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Regularity, Decay, Differentiability for Solutions to Conservation Laws and Structural Properties for Conservation Laws with Discontinuous Flux

Ph.D. Thesis

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Contents

Introduction

In this thesis we study different problems related to the theory of conservation laws. **Part 1** consists of four chapters (**Chapters 1, 2, 3** and **4**) and deals with problems of regularity and decay of solutions, as well as differentiability properties of the solution operator.

In **Chapter 1** we introduce a Lagrangian representation for multidimensional scalar balance laws, in the framework of solutions with finite entropy production. We then use the representation to prove that, in the one-dimensional case and for a class of genuinely nonlinear 2×2 systems of conservation laws, including the isentropic system of gas dynamics with exponent $\gamma = 3$, the entropy dissipation measures are concentrated on a 1-rectifiable set. Moreover, regularity results are proved for the isentropic system.

In **Chapter 2** we consider 2×2 systems of conservation laws. We observe that bounded vanishing viscosity solutions of 2×2 systems obtained with the compensated compactness method satisfy a pair of (nonlocal) kinetic equations, and we use it to obtain a dispersive estimate in the case of genuinely nonlinear systems.

In **Chapter 3** we consider the problem of endowing the semigroup operator associated to a scalar conservation law with a differential structure. We prove that first order perturbations satisfy a continuity equation, and we observe that this is not enough to define a duality with integral functionals. We then introduce a finer framework, which is the correct one for computing variations of this type of functionals.

In **Chapter 4** we introduce a class of intermediate domains $\mathcal{P}_\alpha$, $0 < \alpha < 1$, lying between $\mathbf{L}^\infty$ and BV for which the BV norm of solutions decays like $t^{-\alpha}$. A key ingredient of the analysis is a "Fourier-type" decomposition of functions of $\mathcal{P}_\alpha$ into components which oscillate more and more rapidly.

The second part, **Part 2**, focuses on various aspects of scalar conservation laws with discontinuous flux. We introduce a notion of backward operator, and we completely classify the attainable states at a positive time $T > 0$. We also prove new regularity and stability properties of solutions.

We now describe more in detail the content of each chapter.

Part I

Lagrangian Representation and Applications to Regularity

In **Part 1**, **Chapter 1**, we start by considering a scalar, multidimensional conservation law, which is, in an open subset $\Omega \subset \mathbb{R} \times \mathbb{R}^d$, the PDE

$$\partial_t u + \mathrm{div}_x \, \boldsymbol{f}(u) = 0 \qquad \text{in } \mathscr{D}'_{t,x}(\Omega), \qquad \boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d. \tag{1}$$

When (1) is supplemented with a source term in the right hand side, we speak more generally of *balance* law.

In **Section 1.1**, we start by recalling the notion of finite entropy (or finite energy) solution: a *finite entropy solution* to (1) (Definition 1.1.1) is a function $u : \Omega \to \mathbb{R}$ such that the distribution $\mu_\eta \in \mathscr{D}'_{t,x}$ defined by

$$\mu_\eta \doteq \partial_t \eta(u) + \mathrm{div}_x \, \boldsymbol{q}(u)$$

is a locally finite measure in Ω for every entropy-entropy flux pair $\eta, \boldsymbol{q}$, that is, every pair of smooth functions such that $\boldsymbol{q}' = \eta' \boldsymbol{f}'$. Notice that, without additional requirements, a finite entropy solution only satisfies a balance law, i.e. we need to consider a measure source term in (1). A by now-classical result first proved in [**80**] for entropy solution (with nonpositive entropy production), and extended in [**56**] for finite entropy solutions, states, in a slightly modified form, that every finite entropy solution $u \geq 0$ (since solutions to a scalar conservation law satisfy a maximum principle, up to a translation this is not a restrictive assumption if we work with $\mathbf{L}^\infty$ initial data) satisfies a kinetic formulation of the form

$$\partial_t \chi + \boldsymbol{f}'(v) \partial_x \chi = \partial_v \mu_1 + \mu_0, \qquad \chi(t,x,v) \doteq \mathbf{1}_{\mathrm{hyp}\, u}(t,x,v) \tag{2}$$

where μ_1, μ_0 are locally finite measures in $\mathscr{M}(\Omega \times \mathbb{R})$ and $\mathrm{hyp}\, u$ denotes the hypograph of the function $u : \Omega \to \mathbb{R}$. Although entropy solutions provide a satisfactory mathematical theory, solutions with finite entropy production arise as interesting models in different areas of physics, such as micromagnetism, liquid crystals, thin film-blistering. We refer to [**57**] for more details on this topic.

In **Section 1.2** we introduce the Lagrangian representation for finite entropy solutions, that is one of the main tools used throughout the chapter. This kind of representation can be thought as the development of the method of characteristics for finite entropy solutions but at the level of the kinetic formulation (2), rather then the original equation (1). It was introduced in [**34**] in the entropic setting, and in [**84**] in the setting of finite entropy solutions without source terms. In both cases the existence of the Lagrangian representation is proved via an explicit construction resembling Brenier's *transport-collapse* scheme [**39**], of which the Lagrangian representation is the measure-theoretic counterpart. In our setting, a Lagrangian representation for a finite entropy solution u is a measure $\boldsymbol{\omega} \in \mathscr{M}^+(\Gamma)$, where Γ is a space of curves:

$$\Gamma \doteq \Big\{ (I_\gamma, \gamma) \;\big|\; \gamma = (\gamma^x, \gamma^v) : I_\gamma = (t^1_\gamma, t^2_\gamma) \to \mathbb{R}^d_x \times \mathbb{R}^+_v, \;\; \gamma^x \text{ is Lipschitz, } \gamma^v \text{ is in } BV(I_\gamma) \Big\}$$

that satisfies the following properties: $\boldsymbol{\omega}$ is concentrated on *characteristic curves*, i.e. curves such that $\dot\gamma^x(t) = \boldsymbol{f}'(\gamma^v(t))$ for a.e. $t \in I_\gamma$, the function χ can be recovered as the superposition of the curves selected by $\boldsymbol{\omega}$: for a.e. $t > 0$

$$\chi(t,\cdot,\cdot) \cdot \mathscr{L}^d = (e_t)_\sharp \boldsymbol{\omega}, \qquad e_t : \Gamma \to \mathbb{R}, \quad e_t((\gamma, I_\gamma)) = \gamma(t)$$

plus additional bounds (see (3) of Definition 1.2.2), essentially related to the variation of the map $t \mapsto \gamma^v(t)$ and to the starting/ending points of the intervals $I_\gamma = (t^1_\gamma, t^2_\gamma)$. In our setting, with the presence of a source term in (2), there are no additional key difficulties, other then observing that, while the variation measure $D\gamma^v$ will be related to the measure μ_1 in (2), the starting/ending points t^1_γ, t^2_γ must be related to the source term μ_0 (for example, if $\mu_0 = 0$, then $t^1_\gamma = 0$ and $t^2_\gamma = +\infty$ for $\boldsymbol{\omega}$-a.e. $\gamma \in \Gamma$). In fact, a key feature of the Lagrangian representation is that it allows to represent the dissipation measure (see the representation formula (1.2.10)).

In order to obtain a finer control on the representation, we use another approach to prove its existence. In Section 1.2 the main result is Theorem 1.2.15, in which we rely on a theorem of Smirnov [**96**] and on a reparametrization argument to obtain a Lagrangian representation starting from a solution to (2).

In **Section 1.3** we focus on the problem of concentration of entropy dissipation for finite entropy solutions, in the 1-d case. The starting point of the literature on this problem is the paper [**56**], in which it is proved that the set $\boldsymbol{J}$ defined by

$$\boldsymbol{J} \doteq \left\{ (t,x) \in \Omega \;\middle|\; \limsup_{r \downarrow 0} \frac{\nu\left(\mathrm{B}_r(t,x)\right)}{r^d} > 0 \right\}, \qquad \nu = (p_{t,x})_\sharp |\mu_1|$$

is d-rectifiable, where $p_{t,x}$ is the canonical projection on the t,x variables and $(p_{t,x})_\sharp|\mu_1|$ denotes the usual push-forward measure defined by

$$(p_{t,x})_\sharp|\mu_1|(A) \doteq |\mu_1|(A \times \mathbb{R}) \qquad \forall \text{ measurable} \quad A \subset \mathbb{R}^2.$$

This obviously proves that $\nu \llcorner \boldsymbol{J}$ is concentrated on a d-dimensional rectifiable set, but a-priori the possible presence of higher dimensional parts of ν prevents to prove that the full measure ν is concentrated on $\boldsymbol{J}$, and this problem remains open. Partial results in one dimension are obtained in [**36**], [**57**], and in general dimension in [**95**]. A first, complete answer came in [**37**] for *entropic* solutions in 1-dimension, for general Lipschitz fluxes, in which the authors prove that the above concentration property holds. In the setting of finite entropy solutions in one dimension, for the Burgers equation $f(u) = u^2/2$, in [**86**] the same result is obtained. The same question is relevant for systems of 2×2 of conservation laws, for solutions obtained via the compensated compactness method (in Section 1.5 we answer to this question for a very special class of systems, including the isentropic system of gas dynamics with $\gamma = 3$).

The main result of this section is the extension of the rectifiability result of [**86**], using the Lagrangian representation to prove that the same result holds for fluxes that are weakly genuinely nonlinear:

THEOREM 1. *Let $d = 1$. Assume that f satisfies $\mathscr{L}^1\big(\{v : f''(v) = 0\}\big) = 0$ and let u be a finite entropy solution to* (1). *Then there are countably many Lipschitz curves $\{\sigma_i\}_{i \in \mathbb{N}} \subset \mathrm{Lip}(\mathbb{R};\mathbb{R})$ such that ν is concentrated on*

$$\boldsymbol{J}' \doteq \bigcup_{i \in \mathbb{N}} \Big\{ (t,x) \in \Omega \,|\, x = \sigma_i(t) \Big\}.$$

In **Section 1.4** we apply the Lagrangian representation to prove a regularity result for Burgers equation with a measure source term. In particular, assume that u is a finite entropy solution to Burgers equation $f(u) = u^2/2$. Assume that the function χ satisfies (2) with a measure μ_1 that has a definite sign (in particular, if $\mu_1 \geq 0$, this corresponds to the situation of an entropic solution with source term). Then Theorem 1.4.2 shows that u belongs to the Besov space $B^{1/2,1}_{\infty,loc}(\Omega)$.

In **Section 1.5** we consider 2×2 systems that satisfy the following assumption: the i-th eigenvalue λ_i depends only on the i-th Riemann invariant: letting w, z be Riemann invariants, with corresponding eigenvalues λ_1, λ_2, we assume

$$\begin{aligned} &\lambda_1(w,z) \equiv \lambda_1(w) \qquad \forall\, w, z \in \mathcal{W} \\ &\lambda_2(w,z) \equiv \lambda_2(z) \qquad \forall\, w, z \in \mathcal{W} \\ &\mathscr{L}^1\big(\{v \mid \lambda_i'(v) = 0\}\big) = 0 \qquad \text{for } i = 1, 2. \end{aligned} \tag{3}$$

This is the class of systems such that for smooth solutions the system completely decouples into two scalar, independent, equations and such that the eigenvalues are weakly genuinely nonlinear. An example of this systems is the system of isentropic gas dynamics with $\gamma = 3$:

$$\begin{aligned} &\partial_t \rho + \partial_x \rho u = 0, \\ &\partial_t(\rho u) + \partial_x \left(\rho u^2 + \rho^3/12\right) = 0. \end{aligned} \tag{4}$$

to which various authors dedicated some attention due to its particular structure ([**100**], [**69**] and some results of [**79**]). Here this systems is inserted in the general class of systems satisfying (3), and the following Theorem is proved.

THEOREM 2. *Assume that the system satisfies assumption* (3)*. Let U be a general vanishing viscosity solution obtained with the compensated compactness method (so that only a-priori $\mathbf{L}^\infty$ bounds are assumed). Then there are countably many Lipschitz curves* $\{\sigma_i\}_{i\in\mathbb{N}} \subset \mathrm{Lip}(\mathbb{R};\mathbb{R})$ *such that for every entropy η, the measure*

$$\mu_\eta = \partial_t \eta(U) + \partial_x q(U)$$

is concentrated on

$$\boldsymbol{J}' \doteq \bigcup_{i\in\mathbb{N}} \Big\{(t,x) \in \Omega \,|\, x = \sigma_i(t)\Big\}.$$

Moreover, if $v \mapsto \lambda_i(v)$ is uniformly increasing, then the i-th Riemann Invariant belongs to $B^{1/2,1}_{\infty,loc}$.

The proof of the theorem is based on the observation (see Section 1.5 for more details) that the Riemann invariants of solutions obtained with the vanishing viscosity method are finite entropy solutions, with non-zero source terms.

Kinetic Formulation and Decay for 2×2 Systems

In **Chapter 2** we study 2×2 systems in $\mathbf{L}^\infty$. In this setting, under *a priori* $\mathbf{L}^\infty$ bounds on the vanishing viscosity approximations, starting from the work of DiPerna (see e.g. [**59**]) using the compensated compactness method, it is possible to prove existence of solutions in $\mathbf{L}^\infty$. However, results about structure and regularity of this solutions are completely absent from the literature and at the present moment seem difficult to achieve. Regularity results at the moment exist only in the setting of solutions with small *oscillation* (i.e. the initial datum U_0 satisfies $\|U_0 - \widehat{U}\|_{\mathbf{L}^\infty} < \epsilon$ for some constant $\widehat{U}$) starting from the work of Glimm and Lax [**68**] (see also [**35**]). In those papers it is proved the existence of solutions whose total variation decays, in particular at any given time $t > 0$ the solution $U(t, \cdot)$ is in BV. Additionally, in the context of solutions with small oscillations, a recent and notable result [**67**] demonstrates that solutions constructed with the front tracking method retain fractional-BV regularity.

In this Chapter, we consider solutions in $\mathbf{L}^\infty$, without bounds on the initial oscillation. In this setting, tools such as the front tracking method, very effective in the small $\mathbf{L}^\infty$

setting, cannot be exploited (due to the fact that it is likely that the total variation of a solution can be infinite, see [**43**]). It seems that different methods and techniques should be developed to deal with merely bounded solutions.

Motivated by this observation, in **Section 2.1**, we first observe that the Riemann invariants associated to a vanishing viscosity solution (obtained via compensated compactness) satisfy a pair of (nonlocal) kinetic equations. A fundamental block is the construction of singular entropies, performed by Perthame and Tzavaras in [**91**] and [**97**], that we briefly recall here. To build singular entropies (say, of the first type) one constructs, for every $\xi \in [\underline{w}, \overline{w}]$ (here we are assuming for simplicity that in Riemann invariants our domain is a square $\mathcal{W} = [\underline{w}, \overline{w}] \times [\underline{z}, \overline{z}]$), smooth entropies $\mathbf{\Theta}[\xi] : \mathcal{W} \to \mathbb{R}$ that can be *cut* along a curve $\{(w, z) \mid w = \xi\}$: by this we mean that the functions

$$\boldsymbol{\chi}[\xi](w, z) \doteq \mathbf{\Theta}[\xi](w, z) \cdot \mathbf{1}_{\{w \geq \xi\}}(w, z)$$

and

$$\widetilde{\boldsymbol{\chi}}[\xi](w, z) \doteq \mathbf{\Theta}[\xi](w, z) \cdot \mathbf{1}_{\{w \leq \xi\}}(w, z)$$

will still be (discontinuous) weak entropies (satisfying the entropy equation in the sense of distributions), for some discontinuous entropy fluxes $\boldsymbol{\psi}[\xi], \widetilde{\boldsymbol{\psi}}[\xi]$. The main theorem of the section (Theorem 2.2.4) is stated informally below.

Theorem 3. *Let U be a vanishing viscosity solution to a strictly hyperbolic 2×2 system obtained with the compensated compactness method. Then the following kinetic formulation holds*

$$\partial_t \boldsymbol{\chi}[\xi](U) + \partial_x \boldsymbol{\psi}[\xi](U) = \partial_\xi \mu_1 + \mu_0 \qquad \text{in } \mathscr{D}'_{t,x,\xi}.$$

where μ_1, μ_0 are locally finite measures, and μ_1 is positive.

In **Section 2.3**, we use the kinetic formulation to obtain a dispersive estimate on vanishing viscosity solutions for genuinely nonlinear systems.

Theorem 4. *Let $U : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{U} \subset \mathbb{R}^2$ be a bounded vanishing viscosity solution to a genuinely nonlinear system. Assume that for some $\widehat{U} \in \mathcal{U}$ it holds $U(0, \cdot) - \widehat{U} \in \mathbf{L}^1$. Then, with $w(\widehat{U}) = \widehat{w}$, we have the estimate*

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(w(t, x) - \widehat{w}\right)^4 \mathrm{d}x\, \mathrm{d}t \leq \mathcal{O}(1) \cdot \|U - \widehat{U}\|_{\mathbf{L}^1} \Big(1 + \|U - \widehat{U}\|_{\mathbf{L}^1}\Big).$$

Differential Structure for Scalar Conservation Laws

In **Chapter 3** we consider the problem of endowing the semigroup associated to a scalar conservation law with a differential structure. We refer to Section 3.1 for a detailed explanation of the results of this chapter, here we present only a brief summary. Let $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$, $u \geq 0$, be the entropy solution to the Cauchy problem

$$\begin{aligned} &\partial_t u \,+\, \partial_x f(u) = 0, \qquad \text{in } \mathscr{D}'_{t,x} \\ &u(0, \cdot) = u_0 \in \mathbf{L}^\infty(\mathbb{R}). \end{aligned} \tag{5}$$

The flux is any $\mathscr{C}^1$ function $f : \mathbb{R} \to \mathbb{R}$ such that f does not have nontrivial linear interval in which it is affine. One of the possible motivations for the problems considered in this chapter is the derivation of necessary conditions for minimizers of integral functionals defined on solutions to scalar conservation laws. Let us illustrate this point with an example: let $\mathscr{G} : \mathbf{L}^\infty((0,T) \times \mathbb{R})) \to \mathbb{R}$ be a functional of the form

$$\mathscr{G}(u) \doteq \int_0^T \int_{\mathbb{R}} g(t,x,u(t,x))\,\mathrm{d}x\,\mathrm{d}t \tag{6}$$

and let us look for necessary conditions for optimality by computing the variation of $\mathscr{G}$ in u along a sequence of perturbations u^h such that

$$\sup_h h^{-1}\|u^h(0,\cdot) - u(0,\cdot)\|_{\mathbf{L}^1} < +\infty.$$

We consider two different but related objects: the perturbation

$$\rho^h \doteq \frac{u^h - u}{h} \cdot \mathscr{L}^2$$

and its lift in the kinetic variable (cfr. (2))

$$\nu_h \doteq \frac{1}{h}\big(\chi^h - \chi\big) \cdot \mathscr{L}^3, \qquad \chi^h(t,x,v) = \mathbf{1}_{\mathrm{hyp}\ u^h}(t,x,v).$$

Assume that both the limits ρ, ν of ρ^h and ν_h exist weakly in the sense of measures as $h \to 0^+$. We then claim that to compute the variation of $\mathscr{G}$ along such a sequence of perturbations it is necessary not only the limiting measure ρ, but also the finer information given by ν. Letting $\widehat{\rho} = (p_{t,x})_\sharp|\nu|$ we disintegrate

$$\nu = a_{t,x} \otimes \widehat{\rho}.$$

Then we compute

$$\begin{aligned}\lim_{h\to0+}\frac{1}{h}\Big(\mathscr{G}(u^h) - \mathscr{G}(u)\Big) &= \lim_{h\to0^+}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_v g(t,x,v)\,\mathrm{d}\nu_h(t,x,v) \\ &= \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_v g(t,x,v)\,\mathrm{d}a_{t,x}(v)\,\mathrm{d}\widehat{\rho}(t,x).\end{aligned} \tag{7}$$

We see that to compute the variation the additional information about the disintegration $a_{t,x}$ is needed. In this Chapter we then study two main problems:

(1) Derive an evolution equation for $\rho, \widehat{\rho}$.
(2) Study the structure of the disintegrations $a_{t,x}$, for general perturbations.

As of problem (1), formally, for smooth solutions, any limit measure ρ satisfies the continuity equation

$$\partial_t\rho + \partial_x(f'(u)\rho) = 0.$$

We thus expect then when shocks are present, one should replace f' by the speed of the shock in the continuity equation. In particular, for every (t,x) we define the *characteristic speed*

$$\lambda(t,x) = \begin{cases} f'(u), & \text{if } u(t,x-) = u(t,x+), \\ \frac{f(u^+)-f(u^-)}{u^+-u^-}, & \text{if } u \text{ has a jump } u(t,x-) = u^- \neq u^+ = u(t,x+). \end{cases} \tag{8}$$

Then, stated informally, we have

THEOREM 5. *Any perturbation ρ solves the Cauchy problem for the continuity equation*

$$\begin{aligned} &\partial_t \rho + \partial_x(\lambda\rho) = 0, \qquad \text{in } \mathscr{D}'_{t,x}, \\ &\rho(0,\cdot) \;=\; \rho_0. \end{aligned} \tag{9}$$

Theorem 5 is essentially a consequence of the Kuratowski convergence of the graphs of u^h to the graph of u, and is proved in §3.4: a fundamental tool is the one of *admissible boundary*, developed in [**37**].

Now we can use information that we have about problem (1) to study problem (2). At the end, the main theorem can be stated informally as follows.

THEOREM 6. *For $\widehat{\rho}$ -a.e. (t,x) continuity point of u, it holds $a_{t,x} = \delta_{u(t,x)}$. For $\widehat{\rho}$ -a.e. (t,x) of jump of u (i.e. $u^- = (t,x-) \neq u(t,x+) = u^+$) the disintegration $a_{t,x}$ is an absolutely continuous measure in the open interval $I_{t,x} \doteq (u^- \wedge u^+, u^- \vee u^+)$:*

$$a_{t,x} = \mathsf{g} \cdot \mathscr{L}^1 \qquad \text{in } I_{t,x}.$$

Most importantly, g is a nonincreasing (BV) function, and the derivative measure $D\mathsf{g}$ is concentrated on the set $\mathscr{K}$ where the flux f touches the chord between u^-, u^+:

$$\Big\{ v \in I(u^+, u^-) \;\Big|\; f(v) - f(u^+) - \lambda(v - u^+) = 0 \Big\}.$$

We notice that Theorem 6 can be seen as a *generalized shift differentiability theorem.* In fact, we will prove in Theorem 3.1.11 that from Theorem 6 it follows that, at the first order, asymptotically near a shock point (t,x) of u, the perturbation u^h looks like a "composite shift" (see Definition 3.1.10). In particular, we prove the following Theorem, which completely classifies the blow-ups:

$$U(t,x) = \lim_{h\to 0^+} u^h\big(h(t-\bar{t}),\, h(x-\overline{x})\big) \tag{10}$$

Crucially, here we are rescaling with rate h, since $\|u^h(t) - u(t)\|_{\mathbf{L}^1} \sim h$.

THEOREM 7. *For $\widehat{\rho}$-almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ it holds*

$$u^h\big(h(t-\bar{t}),\, h(x-\overline{x})\big) \longrightarrow U_{\widehat{\rho}(\{(t,x)\})}[\mathsf{g}_{(t,x)}] \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R}^2)$$

where $U_{\widehat{\rho}(\{(t,x)\})}[\mathsf{g}_{(t,x)}]$ is the composite shift with parameters $\widehat{\rho}(\{(t,x)\}), \mathsf{g}_{(t,x)}$ (see Definition 3.1.10).

Differentiability results for the semigroup associated to scalar conservation laws that are analogous to Theorem 3.1.1 are present in the literature only in the case of a convex flux ([**98**], [**44**], [**45**]). In this case one can take advantage of the fact that the flow generated by λ is monotone (as in [**45**]) or Hamilton-Jacobi formulations ([**98**]) to obtain that the limiting ρ is a duality solution (see again [**45**] and the references therein) with a monotone vector field λ. Instead in [**44**] a finer differentiability is studied, more similar to our Theorem 3.1.6: they prove that the semigroup associated to a convex flux is "shift differentiable". In essence, this means that locally near shock points, every perturbation, at the first order, is a translation of the limiting shock in some direction, we refer to [**44**] for more details.

The Chapter is organized as follows.

In **Section 3.1** we provide a detailed road map of the results and a sketch of the proofs.

In **Section 3.3** we recall some objects and results introduced in [**37**], related to the concept of admissible boundaries, and to their properties. We then use these results to study graph-convergence properties of entropy solutions with the topology induced by the Kuratowski convergence for sets.

In **Section 3.4** we prove Theorem 5. The fundamental tool is the graph convergence result proved in the previous section.

In **Section 3.5** we prove Theorem 6. This is the most technical part of the chapter. We use the results of Section 3.4 to study various properties of the disintegration $\{a_{t,x}\}$, that at the end will yield Theorem 6.

In **Section 3.6** we prove Theorem 7. The results of Section 3.5 are used to identify the candidate blow-up limits (composite shifts). Here we show how the properties of the disintegration $\{a_{t,x}\}$ established in Section 3.5 transfer in a natural way to the blow up limits U (10). A final argument, again using admissible boundaries (Section 3.3), is the key point to conclude the proof of Theorem 7.

Intermediate Domains for Scalar Conservation Laws

In **Chapter 4** we consider a scalar conservation law

$$u_t + f(u)_x = 0, \tag{11}$$

with strictly convex flux. By a classical result [**51, 76**], there exists a contractive semigroup $S : \mathbf{L}^1(\mathbb{R}) \times \mathbb{R}_+ \mapsto \mathbf{L}^1(\mathbb{R})$ such that, for every initial datum

$$u(0, \cdot) = \overline{u} \in \mathbf{L}^1(\mathbb{R}), \tag{12}$$

the trajectory $t \mapsto u(t) = S_t\overline{u}$ is the unique entropy weak solution of the Cauchy problem.

It is well known that, even for smooth initial data, the solution can develop shocks in finite time. Taking an abstract point of view, consider the operator $Au \doteq \frac{\partial}{\partial x} f(u)$ which generates the semigroup. Then there exists data $\overline{u} \in Dom(A)$ in the domain of the generator, such that $S_\tau\overline{u} \notin Dom(A)$ for some $\tau > 0$. In other words, the domain of the generator is not positively invariant. To address this issue, the paper [**52**] introduced a definition of "generalized domain" $\mathcal{D}$ for the operator. This consists of all initial data $\overline{u}$ for which the map $t \mapsto S_t\overline{u}$ is globally Lipschitz continuous. Notice that for the conservation law (11) one has

$$\mathbf{L}^1 \cap BV \subseteq \mathcal{D}.$$

Using the fact that the semigroup is contractive, it is easy to show that this generalized domain is positively invariant. Indeed, the quantity

$$\limsup_{\varepsilon \to 0+} \frac{\left\| S_{t+\varepsilon}\overline{u} - S_t\overline{u} \right\|_{\mathbf{L}^1}}{\varepsilon}$$

is a non-increasing function of t. The aim of this chapter is to study intermediate domains

$$\mathcal{D}_\alpha \subset \mathbf{L}^1(\mathbb{R}), \qquad 0 < \alpha < 1, \tag{13}$$

related to the decay properties of the corresponding trajectories of (11). As in [**7**] we define

$$\mathcal{D}_\alpha \doteq \left\{ \overline{u} \in \mathbf{L}^1(\mathbb{R})\,;\, \sup_{0<t<1} t^{-\alpha} \left\| S_t\overline{u} - \overline{u} \right\|_{\mathbf{L}^1} < +\infty \right\}. \tag{14}$$

We also consider the domains (slightly different from the ones considered in [**7**])

$$\widetilde{\mathcal{D}}_\alpha \doteq \Big\{ \overline{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})\,;\ \sup_{0<t<1} t^{1-\alpha} \cdot \text{Tot.Var.}\big\{S_t \overline{u}\big\} < +\infty \Big\}. \tag{15}$$

The domains $\mathcal{D}_\alpha$ arise naturally in connection with balance laws:

$$u_t + f(u)_x = g(t,x). \tag{16}$$

Indeed, as shown in [**7**], one has

Proposition 1. *Let $f \in \mathcal{C}^2$ with $f''(u) \geq c > 0$ for all $u \in \mathbb{R}$. Consider a compactly supported solution $u = u(t,x)$ of (16), and assume that the source term satisfies*

$$\big\| g(t,\cdot) \big\|_{\mathbf{L}^1} \leq C \qquad\qquad \forall t \in [0,T]. \tag{17}$$

Then for every $0 < t \leq T$, one has $u(t,\cdot) \in \mathcal{D}_{1/2}$.

We remark, however, that some of the functions $u(t,\cdot)$ can be unbounded. In particular, they can have infinite total variation.

In the theory of linear analytic semigroups [**71, 81**], intermediate domains arise naturally as domains of fractional powers of sectorial operators. The faster decay of solutions is usually related to higher Sobolev regularity of the initial data. In particular, this theory applies to semilinear equations of the form

$$u_t - \Delta u = F(x, u, \nabla u), \qquad\qquad u(0,\cdot) = \overline{u}. \tag{18}$$

Under natural assumptions (see [**71**] for details), this Cauchy problem is well posed provided that the initial datum $\overline{u}$ lies in the domain of some fractional power $(-\Delta)^\alpha$ of the generator.

Our eventual goal is to develop a similar theory of intermediate domains for nonlinear semigroups generated by conservation laws. In particular, we conjecture that the Cauchy problem for a genuinely nonlinear 2×2 hyperbolic system with $\mathbf{L}^\infty$ initial data [**35, 68**] is well posed within an intermediate domain such as (14) or (15).

As a first step in this research program, here we focus our attention on the scalar conservation law (11), seeking conditions on the initial data $\overline{u} \in \mathbf{L}^1(\mathbb{R})$ that will imply $\overline{u} \in \mathcal{D}_\alpha$ or $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$, respectively. Assumptions that imply $\overline{u} \in \mathcal{D}_\alpha$ can be readily formulated in terms of fractional Sobolev regularity. On the other hand, conditions that guarantee a faster decay rate of the total variation are more subtle. Here we consider the assumption

($\mathbf{P}_\alpha$) *For every $\lambda \in\,]0,1]$, there exists an open set $\boldsymbol{V}(\lambda) \subset \mathbb{R}$ such that the following holds.*

$$\mathscr{L}^1\big(\boldsymbol{V}(\lambda)\big) \leq C\,\lambda^\alpha, \tag{19}$$

$$\textit{Tot.Var.}\big\{\overline{u}\,;\mathbb{R}\setminus \boldsymbol{V}(\lambda)\big\} \leq C\,\lambda^{\alpha-1}, \tag{20}$$

for some constant C independent of λ.

Roughly speaking, $\overline{u}$ can have unbounded variation, but most of its variation should be concentrated on a set with small Lebesgue measure. Our main result establishes is the following.

Theorem 8. *We have the following two cases:*

(1) If $1/2 < \alpha < 1$, then

$$\overline{u} \text{ satisfies } (\mathbf{P}_\alpha) \qquad \Longrightarrow \qquad \overline{u} \in \widetilde{\mathcal{D}}_\alpha \tag{21}$$

(2) If $0 < \alpha \leq 1/2$, the above implication is false.

The proof of (21) relies on a structural result for functions satisfying $(\mathbf{P}_\alpha)$, which is of independent interest. Indeed, Theorem 4.4.1 provides a nonlinear "Fourier-type" decomposition of such functions, in components which oscillate more and more rapidly.

In addition to genuinely nonlinear conservation laws, several other examples of nonlinear semigroups with regularizing properties are known in the literature, see in particular [**28, 32, 53, 93**].

Chapter 4 is organized as follows.

In **Section 4.1** we describe a general class of metric interpolation spaces, for functions defined on a set $\Omega \subseteq \mathbb{R}^N$. This yields a natural way to formulate conditions such as $(\mathbf{P}_\alpha)$, in a general setting.

Section 4.2 contains some examples. The first one (Fig. 1) shows how to construct an initial datum $\overline{u}$ with unbounded variation, such that $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$, for any given $0 < \alpha < 1$. We then consider initial data consisting of a packet of triangular waves (Fig. 3). By suitably choosing the size and distance of these triangular blocks we show that, if $0 < \alpha \leq 1/2$, then there exists an initial datum $\overline{u}$ that satisfies $(\mathbf{P}_\alpha)$ and yet $\overline{u} \notin \widetilde{\mathcal{D}}_\beta$ for any $\beta \in]0, 1[$. As stated in Proposition 4.2.2, the implication (21) thus cannot hold for $\alpha \leq 1/2$.

Section 4.3 is concerned with the intermediate domain $\mathcal{D}_\alpha$. For $0 < \alpha < 1$ we prove that any one of the conditions: (i) $\overline{u}$ lies in the fractional Sobolev space $W^{\alpha,1}(\mathbb{R})$, or (ii) $\overline{u}$ satisfies $(\mathbf{P}_\alpha)$, implies $\overline{u} \in \mathcal{D}_\alpha$. These results are valid for any flux $f \in \mathbf{C}^1$, not necessarily convex.

Section 4.4 establishes further properties of functions which satisfy $(\mathbf{P}_\alpha)$, proving a useful decomposition result, stated in Theorem 4.4.1. Finally, in Section 4.5 we prove our main theorem, showing that for $1/2 < \alpha < 1$ the property $(\mathbf{P}_\alpha)$ implies that $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$. To simplify the exposition, the proofs will first be given for Burgers' equation. In Remark 4.5.2 we observe that all results remain valid for a conservation law with uniformly convex flux.

Part II

Conservation Laws with Discontinuous Flux

In Part II, we consider the initial value problem for the scalar conservation law in one space dimension

$$u_t + f(x,u)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{22}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{23}$$

where $u = u(x,t)$ is the state variable and the flux f is a space discontinuous function given by

$$f(x,u) = \begin{cases} f_l(u), & x < 0, \\ f_r(u), & x > 0\,. \end{cases} \tag{24}$$

We assume that $f_l, f_r : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable, uniformly convex maps that satisfy

$$f_l''(u),\ f_r''(u) \geq a > 0 \tag{25}$$

with critical points respectively θ_l, θ_r. Conservation laws with discontinuous flux serve as mathematical models for: oil reservoir simulation [**65, 66**]; traffic flow dynamics with roads of varying amplitudes or surface conditions [**88**]; radar shape-from-shading problems [**90**]; blood flow in endovascular treatments [**63, 47**]; and for many other different applications (see [**9**] and references therein).

We recall that problems of this type do not possess classical solutions globally defined in time (even in the continuous flux case when $f_l = f_r$), since, regardless of how smooth the initial data are, they can develop discontinuities (shocks) in finite time because of the nonlinearity of the equation. To achieve existence results, one has to look for weak distributional solution that, for sake of uniqueness, satisfy the classical Kruzkov entropy inequalities away from the point of flux discontinuity, and a further interface entropy condition at the flux-discontinuity interface $x = 0$.

Various type of interface-entropy conditions have been introduced in the literature according with the different physical phenomena modelled by (22) (see [**20, 23**]). Here, as in [**9**], for modellization and control treatment reasons we employ an admissibility criterion involving the so-called *interface connection* (A, B), which yields the Definition 5.1.2 of *AB-entropy solution* (cfr.[**4, 27**]). A connection (A, B) is a pair of states connected by a stationary weak solution of (22), taking values A for $x < 0$, and B for $x > 0$, which has characteristics diverging from (or parallel to) the flux-discontinuity interface $x = 0$ (see Definition 5.1.1). The admissibility criterion for an *AB-entropy solution* can be equivalently formulated in terms of an interface entropy condition or of Kruzkov-type entropy inequalities adapted to the particular connection (A, B) taken into account (cfr. [**4, 25, 27**]). Relying on these extended entropy inequalities and using an adapted version of the Kruzkov doubling of variables argument, one can establish $\mathbf{L^1}$-stability and uniqueness of AB-entropy solutions to the Cauchy problem (22)-(23) (see [**27, 64**]). We shall adopt the semigroup notation $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$ for the unique solution of (22)-(23).

Here we are concerned as in [**2, 9**] with a controllability problem for (22) where one regards the initial data as controls and study the corresponding *attainable set* at a fixed time $T > 0$:

$$\mathcal{A}^{[AB]}(T) \doteq \left\{ \mathcal{S}_T^{[AB]+} u_0 \ : \ u_0 \in \mathbf{L}^\infty(\mathbb{R}) \right\}. \tag{26}$$

In the same spirit of [**60, 61, 70, 78**] we introduce a *backward solution operator* (see Definition 5.1.16)

$$\mathcal{S}_T^{[AB]-} : \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R}), \qquad \omega \mapsto \mathcal{S}_T^{[AB]-} \omega \,, \tag{27}$$

and we characterize the attainable targets for (22) at a time horizon $T > 0$ as fixed-points of the composition *backward-forward operator* $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}$, as stated in our first main result:

THEOREM 9. *Let f be a flux as in* (24) *satisfying the assumption* (25)*, and let* (A, B) *be a connection. Then, for every $T > 0$, and for any $\omega \in \mathbf{L}^\infty(\mathbb{R})$, the following conditions are equivalent.*

(1) $\omega \in \mathcal{A}^{[AB]}(T)$,

(2) $\omega = \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-} \omega$.

Moreover, if (A,B) is a non critical connection, i.e. if $A \neq \theta_l, B \neq \theta_r$, then the condition (2) is equivalent to

(1)' $\omega \in \mathcal{A}_{bv}^{[AB]}(T)$, where

$$\mathcal{A}_{bv}^{[AB]}(T) \doteq \left\{ \mathcal{S}_T^{[AB]+} u_0 \ : \ u_0 \in BV_{loc}(\mathbb{R}) \right\}, \tag{28}$$

and it holds true

$$\mathcal{A}^{[AB]}(T) = \mathcal{A}_{bv}^{[AB]}(T). \tag{29}$$

Clearly the main content of Theorem 9 are the implication (1) $\Longrightarrow$ (2) and (1)' $\Longrightarrow$ (2), since the reverse implications are straightforward once we define the backward operator $\mathcal{S}_T^{[AB]-}$ and verify that, in the case of a non critical connection, one has $\mathcal{S}_T^{[AB]-} u_0 \in BV_{loc}(\mathbb{R})$ for all $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. This last property is an immediate consequence of Proposition 5.5.1, since the backward operator $\mathcal{S}_T^{[AB]-}$ is defined in terms of the forward operator $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}$ (see Definition 5.1.16).
The proof of (1) $\Longrightarrow$ (2) and (1)' $\Longrightarrow$ (2) are obtained in two steps:

(I) First, we show that any attainable profile $\omega \in \mathcal{A}^{[AB]}(T)$ belongs to a class of functions $\mathscr{A} \subset BV_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ which satisfy suitable Oleinik-type inequalities and pointwise constraints related to the (A,B)-connection in intervals containing the origin (see Theorem 5.3.3, 5.3.11, 5.3.14).

(II) Next, we prove that any element of $\mathscr{A}$ is a fixed point of the composition backward-forward operator $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}$. Namely, for any given $\omega \in \mathscr{A}$ we construct an initial datum $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ such that $\omega = \mathcal{S}_T^{[AB]+} u_0$, and then we show that indeed $u_0 = \mathcal{S}_T^{[AB]-} \omega$.

These two steps are firstly carried out in the case of a non critical connection (A,B) and of attainable profiles $\omega \in \mathcal{A}^{[AB]}(T) \cap BV_{loc}(\mathbb{R})$. The proofs are obtained exploiting as in [**9**] the theory of generalized characteristics by Dafermos [**55**], applied to the setting of discontinuous flux, and relying on the duality property of the backward and forward solution operators. Next, we address the case of a critical connection and of attainable profiles $\omega \in \mathcal{A}^{[AB]}(T)$ relying on the $\mathbf{L^1}$-stability of the map $(A,B,u_0) \mapsto \mathcal{S}_t^{[AB]+} u_0$ (see Theorem 5.1.8).

Some remarks are here in order.

- The results of Theorem 9 extend to the present setting of space discontinuous fluxes the similar *characterization of attainable profiles in terms of the backward solution operator* obtained in [**50**, Theorem 3.1, Corollary 3.2] and [**70**, Corollary 1] for conservation laws with strictly convex flux independent on the space variable.
- The characterization of $\mathcal{A}^{[AB]}(T)$ obtained here unveils the presence of *a class of attainable states for non critical* connections that were *not detected in* [**2, 9**], see Remark 5.3.18.
- The characterization of attainable profiles for (22), (24) in terms of unilateral constraints and Oleinik-type estimates provides a powerful tool to investigate regularity properties of the solutions to (22), (24). In particular, we build on such a characterization to derive *uniform BV bounds* on AB-entropy solutions with initial datum in $\mathbf{L}^\infty$ (in the case of non critical connections), and *on the flux of AB-entropy solutions* (for

general connections). This is a fairly non-trivial result since it is well known [**1**] that the total variation of AB-entropy solutions may well blow up in a neighborhood of the flux-discontinuity interface $x = 0$. Thanks to these uniform BV bounds, we can then establish the $\mathbf{L^1_{loc}}$*-Lipschitz continuity in time of AB-entropy solutions*.

- The proof that Theorem 9 holds for critical connections once we know that Theorem 9 is verified by non critical connections relies on a perturbation argument for attainable profiles. This construction yields an *approximate controllability* result since it provides a general explicit procedure to approximate an attainable profile for a critical connection by attainable profiles for non critical connections.

Note furthermore that, by the backward-non uniqueness of (22) (due to the possible presence of shocks in its solutions), there may exist in general multiple initial data u_0 that are steered by (22) to $\omega \in \mathcal{A}^{[AB]}(T)$. In fact, an important control problem related to the one considered here is the inverse design, which has the goal to reconstruct the set of initial data u_0 evolving to a given attainable target ω (see [**50, 70, 78**] for conservation laws with convex flux independent on the space variable, and [**61**] for Hamilton-Jacobi equations with convex Hamiltonian). On the other hand, when a target state ω is not attainable at time $T > 0$, the image of ω through the backward-forward operator $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}$ represents a natural candidate to construct a reachable function which is "as close as possible" (in an appropriate sense) to the observed state ω (see [**60**] in the case of Hamilton-Jacobi and Burgers equations).

The results of the present chapter provide a key building block to address both of these problems, namely the characterization of the aforementioned set of initial data leading to a given attainable target ω for (22), and the analysis of the properties of the backward-forward operator $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}$ related to optimization problems for unattainable target profiles, which are pursued in the forthcoming paper [**19**].

In the case of non-convex flux, an explicit characterizations of the attainable set in terms of Oleinik-type estimates seems difficult to obtain and only partial results are present in the literature, see for example [**21**]. For systems of conservation laws, the problem has been considered in [**22**] (triangular systems) and in [**48**] (chromatography system). For a characterization of the attainable set in terms of fixed points of a backward-forward operator, a key point would be to provide a proper definition of backward operator in these more general contexts, which is lacking at the moment, making also the analysis of the inverse design problem nontrivial.

