mathrm{Mod}(R)$. For example, we can recover $\mathrm{Spec}(R)$ from the set of the indecomposable injective R-modules.

Proposition 2.0.2 ([Mat58, Proposition 3.1])**.** *Let R be a commutative noetherian ring. Then the assignment*

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Mod}(R)$$
$$\mathfrak{p} \longmapsto E(R/\mathfrak{p})$$

where $E(R/\mathfrak{p})$ is the injective envelope of the module $R/\mathfrak{p}$, defines a bijection between $\mathrm{Spec}(R)$ *and the set of isoclasses of indecomposable injective R-modules.*

Given a prime ideal $\mathfrak{p}$ of R, let $R_\mathfrak{p}$ denote the localization of R at the multiplicative set $R \setminus \mathfrak{p}$ and $\kappa(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ the residue field at $\mathfrak{p}$. Consider the left derived functors

$$(\_)_\mathfrak{p} := \_ \otimes_R R_\mathfrak{p} : \mathcal{D}(R) \to \mathcal{D}(R) \text{ and } \_ \otimes^{\mathbf{L}}_{\mathrm{R}} \kappa(\mathfrak{p}) : \mathcal{D}(R) \to \mathcal{D}(R)$$

Definition 2.0.3. Let R be a commutative noetherian ring and $X \in \mathcal{D}(R)$. We define:

- The *big support* of X as $\mathrm{Supp}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid X_\mathfrak{p} \neq 0\}$.
- The (*small*) *support* of X as $\mathrm{supp}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} X \neq 0\}$.

The (big) support of a subcategory $\mathcal{S} \subseteq \mathcal{D}(R)$ is the union of the (big) supports of its objects.

Lemma 2.0.4. *Let R be a commutative noetherian ring.*

1. *For any $X \in \mathcal{D}(R)$,* $\mathrm{Supp}_R(X)$ *is specialization closed and it is closed if $X \in \mathcal{D}^c(R)$.*
2. *For any $X \in \mathcal{D}(R)$,* $\mathrm{supp}_R(X) \subseteq \mathrm{Supp}_R(X)$ *and equality holds if $X \in \mathcal{D}^c(R)$.*
3. *For every prime $\mathfrak{p} \in \mathrm{Spec}(R)$:*
   (a) $\mathrm{supp}_R(R/\mathfrak{p}) = \mathrm{Supp}_R(R/\mathfrak{p}) = V(\mathfrak{p})$.
   (b) $\mathrm{supp}_R(R_\mathfrak{p}) = \mathrm{Supp}_R(R_\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.
   (c) $\mathrm{supp}_R(\kappa(\mathfrak{p})) = \{\mathfrak{p}\}$ *and* $\mathrm{Supp}_R(\kappa(\mathfrak{p})) = V(\mathfrak{p})$ *and the same hold for $E(R/\mathfrak{p})$.*

2.1 Localizing subcategories

Via the assignment of support, $\mathrm{Spec}(R)$ turns out to control the localizing and smashing subcategories of $\mathcal{D}(R) := \mathcal{D}(\mathrm{Mod}(R))$. This is due to Neeman [Nee92a].

For a subset $P \subseteq \mathrm{Spec}(R)$, we write $\mathrm{supp}_R^{-1}(P)$ for the full subcategory of $\mathcal{D}(R)$ whose objects have small support contained in P.

Theorem 2.1.1 ([Nee92a, Theorem 2.8]). *Let R be a commutative noetherian ring. Then the following assignments form an order-preserving bijection between localizing subcategories of $\mathcal{D}(R)$ and subsets of $\mathrm{Spec}(R)$ ordered by inclusion:*

$$\begin{aligned} \mathrm{Loc}(\mathcal{D}(R)) &\longleftrightarrow \mathbf{P}(\mathrm{Spec}(R)) \\ \mathcal{L} &\longmapsto \mathrm{supp}_R(\mathcal{L}) \\ \mathrm{supp}_R^{-1}(P) = \mathsf{loc}_R\langle \kappa(\mathfrak{p}) \mid \mathfrak{p} \in P\rangle &\longleftarrow\!\shortmid P \end{aligned}$$

In particular, localizing subcategories of $\mathcal{D}(R)$ are Bousfield localizing.

Recall that the derived category of R-modules $\mathcal{D}(R)$ is a compactly generated triangulated category whose compact objects $\mathcal{D}^c(R) = \mathcal{D}^b(\mathrm{proj}\, R)$ are bounded complexes of finitely generated projective R-modules.

Definition 2.1.2. For an ideal $\mathfrak{a}$ in R, let n be the minimal number of generators of $\mathfrak{a}$ and $\{a_1, \dots, a_n\}$ be a set of generators. The *Koszul complex* (on R) at $\mathfrak{a}$ is defined to be

$$K(\mathfrak{a}) := \bigotimes_{i=1}^{n} R \xrightarrow{a_i\cdot} R$$

where the complex $R \xrightarrow{a_i\cdot} R$ is concentrated in degrees -1 and 0 and the map is given by the multiplication by a_i.

Remark 2.1.3.

- Although, in some situations, the definition may be independent of the choice of generators (see [CFH24, 14.4.29]), in general this is not the case, so for each ideal $\mathfrak{a}$ we choose a set of generators of minimal cardinality.

- By definition, $K(\mathfrak{a})$ is a complex of finitely generated free R-modules concentrated in degrees $[-n, 0]$, thus it is compact. In particular, $K(\mathfrak{a})^{-i} = R^{\binom{n}{i}}$ for any $0 \le i \le n$ and the differentials are represented by matrices with entries in $\mathfrak{a}$. Thus, it follows that, $\mathrm{supp}_R(K(\mathfrak{a})) = \mathrm{Supp}_R(K(\mathfrak{a})) = V(\mathfrak{a})$ (see also [CFH24, 15.1.10]).

Theorem 2.1.4 ([Nee92a, Theorems 1.5, 3.3]). *Let R be a commutative noetherian ring. Then the assignments of Theorem 2.1.1 restrict to an order-preserving bijection between thick subcategories of $\mathcal{D}^c(R)$ and specialization-closed subsets of $\mathrm{Spec}(R)$ ordered by inclusion:*

$$\begin{aligned} \mathrm{Thick}(\mathcal{D}^c(R)) &\longleftrightarrow \mathbf{V}(\mathrm{Spec}(R)) \\ \mathcal{S} &\longmapsto \mathrm{supp}_R(\mathcal{S}) = \mathrm{Supp}_R(\mathcal{S}) \\ \mathrm{supp}_R^{-1}(V) = \mathsf{thick}_R\langle K(\mathfrak{p}) \mid \mathfrak{p} \in P\rangle &\longleftarrow\!\shortmid V \end{aligned}$$

Moreover, the derived category $\mathcal{D}(R)$ satisfies the stable telescope conjecture.

2.2 Compactly generated t-structures

Just like in the case of localizing subcategories, Spec(R) turns out to classify also compactly generated t-structures of $\mathcal{D}(R)$. This is due to Alonso, Jeremías and Saorín [AJS10].

Definition 2.2.1. A *filtration* of specialization closed subsets of Spec(R) is a non-increasing sequence

$$\dots \supseteq V_{n-1} \supseteq V_n \supseteq V_{n+1} \supseteq \dots$$

where $V_i \subseteq \operatorname{Spec}(R)$ is specialization closed and we denote by $\operatorname{Filt}(\mathbf{V}(\operatorname{Spec}(R)))$ the set of filtrations of specialization closed subsets of Spec(R).

Note that we can equivalently define them as a poset homomorphisms, i.e. an order-preserving maps, $\sigma \in \operatorname{Hom}_{\mathrm{Pos}}(\mathbb{Z}^{\mathrm{op}}, \mathbf{V}(\operatorname{Spec}(R)))$ by letting $\sigma(n) = V_n$.

Theorem 2.2.2 ([AJS10, Theorem 3.11])**.** *Let R be a commutative noetherian ring. Then the following assignments form an order-preserving bijection between compactly generated t-structures of $\mathcal{D}(R)$ and filtrations of specialization closed subsets of* Spec(R) *ordered by degreewise inclusion:*

$$\begin{gathered} \operatorname{Aisle}_{\mathrm{cg}}(\mathcal{D}(R)) \longleftrightarrow \operatorname{Filt}(\mathbf{V}(\operatorname{Spec}(R))) \\ \mathcal{U} \longmapsto (\sigma_{\mathcal{U}} : n \mapsto \{\mathfrak{p} \in \operatorname{Spec}(R) \mid R/\mathfrak{p}[-n] \in \mathcal{U}\}) \\ \mathcal{U}_\sigma := \mathsf{aisle}_R\langle R/\mathfrak{p}[-n] \mid \mathfrak{p} \in \sigma(n), n \in \mathbb{Z}\rangle \longleftarrow\!\shortmid \sigma \end{gathered}$$

In particular:

1. *A set of compact generators for $\mathcal{U}_\sigma$ is given by $\mathcal{X}_\sigma = \{K(\mathfrak{p})[-n] \mid \mathfrak{p} \in \sigma(n), n \in \mathbb{Z}\}$.*

2. *In terms of support, the images of the assignments turn out to be*

$$\sigma_{\mathcal{U}} : n \mapsto \operatorname{supp}_R(\mathcal{U}[n] \cap \operatorname{Mod}(R)) \text{ and } \mathcal{U}_\sigma = \mathsf{aisle}_R \left\langle \bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\leq n}(R) \cap \operatorname{supp}_R^{-1}(\sigma(n)) \right\rangle$$

3. *The coaisle associated to $\mathcal{U}_\sigma$ is*

$$\mathcal{V}_\sigma := \{X \in \mathcal{D}(R) \mid \Gamma_{\sigma(n)} X \in \mathcal{D}^{>n}(R), n \in \mathbb{Z}\}$$

where $\Gamma_V : \mathcal{D}(R) \to \mathcal{D}(R)$ denotes the local cohomology functor on V (see Remark 4.1.4).

Unlike the situation in Theorem 2.1.4, even if the descriptions in terms of support and Koszul complexes agree, the residue fields $\kappa(\mathfrak{p})$ are not the right objects to classify compactly generated t-structures.

Example 2.2.3. For example, when $R = \mathbb{Z}$, the aisle $\mathcal{U} := \mathsf{aisle}_{\mathbb{Z}}\langle \mathbb{Q} \rangle$ is not compactly generated in $\mathcal{D}(\mathbb{Z})$. Otherwise it would be associated to the filtration $\sigma_{\mathcal{U}}(n) = \operatorname{Spec}(R)$ for $n \leq 0$, and $\sigma_{\mathcal{U}}(n) = \emptyset$ for $n > 0$ and we would have $\mathcal{U} = \mathcal{D}^{\leq 0}(R)$. Note that this is impossible because $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, i.e. $\mathbb{Z}[0] \in \mathcal{U}^{\perp}$.

2.2.1 Telescope conjecture

Thanks to the classification of compactly generated t-structures, it is possible to show that the category $\mathcal{D}(R)$ satisfies the telescope conjecture. This is due to Hrbek and Nakamura [HN21].

In particular, it can be shown that the homotopically smashing t-structures of $\mathcal{D}(R)$ are cogenerated by shifts of indecomposable injective modules, in such way that the corresponding aisles are determined by specialization closed subsets.

Lemma 2.2.4 ([HN21, Corollary 2.7, Lemmas 2.8, 2.9, 2.13])**.** *Let R be a commutative noetherian ring and $(\mathcal{U}, \mathcal{V})$ a homotopically smashing t-structure in $\mathcal{D}(R)$. Then:*

1. *For any $n \in \mathbb{Z}$ and any $\mathfrak{p} \in \operatorname{Spec}(R)$, either $\kappa(\mathfrak{p})[-n] \in \mathcal{U}$ or $E(R/p)[-n] \in \mathcal{V}$.*
2. *For any two prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$, $E(R/\mathfrak{p})[-n] \in \mathcal{V}$ implies $E(R/\mathfrak{q})[-n] \in \mathcal{V}$.*
3. *$\mathcal{V} = \mathsf{cosusp}_R^{\Pi,\mathbf{hs}}\langle E(R/\mathfrak{p})[-n] \mid E(R/\mathfrak{p})[-n] \in \mathcal{V}, \mathfrak{p} \in \operatorname{Spec}(R), n \in \mathbb{Z}\rangle$.*

In particular, $\operatorname{supp}_R(\mathcal{U}[n] \cap \operatorname{Mod}(R))$ is specialization closed for any $n \in \mathbb{Z}$.

Theorem 2.2.5 ([HN21, Theorem 1.1])**.** *Let R be a commutative noetherian ring. Then any homotopically smashing t-structure in $\mathcal{D}(R)$ is compactly generated, i.e. the derived category $\mathcal{D}(R)$ satisfies the telescope conjecture.*

Chapter 3

Tensor Triangulated Categories

We now present some preliminaries on *tensor triangulated categories*. For a more detailed introduction to this topic we refer the reader to Balmer [Bal05], for the "small" context, and to Balmer and Favi [BF11], for the "big" context.

Definition 3.0.1. A *tensor triangulated category* (*tt-category*) is a triple $(\mathcal{T}, \otimes, \mathbf{1})$ consisting of a triangulated category $\mathcal{T}$, a symmetric monoidal bitriangulated functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with unit $\mathbf{1} \in \mathcal{T}$. In particular, the axioms are the following:

(TT1) (*Symmetric*) For any two objects $X, Y \in \mathcal{T}$, it holds that $X \otimes Y \cong Y \otimes X$;

(TT2) (*Associative*) For any three objects $X, Y, Z \in \mathcal{T}$, it holds that $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$;

(TT3) (*Unital*) For any object $X \in \mathcal{T}$, $X \otimes \mathbf{1} \cong \mathbf{1} \otimes X \cong X$;

(TT4) (*Bitriangulated*) For any object $X \in \mathcal{T}$, the functors $X \otimes \_$ and $\_ \otimes X$ are triangulated.

Definition 3.0.2. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tt-category. A thick subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick tensor-ideal* if $\mathcal{T} \otimes \mathcal{S} \subseteq \mathcal{S}$. A thick tensor-ideal $\mathcal{P}$ of $\mathcal{T}$ is called *prime* if:

(PT1) It is a proper subcategory, i.e. $\mathcal{P} \subsetneq \mathcal{T}$;

(PT2) For any $X, Y \in \mathcal{T}$ such that $X \otimes Y \in \mathcal{P}$, either $X \in \mathcal{P}$ or $Y \in \mathcal{P}$, or equivalently, the collection of objects $\mathcal{P}^{\complement} := \mathcal{T} \setminus \mathcal{P}$ is closed under tensor product of its elements.

Notation 3.0.3. Given a set of objects $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\mathsf{thick}_{\otimes}\langle \mathcal{X} \rangle$ the smallest thick tensor-ideal of $\mathcal{T}$ containing $\mathcal{X}$ and we call it the thick tensor-ideal *generated by* $\mathcal{X}$. Moreover, we denote by $\mathrm{Thick}_{\otimes}(\mathcal{T})$ the lattice of thick tensor-ideals of $\mathcal{T}$ ordered by inclusion.

Remark 3.0.4. If $\mathcal{T}$ is generated as a thick subcategory by the tensor unit $\mathbf{1}$, then any thick subcategory of $\mathcal{T}$ is automatically a tensor-ideal.

3.1 Small tt-categories and Balmer spectrum

In this section, $\mathcal{T}$ denotes a *small tt-category* that is an essentially small tensor triangulated category. The set of prime thick tensor-ideals of $\mathcal{T}$, will be called the *Balmer spectrum* of $\mathcal{T}$ and denoted by $\mathrm{Spc}(\mathcal{T})$. For a thick tensor-ideal $\mathcal{S} \subseteq \mathcal{T}$, we write

$$V(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}$$

The *Zariski topology* on $\mathrm{Spc}(\mathcal{T})$ is the topology whose closed subsets are the sets of the form $V(\mathcal{S})$ for all thick tensor-ideals $\mathcal{S} \subseteq \mathcal{T}$. In particular, a basis of open subsets is given by

$$\{\mathcal{O}(X) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid X \in \mathcal{P}\} \mid X \in \mathcal{T}\}$$

Definition 3.1.1. Let $\mathcal{T}$ be a small tt-category and $X \in \mathcal{T}$, we define the *support* of X as

$$\operatorname{supp}_{\mathcal{T}}(X) := \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid X \notin \mathcal{P}\}$$

The support of a subcategory $\mathcal{S} \subseteq \mathcal{T}$ is the union of the supports of its objects.

The Balmer spectrum and the support defined above turn out to have some universal property.

Definition 3.1.2. A *support data* on a small tt-category $(\mathcal{T}, \otimes, \mathbf{1})$ is a pair (S, σ) where S is a topological space and σ is an assignment which maps any object $X \in \mathcal{T}$ to a closed subset $\sigma(X) \subseteq \mathsf{S}$ satisfying the following properties:

(SD1) $\sigma(0) = \emptyset$ and $\sigma(\mathbf{1}) = \mathsf{S}$;

(SD2) $\sigma(X \oplus Y) = \sigma(X) \cup \sigma(Y)$;

(SD3) $\sigma(\Sigma X) = \sigma(X)$;

(SD4) $\sigma(X) \subseteq \sigma(Y) \cup \sigma(Z)$ for any distinguished triangle $X \to Y \to Z \xrightarrow{+}$;

(SD5) $\sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$.

Proposition 3.1.3 ([Bal05, Lemma 2.6, Theorem 3.2])**.** *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a small tt-category. Then the pair $(\operatorname{Spc}(\mathcal{T}), \operatorname{supp}_{\mathcal{T}})$ is the final support data on $\mathcal{T}$, i.e. it is a support data and for any support data (S, σ) on $\mathcal{T}$ there exists a unique continuous map $f : \mathsf{S} \to \operatorname{Spc}(\mathcal{T})$ such that $\sigma(X) = f^{-1}(\operatorname{supp}_{\mathcal{T}}(X))$ for any object $X \in \mathcal{T}$.*

The crucial property of this theory is that this support data classifies the radical thick tensor-ideals of $\mathcal{T}$.

Definition 3.1.4. For a tt-category $\mathcal{T}$, the radical $\operatorname{rad}(\mathcal{S})$ of a thick tensor-ideal $\mathcal{S} \subseteq \mathcal{T}$ is defined to be

$$\operatorname{rad}(\mathcal{S}) := \{X \in \mathcal{T} \mid X^{\otimes n} \in \mathcal{S} \textit{ for some } n \geq 0\}$$

A thick tensor-ideal $\mathcal{R}$ is called *radical* if $\mathcal{R} = \operatorname{rad}(\mathcal{R})$ and we denote by $\operatorname{Rad}\operatorname{Thick}_{\otimes}(\mathcal{T})$ the lattice of radical thick tensor-ideals of $\mathcal{T}$ ordered by inclusion.

Definition 3.1.5. For a topological space S, a subset $V \subseteq \mathsf{S}$ is called *specialization closed* if it is of the form $V = \bigcup_{i \in I} V_i$ where all V_i are closed and *Thomason subset* if moreover all V_i have quasi-compact complement. We write $\mathbf{V}(\mathsf{S})$ and $\mathbf{Th}(\mathsf{S})$ for the set of specialization closed and Thomason subsets of S, respectively. Moreover, a topological space S is called *noetherian* if it satisfies the ascending chain condition on open subsets, or equivalently, if any open subset is quasi-compact and *sober* if any non-empty irreducible closed subset $V \subseteq \mathsf{S}$ has a unique generic point, i.e. $V = \overline{\{s\}}$ for a unique point $s \in \mathsf{S}$.

Proposition 3.1.6 ([Bal05, Proposition 4.4, Remark 4.11])**.** *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a small tt-category. Then the following hold:*

1. *Any thick tensor-ideal of $\mathcal{T}$ is radical if and only if for any object X in $\mathcal{T}$ it holds that $X \in \mathsf{thick}_{\otimes}\langle X \otimes X \rangle$.*

2. *If $\operatorname{Spc}(\mathcal{T})$ is a noetherian topological space, then a subset $V \subseteq \operatorname{Spc}(\mathcal{T})$ is a Thomason subset if and only if it is specialization closed.*

Theorem 3.1.7 ([Bal05, Theorem 4.10]). *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a small tt-category. Then the following assignments form an order-preserving bijection between radical thick tensor-ideals of $\mathcal{T}$ and Thomason subsets of* $\mathrm{Spc}(\mathcal{T})$ *ordered by inclusion:*

$$\begin{aligned} \mathrm{Rad\,Thick}_{\otimes}(\mathcal{T}) &\longleftrightarrow \mathbf{Th}(\mathrm{Spc}(\mathcal{T})) \\ \mathcal{R} &\longmapsto \mathrm{supp}_{\mathcal{T}}(\mathcal{R}) \\ \mathrm{supp}_{\mathcal{T}}^{-1}(V) &\longleftarrow\!\shortmid V \end{aligned}$$

In particular, if any thick tensor-ideal of $\mathcal{T}$ is radical and $\mathrm{Spc}(\mathcal{T})$ *is a noetherian topological space, these extend to an order-preserving bijection between* $\mathrm{Thick}_{\otimes}(\mathcal{T})$ *and* $\mathbf{V}(\mathrm{Spc}(\mathcal{T}))$.

The equivalence between the Balmer spectrum and some already-known "spectra", such as $\mathrm{Spc}(\mathcal{D}^{\mathrm{perf}}(X)) \cong X$ for any topological noetherian scheme X or $\mathrm{Spc}(\mathrm{stab}(\kappa G)) \cong \mathrm{Proj}(H^{\bullet}(G, \kappa))$ for any finite group scheme G and field κ, comes from the following – quite technical – theorem.

Theorem 3.1.8 ([Bal05, Theorem 5.2]). *Suppose that* (S, σ) *is a* classifying support data *on $\mathcal{T}$, i.e.* S *is a noetherian sober topological space and σ gives a bijection between* $\mathrm{Rad\,Thick}_{\otimes}(\mathcal{T})$ *and* $\mathbf{V}(\mathsf{S})$. *Then the canonical map $f : \mathsf{S} \to \mathrm{Spc}(\mathcal{T})$ of Proposition 3.1.3 is a homeomorphism.*

3.2 Big tt-categories: stratification and tensor-t-structures

By a *big tt-category* we mean a compactly generated tensor triangulated category and we keep this hypothesis for the rest of this section.

Definition 3.2.1. A tensor triangulated category $\mathcal{T}$ is called *compactly generated* if:

(CG0) $\mathcal{T}$ is a compactly generated triangulated category;

(CG1) The tensor product $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ preserves coproducts;

(CG2) $\mathcal{T}^c$ is a tt-subcategory, i.e. the tensor product restricts to $\otimes : \mathcal{T}^c \times \mathcal{T}^c \to \mathcal{T}^c$ and $\mathbf{1} \in \mathcal{T}^c$.

For a big tt-category $\mathcal{T}$, we call Balmer spectrum of $\mathcal{T}$ the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ of its compact objects.

Note that, in this setting, by Brown representability (Theorem 1.0.7), the tensor product $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a *closed* monoidal product, that is the functor $X \otimes \_$ admits a right adjoint $[X, \_] : \mathcal{T} \to \mathcal{T}$ for all $X \in \mathcal{T}$. We call the bifunctor $[\_, \_] : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ the *internal hom* of $\mathcal{T}$. Denoting by $\eta_{X,Y} : Y \to [X, X \otimes Y]$ and $\varepsilon_{X,Y} : X \otimes [X, Y] \to Y$ be the unit and the counit of the tensor-hom adjunction, and by $X^{\vee}$ the object $[X, \mathbf{1}]$, the *evaluation map* $e_{X,Y} : X^{\vee} \otimes Y \to [X, Y]$ is the image of id_Y through the composite

$$\mathrm{Hom}_{\mathcal{T}}(\mathbf{1} \otimes Y, Y) \xrightarrow{\_\circ(\varepsilon_{X,\mathbf{1}} \otimes \mathrm{id}_Y)} \mathrm{Hom}_{\mathcal{T}}(X \otimes X^{\vee} \otimes Y, Y) \cong \mathrm{Hom}_{\mathcal{T}}(X^{\vee} \otimes Y, [X, Y])$$

Definition 3.2.2. A compactly generated tt-category $\mathcal{T}$ is called *rigidly-compactly generated* if:

(RC1) The internal hom restricts to $[\_, \_] : \mathcal{T}^c \times \mathcal{T}^c \to \mathcal{T}^c$

(RC2) $\mathcal{T}^c$ is a *rigid* tt-category, i.e. for any $X, Y \in \mathcal{T}^c$ the *evaluation map*

$$e_{X,Y} : X^{\vee} \otimes Y \longrightarrow [X, Y]$$

is an isomorphism.

The objects $X \in \mathcal{T}$ satisfying the property (RC2) for any $Y \in \mathcal{T}$ are called *rigid* objects of $\mathcal{T}$. Note that, by [HPS97, Theorem 2.1.3 (d)], the condition (RC2) implies that compact objects and rigid objects coincide in $\mathcal{T}$.

Remark 3.2.3. Any rigid object X is *dualizable*, i.e. $X \cong (X^\vee)^\vee$. Indeed, let $\rho : X \to (X^\vee)^\vee$ be the image of the composite $X^\vee \otimes X \cong X \otimes X^\vee \xrightarrow{\varepsilon_{X,\mathbf{1}}} \mathbf{1}$ through the adjunction isomorphism $\mathrm{Hom}_\mathcal{T}(X^\vee \otimes X, \mathbf{1}) \cong \mathrm{Hom}_\mathcal{T}(X, (X^\vee)^\vee)$, then the inverse ρ^{-1} is given by the composite

$$\mathbf{1} \otimes (X^\vee)^\vee \xrightarrow{(e^{-1}_{X,X} \circ \eta_{X,\mathbf{1}}) \otimes \mathrm{id}_{(X^\vee)^\vee}} X^\vee \otimes X \otimes (X^\vee)^\vee \cong X \otimes X^\vee \otimes (X^\vee)^\vee \xrightarrow{\mathrm{id}_X \otimes \varepsilon_{X^\vee,\mathbf{1}}} X \otimes \mathbf{1}$$

In particular, this shows also that any rigid object X is a retract of $X^\vee \otimes X \otimes X$ and thus, by Proposition 3.1.6 (1), any thick tensor-ideal of a rigid small tt-category is radical.

Definition 3.2.4. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a big tt-category. Then a thick tensor-ideal $\mathcal{S}$ of $\mathcal{T}$ is called *localizing* (resp. *Bousfield localizing, smashing*) *tensor-ideal* if $\mathcal{S}$ is a localizing (resp. Bousfield localizing, smashing) subcategory.

Notation 3.2.5. Given a set of objects $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\mathsf{loc}_\otimes\langle\mathcal{X}\rangle$ the smallest localizing tensor-ideal of $\mathcal{T}$ containing $\mathcal{X}$ and we call it the localizing tensor-ideal *generated by* $\mathcal{X}$. Moreover, we denote by $\mathrm{Loc}_\otimes(\mathcal{T})$ (resp. $\mathrm{Smash}_\otimes(\mathcal{T})$) the lattice (resp. poset) of localizing (resp. smashing) tensor-ideals of $\mathcal{T}$ ordered by inclusion. Recall that $\mathrm{Smash}_\otimes(\mathcal{T})$ forms a lattice when $\mathcal{T}$ is rigidly-compactly generated (see [BF11, Proposition 3.11]).

Remark 3.2.6. If $\mathcal{T}$ is generated as a localizing subcategory by the tensor unit $\mathbf{1}$, then any localizing subcategory of $\mathcal{T}$ and any thick subcategory of $\mathcal{T}^c$ are automatically tensor-ideals.

3.2.1 Balmer-Favi Support

In the context of rigidly-compactly generated tt-categories, Balmer and Favi in [BF11] developed a theory of support for not necessarily compact objects which extends the notion of support described in the previous section. However, this does not satisfy any universal property.

For the rest of this sub-section we restrict ourselves to the context of rigidly-compactly generated tt-categories. The next proposition already relies crucially on the rigidity assumption, see Example 8.1.9 for a counterexample in a non-rigid context.

Proposition 3.2.7 ([BF11, Theorem 2.13], [HPS97, Definition 3.3.2])**.** *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category and $\mathcal{S} \subseteq \mathcal{T}$ be a Bousfield localizing tensor-ideal. Let $\Delta_\mathcal{S} : \Gamma_\mathcal{S} \to \mathrm{id}_\mathcal{T} \to \mathrm{L}_\mathcal{S} \xrightarrow{+}$ be the localization triangle for $\mathcal{S}$ (see Proposition 1.1.9). Then the following conditions are equivalent:*

1. *$\mathcal{S}$ is a smashing tensor-ideal.*
2. *$\mathcal{S}^\perp$ is a (localizing) tensor-ideal.*
3. *There is an isomorphism of functors $\mathrm{L}_\mathcal{S} \cong \mathrm{L}_\mathcal{S}\mathbf{1} \otimes \_$.*
4. *There is an isomorphism of triangles $\Delta_\mathcal{S}(\_) \cong \Delta_\mathcal{S}(\mathbf{1}) \otimes \_$.*

This characterization of the smashing tensor-ideals allows us to equivalently consider smashing tensor-ideals and *tensor-idempotent triangles.*

Definition 3.2.8 ([BF11, Proposition 3.1])**.** Let $\mathcal{T}$ be a rigidly-compactly generated tt-category. A distinguished triangle $\Delta = e \xrightarrow{\gamma} \mathbf{1} \xrightarrow{\lambda} f \xrightarrow{+}$ is called *tensor-idempotent* if the following equivalent conditions hold:

- $\gamma \otimes \mathrm{id}_e : e \otimes e \to e$ is an isomorphism.
- $e \otimes f = 0$.
- $\lambda \otimes \mathrm{id}_f : f \to f \otimes f$ is an isomorphism.

Moreover, in this case, $\gamma : e \to \mathbf{1}$ will be called a *left idempotent* and $\lambda : \mathbf{1} \to f$ a *right idempotent.*

Theorem 3.2.9 ([BF11, Theorem 3.5]). *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category. Then the following assignments form a bijection between smashing tensor-ideals and tensor-idempotent triangles of $\mathcal{T}$:*

$$\mathrm{Smash}_{\otimes}(\mathcal{T}) \longleftrightarrow \mathbf{D}(\mathcal{T})$$
$$\mathcal{S} \longmapsto \Delta_{\mathcal{S}}(\mathbf{1})$$
$$\ker(f \otimes \_) = \mathrm{im}(e \otimes \_) \longleftarrow\!\shortmid \Delta = e \to \mathbf{1} \to f \xrightarrow{+}$$

In particular, $\Delta_{\mathcal{S}}(\mathbf{1}) := e_{\mathcal{S}} \to \mathbf{1} \to f_{\mathcal{S}} \xrightarrow{+}$ is the unique tensor-idempotent triangle such that $e_{\mathcal{S}} \in \mathcal{S}$ and $f_{\mathcal{S}} \in \mathcal{S}^{\perp}$ and for any tensor-idempotent triangle $\Delta = e \to \mathbf{1} \to f \xrightarrow{+}$ it holds that $\ker(f \otimes \_)^{\perp} = \mathrm{im}(e \otimes \_)^{\perp} = \ker(e \otimes \_) = \mathrm{im}(f \otimes \_)$.

Remark 3.2.10. Note that for any Thomason subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$, by Proposition 1.1.14 and Theorem 3.1.7, we have a smashing tensor-ideal $\mathsf{loc}_{\otimes} \langle \mathrm{supp}_{\mathcal{T}}^{-1}(V) \rangle$ and thus colocalization and localization functors

$$\Gamma_V \cong \Gamma_V \mathbf{1} \otimes \_ = e_V \otimes \_ \quad \text{and} \quad \mathrm{L}_V \cong \mathrm{L}_V \mathbf{1} \otimes \_ = f_V \otimes \_$$

These properties of tensor-idempotent triangles, together with nice properties of the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$, lead to the definition of *tensor-idempotent residue objects* and the notion of *Balmer-Favi support.*

Lemma 3.2.11 ([BF11, Lemma 7.8, Corollary 7.14]). *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category and, for any prime thick tensor-ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, let*

$$V_{\mathcal{P}} := \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}^c) \mid \mathcal{Q} \subseteq \mathcal{P}\} \text{ and } Z_{\mathcal{P}} := \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}^c) \mid \mathcal{P} \nsubseteq \mathcal{Q}\}$$

If $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space, then:

1. *Both $V_{\mathcal{P}}$ and $Z_{\mathcal{P}}$ are specialization closed subsets of $\mathrm{Spc}(\mathcal{T}^c)$ and $V_{\mathcal{P}} \cap Z_{\mathcal{P}}^{\complement} = \{\mathcal{P}\}$.*
2. *The tensor-idempotent object $g(\mathcal{P}) := e_{V_{\mathcal{P}}} \otimes f_{Z_{\mathcal{P}}}$ is non-zero.*

Definition 3.2.12. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category such that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space. For any thick tensor-ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, we call $g(\mathcal{P})$ *tensor-idempotent residue object at $\mathcal{P}$* and we define $\Gamma_{\mathcal{P}} := \Gamma_{V_{\mathcal{P}}} \circ \mathrm{L}_{Z_{\mathcal{P}}}$. In particular, it holds that

$$\Gamma_{\mathcal{P}} \cong \Gamma_{V_{\mathcal{P}}} \mathbf{1} \otimes \mathrm{L}_{Z_{\mathcal{P}}} \mathbf{1} \otimes \_ = e_{V_{\mathcal{P}}} \otimes f_{Z_{\mathcal{P}}} \otimes \_ =: g(\mathcal{P}) \otimes \_$$

We call $\Gamma_{\mathcal{P}}\mathcal{T}$ the *stalk subcategory of $\mathcal{T}$ at $\mathcal{P}$* and for any object $X \in \mathcal{T}$, we define the *Balmer-Favi support of X* as

$$\mathrm{Supp}_{\mathcal{T}}(X) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \mid \Gamma_{\mathcal{P}} X \neq 0\}$$

Remark 3.2.13 ([BF11, Proposition 7.18]). For any specialization closed subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$, denoting by $\mathcal{V} = \mathsf{loc}_{\otimes} \langle \mathrm{supp}_{\mathcal{T}}^{-1}(V) \rangle$ its associated smashing tensor-ideal, we have that $\mathrm{Supp}_{\mathcal{T}}(e_{\mathcal{V}}) = V$ and $\mathrm{Supp}_{\mathcal{T}}(f_{\mathcal{V}}) = V^{\complement}$. In particular, for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ and $X \in \mathcal{T}$, $\mathrm{Supp}_{\mathcal{T}}(\Gamma_{\mathcal{P}} X) \subseteq \{\mathcal{P}\}$ and the equality holds if and only if $\Gamma_{\mathcal{P}} X \neq 0$, e.g. $\mathrm{Supp}_{\mathcal{T}}(g(\mathcal{P})) = \{\mathcal{P}\}$.

This notion of support extends the support of compact objects introduced earlier and satisfies *almost* every property expected of a support datum on a big tt-category.

Proposition 3.2.14 ([BF11, Proposition 7.17, Theorem 7.22]). *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category such that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space, then the Balmer-Favi support satisfies the following properties:*

1. *For every compact object $X \in \mathcal{T}^c$ one has $\mathrm{Supp}_{\mathcal{T}}(X) = \mathrm{supp}_{\mathcal{T}}(X)$;*

2. $\mathrm{Supp}_{\mathcal{T}}(0) = \emptyset$ *and* $\mathrm{Supp}_{\mathcal{T}}(\mathbf{1}) = \mathrm{Spc}(\mathcal{T}^c)$

3. $\mathrm{Supp}_{\mathcal{T}}(\coprod_{i\in I} X_i) = \bigcup_{i\in I} \mathrm{Supp}_{\mathcal{T}}(X_i)$

4. $\mathrm{Supp}_{\mathcal{T}}(\Sigma X) = \mathrm{Supp}_{\mathcal{T}}(X)$*;*

5. $\mathrm{Supp}_{\mathcal{T}}(X) \subseteq \mathrm{Supp}_{\mathcal{T}}(Y) \cup \mathrm{Supp}_{\mathcal{T}}(Z)$ *for any distinguished triangle* $X \to Y \to Z \xrightarrow{+}$ *;*

6. $\mathrm{Supp}_{\mathcal{T}}(X \otimes Y) \subseteq \mathrm{Supp}_{\mathcal{T}}(X) \cap \mathrm{Supp}_{\mathcal{T}}(Y)$ *and the equality holds if X or Y is compact.*

Due to the failure of the equality in (6), known as the *tensor product formula*, this notion of support could be regarded as non-ideal. Consequently, many authors continue to explore alternative notions of support for big objects, such as the *homological support* (see [Bal20] for further details).

At present, none of these supports has been shown to be universal in any sense analogous to Proposition 3.1.3. However, we will see that, the Balmer-Favi support turns out to be the unique *stratifying* support data, in a sense similar to Theorem 3.1.8.

3.2.2 Stratification

Among the many applications of the Balmer-Favi support theory, one of the most significant is that, under additional assumptions on $\mathcal{T}$, it yields a classification of the localizing tensor-ideals of $\mathcal{T}$. This is shown by Barthel, Heard, and Sanders in [BHS23]. We assume through the section that the Balmer spectrum $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space, even if the generality of *loc. cit.* is wider.

Definition 3.2.15. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category. We say that $\mathcal{T}$ satisfies:

- The *local-to-global principle* if, for any object $X \in \mathcal{T}$, it holds that

$$\mathsf{loc}_{\otimes}\langle X\rangle = \mathsf{loc}_{\otimes}\langle \Gamma_{\mathcal{P}} X \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)\rangle$$

- The *minimality of stalk subcategories* if, for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, $\Gamma_{\mathcal{P}}\mathcal{T} = \mathsf{loc}_{\otimes}\langle g(\mathcal{P})\rangle$ is a minimal localizing tensor-ideal of $\mathcal{T}$, i.e. it is generated by any non-zero object it contains.

Remark 3.2.16 ([BHS23, Theorem 3.22]). Whenever $\mathrm{Spc}(\mathcal{T}^c)$ is noetherian, $\mathcal{T}$ satisfies the local-to-global principle.

