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# BGG Decomposition for de Rham Sheaves on the Modular Elliptic Curve

**Ph.D. Thesis**

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# Abstract

In this thesis we present a joint work with F. Andreatta and A. Iovita about a BGG decomposition of the de Rham sheaves $\mathbb{W}_\kappa$ defined over the modular elliptic curves. In Chapter 1 we study the infinitesimal site of smooth rigid analytic varieties and we define the linearization and delinearization functors. In Chapter 2 we introduce the BGG decomposition for some infinite dimensional $\mathfrak{g}$–modules, where $\mathfrak{g}$ is a semisimple Lie algebra. Thanks to this decomposition we compute the de Rham cohomology of the sheaves $\mathbb{W}_\kappa$. The techniques presented could be used in order to study infinite dimensional $\mathfrak{g}$–modules over more general Shimura varieties.

# Contents

# Introduction

The main goal of this thesis is to study the Berstein–Gelfand–Gelfand (BGG) decomposition for certain de Rham sheaves on a specific $p$-adic rigid analytic open subvariety of a modular elliptic curve, via a linearization and delinearization method. This method should give a way to decompose de Rham sheaves defined over more general Shimura varieties. In Chapter 1 we get a technical result about a delinearization functor over smooth rigid analytic varieties defined over a complete $p$-adic field $K$. In Chapter 2 we describe an ongoing project with F. Andreatta and A. Iovita about a decomposition of de Rham sheaves defined over the modular elliptic curve.

The BGG decomposition was first presented in the article [BGG75] for a finite dimensional $\mathbb{C}$-vector space $V^\vee$, with an action of a semisimple Lie algebra $\mathfrak{g}$. This theory starts by building a resolution of $V^\vee$ using the $\mathfrak{g}$-action and a parabolic sub-algebra $\mathfrak{p}$ of $\mathfrak{g}$. The resolution takes the following form

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left( V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{\bullet} (\mathfrak{g}/\mathfrak{p}) \right) \longrightarrow V^\vee;$$

the complex on the left is called the Koszul complex. If the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{p}$ one gets

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left( V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{k} (\mathfrak{g}/\mathfrak{p}) \right) \cong \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{k} \mathfrak{n}^-.$$

The objects of the Koszul complex are built using the parabolic induction, where only the $\mathfrak{p}$-action on $V^\vee$ is involved; the maps of the Koszul complex are given by the $\mathfrak{n}^-$-action on $V^\vee$. The Koszul complex is exact, hence it is a resolution of $V^\vee$.

At this point we assume that there is a reductive, algebraic group $G$ and a parabolic subgroup $P$ of it such that $\mathfrak{g}$ is a semisimple Lie sub-algebra of $\mathrm{Lie}(G)$ and that $\mathfrak{n}^-$ is commutative, *i.e.* $\mathcal{U}(\mathfrak{n}^-) \cong \mathrm{Sym}_{\mathbb{C}} \mathfrak{n}^-$.

Now we dualize the complex above and we look at it as a complex of sheaves on the projective variety $G/P$, we get the exact sequence

$$\mathcal{O}_{G/P} \otimes_{\mathbb{C}} V \to \mathcal{O}_{G/P} \otimes_{\mathbb{C}} \left( \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left( V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{\bullet} (\mathfrak{g}/\mathfrak{p}) \right) \right)^\vee.$$

The complex $\mathcal{O}_{G/P} \otimes_{\mathbb{C}} \left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (V^\vee \otimes_{\mathbb{C}} \wedge_{\mathbb{C}}^{\bullet}(\mathfrak{g}/\mathfrak{p}))\right)^\vee \cong L(V \otimes_{\mathbb{C}} \Omega^{\bullet}_{(G/P)/\mathbb{C}})$ is the linearized de Rham complex of the $\mathcal{O}_{G/P}$-module $\mathcal{O}_{G/P} \otimes_{\mathbb{C}} V$, where the connection is defined by the $\mathfrak{n}^-$-action on $V^\vee$. The BGG decomposition gives us a way in order to "cut" the Koszul complex and then the linearized de Rham complex. Now we want to explain how this decomposition works; in order to simplify the exposition we make the assumption that $V^\vee$ is irreducible; in the rest of the manuscript this assumption will be removed. Fix a maximal Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{p}$; then $\mathfrak{h}$ acts on $V^\vee$ via a character $\chi_\kappa : \mathfrak{h} \to \mathbb{C}$ and we can consider the resolution

$$\left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{\bullet}(\mathfrak{g}/\mathfrak{p})\right)\right)_{\chi_\kappa} \longrightarrow V^\vee.$$

of generalized $\chi_\kappa$-eigenspaces for the action of the center $Z(\mathfrak{g})$ of the universal envelopping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$; this is the BGG complex. In [BGG75] it is showed that the BGG complex is also exact and there is an explicit description of each object in the resolution. For more details see Appendix B.

Then we get a resolution of the $\mathcal{O}_{G/P}$-module $\mathcal{O}_{G/P} \otimes_{\mathbb{C}} V$ given by

$$\mathcal{O}_{G/P} \otimes_{\mathbb{C}} V \to \mathcal{O}_{G/P} \otimes_{\mathbb{C}} \left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(V^\vee \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^{\bullet}(\mathfrak{g}/\mathfrak{p})\right)\right)_{\chi_\kappa}^{\vee},$$

and $\mathcal{O}_{G/P} \otimes_{\mathbb{C}} \left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (V^\vee \otimes_{\mathbb{C}} \wedge_{\mathbb{C}}^{\bullet}(\mathfrak{g}/\mathfrak{p}))\right)_{\chi_\kappa}^{\vee} \cong L\left((V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/\mathbb{C}})_{\chi_\kappa}\right)$. The BGG theory works for split semisimple Lie algebra $\mathfrak{g}$ and split $\mathfrak{g}$-modules $V^\vee$ defined for any finite dimensional $K$-module, where $K$ is a field of characteristic 0. The study of the infinitesimal site done by Grothendieck in [Gro68] and Berthelot–Ogus in [BO78] allows us to say that there is a de-linearization functor $u_*$ over the scheme $G/P$ and

$$u_* L(V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/K})_{\mathrm{inf}} \cong V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/K},$$

where $L(V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/K})_{\mathrm{inf}}$ is a sheaf over the infinitesimal topos associated to the sheaf $L(V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/K})$. One can show that the cohomology of the complex $V \otimes_{\mathbb{C}} \Omega^\bullet_{(G/P)/K}$, *i.e.* the de Rham cohomology of the sheaf $V \otimes_K \mathcal{O}_{G/P}$, is the cohomology of the complex

$$u_* L\left((V \otimes_K \Omega^\bullet_{(G/P)/K})_{\chi_\kappa}\right)_{inf} \cong (V \otimes_K \Omega^\bullet_{(G/P)/K})_{\chi_\kappa}.$$

Moreover if there is an étale map $f : U \to G/P$ one can perform the same computations for the de Rham cohomology of the sheaf $f^*\left(V \otimes_K \mathcal{O}_{G/P}\right) = V \otimes_K \mathcal{O}_U$. The BGG theory presented above works only for finite dimensional $\mathfrak{g}$-modules and over algebraic varieties.

In this thesis we describe a new method by F. Andreatta and A. Iovita in order to get the BGG decomposition of some infinite dimensional $\mathfrak{g}$-modules. Since we will apply the BGG decomposition with $\mathfrak{g} = \mathfrak{sl}_2$ we suppose that $\mathfrak{p} = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. We consider a $\mathfrak{g}$-module $V = \mathfrak{W}_\kappa^{K,alg}$ with an increasing filtration $\{F_i\}_{i\in\mathbb{N}}$ such that $\mathfrak{W}_\kappa^{K,alg} = \varinjlim_{i\in\mathbb{N}} F_i$. We assume that the $\mathfrak{p}$-action preserves the filtration and the $\mathfrak{n}^-$-action increases the filtration by one. Firstly we reduce to the finite dimensional case looking at the $\mathfrak{g}$-module $F_s^\vee$ getting a resolution of the form

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(F_s^\vee \otimes_K \bigwedge_K^{\bullet}(\mathfrak{g}/\mathfrak{p})\right) \longrightarrow F_s^\vee.$$

Thanks to the Lemma 2.1.3 one shows that the BGG decomposition is well defined on the limit on $s$ and we get the two sequences

$$\mathcal{F}^\bullet := \lim_{s\in\mathbb{N}} \left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} F_{s+\bullet}^\vee \otimes_K \bigwedge_K^{\bullet}(\mathfrak{g}/\mathfrak{p})\right)_{\chi_\kappa} \subset \lim_{s\in\mathbb{N}} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(F_{s+\bullet}^\vee \otimes_K \bigwedge_K^{\bullet}(\mathfrak{g}/\mathfrak{p})\right) =: \mathcal{C}^\bullet.$$

Moreover we are interested in the other limit:

$$\mathcal{D}^\bullet := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(\lim_{s\in\mathbb{N}} (F_s^\vee) \otimes_K \bigwedge_K^{k}(\mathfrak{g}/\mathfrak{p})\right) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \left(\mathfrak{W}_\kappa^{K,alg,\vee} \otimes_K \bigwedge_K^{k}(\mathfrak{g}/\mathfrak{p})\right).$$

The Theorem 2.1.1 says that the composition map $\gamma^\bullet:\ \mathcal{D}^\bullet \to \mathcal{C}^\bullet \to \mathcal{C}^\bullet/\mathcal{F}^\bullet = \mathcal{C}^\bullet_{\chi\neq\chi_\kappa}$ vanishes on cohomology. In Subsection 2.1.2 we show that also the dual map $\gamma^{\bullet,\vee}$ is zero on the cohomology groups. Finally in Subsection 2.1.3 we get that also the continuous dual map

$$\gamma^{\bullet,\vee,cont}: \varinjlim_{s\in\mathbb{N}} L(F_{s+\bullet}\otimes_K \Omega^\bullet_{(G/B)/K})_{\chi\neq\chi_\kappa} = \varinjlim_{s\in\mathbb{N}} \left(\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\left(F^\vee_{s+\bullet}\otimes_K \bigwedge_K^\bullet(\mathfrak{g}/\mathfrak{b})\right)\right)^\vee_{\chi\neq\ \chi_\kappa} =: \left(\mathcal{C}^\bullet/\mathcal{F}^\bullet\right)^{\vee,cont}$$

$$\longrightarrow L\left(\mathfrak{W}^{K,\mathrm{alg}}_\kappa\otimes_K \Omega^\bullet_{(G/B)/K}\right) = \lim_{n\in\mathbb{N}} \varinjlim_{s\in\mathbb{N}} \left(\mathcal{U}(\mathfrak{g})^{\leq n-\bullet}\otimes_{\mathcal{U}(\mathfrak{b})}\left(F^\vee_{s+\bullet}\otimes_K \bigwedge_K^\bullet(\mathfrak{g}/\mathfrak{b})\right)\right)^\vee =: \mathcal{D}^{\bullet,\vee,cont}$$

vanishes on the cohomology groups. In Chapter 1 we make some explicit computations that allow us to write

$$\mathfrak{W}^{K,\mathrm{alg}}_\kappa\otimes_K \Omega^\bullet_{(G/B)/K} \cong u_*\varinjlim_{s\in\mathbb{N}} L(F_{s+\bullet}\otimes_K \Omega^\bullet_{(G/B)/K})_{inf} \cong u_* L\left(\mathfrak{W}^{K,\mathrm{alg}}_\kappa\otimes_K \Omega^\bullet_{(G/B)/K}\right)_{inf}.$$

We get that the de Rham cohomology of $\mathfrak{W}^{K,\mathrm{alg}}_\kappa$ is the direct sum of the cohomologies of $u_*\mathcal{F}^{\bullet,\vee}$ and $u_*\left(\mathcal{C}^\bullet/\mathcal{F}^\bullet\right)^{\vee,cont}$ and the map

$$u_*\gamma^{\bullet,\vee,cont}: u_*\left(\mathcal{C}^\bullet/\mathcal{F}^\bullet\right)^{\vee,cont}\to \mathfrak{W}^{K,\mathrm{alg}}_\kappa\otimes_K \Omega^\bullet_{(G/B)/K}$$

is zero on cohomology, hence our main Theorem

$$H^\bullet\left(G/B, u_*\mathcal{F}^{\bullet,\vee}\right)\cong H^\bullet_{dR}\left(G/B,\mathfrak{W}^{K,\mathrm{alg}}_\kappa\right).$$

Another consequence of the Lemma 2.1.3 is that $u_*\mathcal{F}^{\bullet,\vee}$ is finite dimensional, hence we cut the de Rham complex of the infinite dimensional $\mathfrak{g}$-module $\mathfrak{W}^{K,\mathrm{alg}}_\kappa$ getting a quasi isomorphic complex of finite dimensional modules. If one takes as before a finite $\mathfrak{g}$-module $V$ one gets that $\mathcal{C}^\bullet=\mathcal{D}^\bullet$, hence the quasi isomorphism

$$L\left((V\otimes_K \Omega^\bullet_{(G/P)/K})_\chi\right)_{inf}\sim L\left((V\otimes_K \Omega^\bullet_{(G/P)/K})\right)_{inf}.$$

In our infinite dimensional case we need to apply the delinearization functor $u_*$ in order to say that these complexes are quasi isomorphic.

Our application is related to modular elliptic curves, set $G:=\mathrm{GL}_2$ and $P=B$ be the Borel subgroup of upper triangular matrices. Let $\mathfrak{X}$ be an overconvergent locus of the modular elliptic curve defined over $\mathrm{Sp}(K)$, where $K$ is a finite extension of $\mathbb{Q}_p$. Locally on $\mathfrak{X}$ we find étale morphisms $\pi_{GM}:\mathfrak{U}\to G/P$, where $\{\mathfrak{U}\}_{\mathfrak{U}\in\mathcal{U}}$ is an affinoid covering of $\mathfrak{X}$ (see Theorem 2.3.2). We consider the $\mathcal{O}_{\mathfrak{X}}$ modules $\mathbb{W}^{\mathrm{alg}}_\kappa$ that $p$-adic interpolate the sheaves $\mathrm{Sym}^k(H_E)$, where $H_E$ is the de Rham cohomology of the Hodge bundle. These sheaves have an integrable connection and they are introduced in [AIP18] and [AI21]. We show that for any $\mathfrak{U}\in\mathcal{U}$ we have $\pi^*_{GM}\left(\mathfrak{W}^{K,\mathrm{alg}}_\kappa\otimes_K\mathcal{O}_{G/P}\right)\cong\left(\mathbb{W}^{\mathrm{alg}}_\kappa\right)_{|\mathfrak{U}}$ for a $K$-module $\mathfrak{W}^{K,\mathrm{alg}}_\kappa$ with the properties as before, see the Theorem 2.3.4. The $\mathcal{O}_{\mathfrak{X}}$-module $\mathbb{W}^{\mathrm{alg}}_\kappa$ is endowed with an exhausting filtration $\{Fil_i\mathbb{W}^{\mathrm{alg}}_\kappa\}_{i\in\mathbb{N}}$ and a $\mathfrak{g}$-action; the filtrations and the actions on $\mathbb{W}^{\mathrm{alg}}_\kappa$ and $\mathfrak{W}^{K,\mathrm{alg}}_\kappa$ are compatible respect to the morphisms $\pi_{GM}$. All these properties and the BGG decomposition for infinite dimensional $\mathfrak{g}$-modules that we have explained show that if $\kappa\notin\mathbb{N}$

$$\left(\left(\mathbb{W}^{alg}_\kappa\right)_{|\mathfrak{U}}\xrightarrow{\nabla}\left(\mathbb{W}^{alg}_\kappa\right)_{|\mathfrak{U}}\otimes_{\mathcal{O}_{\mathfrak{U}}}\Omega^1_{\mathfrak{U}/K}\right)\sim\left(0\to(\omega_{\mathbb{E}})^{\kappa+2}_{|\mathfrak{U}}\right)$$

is a quasi isomorphism; where $\omega^{\kappa+2}_{\mathbb{E}}$ is the line bundle of modular forms of weight $\kappa+2$ and the complex on the right is computed locally via the BGG decomposition. We recover the Corollary 3.35 of [AI21]. We denote with $\mathbb{W}_\kappa:=\widehat{\mathbb{W}^{alg}_\kappa}$ the $p$-adic closure of $\mathbb{W}^{alg}_\kappa$. The finite slope part of the de Rham cohomology of $\mathbb{W}^{alg}_\kappa$ and

$\mathbb{W}_\kappa$ are equal via the Lemma 3.33 [AI21], *i.e.*

$$H^1_{dR}(\mathfrak{U}, \mathbb{W}_\kappa)^{\leqslant h} \cong H^1_{dR}(\mathfrak{U}, \mathbb{W}_\kappa^{alg})^{\leqslant h}$$

for any $h \in \mathbb{N}$, hence we compute the finite slope part of the de Rham cohomology of the sheaf $\mathbb{W}_\kappa$.

Our result is local on $\mathfrak{X}$; in order to get a global result it suffices to check that the BGG resolutions on different $\mathfrak{U}$ glue. This is true since the generalized $\chi_\kappa$-eigenspace does not depend on the choosen local trivialization of $\mathbb{W}_\kappa$.

In the computation of the de Rham cohomology that we described above we needed to use the delinarization functor over an overconvergent locus of the modular elliptic curve, which is a rigid analytic variety. Now we explain some history and the technical result that we get in Chapter 1 about the infinitesimal topos.

The infinitesimal site was introduced by A. Grothendieck in [Gro68] over schemes of zero characteristic. He used this site in order to give an algebraic interpretation of the de Rham cohomology: this cohomology has not nice properties in positive characteristic; hence he develops the infinitesimal cohomology for schemes with arbitrary characteristic that, in zero characteristic, is compatible with the de Rham cohomology; see p.352 of [Gro68] and Berthelot–Ogus [BO78]. Grothendieck showed that the infinitesimal cohomology of a sheaf with connection is equal to its de Rham cohomology; we refer to this result as the Grothendieck comparison Theorem. The main result of Chapter 1 is an analogue of the Grothendieck comparison Theorem in the rigid analytic setting. More precisely let $K$ be a complete field over $\mathbb{Q}_p$ and $X$ be a smooth rigid analytic variety over $S = \mathrm{Sp}(K)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, we say that $\mathcal{F}$ is **flat** if for any exact sequence of coherent $\mathcal{O}_X$-modules $\mathcal{R}^\bullet$, also the sequence $\mathcal{R}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact. A connection on $\mathcal{F}$ is a morphism

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

that satisfies the Leibnitz rule, see §1.2.2. We use the definition of the infinitesimal site for rigid analytic varieties given by H. Guo in [Guo21]. From an $\mathcal{O}_X$-module $\mathcal{F}$ with connection $\nabla$ we associate an $\mathcal{O}_{X_{\mathrm{inf}}}$-module $\mathcal{F}_{inf}$: a sheaf on the infinitesimal site of $X$. We build a linearization functor $L(-)_{inf}$, that is an analogue of the Grothendieck linearization functor in this setting. The linearization functor sends $\mathcal{O}_X$-modules to $\mathcal{O}_{X_{\mathrm{inf}}}$-modules and differential morphisms to linear morphisms, hence it "linearizes" the morphisms. Our main Theorem of the first Chapter is the Theorem 1.3.28:

**Theorem.** *For any flat $\mathcal{O}_X$-module $\mathcal{F}$ with integrable connection $\nabla$ the de Rham complex associated to $\nabla$ and the linearized de Rham complex have the same cohomology, i.e. there is a natural isomorphism*

$$H^\bullet\left(X, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\right) \cong H^\bullet\left(X_{\mathrm{inf}}, L\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\right)_{inf}\right) \cong H^\bullet(X_{\mathrm{inf}}, \mathcal{F}_{inf}).$$

Moreover the Proposition 1.3.12 shows that for any $\mathcal{O}_X$-module $\mathcal{F}$ with integrable connection $\nabla$ the complexes of $\mathcal{O}_{X_{\mathrm{inf}}}$-modules

$$\left(\mathcal{F}_{inf} \hat{\otimes}_{\mathcal{O}_{X_{\mathrm{inf}}}} L(\Omega^\bullet_{X/S})_{inf},\ \mathrm{id}_{\mathcal{F}_{inf}} \hat{\otimes} L(\mathrm{d}^\bullet)_{inf}\right) \cong \left(L\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\right),\ L(\nabla^\bullet)_{inf}\right) \tag{1}$$

are isomorphic. Hence the main Theorem shows that the de Rham cohomology does not depend on the connection $\nabla$, since the equation (1) shows that the cohomology of the linearized de Rham complex in the infinitesimal site does not depend on $\nabla$.

The strategy of the proof of the main Theorem is to show that the linearized sheaves are acyclic with respect to the delinearization functor $u_*$ introduced in §1.3. Due to the flatness of $\mathcal{F}$ one gets that the sequence

$$0 \to \mathcal{F}_{inf} \to L\left(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\right)_{inf} \to 0$$

is exact, hence

$$H^\bullet\left(X_{\inf},\mathcal{F}_{inf}\right)\cong H^\bullet\left(X_{\inf},L\left(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/S}\right)_{inf}\right)\cong H^\bullet\left(X,u_*L\left(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/S}\right)_{inf}\right).$$

Via the flatness of $\mathcal{F}$ and an application of the Mittag–Leffler condition we get that

$$u_*L(\mathcal{F}\otimes\Omega^\bullet_{X/S})_{inf}\cong\mathcal{F}\otimes\Omega^\bullet_{X/S}$$

and the main Theorem is proved. We followed the techniques used in [BO78]. It should be possible to give a homotopic proof following [BdJ11].

What is new in this thesis is the description of a method in order to get a BGG decomposition of the de Rham complex of some infinite dimensional $\mathfrak{g}$-modules defined over smooth rigid analytic varieties locally étale over $G/P$.

Chapter 1 is the main contribute of the author: there is a description of some properties of the infinitesimal site over smooth rigid analytic varieties and the delinearization functor; this Chapter is useful in order to have an analogue of the Grothendieck delinearization functor in this rigid analytic setting.

Chapter 2 describes the ongoing project with F. Andreatta and A. Iovita: we get a BGG decomposition of some infinite dimensional $\mathfrak{g}$-modules and we apply this decomposition in order to compute the de Rham cohomology of some sheaves defined over some overconvergent locus of the modular elliptic curve. We believe that the methods involved could be generalized to some sheaves with connection over some PEL Shimura varieties.

In Appendix A we prove some technical Lemmas; in the Appendix B we describe the classical BGG decomposition of a finite dimensional $\mathfrak{g}$-module and in Appendix C we resume the vector bundle with marked sections (VBMS) construction.

# Chapter 1

# Infinitesimal cohomology

In this Chapter we describe the (small) infinitesimal site relative to a smooth rigid space $X$ defined over $S = \mathrm{Sp}(K)$, where $K$ is a complete $p$-adic field. We are interested, following Grothendieck's work, in the description of the de Rham cohomology of a flat $\mathcal{O}_X$-module $\mathcal{F}$ with an integrable connection $\nabla$ in terms of the cohomology of the linearized De Rham complex

$$L(\mathcal{F})_{\mathrm{inf}} \xrightarrow{L(\nabla)_{\mathrm{inf}}} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})_{\mathrm{inf}} \xrightarrow{L(\nabla^1)_{\mathrm{inf}}} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^2_{X/S})_{\mathrm{inf}} \xrightarrow{L(\nabla^2)_{\mathrm{inf}}} \dots$$

in the infinitesimal site.

## 1.1 Infinitesimal category

Let $K$ be a complete $p$-adic field, we denote $S := \mathrm{Sp}(K)$, let $X$ be a smooth rigid space defined over $S$. We recall some definitions.

**Definition 1.1.1.** Let $g : U \to T$ be a morphism of rigid spaces over $K$, we say that $g$ is:

- **nilpotent** if it is a closed immersion defined by a coherent ideal $I \subset \mathcal{O}_T$ that is locally nilpotent, *i.e.* for each $t \in T$ there is an open $t \in V \subset T$ and an $n \in \mathbb{N}$ s.t. $I^n_{|V} = 0$;
- **nilpotent of order** $N \in \mathbb{N}$ if it is a closed immersion defined by a coherent ideal $I \subset \mathcal{O}_T$ with $I^N = 0$.

Observe that if $T$ is quasi-compact, then you can cover $T$ with a finite cover where $g$ is nilpotent of a certain finite order, so you get that $g$ has finite order. Now we define the (small) infinitesimal category over $X$ following the Definition 2.1.2 in [Guo21].

**Definition 1.1.2.** The **infinitesimal category over** $X$ is the category, denoted by $(X/S)_{\mathrm{inf}}$, described by the following objects and arrows:

- The objects are denoted by $(U, T)$ where $U, T$ are rigid spaces (over $S$) equipped with a nilpotent $S$-morphism $g : U \to T$ and an open embedding $i : U \to X$ defined over $S$;
- The arrows from $(U_1, T_1)$ to $(U_2, T_2)$ are pair of $S$-morphisms $(\alpha, \beta) : (U_1, T_1) \to (U_2, T_2)$ s.t. $\alpha$ is an open immersion and the following diagram commutes
$$\begin{tikzcd} & X \\ U_1 \arrow[ur, "i_1"] \arrow[r, "\alpha"] \arrow[d, "g_1"] & U_2 \arrow[u, "i_2"'] \arrow[d, "g_2"] \\ T_1 \arrow[r, "\beta"] & T_2 \end{tikzcd} .$$

Before introducing Grothendieck topology in these categories we prove some nice properties.

**Lemma 1.1.3.** *The category $(X/S)_{\mathrm{inf}}$ has equalizers and fiber products.*

*Proof.* Let $W_i = (U_i, T_i) \in (X/S)_{\mathrm{inf}}$ with $i = 0, 1, 2$ and $(\alpha_i, \beta_i) : (U_i, T_i) \to (U_0, T_0)$ with $i = 1, 2$. We claim that $W_1 \times_{W_0} W_2 = (U_1 \cap U_2, T_1 \times_{T_0} T_2)$.

Firstly we have to show that $(U, T) := (U_1 \cap U_2, T_1 \times_{T_0} T_2)$ is in the category. For $i = 1, 2$ let us denote $p_i^U : U \to U_i$ and $p_i^T : T \to T_i$ the maps induced by the fiber product as rigid spaces.

The following diagrams are commutative:

$$
\begin{tikzcd}
 & U \arrow[dl, "p_1^U"'] \arrow[d, dashed, "g"] \arrow[dr, "p_2^U"] & \\
U_1 \arrow[d, "f_1"] \arrow[dr, bend left] & T \arrow[dl, "p_1^T"'] \arrow[dr, "p_2^T"] & U_2 \arrow[d, "f_2"] \arrow[dl, bend right] \\
T_1 \arrow[dr] & U_0 \arrow[d, "f_0"] & T_2 \arrow[dl] \\
 & T_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook] \arrow[dr] & U_2 \arrow[d, hook] \arrow[dr] & \\
U_1 \arrow[r, hook] \arrow[rr, bend right] & U_0 \arrow[r, hook] & X
\end{tikzcd}
.
$$

Then there is a unique induced map $g : U \to T$ s.t. that the diagram is still commutative. Once we show that $g$ is nilpotent we will have a commutative diagram in the category $(X/S)_{\mathrm{inf}}$; the fact that this is the fiber product $W_1 \times_{W_0} W_2$ could be checked rephrasing that $U = U_1 \times_{U_0} U_2$ and $T = T_1 \times_{T_0} T_2$ are fibered products of rigid spaces.

We have to show that $g : U \to T$ is nilpotent.

We can check locally on $T$ that it is a nilpotent map. Since $f_1, f_2, f_3$ are nilpotent, we can cover $T_0, T_1$ and $T_2$ with affinoid opens $\mathrm{Sp}(R_i)$ where $f_i$ corresponds to the projection for some nilpotent ideal $I_i$, say $I_i^n = 0$. Then $T$ is covered by $\mathrm{Sp}(R_1 \hat{\otimes}_{R_0} R_2)$ and the map $g$ corresponds locally on $T$ to the projection

$$\phi : R_1 \hat{\otimes}_{R_0} R_2 \to R_1/I_1 \hat{\otimes}_{R_0/I_0} R_2/I_2.$$

Since each map is surjectve we get that $g$ is closed. The first tensor product is along the morphism $\phi_i : R_0 \to R_i$ corresponding to $\beta_i$ and the other along $\phi_i \mathrm{mod} I_i$ corresponding to $\alpha_i$. The kernel of $\phi$ is $I_1 \hat{\otimes}_{R_0} R_2 + R_1 \hat{\otimes}_{R_0} I_2$ and it is nilpotent via the Lemma A.0.2, then $(U, T) \in (X/S)_{\mathrm{inf}}$ is the fiber product we looked for.

Now we prove that the category $(X/S)_{\mathrm{inf}}$ has equalizers. Let

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2) : (U_1, T_2) \rightrightarrows (U_2, T_2) \ ,$$

we claim that the pair $(U, T) := (\mathrm{Eq}(\alpha_1, \alpha_2), \mathrm{Eq}(\beta_1, \beta_2))$ formed by equalizers as rigid spaces is the equalizer for the diagram above. Observe that $U = U_1$ and denote $\beta : T \to T_1$ the natural map.

Since the diagram of the equalizer of the $U_i$ maps into the diagram of the equalizer for $T_i$, there is a map $g : U \to T$. We have to show, as before, that $g$ is nilpotent. In order to show that $g$ is closed, observe that one can build the equalizers as the fibered product

$$
\begin{tikzcd}
U \arrow[r] \arrow[d] & U_1 \arrow[d, "\alpha"] \\
U_2 \arrow[r, "\Delta_{U_2}"] & U_2 \times_S U_2
\end{tikzcd}
$$

where $\Delta_{U_2}$ is the diagonal map of $U_2$ and $\alpha = (\alpha_1, \alpha_2)$. One can proceed similarly for $T$. As before, in order to show that $g$ is nilpotent, we have to work locally on $T$. Let's work locally and assume that $T_i$ are the affinoids $\mathrm{Sp}(R_i)$, the maps $g_i$ correspond to quotient by the ideal $I_i$ with $I_i^n = 0$ and the maps $\alpha_1, \alpha_2$ correspond to $\phi_1, \phi_2 : R_2 \to R_1$. Then $g$ corresponds to the map

$$R_2 \hat{\otimes}_{R_2 \hat{\otimes}_K R_2} R_1 \to R_2/I_2 \hat{\otimes}_{R_2/I_2 \hat{\otimes} R_2/I_2} R_1/I_1$$

where the $R_2 \hat{\otimes}_K R_2$-module structure is induced by:

$$(a \otimes b) \cdot (c \otimes d) = (abc \otimes d) = c \otimes (\phi_1(a)\phi_2(b)d).$$

Since the map is surjective, we have that $g$ is closed and, as before, $g$ is nilpotent via the Lemma A.0.2.

We proved that $(U, T) \in (X/S)_{\mathrm{inf}}$. It is an equalizer since $U$ and $T$ are equalizers in the category of rigid spaces. □

Now we can define coverings on the category $(X/S)_{\mathrm{inf}}$ and the topos associated.

**Definition 1.1.4.** A **covering in the small infinitesimal site** is a collection of morphism $\{(\alpha_i, \beta_i) : (U_i, T_i) \to (U, T)\}_{i \in I}$ in the category $(X/S)_{\mathrm{inf}}$ s.t. the collections $\{\alpha_i\}_{i \in I}$ and $\{\beta_i\}_{i \in I}$ are open admissible coverings in the site of rigid spaces over $S$.

**Lemma 1.1.5.** *The coverings on $(X/S)_{\mathrm{inf}}$ describe a topology.*

*Proof.* This fact follows via the explicit description of the fiber products in the Lemma 1.1.3 and the property of coverings in the rigid category over $S$. □

We denote as $X_{\mathrm{inf}}$ the (small) infinitesimal site, *i.e.* the category of sheaves over $(X/S)_{\mathrm{inf}}$.

### 1.1.1 Pre-sheaves and sheaves in the infinitesimal topos

We give a description of the (small) infinitesimal topos. Firstly we would like to introduce some notations that (we hope) could simplify the reading of this Section.

In this section we will denote an object $g = (U, T) \in (X/S)_{\mathrm{inf}}$ as the morphism $g : U \to T$ attached to it, moreover we denote $U =: s(g)$ and $T =: t(g)$ where $s$ stands for source and $t$ stands for target. For a morphism $\gamma : g_1 \to g_2$ we denote

$$\gamma_s : s(g_1) \to s(g_2) \quad \text{and} \quad \gamma_t : t(g_1) \to t(g_2)$$

the morphisms between sources and between targets.

**Lemma 1.1.6.** *The data of a presheaf $\mathcal{F} \in PSh((X/S)_{\mathrm{inf}})$ is equivalent to:*

- *a collection of presheaves $\{\mathcal{F}_g\}_{g \in (X/S)_{\mathrm{inf}}}$, where $\mathcal{F}_g$ is a presheaf over $t(g)$ for each $g \in (X/S)_{\mathrm{inf}}$;*
- *a collection of morphisms $\{\phi_\gamma : \gamma_t^{-1}\mathcal{F}_{g_2} \to \mathcal{F}_{g_1}\}_{\gamma : g_1 \to g_2}$, where $\gamma$ varies between all the morphisms in $(X/S)_{\mathrm{inf}}$.*

*with the following properties:*

1. *if* $\gamma = \gamma_2 \circ \gamma_1$*, then* $\phi_{\gamma_1} \circ \left(\gamma_{1,t}^{-1}\phi_{\gamma_2}\right) = \phi_\gamma$*, i.e.*

$$\begin{tikzcd} \gamma_{1,t}^{-1}\left(\gamma_{2,t}^{-1}\mathcal{F}_{g_3}\right) \arrow[r, "\gamma_{1,t}^{-1}\phi_{\gamma_2}"] \arrow[d, "\cong"] & \gamma_{1,t}^{-1}\mathcal{F}_{g_2} \arrow[r, "\phi_{\gamma_1}"] & \mathcal{F}_{g_1} \\ (\gamma_t)^{-1}\mathcal{F}_{g_3} \arrow[urr, "\phi_\gamma"] & & \end{tikzcd}$$

*is a commutative diagram, where* $\gamma_1 : g_1 \to g_2$ *and* $\gamma_2 : g_2 \to g_3$*;*

2. $\phi_{id_g} = id_{\mathcal{F}_g}$.

*Moreover* $\mathcal{F}$ *is a sheaf over the small infinitesimal topos if and only if each* $\mathcal{F}_g$ *is a sheaf over* $t(g)$*.*

*Proof.* Given an object $g = (U,T) \in (X/S)_{\text{inf}}$ and an open $T_1 \subset T$, we can build $U_1 := U \times_T T_1$. The natural map $U_1 \to U$ is an open immersion and $(U_1, T_1) \in (X/S)_{\text{inf}}$.

Let $\mathcal{F}$ be a presheaf on $(X/S)_{\text{inf}}$, we build the families of presheaves and of morphisms as in the statement. For a $g = (U,T) \in (X/S)_{\text{inf}}$ we define the presheaf $\mathcal{F}_g$ as the functor

$$T_1 \in \text{Open}^{opp}(T) \mapsto \mathcal{F}(U_1, T_1) \in Set$$

where $U_1 := U \times_T T_1$. The restriction morphism of $\mathcal{F}_g$ corresponding to an open inclusion $\beta : T_1 \hookrightarrow T_2 \in \text{Open}(T)$ is $\mathcal{F}(\alpha, \beta)$, where $\alpha := \text{id}_U \times_T \beta$:

$$\begin{tikzcd} U_1 := U \times_T T_1 \arrow[r, "\alpha"] \arrow[d] & U_2 := U \times_T T_2 \arrow[d] \\ T_1 \arrow[r, "\beta"] & T_2 \end{tikzcd}.$$

We have to verify that $\mathcal{F}_g$ is actually a functor (hence a presheaf). The restriction corresponding to an identity $\beta = \text{id}_{T_1} : T_1 = T_1 \in \text{Open}(T)$, corresponds to $\mathcal{F}(\text{id}_{(U_1,T_1)}) = \text{id}_{\mathcal{F}(U_1,T_1)} = \text{id}_{\mathcal{F}_g(T_1)}$, where $U_1 := U \times_T T_1$ as above, via the functoriality of $\mathcal{F}$. The cocycle condition follows directly from the cocycle condition on $\mathcal{F}$. For a morphism $(\alpha, \beta) = \gamma : g = (U,T) \to g' = (U',T') \in (X/S)_{\text{inf}}$ we want to build $\phi_\gamma : \beta^{-1}\mathcal{F}_{g'} \to \mathcal{F}_g$. Via adjunction we can also build a map $\hat{\phi} : \mathcal{F}_{g'} \to \beta_*\mathcal{F}_g$ and two conditions on the family become (notation as in the statement):

- $\hat{\phi}_{\gamma_2} \circ \left((\gamma_{2,t})_* \hat{\phi}_{\gamma_1}\right) = \hat{\phi}_{\gamma_2 \circ \gamma_1}$;
- $\hat{\phi}_{\text{id}_g} = \text{id}_{\mathcal{F}_g}$.

Let $T_1' \subset T'$ be an open, then $T_1 := \beta^{-1}(T_1') = T_1' \times_{T'} T$ is an open of $T$; moreover $U_1' := U' \times_{T'} T_1'$ and $U_1 := U \times_{U'} U_1'$ are open of $X$. We can define

$$\left(\hat{\phi}_\gamma\right)_{T_1'} : \mathcal{F}_{g'}(T_1') = \mathcal{F}(U_1', T_1') \xrightarrow{\mathcal{F}(\gamma')} \mathcal{F}(U_1, T_1) = \mathcal{F}_g(T_1) = (\beta_*\mathcal{F}_g)(T_1')$$

where $\gamma'$ is the restriction of $\gamma$. The fact that the morphism is a pre-sheaf morphism (compatible with the restrictions) follows by the cocycle condition of $\mathcal{F}$. The cocycle condition $\hat{\phi}_{\gamma_2} \circ \left((\gamma_{2,t})_* \hat{\phi}_{\gamma_1}\right) = \hat{\phi}_{\gamma_2 \circ \gamma_1}$ and the equality $\hat{\phi}_{\text{id}_g} = \text{id}_{\mathcal{F}_g}$ follow via the functoriality of $\mathcal{F}$. We have just verified that given a presheaf we obtain the data as in the statement.

Now let $\{\mathcal{F}_g\}_g$ and $\{\phi_\gamma\}_\gamma$ as in the statement, we will define a presheaf $\mathcal{F}$ on $(X/S)_{\text{inf}}$. Given an object $(U,T) \in (X/S)_{\text{inf}}$, we define $\mathcal{F}(U,T) := \mathcal{F}_{(U,T)}(T)$. Given a morphism $(\alpha, \beta) : (U_1, T_1) \to (U_2, T_2)$, we define the restriction

$$\mathcal{F}(\alpha,\beta) : \mathcal{F}(U_2,T_2) = \mathcal{F}_{(U_2,T_2)}(T_2) \overset{\hat{\phi}_{(\alpha,\beta),T_2}}{\longrightarrow} \left(\beta_* \mathcal{F}_{(U_1,T_1)}\right)(T_2) = \mathcal{F}(U_1,T_1)$$

where $\hat{\phi}_{(\alpha,\beta)} : \mathcal{F}_{(U_1,T_1)} \to \beta_* \mathcal{F}_{(U_2,T_2)}$ is the morphism of presheaves on $T_2$ that (by adjunction) corresponds to the morphism $\phi_{(\alpha,\beta)}$ on $T_1$ given by the data. The fact that $\mathcal{F}$ is a functor (cocycle condition and restriction corresponding to the identity morphism is the identity) follows by the two conditions on the collection of morphisms. These associations are the inverse of each other.

The last thing that we have to prove is that the association sends a sheaf over $(X/S)_{\text{inf}}$ to a collection of sheaves and vice versa sends a collection of sheaves to a sheaf over $(X/S)_{\text{inf}}$.

Let $\mathcal{F}$ be a sheaf over $(X/S)_{\text{inf}}$, let $(U',T') \in (X/S)_{\text{inf}}$ and $\{T_i\}_{i\in I}$ an admissible open cover of $T$, where $T$ is an open of $T'$; we get that $\{U_i := U' \times_{T'} T_i = U \times_T T_i\}_{i\in I}$ is an open covering of $U := U' \times_{T'} T$. We have to verify the sheaf condition for $\mathcal{F}_{(U,T)}$. Since $\mathcal{F}$ is a sheaf we get that the first row of the following diagram is an equalizer

$$\begin{tikzcd}
\mathcal{F}(U,T) \arrow[r] \arrow[d, equal] & \prod_{i\in I} \mathcal{F}(U_i,T_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{(j,k)\in I^2} \mathcal{F}(U_j \cap U_k, T_j \cap T_k) \arrow[d, equal] \\
\mathcal{F}_{(U',T')}(T) \arrow[r] & \prod_{i\in I} \mathcal{F}_{(U',T')}(T_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{(j,k)\in I^2} \mathcal{F}_{(U',T')}(T_j \cap T_k)
\end{tikzcd} .$$

Then also the second row is an equalizer and $\mathcal{F}_{(U',T')}$ is a sheaf over $T'$. The converse is similar.

□

If we want to specify that $\phi_\gamma$ is attached to the sheaf $\mathcal{F}$ we write $\phi_\gamma^{\mathcal{F}}$.

### 1.1.2 Envelop

Usually the global sections of a sheaf $\mathcal{F}$ over a topos $\mathcal{C}$ are defined as the elements of $\mathcal{F}(\mathbb{1}_{\mathcal{C}})$, where $\mathbb{1}_{\mathcal{C}}$ is the terminal object of $\mathcal{C}$. Observe that in general the category $(X/S)_{\text{inf}}$ has not the terminal object; in this case one can define global sections of a sheaf $\mathcal{F}$ as

$$\Gamma(X_{\text{inf}}, \mathcal{F}) := \operatorname{Hom}_{X_{\text{inf}}}(\mathbb{1}, \mathcal{F})$$

where $\mathbb{1}$ is the terminal object in $X_{\text{inf}}$, *i.e.* is the sheafification of $\mathbb{1}^{pre}$, where

$$\mathbb{1}^{pre}(U,T) = \{*\} \quad \text{for each } (U,T) \in (X/S)_{\text{inf}} .$$

In this section we define the envelop of a closed morphism and we look for the basic properties of the envelop. The envelop construction plays a central role in this theory since it provides a hypercovering of the sheaf $\mathbb{1}$, hence a way to compute global sections of a given sheaf. Moreover via the envelop one can see that there are finite products in the ind category of $(X/S)_{\text{inf}}$.

Let $j : X \to Y$ be a closed immersion, then there is an ideal $J \subset \mathcal{O}_Y$ s.t. $j^{-1}\mathcal{O}_Y/J \cong \mathcal{O}_X$ via the morphism $j$. Let $P^n_{X/Y}$ be the rigid space given by $|X|$ as space with a Grothendieck topology and with ring of definition $\mathcal{O}_Y/J^{n+1}$. Observe that $X$ is isomorphic (via $j$) to $P^0_{X/Y}$ as rigid space, moreover the morphism $X \to P^n_{X/Y}$ is nilpotent of order $n+1$, hence $(X, P^n_{X/Y})$ is an object in the category $(X/S)_{\text{inf}}$.

**Definition 1.1.7.** The object

$$D^n_X(Y) := (X, P^n_{X/Y}) \in (X/S)_{\text{inf}}$$

is called the $n-th$ **envelop** of $X$ in $Y$.

We have already seen that $P^0_{X/Y} = X$, moreover if $X = Y$ and $j = \mathrm{id}_X$, then $P^n_{X/Y} = X$ for any $n \in \mathbb{N}$ and $D^n_X(X) = (X, X) \in (X/S)_{\mathrm{inf}}$. An interesting example is given by taking the $n$-th envelop of the diagonal morphism

$$\Delta_k : X \longrightarrow X^{\times_S^{k+1}}.$$

Remember that $X$ is smooth, hence $\Delta_k$ is a closed immersion for all $k \in \mathbb{N}$. We will denote $P^n_X(k) := P^n_{X/X^{\times_S^{k+1}}}$ and $D^n_X(k) := (X, P^n_X(k))$. Moreover $I(k) \subset \mathcal{O}_{X^{\times_S^{k+1}}}$ will denote the ideal defined via the closed immersion of $X$ in $X^{\times_S^{k+1}}$ and

$$\mathcal{P}^n_X(k) := \Delta_k^{-1}\left(\mathcal{O}_{X^{\times_S^{k+1}}}/I(k)^{n+1}\right)$$

denote the structural sheaf of ring of $P^n_X(k)$; we see it $\mathcal{P}^n_X(k)$ over $X$ since the topologies of $P^n_X(k)$ and $X$ are equal. For each $i \in \mathbb{N}_{\leqslant k}$ let us denote with $p^n_i(k)$ the morphism

$$\begin{array}{ccc} P^n_X(k) & \xrightarrow{p^n_i(k)} & P^n_X(0) \\ \big\downarrow & & \big\| \\ X^{\times_S^{k+1}} & \xrightarrow{p_i} & X \end{array}$$

induced by the $i-th$ projection $p_i$. If $k = 1$ we simplify the notation:

$$I := I(1), \qquad P^n_X := P^n_X(1) \quad \text{and} \quad \mathcal{P}^n_X := \mathcal{P}^n_X(1).$$

For $i, j \in \mathbb{N}_{\leqslant 2}$ we denote $p^n_{i,j}(2)$ the morphism

$$\begin{array}{ccc} P^n_X(2) & \xrightarrow{p^n_{i,j}(2)} & P^n_X \\ \big\downarrow & & \big\downarrow \\ X \times_S X \times_S X & \xrightarrow{p_{i,j}} & X \times_S X \end{array}$$

induced via the projection on the $i-th$ and $j-th$ coordinates (where the order does matter).