The Chapter is organized as follows. In **Section 5.1** we recall the definitions of interface connection (A, B), of AB-entropy solution and of AB-backward solution operator. We also collect the stability properties of the $\mathbf{L^1}$-contractive semigroup of AB-entropy solutions. In **Section 5.2** we establish the duality property of the backward and forward solution operators, which constitutes a fundamental ingredient of the proof of Theorem 9. **Section 5.3** collects the precise statements of the results on the characterization of the attainable set $\mathcal{A}^{[AB]}(T)$ via Oleinik-type inequalities and state constraints. We also include the statement of Theorem 5.3.17 which contains the equivalence of conditions (1), (2) of Theorem 9 with the characterization of $\mathcal{A}^{[AB]}(T)$ in terms of Oleinik-type inequalities and unilateral constraints. In **Section 5.4** we carry out the rather technical and involved proof of Theorem 5.3.17. At the beginning of the section, for reader's convenience, we provide a roadmap of the proof of Theorem 5.3.17, where we also highlight the key innovative parts of the argument. In **Section 5.6** we establish the $\mathbf{L^1}$-stability properties of the semigroup of AB-entropy solutions with respect to time and with respect to the connections. We also derive uniform BV bounds on AB-entropy solutions in the case of non critical connections, and on the flux of AB-entropy solutions for general connections. In **Section 5.7** we provide, for sake of completeness, a simple proof of the non existence of rarefactions emanating from the interface $x = 0$, which is a distinctive feature of AB-entropy solutions. Finally, in **Section 5.8** we derive some lower/upper $\mathbf{L^1}$-semicontinuity property for solutions to conservation laws, used to recover the proof of Theorem 5.3.17 in the case of critical connections once we know the validity of Theorem 5.3.17 for non critical connections.

We conclude the introduction with some references:

- **Chapter 1, 3** are contained in [**13**], [**14**],
- The kinetic formulation of **Chapter 2** is part of [**15**],
- **Chapter 4** is contained in [**8**],
- **Chapter 5** is contained in [**11**] (see also the related [**12**]).

Part 1

Regularity, Decay, Differentiability for Solutions to Conservation Laws

CHAPTER 1

Lagrangian Representation and Applications to Regularity

1.1. Kinetic Formulation of Finite Entropy Solutions

Let a smooth *flux* function $\boldsymbol{f}:\mathbb{R}\to\mathbb{R}^d$ be given. Given an open set $\Omega\subset\mathbb{R}^{d+1}$, our aim is to study *balance laws* of the form

$$\partial_t u+\operatorname{div}_x \boldsymbol{f}(u)\;=\;\nu \qquad \text{in } \mathscr{D}'_{t,x}(\Omega). \tag{1.1.1}$$

In such case we say that u is a distributional (weak) solution to the balance law with flux $\boldsymbol{f}$, and measure source term ν. Most of the times we will take $\Omega=[0,T]\times\mathbb{R}$ for some $T>0$. Among weak solutions of (1.1.1), a particularly relevant class is the one of *finite entropy* (or finite energy) solutions. We remark that, since we will always consider solutions in $\mathbf{L}^\infty$, it is not restrictive, up to a translation and a dilation, to assume that any solution takes values in the interval $[0,1]$.

Definition 1.1.1. We say that $u\in\mathbf{L}^\infty(\Omega;[0,1])$ is a *finite entropy* solution with flux $\boldsymbol{f}$ if for every entropy-entropy flux pair $(\eta,\boldsymbol{q})$, with $\eta\in\mathscr{C}^2(\mathbb{R})$, there exists a locally finite measure $\mu_\eta\in\mathscr{M}(\Omega)$ such that

$$\partial_t\eta(u)+\operatorname{div}_x \boldsymbol{q}(u)\;=\;\mu_\eta \qquad \text{in } \mathscr{D}'_{t,x}(\Omega). \tag{1.1.2}$$

Since in particular the pair $(\eta_0,\boldsymbol{q}_0)$ defined by $\eta_0(u)=u$, $\boldsymbol{q}_0(u)=\boldsymbol{f}(u)$ is an entropy-entropy flux pair, if u is a finite entropy solution we find that u solves (1.1.1) with $\nu=\mu_{\eta_0}$.

Remark 1.1.2. The more classical concept of entropy solution to the conservation law

$$\partial_t u+\operatorname{div}_x \boldsymbol{f}(u)=0$$

can be obtained by requiring, in addition, that $\mu_{\eta_0}=0$ and that for every *convex* entropy η, the corresponding entropy measure μ_η is nonpositive.

In [**80**], Lions, Perthame and Tadmor characterize entropy solutions via a *kinetic formulation.* In particular, they prove that u is an entropy solution to a conservation law if and only if, with

$$\chi[u](t,x,v)\equiv\chi(t,x,v)\doteq\begin{cases}1 & \text{if } 0\le v\le u(t,x),\\ 0 & \text{otherwise}\end{cases}$$

then there exists a *positive* measure $\mu\in\mathscr{M}^+(\Omega)$ such that

$$\partial_t\chi+\boldsymbol{f}'(v)\cdot\nabla_x\chi=\partial_v\mu \qquad \text{in } \mathscr{D}'(\Omega\times\mathbb{R}). \tag{1.1.3}$$

In [**56**], it is shown how to adapt the kinetic formulation to finite entropy solutions as in Definition 1.1.1. In particular, the defect measure μ does not have a definite sign, and the presence of a source term is encoded in the fact that μ need not have compact support in

"v". Here we use a slightly different formulation, that immediately follows from the one of [**56**]. We write its proof for completeness.

PROPOSITION 1.1.3 ([**56**]). *A function* $u \in \mathbf{L}^\infty(\Omega;[0,1])$ *is a finite entropy solution in the sense of Definition 1.1.1 if and only if, with there exist locally finite measures* $\mu_0, \mu_1 \in \mathscr{M}(\Omega \times \mathbb{R})$ *with* $\operatorname{supp} \mu_i \subset \Omega \times [0,1]$, *such that*

$$\partial_t \chi + \boldsymbol{f}'(v) \cdot \nabla_x \chi = \partial_v \mu_1 + \mu_0 \qquad in\ \mathscr{D}'(\Omega \times \mathbb{R}). \tag{1.1.4}$$

REMARK 1.1.4. Clearly the measures μ_0, μ_1 such that (1.1.4) holds are not unique. It is possible to choose the measures μ_1, μ_0 such that their support is contained in the boundary of the hypograph of u:

$$\operatorname{supp} \mu_i \subset \partial \operatorname{hyp} u \qquad i = 0, 1.$$

However, later on we will need to make an ever finer choice on the measures μ_0, μ_1, that in particular implies the above condition.

PROOF. We first show that if u satisfies (1.1.4), then u is a finite entropy solution. Let $(\eta, \boldsymbol{q})$ be any entropy-entropy flux pair. Let $U \Subset \Omega$ and for every test function $\phi \in \mathscr{D}(\Omega)$, with $\operatorname{supp} \phi \subset U$, we can compute, omitting the variables (t,x),

$$\begin{aligned}\int_\Omega \phi_t\, \eta(u) + \nabla_x \phi \cdot \boldsymbol{q}(u) \,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \int_{\mathbb{R}} \chi(v) \Big(\phi_t \eta'(v) + \nabla_x \phi\, \eta'(v)\ f'(v) \Big) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \int_{\mathbb{R}} \phi\, \eta'\ \mathrm{d}\mu_0(t,x,v) - \int_\Omega \int_{\mathbb{R}} \phi\, \eta''\ \mathrm{d}\mu_1(t,x,v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In particular, letting $C_U = \|\mu_0\|_{\mathscr{M}(U)} + \|\mu_1\|_{\mathscr{M}(U)}$ and defining the distribution

$$T_\eta \doteq \partial_t \eta(u) + \nabla_x \boldsymbol{q}(u)$$

we obtain the bound

$$\langle T_\eta,\, \phi \rangle \le C_U\, \|\phi\|_{C^0}\, \|\eta'\|_{C^1} \qquad \forall\, \phi \in \mathscr{D}(\Omega) \qquad \text{with } \operatorname{supp} \phi \subset U. \tag{1.1.5}$$

By Riesz theorem, T_η can be identified with a locally finite measure.

Conversely, assume that u is a finite entropy solution as in Definition 1.1.1. Define a distribution $T \in \mathscr{D}'(\Omega \times \mathbb{R})$ by

$$\langle T, \phi\, \varrho \rangle \doteq \int_\Omega \phi_t \eta_\varrho(u)\ +\ \nabla_x \phi \cdot \boldsymbol{q}_\varrho(u) \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \phi \in \mathscr{D}(\Omega), \quad \varrho \in \mathscr{D}(\mathbb{R})$$

where we define the entropy-entropy flux pair associated to ϱ

$$\eta_\varrho(v) \doteq \int_0^v \int_0^z \varrho(w)\,\mathrm{d}w\,\mathrm{d}z, \qquad \boldsymbol{q}_\varrho(v) \doteq \int_0^v \boldsymbol{f}'(z) \eta'_\varrho(z)\,\mathrm{d}z.$$

We again consider any $U \Subset \Omega$ and for any $\phi \in \mathscr{D}(U)$ we define a linear functional $L_\phi : \mathbf{C}(\mathbb{R}) \to \mathbb{R}$ by

$$L_\phi(\varrho) \doteq \int_U \phi_t \eta_\varrho(u)\ +\ \nabla_x \phi \cdot \boldsymbol{q}_\varrho(u)\,\mathrm{d}x\,\mathrm{d}t.$$

Each functional L_ϕ is bounded, and therefore also continuous, since it holds

$$|L_\phi(\varrho)| \le C_{U,\phi}\, \|\varrho\|_{C^0} \qquad \forall \varrho \in \mathbf{C}(\mathbb{R})$$

for some constant $C_{U,\phi}$ depending only the set U and the $\mathbf{C}^1$ norm of the function ϕ. Since u is a finite entropy solution, we deduce that the family of functionals L_ϕ is pointwisely bounded, because

$$\sup_{\substack{\phi\in\mathscr{D}(U)\\ |\phi|\le 1}} |L_\phi(\varrho)| \le \int_U \mathrm{d}|\mu_{\eta_\varrho}|.$$

Therefore, by the uniform boundedness principle, the family L_ϕ is uniformly (norm) bounded, that is

$$\sup_{\substack{\phi\in\mathscr{D}(U)\\ |\phi|\le 1,\,|\varrho|\le 1}} |L_\phi(\varrho)| = \sup_{\substack{\phi\in\mathscr{D}(U)\\ |\phi|\le 1,\,|\varrho|\le 1}} |\langle T, \phi\varrho\rangle| \le C_U. \tag{1.1.6}$$

We also notice that by definition of L_ϕ, it holds

$$L_\phi(\varrho) = 0 \qquad \forall \varrho \in \mathbf{C}(\mathbb{R}) \quad \text{with } \operatorname{supp}\varrho \subset \mathbb{R}\setminus[0,1]. \tag{1.1.7}$$

Combining (1.1.6), (1.1.7) with the Riesz representation theorem, we then obtain the existence of a locally finite measure $\mu_1 \in \mathscr{M}(\Omega\times\mathbb{R})$, with $\operatorname{supp}\mu_1 \subset \Omega\times[0,1]$, such that

$$\langle T, \phi\varrho\rangle = \int_\Omega\int_{\mathbb{R}} \varphi(t,x)\varrho(v)\,\mathrm{d}\mu_1(t,x,v) \qquad \forall\phi\in\mathscr{D}(\Omega), \quad \varrho\in\mathscr{D}(\mathbb{R}).$$

Finally, we calculate, for any $\phi\in\mathscr{D}(\Omega)$ and $\eta'\in\mathscr{D}(\mathbb{R})$,

$$\begin{aligned}\Big\langle \partial_t\chi + \boldsymbol{f}'(v)\nabla_x\chi,\, \phi\eta'\Big\rangle &= \Big\langle \partial_t\chi + \boldsymbol{f}'(v)\nabla_x\chi,\, \phi(\eta'-\eta'(0))\Big\rangle \\ &\quad + \Big\langle \partial_t\chi + \boldsymbol{f}'(v)\nabla_x\chi,\, \phi\eta'(0)\Big\rangle \\ &= -\int_\Omega\int_{\mathbb{R}} \phi(t,x)\eta''(v)\,\mathrm{d}\mu_1 + \eta'(0)\int_\Omega \phi(t,x)\,\mathrm{d}\mu_{\eta_0}.\end{aligned}$$

The result follows by setting $\mu_0 = \mu_{\eta_0}\times\delta_0$. □

1.2. Lagrangian Representation

In this section we introduce the concept of *Lagrangian representation* of finite entropy solution to the balance law (1.1.1), which is the main tool that we will use throughout the rest of this chapter. This kind of representation was introduced for the first time in [**34**] for entropy solutions to scalar conservation laws:

$$\partial_t u + \operatorname{div}_x \boldsymbol{f}(u) \;=\; 0. \tag{1.2.1}$$

The approach is inspired by the so-called *transport-collapse* scheme, introduced by Y. Brenier in [**39**]. Brenier's idea is to approximate any solution of (1.2.1) via an operator-splitting method. First, one fixes a small parameter Δt. Then define a *free transport* operator, called $\mathsf{Tr}_{\Delta t}$, defined on sets that are hypograph of functions $u:\mathbb{R}^d\to[0,1]$, by

$$\mathsf{Tr}_{\Delta t}\big[\operatorname{hyp} u\big] \doteq \Big\{(x,v) \;\big|\; v \le u\big(x - \Delta t\boldsymbol{f}'(v)\big)\Big\} \subset \mathbb{R}^d\times[0,1].$$

The key point is that now the transported set as defined above need not to be an hypograph of a function: to restore this property, one defines a *collapse* operator, defined on sets $E\subset\mathbb{R}^d\times[0,1]$, by

$$\mathsf{C}[E] \doteq \Big\{(x,v) \;\big|\; x\in\mathbb{R}^d, \quad v\in[0,\mathscr{H}^1(E_x)]\Big\}$$

where $E_x = \{v \in [0,1] \mid (x,v) \in E\}$ is the x-section of E. Note that now, $\mathsf{C}[E]$ is an hypograph of a function. Starting with an initial datum $u_0 \in \mathbf{L}^\infty(\mathbb{R}^d)$, one then defines approximate solutions $u_{\Delta t}$ to (1.2.1), that at some time $t > 0$ are defined by

$$u_{\Delta t}(t) \doteq \mathrm{hyp}^{-1} \circ \underbrace{\mathsf{C} \circ \mathsf{Tr}_{\Delta t} \circ \ldots \circ \mathsf{C} \circ \mathsf{Tr}_{\Delta t}}_{k \text{ times}} \left[\mathrm{hyp} u_0\right], \qquad k = \left\lfloor \frac{t}{\Delta t} \right\rfloor.$$

As shown in [**39**], the approximate solutions $u_{\Delta t}$ converge to the unique entropy solution of (1.2.1) with initial datum u_0.

THEOREM 1.2.1 ([**39**]). *For every $u_0 \in \mathbf{L}^\infty(\mathbb{R}^d; [0,1])$, one has the convergence*

$$u_{\Delta t}(t) \longrightarrow S_t u_0 \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R}^d) \qquad \text{as } \Delta t \to 0^+.$$

The Lagrangian representation of [**34**] is a far reaching, measure-theoretic version, of the transport-collapse scheme of Brenier, that allows one to track the movement of each particle of the hypograph of an entropy solution u. In fact, it can be thought as the method of characteristics but applied at the level of the kinetic equation (1.1.3). In [**34**], the authors use the Lagrangian representation to prove that continuous solutions to the conservation law (1.2.1) do not dissipate any entropy.

In [**84**] the Lagrangian representation is introduced also for solutions with finite entropy production to the conservation law (1.2.1) (i.e., no source terms are present, but the measures μ_η can change sign) and later on, in [**86**], it is used to prove that, in the one dimensional case and for Burgers equation $f(u) = u^2/2$, the entropy dissipation measures are concentrated on a 1-rectifiable set. In this case the existence of a Lagrangian representation was proved via an explicit approximation procedure, similar to the transport-collapse scheme. In our setting, to handle the source terms, we adopt a different approach based on Smirnov's Theorem on the representation of 1-currents. This method provides finer control over the behavior of the Lagrangian curves, which will be essential in the subsequent analysis.

1.2.1. Definition of the Lagrangian Representation. In this subsection we define what is a Lagrangian representation for a finite entropy solution to the balance law (1.1.1). In the following, we denote by Γ the space of curves

$$\Gamma \doteq \left\{ (I_\gamma, \gamma) \mid \gamma = (\gamma^x, \gamma^v) : I_\gamma \to \mathbb{R}_x \times \mathbb{R}^+_v, \;\; \gamma^x \text{ is continuous and } \gamma^v \text{ is in } BV(I_\gamma) \right\}$$

where $I_\gamma = (t^1_\gamma, t^2_\gamma)$ is the domain of definition of a curve γ.

DEFINITION 1.2.2. A *Lagrangian representation* of a finite entropy solution u to (1.1.1) is a positive finite measure $\boldsymbol{\omega} \in \mathscr{M}^+(\Gamma)$ such that:

(1) it holds

$$\chi \cdot \mathscr{L}^{d+2} = \int_\Gamma (\mathrm{id}, \gamma)_\sharp \mathscr{L}^1 \llcorner I_\gamma \, \mathrm{d}\boldsymbol{\omega}(\gamma) \tag{1.2.2}$$

(2) ω is concentrated on the set of curves $\gamma = (\gamma^x, \gamma^v) \in \Gamma$ such that

$$\dot{\gamma}^x = f'(\gamma^v(t)), \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in I_\gamma, \tag{1.2.3}$$

(3) the following integral bound holds: for every compact $K \subset \mathbb{R}^d$ and $T > 0$, one has

$$\begin{aligned} \int_\Gamma \text{Tot.Var.}\big(\gamma^v,\ [t^1_\gamma, T)\big) \cdot \mathbf{1}_{\{\gamma^x(t^1_\gamma)\in K\}}(\gamma)\ \mathrm{d}\boldsymbol{\omega}(\gamma) < C_{K,T} \\ \sum_{i=1}^2 \boldsymbol{\omega}\Big(\big\{\gamma \in \Gamma \mid (t, \gamma^x(t^i_\gamma)) \in (0,T)\times K\big\}\Big) \le \widetilde{C}_{K,T} \end{aligned} \tag{1.2.4}$$

The link between the notion of kinetic solution and Lagrangian representation is given by the following proposition.

Theorem 1.2.3. *A function $u \in \mathbf{L}^\infty(\Omega,\ [0,1])$ is a finite entropy solution of* (1.1.1) *if and only if admits a Lagrangian representation.*

We prove that if u admits a Lagrangian representation, then u is a finite entropy solution. The opposite implication is deeper: the proof, which relies on the structure of 1-dimensional normal currents in $\mathbb{R}^n$, is carried out in the next subsection (see Theorem 1.2.15).

Proof. Assume that u admits a Lagrangian representation $\boldsymbol{\omega}$. We have to show the existence of a pair of locally finite measures $\mu_0, \mu_1 \in \mathscr{M}(\Omega\times\mathbb{R})$ of Radon measures such that (1.1.4) holds. To do so, we consider the distribution

$$S \doteq \partial_t \chi + \boldsymbol{f}'(v)\cdot\nabla_x\chi$$

and calculate, for every test functions $\eta' \in \mathscr{D}(\mathbb{R})$ and $\phi \in \mathscr{D}(\Omega)$,

$$\begin{aligned} -\langle S, \eta'\phi\rangle &= \int_\Omega\int_\mathbb{R} \eta'(v)\chi(t,x,v)[\phi_t(t,x) + \boldsymbol{f}'(v)\cdot\nabla_x\phi(t,x)]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\overset{\text{by (1.2.2)}}{=} \int_\Gamma\int_{I_\gamma} \eta'(\gamma^v(t))[\phi_t(t,\gamma^x(t)) + \boldsymbol{f}'(\gamma^v(t))\cdot\nabla_x\phi(t,\gamma^x(t))]\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\omega}(\gamma) \\ &\overset{\text{by (1.2.3)}}{=} \int_\Gamma\int_{I_\gamma} \eta'(\gamma^v(t))[\phi_t(t,\gamma^x(t)) + \dot{\gamma}^x(t)\cdot\nabla_x\phi(t,\gamma^x(t))]\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\omega}(\gamma) \\ &= \int_\Gamma\int_{I_\gamma} \eta'(\gamma^v(t))\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,\gamma^x(t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\omega}(\gamma). \end{aligned} \tag{1.2.5}$$

Now we compute more explicitly the integrand in the last line: in particular, since $\eta'\phi$ has compact support, by (3) we deduce that for almost every γ the function $t \mapsto g_\gamma(t) \doteq \eta'(\gamma^v(t))\phi(t,\gamma^x(t))$ is in $BV(I_\gamma)$. Let J_γ be the jump set of γ^v, defined by

$$J_\gamma \doteq \big\{t \in I_\gamma \mid \gamma^v(t+) \neq \gamma^v(t-)\big\}.$$

By Volpert's chain rule we obtain that $Dg_\gamma \in \mathscr{M}(I_\gamma)$ is given by

$$\begin{aligned} Dg_\gamma = \eta'(\gamma^v(t))\,\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,\gamma^x(t))\cdot\mathscr{L}^1 \,+\, \eta''(\gamma^v(t))\phi(t,\gamma^x(t))\cdot\widetilde{D}\gamma^v(t) \\ +\sum_{t\in J_\gamma}\big(\eta'(\gamma^v(t+)) - \eta'(\gamma^v(t-))\big)\cdot\phi(t,\gamma^x(t))\cdot\delta_t \end{aligned}$$

where $\widetilde{D}\gamma^v$ denotes the diffuse part of the derivative of γ^v. Therefore, using the chain rule in (1.2.5), we obtain

$$\begin{aligned}\int_{I_\gamma} \eta'(\gamma^v(t))\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,\gamma^x(t))\,\mathrm{d}t = &\int_{I_\gamma} \mathrm{d}Dg_\gamma(t)\\ &-\int_{I_\gamma} \eta''(\gamma^v(t))\phi(t,\gamma^x(t))\,\mathrm{d}\widetilde{D}\gamma^v(t)\\ &-\sum_{t\in J_\gamma}(\eta'(\gamma^v(t+))-\eta'(\gamma^v(t-)))\cdot\phi(t,\gamma^x(t)).\end{aligned} \tag{1.2.6}$$

For every $\gamma \in \Gamma$, define the measure $\mu_1^\gamma \in \mathscr{M}(\Omega\times\mathbb{R})$ as

$$\begin{aligned}\mu_1^\gamma \doteq &-(\mathbb{I},\gamma)_\sharp \widetilde{D}\gamma^v\\ &-\mathscr{H}^1\llcorner\{(t,x,v)\mid \gamma^v(t-)\le v\le \gamma^v(t+),\quad t\in J_\gamma\}\\ &+\mathscr{H}^1\llcorner\{(t,x,v)\mid \gamma^v(t+)\le v\le \gamma^v(t-),\quad t\in J_\gamma\},\end{aligned} \tag{1.2.7}$$

and the measure $\mu_0^\gamma \in \mathscr{M}(\Omega\times(0,1))$ as

$$\mu_0^\gamma \doteq \delta_{(t^1_\gamma,\gamma(t^1_\gamma))} - \delta_{(t^2_\gamma,\gamma(t^2_\gamma))}. \tag{1.2.8}$$

Then inserting (1.2.6) in the last line of (1.2.5), we obtain

$$-\langle S,\eta'\phi\rangle \;=\int_\Gamma \langle -\mu_0^\gamma,\ \eta'\phi\rangle\,\mathrm{d}\boldsymbol{\omega}(\gamma) + \int_\Gamma \langle \mu_1^\gamma,\ \eta''\phi\rangle\,\mathrm{d}\boldsymbol{\omega}(\gamma). \tag{1.2.9}$$

This shows that setting

$$\mu_1 := \int_\Gamma \mu_1^\gamma\,\mathrm{d}\boldsymbol{\omega}(\gamma), \qquad \mu_0 := \int_\Gamma \mu_0^\gamma\,\mathrm{d}\boldsymbol{\omega}(\gamma) \tag{1.2.10}$$

it holds

$$S = \partial_v\mu_1 + \mu_0, \qquad \text{in } \mathscr{D}'(\Omega\times(0,1)) \tag{1.2.11}$$

which proves the claim. Finally, notice that μ_1,μ_0 are locally finite measures, by assumption (3) in Definition 1.2.2. Therefore u is a finite entropy solution by Proposition 1.1.3. □

Motivated by the above discussion, we give the following definition.

Definition 1.2.4. We say that a Lagrangian representation $\boldsymbol{\omega}$ *induces* a pair (μ_0,μ_1) if (1.2.10) holds, with $\mu_1^\gamma,\mu_0^\gamma$ defined in(1.2.7), (1.2.8), *and moreover*

$$|\mu_1| = \int_\Gamma |\mu_1^\gamma|\,\mathrm{d}\boldsymbol{\omega}(\gamma), \qquad |\mu_0| = \int_\Gamma |\mu_0^\gamma|\,\mathrm{d}\boldsymbol{\omega}(\gamma) \tag{1.2.12}$$

Condition (1.2.12) says that there are no cancellations (e.g. no curves are created in the same point where others are destroyed).

Remark 1.2.5. Not every couple (μ_1,μ_0) is induced by a Lagrangian representation. For example, every couple which (μ_1,μ_0) corresponds to $\chi=0$, i.e. $\mu_0+\partial_v\mu_1=0$, is not induced by any Lagrangian representation.

1.2.2. Existence of Lagrangian Representations. In this subsection we use a Theorem of Smirnov [**96**] about the decomposition of 1-currents into currents of the form $[\![\gamma]\!]$ to derive "generalized" Lagrangian representation for *all* couples μ_1, μ_0. As a by product, we obtain a proof of the remaining implication of Proposition 1.2.3.

1.2.2.1. *Preliminaries About the Theory of Currents.* For an introduction to the subject we refer for example to [**75**]. The space $\mathscr{D}_k(\mathbb{R}^d)$ of *k-dimensional currents* is the dual of the space $\mathscr{D}_k(\mathbb{R}^d)$ of all smooth k-differential forms with compact support in $\mathbb{R}^d$.

Given a k-current $\mathsf{T} \in \mathscr{D}_k(\mathbb{R}^d)$, its *mass* $\mathbb{M}(\mathsf{T})$ is defined as

$$\mathbb{M}(\mathsf{T}) = \sup\{\langle \mathsf{T} \,, \omega\rangle \mid \omega \in \mathscr{D}^k(\mathbb{R}^d), \quad |\omega| \le 1\}$$

Remark 1.2.6 (1-currents with finite mass). The space of 1-currents with finite mass can be identified with Radon vector measures $\mathsf{T} \in \mathscr{M}(\mathbb{R}^d)^d$, and we will often identify a 1-current with finite mass with the associated vector measure. The duality of a 1-current with finite mass $\mathsf{T} = (\mathsf{T}_1, \dots, \mathsf{T}_d)$ with a vector field $\Phi = (\Phi^1, \dots, \Phi^d) \in \mathbf{C}_c^\infty(\mathbb{R}^d,\ \mathbb{R}^d)$ is

$$\langle \mathsf{T}\ ,\ \Phi\rangle = \sum_{i=1}^n \int_{\mathbb{R}^d} \Phi^i \, \mathrm{d}\mathsf{T}_i$$

Analogously, the space of 0-currents with finite mass can be identified with the space of Radon measures on $\mathbb{R}^d$.

Example 1.2.7 (Rectifiable k-currents). Let a k-rectifiable set $M \subset \mathbb{R}^d$ oriented by a unit k-vector τ, and a measurable multiplicity function θ be given. Then we denote by $[\![M,\ \tau,\ \theta]\!]$ the k-current defined by

$$\langle [\![M,\ \tau,\ \theta]\!],\ \omega\rangle = \int_M \theta(x)\langle \tau(x),\ \omega(x)\rangle \, \mathrm{d}\mathscr{H}^k(x), \qquad \forall\, \omega \in \mathscr{D}^k(\mathbb{R}^d)$$

The *boundary* of a k-current T is a $k-1$-current $\partial\mathsf{T} \in \mathscr{D}_{k-1}(\mathbb{R}^d)$ defined as

$$\langle \partial\mathsf{T}\ ,\ \omega\rangle = \langle \mathsf{T}\ ,\ d\omega\rangle, \qquad \forall\, \omega \in \mathscr{D}^{k-1}(\mathbb{R}^d)$$

where $d : \mathscr{D}^{k-1}(\mathbb{R}^d) \to \mathscr{D}^k(\mathbb{R}^d)$ is the De Rham's-Cartan differential.

Definition 1.2.8 (Normal Currents). We say that a k-current $\mathsf{T} \in \mathscr{D}_k(\mathbb{R}^d)$ is *normal* if both T and $\partial\mathsf{T}$ are of finite mass.

Definition 1.2.9 (A-cyclic currents). Let $\mathsf{T} \in \mathscr{D}_k(\mathbb{R}^d)$ be a k-current. We say that

(1) A current $\mathsf{C} \in \mathscr{D}_k(\mathbb{R}^d)$ is a *subcurrent* of T, and we write $\mathsf{C} \le \mathsf{T}$, if $\mathbb{M}(\mathsf{T}) = \mathbb{M}(\mathsf{T} - \mathsf{C}) + \mathbb{M}(\mathsf{C})$.
(2) A current $\mathsf{C} \le \mathsf{T}$ is a *cycle* of T if $\partial\mathsf{C} = 0$.
(3) T is a-cyclic if its only cycle is $\mathsf{C} = 0$.

Remark 1.2.10. In general, given $\mathsf{T}, \mathsf{C} \in \mathscr{D}_k(\mathbb{R}^d)$, it is clear by the definition of mass of a current that it holds $\mathbb{M}(\mathsf{T}) \le \mathbb{M}(\mathsf{T} - \mathsf{C}) + \mathbb{M}(\mathsf{C})$. Therefore C is a subcurrent of T if and only if $\mathbb{M}(\mathsf{T}) \ge \mathbb{M}(\mathsf{T} - \mathsf{C}) + \mathbb{M}(\mathsf{C})$. In particular, if T is of finite mass, C must be of finite mass.

Example 1.2.11 (Structure of 1-subcurrents). Let $\mathsf{T} \in \mathscr{D}_1(\mathbb{R}^d)$ be a 1-current with finite mass, and let $\|\mathsf{T}\| \in \mathscr{M}^+(\mathbb{R}^d)$ be its total variation measure. By the polar decomposition of vector valued measures (see [**6**]) there exists a unit measurable vector field

$\vec{S} : \mathbb{R}^d \to \mathbb{R}^d$, defined $\|\mathsf{T}\|$-a.e., such that $\mathsf{T} = \vec{S} \cdot \|\mathsf{T}\|$. It is easy to see that every subcurrent $\mathsf{C} \in \mathscr{D}_1(\mathbb{R}^d)$ of T has the form

$$\mathsf{C} = \vec{S} \cdot \|\mathsf{C}\|\,, \qquad \|\mathsf{C}\| \le \|\mathsf{T}\| \quad \text{as measures}$$

EXAMPLE 1.2.12 (1-currents associated with Lipschitz curves). The most important example for the following of this section is the case of 1-rectifiable currents

$$[\![\gamma]\!] := [\![\gamma[(0,1)],\ \dot\gamma \cdot |\dot\gamma|^{-1},\ \mathscr{H}^0\big(\gamma^{-1}(x)\big)]\!] \in \mathscr{D}_1(\mathbb{R}^d)$$

associated with a Lipschitz curve $\gamma : (0,1) \to \mathbb{R}^d$. Notice that $|\dot\gamma(x)| > 0$ for $\gamma[(0,1)]\llcorner\mathscr{H}^1$-a.e. x, so that the tangent unit vectors $\dot\gamma \cdot |\dot\gamma|^{-1}$ are well defined, while $\mathscr{H}^0\big(\gamma^{-1}(x)\big)$ counts how many times the curve γ passes through the point x. The action of $[\![\gamma]\!]$ on 1-forms (which can be identified with vector fields $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$) can be explicitly written as (see the Area Formula in [**6**])

$$\begin{aligned}\langle [\![\gamma]\!],\ \Phi\rangle &= \int_{\gamma((0,1))} \mathscr{H}^0\big(\gamma^{-1}(x)\big) \cdot \Big\langle \frac{\dot\gamma}{|\dot\gamma|},\ \Phi \Big\rangle \cdot \mathrm{d}\mathscr{H}^1 \\ &= \int_0^1 \langle \Phi(\gamma(\tau)),\ \dot\gamma(\tau)\rangle \,\mathrm{d}\tau, \qquad \forall\ \Phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d)\end{aligned}$$

In particular $[\![\gamma]\!]$ has finite mass since

$$\langle [\![\gamma]\!],\ \Phi\rangle \le \int_0^1 |\dot\gamma(\tau)| \,\mathrm{d}\tau = \mathscr{H}^1(\gamma((0,1))), \qquad \forall\ \Phi \in \mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d), \quad |\Phi| \le 1$$

The boundary of the 1-current $[\![\gamma]\!] \in \mathscr{D}_1(\mathbb{R}^d)$, satisfies

$$\langle \partial[\![\gamma]\!],\ \phi\rangle = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\phi(\gamma(\tau)) \,\mathrm{d}\tau = \phi(\gamma(1)) - \phi(\gamma(0)), \qquad \forall\ u \in \mathcal{C}^\infty_c(\mathbb{R}^d)$$

so that $\partial[\![\gamma]\!] = \delta_{\gamma(1)} - \delta_{\gamma(0)} \in \mathscr{M}(\mathbb{R}^d)$. Therefore $[\![\gamma]\!]$ is a normal 1-current. Moreover, a 1-current $[\![\gamma]\!] \in \mathscr{D}_1(\mathbb{R}^d)$ is a cycle if and only if $\gamma(1) = \gamma(0)$, otherwise it is an a-cyclic 1-current.

1.2.2.2. *Smirnov's Theorem for Normal Currents.* The following Theorem of Smirnov yields a decomposition of 1-normal currents into superposition of currents associated to Lipschitz curves.

THEOREM 1.2.13 (Smirnov, [**96**]). *Let $\mathsf{T} \in \mathscr{D}_1(\mathbb{R}^d)$ be a normal, a-cyclic 1-current in $\mathbb{R}^d$. Then there exists a positive measure $\boldsymbol{\eta} \in \mathscr{M}^+(\mathrm{Lip}((0,1);\mathbb{R}^d))$ such that*

$$\langle \mathsf{T}, \omega\rangle = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \langle [\![\gamma]\!], \omega\rangle \,\mathrm{d}\boldsymbol{\eta}(\gamma), \qquad \forall\ \omega \in \mathscr{D}^1(\mathbb{R}^n) \tag{1.2.13}$$

Furthermore, the mass of T decomposes as

$$\mathbb{M}(\mathsf{T}) = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \mathbb{M}([\![\gamma]\!]) \,\mathrm{d}\boldsymbol{\eta}(\gamma) \tag{1.2.14}$$

and the boundary measure ∂T decomposes as

$$(\partial T)^+ = (e_1)_\sharp \boldsymbol{\eta}, \qquad (\partial T)^- = (e_0)_\sharp \boldsymbol{\eta} \tag{1.2.15}$$

REMARK 1.2.14. Let $\mathsf{T} = \vec{S} \cdot \|\mathsf{T}\|$ be the polar decomposition of the vector measure T. From the formula of decomposition of the mass it follows that the curves on which $\boldsymbol{\eta}$ is concentrated are such that $\vec{S}(\gamma(\tau))$ is parallel to $\dot{\gamma}(\tau)$ for $\mathscr{L}^1$ a.e. τ. In fact, since T is of finite mass, we can use $\vec{S}$ as a test vector field in equation (1.2.13) (see for example [**75**]). This yields

$$\mathbb{M}(\mathsf{T}) = \langle \mathsf{T},\ \vec{S}\rangle = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \langle [\![\gamma]\!], \vec{S}\rangle \,\mathrm{d}\boldsymbol{\eta}(\gamma) = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \int_0^1 \langle \dot{\gamma}(\tau),\ \vec{S}(\gamma(\tau))\rangle \,\mathrm{d}\tau \,\mathrm{d}\boldsymbol{\eta}(\gamma) \tag{1.2.16}$$

But on the other hand, by equation (1.2.14), it also holds

$$\mathbb{M}(\mathsf{T}) = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \mathbb{M}([\![\gamma]\!]) \,\mathrm{d}\boldsymbol{\eta}(\gamma) = \int_{\mathrm{Lip}((0,1);\mathbb{R}^d)} \int_0^1 |\dot{\gamma}(\tau)| \,\mathrm{d}\tau \,\mathrm{d}\boldsymbol{\eta}(\gamma) \tag{1.2.17}$$

Since $\vec{S}$ is a unitary vector, it holds

$$\langle \dot{\gamma}(\tau),\ \vec{S}(\gamma(\tau))\rangle \leq |\dot{\gamma}(\tau)|, \qquad \text{for } \mathscr{L}^1 \text{ a.e. } \tau \in (0,1)$$

Therefore combining (1.2.16), (1.2.17) one obtains that $\dot{\gamma}(\tau)$ must be parallel to $\vec{S}$ for $\mathcal{L}^1$ a.e. $\tau \in (0,1)$, for $\boldsymbol{\eta}$ a.e. $\gamma \in \Gamma$.

We fix some notation that we will need for the following proposition. We let Γ be as above, and as usual elements of Γ will be denoted by $\gamma = (\gamma^x, \gamma^v)$. Instead, we let

$$\widetilde{\Gamma} = \mathrm{Lip}((0,1),\, \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}).$$

Elements of $\widetilde{\Gamma}$ are denoted by

$$\Theta(\tau) \doteq \big(\Theta^t(\tau), \Theta^x(\tau), \Theta^v(\tau)\big) \in \Omega \times \mathbb{R}, \quad \tau \in (0,1).$$

THEOREM 1.2.15. *Let u be a finite entropy solution and let $(\mu_0, \mu_1) \in \mathscr{M}(\Omega \times \mathbb{R})^2$ be as in Proposition 1.1.3. Then there exists a Lagrangian representation $\boldsymbol{\omega} \in \mathscr{M}^+(\Gamma)$ of u. Moreover, letting $(\widetilde{\mu}_0, \widetilde{\mu}_1)$ be the pair induced by $\boldsymbol{\omega}$ as in Definition 1.2.4, it holds $|\widetilde{\mu}_0| \leq |\mu_0|$ and*

$$\widetilde{\mu}_1^+ \leq \mu_1^+, \qquad \widetilde{\mu}_1^- \leq \mu_1^-, \qquad \textit{as measures.} \tag{1.2.18}$$

PROOF. **1.** Define the 1-current $\mathsf{T} \in \mathscr{D}_1(\Omega \times \mathbb{R})$ as

$$\mathsf{T} = \Big(\chi \cdot \mathscr{L}^{d+2}, \quad \chi \cdot \boldsymbol{f}'(v) \llcorner \mathscr{L}^{d+2}, \quad -\mu_1\Big)^T \in \mathscr{D}_1(\Omega \times \mathbb{R}). \tag{1.2.19}$$

In this step we prove that T is a normal, a-cyclic current. When testing T against a smooth vector field, we obtain

$$\begin{aligned} \langle \mathsf{T}\ ,\ \Phi\rangle &= \int_{\Omega\times\mathbb{R}} \chi \cdot [\Phi^t + \Phi^x \cdot \boldsymbol{f}'(v)] \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v - \int_{\Omega\times\mathbb{R}} \Phi^v \,\mathrm{d}\mu_1 \\ &\leq \sqrt{1 + |\boldsymbol{f}'(v)^2|} \cdot \int_{\Omega\times\mathbb{R}} \chi \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v + \|\mu_1\|_{\mathscr{M}}, \\ &\forall\, \Phi \in \mathbf{C}_c^\infty(\Omega \times \mathbb{R}, \mathbb{R}^{d+2}), \qquad |\Phi| \leq 1. \end{aligned} \tag{1.2.20}$$

The boundary of T is, by equation (1.1.4), the measure μ_0, plus the boundary terms at $t = 0$ and $t = T$. In fact it holds

$$\begin{aligned}\langle \partial \mathsf{T}\ ,\ \phi\rangle &= \int_{\Omega\times\mathbb{R}} \chi\cdot[\partial_t\phi + \nabla\phi_x\cdot \boldsymbol{f}'(v)]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v - \int_{\Omega\times\mathbb{R}}\partial_v\phi\,\mathrm{d}\mu_1\\ &= -\int_{\Omega\times\mathbb{R}}\phi\,\mathrm{d}\mu_0 + \int_{\mathbb{R}^{d+1}}\chi(T)\cdot\phi(T) - \chi(0)\cdot\phi(0)\,\mathrm{d}x\,\mathrm{d}v\\ &= -\left\langle \phi,\mu_0\right\rangle + \left\langle \delta_T\otimes\chi(t)\cdot\mathscr{L}^{d+1} - \delta_0\otimes\chi(0)\cdot\mathscr{L}^{d+1}\ ,\ \phi\right\rangle,\\ &\qquad \forall\,\phi\in\mathbf{C}^\infty_c(\Omega\times\mathbb{R}).\end{aligned} \tag{1.2.21}$$

Therefore T is a normal 1-current.

Finally, we prove that T is a-cyclic. Let $\mathsf{T} = \vec{S}\cdot\|\mathsf{T}\|$ be the polar decomposition of the current (vector measure) T. Moreover, we choose $A\subset\Omega\times\mathbb{R}$ measurable such that $\mu_1 = \mu_1^a + \mu_1^s = \mu_1\llcorner A + \mu_1\llcorner A^c$ is the decomposition of μ_1 with respect to the Lebesgue measure into absolutely continuous and singular part respectively. In particular, we can choose A such that $\mathscr{L}^{d+2}(A^c) = 0$. We let $\rho:\Omega\times\mathbb{R}\to\mathbb{R}$ be the density of μ_1^a, i.e. $\mu_1^a = \rho\mathscr{L}^{d+2}$. Then the unit vector $\vec{S}$ is given by

$$(\vec{S}^t,\vec{S}^x,\vec{S}^v) = \vec{S} = \begin{cases}\left(\sqrt{1+|\boldsymbol{f}'(v)|^2+\rho^2}\right)^{-1}(1,\boldsymbol{f}'(v),\rho(t,x,v))^T, & \text{in } A\\ \\ (0,0,\sigma)^T, & \text{in } A^c\end{cases}$$

where $\mu_1^s = \sigma\,\|\mu_1^s\|$, $\sigma\in\{-1,1\}$ μ_1^s-a.e., is the polar decomposition of μ_1^s.