Theorem 3.2.17 ([BHS23, Theorem 4.1])**.** *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a rigidly-compactly generated tt-category. Then the following are equivalent:*

1. *$\mathcal{T}$ satisfies the local-to-global principle and the minimality of stalk subcategories.*

2. *The following assignments form an order-preserving bijection between localizing tensor-ideals of $\mathcal{T}$ and subsets of $\mathrm{Spc}(\mathcal{T}^c)$ ordered by inclusion:*

$$\begin{gathered} \mathrm{Loc}_{\otimes}(\mathcal{T}) \longleftrightarrow \mathbf{P}(\mathrm{Spc}(\mathcal{T}^c)) \\ \mathcal{L} \longmapsto \mathrm{Supp}_{\mathcal{T}}(\mathcal{L}) \\ \mathrm{Supp}_{\mathcal{T}}^{-1}(P) = \mathsf{loc}_{\otimes}\langle g(\mathcal{P}) \mid \mathcal{P} \in P\rangle \longleftarrow\!\shortmid P \end{gathered}$$

Definition 3.2.18. A rigidly-compactly generated tt-category $(\mathcal{T}, \otimes, \mathbf{1})$ is said to be *stratified* (*by the Balmer-Favi support*) if the equivalent conditions of Theorem 3.2.17 hold.

As promised, we now state the analogue of Theorem 3.1.8 for the Balmer-Favi support, which clarifies in what sense it is the unique *stratifying* support datum (whenever this is the case).

Theorem 3.2.19 ([BHS23, Theorem 7.6, Theorem 8.2]). *Suppose that* (S,σ) *is a* stratifying support data *on* $\mathcal{T}$, *i.e.* S *is a noetherian sober topological space,* σ *maps any object* $X\in\mathcal{T}$ *to a subset of* $\mathcal{S}$ *satisfying the properties in Proposition 3.2.14 (2-6) and gives a bijection between* $\mathrm{Loc}_{\otimes}(\mathcal{T})$ *and* $\mathbf{P}(\mathsf{S})$ *which restricts to a bijection between* $\mathrm{Thick}_{\otimes}(\mathcal{T}^c)$ *and* $\mathbf{V}(\mathsf{S})$. *Then the canonical map* $f:\mathsf{S}\to\mathrm{Spc}(\mathcal{T})$ *of Proposition 3.1.3 is a homeomorphism. Moreover, any stratifying support data satisfies the tensor product formula.*

3.2.3 Tensor-t-structures and tensor telescope conjecture

Before leaving the realm of big tt-categories we want to recall some definitions and preliminary results regarding *tensor-t-structure*. Our references for this sub-section are [DS23a, DS23b]. In this subsection $\mathcal{T}$ is a (not necessarily rigidly) compactly generated tt-category. While rigidity is assumed in the references, it does not play a role in the results we report here.

Definition 3.2.20. Let $(\mathcal{T},\otimes,\mathbf{1})$ be a compactly generated tt-category given with a suspended subcategory $\mathcal{T}^{\leq 0}$ satisfying

$$\mathcal{T}^{\leq 0}\otimes\mathcal{T}^{\leq 0}\subseteq\mathcal{T}^{\leq 0} \text{ and } \mathbf{1}\in\mathcal{T}^{\leq 0}$$

A suspended subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *tensor-suspended* (*with respect to* $\mathcal{T}^{\leq 0}$) if $\mathcal{T}^{\leq 0}\otimes\mathcal{S}\subseteq\mathcal{S}$.

Notation 3.2.21. Given a set of objects $\mathcal{X}\subseteq\mathcal{T}$, we denote by $\mathsf{susp}_{\otimes}\langle\mathcal{X}\rangle$ the smallest tensor-suspended subcategory containing $\mathcal{X}$ and we call it the tensor-suspended subcategory *generated by* $\mathcal{X}$. We will use the decoration $\amalg$ in the apex to mean that the subcategory is the smallest among the cocomplete ones. Moreover, we denote by $\mathrm{Susp}_{\otimes}(\mathcal{T})$ the lattice of tensor-suspended subcategories of $\mathcal{T}$ ordered by inclusion.

Definition 3.2.22. A t-structure $(\mathcal{U},\mathcal{V})$ of $\mathcal{T}$ is called a *tensor-t-structure* (*with respect to* $\mathcal{T}^{\leq 0}$) if the aisle $\mathcal{U}$ is a tensor-suspended subcategory – or equivalently if $[\mathcal{T}^{\leq 0},\mathcal{V}]\subseteq\mathcal{V}$. In this case $\mathcal{U}$ and $\mathcal{V}$ are called *tensor-aisle* and *tensor-coaisle*, respectively.

Proposition 3.2.23 ([DS23a, Lemma 3.5, Propositions 3.6, 3.11]). *If* $\mathcal{T}^{\leq 0}$ *is generated as a suspended subcategory by a set of objects* $\mathcal{X}\subseteq\mathcal{T}$, *then a suspended subcategory* $\mathcal{S}$ *of* $\mathcal{T}$ *is tensor-suspended if and only if* $\mathcal{X}\otimes\mathcal{S}\subseteq\mathcal{S}$. *In particular:*

1. *If* $\mathcal{T}^{\leq 0}=\mathsf{susp}_{\mathcal{T}}\langle\mathbf{1}\rangle$, *any suspended subcategory is tensor-suspended and consequently any t-structure is a tensor-t-structure.*
2. *If* $\mathcal{T}^{\leq 0}$ *is compactly generated, for any set of compact objects* $\mathcal{X}\subseteq\mathcal{T}^c$, $\mathsf{susp}^{\amalg}_{\otimes}\langle\mathcal{X}\rangle$ *is a tensor-aisle.*

When $\mathcal{T}^{\leq 0}$ is a compactly generated suspended subcategory, we say that a tensor-t-structure $(\mathcal{U},\mathcal{V})$ is *compactly generated* if the tensor-aisle $\mathcal{U}=\mathsf{susp}^{\amalg}_{\otimes}\langle\mathcal{X}\rangle$ is generated by a set of compact objects $\mathcal{X}\subseteq\mathcal{T}^c$ and we denote by $\mathrm{Aisle}_{\mathsf{cg}_{\otimes}}(\mathcal{T})$ the lattice of compactly generated tensor-aisle of $\mathcal{T}$ ordered by inclusion.

Theorem 3.2.24 ([DS23b, Theorem 15 (i)]). *If* $\mathcal{T}^{\leq 0}$ *is a compactly generated suspended subcategory, then the following assignments form an order-preserving bijection between tensor-suspended subcategories of* $\mathcal{T}^c$ *and compactly generated tensor-aisles of* $\mathcal{T}$*:*

$$\begin{aligned}\mathrm{Susp}_{\otimes}(\mathcal{T}^c)&\longleftrightarrow\mathrm{Aisle}_{\mathsf{cg}_{\otimes}}(\mathcal{T})\\ \mathcal{S}&\longmapsto{}^{\perp}(\mathcal{S}^{\perp})\\ \mathcal{U}\cap\mathcal{T}^c&\longleftarrow\!\shortmid\,\mathcal{U}\end{aligned}$$

When $(\mathcal{T},\otimes,\mathbf{1})$ is algebraic, we say that a tensor-t-structure $(\mathcal{U},\mathcal{V})$ is *homotopically smashing* if the tensor-coaisle $\mathcal{V}$ is closed under directed homotopy colimits. It is clear that, by Proposition 1.1.14, any compactly generated tensor-t-structure is homotopically smashing.

Definition 3.2.25. A compactly generated algebraic tt-category $(\mathcal{T}, \otimes, \mathbf{1})$, together with a fixed compactly generated suspended subcategory $\mathcal{T}^{\leq 0}$, is said to satisfy the *tensor telescope conjecture* (*with respect to* $\mathcal{T}^{\leq 0}$) if any homotopically smashing tensor-t-structure (with respect to $\mathcal{T}^{\leq 0}$) is compactly generated.

This variation of the *telescope conjecture* is far less studied in the literature. A positive example can be found in [DS23a], where it is proved that the derived category of quasi-coherent sheaves $\mathcal{D}(\mathrm{Qcoh}(X))$ for a separated noetherian scheme X satisfies it. We will see in Chapter 8 that, for any finite acyclic quiver Q and commutative noetherian ring R, the *tensor telescope conjecture* holds in $\mathcal{D}(RQ)$ with respect to a wide family of suspended subcategories $\mathcal{T}^{\leq 0}$.

3.3 Actions of tt-categories

We dedicate the last section of this chapter to *actions of tt-categories* on triangulated categories. The readers who may thought to tensor triangulated categories as some exotic rings, can now think that we are going to study the corresponding module theory. Standard references for this topic are Stevenson [Ste13, Ste18]. We fix throughout this section a rigidly-compactly generated tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ with a noetherian spectrum $\mathrm{Spc}(\mathcal{T}^c)$ and a compactly generated triangulated category $\mathcal{K}$.

Definition 3.3.1. An *action of* $\mathcal{T}$ *on* $\mathcal{K}$ is a bitriangulated functor

$$\odot : \mathcal{T} \times \mathcal{K} \longrightarrow \mathcal{K}$$

which is coproduct-preserving in both variables, together with two natural isomorphisms α and υ, called *associator* and *unitor* respectively, defined for all $X, Y \in \mathcal{T}$ and $A \in \mathcal{K}$ as

$$\alpha_{X,Y,A} : (X \otimes Y) \odot A \longrightarrow X \odot (Y \odot A) \text{ and } \upsilon_A : \mathbf{1} \odot A \longrightarrow A$$

and satisfying the following conditions:

(TA1) The associator α makes the following diagram commute for all $X, Y, Z \in \mathcal{T}$ and $A \in \mathcal{K}$

$$\begin{array}{ccccc} & & X \odot (Y \odot (Z \odot A)) & & \\ & \nearrow^{\alpha_{X,Y,Z\odot A}} & & \nwarrow^{\mathrm{id}_X \odot \alpha_{Y,Z,A}} & \\ (X \otimes Y) \odot (Z \odot A) & & & & X \odot ((Y \otimes Z) \odot A) \\ \uparrow{\scriptstyle \alpha_{X\otimes Y,Z,A}} & & & & \uparrow{\scriptstyle \alpha_{X,Y\otimes Z,A}} \\ ((X \otimes Y) \otimes Z) \odot A & & = & & (X \otimes (Y \otimes Z)) \odot A \end{array}$$

(TA2) The unitor υ makes the following diagram commute for all $X \in \mathcal{T}$ and $A \in \mathcal{K}$

$$\begin{array}{ccccc} & & X \odot A & & \\ & \nearrow^{\upsilon_{X\odot A}} & & \nwarrow^{\mathrm{id}_X \odot \upsilon_A} & \\ \mathbf{1} \odot (X \odot A) & & & & X \odot (\mathbf{1} \odot A) \\ \uparrow{\scriptstyle \alpha_{\mathbf{1},X,A}} & & & & \uparrow{\scriptstyle \alpha_{X,\mathbf{1},A}} \\ (\mathbf{1} \otimes X) \odot A & & = & & (X \otimes \mathbf{1}) \odot A \end{array}$$

(TA3) For every $A \in \mathcal{K}$ and $r, s \in \mathbb{Z}$ the following diagram commute

$$\begin{array}{ccccc} \Sigma^r_{\mathcal{T}}\mathbf{1} \odot \Sigma^s_{\mathcal{K}} A & = & \Sigma^r_{\mathcal{K}}(\mathbf{1} \odot \Sigma^s_{\mathcal{K}} A) & \xrightarrow{\Sigma^r \upsilon_{\Sigma^s A}} & \Sigma^{r+s}_{\mathcal{K}} A \\ \| & & & & \downarrow{\scriptstyle (-1)^{rs}} \\ \Sigma^s_{\mathcal{K}}(\Sigma^r_{\mathcal{T}}\mathbf{1} \odot A) & = & \Sigma^{r+s}_{\mathcal{K}}(\mathbf{1} \odot A) & \xrightarrow{\Sigma^{r+s} \upsilon_A} & \Sigma^{r+s}_{\mathcal{K}} A \end{array}$$

Definition 3.3.2. A thick (resp. localizing, smashing) subcategory $\mathcal{S}$ of $\mathcal{K}$ is called *thick* (resp. *localizing, smashing*) *$\mathcal{T}$-submodule* if $\mathcal{T} \odot \mathcal{S} \subseteq \mathcal{S}$.

Notation 3.3.3. Given a set of objects $\mathcal{A} \subseteq \mathcal{K}$, we denote by $\mathsf{thick}_\odot\langle\mathcal{A}\rangle$ (resp. $\mathsf{loc}_\odot\langle\mathcal{A}\rangle$) the smallest thick (resp. localizing) $\mathcal{T}$-submodule of $\mathcal{K}$ containing $\mathcal{A}$ and we call it the thick (resp. localizing) $\mathcal{T}$-submodule *generated by* $\mathcal{A}$. Moreover, we denote by $\mathrm{Thick}_\odot(\mathcal{K})$ (resp. $\mathrm{Loc}_\odot(\mathcal{K}), \mathrm{Smash}_\odot(\mathcal{K})$) the lattice of thick (resp. localizing, smashing) $\mathcal{T}$-submodule of $\mathcal{K}$ ordered by inclusion.

Lemma 3.3.4 ([Ste13, Lemmas 3.12, 3.13]). *For any collections of objects $\mathcal{X} \subseteq \mathcal{T}$ and $\mathcal{A} \subseteq \mathcal{K}$, we have an equality of localizing $\mathcal{T}$-submodules*

$$\mathsf{loc}_\odot\langle\mathcal{X} \odot \mathcal{A}\rangle = \mathsf{loc}_\otimes\langle\mathcal{X}\rangle \odot \mathsf{loc}_\mathcal{K}\langle\mathcal{A}\rangle = \mathsf{loc}_\mathcal{K}\langle T \odot (X \odot A) \mid T \in \mathcal{T}, X \in \mathcal{X}, A \in \mathcal{A}\rangle$$

Moreover, if $\mathcal{T}$ is generated as a localizing subcategory by the tensor unit $\mathbf{1}$, then any localizing subcategory of $\mathcal{K}$ is a $\mathcal{T}$-submodule.

Recall from Remark 3.2.10 that for any specialization closed subset $V \subseteq \mathrm{Spc}(\mathcal{T})$ we have colocalization and localization functors $\Gamma_V \cong \Gamma_V\mathbf{1} \otimes \_$ and $\mathrm{L}_V \cong \mathrm{L}_V\mathbf{1} \otimes \_$. By abuse of notation, we denote similarly the functors

$$\Gamma_V := \Gamma_V\mathbf{1} \odot \_, \mathrm{L}_V := \mathrm{L}_V\mathbf{1} \odot \_ : \mathcal{K} \longrightarrow \mathcal{K}$$

Proposition 3.3.5 ([Ste13, Lemmas 4.3, 4.4, Corollary 4.11]). *For any specialization closed subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$, we have that:*

1. *$\Gamma_V\mathcal{K}$ is a compactly generated localizing subcategory and so it is smashing.*
2. *$\mathrm{L}_V\mathcal{K} = (\Gamma_V\mathcal{K})^\perp$ is the corresponding Bousfield localizing and colocalizing subcategory.*

In particular, for any complex $A \in \mathcal{K}$, we can find a distinguished triangle

$$\Gamma_V A \xrightarrow{\gamma_A} A \xrightarrow{\lambda_A} \mathrm{L}_V A \xrightarrow{+}$$

Definition 3.3.6. Recall from Definition 3.2.12 that for any thick tensor-ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, we have a functor $\Gamma_\mathcal{P} \cong g(\mathcal{P}) \otimes \_$. By abuse of notation, we denote similarly the functor

$$\Gamma_\mathcal{P} := g(\mathcal{P}) \odot \_ : \mathcal{K} \longrightarrow \mathcal{K}$$

We call $\Gamma_\mathcal{P}\mathcal{K}$ the *stalk subcategory of $\mathcal{K}$ at $\mathcal{P}$* and, for $A \in \mathcal{K}$, define the *support of A on $\mathcal{T}$* as

$$\mathrm{Supp}_\mathcal{T}(A) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \mid \Gamma_\mathcal{P} A \neq 0\}$$

Proposition 3.3.7 ([Ste13, Proposition 5.7, Theorem 6.9 (ii)]). *The notion of support introduced above satisfies the following properties:*

1. *If $\mathcal{T}$ is an algebraic tt-category, then the support* detects vanishing of objects, *i.e.*

$$\mathrm{Supp}_\mathcal{T}(A) = \emptyset \text{ if and only if } A = 0$$

2. $\mathrm{Supp}_\mathcal{T}(\coprod_{i\in I} A_i) = \bigcup_{i\in I} \mathrm{Supp}_\mathcal{T}(A_i)$;
3. $\mathrm{Supp}_\mathcal{T}(\Sigma A) = \mathrm{Supp}_\mathcal{T}(A)$;
4. $\mathrm{Supp}_\mathcal{T}(A) \subseteq \mathrm{Supp}_\mathcal{T}(B) \cup \mathrm{Supp}_\mathcal{T}(C)$ *for any distinguished triangle* $A \to B \to C \xrightarrow{+}$;
5. *For any specialization closed subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$*

$$\mathrm{Supp}_\mathcal{T}(\Gamma_V\mathcal{K}) \subseteq V \text{ and } \mathrm{Supp}_\mathcal{T}(\mathrm{L}_V\mathcal{K}) \subseteq V^\complement$$

Proposition 3.3.8. *If $\mathcal{T}$ is algebraic, for any $V \in \mathbf{V}(\mathrm{Spc}(\mathcal{T}^c))$ and $A \in \mathcal{K}$, we have that:*

1. $\mathrm{Supp}_{\mathcal{T}}(A) \subseteq V$ *if and only if* $\mathrm{L}_V A = 0$ *if and only if* $\gamma_A : \Gamma_V A \to A$ *is an isomorphism.*
2. $\mathrm{Supp}_{\mathcal{T}}(A) \cap V = \emptyset$ *if and only if* $\Gamma_V A = 0$ *if and only if* $\lambda_A : A \to \mathrm{L}_V A$ *is an isomorphism.*
3. *For any* $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, $\mathrm{Supp}_{\mathcal{T}}(A) \subseteq \{\mathcal{P}\}$ *if and only if* $A \in \Gamma_{\mathcal{P}}\mathcal{K}$ *if and only if* $\Gamma_{\mathcal{P}} A \cong A$.

Proof. Point (1) and (2) follow from Proposition 3.3.7, by considering the triangle

$$\Gamma_V A \xrightarrow{\gamma_A} A \xrightarrow{\lambda_A} \mathrm{L}_V A \xrightarrow{+}$$

(3) By definition, the support of A is contained in $\{\mathcal{P}\}$ if and only if $\Gamma_{\mathcal{P}'} A = 0$ for any $\mathcal{P}' \neq \mathcal{P}$. In this case, by Remark 3.3.11, A lies in the localizing subcategory generated by $\Gamma_{\mathcal{P}} A$ and thus in $\Gamma_{\mathcal{P}}\mathcal{K}$. Moreover, if $A \cong \Gamma_{\mathcal{P}} B$ for some $B \in \mathcal{K}$, by Remark 3.2.13, the support of A is contained in $\{\mathcal{P}\}$ and, by idempotence of $g(\mathcal{P})$, we have that $\Gamma_{\mathcal{P}} A \cong A$. □

We now turn to the study of *stratification* for actions of tensor triangulated categories.

Definition 3.3.9. Thanks to this notion of support and its properties we can define the following assignments:

$$\mathrm{Loc}_{\odot}(\mathcal{K}) \longleftrightarrow \mathbf{P}(\mathrm{Spc}(\mathcal{T}^c))$$
$$\sigma : \mathcal{L} \longmapsto \mathrm{Supp}_{\mathcal{T}}(\mathcal{L})$$
$$\mathrm{Supp}_{\mathcal{T}}^{-1}(P) \longleftarrow\!\shortmid P : \tau$$

In particular, if the support *detects vanishing of objects*, restricting the image of τ to $\mathcal{K}^c$ gives an assignment

$$\mathcal{K}^c \cap \tau : \mathbf{V}(\mathrm{Spc}(\mathcal{T}^c)) \longrightarrow \mathrm{Thick}_{\odot}(\mathcal{K}^c)$$

Note that a priori there is not a well-defined map on the other way. If one assumes that compact objects of $\mathcal{K}$ have specialization closed support then also σ restricts.

Whether the above assignments yield a classification of the localizing (resp. thick) $\mathcal{T}$-submodules of $\mathcal{K}$ (resp. $\mathcal{K}^c$) is not fully understood in general. Nevertheless, considering relative versions of the *local-to-global principle* and of the *minimality of stalk subcategories* can lead to partial results.

Definition 3.3.10. We say that:

- The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the *local-to-global principle* if, for any object $A \in \mathcal{K}$, it holds that
$$\mathsf{loc}_{\odot}\langle A\rangle = \mathsf{loc}_{\odot}\langle \Gamma_{\mathcal{P}} A \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)\rangle$$
- The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the *minimality of stalk subcategories* if, for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, $\Gamma_{\mathcal{P}}\mathcal{K}$ is a minimal localizing $\mathcal{T}$-submodule.
- The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the *telescope conjecture* if any smashing $\mathcal{T}$-submodule arises as a compactly generated localizing $\mathcal{T}$-submodule.

Remark 3.3.11 ([Ste13, Theorem 6.9]). We have the following chain of implications: $\mathcal{T}$ is an algebraic tt-category $\Longrightarrow$ The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle $\Longrightarrow$ The support detects vanishing of objects.

Proposition 3.3.12 ([Ste13, Proposition 6.3])**.** *If the local-to-global principle holds for the action of $\mathcal{T}$ on $\mathcal{K}$ then the composition $\sigma \circ \tau$ is the identity on $\mathbf{P}(\sigma(\mathcal{K}))$. In particular, τ restricts to an injective map $\mathbf{P}(\sigma(\mathcal{K})) \hookrightarrow \mathrm{Loc}_{\odot}(\mathcal{K})$.*

Note that a priori this does not imply that the same results holds for the restrictions between specialization closed subsets of $\mathrm{Spc}(\mathcal{T}^c)$ and thick $\mathcal{T}$-submodules of $\mathcal{K}^c$. For this, one must additionally assume that the compact objects of $\mathcal{K}$ have specialization-closed support, and that for each irreducible closed subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$ there exists a compact object of $\mathcal{K}$ whose support is exactly $V \cap \sigma(\mathcal{K})$. Under these assumptions, $\sigma \circ (\mathcal{K}^c \cap \tau)$ is the identity on $\mathbf{V}(\sigma(\mathcal{K}^c))$, and thus τ restricts to an injective map $\mathbf{V}(\sigma(\mathcal{K}^c)) \hookrightarrow \mathrm{Thick}_{\odot}(\mathcal{K}^c)$. In this ideal situation, we can say much more:

Theorem 3.3.13 ([Ste13, Theorem 7.5]). *Assume that $\mathcal{T}$ is an algebraic tt-category, compact objects of $\mathcal{K}$ have specialization closed support and for each irreducible closed subset $V \subseteq \mathrm{Spc}(\mathcal{T}^c)$ there exists a compact object of $\mathcal{K}$ whose support is precisely $V \cap \sigma(\mathcal{K})$. If moreover, σ and τ define a bijection between $\mathbf{P}(\sigma(\mathcal{K}))$ and $\mathrm{Loc}_{\odot}(\mathcal{K})$) (i.e. $\tau \circ \sigma$ is the identity), then the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the telescope conjecture.*

When the acting tt-category $\mathcal{T}$ is the derived category $\mathcal{D}(R)$ of a commutative noetherian ring R, Benson, Iyengar, and Krause established relative versions of the stratification and telescope conjecture under milder assumptions – see [Ste13, Section 9] for the equivalence between this setting and that of [BIK11]. Recall from Lemma 3.3.4 that, in this case, all localizing subcategories of $\mathcal{K}$ are $\mathcal{D}(R)$-submodules.

Theorem 3.3.14 ([BIK11, Proposition 3.6, Theorems 4.2, 6.1, 6.3]).

1. *The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the local-to-global principle if and only if the following map is bijective:*
$$\mathrm{Loc}(\mathcal{K}) \longrightarrow \prod_{\mathcal{P}\in\sigma(\mathcal{K})} \mathrm{Loc}(\Gamma_{\mathcal{P}}\mathcal{K})$$
$$\mathcal{L} \longmapsto (\Gamma_{\mathcal{P}}\mathcal{L})_{\mathcal{P}\in\sigma(\mathcal{K})}$$

2. *The action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the minimality of stalk subcategories if and only if the following map is bijective:*
$$\prod_{\mathcal{P}\in\sigma(\mathcal{K})} \mathrm{Loc}(\Gamma_{\mathcal{P}}\mathcal{K}) \longrightarrow \mathbf{P}(\sigma(\mathcal{K}))$$
$$(\mathcal{L}_{\mathcal{P}})_{\mathcal{P}\in\sigma(\mathcal{K})} \longmapsto \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \mid \mathcal{L}_{\mathcal{P}} \neq 0\}$$

Thus, if $\mathcal{K}$ satisfies both (1) and (2), σ and τ provide a bijection between localizing subcategories of $\mathcal{K}$ and subsets of $\sigma(\mathcal{K})$.

3. *If $\mathcal{K}$ satisfies (1) and (2) and is a noetherian R-linear category, i.e.* $\mathrm{Hom}_{\mathcal{K}}(C, D)$ *is a finitely generated R-module for any compact objects $C, D \in \mathcal{K}^c$, the bijection restricts between thick subcategories of $\mathcal{K}^c$ and specialization closed subsets of $\sigma(\mathcal{K}^c)$.*

In this case, the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the telescope conjecture.

In view of Theorem 3.3.14 (1), given the generality with which the local-to-global principle holds and the specificity required for the minimality of stalk subcategories, it is natural to focus just on the map $\mathrm{Loc}(\mathcal{K}) \longrightarrow \prod_{\mathcal{P}\in\sigma(\mathcal{K})} \mathrm{Loc}(\Gamma_{\mathcal{P}}\mathcal{K})$ and to seek deeper classification result – or even approach the telescope conjecture – starting from this perspective. In the next chapter, we will see that this strategy indeed succeeds for the action of $\mathcal{T} = \mathcal{D}(R)$ on the derived category $\mathcal{K} = \mathcal{D}(RQ)$ of representations of a Dynkin quiver Q over R.

Chapter 4

Representations of Small Categories I

Given a small category C and a ring R, the category $\mathrm{Mod}_R(C)$ of left C-modules over R is the category of functors $\mathrm{Fun}(C, \mathrm{Mod}(R))$ or, equivalently, the category of R-linear functors $\mathrm{Fun}_R(RC, \mathrm{Mod}(R))$ (see [AS16, Lemma 2.7]), where RC is the R-linearization of the small category C, i.e. the category with the same objects as C and with hom-set $\mathrm{Hom}_{RC}(c, c')$ equal to the free R-module with basis $\mathrm{Hom}_C(c, c')$, for any two objects $c, c' \in C$.

Lemma 4.0.1 ([Pop73, III 4.2, 4.6, 5.2])**.** *The functor category $\mathrm{Mod}_R(C)$ is a Grothendieck abelian category with exact products and a set of small projective generators. In particular, it has also exact coproducts, an injective cogenerator and a projective generator.*

As for any Grothendieck abelian category, we can construct the homotopy category $\mathcal{K}(RC)$ and the derived category $\mathcal{D}(RC)$ (see Section 1.2).

Lemma 4.0.2. *We have that the following holds:*

1. *The derived category $\mathcal{D}(RC)$ is compactly generated. In particular, $\mathcal{D}^c(RC) = \mathcal{K}^b(\mathrm{proj}\, RC)$ and $\mathcal{D}(RC) = \mathsf{loc}_{RC}\langle \mathrm{proj}\, RC\rangle$.*

2. *The homotopy category $\mathcal{K}(RC)$ has enough K-injectives and K-projectives.*

Proof. (1) Let $\mathcal{P}$ be the set of all shifts of the small projective generators of $\mathrm{Mod}_R(C)$. Note that $\mathcal{P}$ is a set of compact objects in $\mathcal{D}(RC)$ and $\mathcal{P}^{\perp} = 0$. Then we have that $\mathsf{loc}_{RC}\langle \mathrm{proj}\, RC\rangle = {}^{\perp}(\mathcal{P}^{\perp}) = \mathcal{D}(RC)$. The characterization of compact objects follows by [Miy00a, Proposition 16.4].
(2) This follows by Lemma 4.0.1 and Proposition 1.2.12 (2,3). □

Proposition 4.0.3. *For any small category C and rings R and S, the following holds:*

1. *Any additive functor $F : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$*
   (a) *Extends to an additive functor $\bar{F} : \mathrm{Mod}_R(C) \to \mathrm{Mod}_S(C)$;*
   (b) *Admits a right and a left derived functor $\mathbf{R}\bar{F}, \mathbf{L}\bar{F} : \mathcal{D}(RC) \to \mathcal{D}(SC)$.*

2. *Any adjoint pair $F \dashv G : \mathrm{Mod}(R) \to \mathrm{Mod}(S)$*
   (a) *Extends to an adjoint pair $\bar{F} \dashv \bar{G} : \mathrm{Mod}_R(C) \to \mathrm{Mod}_S(C)$;*
   (b) *Extends to an adjoint pair $\mathbf{L}\bar{F} \dashv \mathbf{R}\bar{G} : \mathcal{D}(RC) \to \mathcal{D}(SC)$.*

Proof. (1) For any $X, X' \in \mathrm{Mod}_R(C)$ and any natural transformation $f : X \to X'$, define $\bar{F}X := F \circ X$ and $\bar{F}f : \bar{F}X \to \bar{F}X'$ such that $(\bar{F}f)_c := F(f_c)$ for any $c \in C$. Then (b) follows by Proposition 1.2.17.
(2) Let $\eta : 1_{\mathrm{Mod}(R)} \to GF$ and $\varepsilon : FG \to 1_{\mathrm{Mod}(S)}$ be the unit and the counit of the adjunction, i.e. natural transformations satisfying, for any $M \in \mathrm{Mod}(R)$ and $N \in \mathrm{Mod}(S)$, the identities

$$\varepsilon_{F(M)} \circ F(\eta_M) = \mathrm{id}_{F(M)} \text{ and } G(\varepsilon_N) \circ \eta_{G(N)} = \mathrm{id}_{G(N)}$$

Define the natural transformations $\bar{\eta}: 1_{\mathrm{Mod}_R(C)} \to \bar{G}\bar{F}$ and $\bar{\varepsilon}: \bar{F}\bar{G} \to 1_{\mathrm{Mod}_S(C)}$ such that, for any $X \in \mathrm{Mod}_R(C)$ and $Y \in \mathrm{Mod}_S(C)$, $(\bar{\eta}_X)_c := \eta_{X(c)}$ and $(\bar{\varepsilon}_Y)_c := \varepsilon_{Y(c)}$ for any $c \in C$. It is straightforward to check that $\bar{\eta}$ and $\bar{\varepsilon}$ are well defined, i.e. for any $\alpha : c \to d$ the diagrams

$$\begin{array}{ccc} X(c) & \xrightarrow{(\bar{\eta}_X)_c} & \bar{G}\bar{F}X(c) \\ \Big\downarrow{\scriptstyle X(\alpha)} & & \Big\downarrow{\scriptstyle \bar{G}\bar{F}X(\alpha)} \\ X(d) & \xrightarrow{(\bar{\eta}_X)_d} & \bar{G}\bar{F}X(d) \end{array} \quad \text{and} \quad \begin{array}{ccc} \bar{G}\bar{F}Y(c) & \xrightarrow{(\bar{\varepsilon}_Y)_c} & Y(c) \\ \Big\downarrow{\scriptstyle \bar{G}\bar{F}Y(\alpha)} & & \Big\downarrow{\scriptstyle Y(\alpha)} \\ \bar{G}\bar{F}Y(d) & \xrightarrow{(\bar{\varepsilon}_Y)_d} & Y(d) \end{array}$$

commute and for any $f : X \to X'$ and $g : Y \to Y'$ the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\bar{\eta}_X} & \bar{G}\bar{F}X \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \bar{G}\bar{F}f} \\ X' & \xrightarrow{\bar{\eta}_{X'}} & \bar{G}\bar{F}X' \end{array} \quad \text{and} \quad \begin{array}{ccc} \bar{F}\bar{G}Y & \xrightarrow{\bar{\varepsilon}_Y} & Y \\ \Big\downarrow{\scriptstyle \bar{F}\bar{G}g} & & \Big\downarrow{\scriptstyle g} \\ \bar{F}\bar{G}Y' & \xrightarrow{\bar{\varepsilon}_{Y'}} & Y' \end{array}$$

commute, and satisfy the triangle identities

$$\bar{\varepsilon}_{\bar{F}X} \circ \bar{F}\bar{\eta}_X = \mathrm{id}_{\bar{F}(X)} \text{ and } \bar{G}\bar{\varepsilon}_Y \circ \bar{\eta}_{\bar{G}Y} = \mathrm{id}_{\bar{G}(Y)}$$

Thus, $\bar{\eta}$ and $\bar{\varepsilon}$ are the unit and the counit of the adjunction $\bar{F} \dashv \bar{G}$. Then (b) follows by [Mil, Theorem 1.7.1]. □

When R is a commutative ring, as it will be from now on, there are two actions of the category $\mathrm{Mod}(R)$ on $\mathrm{Mod}_R(C)$ which play a relevant role. These are given by the bifunctors

$$\_ \otimes_R \_, \mathrm{Hom}_R(\_, \_) : \mathrm{Mod}(R) \times \mathrm{Mod}_R(C) \to \mathrm{Mod}_R(C)$$

where for every $M \in \mathrm{Mod}(R)$, $X \in \mathrm{Mod}_R(C)$ and $c \in C$, they are defined as

$$(M \otimes_R X)(c) := M \otimes_R X(c) \text{ and } \mathrm{Hom}_R(M, X)(c) := \mathrm{Hom}_R(M, X(c))$$

Remark 4.0.4. These actions extend to the homotopy categories in the standard way, i.e. by taking, for any R-complex M and RC-complex X, the II-totalization of the double complex $M^i \otimes_R X^j$ and the Π-totalization of the double complex $\mathrm{Hom}_R(M^{-i}, X^j)$.

Note that, K-projective complexes of $\mathcal{K}(R)$ are exactly those P which make the functor $\mathrm{Hom}_R(P, \_) : \mathcal{K}(RC) \to \mathcal{K}(RC)$ preserve acyclicity. Analogously, we call a complex $F \in \mathcal{K}(R)$ *K-flat* if the functor $F \otimes_R \_ : \mathcal{K}(RC) \to \mathcal{K}(RC)$ preserves acyclicity. Moreover, we say that a complex $X \in \mathcal{K}(RC)$ is *R-K-flat* (resp. *R-K-injective*) if the functor $\_ \otimes_R X : \mathcal{K}(R) \to \mathcal{K}(RC)$ (resp. $\mathrm{Hom}_R(\_, X) : \mathcal{K}(R) \to \mathcal{K}(RC)$) sends acyclic R-complexes to acyclic RC-complexes.

Define a functor $X \in \mathrm{Mod}_R(C)$ to be *objectwise flat* (resp. *injective*) if $X(c)$ is a flat (resp. injective) R-module for any $c \in C$. The next lemma shows some examples of these complexes and the relationship between them.

Lemma 4.0.5. *The following holds:*

1. *K-projective complexes of $\mathcal{K}(R)$ and bounded above complexes of flat modules are K-flat;*

2. *K-projective complexes of $\mathcal{K}(RC)$ and bounded above complexes of objectwise flat functors are R-K-flat;*

3. *K-injective complexes of $\mathcal{K}(RC)$ and bounded below complexes of objectwise injective functors are R-K-injective.*

Proof. Let us prove (3), the others follow similarly. Consider the class

$$\{Y \in \mathcal{K}(RC) \mid \mathrm{Hom}_R(\_, Y) \text{ preserves acyclicity}\}$$

It is a triangulated subcategory of $\mathcal{K}(RC)$ closed under products and containing stalk complexes of objectwise injective functors. In particular, it contains stalk complexes of injective functors and so, by Proposition 1.2.12 (2), it contains K-injective complexes of $\mathcal{K}(RC)$. Moreover, since bounded below complexes of objectwise injective functors can be reconstructed, via extensions and products, from bounded ones, they are contained in the above class too. □

Theorem 4.0.6. *We have that:*

1. *(a) The bifunctor* $\_ \otimes_R \_$ *admits a left-derived functor* $\_ \otimes^{\mathbf{L}}_{\mathrm{R}} \_$*, which can be computed, independently up to isomorphism, either using K-flat resolutions in the first variable or R-K-flat resolutions in the second;*

   (b) The bifunctor $\mathrm{Hom}_R(\_,\_)$ *admits a right-derived functor* $\mathbf{R}\mathrm{Hom}_R(\_,\_)$*, which can be computed, independently up to isomorphism, either using K-projective resolutions in the first variable or R-K-injective resolutions in the second;*

2. *For any two commutative rings R and S and any R-S-bicomplex M, the functors* $M \otimes^{\mathbf{L}}_S \_$ *and* $\mathbf{R}\mathrm{Hom}_R(M, \_)$ *form an adjoint pair. In particular, for any SC-complex X and RC-complex Y there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(RC)}(M \otimes^{\mathbf{L}}_S X, Y) \cong \mathrm{Hom}_{\mathcal{D}(SC)}(X, \mathbf{R}\mathrm{Hom}_R(M, Y))$$

Proof. (1) By Lemma 4.0.2 (2) and Lemma 4.0.5, $\mathcal{K}(R)$ admits K-flat (resp. K-projective) resolutions and $\mathcal{K}(RC)$ admits R-K-flat (resp. R-K-injective) resolutions and, for any quasi-isomorphisms $f : M \to M'$ between K-flat (resp. K-projective) complexes and $g : X \to X'$ between R-K-flat (resp. R-K-injective) complexes, the morphism $f \otimes_R g$ (resp $\mathrm{Hom}_R(f, g)$) is a quasi-isomorphism. Thus, the statement follows from [Yek19, Theorem 9.3.16] (resp. [Yek19, Theorem 9.3.11, Remark 9.3.18]). Moreover, denoting by $\mathbf{f}M \to M$ and $\mathbf{f_R}X \to X$ a K-flat resolution of M and a R-K-flat resolution of X (resp. $\mathbf{p}M \to M$ and $X \to \mathbf{i_R}X$ for K-projective and R-K-injective), we have that

$$(M \otimes^{\mathbf{L}}_{\mathrm{R}} \_)(X) := M \otimes_R \mathbf{f_R}X \cong \mathbf{f}M \otimes_R \mathbf{f_R}X \cong \mathbf{f}M \otimes_R X =: (\_ \otimes^{\mathbf{L}}_{\mathrm{R}} X)(M)$$

(resp. $\mathbf{R}\mathrm{Hom}_R(M, \_)(X) \cong \mathrm{Hom}_R(\mathbf{p}M, \mathbf{i_R}X) \cong \mathbf{R}\mathrm{Hom}_R(\_, X)(M)$).
(2) Follows from Proposition 4.0.3. □

Remark 4.0.7. The associativity and adjunction isomorphisms also hold, i.e. for any R-S-bicomplex M, R-complex L, S-complex N, SC-complex X and RC-complex Y we have the natural isomorphisms

$$L \otimes^{\mathbf{L}}_{\mathrm{R}} (M \otimes^{\mathbf{L}}_S X) \cong (L \otimes^{\mathbf{L}}_{\mathrm{R}} M) \otimes^{\mathbf{L}}_S X$$

$$\mathbf{R}\mathrm{Hom}_R(M \otimes^{\mathbf{L}}_S N, Y) \cong \mathbf{R}\mathrm{Hom}_S(N, \mathbf{R}\mathrm{Hom}_R(M, Y))$$

Proof. The isomorphisms trivially hold for the non-derived functors $\_ \otimes \_$ and $\mathrm{Hom}(\_\ \_)$ because they hold in the respective module categories. Then, the isomorphisms above are induced by taking the appropriate resolutions, in particular

$$\mathbf{p}L \otimes_R (M \otimes_S \mathbf{f_S}X) \cong (\mathbf{p}L \otimes_R M) \otimes_S \mathbf{f_S}X$$

$$\mathrm{Hom}_R(M \otimes_S \mathbf{p}N, \mathbf{i_R}Y) \cong \mathrm{Hom}_S(\mathbf{p}N, \mathrm{Hom}_R(M, \mathbf{i_R}Y))$$

□

The next lemma shows how t-structures in $\mathcal{D}(RC)$ are affected by the actions of $\mathcal{D}(R)$ described before.