Coming back to the general case, if $j : X \to Y$ is a closed immersion defined via the ideal $J \subset \mathcal{O}_Y$, then for any $n \leqslant m$ there is a map (identity between the topologies and projection by $J^{n+1}$ as morphism between the sheaves)

$$P^n_{X/Y} \longrightarrow P^m_{X/Y}.$$

We get an inductive system $\{D^n_X(Y)\}_{n \in \mathbb{N}}$; we would like to define the envelop of $X$ in $Y$ as the colimit of the $D^n_X(Y)$, but in general this colimit does not exist in our site. Then we consider the Yoneda embedding and we compute the colimit in the infinitesimal topos.

**Definition 1.1.8.** The **envelop** of $X$ in $Y$ is

$$D_X(Y) := \underset{\substack{\longrightarrow \\ n \in \mathbb{N}}}{\mathrm{colim}}^{X_{\mathrm{inf}}} h_{D^n_X(Y)} \in X_{\mathrm{inf}}$$

where $h_{D^n_X(Y)} := \mathrm{Hom}_{(X/S)_{\mathrm{inf}}}(-, D^n_X(Y))$.

The overscript in the colimit stands to remember in which category the colimit is computed. Observe that one can compute colimits of sheaves as colimit of presheaves and then sheafify, moreover the colimit of presheaves is easy to compute. More concretely: for each object $(U, T) \in (X/S)_{\mathrm{inf}}$ there is a covering

$\{g_i = (U_i, T_i)\}$ with

$$D_X(Y)(U,T) = \left(\varinjlim_{n\in\mathbb{N}}{}^{\mathrm{PSh}((X/S)_{\inf})} h_{D_X^n(Y)}\right)(g_i) = \varinjlim_{n\in\mathbb{N}}{}^{\mathrm{Set}} \mathrm{Hom}_{(X/S)_{\inf}}(g_i, D_X^n(Y)).$$

Now we would like to give another description of the envelop $D_X(Y)$. Let $g : U \to T$ be an object in the infinitesimal category, we define

$$\mathrm{Hom}_{(X/S)_{\inf}}((U,T),(X,Y)) \coloneqq \left\{ \beta \in \mathrm{Hom}_S(T,Y) \mid \begin{tikzcd} U \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "j"] \\ T \arrow[r, "\beta"] & Y \end{tikzcd} \text{ is commutative} \right\},$$

One can verify that this functor, with the obvious restriction maps, is a sheaf in the infinitesimal topos. We denote these sheaves as

$$\mathrm{Hom}_{(X/S)_{\inf}}(\,-\,,(X,Y)).$$

We remark that the pair $(X,Y)$ is not in general in the site, as we do not require that the morphism $f$ is nilpotent.

**Lemma 1.1.9.** *If $j : X \to Y$ is a closed immersion, then*

$$D_X(Y) \cong \mathit{Hom}_{(X/S)_{\inf}}(\,-\,,(X,Y))$$

*Proof.* Let's denote $D_X^{pre}(Y) = \varinjlim_{n\in\mathbb{N}}{}^{\mathrm{PSh}(X_{\inf})} h_{D_X^n(Y)}$, the sheafification of the presheaf $D_X^{pre}(Y)$ is $D_X(Y)$. Moreover for each $(U,T) \in (X/S)_{\inf}$ one has that

$$D_X^{pre}(Y)(U,T) = \varinjlim_{n\in\mathbb{N}}{}^{Set} \mathrm{Hom}\left((U,T),(X,P^n_{X/Y})\right).$$

The commutative diagrams

$$\begin{tikzcd} P^n_{X/Y} \arrow[r, hook] \arrow[d] & Y \\ P^{n+1}_{X/Y} \arrow[ur, hook] & \end{tikzcd}$$

induce a morphism

$$D_X^{pre}(Y)(U,T) \to \mathrm{Hom}\left((U,T),(X,Y)\right).$$

This is functorial in $(U,T)$, hence we get a morphism of pre-sheaves

$$D_X^{pre}(Y) \to \mathrm{Hom}\left(\,-\,,(X,Y)\right).$$

For the universal property of the sheafification we get a natural morphism of sheaves

$$D_X(Y) \to \mathrm{Hom}\left(\,-\,,(X,Y)\right).$$

We have to show that this morphism is an isomorphism of sheaves, *i.e.* that the morphism of pre-sheaves is locally an isomorphism.

Let $(U,T) \in (X/S)_{\inf}$. Take $\beta \in \mathrm{Hom}\left((U,T),(X,Y)\right)$, *i.e.* a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"] & X \arrow[d, "j"] \\ T \arrow[r, "\beta"] & Y \end{tikzcd}.$$

We want to say that locally the map $\beta$ factors uniquely through $P^n_{X/Y}$ for some $n \in \mathbb{N}$. Let $\{(U_i, T_i)\}_{i\in I}$ be a covering of $(U, T)$ s.t. for any $i \in I$, $T_i = \mathrm{Sp}(R_i)$, $U_i = \mathrm{Sp}(R_i/I_i)$, $I^{n_i+1} = 0$, where $I_i$ is the ideal of the map $U_i \to T_i$. We have a diagram

$$\begin{tikzcd} U_i \arrow[r, "\alpha"] \arrow[dd, "g_i"] & X \arrow[d, "i"] \arrow[dd, bend left, "j"] \\ & P^{n_i}_{X/Y} \arrow[d, "h"] \\ T_i \arrow[r, "\beta_i"] & Y \end{tikzcd} .$$

Due to the commutativity of the diagram, the ideal $J \subset \mathcal{O}_Y$ is sent to the ideal $I_i \subset \mathcal{O}_{T_i}$, hence $J^{n_i+1}$ is sent to the ideal $I^{n_i+1} = 0 \subset \mathcal{O}_{T_i}$. Hence there is a unique morphism $\mathcal{P}^n_{X/Y} = j^{-1}\mathcal{O}_Y/J^{n_i+1} \to \beta_{i,*}\tilde{R}_i$ and a unique corresponding morphism of rigid spaces

$$T_i \longrightarrow P^{n_i}_{X/Y}$$

s.t. the diagram above is still commutative. Hence we showed that for any $(U, T)$ there is a cover $(U_i, T_i)$ s.t.

$$D^{pre}_X(Y)(U_i, T_i) \to \mathrm{Hom}\left((U_i, T_i), (X, Y)\right)$$

is an isomorphism, hence

$$D_X(Y) \to \mathrm{Hom}\left(\ -\ , (X, Y)\right)$$

is an isomorphism of sheaves.

□

### 1.1.3 Global sections of a sheaf

Observe that the two projection maps $p^n_0(1), p^n_1(1) : P^n_X(1) \rightrightarrows X$ make the following diagram commute for any $n_1 \leqslant n_2$ and $i = 0, 1$:

$$\begin{tikzcd} P^{n_1}_X \arrow[r, "p^{n_1}_i(1)"] \arrow[d] & X \arrow[d] \\ P^{n_2}_X \arrow[r, "p^{n_2}_i(1)"] & X \end{tikzcd} ,$$

hence we get two induced projection maps

$$D_X(1) := D_X(X \times_S X) \rightrightarrows (X, X).$$

These projection maps correspond to the two projection maps

$$\mathrm{Hom}_{(X/S)_{\mathrm{inf}}}\left(-, (X, X \times_S X)\right) \rightrightarrows \mathrm{Hom}\left(-, (X, X)\right)$$

**Lemma 1.1.10.** *The canonical morphism*

$$D_X(0) = Hom_{(X/S)_{\mathrm{inf}}}\left(-, (X, X)\right) \to \mathbb{1}$$

*is an epimorphism of sheaves, in particular*

$$\mathbb{1} = CoEq^{X_{\mathrm{inf}}}\left(D_X(1) \rightrightarrows D_X(0)\right),$$

*where the two morphisms are the two projections.*

*Proof.* Showing that a morphism of sheaves is an epimorphism requires to show (by definition) that for any object $(U,T) \in (X/S)_{\text{inf}}$ there is a covering $\{(U_i,T_i)\}_{i\in I}$ s.t. the sheaf morphism computed at $(U_i,T_i)$ is surjective.

Let $(U,T)$ be an object in $(X/S)_{\text{inf}}$, then there is a covering $\{(U_i,T_i)\}_{i\in I}$ s.t. for each $i \in I$ the map $U_i \to T_i$ is nilpotent of finite order $n_i$ and we have a diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & X \arrow[d, equal] \\ T_i & X \end{tikzcd} .$$

By smoothness of $X$ we get that there is a map

$$g_i : T_i \longrightarrow X$$

s.t. the diagram is still commutative; via the Lemma 1.1.9 we get an element $g_i \in D_X(0)(U_i,T_i)$, therefore the unique map into $\mathbb{1}(U_i,T_i) = \{*\}$ is surjective for each $i \in I$. We get that $D_X(Y) \to \mathbb{1}$ is an epimorphism of sheaves.

Observe that by Lemma 1.1.9

$$D_X(1) := D_X(X \times_S X) \cong \mathrm{Hom}_{(X/S)_{\text{inf}}}(\,-\,, X \times_S X) \cong \mathrm{Hom}_{(X/S)_{\text{inf}}}(\,-\,, X)^2 \cong D_X(0) \times_{\mathbb{1}}^{X_{\text{inf}}} D_X(0),$$

where we used that the product in a category with terminal object is the fibered product in this category along the terminal object. Since $D_X(0) \to \mathbb{1}$ is an epimorphism, then

$$\mathbb{1} \cong \mathrm{CoEq}^{X_{\text{inf}}}\left(D_X(0) \times_{\mathbb{1}}^{X_{\text{inf}}} D_X(0) \rightrightarrows D_X(0)\right) \cong \mathrm{CoEq}^{X_{\text{inf}}}\left(D_X(1) \rightrightarrows D_X(0)\right)$$

and we conclude the proof. □

**Definition 1.1.11.** Given a sheaf $\mathcal{F}$ in the topos $X_{\text{inf}}$ we define the **global sections of $\mathcal{F}$** as the set

$$\Gamma(X_{\text{inf}}, \mathcal{F}) := \mathrm{Hom}_{X_{\text{inf}}}(\mathbb{1}, \mathcal{F})$$

**Proposition 1.1.12.** *Let $\mathcal{F}$ be a sheaf in the topos $X_{\text{inf}}$, there is a canonical isomorphism*

$$\Gamma(X_{\text{inf}}, \mathcal{F}) \cong \lim_{n\in\mathbb{N}} Eq^{Set}\left(\mathcal{F}(X,X) \rightrightarrows \mathcal{F}(X, P_X^n)\right)$$

*where the morphisms in the equalizer are induced by the two projections maps.*

*Proof.* We have this chain of natural isomorphisms

$$\begin{aligned}\Gamma(X_{\text{inf}}, \mathcal{F}) = \mathrm{Hom}_{X_{\text{inf}}}(\mathbb{1}, \mathcal{F}) &\cong \mathrm{Hom}_{X_{\text{inf}}}\left(\mathrm{CoEq}^{X_{\text{inf}}}\left(D_X(X \times_S X) \rightrightarrows D_X(0)\right), \mathcal{F}\right) \\ &\cong \mathrm{Hom}_{X_{\text{inf}}}\left(\varinjlim_{n\in\mathbb{N}} \mathrm{CoEq}^{X_{\text{inf}}}\left(h_{(X,P_X^n)} \rightrightarrows h_{(X,X)}\right), \mathcal{F}\right) \\ \cong \lim_{n\in\mathbb{N}} \mathrm{Eq}^{Set}\left(\mathrm{Hom}_{X_{\text{inf}}}\left(h_{(X,X)}, \mathcal{F}\right) \rightrightarrows \mathrm{Hom}_{X_{\text{inf}}}\left(h_{(X,P_X^n)}, \mathcal{F}\right)\right) &\cong \lim_{n\in\mathbb{N}} \mathrm{Eq}^{Set}\left(\mathcal{F}(X,X) \rightrightarrows \mathcal{F}(X, P_X^n)\right)\end{aligned}$$

where the two morphisms in the equalizers are the two restriction morphisms of $\mathcal{F}$ induced by the two projection maps

$$(\mathrm{id}_X, p_0^n(1)),\ (\mathrm{id}_X, p_1^n(1)) : D_X^n(1) = \left(X, P_{X/Y}^n\right) \rightrightarrows (X, X) = D_X^n(0)$$

□

If $X$ is not smooth but there is a smooth variety $Y$ and a closed morphism $X \to Y$ one can compute global sections of a sheaf $\mathcal{F} \in X_{\text{inf}}$ as

$$\Gamma(X_{\text{inf}}, \mathcal{F}) \cong \lim_{n \in \mathbb{N}} \mathrm{Eq}^{Set}\left(\mathcal{F}(D_X^n(Y)) \rightrightarrows \mathcal{F}(D_X^n(Y \times_S Y)\right).$$

For the details see the Lemma 2.2.6 of [Guo21].

## 1.2 Crystals

In this Section we introduce the notions of quasi coherent sheaf and crystal in the infinitesimal topos. We use the description of sheaves over the infinitesimal site done in the Lemma 1.1.6. We also introduce the notions of differential operators and connections in order to give a more computable description of the global sections of a crystal.

Let $f : X \to S$ be a smooth rigid space over $S = \mathrm{Sp}(K)$.

**Definition 1.2.1.** Let $\mathcal{O}_{X_{\text{inf}}}$ be the **structural sheaf** in the topos $X_{\text{inf}}$ defined by $\mathcal{O}_{X_{\text{inf}},(U,T)} := \mathcal{O}_T$ together with the restriction morphisms

$$\phi^{\mathcal{O}}_{(\alpha,\beta)} : \beta^{-1}\mathcal{O}_{T_2} \to \mathcal{O}_{T_1}$$

induced by $\beta$ for any $(\alpha, \beta) : (U_1, T_1) \to (U_2, T_2) \in (X/S)_{\text{inf}}$.

**Definition 1.2.2.** An $\mathcal{O}_{X_{\text{inf}}}$**-module** is a sheaf $\mathcal{F} \in X_{\text{inf}}$ s.t. for each $(U, T) \in (X/S)_{\text{inf}}$, the sheaf $\mathcal{F}_{(U,T)}$ is an $\mathcal{O}_T$-module and for each $(\alpha, \beta) : (U_1, T_1) \to (U_2, T_2) \in (X/S)_{\text{inf}}$, the morphism

$$\phi^{\mathcal{F}}_{(\alpha,\beta)} : \beta^{-1}\mathcal{F}_{(U_2,T_2)} \to \mathcal{F}_{(U_1,T_1)}$$

is $\beta^{-1}\mathcal{O}_{T_2}$-linear; where $\mathcal{F}_{(U_1,T_1)}$ is a $\beta^{-1}\mathcal{O}_{T_2}$-module via the ring morphism $\phi^{\mathcal{O}}_{(\alpha,\beta)}$ induced by $\beta$ and the $\mathcal{O}_{T_1}$-module structure.

A **quasi-coherent/coherent** sheaf in the topos $X_{\text{inf}}$ is an $\mathcal{O}_{X_{\text{inf}}}$-module $\mathcal{F}$ where $\mathcal{F}_{(U,T)}$ is a quasi coherent/coherent $\mathcal{O}_T$-module for each $(U, T) \in (X/S)_{\text{inf}}$.

Observe that for an $\mathcal{O}_{X_{\text{inf}}}$-module $\mathcal{F}$ the restriction morphisms induce the morphisms

$$\begin{aligned} \varphi^{\mathcal{F}}_{(\alpha,\beta)} : \beta^*\mathcal{F}_{(U_2,T_2)} = \beta^{-1}\mathcal{F}_{(U_2,T_2)} \otimes_{\beta^{-1}\mathcal{O}_{T_2}} \mathcal{O}_{T_1} &\longrightarrow \mathcal{F}_{(U_1,T_1)} \\ (x \otimes a) &\longmapsto a\phi^{\mathcal{F}}_{(\alpha,\beta)}(x) \end{aligned}$$

of $\mathcal{O}_{T_1} - module$

**Definition 1.2.3.** A **crystal** in $X_{\text{inf}}$ is a coherent $\mathcal{O}_{X_{\text{inf}}}$-module $\mathcal{F} \in X_{\text{inf}}$ s.t. the morphisms $\varphi^{\mathcal{F}}_{(\alpha,\beta)}$ are isomorphisms for each morphism $(\alpha, \beta) \in (X/S)_{\text{inf}}$.

Let $\mathcal{F} \in X_{\text{inf}}$ be a crystal, then the morphisms (for $i = 0, 1$)

$$(\mathrm{id}_X, p_i^n(1)) : D_X^n(1) \rightrightarrows D_X^n(0) = (X, X)$$

induce $\mathcal{P}_X^n$-linear isomorphisms (by definition of a crystal)

$$\epsilon_n^{\mathcal{F}} := \left(\varphi_{p_{1,n}}^{\mathcal{F}}\right)^{-1} \circ \varphi_{p_{0,n}}^{\mathcal{F}} : p_0^n(1)^*\mathcal{F}_{(X,X)} \xrightarrow{\cong} \mathcal{F}_{D_X^n(1)} \xrightarrow{\cong} p_1^n(1)^*\mathcal{F}_{(X,X)}.$$

It is good to have in mind that, as $\mathcal{P}_X^n$-module

$$p_0^n(1)^*\mathcal{F}_{(X,X)} = \mathcal{F}_{(X,X)} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \quad \text{and} \quad p_1^n(1)^*\mathcal{F}_{(X,X)} = \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F}_{(X,X)}.$$

**Remark 1.2.4.** If $\mathcal{F} \in X_{\text{inf}}$ is a crystal and $X$ is smooth over $S$, then

$$\Gamma\left(X_{\text{inf}}, \mathcal{F}\right) = \{x \in \mathcal{F}(X,X) \mid \epsilon_n^{\mathcal{F}}(p_0^n(1)^*x) = p_1^n(1)^*x \text{ for all } n \in \mathbb{N}\}.$$

Observe that the Remark follows from the Proposition 1.1.12, indeed for any $n \in \mathbb{N}_{>0}$

$$\{x \in \mathcal{F}(X,X) \mid \epsilon_n^{\mathcal{F}}(p_0^n(1)^*x) = p_1^n(1)^*x\} \subset \{x \in \mathcal{F}(X,X) \mid \epsilon_{n-1}^{\mathcal{F}}(p_0^{n-1}(1)^*x) = p_1^{n-1}(1)^*x\},$$

because $\epsilon_{n-1}^{\mathcal{F}} = \epsilon_n^{\mathcal{F}} \mod I^n$.

Now we would like to prove that for a crystal over a smooth adic space $X$ the global sections are

$$\Gamma\left(X_{\text{inf}}, \mathcal{F}\right) = \{x \in \mathcal{F}(X,X) \mid \epsilon_{\mathcal{F}}^1(p_0^1(1)^*x) = p_1^1(1)^*x\}. \tag{1.1}$$

*i.e.* if $x \in \mathcal{F}(X,X)$ satisfies $\epsilon_n^{\mathcal{F}}(p_0^n(1)^*x) = p_1^n(1)^*x$ with $n = 1$, then it satisfies the condition for all $n \in \mathbb{N}$. In order to proof the formula above we have to define what a differential operator of a certain order is and investigate some properties.

### 1.2.1 Differential operators over a smooth rigid space

Here we will define what a stratification $\{\epsilon_n^{\mathcal{F}}\}_{n\in\mathbb{N}}$ over an $\mathcal{O}_X$-module $\mathcal{F}$ is and differential operators of certain orders between two $\mathcal{O}_X$-modules. Moreover given a stratification we will define a ring homomorphism

$$\operatorname{Diff}^{\bullet}\left(\mathcal{O}_X, \mathcal{O}_X\right) \longrightarrow \operatorname{Diff}^{\bullet}\left(\mathcal{F}, \mathcal{F}\right).$$

We will use these tools in order to prove the formula (1.1) that computes global sections of a crystal. We will use the infinitesimal neighbourhood of $X$ in the self product of $X$, moreover we denote

$$P := \mathcal{O}_X \hat{\otimes}_{f^{-1}\mathcal{O}_S} \mathcal{O}_X = \Delta^{-1}\mathcal{O}_{X\times_S X}, \quad P(k) := \mathcal{O}_X^{\hat{\otimes}_{f^{-1}\mathcal{O}_S}^{k+1}} = \Delta_k^{-1}\mathcal{O}_{X^{\times_S^{k+1}}},$$

they are sheaves of rings over $X$. With a little abuse of notation we denote $I(k) := \Delta_k^{-1}I(k) \subset P(k)$ and $I := I(1)$. Recall that for any $k, n \in \mathbb{N}$,

$$\mathcal{P}_X^n(k) = P(k)/I(k)^{n+1}, \quad \mathcal{P}_X^n = \mathcal{P}_X^n(1), \quad \mathcal{P}_X := \lim_n \mathcal{P}_X^n.$$

$\mathcal{P}_X^n(k)$ have a structure of $P(k)$-algebra and $k+1$ structures of $\mathcal{O}_X$-algebra given by the multiplication by elements of $\mathcal{O}_X$ in the different components, *i.e.* if $i \in \mathbb{N}_{\leqslant k}$ and $p_i : X^{\times_S^{k+1}} \to X$ is the projection on the $i-th$ component, then the morphism

$$\mathcal{O}_X = \Delta_k^{-1}p_i^{-1}\mathcal{O}_X \longrightarrow \Delta_k^{-1}\mathcal{O}_{X^{\times_S^{k+1}}} = P(k)$$

gives the $i-th$ structure. We will refer to the left and right structure of $P$ as (respectively) the $0th$ and the $1st$ structure. The sheaf of differentials is $\Omega^1_{X/S} := I/I^2$.

Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules, then they are $f^{-1}\mathcal{O}_S$-modules. Let $D : \mathcal{F} \to \mathcal{G}$ be an $f^{-1}\mathcal{O}_S$-linear morphism of sheaves.

When we write $P \otimes_{\mathcal{O}_X} \mathcal{F}$ or $\mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F}$ we tensor w.r.t. the right $\mathcal{O}_X$-module structure of $P$, $\mathcal{P}^n_X$ and we use the left $\mathcal{O}_X$-module structure of $P$, $\mathcal{P}^n_X$ in order to consider these sheaves as $\mathcal{O}_X$-module. Let

$$\begin{aligned} t_{\mathcal{F}} : \mathcal{F} &\longrightarrow P \otimes_{\mathcal{O}_X} \mathcal{F} \\ x &\longmapsto (1 \otimes 1) \otimes x \end{aligned}.$$

$t_{\mathcal{F}}$ is $\mathcal{O}_X$-linear w.r.t. the right module structure of $P \otimes_{\mathcal{O}_X} \mathcal{F}$. Observe that $D$ is $\mathcal{O}_X$-linear if and only if it factors through $P/I \otimes_{\mathcal{O}_X} \mathcal{F}$, *i.e.* if an $\mathcal{O}_X$-linear dot arrow exists in the diagram below.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "D"] \arrow[d, "t_{\mathcal{F}}"] & \mathcal{G} \\ P \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & P/I \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{F} \arrow[u, dashed] \end{tikzcd}.$$

A differential operator is a quasi $\mathcal{O}_X$-linear morphism, in the sense that it is a morphism that factors through a higher power of $I$.

**Definition 1.2.5.** An $f^{-1}\mathcal{O}_S$-linear morphism of $\mathcal{O}_X$-modules $D : \mathcal{F} \to \mathcal{G}$ is a **differential operator** of order $n$ if $D$ factors through $\mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F}$, *i.e.* if an $\mathcal{O}_X$-linear map $\overline{D}^n : \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \to G$ exists s.t. the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "D"] \arrow[d, "t_F"] & \mathcal{G} \\ P \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[u, "\overline{D}^n"'] \end{tikzcd}$$

is commutative. We denote the set of such operators as $\mathrm{Diff}^n(\mathcal{F}, \mathcal{G})(X)$ and

$$\mathrm{Diff}^{\bullet}(\mathcal{F}, \mathcal{G})(X) := \bigcup_{n \in \mathbb{N}} \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})(X)$$

Observe that if $\overline{D}^n$ exists it is unique, indeed $\overline{D}^n((a \otimes b) \otimes x) = a\overline{D}^n(t_{\mathcal{F}}(bx)) = aD(bx)$.

**Remark 1.2.6.** There is an isomorphism

$$\begin{aligned} \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})(X) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \\ D &\longmapsto \overline{D}^n \end{aligned}$$

with inverse $\overline{D}^n \mapsto \overline{D}^n \circ t^n_F$.

**Remark 1.2.7.** The rules

$$U \mapsto \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})(U) := \mathrm{Diff}^n(\mathcal{F}_{|U}, \mathcal{G}_{|U})(U)$$

and

$$U \mapsto \mathrm{Diff}^{\bullet}(\mathcal{F}_{|U}, \mathcal{G}_{|U})(U), \tag{1.2}$$

where $U \subset X$ is any open, define sheaves of $\mathcal{O}_X$-algebras $\mathrm{Diff}^n(\mathcal{F}, \mathcal{G})$ and $\mathrm{Diff}^{\bullet}(\mathcal{F}, \mathcal{G})$, where the $\mathcal{O}_X$-algebra's structure is induced via the $\mathcal{O}_X$-module structure on $\mathcal{G}$.

**Lemma 1.2.8.** *Since $X$ is smooth, there is an affinoid open covering $\{U_i = \mathrm{Sp}(A_i)\}$ of $X$ with each $A_i$ of the following form*

$$A = \frac{K < x_1, \ldots, x_d, y_1, \ldots, y_s >_{T_1,\ldots,T_s}}{(f_1, \ldots, f_{s_i})}, \quad \det\left(\frac{\partial f_k}{\partial y_j}\right)_{(k,j)} \in A^\times, \quad T_1, \ldots, T_s \in K < x_1, \ldots, x_d >;$$

*in particular* $\Omega^1_{U_i/S} = \mathcal{O}_U \,\mathrm{d}\, x_1 \oplus \ldots \mathcal{O}_U \,\mathrm{d}\, x_d$ *and* $\mathcal{P}^n_{U_i} \cong \mathcal{O}_{U_i}[\mathrm{d}\, x_1, \ldots \mathrm{d}\, x_d]^{\leqslant n}$*; where* $\mathrm{d}\, x_i := (1 \otimes x_i - x_i \otimes 1) \mathrm{mod}\, I^2$.

*Proof.* Since $X$ is smooth over $S$, then (by Corollary 1.6.10, Proposition 1.7.1, Corollary 1.7.2 in [Hub96] ) there is an open affinoid covering $\{U = \mathrm{Sp}(A)\}_{U \in \mathcal{U}}$ of $X$ s.t. $A$ is in the same form as in the statement. If $A = K < x_1, \ldots, x_d >$, then $A \hat{\otimes}_K A = K < x_1, \ldots, x_d, x'_1, \ldots, x'_d >$ and

$$x'_i = x_i + (x'_i - x_i) = x_i + \mathrm{d}\, x_i;$$

hence

$$K < x_1, \ldots, x_d, x'_1, \ldots, x'_d >= K < x_1, \ldots, x_d, \mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d >$$

and going modulo $I^{n+1}$ one gets the result for this special case.

For the general case, observe that the map $K < x_1, \ldots, x_d > \to A$. is étale, then $\Omega^1_{A/K} = \oplus_{i=1}^d A \,\mathrm{d}\, x_i$. In particular one can express each $\mathrm{d}\, y_j \mod I^2$ in terms of $\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d$. By induction we get that

$$\mathrm{d}\, y_j \in P(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d) + Q(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d, \mathrm{d}\, y_1, \ldots, \mathrm{d}\, y_s) + I^{n+1}$$

where $P$ and $Q$ are polynomials with coefficients in $A$ and $Q$ is homogeneus of degree $n$. Using that we can express each $\mathrm{d}\, y_j$ in terms of $\mathrm{d}\, x_i$ and that $Q$ has no terms of degree $\leqslant n$ we conclude that

$$\mathrm{d}\, y_j \in P'(\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d) + I^{n+1},$$

then $\mathcal{P}^n_{U_i} \cong \mathcal{O}_{U_i}[\mathrm{d}\, x_1, \ldots, \mathrm{d}\, x_d]^{\leqslant n}$. □

**Lemma 1.2.9.** *For each $n, m \in \mathbb{N}$ the map of $\mathcal{O}_X$-algebras (with respect to both left or both right $\mathcal{O}_X$-module structures)*

$$\delta : P \longrightarrow P \otimes_{\mathcal{O}_X} P$$

$$a \otimes b \longmapsto (a \otimes 1) \otimes (1 \otimes b)$$

*induces an $\mathcal{O}_X$-linear map (with respect to both left or both right $\mathcal{O}_X$-module structures )*

$$\delta^{n,m} : \mathcal{P}^{n+m}_X \longrightarrow \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{P}^m_X.$$

*Proof.* One has to show that $I^{n+m+1}$ is in the kernel of the composition

$$P \xrightarrow{\delta} P \otimes_{\mathcal{O}_X} P \to \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{P}^m_X,$$

where the second map is the projection morphism. The ideal $I^{n+m+1}$ is generated by elements $\prod_{i=0}^{n+m} d' a_i$ with $d' a_i = (1 \otimes a_i) - (a_i \otimes 1)$. (Observe that $d' a$ modulo $I^2$ is $\mathrm{d}\, a \in I/I^2 = \Omega^1_{X/S}$, this justifies the notation).

Moreover

$$\delta(d' a_i) = ((1 \otimes 1 \otimes 1 \otimes a_i) - (1 \otimes 1 \otimes a_i \otimes 1)) + ((1 \otimes a_i \otimes 1 \otimes 1) - (a_i \otimes 1 \otimes 1 \otimes 1)),$$

we used that the tensor product in the middle is over $\mathcal{O}_X$, then $1 \otimes a_i \otimes 1 \otimes 1 = 1 \otimes 1 \otimes a_i \otimes 1$.

So

$$\delta\left(\prod_{i=0}^{n+m+1} d'a_i\right) = \prod_{i=0}^{n+m+1} \delta(d'a_i) = \prod_{i=0}^{n+m+1} (1 \otimes 1 \otimes d'a_i) + (d'a_i \otimes 1 \otimes 1)$$

If one expands the product each term has at least $n+1$ of $d'a_i$ on the right or $m+1$ of $d'a_i$ on the left, then

$$\delta\left(\prod_{i=0}^{n+m+1} d'a_i\right) \in I^n \otimes_{\mathcal{O}_X} P + P \otimes_{\mathcal{O}_X} I^m$$

is sent to zero via the projection $P \otimes_{\mathcal{O}_X} P \to \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{P}_X^m$. □

Thanks to this Lemma we can say that a composition of two differential operators $D_1, D_2$ of order $n_1, n_2$ is a differential operator of order $n_1 + n_2$ and we can give a description of $\overline{D_2 \circ D_1}^{n_1+n_2}$ in terms of $\overline{D_1}^{n_1}$ and $\overline{D_2}^{n_2}$.

**Lemma 1.2.10.** *If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are three $\mathcal{O}_X$-modules, for each $n, m \in \mathbb{N}$ the composition map*

$$\mathrm{Diff}^n(\mathcal{F}, \mathcal{G}) \times \mathrm{Diff}^m(\mathcal{G}, \mathcal{H}) \to \mathrm{Diff}^{n+m}(\mathcal{F}, \mathcal{H})$$

*sending $(D_2, D_1)$ to $D_2 \circ D_1$ is well defined. If $D_1$ and $D_2$ are global sections $\overline{D_2 \circ D_1}^{n_1+n_2}$ is the composition*

$$\mathcal{P}_X^{n+m} \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\delta^{n,m} \otimes id_{\mathcal{F}}} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{P}_X^m \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{id_{\mathcal{P}_X^n} \otimes \overline{D_1}^m} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{G} \xrightarrow{\overline{D_2}^n} \mathcal{H}.$$

*if $D_1, D_2$ are defined on an open $U \subset X$, then*

$$\overline{D_2 \circ D_1}^{n_1+n_2} = \overline{D_2}^n \circ \left(id_{\mathcal{P}^n_{X|U}} \otimes \overline{D_1}^m\right) \circ \left(\delta_U^{n,m} \otimes id_{\mathcal{F}_{|U}}\right).$$

*Proof.* Without loss of generality we can prove the Lemma for global sections $D_1 \in \mathrm{Diff}^m(\mathcal{F}, \mathcal{G})(X)$ and $D_2 \in \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})(X)$. Let's consider the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[rrr] \arrow[d] & & & \mathcal{H} \\
\mathcal{P}_X^{n+m} \otimes \mathcal{F} \arrow[r, "\delta^{n,m} \otimes \mathrm{id}_{\mathcal{F}}"] & \mathcal{P}_X^n \otimes \mathcal{P}_X^m \otimes \mathcal{G} \arrow[r, "\mathrm{id}_{\mathcal{P}_X^n} \otimes \overline{D_1}^m"] & \mathcal{P}_X^n \otimes \mathcal{H} \arrow[ur, bend right, "\overline{D_2}^n"] &
\end{tikzcd}.$$

It is commutative, indeed

$$\begin{aligned}
a(D_2 \circ D_1)(bx) &= \overline{D_2}^n \left((a \otimes 1) \otimes \overline{D_1}^m (1 \otimes b \otimes x)\right) \\
&= \overline{D_2}^n \circ \left(\mathrm{id}_{\mathcal{P}_X^n} \otimes \overline{D_1}^m\right) ((a \otimes 1) \otimes (1 \otimes b) \otimes x) \\
&= \overline{D_2}^n \circ \left(\mathrm{id}_{\mathcal{P}_X^n} \otimes \overline{D_1}^m\right) \circ (\delta^{n,m} \otimes \mathrm{id}_{\mathcal{F}}) (a \otimes b \otimes x)
\end{aligned}$$

and the Lemma follows. □

**Definition 1.2.11.** A **stratification** on an $\mathcal{O}_X$-module $\mathcal{F}$ is a collection $\{\epsilon_n\}_{n \in \mathbb{N}}$ of $P$-linear isomorphism

$$\epsilon_n : p_1^n(1)^* \mathcal{F} = \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n = p_0^n(1)^* \mathcal{F}$$

with the following properties:

- (compatibility) for all $n \leqslant m$ the following diagram commutes

$$\begin{tikzcd} \mathcal{P}_X^m \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\epsilon_m"] \arrow[d] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^m \arrow[d] \\ \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\epsilon_n"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \end{tikzcd},$$

  where the vertical maps are induced by the projections;

- (cocycle condition) for all $n \in \mathbb{N}$

$$\left(p_{0,1}^n(2)^*\epsilon_n\right) \circ \left(\left(p_{1,2}^n(2)^*\epsilon_n\right) = p_{0,2}^n(2)^*\epsilon_n. \tag{1.3}$$

- (identity) $\epsilon_0$ is the identity.

The cocycle condition is well defined because

$$p_{1,2}^n(2)^* p_0^n(1)^* \mathcal{F} = \left(p_0^n(1) \circ p_{1,2}^n(2)\right)^* \mathcal{F} = \left(p_1^n(1) \circ p_{0,1}^n(2)\right)^* \mathcal{F} = p_{0,1}^n(2)^* p_1^n(1)^* \mathcal{F},$$

and similarly

$$p_{1,2}^n(2)^* p_1^n(1)^* = p_{0,2}^n(2)^* p_1^n(1)^* \quad \text{and} \quad p_{0,1}^n(2)^* p_0^n(1)^* = p_{0,2}^n(2)^* p_0^n(1)^*.$$

The identity condition is well defined through the identification $\mathcal{P}_X^0 = \mathcal{O}_X$.

Given a stratification $\{\epsilon_n\}_{n\in\mathbb{N}}$ on $\mathcal{F}$ we want to define a ring homomorphism

$$\mathrm{Diff}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}^\bullet(\mathcal{F}, \mathcal{F}).$$

Let $\partial \in \mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X)(X)$, then we can consider the map

$$\begin{tikzcd} \mathcal{F} \arrow[d, "t_{\mathcal{F}}"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{F} \\ \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\epsilon_n"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \arrow[u, "\mathrm{id}_{\mathcal{F}}\otimes\overline{\partial}^n"'] \end{tikzcd}.$$

By definition $\epsilon_n$ is $P$-linear, then

$$\overline{\nabla^n(\partial)} \coloneqq \left(\mathrm{id}_{\mathcal{F}} \otimes \overline{\partial}^n\right) \circ \epsilon_n$$

is $\mathcal{O}_X$-linear w.r.t. the left structure, then

$$\nabla^n(\partial) \coloneqq \overline{\nabla^n(\partial)} \circ t_{\mathcal{F}} \in \mathrm{Diff}^n(\mathcal{F}, \mathcal{F})(X).$$

We can do the same on open subsets $U \subset X$, hence we just defined a morphism

$$\nabla^n : \mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}^n(\mathcal{F}, \mathcal{F}).$$

Via the compatibility of $\{\epsilon_n\}_{n\in\mathbb{N}}$ the $\nabla^n$ agrees and there is a morphism

$$\nabla : \mathrm{Diff}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}^\bullet(\mathcal{F}, \mathcal{F}).$$

**Proposition 1.2.12.** *Given a stratification $\{\epsilon_n\}_{n\in\mathbb{N}}$ over an $\mathcal{O}_X$-module $\mathcal{F}$, the morphism*

$$\nabla : \mathrm{Diff}^{\bullet}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}^{\bullet}(\mathcal{F}, \mathcal{F})$$

*defined above is a morphism of $\mathcal{O}_X$-algebras, where the $\mathcal{O}_X$-modules structures are the ones defined in the Remark 1.2.7.*

*Proof.* As before we prove the statement globally and one can use the same proof for any open $U \subset X$. Let $\partial \in \mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X)(X)$. The map $\nabla_X$ is a morphism of $\mathcal{O}_X(X)$-modules since $\overline{\partial}^n$ is (left-)$\mathcal{O}_X$-linear and $\epsilon_n$ is $P$-linear.

Showing that $\nabla$ is an algebra morphism requires the cocycle condition on $\{\epsilon_n\}_{n\in\mathbb{N}}$. Let $\partial_1, \partial_2 \in \mathrm{Diff}^{\bullet}(\mathcal{O}_X, \mathcal{O}_X)(X)$ of order $n$ and $m$. Let's consider the following diagram

$$\begin{tikzcd}
\mathcal{P}_X^{n+m} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\delta^{n,m}"] \arrow[dd, "\epsilon_{n+m}"] & \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{P}_X^m \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d, "\mathrm{id}_{\mathcal{P}_X^n} \otimes \epsilon_m"] & & \\
 & \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^m \arrow[r, "\mathrm{id}_{(\mathcal{P}_X^n \otimes \mathcal{F})} \otimes \overline{\partial_2}^m"] \arrow[d, "\epsilon_n \otimes \mathrm{id}_{\mathcal{P}_X^m}"] & \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d, "\epsilon_n"] & \\
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^{n+m} \arrow[r, "\mathrm{id}_{\mathcal{F}} \otimes \delta^{n,m}"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{P}_X^m \arrow[r, "\mathrm{id}_{\mathcal{F} \otimes \mathcal{P}_X^n} \otimes \overline{\partial_2}^m"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \arrow[r, "\mathrm{id}_{\mathcal{F}} \otimes \overline{\partial_1}^n"] & \mathcal{F}
\end{tikzcd}.$$

By the Lemma 1.2.10 applied to $\partial_1 \circ \partial_2$ and $\nabla(\partial_1) \circ \nabla(\partial_2)$ we know that the path on the bottom of the diagram is $\nabla(\partial_1 \circ \partial_2)$ and the path on the upper part of the diagram is $\nabla(\partial_1) \circ \nabla(\partial_2)$. Hence we have to check that the two squares in the diagram are commutative.

The first square is commutative by the compatibility and the cocycle condition on the stratification.

The second square is commutative because $\epsilon_n$ is right (and left) $\mathcal{O}_X$-linear. □

**Lemma 1.2.13.** *The $\mathcal{O}_X$-algebra $\mathrm{Diff}^{\bullet}(\mathcal{O}_X, \mathcal{O}_X)$ is locally generated by the elements of degree 1.*

*Proof.* Since the statement is local, let's assume that we have a system of coordinates $x_1, \ldots, x_d$. We get that

$$\mathcal{P}_X^n \cong \bigoplus_{k=0}^{n} Sym^k(I/I^2) = \mathcal{O}_X[\mathrm{d}\,x_1, \ldots \mathrm{d}\,x_d]^{deg \leqslant n}.$$

Remember that

$$\mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^n, \mathcal{O}_X)$$

as $\mathcal{O}_X$-modules. For a $q \in \mathbb{N}^d$ let's denote $\overline{D_q} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}_X^n, \mathcal{O}_X)$ the dual element of $\mathrm{d}\,x^q := (\mathrm{d}\,x_1)^{q_1} \ldots (\mathrm{d}\,x_d)^{q_d} \in \mathcal{P}_X^n$ and $\partial_{x_i}$ the dual element of $\mathrm{d}\,x_i$.

Via the isomorphism above $\mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X)$ is generated as module (hence as algebra) by all the $D_q$ with

$$|q|_1 := \sum_{i=1}^{d} q_i \leqslant n.$$

Define $q! := \prod_{i=1}^{d} q_i!$. One can check via the composition formula 1.2.10, as in [BO78], that

$$\overline{D_q \circ D_{q'}}(\mathrm{d}\,x^{q''}) = \frac{(q')!(q'+q)!}{q!} \overline{D_{q+q'}}(\mathrm{d}\,x^{q''}).$$

So

$$D_q \circ D_{q'} = \binom{q+q'}{q} D_{q+q'}.$$

Then the elements $\partial_{x_i}$ commute and by induction one get that for any $q \in \mathbb{N}^d$

$$D_q = \frac{1}{q!} \prod_{i=1}^{n} \partial_{x_i}^{q^i},$$

and $\mathrm{Diff}^1(\mathcal{O}_X, \mathcal{O}_X)$ generates the whole algebra of differential operators as stated. □

### 1.2.2 Connections

**Definition 1.2.14.** A **connection** on an $\mathcal{O}_X$-module $\mathcal{F}$ is an $f^{-1}\mathcal{O}_S$-linear morphism

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

s.t. $\nabla(ax) = a\nabla x + x \otimes \mathrm{d}\, a$ for each $x \in \mathcal{F}(U)$, $a \in \mathcal{O}_X(U)$ and $U \subset X$ open.

In what follows we will not write $U$ in order to simplify the notation. Observe that a connection is a differential of order 1:

$$\nabla(abx) = a\nabla(bx) + b\nabla(ax) - ab\nabla(x),$$

hence

$$\overline{\nabla}(ab \otimes x - a \otimes bx - b \otimes ax + 1 \otimes abx) = 0$$

and $\overline{\nabla}(I^2 \otimes_{\mathcal{O}_X} \mathcal{F}) = 0$.

**Definition 1.2.15.** For each $k \in \mathbb{N}_{>0}$ define

$$\begin{aligned} \nabla^k : \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} &\longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \\ x \otimes \omega &\longmapsto \nabla(x) \wedge \omega + x \otimes \mathrm{d}^k\, \omega \end{aligned}.$$

The map $K \coloneqq \nabla^1 \circ \nabla$ is called the **curvature** of $\nabla$. The connection $\nabla$ is said to be **integrable** if $K = 0$.