As a preliminary step we prove that for every subcurrent $\mathsf{C}\le\mathsf{T}$ such that $\mathsf{C} = \mathsf{C}_1 + \mathsf{C}_2$ with $0\neq\mathsf{C}_1\le\mathsf{T}\llcorner A$ and $\mathsf{C}_2\le\mathsf{T}\llcorner A^c$, then $\partial\mathsf{C}\neq 0$. Let $R>0$ be such that $\operatorname{supp}\mathsf{T}\subset[0,T]\times B_R(0)\subset\Omega\times\mathbb{R}$. Choose a test function $\phi\in\mathbf{C}^\infty_c(\Omega\times\mathbb{R})$ such that

$$\phi(t,x,v) = t,\quad \text{if } (t,x,v)\in[0,T]\times B_R(0),\qquad \phi(t,x,v) = 0,\quad \text{if } (t,x,v)\in[0,T]\times B_{2R}(0)^c$$

Then it holds

$$\langle \mathsf{C}\ ,\ \nabla\phi\rangle = \int_{\Omega\times\mathbb{R}}\vec{S}^t\,\mathrm{d}\,\|\mathsf{C}_1\| > 0$$

because $\vec{S}^t(x) > 0$ for$\|\mathsf{C}_1\|$ a.e. $x\in\Omega\times\mathbb{R}$. To conclude that T is a-cyclic, we need to prove that for every $0\neq\mathsf{C}\le\mathsf{T}\llcorner A^c$, it holds $\partial\mathsf{C}\neq 0$. In this case, one has $\mathsf{C} = (0,0,-\hat{\mu}_1)$, with $\hat{\mu}_1\le\mu_1$ (as 0-currents), and $\partial\mathsf{C} = 0$ corresponds to $\partial_v\hat{\mu}_1 = 0$ in the sense of distributions. But since $\hat{\mu}_1$ is compactly supported in v, this implies $\hat{\mu}_1 = 0$, and therefore $\mathsf{C} = 0$, as wanted.

2. Thanks to Step 1, we can apply Smirnov Theorem 1.2.13, which gives a measure $\boldsymbol{\eta}\in\mathscr{M}^+(\widetilde{\Gamma})$ such that (1.2.13) holds. Expanding the relation, this means that for every vector field $\Phi = (\Phi^t,\Phi^x,\Phi^v)\in\mathbf{C}^\infty_c(\Omega\times\mathbb{R})$, we have

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}}\chi\cdot[\Phi^t + (\boldsymbol{f}'(v)\cdot\Phi^x)]\,\mathrm{d}\mathcal{L}^{d+2} - \int_{\Omega\times\mathbb{R}}\Phi^v\,\mathrm{d}\mu_1\\ &= \int_{\widetilde{\Gamma}}\int_0^1\dot{\Theta}(\tau)\cdot\Phi(\Theta(\tau))\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\eta}(\Theta).\end{aligned} \tag{1.2.22}$$

By Remark 1.2.14 the measure $\boldsymbol{\eta} \in \mathscr{M}^+(\widetilde{\Gamma})$ given by Smirnov Theorem is concentrated on curves $\Theta \in \widetilde{\Gamma}$ such that $\dot{\Theta}(\tau)$ is parallel to $\vec{S}(\Theta(\tau))$ for a.e. $\tau \in (0,1)$. In particular, this means either Θ travels vertically along v, or the velocity of the first component positive.

3. Given the measure $\boldsymbol{\eta} \in \mathscr{M}^+(\widetilde{\Gamma})$ in Step 2., in order to get a Lagrangian representation in the sense of Definition 1.2.2, we want to eliminate a specific set of "bad" curves. In particular, define $\Delta \subset \widetilde{\Gamma}$ as the set of curves such that $\Theta_t((0,1))$ is a singleton. These are the curves whose support is contained on a single fiber $\{(t,x)\} \times \mathbb{R}_v \subset \Omega \times \mathbb{R}$. The new measure will be defined by

$$\widetilde{\boldsymbol{\eta}} = \boldsymbol{\eta} \llcorner (\widetilde{\Gamma} \setminus \Delta).$$

Notice that in any case, by the shape of the vector field $\vec{S}$, all the curves satisfy $\dot{\Theta}^t(\tau) \geq 0$ for a.e. $\tau \in (0,1)$. For curves in $\widetilde{\Gamma} \setminus \Delta$ such that the first component Θ^t is non-decreasing, it is well defined a *reparametrization map* $\mathcal{R} : \widetilde{\Gamma} \setminus \Delta \to \Gamma$ defined as

$$\mathcal{R}(\Theta)(t) \doteq \Theta(\tau(t)), \qquad \tau(t) \doteq \inf\{\tau \in (0,1) \ : \ t < \Theta^t(\tau)\}. \tag{1.2.23}$$

Here $\tau(t)$ is just the pseudo-inverse of the increasing Lipschitz function $\Theta^t(\tau)$. The domain of the curve $\mathcal{R}(\Theta)$ will be $\mathrm{int}[\Theta^t((0,1))]$ (the interior). Therefore we are now allowed to consider the pushforward measure

$$\boldsymbol{\omega} \doteq \mathcal{R}_\sharp \widetilde{\boldsymbol{\eta}} \in \mathscr{M}^+(\Gamma)$$

which is a good candidate to be a Lagrangian representation in the sense of Definition 1.2.2.

4. Finally, we prove that $\boldsymbol{\omega}$ is a Lagrangian representation. We verify that $\boldsymbol{\omega}$ is a Lagrangian representation. The second condition in Definition 1.2.2 is trivial. For the first condition, we need to verify that

$$\int_\Gamma \left[(\mathbb{I},\gamma)_\sharp \mathscr{L}^1 \llcorner I_\gamma\right] \mathrm{d}\mathcal{R}_\sharp \widetilde{\boldsymbol{\eta}}(\gamma) = \chi \cdot \mathscr{L}^{d+2}. \tag{1.2.24}$$

Therefore consider any test function $\phi \in \mathbf{C}^\infty_c(\Omega \times \mathbb{R})$ and calculate

$$\begin{aligned}
\int_\Gamma \int_{\Omega \times \mathbb{R}} \phi(t,x,v)\, \mathrm{d}\left[(\mathbb{I},\gamma)_\sharp \mathscr{L}^1 \llcorner I_\gamma\right](t,x,v)\, \mathrm{d}\mathcal{R}_\sharp \widetilde{\boldsymbol{\eta}}(\gamma) &= \int_\Gamma \int_{I_\gamma} \phi(t,\gamma(t))\, \mathrm{d}t\, \mathrm{d}\mathcal{R}_\sharp \widetilde{\boldsymbol{\eta}}(\gamma) \\
&= \int_\Gamma \int_{I_\gamma} \phi(\Theta(\tau(t)))\, \mathrm{d}t\, \mathrm{d}\widetilde{\boldsymbol{\eta}}(\Theta) \\
&= \int_\Gamma \int_0^1 \phi(\Theta(\tau))(\Theta^t)'(\tau)\, \mathrm{d}\tau\, \mathrm{d}\widetilde{\boldsymbol{\eta}}(\Theta) \\
&= \int_\Gamma \int_0^1 \phi(\Theta(\tau))(\Theta^t)'(\tau)\, \mathrm{d}\tau\, \mathrm{d}\boldsymbol{\eta}(\Theta),
\end{aligned} \tag{1.2.25}$$

where the last equality holds because

$$\int_\Delta \int_0^1 (\Theta^t)'(\tau)\, \mathrm{d}\tau\, \mathrm{d}\boldsymbol{\eta} = \int_\Delta \int_0^1 (\Theta^t)'(\tau)\, \mathrm{d}\tau\, \mathrm{d}\widetilde{\boldsymbol{\eta}} = 0$$

since $(\Theta^t)'(\tau) = 0$ for every $\tau \in (0,1)$ if $\Theta \in \Delta$, therefore we are allowed to substitute $\widetilde{\boldsymbol{\eta}}$ with $\boldsymbol{\eta}$. Moreover, for $\boldsymbol{\eta}$-a.e. $\Theta \in \widetilde{\Gamma}$, by the last observation in Step 2., it holds

$$(\Theta^t)'(\tau) \cdot \mathbf{1}_{A^c}(\Theta(\tau)) = 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } \tau \in (0,1)$$

Therefore we obtain

$$\int_\Gamma \int_0^1 \phi(\Theta(\tau))(\Theta^t)'(\tau)\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\eta}(\Theta) = \int_\Gamma \int_0^1 \phi(\Theta(\tau))(\Theta^t)'(\tau)\cdot \mathbf{1}_A(\Theta(\tau))\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\eta}(\Theta). \quad (1.2.26)$$

Define the vector field

$$\Phi(t,x,v) = (\phi(t,x,v),0,0)\cdot \mathbf{1}_A(t,x,v) \in \mathbf{C}_c^\infty([0,T]\times\mathbb{R}^{d+1}).$$

Using Φ as a test function in the representation of Step 2, we obtain

$$\int_\Gamma \int_0^1 \phi(\Theta(\tau))(\Theta^t)'(\tau)\cdot \mathbf{1}_A(\Theta(\tau))\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\eta}(\Theta) = \langle \mathsf{T}, \Phi\rangle = \langle \chi\cdot\mathscr{L}^{d+2}, \phi\rangle \qquad (1.2.27)$$

as wanted. □

1.2.3. Epigraph and Simultaneous Lagrangian Representations. We conclude this section with the proof of some useful properties of Lagrangian representations.

1.2.3.1. *Lagrangian representation for the epigraph.* Given a finite entropy solution of the balance law (1.1.1), we constructed a Lagrangian representation for the function χ, which is the characteristic function of the hypograph of u. In an entirely similar way, we can give the definition of Lagrangian representation for the *epigraph* of u.

DEFINITION 1.2.16. We say that $\boldsymbol{\omega}_e$ is a Lagrangian representation for the epigraph of u if conditions (2), (3) of Definition 1.2.2 hold, and condition (1) is replaced by:

$$\chi_e\cdot\mathscr{L}^{d+2} = \int_\Gamma (\mathrm{id},\gamma)_\sharp \mathscr{L}^1 \llcorner I_\gamma\,\mathrm{d}\boldsymbol{\omega}_e(\gamma) \qquad (1.2.28)$$

where

$$\chi_e \doteq \begin{cases} 1, & \text{if } u(t,x)\le v < 1,\\ 0, & \text{if } 0 \le u(t,x) < v \le 1,\end{cases} \qquad (1.2.29)$$

The existence of such a representation follows from Proposition 1.2.15: in fact, define

$$\widetilde{u}(t,x) \doteq 1 - u(t,x), \qquad \boldsymbol{g}(v) \doteq -\boldsymbol{f}(1-v)$$

Then, if $\widetilde{\chi}(t,x,v) = \mathbf{1}_{\mathrm{hyp}\,\widetilde{u}}(t,x,v)$, $\widetilde{u}$ is a solution of the kinetic equation

$$\partial_t\widetilde{\chi} + \boldsymbol{g}'(v)\cdot\nabla_x\widetilde{\chi} = \partial_v\big[\alpha_\sharp\mu_1\big] - \alpha_\sharp\mu_0, \qquad \text{in } \mathscr{D}'_{t,x,v}$$

where $\alpha : \Omega\times(0,1)\mapsto\Omega\times(0,1)$ is defined as $\alpha(t,x,v) = \alpha(t,x,1-v)$. Proposition 1.2.15 then yields a Lagrangian representation of $\widetilde{u}$, which we call $\widetilde{\boldsymbol{\omega}}$. Now, letting the map $T:\Gamma\to\Gamma$ be defined as

$$T(\gamma)(t) = \alpha(\gamma(t)), \qquad t\in I_\gamma$$

it is easy to see that the measure

$$\boldsymbol{\omega}_e \doteq T_\sharp\widetilde{\boldsymbol{\omega}}$$

is a Lagrangian representation of the epigraph of u.

1.2.3.2. *Simultaneous Lagrangian Representations.* Let u a kinetic solution be given. To keep the notation as clear as possible, when we need to consider simultaneously Lagrangian representations of the hypograph and of the epigraph, we will call them $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_e$, respectively. Moreover, we will let

$$\chi_h \doteq \mathbf{1}_{\mathrm{hyp}\, u}, \qquad \chi_e \doteq \mathbf{1}_{\mathrm{epi}\, u}$$

It is clear that, if χ_h satisfies (1.1.4) for some μ_0, μ_1, then $\chi_e = 1 - \chi_h$ satisfies (1.1.4) with $(-\mu_0, -\mu_1)$. Motivated by this observation, we give the following definition.

DEFINITION 1.2.17. We say that a pair (μ_0, μ_1) is *simultaneously induced* by Lagrangian representations $\boldsymbol{\omega}_h$, $\boldsymbol{\omega}_e$ of the hypograph and of the epigraph respectively if (μ_0, μ_1) is induced by $\boldsymbol{\omega}_h$ and $(-\mu_0, -\mu_1)$ is induced by $\boldsymbol{\omega}_e$.

It is useful to introduce the following relation $\preceq$ between pairs (μ_0, μ_1) that belong to the set

$$\boldsymbol{P}_{\chi_h} := \big\{(\mu_0, \mu_1) \in \mathscr{M}(\Omega \times \mathbb{R}) \mid (1.1.4) \text{ holds}\big\}.$$

DEFINITION 1.2.18. Given (μ_0, μ_1) and $(\widehat{\mu}_0, \widehat{\mu}_1)$ in $\boldsymbol{P}_\chi$, we write

$$(\widehat{\mu}_0, \widehat{\mu}_1) \preceq (\mu_0, \mu_1) \iff \begin{cases} \widehat{\mu}_1^+ \le \mu_1^+, \qquad \widehat{\mu}_1^- \le \mu_1^- \qquad \text{as measures} \\ |\widehat{\mu}_0| \le |\mu_0| \end{cases} \tag{1.2.30}$$

Note that $\preceq$ is the natural relation respected by the application of Theorem 1.2.15. This allows to prove the following proposition, which states the existence of simultaneously induced pairs.

PROPOSITION 1.2.19. *Let u be a finite entropy solution. Every pair $(\widehat{\mu}_0, \widehat{\mu}_1) \in \boldsymbol{P}_\chi$ which is minimal with respect to $\preceq$ is simultaneously induced by some $\boldsymbol{\omega}_h, \boldsymbol{\omega}_e$.*

PROOF. Let $(\widehat{\mu}_0, \widehat{\mu}_1)$ be a minimal element of $\boldsymbol{P}_{\chi_h}$ for the relation $\preceq$ (for example, a minimizer for the function $(\mu_0, \mu_1) \mapsto \|\mu_1\|_{\mathscr{M}}$). Now apply twice Proposition 1.2.15 first to u with the pair $(\widehat{\mu}_0, \widehat{\mu}_1)$ and then to $\widetilde{u}$ (with the reversed flux $\boldsymbol{g}(v) = -\boldsymbol{f}(1-v)$) with the pair $(-\widehat{\mu}_0, -\widehat{\mu}_1)$. The new couple of pairs induced by the Lagrangian representations $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_e$ must coincide with the starting one, by minimality for the relation $\preceq$. □

REMARK 1.2.20. We proved that any minimal pair (μ_0, μ_1) for the relation "$\preceq$" is simultaneously induced. This proves that if a pair is efficient enough, then it is simultaneously induced. A partial converse holds: if a pair is simultaneously induced by $\boldsymbol{\omega}_h$, $\boldsymbol{\omega}_e$, then it is "efficient" in the sense that

$$\mathrm{supp}\, \mu_0 \,\cup\, \mathrm{supp}\, \mu_1 \subset \,\partial\, (\mathrm{hyp}\, u).$$

1.2.3.3. *Good Selection of Curves.* Given a Lagrangian representation $\boldsymbol{\omega} \in \mathscr{M}^+(\Gamma)$, we can select a good set of curves on which it is concentrated. In this direction, with the same proof contained in [**86**], we have the following Lemma.

LEMMA 1.2.21. *For $\boldsymbol{\omega}_h$-a.e. $\gamma \in \Gamma$ it holds that for $\mathscr{L}^1$-a.e. $t \in [0, T]$*

(1) $(t, \gamma^x(t))$ is a Lebesgue point of u;

(2) $\gamma^v(t) < u(t, \gamma^x(t))$

We denote by Γ_h the set of curves $\gamma \in \Gamma$ such that the two properties above hold. Similarly, for $\boldsymbol{\omega}_e$-a.e. $\gamma \in \Gamma$ it holds that for $\mathscr{L}^1$-a.e. $t \in [0, T]$

(1) $(t,\gamma^x(t))$ *is a Lebesgue point of* u*;*
(2) $\gamma^v(t) > u(t,\gamma^x(t))$

and we denote by Γ_e *the set of curves* $\gamma \in \Gamma$ *such that the two properties above hold.*

1.3. Structure of the Kinetic Measures

1.3.1. Rectifiability of $\boldsymbol{J}$. Throughout this section, we will use the following structural assumption about the nonlinearity of the flux $\boldsymbol{f}$.

DEFINITION 1.3.1. We say that a flux $\boldsymbol{f} : [0,1] \to \mathbb{R}^d$ is *weakly genuinely nonlinear* if

$$\mathscr{L}^1\Big(\big\{v \in (0,1) \mid \tau + \xi \cdot \boldsymbol{f}'(v) = 0\big\}\Big) = 0, \qquad \forall\, (\xi,\tau) \in \mathbb{S}^{d+1} \tag{1.3.1}$$

In [**56**], the structure of the dissipation measures μ_1 has been studied in general dimension, when $\mu_0 = 0$. These results directly apply also to the case when a source term μ_0 is present, although some extra care must be used when choosing the representative $(\mu_0,\mu_1) \in \mathscr{P}_\chi$. We summarize below the results that are obtained directly from [**56**], in our setting. We first recall some definitions.

DEFINITION 1.3.2. In connection to a pair $(\mu_0,\mu_1) \in \boldsymbol{P}_\chi$,

(1) we denote by ν_0, ν_1 the (t,x) marginals of the total variation of μ_1, μ_0:

$$\nu_0 \doteq [p_{t,x}]_\sharp \left\|\mu_0\right\|, \qquad \nu_1 \doteq [p_{t,x}]_\sharp \left\|\mu_1\right\| \tag{1.3.2}$$

where $p_{t,x} : \Omega \times (0,1) \to \Omega$ is the natural projection on the (t,x) variables. We also let

$$\nu \doteq \nu_0 + \nu_1.$$

(2) we denote by $\boldsymbol{J} \subset \Omega$ the set of points (t,x) of positive $\mathscr{H}^d$ density of ν:

$$\boldsymbol{J} \doteq \left\{(t,x) \in (0,T) \times \mathbb{R}^d \;\middle|\; \limsup_{r\downarrow 0} \frac{\nu\left(\mathrm{B}_r(t,x)\right)}{r^d} > 0\right\} \tag{1.3.3}$$

(3) we say that $u : \Omega \to (0,1)$ has vanishing mean oscillation at a point (t,x) if

$$\lim_{r\downarrow 0} r^{-d-1} \int_{\mathrm{B}_r(t,x)} |u(s,y) - \overline{u}_r(t,x)|\,\mathrm{d}s\,\mathrm{d}y = 0 \tag{1.3.4}$$

where $\overline{u}_r(t,x)$ is the mean of u in the ball $\mathrm{B}_r(t,x)$.

(4) Let $J \subset \mathbb{R}^{d+1}$ be a rectifiable set of dimension $n-1$ with unit normal $\vec{\boldsymbol{n}}$. We call two Borel functions $u^-, u^+ : J \to \mathbb{R}$ left and right traces of u on J with respect to $\vec{\boldsymbol{n}}$ if, for $\mathscr{H}^d$-a.e. $(s,y) \in J$,

$$\lim_{r\downarrow 0} \frac{1}{r^n}\left(\int_{B_r^-(s,y)} |u(t,x) - u^-(s,y)|\,\mathrm{d}t\,\mathrm{d}x + \int_{B_r^+(s,y)} |u(t,x) - u^+(s,y)|\,\mathrm{d}t\,\mathrm{d}x\right) = 0,$$

where $B_r^\pm(s,y) := \{(t,x) \in B_r(s,y) \mid \pm\big((t,x)-(s,y)\big)\cdot \vec{\boldsymbol{n}}(s,y) > 0\}$.

Summarizing, in our context, the results of [**56**], yields

THEOREM 1.3.3 ([**56**]). *Let* $d \in \mathbb{N}$ *and* $\boldsymbol{f}$ *be weakly genuinely nonlinear. Let* u *be a finite entropy solution of* (1.1.1)*, and let* (μ_0,μ_1) *be a minimal pair in* $\boldsymbol{P}_\chi$*. Then the set* $\boldsymbol{J}$ *is d-rectifiable and*

(1) u has vanishing mean oscillation at every point $(t,x) \in \boldsymbol{J}^c$*,*
(2) u has left and right traces on $\boldsymbol{J}$*.*
(3) $\mu_\eta \llcorner \boldsymbol{J} = ((\eta(u^+), q(u^+)) - (\eta(u^-), q(u^-))) \cdot \vec{\boldsymbol{n}} \llcorner \boldsymbol{J}$*, where* $u^\pm$ *denotes the traces on* J *and* $\vec{\boldsymbol{n}}$ *denotes the normal to* J*.*

For BV solutions (1) and (3) can be improved to

(1') every $(t,x) \notin J$ is a Lebesgue point;
(3') $\mu_\eta = ((\eta(u^+), q(u^+)) - (\eta(u^-), q(u^-))) \cdot \vec{\boldsymbol{n}} \llcorner \boldsymbol{J}$.

Establishing $(1')$ for general weakly genuinely nonlinear fluxes is an open problem at the time of writing, but in [**95**] the author considered the case of entropy solutions with a power-type nonlinearity assumption on $\boldsymbol{f}$, and in this setting he proved that every point $(t,x) \notin J$ is a continuity point, providing a positive answer about (1') in this particular case. Moreover, in [**84**] the author showed that, for general finite entropy solutions, the set of non-Lebesgue points has Hausdorff dimension at most d, also providing a partial answer to (1). Also, for entropy solutions in one space dimension both (1') and (3') have affirmative answers: see [**37**].

In the following section, we prove that property (3') holds for finite entropy solutions in one space dimension.

1.3.2. The One-Dimensional Case. An important point that is left open in [**56**] is the structure of that part of ν_1 that is more diffuse than the d dimensional Hausdorff measure. A first complete answer for entropic solutions in one space dimension came with [**37**], where the authors prove that the entropy production is concentrated on a 1-rectifiable set. In [**86**], for case of Burgers equation $f(u) = u^2/2$ using the Lagrangian representation, it is proved that ν_1 is concentrated on the set $\boldsymbol{J}$, i.e. ν_1 does not posses lower dimensional parts, also for finite entropy solutions. Aim of this section is to extend the result to general genuine nonlinear fluxes in one space dimension. In particular, we will prove the following Theorem.

THEOREM 1.3.4. *Let* $d = 1$ *and* f *satisfy* (1.3.1)*. Let* u *be a finite entropy solution of* (1.1.1)*, and let* (μ_0, μ_1) *be a minimal pair in* $\boldsymbol{P}_\chi$*. Then, in addition to Theorem 1.3.3,* ν_1 *is concentrated on* $\boldsymbol{J}$*, i.e.*

$$\nu_1 = [\nu_1] \llcorner \boldsymbol{J} \tag{1.3.5}$$

For convenience, recall that $\Omega = (0,T) \times \mathbb{R}$, and in the following we denote $X = \Omega \times (0,1)$. The strategy of the proof is as follows. From Proposition 1.2.19 we know that if (μ_0, μ_1) is a minimal pair, then it is simultaneously induced by some $\boldsymbol{\omega}_h, \boldsymbol{\omega}_e$ Lagrangian representations of the hypograph and the epigraph of u, respectively. The rectifiability of the full measure ν_1 will follow by the rectifiability of both the measures $[p_{t,x}]_\sharp \mu_1^+$ and $[p_{t,x}]_\sharp \mu_1^-$. Therefore first we prove the result for $[p_{t,x}]_\sharp \mu_1^+$, and the other case will follow by symmetry.

1. We introduce the measures $\boldsymbol{\omega}_h \otimes \mu_1^{\gamma,+}$, $\boldsymbol{\omega}_e \otimes \mu_1^{\gamma,-} \in \mathscr{M}(\Gamma \times X)$, i.e. the measures defined as, for $G \in \mathcal{B}(\Gamma)$, $A \in \mathcal{B}(X)$,

$$\begin{aligned} \boldsymbol{\omega}_h \otimes \mu_1^{\gamma,+}(G \times A) &:= \int_G \mu_1^{\gamma,+}(A) \, \mathrm{d}\boldsymbol{\omega}_h(\gamma), \\ \boldsymbol{\omega}_e \otimes \mu_1^{\gamma,-}(G \times A) &:= \int_G \mu_1^{\gamma,-}(A) \, \mathrm{d}\boldsymbol{\omega}_e(\gamma). \end{aligned} \tag{1.3.6}$$

Since (μ_0, μ_1) is simultaneously induced by $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_e$, it will be possible to choose a *transport plan* $\pi^+ \in \mathscr{M}((\Gamma \times X)^2)$ which transports $\boldsymbol{\omega}_h \otimes \mu_1^{\gamma,+}$ to $\boldsymbol{\omega}_e \otimes \mu_1^{\gamma,-}$ and which is concentrated on the set

$$\begin{aligned} \mathcal{G} := \Big\{ &(\gamma, t, x, v, \gamma', t', x', v') \in (\Gamma \times X)^2 \ \big| \\ &(t, x) = (t', x'), \quad v = v', \quad v \in [\gamma^v(t+), \gamma^v(t-)] \cap [\gamma'^{,v}(t-), \gamma'^{,v}(t+)] \Big\} \end{aligned} \tag{1.3.7}$$

This follows from the following Lemma.

LEMMA 1.3.5 ([**86**], Lemma 8). *Denote by $P_1, P_2 : (\Gamma \times X)^2 \to (\Gamma \times X)$ the standard projections. Then there exists a plan $\pi^+ \in \mathscr{M}((\Gamma \times X)^2)$ with marginals*

$$\begin{aligned} [P_1]_\sharp \pi^+ &= \boldsymbol{\omega}_h \otimes \mu_1^{+,\gamma}, \\ [P_2]_\sharp \pi^+ &= \boldsymbol{\omega}_e \otimes \mu_1^{-,\gamma} \end{aligned} \tag{1.3.8}$$

concentrated on the set $\mathcal{G}$ defined in (1.3.7).

2. In this step, we decompose the plan π^+ constructed in Step 1 into a countable sum of components. First, consider the set $I = \{v \ : \ f''(v) = 0\}$, and its complement $(0,1) \setminus I$. Since $(0,1) \setminus I$ is an open set, we can write it as the union of countably many disjoint intervals A_l:

$$(0,1) \setminus I = \bigcup_{l=1}^{\infty} A_l, \qquad A_l \subset (0,1) \quad \text{disjoint intervals}$$

Then we will "decompose" π^+ as

$$\pi^+ \leq \sum_{l \in \mathbb{N}} \pi_l^+ + \pi_J^+ \tag{1.3.9}$$

where, loosely speaking, π_l^+ is supported in the set $(\gamma, t, x, v, \gamma', t, x, v')$ where at the point (t, x) the curves γ, γ' jump inside the same interval A_l, while π_J^+ represents the remaining curves. In the following of this step we formalize this discussion.

By definition, the measure $\boldsymbol{\omega}_h \otimes \mu_1^{+,\gamma}$ (recall also (1.2.7)) is concentrated on the set

$$\mathcal{G}_h^+ := \Big\{ (\gamma, t, x, v) \in \Gamma \times X \ \Big| \ \gamma^x(t) = x, \ v \in [\gamma^v(t+), \gamma^v(t-)] \Big\}. \tag{1.3.10}$$

Analogously, the measure $\boldsymbol{\omega}_e \otimes \mu_1^{-,\gamma}$ is concentrated on the set

$$\mathcal{G}_e^- := \Big\{ (\gamma', t', x', v') \in \Gamma \times X \ \Big| \ \gamma^x(t) = x, \ v \in [\gamma^v(t-), \gamma^v(t+)] \Big\}. \tag{1.3.11}$$

We define the sets, for each $l \in \mathbb{N}$,

$$\begin{aligned} \mathcal{G}_{h,l}^+ &:= \Big\{ (\gamma, t, x, v) \in \mathcal{G}_h^+ \ \Big| \ \gamma(t-), \gamma(t+) \in A_l \Big\}, \\ \mathcal{G}_{e,l}^- &:= \Big\{ (\gamma', t', x', v') \in \mathcal{G}_e^- \ \Big| \ \gamma(t-), \gamma(t+) \in A_l \Big\}. \end{aligned} \tag{1.3.12}$$

and the sets

$$\begin{aligned} \mathcal{G}_{h,J}^+ &:= \Big\{ (\gamma, t, x, v) \in \mathcal{G}_h^+ \ \Big| \ \gamma(t-) > \gamma(t+) \Big\}, \\ \mathcal{G}_{e,J}^- &:= \Big\{ ((\gamma', t', x', v') \in \mathcal{G}_e^- \ \Big| \ \gamma(t-) < \gamma(t+) \Big\}. \end{aligned} \tag{1.3.13}$$

Finally we define

$$\pi_l^+ := \pi^+ \llcorner (\mathcal{G}_{h,l}^+ \times \mathcal{G}_{e,l}^-), \qquad \pi_J^+ := \pi^+ \llcorner (\mathcal{G}_h^+ \times \mathcal{G}_{e,J}^-) \cup (\mathcal{G}_{h,J}^+ \times \mathcal{G}_e^-) \tag{1.3.14}$$

By Step 1 it holds

$$\pi^+ \leq \sum_{l \in \mathbb{N}} \pi_l^+ + \pi_J^+ + \pi_I^+$$

where

$$\pi_I^+ := \pi^+ \llcorner (\mathcal{G}_{h,I}^+ \times \mathcal{G}_{e,I}^-)$$

$$\mathcal{G}_{h,I}^+ = \left\{ (\gamma, t, x, v) \in \mathcal{G}_h^+ \middle| \gamma(t-) = \gamma(t+) \in I \right\}$$
$$\mathcal{G}_{e,I}^- = \left\{ (\gamma', t', x', v') \in \mathcal{G}_h^+ \middle| \gamma(t-) = \gamma(t+) \in I \right\}$$

We claim that $\pi_I^+ = 0$ so that actually

$$\pi^+ \leq \sum_{l \in \mathbb{N}} \pi_l^+ + \pi_J^+$$

This follows by the following Lemma about functions of bounded variation.

LEMMA 1.3.6. *Let $v : (a,b) \to (0,1)$ be a BV function. Then, if $\widetilde{D}v$ is the diffuse part of the measure Dv, for any set $I \subset (0,1)$ of zero $\mathscr{L}^1$ measure, it holds*

$$\widetilde{D}v(v^{-1}(I)) = 0$$

3. Using the construction of Step 2, we prove that for every $l \in \mathbb{N}$, the measure

$$\nu_{1,l}^+ := [P_{t,x}]_\sharp \pi_l^+ \tag{1.3.15}$$

is concentrated on a 1-rectifiable set, where $P_{t,x} : (\Gamma \times X)^2 \to (0,T) \times \mathbb{R}$ is the projection on the first two variables $(\gamma, t, x, v, \gamma', t', x', v') \mapsto (t, x)$.

We fix $l \in \mathbb{N}$, and prove that ν_l^+ is concentrated on a 1-rectifiable set. Without loss of generality we can assume that $A_l \subset (0,1)$ is such that $f''(v) > 0$ for every $v \in A_l$. The other case is completely symmetric. We start with some preliminary results, that are proved in [**86**].

LEMMA 1.3.7. *Let $(\bar{\gamma}, I_{\bar{\gamma}}) \in \Gamma_h$ and let $(a,b) \subset I_\gamma$ be such that $\bar{\gamma}^v((a,b)) \subset A_l$. Let moreover $G \subset \Gamma_e$ be a set of curves (γ, I_γ) such that there exist $a < s_\gamma^1 < s_\gamma^2 < b$ with*

$$\gamma(s_\gamma^1) > \bar{\gamma}(s_\gamma^1), \qquad \gamma(s_\gamma^2) < \bar{\gamma}(s_\gamma^2), \qquad \gamma^v(s_\gamma^1, s_\gamma^2) \subset A_l \tag{1.3.16}$$

Then

$$\boldsymbol{\omega}_e(G) = 0 \tag{1.3.17}$$

The proof of the Lemma can be found in [**86**] in the case of Burgers equation, but it is the same for strictly convex fluxes, therefore we omit the proof.

Let $G_{\bar{t},\bar{x}}^l \subset \Gamma_h$ be the set of curves (γ, I_γ) such that $\bar{t} \in I_\gamma$, $\gamma^v(\bar{t}) \in A_l$, and $\gamma^x(\bar{t}) < \bar{x}$. Analogously, let $G_{\bar{t},\bar{x}}^r \subset \Gamma_e$ be the set of curves (γ, I_γ) such that $\bar{t} \in I_\gamma$, $\gamma^v(\bar{t}) \in A_l$, and $\gamma^x(\bar{t}) > \bar{x}$.

Now we construct the candidate Lipschitz curves on which ν_l^- is concentrated.

Lemma 1.3.8. *To each* (γ, I_γ) *associate the time* $t^l_\gamma \in (\bar t, t^2_\gamma]$ *defined as*

$$t^l_\gamma := \sup\big\{t \in [\bar t, t^2_\gamma) \;\big|\; \gamma^v(s) \in A_l \;\forall\, s \in (\bar t, t)\big\}$$

Define the curve

$$\tilde f^l_{\bar t,\bar x}(t) := \sup_{\{\gamma \in G^l_{\bar t,\bar x}\,:\, t \in (\bar t,\, t^l_\gamma)\}} \gamma^x(t), \qquad t \in [\bar t, T)$$

and its upper Lipschitz envelope $f^l_{\bar t,\bar x} : [\bar t, T) \to \mathbb{R}$

$$f^l_{\bar t,\bar x} := \inf\big\{f \;\big|\; f : [\bar t, T] \to \mathbb{R} \text{ is } |f'|\text{-Lipschitz and } f \geq \tilde f^l_{\bar t,\bar x} \text{ in } [\bar t, T)\big\} \tag{1.3.18}$$

Then, for every $t \in [\bar t, T)$, *it holds*

$$\begin{aligned} &\boldsymbol{\omega}_h\Big(\big\{\gamma \in G^l_{\bar t,\bar x} \;\big|\; \gamma^x(t) > f^l_{\bar t,\bar x}(t), \quad t \in [\bar t, t^l_\gamma)]\big\}\Big) = 0 \\ &\boldsymbol{\omega}_e\Big(\big\{\gamma \in G^r_{\bar t,\bar x} \;\big|\; \gamma^x(t) < f^l_{\bar t,\bar x}(t), \quad t \in [\bar t, t^l_\gamma)\big\}\Big) = 0. \end{aligned} \tag{1.3.19}$$

The following Lemma is a general result about functions of bounded variation.

Lemma 1.3.9. *Let* $v : (a,b) \to \mathbb{R}$ *be a BV function and denote by* D^-v *the negative part of the measure* Dv. *Then for* $\tilde D^-v$*-a.e.* $\bar x \in (a,b)$ *there exists a* $\delta > 0$ *such that*

$$v(x) > v(\bar x) \quad \forall\, x \in (\bar x - \delta, \bar x), \qquad v(x) < v(\bar x) \quad \forall\, x \in (\bar x, \bar x + \delta)$$

Now that we have all the elements, we divide the proof of Step 3 into further substeps.

3.1. Fix $(\bar t, \bar x) \in (0,T) \times \mathbb{R}$. Define the sets $\mathcal{G}^{h,l}_{\bar t,\bar x}$, $\mathcal{G}^{e,l}_{\bar t,\bar x} \subset \Gamma \times X$ as

$$\begin{aligned} \mathcal{G}^{h,l}_{\bar t,\bar x} &:= \big\{(\gamma, t, x, v) \in \Gamma_h \times X \;:\; t \in (\bar t, t^{l,\bar t}_\gamma), \quad \gamma(\bar t) < \bar x\big\} \\ \mathcal{G}^{e,l}_{\bar t,\bar x} &:= \big\{(\gamma, t, x, v) \in \Gamma_e \times X \;:\; t \in (\bar t, t^{l,\bar t}_\gamma), \quad \gamma(\bar t) > \bar x\big\}. \end{aligned} \tag{1.3.20}$$

Define the measure

$$\pi^+_{l,\bar t,\bar x} := \pi^+_l \llcorner \big(\mathcal{G}^{h,l}_{\bar t,\bar x} \times \mathcal{G}^{e,l}_{\bar t,\bar x}\big)$$

The main contribution of this step is to show that, if $\mathscr{F}^l_{\bar t,\bar x} := (\mathrm{id}_t, f^l_{\bar t,\bar x})(\bar t, T) \subset (0,T) \times \mathbb{R}$, then

$$\text{the measure } (P_{t,x})_\sharp \pi^+_{l,\bar t,\bar x} \text{ is concentrated on } \mathscr{F}^l_{\bar t,\bar x}. \tag{1.3.21}$$

Let $\Omega^{l,\pm}_{\bar t,\bar x}$ be the two connected components of $(0,T) \times \mathbb{R} \setminus \mathscr{F}^l_{\bar t,\bar x}$, the left $(-)$ and the right $(+)$ one, respectively. By definition of $\mathcal{G}^{h,l}_{\bar t,\bar x}$ and of $\mathscr{F}^l_{\bar t,\bar x}$, it holds

$$\boldsymbol{\omega}_h \otimes \mu^+_\gamma \llcorner \mathcal{G}^{h,l}_{\bar t,\bar x}\big(\Omega^{l,+}_{\bar t,\bar x} \times (0,1)\big) = 0 \tag{1.3.22}$$

Moreover, one has

$$\begin{aligned} [P_{t,x}]_\sharp \pi^+_{l,\bar t,\bar x} &\leq [P_{t,x}]_\sharp \pi^+_l \llcorner \big(\mathcal{G}^{h,l}_{\bar t,\bar x} \times (\Gamma \times X)\big) \\ &= [p_{t,x}]_\sharp \big(\boldsymbol{\omega}_h \otimes \mu^+_\gamma \llcorner \mathcal{G}^{h,l}_{\bar t,\bar x}\big) \end{aligned} \tag{1.3.23}$$

It follows from (1.3.22), (1.3.23) that

$$[P_{t,x}]_\sharp \pi^+_{l,\bar t,\bar x}(\Omega^{l,+}_{\bar t,\bar x}) = 0 \tag{1.3.24}$$

In an entirely similar way, we prove

$$[P_{t',x'}]_\sharp \pi^+_{l,\bar t,\bar x}(\Omega^{l,-}_{\bar t,\bar x}) = 0 \tag{1.3.25}$$

Finally, we claim that $[P_{t',x'}]_\sharp \pi^+_{l,\bar t,\bar x} = [P_{t,x}]_\sharp \pi^+_{l,\bar t,\bar x}$. In fact, the transport plan

$$[P_{t,x} \times P_{t',x'}]_\sharp \pi^+_l \in \mathscr{M}\big(((0,T)\times\mathbb{R})^2\big)$$

is concentrated on the graph of the identity, because π^+ is supported in $\mathcal{G}$. Therefore we conclude that

$$[P_{t,x}]_\sharp \pi^+_{l,\bar t,\bar x}((0,T)\times\mathbb{R}\setminus(\Omega^{l,+}_{\bar t,\bar x}\cup\Omega^{l,+}_{\bar t,\bar x})) = 0 \tag{1.3.26}$$

which means that $[P_{t,x}]_\sharp \pi^+_{l,\bar t,\bar x}$ is concentrated on $\mathscr{F}^l_{\bar t,\bar x}$.

3.2. We prove that for π^+_l-a.e. pair $(\gamma,t,x,v,\gamma',t',x',v')$ there exists a $\delta>0$ such that

(1) for every $s\in[t-\delta,t)$ it holds $\gamma^x(s)<\gamma'^x(s)$,
(2) $\gamma^v(s)\in A_l$ for all $s\in(t-\delta,t]$,
(3) $\gamma'^{,v}(s)\in A_l$ for all $s\in(t'-\delta,t']$.

In order to prove it, we proceed as follows. By definition of π^+_l, it holds

for π^+_l-a.e. pair $(\gamma,t,x,v,\gamma',t',x',v')$ it holds:

$$\begin{cases}\gamma^v(t-)\geq\gamma^v(t+) \text{ and } \gamma^v(t-),\gamma^v(t+)\in A_l\\ \gamma'^{,v}(t-)\leq\gamma'^{,v}(t+) \text{ and } \gamma'^{,v}(t-),\gamma'^{,v}(t+)\in A_l\end{cases} \tag{1.3.27}$$

Therefore there exists a $\delta_{2,3}>0$ such that (2), (3), are satisfied. To prove (1), we first prove the following claim:

$$\text{for } \mu^-_\gamma\text{-a.e. } (t,x,v)\text{, there exists a } \delta>0 \text{ such that } \gamma^v(s)>v \text{ for every } s\in(t-\delta,t) \tag{1.3.28}$$

An application of Lemma 1.3.9 provides a $\tilde D^-\gamma$-negligible subset $N_\gamma\subset I_\gamma$ such that for every $t\in I_\gamma\setminus N_\gamma$ there exists a δ such that $\gamma^v(s)>\gamma^v(t+)$ for every $s\in(t-\delta,t)$. Moreover, for every $t\in I_\gamma$ in which γ^v has a negative jump, for each $v\in[\gamma^v(t+),\gamma^v(t-))$ there exists a $\delta>0$ such that for every $s\in(t-\delta,t)$ it holds $\gamma^v(s)>v$. Let

$$E_\gamma=\{(t,x,v)\ :\ \gamma^v(t-)=v>\gamma^v(t+)\}$$

Since E_γ is at most countable and μ^+_γ has no atoms, it follows that $\mu^+_\gamma\big(E_\gamma\big)=0$. Therefore

$$\mu^+_\gamma\big(E_\gamma\cup(\mathbb{I},\gamma)(N_\gamma)\big)=0$$

and (1.3.28) is proved. Therefore, using also (2), we obtain the following statement: for $\boldsymbol{\omega}_h\otimes\mu^+_\gamma$-a.e. (γ,t,x,v), there exists a $\delta_h>0$ ($\delta_h<\delta_2$) such that

$$\text{for every } s\in(t-\delta_h,t) \text{ it holds } \gamma^v(s)>v\text{, and hence } \gamma^x(s)<x-(t-s)f'(v) \tag{1.3.29}$$

where we used the strict convexity of f in A_l and the characteristic equation for γ (1.2.3). In an entirely analogous way we prove the symmetric statement: for $\boldsymbol{\omega}_e\otimes\mu^-_\gamma$-a.e. (γ,t,x,v), there exists a $\delta_e>0$ ($\delta_e<\delta_3$) such that

$$\text{for every } s\in(t-\delta_h,t) \text{ it holds } \gamma^v(s)<v\text{, and hence } \gamma^x(s)>x-(t-s)f'(v) \tag{1.3.30}$$

To conclude is sufficient to notice that since π^+ is concentrated on $\mathcal{G}$ (1) holds with $\delta = \min(\delta_h, \delta_e)$ for π^+_l-a.e. $(\gamma, t, x, v, \gamma', t', x', v')$.