Lemma 4.0.8. *For any object $X \in \mathcal{D}(RC)$ the following holds:*

1. *The class $\{M \in \mathcal{D}(R) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} X \in \mathsf{susp}^{\Pi}_{RC}\langle X\rangle\}$ is a cocomplete suspended subcategory of $\mathcal{D}(R)$ containing $\mathcal{D}^{\leq 0}(R)$;*
2. *The class $\{M \in \mathcal{D}(R) \mid \mathbf{R}\mathrm{Hom}_R(M, X) \in \mathsf{cosusp}^{\Pi}_{RC}\langle X\rangle\}$ is a cocomplete suspended subcategory of $\mathcal{D}(R)$ containing $\mathcal{D}^{\leq 0}(R)$;*
3. *The class $\{M \in \mathcal{D}(R) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} X \in \mathsf{cosusp}^{\mathbf{hs}}_{RC}\langle X\rangle\}$ is a homotopically smashing cosuspended subcategory of $\mathcal{D}(R)$ containing $\mathcal{D}^{[0,n]}(\mathrm{Flat}(R))$ for any $n \geq 0$;*
4. *If X is compact, the class $\left\{M \in \mathcal{D}(R) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} X \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC}\langle X\rangle\right\}$ is a homotopically smashing complete cosuspended subcategory of $\mathcal{D}(R)$.*

Proof. (1-2) The first two classes are trivially closed under positive shifts, extensions, coproducts and contain the stalk complex $R[0]$. Thus, by Theorem 1.3.6 (1), they contain the standard aisle $\mathcal{D}^{\leq 0}(R)$.
(3) It is also clear that the third class is closed under negative shifts, extensions and directed homotopy colimits and contains the stalk complex $R[0]$. Thus, by Lazard's theorem (see [CFH24, 5.5.7]), it contains the stalk complex $F[0]$ for any flat R-module F. By closure under negative shifts and extensions, it contains all bounded complexes of flat modules concentrated in non-negative degrees.
(4) If X is compact, it is isomorphic to a bounded complex of objectwise finitely presented functors and so the functor $\_ \otimes^{\mathbf{L}}_{\mathrm{R}} X$ commutes with products (see [CFH24, 3.1.31]). □

Proposition 4.0.9. *For any t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathcal{D}(RC)$ the following holds:*

1. *$\mathcal{D}^{\leq 0}(R) \otimes^{\mathbf{L}}_{\mathrm{R}} \mathcal{U} \subseteq \mathcal{U}$;*
2. *$\mathbf{R}\mathrm{Hom}_R(\mathcal{D}^{\leq 0}(R), \mathcal{V}) \subseteq \mathcal{V}$;*
3. *If $\mathcal{V}$ is homotopically smashing, $\mathcal{D}^{[0,n]}(\mathrm{Flat}(R)) \otimes^{\mathbf{L}}_{\mathrm{R}} \mathcal{V} \subseteq \mathcal{V}$ for any $n \geq 0$.*

Proof. It follows by Lemma 4.0.8 (1), (2), (3) respectively. □

4.1 Support theory

The tensor-action $\_ \otimes^{\mathbf{L}}_{\mathrm{R}} \_ : \mathcal{D}(R) \times \mathcal{D}(RC) \to \mathcal{D}(RC)$ satisfies the axioms of an action of a tt-category on a compactly generated triangulated category (see Section 3.3). Let us now recall some notions of support theory for $\mathcal{D}(RC)$.

Recall that a subset V of $\mathrm{Spec}(R)$ is called *specialization closed* if for any $\mathfrak{p} \in V$ and $\mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ it holds that $\mathfrak{q} \in V$. For any specialization closed subset $V \subseteq \mathrm{Spec}(R)$, there are the localization and colocalization functors

$$\mathrm{L}_V := \mathrm{L}_V R \otimes^{\mathbf{L}}_{\mathrm{R}} \_ \text{ and } \Gamma_V := \Gamma_V R \otimes^{\mathbf{L}}_{\mathrm{R}} \_$$

whose essential images form a compactly generated stable t-structure $(\Gamma_V \mathcal{D}(RC), \mathrm{L}_V \mathcal{D}(RC))$. Given a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, we can consider the specialization closed subsets

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\} \text{ and } Z(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$$

then define the functor

$$\Gamma_{\mathfrak{p}} := \mathrm{L}_{Z(\mathfrak{p})} \circ \Gamma_{V(\mathfrak{p})} \cong \Gamma_{V(\mathfrak{p})} \circ \mathrm{L}_{Z(\mathfrak{p})}$$

We call $\Gamma_{\mathfrak{p}}\mathcal{D}(RC)$ the *stalk subcategory at* $\mathfrak{p}$ and, for a complex $X \in \mathcal{D}(RC)$, we define the support of X over R as

$$\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Gamma_{\mathfrak{p}}X \neq 0\}$$

Remark 4.1.1. This notion of support agrees with the usual one of homological algebra. Indeed, by [Ste18, Lemma 3.22], for any complex $X \in \mathcal{D}(RC)$ we have that

$$\mathrm{supp}_R(X) = \left\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} X \neq 0\right\}$$

For the specialization closed subsets $V(\mathfrak{p})$ and $Z(\mathfrak{p})$, we can find explicit descriptions of the functors $\mathrm{L}_{Z(\mathfrak{p})}$ and $\Gamma_{V(\mathfrak{p})}$ (see [Ste18, Example 2.24]) through the tensor-idempotent objects

$$\mathrm{L}_{Z(\mathfrak{p})}R \cong R_{\mathfrak{p}} \text{ and } \Gamma_{V(\mathfrak{p})}R \cong K_\infty(\mathfrak{p})$$

where the complex $K_\infty(\mathfrak{p})$ is the *infinite Koszul complex* at $\mathfrak{p}$ (also known as stable Koszul complex or Čech complex and denoted by $\check{C}(\mathfrak{p})$, as in [CFH24]).

Definition 4.1.2. For an ideal $\mathfrak{a}$ in R, let n be the minimal number of generators of $\mathfrak{a}$ and $\{a_1, \dots, a_n\}$ be a set of generators. The *infinite Koszul complex* (on R) at $\mathfrak{a}$ is defined to be

$$K_\infty(\mathfrak{a}) = \bigotimes_{i=1}^{n} R \xrightarrow{\rho_i} R[a_i^{-1}]$$

where the complex $R \xrightarrow{\rho_i} R[a_i^{-1}]$ is concentrated in degrees 0 and 1 and the map ρ_i is given by $r \mapsto \frac{r}{1}$. Note that $K_\infty(\mathfrak{a})$ is a bounded complex of flat R-modules concentrated in degrees $[0, n]$ and, by [CFH24, 11.4.16], the definition is independent, up to quasi-isomorphism, of the choice of the set of generators.

Remark 4.1.3. We now wish to highlight some key properties of the (infinite) Koszul complexes:

- For any prime ideal $\mathfrak{p}$, it holds

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} K(\mathfrak{a}) = \begin{cases} \bigoplus_{i=0}^{n} \kappa(\mathfrak{p})^{\binom{n}{i}}[i] & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

  Indeed, by Remark 2.1.3 the complex is non-zero if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ and in this case, tensoring by $\kappa(\mathfrak{p})$ annihilates multiplications by elements of $\mathfrak{a}$, so it annihilates all the differentials of $K(\mathfrak{a})$.

- [CFH24, 14.3.2] Known as *self-duality* property: there is an isomorphism of functors

$$\mathbf{R}\mathrm{Hom}_R(K(\mathfrak{a}), \_) \cong K(\mathfrak{a})[-n] \otimes^{\mathbf{L}}_{\mathrm{R}} \_$$

- [CFH24, 11.4.12] Let $\mathfrak{a}^{(t)}$ denote the ideal $\left(a_1^t, \dots, a_n^t\right)$, then

$$K_\infty(\mathfrak{a}) \cong \underrightarrow{\mathrm{hocolim}}_{t\in\mathbb{N}} K\left(\mathfrak{a}^{(t)}\right)[-n]$$

Remark 4.1.4. Actually, by [Hrb25, Theorem 1.6], we can find an explicit description of the functor Γ_V for any specialization closed subset V. Indeed, writing $V = \bigcup_{i\in I} V(\mathfrak{p}_i)$ as an union of Zariski closed subsets of $\mathrm{Spec}(R)$ and denoting by $\mathcal{F}$ the lattice of finite subsets of I and, for any $F \in \mathcal{F}$, by $V_F = \bigcup_{i\in F} V(\mathfrak{p}_i)$, in *loc. cit.* it is proved the isomorphism of triangles

$$\begin{array}{ccccccc} \Gamma_V R & \longrightarrow & R & \longrightarrow & \mathrm{L}_V R & \xrightarrow{+} & \\ \| \wr & & \| & & \| \wr & & \\ \underrightarrow{\mathrm{hocolim}}_{F\in\mathcal{F}} \Gamma_{V_F} R & \longrightarrow & R & \longrightarrow & \underrightarrow{\mathrm{hocolim}}_{F\in\mathcal{F}} \mathrm{L}_{V_F} R & \xrightarrow{+} & \end{array}$$

Note that V_F is equal to the Zariski closed $V(\mathfrak{a}_F)$ corresponding to the ideal $\mathfrak{a}_F = \prod_{i\in F} \mathfrak{p}_i$. Thus, since $\Gamma_{V_F} R \cong K_\infty(\mathfrak{a}_F)$, we get that

$$\Gamma_V \cong \underrightarrow{\text{hocolim}}_{F\in\mathcal{F}} K_\infty(\mathfrak{a}_F) \otimes^{\mathbf{L}}_{\mathrm{R}} \_ \cong \underrightarrow{\text{hocolim}}_{(F,t)\in\mathcal{F}\times\mathbb{N}} \mathbf{R}\mathrm{Hom}_R\left(K\left(\mathfrak{a}_F^{(t)}\right), \_\right)$$

4.2 Stratification and telescope conjecture

Since we are in the situation in which $\mathcal{D}(R)$ acts on a compactly generated triangulated category, we can consider the stratification theory presented in [BIK11] (see end of Section 3.3). However, while the local-to-global principle holds in $\mathcal{D}(RC)$, for any commutative noetherian ring R and small category C by Remark 3.3.11, the minimality fails even in $\mathcal{D}(\mathbb{K}Q)$, for a field $\mathbb{K}$ and a Dynkin quiver Q (see [BIK11, Example 4.6]). Thus we only have that the following map is a bijection:

$$\begin{aligned} \mathrm{Loc}(\mathcal{D}(RC)) &\longrightarrow \prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Loc}(\Gamma_\mathfrak{p}\mathcal{D}(RC)) \\ \mathcal{L} &\longmapsto (\Gamma_\mathfrak{p}\mathcal{L})_{\mathfrak{p}\in\mathrm{Spec}(R)} \end{aligned}$$

In order to obtain a better classification of the localizing subcategories of $\mathcal{D}(RC)$ and then prove the stable telescope conjecture, the authors of [AS16, Section 4] replaced the minimality of stalk subcategories with the following equivalent conditions:

(M) For any $\mathfrak{p} \in \mathrm{Spec}(R)$ and $X \in \Gamma_\mathfrak{p}\mathcal{D}(RC)$, $\mathsf{loc}_{RC}\langle X\rangle = \mathsf{loc}_{RC}\left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} X\right\rangle$;

(M') For any $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a bijection between $\mathrm{Loc}(\Gamma_\mathfrak{p}\mathcal{D}(RC))$ and $\mathrm{Loc}(\mathcal{D}(\kappa(\mathfrak{p})C))$.

Even if these conditions are not equivalent to the minimality in [BIK11] for a general small category C, they are in the case of $\mathcal{D}(R)$ and so, in a sense, they transfer the notion of minimality from the stratified categories to our context; so, it still makes sense to call them minimality conditions in $\mathcal{D}(RC)$.

Theorem 4.2.1 ([AS16, Theorems 4.1, 4.2, Corollary 4.3]). *For any commutative noetherian ring R and small category C, the category $\mathcal{D}(RC)$ satisfies both* (M) *and* (M')*, and the following assignments form a bijection*

$$\begin{aligned} \mathrm{Loc}(\mathcal{D}(RC)) &\longrightarrow \prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Loc}(\mathcal{D}(\kappa(\mathfrak{p})C)) \\ \mathcal{L} &\longmapsto \left(\mathsf{add}_{\kappa(\mathfrak{p})C}\left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \mathcal{L}\right\rangle\right)_{\mathfrak{p}\in\mathrm{Spec}(R)} \\ \mathsf{loc}_{RC}\langle \ell(\mathfrak{p}) \mid \mathfrak{p}\in\mathrm{Spec}(R)\rangle &\longleftarrow (\ell(\mathfrak{p}))_{\mathfrak{p}\in\mathrm{Spec}(R)} \end{aligned}$$

Remark 4.2.2 ([AS16, Lemma 6.9]). By the characterization of the functor $\Gamma_\mathfrak{p}$ as $K_\infty(\mathfrak{p})_\mathfrak{p} \otimes_R \_$, it follows that the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RC)$ is a compactly generated triangulated category with $(\Gamma_\mathfrak{p}\mathcal{D}(RC))^c = \Gamma_\mathfrak{p}\mathcal{D}(RC) \cap \mathcal{D}^c(R_\mathfrak{p}C)$.

Thus it makes sense to ask wether stalk subcategories satisfy the stable telescope conjecture.

Theorem 4.2.3 ([AS16, Proposition 6.10, Corollary 6.12]). *For any commutative noetherian ring R, small category C and prime ideal $\mathfrak{p}$ such that $R_\mathfrak{p}$ is regular, the bijection of* (M') *restricts to a bijection between thick subcategories of $(\Gamma_\mathfrak{p}\mathcal{D}(RC))^c$ and thick subcategories of $\mathcal{D}^c(\kappa(\mathfrak{p})C)$. Moreover, for any quiver Q, the category $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$ satisfies the stable telescope conjecture.*

We will see later in Theorem 6.2.4 that the telescope conjecture for $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$ also holds without the regularity assumption on $R_\mathfrak{p}$.

4.2.1 Representations of Dynkin quivers

When the small category C is the free category on a finite quiver Q, the category $\mathrm{Mod}_R(C)$ is equivalent to the category of modules $\mathrm{Mod}(RQ)$, where RQ is the free R-algebra on the set of paths of Q with the composition of paths as product. In this case, the functors $X : C \to \mathrm{Mod}(R)$ can be seen as representations of the quiver $Q = (Q_0, Q_1)$ over the ring R, i.e. $X = (X_i, X_\alpha)_{i\in Q_0,\alpha\in Q_1}$ where X_i is an R-module for any vertex $i \in Q_0$ and $X_\alpha : X_i \to X_j$ is an R-linear map for any arrow $\alpha : i \to j \in Q_1$.

When Q is a Dynkin quiver and $R = \mathbb{K}$ a field, the path algebra $\mathbb{K}Q$ is a representation-finite hereditary algebra and it is well-known that the indecomposable modules do not depend on the field, be it algebraically closed or not (see, for example, [DDPW08, Theorem 1.23]). In this case also the localizing subcategories of $\mathcal{D}(\mathbb{K}Q)$ are independent of the field $\mathbb{K}$. In particular, they are in bijection with the lattice $\mathbf{Nc}(Q)$ of noncrossing partitions of Q (see [IT09, Theorem 1.1] or [Kra12, Theorem 6.10]).

Theorem 4.2.4 ([AS16, Corollary 5.1])**.** *For any commutative noetherian ring R and Dynkin quiver Q, there is an order-preserving bijection between localizing subcategories of $\mathcal{D}(RQ)$ and set-theoretic functions from* $\mathrm{Spec}(R)$ *to* $\mathbf{Nc}(Q)$*:*

$$\mathrm{Loc}(\mathcal{D}(RQ)) \longleftrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Spec}(R), \mathbf{Nc}(Q))$$

In order to restrict this bijection to the lattice of thick subcategories of $\mathcal{D}^c(RQ)$, we introduce a class of RQ-modules which, in a sense, play the role of the indecomposables in $\mathrm{Mod}(\mathbb{K}Q)$.

Definition 4.2.5. An RQ-module is called (*free*) *lattice* if it is vertexwise finitely generated projective (resp. free) over R. A lattice X is called *rigid* if $\mathrm{Ext}^1_{RQ}(X, X) = 0$ and *exceptional* if it is rigid and $\mathrm{End}_{RQ}(X) \cong R$. A lattice X is said to have a *rank vector* $d \in \mathbb{N}^{|Q_0|}$ if for every $i \in Q_0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ there is an isomorphism $\kappa(\mathfrak{p}) \otimes_R X_i \cong \kappa(\mathfrak{p})^{d_i}$. Obviously, a free lattice has always a rank vector d with $d_i = \mathrm{rank}(X_i)$.

Remark 4.2.6 ([CB24, Lemma 3.1 (i)]). RQ-lattices have projective dimension less or equal then 1. In particular, stalk complexes of RQ-lattices are compact in $\mathcal{D}(RQ)$. Moreover, by Lemma 4.0.5, they are R-K-projective. Thus, by Theorem 4.0.6, any time the functor $\otimes^{\mathbf{L}}_{\mathrm{R}}$ is applied to an RQ-lattice, it will be computed as the standard vertexwise tensor product.

In view of the fact that, for any field $\mathbb{K}$, the indecomposable $\mathbb{K}Q$-modules are in bijection with the real Schur roots of Q, we can restate [CB24, Theorem B] as follows.

Theorem 4.2.7. *For any field $\mathbb{K}$, commutative ring R and Dynkin quiver Q, there exists an indecomposable $\mathbb{K}Q$-module L with dimension vector d if and only if there exists an exceptional RQ-lattice X with rank vector d. In this case, there exists a unique exceptional free lattice $\widetilde{L}$ with rank vector d, and any rigid RQ-lattice of rank vector d is exceptional.*

We will denote by $\widetilde{\ }: \mathrm{ind}(\mathbb{K}Q) \longrightarrow \mathrm{lat}(RQ)$ the assignment in the theorem and we will refer to it as *lattice lift.* Since the indecomposable objects of $\mathcal{D}(\mathbb{K}Q)$ are shifts of indecomposable $\mathbb{K}Q$-modules, given a class of objects $\mathcal{X} \subseteq \mathcal{D}(\mathbb{K}Q)$ closed under summands, by abuse of notation, we write

$$\widetilde{\mathcal{X}} = \left\{ \widetilde{L}[n] \mid L \in \mathrm{ind}(\mathbb{K}Q) \text{ and } L[n] \in \mathcal{X} \right\}$$

for the lattice lift of the indecomposables in $\mathcal{X}$. Then we have the following:

Theorem 4.2.8 ([AS16, Corollaries 5.10, 5.11])**.** *For any commutative noetherian ring R and Dynkin quiver Q, the following assignments form an order-preserving bijection between thick subcategories of $\mathcal{D}^c(RQ)$ and poset homomorphisms from* $\mathrm{Spec}(R)$ *to* $\mathbf{Nc}(Q)$*:*

$$\mathrm{Thick}(\mathcal{D}^c(RQ)) \longleftrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathbf{Nc}(Q))$$

$$\mathcal{S} \longmapsto \left(\sigma_{\mathcal{S}} : \mathfrak{p} \mapsto \mathsf{loc}_{\kappa(\mathfrak{p})Q} \left\langle L \in \mathrm{ind}(\mathcal{D}(\kappa(\mathfrak{p})Q)) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{L} \right\rangle\right)$$

$$\mathcal{S}_\sigma := \mathsf{thick}_{RQ} \left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R) \right\rangle \longleftarrow \sigma$$

In particular, a set of compact generators for $\mathcal{S}_\sigma$ is given by $\mathcal{X}_\sigma = \left\{ K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R) \right\}$. *Moreover, the derived category $\mathcal{D}(RQ)$ satisfies the stable telescope conjecture.*

Note that the category $\mathcal{D}(R)$ is equivalent to the derived category of representations of the quiver A_1, with one vertex and no arrows, over R and the lattice of noncrossing partitions $\mathbf{Nc}(A_1)$ is isomorphic to the lattice $\{0, 1\}$ with the obvious order. Then considering maps from $\mathrm{Spec}(R)$ to $\{0, 1\}$ as characteristic functions of subsets of $\mathrm{Spec}(R)$, gives a bijection between $\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Spec}(R), \{0, 1\})$ and $\mathbf{P}(\mathrm{Spec}(R))$ and between $\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \{0, 1\})$ and $\mathbf{V}(\mathrm{Spec}(R))$. In this framework, the classifications in Theorem 4.2.4 and Theorem 4.2.8 extends the classifications in Theorem 2.1.1 and Theorem 2.1.4, respectively.

Part II

Original Contributions

Chapter 5

Representations of Small Categories II

The two sections of this chapter are respectively part of the author's works [Sab25a] and [Sab25b].

5.1 Local-to-global and minimality: semi-stable version

The aim of this section is to generalize two properties of localizing subcategories of $\mathcal{D}(RC)$ – the *local-to-global principle* and the *minimality of stalk subcategories* – to the setting of homotopically smashing cosuspended subcategories. These properties will allow us to reduce the problem of generating homotopically smashing coaisles, first from $\mathcal{D}(RC)$ to the stalk subcategories $\Gamma_\mathfrak{p}\mathcal{D}(RC)$, and then to the better understood derived categories $\mathcal{D}(\kappa(\mathfrak{p})C)$.

We recall the *stable* versions here:

(LTG) For any $X \in \mathcal{T}$, $\mathsf{loc}_\mathcal{T}\langle X\rangle = \mathsf{loc}_\mathcal{T}\langle \Gamma_\mathfrak{p} X \mid \mathfrak{p} \in \mathrm{Spec}(R)\rangle$;

(M) For any $\mathfrak{p} \in \mathrm{Spec}(R)$ and $X \in \Gamma_\mathfrak{p}\mathcal{D}(RC)$, $\mathsf{loc}_{RC}\langle X\rangle = \mathsf{loc}_{RC}\left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} X\right\rangle$;

(M') For any $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a bijection between $\mathrm{Loc}(\Gamma_\mathfrak{p}\mathcal{D}(RC))$ and $\mathrm{Loc}(\mathcal{D}(\kappa(\mathfrak{p})C))$.

5.1.1 Local-to-global principle

In the following we will generalize (LTG) from localizing subcategories to cosuspended subcategories closed under homotopy colimits, thus avoiding the closure under positive shifts. Note that, while in the case of stable t-structures (LTG) is used to study localizing subcategories, i.e. the left side of the pair, our generalization also applies to any homotopically smashing t-structures, but on the side of the coaisles.

Let us start defining inductively a transfinite collection $\{W_\alpha\}$ of specialization closed subsets of $\mathrm{Spec}(R)$. Denoting by $\max W$ the set of maximal elements of W, let:

- $W_0 = \max \mathrm{Spec}(R)$;
- $W_{\alpha+1} = W_\alpha \cup \max(\mathrm{Spec}(R) \setminus W_\alpha)$, for any successor ordinal $\alpha + 1$;
- $W_\lambda = \bigcup_{\beta<\lambda} W_\beta$, for any limit ordinal λ.

In this way, at some point, we reach the least ordinal δ satisfying $W_\delta = \mathrm{Spec}(R)$.

Lemma 5.1.1. *Let $Y \in \mathcal{D}(RC)$, then:*

1. *For any successor ordinal $\alpha + 1$, there exists a distinguished triangle*

$$\Gamma_{W_\alpha} Y \longrightarrow \Gamma_{W_{\alpha+1}} Y \longrightarrow \bigoplus_{\mathfrak{p}\in W_{\alpha+1}\setminus W_\alpha} \Gamma_\mathfrak{p} Y \xrightarrow{+}$$

2. For any limit ordinal λ*, we have that* $\Gamma_{W_\lambda} Y \cong \varinjlim\text{hocolim}_{\beta<\lambda} \Gamma_{W_\beta} Y$.

Proof. (1) By [HNŠ24, Remark 4.2], for each ordinal $\alpha + 1$, the following is a distinguished triangle in $\mathcal{D}(R)$

$$\Gamma_{W_\alpha} R \longrightarrow \Gamma_{W_{\alpha+1}} R \longrightarrow \bigoplus_{\mathfrak{p}\in W_{\alpha+1}\setminus W_\alpha} \Gamma_{\mathfrak{p}} R \xrightarrow{+}$$

(beware of the change of notation in the reference: the functors $\mathbf{R}\Gamma_V$ and $\mathbf{R}\Gamma_{\mathfrak{p}}$ there stand for Γ_V and $\Gamma_{V(\mathfrak{p})}$ here). Then, applying $\_ \otimes^{\mathbf{L}}_{\mathrm{R}} Y$ to it, we get the triangle in the statement.
(2) By [Ste13, Lemma 6.6] (see also Remark 4.1.4), we have that $\Gamma_{W_\lambda} R \cong \varinjlim\text{hocolim}_{\beta<\lambda} \Gamma_{W_\beta} R$, then the result follows applying $\_ \otimes^{\mathbf{L}}_{\mathrm{R}} Y$ to both sides. □

Theorem 5.1.2 (Local-to-global Principle). *For any complex* $Y \in \mathcal{D}(RC)$*, we have that*

$$\mathsf{cosusp}^{\mathbf{hs}}_{RC} \langle Y \rangle = \mathsf{cosusp}^{\mathbf{hs}}_{RC} \langle \Gamma_{\mathfrak{p}} Y \mid \mathfrak{p} \in \mathrm{Spec}(R) \rangle$$

Proof. By Lemma 5.1.1, any complex Y is isomorphic to $\Gamma_{W_\delta} Y$, where δ is the least ordinal such that $W_\delta = \mathrm{Spec}(R)$. In particular, Y can be reconstructed, through extensions, coproducts and direct homotopy colimits, from the complexes $\{\Gamma_{\mathfrak{p}} Y\}_{\mathfrak{p}\in\mathrm{Spec}(R)}$. Thus, the containment from left to right follows. On the other hand, $\Gamma_{\mathfrak{p}} Y := (K_\infty(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} R_{\mathfrak{p}}) \otimes^{\mathbf{L}}_{\mathrm{R}} Y$ is given by tensoring Y with a bounded complex of flat modules concentrated in non-negative degrees and so, by Lemma 4.0.8 (3), it is contained in $\mathsf{cosusp}^{\mathbf{hs}}_{RC} \langle Y \rangle$ for any prime ideal $\mathfrak{p}$. □

Note that the last containment is the one that forces us to switch to coaisles, in fact aisles of non-stable t-structure are never closed under tensor products with complexes concentrated in non-negative degrees.

5.1.2 Minimality of stalk subcategories

The following theorem is the analogue of (M) for cosuspended subcategories closed under products and directed homotopy colimits. Unlike the theorem above, this is not a generalization of (M), in fact it does not apply to localizing subcategories, since they are not closed under products in general. However, we will prove in Chapter 6 that this result implies (M') for homotopically smashing coaisles in the context of *quiver* representations, i.e. for any quiver Q and $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a bijection between $\mathrm{Coaisle}_{\mathrm{hs}}(\Gamma_{\mathfrak{p}}\mathcal{D}(RQ))$ and $\mathrm{Coaisle}_{\mathrm{hs}}(\mathcal{D}(\kappa(\mathfrak{p})Q))$ for any finite quiver Q (see Theorem 6.2.4).

Theorem 5.1.3 (Minimality of Stalk Subcategories). *For any complex* $Y \in \Gamma_{\mathfrak{p}}\mathcal{D}(RC)$*, we have that*

$$\mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC} \langle Y \rangle = \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y) \rangle$$

Proof. The containment from right to left follows from Lemma 4.0.8 (2). For the other direction, consider the class

$$\mathcal{M} = \left\{ M \in \mathcal{D}(R_{\mathfrak{p}}) \mid \mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y) \rangle \right\}$$

It is a cocomplete suspended subcategory of $\mathcal{D}(R_{\mathfrak{p}})$ containing $\kappa(\mathfrak{p}) \cong R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. Thus, by Theorem 1.3.6 and [AJS10, Proposition 2.4] (1), it contains $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^{(t)}$ for any $t \geq 1$ and, by [Hrb20, Lemma 5.3], it contains the Koszul complexes $K^{R_{\mathfrak{p}}}\left(\mathfrak{p}_{\mathfrak{p}}^{(t)}\right) \cong K\left(\mathfrak{p}^{(t)}\right)_{\mathfrak{p}}$, where the first complex is computed over R_p and the second over R.
Consider now the class

$$\left\{ M \in \mathcal{D}(R_{\mathfrak{p}}) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} Y \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y) \rangle \right\}$$

It is a homotopically smashing cosuspended subcategory of $\mathcal{D}(R_\mathfrak{p})$. By self-duality of Koszul complexes in Remark 4.1.3, $\mathbf{R}\mathrm{Hom}_R(K\left(\mathfrak{p}^{(t)}\right)_\mathfrak{p}, Y) \cong K\left(\mathfrak{p}^{(t)}\right)_\mathfrak{p}[-n] \otimes^{\mathbf{L}}_{\mathrm{R}} Y$, thus it contains the complex $K\left(\mathfrak{p}^{(t)}\right)_\mathfrak{p}[-n]$ for any $t \geq 1$. Since it is closed under directed homotopy colimits, by Remark 4.1.3, it contains the the complex $K_\infty(\mathfrak{p})_\mathfrak{p} = \Gamma_\mathfrak{p} R$ and we can conclude that $Y \cong \Gamma_\mathfrak{p} R \otimes^{\mathbf{L}}_{\mathrm{R}} Y \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y)\rangle$. □

Remark 5.1.4. Note that, we use products only in the first part of the proof to make $\mathcal{M}$ cocomplete and then apply Theorem 1.3.6 (1). It turns out that, with a bit more of work, we can prove that the inclusion $Y \in \mathsf{cosusp}^{\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y)\rangle$ holds without closure under products. Indeed, in this case, the class

$$\mathcal{M} = \left\{ M \in \mathcal{D}(R_\mathfrak{p}) \mid \mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathsf{cosusp}^{\mathbf{hs}}_{RC} \langle \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), Y)\rangle \right\}$$

is just a suspended subcategory of $\mathcal{D}(R_\mathfrak{p})$ containing $\kappa(\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. For any $t \geq 1$, the Koszul complex $K^{R_\mathfrak{p}}\left(\mathfrak{p}_\mathfrak{p}^{(t)}\right)$ is a bounded complex of finite free $R_\mathfrak{p}$-modules concentrated in degrees $[-n, 0]$ and, for any $i \in [-n, 0]$, its cohomology module H^i is a finitely generated $R_\mathfrak{p}$-module annihilated by $\mathfrak{p}_\mathfrak{p}^{(t)}$ (see [CFH24, 11.4.6 (a)]). In particular, each H^i admits a finite filtration with composition factors isomorphic to $\kappa(\mathfrak{p})$ (see [Sta, Lemma 10.62.1-2-4]). It follows that, for any i, the module H^i is contained in $\mathcal{M}$ and, by closure under extensions, so does the complex $K^{R_\mathfrak{p}}\left(\mathfrak{p}_\mathfrak{p}^{(t)}\right) \cong K\left(\mathfrak{p}^{(t)}\right)_\mathfrak{p}$. Then we can conclude as in the proof of Theorem 5.1.3.

5.2 Tensor triangulated structure

Given two RQ-modules $X = (X_i, X_\alpha)$ and $Y = (Y_i, Y_\alpha)$, we define a symmetric tensor product as $X \boxtimes_{RQ} Y = (X_i \otimes_R Y_i, X_\alpha \otimes_R Y_\alpha)$. In the following we will consider the tt-category $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$, where $\mathcal{D}(RQ)$ is the derived category of the module category $\mathrm{Mod}(RQ)$, the symmetric monoidal structure is given by the left derived functor of $\boxtimes_{RQ}$ and the tensor unit $\mathbf{U}(= \mathbf{U}[0])$ is the stalk complex defined by $\mathbf{U}_i = R$ and $\mathbf{U}_\alpha = \mathrm{id}_R$ for any $i \in Q_0$ and $\alpha \in Q_1$.

Remark 5.2.1. Note that the tensor unit is not a generator of the triangulated category, indeed $\mathsf{thick}_{RQ}\langle \mathbf{U}\rangle \neq \mathcal{D}(RQ)$. There are many results available for tt-categories $(\mathcal{T}, \otimes, \mathbf{1})$ such that $\mathsf{thick}_{\mathcal{T}}\langle \mathbf{1}\rangle = \mathcal{T}$, which we can not therefore use in our context. In particular, a localizing (resp. thick) subcategory of $\mathcal{D}(RQ)$ (resp. $\mathcal{D}^c(RQ)$) will not be automatically a localizing (resp. thick) tensor-ideal of $\mathcal{D}(RQ)$ (resp. $\mathcal{D}^c(RQ)$).

Proposition 5.2.2. *The tt-category $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ is compactly generated. In particular, $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$ is an essentially small tt-category.*

Proof. By Lemma 4.0.2, the derived category $\mathcal{D}(RQ)$ is a compactly generated triangulated category with compact objects $\mathcal{K}^b(\mathrm{proj}\, RQ)$ and the tensor product $\boxtimes_{RQ}$ clearly commutes with coproducts. Recall from [EE05, Theorem 3.1] that an RQ-module P is projective if and only if, for any $i \in Q_0$, P_i is projective and the map

$$\bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} P_{s(\alpha)} \to P_i$$

is a split monomorphism – where $s(\alpha)$ and $t(\alpha)$ denote respectively the source and the target of the arrow α. It follows easily that, given two projective RQ-module P and P', the tensor product $P \boxtimes_{RQ} P'$ is again projective. Since for any two compact objects $X, X' \in \mathcal{D}^c(RQ)$, the product $X \boxtimes^{\mathbf{L}}_{RQ} X'$ is isomorphic to the $\oplus$-totalization of the double complex $X^i \boxtimes_R X'^j$, it is compact too. Finally, by Remark 4.2.6, the tensor unit $\mathbf{U} = \mathbf{U}[0]$ is compact. □

By Brown representability (Theorem 1.0.7), it follows from Proposition 5.2.2 that the category $\mathcal{D}(RQ)$ is endowed with an internal hom, which we will denote by

$$\mathcal{H}\mathrm{om}_{RQ}(\_,\_) : \mathcal{D}(RQ) \times \mathcal{D}(RQ) \to \mathcal{D}(RQ)$$

For any $i \in Q_0$, let $P(i)$ be the projective RQ-module such that, for any vertex k, $P(i)_k$ is the free R-module with basis the set of all paths in Q from i to k and, for any arrow $\beta : k \to \ell$, $P(i)_\beta : P(i)_k \to P(i)_\ell$ is given by left multiplication by β; for any $\alpha : i \to j \in Q_1$, let $\iota_\alpha : P(j) \to P(i)$ be the monomorphism given by right multiplication by α. Then:

Theorem 5.2.3. *Given left RQ-modules X and Y, the internal hom is defined by*

$$\mathcal{H}\mathrm{om}_{RQ}(X,Y) := (\mathrm{Hom}_{RQ}(X \boxtimes_{RQ} P(i), Y),\ \_ \circ (\mathrm{id}_X \boxtimes_{RQ} \iota_\alpha))_{i\in Q_0,\, \alpha\in Q_1}$$

Proof. By [Ste75, Proposition IV.10.1], for any left RQ-module X, the functor $X \boxtimes_{RQ} \_$ is isomorphic to the functor $(X \boxtimes_{RQ} RQ) \otimes_{RQ} \_$. Here the module $X \boxtimes_{RQ} RQ$ is seen as an RQ-bimodule where, denoting by ρ_ω the right multiplication in RQ by a path ω, the right RQ-module structure of $X \boxtimes_{RQ} RQ$ is given by $\cdot\omega := \mathrm{id}_X \boxtimes_{RQ} \rho_\omega$. Thus, the functor $X \boxtimes_{RQ} \_$ is left adjoint to the functor $\mathrm{Hom}_{RQ}(X \boxtimes_{RQ} RQ, \_)$. For any left RQ-module Y, the left RQ-module structure on $\mathrm{Hom}_{RQ}(X \boxtimes_{RQ} RQ, Y)$ is defined by $\omega \cdot (h_i)_{i\in Q_0} = (h_i \circ (\mathrm{id}_X \boxtimes_{RQ} \rho_\omega))_{i\in Q_0}$. In particular, at any vertex $i \in Q_0$ it is represented by the R-module $\varepsilon_i \cdot \mathrm{Hom}_{RQ}(X \boxtimes_{RQ} RQ, Y) = \mathrm{Hom}_{RQ}(X \boxtimes_{RQ} RQ \cdot \varepsilon_i, Y) = \mathrm{Hom}_{RQ}(X \boxtimes_{RQ} P(i), Y)$. □

Proposition 5.2.4. *The internal hom restricts to $\mathcal{D}^c(RQ)$.*

Proof. For any two finitely generated projective RQ-modules P and P', by Proposition 5.2.2 the RQ-module $P \boxtimes_{RQ} P(i)$ is finitely generated projective too, and, by [CB24, Theorem A], $\mathrm{Hom}_{RQ}(P \boxtimes_{RQ} P(i), P')$ is a finitely generated projective R-module. Thus, $\mathcal{H}\mathrm{om}_{RQ}(P, P')$ is an RQ-lattice and, by Remark 4.2.6, it is compact in $\mathcal{D}(RQ)$. Since, for any two compact objects $X, X' \in \mathcal{D}^c(RQ)$, the complex $\mathcal{H}\mathrm{om}_{RQ}(X, X')$ is isomorphic to the Π-totalization of the double complex $\mathcal{H}\mathrm{om}_{RQ}(X^{-i}, X'^j)$ and only finite coproducts are involved, it is compact. □

We will prove in the next proposition that the tt-category $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$ is not rigid whenever the quiver Q is connected, has at least two vertices, and admits a source – which is always the case for finite acyclic quivers. This result generalizes [Ito23, Remark 2.21] and [San13, Example 6.13] to any commutative noetherian ring R.