By explicit computations, similar to the ones given before for $\nabla$, the morphisms $\nabla^k$ are differentials of order 1 for each $k \in \mathbb{N}_{>0}$. Locally, where there are coordinates $x_1, \dots, x_d$ on $U \subset X$, the sheaf $\Omega^1_{U/S} \cong \bigoplus_{i=1}^d \mathcal{O}_U\, \mathrm{d}\, x_i$ and the connection $\nabla_{|_U} : \mathcal{F}_{|_U} \to \mathcal{F}_{|_U} \otimes_{\mathcal{O}_U} \Omega^1_{U/S}$ is given by

$$\nabla_{|_U}(x) = \sum_{i=1}^{d} \nabla_{\partial x_i}(x)\, \mathrm{d}\, x_i$$

for some maps $\nabla_{\partial x_i} : \mathcal{F}_{|_U} \to \mathcal{F}_{|_U}$ uniquely determined by the coordinates. We can also write these local formulas for $\nabla^k$:

$$\nabla^k_{|_U}(x \otimes \omega) = \sum_{i=1}^{d} \nabla_{\partial x_i}(x)\ \mathrm{d}\, x_i \wedge \omega + x\ \mathrm{d}^k\, \omega.$$

Locally the connection is given by

$$K_{|_U}(x) = \nabla^1\left(\sum_{i=1}^{d} \nabla_{\partial x_i}(x)\ \mathrm{d}\, x_i\right) = \sum_{i,j=1}^{d} \nabla_{\partial x_j} \circ \nabla_{\partial x_i}(x)\ \mathrm{d}\, x_j \wedge \mathrm{d}\, x_i = \sum_{j=1}^{d}\sum_{i=1}^{d} [\nabla_{\partial x_j}, \nabla_{\partial x_i}]\ \mathrm{d}\, x_j \wedge \mathrm{d}\, x_i.$$

Then $K_{|_U} = 0$ if and only if $\nabla_{\partial x_i}$ commutes. Via the local formulas of $\nabla^k_{|_U}$ one can check that if $K_{|_U} = 0$, then $\nabla^{k+1}_{|_U} \circ \nabla^k_{|_U} = 0$.It can be checked on elements of the form $x\ \mathrm{d}\, x_{i_1} \wedge \cdots \wedge \mathrm{d}\, x_{i_k}$, so that $\mathrm{d}^k\, \omega$ vanishes. Hence if

$\nabla$ is an integrable connection over $\mathcal{F}$, we get a complex

$$\mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla^1} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} \xrightarrow{\nabla^2} \dots \ .$$

Now we want to define the stratification attached to an integrable connection. Let's consider the map

$$\begin{aligned} \epsilon_1 : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F} &\longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X \\ (a \otimes b)x &\longmapsto (a \otimes b)\left(\nabla x + x \otimes (1 \otimes 1)\right) = abx \oplus ((a \otimes b)\nabla x \ + ax \ \mathrm{d}\, b) \end{aligned} .$$

**Lemma 1.2.16.** *If $\nabla$ is an integrable connection, the $P$-linear isomorphism*

$$\epsilon_1 : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X,$$

*satisfies the cocycle condition for a stratification (1.3), where $n = 1$.*

*Proof.* The fact that $\epsilon_1$ is an isomorphism could be seen by the explicit computations above: $\epsilon_{1|\mathcal{F}}(x) = x \oplus \nabla x$ and $\epsilon_{1|I/I^2 \otimes_{\mathcal{O}_X} \mathcal{F}}(\mathrm{d}\, a \otimes y) = 0 \oplus y \otimes \mathrm{d}\, a$, then one can write explicitly the (local) inverse as

$$\epsilon_1^{-1}(x \oplus y \otimes \mathrm{d}\, a) = x \oplus \left(\mathrm{d}\, a \otimes y - \sum_{i=1}^{d} \mathrm{d}\, x_i \otimes \nabla_{\partial_{x_i}}(x)\right),$$

where $x_1, \dots, x_d$ are local coordinates, indeed

$$\epsilon_1\left(x \oplus (\mathrm{d}\, a \otimes y - \sum_{i=1}^{d} \mathrm{d}\, x_i \otimes \nabla_{\partial_{x_i}}(x))\right) = x \oplus \nabla x + 0 \oplus y \otimes \mathrm{d}\, a - 0 \oplus \sum_{i=1}^{d} x \otimes \mathrm{d}\, x_i = x \oplus y \otimes \mathrm{d}\, a.$$

The $P$-linearity is immediate by definition. The cocycle condition could be checked locally, so let's assume that $X$ has coordinates $x_1, \dots, x_d$, then we have to check that the following diagram commutes (notations as in (1.3) and in §1.1.2)

$$\begin{tikzcd} (P \otimes_{\mathcal{O}_X} P \otimes_{\mathcal{O}_X} \mathcal{F})/I(2) \arrow[r, "p^1_{1,2}(2)^*\epsilon_1"] \arrow[d, equal] & (P \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} P)/I(2) \arrow[r, "p^1_{0,1}(2)^*\epsilon_1"] & p^1_{0,1}(2)^* p^1_0(1)^*\mathcal{F} \arrow[d, equal] \\ (P \otimes_{\mathcal{O}_X} P \otimes_{\mathcal{O}_X} \mathcal{F})/I(2) \arrow[d, equal] & & (\mathcal{F} \otimes_{\mathcal{O}_X} P \otimes_{\mathcal{O}_X} P)/I(2) \arrow[d, equal] \\ p^1_{0,2}(2)^* p^1_1(1)^*\mathcal{F} \arrow[rr, "p^1_{0,2}(2)^*\epsilon_1"] & & p^1_{0,2}(2)^* p^1_0(1)^*\mathcal{F} \end{tikzcd} \ .$$

Since the morphisms are $P \otimes_{\mathcal{O}_X} P$-linear we can check the equality on elements of the form $(1 \otimes 1) \otimes (1 \otimes 1) \otimes x$, with $x \in \mathcal{F}$. Let's denote $\mathbb{1} := 1 \otimes 1 \in P$. Following the second row we get

$$\begin{aligned} \mathbb{1} \otimes \mathbb{1} \otimes x \mapsto x \otimes \mathbb{1} \otimes \mathbb{1} + \sum_{i=1}^{d} \left(\nabla_{\partial_{x_i}} x\right) \otimes (\mathbb{1} \otimes 1 \otimes x_i - x_i \otimes 1 \otimes \mathbb{1}) \\ = x \otimes \mathbb{1} \otimes \mathbb{1} + \sum_{i=1}^{d} \left(\nabla_{\partial_{x_i}} x\right) \otimes (\mathbb{1} \otimes 1 \otimes x_i - x_i \otimes 1 \otimes \mathbb{1}) . \end{aligned}$$

The first row is a similar computation, observing that $\zeta_j \otimes \zeta_i \in I(2)$, where $\zeta_k = 1 \otimes x_k - x_k \otimes 1 \in P$, we get

$$\mathbb{1} \otimes \mathbb{1} \otimes x \mapsto x \otimes \mathbb{1} \otimes \mathbb{1} + \sum_{i=1}^{d} \left(\nabla_{\partial_{x_i}} x\right) \otimes (\zeta_i \otimes \mathbb{1} + \mathbb{1} \otimes \zeta_i) .$$

Hence the difference is 0 modulo $I(2)$. □

**Lemma 1.2.17.** *If $\nabla$ is an integrable connection we can define $\mathcal{O}_X$-linear isomorphisms*

$$\epsilon_n : \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_X \ ,$$

*for each $n \in \mathbb{N}$, where $\epsilon_0 = id_{\mathcal{F}}$ and $\epsilon_1$ is the map described above.*

*Locally, on $U \subset X$ with coordinates $x_1, \dots, x_d$ the morphism $\epsilon_n$ is defined by*

$$\begin{aligned} \epsilon_n : \mathcal{P}^n_U \otimes_{\mathcal{O}_U} \mathcal{F} &\longrightarrow \mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{P}^n_U \\ (a \otimes b) \otimes x &\longmapsto a \left( \sum_{\substack{0 \leqslant |q|_1 \leqslant n \\ q \in \mathbb{N}^d}} \nabla_{D_q} \zeta^q \right) b \end{aligned} .$$

*Proof.* Let $\partial \in \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{P}^1_X, \mathcal{O}_X\right) = \mathrm{Diff}^1\left(\mathcal{O}_X, \mathcal{O}_X\right)$, then we get a map

$$\nabla_\partial : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\epsilon_1} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X \xrightarrow{\mathrm{id}_{\mathcal{F}} \otimes \partial} \mathcal{F}.$$

We get an $\mathcal{O}_X$-linear morphism

$$\mathrm{Diff}^1\left(\mathcal{O}_X, \mathcal{O}_X\right) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{F}\right) = \mathrm{Diff}^1\left(\mathcal{F}, \mathcal{F}\right).$$

Locally, on $U \subset X$ with coordinates $x_1 \dots x_d$, we can extend this morphism to a unique $\mathcal{O}_X$-algebra morphism

$$\mathrm{Diff}^\bullet\left(\mathcal{O}_X, \mathcal{O}_X\right)(U) \longrightarrow \mathrm{Diff}^\bullet\left(\mathcal{F}, \mathcal{F}\right)(U)$$

sending $D_q = \frac{1}{q!} \prod_{i=1}^d \partial_{x_i}^{q_i}$ to the element $\frac{1}{q!} \prod_{i=1}^d \nabla_{\partial_{x_i}}^{q_i}$, since the elements $\nabla_{\partial_{x_i}}$ commute and we use that $\mathrm{Diff}^\bullet\left(\mathcal{O}_X, \mathcal{O}_X\right)$ is generated in degree 1. In order to see that this morphism extends to a unique morphism

$$\nabla_- : \mathrm{Diff}^\bullet\left(\mathcal{O}_X, \mathcal{O}_X\right) \longrightarrow \mathrm{Diff}^\bullet\left(\mathcal{F}, \mathcal{F}\right),$$

we have to motivate that the local morphism does not depend on the choice of coordinates. The point is showing that if $\partial_1, \partial_2, \partial_1', \partial_2' \in \mathrm{Diff}^1\left(\mathcal{O}_X, \mathcal{O}_X\right)$ with $\partial_1 \circ \partial_2 = \partial_1' \circ \partial_2'$, then $\nabla_{\partial_1} \circ \nabla_{\partial_2} = \nabla_{\partial_1'} \circ \nabla_{\partial_2'}$. And this could be checked by the formula 1.2.10 using the cocycle condition for $\epsilon_1$ proved in the previous Lemma.

Let's consider the map

$$\begin{tikzcd} \mathcal{F} \arrow[r, "ev"] \arrow[d, "(\epsilon_n)_{|\mathcal{F}}"'] & \mathrm{Hom}_{\mathcal{O}_X}\left(\mathrm{Diff}^n\left(\mathcal{F}, \mathcal{F}\right), \mathcal{F}\right) \arrow[d, "\nabla_-^*"] \\ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_X \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{O}_X}\left(\mathrm{Diff}^n\left(\mathcal{O}_X, \mathcal{O}_X\right), \mathcal{F}\right) \end{tikzcd} .$$

The first map is the evaluation map and the third map is the isomorphism given by the fact that $\mathcal{P}^n$ is locally free. We claim that $(\epsilon_n)_{|\mathcal{F}}$ is linear with respect to the right structure, then it extends $\mathcal{P}^n_X$-linearly to a map

$$\epsilon_n : \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_X.$$

The fact that $(\epsilon_n)_{|\mathcal{F}}$ is linear and that $\epsilon_n$ is an isomorphism could be checked locally. Let's assume that $X$

has coordinates $x_1, \dots, x_d$, we use notations as in 1.2.13. Let $\zeta_i := \mathrm{d}\, x_i$, hence

$$\epsilon_n(ax) = \sum_{\substack{0\leqslant |q|_1\leqslant n\\ q\in\mathbb{N}^d}} \nabla_{D_q}(ax)\zeta^q = \sum_{\substack{0\leqslant |q|_1\leqslant n\\ q\in\mathbb{N}^d}} \nabla_{D_q}(x)\zeta^q a = \epsilon_n(x)a.$$

So the morphism $(\epsilon_n)_{|\mathcal{F}}$ is linear as claimed. It follows by definition, or by the local formulas for $\epsilon_n(ax)$ with $a = 1$, that $\epsilon_n$ modulo $I^n$ is $\epsilon_{n-1}$, *i.e.* that $\{\epsilon_n\}_{n\in\mathbb{N}}$ is compatible. Since $\epsilon_n$ is $P$-linear, the restriction

$$\rho_n := (\epsilon_n)_{|Sym^n(I/I^2)\otimes_{\mathcal{O}_X}\mathcal{F}} : Sym^n(I/I^2)\otimes_{\mathcal{O}_X}\mathcal{F} \to \mathcal{F}\otimes_{\mathcal{O}_X} Sym^n(I/I^2)$$

is well defined. By induction it suffices to prove that $\rho_n$ is an isomorphism. But for an element $\zeta^q$ with $|q|_1 = n$

$$\rho_n(\zeta^q x) = \sum_{0\leqslant |q'|_1\leqslant n} \nabla_{D_{q'}}(x)\zeta^{q+q'} = x\zeta^q.$$

where the last equality follows by the fact that if a polynomial has degree $> n$, then it is zero in $\mathcal{P}^n_X$. So $\rho_n$ is the identity and the Lemma follows. □

**Lemma 1.2.18.** *Given an integrable connection $\nabla$, the collection of isomorphisms defined in the Lemma 1.2.17 $\{\epsilon_n\}_{n\in\mathbb{N}_{>0}} \cup \{\epsilon_0 := id_{\mathcal{F}}\}$ is a stratification on $\mathcal{F}$.*

*Proof.* The conditions of the stratification could be checked locally, then we can suppose that $X$ has coordinates $x_1, \dots x_d$. The identity and compatibility conditions follow immediately via the local computation of $\epsilon_n$ given above. We have to verify the cocycle condition for $n > 1$ (for $n = 1$ we already proved the cocycle condition in the Lemma 1.2.16). We proceed via induction on $n$. Via direct computations

$$p^n_{0,2}(2)\epsilon_n(\mathbb{1}\otimes\mathbb{1}\otimes x) = \sum_{0\leqslant |q|_1\leqslant n} \nabla_{D_q} x\ \zeta^q_{0,2}$$

where we denote $\zeta^q_{0,2} := \prod_{i=1}^d (\mathbb{1}\otimes 1\otimes x_i - x_i\otimes 1\otimes\mathbb{1})^{q_i}$ and

$$\left(p^n_{0,1}(2)^*(\epsilon_n)\circ p^n_{1,2}(2)^*(\epsilon_n)\right)(\mathbb{1}\otimes\mathbb{1}\otimes x) = \sum_{0\leqslant |q|_1, |q'|_1\leqslant n} \left(\nabla_{D_{q''}}\circ\nabla_{D_{q'}}\right)(x)\ \zeta^q\otimes\zeta^{q'}$$

Observe that $\zeta^q\otimes\zeta^{q'} = 0$ if $n \leqslant |q|_1 + |q'|_1$. Via induction we have to show that

$$\sum_{|q''|_1+|q'|_1=n} \left(\nabla_{D_{q''}}\circ\nabla_{D_{q'}}\right)(x)\ \zeta^q\otimes\zeta^{q'} = \sum_{|q|_1=n} \nabla_{D_q} x\ \zeta^q_{0,2}$$

But $\mathbb{1}\otimes 1\otimes x_i - x_i\otimes 1\otimes\mathbb{1} = \mathbb{1}\otimes \mathrm{d}\, x_i + \mathrm{d}\, x_i\otimes\mathbb{1}$, hence

$$\begin{aligned}
\sum_{|q|_1=n} \nabla_{D_q} x\ \zeta^q_{0,2} &= \sum_{|q|_1=n} \nabla_{D_q} x \prod_{i=1}^d (\mathbb{1}\otimes\mathrm{d}\, x_i + \mathrm{d}\, x_i\otimes\mathbb{1})^{q_i} \\
&= \sum_{|q|_1=n} \nabla_{D_q} x \sum_{|q'|_1+|q''|_1=n} \frac{q!}{q'!\ \cdot\ q''!} \prod_{i=1}^d (\mathbb{1}\otimes\mathrm{d}\, x_i)^{q'_i}(\mathrm{d}\, x_i\otimes\mathbb{1})^{q''_i} \\
&= \sum_{|q'|_1+|q''|_1=n} \frac{q!}{q'!\ \cdot\ q''!} \nabla_{D_{q'+q''}} x \prod_{i=1}^d (\mathbb{1}\otimes\mathrm{d}\, x_i)^{q'_i}(\mathrm{d}\, x_i\otimes\mathbb{1})^{q''_i} \\
&= \sum_{|q''|_1+|q'|_1=n} \left(\nabla_{D_{q''}}\circ\nabla_{D_{q'}}\right)(x)\ \zeta^q\otimes\zeta^{q'}\ .
\end{aligned}$$

□

### 1.2.3 Global sections of a crystal

**Proposition 1.2.19.** *If $f : X \to S$ is a smooth rigid space and $\mathcal{F} \in X_{\inf}$ is a crystal, then*

$$\Gamma\left(X_{\inf}, \mathcal{F}\right) = \{x \in \mathcal{F}(X, X) \mid \epsilon_1^{\mathcal{F}}(p_0^1(1)^* x) = p_1^1(1)^* x\}.$$

*Proof.* We have to show that for any $x \in \mathcal{F}(X, X)$, if $\epsilon_1^{\mathcal{F}}(p_0^1(1)^* x) = p_1^1(1)^* x$, then $\epsilon_n^{\mathcal{F}}(p_0^n(1)^* x) = p_1^n(1)^* x$ for every $n \in \mathbb{N}_{>1}$.

**I step.** Observe that $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a stratification on $\mathcal{F}_{(X,X)}$, *i.e.*

- (identity) $\epsilon_0$ is the restriction associated to $\mathrm{id}_{(X,X)}$, then it must be the identity of $\mathcal{F}_{(X,X)}$;

- (compatibility) holds via the cocycle condition of the restriction morphisms of $\mathcal{F}$ applied to the following morphisms for $i = 0, 1$.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[d, equal] \\ P_X^n \arrow[r] \arrow[rr, bend right, "p_i^n(1)"'] & P_X^m \arrow[r, "p_i^m(1)"] & X \end{tikzcd}$$

- (cocycle condition): via the cocycle condition of the restriction morphisms it follows that the diagram

$$\begin{tikzcd} p_2^n(2)^* \mathcal{F}_{(X,X)} \arrow[r, "\cong"] \arrow[dr] & \mathcal{F}_{D_X^n(2)} & p_0^n(2)^* \mathcal{F}_{(X,X)} \arrow[l, "\cong"'] \\ & p_1^n(2)^* \mathcal{F}_{(X,X)} \arrow[u, "\cong"] \arrow[ur] & \end{tikzcd}$$

  is commutative and the cocycle condition on $\{\epsilon_n\}_{n \in \mathbb{N}}$ follows by the following equalities

$$p_2^n(2)^* = p_{1,2}^n(2)^* p_1^n(1)^* = p_{0,2}^n(2)^* p_1^n(1)^*;$$
$$p_1^n(2)^* = p_{1,2}^n(2)^* p_0^n(1)^* = p_{0,1}^n(2)^* p_1^n(1)^*;$$
$$p_0^n(2)^* = p_{0,1}^n(2)^* p_0^n(1)^* = p_{0,2}^n(2)^* p_0^n(1)^*.$$

Since $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a stratification on $\mathcal{F}_{(X,X)}$, via the Proposition 1.2.12, there is a homomorphism of sheaves of $\mathcal{O}_X$-algebras

$$\nabla_{-} : \mathrm{Diff}^{\bullet}\left(\mathcal{O}_X, \mathcal{O}_X\right) \longrightarrow \mathrm{Diff}^{\bullet}\left(\mathcal{F}_{(X,X)}, \mathcal{F}_{(X,X)}\right).$$

**II step.** For each open $U \subset X$, the following equality holds:

$$\{x \in \mathcal{F}_{(X,X)}(U) \mid \epsilon_1^{\mathcal{F}}(p_0^1(1)^* x) = p_1^1(1)^* x\} = \bigcap_{\partial \in D^1(U)} \{x \in \mathcal{F}_{(X,X)}(U) \mid \left(\nabla_{\partial}\right)(x) = 0\}$$

where $D^1(U) := \mathrm{Diff}^1(\mathcal{O}_X, \mathcal{O}_X)(U) \setminus \mathrm{Diff}^0(\mathcal{O}_X, \mathcal{O}_X)(U)$.

Since the two sets (letting $U$ varying) are defined by local conditions, they define subsheaves of $\mathcal{F}$ and we can check the equality locally. Let $U$ be an open where there are $y_1, \ldots, y_d \in \mathcal{O}_X(U)$ coordinates on $U$, without loss of generality we assume that $X = U$. Since $\mathrm{d}\, y_i, 1 \otimes 1$ is a basis of $\mathcal{P}_X^1$, we can write

$$\epsilon_1(p_0^1(1)^* x) = x_0 + \sum_{i=1}^{d} x_i \otimes \mathrm{d}\, y_i.$$

Observe that $x = \epsilon_0(x) = x_0$ since $\epsilon_0$ is the identity on $\mathcal{F}$ and $\epsilon_1 \equiv \epsilon_0$ modulo $I$. Moreover

$$(\nabla_{\partial_{y_i}})(x) = \overline{\partial_{y_j}}^{1}\left(x + \sum_{i=1}^{d} x_i \otimes \mathrm{d}\, y_i\right) = x_j.$$

Then $(\nabla_\partial)(x) = 0$ for each $\partial \in D^1(X)$ if and only if $x_i = 0$ for each $i = 1, \dots, d$ and the 2nd step follows.

**III step.** For each $n \in \mathbb{N}_{>0}$ and any open $U \subset X$

$$\{x \in \mathcal{F}_{(X,X)}(U) \mid \epsilon_n(p_0^n(1)^* x) = p_1^n(1)^* x\} = \{x \in \mathcal{F}_{(X,X)}(U) \mid \epsilon_1(p_0^1(1)^* x) = p_1^1(1)^* x\}.$$

One inclusion is true in general (by the compatibility condition).

For the other inclusion as in the previous step we can work locally where we have coordinates $y_1, \dots, y_d$. We shall show this fact by induction on $n \in \mathbb{N}_{>1}$. Fix an $x \in \mathcal{F}_{(X,X)}(X)$ with $\epsilon_1(p_0^1(1)^* x) = p_1^1(1)^* x$. Denote

$$\epsilon_n(p_0^1(1)^* x) = x + \sum_{|q|_1 = n} x_q \otimes (\mathrm{d}\, y)^q.$$

where $q \in \mathbb{N}^d$ and $(\mathrm{d}\, y)^q \coloneqq (\mathrm{d}\, y_1)^{q_1} \dots (\mathrm{d}\, y_1)^{q_1}$ via the identification

$$\mathcal{P}_X^n \cong \mathcal{O}_X[\mathrm{d}\, y_1, \dots, \mathrm{d}\, y_d]^{deg \leq n}.$$

But now observe that for each $q_* \in \mathbb{N}^d$ with $|q_*|_1 = n$

$$x_{q_*} = \overline{D_{q_*}}^{n}\left(x + \sum_{|q|_1 = n} x_q \otimes (\mathrm{d}\, y)^q\right) = (\nabla_{D_{q_*}})(x) = (\nabla_{\frac{1}{q_*!}\prod_{i=1}^d \partial_{y_i}^{q_{*,i}}})(x) = \frac{1}{q_*!}\left(\prod_{i=1}^{d} (\nabla_{\partial_{y_i}})^{q_{*,i}}\right)(x) = 0$$

where we used that $D_{q_*}(\mathrm{d}\, y)^q = \delta_q^{q_*} \cdot (\mathrm{d}\, y)^{q_*}$ for each $q \in \mathbb{N}^d$ with $|q|_1 = |q_*|_1$; the fact that $\mathrm{Diff}^n(\mathcal{O}_X, \mathcal{O}_X)$ is (locally) generated as $\mathcal{O}_X$-algebra in degree 1; the fact that $\nabla_-$ is a ring homomorphism and the $II$ step. □

## 1.3 Linearization and delinearization

The linearization was introduced by Grothendieck in [Gro68] and developed by Berthelot–Ogus in [BO78] for schemes of arbitrary characteristic with power structures. We follow these works in order to associate to an $\mathcal{O}_X$-module $\mathcal{F}$ with integrable connection a crystal $\mathcal{F}_{inf}$ and the linearized de Rham complex. We will show that also in our setting there is a “de-linearization” functor $u_*$ that brings the linearized de Rham complex into the de Rham complex that preserves the cohomology.

### 1.3.1 Linearization

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As before $I \subset P \coloneqq \mathcal{O}_X \hat{\otimes}_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$ is the ideal corresponding to the diagonal immersion.

**Definition 1.3.1.** The **linearization of** $\mathcal{F}$ is the $P$-module

$$L(\mathcal{F}) \coloneqq \lim_{n \in \mathbb{N}} L_n(\mathcal{F}),$$

where the projective system is $\{L_n(\mathcal{F}) \coloneqq \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F}\}_{n \in \mathbb{N}}$ with morphisms

$$L_{n+1}(\mathcal{F}) = P/I^{n+2} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow P/I^{n+1} \otimes_{\mathcal{O}_X} \mathcal{F} = L_n(\mathcal{F}).$$

**Remark 1.3.2.** As we observed locally, where $x_1, \dots, x_d$ are coordinates of $X$, we can write

$$\mathcal{P}_X^n = \mathcal{O}_X[\zeta_1, \dots, \zeta_d]^{deg \leqslant n},$$

where $\zeta_i = 1 \otimes x_i - x_i \otimes 1 \in I \subset P$ and the map

$$\mathcal{P}_X^{n+s} \to \mathcal{P}_X^n$$

corresponds to the projection of polynomials of degree $\leqslant n+s$ into polynomials of degree $\leqslant n$. Hence

$$\mathcal{P}_X := \lim_{n \in \mathbb{N}} \mathcal{P}_X^n = \mathcal{O}_X[[\zeta_1, \dots, \zeta_d]].$$

Observe that $L(\mathcal{F}) = \mathcal{P}_X \hat{\otimes}_{\mathcal{O}_X} \mathcal{F}$, where the completion is taken with respect to the the $(\zeta_1, \dots, \zeta_d)$-topology.

**Lemma 1.3.3.** *If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of degree $n \in \mathbb{N}$, then there is a $P$-linear map*

$$L(D) : L(\mathcal{F}) \to L(\mathcal{G})$$

*limit of the morphisms $L_k(D)$ for $k \in \mathbb{N}$, defined by*

$$\left( id_{\mathcal{P}_X^k} \otimes \overline{D}^n \right) \circ \left( \delta^{k,n} \otimes id_F \right) : \mathcal{P}_X^{k+n} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G}.$$

*Proof.* Since

$$\mathrm{Hom}_P \left( L(\mathcal{F}), \lim_{k \in \mathbb{N}} \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G} \right) = \lim_{k \in \mathbb{N}} \mathrm{Hom}_P \left( L(\mathcal{F}), \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G} \right)$$

we have to show that there are compatible $P$-linear maps $L(\mathcal{F}) \to \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G}$. We consider the $P$-linear morphisms as in the statement:

$$L(\mathcal{F}) \to \mathcal{P}_X^{k+n} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G}.$$

Let $k, s \in \mathbb{N}$, then the diagram

$$\begin{tikzcd} L(\mathcal{F}) \arrow[r] \arrow[dr] & \mathcal{P}_X^{k+s+n} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[d] \\ & \mathcal{P}_X^{k+n} \otimes_{\mathcal{O}_X} \mathcal{F} \end{tikzcd}$$

commutes by definition of limit. Let's consider the following diagrams

$$\begin{tikzcd} \mathcal{P}_X^{k+s+n} \arrow[r, "\delta^{k+s,n}"] \arrow[d] & \mathcal{P}_X^{k+s} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \arrow[d] \\ \mathcal{P}_X^{k+n} \arrow[r, "\delta^{k,n}"] & \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}_X^{k+s} \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\mathrm{id}_{\mathcal{P}^{k+s}} \otimes \overline{D}^n"] \arrow[d] & \mathcal{P}_X^{k+s} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[d] \\ \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{P}_X^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r, "\mathrm{id}_{\mathcal{P}^k} \otimes \overline{D}^n"] & \mathcal{P}_X^k \otimes_{\mathcal{O}_X} \mathcal{G} \end{tikzcd}$$

where the vertical maps are induced by projections. Observe that the two diagrams are commutative and this system gives us a $P$-linear map

$$L(\mathcal{F}) \longrightarrow L(\mathcal{G}).$$

$\square$

**Remark 1.3.4.** Observe that if $D \in \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})$, then $D \in \mathrm{Diff}^{n+k}(\mathcal{F}, \mathcal{G})$ for any $k \in \mathbb{N}$. At this point one should check that $\overline{D}^n$ and $\overline{D}^{n+k}$ give the same map $L(D) : L(\mathcal{F}) \to L(\mathcal{G})$.

**Lemma 1.3.5.** *If $\mathcal{F},\mathcal{G},\mathcal{H}$ are $\mathcal{O}_X$-modules, $D_1 \in \mathrm{Diff}^{n_1}(\mathcal{F},\mathcal{G})$ and $D_2 \in \mathrm{Diff}^{n_2}(\mathcal{G},\mathcal{H})$, then*

$$L(D_2 \circ D_1) = L(D_2) \circ L(D_1) : L(\mathcal{F}) \longrightarrow L(\mathcal{H}).$$

*Proof.* This Lemma is a consequence of the formula computed in the Lemma 1.2.10 and the description of $L(D_i)$ in terms of $\overline{D_i}^{n_i}$. □

**Corollary 1.3.6.** *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with an integrable connection $\nabla$, then there is a complex*

$$L(\mathcal{F}) \xrightarrow{L(\nabla)} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \xrightarrow{L(\nabla^1)} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^2_{X/S}) \xrightarrow{L(\nabla^2)} \dots$$

Now we would like to show that this complex is exact. In order to do this let's consider the map

$$\mathrm{d} : \mathcal{O}_X \to \Omega^1_{X/S},$$

it is a connection:

$$\mathrm{d}(ab) = a\,\mathrm{d}\,b + b\,\mathrm{d}\,a \in \Omega^1_{X/S} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/S},$$

moreover it is integrable since $\mathrm{d}^1\,\mathrm{d}\,a = \mathrm{d}^1(1 \otimes \mathrm{d}\,a) = \mathrm{d}\,1 \wedge \mathrm{d}\,a = 0$.

**Lemma 1.3.7.** *For any $k \in \mathbb{N}$, locally on $X$, where there are coordinates $x_1,\dots,x_d$, the map*

$$L(\mathrm{d}) : \mathcal{P}_X \hat{\otimes}_{\mathcal{O}_X} \Omega^k_{X/S} \to \mathcal{P}_X \hat{\otimes}_{\mathcal{O}_X} \Omega^{k+1}_{X/S}$$

*is the completion of the map*

$$\begin{array}{ccc} \mathcal{O}_X[[\zeta_1,\dots,\zeta_d]] \otimes_{\mathcal{O}_X} \Omega^k_{X/S} & \longrightarrow & \mathcal{O}_X[[\zeta_1,\dots,\zeta_d]] \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \\ f \otimes \omega & \longmapsto & f \otimes \mathrm{d}^k\,\omega + \sum_{i=1}^d \left(\frac{\partial}{\partial \zeta_i} f\right) \otimes \mathrm{d}\,x_i \wedge \omega \end{array}$$

*where $\zeta_i = (1 \otimes x_i) - (x_i \otimes 1)$, we identified $\mathcal{P}_X$ with a submodule of the polynomial ring as in 1.3.2 and $\mathrm{d}^0 = d$.*

*Proof.* We prove the Lemma for $k > 0$, for $k = 0$ the proof is the same but the notation does not agree. In order to simplify the notation sometimes we write $a \otimes b\omega$: we do not write the tensors over $\mathcal{O}_X$ for the elements. Observe that the map $L(d)$ is, by definition, the limit of

$$\begin{array}{ccccc} \phi_n : \mathcal{P}^{n+1}_X \otimes_{\mathcal{O}_X} \Omega^k_{X/S} & \xrightarrow{\delta^{n,1}} & \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \Omega^k_{X/S} & \xrightarrow{\mathrm{id}_{\mathcal{P}^n_X} \otimes \overline{\mathrm{d}}^k} & \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \\ (a \otimes b) \otimes \omega & \longmapsto & (a \otimes 1) \otimes (1 \otimes b) \otimes \omega & \longmapsto & (a \otimes 1) \otimes \mathrm{d}^k(b\omega) \end{array}$$

Hence $L(d^k)(a \otimes \omega) = (a \otimes 1)\,\mathrm{d}^k\,\omega$, and we proved the formula for polynomials of degree 0.

Let $\omega = \mathrm{d}\,x_{i_1} \wedge \mathrm{d}\,x_{i_2} \cdots \wedge \mathrm{d}\,x_{i_k}$ with $1 \leqslant i_1, i_2, \dots, i_k \leqslant d$, recall that $\zeta_i = 1 \otimes x_i - x_i \otimes 1$. Then

$$L(d^k)(\zeta_i\omega) = (1 \otimes 1)\,\mathrm{d}\,x_i\omega - (x_i \otimes 1)\,\mathrm{d}\,1 \wedge \omega = (1 \otimes 1)\,\mathrm{d}\,x_i \wedge \omega.$$

Now observe that if $y_1 = a_1 \otimes a_2,\ y_2 = a_2 \otimes b_2 \in \mathcal{P}^n_X$

$$L(d^k)(y_1y_2\omega) = a_1a_2 \otimes b_1\,\mathrm{d}\,b_2 \wedge \omega + a_1a_2 \otimes b_2\,\mathrm{d}\,b_1 \wedge \omega = y_1L(d^k)(y_2\omega) + y_2L(d^k)(y_1\omega).$$

Hence by induction on $l \in \mathbb{N}$ and the previous computations

$$L(d^k)(\zeta_i^l \omega) = l\zeta_i^{l-1} L(\zeta_i \omega) = l\zeta_i^{l-1} \,\mathrm{d}\, x_i \wedge \omega.$$

Then

$$\begin{aligned} L(d^k)(\prod_{i=1}^k \zeta_i^{l_i} \omega) &= \prod_{i=2}^k \zeta_i^{l_i} L(d^k)(\zeta_1^{l_1}\omega) + \zeta_1^{l_1} L(d^k)(\prod_{i=2}^k \zeta_i^{l_i}\omega) \\ &= \cdots = \sum_{i=1}^k \prod_{i\neq j=1}^k \zeta_j^{l_j} L(d^k)(\zeta_i^{l_i}\omega) = \sum_{i=1}^k \left( \prod_{i\neq j=1}^k \zeta_j^{l_j} \frac{\partial \zeta_i^{l_i}}{\partial \zeta_i} \right) \mathrm{d}\, x_i \wedge \omega \\ &= \sum_{i=1}^k \left( \frac{\partial}{\partial \zeta_i} \prod_{j=1}^k \zeta_j^{l_j} \right) \mathrm{d}\, x_i \wedge \omega. \end{aligned} \tag{1.4}$$

But each element is a sum of elements of the form $f \otimes \omega$, where $f$ is a monomial and $\omega$ as above. Since the expression in the statement and the map are additive we conclude the proof of the Lemma. □

Observe that $a \in \mathcal{O}_X$ inside $\mathcal{P}_X^n$ is seen as $a \otimes 1$, since in the other way the inclusion would not be $\mathcal{O}_X$-left-linear.

**Proposition 1.3.8.** *The complex*

$$L(\mathcal{O}_X) \xrightarrow{L(\mathrm{d})} L(\Omega^1_{X/S}) \xrightarrow{L(\mathrm{d}^1)} L(\Omega^2_{X/S}) \xrightarrow{L(\mathrm{d}^2)} \ldots$$

*is exact, moreover*

$$\mathrm{Ker}(L(\mathrm{d})) = \mathcal{O}_X.$$

*Proof.* The statement is local, hence we can suppose that $x_1, \ldots, x_d$ are coordinates of $X$.

Maybe one can follow §6.12 in [BO78]; we will give another (more explicit, but longer) proof. Observe that the projective system $\mathcal{P}_X^n \otimes_{\mathcal{O}_X} \Omega^k_{X/S}$ is done by surjective maps (the projections). Hence it satisfies the Mittag–Leffler condition and we can prove the exactness of the sequence for any $n \in \mathbb{N}_{>1}$. Let $k \in \mathbb{N}$

**Step I.** Consider the sequence

$$\mathbb{C}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+2} \otimes_{\mathbb{C}} \bigwedge^k \mathbb{C}^d \to \mathbb{C}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+1} \otimes_{\mathbb{C}} \bigwedge^{k+1} \mathbb{C}^d \to \mathbb{C}[\zeta_1, \ldots, \zeta_d]^{\leqslant n} \otimes_{\mathbb{C}} \bigwedge^{k+2} \mathbb{C}^d.$$

We claim that this sequence is exact. But over the complex numbers any polynomial is a holomorphic function on the simple connected domain $\mathbb{C}^d$, hence an element $f\omega_1 \in \mathbb{C}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+1} \otimes_{\mathbb{C}} \bigwedge^{k+1} \mathbb{C}^d$ is closed if and only if there is a holomorphic function $g : \mathbb{C}^d \to \mathbb{C}$ and $\omega_2 \in \bigwedge^k \mathbb{C}^d$ with $L_{\mathbb{C}}(d^k)(g\omega_2) = f\omega_1$. But if we expand $g$ at $0 \in \mathbb{C}^d$ we see that it must be a polynomial, since its derivatives are polynomial. Hence by complex analysis we conclude that also the sequence above is exact.

**Step II.** Let's analyze the situaton over $\mathbb{Q}$. We claim that the sequence

$$\mathbb{Q}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+2} \otimes_{\mathbb{Q}} \bigwedge^k \mathbb{Q}^d \to \mathbb{Q}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+1} \otimes_{\mathbb{Q}} \bigwedge^{k+1} \mathbb{Q}^d \to \mathbb{Q}[\zeta_1, \ldots, \zeta_d]^{\leqslant n} \otimes_{\mathbb{Q}} \bigwedge^{k+2} \mathbb{Q}^d$$

is exact. Consider a closed element $f\omega_1 \in \mathbb{Q}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+1} \otimes_{\mathbb{Q}} \bigwedge^{k+1} \mathbb{Q}^d$; then it is reached by an element $g \otimes \omega_2 \in \mathbb{C}[\zeta_1, \ldots, \zeta_d]^{\leqslant n+2} \otimes_{\mathbb{Q}} \bigwedge^k \mathbb{Q}^d$ by the previous step. But looking at the Taylor expansion of $g$ one gets that $g - g(0)$ must have rational coefficients, since his derivatives have rational coefficients and we conclude the II step.

**III Step.** Now we do the general case. Observe that $\Omega^1_{X/S} \cong \mathcal{O}_X^d$ since we are working locally, hence

$$\mathcal{O}_X \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_1, \dots, \zeta_d]^{\leqslant n+1} \otimes_{\mathbb{Q}} \bigwedge^k \mathbb{Q}^d \cong \mathcal{O}_X[\zeta_1, \dots, \zeta_d]^{\leqslant n+2} \otimes_{\mathcal{O}_X} \Omega^k_{X/S}$$

and by the previous step we conclude that

$$\mathcal{O}_X[\zeta_1, \dots, \zeta_d]^{\leqslant n+2} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \to \mathcal{O}_X[\zeta_1, \dots, \zeta_d]^{\leqslant n+1} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \to \mathcal{O}_X[\zeta_1, \dots, \zeta_d]^{\leqslant n} \otimes_{\mathcal{O}_X} \Omega^{k+2}_{X/S}$$

is locally exact, since we are working in characteristic 0.

The statement on the kernel is immediate: if $f \in \mathcal{O}_X[[\zeta_1, \dots, \zeta_d]]$ is such that $\frac{\partial}{\partial \zeta_i} f = 0$ for any $i = 1, \dots, d$, then $f$ is constant, *i.e.* $f \in \mathcal{O}_X$. Here we are using again that $\mathcal{O}_X$ is a $\mathbb{Q}$-algebra. $\square$

The same statement holds for any flat $\mathcal{O}_X$-module with integrable connection.

**Proposition 1.3.9.** *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with integrable connection $\nabla$, then the complex*

$$L(\mathcal{F}) \xrightarrow{L(\nabla)} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) \xrightarrow{L(\nabla^1)} L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^2_{X/S}) \xrightarrow{L(\nabla^2)} \dots$$

*is isomorphic to the complex*

$$\mathcal{F} \hat{\otimes}_{\mathcal{O}_X} L(\mathcal{O}_X) \xrightarrow{id_{\mathcal{F}} \otimes L(\mathrm{d})} \mathcal{F} \hat{\otimes}_{\mathcal{O}_X} L(\Omega^1_{X/S}) \xrightarrow{id_{\mathcal{F}} \otimes L(\mathrm{d}^1)} \mathcal{F} \hat{\otimes}_{\mathcal{O}_X} L(\Omega^2_{X/S}) \xrightarrow{id_{\mathcal{F}} \otimes L(\mathrm{d}^2)} \dots .$$

*In particular if $\mathcal{F}$ is flat the two complexes are exact with kernel*

$$\mathrm{Ker}(L(\nabla)) \cong \mathcal{F}.$$

*Proof.* Let $n \in \mathbb{N}$, by Lemma 1.2.18 we know that there is a stratification $\{\epsilon_n\}_{n \in \mathbb{N}}$ on $\mathcal{F}$ attached to the connection $\nabla$, *i.e.* a compatible system of $P$-linear isomorphisms where

$$\epsilon_n : \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_X$$

with $\epsilon_0 = \mathrm{id}_{\mathcal{F}}$ and cocycle condition. The map $\epsilon_1$ is defined as

$$\begin{aligned} \epsilon_1 : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F} &\longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X \\ (a \otimes b) \otimes x &\longmapsto (a \otimes b) \cdot (\nabla(x) + x \otimes 1 \otimes 1) \end{aligned}.$$

Now we want to show that the maps $\epsilon_n$ induce a map between the two complexes, *i.e.* that

$$\begin{tikzcd} \mathcal{P}^{n+1}_X \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \arrow[r, "L_n(\nabla^k)"] \arrow[d, "\epsilon_{n+1} \otimes \mathrm{id}_{\Omega^k_{X/S}}"] & \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \arrow[d, "\epsilon_n \otimes \mathrm{id}_{\Omega^{k+1}_{X/S}}"] \\ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^{n+1}_X \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \arrow[r, "\mathrm{id}_{\mathcal{F}} \otimes L_n(\mathrm{d}^k)"] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S} \end{tikzcd}$$

is commutative for each $n, k \in \mathbb{N}$. Once we show this commutativity, since $\epsilon_n$ is an isomorphism we conclude the first statement of the Proposition.

The statement now is local, hence we can suppose that $X$ has coordinates $x_1, \dots, x_d$. Let $y := \mathbb{1} \otimes x \otimes \omega$

where $x \in \mathcal{F}$ and $\omega = \mathrm{d}\, x_{i_1} \wedge \ldots \mathrm{d}\, x_{i_k}$. Then

$$\left(\epsilon_n \otimes \mathrm{id}_{\Omega^{k+1}_{X/S}}\right) \circ L_n(\nabla^k)(y) = \left(\epsilon_n \otimes \mathrm{id}_{\Omega^{k+1}_{X/S}}\right)\left(\sum_{i=1}^{d} \mathbb{1} \otimes (\nabla_{\partial_{x_i}} x)\ \mathrm{d}\, x_i \wedge \omega\right)$$
$$= \sum_{i=1}^{d} \sum_{0 \leqslant |q|_1 \leqslant n} \frac{1}{q!}\left(\prod_{j=1}^{d} \nabla_{\partial_{x_j}}^{q_j + \delta_i^j}\right)(x)\ \zeta^q \otimes\ \mathrm{d}\, x_i \wedge \omega$$

and

$$\left(\mathrm{id}_{\mathcal{F}} \otimes L_n(\mathrm{d}^k)\right) \circ \left(\epsilon_{n+1} \otimes \mathrm{id}_{\Omega^k_{X/S}}\right)(y) = \left(\mathrm{id}_{\mathcal{F}} \otimes L_n(\mathrm{d}^k)\right)\left(\sum_{0 \leqslant |q|_1 \leqslant n+1}\left(\prod_{j=1}^{d} \nabla_{\partial_{x_j}}^{q_j}\right)(x)\ \zeta^q \otimes \omega\right)$$
$$= \sum_{1 \leqslant |q|_1 \leqslant n+1} \sum_{i=1}^{d} \frac{1}{q!}\left(\prod_{j=1}^{d} \nabla_{\partial_{x_j}}^{q_j}\right)(x)\ \frac{\partial \zeta^q}{\partial_{\zeta_{q_i}}} \otimes \mathrm{d}\, x_i\ \wedge\ \omega.$$

The two expressions are equal (via a change of variable $q$ in the summation).

For the last two assertions observe that $\mathcal{F}$ is flat, hence locally (we tensor a s.e.s. used in the proof of Proposition 1.3.8 with $\mathcal{F}$) the s.e.s.