3.3. From Step 3.2, we deduce that for π^+_l-a.e. pair $(\gamma, t, x, v, \gamma', t', x', v')$, there exists a rational pair $(\bar{t}, \bar{x}) \in ((0,T) \cap \mathbb{Q}) \times \mathbb{Q}$ such that

$$(\gamma, t, x, v, \gamma', t', x', v') \in \mathcal{G}^{h,l}_{\bar{t},\bar{x}} \times \mathcal{G}^{e,l}_{\bar{t},\bar{x}}$$

This shows that π^+_l is concentrated on

$$\bigcup_{\substack{\bar{t}\in(0,T)\cap\mathbb{Q}\\ \bar{x}\in\mathbb{Q}}} \mathcal{G}^{h,l}_{\bar{t},\bar{x}} \times \mathcal{G}^{e,l}_{\bar{t},\bar{x}}$$

Since holds

$$[P_{t,x}]_\sharp \pi^+_l = \sum_{\substack{\bar{t}\in(0,T)\cap\mathbb{Q}\\ \bar{x}\in\mathbb{Q}}} [P_{t,x}]_\sharp \Big(\pi^+_l \llcorner \mathcal{G}^{h,l}_{\bar{t},\bar{x}} \times \mathcal{G}^{e,l}_{\bar{t},\bar{x}}\Big) \tag{1.3.31}$$

by Step 3.1 it follows that $[P_{t,x}]_\sharp \pi^+_l$ is concentrated on the 1-rectifiable set

$$\bigcup_{\substack{\bar{t}\in(0,T)\cap\mathbb{Q}\\ \bar{x}\in\mathbb{Q}}} \mathscr{F}^l_{\bar{t},\bar{x}} \subset (0,T)\times\mathbb{R}$$

4. As a final step, we prove that the remaining part

$$\nu^+_{1,J} := [P_{t,x}]_\sharp \pi^+_J \tag{1.3.32}$$

is concentrated on the set J of Theorem 1.3.3, and therefore is concentrated on a 1-rectifiable set.

For $\bar{v} \in (0,1)$, we introduce the following functions which measure the nonlinearity of f near a point $\bar{v}$. For $\delta > 0$, define

$$\mathfrak{h}^-(\bar{v},\delta) := \sup\Big\{h > 0 \;\Big|\; \mathscr{L}^1\big((\bar{v}-\delta,\bar{v}) \cap \{v \;:\; |f'(\bar{v}) - f'(v)| > 2h\}\big) > h\Big\} > 0$$

$$\mathfrak{h}^+(\bar{v},\delta) := \sup\Big\{h > 0 \;\Big|\; \mathscr{L}^1\big((\bar{v},\bar{v}+\delta) \cap \{v \;:\; |f'(\bar{v}) - f'(v)| > 2h\}\big) > h\Big\} > 0$$

If the flux is genuinely nonlinear in the sense of Definition 1.3.1, it holds

$$0 < \mathfrak{h}^\pm(\bar{v},\delta) < \delta, \qquad \delta > 0 \tag{1.3.33}$$

For example, if $f(v) = v^2$, then for every $\bar{v} \in (0,1)$ one has $\mathfrak{h}^\pm(\bar{v},\delta) = \delta/3$.

Lemma 1.3.10. *Let $(\gamma, I_\gamma) \in \Gamma_h$, let $\bar{t} \in I_\gamma$ and set $\bar{x} = \gamma^x(\bar{t})$. Let $\bar{v} = \gamma^v(\bar{t}-) \vee \gamma^v(\bar{t}+)$. Then there exists a constant c depending only on $\|f''\|_\infty$ such that for every $\delta \in (0,1)$ at least one of the following holds true.*

$$\begin{aligned}
&\liminf_{r\downarrow 0} \frac{\mathscr{L}^2\big\{(t,x) \in B_R(\bar{t},\bar{x}) \;\big|\; u(x) > \bar{u} - \delta\big\}}{r^2} > c\cdot\mathfrak{h}^-(\bar{v},\delta)\\
&\limsup_{r\downarrow 0} \frac{\nu_0(B_R(\bar{t},\bar{x}))}{r} > c\cdot\mathfrak{h}^-(\bar{v},\delta)^2\\
&\limsup_{r\downarrow 0} \frac{\nu_1(B_R(\bar{t},\bar{x}))}{r} > c\cdot\mathfrak{h}^-(\bar{v},\delta)^3.
\end{aligned} \tag{1.3.34}$$

PROOF. Without loss of generality, we assume that $\bar{v} = \bar{\gamma}^v(\bar{t}-)$. We let $\delta_1 > 0$ be such that for every $t \in (\bar{t} - \delta_1, \bar{t})$ it holds

$$|f'(\bar{\gamma}^v(t)) - f'(\bar{v})| < \mathfrak{h}^-(\bar{v}, \delta)/2, \qquad \bar{\gamma}^v(t) > \bar{v} - \mathfrak{h}^-(\bar{v}, \delta)/2$$

Moreover, for $\bar{r}$ small, and for every $r < \bar{r}$, the curve $(t^1_{\bar{\gamma}}, \bar{t}) \ni t \mapsto (t, \bar{\gamma}^x(t))$ has a unique intersection with $\partial B_r((\bar{t}, \bar{x}))$, at a point that we call t_r.

1. (Existence of a big transversal interval). In this step we show the existence of an interval $J \subset (\bar{v}-\delta, \bar{v}-\mathfrak{h}^-(\bar{v}, \delta)/4)$ of length $\sim \mathfrak{h}^-(\bar{v}, \delta)$ and in which f' is $\sim \mathfrak{h}^-(\bar{v}, \delta)$-distant from $f'(\bar{v})$ (see (1.3.35) for the precise statements).

Define the set

$$I := \big\{ v \in (\bar{v} - \delta, \bar{v} - \mathfrak{h}^-(\bar{v}, \delta)/4) \ : \ |f'(\bar{v}) - f'(v)| \geq 2\mathfrak{h}^-(\bar{v}, \delta) \big\}$$

and

$$\tilde{I} := \big\{ v \in (\bar{v} - \delta, \bar{v} - \mathfrak{h}^-(\bar{v}, \delta)/4) \ : \ |f'(\bar{v}) - f'(v)| > \mathfrak{h}^-(\bar{v}, \delta) \big\}$$

and notice that by definition of $\mathfrak{h}^-(\bar{v}, \delta)$ they are not empty. Write the open set $\tilde{I}$ as the union of its connected components (open intervals)

$$\tilde{I} = \bigcup_{n=1}^{\infty} I_n$$

and let $\{I_k\}_{k \in \mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}$, be the set of the intervals I_k which intersect the set I. Notice that there are at most a finite number of them because for any I_k, $k \in \mathcal{I}$, such that there exist $k_1, k_2 \in \mathcal{I}$ with $\emptyset \neq I_{n_{k_1}} < I_{n_k} < I_{n_{k_1}} \neq \emptyset$ (here if I_1, I_2 are two intervals by $I_1 < I_2$ we mean $\max I_1 < \min I_2$), it holds

$$|I_k| > \frac{1}{\|f''\|_\infty} \cdot \mathfrak{h}^-(\bar{v}, \delta)$$

Therefore up to a relabeling, we call them $\tilde{I}_1, \ldots, \tilde{I}_{\tilde{n}}$, and without loss of generality assume

$$\tilde{I}_1 < \ldots < \tilde{I}_{\tilde{n}}$$

Notice that the intermediate ones (if they exist) satisfy

$$|\tilde{I}_i| > \frac{1}{\|f''\|_\infty} \cdot \mathfrak{h}^-(\bar{v}, \delta), \qquad i = 2, \ldots, \tilde{n} - 1$$

If $\tilde{I}_1, \tilde{I}_n$ are small, we remove them: in particular, denote

$$\mathcal{K} := \big\{ i \in \{0, \tilde{n}\} \mid |\tilde{I}_i| < \mathfrak{h}^-(\bar{v}, \delta)/4 \big\}$$

and define the new set of intervals

$$\{I_1, \ldots, I_n\} = \mathfrak{I} := \{\tilde{I}_i \ : \ i \in \{1, \ldots, \tilde{n}\} \setminus \mathcal{K}\}$$

Notice that this new set of intervals is not empty: in fact, the starting set of intervals $\tilde{I}_1, \ldots \tilde{I}_{\tilde{n}}$ satisfies $|\tilde{I}_1| + \ldots + |\tilde{I}_{\tilde{n}}| > \frac{3}{4}\mathfrak{h}^-(\bar{v}, \delta)$ (by definition of $\mathfrak{h}^-(\bar{v}, \delta)$), therefore the new set of intervals $I_1, \ldots, I_n$ satisfies

$$|I_1| + \ldots + |I_n| \geq \frac{1}{4}\mathfrak{h}^-(\bar{v}, \delta)$$

Choose any of these intervals, and call it J. By construction, we have

$$\begin{aligned}&|f'(\bar v)-f'(v)|>\mathfrak{h}^-(\bar v,\delta), \qquad \forall v\in J\\ &|J|\geq \frac{1}{\max\{4,\|f''\|\}}\mathfrak{h}^-(\bar v,\delta).\end{aligned} \tag{1.3.35}$$

2. (Transversality argument). Since $\bar\gamma\in\Gamma_h$ and $\bar\gamma^v(t)>\bar v-\mathfrak{h}^-(\bar v,\delta)/4$, it holds, for some $\varepsilon>0$ possibly depending on r, that

$$\mathscr{L}^2\left\{(t,x)\in S_{\varepsilon,r}\;\middle|\;u(t,x)>\bar v-\frac{\mathfrak{h}^-(\bar v,\delta)}{4}\right\}\geq\varepsilon r,\quad \text{where } S_{\varepsilon,r}:=(\mathrm{id}_t,\gamma^x)((t_r,\bar t))+B_\varepsilon(0). \tag{1.3.36}$$

For every $(\gamma,I_\gamma)\in\Gamma$ consider the nontrivial interiors $(t_j^{\gamma,-},t_j^{\gamma,+})_{j=1}^{N_\gamma}$ of the connected components of $(\gamma^v)^{-1}(J)$ which intersect

$$(\mathrm{id}_t,\gamma)^{-1}\big(S_{\varepsilon,r}\times\bar J\big)\subset I_\gamma$$

where $\bar J$ is the central interval of J of length $|J|/3$. Notice that we have the estimate

$$N_\gamma\leq 1+\frac{3}{|J|}\mathrm{Tot.Var.}\gamma^v$$

For every $j\in\mathbb{N}$, consider the set

$$\Gamma_j:=\{(\gamma,I_\gamma)\ :\ N_\gamma\geq j\}$$

and consider the measurable restriction map

$$R_j:\Gamma_j\to\Gamma,\quad (\gamma,I_\gamma)\mapsto(\gamma,(t_j^{\gamma,-},t_j^{\gamma,+}))$$

Define the measure

$$\tilde\omega_h:=\sum_{j=1}^{\infty}(R_j)_\sharp(\omega\llcorner\Gamma_j)$$

By an elementary transversality argument, for $\tilde\omega_h$-a.e. curve (γ,I_γ) it holds

$$\mathscr{L}^1\left\{t\in I_\gamma\;\middle|\;\gamma(t)\in S_{\varepsilon,r}\times\bar J\right\}\leq\frac{2\varepsilon}{\mathfrak{h}^-(\bar v,\delta)} \tag{1.3.37}$$

By construction, it holds

$$\int_\Gamma(\mathrm{id}_t,\gamma)_\sharp\mathscr{L}^1\llcorner I_\gamma\,\mathrm{d}\tilde\omega_h\geq\mathscr{L}^3\llcorner\left\{(t,x,v)\in S_{\varepsilon,r}\times\bar J\;\middle|\;u(t,x)>v\right\} \tag{1.3.38}$$

The measure of the set in the right hand side of (1.3.38) is at least $r\varepsilon|\bar J|$, therefore combining (1.3.37), (1.3.38), we obtain

$$\tilde\omega_h(\Gamma)\geq\frac{1}{2}r|\bar J|\mathfrak{h}^-(\bar v,\delta) \tag{1.3.39}$$

We let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \big\{(\gamma, I_\gamma) \,\big|\, |I_\gamma| \geq r\big\}$$

$$\Gamma_2 = \big\{(\gamma, I_\gamma) \,\big|\, |I_\gamma| < r, \quad \gamma^v(\partial I_\gamma) \cap \partial J \neq 0\big\}$$

$$\Gamma_3 = \big\{(\gamma, I_\gamma) \,\big|\, |I_\gamma| < r, \quad \gamma^v(\partial I_\gamma) \cap \partial J = 0\big\}$$

For $\tilde{\omega}_h$-a.e. $(\gamma, I_\gamma) \in \Gamma_1$ it holds

$$\mathscr{L}^1\big\{t \in I_\gamma \,\big|\, (t, \gamma(t)) \in B_{2r}(\bar{t}, \bar{x}) \times J\big\} \geq r$$

For $\tilde{\omega}_h$-a.e. $(\gamma, I_\gamma) \in \Gamma_2$ it holds

$$\gamma(I_\gamma) \subset B_{2r}(\bar{t}, \bar{x}) \times J, \qquad \text{Tot.Var.}\gamma^v > |J|/3$$

For $\tilde{\omega}_h$-a.e. $(\gamma, I_\gamma) \in \Gamma_3$ it holds

$$(t^1_\gamma, \gamma(t^1_\gamma)), \quad (t^2_\gamma, \gamma(t^2_\gamma)) \in B_{2r}(\bar{t}, \bar{x}) \times J$$

Then it follows that one of these condition holds:

$$\tilde{\omega}_h(\Gamma_1) \geq \frac{r|\bar{J}|\mathfrak{h}^-(\bar{v}, \delta)}{3}, \quad \tilde{\omega}_h(\Gamma_2) \geq \frac{r|\bar{J}|\mathfrak{h}^-(\bar{v}, \delta)}{3}, \quad \tilde{\omega}_h(\Gamma_3) \geq \frac{r|\bar{J}|\mathfrak{h}^-(\bar{v}, \delta)}{3}. \tag{1.3.40}$$

If the first condition holds, we deduce that

$$\mathscr{L}^2\Big(\big\{(t, x) \in B_{2r}(\bar{t}, \bar{x}) \,\big|\, u(t, x) > \bar{v} - \delta\big\}\Big) \geq \frac{1}{3} r^2 \mathfrak{h}^-(\bar{v}, \delta)$$

If the second condition holds, we deduce that

$$|\mu_1|(B_{2r}(\bar{t}, \bar{x}) \times J) > \frac{1}{27 \max\{4, \|f''\|_\infty\}^2} r \mathfrak{h}^-(\bar{v}, \delta)^3$$

If the third condition holds, we deduce that

$$|\mu_0|(B_{2r}(\bar{t}, \bar{x}) \times J) > \frac{1}{9 \max\{4, \|f''\|_\infty\}} r \mathfrak{h}^-(\bar{v}, \delta)^2$$

This proves the result. □

The symmetric statement holds for the epigraph: the proof is identical, therefore is omitted.

Lemma 1.3.11. *Let $(\gamma, I_\gamma) \in \Gamma_e$, let $\bar{t} \in I_\gamma$ and set $\bar{x} = \gamma^x(\bar{t})$. Let $\bar{v} = \gamma^v(\bar{t}-) \wedge \gamma^v(\bar{t}+)$. Then there exists an absolute constant c depending only on $\|f''\|_\infty$ such that for every $\delta \in (0, 1)$ at least one of the following holds true.*

$$\begin{aligned} &\liminf_{r \downarrow 0} \frac{\mathscr{L}^2\big\{(t, x) \in B_R(\bar{t}, \bar{x}) \,\big|\, u(x) < \bar{u} - \delta\big\}}{r^2} > c \cdot \mathfrak{h}^+(\bar{v}, \delta) \\ &\limsup_{r \downarrow 0} \frac{\nu_0(B_R(\bar{t}, \bar{x}))}{r} > c \cdot \mathfrak{h}^+(\bar{v}, \delta)^2 \\ &\limsup_{r \downarrow 0} \frac{\nu_1(B_R(\bar{t}, \bar{x}))}{r} > c \cdot \mathfrak{h}^+(\bar{v}, \delta)^3. \end{aligned} \tag{1.3.41}$$

The following proposition concludes the proof of Step 4.

Proposition 1.3.12. *For $\nu^+_{1,J}$-a.e. $(t,x) \in (0,T)\times\mathbb{R}$, it holds*

$$\limsup_{r\downarrow 0} \frac{\nu(B_r(t,x))}{r} > 0 \tag{1.3.42}$$

In particular, the measure $\nu^+_{1,J}$ is concentrated on the 1-rectifiable set $\boldsymbol{J}$ of Theorem 1.3.3.

Proof. For $\nu^+_{1,J}$ a.e. (t,x) one of the following holds:

(1) there exists $(\gamma,t,x,v)\in\mathcal{G}^+_{h,J}$ and $(\gamma',t',x',v')\in\mathcal{G}^-_e$ such that

$$(t,x)=(t',x') \quad\text{and}\quad \gamma^v(t-) > v = v' \geq \gamma'^{,v}(t-)$$

(2) there exists $(\gamma,t,x,v)\in\mathcal{G}^-_h$ and $(\gamma',t',x',v')\in\mathcal{G}^-_{e,J}$ such that

$$(t,x)=(t',x') \quad\text{and}\quad \gamma'^{,v}(t-) < v = v' \leq \gamma^v(t-)$$

Since the proof is symmetrical, assume the first condition holds. We apply Lemma 1.3.10 to the curve γ and Lemma 1.3.11 to the curve γ' with

$$\delta = \frac{1}{3}\{|\gamma^v(t-)-v|\}$$

Then either the second or the third condition holds in at least one of the two Lemma 1.3.10 , 1.3.11, or the first condition holds in both Lemmas. But in this case, (t,x) cannot be a point of vanishing mean oscillation. Therefore by Theorem 1.3.3, it must holds $(t,x)\in J$. □

5. From the previous steps, we conclude that ν^+_1 is concentrated on a 1-rectifiable set. In fact, one has

$$\nu^+_1 = (p_{t,x})_\sharp\big(\boldsymbol{\omega}_h\otimes\mu^{\gamma,+}_1\big) \leq (P_{t,x})_\sharp\pi^+ = \nu^+_{1,J} + \sum_{l\in\mathbb{N}}\nu^+_{1,l}$$

From Step 3 and Step 4 we deduce that ν^+_1 is concentrated on a 1-rectifiable set. The same argument holds for ν^-_1, therefore the proof is completed.

1.4. Regularity of Burgers' Equation

In this section we provide a first application of the Lagrangian representation to obtain a regularity result for Burger's equation with a measure source term. We first recall the Definition of the Besov spaces $B^{\alpha,p}_{\infty,\mathrm{loc}}(\mathbb{R})$.

Definition 1.4.1. A function $u:\mathbb{R}\to\mathbb{R}$ belongs to $B^{\alpha,p}_{\infty,\mathrm{loc}}(\mathbb{R})$ if

$$\|u\|^p_{B^{\alpha,p}_\infty(K)} := \sup_{h>0}\int_K \left|\frac{u(x+h)-u(x)}{h^\alpha}\right|^p \mathrm{d}x < +\infty, \quad \forall\, K\subset\mathbb{R} \quad \text{compact.} \tag{1.4.1}$$

Then we have the following regularity result in terms of Besov spaces.

Theorem 1.4.2. *Let u be a finite entropy solution to Burgers equation such that in addition μ_1 given by Proposition 1.1.3 is a signed measure. Then for every $K\subset\mathbb{R}$ compact and $\delta>0$ there exists a constant C_K such that*

$$\int_\delta^T \|u(t)\|_{B^{1/2,1}_\infty(K)}\,\mathrm{d}t < \frac{C_K}{\max\{\delta,1\}}$$

Remark 1.4.3. One can prove, using a regularization procedure, that if $u \in B^{1/2,2}_{\infty,\mathrm{loc}}$ (instead of just $B^{1/2,1}_{\infty,\mathrm{loc}}$) , then $\chi = \mathbf{1}_{\mathrm{hyp}\, u}(t,x,v)$ satisfies a kinetic equation of the form

$$\partial_t \chi + v\, \partial_x \chi = \mu, \qquad \mu \in \mathscr{M}((0,T)\times\mathbb{R}\times(0,1)). \tag{1.4.2}$$

In turn, this would imply the existence of a Lagrangian representation $\boldsymbol{\omega}$ for χ whose curves are segments in the (t,x,v) space with a constant v-component.

Proof. Up to a symmetry in the v, we can assume that μ_1 is a positive measure.

Moreover, up to choosing a different pair $(\hat\mu_0, \hat\mu_1)$ smaller than (μ_0,μ_1) for $\preceq$, we can assume that $\boldsymbol{\omega}_h$ is a Lagrangian representation of the hypograph of u (Definition 1.2.2) that induces (μ_0,μ_1) as in Definition 1.2.4. Thanks to Proposition 1.2.19, we can also assume that (μ_0,μ_1) is simultaneously induced (Definition 1.2.17) by $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_e$, where $\boldsymbol{\omega}_e$ is a Lagrangian representation of the epigraph of u (Definition 1.2.16). This second step is not really necessary but makes the proof easier. A key point is that these operation preserve the sign of μ_1, by definition of the relation $\preceq$. Finally, we denote by Γ_h, Γ_e the set of curves selected by Lemma 1.2.21.

1. Fix $\Delta t > 0$ and for $t \geq \Delta t$ consider the set of curves

$$\Gamma_h^{t,\Delta t} := \Big\{\gamma_h \in \Gamma_h \quad \Big| \quad t^-_{\gamma_h} \leq t - \Delta t\Big\}$$

$$\Gamma_e^{t,\Delta t} := \Big\{\gamma_e \in \Gamma_e \quad \Big| \quad t^-_{\gamma_e} \leq t - \Delta t\Big\}$$

Define the measures

$$\begin{aligned} \chi^t_{a^{\Delta t}} &:= e_{t\sharp}\,(\boldsymbol{\omega}_h \llcorner \Gamma_h^{t,\Delta t}) \leq \chi_h(t,\cdot,\cdot)\mathscr{L}^2 \llcorner (\mathbb{R}\times(0,1)) \quad \text{in } \mathscr{M}(\mathbb{R}\times(0,1)), \\ \chi^t_{b^{\Delta t}} &:= e_{t\sharp}\,(\boldsymbol{\omega}_e \llcorner \Gamma_e^{t,\Delta t}) \leq \chi_e(t,\cdot,\cdot)\mathscr{L}^2 \llcorner (\mathbb{R}\times(0,1)) \quad \text{in } \mathscr{M}(\mathbb{R}\times(0,1)). \end{aligned} \tag{1.4.3}$$

Finally, we define the functions

$$\begin{aligned} a^{\Delta t}(t,x) &:= \sup\Big\{v \in (0,1) \;\Big|\; (v,x) \in \operatorname{supp} \chi^t_{a^{\Delta t}}\Big\}, \qquad \forall\, x \in \mathbb{R} \\ b^{\Delta t}(t,x) &:= \inf\Big\{v \in (0,1) \;\Big|\; (v,x) \in \operatorname{supp} \chi^t_{b^{\Delta t}}\Big\}, \qquad \forall\, x \in \mathbb{R}. \end{aligned} \tag{1.4.4}$$

Notice that the L^1 distance between $a^{\Delta t}(t,\cdot)$ and $u(t,\cdot)$ can be estimated in terms of the source μ_0. In fact, by definition of $a^{\Delta t}$, we have

$$\begin{aligned} \int_{\mathbb{R}} |a^{\Delta t}(t,x) - u(t,x)|\, \mathrm{d}x &= \boldsymbol{\omega}_h \Big\{\gamma_h \in \Gamma_h \quad \Big| \quad t \in I_{\gamma_h}, \quad t^-_{\gamma_h} \in (t-\Delta t, t)\Big\} \\ &\leq \mu_0^+((t-\Delta t,t)\times\mathbb{R}\times(0,1)) \end{aligned} \tag{1.4.5}$$

which implies, integrating in $(0,T)$ and using Fubini's Theorem:

$$\int_0^T \int_{\mathbb{R}} |a^{\Delta t}(t,x) - u(t,x)|\, \mathrm{d}x\, \mathrm{d}t \leq \Delta t \cdot |\mu_0^+|. \tag{1.4.6}$$

Entirely symmetrical statements holds for the distance of u from $b^{\Delta t}$:

$$\begin{aligned}&\int_{\mathbb{R}} |a^{\Delta t}(t,x) - u(t,x)|\,\mathrm{d}x \le \mu_0^-((t-\Delta t, t)\times\mathbb{R}\times(0,1)),\\&\int_0^T\int_{\mathbb{R}} |b^{\Delta t}(t,x) - u(t,x)|\,\mathrm{d}x\,\mathrm{d}t \le \Delta t\cdot|\mu_0^-|.\end{aligned}\tag{1.4.7}$$

By triangular inequality, the difference $a^{\Delta t}(t,x) - b^{\Delta t}(t,x)$ lies in L^1 as well and

$$\begin{aligned}&\int_{\mathbb{R}} |a^{\Delta t}(t,x) - b^{\Delta t}(t,x)|\,\mathrm{d}x\,\mathrm{d}t \le \|\mu_0\|((t-\Delta t, t)\times\mathbb{R}\times(0,1)),\\&\int_0^T\int_{\mathbb{R}} |a^{\Delta t}(t,x) - b^{\Delta t}(t,x)|\,\mathrm{d}x\,\mathrm{d}t \le \Delta t|\mu_0|.\end{aligned}\tag{1.4.8}$$

2. Now fix $x<y$ and $\varepsilon>0$ small. Take a curve $\bar\gamma_h \in \Gamma_h^{t,\Delta t}$ such that $|\bar\gamma_h(t) - (y, a^{\Delta t}(t,y))| \le \varepsilon$. Since μ_1 is positive and $(y, a^{\Delta t}(t,y)) \in \operatorname{supp}\chi^t_{a^{\Delta t}}$, by (1.2.7) we can assume that $t\mapsto \bar\gamma_h^v(t)$, $t\in I_{\bar\gamma_h}$, is decreasing. By definition of $b^{\Delta t}$ there exist positive measure set of curves $G\subset\Gamma_e^{t,\Delta t}$ such that

$$|\gamma_e(t) - (x, b^{\Delta t}(t,x))| \le \varepsilon.$$

By Lemma 1.3.7, it holds

$$\omega_e\Big\{\gamma_e\in G \quad\Big|\quad \gamma_e(t-\Delta t) > \bar\gamma_h(t-\Delta t) \quad \text{and} \quad \gamma_e(t) < \bar\gamma_h(t)\Big\} = 0 \tag{1.4.9}$$

Moreover, since $\mu_1\ge 0$ and again by (1.2.7), $I_{\gamma_e}\ni t\mapsto \gamma_e^v(t)$ is increasing for ω_e-a.e. $\gamma_e\in G$. Therefore, thanks to the characteristic equation (1.2.3), it holds

$$\begin{aligned}&\bar\gamma_h^x(t-\Delta t) < y - \Delta t\cdot a^{\Delta t}(t,y) + 2\varepsilon\\&\gamma_e^x(t-\Delta t) > x - \Delta t\cdot b^{\Delta t}(t,x) - 2\varepsilon, \quad \text{for } \omega_e\text{-a.e. } \gamma_e\in G.\end{aligned}\tag{1.4.10}$$

Then, combining (1.4.9) with (1.4.10) and letting $\varepsilon\to 0$ we obtain

$$a^{\Delta t}(t,y) - b^{\Delta t}(t,x) \le \frac{y-x}{\Delta t}, \qquad \text{for every } x<y \tag{1.4.11}$$

Setting $h = y-x$ and integrating in an interval $[-M,M]$, for some $M>0$, we obtain

$$\int_{-M}^{M} (a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x))^+\,\mathrm{d}x \le 2Mh\Delta t^{-1}. \tag{1.4.12}$$

Moreover, combining with the inequality above the L^∞ bounds for $a^{\Delta t}, b^{\Delta t}$, and the inequality (1.4.8) of Step 1, we obtain

$$\begin{aligned}-\int_{-M}^{M} (a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x))^-\,\mathrm{d}x &= -\int_{-M}^{M} (a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x))^+\,\mathrm{d}x\\&\quad+\int_{-M}^{M} (a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x))\,\mathrm{d}x\\&\ge -2Mh\Delta t^{-1} - 2h - \|\mu_0\|(t-\Delta t, t)\end{aligned}\tag{1.4.13}$$

which in turn yields the bound

$$\int_{-M}^{M} |a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x)| \,\mathrm{d}x \le C\Big(h\Delta t^{-1} + \|\mu_0\|(t-\Delta t,t)\Big), \qquad \text{for all } t>0 \text{ and } \Delta t < t \tag{1.4.14}$$

with C depending only on M.

Step 3. Fix $\delta > 0$ and $h^{1/2} < \delta$. Then, for every $\Delta t < \delta$, we estimate the L^1 norm of the difference $u(t,x) - u(t,x+h)$ by

$$\begin{aligned}
\int_\delta^T \int_{-M}^{M} |u(t,x) - u(t,x+h)| \,\mathrm{d}x\,\mathrm{d}t \le& \int_\delta^T \int_{-M}^{M} |u(t,x) - b^{\Delta t}(t,x)| \,\mathrm{d}x\,\mathrm{d}t + \\
&+ \int_\delta^T \int_{-M}^{M} |b^{\Delta t}(t,x) - a^{\Delta t}(t,x+h)| \,\mathrm{d}x\,\mathrm{d}t + \\
&+ \int_\delta^T \int_{-M}^{M} |a^{\Delta t}(t,x+h) - u(t,x+h)| \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\Delta t \cdot |\mu_0| + \int_\delta^T \int_{-M}^{M} |a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x)| \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{1.4.15}$$

where the inequality in the last line follows by (1.4.6), (1.4.7). By (1.4.14), since for $t \in (\delta, T)$ one has $\Delta t < \delta < t$, we obtain

$$\int_\delta^T \int_{\mathbb{R}} |a^{\Delta t}(t,x+h) - b^{\Delta t}(t,x)| \,\mathrm{d}x\,\mathrm{d}t \le C(h\Delta t^{-1} + \Delta t)$$

so that in total, for another constant C depending on M, T and $|\mu_0|$, we obtain

$$\int_\delta^T \int_{-M}^{M} |u(t,x) - u(t,x+h)| \,\mathrm{d}x\,\mathrm{d}t \le C(h\Delta t^{-1} + \Delta t), \qquad \forall\ \Delta t < \delta \tag{1.4.16}$$

Choosing $\Delta t = h^{1/2}$ (which is possible since $h^{1/2} < \delta$ by assumption), we obtain

$$\int_\delta^T \int_{-M}^{M} \frac{|u(t,x) - u(t,x+h)|}{h^{1/2}} \,\mathrm{d}x\,\mathrm{d}t < 2C, \qquad \forall h \quad \text{such that } h^{1/2} < \delta \tag{1.4.17}$$

Instead if $h^{1/2} \ge \delta$, we obtain trivially

$$\int_\delta^T \int_{-M}^{M} \frac{|u(t,x) - u(t,x+h)|}{h^{1/2}} \,\mathrm{d}x\,\mathrm{d}t \le \frac{2M}{\delta} \tag{1.4.18}$$

This proves that

$$\sup_{h>0} \int_\delta^T \int_K \frac{|u(t,x) - u(t,x+h)|}{h^{1/2}} \,\mathrm{d}x\,\mathrm{d}t \le \frac{C_K}{\delta}, \qquad \forall \quad K \subset \mathbb{R} \text{ compact.} \tag{1.4.19}$$

□

We conclude by quoting a result about the existence of characteristics. It is proved in [**49**] when $\mu_0 = 0$, but the proof does not depend on the presence of a source term.

PROPOSITION 1.4.4 ([**49**]). *Let $u : \Omega \to \mathbb{R}$ be a finite entropy solution, with $d = 1$ and with strictly convex flux f. For any $x_0 \in \mathbb{R}$, there exists a generalized characteristic of u starting at x_0, that is, a Lipschitz curve $x\colon [t_0, T] \to \mathbb{R}$ such that $x(0) = x_0$ and*

$$x'(t) = f'(u^{\pm}(t, x(t))) \qquad \text{for a.e. } t \in [0, T] \text{ s.t. } u^+(t, x(t)) = u^-(t, x(t)),$$

where $u^{\pm}(t, x(t))$ denote the left and right traces of u along $(t, x(t))$.

1.5. Applications to Some 2×2 Systems

1.5.1. Preliminaries about 2×2 Systems of Conservation Laws. A 2×2 system of conservation laws is a system of two equations of the form

$$U(t, x)_t + f(U(t, x))_x = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R} \tag{1.5.1}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is an open bounded connected set $\mathcal{U}$, $U = (u_1, u_2) \in \mathcal{U} \subset \mathbb{R}^2$ is a state vector of conserved quantities and the flux f is a smooth function $f : \mathcal{U} \to \mathbb{R}^2$. The system (1.5.1) is called strictly hyperbolic if the matrix ∇f has distinct real eigenvalues

$$\lambda_1(U) < \lambda_2(U) \qquad \forall\, U \in \mathcal{U}$$

with corresponding eigenvectors $r_1(U), r_2(U)$. We also let ℓ_1, ℓ_2 be the corresponding left eigenvectors, normalized so that

$$\ell_i(U) \cdot r_i(U) = \delta_{i,j} \qquad \forall\, U \in \mathcal{U}.$$

We say that the i-th field is *genuinely nonlinear* if

$$\nabla \lambda_i(U) \cdot r_i(U) \geq c > 0 \qquad \forall\, U \in \mathcal{U}.$$

It is well known that weak solutions to hyperbolic systems of conservation laws are not unique, therefore in order to select physically relevant solutions, one is usually interested only in entropic solutions of (1.5.1).

DEFINITION 1.5.1 (Entropies). A scalar function $\eta : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ is an *entropy* for (1.5.1) if there exists some $q : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla \eta \cdot \nabla f = \nabla q \tag{1.5.2}$$

Admissible solutions of (1.5.1) will be the ones that dissipate the family of *convex* entropies:

DEFINITION 1.5.2 (Entropy solutions). A function $U : (0, T) \times \mathbb{R} \to \mathcal{U}$ is called an *entropy weak solution* of (1.5.1) if it satisfies

$$\partial_t \eta + \partial_x q \leq 0 \qquad \text{in } \mathscr{D}'((0, T) \times \mathbb{R}) \tag{1.5.3}$$

for all convex entropies η.

The relevance of this definition lies in the fact that the viscous approximations to (1.5.1)

$$U^{\epsilon}(t, x)_t + f(U^{\epsilon}(t, x))_x = \epsilon U^{\epsilon}_{xx} \tag{1.5.4}$$

produce entropy admissible weak solutions of (1.5.1) in the limit $\epsilon \to 0^+$. Since the system (1.5.1) is a 2×2 system, it admits at least one *uniformly* convex entropy $E : \mathcal{U} \to \mathbb{R}$, i.e. an entropy E such that for some $\alpha > 0$

$$\alpha \operatorname{Id} \leq \nabla^2 E(U) \qquad \forall\, U \in \mathcal{U} \tag{1.5.5}$$

in the sense of symmetric matrices. We have the following classical energy bound on vanishing viscosity solutions.

PROPOSITION 1.5.3. *Let $U_0 \in \mathbf{L}^\infty$ with $\operatorname{Im} U_0 \subset \mathcal{U}$. Let U^ϵ be a sequence of solutions of (1.5.4) with $\operatorname{Im} U^\epsilon \subset \mathcal{U}$, $U^\epsilon(0,x) = U_0(x)$. Then for every compact $K \subset \mathbb{R}^+ \times \mathbb{R}$ there exists a constant C_K such that it holds*

$$\sup_{\epsilon > 0} \iint_K \left(\sqrt{\epsilon}\, U_x^\epsilon\right)^2 \mathrm{d}x\, \mathrm{d}t \le C_K. \tag{1.5.6}$$

PROOF. We define a trapezoid of the form, for $M > 0$,

$$\Delta_M \doteq \left\{(t,x) \mid M + \widehat{\lambda} t \le x \le M - \widehat{\lambda} t, \quad t > 0\right\}$$

where $\widehat{\lambda}$ is a big positive constant. Let η be any entropy for (1.5.1) and take the scalar product of (1.5.4) with $\nabla \eta(U^\epsilon)$ and use the chain rule to obtain

$$\eta(U^\epsilon)_t + q(U^\epsilon)_x = \epsilon\, \nabla \eta(U^\epsilon) U^\epsilon_{xx} = \epsilon\, \eta(U^\epsilon)_{xx} - \epsilon \left\langle \nabla^2 \eta(U^\epsilon) \cdot U_x^\epsilon\,,\, U_x^\epsilon \right\rangle \tag{1.5.7}$$

Let E be a positive uniformly convex entropy (satisfying (1.5.5)). Integrating in x and in t, we obtain the bound

$$\epsilon \iint_{\Delta_M} \left\langle \nabla^2 E(U^\epsilon) \cdot U_x^\epsilon\,,\, U_x^\epsilon \right\rangle \mathrm{d}x\, \mathrm{d}t \le \int_{-M}^{M} E(U^\epsilon(0,x))\, \mathrm{d}x.$$

Then, by (1.5.5), we obtain that

$$\iint_{\Delta_M} \left(\sqrt{\epsilon}\, U_x^\epsilon\right)^2 \mathrm{d}x\, \mathrm{d}t \le \frac{1}{\alpha} \cdot \int_{-M}^{M} E(U^\epsilon(0,x))\, \mathrm{d}x \tag{1.5.8}$$

Therefore $\left(\sqrt{\epsilon}\, U_x^\epsilon\right)^2$ is locally uniformly bounded as a measure, i.e. for every compact $K \subset \mathbb{R}^+ \times \mathbb{R}$ there exists a constant C_K such that it holds

$$\sup_{\epsilon > 0} \iint_K \left(\sqrt{\epsilon}\, U_x^\epsilon\right)^2 \mathrm{d}x\, \mathrm{d}t \le C_K. \tag{1.5.9}$$

as wanted.

□

REMARK 1.5.4. If there exists $\widehat{U}$ such that $U_0 - \widehat{U} \in \mathbf{L}^2$, then choosing the entropy E such that $E(\widehat{U}) = 0$ and passing to the limit for $M \to +\infty$ in (1.5.8)

$$\sup_{\epsilon} \int_{\mathbb{R}^+ \times \mathbb{R}} \left(\sqrt{\epsilon}\, U_x^\epsilon\right)^2 \mathrm{d}x\, \mathrm{d}t < \mathcal{O}(1) \cdot \|U - \widehat{U}\|^2_{\mathbf{L}^2}. \tag{1.5.10}$$

1.5.2. A Class of 2×2 Systems. Being a 2×2 system, (1.5.1) admits a coordinate system of Riemann invariants w, z. The latter are functions defined by

$$\nabla w(U) = \ell_1(U), \qquad \nabla z(U) = \ell_2(U) \qquad \forall U \in \mathcal{U}. \tag{1.5.11}$$

Without loss of generality, we can assume that $w(U), z(U) \ge 0$ for all $U \in \mathcal{U}$. Then, for smooth solutions, the system (1.5.1) can be diagonalized via the mapping

$$U \,:\, \mathcal{W} \to \mathcal{U}, \qquad \mathcal{W} \doteq (w,z)(\mathcal{U}) \subset \mathbb{R}^2$$

defined by $w(U(w,z)) = w$, $z(U(w,z)) = z$. In fact, if U is smooth, take the scalar product of (1.5.1) with ∇w, ∇z to obtain

$$\begin{aligned} & w(U)_t + \lambda_1(U)\, w(U)_x = 0, \\ & z(U)_t + \lambda_2(U)\, z(U)_x = 0, \end{aligned} \qquad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Similarly, the viscous system (1.5.4) can be diagonalized

$$\begin{aligned} & w(U^\epsilon)_t + \lambda_1(U^\epsilon)\, w(U^\epsilon)_x = \epsilon\, w(U^\epsilon)_{xx} - \epsilon\, \langle \nabla^2 w(U^\epsilon) \cdot U^\epsilon_x\,,\, U^\epsilon_x \rangle, \\ & z(U^\epsilon)_t + \lambda_2(U^\epsilon)\, z(U^\epsilon)_x = \epsilon\, z(U^\epsilon)_{xx} - \epsilon\, \langle \nabla^2 z(U^\epsilon) \cdot U^\epsilon_x\,,\, U^\epsilon_x \rangle. \end{aligned} \tag{1.5.12}$$

Thanks to the bound (1.5.6), the second term in the right hand side is locally uniformly bounded in $\mathscr{M}(\mathbb{R}^+ \times \mathbb{R})$. Consider the following assumption on the system (1.5.1):

$$\begin{aligned} & \lambda_1(w,z) \equiv \lambda_1(w) \qquad \forall\, w,z \in \mathcal{W} \\ & \lambda_2(w,z) \equiv \lambda_2(z) \qquad \forall\, w,z \in \mathcal{W} \\ & \mathscr{L}^1\big(\{v \mid \lambda_i'(v) = 0\}\big) = 0 \qquad \text{for } i = 1,2. \end{aligned} \tag{1.5.13}$$

where with an abuse of notation we regarded the eigenvectors as functions of the Riemann invariants. A notable example in this class is the isentropic system of gas dynamics with $\gamma = 3$, which reads

$$\begin{aligned} & \partial_t \rho + \partial_x \rho u = 0, \\ & \partial_t(\rho u) + \partial_x \left(\rho u^2 + \rho^3/12\right) = 0, \end{aligned} \tag{1.5.14}$$

where we assume that $(\rho, m) \doteq (\rho, \rho u)$ takes values in a bounded connected open set $\mathcal{U} \subset (\rho_{\min}, +\infty) \times \mathbb{R}$ for some $\rho_{\min} > 0$. Then, for such systems, we have the following Proposition.

Proposition 1.5.5. *Assume that* (1.5.13) *holds. Then any weak entropy solution* U *to* (1.5.1) *which is pointwise a.e. limit of vanishing viscosity approximations* (1.5.4) *is such that its Riemann invariants* $w, z : \Omega \to \mathbb{R}$ *are finite entropy solution* (1.1.1)*. Moreover, the measure* μ_1 *in Proposition 1.1.3 can be chosen to be a signed measures.*

Remark 1.5.6. In particular, it follows that solutions U as above satisfy the same structural properties of scalar finite entropy solutions stated in Theorem 1.3.3.