Proposition 5.2.5. *For a non-trivial connected finite acyclic quiver Q, the tt-category $\mathcal{D}^c(RQ)$ is not rigid.*

Proof. Any finite acyclic quiver has a source, say $s \in Q_0$. Let $U(s)$ be the representation such that, $U(s)_s = R$ and $U(s)_k = 0$ for any $k \neq s$, note that it is compact by Remark 4.2.6. Since s is a source, the product $U(s) \boxtimes_{RQ} P(i)$ is equal to $U(s)$ if $i = s$ or 0 otherwise. Thus, since the quiver is non-trivial and connected, $\mathcal{H}\mathrm{om}_{RQ}(U(s), \mathbf{U}) = 0$, while $\mathcal{H}\mathrm{om}_{RQ}(U(s), U(s)) = U(s)$. It follows that $U(s)$ is not rigid, i.e. $\mathcal{H}\mathrm{om}_{RQ}(U(s), \mathbf{U}) \boxtimes_{RQ} U(s) \ncong \mathcal{H}\mathrm{om}_{RQ}(U(s), U(s))$. □

There are many results available for rigid tt-categories that we cannot apply in our context; in particular, not every thick tensor-ideal of $\mathcal{D}^c(RQ)$ is automatically radical. Nevertheless, even though $\mathcal{D}^c(RQ)$ is not rigid, we will show in Proposition 8.1.5 that every thick tensor-ideal is indeed radical, so the Balmer spectrum still provides a classification of all thick tensor-ideals. For further distinctions from the rigid setting, see Section 8.1.2.

Chapter 6

Representations of Quivers over Artinian Rings

Let $\mathcal{D}(RC)$ be the derived category of representations of a small category C over a commutative noetherian ring R. We study the homotopically smashing t-structures on this category. Specifying our discussion to the stalk categories $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ for a finite quiver Q and a prime ideal $\mathfrak{p}$ of R, we prove the telescope conjecture for the derived category of representations of finite quivers over artinian rings. More generally, we prove the same result also outside of the noetherian context, for representations of finite quivers over commutative perfect rings.

The material in this chapter is part of a joint work with Michal Hrbek [HS25].

6.1 t-Structures with definable coaisle

Under mild assumptions, the coaisles of homotopically smashing t-structures turn out to be *definable* in the following sense.

Definition 6.1.1. A subcategory of a compactly generated triangulated category $\mathcal{T}$ is called *definable* if it is of the form

$$\mathcal{I}^{\perp} := \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(f, X) = 0 \text{ for any } f \in \mathcal{I}\}$$

for a set of maps between compact objects $\mathcal{I} \subseteq \mathrm{Mor}(\mathcal{T}^c)$.

Note that definable subcategories are not assumed a priori to satisfy any closure conditions such as closure under shifts or extensions. However, these can be imposed by requiring corresponding properties on the set $\mathcal{I}$. In particular, by [SŠ23, Proposition 8.17], definable subcategories are in bijection with the so-called *saturated ideals* $\mathcal{I}$ of $\mathcal{T}^c$. Moreover, by the arguments in [SŠ23, Corollary 8.19], *idempotent* saturated ideals give rise to extension-closed definable subcategories (although it is not yet known whether this correspondence is bijective). Finally, closure under shifts can also be recovered: indeed, by [SŠ23, Theorem 8.16] (resp. [Kra05, Corollary 12.5]), cosuspended (resp. triangulated) definable subcategories are in bijection with *suspended* (resp. *exact*) ideals. In the latter case, the definable subcategories fit in the right-hand sides of (stable) t-structures.

The following shows that in our setting these t-structures are precisely the homotopically smashing ones. Because of this property, it may happen that we will refer to homotopically smashing t-structures also as t-structures with *definable coaisles*.

Proposition 6.1.2 ([SŠ23, LV20])**.** *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category, then a subcategory $\mathcal{V} \subseteq \mathcal{T}$ is the coaisle of a homotopically smashing t-structure if and only if it is a cosuspended definable subcategory.*

Proof. By [SŠ23, Remark 8.9, (3)⇔(2')], we have that any homotopically smashing t-structure has a definable coaisle, which by hypothesis is also cosuspended. Moreover, by [LV20, Theorem 4.7, Proposition 5.10], any cosuspended definable subcategory is the coaisle of a t-structure and it is closed under directed homotopy colimits. □

Next remark clarifies why we should not care about the ambiguity of Definition 1.2.22 in the following.

Remark 6.1.3. By Proposition 6.1.2, a subcategory $\mathcal{V} \subseteq \mathcal{T}$ of a compactly generated algebraic triangulated category is a homotopically smashing coaisle if and only if it is the right orthogonal to a suspended ideal $\mathcal{I} \subseteq \mathcal{T}^c$. In this case, the property of being closed under directed homotopy colimits does not depend on the choice of a dg-enhancement of $\mathcal{T}$, as it only depends on the full subcategory of compact objects $\mathcal{T}^c$. For the same reason, when studying cosuspended subcategories closed under directed homotopy colimits, the choice of dg-enhancement is irrelevant, provided the subcategory is also definable.

In view of Proposition 6.1.2, the (stable) telescope conjecture can be reformulated in terms of suspended (resp. exact) ideals (we refer to [Kra05, Theorem 13.4] for the stable case).

Proposition 6.1.4. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. Then the following are equivalent:*

1. *Every homotopically smashing t-structure of $\mathcal{T}$ is compactly generated;*
2. *Every suspended ideal in $\mathcal{T}^c$ is generated by identity maps.*

Proof. By [SŠ23, Theorem 8.16], any definable coaisle is right orthogonal to a set of compact objects if and only if for any suspended ideal $\mathcal{I} \subseteq \mathcal{T}^c$ there exists a set of compact objects $\mathcal{S} \subseteq \mathcal{T}^c$ such that $\mathcal{I}^\perp = \mathcal{S}^\perp$. In this case, any map in $\mathcal{I}$ factors through an object of $\mathsf{aisle}_{\mathcal{T}}\langle \mathcal{S} \rangle$ and thus, by [SŠ23, Proposition 8.27], through an object of $\mathsf{aisle}_{\mathcal{T}}\langle \mathcal{S} \rangle \cap \mathcal{T}^c$. Therefore, the identities on these objects generate $\mathcal{I}$. On the other hand, if $\mathcal{I}$ is generated by identity maps we can take $\mathcal{S}$ to be the set of compacts such that $\{\mathrm{id}_S\}_{S \in \mathcal{S}}$ generates $\mathcal{I}$. □

6.1.1 Representations of small categories

Let us now set the background for the study of homotopically smashing t-structures over $\mathcal{D}(RC)$ by considering their "local pieces" over $\mathcal{D}(\kappa(\mathfrak{p})C)$, for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$. In particular, we will show that any definable coaisle of $\mathcal{D}(RC)$ can be decomposed into a collection of definable coaisles of $\mathcal{D}(\kappa(\mathfrak{p})C)$ indexed over $\mathrm{Spec}(R)$ and that this assignment is injective.

For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, we define the adjoint triple $\varphi_{\mathfrak{p}}^* \dashv \varphi_{\mathfrak{p}} \dashv \varphi_{\mathfrak{p}}^!$, denoting respectively the base change, the forgetful functor, and the cobase change at $\mathfrak{p}$ as follows:

$$\varphi_{\mathfrak{p}}^* := {}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{p})_R \otimes^{\mathbf{L}}_{\mathrm{R}} \_ \colon \mathcal{D}(RC) \to \mathcal{D}(\kappa(\mathfrak{p})C), \qquad \mathcal{D}(\kappa(\mathfrak{p})C) \xrightarrow{\ \varphi_{\mathfrak{p}}\ } \mathcal{D}(RC), \qquad \varphi_{\mathfrak{p}}^! := \mathbf{R}\mathrm{Hom}_R({}_R\kappa(\mathfrak{p})_{\kappa(\mathfrak{p})}, \_) \colon \mathcal{D}(RC) \to \mathcal{D}(\kappa(\mathfrak{p})C)$$

By uniqueness of adjoints, the forgetful functor $\varphi_{\mathfrak{p}}$ can be either identified with ${}_R\kappa(\mathfrak{p})_{\kappa(\mathfrak{p})} \otimes^{\mathbf{L}}_{\kappa(\mathfrak{p})} \_$ or $\mathbf{R}\mathrm{Hom}_{\kappa(\mathfrak{p})}({}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{p})_R, \_)$. In particular, it preserves coproducts and so, by Theorem 1.0.7, the functor $\varphi_{\mathfrak{p}}^*$ preserves compact objects.

Proposition 6.1.5. *For any prime ideal $\mathfrak{p}$ of R, the following hold:*

1. $\varphi_{\mathfrak{p}} \circ \varphi_{\mathfrak{p}}^* \cong \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \_$ *and* $\varphi_{\mathfrak{p}} \circ \varphi_{\mathfrak{p}}^! \cong \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \_)$*;*

2. *If R is noetherian, it holds that*

$$\varphi_{\mathfrak{p}}^{*} \circ \varphi_{\mathfrak{p}}(X) \cong \bigoplus_{i\geq 0} X^{n_i}[i] \text{ and } \varphi_{\mathfrak{p}}^{!} \circ \varphi_{\mathfrak{p}}(X) \cong \prod_{i\geq 0} X^{n_i}[-i]$$

for some $n_i \geq 0$ and $n_0 = 1$;

3. *For any ideal $\mathfrak{q} \neq \mathfrak{p}$: $\varphi_{\mathfrak{q}}^{*} \circ \varphi_{\mathfrak{p}} = 0$ and $\varphi_{\mathfrak{q}}^{!} \circ \varphi_{\mathfrak{p}} = 0$.*

Proof. (1) By associativity and adjunction isomorphisms in Remark 4.0.7, we have that

$$\varphi_{\mathfrak{p}} \circ \varphi_{\mathfrak{p}}^{*} \cong \left(\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})}^{\mathbf{L}} \kappa(\mathfrak{p})\right) \otimes_{\mathrm{R}}^{\mathbf{L}} \_ \text{ and } \varphi_{\mathfrak{p}} \circ \varphi_{\mathfrak{p}}^{!} \cong \mathbf{R}\mathrm{Hom}_R\left(\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})}^{\mathbf{L}} \kappa(\mathfrak{p}), \_\right)$$

where, trivially, $\kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})}^{\mathbf{L}} \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})$.
(2) By associativity and adjunction isomorphisms in Remark 4.0.7, we have that

$$\varphi_{\mathfrak{p}}^{*} \circ \varphi_{\mathfrak{p}} \cong \left(\kappa(\mathfrak{p}) \otimes_{\mathrm{R}}^{\mathbf{L}} \kappa(\mathfrak{p})\right) \otimes_{\kappa(\mathfrak{p})}^{\mathbf{L}} \_ \text{ and } \varphi_{\mathfrak{p}}^{!} \circ \varphi_{\mathfrak{p}} \cong \mathbf{R}\mathrm{Hom}_{\kappa(\mathfrak{p})}\left(\kappa(\mathfrak{p}) \otimes_{\mathrm{R}}^{\mathbf{L}} \kappa(\mathfrak{p}), \_\right)$$

where $\kappa(\mathfrak{p}) \otimes_{\mathrm{R}}^{\mathbf{L}} \kappa(\mathfrak{p})$ is computed by taking a flat resolution $\mathbf{f}\kappa(\mathfrak{p})$ of $\kappa(\mathfrak{p})$ over R. In fact, we can choose $\mathbf{f}\kappa(\mathfrak{p})$ to be a resolution of $\kappa(\mathfrak{p})$ by finite rank free $R_{\mathfrak{p}}$-modules such that $\kappa(\mathfrak{p}) \otimes_R \mathbf{f}\kappa(\mathfrak{p})$ has zero differentials. Indeed, we can let $\mathbf{f}\kappa(\mathfrak{p})$ be the minimal projective resolution of the finitely generated $R_{\mathfrak{p}}$-module $\kappa(\mathfrak{p})$ over the local ring $R_{\mathfrak{p}}$. The rest follows by letting n_i be the dimension of the $(-i)$-th cohomology of $\kappa(\mathfrak{p}) \otimes_{\mathrm{R}}^{\mathbf{L}} \kappa(\mathfrak{p})$.
(3) By Lemma 2.0.4 (3c), $\kappa(\mathfrak{q}) \otimes_{\mathrm{R}}^{\mathbf{L}} \kappa(\mathfrak{p}) = 0$. □

Denote the disjoint union of the collections of definable coaisles of $\mathcal{D}(\kappa(\mathfrak{p})C)$ by

$$\mathfrak{C}_{\mathrm{Def}} = \bigsqcup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(\kappa(\mathfrak{p})C)).$$

By a map $\mathsf{f} : \mathrm{Spec}(R) \longrightarrow \mathfrak{C}_{\mathrm{Def}}$ we will always implicitly mean that $\mathsf{f}(\mathfrak{p}) \in \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(\kappa(\mathfrak{p})C))$, i.e. we will only consider sections of the map $\pi : \mathfrak{C}_{\mathrm{Def}} \to \mathrm{Spec}(R)$ assigning to a definable coaisle of $\mathcal{D}(\kappa(\mathfrak{p})C)$ the prime $\mathfrak{p}$. Note that, by elementary set theory, considering these maps is equivalent to considering collections of definable coaisles of $\mathcal{D}(\kappa(\mathfrak{p})C)$ indexed over $\mathrm{Spec}(R)$.

Theorem 6.1.6.

1. *For any definable coaisle $\mathcal{V} = \mathcal{I}^{\perp}$ of $\mathcal{D}(RC)$ and any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, it holds that $\mathsf{add}_{\kappa(\mathfrak{p})C}\langle\varphi_{\mathfrak{p}}^{!}\mathcal{V}\rangle = (\varphi_{\mathfrak{p}}^{*}\mathcal{I})^{\perp}$. Furthermore, setting $F(\mathcal{V})(\mathfrak{p}) := \mathsf{add}_{\kappa(\mathfrak{p})C}\langle\varphi_{\mathfrak{p}}^{!}\mathcal{V}\rangle$ gives rise to a well-defined assignment*

$$F : \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(RC)) \longrightarrow \{\mathrm{Spec}(R) \to \mathfrak{C}_{\mathrm{Def}}\}.$$

2. *For a map $\mathsf{v} : \mathrm{Spec}(R) \longrightarrow \mathfrak{C}_{\mathrm{Def}}$, let $G(\mathsf{v}) := \mathsf{cosusp}_{RC}^{\mathbf{hs}}\langle\varphi_{\mathfrak{p}}\mathsf{v}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\rangle$. Then, for any definable coaisle $\mathcal{V} \subseteq \mathcal{D}(RC)$ we have $G \circ F(\mathcal{V}) = \mathcal{V}$. In particular, F is injective.*

Proof. (1) Let us first prove the equality. For any morphism in $\mathcal{I}$, say $f : A \to B$, and $Y \in \mathcal{V}$, by adjunction, we have the following commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})C)}\left(\varphi_{\mathfrak{p}}^{*}B, \varphi_{\mathfrak{p}}^{!}Y\right) & \xrightarrow{\mathrm{Hom}(\varphi_{\mathfrak{p}}^{*}f, \varphi_{\mathfrak{p}}^{!}Y)} & \mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})C)}\left(\varphi_{\mathfrak{p}}^{*}A, \varphi_{\mathfrak{p}}^{!}Y\right) \\
\wr\| & & \wr\| \\
\mathrm{Hom}_{\mathcal{D}(RC)}\left(B, \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}Y\right) & \xrightarrow{\mathrm{Hom}(f, \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}Y)} & \mathrm{Hom}_{\mathcal{D}(RC)}\left(A, \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}Y\right)
\end{array}$$

In particular, $\mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})C)}\left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}, \varphi_{\mathfrak{p}}^{!}\mathcal{V}\right) = 0$ if and only if $\mathrm{Hom}_{\mathcal{D}(RC)}\left(\mathcal{I}, \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{V}\right) = 0$. Then, since by Proposition 6.1.5 (1) $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{V} \cong \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \mathcal{V})$ and, by Proposition 4.0.9 (2) $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \mathcal{V})$ is contained in $\mathcal{V}$, it follows that $\mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})C)}\left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}, \varphi_{\mathfrak{p}}^{!}\mathcal{V}\right) = 0$. Thus, since $\left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}\right)^{\perp}$ is closed under summands, we get that $\mathsf{add}_{\kappa(\mathfrak{p})C}\left\langle \varphi_{\mathfrak{p}}^{!}\mathcal{V}\right\rangle \subseteq \left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}\right)^{\perp}$. For the other containment, let $Y \in \left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}\right)^{\perp}$. By adjunction, we have that $\varphi_{\mathfrak{p}}Y \in \mathcal{I}^{\perp} = \mathcal{V}$, thus $\varphi_{\mathfrak{p}}^{!}\varphi_{\mathfrak{p}}Y$ lies in $\varphi_{\mathfrak{p}}^{!}\mathcal{V}$ and so, by Proposition 6.1.5 (2), Y belongs to $\mathsf{add}_{\kappa(\mathfrak{p})C}\left\langle \varphi_{\mathfrak{p}}^{!}\mathcal{V}\right\rangle$. Now let us prove the well definiteness of the assignment, i.e. that $F(\mathcal{V})(\mathfrak{p})$ is a definable coaisle of $\mathcal{D}(\kappa(\mathfrak{p})C)$. Since it is evidently definable and closed under summands and negative shifts, it is sufficient to check the closure under extension. Without loss of generality we consider a triangle

$$\varphi_{\mathfrak{p}}^{!}X \longrightarrow Y \longrightarrow \varphi_{\mathfrak{p}}^{!}Z \xrightarrow{+}$$

with $X, Z \in \mathcal{V}$ and, applying the forgetful functor $\varphi_{\mathfrak{p}}$, as above, we get that $\varphi_{\mathfrak{p}}Y \in \mathcal{V}$ and so Y is in $\mathsf{add}_{\kappa(\mathfrak{p})C}\left\langle \varphi_{\mathfrak{p}}^{!}\mathcal{V}\right\rangle$.
(2) Notice that, for any definable coaisle $\mathcal{V}$ of $\mathcal{D}(RQ)$

$$G \circ F(\mathcal{V}) = \mathsf{cosusp}_{RQ}^{\mathbf{hs}}\left\langle \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{V} \mid \mathfrak{p} \in \mathrm{Spec}(R)\right\rangle$$

Thus, it is contained in $\mathcal{V}$. On the other hand, for any complex Y in $\mathcal{V}$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, by Proposition 4.0.9 (3), the complex $\Gamma_{\mathfrak{p}}Y$ is also in $\mathcal{V}$, thus $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \Gamma_{\mathfrak{p}}Y) = \varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\Gamma_{\mathfrak{p}}Y$ is contained in $G \circ F(\mathcal{V})$ and, by minimality and local-to-global principles (Theorems 5.1.2, 5.1.3), $Y \in G \circ F(\mathcal{V})$. □

Remark 6.1.7. Actually, one can prove that the injection F is just a "shadow" of two different, more general, Galois connections. Indeed, defining

$$\mathfrak{C} = \bigsqcup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Cosusp}(\mathcal{D}(\kappa(\mathfrak{p})C)) \text{ and } \mathfrak{D} = \bigsqcup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Def}(\mathcal{D}(\kappa(\mathfrak{p})C)),$$

where $\mathrm{Def}(\mathcal{T})$ denotes the set of all definable subcategories of $\mathcal{T}$, the following assignments form two Galois connections

$$\begin{array}{cc}
\mathrm{Cosusp}(\mathcal{D}(RC)) \longleftrightarrow \{\mathrm{Spec}(R) \longrightarrow \mathfrak{C}\} & \mathrm{Def}(\mathcal{D}(RC)) \longleftrightarrow \{\mathrm{Spec}(R) \longrightarrow \mathfrak{D}\} \\
\mathcal{B} \xmapsto{F^{\mathfrak{C}}} \left(\mathfrak{p} \mapsto \mathsf{cosusp}_{\kappa(\mathfrak{p})C}\left\langle \varphi_{\mathfrak{p}}^{!}\mathcal{B}\right\rangle\right) & \left\{f \in \mathcal{D}^{c}(RC) \mid \varphi_{\mathfrak{p}}^{*}f \in \mathsf{i}(\mathfrak{p})\right\}^{\perp} \xleftarrow{G^{\mathfrak{D}}} \mathsf{i}^{\perp} \\
\left\{X \in \mathcal{D}(RC) \mid \varphi_{\mathfrak{p}}^{!}X \in \mathsf{b}(\mathfrak{p})\right\} \xleftarrow{G^{\mathfrak{C}}} \mathsf{b} & \mathcal{I}^{\perp} \xmapsto{F^{\mathfrak{D}}} \left(\mathfrak{p} \mapsto \left(\varphi_{\mathfrak{p}}^{*}\mathcal{I}\right)^{\perp}\right)
\end{array}$$

In particular:

- For any cosuspended subcategory $\mathcal{B} \subseteq \mathcal{D}(RC)$ and any map $\mathsf{b} : \mathrm{Spec}(R) \to \mathfrak{C}$, it holds that $F^{\mathfrak{C}}(\mathcal{B}) \subseteq \mathsf{b}$ if and only if $\mathcal{B} \subseteq G^{\mathfrak{C}}(\mathsf{b})$;
- For any definable subcategory $\mathcal{I}^{\perp} \subseteq \mathcal{D}^{c}(RC)$ and any map $\mathsf{i}^{\perp} : \mathrm{Spec}(R) \to \mathfrak{D}$, it holds that $G^{\mathfrak{D}}(\mathsf{i}^{\perp}) \subseteq \mathcal{I}^{\perp}$ if and only if $\mathsf{i}^{\perp} \subseteq F^{\mathfrak{D}}(\mathcal{I}^{\perp})$.
- If further restricted to complete cosuspended subcategories and, respectively, definable subcategories closed under positive shifts, they give two Galois insertions

$$\mathrm{Cosusp}_{\Pi}(\mathcal{D}(RC)) \hookleftarrow \{\mathrm{Spec}(R) \longrightarrow \mathfrak{C}_{\Pi}\} \qquad \mathrm{Def}_{[\geq 0]}(\mathcal{D}(RC)) \hookrightarrow \{\mathrm{Spec}(R) \longrightarrow \mathfrak{D}_{[\geq 0]}\}$$

What Theorem 6.1.6 shows is that the a priori distinct assignments $F^{\mathfrak{C}}$ and $F^{\mathfrak{D}}$ coincide on subcategories that are both cosuspended and definable, and in this case, they share the same left inverse G.

6.2 Representations of quivers and telescope conjecture

Recall that, when the small category C is the free category on a finite quiver Q, the category $\mathrm{Mod}_R(C)$ is equivalent to the category of modules $\mathrm{Mod}(RQ)$ and the functors $X : C \to \mathrm{Mod}(R)$ can be seen as representations $X = (X_i, X_\alpha)_{i\in Q_0,\alpha\in Q_1}$ where X_i is an R-module and X_α a compatible R-linear map. In this case, by [Ben95, Theorem 4.1.4], under the assumption that Q is finite, for any field $\mathbb{K}$ the path algebra $\mathbb{K}Q$ is hereditary.

The aim of this section is to prove that, in this setting, for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, the assignment of Theorem 6.1.6 restricts to a bijection between the class of definable coaisles of the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$ and the class of definable coaisles of $\mathcal{D}(\kappa(\mathfrak{p})Q)$. Moreover, in virtue of these bijections, the former category satisfies the telescope conjecture.

Remark 6.2.1. By Definition 1.2.20, it is clear that full triangulated subcategories of algebraic triangulated categories are still algebraic. Thus, by [AS16, Lemma 6.9], for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, the stalk subcategory $\Gamma_\mathfrak{p}\mathcal{D}(RQ)$ is a compactly generated algebraic triangulated category such that

$$(\Gamma_\mathfrak{p}\mathcal{D}(RQ))^c = \Gamma_\mathfrak{p}\mathcal{D}(RQ) \cap \mathcal{D}^c(R_\mathfrak{p}Q)$$

6.2.1 Lift of Compacts

First, we want to show a construction for "lifting" a compact object $S \in \mathcal{D}(\kappa(\mathfrak{p})Q)$ to a compact object $\widetilde{S} \in \Gamma_\mathfrak{p}\mathcal{D}(RQ)$, in such a way that its "restriction" $\kappa(\mathfrak{p}) \otimes_R \widetilde{S}$ is an object closely related to the original object S.

Proposition 6.2.2. *Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, for any compact object $S \in \mathcal{D}(\kappa(\mathfrak{p})Q)$ there exists:*

1. $\widehat{S} \in \mathcal{D}^c(R_\mathfrak{p}Q)$ *such that* $\kappa(\mathfrak{p}) \otimes_R \widehat{S} \cong S$*;*
2. $\widetilde{S} \in (\Gamma_\mathfrak{p}\mathcal{D}(RQ))^c$ *such that* $\kappa(\mathfrak{p}) \otimes_R \widetilde{S} \in \mathsf{add}_{\kappa(\mathfrak{p})Q}\langle S[\geq 0]\rangle$.

Proof. Since the path algebra $\kappa(\mathfrak{p})Q$ is hereditary, any complex $X \in \mathcal{D}(\kappa(\mathfrak{p})Q)$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} H^i(X)[-i]$, see [Kra07, §1.6]. So instead of starting with a complex $S \in \mathcal{D}^c(\kappa(\mathfrak{p})Q)$, without loss of generality, we can assume $S \in \mathrm{mod}(\kappa(\mathfrak{p})Q)$ to be a finitely presented module and then extend the construction to complexes by direct sums and shifts.
(1) For such $S \in \mathrm{mod}(\kappa(\mathfrak{p})Q)$, we have an exact sequence

$$0 \longrightarrow P \xrightarrow{\ A\ } \kappa(\mathfrak{p})Q^n \longrightarrow S \longrightarrow 0$$

where $P = (\kappa(\mathfrak{p})^{m_i}, P_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$ is a finitely generated projective module. In particular, this exact sequence induces a distinguished triangle

$$P \xrightarrow{\ A\ } \kappa(\mathfrak{p})Q^n \longrightarrow S \xrightarrow{\ +\ }$$

By the Krull-Schmidt property, P is isomorphic to a direct sum of indecomposable projective modules which, by [Rin98], admit a presentation by $0,1$-matrices on the arrows (see Remark 7.1.1). In particular, the entries of P_α involve just zero and identity maps for any $\alpha \in Q_1$, thus P can be lifted to a finitely generated projective $R_\mathfrak{p}Q$-module $\widehat{P} = (R_p^{m_i}, P_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$. By [CB24, Theorem 4.1], the morphism A, given by a collection of matrices $(A_i)_{i\in Q_0}$ with entries in $\kappa(\mathfrak{p})$, lifts to a morphism $\widehat{A} : \widehat{P} \to R_\mathfrak{p}Q^n$, given by a collection of matrices $(\widehat{A}_i)_{i\in Q_0}$ with entries in $R_\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \otimes_R \widehat{A} = A$. We can complete $\widehat{A}$ to a triangle in $\mathcal{D}^c(R_\mathfrak{p}Q)$

$$\widehat{P} \xrightarrow{\ \widehat{A}\ } R_\mathfrak{p}Q^n \longrightarrow \widehat{S} \xrightarrow{\ +\ } \qquad (\star)$$

and, by construction, we get that $\kappa(\mathfrak{p}) \otimes_R \widehat{S}$ is quasi-isomorphic to S.

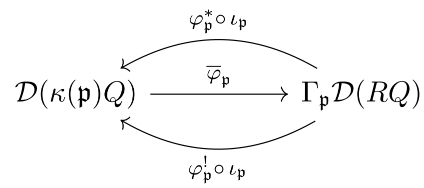
(2) Let $K(\mathfrak{p})$ be the Koszul complex at $\mathfrak{p}$. Since it is compact in $\mathcal{D}(R)$, the complexes $K(\mathfrak{p})\otimes_R \widehat{P}$ and $K(\mathfrak{p})\otimes_R R_{\mathfrak{p}}Q^n$ are compact in $\mathcal{D}(R_{\mathfrak{p}}Q)$ and, by Remark 4.1.3, are supported on $\{\mathfrak{p}\}$, thus they belong to $\Gamma_{\mathfrak{p}}\mathcal{D}(R_{\mathfrak{p}}Q)$ (see [Sab25a, Proposition 1.13 (3)]). By Remark 6.2.1, it follows that both the objects are compact in $\Gamma_{\mathfrak{p}}\mathcal{D}(R_{\mathfrak{p}}Q)$. Define $\widetilde{S} := K(\mathfrak{p})\otimes_R \widehat{S}$ and note that, by tensoring the triangle $(\star)$ by $K(\mathfrak{p})$, we get that $\widetilde{S}$ lies in $(\Gamma_{\mathfrak{p}}\mathcal{D}(R_{\mathfrak{p}}Q))^c$. Moreover, by Remark 4.1.3, we have that

$$\kappa(\mathfrak{p})\otimes_R \widetilde{S} = \bigoplus_{i=0}^{n} \kappa(\mathfrak{p})^{\binom{n}{i}}[i]\otimes_R \widehat{S} = \bigoplus_{i=0}^{n} S^{\binom{n}{i}}[i].$$

In particular, $\kappa(\mathfrak{p})\otimes_R \widetilde{S}\in \mathsf{add}_{\kappa(\mathfrak{p})Q}\langle S[\geq 0]\rangle$. □

6.2.2 Telescope conjecture

The adjoint triple $\varphi_{\mathfrak{p}}^{*}\dashv\varphi_{\mathfrak{p}}\dashv\varphi_{\mathfrak{p}}^{!}$ introduced in the previous section restricts to an adjunction



where $\iota_{\mathfrak{p}}:\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)\hookrightarrow\mathcal{D}(RQ)$ denotes the full embedding of the stalk subcategory at $\mathfrak{p}$. Since the support of $\kappa(\mathfrak{p})$ is equal to $\{\mathfrak{p}\}$, the essential image of $\varphi_{\mathfrak{p}}$ is contained in $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$, here $\overline{\varphi}_{\mathfrak{p}}$ denotes the corestriction. To justify the adjunction, first notice that $\varphi_{\mathfrak{p}}=\iota_{\mathfrak{p}}\circ\overline{\varphi}_{\mathfrak{p}}$, then for any $X,Z\in\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ and $Y\in\mathcal{D}(\kappa(\mathfrak{p})Q)$, we have isomorphisms

$$\mathrm{Hom}_{\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)}(X,\overline{\varphi}_{\mathfrak{p}}Y)\cong\mathrm{Hom}_{\mathcal{D}(RQ)}(\iota_{\mathfrak{p}}X,\iota_{\mathfrak{p}}\overline{\varphi}_{\mathfrak{p}}Y)\cong\mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})Q)}(\varphi_{\mathfrak{p}}^{*}\iota_{\mathfrak{p}}X,Y)$$

$$\mathrm{Hom}_{\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)}(\overline{\varphi}_{\mathfrak{p}}Y,Z)\cong\mathrm{Hom}_{\mathcal{D}(RQ)}(\iota_{\mathfrak{p}}\overline{\varphi}_{\mathfrak{p}}Y,\iota_{\mathfrak{p}}Z)\cong\mathrm{Hom}_{\mathcal{D}(\kappa(\mathfrak{p})Q)}(Y,\varphi_{\mathfrak{p}}^{!}\iota_{\mathfrak{p}}Z)$$

By abuse of notation, we will denote this adjoint triple using the same notation as the one in the previous section.

Note that to study the closure property of t-structures in $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ under the composite $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{*}$ and $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}$, we can not directly use the same arguments of Proposition 4.0.9. However we still get an analogous result.

Lemma 6.2.3. *For any t-structure $(\mathcal{U},\mathcal{W})$ in $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$, it holds that:*

1. $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{*}\mathcal{U}=\kappa(\mathfrak{p})\otimes_{\mathrm{R}}^{\mathbf{L}}\mathcal{U}\subseteq\mathcal{U}$;
2. $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{W}=\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}),\mathcal{W})\subseteq\mathcal{W}$.

Proof. (1) Recall that $\mathcal{U}$ is a suspended subcategory of $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ closed under coproducts. Since $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ is a localizing subcategory of $\mathcal{D}(RQ)$, $\mathcal{U}$ is still suspended and closed under coproducts in $\mathcal{D}(RQ)$. Thus, by Lemma 4.0.8 (1), $\kappa(\mathfrak{p})\otimes_{\mathrm{R}}^{\mathbf{L}}\mathcal{U}\subseteq\mathcal{U}$.
(2) We have $\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{W}\subseteq\mathcal{W}$ if and only if $\mathrm{Hom}_{\mathcal{D}(RQ)}({}^{\perp}\mathcal{W},\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{!}\mathcal{W})=0$. By adjuction, this holds if and only if $\mathrm{Hom}_{\mathcal{D}(RQ)}(\varphi_{\mathfrak{p}}\varphi_{\mathfrak{p}}^{*}{}^{\perp}\mathcal{W},\mathcal{W})=0$, which holds by point (1). □

Now we want to prove that, for any prime $\mathfrak{p}\in\mathrm{Spec}(R)$, the assignment of Theorem 6.1.6 restricts to a bijection between definable coaisles of the stalk subcategory $\Gamma_{\mathfrak{p}}(\mathcal{D}(RQ))$ and the ones of $\mathcal{D}(\kappa(\mathfrak{p})Q)$. Recall form Remark 6.2.1, that stalk subcategories are compactly generated algebraic triangulated categories, thus the equivalence between homotopically smashing coaisles and cosuspended definable subcategories, mentioned in Proposition 6.1.2, also holds in $\Gamma_{\mathfrak{p}}(\mathcal{D}(RQ))$.

Theorem 6.2.4. *Let R be a commutative noetherian ring and Q a finite quiver. Then, for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, there is an order-preserving bijection*

$$F_{\mathfrak{p}} : \mathrm{Coaisle}_{\mathrm{Def}}(\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)) \longrightarrow \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(\kappa(\mathfrak{p})Q))$$
$$\mathcal{W} = \mathcal{J}^{\perp} \mapsto \mathsf{add}_{\kappa(\mathfrak{p})Q}\left\langle \varphi_{\mathfrak{p}}^{!}\mathcal{W}\right\rangle = \left(\varphi_{\mathfrak{p}}^{*}\mathcal{J}\right)^{\perp}$$

with inverse $G_{\mathfrak{p}} : \mathsf{w} \mapsto \mathsf{cosusp}^{\mathbf{hs}}_{RQ}\langle\varphi_{\mathfrak{p}}\mathsf{w}\rangle$. Moreover, any definable coaisle in $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ is compactly generated, in particular, the stalk subcategories $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ satisfy the telescope conjecture.

Proof. The assignment $F_{\mathfrak{p}}$ is well defined and injective. Indeed, for a definable coaisle $\mathcal{W} = \mathcal{J}^{\perp}$ of $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$, we can prove that $\mathsf{add}_{\kappa(\mathfrak{p})Q}\langle\varphi_{\mathfrak{p}}^{!}\mathcal{W}\rangle = \left(\varphi_{\mathfrak{p}}^{*}\mathcal{J}\right)^{\perp}$ and $\mathcal{W} = G_{\mathfrak{p}} \circ F_{\mathfrak{p}}(\mathcal{W})$ following the proof of Theorem 6.1.6, which still holds in this context by Lemma 6.2.3.
Let us prove that $F_{\mathfrak{p}}$ is surjective. Let w be a definable coaisle of $\mathcal{D}(\kappa(\mathfrak{p})Q)$, by the telescope conjecture for hereditary rings [AH21, Theorem 3.11] and its characterization in Proposition 6.1.4, there is a set s of identity maps between compact objects of $\mathcal{D}(\kappa(\mathfrak{p})Q)$ such that $\mathsf{w} = \mathsf{s}^{\perp}$. Denote by $\mathcal{S} = \{\mathrm{id}_{\widetilde{S}} \mid \mathrm{id}_S \in \mathsf{s}\}$, the set of identity maps between the lifts $\widetilde{S} \in (\Gamma_{\mathfrak{p}}\mathcal{D}(RQ))^c$ (see Proposition 6.2.2) for every S such that $\mathrm{id}_S \in \mathsf{s}$. Since s is closed under positive shifts, $\mathcal{S}^{\perp}$ is a (definable) coaisle of $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ and, by Proposition 6.2.2 (2), $\mathsf{s}^{\perp} = (\varphi_{\mathfrak{p}}^{*}\mathcal{S})^{\perp}$, i.e. $\mathsf{w} = F_{\mathfrak{p}}(\mathcal{S}^{\perp})$.
As for the last part, it follows from the bijection that any definable coaisle $\mathcal{W}$ of $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$ is of the form $\mathcal{W} = G_{\mathfrak{p}}(\mathsf{s}^{\perp})$, for some set s of identity maps between compact objects of $\mathcal{D}(\kappa(\mathfrak{p})Q)$. Moreover, $G_{\mathfrak{p}}(\mathsf{s}^{\perp}) = G_{\mathfrak{p}} \circ F_{\mathfrak{p}}(\mathcal{S}^{\perp}) = \mathcal{S}^{\perp}$, for a set $\mathcal{S}$ of identity maps between compact objects of $\Gamma_{\mathfrak{p}}\mathcal{D}(RQ)$, and thus it is compactly generated. □

Let us now restrict the discussion to commutative artinian rings. In particular, for such a ring R, the prime spectrum is finite and discrete, consisting of finitely many maximal ideals $\operatorname{Spec}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ and the ring decomposes into the direct product of its localizations $R \cong \bigoplus_{i=1}^{n} R_{\mathfrak{m}_i}$ (see [AM69, Theorems 8.3, 8.5, 8.7]).