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\zeta_1, \ldots, \zeta_d]^{\leqslant n+2} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \overset{id_{\mathcal{F}} \otimes L_{n+1}(d^k)}{\longrightarrow} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\zeta_1, \ldots, \zeta_d]^{\leqslant n+1} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S}$$
$$\overset{id_{\mathcal{F}} \otimes L_n(d^{k+1})}{\longrightarrow} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\zeta_1, \ldots, \zeta_d]^{\leqslant n} \otimes_{\mathcal{O}_X} \Omega^{k+2}_{X/S}$$

is exact for each $n, k \in \mathbb{N}$. Hence

$$0 \to \mathrm{Ker}(L_{n+1}(d^k)) \hookrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\zeta_1, \ldots, \zeta_d]^{\leqslant n+2} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \to \mathrm{Ker}(L_n(d^{k+1})) \to 0$$

is exact for each $n, k \in \mathbb{N}$, since $\mathrm{Ker}(L_{n+1}(d^k))$ satisfies the M–L condition, kernel and limit commute, then we can conclude. □

### 1.3.2 Crystal associated to a sheaf with stratification and the linearization functor

In this section we describe how to build crystal $\mathcal{F}_{\mathrm{inf}}$ on $(X/S)_{\mathrm{inf}}$ by the data of an $\mathcal{O}_X$-module $\mathcal{F}$ with a stratification $\{\epsilon_n\}_{n \in \mathbb{N}}$. Moreover we describe the linearization functor $L(-)_{\mathrm{inf}}$ and some relations between the sheaves $\mathcal{F}_{\mathrm{inf}}$ and $L(\mathcal{F})_{\mathrm{inf}}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with a stratification $\{\epsilon_n\}_{n \in \mathbb{N}}$. We have already seen in Lemma 1.1.10 that $(X, X)$ covers $\mathbb{1}$, *i.e.* that for any object $(U, T) \in (X/S)_{\mathrm{inf}}$ there is a cover $\{(U_i, T_i)\}_{i \in I}$ with maps $g_i : T_i \to X$ s.t. the following diagram commutes

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & X \\ T_i \arrow[ur, "g_i"] & \end{tikzcd}$$

for any $i \in I$. Hence locally we can define the sheaf $\mathcal{F}_{\mathrm{inf},(U_i,T_i)} \coloneqq g_i^* \mathcal{F}$; we have to use the stratification in order to glue these sheaves. We can suppose that the map $g_i$ is nilpotent of order $n_i \in \mathbb{N}$. Let $i, j \in I$,

$U_{i,j} := U_i \cap U_j$, $T_{i,j} := T_i \cap T_j$ and $n_i, n_j \leqslant n \in \mathbb{N}$, then there is a diagram

$$\begin{tikzcd} & U_{i,j} \arrow[r, "\alpha_{i,j}"] & T_{i,j} \arrow[dll, "g_i"'] \arrow[d, "g_{i,j}"] \arrow[drr, "g_j"] & & \\ X & & X \times_S X \arrow[ll, "p_0"] \arrow[rr, "p_1"] & & X \end{tikzcd}$$

the map $g_{i,j}$ splits through a map $h_{i,j} : T_{i,j} \to P^n_X$, the proof of this fact is similar to the proof of the Lemma 1.1.9: using that $g_i \circ \alpha_{i,j} = g_j \circ \alpha_{i,j}$ the elements of $I$ are sent to elements in $\mathcal{O}_{T_{i,j}}$ that differs via an $n$-nilpotent element, hence the two morphisms coincide modulo $I^n$. For any $n_i, n_j \leqslant n \in \mathbb{N}$ we get a diagram

$$\begin{tikzcd} & U_{i,j} \arrow[r, "\alpha_{i,j}"] & T_{i,j} \arrow[dll, "g_i"'] \arrow[d, "g^n_{i,j}"] \arrow[drr, "g_j"] & & \\ X & & P^n_X \arrow[ll, "p^n_0(1)"] \arrow[rr, "p^n_1(1)"] & & X \end{tikzcd}.$$

We can glue the sheaf $\mathcal{F}_{\mathrm{inf},(U_i,T_i)}$ over $T_i$ and the sheaf $\mathcal{F}_{\mathrm{inf},(U_j,T_j)}$ over $T_j$ via the isomorphism given by the stratification:

$$\psi_{i,j} : g_i^*\mathcal{F} = (g^n_{i,j})^* p^n_0(1)^*\mathcal{F} \xrightarrow{(g^n_{i,j})^*\epsilon_n} (g^n_{i,j})^* p^n_1(1)^*\mathcal{F} = g_j^*\mathcal{F}.$$

The compatibility condition ensures that this procedure could be done with any $n \in \mathbb{N}$ bigger than $n_i, n_j$, the cocycle condition ensures that the composition of two gluing is the right gluing, the identity condition (together with the cocycle condition) ensures that $\psi_{i,j} = \psi^{-1}_{j,i}$. Hence we can glue these sheaves to a unique sheaf $\mathcal{F}_{\mathrm{inf},(U,T)}$. For any map $(\alpha,\beta) : (U_1,T_1) \to (U_2,T_2)$ there is a canonical isomorphism $\beta^*\mathcal{F}_{\mathrm{inf},(U_1,T_1)} \cong \mathcal{F}_{\mathrm{inf},(U_2,T_2)}$: if locally on $T_2$ we have the maps $g_i^{(2)} : T_{2,i} \to X$, then we get maps $g_i^{(1)} := g_i^{(2)} \circ \beta_i$ and $\mathcal{F}_{\mathrm{inf},(U_{1,i},T_{1,i})} \cong (g_i^{(1)})^*\mathcal{F} \cong \beta_i^*\mathcal{F}_{\mathrm{inf},(U_{2,i},T_{2,i})}$ by definition; where $\beta_i$ is the right restriction of $\beta$, $\{(U_{2,i},T_{2,i})\}$ is a cover as above of $(U_2,T_2)$ and $\{(U_{1,i},T_{1,i}) := \big(\alpha^{-1}(U_{2,i}), \beta^{-1}(T_{2,i})\big)\}$.

We define the restriction morphism as the composition

$$\beta^{-1}\mathcal{F}_{\mathrm{inf},(U_2,T_2)} \to \beta^{-1}\mathcal{F}_{\mathrm{inf},(U_2,T_2)} \otimes_{\beta^{-1}\mathcal{O}_{T_1}} \mathcal{O}_{T_2} \cong \beta^*\mathcal{F}_{\mathrm{inf},(U_2,T_2)}.$$

One can check that the conditions of the Lemma 1.1.6 are satisfied (via the identity, cocycle condition of the stratification).

We denote with $\mathcal{O}^{diff}_X - mod$ the category of $\mathcal{O}_X$-modules $\mathcal{F}$ with differential maps of a certain order as morphisms. Now we can prove the following Lemma:

**Lemma 1.3.10.** *If $\mathcal{F}$ is an $\mathcal{O}_X$-module with a stratification $\{\epsilon_n\}_{n\in\mathbb{N}}$, then there is an associated crystal $\mathcal{F}_{\mathrm{inf}}$ in the infinitesimal site. Moreover there is a linearization functor*

$$L(-)_{\mathrm{inf}} : \mathcal{O}^{diff}_X - mod \longrightarrow \mathcal{O}_{X_{\mathrm{inf}}} - mod$$

*Proof.* The constriction of the sheaf $\mathcal{F}_{\mathrm{inf}}$ is the presented above.

Now we build the linearization functor. Firstly for each $n \in \mathbb{N}$ we define $L(\mathcal{F})_{n,\mathrm{inf}} \in \mathcal{O}_{X_{\mathrm{inf}}}$, then we will define

$$L(\mathcal{F})_{\mathrm{inf}} := \lim_{n\in\mathbb{N}} L(\mathcal{F})_{n,\mathrm{inf}}.$$

In order to build the sheaves $L(\mathcal{F})_{n,\mathrm{inf}}$ we use the construction made in the Lemma 1.3.10. Let $\mathcal{F}$ be an

$\mathcal{O}_X$-module, let's consider his $n$-linearization $L(\mathcal{F})_n := \mathcal{P}^n_X \otimes_{\mathcal{O}_X} \mathcal{F}$, it has a canonical stratification given by

$$\begin{aligned} \epsilon^{\mathcal{F},can}_{m,n} : \mathcal{P}^m_X \otimes_{\mathcal{O}_X} L(\mathcal{F})_n &\xrightarrow{\qquad\cong\qquad} L(\mathcal{F})_n \otimes_{\mathcal{O}_X} \mathcal{P}^m_X \\ (a\otimes b)\otimes(c\otimes d)x &\longmapsto (a\otimes dx)\otimes(1\otimes bc) = (1\otimes dx)\otimes(a\otimes bc) \end{aligned}.$$

Via this canonical stratification we get the crystal $L(\mathcal{F})_{n,\mathrm{inf}}$. Moreover via a differential map $D : \mathcal{F} \to \mathcal{G}$ of order $n$ we get a linearized map $L(D)_n : L(\mathcal{F})_n \to L(\mathcal{G})_n$. For any tickening $(U,T)$ with a section $g : T \to X$ we can define

$$L(D)_{n,\mathrm{inf},(U,T)} := g^* L(D)_n : L(\mathcal{F})_{n,\mathrm{inf},(U,T)} = g^* L(\mathcal{F}) \longrightarrow L(\mathcal{G})_{n,\mathrm{inf},(U,T)} = g^* L(\mathcal{G})$$

moreover the following diagram commutes

$$\begin{tikzcd} p^m_0(1)^* L(\mathcal{F})_n \arrow[r, "\epsilon^{\mathcal{F},can}_{m,n}"] \arrow[d, "p^m_0(1)^* L(D)_n"] & p^m_1(1)^* L(\mathcal{F})_n \arrow[d, "p^m_1(1)^* L(D)_n"] \\ p^m_0(1)^* L(\mathcal{G})_n \arrow[r, "\epsilon^{\mathcal{G},can}_{m,n}"] & p^m_1(1)^* L(\mathcal{G})_n \end{tikzcd},$$

indeed following the upper path we get

$$(a\otimes bx)\otimes(c\otimes d) \mapsto (a\otimes D(bx))\otimes(c\otimes d) \mapsto (ac\otimes d)\otimes(1\otimes D(bx))$$

and following the other path we get

$$(a\otimes bx)\otimes(c\otimes d) \mapsto (ac\otimes d)\otimes(1\otimes bx) \mapsto (ac\otimes d)\otimes(1\otimes D(bx)).$$

Hence the morphism $L(D)_{n,\mathrm{inf}}$ glues on intersections and we can define $L(D)_{\mathrm{inf}}$ as the limit of the maps

$$L(\mathcal{F})_{\mathrm{inf}} \longrightarrow L(\mathcal{F})_{n,\mathrm{inf}} \longrightarrow L(\mathcal{G})_{n,\mathrm{inf}}.$$

□

Observe that if $\mathcal{F}$ is an $\mathcal{O}_X$-module with an integrable connection $\nabla$, we have a stratification on $\mathcal{F}$ given by the connection (see the Lemma 1.2.17). This observation allows us to define $\mathcal{F}_{\mathrm{inf}} \in X_{\mathrm{inf}}$ via the construction of the Lemma 1.3.10. Now we can define a linearization of $\mathcal{F}_{\mathrm{inf}}$ in the infinitesimal topos.

**Definition 1.3.11.** If $\mathcal{F}$ is an $\mathcal{O}_X$-module with an integrable connection $\nabla$, the **linearization of** $\mathcal{F}$ in the infinitesimal topos is the sheaf

$$\mathcal{F}_{\mathrm{inf}} \hat{\otimes} L(\mathcal{O}_X) := \lim_{n\in\mathbb{N}} (\mathcal{F} \otimes_{\mathcal{O}_X} L(\mathcal{O}_X)_n)_{\mathrm{inf}}$$

where the stratification is given by the isomorphisms

$$\mathcal{P}^m_X \otimes_{\mathcal{O}_X} (\mathcal{F} \otimes_{\mathcal{O}_X} L(\mathcal{O}_X)_n) \xrightarrow{\epsilon_m \otimes \mathrm{id}_{\mathcal{P}^n_X}} \mathcal{F} \otimes (\mathcal{P}^m_X \otimes L(\mathcal{O}_X)_n) \xrightarrow{\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}_{m,n}} (\mathcal{F} \otimes L(\mathcal{O}_X)_n) \otimes \mathcal{P}^m_X$$

and the limit is done in the category of sheaves in $X_{\mathrm{inf}}$.

Analogously one can define for any $k \in \mathbb{N}$ the sheaf

$$\mathcal{F}_{\inf}\hat{\otimes} L(\Omega^k_{X/S})_{\inf} := \lim_{n\in\mathbb{N}} \left(\mathcal{F} \otimes_{\mathcal{O}_X} L(\Omega^k_{X/S})_n\right)_{\inf}$$

where the stratification isomorphisms (as before) are given by the composition of the stratification on $\mathcal{F}$ given by the connection and the canonical stratifications on $L(\Omega^k_{X/S})_n$.

**Proposition 1.3.12.** *For any $\mathcal{O}_X$-module $\mathcal{F}$ with connection $\nabla$, there are two complexes of sheaves $\mathcal{F}_{\inf}\hat{\otimes} L(\Omega^\bullet_{X/S})_{\inf}$ and $L(\mathcal{F}\otimes\Omega^\bullet_{X/S})_{\inf}$ that are isomorphic.*

*Proof.* The first complex is given via the (limit) maps of the following morphisms

$$\mathcal{F}_{\inf}\hat{\otimes} L(\Omega^k_{X/S})_{\inf} \to \left(\mathcal{F}\otimes L(\Omega^k_{X/S})_{n+1}\right)_{\inf} \overset{(id_{\mathcal{F}}\otimes L(d^k)_n)_{\inf}}{\longrightarrow} \left(\mathcal{F}\otimes L(\Omega^{k+1}_{X/S})_{n+1}\right)_{\inf}.$$

In order to show that this morphism is well defined we have to check that it glues, *i.e.* that the maps "commutes with the stratification", *i.e.* that the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}^m_X\otimes\mathcal{F}\otimes L(\Omega^k_{X/S})_{n+1} \arrow[r, "\epsilon_m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\delta^{n,1}\otimes\mathrm{id}"] & \mathcal{F}\otimes\mathcal{P}^n_X\otimes L(\Omega^k_{X/S})_{n+1} \arrow[r, "\mathrm{id}_{\mathcal{F}}\otimes\epsilon^{can}_{m,n}"] & \left(\mathcal{F}\otimes L(\Omega^k_{X/S})_{n+1}\right)\otimes\mathcal{P}^m_X \arrow[d, "\mathrm{id}\otimes\delta^{n,1}\otimes\mathrm{id}"] \\
\mathcal{P}^m_X\otimes\mathcal{F}\otimes\mathcal{P}^n_X\otimes\mathcal{P}^1_X\otimes\Omega^k_{X/S} \arrow[d, "\mathrm{id}\otimes L(d^k)_1"] & & \left(\mathcal{F}\otimes\mathcal{P}^n_X\otimes\mathcal{P}^1_X\otimes\Omega^k_{X/S}\right)\otimes\mathcal{P}^m_X \arrow[d, "\mathrm{id}\otimes L(d^k)_1\otimes\mathrm{id}"] \\
\mathcal{P}^m_X\otimes\mathcal{F}\otimes\mathcal{P}^n_X\otimes\Omega^{k+1}_{X/S} \arrow[r, "\epsilon_m\otimes\mathrm{id}"] & \mathcal{F}\otimes\mathcal{P}^m_X\otimes L(\Omega^{k+1}_{X/S})_n \arrow[r, "\mathrm{id}\otimes\epsilon^{can}_{m,n}"] & \left(\mathcal{F}\otimes L(\Omega^{k+1}_{X/S})_n\right)\otimes\mathcal{P}^m_X
\end{tikzcd}$$

where $\{\epsilon_m\}_{m\in\mathbb{N}}$ is the stratification attached to $\mathcal{F}$ relative to $\nabla$. The commutativity could be checked locally where $X$ has coordinates $x_1,\dots,x_d$ and we denote $\zeta_i := \mathrm{d}\,x_i$ and $\zeta^q = \prod_{i=1}^d \zeta_i^{q_i}$ for any $q\in\mathbb{N}^d$. We denote $\mathbb{1} := 1\otimes 1$ and $D_q := \frac{1}{q!}\prod_{i=1}^d \partial_{x_i}^{q_i}$. Observe that all the maps are $\mathcal{O}_X$-linear with respect to the left multiplication and

$$(1\otimes a)\otimes x\otimes(b\otimes c)\otimes w = (1\otimes 1)\otimes abx\otimes(1\otimes 1)\otimes cw \in \mathcal{P}^m_X\otimes\mathcal{F}\otimes L(\Omega^k_{X/S})_{n+1},$$

computing the arrows of the diagram one gets

$$\begin{tikzcd}
\mathbb{1}\otimes x\otimes\mathbb{1}\otimes\omega \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{q\in\mathbb{N}^d,|q|\leqslant n}\left(\nabla_{D_q}x\right)\otimes\zeta^q\otimes\mathbb{1}\otimes\omega \arrow[r, mapsto] & \sum_q\left(\left(\nabla_{D_q}x\right)\otimes\mathbb{1}\otimes\omega\right)\otimes\zeta^q \arrow[d, mapsto] \\
\mathbb{1}\otimes x\otimes\mathbb{1}\otimes\mathrm{d}^k\omega \arrow[r, mapsto] & \sum_q\left(\nabla_{D_q}x\right)\otimes\zeta^q\otimes\mathbb{1}\otimes\mathrm{d}^k\omega \arrow[r, mapsto] & \sum_q\left(\left(\nabla_{D_q}x\right)\otimes\mathbb{1}\otimes\mathrm{d}^k\omega\right)\otimes\zeta^q
\end{tikzcd}.$$

Hence the complex $\mathcal{F}_{\inf}\hat{\otimes} L(\Omega^\bullet_{X/S})_{\inf}$ is well defined.

The complex $L(\mathcal{F}\otimes\Omega^\bullet_{X/S})_{\inf}$ is well defined via the functoriality of $L(-)_{\inf}$ and the fact that $\nabla^k$ are differential morphisms (of degree 1).

Now we want to check that the two complexes are isomorphic. Remember that

$$L(\mathcal{F}\otimes\Omega^k_{X/S})_{\inf} = \lim_{n\in\mathbb{N}} L(\mathcal{F}\otimes\Omega^k_{X/S})_{n,\inf}, \quad \mathcal{F}_{\inf}\hat{\otimes} L(\Omega^k_{X/S})_{\inf,(U,T)} = \lim_{n\in\mathbb{N}} \mathcal{F}_{\inf}\otimes L(\Omega^k_{X/S})_{n,\inf},$$

where the two crystals on the limits are built via their stratification as in Lemma 1.3.10. Then it suffices to show that for each $n\in\mathbb{N}$ there is an isomorphism

$$L(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^k_{X/S})_n \cong \mathcal{F}\otimes_{\mathcal{O}_X} L(\Omega^k_{X/S})_n$$

commuting with the two stratifications and then check that this isomorphism commutes with the complex maps. The isomorphism is the following:

$$L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^k_{X/S})_n = \mathcal{P}^n_X \otimes \mathcal{F} \otimes \Omega^k_{X/S)} \xrightarrow{\epsilon_n \otimes \mathrm{id}_{\Omega^k}} \mathcal{F} \otimes \mathcal{P}^n_X \otimes \Omega^k_{X/S)} = \mathcal{F} \otimes L(\Omega^k_{X/S})_n \,.$$

In order to check the compatibility of the stratification with the isomorphism we have to show that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{P}^n_X \otimes L\left(\mathcal{F} \otimes \Omega^k_{X/S}\right)_m \arrow[r, "\epsilon^{can}_{m,n}"] \arrow[d, "\mathrm{id}_{\mathcal{P}^m_X} \otimes \epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S}}"] & L\left(\mathcal{F} \otimes \Omega^k_{X/S}\right)_m \otimes \mathcal{P}^n_X \arrow[drr, "\epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S} \otimes \mathcal{P}^n_X}"] & & \\
\mathcal{P}^n_X \otimes \mathcal{F} \otimes L\left(\mathcal{P}^m_X \otimes \Omega^k_{X/S}\right) \arrow[r, "\epsilon_n \otimes \mathrm{id}"] & \mathcal{F} \otimes \mathcal{P}^n_X \otimes \mathcal{P}^m_X \otimes \Omega^k_{X/S} \arrow[rr, "\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}"] & & \left(\mathcal{F} \otimes \mathcal{P}^m_X \otimes \Omega^k_{X/S}\right) \otimes \mathcal{P}^n_X
\end{tikzcd}$$

The commutativity could be checked locally and could be checked in the case $n = m$ via the compatibility condition on the stratifications and the surjectivity of the projection morphisms. We use the same notations as above for coordinates and differentials and as in the Lemma 1.2.18 for the elements $\zeta^q_{0,2}$. Computing the two maps one gets

$$\begin{aligned}
&\left(\epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S} \otimes \mathcal{P}^n_X}\right) \circ \epsilon^{can}_{m,n} \left(\mathbb{1} \otimes \mathbb{1} \otimes (x \otimes \omega)\right) \\
&\qquad = \left(\epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S} \otimes \mathcal{P}^n_X}\right)\left(\mathbb{1} \otimes (x \otimes \omega)\right) \otimes \mathbb{1} = \left(\sum_{q \in \mathbb{N}^d, |q| \leqslant n} \left(\nabla_{D_q} x\right) \otimes \zeta^q \otimes \omega\right) \otimes \mathbb{1}
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}) \circ (\epsilon_n \otimes \mathrm{id}) \circ \left(\mathrm{id}_{\mathcal{P}^m_X} \otimes \epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S}}\right)\left(\mathbb{1} \otimes \mathbb{1} \otimes (x \otimes \omega)\right) \\
&\qquad = (\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}) \circ (\epsilon_n \otimes \mathrm{id}) \left(\mathbb{1} \otimes \sum_{q_1 \in \mathbb{N}^d, |q_1| \leqslant n} \left(\nabla_{D_{q_1}} x\right) \otimes \zeta^{q_1} \otimes \omega\right) \\
&\qquad = (\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}) \left(\sum_{|q_1| \leqslant n, |q_2| \leqslant m} \left(\nabla_{D_{q_2}} \nabla_{D_{q_1}} x\right) \otimes \zeta^{q_1} \otimes \zeta^{q_2} \otimes \omega\right) \\
&\qquad = (\mathrm{id}_{\mathcal{F}} \otimes \epsilon^{can}) \left(\sum_{q \in \mathbb{N}^d, |q| \leqslant n} \left(\nabla_{D_q} x\right) \otimes \zeta^q_{0,2} \otimes \omega\right) = \left(\sum_{q \in \mathbb{N}^d|, q| \leqslant n} \left(\nabla_{D_q} x\right) \otimes \zeta^q \otimes \omega\right) \otimes \mathbb{1}
\end{aligned}$$

where we used the computation in the proof of the Lemma 1.2.18 for the equality

$$\sum_{|q_1| \leqslant n, |q_2| \leqslant m} \left(\nabla_{D_{q_2}} \nabla_{D_{q_1}} x\right) \otimes \zeta^{q_1} \otimes \zeta^{q_2} \otimes \omega = \sum_{q \in \mathbb{N}^d, |q| \leqslant n} \left(\nabla_{D_q} x\right) \otimes \zeta^q_{0,2} \otimes \omega.$$

The last thing that we have to show is that the isomorphisms just built commute with the two complexes. This could be checked locally and only on $(X, X)$, the diagram over a tickening is (locally) the pullback of the diagram over $(X, X)$. The diagram is the following:

$$\begin{tikzcd}
\mathcal{P}^{n+1}_X \otimes \mathcal{F} \otimes \Omega^k_{X/S} \arrow[r, "L(\nabla^k)_{m+1}"] \arrow[d, "\epsilon_m \otimes \mathrm{id}_{\Omega^k_{X/S}}"] & \mathcal{P}^m \otimes \mathcal{F} \otimes \Omega^{k+1}_{X/S} \arrow[d, "\epsilon_m \otimes \mathrm{id}_{\Omega^{k+1}_{X/S}}"] \\
\mathcal{F} \otimes \mathcal{P}^{m+1}_X \otimes \Omega^k_{X/S} \arrow[r, "\mathrm{id}_{\mathcal{F}} \otimes L(d^i)_m"] & \mathcal{F} \otimes \mathcal{P}^m_X \otimes \Omega^{k+1}_{X/S}
\end{tikzcd} .$$

The commutativity of this diagram follows by the computations done in the Proposition 1.3.9. □

We can define the complex $L(\Omega^\bullet_{X/S})_{\mathrm{inf}}$; as in the Proposition 1.3.9 we get an exact sequence.

**Theorem 1.3.13.** *The sequence*

$$0 \to \mathcal{O}_{X_{\mathrm{inf}}} \to L(\Omega^\bullet_{X/S})_{\mathrm{inf}}$$

*is exact, moreover if $\mathcal{F}$ is a flat $\mathcal{O}_X$-module with a connection $\nabla$, the sequences*

$$0 \to \mathcal{F}_{\mathrm{inf}} \to \mathcal{F}_{\mathrm{inf}} \hat{\otimes} L(\Omega^\bullet_{X/S})_{\mathrm{inf}} \quad \text{and} \quad 0 \to \mathcal{F}_{\mathrm{inf}} \to L(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S})_{\mathrm{inf}}$$

*are exact.*

*Proof.* A sequence in the infinitesimal site is exact if and only if the sequence of sheaves associated to any tickening $(U,T)$ is exact in the rigid topos of $T$. We can work locally, where $(U,T)$ has a section $g : T \to X$ and the sequence becomes

$$0 \to g^*\mathcal{O}_X \to g^*L(\mathcal{O}_{X/S}) \overset{g^*L(\mathrm{d})}{\to} g^*L(\Omega^1_{X/S}) \overset{g^*L(\mathrm{d}^1)}{\to} g^*L(\Omega^2_{X/S}) \to \cdots \quad .$$

Locally, where we have coordinates $x_1, \ldots, x_d$ of $X$, the sequence is

$$0 \to \mathcal{O}_T \to \mathcal{O}_T[[\zeta_1, \ldots, \zeta_d]] \to \mathcal{O}_T[[\zeta_1, \ldots, \zeta_d]] \hat{\otimes}_{\mathcal{O}_X} \Omega^1_{X/S} \to \cdots \quad .$$

Via the explicit computations done in the Proposition 1.3.8 the complex above is exact.

In order to see that

$$0 \to \mathcal{F}_{\mathrm{inf}} \to \mathcal{F}_{\mathrm{inf}} \hat{\otimes} L(\Omega^\bullet_{X/S})_{\mathrm{inf}}$$

is exact; one can work locally and, as in the Proposition 1.3.9, this sequence is limit of exact sequences that (locally) satisfy the M–L condition. □

### 1.3.3 Delinearization and the comparison Theorem

The main goal of this section is to introduce a morphism of topoi

$$u_{X/S} : X_{\mathrm{inf}} \longrightarrow X_{\mathrm{rig}}$$

and to show that for a flat $\mathcal{O}_X$-module $\mathcal{F}$ with an integrable connection there is a canonical isomorphism

$$H^i\left(X_{\mathrm{inf}}, \mathcal{F}_{\mathrm{inf}}\right) \cong H^i_{dR}\left(X, (\mathcal{F}, \nabla)\right).$$

Firstly we define what a morphism of topoi is.

**Definition 1.3.14.** Given two topoi $\mathcal{C}$ and $\mathcal{D}$, a **morphism of topoi** $f : \mathcal{C} \to \mathcal{D}$ is an adjunction pair $f^* \leftrightarrows f_*$ with

$$f_* : \mathcal{C} \to \mathcal{D}, \quad f^* : \mathcal{D} \to \mathcal{C}$$

such that $f^*$ commutes with finite limits.

Observe that an adjunction pair is s.t. $f^*$ commutes with arbitrary co-limits and $f_*$ commutes with arbitrary limits. We denote with

$$X_{\mathrm{inf},X} := \mathrm{Sh}\left((X/S)_{\mathrm{inf},(X,X)}\right),$$

where $((X/S)_{\mathrm{inf},(X,X)})$ is the category of the morphisms $g : (U,T) \to (X,X)$ in $(X/S)_{\mathrm{inf}}$ and a covering of $g$ is a collection of morphisms $\{g_i \to g\}_{i\in I}$, where $g_i : (U_i,T_i) \to (X,X)$ such that $(U_i,T_i)_{i\in I}$ is an admissible covering of $(U,T)$ in $(X/S)_{\mathrm{inf}}$. A sheaf $\mathcal{G} \in X_{\mathrm{inf},X}$ is a compatible collection of sheaves $\mathcal{G}_g$ on $T$, where $T$ and $g$ vary between all the morphisms of thickenings $g : (U,T) \to (X,X)$.

**Definition 1.3.15.** Let $\varphi : X_{\mathrm{inf},X} \to X_{\mathrm{rig}}$ be the morphism of topoi given by

$$\begin{array}{ccc} \varphi^{-1} : X_{\mathrm{rig}} & \longrightarrow & X_{\mathrm{inf},X} \\ \mathcal{E} & \longmapsto & \{\beta^{-1}\mathcal{E}\}_{(\alpha,\beta):(U,T)\to(X,X)} \end{array} \qquad \begin{array}{ccc} \varphi_* : X_{\mathrm{inf},X} & \longrightarrow & X_{\mathrm{rig}} \\ \mathcal{G} = \{\mathcal{G}_g\}_{g:(U,T)\to(X,X)} & \longmapsto & \mathcal{G}_{\mathrm{id}_{(X,X)}} \end{array}.$$

Let $j : X_{\mathrm{inf},X} \to X_{\mathrm{inf}}$ be the morphism of topoi defined by

$$\begin{array}{ccc} j^* : X_{\mathrm{inf}} & \longrightarrow & X_{\mathrm{inf},X} \\ \mathcal{F} = \{\mathcal{F}_{(U,T)}\}_{(U,T)} & \longmapsto & \{\mathcal{F}_{(U,T)}\}_{(U,T)\to(X,X)} \end{array}$$

and the sheafification of

$$\begin{array}{ccc} \tilde{j}'_* : X_{\mathrm{inf},X} & \longrightarrow & \mathrm{PSh}\,(X/S)_{\mathrm{inf}} \\ \mathcal{G} = \{\mathcal{G}_g\}_{g:(U,T)\to(X,X)} & \longmapsto & \{\prod\limits_{g:(U,T)\to(X,X)} \mathcal{G}_g\}_{(U,T)} \end{array}.$$

We denote the sheafification of $\tilde{j}'_*$ as $j'_*$. Let $u : X_{\mathrm{inf}} \to X_{\mathrm{rig}}$ the morphism of topoi given by

$$\begin{array}{ccc} u^* : X_{\mathrm{rig}} & \longrightarrow & X_{\mathrm{inf}} \\ \mathcal{E} & \longmapsto & \{t_*\mathcal{E}_{|_U}\}_{t:U\to T} \end{array} \qquad \begin{array}{ccc} u_* : X_{\mathrm{inf}} & \longrightarrow & X_{\mathrm{rig}} \\ \mathcal{F} = \{\mathcal{F}_{(U,T)}\}_{(U,T)} & \longmapsto & \{\Gamma(U_{\mathrm{inf}}, \mathcal{F}_{|_{U_{\mathrm{inf}}}})\}_U \end{array}.$$

**Remark 1.3.16.** The pairs in the above definitions define morphisms of topoi, indeed

- An $f \in Hom_{X_{\mathrm{inf},X}}(\varphi^{-1}\mathcal{E},\mathcal{F})$ corresponds to a compatible choice of morphisms $f_g : \beta^{-1}\mathcal{E} \to \mathcal{F}_g$ for each $g = (\alpha,\beta) : (U,T) \to (X,X)$, the cocycle condition says that for a morphism $g$ as before, the following diagram should commute:

$$\begin{array}{ccc} \beta^{-1}\mathcal{E} & \xrightarrow{\beta^{-1} f_{id_{(X,X)}}} & \beta^{-1}\mathcal{F}_{id_{(X,X)}} \\ \| & & \downarrow res^{\mathcal{F}} \\ \beta^{-1}\mathcal{E} & \xrightarrow{\quad f_g \quad} & \mathcal{F}_g \end{array},$$

  hence $f_g$ is determined via the restriction morphism of $\mathcal{F}$ and $f_{id_{(X,X)}}$ and the association $f \mapsto f_{id_{(X,X)}}$ gives the adjunction;

- For the map $j$ one can refer to the Proposition 5.23 in [BO78]. More explicitly if $f \in Hom_{X_{\mathrm{inf},X}}(j^{-1}G,\mathcal{F})$, *i.e.* a collection of compatible morphism $f_g : \mathcal{G}_{(U,T)} \to \mathcal{F}_g$, we can (locally, where we have sections $g$) build the family of morphisms $h_{(U,T)} := \prod\limits_g f_g : \mathcal{G}_{(U,T)} \to \prod\limits_{g:(U,T)\to(X,X)} \mathcal{F}_g$. Since locally any tickening $(U,T)$ has at least a section $g : (U,T) \to (X,X)$, then this procedure gives a bijection $Hom_{X_{\mathrm{inf},X}}(j^{-1}\mathcal{G},\mathcal{F}) \cong Hom_{X_{\mathrm{inf}}}(\mathcal{G}, j'_*\mathcal{F})$;

- Every $f \in Hom_{X_{\mathrm{rig}}}(\mathcal{E}, u_*\mathcal{F})$ is a compatible collection of morphisms $f_U : \mathcal{E}_{|_U} \to (u_*\mathcal{F})_{|_U}$. For each nilpotent tickening $t: U \to T$ there is an arrow

$$Hom_{(-)_{\inf}}\left(\mathbb{1}, \mathcal{F}_{(-)_{\inf}}\right) \to Hom_{(-)_{\inf}}\left((h_{(U,T)})_{|(-)_{\inf}}, \mathcal{F}_{(-)_{\inf}}\right)$$

of sheaves in $U_{\mathrm{rig}}$, remember that $h_{(U,T)}$ denotes the sheaf associated to $(U,T)$ via the Yoneda's embedding. Via the pushforward $t_*$ we get a morphism

$$s_{(U,T)} : (u^{-1}\mathcal{E})_{(U,T)} = t_*\mathcal{E}_{|U} \xrightarrow{t_* f_U} t_* Hom_{(-)_{\inf}}\left(\mathbb{1}, \mathcal{F}_{(-)_{\inf}}\right) \to \mathcal{F}_{(U,T)}.$$

This association is bijective since $t$ is nilpotent, hence $t_* : U_{\mathrm{rig}} \to T_{\mathrm{rig}}$ is an equivalence of categories. The inverse is given by taking a morphism $s = \{s_{(U,T)}\}$. For any section $x \in \mathcal{E}(U)$ we get an element $x_{(U,T)} := s_{(U,T)}(x) \in \mathcal{F}(U,T)$ for each $(V,T)$ and $V \subset U$, this family $x_{(U,T)}$ gives a morphism $f_U(x) \in Hom_{U_{\inf}}\left(\mathbb{1}, \mathcal{F}_{|U_{\inf}}\right)$. Varying $U \subset X$, $x \in \mathcal{E}(U)$ we get the morphism $f : \mathcal{E} \to u_*\mathcal{F}$ associated to $s$.

We get a diagram

$$\begin{tikzcd} X_{\inf,X} \arrow[r, "\varphi"] \arrow[d, "j"] & X_{\mathrm{rig}} \\ X_{\inf} \arrow[ur, "u"] & \end{tikzcd}.$$

One should check that $u^*, j^*, \varphi^{-1}$ commute with finite limits, this could be done using that (notations as before) $t_*$ is an equivalence of category and $\beta^{-1}$ commutes with finite limits.

**Lemma 1.3.17.** *The diagram above is commutative.*

*Proof.* If an adjoint functor exists it is unique (up to natural isomorphism), then we can reduce to compute that $(u \circ j)^* = \varphi^{-1}$.

Let $\mathcal{E} \in X_{\mathrm{rig}}$, then for any $g = (\alpha, \beta) : (U,T) \to (X,X)$

$$((u \circ j)^*\mathcal{E})_g = (j^* u^* \mathcal{E})_g = (u^*\mathcal{E})_{(U,T)} = t_*\mathcal{E}_{|U},$$

where $t: U \to T$. Moreover

$$(\varphi^{-1}\mathcal{E})_g = \beta^{-1}\mathcal{E}.$$

Since $t$ is a nilpotent tickening $t^{-1}$ and $t_*$ are equivalences of categories and

$$(\varphi^{-1}\mathcal{E})_g = t_* t^{-1} \beta^{-1} \mathcal{E} = t_*\mathcal{E}_{|U} = ((u \circ j)^*\mathcal{E})_g .$$

□

**Definition 1.3.18.** Let $j_!^{Ab} : X_{\inf,X}^{Ab} \to X_{\inf}^{Ab}$ be the functor that associates to an abelian sheaf $\mathcal{G} \in X_{\inf,X}^{Ab}$ the abelian sheaf defined by

$$(j_!^{Ab}\mathcal{G})_{(U,T)} := \bigoplus_{g:(U,T)\to(X,X)} \mathcal{G}_g .$$

Let $j_! : X_{\inf,X} \to X_{\inf}$ be the functor that associates to a sheaf $\mathcal{G} \in X_{\inf,X}$ the sheaf defined by

$$(j_!\, \mathcal{G})_{(U,T)} := \coprod_{g:(U,T)\to(X,X)} \mathcal{G}_g .$$

Observe that the functors $j'_*$ and $j^*$ restrict to a pair of functors

$$j^* : X_{\inf}^{Ab} \leftrightarrows X_{\inf,X}^{Ab} : j'_*$$

that are adjoint. Observe that $j_!^{Ab} \leftrightarrows j^*$ are adjoint. Indeed a morphism $f \in Hom_{X_{\inf}^{Ab}}\left(j_!^{Ab}\mathcal{G}, \mathcal{F}\right)$ is a collection of morphisms $f_{(U,T)} : \oplus_g \mathcal{G}_g \to \mathcal{F}_{(U,T)}$ and it corresponds to a collection of morphisms $f_g : \mathcal{G}_g \to \mathcal{F}_{(U,T)} = (j^*\mathcal{F})_g$.

Analogously $j_! \leftrightarrows j^*$.

**Remark 1.3.19.** $j_!^{Ab} \leftrightarrows j^* \leftrightarrows j'_*$, hence the functor $j^*$ commutes with limits and colimits, then $j^*$ is exact.

**Lemma 1.3.20.** *If $\mathcal{E} \in X_{\inf}^{Ab}$ is an abelian sheaf, then for each $i \in \mathbb{N}$*

$$H^i\left(X_{\inf,X}, j^*\mathcal{E}\right) \cong H^i\left((X,X), \mathcal{E}\right).$$

*Proof.* For $i = 0$ the proof is an easy computation:

$$\Gamma\left((X,X), \mathcal{E}\right) = Hom_{X_{\inf}}\left(h_{(X,X)}, \mathcal{E}\right) = Hom_{X_{\inf}}\left(j_! \mathbb{1}, \mathcal{E}\right) \cong Hom_{X_{\inf,X}}\left(\mathbb{1}, j^*\mathcal{E}\right) = \Gamma\left(X_{\inf,X}, j^*\mathcal{E}\right).$$

For $i > 0$ observe that $j_!^{Ab}$ is exact, hence his right adjoint $j^*$ takes injectives to injectives and it is exact by the Remark 1.3.19, hence we can take an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{E}$ and

$$H^i\left((X,X), \mathcal{E}\right) = H^i\left(\mathcal{I}^\bullet(X,X)\right)$$

moreover (by the previous observations) $j^*\mathcal{I}^\bullet$ is an injective resolution of $j^*\mathcal{E}$ and

$$H^i\left(X_{\inf,X}, j^*\mathcal{E}\right) = H^i\left((j^*\mathcal{I}^\bullet)(id_{(X,X)})\right) = H^i\left(\mathcal{I}^\bullet(X,X)\right).$$

In the last equality we used the computation with $i = 0$. □

Observe that $\varphi_*, \varphi^{-1}$ restrict to a pair of functors between abelian sheaves (that we also denote with $\varphi_*$ and $\varphi^{-1}$). Moreover also the pair

$$\varphi^{-1} : X_{\mathrm{rig}}^{Ab} \leftrightarrows X_{\inf,X}^{Ab} : \varphi_*$$

is a pair of adjunction.

**Proposition 1.3.21.** *For any $\mathcal{F} \in X_{\inf}$ we have that $\varphi_* j^*\mathcal{F} \cong \mathcal{F}_{(X,X)}$, moreover $\varphi_*$ is exact and for any abelian sheaf $\mathcal{F} \in X_{\inf}^{Ab}$*

$$H^i\left((X,X), \mathcal{F}\right) \cong H^i\left(X_{\mathrm{rig}}, \mathcal{F}_{(X,X)}\right).$$

*Proof.* For $\mathcal{F} \in X_{\inf}$, then

$$\varphi_* j^*\mathcal{F} = (j^*\mathcal{F})_{id_{(X,X)}} = \mathcal{F}_{(X,X)}$$

$\varphi_*$ preserves limits since it is right adjoint to the functor $\varphi^{-1}$. Moreover $\varphi_*$ preserves epimorphisms: $f : \mathcal{F}_1 \to \mathcal{F}_2$ is an epimorphism of abelian sheaves $\mathcal{F}_1, \mathcal{F}_2 \in X_{\inf,X}^{Ab}$ iff for each $g : (U,T) \to (X,X)$ the map $f_g$ is an epimorphism of sheaves in $T_{\mathrm{rig}}$, hence if $f$ is an epimorphism, then $\varphi_* f = f_{id_{(X,X)}}$ is an epimorphism too. Hence $\varphi_*$ preserves limits and epimorphisms, then it preserves all exact sequences and it is exact.

For the last part of the statement let $\mathcal{F} \in X_{\inf}^{Ab}$, observe that $\varphi^{-1}$ commutes with colimits (since it is left adjoint) and finite limits, hence it is exact. Hence $\varphi_*$ brings injectives into injectives and it is exact, then the spectral sequence

$$R^p\Gamma\left(X_{\mathrm{rig}}, -\right) \circ R^q\varphi_* \Rightarrow R^{p+q}\left(\varphi_* \circ \Gamma\left(X_{\mathrm{rig}}, -\right)\right) = R^{p+q}\left(\Gamma\left(X_{\inf,X}, -\right)\right).$$

degenerates to

$$\left(R^i\Gamma\left(X_{\mathrm{rig}}, -\right)\right) \circ \varphi_* \cong R^i\left(\Gamma\left(X_{\inf,X}, -\right)\right).$$

Hence for any $\mathcal{F} \in X_{\inf}^{Ab}$

$$H^i((X,X),\mathcal{F}) \cong H^i(X_{\inf,X}, j^{-1}\mathcal{F}) \cong H^i(X_{\mathrm{rig}}, \varphi_* j^{-1}\mathcal{F}).$$

□

**Corollary 1.3.22.** *For any $\mathcal{G} \in X_{\inf,X}^{Ab}$ the abelian sheaf $j'_*\mathcal{G}$ is acyclic for $u_*$.*

*Proof.* The functor $j^*$ commutes with colimits and finite limits, then it is exact, hence $j'_*$ brings injective objects to injective objects and there is a spectral sequence

$$R^p u_* R^q j'_* \Rightarrow R^{p+q}(u \circ j')_* \cong R^{p+q}\varphi_*.$$

But $j'_*$ is also exact via the explicit description and the fact that locally any $(U,T)$ has a section $(U,T) \to (X,X)$. Moreover $\varphi_*$ is exact, hence the spectral sequence degenerates to

$$\left(R^i u_*\right) j'_*\mathcal{G} = 0$$

for any $i > 0$ and $\mathcal{G} \in X_{\inf,X}$. □

Now we give another description of the functor $j'_*$.

**Definition 1.3.23.** Let $j_* : X_{\inf,X} \longrightarrow X_{\inf}$ be the functor that sends a sheaf $\mathcal{G} \in X_{\inf,X}$ to the sheaf

$$j_*\mathcal{G} := \{\lim_{n\in\mathbb{N}} (p_T^n)_* \, \mathcal{G}_{p_X^n}\}$$

where $p_T^n : P^n_{U/U\times_S T} \to T$ and $p_X^n : D_U^n(U \times_S T) \to (X,X)$ are the two projections.

**Lemma 1.3.24.** *There is a natural isomorphism $j_* \cong j'_*$; in particular for any $\mathcal{G} \in X_{\inf,X}^{Ab}$, the sheaf $j_*\mathcal{G}$ is acyclic for $u_*$.*

*Proof.* Firstly we want to describe, for a thickening $(U,T)$, the sheaf $j^*h_{(U,T)}$. Let $g : (U',T') \to (X,X) \in (X/S)_{\inf|_X}$, then the set

$$\left(j^*h_{(U,T)}\right)(U',T') = Hom_{(X/S)_{\inf}}((U',T'),(U,T))$$

corresponds to the set of commutative diagrams between $(U',T') \to (X,X)$ and $(U, U\times_S T) \to (X,X)$; thanks to the Lemma 1.1.9 we get

$$\left(j^*h_{(U,T)}\right)(U',T') \cong \operatorname*{\underrightarrow{colim}}_{n\in\mathbb{N}} Hom_{(X/S)_{\inf|_X}}\left((U',T') \to (X,X), D_U^n(U\times_S T) \to (X,X)\right).$$

Then $j^*h_{(U,T)} = \operatorname*{\underrightarrow{colim}}_{n\in\mathbb{N}} Hom_{(X/S)_{\inf|_X}}(- \,, D_U^n(U\times_S T) \to (X,X))$ is colimit of representable objects.

Let $\mathcal{G} \in X_{\inf,X}$, then for any thickening $(U,T)$ we get

$$\begin{aligned}(j'_*\mathcal{G})(U,T) &\cong Hom_{X_{\inf}}\left(h_{(U,T)}, j_*\mathcal{G}\right) \cong Hom_{X_{\inf,X}}(j^{-1}h_{(U,T)},\mathcal{G}) \\ &= \lim_{n\in\mathbb{N}} \mathcal{G}\left(D_U^n(U\times_S T) \to (X,X)\right) = (j_*\mathcal{G})(U,T).\end{aligned}$$

□

Observe that $\varphi_*$ restricts to a functor

$$\varphi_* : \mathcal{O}_{X_{\inf,X}} - mod \longrightarrow \mathcal{O}_{X_{\mathrm{rig}}} - mod\ ,$$

we can define $\varphi^*$, an analogue of $\varphi^{-1}$.

**Definition 1.3.25.** Let $\varphi^* : \mathcal{O}_{X_{\mathrm{rig}}} - mod \longrightarrow \mathcal{O}_{X_{\mathrm{inf},X}} - mod$ be the functor that associates to an $\mathcal{O}_X$-module $\mathcal{E}$ the $\mathcal{O}_{X_{\mathrm{inf},X}}$-module defined by

$$(\varphi^*\mathcal{E})_g := \beta^*\mathcal{E}$$

where $g = (\alpha, \beta) : (U, T) \to (X, X)$.