Proof. We prove the Proposition for w, the proof for the other Riemann invariant z being entirely similar. Assume that $w^\epsilon \to w$ strongly in $\mathbf{L}^1(\Omega)$. Define

$$\chi(t,x,v) \doteq \mathbf{1}_{\operatorname{hyp} w}(t,x,v)$$

and the distribution

$$T \doteq \partial_t \chi + \lambda_1(v) \partial_x \chi \qquad \text{in } \mathscr{D}'_{t,x,v}.$$

We test T against a test function of the form

$$\phi(t,x,v) \doteq \varphi(t,x) \cdot \eta'(v), \qquad \eta'' \geq 0$$

where φ, is smooth and compactly supported. We obtain

$$\begin{aligned}\langle T\,,\,\phi\rangle &= -\int_0^T\int_{\mathbb{R}}\left[\varphi_t\int_v \eta'(v)\chi(v)\,\mathrm{d}v + \varphi_x\int_v \lambda_1(v)\eta'(v)\,\mathrm{d}v\right]\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{R}}\left[\varphi_t\eta(w)+\varphi_x q(w)\right]\mathrm{d}x\,\mathrm{d}t\\ &= -\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{R}}\left[\varphi_t\eta(w^\epsilon)+\varphi_x q(w^\epsilon)\right]\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{1.5.15}$$

where q is defined by $q' = \lambda_1\,\eta'$ and we assumed without loss of generality $\eta(0) = q(0) = 0$. By multiplying (1.5.12) by $\eta'(w^\epsilon)$ we obtain

$$\eta(w^\epsilon)_t + q(w^\epsilon) = \epsilon\eta(w^\epsilon)_{xx} - \epsilon\,\eta''(w^\epsilon)\,(w^\epsilon)^2_x - \epsilon\,\eta'(w^\epsilon)\langle\nabla^2 w(U^\epsilon)\cdot U^\epsilon_x\,,\,U^\epsilon_x\rangle \tag{1.5.16}$$

The first term in the right hand side converges to zero, while the second and the third terms can be estimated by

$$\epsilon\,\eta''(w^\epsilon)\,(w^\epsilon)^2_x \le C_1\cdot(\sqrt{\epsilon}\,U^\epsilon_x)^2\cdot\max\,\eta'',\qquad \epsilon\,\eta'(w^\epsilon)\langle\nabla^2 w(U^\epsilon)\cdot U^\epsilon_x\,,\,U^\epsilon_x\rangle \le C_2\cdot(\sqrt{\epsilon}\,U^\epsilon_x)^2\cdot\max\eta'.$$

where C_1, C_2 depend only on the flux f. Moreover, the first term is positive, since $\eta'' \ge 0$. Define the map $(\mathrm{id}, w^\epsilon) : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$ by $(\mathrm{id}, w^\epsilon)(t,x) \doteq (t, x, w^\epsilon(t,x))$. Then, the bound (1.5.6) above shows that the measures

$$\mu^\epsilon_1 \doteq (\mathrm{id}, w^\epsilon)_\sharp\Big[(\sqrt{\epsilon}w^\epsilon_x)^2\cdot\mathscr{L}^2\Big],\qquad \mu^\epsilon_0 \doteq (\mathrm{id}, w^\epsilon)_\sharp\Big[\langle\epsilon\nabla^2 w(U^\epsilon)\cdot U^\epsilon_x\,,\,U^\epsilon_x\rangle\cdot\mathscr{L}^2\Big]$$

are locally uniformly bounded in ϵ, i.e. for every compact set $K\subset\Omega$ it holds

$$\sup_\epsilon|\mu^\epsilon_i|(K) < +\infty \qquad \text{for } i = 0, 1.$$

Furthermore, combining (1.5.15) with (1.5.16) we find that

$$T = \lim_{\epsilon\to 0}\left(\partial_v\mu^\epsilon_1 + \mu^\epsilon_0\right)\qquad \text{in } \mathscr{D}'_{t,x,v}.$$

Since μ_0, μ_1 are locally uniformly bounded, up to subsequences we can assume that $\mu^\epsilon_1 \rightharpoonup \mu_1$ weakly in the sense of measures, with μ_1 a locally finite measure, and the same for μ_0. Moreover, μ_1 is positive. Then we conclude that

$$T = \partial_v\mu_1 + \mu_0,\qquad \mu_0, \mu_1 \in \mathscr{M}_{t,x,v},\qquad \mu_1 \ge 0$$

as wanted. □

Therefore, from Theorem 1.3.4 and from Proposition 1.5.5, the following theorem follows immediately

COROLLARY 1.5.7. *Assume that f satisfies* (1.5.13). *Then any weak entropy solution U to* (1.5.1) *which is pointwise a.e. limit of vanishing viscosity approximations* (1.5.4) *satisfies the following: for every entropy-entropy flux (η, q) the entropy dissipation measure*

$$\mu_\eta \doteq \eta(U)_t + q(U)_x \in \mathscr{M}_{t,x}$$

is concentrated on a 1-rectifiable set.

By Proposition 1.4.4, we also have the following result about regularity and the existence of characteristics under the assumption that $v \mapsto \lambda_i(v)$ are strictly convex (in particular this is satisfied by the system of isentropic gas dynamics (1.5.14)).

COROLLARY 1.5.8. *Assume that f satisfies* (1.5.13) *and that $v \mapsto \lambda_i(v)$ are strictly convex. Then U belongs to the Besov space $B^{1/2,1}_{\infty,\mathrm{loc}}(K)$, for every $K \subset \Omega$ compact. Moreover, for any $x_0 \in \mathbb{R}$, there exists 1 and 2 generalized characteristic of U starting at x_0, that is, Lipschitz curves $x^i : [t_0, T] \to \mathbb{R}$, $i = 1, 2$ such that $x^i(0) = x_0$ and*

$$(x^1)'(t) = \lambda_1(w(t, x(t))) \qquad \text{for a.e. } t \in [0, T] \text{ s.t. } w(t, x(t)-) = w(t, x(t)+),$$
$$(x^2)'(t) = \lambda_2(z(t, x(t))) \qquad \text{for a.e. } t \in [0, T] \text{ s.t. } z(t, x(t)-) = z(t, x(t)+),$$

REMARK 1.5.9. The results of this section can be extended with no additional difficulty to $n \times n$ rich systems such that the eigenvalues satisfy the assumption (1.5.13).

CHAPTER 2

Kinetic Formulation and Decay for 2×2 Systems

2.1. A Kinetic-Type Equation for General 2×2 systems

In this section, we establish a system of two kinetic equations that is satisfied by the Riemann invariants of all vanishing viscosity solutions obtained via the compensated compactness method (see Proposition 1.5.3). The derivation does not impose specific assumptions on the system under consideration, nor does it require any smallness conditions on the solutions, other than their boundedness in $\mathbf{L}^\infty$.

This derivation is strongly inspired by the construction of singular entropies developed in the works of Perthame and Tzavaras [**91**] and [**97**], where, among other things, the authors derived a kinetic formulation for the system of elastodynamics. The authors exploited specific properties of the systems, allowing them to identify families of convex entropies, which in turn provided a kinetic formulation that characterizes the entropy solutions. A similar kinetic formulation was previously developed by Lions, Perthame, and Tadmor [**79**] for the system of isentropic gas dynamics, where the authors also utilized the specific structure of the system to derive a corresponding kinetic equation.

We show that by potentially introducing a source term, consistent with the framework established in previous sections, it is possible to derive kinetic equations that are satisfied by all vanishing viscosity solutions to 2×2 systems. As expected, the resulting kinetic equation will be *non-local*, a characteristic feature shared with all the kinetic equations derived for 2×2 systems in the mentioned works.

Although our kinetic equation does not characterize entropy solution, we would expect that some of the regularity properties of solutions to 2×2 systems are retained by solutions to the kinetic equation. As a first application, we prove a result about the decay of $\mathbf{L}^4$ norms of solutions for a class of genuinely non-linear systems.

2.2. Entropies, Kinetic Formulation

2.2.1. Construction of Singular Entropies. In this subsection we recall the construction of singular entropies performed in [**91**], [**97**]. From now on, it is convenient to work on a domain that is a square in the Riemann Invariants:

$$\mathcal{W} = \big[\underline{w},\, \bar{w}\big] \times \big[\underline{z},\, \bar{z}\big]$$

and to employ a relaxed concept of entropy-entropy flux pair. In particular, a *weak entropy-entropy flux pair* is a pair of functions $\eta, q : \mathcal{U} \to \mathbb{R}$ that solves in the sense of distribution

$$\nabla q(U) \,-\, \mathrm{D}f(U)\, \nabla \eta(U) = 0 \qquad \text{in } \mathscr{D}'(\mathcal{U}). \tag{2.2.1}$$

Let g, h be the unique solutions to

$$h_w = \frac{\lambda_{2w}}{\lambda_1 - \lambda_2} h, \qquad g_z = -\frac{\lambda_{1z}}{\lambda_1 - \lambda_2} g, \qquad h(\underline{w}, z) = 1, \qquad g(w, \underline{z}) = 1. \tag{2.2.2}$$

They can be computed explicitly as

$$g(w,z) = \exp\left[\int_{\underline{z}}^{z} -\frac{\lambda_{1z}(w,y)}{\lambda_1(w,y) - \lambda_2(w,y)}\,\mathrm{d}y\right]$$
$$h(w,z) = \exp\left[\int_{\underline{w}}^{w} \frac{\lambda_{2w}(y,z)}{\lambda_1(y,z) - \lambda_2(y,z)}\,\mathrm{d}y\right].$$

and they are uniformly positive on $\mathcal{W}$. It is then classical (see e.g. [**94**, Section 9.3]) that η is a smooth entropy if and only if, in Riemann coordinates,

$$\eta_{wz} = \frac{g_z}{g}\eta_w + \frac{h_w}{h}\eta_z \qquad \text{in } \mathcal{W}.$$

Following [**91**], we first construct a family of *smooth* entropies $\mathbf{\Theta}[\xi, b_0](w,z)$, depending on two parameters: a scalar $\xi \in [\underline{w}, \bar{w}]$ and a smooth function $b_0 : [\underline{w}, \bar{w}] \to \mathbb{R}$. These entropies are constructed so that they can be "cut" along a line $\{w = \xi\}$. By this we mean that

$$\boldsymbol{\chi}[\xi, b_0](w,z) \doteq \mathbf{\Theta}[\xi, b_0](w,z) \cdot \mathbf{1}_{\{w \geq \xi\}}(w,z) \tag{2.2.3}$$

and

$$\widetilde{\boldsymbol{\chi}}[\xi, b_0](w,z) \doteq \mathbf{\Theta}[\xi, b_0](w,z) \cdot \mathbf{1}_{\{w \leq \xi\}}(w,z) \tag{2.2.4}$$

will still be (discontinuous) weak entropies.

DEFINITION 2.2.1. We denote by $\mathbf{\Theta}[\xi, b_0]$ the entropy constructed as the unique solution to the Goursat-boundary value problem (see Figure 1)

$$\begin{cases} \mathbf{\Theta}_{wz} = \frac{g_z}{g}\mathbf{\Theta}_w + \frac{h_w}{h}\mathbf{\Theta}_z, & \text{in } \mathcal{W} \\ \mathbf{\Theta}(w, \underline{z}) = b_0(w), & \forall\, w \in [\underline{w}, \bar{w}] \\ \mathbf{\Theta}(\xi, z) = b_0(\xi) g(\xi, z) & \forall\, z \in [\underline{z}, \bar{z}]. \end{cases}$$

Since $g(\xi, \underline{z}) = 1$, the two boundary conditions are compatible (continuous) at the point $(\xi, \underline{z})$, by construction. Since h, g are smooth and bounded away from zero, the existence of a unique, smooth solution $\mathbf{\Theta}$ to the above boundary value problem is standard. For a proof of this fact, see e.g. [**94**, Section 9.3], in which it is proved that solutions to the Goursat problem are at least as smooth as the data and as the coefficients $g_z/g, h_w/h$. Moreover, it also follows that

$$w, z, \xi \mapsto \mathbf{\Theta}[\xi, b_0](w,z)$$

is smooth as a function of three variables w, z, ξ. Now for fixed ξ, b_0 we consider the entropy flux $\mathbf{\Xi} \equiv \mathbf{\Xi}[\xi, b_0]$ associated with $\mathbf{\Theta} \equiv \mathbf{\Theta}[\xi, b_0]$: we have that

$$\mathbf{\Xi}_z(\xi, z) = \lambda_2(\xi, z)\mathbf{\Theta}_z(\xi, z) = -\frac{\lambda_2(\xi, z)\lambda_{1z}(\xi, z)}{\lambda_1(\xi, z) - \lambda_2(\xi, z)}\mathbf{\Theta}(\xi, z) = (\lambda_1(\xi, z)\mathbf{\Theta}(\xi, z))_z$$

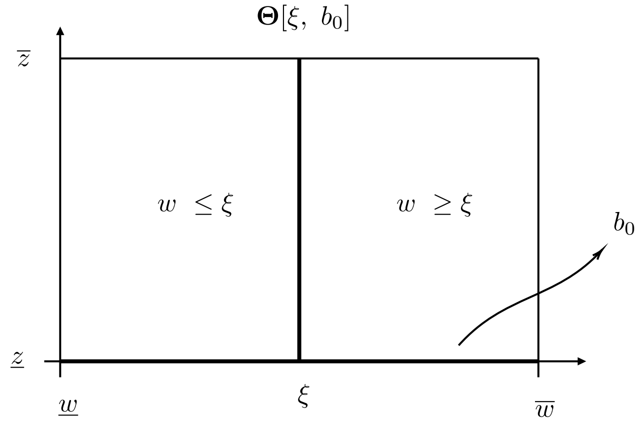

FIGURE 1. Goursat problem for the entropy $\mathbf{\Theta}[\xi, b_0]$. The data are given along the thick lines.

where the first equality follows from applying (2.2.1) to $\mathbf{\Theta}, \mathbf{\Xi}$, and by taking the scalar product with $r_2(U)$, while the second equality follows from the fact that $\mathbf{\Theta}(\xi, z) = b_0(\xi) g(\xi, z)$ for every $z \in [\underline{z}, \bar{z}]$ and by (2.2.2). Therefore up to an additive constant in the entropy flux we can assume that

$$\mathbf{\Xi}(\xi, z) = \lambda_1(\xi, z)\mathbf{\Theta}(\xi, z) \qquad \forall\, z \in [\underline{z}, \bar{z}]. \tag{2.2.5}$$

Thanks to (2.2.5), we see that $(\boldsymbol{\chi}, \boldsymbol{\psi})$, and $(\widetilde{\boldsymbol{\chi}}, \widetilde{\boldsymbol{\psi}})$, where

$$\boldsymbol{\psi}[\xi, b_0] \doteq \mathbf{\Xi}[\xi, b_0] \cdot \mathbf{1}_{\{w \geq \xi\}}, \quad \widetilde{\boldsymbol{\psi}}[\xi, b_0] \doteq \mathbf{\Xi}[\xi, b_0] \cdot \mathbf{1}_{\{w \leq \xi\}} \tag{2.2.6}$$

are entropy-entropy flux pair solving (2.2.1).

The entropies $\mathbf{\Theta}[\xi, b_0]$ depend on a number $\xi \in [\underline{w}, \bar{w}]$ and on a function b_0. To obtain a "one dimensional" kinetic formulation for the first Riemann Invariant, for every ξ we need to make a choice of b_0. Following [**91**], we choose

$$b_0(w) = 1 \qquad \forall\, w \in [\underline{w}, \bar{w}]$$

and with this choice we rename the entropy $\mathbf{\Theta}$ omitting the dependence on b_0, which is now fixed:

$$\mathbf{\Theta}[\xi](w, z) \equiv \mathbf{\Theta}[\xi, 1](w, z) \qquad \forall\, \xi \in [\underline{w}, \bar{w}]$$

and the same for $\boldsymbol{\chi}[\xi], \boldsymbol{\psi}[\xi] \equiv \boldsymbol{\chi}[\xi, 1], \boldsymbol{\psi}[\xi, 1]$. The following proposition contains some structural results for the entropies $\boldsymbol{\chi}$.

PROPOSITION 2.2.2. *There exists positive $\bar{r}, c > 0$ such that, for every $(w, z) \in \mathcal{W}$, the following holds:*

(1) Strict positivity of the entropies:

$$\boldsymbol{\chi}[\xi](w, z) \geq c > 0, \qquad \forall \quad \xi, w, z \text{ such that } \quad \xi \leq w \leq \xi + \bar{r}.$$

(2) If λ_1 is genuinely nonlinear, then we have the monotonicity of the kinetic speed:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\boldsymbol{\lambda}_1[\xi](w, z) \geq c > 0 \qquad \forall\, \xi \leq w \leq \xi + \bar{r}$$

where

$$\boldsymbol{\lambda}_1[\xi](w,z) \doteq \frac{\boldsymbol{\psi}[\xi](w,z)}{\boldsymbol{\chi}[\xi](w,z)} \qquad \forall\, \xi \le w \le \xi + \bar{r}.$$

REMARK 2.2.3. The numbers $\bar{r}, c$ are uniform in the choice of $\xi \in [\underline{w}, \bar{w}]$.

PROOF. Fix ξ. Since the entropy $\boldsymbol{\chi}[\xi]$ is uniformly positive along the boundary data curve $\{(w,z) \in \mathcal{W} \mid w = \xi\}$, there exists $\delta(\xi) > 0, c_1 > 0$ such that

$$\boldsymbol{\chi}[\xi](w,z) \ge c_1 > 0, \qquad \forall\, (w,z) \in \mathcal{W}, \qquad \xi \le w \le \xi + \bar{r}.$$

Then, since the function $(\xi, w, z) \mapsto \boldsymbol{\Theta}[\xi](w,z)$ is in particular continuous and since $\xi \in [\underline{w}, \bar{w}]$ which is compact, there exists uniform $r, c > 0$ (not dependent on ξ) such that (1) holds. Furthermore, for every $w \ge \xi$, the entropy flux $\boldsymbol{\psi}[\xi]$ associated to $\boldsymbol{\chi}[\xi]$ can be computed as

$$\begin{aligned} \boldsymbol{\psi}[\xi](w,z) &= \lambda_1(w,z)\boldsymbol{\chi}[\xi](\xi,z) + \int_\xi^w \lambda_1(v,z)\boldsymbol{\chi}_w[\xi](v,z)\,\mathrm{d}v \\ &= \lambda_1(w,z)\boldsymbol{\chi}[\xi](w,z) - \int_\xi^w \lambda_{1w}(v,z)\boldsymbol{\chi}[\xi](v,z)\,\mathrm{d}v. \end{aligned} \tag{2.2.7}$$

where the first equality follows from the fundamental theorem of calculus and (2.2.1), and the second follows by integrating by parts. Therefore, if the first eigenvalue is genuinely nonlinear the kinetic speed $\boldsymbol{\lambda}_1[\xi](w,z)$ is monotonically increasing in ξ if ξ is close to w: in particular, for some $c_2 > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\boldsymbol{\lambda}_1[\xi](w,z) \ge c_2 > 0 \qquad \forall\, (w,z) \in \mathcal{W}, \qquad \xi \le w \le \xi + \bar{r}.$$

The existence of uniform r, c such that (2) holds is again ensured by the smoothness of all the functions involved. □

A completely symmetric construction can be made for entropies that can be cut along the second Riemann invariant; for these entropies, for $\zeta \in [\underline{z}, \bar{z}]$, we let $\boldsymbol{\upsilon}[\zeta](w,z)$ be entropy symmetric to $\boldsymbol{\chi}[\xi](w,z)$, and $\boldsymbol{\varphi}[\zeta](w,z)$ for the corresponding entropy flux, symmetric to $\boldsymbol{\psi}[\xi](w,z)$.

2.2.2. Kinetic Formulation. We can now state the main theorem of this section, according to which if a function $U : \Omega \to \mathcal{U}$ is a vanishing viscosity solution, then it satisfies a suitable pair of kinetic-type equations. In the following, given a function $U : \Omega \to \mathcal{U}$, we define

$$\begin{aligned} \boldsymbol{\chi}_U(t,x,\xi) &\doteq \boldsymbol{\chi}[\xi](U(t,x)) \qquad \forall\, (t,x,\xi) \in \Omega \times (\underline{w}, \bar{w}), \\ \boldsymbol{\upsilon}_U(t,x,\zeta) &\doteq \boldsymbol{\upsilon}[\zeta](U(t,x)) \qquad \forall\, (t,x,\zeta) \in \Omega \times (\underline{z}, \bar{z}) \end{aligned} \tag{2.2.8}$$

and analogously for $\boldsymbol{\psi}_U$ and $\boldsymbol{\varphi}_U$.

THEOREM 2.2.4. *Let $U : \Omega \to \mathcal{U}$ be a vanishing viscosity solution to* (1.5.1)*, and assume that* (1.5.1) *admits a uniformly convex entropy. Then there are locally finite measure $\mu_0, \mu_1 \in \mathscr{M}(\Omega \times (\underline{w}, \bar{w}))$ and $\nu_0, \nu_1 \in \mathscr{M}(\Omega \times (\underline{z}, \bar{z}))$ such that*

$$\partial_t \boldsymbol{\chi}_U(t,x,\xi) + \partial_x \boldsymbol{\psi}_U(t,x,\xi) = \partial_\xi \mu_1 + \mu_0 \qquad \text{in } \mathscr{D}'\big(\Omega \times (\underline{w}, \bar{w})\big) \tag{2.2.9}$$

$$\partial_t \boldsymbol{\upsilon}_U(t,x,\zeta) + \partial_x \boldsymbol{\varphi}_U(t,x,\zeta) = \partial_\zeta \nu_1 + \nu_0 \qquad \text{in } \mathscr{D}'\big(\Omega \times (\underline{z}, \bar{z})\big) \tag{2.2.10}$$

Moreover μ_1 and ν_1 are positive measures, and for some constant $C > 0$, we have

$$(\mathtt{p}_{t,x})_\sharp|\mu_0| + (\mathtt{p}_{t,x})_\sharp|\nu_0| \leq C\,(\mathtt{p}_{t,x})_\sharp\mu_1 + (\mathtt{p}_{t,x})_\sharp\nu_1.$$

Here $\mathtt{p}_{t,x}$ denotes the canonical projection on the t, x variables. We recall that given a measurable map $f : X \to Y$ between measure spaces X, Y, for any $\mu \in X \times Y$ the pushforward measure $f_\sharp\mu \in \mathscr{M}(X)$ is defined by

$$f_\sharp\mu(A) = \mu(f^{-1}(A)) \qquad \forall \text{ measurable } A \subset X.$$

Proof. We prove the Theorem for (2.2.9), since (2.2.10) follows from an entirely symmetrical argument.

1. For every smooth $\xi \mapsto \varrho(\xi)$, we can consider a smooth entropy η_ϱ where the entropy $\boldsymbol{\chi}[\xi]$ appears with density $\varrho(\xi)$:

$$\eta_\varrho(U) \doteq \int_{\mathbb{R}} \boldsymbol{\chi}[\xi](U)\varrho(\xi)\,\mathrm{d}\xi, \qquad q_\varrho(U) \doteq \int_{\mathbb{R}} \boldsymbol{\psi}[\xi](U)\varrho(\xi)\,\mathrm{d}\xi. \tag{2.2.11}$$

Then η_ϱ, q_ϱ is a smooth entropy-entropy flux pair. In fact, clearly is a solution of (2.2.1), since each $\boldsymbol{\chi}[\xi], \boldsymbol{\psi}[\xi]$ is, and the equation is linear. The fact that it is smooth comes from the fact that ϱ is smooth since an explicit calculation yields that the gradient of η_ϱ is

$$\nabla\eta_\varrho(U) = \int_{\underline{w}}^{w(U)} \nabla\boldsymbol{\Theta}[\xi](U)\varrho(\xi)\,\mathrm{d}\xi + \varrho(w(U)) \cdot \boldsymbol{\Theta}[w(U)](U) \cdot \nabla w(U) \qquad \forall\, U \in \mathcal{U}$$

where we recall that $w : \mathcal{U} \to [\underline{w}, \bar{w}]$ is the map that sends U to its first Riemann invariant $w(U)$.

2. Now multiply from the left equation (1.5.4) by $\nabla\eta_\varrho(U^\epsilon)$ to obtain

$$\begin{aligned}\nabla\eta_\varrho(U^\epsilon)\big[U^\epsilon_t + f(U^\epsilon)_x\big] &= \epsilon\nabla\eta_\varrho(U)U^\epsilon_{xx}\\ &= \epsilon\int_{\underline{w}}^{w(U^\epsilon)} \nabla\boldsymbol{\Theta}[\xi](U)\varrho(\xi)\,\mathrm{d}\xi\, U^\epsilon_{xx} + \epsilon\varrho(w(U^\epsilon))\boldsymbol{\Theta}[w(U^\epsilon)](U^\epsilon)\nabla w(U^\epsilon)U^\epsilon_{xx}\end{aligned} \tag{2.2.12}$$

where from now on the symbol ∇ will be reserved to denote the gradient of a function in the U variable. We calculate the first term:

$$\begin{aligned}\epsilon\int_{\underline{w}}^{w(U^\epsilon)} \nabla\boldsymbol{\Theta}[\xi](U)\varrho(\xi)\,\mathrm{d}\xi\, U^\epsilon_{xx} &= \Big[\epsilon\int_{\underline{w}}^{w(U^\epsilon)} \nabla\boldsymbol{\Theta}[\xi](U^\epsilon)\varrho(\xi)\,\mathrm{d}\xi\, U^\epsilon_x\Big]_x\\ &\quad - \epsilon\Big[\int_{\underline{w}}^{w(U^\epsilon)} \nabla\boldsymbol{\Theta}[\xi](U^\epsilon)\varrho(\xi)\,\mathrm{d}\xi\Big]_x U^\epsilon_x.\end{aligned} \tag{2.2.13}$$

and the second term:

$$\begin{aligned}\epsilon\varrho(w(U^\epsilon))\boldsymbol{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)U^\epsilon_{xx} &= \Big[\epsilon\varrho(w(U^\epsilon))\boldsymbol{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)U^\epsilon_x\Big]_x\\ &\quad - \Big[\epsilon\varrho(w(U^\epsilon))\boldsymbol{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big]_x U^\epsilon_x.\end{aligned} \tag{2.2.14}$$

The second term in the right hand side of (2.2.14) can be calculated as

$$
\begin{aligned}
\Big[\epsilon\varrho(w(U^\epsilon))\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big]_x U^\epsilon_x &= \Big[\epsilon\varrho'(w(U^\epsilon))\partial_x w(U^\epsilon)\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big]\cdot U^\epsilon_x \\
&\quad + \epsilon\varrho(w(U^\epsilon))\langle \mathrm{D}\Big(\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big)U^\epsilon_x,\, U^\epsilon_x\rangle \\
&= \varrho'(w(U^\epsilon))\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\Big[\sqrt{\epsilon}\partial_x w(U^\epsilon)\Big]^2 \\
&\quad + \epsilon\varrho(w(U^\epsilon))\langle \mathrm{D}\Big(\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big)U^\epsilon_x,\, U^\epsilon_x\rangle
\end{aligned}
\tag{2.2.15}
$$

Therefore we have

$$
\begin{aligned}
\eta_\varrho(U^\epsilon)_t + q_\varrho(U^\epsilon)_x &= \nabla\eta_\varrho(U^\epsilon)\big[U^\epsilon_t + f(U^\epsilon)_x\big] \\
&= -\epsilon\Big[\int_{\underline{w}}^{w(U^\epsilon)}\nabla\mathbf{\Theta}[\xi](U^\epsilon)\varrho(\xi)\,\mathrm{d}\xi\Big]_x U^\epsilon_x \\
&\quad - \varrho'(w(U^\epsilon))\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\Big[\sqrt{\epsilon}\partial_x w(U^\epsilon)\Big]^2 \\
&\quad - \epsilon\varrho(w(U^\epsilon))\langle \mathrm{D}\Big(\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big)U^\epsilon_x,\, U^\epsilon_x\rangle \\
&\quad + g^\epsilon_\varrho
\end{aligned}
\tag{2.2.16}
$$

where

$$
g^\epsilon_\varrho \doteq \Big[\epsilon\int_{\underline{w}}^{w(U^\epsilon)}\nabla\mathbf{\Theta}[\xi](U^\epsilon)\varrho(\xi)\,\mathrm{d}\xi\, U^\epsilon_x\Big]_x + \Big[\epsilon\varrho(w(U^\epsilon))\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)U^\epsilon_x\Big]_x.
$$

We notice that g^ϵ_ϱ is going to zero in distributions as $\epsilon\to 0^+$. In fact, using (1.5.6), we deduce that for every compact $K\subset\Omega$

$$
\begin{aligned}
\Big\|\epsilon\int_{\underline{w}}^{w(U^\epsilon)}\nabla\mathbf{\Theta}[\xi](U^\epsilon)\varrho(\xi)\,\mathrm{d}\xi\, U^\epsilon_x\Big\|_{\mathbf{L}^1(K)} &= \mathcal{O}(1)\cdot\|\varrho\|_{\mathbf{C}^0}\cdot\sqrt{\epsilon}\,\big\|\sqrt{\epsilon}U^\epsilon_x\big\|_{\mathbf{L}^1(K)} \\
&= \mathcal{O}(1)\cdot\|\varrho\|_{\mathbf{C}^0}\cdot\sqrt{\epsilon}\,\big\|\sqrt{\epsilon}U^\epsilon_x\big\|_{\mathbf{L}^2(K)} \\
&= \mathcal{O}(1)\cdot\|\varrho\|_{\mathbf{C}^0}\cdot\sqrt{\epsilon}\cdot C_K^{\frac{1}{2}} \longrightarrow 0 \quad \text{as } \epsilon\to 0^+.
\end{aligned}
$$

where $\mathcal{O}(1)$ is a constant depending only on the compact set K. The same estimate shows that also the second term in g^ϵ is going to zero in distributions.

3. Define the distribution $T^\epsilon\in\mathscr{D}'(\Omega\times(\underline{w},\bar{w}))$

$$
\begin{aligned}
\langle T^\epsilon,\,\varphi\varrho\rangle &\doteq -\iiint_{\Omega\times\mathbb{R}}\big[\varphi_t(t,x)\boldsymbol{\chi}_{U^\epsilon}(t,x,\xi) + \varphi_x(t,x)\boldsymbol{\psi}_{U^\epsilon}(t,x,\xi)\big]\varrho(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint_\Omega \varphi\big(\partial_t\eta_\varrho(U^\epsilon) + \partial_x q_\varrho(U^\epsilon)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all smooth $\varphi(t,x)$, $\varrho(\xi)$, this is sufficient because by the Stone–Weierstrass theorem finite sums $\sum_{i=1}^N\varphi_i\varrho_i$ are dense in C^k for every $k\in\mathbb{N}$. Notice that since $U^\epsilon\to U$ in $\mathbf{L}^1_{loc}$, also $\boldsymbol{\chi}_{U^\epsilon},\boldsymbol{\psi}_{U^\epsilon}$ converge in $\mathbf{L}^1_{loc}$ to $\boldsymbol{\chi}_U,\boldsymbol{\psi}_U$ and T^ϵ converges to the left hand side of (2.2.9) in the sense of distributions.

Thanks to (2.2.16), we have that

$$T^\epsilon = \mu_0^\epsilon + \partial_\xi \mu_1^\epsilon + f^\epsilon$$

where f^ϵ is going to zero in distributions, μ_0^ϵ, μ_1^ϵ are locally uniformly bounded measures, and in particular:

(1) f^ϵ is defined by

$$\langle f^\epsilon, \varphi\varrho\rangle \doteq \langle g_\varrho^\epsilon, \varphi\rangle \qquad \forall \text{ smooth } \varphi(t,x), \varrho(\xi);$$

(2) μ_1^ϵ accounts for the third line in (2.2.16) and

$$\mu_1^\epsilon \doteq (\mathrm{id}, w^\epsilon)_\sharp \Big[\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\big(\sqrt{\epsilon}w(U^\epsilon)_x\big)^2 \cdot \mathscr{L}^2\Big] \in \mathscr{M}^+_{t,x,\xi}$$

where

$$(\mathrm{id}, w^\epsilon) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R} \times [\underline{w}, \bar{w}], \qquad (\mathrm{id}, w^\epsilon)(t,x) \doteq (t, x, w^\epsilon(t,x)).$$

In particular μ_1^ϵ is positive (because by definition $\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon) > 0$) and it satisfies the bound

$$|\mu_1^\epsilon|(K) \le \sup|\mathbf{\Theta}| \cdot \sup|\nabla w|^2 \cdot \big(\sqrt{\epsilon}U_x^\epsilon\big)^2 = \mathcal{O}(1)\cdot C_K \tag{2.2.17}$$

where $\mathcal{O}(1)$ is independent on ϵ, and the last equality follows from (1.5.6).

(3) μ_0^ϵ accounts for the second and the forth lines of (2.2.16) and is given by

$$\begin{aligned}\mu_0^\epsilon \doteq -\epsilon(\mathrm{id}, w^\epsilon)_\sharp\Big[&\langle \mathrm{D}\Big(\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\cdot\nabla w(U^\epsilon)\Big)U_x^\epsilon,\, U_x^\epsilon\rangle \\ &+ \langle \nabla\mathbf{\Theta}[w(U^\epsilon)](U^\epsilon)\otimes\nabla w(U^\epsilon)\cdot U_x^\epsilon,\, U_x^\epsilon\rangle\Big]\cdot\mathscr{L}^2 \\ &- \big\langle \nabla^2\mathbf{\Theta}[\xi](U^\epsilon)U_x^\epsilon,\, U_x^\epsilon\big\rangle \cdot \mathscr{L}^3 \llcorner \{\xi \ge w(U^\epsilon)\}.\end{aligned}$$

The same type of estimate leading to (2.2.17) shows also that

$$|\mu_0^\epsilon|(K) = \mathcal{O}(1)\cdot C_K$$

independently of ϵ. Therefore up to subsequences the measures μ_0^ϵ, μ_1^ϵ weakly converge to limiting measures μ_0 and $\mu_1 \ge 0$ that satisfy (2.2.9). □

REMARK 2.2.5. The measures μ_1, μ_0 (and ν_1, ν_0) are not uniquely determined by the left hand sides of (2.2.9), (2.2.10).

2.3. Dispersive Estimates

Using the kinetic formulation, in this section we obtain a decay estimate of dispersive character for all vanishing viscosity solutions to genuinely nonlinear systems. The only assumption is boundedness, so that the results of the previous section can be applied.

The dispersion mechanism is induced by Proposition 2.2.2, in particular by the fact that the speed $\boldsymbol{\lambda}_1[\xi](w,z)$ is monotone (in a uniform way) with respect to ξ, when ξ is close to w. We start by defining appropriate localizations of $\boldsymbol{\chi}$, that are needed to obtain a localized estimate in strips $\mathbb{R}^+ \times \mathbb{R} \times [\ell, \ell + \mathsf{r}]$ of width $\mathsf{r} > 0$ where Proposition 2.2.2 holds. Given $\ell \in [\underline{w}, \overline{w} - \mathsf{r}]$, we define

$$\boldsymbol{\chi}^{\mathsf{r},\ell}(t,x,\xi) \doteq \boldsymbol{\chi}[\xi](U(t,x)) \cdot \mathbf{1}_{\{\ell \le \xi \le \ell+\mathsf{r}\}}(\xi). \tag{2.3.1}$$

In the following Lemma, we observe that $\boldsymbol{\chi}^{\mathsf{r},\ell}$ satisfies a kinetic equation with a monotone speed. One has to add additional source terms to take into account the flux through the boundaries of the strip, namely $\mathbb{R}^+ \times \mathbb{R} \times \{\ell\}$ and $\mathbb{R}^+ \times \mathbb{R} \times \{\ell + \mathsf{r}\}$.

LEMMA 2.3.1. *In the notation introduced above, $\boldsymbol{\chi}^{\mathsf{r},\ell}$ satisfies, for some positive, locally finite measures $\mathsf{f}^{\mathrm{in}}, \mathsf{f}^{\mathrm{out}}$,*

$$\begin{aligned}\partial_t \boldsymbol{\chi}^{\mathsf{r},\ell}[\xi](U) + \partial_x\big(\boldsymbol{\lambda}_1[\xi](U)\cdot\boldsymbol{\chi}^{\mathsf{r},\ell}[\xi](U)\big) &= \partial_\xi[\mu_1 \cdot \mathbf{1}_{\{\ell<\xi<\ell+\mathsf{r}\}}] \\ &\quad + \mu_0 \cdot \mathbf{1}_{\{\ell<\xi<\ell+\mathsf{r}\}} + \mathsf{f}_{\mathrm{in}} - \mathsf{f}_{\mathrm{out}} \qquad in\ \mathscr{D}'_{t,x,\xi}.\end{aligned} \tag{2.3.2}$$

Moreover, if there is $\widehat{U} \in \mathcal{U}$ such that $U_0 - \widehat{U} \in \mathbf{L}^1(\mathbb{R})$, it holds

$$\iint_{\mathbb{R}^+\times\mathbb{R}} \mathrm{d}\,\mathsf{f}^{\mathrm{in}}(t,x) \le \mathscr{O}(1)\cdot\|U_0 - \widehat{U}\|_{\mathbf{L}^1}, \qquad \iint_{\mathbb{R}^+\times\mathbb{R}} \mathrm{d}\,\mathsf{f}^{\mathrm{out}}(t,x) \le \mathscr{O}(1)\cdot\|U_0 - \widehat{U}\|_{\mathbf{L}^1} \tag{2.3.3}$$

PROOF. For $0 < \epsilon < \delta$, define an ϵ^{-1}-Lipschitz function h_ϵ by

$$h_\epsilon(\xi) \doteq \begin{cases} 0 & \text{if } \xi \le \ell \\ (\xi-\ell)\epsilon^{-1} & \text{if } \ell \le \xi \le \ell+\epsilon \\ 1 & \text{if } \ell+\epsilon \le \xi \le \ell-\epsilon \\ (\ell+\mathsf{r}-\xi)\epsilon^{-1} & \text{if } \ell+\mathsf{r}-\epsilon \le \xi \le \ell+\mathsf{r} \\ 0 & \text{if } \xi \ge \ell+\mathsf{r}. \end{cases}.$$

For any test function $\varphi(t,x,\xi)$ we compute

$$\begin{aligned}\iiint \left[\varphi_t \boldsymbol{\chi}^{\mathsf{r},\ell} + \varphi_x\left(\boldsymbol{\lambda}_1 \boldsymbol{\chi}^{\mathsf{r},\ell}\right)\right] \mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\epsilon\to 0^+} \iiint \left[\varphi_t h_\epsilon \boldsymbol{\chi}^{\mathsf{r},\ell} + \varphi_x h_\epsilon\left(\boldsymbol{\lambda}_1 \boldsymbol{\chi}^{\mathsf{r},\ell}\right)\right] \mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\epsilon\to 0^+} \iiint -(\varphi h'_\epsilon + \partial_\xi \varphi h_\epsilon)\,\mathrm{d}\mu_1 + \iiint \varphi h_\epsilon\,\mathrm{d}\mu_0.\end{aligned} \tag{2.3.4}$$

We now compute

$$-\iiint \varphi h'_\epsilon\,\mathrm{d}\mu_1 = -\frac{1}{\epsilon}\iiint \varphi \mathbf{1}_{\{\ell\le\xi\le\ell+\epsilon\}}(\xi)\,\mathrm{d}\mu_1 + \frac{1}{\epsilon}\iiint \varphi\mathbf{1}_{\{\ell+\mathsf{r}-\epsilon\le\xi\le\ell+\mathsf{r}\}}(\xi)\,\mathrm{d}\mu_1$$

We now compute the limits in the right hand side. We start with the first term. Define a test function

$$g_\epsilon(\xi) = \begin{cases} 0 & \text{if } \xi \le \ell \\ (\xi-\ell)\epsilon^{-1} & \text{if } \ell \le \xi \le \ell+\epsilon \\ 1 & \text{if } \ell+\epsilon \le \xi. \end{cases}$$

Now, for every $\varphi \in \mathcal{C}^1_c(\mathbb{R}^+_t \times \mathbb{R}_x)$ and $\rho \in \mathcal{C}^1(\mathbb{R}_\xi)$, we test $\varphi\rho g_\epsilon$ against (2.2.9), so that we obtain (omitting the dependence on (t,x)):

$$\iiint (\varphi\rho(\xi))g'_\epsilon\,\mathrm{d}\mu_1 = \iiint (\varphi\rho(\xi))g_\epsilon\,\mathrm{d}\mu_0 + \iiint \big(\varphi_t \boldsymbol{\chi}[\xi](U) + \varphi_x \boldsymbol{\psi}[\xi](U)\big)\rho(\xi)g_\epsilon(\xi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi$$

Now taking the limit we obtain

$$\lim_{\epsilon\to 0^+}\iiint(\varphi\rho)\,\mathrm{d}\Big(\frac{1}{\epsilon}\mu_1\llcorner(\ell,\ell+\epsilon)\Big)=\iiint\varphi\rho\mathbf{1}_{\xi>\ell}(\xi)\,\mathrm{d}\mu_0 \\ +\iiint\big(\varphi_t\boldsymbol{\chi}[\xi](U)+\varphi_x\boldsymbol{\psi}[\xi](U)\big)\rho(\xi)\mathbf{1}_{\xi>\ell}(\xi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi. \tag{2.3.5}$$

Therefore we deduce that the limit

$$\mathsf{f}_{\mathrm{out}}\doteq\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\mu_1\llcorner(\ell,\ell+\epsilon)$$

exists in the sense of distributions; moreover, since μ_1 it is a positive measure, the limit must be a positive distribution and therefore is a locally finite measure $\mathsf{f}_{\mathrm{out}}\in\mathscr{M}(\mathbb{R}^3)$.

Next, we estimate the norm of $\mathsf{f}_{\mathsf{out}}$ when $U_0-\widehat{U}\in\mathbf{L}^1$. We assume, without loss of generality, that $\ell\geq\widehat{w}=w(\widehat{U})$; the other case will be symmetric. We choose (appropriate regularizations of)

$$\varphi(t,x)=\mathbf{1}_{(0,T)\times\mathbb{R}}(t,x)\qquad\forall\,t,x\in\mathbb{R}^+\times\mathbb{R},\qquad\rho(\xi)=1\qquad\forall\,\xi\in\mathbb{R}$$

and we compute, using (2.3.5)

$$\iiint\mathrm{d}\mathsf{f}_{\mathsf{out}}(x,t,\xi)=\lim_{T\to\infty}|\mu_0|((0,T)\times\mathbb{R}^2))-\iint_{\xi\geq\ell}(\boldsymbol{\chi}[\xi](U(T,x))\,\mathrm{d}x\,\mathrm{d}\xi \\ +\iint_{\xi\geq\ell}(\boldsymbol{\chi}[\xi](U(0,x))\,\mathrm{d}x\,\mathrm{d}\xi\leq\mathscr{O}(1)\cdot\|U-\widehat{U}\|_{\mathbf{L}^1}$$

where, to estimate the size of $\mu_0(\mathbb{R}^3)$, we used the bounds in Theorem 2.2.4 in the case $U-\widehat{U}\in\mathbf{L}^2$.