Corollary 6.2.5. *For any commutative artinian ring R and finite quiver Q, the derived category $\mathcal{D}(RQ)$ satisfies the telescope conjecture.*

Proof. Let $R \cong \bigoplus_{i=1}^{n} R_{\mathfrak{m}_i}$ be a commutative artinian ring. Note that, for each $\mathfrak{m} \in \operatorname{Spec}(R)$, $R_{\mathfrak{m}}$ is a 0-dimensional local ring and thus every $R_{\mathfrak{m}}Q$-module is supported on $\{\mathfrak{m}\}$, i.e. $\mathcal{D}(R_{\mathfrak{m}}Q) \cong \Gamma_{\mathfrak{m}}\mathcal{D}(RQ)$ (see Proposition 3.3.8 (3)). The decomposition $R \cong \bigoplus_{i=1}^{n} R_{\mathfrak{m}_i}$ of the ring induces a decomposition $\mathrm{Mod}_R(C) \cong \mathrm{Mod}_{R_{\mathfrak{m}_1}}(C) \times \ldots \times \mathrm{Mod}_{R_{\mathfrak{m}_n}}(C)$ of the module categories. It follows that the derived category $\mathcal{D}(RQ)$ is equivalent to the product category $\Gamma_{\mathfrak{m}_1}\mathcal{D}(RQ) \times \ldots \times \Gamma_{\mathfrak{m}_n}\mathcal{D}(RQ)$ and so, by Theorem 6.2.4, it satisfies the telescope conjecture. □

Example 6.2.6 (Representations over truncated polynomial algebras).
An application of Corollary 6.2.5 to the representation theory of finite-dimensional algebras is provided by representations of finite acyclic quivers Q over truncated polynomial algebras

$$\Lambda := \mathbb{K}[x_1, \dots, x_m]/\left(x_i^{n_i} \mid 1 \le i \le m\right)$$

where $\mathbb{K}$ is a field and $n_i \ge 2$ for all $1 \le i \le m$. In this case ΛQ is again a finite-dimensional algebra, and hence isomorphic to a path algebra $\mathbb{K}Q'/I$ for some quiver Q' and admissible ideal I, where

$$Q'_0 = Q_0,\ Q'_1 = Q_1 \cup \left\{l_{1,k}, \dots, l_{m,k} \mid k \in Q'_0\right\} \text{ and}$$
$$I = \left(\alpha l_{i,k} - l_{i,h}\alpha,\ l_{i,k}^{n_i} \mid 1 \le i \le m,\ k \in Q'_0,\ \alpha : k \to h \in Q'_1\right)$$

with $l_{i,k}$ denoting a loop at the vertex k for any $1 \le i \le m$. Therefore, these algebras belong among the few known examples of non-hereditary finite dimensional algebras satisfying the telescope conjecture.

For the so-called algebra of dual numbers $\Lambda = \mathbb{K}[x]/(x^2)$, as discussed in [RZ17], representations of a finite quiver Q over Λ coincides with the category of differential $\mathbb{K}Q$-modules $\mathrm{Diff}(\mathbb{K}Q)$. Thus, in this framework, Corollary 6.2.5 proves the telescope conjecture for the derived category $\mathcal{D}(\mathrm{Diff}(\mathbb{K}Q))$, or equivalently, for the J-shaped derived category $\mathcal{D}_J(\mathbb{K}Q)$ (see for example [HJ25, Section 4]), where J denotes the Jordan quiver $\overset{1}{\bullet} \circlearrowleft^{\delta}$ with the relation $\delta^2 = 0$.

6.A Towards non-noetherianity

We want to dedicate this section to extending the previous results out of the noetherian world. To do this, we focus our attention on commutative rings which decompose into a product of 0-dimensional local rings which satisfy a minimality condition as Theorem 5.1.3, the *perfect rings*. Recall that a commutative ring R is *semi-local* if it has only finitely many maximal ideals, and *semi-artinian* if it admits a (possibly infinite) composition series – that is, if R as a regular R-module can be obtained as a transfinite extension of simple R-modules. Note that a semi-artinian ring is artinian if and only if it is noetherian. Moreover, an ideal I of R is said to be *T-nilpotent* if, for every sequence $r_1, r_2, \dots$ in I, there exists an $n > 0$ such that $r_1 r_2 \dots r_n = 0$.

Definition 6.A.1 ([BG21, Proposition 2.6])**.** A commutative ring R, is called *perfect* if one of the following equivalent properties is satisfied:

(i) R is semi-local and semi-artinian;

(ii) R is a finite direct product of local rings with T-nilpotent maximal ideals;

(iii) Every R-module has a projective cover.

In particular, if R is a commutative perfect ring then, due to its semi-artinianity, it is 0-dimensional (see [KT18, Example 2.7]) and, due to its semi-local condition, one has $\mathrm{Spec}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ where the $\mathfrak{m}_i$ are the maximal ideals of R. Moreover, by condition (ii), it follows that a commutative perfect ring is isomorphic to a finite direct product of local perfect rings each of them being the localization at a maximal ideal, i.e. $R \cong \bigoplus_{i=1}^n R_{\mathfrak{m}_i}$. Thus, for now, let us focus on local perfect rings.

Let us show that, for a local perfect ring $(R, \mathfrak{m}, \kappa)$, the algebra RQ satisfies a minimality condition for homotopically smashing complete cosuspended subcategories, as in Theorem 5.1.3.

Proposition 6.A.2. *Let $(R, \mathfrak{m}, \kappa)$ be commutative local perfect ring and Q be a finite quiver. Then, for any complex $Y \in \mathcal{D}(RC)$, it holds that*

$$\mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle Y \rangle = \mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle \mathbf{R}\mathrm{Hom}_R(\kappa, Y) \rangle$$

Proof. The proof follows as the one of Theorem 5.1.3. Indeed, by semi-artinianity, the ring R itself is a transfinite extension of copies of the residue field κ. So, since κ is contained in the cocomplete suspended subcategory

$$\mathcal{M} = \left\{ M \in \mathcal{D}(R) \mid \mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle \mathbf{R}\mathrm{Hom}_R(\kappa, Y) \rangle \right\}$$

this contains also R, and we can conclude that $Y \cong \mathbf{R}\mathrm{Hom}_R(R, Y) \in \mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle \mathbf{R}\mathrm{Hom}_R(\kappa, Y) \rangle$. □

We will see that the bijection of Theorem 6.2.4, holds also for a local perfect ring R but before doing that let us briefly recall some results of the previous sections in the present context.

Proposition 6.A.3. *Let $(R, \mathfrak{m}, \kappa)$ be a commutative local perfect ring, then:*

1. *The adjoint triple* $\varphi_{\mathfrak{m}}^* \dashv \varphi_{\mathfrak{m}} \dashv \varphi_{\mathfrak{m}}^!$ *satisfies:*
   (a) $\varphi_{\mathfrak{m}} \circ \varphi_{\mathfrak{m}}^! \cong \mathbf{R}\mathrm{Hom}_R(\kappa, \_)$;
   (b) $\varphi_{\mathfrak{m}}^! \circ \varphi_{\mathfrak{m}}(X) \cong \prod_{i\geq 0} \kappa^{(\alpha_i)}[i]$ *for some cardinals* α_i.
2. *For any compact object* $S \in \mathcal{D}(\kappa(\mathfrak{p})Q)$ *there exists* $\widehat{S} \in \mathcal{D}^c(RQ)$ *such that* $\kappa \otimes_R \widehat{S} \cong S$.

Proof. (1) Point (a) follows as Proposition 6.1.5 (1). Since R is local and perfect, by condition (iii) of Definition 6.A.1, κ admits a minimal free resolution $\mathbf{f}\kappa$ and $\kappa \otimes_R \mathbf{f}\kappa \cong \bigoplus_{i\geq 0} \kappa^{(\alpha_i)}[i]$ for some cardinals α_i. Thus, (b) follows as Proposition 6.1.5 (2) with the obvious modifications.
(2) The result follows as Proposition 6.2.2 (1). □

Theorem 6.A.4. *Let* $(R, \mathfrak{m}, \kappa)$ *be a commutative local perfect ring and* Q *a finite quiver. Then, there is an order-preserving bijection*

$$F : \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(RQ)) \longrightarrow \mathrm{Coaisle}_{\mathrm{Def}}(\mathcal{D}(\kappa Q))$$
$$\mathcal{W} = \mathcal{J}^{\perp} \mapsto \mathsf{add}_{\kappa Q} \left\langle \varphi_{\mathfrak{m}}^! \mathcal{W} \right\rangle = (\varphi_{\mathfrak{m}}^* \mathcal{J})^{\perp}$$

with inverse $G : \mathsf{w} \mapsto \mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle \varphi_{\mathfrak{m}} \mathsf{w} \rangle$. *Moreover, any definable coaisle in* $\mathcal{D}(RQ)$ *is compactly generated.*

Proof. The well definiteness of the assignment F follows with the same proof of Theorem 6.1.6 (1), using Proposition 6.A.3 (1). As for the injectivity, given a definable coaisle $\mathcal{W} \subseteq \mathcal{D}(RQ)$, the subcategory $G \circ F(\mathcal{W}) = \mathsf{cosusp}_{RQ}^{\Pi,\mathbf{hs}} \langle \varphi_{\mathfrak{m}} \varphi_{\mathfrak{m}}^! \mathcal{W} \rangle$ is contained in $\mathcal{W}$ and, by Proposition 6.A.2, also the reverse containment holds.
The surjectivity of F follows similarly to Theorem 6.2.4. Given a definable coaisle w of $\mathcal{D}(\kappa Q)$, by the telescope conjecture in [AH21, Theorem 3.11] and Proposition 6.1.4, there is a set $\mathsf{s} \subseteq \mathcal{D}^c(\kappa Q)$ of identity maps such that $\mathsf{w} = \mathsf{s}^{\perp}$. Denote by $\mathcal{S} = \left\{ \mathrm{id}_{\widehat{S}} \mid \mathrm{id}_S \in \mathsf{s} \right\}$ the set of identity maps between the lifts $\widehat{S} \in \mathcal{D}^c(RQ))$ for every S such that $\mathrm{id}_S \in \mathsf{s}$. Since s is closed under positive shifts, $\mathcal{S}^{\perp}$ is a (definable) coaisle of $\mathcal{D}(RQ)$ and, by Proposition 6.A.3 (2), $\mathsf{s}^{\perp} = (\varphi_{\mathfrak{p}}^* \mathcal{S})^{\perp}$, i.e. $\mathsf{w} = F_{\mathfrak{p}}(\mathcal{S}^{\perp})$. As for the last part, it follows from the bijection that any definable coaisle $\mathcal{W}$ of $\mathcal{D}(RQ)$ is of the form $\mathcal{W} = G(\mathsf{s}^{\perp})$, for some set of identity maps $\mathsf{s} \subseteq \mathcal{D}^c(\kappa Q)$. Moreover, $G(\mathsf{s}^{\perp}) = G \circ F(\mathcal{S}^{\perp}) = \mathcal{S}^{\perp}$, for a set of identity maps $\mathcal{S} \subseteq \mathcal{D}^c(RQ)$, and thus it is compactly generated. □

So, analogously to Corollary 6.2.5, we can conclude the following.

Corollary 6.A.5. *For any commutative perfect ring* R *and finite quiver* Q, *the derived category* $\mathcal{D}(RQ)$ *satisfies the telescope conjecture.*

Proof. Let R be a commutative perfect ring, then by definition it is equivalent to a direct product of finitely many local perfect rings $R \cong \bigoplus_{i=1}^n R_i$. It follows again that the derived category $\mathcal{D}(RQ)$ is equivalent to the product category $\mathcal{D}(R_1Q) \times \ldots \times \mathcal{D}(R_nQ)$ and so, by Theorem 6.A.4, it satisfies the telescope conjecture. □

Chapter 7

Representations of Dynkin Quivers

Let RQ be the path algebra of a Dynkin quiver Q over a commutative noetherian ring R. We show that any homotopically smashing t-structure in the derived category of RQ is compactly generated. We also give a complete description of the compactly generated t-structures in terms of poset homomorphisms from the prime spectrum of the ring $\mathrm{Spec}(R)$ *to the poset of filtrations of noncrossing partitions of the quiver* $\mathrm{Filt}(\mathbf{Nc}(Q))$. *In the case that R is regular, we also get a complete description of the wide subcategories of the category* $\mathrm{mod}(RQ)$.

The material in this chapter is part of the author's work [Sab25a].

7.1 t-Structures and lattice lifts

In the following, R will be a commutative noetherian ring and Q a Dynkin quiver. Recall from Section 4.2.1 that when $R = \mathbb{K}$ is a field the indecomposable $\mathbb{K}Q$-modules do not depend on the field, be it algebraically closed or not. In this case, also the (compactly generated) t-structures of $\mathcal{D}(\mathbb{K}Q)$ are independent of the field $\mathbb{K}$. In particular, they depend on the lattice $\mathbf{Nc}(Q)$ of noncrossing partitions of Q (see Section 7.A for more details).

Proposition (Corollary 7.A.4)**.** *For any field $\mathbb{K}$ and Dynkin quiver Q, any t-structure is compactly generated and there is an order-preserving bijection between* $\mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$ *and* $\mathrm{Filt}(\mathbf{Nc}(Q))$.

Moreover, by Theorem 4.2.7, for any commutative ring R, we have the *lattice lift* assignment

$$\widetilde{\ } : \mathrm{ind}(\mathcal{D}(\mathbb{K}Q)) \longrightarrow \mathrm{lat}(RQ)[\mathbb{Z}]$$

Remark 7.1.1. We can actually give a concrete description of the lattice lifts. Indeed, by [Rin98], for any field $\mathbb{K}$, the exceptional $\mathbb{K}Q$-modules are tree modules, i.e. they admit a presentation by $0,1$-matrices on the arrows. This applies to the indecomposable modules over Dynkin quivers, thus for an indecomposable $\mathbb{K}Q$-module L of dimension vector d, we can construct $\widetilde{L}$ defining $\widetilde{L}_i = R^{d_i}$ and $\widetilde{L}_\alpha = L_\alpha$ for any $i \in Q_0$ and $\alpha \in Q_1$. In this way, we obtain a free RQ-lattice which, by [CB24, Lemma 3.1 (iv)], is rigid and so, by Theorem 4.2.7, exceptional. Thus, it is the unique lattice lift of L.

The following proposition shows that the hom-orthogonality of two indecomposable $\mathbb{K}Q$-modules is sufficient to detect the hom-orthogonality of their lattice lifts.

Proposition 7.1.2. *For any indecomposable $\mathbb{K}Q$-modules M and L, it holds that:*

1. $\mathrm{Hom}_{RQ}\left(\widetilde{M}, \widetilde{L}\right) = 0$ *if and only if* $\mathrm{Hom}_{\mathbb{K}Q}(M, L) = 0$;
2. $\mathrm{Ext}^1_{RQ}\left(\widetilde{M}, \widetilde{L}\right) = 0$ *if and only if* $\mathrm{Ext}^1_{\mathbb{K}Q}(M, L) = 0$.

Proof. By [CB24, Theorem 4.1], both the R-modules $\operatorname{Hom}_{RQ}\left(\widetilde{M},\widetilde{L}\right)$ and $\operatorname{Ext}^1_{RQ}\left(\widetilde{M},\widetilde{L}\right)$ are finitely generated projective of constant rank equal to the dimension over $\mathbb{K}$ of the vector spaces $\operatorname{Hom}_{\mathbb{K}Q}(M,L)$ and $\operatorname{Ext}_{\mathbb{K}Q}(M,L)$ respectively. Thus, both point 1 and 2 hold. □

Corollary 7.1.3. *For any two indecomposable $\mathbb{K}Q$-complexes M and L:*

$$\operatorname{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M},\widetilde{L}\right)=0 \textit{ if and only if } \operatorname{Hom}_{\mathcal{D}(\mathbb{K}Q)}(M,L)=0$$

In particular, for any class $\mathcal{X}\subseteq\mathcal{D}(\mathbb{K}Q)$, it holds that $\widetilde{\mathcal{X}^{\perp}}\subseteq\widetilde{\mathcal{X}}^{\perp}$.

Proof. Let $M=M'[m]$ and $L=L'[\ell]$, where M' and L' are two indecomposable $\mathbb{K}Q$-modules and $m,\ell\in\mathbb{Z}$. There are standard isomorphisms

$$\operatorname{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M},\widetilde{L}\right)\cong\operatorname{Ext}^{\ell-m}_{RQ}\left(\widetilde{M'},\widetilde{L'}\right) \text{ and } \operatorname{Hom}_{\mathcal{D}(\mathbb{K}Q)}(M,L)\cong\operatorname{Ext}^{\ell-m}_{\mathbb{K}Q}(M',L')$$

By Proposition 7.1.2, the statement holds for $\ell-m=0,1$. Moreover, by Remark 4.2.6 and since $\mathbb{K}Q$ is hereditary, both $\operatorname{Ext}^{\ell-m}$ vanish for $\ell-m<0$ and $\ell-m\geq 2$. The particular part follows by taking $M\in\mathcal{X}$ and $L\in\mathcal{X}^{\perp}$. □

7.1.1 Compactly generated t-structures

In this section we will introduce two assignments, linking the compactly generated aisles of $\mathcal{D}(RQ)$ to the aisles of $\mathcal{D}(\mathbb{K}Q)$, which will lead to the complete classification of the former. In view of Corollary 7.A.4, we can refer to the poset $\operatorname{Aisle}(\mathcal{D}(\mathbb{K}Q))$ without specifying any field. The assignments are the following

$$\begin{gathered}\operatorname{Aisle}_{\mathrm{cg}}(\mathcal{D}(RQ))\longleftrightarrow\operatorname{Hom}_{\mathrm{Pos}}(\operatorname{Spec}(R),\operatorname{Aisle}(\mathcal{D}(\mathbb{K}Q)))\\ \varphi:\mathcal{U}\longmapsto\left(\mathfrak{p}\mapsto\mathsf{aisle}_{\mathbb{K}Q}\left\langle L\in\operatorname{ind}(\mathcal{D}(\mathbb{K}Q))\mid R/\mathfrak{p}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\in\mathcal{U}\right\rangle\right)\\ \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{p}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{\sigma(\mathfrak{p})}\mid\mathfrak{p}\in\operatorname{Spec}(R)\right\rangle\longleftarrow\sigma:\psi\end{gathered}\tag{7.1.1}$$

Lemma 7.1.4. *For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ and indecomposable $\mathbb{K}Q$-complex L, it holds that:*

1. *If $\mathfrak{a}\subseteq\mathfrak{b}$ then $\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{b}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle\subseteq\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle$;*

2. *If $\operatorname{rad}(\mathfrak{a})=\operatorname{rad}(\mathfrak{b})$ then $\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle=\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{b}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle$;*

3. *$\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{ab}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle=\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L},R/\mathfrak{b}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle$;*

4. *$\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle=\mathsf{aisle}_{RQ}\left\langle K(\mathfrak{a})\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle$ and it is compactly generated.*

Proof. (1) Let $\mathfrak{a}\subseteq\mathfrak{b}$, note that, similarly to Lemma 4.0.8, the class

$$\mathcal{M}=\left\{M\in\mathcal{D}(R)\mid M\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\in\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle\right\}$$

is a cocomplete suspended subcategory containing $R/\mathfrak{a}[0]$. By [AJS10, Proposition 2.4 (1)], $R/\mathfrak{b}\in\mathsf{aisle}_R\langle R/\mathfrak{a}\rangle$, so $\mathcal{M}$ contains $R/\mathfrak{b}[0]$.
(2) Let $\operatorname{rad}(\mathfrak{a})=\operatorname{rad}(\mathfrak{b})$, as before, the class

$$\mathcal{M}'=\left\{M\in\mathcal{D}(R)\mid M\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\in\mathsf{aisle}_{RQ}\left\langle R/\mathfrak{b}\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}\right\rangle\right\}$$

it is a cocomplete suspended subcategory containing $R/\mathfrak{b}[0]$ and, by [AJS10, Proposition 2.4 (4)], $\mathsf{aisle}_R\langle R/\mathfrak{a}\rangle=\mathsf{aisle}_R\langle R/\mathfrak{b}\rangle$. So, $R/\mathfrak{b}[0]\in\mathcal{M}$ and $R/\mathfrak{a}[0]\in\mathcal{M}'$ and the thesis follows.

(3) The proof follows analogously to (2) since, by [AJS10, Proposition 2.4 (2)], $\mathsf{aisle}_R \langle R/\mathfrak{ab}\rangle = \mathsf{aisle}_R \langle R/\mathfrak{a}, R/\mathfrak{b}\rangle$.
(4) The proof follows analogously to (2) since, by [Hrb20, Lemma 5.3 (iii)], $\mathsf{aisle}_R \langle R/\mathfrak{a}\rangle = \mathsf{aisle}_R \langle K(\mathfrak{a})\rangle$. Moreover, since the complex $K(\mathfrak{a})$ is compact, by Theorem 4.0.6, the functor $K(\mathfrak{a}) \otimes^{\mathbf{L}}_{\mathrm{R}} \_$ has a coproduct preserving right adjoint, and thus it preserves compactness by Theorem 1.0.7. Since, by Remark 4.2.6, any RQ-lattice is compact, the complex $K(\mathfrak{a}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ is compact in $\mathcal{D}(RQ)$. □

Proposition 7.1.5. *The assignments 7.1.1 are well defined and the composite $\varphi \circ \psi$ gives the identity on the poset* $\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q)))$. *In particular, the assignment ψ is injective.*

Proof. Clearly the assignment φ gives a map from $\mathrm{Spec}(R)$ to $\mathrm{Aisle}(\mathbb{K}Q)$ and, by Lemma 7.1.4 (1), it is also a homomorphism of posets. Indeed, for any two prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$, there is a containment

$$\mathsf{aisle}_{\mathbb{K}Q} \left\langle L \mid R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{U} \right\rangle \subseteq \mathsf{aisle}_{\mathbb{K}Q} \left\langle L \mid R/\mathfrak{q} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{U} \right\rangle$$

As for the assignment ψ, by Lemma 7.1.4 (4), we have the equality

$$\mathsf{aisle}_{RQ} \left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \right\rangle = \mathsf{aisle}_{RQ} \left\langle K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \right\rangle$$

and $K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})}$ is compact, yielding a compactly generated aisle of $\mathcal{D}(RQ)$.
Denoting $\varphi \circ \psi(\sigma)$ by σ', for a prime ideal $\mathfrak{p}$ we have that

$$\sigma'(\mathfrak{p}) = \mathsf{aisle}_{\mathbb{K}Q} \left\langle L \mid R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathsf{aisle}_{RQ} \left\langle R/\mathfrak{q} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \mid \mathfrak{q} \in \mathrm{Spec}(R) \right\rangle \right\rangle$$

Obviously, $\sigma(\mathfrak{p}) \subseteq \sigma'(\mathfrak{p})$. Let L be a generator of $\sigma'(\mathfrak{p})$, by Lemma 7.1.4 (4), we have that

$$K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathsf{aisle}_{RQ} \left\langle K(\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \mid \mathfrak{q} \in \mathrm{Spec}(R) \right\rangle$$

By Theorem 1.3.6 (2), $K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ is constructed trough extensions, positive shifts, direct summands and coproducts, from $\left\{ K(\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \mid \mathfrak{q} \in \mathrm{Spec}(R) \right\}$. Since all these operations commute with the base change $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \_$, it holds that

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \left(K(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right) \in \mathsf{aisle}_{\kappa(\mathfrak{p})Q} \left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \left(K(\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \right) \mid \mathfrak{q} \in \mathrm{Spec}(R) \right\rangle$$

then, by Remark 4.1.3, it follows that

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathsf{aisle}_{\kappa(\mathfrak{p})Q} \left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \mid \mathfrak{q} \subseteq \mathfrak{p} \right\rangle$$

Since aisles are independent of the field, we can conclude that $L \in \mathsf{aisle}_{\mathbb{K}Q} \langle \sigma(\mathfrak{q}) \mid \mathfrak{q} \subseteq \mathfrak{p}\rangle$ and so, since σ is a poset homomorphism, L lies in $\sigma(\mathfrak{p})$. □

In what follows we want to get a better understanding of the aisles of $\mathcal{D}(RQ)$ reached by the assignment ψ in 7.1.1. In particular, we will find a special family of compactly generated aisles from which all the other aisles are built from.

Theorem 7.1.6. *For any specialization closed subset V of* $\mathrm{Spec}(R)$ *and aisle $\mathcal{X}$ of* $\mathcal{D}(\mathbb{K}Q)$, *there is a compactly generated t-structure*

$$(\mathcal{A}^{\mathcal{X}}_V, \mathcal{V}^{\mathcal{X}}_V) := \left(\mathsf{aisle}_{RQ} \left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\mathcal{X}} \mid \mathfrak{p} \in V \right\rangle, \left\{ Y \in \mathcal{D}(RQ) \mid \Gamma_V Y \in \widetilde{\mathcal{X}}^{\perp} \right\} \right)$$

Moreover, $\mathcal{A}^{\mathcal{X}}_V = \mathsf{aisle}_{RQ} \left\langle \widetilde{\mathcal{X}} \right\rangle \cap \mathrm{supp}^{-1}_R(V)$ *and the associated truncation functor is* $\tau^{\leq}_{\widetilde{\mathcal{X}}} \circ \Gamma_V$, *where* $\tau^{\leq}_{\widetilde{\mathcal{X}}}$ *is the truncation functor of* $\mathsf{aisle}_{RQ} \left\langle \widetilde{\mathcal{X}} \right\rangle$.

Proof. Let $\mathcal{U}_V^{\mathcal{X}} = \mathsf{aisle}_{RQ}\left\langle \widetilde{\mathcal{X}} \right\rangle \cap \mathrm{supp}_R^{-1}(V)$, then we will prove that

$$\mathcal{A}_V^{\mathcal{X}} = \mathcal{U}_V^{\mathcal{X}} = {}^{\perp}\mathcal{V}_V^{\mathcal{X}} \text{ and } \mathcal{A}_V^{\mathcal{X}\perp} = \mathcal{U}_V^{\mathcal{X}\perp} = \mathcal{V}_V^{\mathcal{X}} \tag{$\diamond$}$$

The proof is divided into three steps:
(i) We start proving that $\mathcal{A}_V^{\mathcal{X}} \subseteq \mathcal{U}_V^{\mathcal{X}}$. Fix a prime ideal $\mathfrak{p}$ in V. By Proposition 4.0.9 (1), we have that

$$R/\mathfrak{p} \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{\mathcal{X}} \subseteq \mathsf{aisle}_{RQ}\left\langle \widetilde{\mathcal{X}} \right\rangle$$

In addition, since $R_{\mathfrak{q}} \otimes_{\mathrm{R}}^{\mathbf{L}} R/\mathfrak{p} \neq 0$ if and only if $\mathfrak{q} \in V(\mathfrak{p})$, for any $\widetilde{L}$ in $\widetilde{\mathcal{X}}$ and $\mathfrak{q} \notin V(\mathfrak{p})$, it follows that

$$\Gamma_{\mathfrak{q}}(R/\mathfrak{p} \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L}) = \Gamma_{V(\mathfrak{q})}\left((R_{\mathfrak{q}} \otimes_{\mathrm{R}}^{\mathbf{L}} R/\mathfrak{p}) \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L}\right) = 0$$

Thus we can conclude that $\mathrm{supp}_R(R/\mathfrak{p} \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L}) \subseteq V(\mathfrak{p})$ which is contained in V.
(ii) We show that $\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\mathcal{U}_V^{\mathcal{X}}, \mathcal{V}_V^{\mathcal{X}}\right) = 0$. Let X and Y be complexes respectively in $\mathcal{U}_V^{\mathcal{X}}$ and $\mathcal{V}_V^{\mathcal{X}}$. If there exists a morphism $f : X \to Y$, then the functor Γ_V induce the commutative square

$$\begin{array}{ccc} \Gamma_V X & \xrightarrow{\gamma_X} & X \\ \downarrow{\scriptstyle \Gamma_V f} & & \downarrow{\scriptstyle f} \\ \Gamma_V Y & \xrightarrow{\gamma_Y} & Y \end{array}$$

By Proposition 3.3.8 (1), the morphism γ_X is an isomorphism, thus $\Gamma_V X$ lies in the aisle generated by $\widetilde{\mathcal{X}}$ and $\Gamma_V f$ is zero, so must be f.
(iii) We prove the inclusion $\mathcal{A}_V^{\mathcal{X}\perp} \subseteq \mathcal{V}_V^{\mathcal{X}}$. Recall the notation of Remark 4.1.4, write $V = \bigcup_{i\in I} V(\mathfrak{p}_i)$ as a union of Zariski closed subsets, let $\mathcal{F}$ be the lattice of finite subsets of I and, for any $F \in \mathcal{F}$, $\mathfrak{a}_F = \prod_{i\in F} \mathfrak{p}_i$. For any $\widetilde{L}$ in $\widetilde{\mathcal{X}}$ and Y in $\mathcal{A}_V^{\mathcal{X}\perp}$, since $\widetilde{L}$ is compact (Remark 4.2.6), by adjunction in Theorem 4.0.6, we have that

$$\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{L}, \Gamma_V Y\right) \cong \varinjlim_{(F,t)\in\mathcal{F}\times\mathbb{N}} \mathrm{Hom}_{\mathcal{D}(RQ)}\left(K\left(\mathfrak{a}_F^{(t)}\right) \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L}, Y\right)$$

Moreover, since the radical ideal of $\mathfrak{a}_F^{(t)}$ is equal to the radical of $\mathfrak{a}_F$, by Lemma 7.1.4 (2)

$$K\left(\mathfrak{a}_F^{(t)}\right) \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L} \in \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{a}_F \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L} \right\rangle = \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{p}_i \otimes_{\mathrm{R}}^{\mathbf{L}} \widetilde{L} \mid i \in F \right\rangle \subseteq \mathcal{A}_V^{\mathcal{X}}$$

thus $\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{L}, \Gamma_V Y\right) = 0$ and $\Gamma_V Y \in \widetilde{X}^{\perp}$.
The first equality of ($\diamond$) follows by (i), (ii) and the inclusion ${}^{\perp}\mathcal{V}_V^{\mathcal{X}} \subseteq \mathcal{A}_V^{\mathcal{X}}$ from (iii); the second equality of ($\diamond$) follows by the inclusion $\mathcal{U}_V^{\mathcal{X}\perp} \subseteq \mathcal{A}_V^{\mathcal{X}\perp}$ from (i), (ii) and (iii).
Let $\tau_{\mathcal{X}}^{<}$ and $\tau_{\mathcal{X}}^{>}$ be the truncation functors of $\mathsf{aisle}_{RQ}\left\langle \widetilde{X} \right\rangle$ and $\widetilde{X}^{\perp}$, respectively. It remains to prove that the truncation functor associated to $\mathcal{A}_V^{\mathcal{X}}$ is the composite $\tau_{\mathcal{X}}^{<} \circ \Gamma_V$. Consider the truncation triangle of Z with respect to the t-structure $(\mathcal{A}_V^{\mathcal{X}}, \mathcal{V}_V^{\mathcal{X}})$:

$$X_Z \xrightarrow{f} Z \xrightarrow{g} Y_Z \xrightarrow{h} X_Z[1]$$

Since $\Gamma_V Y_Z$ is in $\widetilde{\mathcal{X}}^{\perp}$, the morphism $\Gamma_V Z \xrightarrow{\Gamma_V(g)} \Gamma_V Y_Z$ factors trough $\tau_{\mathcal{X}}^{>}(\Gamma_V Z)$, thus we can consider the octahedral diagram

$$\begin{array}{ccccccc} \tau_{\mathcal{X}}^{<}(\Gamma_V Z) & \longrightarrow & \Gamma_V Z & \longrightarrow & \tau_{\mathcal{X}}^{>}(\Gamma_V Z) & \xrightarrow{+} & \\ \downarrow & & \| & & \downarrow & & \\ \Gamma_V X_Z & \longrightarrow & \Gamma_V Z & \xrightarrow{\Gamma_V(g)} & \Gamma_V Y_Z & \xrightarrow{+} & \\ \downarrow & & \downarrow & & \| & & \\ C & \longrightarrow & \tau_{\mathcal{X}}^{>}(\Gamma_V Z) & \longrightarrow & \Gamma_V Y_Z & \xrightarrow{+} & \end{array}$$

According to the octahedral axiom, the first column is a distinguished triangle. So is its rotation

$$C[-1] \longrightarrow \tau^{\leq}_{\mathcal{X}}(\Gamma_V Z) \longrightarrow \Gamma_V X_Z \xrightarrow{+} C$$

Noting that, from the third row of the octahedron, C is in $\widetilde{\mathcal{X}}^{\perp}$ and $\Gamma_V X_Z \cong X_Z$ is in $\mathsf{aisle}_{RQ}\left\langle \widetilde{\mathcal{X}} \right\rangle$, the triangle splits, i.e. $\tau^{\leq}_{\mathcal{X}}(\Gamma_V Z) \cong C[-1] \oplus X_Z$. This forces C to be also in $\mathsf{aisle}_{RQ}\left\langle \widetilde{X} \right\rangle$, thus it is zero and $\tau^{\leq}_{\mathcal{X}}(\Gamma_V Z) \cong X_Z$. □

Given a homomorphism $\sigma : \mathrm{Spec}(R) \to \mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$ its associated aisle is

$$\mathcal{A}_\sigma := \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R) \right\rangle = \mathsf{aisle}_{RQ}\left\langle \bigcup_{\mathrm{Spec}(R)} \mathcal{A}^{\sigma(\mathfrak{p})}_{V(\mathfrak{p})} \right\rangle$$

In particular, the aisles of the form $\mathcal{A}^{\sigma(\mathfrak{p})}_{V(\mathfrak{p})}$ are a special kind of aisles, from which we can construct the ones given by ψ in 7.1.1. Note that, by Theorem 7.1.6, we can compute also the associated (homotopically smashing) coaisle $\mathcal{V}_\sigma := \mathcal{A}_\sigma^{\perp}$, indeed

$$\mathcal{V}_\sigma = \bigcap_{\mathrm{Spec}(R)} \mathcal{V}^{\sigma(\mathfrak{p})}_{V(\mathfrak{p})} = \left\{ Y \in \mathcal{D}(RQ) \mid \Gamma_{V(\mathfrak{p})} Y \in \widetilde{\sigma(\mathfrak{p})}^{\perp} \text{ for any } \mathfrak{p} \in \mathrm{Spec}(R) \right\}$$

Note that in general the intersection of coaisles is not a coaisle, but this holds for homotopically smashing ones since these are defined just by closure operators. Indeed, by Proposition 6.1.2 and the characterization of definability in [LV20, Theorem 4.7], they are exactly the homotopically smashing complete cosuspended subcategories closed under pure subobjects (we refer to *loc. cit.* for the undefined terminology).

A parallel approach that would allow a classification of the compactly generated t-structures is to classify the homotopically smashing ones and then prove that these are the same, i.e. prove that the telescope conjecture holds.

7.1.2 Homotopically smashing t-structures

In this section we will show that, according to the commutative setting (Section 2.2.1), homotopically smashing coaisles are cogenerated by stalk complexes of vertexwise injective RQ-modules. In particular, given a poset homomorphism $\sigma : \mathrm{Spec}(R) \to \mathrm{Aisle}(\mathbb{K}Q)$, a candidate set to cogenerate the coaisle $\mathcal{V}_\sigma$ is

$$\mathcal{E}_\sigma = \left\{ E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})^{\perp}} \mid \mathfrak{p} \in \mathrm{Spec}(R) \right\}$$

The following is a dual version of Lemma 7.1.4 and shows how, for the generation of coaisles, it is equivalent to consider modules of the form $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ or $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$.

Proposition 7.1.7. *For any prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ and indecomposable $\mathbb{K}Q$-complex L, it holds that:*

1. *If $\mathfrak{q} \subseteq \mathfrak{p}$ then* $\mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ}\left\langle E(R/\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle \subseteq \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ}\left\langle E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle$*;*
2. $\mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ}\left\langle E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle = \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ}\left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle$.

Proof. Firstly note that the proofs of [HN21, Lemma 2.6, Lemma 2.8, Lemma 2.9] holds true even for $\mathcal{V}$ homotopically smashing complete cosuspended subcategory of $\mathcal{D}(R)$, without assuming it is a coaisle. In particular, Lemma 2.2.4 (2,3) still hold for such $\mathcal{V}$, i.e. for any $n \in \mathbb{Z}$ we have that

$$E(R/\mathfrak{p})[-n] \in \mathcal{V} \text{ if and only if } \kappa(\mathfrak{p})[-n] \in \mathcal{V}$$

and, for any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, that

$$E(R/\mathfrak{p})[-n] \in \mathcal{V} \text{ implies } E(R/\mathfrak{q})[-n] \in \mathcal{V}$$

(1) Let $\mathfrak{q} \subseteq \mathfrak{p}$, consider the class

$$\mathcal{M} = \left\{ M \in \mathcal{D}(R) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ} \left\langle E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle \right\}$$

it is a homotopically smashing complete cosuspended subcategory of $\mathcal{D}(R)$ containing $E(R/\mathfrak{p})[0]$, thus, by the above discussion, it contains also $E(R/\mathfrak{q})[0]$.
(2) Consider the class

$$\mathcal{M}' = \left\{ M \in \mathcal{D}(R) \mid M \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ} \left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \right\rangle \right\}$$

it is a homotopically smashing complete cosuspended subcategory of $\mathcal{D}(R)$ containing $\kappa(\mathfrak{p})[0]$. By the above discussion, $\kappa(\mathfrak{p})[0] \in \mathcal{M}$ and $E(R/\mathfrak{p})[0] \in \mathcal{M}'$ and the thesis follows. □

Corollary 7.1.8. *Let $\mathcal{V}$ be a homotopically smashing coaisle of $\mathcal{D}(RQ)$, L an indecomposable $\mathbb{K}Q$-complex and $\mathfrak{p} \in \operatorname{Spec}(R)$, then:*

1. *For any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}$ implies $E(R/\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}$;*
2. *$E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}$ if and only if $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}$.*

Now we present a slight variation of Proposition 7.1.2, showing how the hom-orthogonality of two indecomposable $\mathbb{K}Q$-modules M and L is sufficient to detect the hom-orthogonality in $\mathcal{D}(RQ)$ of $\widetilde{M}$ and $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$.