One can prove that $\varphi^* \leftrightarrows \varphi_*$ following the proof of the adjunction $\varphi^{-1} \leftrightarrows \varphi_*$.

**Theorem 1.3.26.** *For any flat $\mathcal{O}_X$-module $\mathcal{E} \in X_{\mathrm{rig}}$ there is an isomorphism*

$$L(\mathcal{E})_{\mathrm{inf}} \cong j_*\varphi^*\mathcal{E},$$

*in particular $L(\mathcal{E})_{\mathrm{inf}}$ is acyclic for $u_*$.*

*Proof.* For any object $g = (\alpha, \beta) : (U, T) \to (X, X) \in (X/S)_{\mathrm{inf}|X}$ one can define the morphism

$$(j^*L(\mathcal{E})_{\mathrm{inf}})_g = \lim_{n\in\mathbb{N}} \beta^*L(\mathcal{E})_n \to \beta^*L(\mathcal{E})_0 = \beta^*\mathcal{E} = (\varphi^*\mathcal{E})_g\,.$$

Since

$$Hom_{X_{\mathrm{inf}}}(L(\mathcal{E})_{\mathrm{inf}}, j_*\varphi^*\mathcal{E}) \cong Hom_{X_{\mathrm{inf},X}}(j^*L(\mathcal{E})_{\mathrm{inf}}, \varphi^*\mathcal{E})\,,$$

the previous morphism corresponds to a morphism $\mu : L(\mathcal{E})_{\mathrm{inf}} \to j_*\varphi^*\mathcal{E}$, we check (locally) that this morphism is an isomorphism. Let $(U, T) \in (X/S)_{\mathrm{inf}}$ with a section $\beta : T \to X$, then

$$(j_*\varphi^*\mathcal{E})_{(U,T)} = \lim_{n\in\mathbb{N}} (p_T^n)_*\,(\varphi^*\mathcal{E})_{p_X^n} = \lim_{n\in\mathbb{N}} (p_T^n)_*(p_X^n)^*\mathcal{E} = \lim_{n\in\mathbb{N}} \left( \frac{(\mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_U)}{I^{n+1}_{U/U\times T}} \otimes_{\mathcal{O}_X} \mathcal{E}_{|U} \right)$$

where $I^{n+1}_{U/U\times T} = \mathrm{Ker}\,(\mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_U \to \mathcal{O}_U)$. On the other side

$$L(\mathcal{E})_{\mathrm{inf},(U,T)} = \lim_{n\in\mathbb{N}} \left( \frac{(\mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_U)}{I^{n+1}_{U/U\times U}} \otimes_{\mathcal{O}_X} \mathcal{E}_{|U} \right),$$

where $I^{n+1}_{U/U\times U} = \mathrm{Ker}\,(\mathcal{O}_U \otimes_{\mathcal{O}_S} \mathcal{O}_U \to \mathcal{O}_U)$.

The morphism $\mu$

$$\lim_{n\in\mathbb{N}} \left( \frac{(\mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_U)}{I^{n+1}_{U/U\times T}} \otimes_{\mathcal{O}_X} \mathcal{E}_{|U} \right) \longrightarrow \lim_{n\in\mathbb{N}} \left( \frac{(\mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{O}_U)}{I^{n+1}_{U/U\times U}} \otimes_{\mathcal{O}_X} \mathcal{E}_{|U} \right)$$

corresponds to the natural projection map, that via the Corollary A.0.4 is an isomorphism. ☐

**Proposition 1.3.27.** *Given a flat $\mathcal{O}_X$-module $\mathcal{F}$ over $X$ with an integrable connection $\nabla$*

$$u_*(L(\mathcal{F})_{\mathrm{inf}}) \cong \mathcal{F}.$$

*Proof.* The proof is a consequence of the Proposition 1.2.19, 1.3.9. More in detail, denote $F := L(\mathcal{F})$ with the canonical connection; for the Proposition 1.2.19 and by definition of $F_{\mathrm{inf}}$, for any open $U \subset X$, there are canonical identifications

$$\begin{aligned}\Gamma\left(U_{\text{inf}},(F_{\text{inf}})_{|U_{\text{inf}}}\right) &\cong \{x \in (F_{\text{inf}})_{|U_{inf}}(U,U) \mid \epsilon^1_{F_{\text{inf}}}(p^1_0(1)^*x) = p^1_1(1)^*x\} \\ &= \{x \in F_{\text{inf}}(U,U) \mid \epsilon^1_{F_{\text{inf}}}(p^1_0(1)^*x) = p^1_1(1)^*x\} \\ &= \{x \in F_{\text{inf}}(U,U) \mid \epsilon^1_{F}(p^1_0(1)^*x) = p^1_1(1)^*x\}.\end{aligned}$$

But by definition of $\epsilon^1_F$ we have that

$$\Gamma\left(U_{\text{inf}},(F_{\text{inf}})_{|U_{\text{inf}}}\right) \cong \mathrm{Ker}(L(\nabla))(U).$$

This isomorphism is functorial on $U$ and, via the Proposition 1.3.9, we get

$$u_*(F_{\text{inf}}) \cong \mathrm{Ker}(L(\nabla)) \cong \mathcal{F}.$$

□

In the proof we used that the stratification $\{\epsilon^n_{F_{\text{inf}}}\}_{n\in\mathbb{N}}$ attached to the sheaf $(F_{\text{inf}})_{(X,X)} = F$ defined in §1.2 is equal to the stratification $\{\epsilon^n_F\}_{n\in\mathbb{N}}$ that we used to define $F_{\text{inf}}$.

**Theorem 1.3.28.** *If $\mathcal{F}$ is a flat $\mathcal{O}_X$-module with integrable connection $\nabla$, then there is a canonical isomorphism*

$$H^i(X_{\text{inf}},\mathcal{F}_{\text{inf}}) \cong H^i_{dR}(X,\mathcal{F})$$

*Proof.* Observe that if $\mathcal{F}$ is flat, then $\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^k_{X/S}$ is flat too for any $k\in\mathbb{N}$, then via the Proposition 1.3.27 we get

$$u_*L\left(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/S}\right)_{\text{inf}} \cong \mathcal{F}\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/S}.$$

The sequence

$$0\to\mathcal{F}_{\text{inf}}\to L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}$$

is exact via the Theorem 1.3.13, then $\mathcal{F}_{\text{inf}}$ is isomorphic to $L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}$ in the derived category. Moreover via the Theorem 1.3.26 $L(\mathcal{F}\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/S})_{\text{inf}}$ is done by acyclic objects for $u_*$ and the Lerray spectral sequence

$$R^p\Gamma(X,-)R^qu_*\left(L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}\right)\Rightarrow R^{p+q}\Gamma(X_{\text{inf}},-)\left(L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}\right)$$

degenerates, then

$$\begin{aligned}R^i\Gamma(X_{\text{inf}},-)(\mathcal{F}_{\text{inf}}) &\cong R^i\Gamma(X_{\text{inf}},-)\left(L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}\right) \\ &\cong R^i\Gamma(X,-)\circ u_*\left(L\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right)_{\text{inf}}\right)\cong R^i\Gamma(X,-)\left(\mathcal{F}\otimes\Omega^\bullet_{X/S}\right).\end{aligned}$$

□

# Chapter 2

# BGG-decomposition for de Rham sheaves

## 2.1 BGG for infinite dimensional $\mathfrak{g}$-modules

In this Section we describe a modification of the BGG-theory for infinite dimensional modules with a filtration given by the action of $\mathfrak{n}^-$. Let $K$ be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{g}'$ be a finite semisimple split Lie algebra defined over $K$. Choose a maximal toral subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ and a simple basis of roots $\Delta'$ and fix the notation as in the Appendix B with the " ' ". We assume that $\mathfrak{n}'^{,+}$ and $\mathfrak{n}'^{,-}$ are abelian Lie algebras.

Let $S$ be a completed Banach algebra over $K$, suppose that $S$ is a domain and let $\mathfrak{g} := S \otimes_K \mathfrak{g}'$. When we remove the " ' " from an object defined over $K$ we mean that we make the base change to $S$.

Let $\mathfrak{W}_\kappa^{\mathrm{alg}}$ be an $S$-module and a $\mathfrak{g}$-module with an $S$-linear $\mathfrak{g}$-action: $(aX).v = X.(av)$ for any $a \in S$, $v \in \mathfrak{W}_\kappa^{\mathrm{alg}}$, $X \in \mathfrak{g}$. Suppose that $\mathfrak{W}_\kappa^{\mathrm{alg}}$ has an increasing filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \subset \mathfrak{W}_\kappa^{\mathrm{alg}}$$

of $S$-modules that is **exhausting**, *i.e.* $\mathfrak{W}_\kappa^{\mathrm{alg}} = \varinjlim_{i \in \mathbb{N}} F_i$ and that there is a system of projections $\{\mathfrak{W}_\kappa^{\mathrm{alg}} \to F_s\}_{s \in \mathbb{N}}$. Suppose that each $F_i$ is a free $S$-module of finite length. Suppose that the filtration is preserved by $\mathfrak{b}$: for any $X \in \mathfrak{b}$, then $X.F_i \subset F_i$. Assume that $\mathfrak{n}^-$ increases the filtration by one: for any $Y \in \mathfrak{n}^-$ we have $Y.F_i \subset F_{i+1}$. Moreover suppose that for each $i \in \mathbb{N}$ the diagram

$$\begin{tikzcd} Y : \mathfrak{W}_\kappa^{alg} \arrow[r] \arrow[d] & \mathfrak{W}_\kappa^{alg} \arrow[d] \\ Y : F_i \arrow[r] & F_{i+1} \end{tikzcd}$$

induced by the projection commutes. This hypothesis forces $\mathfrak{n}^-$ to increase the filtration: via the projection morphisms $F_{i+1} \cong F_i \oplus F_{i+1}/F_i$; and this hypothesis tell us that $F_{i+1}/F_i \subset F_{i+1}$ is sent into $F_{i+2}/F_{i+1} \subset F_{i+2}$, *i.e.* that "it is no more in $F_{i+1}$".

Let $\kappa : \mathfrak{h} \to S$ be a $S$-linear morphism and suppose that $\mathfrak{W}_\kappa^{\mathrm{alg}}$ has a highest weight vector $v^+ \in F_1 \subset \mathfrak{W}_\kappa^{\mathrm{alg}}$ of weight $\kappa$ (see Definition B.3.4). Moreover suppose that each $F_i$ has a weight decomposition:

$$F_i = \bigoplus_{\lambda : \mathfrak{h} \to S} F_{i,\lambda}.$$

Let $M^s := \mathrm{Hom}_S(F_s, S)$ for any $s \in \mathbb{N}$, observe that

$$M := \mathrm{Hom}_S\left(\mathfrak{W}_\kappa^{\mathrm{alg}}, S\right) = \mathrm{Hom}_S\left(\underset{\longrightarrow\ s\in\mathbb{N}}{\mathrm{colim}} F_s, S\right) = \lim_{s\in\mathbb{N}} M^s.$$

The $S$-module $M$ has a $\mathfrak{g}$-action via $(X.f)(v) := -f(X.v)$ for any $v \in \mathfrak{W}_\kappa^{\mathrm{alg}}$, $f \in M$, $X \in \mathfrak{g}$. In this way $M^s$ is a projective system of $\mathfrak{b}$-modules. For any $Y \in \mathfrak{n}^-$ and $f \in M^s$ one can define the function $Y.f \in M^{s-1}$ via

$$(Y.f)(v_{s-1}) := -f(Y.v_{s-1}) \quad \text{for any} \quad v_{s-1} \in F_{s-1}.$$

The definition of $X.f$ is compatible w.r.t. the projective system.

Since we suppose that $\mathfrak{u}^-$ is abelian we get the exhausting increasing filtration

$$\mathcal{U}(\mathfrak{n}^-)^n := \mathrm{Sym}_S^{\leqslant n} \mathfrak{n}^- \subset \mathrm{Sym}_S^{\bullet} \mathfrak{n}^- = \mathcal{U}(\mathfrak{n}^-).$$

Let $M^{-1} := \{0\}$, for any $j, n, s \in \mathbb{N}$ we consider the map

$$\begin{aligned} d^{n,s,j} : \mathcal{U}(\mathfrak{n}^-)^n \otimes_S M^s \otimes_S \textstyle\bigwedge_S^j \mathfrak{n}^- &\longrightarrow \mathcal{U}(\mathfrak{n}^-)^{n+1} \otimes_S M^{s-1} \otimes_S \textstyle\bigwedge_S^{j-1} \mathfrak{n}^- \\ a \otimes b \otimes (Y_1 \wedge \cdots \wedge Y_j) &\longmapsto \textstyle\sum_{i=1}^j (-1)^i (Y_i.a \otimes b - a \otimes Y_i.b) \otimes (Y_1 \wedge \cdots \wedge Y_{i-1} \wedge Y_{i+1} \wedge \cdots \wedge Y_j) \end{aligned}.$$

Hence for any $s, n \in \mathbb{N}$ we get the complex $\mathcal{U}(\mathfrak{n}^-)^{n-\bullet} \otimes_S M^{s+\bullet} \otimes_S \bigwedge_S^\bullet \mathfrak{n}^-$:

$$0 \to \mathcal{U}(\mathfrak{n}^-)^0 \otimes_S M^n \otimes_S \bigwedge_S^n \mathfrak{n}^- \longrightarrow \cdots \to \mathcal{U}(\mathfrak{n}^-)^{n-1} \otimes_S M^{s+1} \otimes_S \mathfrak{n}^- \to \mathcal{U}(\mathfrak{n}^-)^n \otimes_S M^s$$

In order to get a complex that does not depend on $n, s$ one can do the colimit on $n$ and the limit on $s$; there are two possible ways. The first way is the following:

$$\begin{aligned} \mathcal{D}^\bullet := \underset{\longrightarrow\ n}{\mathrm{colim}} \lim_s \mathcal{U}(\mathfrak{n}^-)^{n-\bullet} \otimes_S M^{s+\bullet} \otimes_S \bigwedge_S^\bullet \mathfrak{n}^- &\cong \underset{\longrightarrow\ n}{\mathrm{colim}}\, \mathcal{U}(\mathfrak{n}^-)^{n-\bullet} \otimes_S M \otimes_S \bigwedge_S^\bullet \mathfrak{n}^- \\ &\cong \mathcal{U}(\mathfrak{n}^-) \otimes_S M \otimes_S \bigwedge_S^\bullet \mathfrak{n}^- \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} M \otimes_S \bigwedge_S^\bullet (\mathfrak{g}/\mathfrak{b}). \end{aligned}$$

Via the Lepowsky–Garland Theorem the complex

$$0 \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} M \otimes_S \bigwedge_S^\bullet (\mathfrak{g}/\mathfrak{b}) \to M \to 0$$

is exact, where the last map $\mathcal{U}(\mathfrak{g}) \otimes_S M \to M$ is defined sending $X \otimes v$ into $X.v$. Hence we get that

$$H^i(\mathcal{D}^\bullet) = \begin{cases} M & \text{if } 0 = i \\ 0 & \text{if } 0 < i \end{cases}.$$

The other way to assemble the complex is doing the limit in the other order:

$$\mathcal{C}^\bullet := \lim_s \underset{\longrightarrow\ n}{\mathrm{colim}}\, \mathcal{U}(\mathfrak{n}^-)^{n-\bullet} \otimes_S M^{s+\bullet} \otimes_S \bigwedge_S^\bullet \mathfrak{n}^- \cong \lim_s \mathcal{U}(\mathfrak{n}^-) \otimes_S M^{s+\bullet} \otimes_S \bigwedge_S^\bullet \mathfrak{n}^-.$$

Now we cannot exchange the projective limit with the tensor product since $\mathcal{U}(\mathfrak{n}^-)$ is not finite over $S$, but there is a natural map between the two limits: $\xi^\bullet : \mathcal{D}^\bullet \to \mathcal{C}^\bullet$.

Observe that if $F_s = \bigoplus_\lambda F_{s,\lambda}$, where each $\lambda$ is a weight $\lambda : \mathfrak{h} \to S$; then $M^s = \bigoplus_\lambda M^s_{-\lambda}$. Hence $v_s^- :=$

$(v^+)^\vee_{|F_s} \in M^s$ is a lowest weight vector of $M^s$ of weight $\kappa^{-1}$. The action of the center $\mathcal{Z}(\mathfrak{g})$ on $F_s$ is given by the character $\chi_\kappa : \mathcal{Z}(\mathfrak{g}) \to S$ (see §B.3.3), hence the action of $\mathcal{Z}(\mathfrak{g})$ on $M^s$ is given by the character $-\chi_\kappa$.

We define $\mathcal{F}^\bullet$ as the sub-complex of $\mathcal{C}^\bullet$ with generalized eigenspace $\chi_\kappa$:

$$\mathcal{F}^i := \mathcal{C}^i_{\chi_\kappa} := \{v \in \mathcal{C}^i \mid \text{for all } \tau \in \mathcal{Z}(\mathfrak{g}) \text{ there is } n \in \mathbb{N} \ s.t. \ (\tau - \chi_\kappa)^n .v = 0\}.$$

Let $\mathcal{G}^\bullet := \mathcal{C}^\bullet / \mathcal{F}^\bullet$. Then we get a commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{D}^\bullet \arrow[r, equal] \arrow[d, "\xi^\bullet"] & \mathcal{D}^\bullet \arrow[r] \arrow[d, "\gamma^\bullet"] & 0 \\ 0 \arrow[r] & \mathcal{F}^\bullet \arrow[r] & \mathcal{C}^\bullet \arrow[r] & \mathcal{G}^\bullet \arrow[r] & 0 \end{tikzcd},$$

where $\gamma^\bullet$ is the composition of $\xi^\bullet$ and the projection on $\mathcal{G}^\bullet$. We want to prove the following Theorem:

**Theorem 2.1.1.** *$H^i(\gamma^\bullet) = 0$ for each $i \in \mathbb{N}$.*

Thanks to this Theorem and the delinearization we will "geometrize" the BGG computations. Indeed, after a "continue" dualization $-^{\vee,cont}$ and a delinearization functor $u_*$, we will get an equality $u_*\mathcal{D}^{\bullet,\vee,cont} = u_*\mathcal{C}^{\bullet,\vee,cont}$. This Theorem will allow us to say that $u_*\mathcal{C}^{\bullet,\vee,cont}$ gives no contribute to the cohomology of $u_*\mathcal{D}^{\bullet,\vee,cont}$, hence that the map $u_*\mathcal{F}^{\bullet,\vee,cont} \to u_*\mathcal{C}^{\bullet,\vee,cont}$ is a quasi isomorphism.

### 2.1.1 Proof of the Theorem 2.1.1

Before proving the Theorem we need some technical results on the modules

$$A_{s,m} := \mathcal{U}(\mathfrak{n}^-) \otimes_S M^s \otimes_S \bigwedge_S^m \mathfrak{n}^- = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \left( M^s \otimes_S \bigwedge_S^m \mathfrak{g}/\mathfrak{b} \right).$$

**Lemma 2.1.2.** *Let $D$ be a finite $\mathcal{U}(\mathfrak{b})$-module free over $S$ with basis $d_1, \dots, d_s$ of weights $\mu_1, \dots, \mu_s : \mathfrak{h} \to S$. Suppose that for any $1 \leqslant i, j \leqslant s$ if $\mu_j < \mu_i$, then $i < j$.*

*Let $E := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} D$ and $E_i \subset E$ be the $\mathcal{U}(\mathfrak{g})$-submodule generated by $1 \otimes d_1, \dots, 1 \otimes d_i$.*

*Then $E_i/E_{i-1} \cong M(\mu_i)$ is the Verma module of weight $\mu_i$. In particular*

$$E = \bigoplus_{\chi \in \{\chi_{\mu_i}\}_{i=1}^n} E_\chi$$

*Proof.* Let $D_i := \bigoplus_{l=1}^i S d_l$. Observe that $\mathcal{U}(\mathfrak{b}) d_l \subset D_l$, since the action of $\mathcal{U}(\mathfrak{b})$ increases the weight, hence it decreases the index of the basis by hypothesis. Hence

$$0 = D_0 \subset D_1 \subset \cdots \subset D_s = D$$

is a filtration of $\mathcal{U}(\mathfrak{b})$-modules with the property that $\mathcal{U}(\mathfrak{n}) D_l \subset D_{l-1}$. Since $D_i$ are $\mathcal{U}(\mathfrak{b})$-modules

$$E_i = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} D_i \cong \mathcal{U}(\mathfrak{n}^-) \otimes_S D_i$$

is a free $\mathcal{U}(\mathfrak{n}^-)$-module with basis $\{1 \otimes d_l\}_{l=1}^i$. Hence

$$E_i / E_{i-1} \cong \mathcal{U}(\mathfrak{n}^-)(1 \otimes d_i)$$

is the Verma module of weight $\mu_i$ and the filtration computes the Jordan–Holder decomposition of the $\mathcal{U}(\mathfrak{g})$-module $E$. □

**Lemma 2.1.3.** *Let $\chi : \mathcal{Z}(\mathfrak{g}) \to S$ be a character. For any $m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ s.t. for any $t, s \in \mathbb{N}$ with $N_m \leqslant s \leqslant t$ the projection*

$$(A_{t,m})_\chi \to (A_{s,m})_\chi$$

*is an isomorphism. Let $(A_m)_\chi := (A_{N,m})_\chi$, then*

$$\mathcal{F}^m = (A_m)_{\chi_\kappa}, \quad \mathcal{G}^m = \lim_{\Delta \underset{finite}{\subset} \Theta} \bigoplus_{\chi \in \Delta} (A_m)_\chi,$$

*where $\Theta$ is the set of all the characters of $\mathcal{Z}(\mathfrak{g})$ with $\chi \neq \chi_\kappa$ and $\Delta$ varies over all the finite subsets of $\Theta$. In particular $\mathcal{C}^\bullet = \mathcal{F}^\bullet \oplus \mathcal{G}^\bullet$.*

*Proof.* $\mathfrak{W}_\kappa^{\mathrm{alg}}$ is a highest weight module; then for any weight $\lambda$ the module $(F_s)_\lambda$ stabilizes in $s \in \mathbb{N}$, *i.e.* there is an $N_\lambda \in \mathbb{N}$ s.t. $(F_s)_\lambda = (F_{N_\lambda})_\lambda$ for any $N_\lambda \leqslant s \in \mathbb{N}$. Hence for any weight $\lambda$ the module $\big(M^s \otimes_{\mathcal{U}(\mathfrak{b})} \bigwedge^m(\mathfrak{g}/\mathfrak{b})\big)_\lambda$ stabilizes in $s$.

Via the Lemma 2.1.2 one gets that the characters that appear in the decomposition of $A_{s,m} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (M^s \otimes_S \bigwedge_S^m \mathfrak{n}^-)$ are the characters associated to the weights of $M^s \otimes_{\mathcal{U}(\mathfrak{b})} \bigwedge_m(\mathfrak{g}/\mathfrak{b})$.

Fixing a character $\chi$ as in the statement of the Lemma, there is a finite number of weights associated to this character $\chi$. Hence the module $(A_{s,m})_\chi$ stabilizes in $s$.

So the module $(A_m)_\chi = \lim_{s \in \mathbb{N}} A_{s,m}$ is well defined and it stabilizes in $s$. We get

$$\mathcal{C}^m = \lim_{s \in \mathbb{N}} A_{s,m} = \lim_{s \in \mathbb{N}} \bigoplus_{\chi \in \Theta \cup \{\chi_\kappa\}} (A_{s,m})_\chi = \lim_{s \in \mathbb{N}} \bigoplus_{\chi \in \Theta_s \cup \{\chi_\kappa\}} (A_m)_\chi = \lim_{\Delta \underset{finite}{\subset} \Theta \cup \{\chi_\kappa\}} \bigoplus_{\chi \in \Delta} (A_m)_\chi,$$

where $\Theta_s = \{\chi \in \Theta \mid (A_{s,m})_\chi = (A_m)_\chi\}$. □

For any character $\chi : \mathcal{Z}(\mathfrak{g}) \to S$ let $I_\chi := \mathrm{Ker}(\chi : \mathcal{Z}(\mathfrak{g}) \to S)$; it is a prime ideal since $S$ is a domain.

**Lemma 2.1.4.** *Let $L := S(Y_1, \ldots, Y_n)$, where $Y_1, \ldots, Y_n$ are transcendental independent variables over $S$ and $n = \dim_K \mathfrak{h}'$. Then there is an element $\tau \in I_{\chi_\kappa} \mathcal{Z}(\mathfrak{g} \otimes_S L)$ s.t. $\tau$ acts invertibly on $\mathcal{G}^i \otimes_S L \otimes_S frac(S)$ for each $0 \leqslant i$.*

*Proof.* Via the Harish–¡Chandra isomorphism $\mathcal{Z}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^W \hookrightarrow \mathrm{Sym}(\mathfrak{h})$, where $W$ is the Weyl group of $\mathfrak{g}$. Then there is a finite map

$$\pi : \mathbb{A}_S^n = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{h})) \to \mathrm{Spec}(\mathcal{Z}(\mathfrak{g})) = \mathbb{A}_S^n/W,$$

Let $\Theta' := \bigcup_{\chi \in \Theta} \pi^{-1}(I_\chi) \cap \mathbb{A}_S^n(S)$ and $\{\mathfrak{P}_i\}_{i=1}^u := \pi^{-1}(I_\chi) \cap \mathbb{A}_S^n(S)$. Observe that there is a section of $\pi$ given by $f \mapsto \frac{1}{\#W} \sum_{\sigma \in W} f^\sigma$; hence $\pi^{-1}(I_\chi) \cap \mathbb{A}_S^n(S) \neq \varnothing$ for each $\chi \in \Theta \cup \{\chi_\kappa\} \subset \mathrm{Spec}(\mathcal{Z}(\mathfrak{g}))(S)$. The proof proceeds by steps.

1. There is $f \in \mathfrak{P}_1 \mathrm{Sym}_L(\mathfrak{h} \otimes_S L) \setminus \bigcup_{\mathfrak{P} \in \Theta'} \mathfrak{P} \mathrm{Sym}_L(\mathfrak{h} \otimes_S L)$.

   Since $\mathfrak{P}_1$ is defined by a section of $S$, if $H_1, \ldots, H_n \in \mathfrak{h}$ is a basis, then

$$\mathfrak{P}_1 \mathrm{Sym}_S(\mathfrak{h}) \cong (X_1, \ldots, X_n) \subset S[X_1, \ldots, X_n] \cong S[Z_1, \ldots, Z_n] \cong \mathrm{Sym}_S(\mathfrak{h}),$$

   where $X_i = Z_i - \chi_\kappa(H_i)$. Let $f := X_1 Y_1 + \cdots + X_n Y_n \in \mathrm{Sym}_S(\mathfrak{h})$. For any $\mathfrak{P} \in \Theta'$ one has that

$$\mathfrak{P} \mathrm{Sym}_S(\mathfrak{h}) \cong (X_1 - \alpha_1, \ldots, X_n - \alpha_n), \quad \alpha_i = (\chi - \chi_\kappa)(H_i).$$

Since $\chi \neq \chi_\kappa$ one gets that $(\alpha_1, \dots, \alpha_n) \in S^n \smallsetminus \{\underline{0}\}$ and

$$f \mod \mathfrak{P}\,\mathrm{Sym}_L(\mathfrak{h} \otimes_S L) = \sum_{i=1}^{n} \alpha_i Y_i \in L = S(Y_1, \dots, Y_n)$$

is not zero since $Y_i$ are independent, hence we finish the first step.

2. There is a $\tau \in I_{\chi_\kappa}\mathcal{Z}(\mathfrak{g} \otimes_S L) \smallsetminus \bigcup_{\chi \in \Theta'} I_\chi \mathcal{Z}(\mathfrak{g} \otimes_S L)$.

   Let $f$ as in the previous step, fix

$$\tau := \prod_{\sigma \in W} f^\sigma \in \mathrm{Sym}_L(\mathfrak{h} \otimes_S L)^W = \mathcal{Z}(\mathfrak{g} \otimes_S L).$$

   If by contradiction $f^\sigma \in \mathfrak{P}\,\mathrm{Sym}_L(\mathfrak{h} \otimes_S L)$ for some $\mathfrak{P} \in \Theta'$, then $f \in \sigma^{-1}(\mathfrak{P})\,\mathrm{Sym}_L(\mathfrak{h} \otimes_S L)$, that is false by the property of $f$, indeed $\pi(\mathfrak{P}) = \pi(\sigma^{-1}(\mathfrak{P}))$. Since $\mathfrak{P}$ is prime we get that $\tau \notin \mathfrak{P}\,\mathrm{Sym}_L(\mathfrak{h} \otimes_S L)$ for each $\mathfrak{P} \in \Theta'$.

   Let $\chi \in \Theta$, let $\mathfrak{P} \in \Theta' \cap \pi^{-1}(I_\chi)$, then $\tau$ could not vanish in $I_\chi \mathcal{Z}(\mathfrak{g} \otimes_S L) = \pi_L(\mathfrak{P}\,\mathrm{Sym}_L(\mathfrak{h} \otimes_S L))$, hence we prove the second step.

3. The action of $\tau$ on $\mathcal{G}^i \otimes_S L$ is invertible for each $0 \leqslant i$.

   Via the Lemma 2.1.3 we know that $\mathcal{G}^i = \lim_{\Delta \underset{finite}{\subset} \Theta} \bigoplus_{\chi \in \Delta} (A_i)_\chi$. For each $\chi \in \Theta$ there is $s \in \mathbb{N}$ s.t. $I_\chi^s (A_i)_\chi = 0$. But $\tau = (\tau - \chi(\tau)) + \chi(\tau)$, where $\chi(\tau) \in L \smallsetminus \{0\}$ by the previous step. Hence $\tau$ acts on $(A_i)_\chi \otimes_S L$ as a sum of a nilpotent and a non-zero element, since $L$ is a domain and $L \otimes_S frac(S)$ is its fraction field, we get that the action of $\tau$ is invertible on $(A_i)_\chi \otimes_S L \otimes_S frac(S)$; by the decomposition of $\mathcal{G}^i$ the Lemma is proved.

□

*Proof.* of the Theorem. Observe that $H^i(\mathcal{D}^\bullet) = 0$ for $0 < i$ and $H^0(\mathcal{D}^\bullet) = M$, then for each $x \in H^i(\mathcal{D}^\bullet)$ there is $n \in \mathbb{N}$ with $I_{\chi_\kappa}^n .x = 0$. Taking $L$ and $\tau \in I_{\chi_\kappa}\mathcal{Z}(\mathfrak{g} \otimes_S L)$ as in the Lemma 2.1.4

$$0 = H^i(\gamma_L)(\tau^n.(x \otimes 1)) = \tau^n.\big(H^i(\gamma)(x) \otimes 1\big)$$

then $\big(H^i(\gamma)(x) \otimes 1\big) = 0 \in H^i(\mathcal{G}^\bullet) \otimes_S frac(S)(Y_1, \dots, Y_n)$, hence

$$\big(H^i(\gamma)(x) \otimes 1\big) = 0 \in H^i(\mathcal{G}^\bullet) \otimes_S frac(S).$$

Via the explicit description of $\mathcal{G}$ as a projective limit of free $S$-modules we get that also $H^i(\gamma)(x) = 0$. □

### 2.1.2 The dual Theorem

Let's consider a morphism between bounded complexes of $S$-modules $\varphi_\bullet : A_\bullet \to B_\bullet$. Suppose that the morphism and the objects are defined over $K$ (see Definition 2.1.6 ); in this Subsection we will prove that if $H^i(\varphi_\bullet) = 0$ for any $i \in \mathbb{Z}$, then also the cohomology of the dual map $\varphi_\bullet^\vee : B_\bullet^\vee \to A_\bullet^\vee$ vanishes, *i.e.* $H^i(\varphi_\bullet^\vee) = 0$ for each $i \in \mathbb{Z}$.

The main consequence of the results in this Subsection is the following Proposition.

**Proposition 2.1.5.** *With the notation as in the previous Subsection,* $H^i(\gamma^{\vee,\bullet}) = 0$ *for each* $i \in \mathbb{N}$.

For technical reasons we need the following definitions.

**Definition 2.1.6.** A morphism between complexes of $S$-modules $\varphi_\bullet : A_\bullet \to B_\bullet$ is **defined over** $K$ if there are complexes of $K$-modules $A'_\bullet$, $B'_\bullet$ with $A_\bullet = A_\bullet \otimes_K S$, $B_\bullet = B'_\bullet \otimes_K S$ and $\varphi_\bullet = \varphi'_\bullet \otimes_K S$ for a $K$-linear morphism $\varphi'_\bullet : A'_\bullet \to B'_\bullet$.

A complex of $S$-modules $(A_\bullet, \mathrm{d}^A_\bullet)$ is **splitting** if for any $i \in \mathbb{Z}$ the morphism

$$\overline{d^A_i} : A_i / \operatorname{Ker}(d^A_i) \to A_{i+1},$$

induced by $d^A_i$, splits over $S$ *i.e.* if there is a projection $p_{i+1} : A_{i+1} \to A_i / \operatorname{Ker}(d^A_i)$ s.t. $p_{i+1} \circ \overline{d^A_i} = \mathrm{id}_{A_i / \operatorname{Ker}(d^A_i)}$.

In order to prove the vanishing of the dual cohomology we introduce a pairing.

**Lemma 2.1.7.** *Let $A_0 \overset{\alpha}{\to} A_1 \overset{\beta}{\to} A_2$ be a complex of $S$-modules. Let $A_2^\vee \overset{\beta^\vee}{\to} A_1^\vee \overset{\alpha^\vee}{\to} A_0^\vee$ be the dual complex, then there is an $S$-linear pairing*

$$< \, , \, >: H(A) \times H(A^\vee) \longrightarrow S$$

*where $H(A) := \frac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\alpha)}$ and $H(A^\vee) := \frac{\operatorname{Ker}(\alpha^\vee)}{\operatorname{Im}(\beta^\vee)}$. If*

$$\begin{array}{ccccc} A_0 & \xrightarrow{\alpha} & A_1 & \xrightarrow{\beta} & A_2 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ B_0 & \xrightarrow{\epsilon} & B_1 & \xrightarrow{\delta} & B_2 \end{array}$$

*is a commutative diagram of $S$-modules, where $\delta \circ \epsilon = 0$; then the maps*

$$H(\varphi) : H(A) \to H(B), \quad H(\varphi^\vee) : H(B^\vee) \to H(A^\vee)$$

*are adjoint w.r.t. the two pairings, i.e.*

$$< H(\varphi)(a), b >_B = < a, H(\varphi^\vee(b)) >_A,$$

*for each $a \in H(A)$ and $b \in H(B^\vee)$.*

*Suppose that the sequence $A_0 \overset{\alpha}{\to} A_1 \overset{\beta}{\to} A_2$ is splitting, we get the following property:*

$$\textit{fix } b \in H(A^\vee), \textit{ if } < a, b >= 0 \textit{ for each } a \in H(A), \textit{ then } b = 0.$$

*Proof.* For any $a \in H(A)$, $b \in H(A^\vee)$ we define $< a, b >=: \phi(x)$ where $x \in A_1$ and $\phi \in A_1^\vee$ are representative elements of $a$ and $b$. Observe that the pairing does not depend on the representative elements: if $\alpha(z) = y \in \operatorname{Im}(\alpha)$ and $\beta^\vee(\eta) = \psi \in \operatorname{Im}(\beta^\vee)$, then

$$(\phi + \psi)(x + y) = \phi(x) + \phi(\alpha(z)) + \eta(\beta(x + y)) = \phi(x)$$

since $x, y \in \operatorname{Ker}(\beta)$ and $\phi \in \operatorname{Ker}(\alpha^\vee)$.

Let $a \in H(A)$, $b \in H(B^\vee)$ and choose some representatives $x \in A_1$ and $\phi \in B_1^\vee$, then

$$< H(\varphi)(a), b >_B = \phi(\varphi_1(x)) = (\varphi_1^\vee(\phi))(x) = < a, H(\varphi^\vee(b)) >_A .$$

Let $a^\vee \in H(A^\vee)$ and $\phi \in A_1^\vee$ be an element representing $a^\vee$. If $< a, a^\vee >= 0$ for any $a \in H(A)$, then $\phi(x) = 0$ for any $x \in \operatorname{Ker}(\beta)$, then $\operatorname{Ker}(\beta) \subset \operatorname{Ker}(\phi)$ and we can define a morphism $\delta' : A_1 / \operatorname{Ker}(\beta) \to S$ s.t.

the following diagram commutes

$$\begin{tikzcd} A_1 \arrow[r, "\phi"] \arrow[d, "\beta"] & S \\ A_1/\operatorname{Ker}(\beta) \arrow[ur, dashed, "\delta'"] & \end{tikzcd}.$$

The fact that the sequence $A_\bullet$ is splitting tell us that there is a decomposition $A_2 \cong A_3 \oplus \operatorname{Im}(A_1) \cong A_3 \oplus A_1/\operatorname{Ker}(\beta)$. Then we can extend $\delta'$ to a $S$-linear function $\delta : A_2 \to S$ and we get that $\phi = \beta^\vee(\delta) \in \operatorname{Im}(\beta^\vee)$ and $a^\vee = 0$. □

We get as corollary the result we claimed at the beginning of this Subsection.

**Corollary 2.1.8.** *Let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism between bounded complexes of $S$-modules with $H^i(f_\bullet) = 0$ for each $i \in \mathbb{Z}$. Suppose that $A_\bullet$ is splitting. Then $H^i(f_\bullet^\vee) = 0$ for each $i \in \mathbb{Z}$.*

*Proof.* We can suppose that the two complexes are concentrated on $0 \leqslant i \leqslant n$, *i.e.* that $A_i = B_i = 0$ for each $i < 0$ and for each $n < i$. By the Lemma 2.1.7 we get two pairings for each $i \in \mathbb{N}$ with $0 \leqslant i \leqslant n$:

$$< \,,\, >_A : H^i(A_\bullet) \times H^{n-i}(A_\bullet^\vee) \longrightarrow S, \quad < \,,\, >_B : H^i(B_\bullet) \times H^{n-i}(B_\bullet^\vee) \longrightarrow S.$$

One can conclude via the property of this pairing and via the adjunction property. Indeed for any $b \in H^{n-i}(B_\bullet^\vee)$ and $a \in H^i(A_\bullet)$, then

$$< a, H^{n-i}(f_\bullet^\vee)(b) >_A = < H^i(f)(a), b >_B = < 0, b >_B = 0$$

hence $H^{n-i}(f_\bullet^\vee)(b) = 0$. □

The proof of the Proposition 2.1.5 is an immediate application of the Corollary 2.1.8.

### 2.1.3 The continuous dual Theorem

For any $s, n \in \mathbb{N}$ we define the complexes

$$B_{n,s}^\bullet := \mathcal{U}(\mathfrak{g})^{\leqslant n-\bullet} \otimes_{\mathcal{U}(\mathfrak{b})} M^{s+\bullet} \otimes_S \bigwedge_S^\bullet(\mathfrak{g}/\mathfrak{b}).$$

By definition and by the Lemma 2.1.3 we have that

$$\mathcal{D}^\bullet = \varinjlim_{n\in\mathbb{N}} \lim_{s\in\mathbb{N}} B_{n,s}^\bullet \quad \text{and} \quad \mathcal{G}^\bullet = \lim_{s\in\mathbb{N}} \bigoplus_{\chi\in\Theta} \Big( \varinjlim_{n\in\mathbb{N}} B_{n,s}^\bullet \Big)_\chi .$$

In the previous Subsection we proved that the map $\mathcal{G}^{\bullet,\vee} \to \mathcal{D}^{\bullet,\vee}$ induces the zero map between the cohomology groups. For geometric reasons that we will explain in §2.3.3 we are interested in the map between the continuous dual with respect to the limit topology, *i.e.* on the map

$$\gamma^{\bullet,\vee,cont} : \varinjlim_{s\in\mathbb{N}} \left( \bigoplus_{\chi\in\Theta} \Big( \varinjlim_{n\in\mathbb{N}} B_{n,s}^\bullet \Big)_\chi \right)^\vee \longrightarrow \lim_{n\in\mathbb{N}} \varinjlim_{s\in\mathbb{N}} \big(B_{n,s}^\vee\big).$$

Observe that there is a diagram of complexes

$$\begin{tikzcd}
\varinjlim_{s\in\mathbb{N}} \left(\bigoplus_{\chi\in\Theta}\left(\varinjlim_{n\in\mathbb{N}} B^{\bullet}_{n,s}\right)_{\chi}\right)^{\vee} \arrow[r, "\gamma^{\bullet,\vee,cont}"] \arrow[d] & \lim_{n\in\mathbb{N}} \varinjlim_{s\in\mathbb{N}} \left(B^{\bullet,\vee}_{n,s}\right) \arrow[d, "c^{\bullet}"] \\
\mathcal{G}^{\bullet,\vee} = \left(\lim_{s\in\mathbb{N}} \bigoplus_{\chi\in\Theta}\left(\varinjlim_{n\in\mathbb{N}} B^{\bullet}_{n,s}\right)_{\chi}\right)^{\vee} \arrow[r, "\gamma^{\bullet,\vee}"] & \mathcal{D}^{\bullet,\vee} = \lim_{n\in\mathbb{N}}\left(\varinjlim_{s\in\mathbb{N}} B^{\bullet}_{n,s}\right)^{\vee}
\end{tikzcd} \tag{2.1}$$

In this Subsection we will prove that the maps induced on the cohomology groups by $\gamma^{\bullet,\vee,cont}$ vanish. We need only to prove that the canonical map $c^{\bullet}$ induces injective morphisms between the cohomology groups; indeed the vanishing of $H^k\left(\gamma^{\bullet,\vee,cont}\right)$ will follow by the diagram above and the vanishing proved in the Proposition 2.1.5 of the cohomology of $\gamma^{\bullet,\vee}$.

**Lemma 2.1.9.** *Let $\{A^{\bullet}_s,\ d^{\bullet}_s\}_{s\in\mathbb{N}}$ be a projective system of complexes of $S$-modules, i.e. $A^{\bullet}_s$ is a complex of $S$-module with morphisms $d^{\bullet}_s$ and for any $s\in\mathbb{N}$ there is a map of complexes $f^{\bullet}_{s+1}: A^{\bullet}_{s+1}\to A^{\bullet}_s$.*

*Suppose that for any $k\in\mathbb{Z}$, $s\in\mathbb{N}$ the map $f^k_{s+1}: A^k_{s+1}\twoheadrightarrow A^k_s$ is surjective and that there is a section $g^k_s: A^k_s\hookrightarrow \lim_{s\in\mathbb{N}} A^k_s$ of the projection map $\lim_{s\in\mathbb{N}} A^k_s\to A^k_s$. Suppose that the sections $g^k_s$ are compatible with the morphisms $d^k_s$, i.e. that the diagrams*

$$\begin{tikzcd}
A^k_s \arrow[r, "d^k_s"] \arrow[d, hook, "g^k_s"] & A^{k+1}_s \arrow[d, hook, "g^{k+1}_s"] \\
\lim_{s\in\mathbb{N}} A^k_s \arrow[r, "\lim_{s\in\mathbb{N}} d^k_s"] & \lim_{s\in\mathbb{N}} A^{k+1}_s
\end{tikzcd}$$

*are commutative. Then the map of complexes $\varinjlim_{s\in\mathbb{N}} (A^{\bullet}_s)^{\vee}\to(\lim_{s\in\mathbb{N}} A_s)^{\vee}$ induces a morphism in cohomology*

$$H^k\left(\varinjlim_{s\in\mathbb{N}} (A^{\bullet}_s)^{\vee}\right)\hookrightarrow H^k\left(\left(\lim_{s\in\mathbb{N}} A^{\bullet}_s\right)^{\vee}\right)$$

*that is injective for any $k\in\mathbb{Z}$.*

*Proof.* Let $k\in\mathbb{Z}$, $s\in\mathbb{N}$ and $\varphi\in\operatorname{Ker}\left(\varinjlim_{s\in\mathbb{N}} d^{k-1,\vee}_s\right)$. Then there is an $s_*\in\mathbb{N}$ such that $\varphi\in\operatorname{Ker}\left(d^{k-1,\vee}_{s_*}\right)$. Suppose that the image of $\varphi$ is zero in cohomology, *i.e.* that there is a $\delta\in\left(\lim_{s\in\mathbb{N}} A^{k+1}_s\right)^{\vee}$ such that the diagram

$$\begin{tikzcd}
\lim_{s\in\mathbb{N}} A^k_s \arrow[r, "\lim_s d^k_s"] \arrow[d, shift left, "p^k_{s_*}"] & \lim_{s\in\mathbb{N}} A^{k+1}_s \arrow[d, shift left, "p^{k+1}_{s_*}"] \arrow[dr, "\delta"] & \\
A^k_{s_*} \arrow[r, "d^k_{s_*}"] \arrow[u, hook, shift left, "g^k_{s_*}"] \arrow[rr, bend right, "\varphi"'] & A^{k+1}_{s_*} \arrow[u, hook, shift left, "g^{k+1}_{s_*}"] & S
\end{tikzcd}$$

commutes. Observe that

$$\varphi=\delta\circ\left(\lim_{s\in\mathbb{N}} d^k_s\right)\circ g^k_{s_*}=\left(\delta\circ g^{k+1}_{s_*}\right)\circ d^k_{s_*}=d^{k,\vee}_{s_*}\left(\delta\circ g^{k+1}_{s_*}\right).$$

Hence $\varphi$ is zero on cohomology and the Lemma is proved. ∎

**Corollary 2.1.10.** *For any $n\in\mathbb{N}$ the map*

$$c^{\bullet}_n:\ \varinjlim_{s\in\mathbb{N}} \left(B^{\bullet,\vee}_{n,s}\right)\to\left(\lim_{s\in\mathbb{N}} B^{\bullet}_{n,s}\right)^{\vee}$$

*induces an injective morphism on the cohomology groups.*

*Proof.* The filtration by hypothesis admits a system of projections $g_i : \mathfrak{W}_\kappa^{alg} \to F_i$. We remind that for any $Y \in \mathfrak{n}^-$ the diagram

$$\begin{tikzcd} Y : \mathfrak{W}_\kappa^{alg} \arrow[r] \arrow[d, "g_i"] & \mathfrak{W}_\kappa^{alg} \arrow[d, "g_{i+1}"] \\ Y : F_i \arrow[r] & F_{i+1} \end{tikzcd}$$

commutes by hypothesis. The projections induce sections $M^s \to M = \lim_{s\in\mathbb{N}} M^s$, hence sections

$$g_s^k : \mathcal{U}(\mathfrak{n}^-)^{\leqslant n-k} \otimes_S M^{s+k} \otimes_S \bigwedge_S^k \mathfrak{n}^- \to \lim_{s\in\mathbb{N}} \mathcal{U}(\mathfrak{n}^-)^{\leqslant n-k} \otimes_S M^{s+k} \otimes_S \bigwedge_S^k \mathfrak{n}^-.$$

Since the Koszul complex $d_{n,s}^{B,k} : B_{n,s}^k \to B_{n,s}^{k+1}$ is defined in terms of the $\mathfrak{n}^-$-action and the projections commute with this action; we get that the sections are compatible with the morphisms $d_{n,s}^{B,k}$ and we can apply the Lemma 2.1.9.