An entirely similar argument proves that

$$\mathsf{f}_{\mathrm{in}}\doteq\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\mathbf{1}_{\{\ell+\mathsf{r}\leq\xi\leq\ell\mathsf{r}+\epsilon\}}(\xi)\cdot\mu_1.$$

and

$$\|\mathsf{f}_{\mathrm{in}}\|_{\mathscr{M}}\leq\mathscr{O}(1)\cdot\|U_0-\widehat{U}\|_{\mathbf{L}^1}.$$

□

In the following Proposition we prove a preliminary decay estimate for the "localized solution" between ℓ and $\ell+\mathsf{r}$:

$$g^{\mathsf{r},\ell}(t,x)\doteq\begin{cases}w(t,x) & \text{if } \ell\leq w(t,x)-\ell\leq\ell+\mathsf{r},\\ 0 & \text{if } w(t,x)\leq\ell,\\ \mathsf{r} & \text{if } w(t,x)>\ell.\end{cases} \tag{2.3.6}$$

The underlying idea is rather easy to visualize, therefore we take a few lines to explain the main point. We introduce a functional $\mathscr{Q}$ whose decrease at time t is at least the $\mathbf{L}^4$ norm of the Riemann invariant in the strip $S\doteq\mathbb{R}\times[\ell,\ell+\mathsf{r}]$. The functional takes the form

$$\mathscr{Q}(t)\doteq\iiiint(\xi-\xi')\,\boldsymbol{\chi}^{\mathsf{r},\ell}(t,x,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,x',\xi')\,\mathrm{d}x'\,\mathrm{d}\xi'\,\mathrm{d}x\,\mathrm{d}\xi.$$

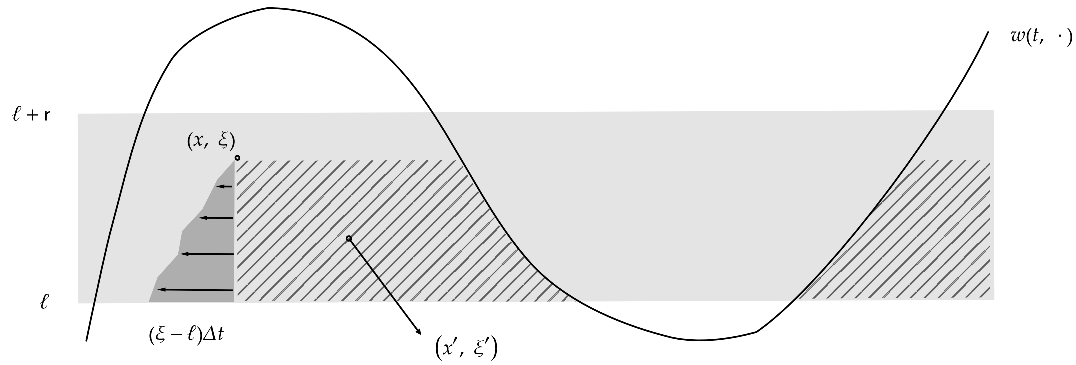

FIGURE 2. For any $(x,\xi) \in \operatorname{hyp} w$, the decrease of $\mathscr{Q}$ is due to the fact that in a small interval of time Δt, that part of $\boldsymbol{\chi}$ contained in a triangle (the dark-shaded area) of area of order $(\xi-\ell)^2\Delta t$ stops to approach the point (x,ξ).

We thinks of two pairs (x,ξ), $(x',\xi') \in S$ as *approaching* if $x < x'$ and $\xi > \xi'$. The functional $\mathscr{Q}$ measures the set of approaching pairs, weighted with the vertical distance $(\xi - \xi')$. Since the speed $\boldsymbol{\lambda}_1[\xi](U)$ is monotone in ξ, and in a first approximation U is locally constant, in a small interval of time Δt, at least a triangle of area $\sim \Delta t \cdot (\xi-\ell)^2$ of points (x',ξ') stop to be approaching points with the point (x,ξ) (see Figure 2). Then, if all the graph of the Riemann invariant w is contained in the strip S, and if $\mu_0, \mu_1 = 0$, we would get

$$\dot{\mathscr{Q}}(t) \leq -C \int_{\mathbb{R}} (w(t,x)-\ell)^4 \,\mathrm{d}x.$$

If the functional $\mathscr{Q}$ is initially bounded, this yields a bound on the time integral of the right hand side above. Of course, errors are present due to the fact that μ_0, μ_1 are not zero, and due to the fact that if $w(t,x) > \ell + \mathsf{r}$, the speed is not monotone anymore. Then, we obtain the following result.

PROPOSITION 2.3.2. *Let $U : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{U}$ be a bounded vanishing viscosity solution to* (1.5.1) *with an initial datum U_0 such that $U_0 - \widehat{U} \in \mathbf{L}^1$. Assume that the eigenvalue λ_1 is genuinely nonlinear. Then, recalling the definition of $g^{\mathsf{r},\ell}$ in* (2.3.6),

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} \big(g^{\mathsf{r},\ell}(t,x)\big)^4 \,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot \Big(\|U_0-\widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0-\widehat{U}\|_{\mathbf{L}^1}\big)^2\Big)& \\ + \mathscr{O}(1)\cdot \mathscr{L}^2\Big(\big\{(t,x)\in\mathbb{R}^+\times\mathbb{R} \mid w(t,x) \geq \ell+\mathsf{r}\big\}\Big).&\end{aligned} \tag{2.3.7}$$

PROOF. First, for a given positive interaction kernel $0 \leq \phi \in \mathbf{C}^1_c(\mathbb{R}^4)$, we define smoothed interactions $\mathscr{Q}_\phi$ as

$$\mathscr{Q}_\phi(t) \doteq \iiiint \phi(x,\xi,x',\xi')\,\boldsymbol{\chi}^{\mathsf{r},\ell}(t,x,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,x',\xi')\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}x'\,\mathrm{d}\xi'.$$

Next, we convolve (2.3.2) with a standard, rescaled at scale δ, smoothing kernel $\varrho_\delta(t,x,\xi)$, to obtain

$$\partial_t\boldsymbol{\chi}^{\mathsf{r},\ell}_\delta + \partial_x\boldsymbol{\psi}^{\mathsf{r},\ell}_\delta = \partial_\xi\mu_1^\delta + \mu_0^\delta + \mathsf{f}^{\mathrm{in}}_\delta - \mathsf{f}^{\mathrm{out}}_\delta \qquad \forall t>0, \quad x,\xi\in\mathbb{R}$$

where we set

$$\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x,\xi) \doteq \boldsymbol{\chi}^{\mathsf{r},\ell} \star \varrho_\delta(t,x,\xi), \quad \boldsymbol{\psi}_\delta^{\mathsf{r},\ell}(t,x,\xi) \doteq \boldsymbol{\psi}^{\mathsf{r},\ell} \star \varrho_\delta(t,x,\xi)$$

$$\mu_i^\delta \doteq \mu_i \star \varrho_\delta(t,x,\xi) \quad \text{for } i = 0, 1, s.$$

Then we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Q}_\phi^\delta(t) &\doteq \frac{\mathrm{d}}{\mathrm{d}t} \iiiint \phi(x,\xi,x',\xi')\, \boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x,\xi)\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x',\xi')\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}x'\,\mathrm{d}\xi' \\
&= \iiiint \phi_x(x,\xi,x',\xi')\boldsymbol{\psi}_\delta^{\mathsf{r},\ell}(t,x,\xi)\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x',\xi') \\
&+ \phi(x,\xi,x',\xi')\left(\partial_\xi \mu_1^\delta(t,x,\xi) + \mu_0^\delta(t,x,\xi) + (\mathsf{f}_\delta^{\mathrm{in}} - \mathsf{f}_\delta^{\mathrm{out}})(t,x,\xi)\right)\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x',\xi')\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}x'\,\mathrm{d}\xi' \\
&\quad \iiiint \phi_{x'}(x,\xi,x',\xi')\boldsymbol{\psi}_\delta^{\mathsf{r},\ell}(t,x',\xi')\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x,\xi) \\
&+ \phi(x,\xi,x',\xi')\left(\partial_{\xi'} \mu_1^\delta(t,x',\xi') + \mu_0^\delta(t,x',\xi') + (\mathsf{f}_\delta^{\mathrm{in}} - \mathsf{f}_\delta^{\mathrm{out}})(t,x',\xi')\right)\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}x'\,\mathrm{d}\xi'
\end{aligned}$$

Estimating the parts related to dissipation and source terms, and using the sign of ϕ and of $\mathsf{f}_\delta^{\mathrm{out}}$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Q}_\phi^\delta(t) \leq \iiiint & \phi_x(x,\xi,x',\xi')\boldsymbol{\psi}_\delta^{\mathsf{r},\ell}(t,x,\xi)\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x',\xi') \\
&+ \phi_{x'}(x,\xi,x',\xi')\boldsymbol{\psi}_\delta^{\mathsf{r},\ell}(t,x',\xi')\boldsymbol{\chi}_\delta^{\mathsf{r},\ell}(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}x'\,\mathrm{d}\xi' \\
&+ \mathcal{O}(1)\cdot\sup\left(|\nabla_{\xi,\xi'}\phi| + |\phi|\right)\cdot\left(\|\mu_1^\delta(t)\|_{\mathscr{M}} + \|\mu_0^\delta(t)\|_{\mathscr{M}} + \|U_0 - \overline{U}\|_{\mathbf{L}^1}\right)
\end{aligned} \tag{2.3.8}$$

Now we choose a sequence of interactions ϕ^ϵ of the following form: set

$$h_\epsilon(r) \doteq \begin{cases} 1 & \text{if } r < 0 \\ 1 - r/\epsilon & \text{if } 0 \leq r \leq \epsilon \\ 0 & \text{if } r > \epsilon \end{cases}$$

and define

$$\phi^\epsilon(x,\xi,x',\xi') \doteq h_\epsilon(x - x')\cdot(\xi - \xi')^+ \qquad \forall (x,\xi,x',\xi') \in \mathbb{R}^4.$$

Next, using ϕ^ϵ in (2.3.8) and passing to the limit as $\epsilon \to 0^+$, and then for $\delta \to 0^+$, we obtain

$$\begin{aligned}
\mathscr{Q}(s) - \mathscr{Q}(0) \leq \int_0^s \iiint & \mathbf{1}_{\{\xi' \leq \xi\}}(\xi - \xi')\big[\boldsymbol{\psi}^{\mathsf{r},\ell}(t,y,\xi')\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi) \\
&- \boldsymbol{\psi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi')\big]\,\mathrm{d}\xi\,\mathrm{d}\xi'\,\mathrm{d}y\,\mathrm{d}t + \mathscr{O}(1)\cdot\|U_0 - \widehat{U}\|_{\mathbf{L}^1}.
\end{aligned} \tag{2.3.9}$$

and using Proposition 2.2.2, we further estimate the right hand side above and obtain

$$\begin{aligned}
\mathscr{Q}(s) - \mathscr{Q}(0) \leq \int_0^s \iiint & \mathbf{1}_{\{\xi' \leq \xi\}}(\xi - \xi')\big[\boldsymbol{\lambda}_1[\xi'](U(t,y)) \\
&- \boldsymbol{\lambda}_1[\xi](U(t,y))\big]\cdot\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi')\cdot\mathbf{1}_{\{w(t,x)\leq \ell+\delta\}}\,\mathrm{d}\xi\,\mathrm{d}\xi'\,\mathrm{d}y\,\mathrm{d}t \\
&+ \mathscr{O}(1)\cdot\mathscr{L}^2\Big(\big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \mid w(t,x) \geq \ell + \delta\big\}\Big) \\
&+ \mathscr{O}(1)\cdot\|U_0 - \widehat{U}\|_{\mathbf{L}^1}.
\end{aligned}$$

where we used the fact that if $\ell \leq w(t,x) \leq \ell + \delta$, then

$$\boldsymbol{\psi}^{\mathsf{r},\ell}(t,y,\xi')\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi) - \boldsymbol{\psi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi') = \big[\boldsymbol{\lambda}_1[\xi'](U(t,y)) - \boldsymbol{\lambda}_1[\xi](U(t,y))\big]\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi').$$

Now Proposition 2.2.2 yields

$$\big[\boldsymbol{\lambda}_1[\xi'](U(t,y)) - \boldsymbol{\lambda}_1[\xi](U(t,y))\big]\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi') \leq -\mathsf{c}(\xi-\xi')\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi)\boldsymbol{\chi}^{\mathsf{r},\ell}(t,y,\xi').$$

therefore, for some positive constant $C > 0$, we obtain

$$\begin{aligned}\mathscr{Q}(s) - \mathscr{Q}(0) \leq -C \int_0^s \int_{\mathbb{R}} (g^{\mathsf{r},\ell}(t,x))^4 \,\mathrm{d}y\,\mathrm{d}t \\ + \mathscr{O}(1)\cdot \mathscr{L}^2\Big(\big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \mid w(t,x) \geq \ell + \delta\big\}\Big) + \mathscr{O}(1)\cdot \|U_0 - \widehat{U}\|_{\mathbf{L}^1}.\end{aligned} \tag{2.3.10}$$

Rearranging, and letting $s \to +\infty$, proves the result. □

We now prove the main theorem of this section.

THEOREM 2.3.3. *Let $U : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{U}$ be a bounded vanishing viscosity solution to* (1.5.1) *with an initial datum U_0 such that $U_0 - \widehat{U} \in \mathbf{L}^1$ for some constant $\widehat{U} \in \mathcal{U}$, with $\widehat{w} = w(\widehat{U})$. Assume that the eigenvalue λ_1 is genuinely nonlinear. Then, it holds*

$$\int_0^{+\infty} \int_{\mathbb{R}} \big(w(t,x) - \widehat{w}\big)^4 \,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot \Big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1}\big)^2\Big). \tag{2.3.11}$$

PROOF. We first prove that

$$\int_0^{+\infty} \int_{\mathbb{R}} \big([w(t,x) - \widehat{w}]^+\big)^4 \,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot \Big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1}\big)^2\Big). \tag{2.3.12}$$

The estimate for the negative part is entirely symmetric and is accordingly omitted. Define $k \in \mathbb{N}, \mathsf{r} > 0$ by

$$k \doteq \left\lceil \frac{\overline{w} - \widehat{w}}{\bar{\mathsf{r}}} \right\rceil, \qquad \mathsf{r} \doteq \frac{\overline{w} - \widehat{w}}{k}.$$

Then $\mathsf{r}\cdot k = \overline{w} - \widehat{w}$ and $0 < \mathsf{r} \leq \bar{\mathsf{r}}$. Define points $\ell_0, \ell_1, \ldots, \ell_{2(k-1)}$ by

$$\ell_0 \doteq \widehat{w}, \qquad \ell_i \doteq \widehat{w} + i\cdot\frac{\mathsf{r}}{2}, \qquad i = 1, \ldots, 2(k-1).$$

Proposition 2.3.2 yields, by setting $\ell = \ell_i$,

$$\begin{aligned}\int_0^{+\infty} \int_{\mathbb{R}} \big(g^{\mathsf{r},\ell_i}(t,x)\big)^4 \,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot \Big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0 - \widehat{U}\|_{\mathbf{L}^1}\big)^2\Big) \\ + \mathscr{O}(1)\cdot \mathscr{L}^2\Big(\big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \mid w(t,x) \geq \ell_i + \mathsf{r}\big\}\Big).\end{aligned} \tag{2.3.13}$$

Now, for every $i = 0, \ldots, 2(k-1) - 1$, using Markov's inequality, we estimate

$$\mathscr{L}^2\Big(\big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \mid w(t,x) \geq \ell_i + \mathsf{r}\big\}\Big) \leq \mathscr{O}(1)\cdot\frac{1}{\mathsf{r}^4}\cdot \int_0^{+\infty} \int_{\mathbb{R}} \big(g^{\mathsf{r},\ell_{i+1}}(t,x)\big)^4 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.3.14}$$

Then, starting from $i = 0$, and applying the estimates above repeatedly, we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(g^{\mathsf{r},\ell_0}(t,x)\right)^4 \mathrm{d}x\,\mathrm{d}t \leq 2(k-1)\cdot \mathscr{O}(1)\cdot \Big(\|U_0-\widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0-\widehat{U}\|_{\mathbf{L}^1}\big)^2\Big) + \mathscr{O}(1)\cdot \mathscr{L}^2\Big(\big\{(t,x)\in\mathbb{R}^+\times\mathbb{R} \mid w(t,x)\geq \ell_{2(k-1)}+\mathsf{r}\big\}\Big). \tag{2.3.15}$$

But now notice that $\ell_{2(k-1)} + \mathsf{r} = \overline{w}$, and by assumption $w(t,x) \leq \overline{w}$ for a.e. (t,x). Moreover, we clearly have

$$\int_0^{+\infty}\int_{\mathbb{R}} \left([w(t,x)-\widehat{w}]^+\right)^4 \mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot\frac{1}{\mathsf{r}^4}\int_0^{+\infty}\int_{\mathbb{R}} \left(g^{\mathsf{r},\ell_0}(t,x)\right)^4 \mathrm{d}x\,\mathrm{d}t.$$

Therefore we conclude that

$$\int_0^{+\infty}\int_{\mathbb{R}} \left([w(t,x)-\widehat{w}]^+\right)^4 \mathrm{d}x\,\mathrm{d}t \leq \mathscr{O}(1)\cdot \Big(\|U_0-\widehat{U}\|_{\mathbf{L}^1} + \big(\|U_0-\widehat{U}\|_{\mathbf{L}^1}\big)^2\Big)$$

and this proves the result. □

CHAPTER 3

A Differential Structure for Scalar Conservation Laws

3.1. Introduction

We consider the following problem: let $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R})$, $u \geq 0$, be the entropy solution to the Cauchy problem

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0, \qquad \text{in } \mathscr{D}'_{t,x} \\ &u(0,\cdot) = u_0 \in \mathbf{L}^\infty(\mathbb{R}). \end{aligned} \tag{3.1.1}$$

The flux is any $\mathscr{C}^1$ function $f : \mathbb{R} \to \mathbb{R}$ such that f does not have nontrivial linear interval in which it is affine. Consider a family $\{v_0^h\}_{h>0} \subset \mathbf{L}^1(\mathbb{R})$ such that for some $\rho_0 \in \mathscr{M}(\mathbb{R})$

$$hv_0^h \in \mathbf{L}^\infty(\mathbb{R}), \qquad v_0^h \rightharpoonup \rho_0 \in \mathscr{M}(\mathbb{R}) \qquad \text{weakly in } \mathscr{M}(\mathbb{R}). \tag{3.1.2}$$

Here $\mathscr{M}(\mathbb{R})$ is the space of finite Radon measures on $\mathbb{R}$ and the weak convergence is in duality with respect to the space of bounded, continuous functions (see §3.2). By the Theorem of Kruzhkov [**76**] , if u^h is the entropy solution to the problem with initial datum

$$u_0^h \doteq u_0 + hv_0^h \tag{3.1.3}$$

then

$$\sup_{h>0} h^{-1}\|u^h(t,\cdot) - u(t,\cdot)\|_{\mathbf{L}^1} \leq \sup_{h>0} \|v_0^h\|_{\mathbf{L}^1}. \tag{3.1.4}$$

Then, since $\|v_0^h\|_{\mathbf{L}^1}$ remains bounded for $h \to 0^+$, $\{h^{-1}(u^h(t)-u(t))\}_h$ is relatively compact in the space of measures endowed with the weak topology, and formally, for smooth solutions, any limit measure ρ satisfies the continuity equation

$$\partial_t \rho + \partial_x (f'(u)\rho) = 0.$$

Recall that a *shock* connecting u^- and u^+ with speed λ is a function of the form

$$\overline{u}(t,x) = \overline{u}_0(x - \lambda t), \qquad \overline{u}_0(x) \doteq \begin{cases} u^-, & \text{if } x \leq 0, \\ u^+, & \text{if } x > 0 \end{cases} \tag{3.1.5}$$

The function $\overline{u}$ is a distributional solution to (3.1.1) if and only if the Rankine-Hugoniot condition holds:

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

We thus expect then when shocks are present, one should replace f' by the speed of the shock in the continuity equation. In particular, for every (t,x) we define the *characteristic speed*

$$\lambda(t,x) = \begin{cases} f'(u), & \text{if } u(t,x-) = u(t,x+), \\ \frac{f(u^+)-f(u^-)}{u^+-u^-}, & \text{if } u \text{ has a jump } u(t,x-) = u^- \neq u^+ = u(t,x+). \end{cases} \tag{3.1.6}$$

As explained in §3.3, a consequence of the results in [**37**] is that the one sided limits $u(t,x\pm)$ can be defined everywhere, so that also the speed $\lambda(t,x)$ can be defined at every point (t,x) (then in particular ρ-almost everywhere), and the product $\lambda\rho \in \mathscr{M}\big(\mathbb{R}^+ \times \mathbb{R}\big)$ is well defined. Given $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, u the entropy solution of the Cauchy problem (3.1.1) and $\rho_0 \in \mathscr{M}(\mathbb{R})$ we define

$$\mathcal{D}(u_0,\rho_0) \doteq \Big\{\rho \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}) \;\Big|\; \exists\, \{v_0^h\}_h \text{ as in (3.1.2) such that } \frac{u^h - u}{h} \rightharpoonup \rho, \\ u^h \text{ entropy solution with } u^h(0,\cdot) = u_0 + h v_0^h\Big\}.$$

THEOREM 3.1.1. *Let u be the entropy solution of* (3.1.1) *with $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Then any $\rho \in \mathcal{D}(u_0,\rho_0)$ solves the Cauchy problem for the continuity equation*

$$\begin{aligned} &\partial_t \rho + \partial_x(\lambda\rho) = 0, \qquad \text{in } \mathscr{D}'_{t,x}, \\ &\rho(0,\cdot) \;=\; \rho_0. \end{aligned} \tag{3.1.7}$$

Theorem 3.1.1 is essentially a consequence of the Kuratowski convergence of the graphs of u^h to the graph of u, and is proved in §3.4.

REMARK 3.1.2. Notice that the convergence $h^{-1}(u^h - u) \rightharpoonup \rho$ holds in $\mathscr{M}(\mathbb{R}^+ \times \mathbb{R})$. Nevertheless, since λ is bounded, the map $t \mapsto \rho_t \doteq \rho(t,\cdot)$ is continuous with respect to the weak topology and there holds

$$\frac{u^h(t,\cdot) - u(t,\cdot)}{h} \rightharpoonup \rho_t \qquad \text{weakly in } \mathscr{M}(\mathbb{R}).$$

REMARK 3.1.3. In an ideal situation the set $\mathcal{D}(u_0,\rho_0)$ would be a singleton, which means, once we fix ρ_0, the perturbation $\rho(t,\cdot)$ is uniquely determined also at later times. It turns out, that this is a too strong of a property to hold, already in the easy case of a convex flux, as shown in the following example. Consider Burgers equation

$$\partial_t\, u \;+\; \partial_x \left(\frac{u^2}{2}\right) \;=\; 0 \tag{3.1.8}$$

with the initial datum

$$u_0(x) \;\doteq\; \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0 \end{cases}$$

and notice that the solution $u(t,x)$ is constant -1 and 1 at the left and at the right of the region $-t \le x \le t$, where u has a centered rarefaction wave $u(t,x) = x/t$. Consider two different sequences of perturbations $v_{\pm,0}^h$ as in (3.1.3) defined by

$$v_{-,0}^h(x) = \frac{1}{\sqrt{h}}\,\mathbf{1}_{(-\sqrt{h},0)}, \qquad\qquad v_{+,0}^h(x) = \frac{1}{\sqrt{h}}\,\mathbf{1}_{(0,\sqrt{h})}.$$

We notice that it holds $v^h_{\pm,0} \to \delta_0$. On the other hand, the corresponding solutions $u^h_\pm$ can be calculated explicitly: for example, one has

$$u^h_-(t,x) = \begin{cases} -1, & \text{if } x \le -\sqrt{h} - t, \\ \frac{x+\sqrt{h}}{t}, & \text{if } -\sqrt{h} - t \le x \le -\sqrt{h} - (1-\sqrt{h})t, \\ -1+\sqrt{h}, & \text{if } -\sqrt{h} - (1-\sqrt{h})t \le x \le -(1-\sqrt{h})t \le x, \\ \frac{x}{t}, & \text{if } -(1-\sqrt{h})t \le x \le t, \\ 1, & \text{if } x \ge t. \end{cases}$$

In particular this implies that

$$\frac{u^h_-(t,\cdot) - u(t,\cdot)}{h} \rightharpoonup \delta_{-t} \qquad \text{weakly in } \mathscr{M}(\mathbb{R}) \text{ for all } t > 0.$$

An analogous computation shows that

$$\frac{u^h_+(t,\cdot) - u(t,\cdot)}{h} \rightharpoonup \delta_{t} \qquad \text{weakly in } \mathscr{M}(\mathbb{R}) \text{ for all } t > 0.$$

This is related to the uniqueness/non-uniqueness of solutions to the continuity equation (3.1.7). It is clear that, for Burgers equation, the only source of non-uniqueness are rarefaction waves starting at $t = 0$. For more general (non-convex) fluxes, the issue is deeper: one can have two different elements in $\rho, \widetilde{\rho} \in \mathcal{D}(u_0, \rho_0)$ and some $\bar{t} > 0$ such that $\rho(t,\cdot) = \widetilde{\rho}(t,\cdot)$ if and only if $t \in (0, \bar{t})$.

Lifting and Disintegration of the Density. The evolution of ρ describes the evolution of a perturbation at the macroscopic level of the conservation law (3.1.1). We aim at a further description of the structure of this perturbation. It is well known that entropy solutions to (3.1.1) satisfy a kinetic equation (see [**80**])

$$\partial_t \chi + f'(v)\partial_x \chi = \partial_v \mu \qquad \text{in } \mathscr{D}'_{t,x,v} \tag{3.1.9}$$

where $\mu \in \mathscr{M}^+(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$

$$\chi(t,x,v) = \mathbf{1}_{\text{hyp}\, u}(t,x,v), \qquad\qquad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}.$$

Here hyp g denotes the hypograph of a function $g \ge 0$:

$$\text{hyp } g \doteq \Big\{ (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^2 \,\Big|\, 0 \le v \le g(t,x) \Big\}.$$

Let also

$$\nu_h \doteq \frac{1}{h}\big(\chi^h - \chi\big) \cdot \mathscr{L}^3, \qquad \chi^h(t,x,v) = \mathbf{1}_{\text{hyp}\, u^h}(t,x,v). \tag{3.1.10}$$

One can recover u from χ (or u^h from χ^h) via integration in the variable "v"

$$\int_{\mathbb{R}} \chi(t,x,v)\,\mathrm{d}v = u(t,x) \qquad \text{for a.e. } t, x.$$

The ν_h are positive measures with mass

$$\int_{(0,T)\times\mathbb{R}\times\mathbb{R}} \mathrm{d}|\nu_h|(t,x,v) = \frac{1}{h}\int_{(0,T)\times\mathbb{R}} |u^h(t,x) - u(t,x)|\,\mathrm{d}x\,\mathrm{d}t \le \sup_h \|v^h_0\|_{\mathbf{L}^1}.$$

As a consequence, since the family $\{v_0^h\}_{h>0}$ is bounded in $\mathbf{L}^1(\mathbb{R})$, the sequence ν_h is relatively compact in $\mathscr{M}(\mathbb{R}^+ \times \mathbb{R}^2)$. Define the set of lifts

$$\widetilde{\mathcal{D}}(u_0, \rho_0) = \Big\{ \nu \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}^2) \;\Big|\; \exists \{v_0^h\}_h \text{ as in (3.1.2) such that } \nu_h \rightharpoonup \nu \\ u^h \text{ entropy solution with } u^h(0,\cdot) = u_0 + h v_0^h \Big\}.$$

We will denote by $\widehat{\rho}$ the positive measure

$$\widehat{\rho} \doteq \lim_{h\to 0} h^{-1}|u^h - u|.$$

The following definition selects pairs $(\widehat{\rho}, \nu)$ that come from the same sequence $\{v_0^h\}$.

Definition 3.1.4 (Compatible Pairs $(\widehat{\rho}, \nu)$). We say that $(\widehat{\rho}, \nu)$ is a *compatible pair* if there exists a sequence $\{v_0^h\}_{h>0}$ such that, if u^h is the entropy solution with $u^h(0,\cdot) = u_0 + h v_0^h$, it holds

$$\widehat{\rho}_h \doteq \frac{|u^h - u|}{h} \cdot \mathscr{L}^2 \rightharpoonup \widehat{\rho}, \qquad \nu^h = \frac{1}{h}\big(\chi^h - \chi\big) \cdot \mathscr{L}^3 \rightharpoonup \nu \qquad \text{in } \mathscr{M}.$$

Since projections commute with weak-limits, for a compatible pair one has $\widehat{\rho} = p_\sharp|\nu|$, $\widehat{\rho}_h = p_\sharp|\nu_h|$ where p is the canonical projection on the space-time variables. Recall that $p_\sharp|\nu|$ is the push-forward of the measure $|\nu|$, defined by $p_\sharp|\nu|(A) = |\nu|(p^{-1}(A))$ for all measurable $A \subset \mathbb{R}^+ \times \mathbb{R}$. In particular, the following diagram holds

$$\begin{array}{ccc} |\nu_h| & \xrightarrow{h\to 0^+} & |\nu| \\ {\scriptstyle p_\sharp}\big\downarrow & & \big\downarrow{\scriptstyle p_\sharp} \\ \widehat{\rho}_h & \xrightarrow{h\to 0} & \widehat{\rho}. \end{array}$$

From now on we then fix a compatible pair $(\widehat{\rho}, \nu)$. If $y = (t,x)$, we denote by $u^\pm(y)$ the left-right limits $u(t, x\pm)$. Since for a compatible pair we have $p_\sharp \nu = \rho$, we can consider the disintegration $\{a_y\}_y$ of ν w.r.t the projection p, as recalled in Theorem 3.2.2

$$\nu(y,v) = a_y(v) \otimes \widehat{\rho}(y) \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}_x \times \mathbb{R}_v). \tag{3.1.11}$$

In Lemma 3.5.2, we show that every measure $\nu \in \widetilde{\mathcal{D}}(u_0, \rho_0)$ is concentrated on the graph of u. This is again a consequence of the Kuratowski convergence of the graph of u^h to the graph of u, and it implies that a_y are probability measures concentrated on the set $\mathrm{conv}\{u^-(y), u^+(y)\}$ for ρ-a.e. y.

As a consequence of the fact that a_y is the disintegration of a limit of multiples of the Lebesgue measure restricted to the region between the graphs of u and of u^h, in Lemma 3.5.3, we show that $\{a_y\}_y$ has the following additional structure: when restricted to the open interval $I(u^-, u^+)$ with endpoints $(u^-(y), u^+(y))$, it is absolutely continuous with non-increasing density $\mathsf{g}_y \in \mathbf{L}^1(I(u^-, u^+))$:

$$(a_y) \llcorner I(u^-, u^+) \;=\; \mathsf{g}_y \cdot \mathscr{L}^1 \qquad D\,\mathsf{g}_y \leq 0 \qquad \text{for } \widehat{\rho}\text{-a.e. } y.$$

Moreover, in Lemma 3.5.5, it is shown via a transversality argument, that for $\widehat{\rho}$ a.e. y

$$f'\big(\max\{u^-, u^+\}\big) \neq \lambda(y) = \frac{f(u^+(y)) - f(u^-(y))}{u^+(y) - u^-(y)} \quad \Longrightarrow \quad [a_y]^+(\max\{u^-, u^+\}) = 0$$

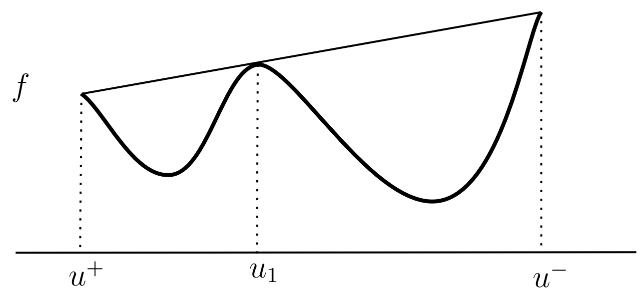

FIGURE 1. Example of the set $\mathscr{I}(u^-,u^+)$ for an admissible shock u^-,u^+. In this case $\mathscr{I}(u^-,u^+) = (u^+,u_1)\cup(u_1,u^-)$.

and that symmetrically

$$f'\Big(\min\{u^-,u^+\}\Big) \neq \lambda(y) = \frac{f(u^+(y)) - f(u^-(y))}{u^+(y)-u^-(y)} \quad \Longrightarrow \quad [a_y]^-(\min\{u^-,u^+\}) = 0$$

Combining Lemma 3.5.3 with Lemma 3.5.5 we thus obtain

$$f'\big(u^-\big), f'(u^+) \neq \lambda(y) \implies \Big(\partial_v\, a_y \,\le\, 0 \quad \text{in } \mathscr{D}'_v\Big) \qquad \text{for } \widehat{\rho}\text{-a.e. } y. \tag{3.1.12}$$

In parallel, combining the information coming from Theorem 3.1.1 with the kinetic equation (3.1.9), we deduce that a_y must be distributed in a way that the mean speed of particles, if we regard a particle located at height v as travelling with speed $f'(v)$, matches the speed given by the continuity equation (3.1.7), which is just the Rankine-Hugoniot speed, obtaining:

$$\int f'(v)\ \mathrm{d}a_y(v) \;=\; \lambda(y) \qquad \text{for } \widehat{\rho}\text{-a.e. } y. \tag{3.1.13}$$

Recall that $\overline{u}$ in (3.1.5) is also an entropy solution (see Definition 3.3.2) if and only if

$$\begin{aligned} u^- > u^+ \quad &\Longrightarrow \quad f(v) \le f(u^+) + \lambda(v-u^+) \quad \forall\, v\in(u^+,u^-)\\ u^- < u^+ \quad &\Longrightarrow \quad f(v) \ge f(u^+) + \lambda(v-u^+) \quad \forall\, v\in(u^+,u^-). \end{aligned} \tag{3.1.14}$$

In such case we say that u^-,u^+ are connected by an entropy admissible shock with speed λ. The following Definition will be fundamental to understand the structure of the measures $a_y(v)$ (and hence of ν).

DEFINITION 3.1.5. Let $u^- > u^+$ be connected by an entropy admissible shock with speed λ. We let

$$\mathscr{I}(u^-,u^+) \;\doteq\; \Big\{v\in(u^+,u^-)\;\Big|\; f(v)-f(u^+)-\lambda(v-u^+)<0\Big\}. \tag{3.1.15}$$

and

$$\mathscr{E}(u^-,u^+) \doteq \Big\{v\in\{u^-,u^+\}\,\Big|\, f'(v)\neq\lambda\Big\}.$$

Finally, we set

$$\mathscr{K}(u^-,u^+) = (u^+,u^-)\setminus\mathscr{I}$$

If $u^- < u^+$, $\mathscr{I}(u^-,u^+)$, $\mathscr{E}(u^-,u^+)$ and $\mathscr{K}(u^-,u^+)$ are defined symmetrically.

Combining now (3.1.12) with (3.1.13), a simple Fubini's argument shows that in order for both of them to be satisfied, the density g_y must almost saturate the constraint (3.1.12) by being constant on each connected component of $\mathscr{I}(y)$

$$[D\,\mathsf{g}_y](\mathscr{I}(y)) \,=\, 0 \qquad \text{for } \rho\text{-a.e. } y, \tag{3.1.16}$$

where

$$\mathscr{I}(y) \doteq \mathscr{I}(u^-(y),u^+(y)) \qquad \mathscr{E}(y) \doteq \mathscr{E}(u^-(y),u^+(y)).$$

For the same reason, it also follows that $a_y(\mathscr{E}(y)) = 0$ for $\widehat{\rho}$-a.e. y. We have thus obtained the following Theorem.

Theorem 3.1.6. *Any compatible pair $(\widehat{\rho},\nu)$ disintegrates*

$$\nu = a_y(v)\otimes\widehat{\rho}(y) \in \mathscr{M}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}), \qquad \operatorname{supp} a_y \subset \overline{I}(u^-(y),u^+(y))$$

where, for $\widehat{\rho}$-almost every y

(1) $(a_y)\llcorner I(u^-(y),u^+(y)) = \mathsf{g}_y\mathscr{L}^1$, *where* $\mathsf{g}_y \in BV_{loc}\big(I(u^-(y),u^+(y))\big)$ *is nonincreasing.*
(2) $D\mathsf{g}_y \in \mathscr{M}\big(I(u^-(y),u^+(y))\big)$ *is concentrated on the set* $\mathscr{K}(u^-(y),u^+(y))$.
(3) $a_y(\mathscr{E}(y)) = 0$.

It is interesting to see how Theorem 3.1.6 specializes in the simple situations of a quadratic or a cubic-like flux.

Corollary 3.1.7 (Convex flux). *Let f be strictly convex or concave. Any compatible pair (ρ,ν) disintegrates*

$$\nu = a_y(v)\otimes\widehat{\rho}(y) \in \mathscr{M}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})$$

where

$$a_y = \pm\frac{1}{|u^-(y)-u^+(y)|}\,\mathscr{L}^1\llcorner I(u^+(y),u^-(y)) \qquad \textit{for } \widehat{\rho}\textit{-almost every } y.$$

In particular, for every $\rho\in\mathcal{D}(u_0,\rho_0)$ there exists a unique lift $\nu\in\widetilde{\mathcal{D}}(u_0,\rho_0)$ such that $(\widehat{\rho},\nu)$ is a compatible pair.

For the cubic flux the set $\mathscr{K}$ is always empty, as for the quadratic flux, but contact discontinuities (i.e. shocks in which $\lambda = f'(u^-)$ or $\lambda = f'(u^+)$) are present, therefore $\mathscr{E}$ is not empty, in general. Therefore specializing Theorem 3.1.6 in this case we obtain

Corollary 3.1.8 (Cubic flux). *Let $f(u) = u^3$. Any compatible pair $(\widehat{\rho},\nu)$ disintegrates*

$$\nu = a_y(v)\otimes\widehat{\rho}(y) \in \mathscr{M}(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})$$

where for some measurable functions $y\mapsto\mathsf{g}_1(y)\in\mathbb{R}$, $y\mapsto\mathsf{g}_\pm(y)\in\mathbb{R}$, we have

$$a_y = \mathsf{g}_1(y)\cdot\mathscr{L}^1\llcorner I(u^+(y),u^-(y)) + \mathsf{g}_+(y)\cdot\delta_{\{u^+\}} + \mathsf{g}_-(y)\cdot\delta_{\{u^-\}} \qquad \textit{for } \rho\textit{-a.e. } y$$

and

$$f'(u^\pm(y)) \neq \lambda(y) \quad\Longrightarrow\quad \mathsf{g}_\pm(y) = 0 \qquad \textit{for } \rho\textit{-a.e. } y.$$

Necessary Conditions for Minimizers. The study of the elements in $\widetilde{\mathcal{D}}$ is also motivated by the fact that they are the objects needed to describe variations of integral functionals. For the sake of illustration we provide an example of such functional below. Let $\mathscr{G} : \mathbf{L}^\infty((0,T)\times\mathbb{R}) \to \mathbb{R}$ be a functional of the form

$$\mathscr{G}(u) \doteq \int_0^T \int_{\mathbb{R}} g(t,x,u(t,x))\,\mathrm{d}x\,\mathrm{d}t \tag{3.1.17}$$

and let us look for necessary conditions for optimality by computing the variation of $\mathscr{G}$ in u along the sequence of perturbations u^h. One has

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h}\Big(\mathscr{G}(u^h) - \mathscr{G}(u)\Big) &= \lim_{h\to 0+} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_v g(t,x,v)\,\mathrm{d}\nu_h(t,x,\cdot)\cdot \mathscr{L}^1(v)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_v g(t,x,v)\,\mathrm{d}a_{t,x}(v)\,\mathrm{d}\rho(t,x). \end{aligned} \tag{3.1.18}$$

To compute the variation one needs to understand what are all the possible "lifts" of ρ in the additional variable v. Theorem 3.1.6 provides a general structure of such a lift.

Remark 3.1.9. Theorem 3.1.6 has a very concrete meaning: assume you are testing the optimality of a shock u^-, u^+. In order to do this, you might perturb it in many ways, for example by "opening" the shock and dividing it into several smaller shock, each corresponding to a connected component $\mathscr{I}(u^-,u^+)$: more components of $\mathscr{I}(u^-,u^+)$ correspond to more possible perturbations. For strictly compressive shocks, i.e. for shocks u^-, u^+ such that $\mathscr{I}(u^-,u^+) = I(u^-,u^+)$, one can produce only one possible perturbation that looks like a "shift" (a translation) of the shock in some direction. This is always the case for convex fluxes.