Lemma 7.1.9. *Let M and L be two indecomposable $\mathbb{K}Q$-modules. Then, for any $\mathfrak{p} \in \operatorname{Spec}(R)$, the following holds*

1. $\operatorname{Hom}_{RQ}\left(\widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) \neq 0$ *if and only if* $\operatorname{Hom}_{\mathbb{K}Q}(M, L) \neq 0$;
2. $\operatorname{Ext}^1_{RQ}\left(\widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) \neq 0$ *if and only if* $\operatorname{Ext}^1_{\mathbb{K}Q}(M, L) \neq 0$.

Proof. (1) Suppose that $\operatorname{Hom}_{\mathbb{K}Q}(M, L) \neq 0$. By [CB24, Theorem 4.1 (ii)] applied to the ring homomorphism $\pi : R \to \kappa(\mathfrak{p})$, the natural morphism $\kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_{RQ}\left(\widetilde{M}, \widetilde{L}\right) \to \operatorname{Hom}_{\kappa(\mathfrak{p})Q}(M, L)$ is an isomorphism. Then, if the latter vector space is non-zero, we can find a non-zero homomorphism $f : M \to L$, which is the image of $1_{\kappa(\mathfrak{p})} \otimes \widetilde{f}$, for a non-zero homomorphism $\widetilde{f} : \widetilde{M} \to \widetilde{L}$. In particular, $\widetilde{f}$ should send some non-zero element of $\widetilde{M}$ to an element of $\widetilde{L}$ which is not annihilated by the tensor product with $\kappa(\mathfrak{p})$. Thus, the composite $\widetilde{M} \xrightarrow{\widetilde{f}} \widetilde{L} \xrightarrow{\pi \otimes \mathrm{id}_L} \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ is a non-zero homomorphism.

For the other direction, take a non-zero homomorphism $\overline{f} : \widetilde{M} \to \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$. Note that the embedding $\operatorname{Mod}(\kappa(\mathfrak{p})Q) \hookrightarrow \operatorname{Mod}(RQ)$ can be identified with the action $\operatorname{Hom}_{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p}), \_)$ and so, by Proposition 4.0.3 (2), it is right adjoint to $\kappa(\mathfrak{p}) \otimes_R \_$. By [RSV00, Corollary 3.2], the morphism $\overline{f}$ factors through $\kappa(\mathfrak{p}) \otimes_R \widetilde{M}$ giving a non-zero morphism $f : \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{M} \to \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ in $\operatorname{Mod}(\kappa(\mathfrak{p})Q)$. Thus, by Proposition 7.1.2 (taking $R = \kappa(\mathfrak{p})$), we have that $\operatorname{Hom}_{\mathbb{K}Q}(M, L) \neq 0$.
(2) Suppose that $\operatorname{Ext}^1_{\mathbb{K}Q}(M, L) \neq 0$. By [Rin98, Section 3] (see also [Wei12, Lemma 3.8]), it has a basis whose elements are extensions of the form

$$\xi := \; 0 \longrightarrow L \xrightarrow{\begin{bmatrix} \mathrm{id}_L \\ 0 \end{bmatrix}} E \xrightarrow{\begin{bmatrix} 0 & \mathrm{id}_M \end{bmatrix}} M \longrightarrow 0 \text{ where } E = \left(L_i \oplus M_i, \begin{bmatrix} L_\alpha & \varepsilon_\alpha \\ 0 & M_\alpha \end{bmatrix} \right)$$

with ε_α represented by 0, 1-matrices, as L_α and M_α (see Remark 7.1.1). We can choose all such matrices over a field $\mathbb{K}$ of characteristic 0 to guarantee that all the commutativity relation between them keep holding over an arbitrary ring, and then consider the following extension

$$0 \longrightarrow \kappa(\mathfrak{p})\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L} \xrightarrow{\begin{bmatrix}\mathrm{id}_L\\ 0\end{bmatrix}} \overline{E} \xrightarrow{\begin{bmatrix}0 & \mathrm{id}_{\widetilde{M}}\end{bmatrix}} \widetilde{M} \longrightarrow 0 \text{ where } \overline{E} = \left(\left(\kappa(\mathfrak{p})\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L}_i\right)\oplus\widetilde{M}_i, \begin{bmatrix}L_\alpha & \varepsilon_\alpha\cdot\pi\\ 0 & M_\alpha\end{bmatrix}\right)$$

Note that this sequence is exact, since it is vertexwise split, and it does not split over RQ. Indeed, if it did, there would exist a morphism $f : \widetilde{M} \to \overline{E}$ given vertexwise by matrices $\begin{bmatrix}f_i^\kappa\\ f_i^R\end{bmatrix}$, where f_i^κ and f_i^R have entries in $\kappa(\mathfrak{p})$ and R respectively, and satisfying the relations

$$\begin{bmatrix}L_\alpha & \varepsilon_\alpha\cdot\pi\\ 0 & M_\alpha\end{bmatrix}\begin{bmatrix}f_i^\kappa\\ f_i^R\end{bmatrix} = \begin{bmatrix}f_j^\kappa\\ f_j^R\end{bmatrix} M_\alpha \text{ and } \begin{bmatrix}0 & \mathrm{id}_{\widetilde{M}_i}\end{bmatrix}\begin{bmatrix}f_i^\kappa\\ f_i^R\end{bmatrix} = \mathrm{id}_{\widetilde{M}_i} \text{ for all } i \in Q_0,\ \alpha \in Q_1$$

In particular, f would satisfy the relations $f_i^R = \mathrm{id}_{\widetilde{M}_i}$ and $L_\alpha \circ f_i^\kappa + \epsilon_\alpha = f_j^\kappa \circ M_\alpha$, i.e. the morphism $\kappa(\mathfrak{p}) \otimes f$ would make ξ split, contradicting the hypothesis.

For the other direction, consider a non-trivial extension $0 \to \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \to P \to \widetilde{M} \to 0$. Since $\widetilde{M}$ is vertexwise free, the sequence is vertexwise split. Thus, applying the functor $\kappa(\mathfrak{p}) \otimes_R \_$ to it, we get the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \kappa(\mathfrak{p})\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L} & \xrightarrow{f} & P & \longrightarrow & \widetilde{M} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
0 & \longrightarrow & \kappa(\mathfrak{p})\otimes^{\mathbf{L}}_{\mathrm{R}}\widetilde{L} & \xrightarrow{h} & \kappa(\mathfrak{p})\otimes_R P & \longrightarrow & \kappa(\mathfrak{p})\otimes_R \widetilde{M} & \longrightarrow & 0
\end{array}$$

Note that if the second row were split, there would be a map $e : \kappa(\mathfrak{p}) \otimes_R P \to \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ such that $e \circ \varphi \circ f = e \circ h = \mathrm{id}_L$. In particular, the first row would split too. Thus we get a non-trivial extension in $\mathrm{Ext}^1_{RQ}\left(\widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right)$. □

Corollary 7.1.10. *For any two indecomposable $\mathbb{K}Q$-complexes M and L, and for any $\mathfrak{p} \in \mathrm{Spec}(R)$:*

$$\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) = 0 \text{ if and only if } \mathrm{Hom}_{\mathcal{D}(\mathbb{K}Q)}(M, L) = 0$$

Proof. Analogous to Corollary 7.1.3. □

The next theorem will prove that the modules of the form $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ which are contained in $\mathcal{V}_\sigma$ are exactly the ones in $\mathcal{E}_\sigma$.

Theorem 7.1.11. *Let $\sigma : \mathrm{Spec}(R) \to \mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$ be a homomorphism of posets, for any prime ideal $\mathfrak{p}$ and indecomposable $\mathbb{K}Q$-complex L, then*

$$E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}_\sigma \text{ if and only if } L \in \sigma(\mathfrak{p})^\perp$$

Proof. Recall that $\mathcal{V}_\sigma = \mathcal{A}_\sigma^\perp$ where $\mathcal{A}_\sigma := \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{q} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{q})} \mid \mathfrak{q} \in \mathrm{Spec}(R)\right\rangle$. Let L be an indecomposable $\mathbb{K}Q$-complex. By Lemma 7.1.4 (4) and Corollary 7.1.8 (2), $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{A}_\sigma^\perp$ if and only if

$$\mathrm{Hom}_{\mathcal{D}(RQ)}\left(K(\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) = 0$$

for any $\mathfrak{q} \in \mathrm{Spec}(R)$ and any $M \in \sigma(\mathfrak{q})$. By adjunction (Theorem 4.0.6) and self-duality of the Koszul complex (Remark 4.1.3), this holds if and only if

$$\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M}, \left(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} K(\mathfrak{q})[-n]\right) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) = 0$$

for any $\mathfrak{q} \in \mathrm{Spec}(R)$ and any $M \in \sigma(\mathfrak{q})$. By Remark 4.1.3, this is always true for $\mathfrak{q} \not\subseteq \mathfrak{p}$, while for $\mathfrak{q} \subseteq \mathfrak{p}$ it is equivalent to

$$\bigoplus_{i=0}^{n} \mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M}, \kappa(\mathfrak{p})^{\binom{n}{i}}[i-n] \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) = 0$$

for any $M \in \sigma(\mathfrak{q})$. Since $\sigma(\mathfrak{q})$ is closed under positive shifts, this is equivalent to

$$\mathrm{Hom}_{\mathcal{D}(RQ)}\left(\widetilde{M}, \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}\right) = 0$$

for any $M \in \sigma(\mathfrak{q})$. Thus, by Corollary 7.1.10, we can conclude that $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{A}_{\sigma}^{\perp}$ if and only if $\mathrm{Hom}_{\mathcal{D}(\mathbb{K}Q)}(M, L) = 0$ for any $M \in \sigma(\mathfrak{q})$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Which means, since σ is a poset homomorphism, for any $M \in \sigma(\mathfrak{p})$. So, the thesis follows. □

Theorem 7.1.12. *For any homotopically smashing coaisle $\mathcal{V} \subseteq \mathcal{D}(RQ)$:*

$$\mathcal{V} = \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ} \left\langle E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V} \mid \mathfrak{p} \in \mathrm{Spec}(R), L \in \mathrm{ind}(\mathcal{D}(\mathbb{K}Q))\right\rangle$$

Proof. Denote the right hand side by $\mathcal{W}$. The containment $\mathcal{W} \subseteq \mathcal{V}$ is clear. On the other hand, if $Y \in \mathcal{V}$, by Proposition 4.0.9 (3), for any prime ideal $\mathfrak{p}$ the complex $\Gamma_{\mathfrak{p}} Y$ lies in $\mathcal{V}$ and so does $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \Gamma_{\mathfrak{p}} Y)$. The latter complex is a direct sum of indecomposable $\kappa(\mathfrak{p})Q$-complexes L_i such that $L_i \cong \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L_i}$ and they are all contained in $\mathcal{V}$. Since, by Proposition 7.1.7 (2) and Corollary 7.1.8 (2), $\mathcal{W} = \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ} \left\langle \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V} \mid \mathfrak{p} \in \mathrm{Spec}(R), L \in \mathrm{ind}(\mathcal{D}(\mathbb{K}Q))\right\rangle$, the complex $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), \Gamma_{\mathfrak{p}} Y)$ lies in $\mathcal{W}$. Thus, by minimality (Theorem 5.1.3), $\Gamma_{\mathfrak{p}} Y \in \mathcal{W}$ for any prime ideal $\mathfrak{p}$ and, by local-to-global principle (Theorem 5.1.2), Y lies in $\mathcal{W}$. □

7.2 Telescope conjecture and classification

Thanks to these last results, we can build an assignment which links homotopically smashing coaisles of $\mathcal{D}(RQ)$ to coaisles of $\mathcal{D}(\mathbb{K}Q)$. Note that we will consider the poset $\mathrm{Coaisle}(\mathcal{D}(\mathbb{K}Q))$ with the reverse order, which is actually isomorphic to $\mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$. The assignment is the following:

$$\begin{gathered} \omega : \mathrm{Coaisle}_{\mathrm{hs}}(\mathcal{D}(RQ)) \longrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Coaisle}(\mathcal{D}(\mathbb{K}Q))^{\mathrm{op}}) \\ \mathcal{V} \longmapsto \left(\omega_{\mathcal{V}} : \mathfrak{p} \mapsto \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{\mathbb{K}Q} \left\langle L \in \mathrm{ind}(\mathcal{D}(\mathbb{K}Q)) \mid E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}\right\rangle\right) \end{gathered}$$

Remark 7.2.1. The assignment ω is well-defined and injective.

- By Remark 1.3.7, for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $\omega_{\mathcal{V}}(\mathfrak{p})$ is a (homotopically smashing) coaisle.
- The map $\omega_{\mathcal{V}}$ is a poset homomorphism. Indeed, for any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, by Proposition 7.1.7 (1) we have that

$$\mathsf{cosusp}^{\Pi,\mathbf{hs}}_{\mathbb{K}Q} \left\langle L \mid E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}\right\rangle \subseteq \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{\mathbb{K}Q} \left\langle L \mid E(R/\mathfrak{q}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{V}\right\rangle$$

  i.e. $\omega_{\mathcal{V}}(\mathfrak{q}) \leq \omega_{\mathcal{V}}(\mathfrak{p})$ in $\mathrm{Coaisle}(\mathcal{D}(\mathbb{K}Q))^{\mathrm{op}}$.
- The assignment ω is injective. Suppose that given two homotopically smashing coaisles $\mathcal{V}$ and $\mathcal{V}'$ we have that $\omega_{\mathcal{V}} = \omega_{\mathcal{V}'}$, then by Theorem 7.1.12 it turns out that

$$\mathcal{V} = \mathsf{cosusp}^{\Pi,\mathbf{hs}}_{RQ} \left\langle E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\omega_{\mathcal{V}}(\mathfrak{p})}\right\rangle = \mathcal{V}'$$

Theorem 7.2.2. *Let R be a commutative noetherian ring and Q a Dynkin quiver, then:*

1. *The following assignments are mutually inverse poset isomorphisms*

$$\begin{gathered}\mathrm{Aisle}_{\mathrm{cg}}(\mathcal{D}(RQ)) \longleftrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))) \\ \varphi : \mathcal{U} \longmapsto \left(\mathfrak{p} \mapsto \mathsf{aisle}_{\mathbb{K}Q}\left\langle L \in \mathrm{ind}(\mathcal{D}(\mathbb{K}Q)) \mid R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{U}\right\rangle\right) \\ \mathsf{aisle}_{RQ}\left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R)\right\rangle \longleftarrow\!\shortmid \sigma : \psi\end{gathered}$$

2. *Any homotopically smashing t-structure in $\mathcal{D}(RQ)$ is compactly generated.*

Proof. (1) By Proposition 7.1.5, it suffices to prove that ψ is surjective. Given a compactly generated aisle $\mathcal{A}$ in $\mathcal{D}(RQ)$, the associated coaisle $\mathcal{V} := \mathcal{A}^{\perp}$ is homotopically smashing. From the above discussion we can associate to it the poset homomorphism $\omega_{\mathcal{V}} : \mathrm{Spec}(R) \to \mathrm{Coaisle}(\mathcal{D}(\mathbb{K}Q))^{\mathrm{op}}$. Define $\sigma \in \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q)))$ to be such that $\sigma(\mathfrak{p}) := {}^{\perp}\omega_{\mathcal{V}}(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$, and consider the compactly generated t-structure $(\mathcal{A}_\sigma, \mathcal{V}_\sigma)$, where $\mathcal{A}_\sigma = \psi(\sigma)$. By Theorem 7.1.11, $\omega_{\mathcal{V}_\sigma}(\mathfrak{p}) = \sigma(\mathfrak{p})^{\perp} = \omega_{\mathcal{V}}(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$. Thus the homomorphism $\omega_{\mathcal{V}_\sigma}$ is precisely $\omega_{\mathcal{V}}$, and so, by injectivity of ω, $(\mathcal{A}, \mathcal{V}) = (\mathcal{A}_\sigma, \mathcal{V}_\sigma)$.
(2) Let $(\mathcal{A}, \mathcal{V})$ be a homotopically smashing t-structure. Following the proof of point (1), we conclude that $(\mathcal{A}, \mathcal{V}) = (\mathcal{A}_\sigma, \mathcal{V}_\sigma)$ for some homomorphism σ. Thus, it is compactly generated. □

7.2.1 Wide subcategories

Finally, we want to end the section by proving that, thanks to Theorem 7.1.12, compactly generated aisles are determined on cohomology. This was already known both for hereditary finite dimensional algebras (by hereditary property) and for commutative noetherian rings (by [AJS10]).

Lemma 7.2.3. *For any prime ideal $\mathfrak{p}$ and indecomposable $\mathbb{K}Q$-module L, the injective dimension of $E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$ over RQ is less or equal then 1.*

Proof. We follow [AHK17, Remark 3.7] to prove the lemma. Given a R-module M and a vertex $i \in Q_0$, define as in the reference the RQ-modules $e^i_\lambda(M)$ as

$$e^i_\lambda(M)_k = \bigoplus_{Q(i,k)} M$$

where $Q(i,k)$ denotes the set of paths from i to k. By [AHK17, Proposition 3.4], applied to the R-module $E(R/\mathfrak{p})$, we get that the injective dimension of the RQ-module $e^i_\lambda(E(R/\mathfrak{p}))$ is less or equal than 1 for any $i \in Q_0$. Then, let $X := E(R/\mathfrak{p}) \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L}$, applying [Miy00b, Corollary 1.3] to the exact sequence

$$0 \longrightarrow \bigoplus_{\alpha \in Q_i} e^{t(\alpha)}_\lambda\left(X_{s(\alpha)}\right) \longrightarrow \bigoplus_{i \in Q_0} e^i_\lambda(X_i) \longrightarrow X \longrightarrow 0$$

we have that the injective dimension of X is less or equal than 1. □

Theorem 7.2.4. *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in the derived category of a ring $\mathcal{D}(A)$. If $\mathcal{U} = {}^{\perp}\mathcal{E}$ for a set $\mathcal{E}$ of A-modules of injective dimension at most 1, then $\mathcal{U}$ is determined on cohomology, i.e. for any $X \in \mathcal{D}(RQ)$, it holds that*

$$X \in \mathcal{U} \text{ if and only if } H^i(X)[-i] \in \mathcal{U} \text{ for any } i \in \mathbb{Z}$$

Proof. Let $E \cong E_0 \overset{d}{\to} E_1$ be a complex of injective A-modules concentrated in degrees 0 and 1. This induce the distinguished triangle

$$E_1[-1] \longrightarrow E \longrightarrow E_0[0] \xrightarrow{\ d\ } E_1[0]$$

Given $X \in \mathcal{D}(A)$, $X \in \mathcal{U}$ if and only if $\mathrm{Hom}_{\mathcal{D}(A)}(X, E[i]) = 0$ for any $i \leq 0$ if and only if $H^i(\mathbf{R}\mathrm{Hom}_A(X,E)) = 0$ for any $i \leq 0$. Noting that for $k = 0, 1$ there is an isomorphism $H^i(\mathbf{R}\mathrm{Hom}_A(X,E_k)) \cong \mathrm{Hom}_A(H^{-i}(X), E_k)$ for any $i \leq 0$, the long exact sequence in cohomology of the complex $\mathbf{R}\mathrm{Hom}_A(X,E)$ looks as

$$\ldots \to \mathrm{Hom}_A(H^1(X), E_0) \xrightarrow{\mathrm{Hom}_A(H^1(X),d)} \mathrm{Hom}_A(H^1(X), E_1) \to \mathrm{Hom}_{\mathcal{D}(A)}(X, E) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_A(H^0(X), E_0) \xrightarrow{\mathrm{Hom}_A(H^0(X),d)} \mathrm{Hom}_A(H^0(X), E_1) \to \ldots$$

It follows that $\mathrm{Hom}_{\mathcal{D}(A)}(X, E[i]) = 0$ for any $i \leq 0$ if and only if $\mathrm{Hom}_A(H^0(X), d)$ is injective and $\mathrm{Hom}_A(H^i(X), d)$ is an isomorphism for any $i > 0$. Consider the complex $\overline{X} = \bigoplus_{i\in\mathbb{Z}} H^i(X)[-i]$ and note that these latter conditions are equivalent to $\mathrm{Hom}_{\mathcal{D}(A)}\left(\overline{X}, E[i]\right) = 0$ for any $i \leq 0$, since the long exact sequence in cohomology would be the same. We can conclude that $X \in \mathcal{U}$ if and only if $\overline{X} \in \mathcal{U}$ and so the statement follows. □

Corollary 7.2.5. *Compactly generated aisles of $\mathcal{D}(RQ)$ are determined on cohomology.*

Proof. Since any compactly generated t-structure of $\mathcal{D}(RQ)$ is homotopically smashing, by Theorem 7.1.12 and Lemma 7.2.3, the aisle is left orthogonal to set of RQ-modules of injective dimension at most 1. Then Theorem 7.2.4 concludes the proof. □

As an application of the previous theorem we can provide a classification of the *wide subcategories* of $\mathrm{mod}(RQ)$, the category of finitely generated modules over RQ. This classification partially merges the one over commutative noetherian rings due to Takahashi in [Tak08] and the one over representation-finite hereditary algebras due to Ingalls and Thomas in [IT09]. This classification holds for rings RQ with R commutative noetherian regular and Q Dynkin.

Definition 7.2.6. Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{W} \subseteq \mathcal{A}$ is a *wide subcategory* if it is closed under kernels, cokernels and extensions.

Theorem 7.2.7. *Let R be a commutative noetherian regular ring and Q a Dynkin quiver. Then, the following maps form a bijection*

$$\mathrm{Wide}(RQ) \longleftrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Wide}(\mathbb{K}Q))$$
$$\mathcal{W} \longmapsto \left(\mathfrak{p} \mapsto \mathsf{wide}_{\mathbb{K}Q}\left\langle L \in \mathrm{ind}(\mathbb{K}Q) \mid R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{W}\right\rangle\right)$$
$$\mathsf{wide}_{RQ}\left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R)\right\rangle \longleftarrow\!\shortmid\ \sigma$$

Proof. By regularity of R, we get that $\mathcal{D}^c(RQ) = \mathcal{D}^b(\mathrm{mod}(RQ))$ (see [CFH24, 20.2.11]) and, by [Nee92b, Lemma 2.2], every thick subcategory $\mathcal{T} \subseteq \mathcal{D}^c(RQ)$ is equal to $\mathsf{loc}_{RQ}\langle\mathcal{T}\rangle \cap \mathcal{D}^c(RQ)$ which is determined on cohomology by Corollary 7.2.5. Thus, by [ZC17, Theorem 2.5], there is a bijection

$$\mathrm{Wide}(RQ) \longleftrightarrow \mathrm{Thick}(\mathcal{D}^c(RQ))$$
$$\mathcal{W} \longmapsto \left\{X \in \mathcal{D}^c(\mathbb{K}Q) \mid H^i(X) \in \mathcal{W} \text{ for any } i \in \mathbb{Z}\right\} \qquad (\dagger)$$
$$\left\{H^0(X) \mid X \in \mathcal{T}\right\} \longleftarrow\!\shortmid\ \mathcal{T}$$

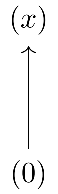
Note that, by virtue of the telescope conjecture and the classification in Theorem 7.2.2, the lattice $\mathrm{Thick}(\mathcal{D}^c(RQ))$ is in bijection with $\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Thick}(\mathcal{D}^c(\mathbb{K}Q)))$. Thus, specifying the above to $R = \mathbb{K}$ (see also [Brü07, Theorem 5.1]) and composing all the maps we obtain the bijection

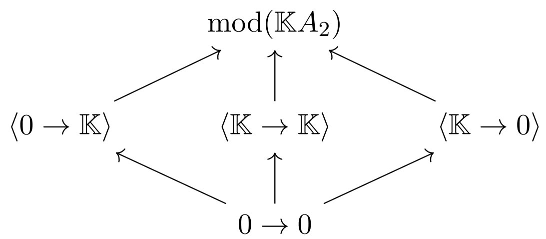
$$\begin{aligned} \mathrm{Wide}(RQ) &\longleftrightarrow \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Wide}(\mathbb{K}Q)) \\ \mathcal{W} &\longmapsto \left(\mathfrak{p} \mapsto \left\{H^0(X) \mid X \in \mathcal{T}_{\mathcal{W}}\right\}\right) \\ \left\{H^0(X) \mid X \in \mathcal{T}_\sigma\right\} &\longleftarrow\!\shortmid \sigma \end{aligned}$$

where $\mathcal{T}_{\mathcal{W}} = \mathsf{thick}_{\mathbb{K}Q} \left\langle L \mid R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{L} \in \mathcal{W} \right\rangle$ and $\mathcal{T}_\sigma = \mathsf{thick}_{RQ} \left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})} \mid \mathfrak{p} \in \mathrm{Spec}(R) \right\rangle$. Since both $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_\sigma$ are cohomology determined and generated by compacts, by [Hrb20, Lemma 5.1], for any complex X in there, $H^0(X)$ is a direct summand of an n-fold extension of finite direct sums of shifts of its generators. Thus, by the bijection (†), the assignments coincide with the ones in the statement. □

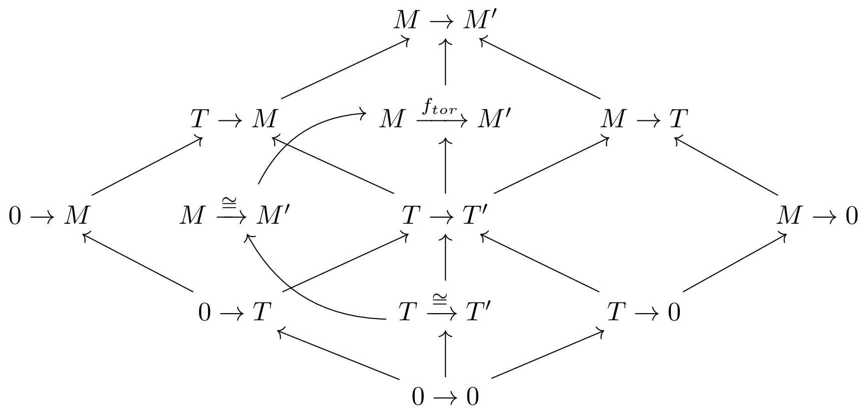
Note that, by [AS25, Proposition 2.3, Theorem 4.6] and Theorem 7.2.7, all the wide subcategories of the category $\mathrm{mod}\, RQ$ will be of the form $\alpha\left(\mathsf{filtgen}\left\langle R/\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} \widetilde{\sigma(\mathfrak{p})}\right\rangle\right)$ for some homomorphism $\sigma : \mathrm{Spec}(R) \to \mathrm{Wide}(\mathbb{K}Q)$ (where $\mathsf{filtgen}$ denotes the closure under quotients and extensions, while for the definition of the map α we refer to Remark 7.A.2).

Example 7.2.8. We can compute the lattice of wide subcategories of the path algebra $\mathbb{C}[[x]]A_2$. Recall that

$\mathrm{Spec}\,\mathbb{C}[[x]] =$ 

and $\mathrm{Wide}(\mathbb{K}A_2) =$ 

Then, denoting by $M \to M'$ a general $\mathbb{C}[[x]]A_2$-module and by T and T' the torsion $\mathbb{C}[[x]]$-modules, the lattice $\mathrm{Wide}(\mathbb{C}[[x]]A_2)$ is



where f_{tor} is a $\mathbb{C}[[x]]$-linear homomorphism with torsion kernel and cokernel and in every vertex we consider the set of all the representations of that kind. For example, taking the homomorphism $\sigma \in \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(\mathbb{C}[[x]]), \mathrm{Wide}(\mathbb{K}Q))$ such that $\sigma((0)) = \langle \mathbb{K} \to \mathbb{K} \rangle$ and $\sigma((x)) = \mathrm{mod}(\mathbb{K}A_2)$, one can check that $\alpha\left(\mathsf{filtgen}\left\langle \mathbb{C}[[x]] \to \mathbb{C}[[x]], \mathrm{mod}(\mathbb{C}A_2)\right\rangle\right) = \left\{M \xrightarrow{f_{tor}} M' \mid M, M' \in \mathrm{mod}(\mathbb{C}[[x]])\right\}$.

7.A t-Structures of Dynkin algebras

This final section is devoted to the classification of the aisles of $\mathcal{D}(\mathbb{K}Q)$, for any field $\mathbb{K}$ and Dynkin quiver Q. We show that the lattice $\mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$ is isomorphic to the lattice $\mathrm{Filt}(\mathbf{Nc}(Q))$ of filtrations (i.e. non-increasing sequences) of noncrossing partitions of Q, and so that it is independent of the field. In particular, this result specifies the classification in [SR19] to algebras of finite representation type.

Lemma 7.A.1. *Let $\mathcal{A}$ be an hereditary abelian category and $\mathcal{S} \subseteq \mathcal{D}(\mathcal{A})$ a suspended subcategory. Then, for any map $f \in \mathrm{Hom}_{\mathcal{A}}(H^n(\mathcal{S}), H^{n+1}(\mathcal{S}))$, it holds that $\ker f \in H^n(\mathcal{S})$ and $\operatorname{coker} f \in H^{n+1}(\mathcal{S})$. In particular,*

$$\ldots \supseteq H^n(\mathcal{S}) \supseteq H^{n+1}(\mathcal{S}) \supseteq \ldots$$

is a filtration of subcategories closed under images, cokernels and extensions (ICE-closed).

Proof. Let $f : A \to B$ with $A \in H^n(\mathcal{S})$ and $B \in H^{n+1}(\mathcal{S})$, then there is a triangle

$$B[-n-1] \longrightarrow \operatorname{cone} f[-n-1] \longrightarrow A[-n] \xrightarrow{+}$$

where $\operatorname{cone} f[-n-1] \cong \ker f[-n] \oplus \operatorname{coker} f[-n-1]$. Since $\mathcal{S}$ is closed under extensions and summands, $\operatorname{cone} f[-n-1] \in \mathcal{S}$ and we get the statement. As for the last part, let $f : C \to D$ in $\mathrm{Hom}(H^n(\mathcal{S}), H^n(\mathcal{S}))$. Since C lies also in $H^{n-1}(\mathcal{S})$, we have that $\operatorname{coker} f \in H^n(\mathcal{S})$ and $\ker f \in H^{n-1}(\mathcal{S})$, so $\operatorname{im} f \cong \operatorname{coker}(\ker f \hookrightarrow C) \in H^n(\mathcal{S})$. The closure under extensions of $H^n(\mathcal{S})$ follows from the one in $\mathcal{S}$. □

Remark 7.A.2. We want to note that some results from [IT09] hold also for ICE-closed subcategories (of artinian, Krull-Schmidt, hereditary abelian categories, e.g. $\mathrm{mod}(\mathbb{K}Q)$). We will call an ICE-closed subcategory $\mathcal{H}$ finitely generated if it contains a finite set of indecomposable objects $\mathcal{I}$, such that $\mathcal{H} \subseteq \mathsf{gen}(\mathcal{I})$, where gen denotes the closure under quotients and finite direct sums. Note that it is always the case in $\mathrm{mod}(\mathbb{K}Q)$ when Q is Dynkin. In particular, the following holds:

- [IT09, Theorem 2.8] A finitely generated ICE-closed subcategory $\mathcal{H}$ has a minimal generator, unique up to isomorphism, which is the direct sum of all its indecomposable split projectives;
- [IT09, Proposition 2.12] For any ICE-closed subcategory $\mathcal{H}$, the class

  $$\alpha(\mathcal{H}) = \{M \in \mathcal{H} \mid \text{for all } f \in \mathrm{Hom}(\mathcal{H}, M),\ \ker f \in \mathcal{H}\}$$

  is a wide subcategory. (See [Sak23, Proposition 2.2] for a proof);
- [IT09, Theorem 2.15] For any finitely generated ICE-closed subcategory $\mathcal{H}$, let

  $$\alpha_s(\mathcal{H}) = \left\{ \begin{aligned} M \in \mathcal{H} \mid \text{for all surjection } f : Z \to M \text{ with} \\ Z \in \mathcal{H} \text{ split projective, } \ker f \in \mathcal{H} \end{aligned} \right\}$$

  Then, $\alpha(\mathcal{H}) = \alpha_s(\mathcal{H})$.

  To make the proof work for $\mathcal{H}$ note that, at the end of their proof, $\ker g'$ is not just a quotient of $\ker g'h$ but, since h is surjective, it is the image of $h_{|\ker g'h} : \ker g'h \to Y'$. Since both $\ker g'h$ and Y' are in $\mathcal{H}$, so does $\ker g'$;
- [IT09, Theorem 2.16] For any finitely generated ICE-closed subcategory $\mathcal{H}$, any split projective U of $\mathcal{H}$ is in $\alpha_s(\mathcal{H}) = \alpha(\mathcal{H})$. In particular, $\mathcal{H} \subseteq \mathsf{gen}(\bigoplus U_i) \subseteq \mathsf{gen}(\alpha(\mathcal{H}))$.

Theorem 7.A.3. *For a field $\mathbb{K}$ and Dynkin quiver Q, the following assignments form a bijection*

$$\begin{aligned} \mathrm{Susp}(\mathcal{D}^c(\mathbb{K}Q)) &\longleftrightarrow \mathrm{Filt}(\mathrm{Wide}(\mathbb{K}Q)) \\ \mathcal{S} &\longmapsto \left(\ldots \supseteq \alpha(H^n(\mathcal{S})) \supseteq \alpha(H^{n+1}(\mathcal{S})) \supseteq \ldots\right) \\ \{X \in \mathcal{D}^c(\mathbb{K}Q) \mid H^n(X) \in \mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}\} &\longleftarrow (\ldots \supseteq \mathcal{W}_n \supseteq \mathcal{W}_{n+1} \supseteq \ldots) \end{aligned}$$

Proof. The first assignment is well defined. By Remark 7.A.2, for any $n \in \mathbb{Z}$ the class $\alpha(H^n(\mathcal{S}))$ is a wide subcategory and, by Lemma 7.A.1, $H^{n+1}(\mathcal{S}) \subseteq \alpha(H^n(\mathcal{S}))$, thus $\alpha(H^{n+1}(\mathcal{S})) \subseteq \alpha(H^n(\mathcal{S}))$. *The second assignment is well defined.* Denote by $\mathcal{S}_{\mathcal{W}}$ the image of the assignment. It is clearly closed under summands and positive shifts, so it remains to show that it is closed under extensions. Let $X, Y \in \mathcal{S}_{\mathcal{W}}$ and consider a distinguished triangle

$$X \longrightarrow Z \longrightarrow Y \xrightarrow{+}$$

It induces the exact sequence in cohomology

$$H^{n-1}(Y) \xrightarrow{f} H^n(X) \longrightarrow H^n(Z) \longrightarrow H^n(Y) \xrightarrow{g} H^{n+1}(X)$$

from which we obtain the short exact sequence

$$0 \longrightarrow \operatorname{coker} f \longrightarrow H^n(Z) \longrightarrow \ker g \longrightarrow 0$$

Since $H^n(X)$ lies in $\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$, on one hand, $\operatorname{coker} f$ is in $\mathsf{gen}(\mathcal{W}_n)$ as well, on the other hand, there exist $G \in \mathcal{W}_{n-1}$ and a surjection $\pi : G \to H^{n-1}(Y)$ such that $f \circ \pi \in \mathrm{Hom}(\mathcal{W}_{n-1}, \mathcal{W}_{n-1})$ and $\operatorname{coker} f = \operatorname{coker}(f \circ \pi_{n-1})$ is in $\mathcal{W}_{n-1}$. Moreover, we will prove later that $\mathcal{W}_n \subseteq \alpha(\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1})$ thus $\ker g$ lies in $\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$. Since, by [IT09, Proposition 2.13], $\mathsf{gen}(\mathcal{W}_n)$ is closed under extensions, it follows that $H^n(Z)$ lies in $\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$.
The identity on the left. Let $\mathcal{S} \subseteq \mathcal{D}^c(\mathbb{K}Q)$ be a suspended subcategory and

$$\mathcal{S}_\alpha := \left\{X \in \mathcal{D}^c(\mathbb{K}Q) \mid H^n(X) \in \mathsf{gen}(\alpha(H^n(\mathcal{S}))) \cap \alpha(H^{n-1}(\mathcal{S}))\right\}$$

We will show that $\mathcal{S} = \mathcal{S}_\alpha$. For any $n \in \mathbb{Z}$ we have that, by Remark 7.A.2, $H^n(\mathcal{S}) \subseteq \mathsf{gen}(\alpha(H^n(\mathcal{S})))$ and, by Lemma 7.A.1, $H^n(\mathcal{S}) \subseteq \alpha(H^{n-1}(\mathcal{S}))$. Thus, $\mathcal{S} \subseteq \mathcal{S}_\alpha$. For the other inclusion, let $X \in \mathcal{S}_\alpha$, then for any $n \in \mathbb{Z}$ there is $G_n \in \alpha(H^n(\mathcal{S}))$ and a surjective map $\pi : G_n \twoheadrightarrow H^n(X)$. In particular, by definition of α, $\ker \pi \in H^n(\mathcal{S}) \subseteq \alpha(H^{n-1}(\mathcal{S}))$. Consider the distinguished triangle

$$G_n[-n] \xrightarrow{\pi[-n]} H^n(X)[-n] \longrightarrow \ker \pi[-n+1] \xrightarrow{+}$$

Since $\mathcal{S}$ is closed under extensions, $H^n(X)[-n]$ lies in $\mathcal{S}$ for any $n \in \mathbb{Z}$, so $X \cong \bigoplus H^n(X)[-n]$ does.
The identity on the right. It suffices to prove that for any filtration of wide subcategories

$$\ldots \supseteq \mathcal{W}_n \supseteq \mathcal{W}_{n+1} \supseteq \ldots$$

we have the equality $\mathcal{W}_n = \alpha\left(\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}\right)$. For the inclusion from left to right, we have that $\mathcal{W}_n \subseteq \mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$ and for any $W \in \mathcal{W}_n$ and $f : A \to W$ with $A \in \mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$, there is $G \in \mathcal{W}_n$ and a surjection $\pi : G \to A$ such that, by surjectivity of π, $\ker f = \operatorname{im}\left(\ker(f \circ \pi) \xrightarrow{\pi} A\right)$. In particular, $\ker f$ is both a quotient of $\ker(f \circ \pi)$, so it is in $\mathsf{gen}(\mathcal{W}_n)$, and the image of a morphism in $\mathcal{W}_{n-1}$, so it is in $\mathcal{W}_{n-1}$. For the other inclusion, let $M \in \alpha\left(\mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}\right)$, in particular there are $G' \in \mathcal{W}_n$ and a surjection $\pi' : G' \to M$ such that, by definition of α, $\ker \pi' \in \mathsf{gen}(\mathcal{W}_n) \cap \mathcal{W}_{n-1}$. Thus, there exist $G'' \in \mathcal{W}_n$ and a surjection $\pi'' : G'' \to \ker \pi'$ such that $M \cong \operatorname{coker}\left(\ker \pi' \xrightarrow{i} G'\right) \cong \operatorname{coker}\left(G'' \xrightarrow{i \circ \pi''} G'\right)$ and so it lies in $\mathcal{W}_n$. □

Corollary 7.A.4. *For any field* $\mathbb{K}$ *and Dynkin quiver* Q*, there is a bijection*

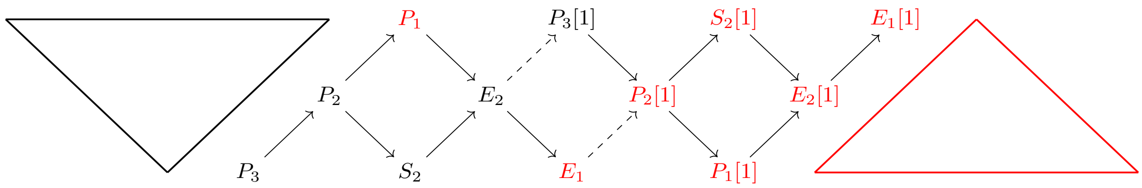
$$\mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q)) \longleftrightarrow \mathrm{Filt}(\mathbf{Nc}(Q))$$

Proof. Note that by [ŠP16, Theorem 4.5 (i)], there is a bijection between $\mathrm{Aisle}(\mathcal{D}(\mathbb{K}Q))$ and $\mathrm{Susp}(\mathcal{D}^c(\mathbb{K}Q))$. Then, by Theorem 7.A.3, this further specifies to a bijection with $\mathrm{Filt}(\mathrm{Wide}(\mathbb{K}Q))$. Finally, by [IT09, Theorem 1.1], the lattice of wide subcategories of $\mathrm{mod}(\mathbb{K}Q)$ is isomorphic to the lattice of noncrossing partitions of Q, thus we get the result. □

Example 7.A.5. The aisle of $\mathcal{D}(\mathbb{K}A_3)$ corresponding to the filtration

$$\mathcal{W}_1 = 0 \subseteq \mathsf{add}(P_1) \subseteq \mathsf{add}(P_2, P_1, E_1) \subseteq \mathrm{mod}(\mathbb{K}A_3) = \mathcal{W}_{-2}$$

is the one generated by the modules in red in the figure below



Chapter 8

The tt-Category of Representations of Quivers

Given a commutative noetherian ring R and a finite acyclic quiver Q, we study the tensor triangulated category $\mathcal{D}(RQ)$ endowed with the vertexwise tensor product. We find a description of the internal hom functor and show that the category is not rigid. We compute its Balmer spectrum and, despite the non-rigidity, we get a classification of all the thick tensor-ideals and a stratification result. After introducing the notion of tensor-t-structure, we give a classification of the compactly generated ones and prove the tensor telescope conjecture.