□

**Lemma 2.1.11.** *Let $\{A_n^\bullet,\ d_n^\bullet\}_{n\in\mathbb{N}}$ be a projective system of complexes of $S$-modules. Suppose that the complex $\lim_{n\in\mathbb{N}} d_n^\bullet$ splits, i.e. for each $k \in \mathbb{Z}$ the short exact sequence*

$$0 \to \operatorname{Ker}\left(\lim_{n\in\mathbb{N}} d_n^k\right) \to \lim_{n\in\mathbb{N}} A_n^k \to \frac{\lim_{n\in\mathbb{N}} A_n^k}{\operatorname{Ker}\left(\lim_{n\in\mathbb{N}} d_n^k\right)} \to 0$$

*splits, i.e. there is a projection $h^k : \lim_{n\in\mathbb{N}} A_n^k \to \operatorname{Ker}\left(\lim_{n\in\mathbb{N}} d_n^k\right)$. Suppose that for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ there is a section $g_n^k : A_n^k \to \lim_{n\in\mathbb{N}} A_n^k$ of the canonical projection. Then*

$$H^k\left(\lim_n A_n^\bullet\right) \hookrightarrow \lim_{n\in\mathbb{N}} H^k(A_n^\bullet)$$

*is an injective morphism.*

*Proof.* Let $x = (x_n)_{n\in\mathbb{N}} \in \operatorname{Ker}\left(\lim_n (d_n^k)\right)$ with $x_n \in \operatorname{Ker}\left(d_n^k\right) \subset A_n^k$ the $n$-th projection of $x$. Suppose that $[x_n]_{n\in\mathbb{N}} \in \lim_{n\in\mathbb{N}} H^k(A_n^\bullet)$ vanishes, *i.e.* that for each $n \in \mathbb{N}$ there is a $z_n \in A_n^{k-1}$ such that $d_n^{k-1}(z_n) = x_n$. We want to show that also $[x] \in H^k(\lim_{n\in\mathbb{N}} A_n^\bullet)$ is zero. A priori the system $z_n$ is not compatible, but we know that the map $\lim_n d_n^{k-1}$ splits, then we can define

$$w_n := p_n^k\left(g_n^k(z_n) - h^k g_n^k(z_n)\right),$$

where $p_n^k : \lim_{n\in\mathbb{N}} A_n^k \to A_n^k$ is the canonical projection. Observe that for each $n \in \mathbb{N}$ we get that $d_n^k(w_n) = x_n$ and the system $w_n$ is a compatible system. Then the element $(w_n)_{n\in\mathbb{N}} \in \lim_{n\in\mathbb{N}} A_n^{k-1}$ is well defined and his image is $x$. □

**Corollary 2.1.12.** *The map*

$$\gamma^{\bullet,\vee,cont} : \underset{\substack{\longrightarrow \\ s\in\mathbb{N}}}{colim} \left(\bigoplus_{\chi\in\Theta}\left(\underset{\substack{\longrightarrow \\ n\in\mathbb{N}}}{colim} B_{n,s}^\bullet\right)_\chi\right)^\vee \longrightarrow \lim_{n\in\mathbb{N}} \underset{\substack{\longrightarrow \\ s\in\mathbb{N}}}{colim} \left(B_{n,s}^\vee\right)$$

*vanishes on the cohomology groups.*

*Proof.* We can apply the Lemma 2.1.11 to the projective system

$$\underset{\longrightarrow}{\mathrm{colim}}_{s\in\mathbb{N}} \left(B^{\bullet}_{n,s}\right)^{\vee} = \left(\mathcal{U}(\mathfrak{n}^{-})^{\leqslant n-\bullet}\right)^{\vee} \otimes_S \left(\mathfrak{W}^{alg}_{\kappa} \otimes_S \left(\bigwedge_S^{\bullet} \mathfrak{n}^{-}\right)^{\vee}\right).$$

Indeed the maps of the complex are defined over $K$, then they split and by the isomorphism $\mathcal{U}(\mathfrak{n}^{-}) \cong \mathrm{Sym}_S(\mathfrak{n}^{-})$ we get the sections $\left(\mathcal{U}(\mathfrak{n}^{-})^{\leqslant n-k}\right)^{\vee} \to (\mathcal{U}(\mathfrak{n}^{-}))^{\vee}$. Then

$$H^k\left(\lim_{n\in\mathbb{N}} \underset{\longrightarrow}{\mathrm{colim}}_{s\in\mathbb{N}} \left(B^{\bullet,\vee}_{n,s}\right)\right) \hookrightarrow \lim_{n\in\mathbb{N}} H^k\left(\underset{\longrightarrow}{\mathrm{colim}}_{s\in\mathbb{N}} \left(B^{\bullet,\vee}_{n,s}\right)\right).$$

Using that the limit commutes with kernels and the fact that $H^k(c^{\bullet}_n)$ is injective all $n,k\in\mathbb{N}$ by the Corollary 2.1.10, one gets that

$$\lim_{n\in\mathbb{N}} H^k\left(\underset{\longrightarrow}{\mathrm{colim}}_{s\in\mathbb{N}} \left(B^{\bullet,\vee}_{n,s}\right)\right) \hookrightarrow \lim_{n\in\mathbb{N}} \left(\underset{\longrightarrow}{\mathrm{colim}}_{s\in\mathbb{N}} B^{\bullet}_{n,s}\right)^{\vee} = \mathcal{D}^{\bullet,\vee}$$

is injective. We can conclude as it was explained at the beginning of this Subsection: we use the diagram (2.1), the vanishing of $H(\gamma^{\bullet,\vee})$ and the injectivity of $H(c^{\bullet})$ that we have just proven. □

## 2.2 A local description of $\mathbb{W}_{\kappa}$

### 2.2.1 The group $\mathcal{G}$

Let $G := \mathrm{GL}_2$, $T\subset G$ be the torus of of diagonal matrices, $B$ the Borel of upper triangular matrices in $G$ and $N$ be the unipotent group of upper triangular matrices with 1 on the diagonal.

Fix $n\in\mathbb{Z}_{>0}$, let $T_n$ be the group scheme over $\mathbb{Z}_p$ defined by

$$T_n(R) := \{\mathrm{diag}(t_1,t_2) \mid t_1 \equiv 1 \mod p^n\}.$$

We define $\mathcal{B}\subset\mathcal{G}$ the $p$-adic analytic groups over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ that represent the following functors:

$$(R,R^+) \mapsto \mathcal{B}(R,R^+) := T_n(R^+)N(R^+) = \begin{pmatrix} 1+p^nR^+ & R^+ \\ 0 & (R^+)^{\times} \end{pmatrix};$$
$$(R,R^+) \mapsto \mathcal{G}(R,R^+) := \left\{ g\in G(R^+) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod p^nR^+ \right\}.$$

### 2.2.2 The $\mathcal{G}$-representation $\mathbb{W}^S_{\kappa}$

Let $K/\mathbb{Q}_p$ be a finite extension, $(S,S^+)$ be a Huber pair over $(K,\mathcal{O}_K)$ with $S = K\otimes_{\mathcal{O}_K} S^+$, $\mathcal{F}^+ := V := S^+e_0 \oplus S^+e_1$ and $\mathcal{X} := \mathrm{Spa}(S,S^+)$. Let $\Lambda := \mathcal{O}_K[[Z_p^{\times}]]$ the Iwasawa algebra and with $\mathcal{W} := \mathrm{Spa}(K\otimes_{\mathcal{O}_K}\Lambda,\Lambda)$, let $\mathcal{X}\to\mathcal{W}$ be a map of adic spaces defined over $K$ s.t. the composition

$$\kappa : \mathbb{Z}_p^{\times} \xrightarrow{\kappa^{\mathrm{univ}}} \mathcal{O}^+_{\mathcal{W}}(\mathcal{W}) \to \mathcal{O}^+_{\mathcal{X}}(\mathcal{X}) = S^+$$

is analytic on $(1+\mathcal{I})\cap\mathbb{Z}_p^{\times} = 1+p^n\mathbb{Z}_p$, where $\alpha S^+ = \mathcal{I}\subset S^+$ is an ideal of definition and $\mathcal{W}$ is the weight space, *i.e.* there is a $u\in S^+$ s.t.

$$\kappa(t) = \exp(u\log(t)) =: t^u \quad \text{for each} \quad t\in 1+p^n\mathbb{Z}_p.$$

We define $s_1 := e_1 \mod p^n$; it is a marked section of $V$ (see Definition C.1.2).

Then we can build the sheaves $f_{V,*}\mathcal{O}_{\mathbb{V}(V)}$ and $f_{0,V,*}\mathcal{O}_{\mathbb{V}_0(V,s)}$. As $S^+$-module they are related to the following morphism

$$\begin{aligned} f_{V,*}\mathcal{O}^+_{\mathbb{V}(V)}(\mathcal{X}) = S^+ < X,Y > &\longrightarrow S^+ < Z,Y >= f_{0,V,*}\mathcal{O}^+_{\mathbb{V}_0(V,s)}(\mathcal{X}) \\ X &\longmapsto 1+\alpha Z \\ Y &\longmapsto Y \end{aligned}.$$

For $a \in 1+\mathcal{I}$ the action $\star$ defined in C.2 is compatible w.r.t. the morphism above. Let $\mathbb{W}^K_\kappa := f_{0,V,*}\mathcal{O}_{\mathbb{V}_0(V,s)}[\kappa] \cong X^u S < \frac{Y}{X} >$. See §C for more details.

Now we describe a $\mathcal{G}(S)$-action on $\mathbb{W}^S_\kappa$. The key idea is that $\mathcal{G}$ acts on the functor $\mathbb{V}(V)$ without changing the value modulo $\mathcal{I}$ on the marked section. Let $\mathcal{Z} = \mathrm{Spa}(R,R^+)$ be an affine formal scheme over $\mathcal{X}$, then $\mathrm{GL}_2(R^+)$ acts on $\mathbb{V}(V)(\mathcal{Z}) = \mathrm{Hom}_{S^+}(V,R^+)$ via the transpose action on $V$. Briefly for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(R^+)$, $f \in \mathbb{V}(V)(\mathcal{Z})$ and $v = xe_1 \oplus ye_2 \in V$ the action is given by

$$(g\cdot f)(v) := f\left(\begin{pmatrix} e_1 & e_2 \end{pmatrix} {}^t g \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Observe that the $\mathrm{GL}_2(R^+)$-action induced on the module $f_{V,*}\mathbb{V}(V) \otimes_{S^+} R^+$ is given by

$$\begin{aligned} g: R^+ < X,Y > &\longrightarrow R^+ < X,Y > \\ X &\longmapsto AX+CY \\ Y &\longmapsto BX+DY \end{aligned}.$$

If $g \in \mathcal{G}(R,R^+)$ and

$$\rho \in \mathbb{V}_0(V,s)(\mathcal{Z}) = \{f \in \mathbb{V}_0(V)(\mathcal{Z}) \mid f(e_1) \equiv 1 \mod p^n R^+\}$$

we get that $g\cdot\rho \in \mathbb{V}_0(V,s)(\mathcal{Z})$. Indeed we can write $\rho = ae_1^\vee + be_2^\vee$ for some $a \in p^nR^+$, $b \in R^+$ and

$$(g\cdot\rho)(e_1) = (Aae_1^\vee + Bbe_2^\vee)(e_1) \equiv Aa \mod p^nR^+ \equiv 1 \mod p^nR^+.$$

Hence we get a $\mathcal{G}$-action on $f_{0,V,*}\mathcal{O}_{\mathbb{V}_0(V,s)}$. Explicitly the $\mathrm{GL}_2(S^+)$-action extends on $f_{V,0,*}\mathcal{O}_{\mathbb{V}_0(V,s)}$ via

$$\begin{aligned} g: S^+ < Z,Y > &\longrightarrow S^+ < Z,Y > \\ Z &\longmapsto AZ + \tfrac{C}{\alpha}Y + \tfrac{A-1}{\alpha} \\ Y &\longmapsto \alpha BZ + DY + B \end{aligned}.$$

As in §C.2, we denote with $\mathcal{T}$ the $p$-adic analytic torus and with $\star$ its action on $f_{0,V,*}\mathcal{O}_{\mathbb{V}_0(V,s)}$ and $f_{V,*}\mathbb{V}(V)$. The $\mathcal{G}$-action preserves the $\star$-action of $\mathcal{T}$: for any $a \in \mathcal{T}(R,R^+)$, $g \in \mathcal{G}(R,R^+)$, $f \in \mathbb{V}_0(V,s)(\mathrm{Spa}(R,R^+))$, one can check that $a\star(g\cdot f) = g\cdot(a\star f)$. Indeed

$$(a\star(g\cdot f))(X,Y) = f(a(AX+CY), a(BX+DY)) = (g\cdot(a\star f))(X,Y),$$

hence the $\mathcal{G}$-action restricts to an action on $\mathbb{W}^S_\kappa$. Now we are interested in the action of the Lie algebra of $\mathcal{G}$

on $\mathbb{W}_\kappa^S$. Observe that

$$\operatorname{Lie}\mathcal{G}\otimes_{S^+} S = \mathfrak{gl}_2(S),$$

since $\mathcal{G}$ is an open of $\mathrm{GL}_2$. We can compute the $\mathfrak{sl}_2(S)$-action on $\mathbb{W}_\kappa^S$ induced by the $\mathcal{G}$-action. Let

$$U^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the action on $f\in\mathbb{W}_\kappa^S$ is given by:

$$U^+.f(X,Y) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\cdot f(X,Y) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} f(X,\lambda X+Y) = \left(X\frac{\partial f}{\partial Y}\right)(X,Y).$$

Using the matrices $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ and $\begin{pmatrix} 1+\lambda & 0 \\ 0 & (1+\lambda)^{-1} \end{pmatrix}$ one gets

$$U^-.f = Y\frac{\partial f}{\partial X}, \quad H.f = X\frac{\partial f}{\partial X} - Y\frac{\partial f}{\partial Y}.$$

One can consider an algebraic and not completed version of $\mathbb{W}_\kappa^S$:

$$\mathbb{W}_\kappa^{S,\mathrm{alg}} := \bigcup_{n\in\mathbb{N}} Fil_n\mathbb{W}_\kappa^S = \bigoplus_{n\in\mathbb{N}} S\,X^{u-n}Y^n = X^u\,S\left[\frac{Y}{X}\right].$$

By an easy computation

$$X\frac{\partial X^u}{\partial X} = X^{u+1}\lim_{\lambda\to 0}\frac{\kappa(X+\lambda)-\kappa(X)}{\lambda\kappa(X)} = X^u\lim_{\lambda\to 0}\frac{\kappa(1+\frac{\lambda}{X})-1}{\frac{\lambda}{X}} = uX^u,$$

where we used that $\left(\frac{\mathrm{d}}{\mathrm{d}t}\kappa\right)(1) = u$. Then $X\frac{\partial X^{u-n}}{\partial X} = (u-n)X^{u-n}$ and

$$H.X^{u-n}Y^n = Y^n(u-n)X^{u-n} - nX^{u-n}Y^n = (u-2n)X^{u-n}Y^n.$$

Then $\mathbb{W}_\kappa^{S,\mathrm{alg}}$ is $\mathfrak{h}$-split over $S$ and it is a direct sum of spaces of weight $(u-2n)$ with $n\in\mathbb{N}$, where we identify $\operatorname{Hom}_S(\mathfrak{h}(S),S)\cong S$ using the basis $\{H\}$ of $\mathfrak{h}$. In particular $\mathbb{W}_\kappa^{S,\mathrm{alg}}$ is the Verma module for $\mathfrak{sl}_2(S)$ with highest weight $u$.

### 2.2.3 The $\mathcal{G}$-representation $\mathfrak{W}_\kappa^S$

Fix the notations as in the pevious Subsection. Let $(R,R^+)$ be a Huber pair over $(S,S^+)$. Denote with $\kappa_{R^+}: 1+p^nR^+\to R^+$ the analytic extension of the character $\kappa$ to $R^+$. For any $\operatorname{diag}(\tau_1,\tau_2)=\tau\in T_n(R^+)$ we can define $\kappa(\tau) := \kappa_{R^+}(\tau_1\tau_2^{-1})$. The character $\kappa$ could be also extended to $\mathcal{B}(R,R^+)$ defining $\kappa(\tau\eta) := \kappa(\tau)$ for any $\eta\in N(R^+)$.

**Definition 2.2.1.** The $\mathcal{G}$-representation representation $\mathfrak{W}_\kappa^S$ is defined as the analytic induction

$$\mathfrak{W}_\kappa^S := \left(Ind_{\mathcal{B}}^{\mathcal{G}}\right)^{an}(\kappa) := \{f:\mathcal{G}\to\mathbb{A}^{1,an} \mid f(b^{-1}g) = \kappa(b)f(g)\ \forall\, b\in\mathcal{B},\ g\in\mathcal{G}\,\text{with}\, f\ \text{analytic}\}.$$

**Proposition 2.2.2.** *There is an isomorphism $\mathfrak{W}_\kappa^S\cong\mathbb{W}_\kappa^S$ as $S$-modules that preserves the $\mathcal{G}$-action.*

*Proof.* Let $\pi_1 : \mathcal{P}_{\mathcal{F}^+,s} \to \mathcal{X} = \mathrm{Spa}(S, S^+)$ be the functor defined by

$$\mathcal{P}_{\mathcal{F}^+,s} : (R, R^+) \mapsto \left\{ (e_1', e_2') \mid \{e_1', e_2'\} \text{ basis of } V \text{ with } \begin{cases} e_1'(s) & \bmod p^n S^+ \equiv 1, \\ e_2'(s) & \bmod p^n S^+ \equiv 0 \end{cases} \right\}.$$

Observe that $(e_1', e_2') \in \mathcal{P}_{\mathcal{F}^+,s}(R, R^+)$ if and only if

$$e_1' \in (1 + p^n R^+) e_1^\vee + R^+ e_2^\vee \quad \text{and} \quad e_2' \in p^n R^+ e_1^\vee + (R^+)^\times e_2^\vee.$$

This functor has a $\mathcal{G}$-action given on points by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (e_1', e_2') := (Ae_1' + Be_2', Ce_1', De_2').$$

In this way $\mathcal{P}_{\mathcal{F}^+,s}$ is a $\mathcal{G}$-torsor over $\mathcal{X}$. Choosing the canonical basis $(e_1^\vee, e_2^\vee)$ we get $\mathcal{P}_{\mathcal{F}^+,s} \cong \mathcal{G}$. Moreover there is a $\mathcal{G}$-equivariant map $\pi_2 : \mathcal{P}_{\mathcal{F}^+,s} \to \mathbb{V}_0(V, s)$ defined by $(e_1', e_2') \mapsto e_1'$. Observe that $\pi_2(e_1', e_2') = \pi_2(e_1'', e_2'')$ if and only if there is a matrix $g$ of the form $g = \begin{pmatrix} 1 & 0 \\ p^n R^+ & (R^+)^\times \end{pmatrix}$ with $(e_1'', e_2'') = g \cdot (e_1', e_2')$. Hence $\mathcal{P}_{\mathcal{F}^+,s}$ is an $\mathcal{N}$-torsor over $\mathbb{V}_0(V, s)$. We get that

$$\mathbb{W}_\kappa^S = \left(\pi_* \mathcal{O}_{\mathbb{V}_0(V,s)}\right)[\kappa] \cong \left(\left((\pi_1)_* \mathcal{O}_{\mathcal{P}_{\mathcal{F}^+,s}}\right)^{\mathcal{N}}\right)[\kappa] \cong \left((\pi_1)_* \mathcal{O}_{\mathcal{G}}^{\mathcal{N}}\right)[\kappa] \cong \mathfrak{W}_\kappa^S.$$

□

Observe that

$$\left(\mathrm{Ind}_{\mathscr{B}}^{\mathscr{G}}\right)^{an}(\kappa) \cong \left(\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\right)^{an}(\kappa) = \mathfrak{W}_\kappa^K$$

where $\mathscr{B} \subset \mathcal{B}$ and $\mathscr{G} \subset \mathcal{G}$ are defined as the subgroups of matrices with determinant 1. The isomorphism is given by the restriction, indeed $\mathcal{G} = \begin{pmatrix} 1 + p^n R^+ & 0 \\ 0 & (R^+)^\times \end{pmatrix} \mathscr{G}$ and for any $f \in \left(\mathrm{Ind}_{\mathscr{B}}^{\mathscr{G}}\right)^{an}(\kappa)$ we can extend the function f on $\mathcal{G}$ as

$$f\left(\begin{pmatrix} 1 + p^n A & 0 \\ 0 & D \end{pmatrix} g\right) := \frac{\kappa(1 + p^n A)}{\kappa(D)} f(g)$$

for any $A, D \in R^+$.

## 2.3 BGG on the modular curve

In this Section we describe how we can use the local BGG decomposition in order to get a global BGG decomposition on the modular curve.

### 2.3.1 Overconvergent modular forms

In this Subsection we want to resume the construction of the sheaves of overconvergent modular forms with $p$-adic weight in order to fix the notations. More completed descriptions are in the paper [AIP18] or in [AI21]. For a nice introduction I suggest also the reading of the PhD thesis [Pan19] of S. Panozzo that I found in the net.

Let $N \geqslant 5$ be a natural number and $p \geqslant 3$ be a prime number with $(N, p) = 1$. We denote $X_1(N)$ the modular curve over $\mathbb{Z}_p$ which classifies generalized elliptic curves with a structure of level $\Gamma_1(N)$. The formal scheme obtained via the formal completion along the special fiber is denoted by $\mathfrak{X}_1(N)$. We denote $\pi : \mathbb{E} \to$

$\mathfrak{X}_1(N)$ the universal generalized elliptic curve. Let $\mathrm{Spf}(R) \subset \mathfrak{X}_1(N)$ be an open where $\omega_{\mathbb{E}} := \pi_* \Omega^1_{\mathbb{E}/\mathfrak{X}_1(N)}$ is trivializable and $\omega_R$ be a basis of $(\omega_{\mathbb{E}})_{|_{\mathrm{Spf}(R)}}$. The ideal $\mathrm{Hdg} \subset \mathcal{O}_{\mathfrak{X}_1(N)}$ is defined, locally on $\mathrm{Spf}(R)$, as the ideal of $R$ generated by $p$ and the Hasse invariant $\mathrm{Ha}(\mathbb{E}_{|_{\mathrm{Spf}(R)}}/\mathrm{Spf}(R), \omega_R)$. For an $r \in \mathbb{N}$ with $1 \leqslant r$, one would like to "select" the elliptic curves with order of vanishing of the Hasse invariant lower than $\frac{1}{r}$: let $\mathfrak{B}$ be the formal admissible blow-up of $\mathfrak{X}_1(N)$ with respect to the ideal $(p, \mathrm{Hdg}^r)$ and $\mathfrak{X}_r \subset \mathfrak{B}$ is defined as the formal open sub-scheme defined by the condition that the ideal $(p, \mathrm{Hdg}^r)$ is generated by $\mathrm{Hdg}^r$. Then $\mathfrak{X}_r$ is a formal model for the rigid open of $\mathfrak{X}_1(N)$ defined by the condition that

$$v_p(\mathrm{Hdg}) \leqslant \frac{v_p(p)}{r}.$$

The universal generalized elliptic curve $\mathbb{E} \to \mathfrak{X}_r$ has a canonical cyclic subgroup $H_m \subset \mathbb{E}[p^m]$ of order $p^m$, where $m \in \mathbb{N}$ depends on $r$, see App. A in [AIP18] for the construction and §3 in [AI21] for the dependence of $m$ on $r$. For example if $2 \leqslant r$ and $5 \leqslant p$ or $4 \leqslant r$ and $p = 3$, then we can take $m = 1$. We denote with $\mathcal{X}_r$ the adic space associated to $\mathfrak{X}_r$, $H_m^D := \mathrm{Hom}(H_m, \mathbb{G}_m)$ the Cartier dual of $H_m$ and $\mathcal{IG}_{m,r} := \mathrm{Isom}(\underline{\mathbb{Z}/p^m\mathbb{Z}}, H_m^D)$ the Igusa variety defined as the space over $\mathcal{X}_r$ of the trivializations of $H_m^D$. Then the map

$$\pi : \mathcal{IG}_{m,r} \longrightarrow \mathcal{X}_r$$

is finite étale and has Galois group $(\mathbb{Z}/p^m\mathbb{Z})^\times$. We denote with $\mathfrak{IG}_{m,r}$ the formal model of $\mathcal{IG}_{m,r}$ obtained by the normalization of $\mathfrak{X}_r$ in $\mathcal{IG}_{m,r}$. The importance of the canonical subgroup is to give a morphism of order $p$ in the elliptic curve that lifts the Frobenius in zero characteristic. We resume some properties proven in [AI21], Appendix A. Observe that in the multiplicative group $\mathbb{G}_m$ there is a canonical differential given by $\frac{\mathrm{d}t}{t}$, hence for a point $P \in H_m^D$ one can define $\mathrm{d}\log(P) := P^* \frac{\mathrm{d}t}{t} \in \omega_{H_m}$. With a little abuse of notation we denote Hdg also the sheaf over $\mathfrak{IG}_{m,r}$ defined by the pullback of the sheaf Hdg over $\mathfrak{X}_r$ via the projection $\mathfrak{IG}_{m,r} \to \mathfrak{X}_r$.

**Proposition 2.3.1.**
- *The canonical subgroup $H_m$ of the universal elliptic curve $\mathbb{E}$ over $\mathfrak{IG}_{m,r}$ is a lifting of Frobenius modulo $\frac{p^m}{\mathrm{Hdg}^{\frac{p^m-1}{p-1}}}$;*
- *There is an isomorphism $\mathbb{E}[p]/H_m \cong H_m^D$ defined by the Weyl pairing $\mathbb{E}[p^m] \times \mathbb{E}[p^m] \to \mu_{p^m}$. We denote with $P^{univ}$ be the canonical section given by the trivialization of $H_m^D$ over $\mathfrak{IG}_{m,r}$;*
- *The inclusion $H_m \subset \mathbb{E}$ induces a map $f : \omega_{\mathbb{E}} \to \omega_{H_m}$ with kernel $\frac{p^m}{\mathrm{Hdg}^{\frac{p^m-1}{p-1}}}\omega_{\mathbb{E}}$, hence we get a canonical section*
$$s' := \mathrm{d}\log(P^{univ}) \in H^0\left(\mathfrak{IG}_{m,r},\ \omega_{\mathbb{E}}/\frac{p^m}{\mathrm{Hdg}^{\frac{p^m-1}{p-1}}}\omega_{\mathbb{E}}\right).$$

We denote $\underline{\beta}_m := \frac{p^m}{\mathrm{Hdg}^{\frac{p^m-1}{p-1}}} \subset \mathcal{O}_{\mathfrak{IG}_{m,r}}$, observe that this ideal generates the $p$-adic topology on $\mathcal{O}_{\mathfrak{IG}_{m,r}}$. The idea now is to define the sheaf of overconvergent modular forms thanks to the VBMS machine. In [AI21] they use the marked section $s'$ , but they have to modify the line bundle $\omega_{\mathbb{E}}$ since the lifting of $s$ does not generate the whole line bundle $\omega_{\mathbb{E}}$. Let $\Omega_{\mathbb{E}}$ be the $\mathcal{O}_{\mathfrak{IG}_{m,r}}$-submodule of $\omega_{\mathbb{E}}$ generated by all the local liftings of $s'$. We denote $s \in H^0\left(\mathfrak{IG}_{m,r},\ \Omega_{\mathbb{E}}/\underline{\beta}_m\Omega_{\mathbb{E}}\right)$ the canonical element determined by $s'$. By definition of $\Omega_{\mathbb{E}}$ the pair $(\Omega_{\mathbb{E}}, s)$ is a sheaf with marked section on $\left(\mathfrak{IG}_{m,r},\ \mathcal{I} := \underline{\beta}_m\right)$. Let $H_{\mathbb{E}}$ be the sheaf on $\mathfrak{IG}_{m,r}$ defined by

$$H_{\mathbb{E}} :\ (E, S) \mapsto H^1_{dR}(E/S),$$

where $E$ is a generalized elliptic curve over $S$ with a $\Gamma_1(N)$-structure and $H^1_{dR}(E/S)$ is the sheaf over $S$ given by the de Rham cohomology of $E/S$. Similarly to the case $\omega_{\mathbb{E}}$ we define a submodule $H_{\mathbb{E}}$. Let

$H^{\#}_{\mathbb{E}} := \Omega_{\mathbb{E}} + \underline{\delta}^p H_{\mathbb{E}} \subset H_{\mathbb{E}}$. Then $(\Omega_{\mathbb{E}}, s) \subset (H^{\#}_{\mathbb{E}}, s)$ is an inclusion of locally free sheaves of rank 1 and 2 with a compatible marked section.

Observe that $\mathfrak{T}^{ext} := \mathbb{Z}_p^\times \left(1 + \underline{\beta}_m\right)$ acts on the morphisms of formal schemes

$$v_H : \mathbb{V}_0\left(H^{\#}_{\mathbb{E}}, s\right) \to \mathfrak{X}_r \quad \text{and} \quad v : \mathbb{V}_0\left(\Omega_{\mathbb{E}}, s\right) \to \mathfrak{X}_r :$$

$\mathbb{Z}_p^\times$ acts by multiplication of the marked section and $1 + \underline{\beta}_m$ acts by multiplication on the choordinate of the marked section as explained in §C.2. Observe that $\mathbb{Z}_p^\times$ could change the marked section modulo $\underline{\beta}_m$, hence it acts non trivially over $\mathfrak{IG}_{m,r}$, instead $1 + \underline{\beta}_m$ acts trivially on the Igusa variety since it does not affect the marked section $s$. The $\mathfrak{T}^{ext}$-action is trivial over $\mathfrak{X}_r$. Let $\kappa : \mathbb{Z}_p^\times \to R^\times$ be an $n$-analytic weight, *i.e.* $R$ is a $p$-adically completed separated ring and $\kappa$ is a group morphism with the property that there is an element $u \in R$ with

$$\kappa(t) = \exp(u \log(t)) =: t^u \quad \text{for every} \quad t \in 1 + p\mathbb{Z}_p.$$

Via the analicity of $\kappa$, it can be extended on $\mathfrak{T}^{ext}$, since the exponential converges on $1 + \underline{\beta}_m$. Then we get two $\mathfrak{T}^{ext}$-actions on

$$v_*\left(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}}, s)} \hat{\otimes}_{\mathbb{Z}_p} R\right), \quad v_{H,*}\left(\mathcal{O}_{\mathbb{V}_0\left(H^{\#}_{\mathbb{E}}, s\right)} \hat{\otimes}_{\mathbb{Z}_p} R\right)$$

and we can define

$$\mathfrak{w}^\kappa := v_*\left(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}}, s)} \hat{\otimes}_{\mathbb{Z}_p} R\right)[\kappa] \subset v_*\left(\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathbb{E}}, s)} \hat{\otimes}_{\mathbb{Z}_p} R\right)$$

the submodule generated by the elements $x \otimes \alpha$ such that $(t \star x) \otimes \alpha = x \otimes (\kappa(t)\alpha)$. Analogously we can define

$$\mathbb{W}_\kappa := v_{H,*}\left(\mathcal{O}_{\mathbb{V}_0\left(H^{\#}_{\mathbb{E}}, s\right)} \hat{\otimes}_{\mathbb{Z}_p} R\right)[\kappa].$$

### 2.3.2 A variant of the Grothendieck–Messing period map

Let $(S, S^+)$ be an Huber pair over a finite extension $K$ of $\mathbb{Q}_p$ as in §2.2.2, $1 \leqslant n \in \mathbb{N}$ and $\kappa : \mathbb{Z}_p^\times \to (S^+)^\times$ be an $n$-analytic weight, *i.e.* group morphism such that there is a $u \in S^+$ with

$$\kappa(t) = t^u \quad \text{for each} \quad t \in 1 + p^n\mathbb{Z}_p.$$

We denote with $\beta : \mathfrak{P}_{(H^{\#}_{\mathbb{E}})^\vee} \to \mathfrak{IG}_{n,r}$ the torsor of local basis $e'_1, e'_2$ of $(H^{\#}_{\mathbb{E}})^\vee$ satisfying $\left(e'_1 \mod \underline{\beta}_n\right)(s) = 1 \in \mathcal{O}_{\mathfrak{IG}_{n,r}}/\underline{\beta}_n$. The locally split filtration $\Omega_{\mathbb{E}} \subset H^{\#}_{\mathbb{E}}$ defines a locally split map $\varphi : \left(H^{\#}_{\mathbb{E}}\right)^\vee \to \Omega^\vee_{\mathbb{E}}$. Let $\mathfrak{U} = \mathrm{Sp}(R)$ be an affinoid open of $\mathfrak{P}_{(H^{\#}_{\mathbb{E}})^\vee}$ where $\beta^*\left(\Omega_{\mathbb{E}}\right)_{|\mathfrak{U}}$ is trivializable and let $x$ be a basis of $\beta^*\left(\Omega_{\mathbb{E}}\right)_{|\mathfrak{U}}$, then one can define

$$[\varphi(e'_1)(x) : \varphi(e'_2)(x)] \in \mathbb{P}^1_{\mathbb{Z}_p}.$$

If one changes the basis this value does not change, hence there is a map

$$\phi : \mathfrak{P}_{(H^{\#}_{\mathbb{E}})^\vee} \to \mathbb{P}^1_{\mathbb{Z}_p}$$

defined on the two coordinates $X_1, X_2$ of $\mathbb{P}^1_{\mathbb{Z}_p}$ by

$$X_1 \mapsto \varphi(e'_1)(x), \quad X_2 \mapsto \varphi(e'_2)(x).$$

Now we consider the generic fibers of the formal schemes above and we get the diagram of adic spaces

$$\begin{tikzcd} \mathcal{P}_{(H^{\#}_{\mathbb{E}})^{\vee}} \arrow[r, "\phi"] \arrow[d, "\beta"] & \mathbb{P}^{1,an} \\ \mathcal{IG}_{n,r} & \end{tikzcd}.$$

**Theorem 2.3.2** (Messing)**.** *Locally on $\mathcal{IG}_{n,r}$ there are sections $g$ of $\beta$ such that $\pi_{GM} := \phi \circ g$ is an étale morphism of adic spaces.*

*Proof.* $\mathcal{IG}_{n,r} \to \mathcal{X}_r \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ is a composition of a finite étale and a smooth morphism, then there is an open affinoid covering $\mathcal{U}'$ of $\mathcal{IG}_{n,r}$ such that for each $U \in \mathcal{U}'$ there is an étale map $\varphi'_U : U \to \mathbb{A}^{d,an}$ (see Corollary 1.6.10, Proposition 1.7.1, Corollary 1.7.2 in [Hub96]) , where $d = 1$ is the dimension of $\mathcal{IG}_{n,r}$. Let $\mathcal{U}$ be an open affinoid covering of $\mathcal{IG}_{n,r}$ that refines $\mathcal{U}'$ such that for any $U \in \mathcal{U}$ the module $H^{\#}_{\mathbb{E}}$ is free of rank 2 on $U$, the inclusion $\Omega^1_{\mathbb{E}} \subset H^{\#}_{\mathbb{E}}$ splits and Hdg is free on $U$. Let $U := \mathrm{Spa}(R, R^+) \in \mathcal{U}$ and $b' := {\varphi'_U}^{\#}(T) \in R$, where $T \in \mathcal{O}^+_{\mathbb{A}^{1,ad}}$ is the coordinate. Let $N \in \mathbb{N}$ be such that $p^n \mid p^N b'$ in $R^+$; we can define $\varphi_U : U \to \mathbb{A}^{1,ad}$ such that ${\varphi_U}^{\#}(T) = p^N b' =: b$. The morphism is still étale: if $\mathrm{d}\, b'$ generates $\Omega^1_{\mathbb{Q}_p/R}$, then also $\mathrm{d}\, b$ generates it. Let $\{\omega, e_2\}$ be a basis on $U$ of $H^{\#}_{\mathbb{E}}$ such that $\omega$ is a basis of $\Omega^1_{\mathbb{E}}$ lifting $s$. Define $e_1 := \omega - be_2$ and let $g$ be the section of $\beta$ defined on $U$ by the basis $\{e_1^{\vee}, e_2^{\vee}\}$ of $\left(H^{\#}_{\mathbb{E}}\right)^{\vee}$. Observe that the basis defines a section since

$$e_1^{\vee}(s) \mod p^n = e_1^{\vee}(\omega) \mod p^n = e_1^{\vee}(e_1 + be_2) \mod p^n \equiv 1.$$

If we consider the affine open on $\mathbb{P}^{1,ad}$ given by the coordinate $T = \frac{X_2}{X_1}$ we get that

$$(\phi \circ g)^{\#}(T) = (\phi \circ g)^{\#}\left(\frac{X_2}{X_1}\right)\frac{e_2^{\vee}(\omega)}{e_1^{\vee}(\omega)} = \frac{b}{1} = b = {\varphi_U}^{\#}(T).$$

Hence $\phi \circ g = \varphi_U$ and $\phi \circ g$ is étale. □

Let $B \subset G = \mathrm{GL}_2$ and $N^- \subset \mathrm{GL}_2$ as in §2.2.1. We denote $\mathfrak{b} := \mathrm{Lie}(B) \subset \mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{n}^- := \mathrm{Lie}(N^-)$.

**Corollary 2.3.3.** *Let $U \subset \mathcal{IG}_{n,r}$ be an open affinoid where there is a section $\pi_{GM}$ given by 2.3.2. Then*

$$\Omega^1_{\mathbb{A}^{1,ad}/\mathbb{Q}_p} \cong (\mathfrak{n}^-)^{\vee} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{A}^{1,ad}} \quad \textit{and} \quad \pi^*_{GM}\left((\mathfrak{n}^-)^{\vee} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{A}^{1,ad}}\right) \cong \Omega^1_{U/\mathbb{Q}_p}.$$

*Proof.* The second isomorphism follows by the first one and by the fact that the map $\pi_{GM}$ is étale; in particular the pullback of the sheaf of differentials via $\pi_{GM}$ is isomorphic to the sheaf of differentials.

Observe that $G/B \cong \mathbb{P}^1$ and $\mathbb{A}^1 \cong \mathbb{G}_a \cong N^- \cong N^-B/B \subset G/B$. Hence the tangent space of $\mathbb{P}^1$ and $\mathbb{A}^1$ at the origin is the $\mathbb{Q}_p$-vector space $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^-$ . Hence the sheaf of differentials is

$$\Omega^1_{\mathbb{A}^{1,ad}/\mathbb{Q}_p} \cong (\mathfrak{n}^-)^{\vee} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{A}^{1,ad}}$$

as stated. □

Thanks to the Grothendieck–Messing map we get a local description of the sheaf $\mathbb{W}_k$ on $\mathcal{IG}_{n,r,S} := \mathcal{IG}_{n,r} \times_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \mathrm{Spa}(S, S^+)$. We denote

$$\pi_0 : \mathbb{V}_0(H^{\#}_{\mathbb{E}}, s) \times_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \mathrm{Spa}(S, S^+) \to \mathcal{IG}_{n,r,S}.$$

**Theorem 2.3.4.** *Let* $\mathfrak{W}_\kappa^{\mathbb{A}^1_S} := \left(\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}\right)^{an}(\kappa) \otimes \mathcal{O}_{\mathbb{A}^{1,ad}_S}$, *where* $\kappa : \mathbb{Z}_p^\times \to (S^+)^\times$ *is an* $n$*-analytic weight, let* $\pi_{GM} : U \to \mathbb{A}^{1,ad}$ *be the Grothendieck–Messing map defined on an open affinoid* $U \subset \mathcal{IG}_{n,r,S}$ *and* $\mathbb{W}_\kappa := \left((\pi_0)_*\left(\mathcal{O}_{\mathbb{V}_0(H^\#_{\mathbb{E}},s)} \otimes_{\mathbb{Q}_p} S\right)\right)[\kappa]$. *Then*

$$\pi_{GM}^*\left(\mathfrak{W}_\kappa^{\mathbb{A}^1_S}\right) \cong (\mathbb{W}_\kappa)_{|U}$$

*as* $\mathfrak{g}$*-modules.*

*Proof.* On $U$ there is a section $g : U \to \mathcal{P}_{\left(H^\#_{\mathbb{E}}\right)^\vee}$ with $\pi_{GM} = \pi \circ g$. Let $U = \mathrm{Spa}(R, R^+)$, the section $g$ gives an $R^+$ basis $\{e_1', e_2'\}$ of $\left(H^\#_{\mathbb{E}}\right)^\vee$ where $e_1'(s) \mod p^n \equiv 1$, hence $g$ gives an isomorphism

$$\left(\left(H^\#_{\mathbb{E}} \otimes_{\mathbb{Z}_p} S^+\right)_{|U}, s_{|U}\right) \cong \left(\pi_{GM}^*\left(S^+\right)^2, e_1\right).$$

Then via the Proposition 2.2.2 and the functoriality of the VBMS construction we conclude that

$$(\mathbb{W}_\kappa)_{|U} \cong \pi_{GM}^*\mathbb{W}_\kappa^{\mathbb{A}^1_S} \cong \pi_{GM}^* f^*\mathbb{W}_\kappa^S \cong \pi_{GM}^* f^*\mathfrak{W}_\kappa^S = \pi_{GM}^*\mathfrak{W}_\kappa^{\mathbb{A}^1_S}$$

where we denoted $\mathbb{W}_\kappa^{\mathbb{A}^1_S}$ the pullback of the $S^+$-module $\mathbb{W}_\kappa^K$, defined in §2.2.2, via the definition morphism $f : \mathbb{A}^{1,ad}_S \to \mathrm{Spa}(S, S^+)$. □

### 2.3.3 Main application

In this subsection we get a covering of the modular curve $\mathcal{X}_{r,S} := \mathcal{X}_r \times_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \mathrm{Spa}(S, S^+)$ where we can compute the de Rham cohomology $H^1_{dR}\left(\mathcal{U}, (\mathbb{W}_\kappa)_{|\mathcal{U}}\right)$; for a technical reason we will restrict our attention to the unnecessary conditon that $S = K$ is a field; now let $S$ be as in §2.2.2. As explained in the Introduction, the strategy is the following:

- trivialize the vector bundle $\mathbb{W}_\kappa$ with a $\mathfrak{g}$-action defined over and his $\mathfrak{g}$-action on a covering of $\mathcal{X}_{r,S}$, getting the vector bundle $\mathbb{W}_\kappa^S$ over $\mathrm{Spa}(S, S^+)$, see the Theorem 2.3.4;
- use the BGG decomposition of the Corollary 2.1.12, getting the linearized de Rham complex and the linearized BGG complex associated to $\mathbb{W}_\kappa^S$ with his $\mathfrak{g}$-action over the space $\mathbb{P}^1 \cong G/B$;
- use the étale Grothendieck–Messing map introduced in the Theorem 2.3.2 in order to bring the linearized information to a covering of $\mathcal{X}_{r,S}$;
- apply the functor $u_*$ getting that the delinearized BGG complex and the de Rham complex associated to $\mathbb{W}_\kappa$ are quasi-isomorphic; in particular we show that we can cut the infinite dimensional vector bundles of the de Rham complex getting a sequence of one finite dimensional (line) bundle.