Asymptotic Structure of the Perturbations, Existence of the Shift. To conclude the paper, we prove that Theorem 3.1.6 allows to detect the asymptotic structure of perturbations near a point $(\bar{t},\overline{x})$. Given a sequence of perturbations u^h as above, consider the rescalings at a point $(\bar{t},\overline{x}) \in \mathbb{R}^+ \times \mathbb{R}$

$$V_h^{(\bar{t},\overline{x})}(t,x) \doteq u^h\big(\bar{t} + h(t-\bar{t}),\, \overline{x} + h(x-\overline{x})\big) \qquad U_h^{(\bar{t},\overline{x})}(x) \doteq V_h^{(\bar{t},\overline{x})}(\bar{t},x) \tag{3.1.19}$$

in which u has a jump $u^- = u(\bar{t},\overline{x}-) \neq u(\bar{t},\overline{x}+) = u^+$ and which is of approximate jump for u, meaning that

$$\lim_{h\to 0} u(\bar{t} + h(t-\bar{t}), \bar{x} + th(x-\overline{x})) \;=\; \overline{u}(x-\lambda t), \qquad \text{in } \mathbf{L}^1_{loc}$$

where

$$\overline{u}_0(x) = u^- \mathbf{1}_{x<0} + u^+ \mathbf{1}_{x>0}.$$

We then consider any limit points $\overline{V}, \overline{U}$ of (3.1.19), and by the results in §3.3, it will follow that $\operatorname{Im}\overline{V}$, $\operatorname{Im}\overline{U} \subset \overline{I}(u^+,u^+)$ for $\mathscr{H}^1$-almost every point $(\bar{t},\overline{x})$. It is shown in Lemma 3.6.1 that the function $\overline{U}$ has the property that its level sets are distributed according to the density $\mathsf{g}_{\bar{t},\overline{x}}$ given by Theorem 3.1.6:

$$\mathscr{L}^1\Big(\big\{x\in\mathbb{R} \;\big|\; \overline{u}_0(x) \le v \le \overline{U}(x)\big\}\Big) \;=\; \rho_{\bar{t}}(\{\overline{x}\})\,\mathsf{g}_{\bar{t},\overline{x}}(v) \qquad \text{for a.e. } v\in\mathbb{R}. \tag{3.1.20}$$

This is in fact the main link between the structure of the disintegration obtained in Theorem 3.1.6 and the structure of blow ups $\overline{V}, \overline{U}$. We then exploit the fact that the

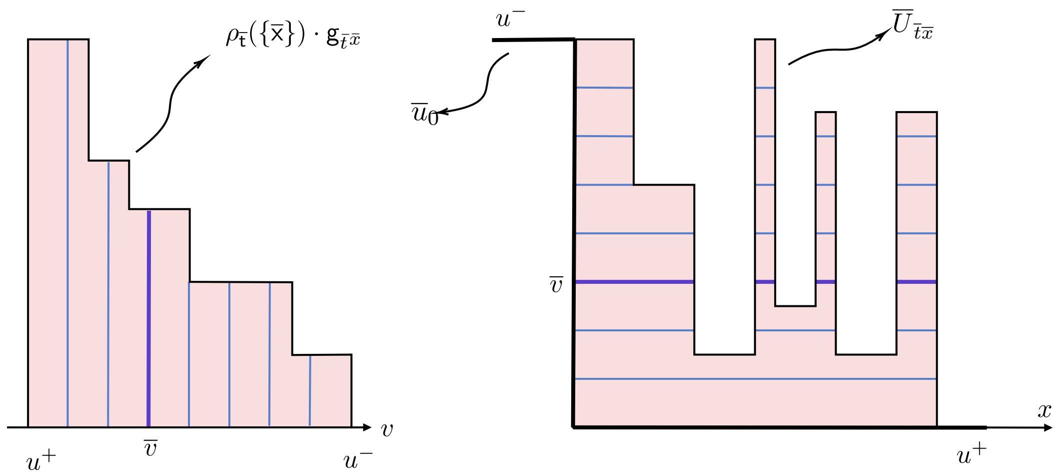

FIGURE 2. Left: Graph of the absolutely continuous part of the disintegration $\rho_{\bar{t}}(\{\bar{x}\}) \cdot \mathsf{g}_{\bar{t}\bar{x}}$. Right: Graphs of the shock $\overline{u}_0$ compared with the graph of the time section $\overline{U}_{\bar{t}\bar{x}}(\bar{t}, \cdot)$. Each vertical blue line in position $\overline{v}$ must have the same length as the sum of the horizontal segments at height $\overline{v}$.

derivative of g is concentrated on $\mathscr{K}$ (point (2) of Theorem 3.1.6) to deduce from (3.1.20) that

$$\operatorname{Im} \overline{U} \subset \mathscr{K} \cup \{u^-, u^+\}. \tag{3.1.21}$$

At this level, all that we know about the function $\overline{U}$ is that its suplevel sets are distributed according to (3.1.20). This amounts to say (see Figure 2) that the length of the vertical blue lines is the same as the sum of the corresponding horizontal segments. The main difficulty of this section is actually to prove that the horizontal segments are connected (and so there is only one segment), which amounts to prove that $\overline{U}$ is a monotone function. This would be trivially true, by the entropy conditions, if we knew a priori that all the shocks in $\overline{V}$ must be entropic at the fixed time section $\overline{V}(0, x) = \overline{U}(x)$, but this is in general not the case for a general entropy solution. To do this, we need to use the additional property (3.1.21) together with $\operatorname{Im} \overline{V} \subset \overline{I}(u^+, u^-)$. In Lemma 3.6.3, we show that in fact $\overline{U}$ must have a very rigid structure: if we also add a condition like $\overline{U}(\pm\infty) = u^\pm$, then (3.1.21) and $\operatorname{Im} \overline{V} \subset \overline{I}(u^+, u^-)$ imply that $x \mapsto \overline{U}(x)$ is a monotone function and that

$$\overline{V}(t, x) \;=\; \overline{U}(x - \lambda t).$$

In particular, all the level sets are connected, and given $\rho(\{x\})$ and g, there exists a unique function $U_{\rho(\{x\})}[\mathsf{g}]$ with connected level sets satisfying condition (3.1.20) (see Figure 3). Below we formalize the results of this discussion.

DEFINITION 3.1.10 (Composite Shifts). Let $u^- > u^+$ be connected by an entropy admissible shock. Let $\rho \in \mathbb{R}^+$ be a positive number and let $0 \leq \mathsf{g} \in BV_{loc}\big((u^+, u^-)\big)$ nonincreasing such that $D\mathsf{g}$ is concentrated in $\mathscr{K}(u^-, u^+)$.

(a) We denote by $\mathsf{g}^{-1} : \big[0, \mathsf{g}(u^+ +)\big] \to [u^+, u^-]$ the *pseudoinverse* of g, defined by

$$\mathsf{g}^{-1}(\vartheta) \doteq \sup \Big\{ v \in [u^+, u^-) \;\Big|\; \mathsf{g}(v+) \geq \vartheta \Big\}.$$

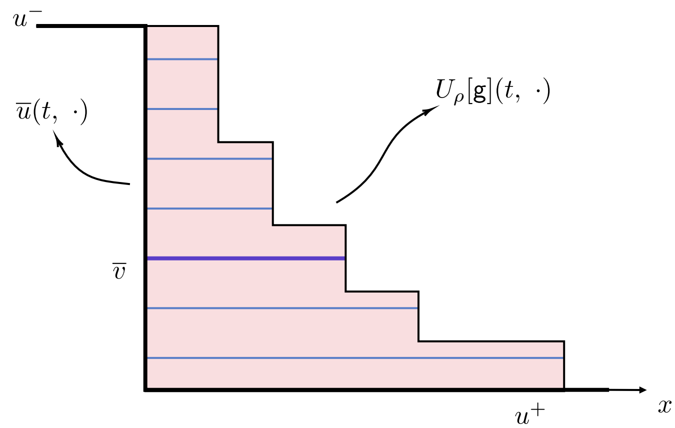

FIGURE 3. Time sections of the function $U_\rho[\mathsf{g}]$. The level sets are all connected.

(b) We define the function $U_\rho[\mathsf{g}]$ as

$$U_\rho[\mathsf{g}](t,x) = \begin{cases} u^-, & \text{if } x < \lambda t, \\ \mathsf{g}^{-1}\Big(\rho^{-1}(x-\lambda t)\Big), & \text{if } \lambda t \leq x \leq \lambda t + \rho \mathsf{g}(u^+), \\ u^+, & \text{if } x > \lambda t + \rho \mathsf{g}(u^+). \end{cases}$$

By the Volpert's chain rule for BV functions and by the choice of g, the function $U_\rho[\mathsf{g}]$ is an entropy solution to (3.1.1) on the whole $\mathbb{R}^2$.

For $u^- < u^+$ the definition is given symmetrically.

THEOREM 3.1.11. *For ρ-almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ it holds*

$$V_h^{(t,x)} \longrightarrow U_{\rho(\{(t,x)\})}[\mathsf{g}_{(t,x)}] \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R}^2)$$

and

$$U_h^{t,x} \longrightarrow U_{\rho(\{(t,x)\})}[\mathsf{g}_{(t,x)}](t,\cdot) \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R}).$$

REMARK 3.1.12. Theorem 3.1.11 can be viewed as a shift differentiability result, for a conservation law with any flux which is not constant on any non-trivial interval. In fact, the mass of the level sets of $U_\rho[\mathsf{g}] - \overline{u}$ (or, the length of the blue lines in Figure 3) is precisely the amount of which one has to "shift" the graph of u to produce a perturbation that is, at first order precision, equivalent to $\{u^h\}_{h>0}$.

3.2. Preliminaries

3.2.1. General Measure Theory. In the following, we consider measures defined on the σ-algebra of Borel sets $\mathcal{B}(X)$ of some complete, separable metric space X. The total variation of a measure μ is defined on every $E \in \mathcal{B}(X)$ by

$$|\mu|(E) \doteq \sup\Big\{ \sum_{i=1}^k |\mu(E_i)| \;\Big|\; E_i \cap E_j = 0 \text{ for } i \neq j, \quad \bigcup_{i=1}^k E_i = E, \quad E_i \in \mathcal{B}(X) \Big\}.$$

We denote by $\mathscr{M}(X)$ the space of finite measures, namely the set of all measures μ such that $|\mu|(X) < \infty$, endowed with the norm $\|\mu\|_{\mathscr{M}} \doteq |\mu|(X)$, and we denote by $\mathscr{M}^+(X)$ the subset of positive measures. By *weak topology* on $\mathscr{M}(X)$ we mean the topology on

$\mathscr{M}(X)$ induced by the duality with the space of bounded and continuous functions $\mathcal{C}_b(X)$. In particular, given a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{M}(X)$, we write $\mu_n \rightharpoonup \mu$ if and only if

$$\lim_{n\to+\infty} \int_X \varphi(x)\, \mathrm{d}\mu_n(x) = \int_X \varphi(x)\, \mathrm{d}\mu(x) \qquad \forall\, \varphi \in \mathcal{C}_b(X)$$

We say that a sequence $\{\mu_n\}_n$ is *tight* if for every $\varepsilon > 0$ there exists a compact set K_ε with

$$\sup_n |\mu_n|(X \setminus K_\varepsilon) \le \varepsilon.$$

We will often use the following classical Theorem.

THEOREM 3.2.1 (Prokhorov). *Let X be a complete, separable metric space. If a sequence $\{\mu_n\} \subset \mathscr{M}(X)$ is tight and uniformly bounded, there exists a subsequence $\{n_k\}_k$ and $\mu \in \mathscr{M}(X)$ with $\mu_{n_k} \rightharpoonup \mu$.*

We say that a measure μ is supported on a set $E \subset X$ if $|\mu|(X \setminus E) = 0$. Given a measurable function $g : X \to \mathbb{R}$, we denote by $g \cdot \mu$ the measure defined by

$$(g \cdot \mu)(E) \doteq \int_E g(x)\, \mathrm{d}\mu(x) \qquad \forall\, E \in \mathcal{B}(X).$$

We will always use the notation $\mathbf{1}_E : X \to \{0, 1\}$ to denote the characteristic function of a set $E \in \mathcal{B}(X)$

$$\mathbf{1}_E(x) \doteq \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$$

and the notation $\mu \llcorner E$ to denote the restriction of the measure μ to the set E

$$\mu \llcorner E \doteq \mathbf{1}_E \cdot \mu.$$

If we are given another metric space Y and a measurable map $g : X \to Y$, we denote by $g_\sharp \mu$ the push-forward of μ by g

$$[g_\sharp \mu](E) \doteq \mu(g^{-1}(E)) \qquad \forall\, E \in \mathcal{B}(Y).$$

A measurable function $g \in BV(\mathbb{R}^n)$ if and only if the distributional gradient Dg belongs to the space of finite measures $\mathscr{M}(\mathbb{R}^n)$. Given a measurable function $f : (a, b) \to \mathbb{R}$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(v) \doteq \mathscr{L}^1\Big(\big\{x \in (a, b) \,|\, f(x) \le v\big\}\Big)$$

is bounded and nondecreasing, hence $g \in BV(\mathbb{R})$. We will sometimes use the following convenient formula to compute the push-forward $f_\sharp \mathscr{L}^1 \llcorner (a, b)$

$$[f_\sharp \mathscr{L}^1 \llcorner (a, b)](E) = \int_E \mathrm{d}\, D\Big[\mathscr{L}^1\Big(\big\{x \in (a, b) \,|\, f(x) \le v\big\}\Big)\Big](v) \qquad \forall\, E \in \mathcal{B}(\mathbb{R}) \quad (3.2.1)$$

which readily follows by observing that for every set E of the form $(-\infty, z]$ it holds

$$[f_\sharp \mathscr{L}^1 \llcorner (a, b)]((-\infty, z]) = \mathscr{L}^1\Big(\big\{x \in (a, b) \;\big|\; f(x) \le z\big\}\Big) = \int_{(-\infty, z]} \mathrm{d}\, Dg(v).$$

We momentarily restrict to the Euclidean setting $X = \mathbb{R}^n$, and we denote by $\mathscr{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathscr{H}^k$ the k-dimensional Hausdorff measure. The next theorem holds in much more generality, but we will use only the following Euclidean version (see for example [**6**]).

THEOREM 3.2.2 (Disintegration of measures). *Let A, B be open sets of $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, $\nu \in \mathscr{M}(A \times B)$, $\pi : A \times B \to A$ the canonical projection on the first factor and $\rho \doteq \pi_\sharp|\nu|$. Then there exists a Borel family $\{\nu_x\}_{x \in A} \subset \mathscr{M}(B)$ such that $\|\nu_x\|_{\mathscr{M}} = 1$ for ρ-a.e. $x \in A$ and ν can be decomposed as $\nu = \nu_x \otimes \rho$, which means*

$$\nu(E) \;=\; \int_A \nu_x(E_x)\,\mathrm{d}\rho(x) \qquad \forall\, E \in \mathcal{B}(A \times B)$$

$$E_x \doteq \{y \in B \;\big|\; (x, y) \in E\}.$$

If $X = \Gamma_T$, where Γ_T is a space of curves defined on sub-intervals $[0, T]$

$$\Gamma_T \;\doteq\; \Big\{\gamma \,\big|\, \gamma : [0, t_\gamma] \to \mathbb{R}, \quad \gamma \text{ Lipschitz curve}\Big\}$$

we endow it with the metric

$$d_\Gamma(\gamma_1, \gamma_2) \;\doteq\; \sup_{(0, t_{\gamma_1} \wedge t_{\gamma_2})} |\gamma_1(t) - \gamma_2(t)| + |t_{\gamma_1} - t_{\gamma_2}|$$

The space Γ_T has natural evaluation maps (or projections) $e_t : \Gamma_t \to \mathbb{R}$ defined by

$$e_t(\gamma) \;\doteq\; \gamma(t) \qquad \forall\, \gamma \in \Gamma_T, \qquad t \in [0, t_\gamma].$$

If $T = +\infty$, we simply let $\Gamma = \Gamma_\infty$.

3.2.2. Continuity Equation, Superposition Principle. Given a bounded measurable vector field $\boldsymbol{b} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, we say that $\gamma : [0, t_\gamma] \to \mathbb{R}^d$ is an *integral curve* of $\boldsymbol{b}$ if

$$\dot\gamma(t) = \boldsymbol{b}(t, \gamma(t)) \qquad \text{for a.e. } t \in \mathbb{R}^+. \tag{3.2.2}$$

We consider measure valued solutions $\mu \in \mathscr{M}^+(\mathbb{R}^+ \times \mathbb{R})$ of the continuity equation with vector field $\boldsymbol{b}$ and measure source $g \in \mathscr{M}^-(\mathbb{R}^+ \times \mathbb{R})$

$$\partial_t \mu + \partial_x\big(\boldsymbol{b}(t, x)\mu\big) = g \qquad \text{in } \mathscr{D}'_{t,x}. \tag{3.2.3}$$

The ordinary differential equation (3.2.2) and (3.2.3) are related by the following Theorem.

THEOREM 3.2.3 (Ambrosio, Smirnov). *Let $\mu \in \mathscr{M}^+(\mathbb{R}^+ \times \mathbb{R})$ solve the continuity equation* (3.2.3)*. Then the following are equivalent:*

(1) There exists a measure $\boldsymbol{\eta} \in \mathscr{M}^+(\Gamma)$ concentrated on integral curves of $\boldsymbol{b}$, satisfying

$$\mu_t \;=\; (e_t)_\sharp \boldsymbol{\eta} \qquad \forall\, t \in \mathbb{R}^+. \tag{3.2.4}$$

(2) μ is a solution of the continuity equation (3.2.3) *for some $g \in \mathscr{M}^-(\mathbb{R}^+ \times \mathbb{R})$.*

3.2.3. Convergence of Sets. We recall the notion of Kuratowski convergence. Let (X, d) be a metric space and $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X.

DEFINITION 3.2.4. We define the *upper limit* and the *lower* limit of the sequence K_n respectively by the formulas

$$\limsup_{n\to+\infty} K_n = \Big\{x \in X \mid \liminf_{n\to+\infty} d(x, K_n) = 0\Big\},$$
$$\liminf_{n\to+\infty} K_n = \Big\{x \in X \mid \limsup_{n\to+\infty} d(x, K_n) = 0\Big\}.$$

We say that the sequence $(K_n)_{n\in\mathbb{N}}$ converges to $K \subset X$ in the sense of Kuratowski if

$$K = \limsup_{n\to+\infty} K_n = \liminf_{n\to+\infty} K_n$$

and we denote it by

$$K_n \xrightarrow{\mathsf{K}} K, \qquad \text{or} \qquad K = \mathsf{K}\!-\!\lim_{n\to\infty} K_n.$$

The following compactness result holds (see [**30**]).

THEOREM 3.2.5 (Zarankiewicz). *Suppose that X is a separable metric space. Then for every sequence K_n of subsets of X there exists a convergent subsequence in the sense of Kuratowski.*

In the following we consider Kuratowski convergence of closed sets arising as boundary of hypographs.

DEFINITION 3.2.6. Let $\Omega \subset \mathbb{R}^d$ be an open set and $u : \Omega \to \mathbb{R}$ be a function. We set

$$\mathcal{G}(u) \doteq \partial\,(\mathrm{hyp}(u))$$

the topological boundary of the hypograph $\mathrm{hyp}(u) \doteq \{(y, v) \in \Omega \times \mathbb{R} : v < u(y)\}$. We refer to $\mathcal{G}(u)$ as the *completed graph* of u.

REMARK 3.2.7. If we denote by $\overline{u}$ and $\underline{u}$ the upper lower semicontinuous envelope and the lower semicontinuous envelope of u, then $\mathcal{G}(u) = \{(y, v) \subset \Omega \times \mathbb{R} : \underline{u}(y) \le v \le \overline{u}(y)\}$. In particular $\mathcal{G}(u)$ reduces to the graph of u for continuous functions.

The Kuratowski convergence is induced by the Hausdorff metric in compact metric spaces (see for example [**31**]). It is easy to check that the following version holds for closed subsets of $\mathbb{R}^d$:

LEMMA 3.2.8. *Let $(K_n)_{n\in\mathbb{N}}$ and K be closed subsets of $\mathbb{R}^d$. Then $K_n \xrightarrow{\mathsf{K}} K$ if and only if for every $R > 0$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n > N$ it holds*

$$K_n \cap B_R \subset K + B_\varepsilon \qquad \textit{and} \qquad K \cap B_R \subset K_n + B_\varepsilon,$$

where B_R, B_ε denote the open ball centered at the origin of radius R and ε respectively.

3.3. Graph Convergence of Entropy Solutions

3.3.1. Structure of Entropy Solutions, Generalized Characteristics and Admissible Boundaries. We consider the following Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}, \end{cases} \tag{3.3.1}$$

where the flux f is a $\mathscr{C}^1$ function and we assume in the whole paper that there are no non-trivial intervals where the restriction of f is affine.

DEFINITION 3.3.1. We say that (η, q) is a convex *entropy-entropy flux pair* if $\eta : \mathbb{R} \to \mathbb{R}$ is convex and $q : \mathbb{R} \to \mathbb{R}$ satisfies $q' = \eta' f'$.

DEFINITION 3.3.2. A bounded function $u \in \mathcal{C}^0([0, +\infty); \mathbf{L}^1_{\text{loc}}(\mathbb{R}))$ is the *entropy solution* of (3.3.1) if it satisfies the initial condition and for all convex entropy-entropy flux pairs (η, q) it holds

$$\mu_\eta \doteq \partial_t \eta(u) + \partial_x q(u) \le 0 \tag{3.3.2}$$

in the sense of distributions.

The well-posedness of the Cauchy problem for entropy solutions with bounded initial data is a classical result [**76**].

See for example [**37**, Corollary 4.6] for the following proposition on the structure of entropy solutions.

PROPOSITION 3.3.3. *Let u be the entropy solution of with initial datum $u_0 \in \mathbf{L}^\infty$. There exists a representative of u such that the following holds. Then for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ the following limits exist:*

$$u^-(t,x) \doteq \lim_{x' \to x^-} u(t,x'), \qquad u^+(t,x) \doteq \lim_{x' \to x^+} u(t,x').$$

Moreover there is a set $J \subset \mathbb{R}^+ \times \mathbb{R}$ and countably many curves $\sigma_j \in \mathrm{Lip}(\mathbb{R}^+)$ such that

(1) the following inclusion holds:

$$J \subset \bigcup_{j \in \mathbb{N}} \{(t, \sigma_j(t)) : t \in \mathbb{R}^+\};$$

(2) u is continuous for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R} \setminus J$;

(3) for every $j \in \mathbb{N}$ and $\mathscr{L}^1$-a.e. $t \in \mathbb{R}^+$ it holds

$$\dot{\sigma_j}(t) = \begin{cases} f'(u^+) & \text{if } u^+ = u^-, \\ \frac{f(u^+) - f(u^-)}{u^+ - u^-} & \text{if } u^+ = u^-, \end{cases}$$

where u^+ and u^- denote respectively $\lim_{x \to \sigma_j(t)^+} u(t,x)$ and $\lim_{x \to \sigma_j(t)^-} u(t,x)$.

(4) for every $j \in \mathbb{N}$ and $\mathscr{L}^1$-a.e. $t \in \mathbb{R}^+$, $(t, \sigma_j(t))$ is point of approximate jump of u (u^-, u^+, λ), which means

$$\lim_{r \to 0+} \frac{1}{r^2} \int_{B_R} |\overline{u}(s,y) - u\big(t + r(s-t), x + r(y-x)\big)| \, \mathrm{d}y \, \mathrm{d}s = 0 \qquad \forall\, R > 0.$$

where $\overline{u}$ is defined as in (3.1.5).

Relying on the previous proposition we can define generalized characteristics associated to entropy solutions of (3.3.1) with initial data in $\mathbf{L}^\infty$.

DEFINITION 3.3.4. We say that $\gamma \in \mathrm{Lip}(\mathbb{R}^+)$ is a generalized characteristic of u if for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}^+$ it holds

$$\dot\gamma(t) = \lambda(t,\gamma(t)), \qquad \text{where} \quad \lambda(t,x) \doteq \begin{cases} f'(u^+) & \text{if } u^+ = u^-, \\ \frac{f(u^+)-f(u^-)}{u^+-u^-} & \text{if } u^+ \neq u^-, \end{cases}$$

and u^+ and u^- denote respectively $\lim_{y\to x^+} u(t,y)$ and $\lim_{x\to x-} u(t,y)$.

In order to describe the characteristics structure of entropy solutions u of (3.3.1), we recall the definition of admissible boundary and two related results from [**37**].

DEFINITION 3.3.5. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be Lipschitz, $\mathsf{w} \in \mathbb{R}$ and $\tau > 0$. Define the domains

$$\Omega_l \doteq \Big\{(t,x) \in [0,\tau]\times\mathbb{R} \;\Big|\; x < \gamma(t)\Big\}, \qquad \Omega_r \doteq \Big\{(t,x) \in [0,\tau]\times\mathbb{R} \;\Big|\; x > \gamma(t)\Big\}$$

We say that (γ,w) is an *admissible boundary* for u in the interval $[0,\tau]$ if the two restrictions $u_l \doteq u_{\llcorner\Omega_l}$ and $u_r \doteq u_{\llcorner\Omega_r}$ solve the initial-boundary value problems

$$\begin{cases} \partial_t u_l + \partial_x f(u_l) = 0 & \text{in } \Omega_l, \\ u(0,\cdot) = u_0(\cdot) & \text{in } (-\infty,\gamma(0)), \\ u(t,\gamma(t)) = \mathsf{w} & \text{in } (0,\tau) \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t u_r + \partial_x f(u_r) = 0 & \text{in } \Omega_r, \\ u(0,\cdot) = u_0(\cdot) & \text{in } (-\infty,\gamma(0)), \\ u(t,\gamma(t)) = \mathsf{w} & \text{in } (0,\tau). \end{cases}$$

where here the boundary condition $u(t,\gamma(t)) = \mathsf{w}$ must be understood in the hyperbolic sense of [**29**]. It is useful to exploit the definition of solution of the initial boundary value problem given in [**29**] to characterize admissible boundaries in terms of entropy inequalities for traces (see [**37**, Definition 2.12]): for every $k \in \mathbb{R}$ we define the entropies

$$\eta_k^+(u) \doteq (u-k)^+, \qquad \eta_k^-(u) \doteq (u-k)^-$$

with the corresponding entropy fluxes

$$q_k^+(u) \doteq \mathbf{1}_{[k,+\infty)}(u)\big(f(u)-f(k)\big), \qquad q_k^-(u) \doteq \mathbf{1}_{(-\infty,k]}(u)\big(f(k)-f(u)\big).$$

Here $\mathbf{1}_E$ denotes the characteristic function of the set E:

$$\chi_E(u) \doteq \begin{cases} 1 & \text{if } u \in E, \\ 0 & \text{if } u \notin E. \end{cases}$$

We have therefore

LEMMA 3.3.6. *Let $\gamma : \mathbb{R}^+ \to \mathbb{R}$ be Lipschitz and let $\mathsf{w} \in \mathbb{R}$, and $\tau \in \mathbb{R}^+$. Then (γ,w) is an admissible boundary for u in the time interval $[0,\tau]$ if and only if the following inequalities hold for $\mathscr{L}^1$-a.e. $t \in (0,\tau)$:*

$$\begin{aligned} -\dot\gamma\,\eta_k^+(u^-) + q_k^+(u^-) \geq 0 \quad \forall\, k \geq \mathsf{w}, \\ -\dot\gamma\,\eta_k^-(u^-) + q_k^-(u^-) \geq 0 \quad \forall\, k \leq \mathsf{w}, \end{aligned} \tag{3.3.3}$$

where $u^- = u^-(t) = \lim_{x\to\gamma(t)^-} u(t,x)$. And symmetrically for $\mathscr{L}^1$ a.e. $t \in (0,\tau)$

$$\begin{aligned} -\dot\gamma\,\eta_k^+(u^+) + q_k^+(u^+) \leq 0 \quad \forall\, k \geq \mathsf{w}, \\ -\dot\gamma\,\eta_k^-(u^+) + q_k^-(u^+) \leq 0 \quad \forall\, k \leq \mathsf{w}, \end{aligned} \tag{3.3.4}$$

where $u^+ = u^+(t) = \lim_{x\to\gamma(t)^+} u(t,x)$.

The following stability result is proven in [**37**, Proposition 2.13].

PROPOSITION 3.3.7. *Let* u^n *be entropy solutions of* (3.3.1) *and* (γ^n, w^n) *admissible boundaries for* u^n *in the time intervals* $[0,\tau^n]$. *Assume that*

- $u^n \to u$ *in* $\mathbf{L}^1(\mathbb{R}^+ \times \mathbb{R})$,
- $\mathsf{w}^n \to \mathsf{w}$ *in* $\mathbb{R}$,
- $\tau^n \to \tau$ *in* $\mathbb{R}$,
- $\gamma^n \to \gamma$ *locally uniformly in* $\mathbb{R}^+$.

Then (γ, w) *is an admissible boundary for* u *in the time interval* $[0,\tau]$.

The next proposition relates $\mathcal{G}(u)$ (recall Definition 3.2.6) with the admissible boundaries for u.

PROPOSITION 3.3.8. *Let* u *be an entropy solution of* (3.3.1). *The following hold:*

(1) *for every* $(t,x,v) \in \mathcal{G}(u)$ *there is an admissible boundary* (γ, w) *for* u *on the time interval* $[0,t]$ *such that* γ *is a generalized characteristic of* u *and it satisfies* $\gamma(t) = x$ *and* $\mathsf{w} = v$;
(2) *if* (γ, w) *is an admissible boundary for* u *in the time interval* $[0,\tau]$, *then for every* $t \in (0,\tau)$ *it holds*
$$\mathsf{w} \in I(u^-(t,\gamma(t)), u^+(t,\gamma(t))). \tag{3.3.5}$$
In particular $(t,\gamma(t),\mathsf{w}) \in \mathcal{G}(u)$.

PROOF. Point (1) is a particular case of [**37**, Theorem 3.8]. We prove Point (2). By Lemma 3.3.6 we have that for $\mathscr{L}^1$-a.e. $t \in (0,\tau)$ the inequalities in (3.3.3) and (3.3.4) hold: we check that they imply (3.3.5).

We consider first the case $u^+ = u^-$ and we denote the common value by u. Assume by contradiction that $\mathsf{w} \neq u$, then for every $k \in I(u,w) \setminus \{u\}$ we have that (3.3.3) implies $\dot\gamma \le \frac{f(u)-f(k)}{u-k}$ and (3.3.4) implies $\dot\gamma \ge \frac{f(u)-f(k)}{u-k}$. Therefore for every $k \in I(w,u) \setminus \{u\}$ we have
$$\dot\gamma = \frac{f(u)-f(k)}{u-k}.$$
Since the left-hand side does not depend on $k \in I(w,u) \setminus \{u\}$, this implies that f is affine in $I(w,u)$, in contradiction with the assumption on f.

We now consider the case $u^+ \neq u^-$, up to removing a negligible set of times $t \in (0,\tau)$ we can assume that $\dot\gamma = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ by the Rankine-Hugoniot conditions. Assume by contradiction that $\mathsf{w} \notin I(u^-,u^+)$. It is not restrictive to assume $u^- < u^+ < \mathsf{w}$, the other cases are analogous: then for every $k \in I(u^+,\mathsf{w})$ the second inequality in (3.3.3) implies that $-\dot\gamma(k-u^-) + f(k) - f(u^-) \ge 0$, namely $(k, f(k))$ lies above or exactly on the secant of f through u^- and u^+. Similarly the second inequality in (3.3.4) implies that $(k,f(k))$ lies below or exactly on the secant of f through u^- and u^+. This means that f must be affine on $I(u,\mathsf{w})$ and that is a contradiction.

The argument above proves (3.3.5) for $\mathscr{L}^1$-a.e. $t \in (0,\tau)$. In order to conclude the proof it remains to extend (3.3.5) for every $t \in (0,\tau)$. We assume by contradiction that there is $t \in (0,\tau)$ for which $\mathsf{w} \notin I(u^-,u^+)$ and we prove that there is a $\delta' > 0$ such that for every $s \in (t, t+\delta')$ the condition (3.3.5) does not hold, providing a contradiction since we know that (3.3.5) holds for $\mathscr{L}^1$-a.e. $t \in (0,\tau)$. Let $\varepsilon \doteq \frac{1}{2}\min\{|u^- - \mathsf{w}|, |u^+ - \mathsf{w}|\}$ and let $\delta > 0$ small enough so that $u(t,\cdot)$ in the interval $(\gamma(t)-\delta, \gamma(t)+\delta)$ takes values in the interval $I_\varepsilon \doteq (\min\{u^-,u^+\}-\varepsilon, \max\{u^-,u^+\}+\varepsilon)$. If we denote by $\hat\lambda \doteq \sup_{[-\|u\|_{L^\infty},\|u\|_{L^\infty}]}|f'|$, by the maximum principle it follows that u takes values in I_ε also in the triangular shaped region

$$\mathcal{T} \doteq \Big\{(s,x) \in (t,+\infty)\times\mathbb{R} : \gamma(t)-\delta+\hat\lambda(s-t) < x < \gamma(t)+\delta-\hat\lambda(s-t)\Big\}.$$

Choosing $\delta' = \frac{\delta}{2\hat\lambda}$ we have that $(s,\gamma(s)) \in \mathcal{T}$ for every $s \in (t, t+\delta)$ so that (3.3.5) cannot hold. This provides the desired contradiction and concludes the proof. □

In the next corollary we compare the time sections of $\mathcal{G}(u)$ with the completed graphs of the time sections of u: more precisely for every $t>0$ we set

$$\mathcal{G}(u)_t \doteq \{(x,v) : (t,x,v) \in \mathcal{G}(u)\}$$

and

$$\mathcal{G}(u(t,\cdot)) = \partial\left(\mathrm{hyp}(u(t,\cdot))\right).$$

COROLLARY 3.3.9. *There is a countable set $\mathcal{N} \subset \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+\setminus\mathcal{N}$ it holds*

$$\mathcal{G}(u)_t = \mathcal{G}(u(t,\cdot)) = \big\{(x,v) : v \in I(u^-(t,x),u^+(t,x))\big\}.$$

In particular for every $t \in \mathbb{R}^+\setminus\mathcal{N}$ it holds

$$\mathcal{G}(u(s,\cdot)) \xrightarrow{\mathsf{K}} \mathcal{G}(u(t,\cdot)) \quad \text{as } s \to t.$$

PROOF. We prove the following one-side limits:

$$\mathcal{G}(u)_t = \mathsf{K}-\lim_{s\to t^-}\mathcal{G}(u(s,\cdot)), \qquad \mathcal{G}(u(t,\cdot)) = \mathsf{K}-\lim_{s\to t^+}\mathcal{G}(u(s,\cdot)). \tag{3.3.6}$$

In particular there is an at most countable set $\mathcal{N}$ such that the two limits are equal for every $t \in \mathbb{R}^+\setminus\mathcal{N}$, so the conclusion follows by (3.3.6).

1. The inclusion $\mathcal{G}(u)_t \subset \mathsf{K}-\liminf_{s\to t^-}\mathcal{G}(u(s,\cdot))$ follows immediately from Proposition 3.3.8.

2. We now prove $\mathcal{G}(u)_t \supset \mathsf{K}-\limsup_{s\to t^-}\mathcal{G}(u(s,\cdot))$. Since $\mathcal{G}(u(s,\cdot)) \subset \mathcal{G}(u)_s$ and $\mathcal{G}(u)$ is closed, then

$$\mathsf{K}-\limsup_{s\to t^-}\mathcal{G}(u(s,\cdot)) \subset \mathsf{K}-\limsup_{s\to t^-}\mathcal{G}(u)_s \subset \mathcal{G}(u)_t.$$

The two inclusions above prove the first equality in (3.3.6).

3. We next prove $\mathcal{G}(u(t,\cdot)) \subset \mathsf{K}-\liminf_{s\to t^+}\mathcal{G}(u(s,\cdot))$. Let $(\bar x,\bar v) \in \mathcal{G}(u(t,\cdot))$; we want to prove that for any sequence $s_n \to t^+$ there is a sequence (x_n,v_n) approaching $(\bar x,\bar v)$ with $(x_n,v_n) \in \mathcal{G}(u(s_n,\cdot))$, namely with $v_n \in I(u^-(s_n,x_n),u^+(s_n,x_n))$. If $(t,\bar x)$ is a continuity point of u, then $\bar v = u(t,\bar x)$ and any sequence $(x_n, u(s_n,x_n))$ with

$$(s_n,x_n) \in \mathbb{R}^+\times\mathbb{R}\setminus J, \quad x_n \to \bar x$$

satisfies the requirement. It remains to consider the case $(t,\bar x) \in J$: now $\bar v \in I(u^-(t,\bar x),u^+(t,\bar x))$. Let $\varepsilon>0$ and let $x^- \in (\bar x-\varepsilon,\bar x)$ be such that

$$(t,x^-) \in \mathbb{R}^+\times\mathbb{R}\setminus J, \qquad \text{and} \qquad |u(t,x^-)-u^-(t,\bar x)| < \varepsilon.$$

Similarly let $x^+ \in (\bar{x}, \bar{x}+\varepsilon)$ be such that

$$(t, x^+) \in \mathbb{R}^+ \times \mathbb{R} \setminus J, \qquad \text{and} \qquad |u(t, x^+) - u^+(t, \bar{x})| < \varepsilon.$$

Since $(t, x^-), (t, x^+)$ are continuity points of u there are two sequences $(x_n^-)_{n\in\mathbb{N}}, (x_n^+)_{n\in\mathbb{N}}$ approaching $\bar{x}^-$ and $\bar{x}^+$ respectively with

$$(s_n, x_n^-), (s_n, x_n^+) \in \mathbb{R}^+ \times \mathbb{R} \setminus J \qquad \text{and} \qquad u(s_n, x_n^\pm) \to u(t, x^\pm).$$

For n sufficiently large we have $x_n^- < x_n^+$ and $|u(s_n, x_n^\pm) - u(t, x^\pm)| < \varepsilon$. Moreover, since $\mathcal{G}(u(s_n, \cdot))$ is connected, for every $v \in I(u(s_n, x_n^+), u(s_n, x_n^-))$ there is $x \in [x_n^-, x_n^+]$ with $(x, v) \in \mathcal{G}(u(s_n, \cdot))$. In particular, by construction we can choose $v_n \in I(u(s_n, x_n^+), u(s_n, x_n^-))$ with $|v_n - \bar{v}| \le 2\varepsilon$ for n large enough. Therefore there is $x_n \in [x_n^-, x_n^+]$ with $(x_n, v_n) \in \mathcal{G}(u(s_n, \cdot))$ and $|x_n - \bar{x}| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary we can conclude by a diagonal argument.

4. Eventually we prove $\mathcal{G}(u(t, \cdot)) \supset \mathsf{K}-\limsup_{s\to t^+} \mathcal{G}(u(s, \cdot))$. By Point (2) in Proposition 3.3.8, it follows that

$$\mathsf{K}-\limsup_{s\to t^+} \mathcal{G}(u)_s \subset \mathcal{G}(u(t, \cdot)).$$

The conclusion follows recalling that $\mathcal{G}(u(s, \cdot)) \subset \mathcal{G}(u)_s$. □

The following Proposition shows that $\mathbf{L}^1_{\mathrm{loc}}$ convergence of bounded entropy solutions implies the stronger Kuratowski convergence of the associated completed graphs.

Proposition 3.3.10. *Let u^h be a sequence of equi-bounded entropy solutions to* (3.3.1) *such that*

$$u^h \to u \quad \textit{strongly in } \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}).$$

Then $\mathcal{G}(u^h) \xrightarrow{\mathsf{K}} \mathcal{G}(u)$ as subsets of $(0, +\infty) \times \mathbb{R} \times \mathbb{R}$. Moreover for every $t \in \mathbb{R}^+ \setminus \mathcal{N}$ it holds $\mathcal{G}(u^h(t, \cdot)) \xrightarrow{\mathsf{K}} \mathcal{G}(u(t, \cdot))$, where $\mathcal{N}$ is defined in Corollary 3.3.9.

Recall that $\mathcal{N}$ does not depend on the sequence u^h, but only on the limiting entropy solution u.

Proof. Up to subsequences, we can assume by Theorem 3.2.5 that $\mathcal{G}(u^h)$ converges to some set $G \subset \mathbb{R}^+ \times \mathbb{R}^2$ in the sense of Kuratowski. The inclusion $\mathcal{G}(u) \subset G$ is a direct consequence of the L^1_{loc} convergence of u^h to u. Indeed let $(\bar{t}, \bar{x}, \bar{v}) \in \mathbb{R}^+ \times \mathbb{R}^2 \setminus G$, then by the definition of Kuratowski convergence there is a neighborhood O of $(\bar{t}, \bar{x}, \bar{v})$ such that for all h small enough

$$O \cap \mathcal{G}(u^h) = \emptyset.$$

Since $u^h \to u$ in L^1_{loc} we deduce that $O \cap \mathcal{G}(u) = \emptyset$, in particular $(\bar{t}, \bar{x}, \bar{v}) \notin \mathcal{G}(u)$.

We now prove that $G \subset \mathcal{G}(u)$: for every $(\bar{t}, \bar{x}, \bar{v}) \in G$, there is a sequence $(\bar{t}^h, \bar{x}^h, \bar{v}^h) \in \mathcal{G}(u^h)$ converging to $(\bar{t}, \bar{x}, \bar{v})$. By Proposition 3.3.8 (Point (1)) there are (γ^h, w^h) admissible boundaries for u^h on time intervals $[0, \bar{t}^h]$ with $\mathsf{w}^h = \bar{v}^h$ and such that γ^h are generalized characteristics with $\gamma^h(\bar{t}^h) = \bar{x}^h$. Since γ^h are generalized characteristics and u^h are equi-bounded, then γ^h are equi-Lipschitz; up to taking a further subsequence we may assume that $\gamma^h \to \gamma$ uniformly on $[0, \bar{t}]$ for some Lipschitz curve γ. By Proposition 3.3.7, $(\gamma, \bar{v})$ is an admissible boundary on $[0, \bar{t}]$ and $\gamma(\bar{t}) = \bar{x}$, and by Proposition 3.3.8 (Point (2)) this implies $(\bar{t}, \bar{x}, \bar{v}) \in \mathcal{G}(u)$.