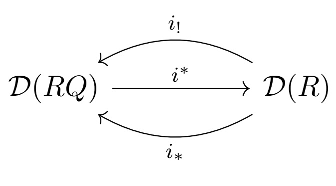
The material in this chapter is part of the author's work [Sab25b].

8.0 Derivators

Before starting this chapter, we introduce some tools from the theory of derivators, which will be extensively used later. Despite this theory being very general, we will stick to our context as much is possible. Our main reference is [Gro13] and we refer to it for any unexplained terminology. By [Gro13, Proposition 1.30, Example 4.2], the assignment

$$\mathbb{D}_R : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT} \text{ such that } \mathbb{D}_R(Q) := \mathcal{D}(RQ) \text{ for any finite quiver } Q$$

forms a strong and stable derivator. In particular, by axiom (Der3) in [Gro13, Definition 1.5], for any vertex $i \in Q_0$ there is an adjoint triple $i_! \dashv i^* \dashv i_*$ as follows



where, by [Gro13, Corollary 4.19], all the functors are triangulated and i^* is the *evaluation functor* at i, induced by precomposition with the injection $\{i\} \hookrightarrow Q$. Note that, since coproducts in $\mathcal{D}(RQ)$ are computed vertexwise and bounded complexes of finitely generated projective RQ-modules are vertexwise bounded complexes of finitely generated projective R-modules, the functor i^* preserves both coproducts and compact objects, thus, by Theorem 1.0.7, $i_!$ preserves compacts and i_* preserves coproducts. Moreover, $i_!$, i^* and i_* (by Theorem 1.0.7) are all left adjoint functors, so they preserve directed homotopy colimits (see [Gro13, Proposition 2.4, Corollary 2.12]).

Remark 8.0.1. By axiom (Der4) in [Gro13, Definition 1.5] (see also [KN13, Lemma 11.1]), for any complex $M \in \mathcal{D}(R)$ and vertices $i, j \in Q_0$ we have that

$$j^* i_! M \cong \coprod_{Q(i,j)} M \text{ and } j^* i_* M \cong \prod_{Q(j,i)} M$$

In particular, if Q is an acyclic quiver, $i^* i_! M \cong M$ and $i^* i_* M \cong M$.

Let Q be a finite and acyclic quiver and, for any vertex $i \in Q_0$, define the RQ-module $U(i)$ such that $U(i)_k$ is equal to R if $i = k$ and 0 otherwise (so $U(i)_\alpha$ is the zero map for any α). For a complex $X \in \mathcal{D}(RQ)$, denote by $X(i)$ the tensor product $U(i) \boxtimes_{RQ} X$ and by $i_\times$ the functor $U(i) \boxtimes_{RQ} i_! : \mathcal{D}(R) \to \mathcal{D}(RQ)$. Notice that, by Proposition 5.2.2, $U(i) \boxtimes_{RQ} \_$ preserves compact objects and so $i_\times$ does.

Lemma 8.0.2. *For any $X, Y \in \mathcal{D}(RQ)$ and $M, N \in \mathcal{D}(R)$ the following holds*

$$i_\times i^* X \cong X(i) \text{ and } j^* i_\times M \cong \begin{cases} M & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

$$i^*(X \boxtimes_{RQ} Y) \cong i^* X \otimes_R i^* Y \text{ and } i_\times(M \otimes_R N) \cong i_\times M \boxtimes_{RQ} i_\times N$$

In particular, we have that $i_\times(i^* X \otimes_R i^* Y) \cong X(i) \boxtimes_{RQ} Y(i)$.

Proof. The first and the second equation easily follow from the definitions.

As for the third one, note that the equality holds in the abelian setting, thus the left derived functors of the compositions $i^*(\_ \boxtimes_{RQ} \_)$ and $i^* \_ \otimes_R i^* \_$ coincide. By the dual of [Mil, Theorem 1.6.1], since i^* is exact, the left derived functor of $i^*(\_ \boxtimes_{RQ} \_)$ is the composition of the left derived functors of i^* and $\boxtimes_{RQ}$. Moreover, since the adjunction $i^* \dashv i_*$ exists also between the homotopy categories ([Mil, Proposition 1.1.3]) and i_* sends acyclic R-complexes to acyclic RQ-complexes, it follows that i^* sends K-projective RQ-complexes to K-projective R-complexes and thus the dual of [Mil, Theorem 1.6.1] also applies to the left derived functor of $i^* \_ \otimes_R i^* \_$.

The fourth equation follows from the fact that $i^* X \cong i^* Y$ implies $X(i) \cong Y(i)$ and the chain of quasi-isomorphisms $i^* i_!(M \otimes_R N) \cong i^* i_! M \otimes_R i^* i_! N \cong i^*(i_! M \boxtimes_R i_! N)$. □

8.1 Balmer spectrum and stratification

Recall from Section 3.1 that the *Balmer spectrum* $\mathrm{Spc}(\mathcal{T})$ of an essentially small tensor triangulated category $\mathcal{T}$ is the set of the prime thick tensor-ideals of $\mathcal{T}$. It has both a poset structure, given by inclusion, and a topological structure, given by the basis of open subsets $\{\mathcal{O}(X) := \{\mathcal{S} \in \mathrm{Spc}(\mathcal{T}) \mid X \in \mathcal{S}\} \mid X \in \mathcal{T}\}$. In this context, a *specialization closed subset* of $\mathrm{Spc}(\mathcal{T})$ is equivalently a lower set (i.e. a subset equal to its downward closure) according to the poset structure, or an arbitrary union of closed subsets for the topological structure. By Theorem 3.1.7, when $\mathrm{Spc}(\mathcal{T})$ is a noetherian topological space, the specialization closed subsets of the Balmer spectrum fully classify the radical thick tensor-ideals of $\mathcal{T}$.

Recall from Section 5.2 that the derived category $\mathcal{D}(RQ)$ can be endowed with the structure of a (non-rigidly) compactly generated tt-category $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ via vertexwise tensor product. In particular, $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$ is a (non-rigid) essentially small tt-category. Finally, recall that the *big support* of a complex $M \in \mathcal{D}(R)$ is defined as

$$\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{R}} M \neq 0\}$$

and we denote by $K(\mathfrak{p})$ the Koszul complex at $\mathfrak{p}$.

Notation 8.1.1. Given a set of objects $\mathcal{X} \subseteq \mathcal{D}(RQ)$, we denote by $\langle \mathcal{X} \rangle$ the smallest thick tensor-ideal of $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ containing $\mathcal{X}$.

The first part of next proposition follows similarly to [LS13, Lemma 2.1.4.1].

Proposition 8.1.2. *For any finite and acyclic quiver Q and $X \in \mathcal{D}(RQ)$, it holds that*

1. $\langle X \rangle = \mathsf{thick}_{RQ} \langle X(i) \mid i \in Q_0 \rangle$;
2. *If X is compact,* $\langle X \rangle = \mathsf{thick}_{RQ} \langle i_\times K(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(i^*X), i \in Q_0 \rangle$.

In particular, any thick tensor-ideal of $\mathcal{D}^c(RQ)$ is generated by complexes of the form $i_\times K(\mathfrak{p})$ for some $i \in Q_0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. (1) Since any finite acyclic quiver has a source and a sink, the RQ-module U admits a filtration whose composition factors are isomorphic to the representations $U(i)$. Tensoring this filtration by a complex $X \in \mathcal{D}(RQ)$ it follows that $X \in \mathsf{thick}_{RQ} \langle X(i) \mid i \in Q_0 \rangle$. Moreover, the latter is a tensor-ideal. Indeed, since thick subcategories of $\mathcal{D}(R)$ are tensor-ideals, for any object $Z \in \mathcal{D}(RQ)$, $i^*Z \otimes_R i^*X$ lies in the thick subcategory generated by i^*X. Thus, since $Z \boxtimes_{RQ} X(i) = Z(i) \boxtimes_{RQ} X(i)$, by Lemma 8.0.2, $Z \boxtimes_{RQ} X(i) = i_\times(i^*Z \otimes_R i^*X)$ which lies in the thick subcategory generated by $i_\times i^*X$, i.e. $\mathsf{thick}_{RQ} \langle X(i) \rangle$. On the other hand, by the closure under tensor product, we have $X(i) = U(i) \boxtimes_{RQ} X \in \langle X \rangle$ for any $i \in Q_0$.
(2) If $X \in \mathcal{D}^c(RQ)$, by Theorem 2.1.4, we have that the thick subcategory $\mathsf{thick}_R\langle i^*X \rangle$ is equal to $\mathsf{thick}_R \langle K(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(i^*X) \rangle$. Then, applying the functor $i_\times$, by Lemma 8.0.2, we can conclude that

$$\mathsf{thick}_{RQ}\langle X(i) \rangle = \mathsf{thick}_{RQ} \langle i_\times K(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(i^*X) \rangle$$

Thus, combining this with point (1) gives the statement.
As for the last part, it is sufficient to note that, given a set of objects $\mathcal{X} \subseteq \mathcal{D}^c(RQ)$, the thick tensor-ideal $\langle \mathcal{X} \rangle$ is equal to $\mathsf{thick}_{RQ} \langle i_\times K(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(i^*X), X \in \mathcal{X}, i \in Q_0 \rangle$. □

For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ and $i \in Q_0$, define the functor $\xi_{\mathfrak{p},i} := R_\mathfrak{p} \otimes_R i^*(\_)$ such that

$$\xi_{\mathfrak{p},i} : \mathcal{D}^c(RQ) \to \mathcal{D}^c(R_\mathfrak{p})$$

$$X \mapsto i^*X_\mathfrak{p}$$

Denote by $\mathcal{S}_{\mathfrak{p},i} := \ker \xi_{\mathfrak{p},i} = \{X \in \mathcal{D}^c(RQ) \mid i^*X_\mathfrak{p} = 0\}$. Note that, since the functor $\xi_{\mathfrak{p},i}$ is exact and commutes with $\boxtimes_{RQ}$, each subcategory $\mathcal{S}_{\mathfrak{p},i}$ is a thick tensor-ideal. In particular, by Proposition 8.1.2 (2):

$$\mathcal{S}_{\mathfrak{p},i} = \mathsf{thick}_{RQ} \langle \{i_\times K(\mathfrak{q}) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\} \cup \{j_\times K(\mathfrak{q}) \mid j \neq i, \mathfrak{q} \in \mathrm{Spec}(R)\} \rangle$$

In [LS13], the authors prove that for any field $\mathbb{K}$, the Balmer spectrum of $(\mathcal{D}(\mathbb{K}Q), \boxtimes_{\mathbb{K}Q})$ is a discrete space consisting of as many points as the vertices of the quiver Q. In the following theorem, we generalize [LS13, Theorem 2.1.5.1] to any commutative noetherian ring, showing that in our context the Balmer spectrum is a disjoint union of as many copies of the prime spectrum $\mathrm{Spec}(R)$ as the number of vertices of Q. Moreover, the following theorem also provides a direct proof, when $Q = A_1$, of the homeomorphism between the Balmer spectrum of $\mathcal{D}^c(R)$ and the prime spectrum of the ring $\mathrm{Spec}(R)$, which originally follows by Theorem 3.1.8.

Theorem 8.1.3. *For any commutative noetherian ring R and finite acyclic quiver Q, we have that*

1. $\mathrm{Spc}(\mathcal{D}^c(RQ)) = \{\mathcal{S}_{\mathfrak{p},i} \mid \mathfrak{p} \in \mathrm{Spec}(R), i \in Q_0\}$;
2. *There is an order-reversing homeomorphism*

$$f : \mathrm{Spc}(\mathcal{D}^c(RQ)) \longrightarrow \mathrm{Spec}(R) \times Q_0$$

i.e. $\mathrm{Spc}(\mathcal{D}^c(RQ)) \overset{f}{\cong}_{\mathrm{Top}} \mathrm{Spec}(R) \times Q_0$ *and* $\mathrm{Spc}(\mathcal{D}^c(RQ)) \overset{f}{\cong}_{\mathrm{Pos}} \mathrm{Spec}(R)^{\mathrm{op}} \times Q_0$.

Proof. (1) Let us start by checking that each thick tensor-ideal $\mathcal{S}_{\mathfrak{p},i}$ is prime by showing that their complements are $\boxtimes$-closed. Given two complexes $X, Y \in \mathcal{D}^c(RQ) \setminus \mathcal{S}_{\mathfrak{p},i}$, since $\mathcal{S}_{\mathfrak{p},i} = \ker \xi_{\mathfrak{p},i}$, the complexes $i^*X_{\mathfrak{p}}$ and $i^*Y_{\mathfrak{p}}$ are non-zero, bounded and vertexwise finitely generated $R_{\mathfrak{p}}$-free. In particular, by the so-called Künneth formula (see [CFH24, 2.5.18 (c)]), the top cohomology of $i^*X_{\mathfrak{p}} \otimes_R i^*Y_{\mathfrak{p}}$ is the tensor product of the top cohomologies of $i^*X_{\mathfrak{p}}$ and $i^*Y_{\mathfrak{p}}$ and so, by the Nakayama's lemma (see [AM69, Exercise 2.3]) it is non-zero. Thus, by Lemma 8.0.2, $i^*(X \boxtimes_{RQ} Y)_{\mathfrak{p}} \neq 0$, i.e. $X \boxtimes_{RQ} Y$ does not lie in $\mathcal{S}_{\mathfrak{p},i}$. Now let us prove that all thick tensor-ideal are of this form. Let $\mathcal{S}$ be a prime thick tensor-ideal of $\mathcal{D}^c(RQ)$, by Proposition 8.1.2 (2) it is generated by a proper subset of $\mathcal{K} = \{i_\times K(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R), i \in Q_0\}$. Let $\mathcal{S}^{\complement} = \mathcal{D}^c(RQ) \setminus \mathcal{S}$ be the complement of $\mathcal{S}$ and consider the set $\mathcal{K} \cap \mathcal{S}^{\complement}$. Since $\mathcal{S}$ is proper, there exists a vertex $\bar{i} \in Q_0$ such that $U(\bar{i}) \in \mathcal{S}^{\complement}$ (otherwise $U \in \mathcal{S}$ and $\mathcal{S} = \mathcal{D}^c(RQ)$). Since $\mathcal{S}^{\complement}$ is $\boxtimes$-closed and $0 \notin \mathcal{S}^{\complement}$, we have that $\mathcal{K} \cap \mathcal{S}^{\complement} \subseteq \{\bar{i}_\times K(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ (indeed for any $j \neq \bar{i}$ and $\mathfrak{q} \in \operatorname{Spec}(R)$, we have $j_\times K(\mathfrak{q}) \boxtimes_{RQ} \bar{i}_\times K(\mathfrak{p}) = 0$). Moreover, since R is noetherian, the set $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \bar{i}_\times K(\mathfrak{p}) \in \mathcal{S}^{\complement}\}$ has a maximal element, say $\bar{\mathfrak{p}}$, and for any $\mathfrak{q} \subseteq \bar{\mathfrak{p}}$ the complex $\bar{i}_\times K(\mathfrak{q}) \in \mathcal{S}^{\complement}$ (otherwise, by Theorem 2.1.4, $K(\bar{\mathfrak{p}}) \in \mathsf{thick}_R\langle K(\mathfrak{q})\rangle$ and so $\bar{i}_\times K(\bar{\mathfrak{p}}) \in \mathcal{S}$). We end by proving that the maximal element $\bar{\mathfrak{p}}$ is unique and so conclude that $\mathcal{S} = \mathcal{S}_{\bar{\mathfrak{p}},\bar{i}}$. Suppose that there is another maximal element $\bar{\mathfrak{q}}$, then:

(i) If $\bar{\mathfrak{p}} + \bar{\mathfrak{q}} = R$, we have $K(\bar{\mathfrak{p}}) \otimes_R K(\bar{\mathfrak{q}}) = 0$. Indeed, by [CFH24, 14.3.1], the tensor product is equal to $K(\bar{\mathfrak{p}} + \bar{\mathfrak{q}})$ which is an acyclic complex by [CFH24, 11.4.17]. In particular, the tensor product $\bar{i}_\times K(\bar{\mathfrak{p}}) \boxtimes_{RQ} \bar{i}_\times K(\bar{\mathfrak{q}}) = 0$ lies in $\mathcal{S}$;

(ii) If $\bar{\mathfrak{p}} + \bar{\mathfrak{q}} \neq R$, we have $K(\bar{\mathfrak{p}}) \otimes_R K(\bar{\mathfrak{q}}) = K(\bar{\mathfrak{p}} + \bar{\mathfrak{q}})$ by [CFH24, 14.3.1]. Let us prove that, even in this case, the tensor product $\bar{i}_\times K(\bar{\mathfrak{p}}) \boxtimes_{RQ} \bar{i}_\times K(\bar{\mathfrak{q}}) = \bar{i}_\times K(\bar{\mathfrak{p}} + \bar{\mathfrak{q}})$ lies in $\mathcal{S}$. Indeed, there exists finitely many prime ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_N$ such that $\bar{\mathfrak{p}} + \bar{\mathfrak{q}} \subseteq \mathfrak{m}_k$ for any $k = 1, \dots, N$ and $V(\bar{\mathfrak{p}} + \bar{\mathfrak{q}}) = V(\mathfrak{m}_1) \cup \dots \cup V(\mathfrak{m}_N)$ (see [Sta, Lemma 10.63.5-6]). So, by maximality of $\bar{\mathfrak{p}}$, it holds that $\bar{i}_\times K(\mathfrak{m}_k) \in \mathcal{S}$ for any $k = 1, \dots, N$ and, by Theorem 2.1.4, the complex $K(\bar{\mathfrak{p}} + \bar{\mathfrak{q}})$ lies in $\mathsf{thick}_R\langle K(\mathfrak{m}_1) \oplus \dots \oplus K(\mathfrak{m}_N)\rangle$. Thus, it follows that $\bar{i}_\times K(\bar{\mathfrak{p}} + \bar{\mathfrak{q}}) \in \mathcal{S}$.

Since $\mathcal{S}^{\complement}$ is $\boxtimes$-closed, both cases lead to a contradiction. Thus, $\mathcal{K} \cap \mathcal{S}^{\complement} = \{\bar{i}_\times K(\mathfrak{q}) \mid \mathfrak{q} \subseteq \bar{\mathfrak{p}}\}$ and so $\mathcal{S} = \mathcal{S}_{\bar{\mathfrak{p}},\bar{i}}$.

(2) Recall that the topology on $\operatorname{Spc}(\mathcal{D}^c(RQ))$ is given by the basis of open subsets $\{\{\mathcal{S}_{\mathfrak{p},i} \mid X \in \mathcal{S}_{\mathfrak{p},i}\} \mid X \in \mathcal{D}^c(RQ)\}$, while the topology on $\operatorname{Spec}(R) \times Q_0$, where Q_0 is taken with discrete topology, is given by the product topology, thus by the basis of open subsets $\{\{\mathfrak{p} \mid r \notin \mathfrak{p}\} \times \{i\} \mid r \in R, i \in Q_0\}$. We want to prove that the bijection given by the assignment $f : \mathcal{S}_{\mathfrak{p},i} \longmapsto (\mathfrak{p}, i)$ is a homeomorphism. Let us start proving the continuity. Given an open set of the basis in the codomain $\mathcal{O} = \{\mathfrak{p} \mid r \notin \mathfrak{p}\} \times \{i\}$ for an element of the ring $r \in R$, by Remark 2.1.3, we have that $\operatorname{Supp}_R(K((r))) = V((r))$. Thus, it holds that

$$\xi_{\mathfrak{q},j}\left(i_\times K((r)) \oplus \left(\textstyle\bigoplus_{\ell \neq i} \ell_\times R\right)\right) = 0 \text{ if and only if } j = i \text{ and } r \notin \mathfrak{q}$$

this implies $f^{-1}(\mathcal{O}) = \left\{\mathcal{S}_{\mathfrak{p},i} \mid i_\times K((r)) \oplus \left(\bigoplus_{\ell \neq i} \ell_\times R\right) \in \mathcal{S}_{\mathfrak{p},i}\right\}$ which is an open set of the basis. Moreover, f is also an open map. Indeed, given an open set of the basis in the domain $\mathcal{O}' = \{\mathcal{S}_{\mathfrak{p},i} \mid X \in \mathcal{S}_{\mathfrak{p},i}\}$, for some complex $X \in \mathcal{D}^c(RQ)$, we have that

$$f(\mathcal{O}') = \{(\mathfrak{p}, i) \mid \xi_{\mathfrak{p},i}(X) = 0\} = \bigcup_{i \in Q_0} \operatorname{Supp}_R(i^*X)^{\complement} \times \{i\}$$

Since the functor i^* preserves compacts and, by Lemma 2.0.4 (1), the support of a compact object of $\mathcal{D}(R)$ is closed in $\operatorname{Spec}(R)$, it follows that $f(\mathcal{O}')$ is an open set. As for the order-reversing property, note that for any $\mathcal{S}_{\mathfrak{q},j} \subseteq \mathcal{S}_{\mathfrak{p},i}$, we have that $\mathcal{K} \cap \mathcal{S}^{\complement}_{\mathfrak{p},i} \subseteq \mathcal{K} \cap \mathcal{S}^{\complement}_{\mathfrak{q},j}$. In particular, by the previous point, it implies that $i = j$ and $\mathfrak{p} \subseteq \mathfrak{q}$, i.e. $(\mathfrak{p}, i) \leq (\mathfrak{q}, j)$. ☐

Example 8.1.4. Let $A_2 = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ and consider the algebra $\mathbb{K}[[x]]A_2$, then we have the posets

$$\mathrm{Spc}(\mathcal{D}^c(\mathbb{K}[[x]]A_2)) = \begin{array}{c} \mathcal{S}_{(0),1} \\ \Big\uparrow \\ \mathcal{S}_{(x),1} \end{array} \cup \begin{array}{c} \mathcal{S}_{(0),2} \\ \Big\uparrow \\ \mathcal{S}_{(x),2} \end{array} \qquad \mathrm{Spec}(\mathbb{K}[[x]]) \times \{1,2\} = \begin{array}{c} ((x),1) \\ \Big\uparrow \\ ((0),1) \end{array} \cup \begin{array}{c} ((x),2) \\ \Big\uparrow \\ ((0),2) \end{array}$$

In this case, we have that $K((x)) \cong \mathbb{K}$, thus the open set of the basis $\{\mathfrak{p} \mid x \notin \mathfrak{p}\} \times \{1\} = \{((0),1)\}$ correspond to the open set of he basis $\{\mathcal{S}_{\mathfrak{p},i} \mid 1_\times \mathbb{K} \oplus 2_\times \mathbb{K}[[x]] \in \mathcal{S}_{\mathfrak{p},i}\} = \{\mathcal{S}_{(0),1}\}$.

8.1.1 Thick tensor-ideals

We saw in Proposition 5.2.5 that the tt-category $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$ is not rigid. In any case, rigidity is not a necessary condition to get a full classification of all the thick tensor-ideals. For this it suffices that all the thick tensor-ideals are radical.

Proposition 8.1.5. *For any $X \in \mathcal{D}^c(RQ)$, it holds that $X \in \langle X \boxtimes_{RQ} X\rangle$. In particular, any thick tensor-ideal is radical.*

Proof. First note that the results holds in $\mathcal{D}^c(R)$, indeed there the tensor products $\otimes$ and $\boxtimes$ coincide and any object is $\otimes$-rigid. Thus, for any $X \in \mathcal{D}^c(RQ)$, we have that $\mathsf{thick}_R\langle i^*X\rangle = \mathsf{thick}_R\langle i^*X \otimes_R i^*X\rangle$ and so $\mathsf{thick}_{RQ}\langle i_\times i^*X\rangle = \mathsf{thick}_{RQ}\langle i_\times(i^*X \otimes_R i^*X)\rangle$. By Lemma 8.0.2, it follows that $\mathsf{thick}_{RQ}\langle X(i)\rangle = \mathsf{thick}_{RQ}\langle X(i) \boxtimes_{RQ} X(i)\rangle$. Combining this with Proposition 8.1.2 (1) we get the statement and, by Proposition 3.1.6 (1), it follows that any thick tensor-ideal in $\mathcal{D}^c(RQ)$ is radical. ☐

It follows the classification of the thick tensor-ideals of $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$. Here we get the result from the general theory of tt-categories (see Section 3.1).

Corollary 8.1.6. *For any commutative noetherian ring R and finite acyclic quiver Q, it holds that*

$$\mathrm{Thick}_\boxtimes(\mathcal{D}^c(RQ)) \cong \mathbf{V}(\mathrm{Spc}(\mathcal{D}^c(RQ)))$$

where $\mathbf{V}(\mathrm{Spc}(\mathcal{D}^c(RQ)))$ is the set of specialization closed subsets of $\mathrm{Spc}(\mathcal{D}^c(RQ))$.

Proof. By Theorem 8.1.3, the Balmer spectrum $\mathrm{Spc}(\mathcal{D}^c(RQ))$ is a finite union of noetherian spaces, indeed $\mathrm{Spec}(R) \times Q_0 \cong \bigsqcup_{Q_0} \mathrm{Spec}(R)$, and so it is noetherian too. Thus, by Theorem 3.1.7 and Proposition 8.1.5, the thick tensor-ideals of $\mathcal{D}^c(RQ)$ are in bijection with the specialization closed subsets of $\mathrm{Spc}(\mathcal{D}^c(RQ))$. ☐

We can also give two alternative classifications of the thick tensor-ideals of $\mathcal{D}^c(RQ)$, following from Corollary 8.1.6. Here, we present a proof using the standard adjunction isomorphism in the category of posets:

$$\mathrm{Hom}_{\mathrm{Pos}}(X \times A, B) \cong \mathrm{Hom}_{\mathrm{Pos}}(A, \mathrm{Hom}_{\mathrm{Pos}}(X, B)) \text{ for any three posets } X, A \text{ and } B$$

given by the assignment $\phi : f \mapsto \phi_f$, where $\phi_f(a)(x) = f((x,a))$, with inverse $\psi : g \mapsto \psi_g$, where $\psi_g((x,a)) = g(a)(x)$. A direct proof also follows from the arguments used in Theorem 8.2.4.

Corollary 8.1.7. *For a commutative noetherian ring R and finite acyclic quiver Q, it holds that*

1. $\mathrm{Thick}_\boxtimes(\mathcal{D}^c(RQ)) \cong \mathrm{Thick}(\mathcal{D}^c(R)) \times Q_0$;
2. $\mathrm{Thick}_\boxtimes(\mathcal{D}^c(RQ)) \cong \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Thick}_\boxtimes(\mathcal{D}(\mathbb{K}Q)))$

Proof. (1) Let $\{0,1\}$ be the poset with two elements and the obvious order, then the specialization closed subsets of $\mathrm{Spc}(\mathcal{D}^c(RQ))$ can be identified with $\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spc}(\mathcal{D}^c(RQ)), \{0,1\})$ by associating to each subset its characteristic function. From the standard adjunction isomorphism in the category of posets, we get that

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R) \times Q_0, \{0,1\}) \cong \mathrm{Hom}_{\mathrm{Pos}}(Q_0, \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \{0,1\}))$$

where, by Theorem 2.1.4, $\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \{0,1\})) \cong \mathrm{Thick}(\mathcal{D}^c(R))$ and thus, since Q_0 is a discrete poset, we get the statement.
(2) Alternatively, applying the other possible adjunction isomorphism we get

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R) \times Q_0, \{0,1\}) \cong \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Hom}_{\mathrm{Pos}}(Q_0, \{0,1\}))$$

where, since Q_0 is a discrete poset, $\mathrm{Hom}_{\mathrm{Pos}}(Q_0, \{0,1\})$ is $\mathbf{P}(Q_0)$, the power set of Q_0. In virtue of the bijection between Serre subcategories and sets of simple modules ([Kan12, Section 8]), $\mathbf{P}(Q_0)$ is isomorphic to the lattice $\mathrm{Serre}(\mathbb{K}Q)$ and thus by Lemma 8.2.3 we get the statement. □

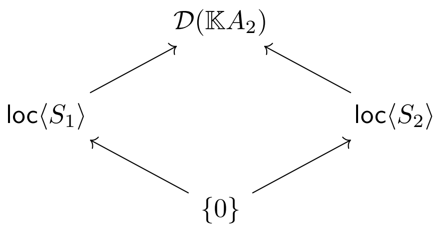
Remark 8.1.8. Comparing Corollary 8.1.7 (2) with the classification in Theorem 4.2.8, where it is proved that $\mathrm{Thick}(\mathcal{D}^c(RQ)) \cong \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathbf{Nc}(Q))$, one can note that the gap between the thick tensor-ideals and thick subcategories in $\mathcal{D}^c(RQ)$ is reflected by the gap between the lattices $\mathbf{P}(Q_0)$ and $\mathbf{Nc}(Q)$, the lattice of noncrossing partitions of Q.

8.1.2 Stratification

As seen in Corollary 8.1.6, although the tt-category $(\mathcal{D}^c(RQ), \boxtimes_{RQ}, \mathbf{U})$ is almost never rigid, we can still recover the classification of all thick tensor-ideals from its Balmer spectrum. However, without rigidity we loose many aspects of the Balmer-Favi theory in Section 3.2, which describes how the Balmer spectrum of the compact objects, $\mathrm{Spc}(\mathcal{T}^c)$, controls the ambient compactly generated tt-category $\mathcal{T}$.

Recall from Theorem 3.2.9 that in a rigidly-compactly generated tt-category there is a bijection between smashing tensor-ideals and tensor-idempotent triangles – a property that crucially depends on rigidity. The next example shows that, for a general compactly generated tt-category, we loose this bijection and that there can be more smashing tensor-ideals than tensor-idempotent triangles.

Example 8.1.9. Let $A_2 = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ and consider the algebra $\mathbb{K}A_2$. Then the lattice of smashing tensor-ideals is



and the truncation triangles of $\mathbf{U} = \mathbb{K} \xrightarrow{1} \mathbb{K}$ with respect to the tensor-ideals $\mathsf{loc}\langle S_1\rangle$ and $\{0\}$ are both $0 \to \mathbf{U} \to \mathbf{U} \xrightarrow{+}$. While the ones with respect to $\mathsf{loc}\langle S_2\rangle$ and $\mathcal{D}(\mathbb{K}A_2)$ are $S_2 \to \mathbf{U} \to S_1 \xrightarrow{+}$ and $\mathbf{U} \to \mathbf{U} \to 0 \xrightarrow{+}$, respectively.

This implies that, a priori, *tensor-idempotent residue objects* may not exist for all points of the spectrum, and consequently, the *Balmer-Favi support* for non-compact objects may not be defined. The major implication of the absence of these notions is the lack of a stratification theory for the classification of localizing tensor-ideals (see Section 3.2.2). Recall that a rigidly-compactly generated tt-category $\mathcal{T}$ is said to be *stratified* (*by the Balmer-Favi support*) if, denoting by $g(\mathcal{P})$ the tensor-idempotent residue object at $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, it equivalently satisfies:

- (*Local-to-global and minimality*) For any object $X \in \mathcal{T}$, $X \in \mathsf{loc}_\otimes \langle g(\mathcal{P}) \otimes X \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \rangle$ and $\mathsf{loc}_\otimes \langle g(\mathcal{P}) \rangle$ is a minimal localizing tensor-ideal for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$;
- (*Stratification*) Taking the Balmer-Favi support and its preimage defines a bijection

$$\mathrm{Loc}_\otimes(\mathcal{T}) \longleftrightarrow \mathbf{P}(\mathrm{Spc}(\mathcal{T}^c))$$

Nevertheless, even in the absence of a general theory, it is still possible – based on the proof of Proposition 8.1.2 and the local-to-global principle in $\mathcal{D}(R)$ – to construct *ad hoc* residue objects $g(\mathfrak{p}, i)$ for all thick prime tensor-ideals $\mathcal{S}_{\mathfrak{p},i}$ and thereby obtain a stratification result (Theorem 8.1.13).

Notation 8.1.10. Given a set of objects $\mathcal{X} \subseteq \mathcal{D}(RQ)$, we denote by $\langle \mathcal{X} \rangle^{\mathrm{II}}$ the smallest localizing tensor-ideal of $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ containing $\mathcal{X}$.

Definition 8.1.11. For a prime thick tensor-ideal $\mathcal{S}_{\mathfrak{p},i}$ in $\mathrm{Spc}(\mathcal{D}^c(RQ))$, we define the tensor-idempotent residue object at $\mathcal{S}_{\mathfrak{p},i}$ to be the complex

$$g(\mathfrak{p}, i) := i_\times K_\infty(\mathfrak{p})_\mathfrak{p}$$

where $K_\infty(\mathfrak{p})$ is the infinite Koszul complex at $\mathfrak{p}$. Consequently, we define the support of an object $X \in \mathcal{D}(RQ)$ as

$$\mathrm{Supp}_{RQ}(X) = \{\mathcal{S}_{\mathfrak{p},i} \in \mathrm{Spc}(\mathcal{D}^c(RQ)) \mid g(\mathfrak{p}, i) \boxtimes_{RQ} X \neq 0\}$$

Remark 8.1.12. It is not clear how to recover this notion of support for a general compactly generated tt-category. Indeed, as we can see in Example 8.1.9, the residue objects do not arise, in any reasonable way, from truncation triangles of the unit $\mathbf{U}$ with respect to some smashing tensor-ideals.

The stratification conditions reported above are satisfied by the tt-category $\mathcal{D}(RQ)$.