Since $\mathcal{IG}_{n,r,S} \to \mathcal{X}_{r,S}$ is a finite torsor and we are working locally, we can study the de Rham cohomology of $\mathbb{W}_\kappa$ directly on $\mathcal{IG}_{n,r,S}$. Let $\mathrm{Spa}(R, R^+) = \mathcal{U} \subset \mathcal{IG}_{n,r,S}$ be an affinoid open such that there is the étale Grothendieck–Messing morphism $\pi_{GM} : \mathcal{U} \to \mathbb{P}^1_S$. Remember that

$$\pi_{GM}^*\mathbb{W}_\kappa^{\mathbb{A}^1_S} \cong (\mathbb{W}_\kappa)_{|\mathcal{U}}$$

In §2.1 we considered the complexes of $S$-modules

$$\mathcal{D}^k := \underset{\longrightarrow\, n\in\mathbb{N}}{\mathrm{colim}}\ \lim_{s\in\mathbb{N}} \mathcal{U}(\mathfrak{g})^{\leq n-k} \otimes_{\mathcal{U}(\mathfrak{b})}\left(M^{s+k} \otimes_S \bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})}\left(M \otimes_S \bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right),$$

$$\mathcal{C}^k := \lim_{s\in\mathbb{N}}\ \underset{\longrightarrow\, n\in\mathbb{N}}{\mathrm{colim}}\ \mathcal{U}(\mathfrak{g})^{\leq n-k} \otimes_{\mathcal{U}(\mathfrak{b})}\left(M^{s+k} \otimes_S \bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right) = \lim_{s\in\mathbb{N}} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})}\left(M^{s+k} \otimes_S \bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right),$$

we built a BGG decomposition $\mathcal{C}^\bullet \cong \mathcal{F}^\bullet \oplus \mathcal{G}^\bullet$ and we showed that the induced map $\gamma^\bullet : \mathcal{G}^\bullet \to \mathcal{D}^\bullet$ is zero on the cohomology groups. Now we consider the continuous dual of $\mathcal{C}^\bullet$, $\mathcal{G}^\bullet$, $\mathcal{F}^\bullet$ and $\mathcal{D}^\bullet$ as we did in §2.1.3. We do not want to take just the dual since $\mathbb{W}_\kappa^{K,alg} \subsetneq \left(\left(\mathbb{W}_\kappa^{K,alg}\right)^\vee\right)^\vee = M^\vee$. Let

$$\mathfrak{D}_{\mathcal{U}}^{k,\vee} := \lim_{n\in\mathbb{N}} \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \left(\mathcal{U}(\mathfrak{g})^{\leqslant n-k}\right)^\vee \otimes_{\mathcal{U}(\mathfrak{b})} \left((M^{s+k})^\vee \otimes_S \left(\bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right)^\vee\right);$$

$$\mathfrak{C}_{\mathcal{U}}^{k,\vee} := \varinjlim_{s\in\mathbb{N}} \lim_{n\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \left(\mathcal{U}(\mathfrak{g})^{\leqslant n-k}\right)^\vee \otimes_{\mathcal{U}(\mathfrak{b})} \left((M^{s+k})^\vee \otimes_S \left(\bigwedge_S^k \mathfrak{g}/\mathfrak{b}\right)^\vee\right).$$

In order to recover the BGG decomposition of $\mathfrak{C}_{\mathcal{U}}^{k,\vee}$ we observe that

$$\begin{aligned}\mathfrak{C}_{\mathcal{U}}^{k,\vee} &\cong \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \left(\mathcal{U}(\mathfrak{g})^\vee \otimes_{\mathcal{U}(\mathfrak{b})} \left((M^{s+k})^\vee \otimes_S\right)\right) = \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S A_{s+k,k}^\vee \\ &\cong \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \left((A_{s+k,k})_{\chi_\kappa} \oplus \left(\bigoplus_{\chi\in\Theta} (A_{s+k,k})_\chi\right)\right)^\vee.\end{aligned}$$

Then we can define

$$\mathfrak{G}_{\mathcal{U}}^{k,\vee} := \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \bigoplus_{\chi\in\Theta} \left((\mathcal{U}(\mathfrak{g}))^\vee \otimes_{\mathcal{U}(\mathfrak{b})} \left(Fil_{s+k} \mathbb{W}_\kappa^{S,alg} \otimes_S \Omega^1_{(G/B)/S}\right)\right)_\chi;$$

$$\mathfrak{F}_{\mathcal{U}}^{k,\vee} := \varinjlim_{s\in\mathbb{N}} \pi_{GM}^* \mathcal{O}_{G/B} \otimes_S \left((\mathcal{U}(\mathfrak{g}))^\vee \otimes_{\mathcal{U}(\mathfrak{b})} \left(Fil_{s+k} \mathbb{W}_\kappa^{S,alg} \otimes_S \Omega^1_{(G/B)/S}\right)\right)_{\chi_\kappa}$$

getting $\mathfrak{C}_{\mathcal{U}}^{k,\vee} \cong \mathfrak{G}_{\mathcal{U}}^{k,\vee} \oplus \mathfrak{F}_{\mathcal{U}}^{k,\vee}$. Observe that

$$\mathfrak{F}_{\mathcal{U}}^{k,\vee} \cong \varinjlim_{s\in\mathbb{N}} L\left(\left(Fil_{s+k}\left(\mathbb{W}_\kappa^{alg}\right)_{|\mathcal{U}} \otimes_S \Omega^k_{\mathcal{U}/S}\right)_{\chi_\kappa}\right) \cong L\left(\left(\left(\mathbb{W}_\kappa^{alg}\right)_{|\mathcal{U}} \otimes_S \Omega^k_{\mathcal{U}/S}\right)_{\chi_\kappa}\right);$$

$$\mathfrak{G}_{\mathcal{U}}^{k,\vee} \cong \bigoplus_{\chi\in\Theta} L\left(\left(\left(\mathbb{W}_\kappa^{alg}\right)_{|\mathcal{U}} \otimes_S \Omega^k_{\mathcal{U}/S}\right)_{\chi}\right).$$

In the isomorphism above reguarding $\mathfrak{G}_{\mathcal{U}}^{k,\vee}$ we used that the limit of the dual is the dual of the colimit, that for any $s, k \in \mathbb{N}$ there are only finite $\chi \in \Theta$ with $(A_{s+k,k})_\chi \neq 0$ and that $(A_{s+k,k})_\chi$ stabilizes in $s$ (see Lemma 2.1.3).

Another reason to consider the continuous dual is that we get some linearized sheaves; hence we can define these sheaves also in the infinitesimal topos associated to a rigid model $\mathfrak{U}$ of $\mathcal{U}$. At this point we should assume that $S = K$ is a field of characteristic zero, since we studied the infinitesimal locus of a smooth rigid variety over such a field. One can remove this condition but there is no time to change it in the Thesis. The full result will hopefully appear in a joint work with F. Andreatta and A. Iovita.

$$\mathfrak{C}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} := \varinjlim_{s\in\mathbb{N}} L(Fil_{s+k} \mathbb{W}_\kappa^S \otimes_S \Omega^k_{\mathfrak{U}/S})_{\mathrm{inf}} \to \mathfrak{D}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} := L\left(\mathbb{W}_\kappa^{S,alg} \otimes_S \Omega^k_{\mathfrak{U}/S}\right)_{\mathrm{inf}}.$$

and

$$\mathfrak{F}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} := L\left(\left(\left(\mathbb{W}_\kappa^{alg}\right)_{|\mathfrak{U}} \otimes_S \Omega^k_{\mathfrak{U}/S}\right)_{\chi_\kappa}\right)_{\mathrm{inf}}; \quad \mathfrak{G}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} := \bigoplus_{\chi\in\Theta} L\left(\left(\left(\mathbb{W}_\kappa^{alg}\right)_{|\mathfrak{U}} \otimes_S \Omega^k_{\mathfrak{U}/S}\right)_{\chi}\right)_{\mathrm{inf}}.$$

We get that $\mathfrak{C}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} \cong \mathfrak{F}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} \oplus \mathfrak{G}_{\mathfrak{U},\mathrm{inf}}^{k,\vee}$. Moreover

$$\mathfrak{C}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} = \mathcal{O}_{\mathfrak{U}}[[\Omega^1_{\mathfrak{U}/S}]] \otimes_{\mathcal{O}_{\mathfrak{U}}} \mathbb{W}_\kappa^{alg} \otimes_{\mathcal{O}_{\mathfrak{U}}} \Omega^k_{\mathfrak{U}/S}$$

is the non complete linearization of $\mathbb{W}_\kappa^{alg}$. Then its delinearization is

$$u_* \mathfrak{D}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} = u_* \mathfrak{C}_{\mathfrak{U},\mathrm{inf}}^{k,\vee} \cong \mathbb{W}_\kappa^{alg} \otimes_{\mathcal{O}_{\mathfrak{U}}} \Omega^k_{\mathfrak{U}/S}.$$

Here we used that $\mathbb{W}_\kappa^{alg} \otimes_{\mathcal{O}_\mathfrak{U}} \Omega^k_{\mathfrak{U}/S}$ is a flat sheaf with an integrable connection together with the Proposition 1.1.12 and the Proposition 1.3.27.

**Proposition 2.3.5.** *The map $\mathfrak{G}^{\bullet,\vee}_{\mathfrak{U},\inf} \to \mathfrak{D}^{\bullet,\vee}_{\mathfrak{U},\inf}$ composition of the embedding $\mathfrak{G}^{\bullet,\vee}_{\mathfrak{U},\inf} \to \mathfrak{C}^{\bullet,\vee}_{\mathfrak{U},\inf}$ and the canonical map $\mathfrak{C}^{\bullet,\vee}_{\mathfrak{U},\inf} \to \mathfrak{D}^{\bullet,\vee}_{\mathfrak{U},\inf}$ vanishes on cohomology.*

*Proof.* Let $(U = \mathrm{Sp}(A), T = \mathrm{Sp}(R))$ be an affinoid pair in the infinitesimal topos over $\mathfrak{U}$, we have to check that the map $\left(\mathfrak{C}^{\bullet,\vee}_{\mathfrak{U},\inf}\right)_{(U,T)} \to \left(\mathfrak{D}^{\bullet,\vee}_{\mathfrak{U},\inf}\right)_{(U,T)}$ is zero on cohomology. Once we prove that the representation $\mathbb{W}_\kappa^{K,alg}$ respects the hypothesis formulated in the Section §2.1, we can apply the Corollary 2.1.12 over the ring $R$.

We know that $\mathbb{W}_\kappa^{S,alg} \cong X^u S\left[\frac{Y}{X}\right]$ has a filtration with

$$Fil_i \mathbb{W}_\kappa^{S,alg} = X^u S\left[\frac{Y}{X}\right]^{\leqslant i}.$$

As we computed in §2.2.2 the action of $\mathfrak{g}$ is given by

$$H.X^{u-n}Y^n = (u-2n)X^{u-n}Y^n; \quad U^+.X^{u-n}Y^n = nX^{u-(n-1)}Y^{n-1}; \quad U^-.X^{u-n}Y^n = (u-n)Y^{n+1}.$$

Hence the $\mathfrak{b}$-action preserves the filtration and the $\mathfrak{n}^-$-action increases the filtration by one. The compatible system of projections $\mathbb{W}_\kappa^{S,alg} \to Fil_i \mathbb{W}_\kappa^{alg}$ is given by the canonical projection

$$g_i : X^u S\left[\frac{Y}{X}\right] \to X^u S\left[\frac{Y}{X}\right]^{\leqslant i}.$$

These projections are compatible with the $\mathfrak{n}^-$-action, indeed

$$g_{i+1}\left(U^-.\sum_{n\in\mathbb{N}} a_n X^{u-n}Y^n\right) = g_{i+1}\left(\sum_{n\in\mathbb{N}}(u-n)a_n X^{u-(n+1)}Y^{n+1}\right) = \sum_{n=0}^{i}(u-n)a_n X^{u-(n+1)}Y^{n+1}$$

and

$$U^-.g_i\left(\sum_{n\in\mathbb{N}} a_n X^{u-n}Y^n\right) = U^-.\sum_{n=0}^{i} a_n X^{u-n}Y^n = \sum_{n=0}^{i}(u-n)a_n X^{u-(n+1)}Y^{n+1}.$$

□

In our $\mathrm{GL}_2$-case, the vanishing of this map follows also by the fact that the complex $\mathfrak{G}^{\bullet,\vee}_{\mathfrak{U},\inf}$ is exact. One can check this exactness by explicit computations of the BGG-resolution.

**Theorem 2.3.6.** *Let $K$ be a $p$-adic field and $\mathfrak{X}$ be the rigid analytic space associated to the modular curve $\mathcal{X}_r \times_{\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \mathrm{Spa}(K,\mathcal{O}_K)$. Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}_K^\times$ be an $n$-analytic map; then there is a covering $\{\mathfrak{U}\}$ of $\mathfrak{X}$ s.t.*

$$H^1_{dR}\left(\mathfrak{U}, \mathbb{W}_\kappa^{alg}\right) \cong H^1\left(\mathfrak{U}, u_*\mathfrak{F}^{\bullet,\vee}_{\mathfrak{U},\inf}\right) \cong H^1\left(\mathfrak{U}, \left((\mathbb{W}_\kappa^{alg})_{|\mathfrak{U}} \otimes_{\mathcal{O}_\mathfrak{U}} \Omega^\bullet_{\mathfrak{U}/K}\right)_{\chi_\kappa}\right)$$

*Proof.* We know that $u_*\mathfrak{D}^{k,\vee}_{\mathfrak{U},\inf} \cong u_*\mathfrak{C}^{k,\vee}_{\mathfrak{U},\inf} \cong \left(u_*\mathfrak{G}^{k,\vee}_{\mathfrak{U},\inf} \oplus \mathfrak{F}^{k,\vee}_{\mathfrak{U},\inf}\right)$ and the map

$$u_*\mathfrak{D}^{k,\vee}_{\mathfrak{U},\inf} \to u_*\mathfrak{G}^{k,\vee}_{\mathfrak{U},\inf}$$

is zero on cohomology by the Proposition 2.3.5. The Theorem follows via the Theorem 1.3.28 and the Proposition 1.3.27. □

Observe that in order to prove the Theorem we used the de Rham cohomology on $\mathbb{W}_\kappa^{alg}$ induced by the

Koszul complex. But the infinitesimal site computations tell us that any connection gives the same de Rham cohomology, since all the linearized connections are isomorphic to the canonical one $\mathrm{id} \hat{\otimes} \mathrm{d}^\bullet$.

# Appendix A

# Algebra Lemmas

**Lemma A.0.1.** *Let $M_1, M_2$ be two $A$-modules, $I \subset A$ an ideal, $I_1 \subset M_1$, $I_2 \subset M_2$ be two $A$-submodules s.t. $I \cdot M_i \subset I_i$, i.e. so that $M_i/I_i$ is an $A/I$-module with the structure induced via the $A$-module structure on $M_i$.*

*Then the kernel of the morphism*

$$\phi : M_1 \otimes_A M_2 \to M_1/I_1 \otimes_{A/I} M_2/I_2$$

*is the $A$-module*

$$\mathrm{Ker}(\phi) = I_1 \otimes_A M_2 + M_1 \otimes I_2$$

*Proof.* Let $K := \mathrm{Ker}(\phi)$ and let's denote $\phi_i : M_i \to M_i/I_i$ the two projections. Then

$$\phi = \left(\mathrm{id}_{M_1/I_1} \otimes \phi_2\right) \circ \left(\phi_1 \otimes \mathrm{id}_{M_2}\right).$$

Since the tensor product is exact on the right we have that the sequences

$$M_1/I_1 \otimes_A I_2 \longrightarrow M_1/I_1 \otimes_A M_2 \longrightarrow M_1/I_1 \otimes_A M_2/I_2$$

$$I_1 \otimes_A M_2 \longrightarrow M_1 \otimes_A M_2 \longrightarrow M_1/I_1 \otimes_A M_2$$

are exact. Let $k \in K$, then $\tilde{k} := \left(\phi_1 \otimes \mathrm{id}_{M_2}\right)(k) \in \mathrm{Ker}(\mathrm{id}_{M_1/I_1} \otimes \phi_2)$, via the first exact sequence above there is an element $k_1 \in M_1 \otimes_A I_2$ s.t. $(\phi_1 \otimes_A \mathrm{id}_{M_2})(k) = \tilde{k}$.

Then $x - k_1 \in \mathrm{Ker}(\phi_1 \otimes \mathrm{id}_{M_2})$ and via the second exact sequence above $k - k_1 = k_2 \in I_1 \otimes_A M_2$ and we conclude that

$$k = k_1 + k_2 \in M_1 \otimes_A I_2 + I_1 \otimes_A M_2.$$

Observe that (in order to simplify the notation) sometimes we have identified the modules $M_1 \otimes_A I_2$ and $I_1 \otimes_A M_2$ with their image in $M_1 \otimes_A M_2$, as in the statement. □

**Lemma A.0.2.** *Let $\phi_i : R_i \to R_i/I_i$ be ring projections with $I_i^n = 0$ for $i = 0, 1, 2$. Let $\beta_i : R_0 \to R_i$ a ring homomorphism s.t. $\beta_i(I_0) \subset I_i$, where $i = 1, 2$. Then the kernel of the map*

$$\phi : R_1 \otimes_{R_0} R_2 \to R_1/I_1 \otimes_{R_0/I_0} R_2/I_2$$

*is nilpotent.*

*Proof.* Via the Lemma A.0.1 we get that the kernel of the map is

$$K := \mathrm{Ker}(\phi) = I_1 \otimes_A R_2 + R_1 \otimes_A I_2.$$

Observe that for $i_1 \in I_1,\ i_2 \in I_2,\ r_1 \in R_1$ and $r_2 \in R_2$

$$(i_1 \otimes r_2)^n = i_1^n \otimes r_2^n = 0 \quad \text{and} \quad (r_1 \otimes i_2)^n = 0.$$

Then $K$ is generated by nilpotent elements, but sum of (commuting) nilpotent elements is nilpotent, then $K$ is nilpotent. □

**Lemma A.0.3.** *If $A$ is a commutative $K$-algebra, $J \subset A$ is an ideal with $J^N = 0$, denote $p : A \to A/J =: B$ the projection and suppose that there is a ring morphism $i : B \to A$, with $p \circ i = id_B$. Denote*

$$I_1 := \mathrm{Ker}\left(B \otimes_K B \xrightarrow{m_B} B\right) \quad \textit{and} \quad I_2 := \mathrm{Ker}\left(A \otimes_K B \xrightarrow{m_B \circ (p \otimes id_B)} B\right),$$

*where $m_B$ is the multiplication map. Then*

$$\lim_{n \in \mathbb{N}} A \otimes_{i,B} \frac{B \otimes_K B}{I_1^n} \cong \lim_{n \in \mathbb{N}} \frac{A \otimes_K B}{I_2^n}.$$

*Proof.* Observe that

$$\frac{A \otimes_K B}{I_2^n} = \frac{A \otimes_{i,B} B \otimes B}{(J \otimes_B B \otimes_K B + A \otimes_B I_1)^n}$$

via the Lemma A.0.1 and

$$A \otimes_{i,B} \frac{B \otimes_K B}{I_1^n} = \frac{A \otimes_{i,B} B \otimes B}{A \otimes_B I_1^n}.$$

When we write quotient by $\mathcal{I}_1^n := A \otimes_B I_1^n$ and $\mathcal{I}_2^n := (J \otimes_B B \otimes_K B + A \otimes_B I_1)^n$ we mean the quotient of $A \otimes_{i,B} B \otimes B$ by the sub-modules generated by their image. We have to prove that $\{\mathcal{I}_1^n\}$ and $\{\mathcal{I}_2^n\}$ are cofinal; via the explicit description $\mathcal{I}_1^n \subset \mathcal{I}_2^n$. But

$$\mathcal{I}_2^n = \sum_{k=0}^{N-1} J^k \otimes_B \otimes_B I_1^{n-k} \subset A \otimes_B I^{n-(N-1)} = \mathcal{I}_1^{n-(N-1)}$$

for any $n \geqslant N-1$, where we used that $J^N = 0$. □

**Corollary A.0.4.** *With the hypothesis of the previous Lemma, for any $B$-module $M$*

$$\lim_{n \in \mathbb{N}} A \otimes_{i,B} \frac{B \otimes_K B}{I_1^n} \otimes_B M \cong \lim_{n \in \mathbb{N}} \frac{A \otimes_K B}{I_2^n} \otimes_B M.$$

# Appendix B

# Lie algebras and the BGG theory

This Chapter is about Lie algebras. We would like to explain how to build the BGG resolution of some $\mathfrak{g}$-modules, where $\mathfrak{g}$ is a split, finite semisimple Lie algebra.

## B.1 Preliminaries on semisimple Lie algebras

In this Section we will recall some basic constructions and results with the Lie algebras in order to fix some notations. A nice introduction of the subject is given in the first three chapters of the book [Hum72a] or in the book [Hal15]. Let $\mathfrak{g}$ be a finite semisimple split Lie algebra over a field $K$ of characteristic zero. Choose a maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{g}_\alpha$ where $\mathfrak{h}^\vee = \mathrm{Hom}_K(\mathfrak{h}, K)$ and

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Let's denote

$$\Phi := \{\alpha \in \mathfrak{h}^\vee \mid \mathfrak{g}_\alpha \neq 0\} \smallsetminus \{0\}$$

the **root system** of $\mathfrak{g}$ associated to $\mathfrak{h}$. Choose an ordered subset of **simple roots** $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$, then one can define

$$\Phi^\pm := \{\alpha \in \Phi \mid \alpha = \pm \sum_{i=1}^{l} \underline{n}_i \alpha_i,\ \underline{n} \in \mathbb{N}^l\}$$

the sets of **positive** and **negative** roots; one has that $\Phi = \Phi^+ \coprod \Phi^-$. The choice of the simple basis gives some canonical Lie sub-algebras of $\mathfrak{g}$:

$$\mathfrak{n}^+ = \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha, \quad \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}.$$

These sub-spaces are Lie sub-algebras since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. For two elements $\lambda, \mu \in \mathfrak{h}^\vee$ we say that $\lambda \leqslant \mu$ if $\mu - \lambda \in \mathbb{N}\Phi^+ = \{\alpha \in \mathfrak{h}^\vee \mid \alpha = \pm \sum_{i=1}^{l} \underline{n}_i \alpha_i,\ \underline{n} \in \mathbb{N}^l\}$.

The Lie algebra structure of $\mathfrak{g}$ gives automatically a representation that is called the **adjoint representation**:

$$\begin{aligned} \mathrm{ad}_\mathfrak{g} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) := \mathrm{End}_K(\mathfrak{g}) \\ X &\longmapsto (Y \mapsto \mathrm{ad}_\mathfrak{g}(X)(Y) := [X, Y]) \end{aligned}.$$

The **Killing form** is the symmetric $K$-bilinear pairing on $\mathfrak{g}$ defined by

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\longrightarrow K \\ (X,Y) &\longmapsto trace(\mathrm{ad}_{\mathfrak{g}}(X) \circ \mathrm{ad}_{\mathfrak{g}}(Y)) \end{aligned}.$$

The restriction of the Killing form to $\mathfrak{h}$ is non degenerated, in particular it induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^\vee$ and a pairing

$$(-,-) : \mathfrak{h}^\vee \times \mathfrak{h}^\vee \to K.$$

For any $\alpha \in \Phi$, let $s_\alpha \in \mathrm{Aut}_K(\mathfrak{h}^\vee)$ be the reflection

$$s_\alpha(\lambda) := \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha = \lambda - < \lambda, \alpha > \alpha, \qquad \lambda \in \mathfrak{h}^\vee,$$

where $< \lambda, \mu > := 2\frac{(\lambda,\mu)}{(\mu,\mu)} \in K$ for any $\lambda, \mu \in \mathfrak{h}^\vee$. Let $W \leqslant \mathrm{Aut}_K(\mathfrak{h}^\vee)$ be the subgroup generated by all the reflections $s_\alpha$, with $\alpha \in \Phi$. This group is the **Weyl group** of $\mathfrak{g}$; observe that $W$ depends on the subgroup $\mathfrak{h}$ but not on the choice of simple roots $\Delta \subset \Phi$. It is a well known fact that $W$ is generated by the elements $s_\alpha$ with $\alpha \in \Delta$. Each element $w \in W$ of the Weyl group has a **length**:

$$l(w) := \min\{n \in \mathbb{N} \mid w = s_{\alpha_1} \circ \cdots \circ s_{\alpha_n} \text{ with } \alpha_i \in \Phi \text{ for all } 1 \leqslant i \leqslant n\},$$

one sets $l(\mathrm{id}_{\mathfrak{h}^\vee}) := 0$. For any $i \in \mathbb{N}$ we denote $W^{(i)} \subset W$ the subset of elements of length $i$.

Let $V$ be a $K$-vector space, we denote with $\mathfrak{gl}(V) := \mathrm{End}_K(V)$ the Lie algebra with the Lie bracket given by

$$[\varphi, \psi] := \varphi \circ \psi - \psi \circ \varphi.$$

A **$\mathfrak{g}$-representation** is a Lie algebra morphism

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V) := \mathrm{End}_K(V)$$

*i.e.* a $K$-linear morphism s.t. $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. Usually we denote $X.v := \phi(X)(v)$ for a $v \in V$ and $X \in \mathfrak{g}$. For any $\lambda \in \mathfrak{h}^\vee$

$$V_\lambda := \{v \in V \mid H.v = \lambda(H)v\}$$

is the subspace of weight $\lambda$. Observe that the notation is coherent with $\mathfrak{g}_\alpha$ if we consider $\mathfrak{g}$ as $\mathfrak{g}$-module via the adjoint representation. Let

$$\prod V := \{\lambda \in \mathfrak{h}^\vee \mid V_\lambda \neq 0\}$$

be the space of weights of $V$ and the **multiplicity** of $\lambda$ is the $K$-dimension of $V_\lambda$.

## B.2 Universal enveloping algebra

The main goal of this Section is to describe briefly the universal enveloping algebra and the Poincaré–Birkhoff–Witt Theorem, usually cited as PBW Theorem. For a more detailed explanation look at the fifth Chapter of [Hum72a]. The idea is building an associative, unitary (non necessary commutative) algebra $\mathcal{U}(\mathfrak{g})$ generated as algebra by 1 and the elements of $\mathfrak{g}$ and that satisfies the relations $XY - YX = [X,Y]$ for any $X, Y \in \mathfrak{g}$.

For any Lie algebra $\mathfrak{l}$ let
$$\mathcal{U}(\mathfrak{l}) := \mathcal{T}(\mathfrak{l})/J,$$
where $\mathcal{T}(\mathfrak{l}) := \bigoplus_{n\in\mathbb{N}} \mathfrak{l}^{\otimes_K n}$ is the **tensor algebra** of $\mathfrak{l}$ and $J$ is the two-sided ideal generated by the relations

$$X \otimes Y - Y \otimes X - [X,Y]$$

for any $X, Y \in \mathfrak{l}$. Observe that there is a canonical $K$-linear morphism $i : \mathfrak{l} \to \mathcal{U}(\mathfrak{l})$ satisfying

$$i([X,Y]) = i(X) \otimes i(Y) - i(Y) \otimes i(X) + J.$$

The center of the universal enveloping algebra is denoted as $\mathcal{U}(\mathfrak{l}) = \mathcal{Z}(\mathcal{U}(\mathfrak{l}))$. The algebra $\mathcal{U}(\mathfrak{l})$ is "universal" with respect to the following property.

**Proposition B.2.1.** *Universal property of the universal enveloping algebra. For any unitary, associative, not necessary commutative $K$-algebra $\mathcal{A}$ and any $K$-linear morphism $\varphi : \mathfrak{l} \to \mathcal{A}$ such that*

$$\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) \quad \textit{for any } X, Y \in \mathfrak{l};$$

*Then there is a unique algebra morphism $\phi : \mathcal{U}(\mathfrak{l}) \to \mathcal{A}$ such that $\phi \circ i = \varphi$ and $\phi(1) = 1_{\mathcal{A}}$.*

Observe that via the universal property of $\mathcal{U}(\mathfrak{l})$ any $\mathfrak{l}$-representation corresponds uniquely to a left $\mathcal{U}(\mathfrak{l})$-module.

The PBW Theorem allows to understand how the algebra $\mathcal{U}(\mathfrak{l})$ is done if we know a basis of the Lie algebra $\mathfrak{l}$.

**Theorem B.2.2.** *Poincaré–Birkhoff–Witt. If $X_1, \dots, X_n$ is an ordered $K$-basis of $\mathfrak{l}$, then the set*

$$\left\{ x_{j(1)} \otimes x_{j(2)} \otimes \ldots \otimes x_{j(k)} + J \in \mathcal{U}(\mathfrak{l}) \mid k \in \mathbb{N},\ j(1) \leqslant j(2) \leqslant \cdots \leqslant j(k) \right\}$$

*is a $K$-basis of $\mathcal{U}(\mathfrak{l})$.*

A nice corollary is that if $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, where $\mathfrak{l}_1, \mathfrak{l}_2$ are Lie sub-algebras of $\mathfrak{l}$, then $\mathcal{U}(\mathfrak{l}) = \mathcal{U}(\mathfrak{l}_1)\ \mathcal{U}(\mathfrak{l}_2)$. In particular if one has the decomposition of a semisimple Lie algebra $\mathfrak{g}$ as in the previous Section $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, then
$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\ \mathcal{U}(\mathfrak{h})\ \mathcal{U}(\mathfrak{n}).$$
Moreover observe that $\mathfrak{h}$ is a commutative Lie algebra, *i.e.* if $[\mathfrak{h}, \mathfrak{h}] = \{0\}$, then $\mathcal{U}(\mathfrak{h}) = \mathrm{Sym}^{\bullet}_K\, \mathfrak{h}$.

## B.3 Verma modules and the $\mathcal{O}$–category

The main goal of this Section is to explain how to build the BGG resolution of some $\mathfrak{g}$–modules built in [BGG75]. In order to do this we investigate some basic properties of the highest weight modules, Verma modules and in general of the $\mathfrak{g}$-modules in the $\mathcal{O}$-category. We follow the exposition of the book [Hum72b] presenting only the results that we need.

### B.3.1 The $\mathcal{O}$-category

Let $\mathfrak{g}$ be a finite semisimple split Lie algebra, we use the notation as in the Section §B.1.

**Definition B.3.1.** An $\mathcal{U}(\mathfrak{g})$–module $V$ is in $\mathcal{O}$ if

**($\mathcal{O}$1):** $V$ is a f.g. $\mathcal{U}(\mathfrak{g})$–module;

**($\mathcal{O}$2):** $V$ is $\mathfrak{h}$–split: $V = \oplus_{\lambda \in \mathfrak{h}^\vee} V_\lambda$;

**($\mathcal{O}$3):** For each $v \in V$, the $K$-vector space $\mathcal{U}(\mathfrak{n}).v$ has finite dimension over $K$.

**Lemma B.3.2.** *If $V \in \mathcal{O}$, then $V$ is a noetherian $\mathcal{U}(\mathfrak{g})$-module, it is finitely generated as $\mathcal{U}(\mathfrak{n}^-)$–module and it satisfies:*

*($\mathcal{O}$4):* $\dim_K V_\lambda$ *is finite for each* $\lambda \in \mathfrak{h}^\vee$*;*

*($\mathcal{O}$5):* $\prod V \subset \bigcup_{\lambda \in S} \lambda - \mathbb{N}\Phi^+$*, where $S$ is a finite subset of $\mathfrak{h}^\vee$.*

*Proof.* Since $V$ is finitely generated as $\mathcal{U}(\mathfrak{g})$–module and $\mathcal{U}(\mathfrak{g})$ is noetherian, then $V$ is noetherian. Moreover, since $V$ is $\mathfrak{h}$-split, one can choose a finite set of $\mathcal{U}(\mathfrak{g})$–generators $\{v_i\}_{i \in I}$ s.t. $v_i$ has weight $\lambda_i \in \mathfrak{h}^\vee$. In addiction the space $\mathcal{U}(\mathfrak{n}).v_i$ is finite dimensional for any $i \in I$, then by PBW $V$ is finitely generated also as $\mathcal{U}(\mathfrak{n}^-)$–module. Let $\{v_j\}_{j \in J}$ be a finite set of generators of $V$ as $\mathcal{U}(\mathfrak{n}^-)$–module s.t. $v_j$ has weight $\lambda_j$. If $Y_1, \ldots, Y_s$ is a $K$-basis of $\mathfrak{n}^-$ of weight $\alpha_1, \ldots, \alpha_s$, then $V$ is generated over $K$ by the vectors $Y_1^{n_1}.Y_2^{n_2}.\ldots.Y_s^{n_s}.v_j$ that have weight $\lambda_j - n_1\alpha_1 - n_2\alpha_2 - \cdots - n_s\alpha_s$ with $n_1, n_2, \ldots, n_s \in \mathbb{N}$. Hence

$$\prod V \subset \bigcup_{j \in J} \lambda_j - \mathbb{N}\Phi^+$$

and there are only finitely many polynomials that bring a vector $v_j$ in a specific weight. □

Observe that the highest $\lambda$–weight module generated by a vector is in the $\mathcal{O}$–category.

**Proposition B.3.3.** *Let $V \in \mathcal{O}$; then for any $v \in V$ the subspace $\mathcal{Z}(\mathfrak{g}).v$ is $K$-finite dimensional. Moreover the category $\mathcal{O}$ is abelian, closed under finite direct sums, submodules and quotients. For any $\mathcal{U}(\mathfrak{g})$–module $L$ that is $K$-finite, the exact functor*

$$L \otimes_K - : \mathcal{O} \to \mathcal{O}$$

*is well defined.*

*Proof.* Look at the proof of the Theorem 1.1 in [Hum72b]. □

**Definition B.3.4.** Let $\lambda \in \mathfrak{h}^\vee$. A $\mathcal{U}(\mathfrak{g})$–module $V \neq 0$ is a **highest $\lambda$–weight module** generated by a vector $v^+$ if:

- $\mathcal{U}(\mathfrak{g}).v^+ = V$;
- $v^+ \in V_\lambda$;
- $\mu \in \prod V$, then $\mu \leqslant \lambda$.

**Proposition B.3.5.** *Any highest $\lambda$–weight module $V$ is in $\mathcal{O}$. Moreover for any $V \in \mathcal{O}$ there is a filtration*

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

*where $V_i/V_{i+1}$ is a highest weight module.*

*Proof.* For the proof we refer to [Hum72b], §1.2. □

### B.3.2 Verma modules

Fix a weight $\lambda \in \mathfrak{h}^\vee$. Let $J(\lambda) \subset \mathcal{U}(\mathfrak{g})$ be the left ideal

$$J(\lambda) := \mathcal{U}(\mathfrak{g})(H - \lambda(H)1)_{H\in\mathfrak{h}} + \mathcal{U}(\mathfrak{g})\,\mathcal{U}(\mathfrak{n}).$$

The **Verma module** of weight $\lambda$ is the $\mathcal{U}(\mathfrak{g})$-module $M(\lambda) := \mathcal{U}(\mathfrak{g})/J(\lambda)$. Let $v_\lambda^+ \in M(\lambda)$ be the class of the element $1 \in \mathcal{U}(\mathfrak{g})$.

Observe that

$$H.v_\lambda^+ = (H - \lambda(H)1)v_\lambda^+ + \lambda(H)v_\lambda^+ = \lambda(H)v_\lambda^+, \quad X.v_\lambda^+ = 0.$$

Hence $v_\lambda^+$ is an element of weight $\lambda$ and, via the PBW Theorem, the module $M(\lambda)$ is $K$–generated by elements $Y.v_\lambda^+$, with $Y \in \mathcal{U}(\mathfrak{n}^-)$. Then

$$\prod M(\lambda) \subset \lambda - \mathbb{N}\Phi^+$$

and $M(\lambda)$ has only weights $\leqslant \lambda$. Hence $M(\lambda)$ is a highest $\lambda$–weight module generated by the vector $v_\lambda^+$, in particular by the Proposition B.3.5 $M(\lambda) \in \mathcal{O}$. The Verma module $M(\lambda)$ has a unique maximal sub-module $N(\lambda) \subset M(\lambda)$ and a unique simple quotient $L(\lambda) = M(\lambda)/N(\lambda)$. Moreover the Verma module is a "universal" highest $\lambda$–weight module in the following sense:

**Lemma B.3.6.** *If $V$ is a highest $\lambda$–weight module generated by $v^+$, then $V$ is a quotient of $M(\lambda)$.*

*Proof.* Let $\varphi : \mathcal{U}(\mathfrak{g}) \to V$ be the $\mathcal{U}(\mathfrak{g})$–linear map sending 1 to $v^+$. This map is surjective since $V$ is $\mathcal{U}(\mathfrak{g})$–generated by $v^+$. Moreover the $\mathcal{U}(\mathfrak{g})$-generators of $J(\lambda)$ are sent to 0:

$$\varphi(H - \lambda(H)1) = H.v^+ - \lambda(H)v^+ = 0, \quad \varphi(X) = X.v^+ = 0.$$

Hence $\varphi(J(\lambda)) = 0$ and there is a surjective $\mathcal{U}(\mathfrak{g})$–linear morphism $M(\lambda) \to V$. □

**Proposition B.3.7.** *If $V \in \mathcal{O}$ is simple, then $V \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$. Moreover for any $\lambda, \mu \in \mathfrak{h}^\vee$*

$$\dim_K \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(L(\mu), L(\lambda)) = \delta_\mu^\lambda.$$

*The simple module $L(\lambda)$ is finite $K$-dimensional if and only if $<\lambda, \alpha> \in \mathbb{N}$ for all $\alpha \in \Phi^+$. This is the case if and only if*

$$\dim_K L(\lambda)_\mu = \dim_K L(\lambda)_{w\mu} \quad \text{for all} \quad \mu \in \mathfrak{h}^\vee, \quad w \in W.$$

*Proof.* See Theorem §1.3 and §1.6 in [Hum72b]. □

Let $V$ be any finite $K$-dimensional $\mathfrak{g}$–module $V$ that splits over $K$, *i.e.* $V = \oplus_{\lambda\in\mathfrak{h}^\vee} V_\lambda$; then $V \in \mathcal{O}$. Moreover up to extending the scalars all the finite dimensional $\mathfrak{g}$–modules split, then this Proposition gives a way in order to characterize all the finite dimensional irreducible representations of $\mathfrak{g}$.

Another way in order to build the Verma module $M(\lambda)$ is the following. Let $K_\lambda$ be the 1–dimensional representation of $\mathfrak{b}$ given by the character $\lambda$:

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \to \mathfrak{h} \xrightarrow{\lambda} K = \mathfrak{gl}(K).$$

Then $K_\lambda$ is a $\mathcal{U}(\mathfrak{b})$–module and one can define

$$M'(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} K_\lambda.$$

It is easy to verify that also $M'(\lambda)$ satisfies the universal property B.3.6, hence $M'(\lambda) \cong M(\lambda)$.

### B.3.3 The central character

Let $V$ be a highest $\lambda$-weight $\mathcal{U}(\mathfrak{g})$–module generated by $v^+ \in V_\lambda$. Let $Z \in \mathcal{Z}(\mathfrak{g})$, then for any $H \in \mathfrak{h}$

$$H.Z.v^+ = Z.H.v^+ = \lambda(H)Z.v^+.$$

Hence $Z.v^+ \in V_\lambda$, but $\dim_K V_\lambda = 1$, then there is $\chi_\lambda(Z) \in K$ s.t. $Z.v^+ =: \chi_\lambda(Z)v^+$. Observe that the map

$$\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to K$$

is a $K$-algebra morphism. Since each highest weight module is a quotient of the Verma module, one gets that the character $\chi_\lambda$ actually does depend only on $\lambda$ and not on the representation $V$. Moreover since each highest weight vector is $\mathcal{U}(\mathfrak{g})$-generated by a single vector $v^+$, we get that the center $\mathcal{Z}(\mathfrak{g})$ acts via $\chi_\lambda$ on the whole space, *i.e.* for each $v \in V$ there is $X \in \mathcal{U}(\mathfrak{g})$ s.t. $v = X.v^+$ and

$$Z.v = Z.X.v^+ = X.Z.v^+ = X.\chi_\lambda(Z)v_+ = \chi_\lambda(Z)v.$$

One could ask when $\chi_\lambda = \chi_\mu$ for some $\lambda, \mu \in \mathfrak{h}^\vee$. The **dot action** of the Weyl group gives us a partial answer. Let $w \in W$,

$$w \cdot \mu := w(\mu + \rho) - \rho \quad \text{and} \quad \rho := \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha.$$

Two weights $\lambda, \mu \in \mathfrak{h}^\vee$ are said to be **linked** if $w \cdot \lambda = \mu$ for some $w \in W$

**Lemma B.3.8.** *$\lambda, \mu \in \mathfrak{h}^\vee$ are linked if and only if $\chi_\lambda = \chi_\mu$.*

*Proof.* Look at Theorem §1.10 in [Hum72b] or Proposition 8.5 in [BGG75]. □

The idea now is that any $V \in \mathcal{O}$ could be written as $V = \oplus_{\lambda \in \mathfrak{h}^\vee} V^{\chi_\lambda}$ where $V^{\chi_\lambda}$ is the generalized eigenspace of $\chi_\lambda$. More precisely:

**Definition B.3.9.** For any $\mathcal{U}(\mathfrak{g})$–module $V$ and for any $K$-algebra morphism $\chi : \mathcal{Z}(\mathfrak{g}) \to K$,

$$V^\chi := \{v \in V \mid \text{ for each } Z \in \mathcal{Z}(\mathfrak{g}),\ \exists\ n \in \mathbb{N}\ s.t.\ (Z - \chi(Z))^n.v = 0\}.$$

Let $\mathcal{O}_\chi$ be the full sub-category of $\mathcal{O}$ of all the objects $V \in \mathcal{O}$ s.t. $V = V^\chi$.

**Proposition B.3.10.** *$\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$, where $\chi$ varies in $\mathrm{Hom}_{K-Alg}(\mathcal{Z}(\mathfrak{g}), K)$. Moreover in the coproduct there are only the $\chi$ of the form $\chi_\lambda$ for some $\lambda \in \mathfrak{h}^\vee$.*

*For any $\lambda \in \mathfrak{h}^\vee$ there is an exact projection functor*

$$\pi_{\chi_\lambda} : V \in \mathcal{O} \mapsto V^{\chi_\lambda} \in \mathcal{O}_{\chi_\lambda}.$$

*Proof.* See Proposition §1.12 in [Hum72b] or Proposition 8.5 in [BGG75]. □

Observe that any highest $\lambda$-weight module $V$ is in $\mathcal{O}_{\chi_\lambda}$.

## B.4 The BGG resolution

Fix $\lambda \in \mathfrak{h}^\vee$ s.t. $< \lambda, \alpha > \in \mathbb{N}$ for each $\alpha \in \Phi^+$. In this Section we will describe the BGG resolution of the $\mathcal{U}(\mathfrak{g})$–module $L(\lambda)$. We skip the proofs that one can find in [BGG75] or in [Hum72b].

The BGG resolution is an exact sequence

$$0 \to C_m \xrightarrow{\delta_m} C_{m-1} \dots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \tag{BGG}$$

with $\mathrm{Coker}(\delta_1) = L(\lambda)$, $C_k = \bigoplus_{w \in W^{(k)}} M(w \cdot \lambda)$ and $m = \dim_K \mathfrak{n}$.

The importance of this resolution is that each $C_k$ is a free $\mathcal{U}(\mathfrak{n}^-)$–module, hence it is a projective resolution of $L(\lambda)$ as $\mathcal{U}(\mathfrak{g}/\mathfrak{b})$–modules.

### B.4.1 Case $\lambda \equiv 0$

Let $k \in \mathbb{N}$, we consider the $\mathfrak{b}$-module $\bigwedge_K^k \mathfrak{g}/\mathfrak{b}$, where the action on the elementary tensors is given by the Leibnitz rule and the adjoint action:

$$X.(X_1 \wedge X_2 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} X_1 \wedge \dots \wedge X_{i-1} \wedge [X, X_i] \wedge X_{i+1} \dots \wedge X_k.$$

Observe that if $X, X_i \in \mathfrak{b}$, then $[X, X_i] \in \mathfrak{b}$, then the action is well defined on the quotient. Observe that if $Y_1, \dots, Y_m$ is a basis of $\mathfrak{n}^- \cong \mathfrak{g}/\mathfrak{b}$ with $Y_i \in \mathfrak{g}_{-\alpha_i}$, then $\{Y_{\underline{i}_1} \wedge \dots \wedge Y_{\underline{i}_k}\}_{1 \leqslant \underline{i}_1 < \underline{i}_2 < \dots < \underline{i}_k \leqslant m}$ gives a basis of $\bigwedge_K^k \mathfrak{g}/\mathfrak{b}$ and the element $Y_{\underline{i}_1} \wedge \dots \wedge Y_{\underline{i}_k}$ has weight $-\sum_{j=1}^k \alpha_{\underline{i}_j}$. Then $\bigwedge_K^k \mathfrak{g}/\mathfrak{b}$ is $\mathfrak{h}$-split with weights that are the sum of distinct roots in $\Phi^-$. If $k = m$, then $\bigwedge_K^m \mathfrak{g}/\mathfrak{b}$ is one-dimensional and the only weight is $-2\rho$, where

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

is the **special weight**. Let $D_k := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \bigwedge_K^k \mathfrak{g}/\mathfrak{b}$, then

$$D_m \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} K_{-2\rho} \cong M(-2\rho); \quad D_0 = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} K_0 \cong M(0).$$

The weak BGG resolution in the $\lambda \equiv 0$ case is given by the maps (see §6.3 of [Hum72b] or §9 in [BGG75]):

$$\begin{aligned} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \textstyle\bigwedge_K^k \mathfrak{g}/\mathfrak{b} &\xrightarrow{\quad\partial_k\quad} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \textstyle\bigwedge_K^{k-1} \mathfrak{g}/\mathfrak{b} \\ X \otimes X_1 \wedge \dots \wedge X_k &\longmapsto \textstyle\sum_{i=1}^k (-1)^{k+1} X \otimes X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots X_k \\ &\quad - \textstyle\sum_{1 \leqslant i < j \leqslant k} (-1)^{i-j} X \otimes [X_i, X_j] \wedge X_1 \wedge \dots \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_k \end{aligned}$$

and $\epsilon : D(0) \cong M(0) \to L(0)$. As usual the hat in a product indicates that the term is omitted.

**Theorem B.4.1.** *The sequence*

$$0 \to D_m \xrightarrow{\partial_m} D_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_1} D_0 \tag{$\star$}$$

*is an exact sequence of $\mathcal{U}(\mathfrak{g})$–modules with* $\mathrm{Coker}(\partial_1) \cong L(0)$ *via* $\epsilon$.

*Proof.* See Theorem 9.1 in [BGG75] setting $\mathfrak{a} = \mathfrak{g}$ and $\mathfrak{p} = \mathfrak{b}$. □

The weak BGG resolution for $\lambda \equiv 0$ is also called **Koszul complex**. It is larger than (*BGG*), in order to obtain a smaller resolution one has use the fact that cutting by the central character is an exact functor that preserves $L(0) \in \mathcal{O}_{\chi_0}$.

**Definition B.4.2.** The **BGG resolution** of $L(0)$ is $\pi_{\chi_0}(\star)$:

$$0 \to D_m^0 \xrightarrow{\delta_m^0} D_{m-1}^0 \xrightarrow{\delta_{m-1}^0} \dots \xrightarrow{\delta_1^0} D_0^0 \tag{0-BGG}$$

where $\delta_k^0 = \pi_{\chi_0}\partial_k$ and $D_k^0 := D_k^{\chi_0} = \pi_{\chi_0} D_k$.

**Theorem B.4.3.** *There is an isomorphism of $\mathcal{U}(\mathfrak{g})$–modules*

$$\bigoplus_{w \in W^{(k)}} M(w \cdot 0) \cong D_k^0$$

*and the resolution (0-BGG) is of the form (BGG) for $\lambda \equiv 0$.*

### B.4.2 General case

Let $\lambda \in \mathfrak{h}^\vee$ s.t. $< \lambda, \alpha > \in \mathbb{N}$ for each $\alpha \in \Phi^+$. There are two possible ways in order to build the resolution with respect to the weight $\lambda$, for more details see §6.2 in [Hum72a]. We present only one of these two ways.

The weak $\lambda$–BGG resolution is (0-BGG)$\otimes_K L(\lambda)$ and the $\lambda$–BGG resolution is defined as the projection on $\mathcal{O}_{\chi_\lambda}$ of the weak $\lambda$–BGG resolution. In formulas let

$$C_k^\lambda := \left(D_k^0 \otimes_K L(\lambda)\right)^{\chi_\lambda}, \quad \delta_k^\lambda := \pi_{\chi_\lambda}\left(\delta_k^0 \otimes_K \mathrm{id}_{L(\lambda)}\right),$$

then the resolution

$$0 \to C_k^\lambda \overset{\delta_m^\lambda}{\to} C_{m-1}^\lambda \overset{\delta_{m-1}^\lambda}{\to} \dots \overset{\delta_1^\lambda}{\to} C_0^\lambda \tag{$\lambda$-BGG}$$

is the claimed (BGG):

**Theorem B.4.4.** *There is an isomorphism of $\mathcal{U}(\mathfrak{g})$–modules*

$$\bigoplus_{w \in W^{(k)}} M(w \cdot \lambda) \cong C_k^\lambda$$

*and the resolution ($\lambda$-BGG) is of the form (BGG).*

*Proof.* See Corollary §6.5 in [Hum72b] or Theorem 10.1 in [BGG75]. □

### B.4.3 The Lepowsky–Garland Theorem

For a $\mathfrak{b}$–module $M$ there is a little modification of the Koszul complex

$$\left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \bigwedge_S^{\bullet}(\mathfrak{g}/\mathfrak{b})\right) \otimes_S M \to M.$$

Indeed one can consider the complex

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \left(M \otimes_S \bigwedge_S^{\bullet}(\mathfrak{g}/\mathfrak{b})\right) \to M$$

where the morphisms are defined using the $\mathcal{U}(\mathfrak{b})$–module structure of $M$:

$$X \otimes Y \otimes X_1 \wedge \dots \wedge X_j \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \left(M \otimes_S \bigwedge_S^{j}(\mathfrak{g}/\mathfrak{b})\right) \mapsto$$
$$\sum_{i=i}^{k} (-1)^k \left(X_i.X \otimes Y - X \otimes X_i.Y\right) \otimes \left(X_1 \wedge \dots \wedge X_{i-1} \wedge X_{i+1} \wedge \dots \wedge X_j\right) \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \left(M \otimes_S \bigwedge_S^{j-1}(\mathfrak{g}/\mathfrak{b})\right).$$

Lepowsky and Garland in the Proposition 1.7 of [GL76] give an isomorphism $\left(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \wedge_S^j(\mathfrak{g}/\mathfrak{b})\right) \otimes_S M \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \left(M \otimes_S \wedge_S^j(\mathfrak{g}/\mathfrak{b})\right)$ and in Proposition 1.9 they say that these isomorphisms induces an isomorphism between the exact sequences

$$\left(\mathcal{U}(\mathfrak{n}^-)\otimes_S \bigwedge_S^{\bullet}(\mathfrak{g}/\mathfrak{b})\otimes_S M \to M\right)\cong\left(\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\left(M\otimes_S \bigwedge_S^{\bullet}(\mathfrak{g}/\mathfrak{b})\right)\to M\right).$$

## B.5 The Koszul and de Rham complexes

Let $G$ be a reductive semisimple affine group over scheme over $K$ and let $\mathfrak{g} := \operatorname{Lie} G$. Choose a maximal torus $T$ and a Borel $B$ in $G$, let $U = U^+ \leqslant B$ be the unipotent radical. This defines also $\mathfrak{h} := \operatorname{Lie} T$, $\mathfrak{b} := \operatorname{Lie} B$ and $\mathfrak{n} := \operatorname{Lie} U$.

Let's consider the complex ($\star$), it is isomorphic as $\mathcal{U}(\mathfrak{g}/\mathfrak{b})$-modules to

$$\mathcal{U}(\mathfrak{g}/\mathfrak{b}) \longleftarrow \mathcal{U}(\mathfrak{g}/\mathfrak{b})\otimes_K \mathfrak{g}/\mathfrak{b} \longleftarrow \mathcal{U}(\mathfrak{g}/\mathfrak{b})\otimes_K \wedge_K^2 \mathfrak{g}/\mathfrak{b}\ldots \longleftarrow \mathcal{U}(\mathfrak{g}/\mathfrak{b})\otimes_K \wedge_K^m \mathfrak{g}/\mathfrak{b} \longleftarrow 0$$

where $m = \dim_K \mathfrak{g}/\mathfrak{b}$. The isomorphism is a consequence of the PBW Theorem and the fact that we do not care about the $\mathfrak{b}$–action. Observe that if $Y_1, \ldots, Y_m$ is a basis of $\mathfrak{g}/\mathfrak{b}$, then $\mathcal{U}(\mathfrak{g}/\mathfrak{b}) = \oplus_{\underline{n}\in\mathbb{N}^m} K\,\underline{Y}^{\underline{n}}$ and

$$\operatorname{Hom}_K\left(\left(\mathcal{U}(\mathfrak{g}/\mathfrak{b})\otimes\bigwedge_K^k \mathfrak{g}/\mathfrak{b}\right),\mathcal{O}_{G/B}\right)\cong \mathcal{O}_{G/B}[[\underline{Y^\vee}]]\otimes_K \bigwedge_K^k(\mathfrak{g}/\mathfrak{b})^\vee = \mathcal{O}_{G/B}[[\Omega^1_{(G/B)/K}]]\otimes_{\mathcal{O}_{G/B}}\Omega^k_{(G/B)/K}.$$

The dual complex $\operatorname{Hom}_K((\star),\mathcal{O}_{G/B})$, as written in §9 Remark 2 of [BGG75], is the linearized de Rham complex for $\mathcal{O}_{G/B}$:

$$\begin{array}{ccc}\mathcal{O}_{G/B}[[\Omega^1_{(G/B)/K}]]\hat{\otimes}_{\mathcal{O}_{G/B}}\Omega^k_{(G/B)/K} & \longrightarrow & \mathcal{O}_{G/B}[[\Omega^1_{(G/B)/K}]]\hat{\otimes}_{\mathcal{O}_{G/B}}\Omega^{k+1}_{(G/B)/K} \\ f\otimes Y_{i_1}^\vee\wedge\cdots\wedge Y_{i_k}^\vee & \longmapsto & \sum_{j=1}^m \frac{\partial f}{\partial Y_j^\vee}\otimes Y_j^\vee\wedge Y_{i_1}^\vee\wedge\cdots\wedge Y_{i_k}^\vee\end{array}.$$

Let $\pi : G/B \to \operatorname{Spec}(K)$, and $W$ be a $K$-finite dimensional irreducible $\mathfrak{g}$–representation, then the dual $W^\vee := \operatorname{Hom}_K(W, K)$ is an irreducible $K$–finite dimensional $\mathfrak{g}$–representation. Let $\lambda$ be the maximal weight of $W^\vee$. Then we can consider the exact sequences

$$(\lambda\text{-BGG}) \subset (0\text{-BGG})\otimes_K W^\vee \subset (\star)\otimes_K W^\vee.$$

All these sequences are quasi-isomorphic and compute the cohomology of $W^\vee$ as $\mathfrak{g}/\mathfrak{b}$–module. Now we can dualize these complexes, applying $\operatorname{Hom}_K\left(-,\mathcal{O}_{G/B}\right)$ getting the following sequences of $\mathcal{O}_{G/B}$-modules

$$L(\Omega^\bullet_{(G/B)/K})\otimes_K W = L(\Omega^\bullet_{(G/B)/K})\otimes_{\mathcal{O}_{G/B}}\mathcal{W}\longrightarrow \operatorname{Hom}_K\left((0\text{-BGG})\otimes_K W^\vee,\mathcal{O}_{G/B}\right)\to \operatorname{Hom}_K\left((\lambda\text{-BGG})\otimes_K W^\vee,\mathcal{O}_{G/B}\right)$$

where $\mathcal{W} := \pi^* W$. The strategy in Chapter 2 is to recover information of the linearized de Rham complex $L\left(\mathcal{W}\otimes_{\mathcal{O}_{G/B}}\Omega^\bullet_{(G/B)/K}\right)$ via the dual BGG resolution $\operatorname{Hom}_K\left((\lambda\text{-BGG})\otimes_K W^\vee,\mathcal{O}_{G/B}\right)$.

# Appendix C

# Sheaf with marked sections

## C.1 The construction

In this section we would like to resume the vector bundle with marked section (VBMS) construction presented in [AI21]. We will work with an adic setting, instead of the formal schemes point of view in the original construction. The two constructions are compatible.

Let $\mathcal{X}$ be an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ with $\mathcal{I} \subset \mathcal{O}^+_{\mathcal{X}}$ an ideal of definition and $n \in \mathbb{N}$ such that $\mathcal{I} \cap \mathbb{Z}_p = p^n\mathbb{Z}_p$. Let $\mathcal{E}^+$ be a locally free $\mathcal{O}^+_{\mathcal{X}}$-module of rank $1 \leqslant r \leqslant 2$. Let $\mathrm{Spa}/\mathcal{X}$ be the category of adic spaces $\gamma : \mathcal{Z} \to \mathcal{X}$ over $\mathcal{X}$ s.t. $\gamma^*\mathcal{I} \subset \mathcal{O}^+_{\mathcal{Z}}$ is an invertible ideal (locally free and locally generated by one element). We suppose that $\mathcal{X}$ has a formal model $\mathfrak{X}$ and the sheaf $\mathcal{E}^+$ is the analytification of a sheaf $\mathfrak{E}$.

Let $\mathcal{E} := \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}^+_{\mathcal{X}}} \mathcal{E}^+$, we associate to the pair $(\mathcal{E}, \mathcal{E}^+)$ the following controvariant functor

$$\begin{aligned} \mathbb{V}(\mathcal{E}) : (\mathrm{Spa}/\mathcal{X}) &\longrightarrow Ab \\ (\gamma : \mathcal{Z} \to \mathcal{X}) &\longmapsto Hom_{\mathcal{O}^+_{\mathcal{Z}}}(\gamma^*\mathcal{E}^+, \mathcal{O}^+_{\mathcal{Z}}) = H^0(\mathcal{Z}, (\gamma^*\mathcal{E}^+)^\vee) \end{aligned}.$$

**Lemma C.1.1.** *The functor $\mathbb{V}(\mathcal{E})$ is representable by an adic space $\mathcal{X}$: the* ***formal vector bundle*** *associated to $(\mathcal{E}, \mathcal{E}^+)$.*

*Proof.* The space that represents the functor $\mathbb{V}(\mathcal{E})$ is

$$\mathrm{Spa}\left(\widehat{Sym}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}), \widehat{Sym}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}^+)\right) \to \mathcal{X},$$

where the hat means that we have to complete by the topologies induced by the $\mathcal{I}$-adic topology. Indeed

$$\begin{aligned} \mathbb{V}(\mathcal{E})(\gamma) &= Hom_{\mathcal{O}_{\mathcal{Z}}}\left(\gamma^*\mathcal{E}^+, \mathcal{O}_{\mathcal{Z}}\right) = Hom_{\mathcal{O}_{\mathcal{X}}}\left(\mathcal{E}^+, \gamma_*\mathcal{O}_{\mathcal{Z}}\right) \\ &= Hom_{\mathcal{O}_{\mathcal{X}}}\left(Sym_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}^+), \gamma_*\mathcal{O}_{\mathcal{Z}}\right) = Hom_{\mathrm{Spa}/\mathcal{X}}\left(\mathcal{Z}, \mathrm{Spa}\left(\widehat{Sym}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}), \widehat{Sym}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}^+)\right)\right). \end{aligned}$$

□

Observe that we can cover $\mathcal{X}$ with admissible opens $\mathcal{U} \subset \mathcal{X}$, where $\mathcal{E}^+_{|\mathcal{U}} = \bigoplus_{i=1}^r \mathcal{O}_{\mathcal{U}} t_i$ and

$$\mathbb{V}(\mathcal{E})_{|\mathcal{U}} = \mathrm{Spa}\left(\mathcal{O}_{\mathcal{U}} << X_i \mid 1 \leqslant i \leqslant r >>, \mathcal{O}^+_{\mathcal{U}} << X_i \mid 1 \leqslant i \leqslant r >>\right).$$

We denote $f_{\mathcal{E}}$ the morphism $f_{\mathcal{E}} : \mathbb{V}(\mathcal{E}) \to \mathcal{X}$

**Definition C.1.2.** A section $s \in H^0(\mathcal{X}, \mathcal{E}^+/\mathcal{I}\mathcal{E}^+)$ is a **marked section** of $\mathcal{E}^+$ if locally on $\mathcal{X}$ this set could be completed to an $\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}$-basis of $\mathcal{E}^+/\mathcal{I}\mathcal{E}^+$.

Observe that if $r = 1$ a marked section is simply a basis of $\mathcal{E}^+/\mathcal{I}\mathcal{E}^+$ and if $r = 2$ a section $s \in H^0(\mathcal{X}, \mathcal{E}^+/\mathcal{I}\mathcal{E}^+)$ is a marked section if and only if the cokernel of the inclusion

$$\bigoplus_{i=1}^{k} \mathcal{O}_{\mathcal{X}}/\mathcal{I}^+ s_i \to \bigoplus_{i=1}^{k} \mathcal{E}^+/\mathcal{I}\mathcal{E}^+$$

is a locally free $\mathcal{O}_X^+/\mathcal{I}\mathcal{O}_X^+$-module. Let $(\mathcal{E}^+, s)$ be a locally free $\mathcal{O}_{\mathcal{X}}^+$-module of rank $r$ with a marked section. Following the Definition 2.3 in [AI21] we define the following functor:

$$\begin{aligned} \mathbb{V}_0(\mathcal{E}, s) : (\mathrm{Spa}/\mathcal{X}) &\longrightarrow Ab \\ (\gamma : \mathcal{Z} \to \mathcal{X}) &\longmapsto \{\rho \in \mathbb{V}(\mathcal{E})(\gamma) \mid \tilde{\rho}(\gamma^* s) = 1\} \end{aligned},$$

where $\tilde{\rho} := \rho \ mod\ \gamma^*\mathcal{I} :\ \gamma^*\mathcal{E}^+/\gamma^*\mathcal{I}\gamma^*\mathcal{E}^+ \to \mathcal{O}_{\mathcal{Z}}/\gamma^*\mathcal{I}\ \mathcal{O}_{\mathcal{Z}}$. Then $\mathbb{V}_0(\mathcal{E}, s)$ is a sub-functor of $\mathbb{V}(\mathcal{E})$.

**Lemma C.1.3.** *The functor $\mathbb{V}_0(\mathcal{E}, s)$ is representable by an adic space over $\mathcal{X}$.*

*Proof.* One can proceed as in Lemma 2.4 in [AI21], consider the ideal $\bar{J} \subset \widehat{\mathrm{Sym}}_{\mathfrak{O}_X/\mathcal{I}}(\mathfrak{E}/\mathcal{I}\mathfrak{E})$ generated by the element $s - 1$. Let $J \subset \widehat{\mathrm{Sym}}_{\mathfrak{O}_X}(\mathfrak{E})$ the pre-image of $\bar{J}$. Then the functor (defined in the same way, but on formal schemes, see loc. cit.) $\mathbb{V}_0(\mathfrak{E}, s)$ is represented by a sub-variety of the blow up in $\mathbb{V}(\mathfrak{E})$ with respect to the ideal $J$. The functor $\mathbb{V}(\mathcal{E}, s)$ is represented by the analytification of $\mathbb{V}_0(\mathfrak{E}, s)$.

Locally if $\mathcal{E}^+_{|\mathcal{U}} = \bigoplus_{i=1}^r \mathcal{O}_{\mathcal{U}} t_i$ is a trivialization of $\mathcal{E}^+$ on $\mathcal{U} \subset \mathcal{X}$ with $s = t_1\ mod\ \mathcal{I}$ and $\mathcal{I}_{|\mathcal{U}} = \alpha\mathcal{O}_{\mathcal{U}}$ is freely generated; then

$$\begin{cases} \mathrm{Spa}(\mathcal{O}_{\mathcal{U}} << Z, X_2 >>, \mathcal{O}_{\mathcal{U}}^+ << Z, X_2 >>) \to \mathcal{U} & \text{if } r = 2; \\ \mathrm{Spa}(\mathcal{O}_{\mathcal{U}} << Z >>, \mathcal{O}_{\mathcal{U}}^+ << Z >>) \to \mathcal{U} & \text{if } r = 1 \end{cases}$$

represents the functor $\mathbb{V}_0(\mathcal{E}, s)_{|\mathcal{U}}$, the map $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_k)_{|\mathcal{U}} \subset \mathbb{V}(\mathcal{E})_{|\mathcal{U}}$ corresponds to the map

$$X_i \mapsto \begin{cases} 1 + \alpha Z & 1 \leqslant i = 1 \\ X_2 & i = r = 2 \end{cases}.$$

□

## C.2 Action of the torus on VBMS

We denote with $\Lambda := \mathbb{Z}_p[[Z_p^\times]]$ the Iwasawa algebra and with $\mathcal{W} := \mathrm{Spa}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda, \Lambda)$ the weight space defined over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let

$$\kappa^{\mathrm{univ}} : a \in \mathbb{Z}_p^\times \mapsto [a] \in \Lambda^\times = \mathcal{O}_{\mathcal{W}}^+(\mathcal{W})$$

be the universal weight. Suppose there is a morphism $\mathcal{X} \to \mathcal{W}$ of adic analytic spaces s.t. the restriction at $1 + p^n\mathbb{Z}_p = (1 + \mathcal{I}) \cap \mathbb{Z}_p^\times$ of the composition

$$\mathbb{Z}_p^\times \xrightarrow{k^{\mathrm{univ}}} \mathcal{O}_{\mathcal{W}}^+(\mathcal{W}) \longrightarrow \mathcal{O}_{\mathcal{X}}^+(\mathcal{X})$$

is an analytic function. Via analycity we can extend this composition to an analytic function

$$\kappa : 1 + \mathcal{I} \to \mathcal{O}_{\mathcal{X}}^+(\mathcal{X}).$$

Moreover for any map of adic space $\mathcal{Z} \to \mathcal{X}$ one can extend $\kappa$ to the ideal $1 + \gamma^*\mathcal{I} \subset \mathcal{O}_{\mathcal{Z}}$.

The $p$-adic analytic torus over $\mathcal{X}$ is the adic space $\mathcal{T}$ that associates to a map $\gamma : \mathcal{Z} \to \mathcal{X}$ of adic spaces s.t. $\gamma^*\mathcal{I}$ is an invertible ideal; the ideal $\mathcal{T}(\gamma) := 1 + \gamma^*\mathcal{I}$. Observe that $\mathcal{T}$ acts on the functor $\mathbb{V}(\mathcal{E})$. For any $\gamma$ as before, $a \in \mathcal{T}(\gamma)$ and $\rho \in \mathbb{V}(\mathcal{E})$, the action is given by the multiplication $a \star \rho = a\rho$. Observe that if $\rho \in \mathbb{V}_0(\mathcal{E}, s)$, then $\overline{a \star \rho}(s) = \bar{a}\bar{\rho}(s) = \bar{\rho}(s) = 1$; hence the action of $\mathcal{T}$ on $\mathbb{V}(\mathcal{E})$ extends to an action on $\mathbb{V}_0(\mathcal{E}, s)$. The induced action on $f_{\mathcal{E},*}\mathcal{O}_{\mathbb{V}(\mathcal{E})}$ locally is given by

$$\begin{array}{ccc} R^+ << X_i \mid 1 \leqslant i \leqslant r >> & \xrightarrow{a\star -} & R^+ << X_i \mid 1 \leqslant i \leqslant r >> \\ f & \longmapsto & f(a\underline{X}) \end{array}$$

and on $f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$

$$\begin{array}{ccc} R^+ << Z >> & \xrightarrow{a\star -} & R^+ << Z >> \\ f & \longmapsto & f(aZ + \frac{a-1}{\alpha}) \end{array}, \qquad \begin{array}{ccc} R^+ << Z, X_2 >> & \xrightarrow{a\star -} & R^+ << Z, X_2 >> \\ f & \longmapsto & f(aZ + \frac{a-1}{\alpha}, aX_2) \end{array},$$

where $\frac{a-1}{\alpha} \in \mathcal{O}_{\mathfrak{U}}$ is well defined since $\mathcal{I}_{\mathfrak{U}}$ is a free $\mathcal{O}_{\mathfrak{U}}$-module generated by $\alpha$ and $a - 1 \in \mathcal{I}_{\mathcal{U}}$. The action extends the previous one since $\alpha Z = X_1 - 1$:

$$a * \alpha Z = \alpha\left(aZ + \frac{a-1}{\alpha}\right) = a\alpha Z + a - 1 = aX_1 - 1 = a * (X_1 - 1).$$

In a less precise way we can deduce the action from the following (not rigorous) equality

$$a * Z = a * \frac{X_1 - 1}{\alpha} = \frac{aX_1 - 1}{\alpha} = aZ + \frac{a-1}{\alpha}.$$

## C.3 Filtration associated to VBMS

Let $\mathcal{E}^+$ be a rank 2 locally free $\mathcal{O}_{\mathcal{X}}^+$-module with a marked section $s \in H^0(\mathcal{X}, \mathcal{E}^+/\mathcal{I}\,\mathcal{E}^+)$. Let $\mathcal{F}^+ \subset \mathcal{E}^+$ a locally free sub-module of rank 1 with the property that $\{s\}$ is a basis of $\mathcal{F}^+/\mathcal{I}\mathcal{F}^+$. Observe that on $f_{\mathcal{E}/\mathcal{F},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E}/\mathcal{F})} = \widehat{\mathrm{Sym}}_{\mathcal{O}^+_{\mathcal{X}}}(\mathcal{E}^+/\mathcal{F}^+)$ there is an increasing filtration given by $\mathrm{Sym}^{\leqslant h}_{\mathcal{O}^+_{\mathcal{X}}}(\mathcal{E}^+/\mathcal{F}^+)$, moreover the inclusion $\mathbb{V}(\mathcal{E}/\mathcal{F}) \hookrightarrow \mathbb{V}_0(\mathcal{E}, s)$ induces a map $f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \to f_{\mathcal{E}/\mathcal{F},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E}/\mathcal{F})}$ and the pre-image of the increasing filtration on $f_{\mathcal{E}/\mathcal{F},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E}/\mathcal{F})}$ gives an increasing filtration on $f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}$

**Proposition C.3.1.** *There are two compatible increasing filtrations $Fil_\bullet f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}$ and $Fil_\bullet f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$ with graded pieces*

$$Gr_h f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \cong f_{\mathcal{F},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}^+_X} Sym^h(\mathcal{E}^+/\mathcal{F}^+)$$
$$Gr_h f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)} \cong f_{\mathcal{F},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}_X} Sym^h(\mathcal{E}/\mathcal{F}).$$

*Moreover if* $\mathrm{Spa}(R, R^+) = \mathcal{U} \subset \mathcal{X}$ *is an open where $\mathcal{F}^+$ has a basis $\{e\}$ lifting the basis $\{s\}$, if $\mathcal{E}^+$ has a basis $\{e, f\}$ and $\mathcal{I}$ is freely generated by an element, then*

$$f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}(\mathcal{U}) = R^+ << Z, Y >> \to R^+ << Y >> = f_{\mathcal{E}/\mathcal{F},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E}/\mathcal{F})}(\mathcal{U})$$

*and the filtration is given by the polynomial degree in $Y$, i.e.*

$$Fil_h f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}(\mathcal{U}) = R^+ << Z, Y >>^{deg_Y \leqslant h} .$$

*And it holds the analogue statement for the filtration on $f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$.*

Via the functoriality of $\mathbb{V}$, $\mathbb{V}_0$ and via the explicit descriptions one gets a diagram

$$\begin{tikzcd} \mathbb{V}(\mathcal{E}/\mathcal{F}) \arrow[r, hook] & \mathbb{V}_0(\mathcal{E},s) \arrow[r] \arrow[d] & \mathbb{V}(\mathcal{E}) \arrow[d] & \\ & \mathbb{V}_0(\mathcal{F},s) \arrow[r] & \mathbb{V}(\mathcal{F}) \arrow[r] & \mathfrak{X} \end{tikzcd} .$$

The two closed inclusions of $\mathbb{V}(\mathcal{E}/\mathcal{F})$ in $\mathbb{V}_0(\mathcal{E},s)$ and $\mathbb{V}(\mathcal{E})$ induce an action on them. Observe that this action is trivial over $\mathbb{V}_0(\mathcal{F},s)$ and $\mathbb{V}(\mathcal{F})$. Moreover locally, where there is a splitting of the sequence $\mathcal{F} \subset \mathcal{E} \to \mathcal{E}/\mathcal{F}$, one has that

$$\mathbb{V}(\mathcal{E}) \cong \mathbb{V}(\mathcal{F}) \times_{\mathfrak{X}} \mathbb{V}(\mathcal{E}/\mathcal{F}), \quad \mathbb{V}_0(\mathcal{E},s) \cong \mathbb{V}_0(\mathcal{F},s) \times_{\mathfrak{X}} \mathbb{V}(\mathcal{E}/\mathcal{F}).$$

hence $\mathbb{V}(\mathcal{E})$ is a $\mathbb{V}(\mathcal{E}/\mathcal{F})$-torsor over $\mathbb{V}(\mathcal{F})$ and $\mathbb{V}_0(\mathcal{E},s)$ is a $\mathbb{V}(\mathcal{E}/\mathcal{F})$-torsor over $\mathbb{V}_0(\mathcal{F},s)$.

## C.4 Connection associated to VBMS

Let $\pi : \mathcal{X} \to \mathcal{S} = \mathrm{Spa}(A, A^+)$, where $A = A^+[\frac{1}{p}]$ and $A^+$ is a $\mathbb{Z}_p$-algebra $\tau$-adically completed and separated, where $\tau \in A^+$. Suppose that $\mathfrak{X} \to \mathrm{Spf}(A^+)$ is a morphism of finite type and $\mathfrak{X}$ has the $\tau$-topology. Suppose that there is an integrable connection $\nabla$ on the sheaf $\mathfrak{E}$ over the formal model $\mathfrak{X}$ of $\mathcal{X}$. Suppose that the section $s$ is **horizontal** w.r.t. the connection $\overline{\nabla}$, *i.e.*

$$\overline{\nabla}(s) := (\nabla \mod \mathcal{I})(s) = 0.$$

Then in §2.4 in [AI21] it is described how to extend the connection to two compatible connections on $f_{\mathfrak{E},*}\mathcal{O}_{\mathbb{V}(\mathfrak{E})}[\frac{1}{p}] \subset f_{\mathfrak{E},*}\mathcal{O}_{\mathbb{V}_0(\mathfrak{E},s)}[\frac{1}{p}]$. We denote with $\nabla'$ and $\nabla_0$ the corresponding connections on $f_{\mathcal{E},*}\mathcal{O}_{\mathbb{V}(\mathcal{E})} \subset f_{\mathcal{E},*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$.

**Proposition C.4.1.** *With the assumptions as before, suppose that $\mathfrak{F} \subset \mathfrak{E}$ is a locally free $\mathcal{O}_{\mathfrak{X}}$-module of rank 1 s.t. the marked sections form an $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}\mathcal{O}_{\mathfrak{X}}$-basis for the sheaf $\mathfrak{F}/\mathcal{I}\mathfrak{F}$. Let $\nabla$ be an integrable connection on $\mathfrak{E}$ s.t. the marked sections are horizontal w.r.t. $\overline{\nabla}$.*

*Then the connection $\nabla$ could be extended canonically on the two sheaves $f_{\mathcal{E},*}\mathcal{O}_{\mathbb{V}(\mathcal{E})}$ and $f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$, i.e. there exist $\nabla', \nabla_0$ s.t. the following diagram is commutative:*

$$\begin{tikzcd} \mathcal{E}^+ \arrow[r, "\nabla"] \arrow[d] & \mathcal{E}^+ \otimes_{\mathcal{O}^+_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{S}} \arrow[d] \\ f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})} \arrow[r, "\nabla'"] \arrow[d] & f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})} \hat{\otimes}_{\mathcal{O}^+_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{S}} \arrow[d] \\ f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \arrow[r, "\nabla_0"] & f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \hat{\otimes}_{\mathcal{O}^+_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{S}} \end{tikzcd} .$$

*Moreover the connection $\nabla_0$ satisfies the Griffith's transversality property w.r.t. the filtration $Fil_\bullet f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$.*

*The induced map $gr'_h(\nabla_0)$ that makes the following diagram commute*

$$\begin{array}{ccc} Gr_h f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} & \xrightarrow{gr_h(\nabla_0)} & Gr_{h+1} f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \hat{\otimes}_{\mathcal{O}^+_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{S}} \\ \downarrow \cong & & \downarrow \cong \\ f_{\mathcal{F},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^h_{\mathcal{O}^+_{\mathcal{X}}}(\mathcal{E}^+/\mathcal{F}^+) & \xrightarrow{gr'_h(\nabla_0)} & f_{\mathcal{F},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{F},s)} \otimes_{\mathcal{O}^+_{\mathcal{X}}} \mathrm{Sym}^{h+1}_{\mathcal{O}^+_{\mathcal{X}}}(\mathcal{E}^+/\mathcal{F}^+) \end{array}$$

*is $Sym^{\bullet}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}/\mathcal{F})$-linear.*

The Proposition is proved for the formal schemes in loc. cit., and we are taking the analytification. We want to resume the local description of the maps in the Proposition.

If $\mathcal{E}^+$ is trivialized on $\mathrm{Spa}(R,R^+) = \mathcal{U} \subset \mathcal{X}$ by $e_1, e_2$, let's denote with $\omega_{s,i} \in \Omega^1_{R/A}$ s.t.

$$\nabla e_s = e_1 \otimes \omega_{s,1} + e_2 \otimes \omega_{s,2} \quad \text{for} \quad s = 1,2..$$

Then the connection is extended on $\left(f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})}\right)_{|\mathcal{U}}$ by the Leibnitz rule and

$$\begin{array}{ccc} R^+ << X_1, X_2 >> & \xrightarrow{\nabla'} & R^+ << X_1, X_2 >> \hat{\otimes}_{R^+} \Omega^1_{R/A} \\ X_s & \longmapsto & X_1 \otimes \omega_{s,1} + X_2 \otimes \omega_{s,2} \end{array}$$

Let $e'_1, e'_2$ be another $\mathcal{O}^+_{\mathcal{X}}$basis of $\mathcal{E}^+$ on $\mathcal{U}$, in order to see that the definition of the connection on $\left(f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})}\right)_{|\mathcal{U}}$ does not depend on the basis of $\mathcal{E}^+$, let $A = (a_{s,i})_{1\leqslant s,i\leqslant 2}$ be the matrix s.t. $A\underline{e} = \underline{e}'$. Then

$$\nabla e'_t = \nabla \sum_{s=1}^{2} a_{t,s} e_s = \sum_{s=1}^{2} \left( \sum_{i=1}^{2} a_{t,s} e_i \otimes \omega_{s,i} \right) + e_s \otimes \mathrm{d}\, a_{t,s}$$

Let's denote $X'_t := \sum_{s=1}^2 a_{t,s} X_s$, then

$$\nabla' X'_t = \nabla' \sum_{s=1}^{2} a_{t,s} X_s = \sum_{s=1}^{2} \left( \sum_{i=1}^{2} a_{t,s} X_i \otimes \omega_{s,i} \right) + X_s \otimes \mathrm{d}\, a_{t,s}.$$

Hence the definition of $\nabla'$ does not depend on the choice of the basis and it glues.

Now we explain how to extend $\nabla$ also to $f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}$. Let $\mathcal{E}^+$ be as before with $s \in H^0(\mathcal{X}, \mathcal{E}^+/\mathcal{I}\mathcal{E}^+)$ marked section. We require that the marked section is horizontal with respect to the connection $\overline{\nabla} := \nabla \mod \mathcal{I}$. Let $e_1, e_2$ be a basis on $\mathrm{Spf}(R,R^+) = \mathcal{U} \subset \mathcal{X}$ s.t. $s = e_1 \mod \mathcal{I}$ and let $\alpha$ be a generator of $\mathcal{I}_{|\mathcal{U}}$. Then

$$\nabla e_s = \begin{cases} \sum_{i=1}^r \alpha e_i \otimes \gamma_{1,i} & s = 1 \\ \sum_{i=1}^r e_i \otimes \omega_{2,i} & s = 2 \end{cases},$$

observe that $\alpha\gamma_{1,i} = \omega_{1,i}$. We can define $\nabla_0$ on $\mathcal{U}$ via the Leibnitz rule and via

$$\begin{array}{ccc} R^+ << Z, X_2 >> & \xrightarrow{\nabla_0(\mathcal{U})} & R^+ << Z, X_2 >> \hat{\otimes}_{R^+} \Omega^1_{R/A} \\ Z & \longmapsto & Z \otimes \gamma_{1,1} + X_2 \otimes \gamma_{1,2} \\ X_2 & \longmapsto & (\alpha Z + 1) \otimes \omega_{2,1} + X_2 \otimes \omega_{2,2} \end{array}.$$

In order to define $\nabla_0 Z$ we used:

$$\nabla' X_1 = \nabla'(X_1 - 1) = \nabla_0(\alpha Z) = \alpha \nabla_0 Z + Z \otimes \mathrm{d}\,\alpha,$$

but $\mathrm{d}\,\alpha = 0 \in \Omega^1_{R/A}$. Indeed $p^r = \alpha^{r'}$ for some $r' \in \mathbb{N}$, since $p \in R^\times$ then also $\alpha$ and $r'$ are invertible in $R$ and

$$0 = \mathrm{d}\,p^r = \mathrm{d}(\alpha^{r'}) = r'\alpha^{r'-1}\,\mathrm{d}\,\alpha.$$

If there is a subsheaf $\mathcal{F} \subset \mathcal{E}$ as in C.3, then the connection $\nabla_0$ satisfies Griffith's transversality property with respect to the filtration $Fil_\bullet f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s)}$, *i.e.*

$$\nabla_0\left(Fil_h f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)}\right) \subset Fil_{h+1} f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E},s)} \hat{\otimes}_{\mathcal{O}^+_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{S}}.$$

Indeed the degree on $X_2$ could increase at most by 1. The morphisms $\nabla_0(\mathcal{U})$ glue on $\mathcal{X}$. Since $\nabla_0$ satisfies the Griffith's transversality property, it induces a morphism on the graded pieces, the fact that the map $gr'_h(\nabla_0)$ is $Sym^\bullet(\mathcal{E}/\mathcal{F})$-linear follows: via induction; via the local explicit computations and via the Leibnitz rule.

## C.5 The sheaf $\mathbb{W}_\kappa(\mathcal{E}, s)$

Let $\kappa : 1 + \mathcal{I} \to \mathcal{O}^\times_{\mathcal{X}}$ be as in §C.2. We defined a $\mathcal{T}$-action on the functors $\mathbb{V}_0(\mathcal{E}, s) \subset \mathbb{V}(\mathcal{E})$. Observe that the $1 + \mathcal{I}$-action on the polynomials of a certain degree $d \in \mathbb{N}$ is the multiplication by the power to the $d$: if $f \in R << X_i \mid 1 \leqslant i \leqslant r >>^{deg X = d}$ and $a \in 1 + \mathcal{I}_{|\mathcal{U}}$, then $a * f = a^d f$. Hence if $\kappa_d : 1 + \mathcal{I} \to \mathcal{O}^\times_{\mathcal{X}}$ is the map defined by $\kappa_d(a) := a^d$, then

$$\mathrm{Sym}^d_{\mathcal{O}^+_{\mathcal{X}}} \mathcal{E}^+ \subset f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})}[\kappa_d] := \{f \in f_{\mathcal{E},*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E})} \mid a * f = \kappa_d(a) f, \text{ for all } a \in (1 + \mathcal{I})\}.$$

This idea gives us a way in order to build a family of sheaves that interpolate the symmetric powers.

Let $\kappa : 1 + \mathcal{I} \to \mathcal{O}^\times_{\mathcal{X}}$ as before. For a sheaf $\mathcal{E}^+$ with a marked section we define the sheaf

$$\mathbb{W}_{\mathcal{E},\kappa} := f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}(\mathcal{E},s)}[\kappa] := \{f \in f_{\mathcal{E},0,*}\mathcal{O}_{\mathbb{V}(\mathcal{E},s)} \mid a * f = \kappa(a) f, \text{ for all } a \in 1 + \mathcal{I}\}.$$

We denote $\mathbb{W}^+_{\mathcal{E},\kappa} := f_{\mathcal{E},0,*}\mathcal{O}^+_{\mathbb{V}(\mathcal{E},s)}[\kappa]$.

The morphism $\kappa$ is analytic, then there is $t \in \mathcal{O}_{\mathcal{X}}$ s.t.

$$\kappa(a) = \exp(t \log(a)) =: a^t \quad \text{for each } a \in 1 + \mathcal{I}.$$

Hence we can define, locally on $\mathcal{X}$, where the ideal $\mathcal{I}$ is generated by a single element $\alpha$, the power series $X^t := \kappa(X) := \kappa(1 + \alpha Z) \in R^+ << Z >>$. For any $u \in \mathcal{O}^+_{\mathcal{X}}$ we can define the character $\kappa_u$ that is the power to $u$, i.e.

$$\begin{aligned} \kappa_u : 1 + \mathcal{I} &\longrightarrow \mathcal{O}^\times_K \\ a &\longmapsto \exp\left(u \log(a)\right) = a^u \end{aligned}.$$

This character converges $\mathcal{I}$-adically since log is well defined on $1 + \mathcal{I}$, moreover $\log(1 + \mathcal{I}) \in \mathcal{I}$ and exp is well defined on $\mathcal{I}$. Observe that $X^{t-u} = X^t X^{-u}$. In particular one has that $X^{-n} \in R^+ << Z >>$ is well defined for any $n \in \mathbb{N}$.

**Example C.5.1.** *If $r = k = 1$, then $\mathbb{W}_{\mathcal{E},\kappa_d} = \mathcal{E}^{\otimes d}$ for any $d \in \mathbb{N}$ and if $\kappa$ is not a $\mathbb{N}$-power one gets that $\mathbb{W}_{\mathcal{E},\kappa}$*

*is locally free of rank* 1 *locally generated by the power series*

$$X^t := \kappa(X) := \kappa(1+\alpha Z) := exp(t\ log(\alpha Z)).$$

*Where exp and log are the well-known power series and* $t \in \mathcal{O}_{\mathcal{X}}$ *s.t.*

$$\kappa(a) = \exp(t \log(a)) \quad \textit{for any} \quad a \in 1 + \mathcal{I}.$$

*The idea is that, thanks to the change of variable* $X_1 = 1 + \alpha Z$ *we can define a* $p$*-adic power of* $X_1$ *as a power series that does not converge in* $X_1$ *but it does in* $Z$.

*Proof.* The fact that $a \star X^t = (aX)^t = \kappa(a)X^t$ is true by definition of $X^t$. We want to show that $X^t$ locally generates the whole module $\mathbb{W}_{\mathcal{E},\kappa}$. The point is to show that

$$R << Z >>^{\star = id} := \{f \in R << Z >> |\ a \star f = f \text{ for all } a \in 1 + \alpha R^+\} = R.$$

Once this equality is proven then for any $f \in R << Z >>$ with $a \star f = \kappa(a) f$, then $f/X^t = fX^{-t} \in R << Z >>^{\star = id} = R$, hence $f \in R\ X^t$.

Let $f \in R << Z >>^{\star = id}$, for any $x \in 1 + \mathcal{I}$ we have that $f(x) = x \star f(1) = f(1)$ and $f$ is constant, hence $f \in R$. □

**Example C.5.2.** *If* $k = 1 < r = 2$, *then* $Fil_d \mathbb{W}_{\mathcal{E},\kappa_d} = \mathrm{Sym}^d \mathcal{E}$ *and if* $\kappa$ *is not a* $\mathbb{N}$*-power* $\mathbb{W}_{\mathcal{E},\kappa}$ *is an infinite dimensional* $\mathcal{O}_{\mathcal{X}}$*-module with an increasing filtration locally isomorphic to*

$$\kappa(X_1) R \left\langle\!\!\left\langle \frac{X_2}{X_1} \right\rangle\!\!\right\rangle \subset R << Z, X_2 >> .$$

*In order to simplify the notation, in this case we denote* $X := X_1$, $Y := X_2$ *and* $V := X_2/X_1$.

*Proof.* The proof is similar to the one for the $r = 1$ case, with the difference that

$$R << Z, Y >>^{\star = \mathrm{id}} = R << Y/X >> = R << V >> .$$

Indeed if $f \in R << Z, Y >>^{\star = \mathrm{id}}$, then $f(x, y) = \frac{1}{x} \star f(x, y) = f(1, \frac{y}{x})$ for any $x \in 1 + \mathcal{I}$ and $y \in \mathcal{O}_{\mathcal{X}}$. □

# Bibliography


[AI21] F. Andreatta and A. Iovita. Triple product p-adic L-functions associated to finite slope p-adic families of modular forms. *Duke Mathematical Journal*, 2021.

[AIP18] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. Le halo spectral. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):603–655, 2018.

[BdJ11] Bhargav Bhatt and Aise Johan de Jong. Crystalline cohomology and de rham cohomology. 2011. https://arxiv.org/abs/1110.5001.

[BGG75] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Differential operators on the base affine space and a study of g-modules. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64. Halsted Press, New York-Toronto, Ont., 1975.

[BO78] P. Berthelot and A. Ogus. *Notes on Crystalline Cohomology.* Princeton University Press, 1978.

[GL76] Howard Garland and James Lepowsky. Lie algebra homology and the Macdonald-Kac formulas. *Invent. Math.*, 34(1):37–76, 1976.

[Gro68] A. Grothendieck. *Crystals and the De Rham Cohomology of Schemes in "Dix esposes sur la cohomologie des schémas".*, chapter IX, pages 306–358. North-Holland Publishing Company, 1968.

[Guo21] H. Guo. Crystalline cohomology of rigid analytic spaces. 2021. https://arxiv.org/abs/2112.14304.

[Hal15] B. C. Hall. *Lie Groups, Lie Algebras, and Representations - Second Edition.* Springer, 2015.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30 of *Aspects of Mathematics.* Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hum72a] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer, 1972.

[Hum72b] J. E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$.* American Mathematical Society, 1972.

[Pan19] S. Panozzo. The halo conjecture. 2019. https://air.unimi.it/retrieve/dfa8b99f-07b7-748b-e053-3a05fe0a3a96/phd_unimi_R11211.pdf.

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