Theorem 8.1.13. *For any finite and acyclic quiver Q, the following statements hold:*

1. *For any $X \in \mathcal{D}(RQ)$, $\langle X \rangle^{\mathrm{II}} = \langle g(\mathfrak{p}, i) \boxtimes_{RQ} X \mid \mathcal{S}_{\mathfrak{p},i} \in \mathrm{Spc}(\mathcal{D}^c(RQ)) \rangle^{\mathrm{II}}$ and $\langle g(\mathfrak{p}, i) \rangle^{\mathrm{II}}$ is a minimal localizing tensor-ideal for any $\mathcal{S}_{\mathfrak{p},i} \in \mathrm{Spc}(\mathcal{D}^c(RQ))$;*
2. *Taking the support and its preimage defines a bijection*

$$\mathrm{Loc}_\boxtimes(\mathcal{D}(RQ)) \longleftrightarrow \{\textit{Subsets of } \mathrm{Spc}(\mathcal{D}^c(RQ))\}$$

Proof. (1) For any $X \in \mathcal{D}(RQ)$, by Proposition 8.1.2 (1), we have that

$$\langle X \rangle^{\mathrm{II}} = \mathsf{loc}_{RQ} \langle X(i) \mid i \in Q_0 \rangle \qquad (\diamond\diamond)$$

Moreover, by the local-to-global principle in $\mathcal{D}(R)$, for any $i \in Q_0$, it holds that

$$\mathsf{loc}_R \langle i^* X \rangle = \mathsf{loc}_R \langle K_\infty(\mathfrak{p})_\mathfrak{p} \otimes_R i^* X \mid \mathfrak{p} \in \mathrm{Spec}(R) \rangle$$

So, applying the functor $i_\times$, by Lemma 8.0.2, we can conclude that

$$\mathsf{loc}_{RQ} \langle X(i) \mid i \in Q_0 \rangle = \mathsf{loc}_{RQ} \langle i_\times K_\infty(\mathfrak{p})_\mathfrak{p} \boxtimes_{RQ} X(i) \mid \mathcal{S}_{\mathfrak{p},i} \in \mathrm{Spc}(\mathcal{D}^c(RQ)) \rangle$$

Noting that $i_\times K_\infty(\mathfrak{p})_\mathfrak{p} \boxtimes_{RQ} X(i) = (g(\mathfrak{p}, i) \boxtimes_{RQ} X)(i)$, by $(\diamond\diamond)$, the above implies that

$$\langle X \rangle^{\mathrm{II}} = \langle g(\mathfrak{p}, i) \boxtimes_{RQ} X \mid \mathcal{S}_{\mathfrak{p},i} \in \mathrm{Spc}(\mathcal{D}^c(RQ)) \rangle^{\mathrm{II}}$$

Moreover, by the minimality in $\mathcal{D}(R)$, for any $\mathfrak{p} \in \operatorname{Spec}(R)$ the subcategory $\mathsf{loc}_R \langle K_\infty(\mathfrak{p})_\mathfrak{p} \rangle$ is minimal in $\mathcal{D}(R)$. Then, for any localizing tensor-ideal $\mathcal{L}$ of $\mathcal{D}(RQ)$, the inclusion $\mathcal{L} \in \langle g(\mathfrak{p}, i) \rangle^{\mathrm{II}}$ implies both $\mathcal{L} = i_\times i^*(\mathcal{L})$ and $i^*(\mathcal{L}) \subseteq \mathsf{loc}_R \langle K_\infty(\mathfrak{p})_\mathfrak{p} \rangle$, respectively by ($\diamond\diamond$) and Lemma 8.0.2. Thus, we can deduce the minimality of the localizing tensor-ideal $\langle g(\mathfrak{p}, i) \rangle^{\mathrm{II}}$.
(2) For any object $X \in \mathcal{D}(RQ)$, by minimality, we have that $\langle g(\mathfrak{p}, i) \boxtimes_{RQ} X \rangle^{\mathrm{II}} = \langle g(\mathfrak{p}, i) \rangle^{\mathrm{II}}$ if $S_{\mathfrak{p},i} \in \operatorname{Supp}_{RQ}(X)$ or zero otherwise. Defining the support of a subcategory $\mathcal{L} \subseteq \mathcal{D}(RQ)$ as the union of the supports of its objects, from the previous point, we have that any localizing tensor-ideal $\mathcal{L}$ is equal to $\langle g(\mathfrak{p}, i) \mid \mathcal{S}_{\mathfrak{p},i} \in \operatorname{Supp}_{RQ}(\mathcal{L}) \rangle$. Noting that the $\operatorname{Supp}_{RQ}(g(\mathfrak{p}, i)) = \{\mathcal{S}_{\mathfrak{p},i}\}$, we have the bijection. □

8.2 Tensor-t-structures

In the following we will consider tensor-t-structures of $\mathcal{D}(RQ)$ with respect to the aisle of the standard t-structure $(\mathcal{D}^{\leq 0}(RQ), \mathcal{D}^{\geq 1}(RQ))$. Observe that, indeed, $\mathbf{U} \in \mathcal{D}^{\leq 0}(RQ)$ and $\mathcal{D}^{\leq 0}(RQ) \boxtimes_{RQ} \mathcal{D}^{\leq 0}(RQ) \subseteq \mathcal{D}^{\leq 0}(RQ)$.

Notation 8.2.1. Given a set of objects $\mathcal{X} \subseteq \mathcal{D}(RQ)$, we denote by $\langle \mathcal{X} \rangle^{\leq}$ the smallest cocomplete tensor-suspended subcategory of $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ with respect to $\mathcal{D}^{\leq 0}(RQ)$ containing $\mathcal{X}$. Recall from Proposition 3.2.23, that it is a tensor-aisle. Similarly, by Theorem 1.3.6 (1), $\mathsf{susp}^{\mathrm{II}}_{RQ} \langle \mathcal{X} \rangle$ is an aisle.

Analogously to Proposition 8.1.2, we have the following.

Proposition 8.2.2. *For any finite and acyclic quiver Q and $X \in \mathcal{D}(RQ)$, it holds that:*

1. $\langle X \rangle^{\leq} = \mathsf{susp}^{\mathrm{II}}_{RQ} \langle X(i) \mid i \in Q_0 \rangle$;
2. *If X is compact,* $\langle X \rangle^{\leq} = \mathsf{susp}^{\mathrm{II}}_{RQ} \langle i_\times K(\mathfrak{p})[-n] \mid \mathfrak{p} \in \operatorname{Supp}_R(H^n(i^*X)), n \in \mathbb{Z}, i \in Q_0 \rangle$.

In particular, any cocomplete tensor-suspended subcategory of $\mathcal{D}^c(RQ)$ is generated by complexes of the form $i_\times K(\mathfrak{p})[-n]$ for some $i \in Q_0$, $\mathfrak{p} \in \operatorname{Spec}(R)$ and $n \in \mathbb{Z}$.

Proof. (1) Noting that all the RQ-modules $U(i)$ belong to the standard aisle $\mathcal{D}^{\leq 0}(RQ)$, the proof follows as the one of Proposition 8.1.2 (1), since there we do not use closure under negative shifts.
(2) If $X \in \mathcal{D}^c(RQ)$, by Theorem 2.2.2, we have that

$$\mathsf{susp}^{\mathrm{II}}_R \langle i^*X \rangle = \mathsf{susp}^{\mathrm{II}}_R \langle K(\mathfrak{p})[-n] \mid \mathfrak{p} \in \operatorname{Supp}_R(H^n(i^*X)), n \in \mathbb{Z} \rangle$$

So, applying the functor $i_\times$, by Lemma 8.0.2, we can conclude that

$$\mathsf{susp}^{\mathrm{II}}_R \langle X(i) \rangle = \mathsf{susp}^{\mathrm{II}}_R \langle i_\times K(\mathfrak{p})[-n] \mid \mathfrak{p} \in \operatorname{Supp}_R(H^n(i^*X)), n \in \mathbb{Z} \rangle$$

and combining this with point (1) gives the statement.
As for the last part, it is sufficient to note that, given a set of objects $\mathcal{X} \subseteq \mathcal{D}^c(RQ)$, the cocomplete tensor-suspended subcategory $\langle \mathcal{X} \rangle^{\leq}$ is equal to

$$\mathsf{susp}^{\mathrm{II}}_{RQ} \langle i_\times K(\mathfrak{p})[-n] \mid \mathfrak{p} \in \operatorname{Supp}_R(H^n(i^*X)), X \in \mathcal{X}, n \in \mathbb{Z}, i \in Q_0 \rangle$$

□

Now we present the analogous of Corollary 8.1.7 for compactly generated tensor-aisles of $\mathcal{D}(RQ)$. We start by proving a version of Theorem 7.A.3 adapted to tensor-aisles.

Lemma 8.2.3. *For any field $\mathbb{K}$ and finite acyclic quiver Q, we have that*

$$\operatorname{Aisle}_\boxtimes(\mathcal{D}(\mathbb{K}Q)) \cong \operatorname{Filt}(\operatorname{Serre}(\mathbb{K}Q))$$

In particular, the thick tensor-ideals of $\mathcal{D}^c(\mathbb{K}Q)$ are in bijection with $\operatorname{Serre}(\mathbb{K}Q)$.

Proof. Since any aisle in $\mathcal{D}(\mathbb{K}Q)$ is compactly generated, by Theorem 3.2.24, it suffices to prove that the following bijection holds

$$\begin{aligned} \mathrm{Susp}_{\boxtimes}(\mathcal{D}^c(\mathbb{K}Q)) &\longleftrightarrow \mathrm{Filt}(\mathrm{Serre}(\mathbb{K}Q)) \\ \mathcal{S} &\longmapsto \big(\ldots \supseteq H^n(\mathcal{S}) \supseteq H^{n+1}(\mathcal{S}) \supseteq \ldots\big) \\ \{X \in \mathcal{D}^c(\mathbb{K}Q) \mid H^n(X) \in \mathcal{Z}_n\} &\longleftarrow (\ldots \supseteq \mathcal{Z}_n \supseteq \mathcal{Z}_{n+1} \supseteq \ldots) \end{aligned}$$

Given a tensor-suspended subcategory $\mathcal{S} \subseteq \mathcal{D}^c(\mathbb{K}Q)$, by Lemma 7.A.1, the class $H^n(\mathcal{S})$ is closed under images, cokernels and extensions for any $n \in \mathbb{Z}$. So, it is sufficient to prove that it is closed also under subobjects to prove that it is a Serre subcategory of $\mathrm{mod}(\mathbb{K}Q)$. Let $M \in H^n(\mathcal{S})$, then $M[-n]$ lies in $\mathcal{S}$ and, by closure under tensor product, all the modules of the form $U(i) \boxtimes_{\mathbb{K}Q} M[-n]$ belongs to $\mathcal{S}$. In particular, all the composition factors of M lies in $H^n(\mathcal{S})$ and so does any of its submodules. Moreover, for a filtration of Serre subcategories $\ldots \supseteq \mathcal{Z}_n \supseteq \mathcal{Z}_{n+1} \supseteq \ldots$, the class $\mathcal{S}_{\mathcal{Z}} = \{X \in \mathcal{D}^c(\mathbb{K}Q) \mid H^n(X) \in \mathcal{Z}_n\}$ is clearly closed under summands and positive shifts, so it remains to show that it is closed under extensions. Let $X, Y \in \mathcal{S}_{\mathcal{Z}}$ and consider a distinguished triangle

$$X \longrightarrow Z \longrightarrow Y \xrightarrow{+}$$

It induces the exact sequence in cohomology

$$H^{n-1}(Y) \xrightarrow{f} H^n(X) \longrightarrow H^n(Z) \longrightarrow H^n(Y) \xrightarrow{g} H^{n+1}(X)$$

from which we obtain the short exact sequence

$$0 \longrightarrow \mathrm{coker}\, f \longrightarrow H^n(Z) \longrightarrow \ker g \longrightarrow 0$$

Since both $\mathrm{coker}\, f$ and $\ker f$ lies in $\mathcal{Z}_n$ so does $H^n(Z)$. Since $\mathbb{K}Q$ is hereditary, it follows that $\mathcal{S}_{\mathcal{Z}} = \mathsf{susp}^{\amalg}_{\mathbb{K}Q}\langle \mathcal{Z}_n[-n] \mid n \in \mathbb{Z}\rangle$ and, since any Serre subcategory $\mathcal{Z}_n$ is uniquely determined by the simple modules it contains ([Kan12, Section 8]), it is actually generated by shifts of simple $\mathbb{K}Q$-modules. In particular, by Proposition 3.2.23, $\mathcal{S}_{\mathcal{Z}}$ is a tensor-suspended subcategory. Then the bijection trivially holds. □

Theorem 8.2.4. *For any commutative noetherian ring R and finite acyclic quiver Q, it holds that*

1. $\mathrm{Aisle}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ)) \cong \mathrm{Aisle}_{\mathrm{cg}}(\mathcal{D}(R)) \times Q_0$;
2. $\mathrm{Aisle}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ)) \cong \mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R), \mathrm{Aisle}_{\boxtimes}(\mathcal{D}(\mathbb{K}Q)))$.

Proof. (1) By Theorem 3.2.24, it suffices to prove that the following bijection holds

$$\begin{aligned} \mathrm{Susp}_{\boxtimes}(\mathcal{D}^c(RQ)) &\longleftrightarrow \mathrm{Susp}(\mathcal{D}^c(R)) \times Q_0 \\ f : \mathcal{S} &\longmapsto (i^*\mathcal{S})_{i \in Q_0} \\ \mathsf{susp}^{\amalg}_{RQ}\langle i_\times \mathcal{A}_i \mid i \in Q_0\rangle &\longleftarrow (\mathcal{A}_i)_{i \in Q_0} : g \end{aligned}$$

Since both functors i^* and $i_\times$ preserve compacts, to show that the assignments are well defined it is sufficient to show that, for any tensor-suspended subcategory $\mathcal{S}$ in $\mathcal{D}^c(RQ)$, $i^*\mathcal{S}$ is extensions closed. Let $X, Z \in \mathcal{S}$ and consider a distinguished triangle $i^*X \to M \to i^*Z \xrightarrow{+}$ in $\mathcal{D}(R)$. In particular, applying the functor $i_\times$, by Lemma 8.0.2, we get that $X(i) \to i_\times M \to Z(i) \xrightarrow{+}$ is a distinguished triangle in $\mathcal{D}(RQ)$ and, by tensor closure of $\mathcal{S}$, $X(i), Z(i) \in \mathcal{S}$. It follows that $i_\times M$ lies in $\mathcal{S}$ and thus $M \cong i^* i_\times M \in i^*\mathcal{S}$.

Moreover, for a tensor-suspended subcategory $\mathcal{S} \subseteq \mathcal{D}^c(RQ)$, we have the equality

$$g \circ f(\mathcal{S}) = \mathsf{susp}^{\amalg}_{RQ}\langle i_\times i^*\mathcal{S} \mid i \in Q_0\rangle$$

which, by Lemma 8.0.2, is equal to $\mathsf{susp}^{\amalg}_{RQ}\langle U(i)\boxtimes_{RQ}\mathcal{S}\mid i\in Q_0\rangle$ and, by Proposition 8.2.2 (1), it is equal to $\mathcal{S}$. While, for a collection of tensor-suspended subcategories $(\mathcal{A}_i)_{i\in Q_0}$ in $\mathcal{D}(R)$, we have

$$f\circ g((\mathcal{A}_i)_{i\in Q_0})=\left(i^*\,\mathsf{susp}^{\amalg}_{RQ}\langle j_\times\mathcal{A}_j\mid j\in Q_0\rangle\right)_{i\in Q_0}$$

which, since we proved that i^* reflects extensions, is equal to $\left(\mathsf{susp}^{\amalg}_{R}\langle i^*j_\times\mathcal{A}_j\mid j\in Q_0\rangle\right)_{i\in Q_0}$ and, by Lemma 8.0.2, it is equal to $(\mathcal{A}_i)_{i\in Q_0}$.
(2) Denote by $L(i)$ the $\mathbb{K}Q$-simple module at a vertex $i\in Q_0$. By Theorem 3.2.24, it suffices to prove that the following bijection holds

$$\begin{gathered}\mathrm{Susp}_{\boxtimes}(\mathcal{D}^c(RQ))\longleftrightarrow\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R),\mathrm{Susp}_{\boxtimes}(\mathcal{D}^c(\mathbb{K}Q)))\\ f:\mathcal{S}\longmapsto\left(\mathfrak{p}\mapsto\mathsf{susp}^{\amalg}_{\mathbb{K}Q}\langle L(i)[n]\mid i_\times K(\mathfrak{p})[n]\in\mathcal{S}\rangle\right)\\ \mathsf{susp}^{\amalg}_{RQ}\langle i_\times K(\mathfrak{p})[n]\mid L(i)[n]\in\sigma(\mathfrak{p}),\mathfrak{p}\in\mathrm{Spec}(R)\rangle\longleftarrow\!\shortmid\ \sigma:g\end{gathered}$$

By Proposition 8.1.2 and Lemma 8.2.3, both $f(\mathcal{U})(\mathfrak{p})$ and $g(\sigma)$ are tensor-suspended subcategories. Moreover by Lemma 7.1.4, for any prime ideals $\mathfrak{q}\subseteq\mathfrak{p}$, we have that $K(\mathfrak{p})\in\mathsf{susp}^{\amalg}_{R}\langle K(\mathfrak{q})\rangle$, thus $i_\times K(\mathfrak{p})[n]\in\mathsf{susp}^{\amalg}_{RQ}\langle i_\times K(\mathfrak{q})[n]\rangle$ and $f(\mathcal{U})$ is a poset homomorphism. Since, by Proposition 8.1.2, any tensor-suspended subcategory of $\mathcal{D}^c(RQ)$ is generated by complexes of the form $i_\times K(\mathfrak{p})[n]$, we have that $g\circ f(\mathcal{S})=\mathcal{S}$. Analogously, in virtue of the bijection between Serre subcategories and sets of simple modules ([Kan12, Section 8]), by Lemma 8.2.3, we have that any tensor-suspended subcategory of $\mathcal{D}^c(\mathbb{K}Q)$ is generated by complexes of the form $L(i)[n]$ and so $f\circ g(\sigma)(\mathfrak{p})=\sigma(\mathfrak{p})$ for any $\mathfrak{p}\in\mathrm{Spec}(R)$. □

Remark 8.2.5. Comparing Theorem 8.2.4 (2) with the classification in Theorem 7.2.2 (1), where the author proves that $\mathrm{Aisle}_{\mathrm{cg}}(\mathcal{D}(RQ))\cong\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R),\mathrm{Filt}(\mathbf{Nc}(Q))$, one can note that the gap between compactly generated tensor-aisles and compactly generated aisles in $\mathcal{D}(RQ)$ is reflected by the gap between the lattices $\mathrm{Serre}(\mathbb{K}Q)\cong\mathbf{P}(Q_0)$ ([Kan12, Section 8]) and $\mathrm{Wide}(\mathbb{K}Q)\cong\mathbf{Nc}(Q)$ ([IT09, Theorem 1.1]).

Now we get the analogous of Corollary 8.1.6, for compactly generated tensor-aisles.

Corollary 8.2.6. *For any commutative noetherian ring R and finite acyclic quiver Q, we have that*

$$\mathrm{Aisle}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ))\cong\mathrm{Filt}(\mathbf{V}(\mathrm{Spc}(\mathcal{D}^c(RQ))))$$

where $\mathbf{V}(\mathrm{Spc}(\mathcal{D}^c(RQ)))$ *is the set of specialization closed subsets of* $\mathrm{Spc}(\mathcal{D}^c(RQ))$.

Proof. Denote by $\mathbf{V}(\mathrm{Spec}(R))$ the set of specialization closed subsets of the prime spectrum of R. By Theorem 2.2.2, the lattice of compactly generated aisles of $\mathcal{D}(R)$ is isomorphic to $\mathrm{Filt}(\mathbf{V}(\mathrm{Spec}(R)))$. Thus, since Q_0 is a discrete poset, by Theorem 8.2.4 (1), the lattice $\mathrm{Aisle}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ))$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{Pos}}(Q_0,\mathrm{Hom}_{\mathrm{Pos}}(\mathbb{Z},\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R),\{0,1\})))$$

while, by Theorem 8.1.3, the lattice $\mathrm{Filt}(\mathbf{V}(\mathrm{Spc}(\mathcal{D}^c(RQ))))$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{Pos}}(\mathbb{Z},\mathrm{Hom}_{\mathrm{Pos}}(\mathrm{Spec}(R)\times Q_0,\{0,1\})))$$

Note that applying the standard adjunction in Pos we can pass from one lattice to the other. □

8.3 Tensor telescope conjecture

In the general setting of representations of an arbitrary quiver over commutative noetherian rings we do not know very much about compactly generated and homotopically smashing t-structures and how they interact. However, restricting to the tensor-compatible ones allows a classification of the former. We will see that this restriction is enough also for proving that any homotopically smashing tensor-t-structure is compactly generated.

Lemma 8.3.1. *For an arbitrary ring A, any homotopically smashing t-structure $(\mathcal{U},\mathcal{V})$ in $\mathcal{D}(A)$ is generated by objects, i.e. there exists a set of objects $\mathcal{X} \subseteq \mathcal{D}(A)$ such that $\mathcal{U} = \mathsf{susp}_A^{\amalg}\langle \mathcal{X}\rangle$.*

Proof. Let us note first that, following the proof of [MZ23, Theorem 3.2 (1)], the homotopically smashing condition implies that the truncation functor $\tau_{\mathcal{U}}^{<}$ of the t-structure $(\mathcal{U},\mathcal{V})$ commutes with directed homotopy colimits. Moreover, by the existence of semi-projective resolutions and [CH15, Theorem 1.1], any complex of A-modules is quasi-isomorphic to a semi-projective complex which is isomorphic to a directed homotopy colimit of compact objects $\{S_i\}_{i\in I}$ in $\mathcal{D}(A)$. Then, for any $U \in \mathcal{U}$, we have that

$$U \cong \tau_{\mathcal{U}}^{<}(U) \cong \tau_{\mathcal{U}}^{<}(\underrightarrow{\mathrm{hocolim}}_{i\in I}\, S_i) \cong \underrightarrow{\mathrm{hocolim}}_{i\in I}\, \tau_{\mathcal{U}}^{<}(S_i)$$

In particular, it follows that $\mathcal{U} = \mathsf{susp}_{RQ}^{\amalg}\langle \tau_{\mathcal{U}}^{<}(S) \mid S \in \mathcal{D}^c(RQ)\rangle$. □

Recall that, by [SŠV23, Theorem A], we can lift any t-structure $(\mathcal{M},\mathcal{N})$ in $\mathcal{D}(R)$ to a t-structure $(\mathcal{M}_Q,\mathcal{N}_Q)$ in $\mathcal{D}(RQ)$ defined as follows

$$\mathcal{M}_Q = \{X \in \mathcal{D}(RQ) \mid j^*X \in \mathcal{M} \text{ for any } j \in Q\}$$

$$\mathcal{N}_Q = \{X \in \mathcal{D}(RQ) \mid j^*X \in \mathcal{N} \text{ for any } j \in Q\}$$

and note that the lifting is compatible with orthogonality, in the sense that $(\mathcal{M}_Q)^{\perp} = (\mathcal{M}^{\perp})_Q$.

In the following proposition, we show how a t-structure $(\mathcal{U},\mathcal{V})$ in $\mathcal{D}(RQ)$ can be restricted to a t-structure in $\mathcal{D}(R)$ preserving the homotopically smashing property. Specifically, the restriction is obtained by restricting the aisle and then taking its right orthogonal in $\mathcal{D}(R)$. As illustrated in Example 8.3.3, the naive attempt to restrict both the aisle and the coaisle fails to even produce a t-structure. Therefore, this construction does not yield an inverse to the lifting in [SŠV23].

Proposition 8.3.2. *For any homotopically smashing t-structure $(\mathcal{U},\mathcal{V})$ of $\mathcal{D}(RQ)$, the pair $\big(i^*\mathcal{U}, (i^*\mathcal{U})^{\perp}\big)$ is an homotopically smashing t-structure in $\mathcal{D}(R)$. Moreover, its associated left truncation functor is given by the composite $i^* \circ \tau_Q^{<} \circ i_*$, where $\tau_Q^{<}$ is the truncation functor associated to the aisle $(i^*\mathcal{U})_Q$.*

Proof. Let $(\mathcal{U},\mathcal{V})$ a homotopically smashing t-structure in $\mathcal{D}(RQ)$. By Lemma 8.3.1, there exists a set of objects $\mathcal{X} \subseteq \mathcal{D}(RQ)$ such that $\mathcal{U} = \mathsf{susp}_{RQ}^{\amalg}\langle X \mid X \in \mathcal{X}\rangle$. By Theorem 1.3.6 (1), it follows that $i^*\mathcal{U} = \mathsf{susp}_R^{\amalg}\langle i^*X \mid X \in \mathcal{X}, i \in Q_0\rangle$ is an aisle in $\mathcal{D}(R)$. We see that $(i^*\mathcal{U})^{\perp}$ is closed under directed homotopy colimits. Indeed, let $\{Y_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}(R)^{\Lambda}$ be a coherent diagram such that $Y_\lambda \in (i^*\mathcal{U})^{\perp}$ for any $\lambda \in \Lambda$. In particular, we have that

$$\mathrm{Hom}_{\mathcal{D}(RQ)}(\mathcal{U}, i_*Y_\lambda) \cong \mathrm{Hom}_{\mathcal{D}(R)}(i^*\mathcal{U}, Y_\lambda) = 0 \text{ for any } \lambda \in \Lambda$$

Since $\mathcal{U}^{\perp}$ is closed under taking directed homotopy colimits and i_* commute with them, it follows that

$$\mathrm{Hom}_{\mathcal{D}(R)}(i^*\mathcal{U}, \underrightarrow{\mathrm{hocolim}}_{\lambda\in\Lambda}\, Y_\lambda) \cong \mathrm{Hom}_{\mathcal{D}(RQ)}(\mathcal{U}, \underrightarrow{\mathrm{hocolim}}_{\lambda\in\Lambda}\, i_*Y_\lambda) = 0$$

Thus $\big(i^*\mathcal{U}, (i^*\mathcal{U})^\perp\big)$ is an homotopically smashing t-structure in $\mathcal{D}(R)$. As for the last part, let $U \in i^*\mathcal{U}$ and $V \in \mathcal{D}(R)$, then we have that $\mathrm{Hom}_{\mathcal{D}(R)}(U, i^* \circ \tau_Q^{\leq} \circ i_* V) \cong \mathrm{Hom}_{\mathcal{D}(RQ)}(i_! U, \tau_Q^{\leq} \circ i_* V)$. Since, by Remark 8.0.1, $i_! U$ lies in $(i^*\mathcal{U})_Q$ and $i^* i_! U \cong U$, the above is further equal to

$$\mathrm{Hom}_{\mathcal{D}(RQ)}(i_! U, i_* V) \cong \mathrm{Hom}_{\mathcal{D}(R)}(i^* i_! U, V) \cong \mathrm{Hom}_{\mathcal{D}(R)}(U, V)$$

Thus, it is clear that the composite $i^* \circ \tau_Q^{\leq} \circ i_*$ is right adjoint to the inclusion $i^*\mathcal{U} \hookrightarrow \mathcal{D}(R)$. □

Example 8.3.3. Consider the algebra $\mathbb{Z}A_2$ and the t-structure $(\mathcal{U}, \mathcal{V})$ generated by the representation $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$. Note that, since there are no non-zero morphisms from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$, the representation $\mathbb{Z} \xrightarrow{1} \mathbb{Z}$ lies in $\mathcal{V}$. In particular, the free module $\mathbb{Z}$ is both in $1^*\mathcal{U}$ and in $1^*\mathcal{V}$, thus $(1^*\mathcal{U}, 1^*\mathcal{V})$ cannot be a t-structure.

We can now prove the tensor telescope conjecture for the tt-category $(\mathcal{D}(RQ), \boxtimes_{RQ}, \mathbf{U})$.

Theorem 8.3.4. *Any homotopically smashing tensor-t-structure of $\mathcal{D}(RQ)$ is compactly generated.*

Proof. Let $(\mathcal{U}, \mathcal{V})$ a homotopically smashing tensor-t-structure in $\mathcal{D}(RQ)$. By Proposition 8.3.2, we have that $\big(i^*\mathcal{U}, (i^*\mathcal{U})^\perp\big)$ is an homotopically smashing t-structure in $\mathcal{D}(R)$, thus by the telescope conjecture in $\mathcal{D}(R)$, we can assume that $i^*\mathcal{U}$ is a compactly generated aisle for any $i \in Q_0$. By the bijection Theorem 8.2.4 (1), $\mathsf{susp}^{\amalg}_{RQ}\langle i_\times i^*\mathcal{U} \mid i \in Q_0\rangle$ is a compactly generated aisle in $\mathcal{D}(RQ)$. By Lemma 8.0.2, we have that the above is equal to $\mathsf{susp}^{\amalg}_{RQ}\langle X(i) \mid X \in \mathcal{U}, i \in Q_0\rangle$ and, by Proposition 8.2.2, it is further equal to $\mathcal{U}$. □

8.3.1 Other tensor-t-structures

Actually, we can strengthen the tensor telescope conjecture of Theorem 8.3.4, by generalizing the result to homotopically smashing tensor-t-structures with respect to a broad class of suspended subcategories $\mathcal{T}^{\leq 0}$, which are strictly contained in the standard aisle.

For an RQ-module C, the *support of C over Q* is the full subquiver of Q with vertices $i \in Q_0$ such that C_i is non-zero and we denote it by $\mathrm{Supp}_Q(C)$.

Definition 8.3.5. We define a *filtration system of the unit* to be a set of R-free RQ-modules $\mathcal{C} = \{C_1, \dots, C_n\}$ such that:

(i) $\mathbf{U} \in \mathsf{filt}\langle\mathcal{C}\rangle$;

(ii) $\mathrm{Supp}_Q(C_i) \cap \mathrm{Supp}_Q(C_j) = \emptyset$ for any $i \neq j$.

Moreover, we say that a filtration system (of the unit) has *Dynkin support* if, for any $k = 1 \dots, n$, the quiver $Q_k := \mathrm{Supp}_Q(C_k)$ is a Dynkin quiver.

Given a filtration system $\mathcal{C}$, we say that a suspended subcategory $\mathcal{S}$ is *$\mathcal{C}$-tensor-suspended* if it is tensor-suspended with respect to $\mathsf{susp}_{RQ}\langle\mathcal{C}\rangle$, i.e. if $\mathcal{C} \boxtimes_{RQ} \mathcal{S} \subseteq \mathcal{S}$. A t-structure is a *$\mathcal{C}$-tensor-t-structure* if the aisle is a $\mathcal{C}$-tensor-suspended subcategory.

Notation 8.3.6. Given a set of objects $\mathcal{X} \subseteq \mathcal{D}(RQ)$, we denote by $\langle\mathcal{X}\rangle^{\mathcal{C}}$ the smallest cocomplete $\mathcal{C}$-tensor-suspended subcategory of $(\mathcal{D}(RQ), \boxtimes^{\mathbf{L}}_{RQ}, \mathbf{U})$ containing $\mathcal{X}$. Recall from Proposition 3.2.23, that it is a tensor-aisle. Moreover, we denote by $\mathrm{Susp}^{\mathcal{C}}_{\boxtimes}(\mathcal{D}^c(RQ))$ the lattice of $\mathcal{C}$-tensor-suspended subcategories of $\mathcal{D}^c(RQ)$ and by $\mathrm{Aisle}^{\mathcal{C}}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ))$ the lattice of compactly generated $\mathcal{C}$-tensor-aisle of $\mathcal{D}(RQ)$.

Example 8.3.7.

1. Any t-structure in $\mathcal{D}(RQ)$ is a $\mathcal{C}$-tensor-t-structure with respect to the trivial filtration system $\mathcal{C} = \{\mathbf{U}\}$. In particular, when Q is a Dynkin quiver, any t-structure is a $\mathcal{C}$-tensor-t-structure with respect to a filtration system with Dynkin support.

2. The tensor-t-structures with respect to the standard aisle $\mathcal{D}^{\leq 0}(RQ)$ are precisely the $\mathcal{C}$-tensor-t-structures with respect to the filtration system $\mathcal{C} = \{U(i) \mid i \in Q_0\}$, which always has Dynkin support.

Remark 8.3.8. Given a filtration system $\mathcal{C}$, any module $C_k \in \mathcal{C}$ defines a fully faithful functor $c_k : Q_k \hookrightarrow Q$ and thus an adjoint triple $c_{k!} \dashv {c_k}^* \dashv c_{k*}$ as follows

$$\mathcal{D}(RQ) \underset{\overset{\longleftarrow}{c_{k*}}}{\overset{\overset{c_{k!}}{\longleftarrow}}{\xrightarrow{\ {c_k}^*\ }}} \mathcal{D}(RQ_k)$$

Since c_k is fully faithful, by the characterization of the projective RQ-modules in [EE05, Theorem 3.1], given a bounded complexes P of finitely generated projective RQ-modules, ${c_k}^*P$ is a bounded complexes of finitely generated projective RQ_k-modules. Thus, ${c_k}^*$ preserves compact objects and, since coproducts in $\mathcal{D}(RQ)$ are computed vertexwise, it also preserves coproducts. In particular, as in Remark 8.0.1, $c_{k!}$ preserves compacts, c_{k*} preserves coproducts and all three $c_{k!}, {c_k}^*, c_{k*}$ preserve directed homotopy colimits. Moreover, by [Gro13, Proposition 1.26], the composite ${c_k}^* c_{k!}$ is natural isomorphic to $\mathrm{id}_{\mathcal{D}(RQ_k)}$. So, we can define the functor

$$c_{k\times} := C_k \boxtimes_{RQ} c_{k!} : \mathcal{D}(RQ_k) \to \mathcal{D}(RQ)$$

In view of Example 8.3.7, the following proposition generalizes Proposition 8.2.2 (1) and Theorem 8.2.4 (1) to $\mathcal{C}$-tensor-t-structures.

Proposition 8.3.9. *For any filtration system of the unit $\mathcal{C} = \{C_1, \ldots, C_n\}$, it holds that:*

1. $\langle X \rangle^{\mathcal{C}} = \mathsf{susp}^{\amalg}_{RQ} \langle C_k \boxtimes_{RQ} X \mid k = 1, \ldots, n \rangle$;

2. $\mathrm{Aisle}^{\mathcal{C}}_{\mathrm{cg}\boxtimes}(\mathcal{D}(RQ)) \cong \prod_{k=1}^{n} \mathrm{Aisle}_{\mathrm{cg}}(\mathcal{D}(RQ_k))$

Proof. (1) Since the tensor unit $\mathbf{U}$ admits a filtration with factors isomorphic to the representations C_k, tensoring this filtration by a complex $X \in \mathcal{D}(RQ)$ it follows that $X \in \mathsf{susp}^{\amalg}_{RQ} \langle C_k \boxtimes_{RQ} X \mid k = 1, \ldots, n \rangle$. Moreover, the latter is a $\mathcal{C}$-tensor-suspended subcategory by Proposition 3.2.23. On the other hand, by closure condition, also the reverse inclusion holds.
(2) By Theorem 3.2.24, it suffices to prove that the following bijection holds

$$\mathrm{Susp}_{\boxtimes}(\mathcal{D}^c(RQ)) \longleftrightarrow \prod_{k=1}^{n} \mathrm{Susp}(\mathcal{D}^c(RQ_k))$$
$$f : \mathcal{S} \longmapsto ({c_k}^* \mathcal{S})_{k=1,\ldots,n}$$
$$\mathsf{susp}^{\amalg}_{RQ} \langle c_{k\times} \mathcal{A}_k \mid k = 1, \ldots, n \rangle \longleftarrow (\mathcal{A}_k)_{k=1,\ldots,n} : g$$

Since both functors ${c_k}^*$ and $c_{k\times}$ preserve compacts and ${c_k}^*$ preserves extensions, shifts and summands, the assignments are well defined. Moreover, for a tensor-suspended subcategory $\mathcal{S} \subseteq \mathcal{D}(RQ)$, the composite $g \circ f(\mathcal{S})$ is $\mathsf{susp}^{\amalg}_{RQ} \langle c_{k\times} {c_k}^* \mathcal{S} \mid k = 1, \ldots, n \rangle$. By Lemma 8.0.2, we have that

$$g \circ f(\mathcal{S}) = \mathsf{susp}^{\amalg}_{RQ} \langle C_k \boxtimes_{RQ} \mathcal{S} \mid k = 1, \ldots, n \rangle$$

which, by point (1), is equal to $\mathcal{S}$. For a family of tensor-ideals $(\mathcal{A}_k)_{k=1,\dots,n}$ with $\mathcal{A}_k \subseteq \mathcal{D}(RQ_k)$, the composite $f \circ g((\mathcal{A}_k)_{k=1,\dots,n})$ is given by $\left(c_k{}^* \,\mathsf{susp}^{\amalg}_{RQ} \langle c_{\ell\times}\mathcal{A}_\ell \mid \ell = 1,\dots,n\rangle\right)_{k=1,\dots,n}$ which, by the properties of $c_k{}^*$ and the fact that the supports of objects in $\mathcal{C}$ are disjoint, is equal to $\left(\mathsf{susp}^{\amalg}_{R} \langle c_k{}^* c_{\ell\times}\mathcal{A}_\ell \mid \ell = 1,\dots,n\rangle\right)_{k=1,\dots,n} = (A_k)_{k=1,\dots,n}$. □

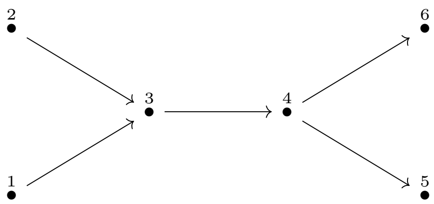
We can now prove the tensor telescope conjecture for $\mathcal{C}$-tensor-t-structures with respect to some filtration system with Dynkin support and so, generalize Theorem 8.3.4.

Theorem 8.3.10. *Given a filtration system with Dynkin support $\mathcal{C}$, any homotopically smashing $\mathcal{C}$-tensor-t-structure of $\mathcal{D}(RQ)$ is compactly generated.*

Proof. Let $(\mathcal{U}, \mathcal{V})$ a homotopically smashing $\mathcal{C}$-tensor-t-structure in $\mathcal{D}(RQ)$. By similar arguments to those in Proposition 8.3.2, we have that $\left(c_k{}^*\mathcal{U}, (c_k{}^*\mathcal{U})^{\perp}\right)$ is a homotopically smashing t-structure in $\mathcal{D}(RQ_k)$ for any $k = 1,\dots,n$, thus by the telescope conjecture in $\mathcal{D}(RQ_k)$ (see Corollary 6.2.5 (2)), we can assume that $c_k{}^*\mathcal{U}$ is a compactly generated aisle for any $k = 1,\dots,n$. By the bijection in Proposition 8.3.9 (2), $\mathsf{susp}^{\amalg}_{RQ}\langle c_{k\times} c_k{}^*\mathcal{U} \mid k = 1,\dots,n\rangle$ is a compactly generated $\mathcal{C}$-tensor-aisle in $\mathcal{D}(RQ)$. Moreover, it is equal to $\mathsf{susp}^{\amalg}_{RQ}\langle C_k \boxtimes_{RQ} \mathcal{U} \mid k = 1,\dots,n\rangle$ and thus, by Proposition 8.3.9 (2), to $\mathcal{U}$. □

The next example illustrates how Theorem 8.3.10 could represent a significant step toward proving the telescope conjecture for representations of finite, acyclic, and simply-laced quivers.

Example 8.3.11. Consider the Euclidean quiver

$\widetilde{D}_5 =$ 

Let D_5 be the full subquiver spanned by the vertices $\{1,2,3,4,5\}$ and P the representation given by $P_i = R$ and $P_\alpha = \mathrm{id}_R$ if $i, \alpha \in D_5$ and $P_6 = 0$, then $\mathcal{C} = \{P, U(6)\}$ is a filtration system with Dynkin support. Note that a suspended subcategory $\mathcal{U}$ is a $\mathcal{C}$-tensor-ideal if and only if for any object $X \in \mathcal{U}$ such that $\mathrm{Supp}_Q(X) \nsubseteq D_5$ then $X(6) \in \mathcal{U}$. Thus, by Theorem 8.3.10, any homotopically smashing t-structure whose aisle satisfies this property is compactly generated.

Moreover, in this example as well as in any Euclidean quiver, a non-trivial filtration systems $\mathcal{C} \neq \{\mathbf{U}\}$ has always Dynkin support. Therefore, for a homotopically smashing t-structure, it is enough to have a $\mathcal{C}$-tensor-aisle for some non-trivial filtration system $\mathcal{C}$ in order to be compactly generated.

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