We then prove that for every $t \in \mathbb{R}^+ \setminus \mathcal{N}$ it holds $\mathcal{G}(u^h(t,\cdot)) \to \mathcal{G}(u(t,\cdot))$. Again we can assume that $\mathcal{G}(u^h(t,\cdot)) \to \tilde{G}$ in the sense of Kuratowski for some $\tilde{G} \subset \mathbb{R}^2$. Since $u^h(t,\cdot) \to u(t,\cdot)$ in $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$, then $\mathcal{G}(u(t,\cdot)) \subset \tilde{G}$. Moreover, since $\mathcal{G}(u^h) \to \mathcal{G}(u)$, then we have $\tilde{G} \subset \mathcal{G}(u)_t$. For $t \in \mathbb{R}^+ \setminus \mathcal{N}$ it holds $\mathcal{G}(u(t,\cdot)) = \mathcal{G}(u)_t$, therefore we deduce that $\tilde{G} = \mathcal{G}(u(t,\cdot))$ and this concludes the proof. □

An immediate consequence of Proposition 3.3.10 is that, if u is continuous on some open set $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$, then u^h converges uniformly to u on compact sets $K \subset \Omega$. The following Lemma shows that if u has shocks, we can obtain a quantitative estimate of the region where the uniform convergence of u^h to u fails, if we assume that

$$\sup_{h>0} h^{-1} \|u_0 - u_0^h\| \le +\infty. \tag{3.3.7}$$

Lemma 3.3.11. *Let u be an entropy solution to* (3.3.1) *and let $(t,x) \in J$ be a point of approximate jump of u (recall Proposition 3.3.3). Assume that* (3.3.7) *holds. Then, for every $\varepsilon > 0$ there exists $r, C, \overline{h} > 0$ for which*

$$\Big\{ y \in B_r(x) \;\Big|\; |u^h(t,y) - u(t,y)| > \varepsilon \Big\} \subset B_{Ch}(x) \qquad \forall\, 0 < h \le \overline{h}. \tag{3.3.8}$$

Proof. First, we choose $r > 0$ small enough so that

$$|u(t,y) - u(t,x-)| \le \frac{\varepsilon}{2} \quad \forall\, y \in (x-r,x), \qquad |u(t,y) - u(t,x+)| \le \frac{\varepsilon}{2} \quad \forall\, y \in (x, x+r). \tag{3.3.9}$$

Assume by contradiction that for every $C, \overline{h} > 0$ (3.3.8) does not hold. Then, for every $j \in \mathbb{N}$, there exists $0 < h_j < j^{-1}$ such that

$$\Big\{ y \in B_r(x) \;\Big|\; |u^{h_j}(t,y) - u(t,y)| > \varepsilon \Big\} \not\subset B_{jh_j}(x).$$

In particular, by Proposition 3.3.8, there exists a point $y_j \in B_r(x) \setminus B_{jh_j}(x)$, together with an admissible boundary (γ_j, w_j) of u^{h_j}, such that

$$\gamma_j(t) = y_j, \qquad |\mathsf{w}_j - u(t, \gamma_j(t)| > \varepsilon.$$

Up to extracting a subsequence, we can assume that the sequence $\{y_j\}_j$ satisfies $y_j > x$ for all $j \in \mathbb{N}$, being the opposite can completely analogous. Using the second in (3.3.9), we thus obtain

$$y_j \in B_r(x) \setminus B_{jh_j}(x), \qquad \gamma_j(t) = y_j, \qquad |\mathsf{w}_j - u(t,x+)| > \frac{\varepsilon}{2} \qquad \forall\, j \in \mathbb{N}. \tag{3.3.10}$$

Next, define a scaling parameter $\ell_j \doteq |y_j - x|$ and consider the rescalings

$$v_j(s,y) \doteq u(t + \ell_j(s-t), x + \ell_j(y-x)), \qquad z_j(s,y) \doteq u^{h_j}(t + \ell_j(s-t), x + \ell_j(y-x)).$$

Since (t,x) is a point of approximate jump of u, the rescalings v_j converge to the function $\overline{u}$ in (3.1.5). On the other hand, $\|u(s,\cdot) - u^{h_j}(s,\cdot)\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot h_j$, where $\mathcal{O}(1)$ is given by (3.3.7), for every $s \in \mathbb{R}^+$, so that

$$\|v_j(s,\cdot) - z_j(s,\cdot)\|_{\mathbf{L}^1} = \ell_j^{-1} \, \|u(\ell_j^{-1}(s-t),\cdot) - u^{h_j}(\ell_j^{-1}(s-t),\cdot)\|_{\mathbf{L}^1} \le \ell_j^{-1}\, h_j \;\le \frac{1}{j}.$$

Therefore by triangular inequality we deduce also that $v_j \to \overline{u}$ as $j \to +\infty$ in $\mathbf{L}^1_{loc}(\mathbb{R}^2)$. The scaled boundary $(\widetilde{\gamma}_j, \mathsf{w}_j)$ defined by

$$\widetilde{\gamma}_j(s) \doteq \ell_j^{-1}\big(\gamma_j(t + s\ell_j) - x\big)$$

is an admissible boundary of the solution z^{h_j}. The important observation is that $\widetilde{\gamma}_j(0) = 1$ for every j, therefore passing to the limit, up to a subsequence, using Proposition 3.3.7, it converges uniformly to an admissible boundary $(\widetilde{\gamma}, \mathsf{w})$ of $\overline{u}$ satisfying

$$\widetilde{\gamma}(0) = 1, \qquad |\mathsf{w}_j - u_+| \geq \frac{\varepsilon}{2}.$$

This is a contradiction with Proposition 3.3.8. □

3.3.2. Riemann Problems with Boundaries. We conclude this section by a discussion on a class of entropy solutions to generalized Riemann problems with moving boundaries.

Let $T > 0$ and consider a domain of the form

$$\Omega = \Big\{(t,x) \in (0,T) \times \mathbb{R} \;\Big|\; \gamma_1(t) < x < \gamma_2(t)\Big\} \tag{3.3.11}$$

where $\gamma_1, \gamma_2 : [0,T] \to \mathbb{R}$ are Lipschitz $\gamma_1 \leq \gamma_2$ and $\gamma_1(0) = \gamma_2(0)$. Given two numbers $a, b \in \mathbb{R}$, there exists a unique entropy solution u in Ω with boundary conditions a on γ_1 and b on γ_2. As shown in [**37**], the solution u admits a representation in terms of the following length minimization problem. For every $(\bar{t}, \bar{x}) \in \Omega$ consider the length minimization problem

$$\min_{\gamma \in \mathcal{A}_{\bar{t},\bar{x}}} \int_0^{\bar{t}} \sqrt{1 + \gamma'(t)^2}\,dt, \quad \text{where} \quad \mathcal{A}_{\bar{t},\bar{x}} = \Big\{\gamma \in \mathrm{Lip}([0,\bar{t}]) \;\Big|\; \gamma_1 \leq \gamma \leq \gamma_2, \gamma(\bar{t}) = \bar{x}\Big\}. \tag{3.3.12}$$

For every $(\bar{t}, \bar{x}) \in \Omega$ the minimizing curve $\gamma^{\bar{t},\bar{x}}$ in (3.3.12) exists and is unique and we set $v(\bar{t}, \bar{x}) = \dot{\gamma}^{\bar{t},\bar{x}}(\bar{t}-)$.

We assume that $a \leq b$, being the opposite case analogous using concave envelopes instead of convex ones. To obtain the entropy solution u from the above minimization problem, first denote by $\mathrm{conv}_{[a,b]} f : [a,b] \to \mathbb{R}$ the convex envelope of f in $[a,b]$ and let $[\lambda^-, \lambda^+]$ the image of its derivative. The function $(\mathrm{conv}_{[a,b]} f)'$ is non decreasing, we denote its pseudo-inverse by $g : [\lambda^-, \lambda^+] \to [a,b]$. For every $(\bar{t}, \bar{x}) \in \Omega$, define

$$u(\bar{t}, \bar{x}) = \begin{cases} a & \text{if } v(\bar{t}, \bar{x}) \leq \lambda^-, \\ g(v(\bar{t}, \bar{x})) & \text{if } v(\bar{t}, \bar{x}) \in (\lambda^-, \lambda^+), \\ b & \text{if } v(\bar{t}, \bar{x}) \geq \lambda^+. \end{cases} \tag{3.3.13}$$

The following Proposition states that u is the unique solution of the boundary value problem in Ω and describes some of its properties.

Proposition 3.3.12 ([**37**]). *The function u defined by* (3.3.13) *is the unique entropy solution of the boundary value problem in Ω.*

Moreover, there exist two Lipschitz curves γ^-, γ^+ such that

(1) for every $t \in [0,T]$, $\gamma_1(t) \leq \gamma^-(t) \leq \gamma^+(t) \leq \gamma_2(t)$;

(2) $u(t,x) = a$ *for every* (t,x) *in*

$$\Omega^- \doteq \{(t,x) \in \Omega \mid \gamma_1(t) < x < \gamma^-(t)\}$$

and $u(t,x) = b$ *for every* (t,x) *in*

$$\Omega^+ \doteq \{(t,x) \in \Omega \mid \gamma^+(t) < x < \gamma_2(t)\};$$

(3) if $\gamma_1(t) < \gamma^-(t) < \gamma_2(t)$ *then* $\dot{\gamma}^-(t) = \lambda^-$ *and similarly if* $\gamma_1(t) < \gamma^+(t) < \gamma_2(t)$ *then* $\dot{\gamma}^+(t) = \lambda^+$*;*

(4) u *and* $f' \circ u$ *are strictly increasing in*

$$\Omega^m := \{(t,x) \in \Omega : \gamma^-(t) < x < \gamma^+(t)\}.$$

Moreover $f' \circ u$ *is locally Lipschitz in* Ω^m*;*

(5) for $\mathscr{L}^1$*-a.e.* $t \in (0,T)$ *such that* $\gamma^-(t) = \gamma_1(t)$*, it holds* $\dot{\gamma}^-(t) \geq \lambda^-$*; similarly for* $\mathscr{L}^1$*-a.e.* $t \in (0,T)$ *such that* $\gamma^+(t) = \gamma_2(t)$*, it holds* $\dot{\gamma}^+(t) \leq \lambda^+$*.*

We conclude the section with a Lemma that refines Proposition 3.3.12 when the boundaries γ_1, γ_2 are part of a complete family of boundaries of an entropy solution $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. For a proof see ([**37**], Remark 4.9).

LEMMA 3.3.13. *Let* u *be an entropy solution to* (3.3.1) *and let* $(\gamma_1, a), (\gamma_2, b) \in \mathcal{K}$ *be admissible boundaries of* u *such that for some* $T > 0$ *it holds* $\gamma_1(t) < \gamma_2(t)$ *for all* $t \in (0,T)$ *and* $\gamma_1(0) = \gamma_2(0)$*. Then, in the notation of Proposition 3.3.12,* $\Omega = \Omega^m$*.*

3.3.3. A Technical Lemma. Let u^-, u^+ be a entropy admissible shock with $u^- > u^+$ and assume that

$$\lambda \doteq \frac{f(u^-) - f(u^+)}{u^- - u^+} < f'(u^-). \tag{3.3.14}$$

Define

$$u_m \doteq \begin{cases} \max \mathscr{K}(u^-, u^+) & \text{if } \mathscr{K}(u^-, u^+) \neq \emptyset, \\ u^+ & \text{otherwise.} \end{cases}$$

By (3.3.14), one has, recalling the definition of $\mathscr{K}(u^-, u^+)$ in Definition 3.1.5, that

$$u_m < u^-. \tag{3.3.15}$$

Moreover, by continuity of f and f', for every $\alpha > 0$ arbitrarily small there exists $\varepsilon > 0$ such that the following holds:

- *for every* $\widetilde{u}^- \in u^- + B_\varepsilon(0)$*, there exists* $\widetilde{u}_m \in u_m + B_\alpha(0)$ *such that* $(\widetilde{u}^-, \widetilde{u}_m)$ *is an entropy admissible shock.*

Therefore from now on fix $\alpha > 0$ and $\varepsilon > 0$ accordingly. We prove a property of the solution $u(t,x) \doteq \mathcal{S}_t \overline{u}(x)$ to the Cauchy problem with an initial datum $\overline{u}(x)$ of the following form:

$$\overline{u}(x) = \begin{cases} u^-(x), & \text{if } x < 0, \\ u^+(x), & \text{if } x \geq 0 \end{cases}$$

where

$$\sup_{x<0} |u^-(x) - u^-| \leq \varepsilon, \qquad \sup_{x>0} |u^+(x) - u^+| \leq \varepsilon.$$

LEMMA 3.3.14. *With the notation introduced above, let $\sigma^-(t)$ denote the minimal forward characteristic starting at $(0,0)$, defined by*

$$\sigma^-(t) \doteq \sup\Big\{x \mid \exists\, (\gamma, \mathsf{w}) \in \mathcal{K} \text{ such that } \gamma(t) = x, \quad \gamma(s) < 0\Big\}.$$

Then

$$u(t, \sigma^-(t)+) \leq u_m + \alpha \qquad \forall\, t > 0.$$

PROOF. In a symmetric way to σ^-, define also the maximal characteristic starting $(0,0)$, that we call σ^+. Then it holds $\sigma^-(0) = \sigma^+(0) = 0$. Then by monotonicity of the family of boundaries, we remark that whenever $\gamma \in \mathcal{K}_\gamma$ and $\sigma^-(t) < \gamma(t) < \sigma^+(t)$, it must hold $\gamma(0) = 0$. By contradiction, assume that for some $t > 0$, it holds

$$u(t, \sigma^-(t)) > u_m + \alpha.$$

Then, by Proposition 3.3.8, there must be an admissible boundary $(\overline{\gamma}, u_m + \alpha)$ with

$$\sigma^-(t) < \overline{\gamma}(t) < \sigma^+(t).$$

Moreover, consider the boundary $(\widetilde{\gamma}, u^- - \varepsilon)$ and notice that $\widetilde{\gamma}(t) \in (\sigma^-(t), \sigma^+(t))$ and $\widetilde{\gamma}(0) = 0$. Then, by Proposition 3.3.12 it must hold $\widetilde{\gamma}(s) = \overline{\gamma}(s)$ for every $s \in (0,t)$, otherwise the solution would be constant in $(\widetilde{\gamma}(s), \overline{\gamma}(s))$, but this is prohibited by Lemma 3.3.13. Then there is an admissible boundary $(\overline{\gamma}, \mathsf{w})$ for every $\mathsf{w} \in (u_m + \alpha, u^- \varepsilon)$. We now show that $\overline{\gamma} = \sigma^-$. In fact take any $t > 0$ and consider a sequence of points $x_n \to \overline{\gamma}(t)^-$, together with boundaries (γ_n, w_n) with $\gamma_n(t) = x_n$. Then, with the same argument as above, we must have $\gamma_n(s) < \overline{\gamma}(s)$ for every $s \in (0,t)$. But by monotonicity of the family of boundaries this implies $\sigma^-(t) = \overline{\gamma}(t)$, as wanted. □

Using finite speed of propagation, the following Corollary readily follows.

COROLLARY 3.3.15. *Assume that u^-, u^+ are as above and that u is an entropy solution to* (3.1.1) *with*

$$u(s, x^-) = u^-, \qquad u(s, x^+) = u^+.$$

Then, if $\sigma^- : [s, +\infty) \to \mathbb{R}$ is the minimal characteristic exiting from (s, x), one has

$$\lim_{t \to s+} u(t, \sigma^-(t)+) \leq u_m.$$

3.4. Evolution of the Perturbation Density

In this section we prove that the perturbation density satisfies a continuity equation with the characteristic speed as velocity field, namely, we will prove Theorem 3.1.1.

Given a sequence u^h of entropy solutions, we consider set-valued velocity fields

$$\mathbf{\Lambda}^h(t,x) \doteq \Big\{f'(\hat{u}) \;\big|\; \hat{u} \in \mathrm{conv}\,\{\overline{u}(t,x), \underline{u}(t,x), \overline{u}^h(t,x), \underline{u}^h(t,x)\}\Big\} \subset \mathbb{R}. \qquad (3.4.1)$$

We also define a set valued vector field associated with the limiting entropy solution u as

$$\mathbf{\Lambda}(t,x) \doteq \Big\{f'(\hat{u}) \;\big|\; \hat{u} \in \mathrm{conv}\,\{\overline{u}(t,x), \underline{u}(t,x)\}\Big\} \subset \mathbb{R}. \qquad (3.4.2)$$

If $\Omega \subset \mathbb{R}^d$ and $\mathbf{\Theta} : \Omega \to \mathbb{R}$ is a set valued map, we define the graph of $\mathbf{\Theta}$ as

$$\mathsf{Graph}(\mathbf{\Theta}) = \{(y, v) \mid y \in \Omega, \quad v \in \mathbf{\Theta}(y)\}.$$

This should not be confused with the completed graph of a function introduced in (3.2.6).

The following Lemma shows that whenever a family of entropy solutions u^h converges to u in $\mathbf{L}^1$, then the graph of $\mathbf{\Lambda}^h$ converges to the graph of $\mathbf{\Lambda}(t,x)$ in the sense of Kuratowski, and is an immediate consequence of the convergence $\mathcal{G}(u^h) \xrightarrow{\mathsf{K}} \mathcal{G}(u)$ and of the continuity of f'.

LEMMA 3.4.1. *Let u^h be a sequence of equi-bounded entropy solutions to* (3.3.1) *such that*

$$u^h \to u \quad \textit{strongly in } \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}).$$

Then $\mathsf{Graph}(\mathbf{\Lambda}_h) \xrightarrow{\mathsf{K}} \mathsf{Graph}(\mathbf{\Lambda})$ *as subsets of* $(0,+\infty) \times \mathbb{R} \times \mathbb{R}$.

PROOF. By Definition of Kuratowski convergence (Definition 3.2.4), we need to show

$$\limsup_{h\to 0+} \mathsf{Graph}\,(\mathbf{\Lambda}_h) \subset \mathsf{Graph}\,(\mathbf{\Lambda}) \subset \liminf_{h\to 0+} \mathsf{Graph}\,(\mathbf{\Lambda}_h).$$

Consider a point $(t,x,v) \in \limsup \mathsf{Graph}\,(\mathbf{\Lambda}_h)$. Then there exists a subsequence $\{h_k\}_k$ and points $(t_k,x_k,v_k) \in \mathsf{Graph}\,(\mathbf{\Lambda}_{h_k})$ such that $(t_k,x_k,v_k) \to (t,x,v)$. By definition of $\mathbf{\Lambda}_{h_k}$, for every k there exist

$$u^1_k \in \mathrm{conv}\,\{\overline{u}(t_k,x_k), \underline{u}(t_k,x_k)\}, \qquad u^2_k \in \mathrm{conv}\,\{\overline{u}^{h_k}(t_k,x_k), \underline{u}^{h_k}(t_k,x_k)\}$$

and $\theta_k \in [0,1]$ such that $v_k = f'\big(\theta_k u^1_k + (1-\theta_k)u^2_k\big)$. Up to subsequences, we can assume that u^1_k, u^2_k, θ_k converge to limiting values u^1, u^2 and θ, respectively. Since f' is a continuous function, passing to the limit we obtain $v = f'\big(\theta u^1 + (1-\theta)u^2\big)$. Since $\mathcal{G}(u^h) \xrightarrow{\mathsf{K}} \mathcal{G}(u)$ (Proposition 3.3.10), we obtain that $u^i \in \mathrm{conv}\,\{\overline{u}(t,x), \underline{u}(t,x)\}$ for $i = 1,2$. This implies that $v = f'(\hat{u})$ where $\hat{u} \in \mathrm{conv}\,\{\overline{u}(t,x), \underline{u}(t,x)\}$, which means $(t,x,v) \in \mathsf{Graph}\,(\mathbf{\Lambda})$. This proves the first inclusion.

For the second inclusion, we first notice that since $\mathcal{G}(u^h) \xrightarrow{\mathsf{K}} \mathcal{G}(u)$, it immediately follows that

$$\begin{aligned}\mathsf{Graph}\,(\mathbf{\Lambda}) &= \big\{(t,x,v) \mid v = f'(\hat{u}), \quad \hat{u} \in \mathrm{conv}\,\{\overline{u}(t,x), \underline{u}(t,x)\}\\ &\subset \liminf_{h\to 0+} \big\{(t,x,v) \mid v = f'(\hat{u}), \quad \hat{u} \in \mathrm{conv}\,\{\overline{u}^h(t,x), \underline{u}^h(t,x)\}\big\}\\ &\subset \liminf_{h\to 0+} \big\{(t,x,v) \mid v = f'(\hat{u}), \quad \hat{u} \in \mathrm{conv}\,\{\overline{u}^h(t,x), \underline{u}^h(t,x), \overline{u}(t,x), \underline{u}(t,x)\}\big\}.\end{aligned}$$

This proves the Lemma. □

The following Lemma is a (classical) stability result for set valued maps.

LEMMA 3.4.2. *Let $\mathbf{\Theta}_k, \mathbf{\Theta} : (0,+\infty) \times \Omega \to [-M,M]$ be set valued maps with convex, compact values. Assume that*

$$\mathsf{Graph}(\mathbf{\Theta}_k) \xrightarrow{\mathsf{K}} \mathsf{Graph}(\mathbf{\Theta}) \qquad \textit{as } k \to +\infty$$

as subsets of $(0,+\infty) \times \Omega \times [-M,M]$. Let $\gamma_k : \mathbb{R}^+ \to \mathbb{R}$ be an integral curve of $\mathbf{\Theta}_k$, that is

$$\dot{\gamma}_k(t) \in \mathbf{\Theta}_k(t, \gamma_h(t)) \qquad \textit{for a.e. } t > 0.$$

Assume that

$$\gamma_k \to \gamma \quad \textit{locally uniformly in } \mathbb{R}^+.$$

Then γ is an integral curve of $\mathbf{\Theta}$.

PROOF. By contradiction, assume that t is a Lebesgue point of $\dot\gamma$ and that

$$\dot\gamma(t) \notin \mathbf{\Theta}(t,\gamma(t)).$$

By assumption, there exists $\overline{k} > 0$ and $\varepsilon > 0$ such that

$$\mathrm{dist}\big(\mathsf{Graph}\mathbf{\Theta}_k,\ (t,\gamma(t),\dot\gamma(t))\big) > \varepsilon \qquad \forall\ k \geq \overline{k}. \tag{3.4.3}$$

In particular,

$$\mathrm{dist}\Big(\bigcup_{k>\overline{k}}\{v \mid (s,y,v)\in \mathsf{Graph}\mathbf{\Theta}_k, \quad (s,y)\in B_{\varepsilon/2}(t,\gamma(t))\},\ \dot\gamma(t)\Big) > \varepsilon/2.$$

In fact, assume that $(s,y)\in B_{\varepsilon/2}(t,\gamma(t))$. By (3.4.3) we deduce that

$$\begin{aligned}\varepsilon < |(s,y,v)-(t,\gamma(t),\dot\gamma(t))| &\leq |(s,y,v)-(s,y,\dot\gamma(t))| + |(s,y,\dot\gamma(t))-(t,\gamma(t),\dot\gamma(t))|\\ &\leq |(s,y,v)-(s,y,\dot\gamma(t))| + \varepsilon/2 = |v-\dot\gamma(t)|+\varepsilon/2.\end{aligned}$$

But now we have for every $s\in(t-\varepsilon/(2M), t+\varepsilon/(2M))$

$$\dot\gamma_k(s)\in C_{\overline{k}} \doteq \bigcup_{k>\overline{k}}\{v \mid (s,y,v)\in\mathsf{Graph}\mathbf{\Theta}_k, \quad (s,y)\in B_{\varepsilon/2}(t,\gamma(t))\} \qquad \forall\ k>\overline{k}.$$

Notice that $C_{\overline{k}}$ is a convex set; since we have the weak convergence

$$\dot\gamma_k \rightharpoonup \dot\gamma \qquad \text{in } (t-\varepsilon/(2M), t+\varepsilon/(2M))$$

we obtain that

$$\dot\gamma(s)\in C_{\overline{k}} \qquad \text{for a.e } s\in(t-\varepsilon/(2M), t+\varepsilon/(2M)).$$

This is a contradiction since t is a Lebesgue point of $\dot\gamma$ and $\dot\gamma(t)\notin C_{\overline{k}}$. □

LEMMA 3.4.3. *Let u^h be a sequence of equi-bounded entropy solutions to* (3.3.1) *such that*

$$u^h \to u \quad \textit{strongly in } \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}).$$

Assume that γ_h are integral curves of $\mathbf{\Lambda}_h$ that converge locally uniformly to γ. Then γ is an integral curve of λ.

PROOF. We use Lemma 3.4.2, and Lemma 3.4.1, to obtain that γ is a integral curve of $\mathbf{\Lambda}$. By the structure of u (see Proposition 3.3.3) and by definition of $\mathbf{\Lambda}$, we readily deduce that

$$\dot\gamma(t)=\lambda(t,\gamma(t)) \qquad \text{for every } t \text{ such that } (t,\gamma(t))\notin J.$$

On the other hand, consider the set $I\subset\mathbb{R}^+$ of times

$$I \doteq \{t\in\mathbb{R}^+ \mid \gamma(t)\in J\}.$$

By Proposition 3.3.3, we know that

$$J\subset\bigcup_{j\in\mathbb{N}}\{(t,\sigma_j(t)) \mid t\in\mathbb{R}^+\}$$

where $\sigma_j:\mathbb{R}^+\to\mathbb{R}$ are countably many Lipschitz curves that satisfy

$$\dot\sigma_j(t)=\lambda(t,\sigma_j(t)) \qquad \text{for a.e. } t\in\mathbb{R}^+.$$

We then partition the set I as

$$I = \bigcup_j I_j, \qquad I_j = \{t \in \mathbb{R}^+ \mid \gamma(t) = \sigma_j(t)\}$$

and deduce that

$$\dot\gamma(t) = \dot\sigma_j(t) = \lambda(t, \sigma_j(t)) = \lambda(t, \gamma(t)) \qquad \text{for a.e. } t \in I_j$$

which implies

$$\dot\gamma(t) = \lambda(t, \gamma(t)) \qquad \text{for a.e. } t \in I.$$

This concludes the proof of the Lemma. □

We can use the previous Lemma to show the main Proposition of this section.

PROPOSITION 3.4.4. *Let u^h be a sequence of equi-bounded entropy solutions to* (3.3.1) *such that*

$$\begin{aligned} h^{-1}\big(u^h(0,\cdot) - u(0,\cdot)\big)^+ &\rightharpoonup \widehat\rho_{0,p} \in \mathscr{M}^+(\mathbb{R}) \\ h^{-1}\big(u^h(0,\cdot) - u(0,\cdot)\big)^- &\rightharpoonup \widehat\rho_{0,n} \in \mathscr{M}^+(\mathbb{R}) \end{aligned}$$

and

$$\begin{aligned} h^{-1}\big(u^h - u\big)^+ &\rightharpoonup \widehat\rho_p \qquad \textit{weakly in } \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}) \\ h^{-1}\big(u^h - u\big)^- &\rightharpoonup \widehat\rho_n \qquad \textit{weakly in } \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}). \end{aligned}$$

Then there is $g \in \mathscr{M}^-(\mathbb{R}^+ \times \mathbb{R})$ such that $\widehat\rho_p, \widehat\rho_n$ solve the Cauchy problems

$$\begin{aligned} \partial_t\widehat\rho_p + \partial_x\big(\lambda\widehat\rho_p\big) = g \qquad &\textit{in } \mathscr{D}'_{t,x}, \qquad \widehat\rho_p(0,\cdot) = \widehat\rho_{0,p} \\ \partial_t\widehat\rho_n + \partial_x\big(\lambda\widehat\rho_n\big) = g \qquad &\textit{in } \mathscr{D}'_{t,x}, \qquad \widehat\rho_n(0,\cdot) = \widehat\rho_{0,n}. \end{aligned} \tag{3.4.4}$$

PROOF. It is enough to prove the result in $[0,T] \times \mathbb{R}$, for every $T > 0$. First, notice that $\rho_h \doteq h^{-1}(u^h - u)$ satisfies the continuity equation

$$\partial_t\rho_h + \partial_x\big(\lambda_h(t,x)\rho_h\big) = 0$$

where

$$\lambda_h(t,x) = \begin{cases} \frac{f(u_h) - f(u)}{u^h - u} & \text{if } u^h \neq u \\ f'(u) & \text{if } u^h = u. \end{cases}$$

The vector field λ_h is defined only on continuity points of u^h, u, but since the density ρ_h is absolutely continuous the product $\lambda_h\rho_h$ is well defined. Then, by the $\mathbf{L}^1$ contraction given by the Theorem of Kruzhkov, the positive and negative parts

$$\widehat\rho_{h,p} \doteq h^{-1}(u^h - u)^+, \qquad \widehat\rho_{h,n} \doteq h^{-1}(u^h - u)^-$$

satisfy the continuity equation

$$\partial_t\widehat\rho_{h,p} + \partial_x\big(\lambda_h(t,x)\widehat\rho_{h,p}\big) = g_h$$

$$\partial_t\widehat\rho_{h,n} + \partial_x\big(\lambda_h(t,x)\widehat\rho_{h,n}\big) = g_h$$

where g_h are negative, finite measures uniformly bounded in h.

Let us focus now on the positive density $\widehat\rho_{h,p}$. By the (2) ⇒ (1) of the superposition principle (Theorem 3.2.3), there exists a measure $\boldsymbol{\eta}^h \in \mathscr{M}^+(\Gamma_T)$ on the space of curves

$$\Gamma_T \doteq \Big\{\gamma : I_\gamma \to \mathbb{R} \;\Big|\; \gamma \quad \text{Lipschitz curve}\Big\}$$

that is concentrated on integral curves of the velocity field λ^h such that

$$(\widehat{\rho}_{h,p})_t = (e_t)_\sharp \boldsymbol{\eta}^h \qquad \text{for all } t > 0.$$

Since the solutions u^h are equi-bounded and f is Lipschitz, then the measures $\boldsymbol{\eta}^h$ are concentrated on a set of uniformly Lipschitz curves with Lipschitz constant at most $L > 0$. We then claim that for every $\varepsilon > 0$, there exists a compact set

$$K \subset \Gamma_T$$

and $\overline{h} > 0$ such that

$$\boldsymbol{\eta}^h(\Gamma_T \setminus K) \leq \varepsilon \qquad \forall\, 0 < h \leq \overline{h}.$$

Since $(e_0)_\sharp \boldsymbol{\eta}^h = v_0^h$ is converging weakly to ρ_0 in $\mathscr{M}(\mathbb{R})$, there exists $M > 0$ and $\overline{h} > 0$ such that

$$(e_0)_\sharp \boldsymbol{\eta}^h(\mathbb{R} \setminus [-M, M]) \leq \varepsilon \qquad \forall\, 0 < h \leq \overline{h}.$$

We then define K as

$$K = \Big\{\gamma \in \Gamma_T \mid \gamma(0) \in [-M, M], \quad \sup_{t\in[0,T]} |\dot{\gamma}| \leq L\Big\}.$$

If $\gamma \in K$, it holds the $\mathbf{L}^\infty$ bound

$$\sup_{t\in[0,T]} |\gamma(t)| \leq |\gamma(0)| + LT$$

therefore, by the Ascoli-Arzelà theorem, the set K is compact in Γ_T. Moreover,

$$\begin{aligned}\boldsymbol{\eta}^h(\Gamma_T \setminus K) &= \boldsymbol{\eta}^h\big\{\gamma \in \Gamma_T \mid |\gamma(0)| > M\big\}\\ &= (e_0)_\sharp \boldsymbol{\eta}^h(\mathbb{R} \setminus [-M, M]) \leq \varepsilon \qquad \forall\, 0 < h \leq \overline{h}.\end{aligned}$$

Therefore we deduce that the measures $\boldsymbol{\eta}^h$ are tight. Then, up to a subsequence, by Prokhorov's Theorem 3.2.1, the measures $\boldsymbol{\eta}^h$ converge weakly to a limiting measure $\boldsymbol{\eta} \in \mathscr{M}^+(\Gamma_T)$ with the property

$$(\widehat{\rho}_{h,p})_t = (e_t)_\sharp \boldsymbol{\eta} \qquad \text{for all } t \in [0, T].$$

From the weak convergence $\boldsymbol{\eta}^h \rightharpoonup \boldsymbol{\eta}$, we deduce that for every γ in the support of $\boldsymbol{\eta}$, there exists a sequence of curves γ^h in the support of $\boldsymbol{\eta}^h$ converging uniformly to γ. Therefore, by Lemma 3.4.3, we deduce that $\boldsymbol{\eta}$ is concentrated on integral curves of the velocity field λ. Again by the superposition principle Theorem 3.2.3 (this time, we use the other implication, (1) $\Rightarrow$ (2)), we deduce that the first in (3.4.4) holds. □

As a Corollary, we obtain the Theorem about the evolution of the perturbation density ρ. In fact, we can decompose $\rho = \widehat{\rho}_p - \widehat{\rho}_n$ and using (3.4.4) we readily obtain:

THEOREM 3.4.5. *Let u^h be a sequence of equibounded entropy solutions to* (3.1.1) *such that* $\sup_h h^{-1}\|u^h(0,\cdot) - u(0,\cdot)\|_{\mathbf{L}^1} < +\infty$. *Assume that*

$$h^{-1}(u^h(0,\cdot) - u(0,\cdot)) \rightharpoonup \rho_0 \qquad \textit{weakly in } \mathscr{M}(\mathbb{R})$$

and that

$$h^{-1}(u^h - u) \rightharpoonup \rho \qquad \textit{weakly in } \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}).$$

Then ρ solves the Cauchy problem

$$\partial_t \rho + \partial_x(\lambda\rho) = 0, \qquad \rho(0,\cdot) = \rho_0. \tag{3.4.5}$$

3.5. Structure of ν

We prove some preliminary Lemmas.

LEMMA 3.5.1. *Let $(\widehat{\rho}, \nu)$ be a compatible pair. Then ν is supported on the closed set $\mathcal{G}(u)$.*

PROOF. Let $\varphi \in \mathcal{C}_c((0,+\infty) \times \mathbb{R} \times (0,1))$ be any continuous function with compact support such that

$$\operatorname{supp} \varphi \subset (0,+\infty) \times \mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{G}(u).$$

Since φ has compact support and $\mathcal{G}(u)$ is locally compact, there exists $\varepsilon > 0$ such that

$$\operatorname{dist}\big(\mathcal{G}(u), \operatorname{supp} \varphi\big) > \varepsilon.$$

By Proposition 3.3.10 and Lemma 3.2.8, there exists $\overline{h}_\varepsilon$ such that

$$\mathcal{G}(u^h) \cap K \subset \mathcal{G}(u)_\varepsilon \qquad \forall\, 0 \le h \le \overline{h}_\varepsilon.$$

where K is a compact set containing the support of φ. In particular this means that

$$\operatorname{supp}\big(\chi^h(t,x,v) - \chi(t,x,v)\big) \subset \mathcal{G}(u)_\varepsilon \qquad \forall\, 0 \le h \le \overline{h}_\varepsilon.$$

But this implies

$$\begin{aligned}\int_{\mathbb{R}^+ \times \mathbb{R} \times (0,1)} \varphi(t,x,v)\, \mathrm{d}\nu(t,x,v) &= \lim_{h \to 0+} \int_{\mathbb{R}^+ \times \mathbb{R} \times (0,1)} \varphi(t,x,v)\big(\chi^h(t,x,v) - \chi(t,x,v)\big)\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t \\ &= \lim_{h \to 0+} \int_{\mathcal{G}(u)_\varepsilon} \varphi(t,x,v)\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t = 0.\end{aligned} \tag{3.5.1}$$

□

An immediate consequence is the following.

COROLLARY 3.5.2. *Let $\{a_{t,x}\}_{t,x}$ be the disintegration associated with a compatible pair $(\widehat{\rho}, \nu)$. Then $a_{t,x}$ is supported on the interval $\overline{I}(u^-(t,x), u^+(t,x))$ for $\widehat{\rho}$-almost every (t,x).*

The following Lemma is a consequence of the fact that ν is the limit of uniform measures ν^h supported on the region between the graph of u and the graph of u^h

LEMMA 3.5.3. *Let $\{a_{t,x}\}_{t,x}$ be the disintegration associated with a compatible pair $(\widehat{\rho}, \nu)$. Then for $\widehat{\rho}$-almost every (t,x) the following holds. There exists a nonincreasing function*

$$\mathsf{g}_{t,x} \,:\, I(u^-, u^+) \to \mathbb{R}$$

such that

$$a_{t,x} \llcorner I(u^-, u^+) = \mathsf{g}_{t,x} \cdot \big[\mathscr{L}^1 \llcorner I(u^-, u^+)\big].$$

In particular $a_{t,x} \llcorner I(u^-, u^+)$ is absolutely continuous with respect to the Lebesgue measure.

REMARK 3.5.4. In the statement we are excluding the endpoints of the interval $I(u^-, u^+)$, for which a separate analysis will be necessary since in principle dirac masses can be present.

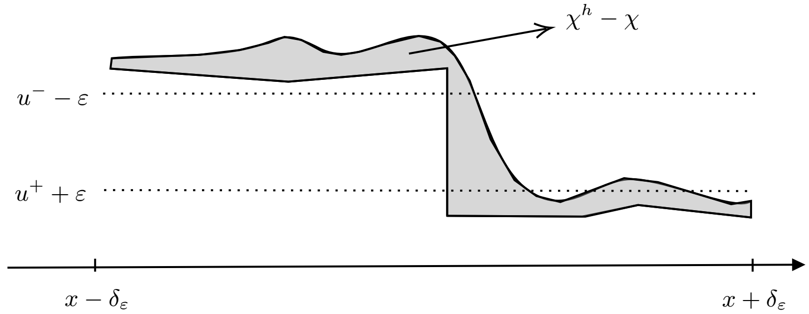

FIGURE 4. The situation considered in Lemma 3.5.3.

PROOF. Let $t \in \mathbb{R}^+ \setminus \mathcal{N}$ be fixed, where $\mathcal{N}$ as in Corollary 3.3.9, and is countable. If $u^-(t,x) = u^+(t,x)$ there is nothing to prove. Otherwise, it means that (t,x) is a shock point. Being the opposite case entirely symmetric assume that $u^- > u^+$. For every ε there exists $\delta_\varepsilon > 0$ such that

$$\begin{aligned} u(t,z) &> u^-(t,x) - \varepsilon \qquad \forall\, z \in (x-\delta_\varepsilon, x), \\ u(t,z) &< u^+(t,x) + \varepsilon \qquad \forall\, z \in (x, x+\delta_\varepsilon). \end{aligned} \tag{3.5.2}$$

Consider any test functions $\varphi(z), \psi(v)$ with $\varphi, \psi \geq 0$,

$$\operatorname{supp}\varphi \subset B_{\delta_\varepsilon}(x), \qquad \operatorname{supp}\psi \subset \big(u^+(t,x)+\varepsilon,\, u^-(t,x)-\varepsilon\big).$$

By (3.5.2) and the assumption on the support of φ, it holds

$$\int_{\mathbb{R}^2} \psi'(v)\varphi(y)\chi(t,y,v)\,\mathrm{d}y = \int \psi'(v)\,\mathrm{d}v \int_{-\infty}^{x} \varphi(y)\,\mathrm{d}y = 0$$

On the other hand, we have

$$\int_{\mathbb{R}^2} \psi'(v)\varphi(y)\chi^h(t,y,v)\,\mathrm{d}v\,\mathrm{d}y = \int_{\mathbb{R}} \varphi(y)\psi(u^h(t,y))\,\mathrm{d}y \geq 0$$

By taking the difference and dividing by h we obtain

$$\int_{\mathbb{R}^2} \psi'(v)\varphi(y)\frac{\chi^h(t,y,v)-\chi(t,y,v)}{h}\,\mathrm{d}y\,\mathrm{d}v \geq 0 \qquad \text{in } \mathscr{D}'_v\big(u^+ + \varepsilon, u^- - \varepsilon\big).$$

We pass to the limit as $h \to 0^+$ to obtain that

$$0 \leq \int \psi'(v)\varphi(y)\,\mathrm{d}\nu_t(y,v) = \widehat{\rho}_t(\{x\}) \int \psi'(v)\,\mathrm{d}a_{t,x}(v) \qquad \forall\, 0 \leq \psi \in \mathcal{C}^1_c(u^+ + \varepsilon, u^- - \varepsilon).$$

Repeating the argument for every $\varepsilon > 0$ proves that $a_{t,x}$ satisfies

$$\big(a_{t,x}(v)\big)' \leq 0 \qquad \text{in } \mathscr{D}'_v(u^+, u^-)$$

which implies the result. □

LEMMA 3.5.5. *Let $I \ni s \mapsto \sigma(s)$ be a shock curve defined in a time interval I and let u^-, u^+ be the left and right traces. The following hold.*

(a) *Let $I^- \subset I$ be defined by*

$$I^- \doteq \left\{ t \in I \;\Big|\; f'(u^-(t)) > \frac{f(u^-(t)) - f(u^+(t))}{u^-(t) - u^+(t)}, \quad u^-(t) > u^+(t) \right\}.$$

Then $a^+_{(t,x)}(\{u^-(t)\}) = 0$ and $a^-_{(t,x)}(\{u^+(t)\}) = 0$ for $\widehat{\rho}$-almost every $(t,x) \in \mathrm{Graph}\,\sigma_{|I^-} \subset \mathbb{R}^+ \times \mathbb{R}$.

(b) *Let $I^+ \subset I$ be defined by*

$$I^+ \doteq \left\{ t \in I \;\Big|\; f'(u^+(t)) < \frac{f(u^-(t)) - f(u^+(t))}{u^-(t) - u^+(t)}, \quad u^+(t) > u^-(t) \right\}.$$

Then $a^+_{(t,x)}(\{u^+(t)\}) = 0$ and $a^-_{(t,x)}(\{u^-(t)\}) = 0$ for $\widehat{\rho}$-almost every $(t,x) \in \mathrm{Graph}\,\sigma_{|I^+} \subset \mathbb{R}^+ \times \mathbb{R}$.

Proof. We prove only point (a), since the proof of (b) is entirely symmetrical. First, defining the measure $\overline{\nu}$ as

$$\overline{\nu} = (a^+_{t,x})_{|\{u^-(t)\}} \otimes \rho_{|\mathrm{Graph}\,\sigma_{|I^-}} \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$$

we note that (a) is equivalent to say that $\overline{\nu} \equiv 0$ (and similarly for $a^-_{t,x}$). From now on we focus only on showing that $\overline{\nu} = 0$, since the analogous statement for $a^-_{t,x}$ follows from a symmetry.

Let $\widetilde{I}^- \subset I^-$ be the set of Lebesgue points of $\mathbf{1}_{I^-}(t)$ and of the traces $u^-(t), u^+(t)$. We show in the Steps below that for every $s \in \widetilde{I}^-$ it holds

$$\lim_{\Delta t \to 0^+} \frac{\overline{\nu}\big([s, s+\Delta t] \times \mathbb{R}^2\big)}{\Delta t} = 0. \tag{3.5.3}$$

This will prove $\overline{\nu} \equiv 0$. From now on, $s \in \widetilde{I}^-$ will be fixed. We define the minimal characteristic starting at $(s, \sigma(s))$ by

$$\sigma^-(t) \doteq \sup \left\{ x \;\big|\; \exists\, (\gamma, \mathsf{w}) \in \mathcal{K} \text{ such that } \gamma(t) = x, \quad \gamma(s) < \sigma(s) \right\}, \qquad t > s$$

and we define the measure

$$\overline{\nu}^- \doteq (a_{t,x})_{|\{u(t,x-)\}} \otimes \rho_{|\mathrm{Graph}\,\sigma^-} \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}).$$

The advantage in working with the measure $\overline{\nu}^-$ rather then with $\overline{\nu}$ is that the left trace of σ^- is taken in $\mathbf{L}^\infty$, as will be clear later in the proof.

1. In this step, we show that σ coincides with σ^- outside a set of times whose measure is asymptotically smaller then Δt:

$$\lim_{\Delta t \to 0+} \Delta t^{-1} \mathscr{L}^1 \left\{ t \in [s, s+\Delta t] \;\big|\; \sigma(t) \neq \sigma^-(t) \right\} = 0. \tag{3.5.4}$$

Since $s \in \widetilde{I}^-$, it is in particular a Lebesgue point of the left trace u^- along the curve σ. By Corollary 3.3.15, we have

$$\lim_{t \to s+} u(t, \sigma^-(t)+) \leq u_m \tag{3.5.5}$$

where u_m is defined by (3.3.15). Since s is a Lebesgue point of u^-, it holds

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_s^{s+\Delta t}|u^-(t)-u^-|\,\mathrm{d}t=0. \tag{3.5.6}$$

For all Δt is sufficiently small, by (3.5.5), we have

$$\left(\sigma(t)\neq\sigma^-(t)\quad\Longrightarrow\quad u^-(t)\le u_m+\frac{|u^--u_m|}{2}\right)\qquad\forall\, t\in(s,s+\Delta t).$$

Therefore by Markov's inequality we get

$$\frac{1}{\Delta t}\cdot\mathscr{L}^1\Big\{t\in[s,s+\Delta t]\;\Big|\;\sigma(t)\neq\sigma^-(t)\Big\}\le\frac{2}{|u^--u_m|}\cdot\frac{1}{\Delta t}\cdot\int_s^{s+\Delta t}|u^-(t)-u^-|\,\mathrm{d}t.$$

Then, using (3.5.6), we obtain (3.5.4).

2. Here we show that

$$\lim_{\Delta t\to 0+}\frac{\overline{\nu}\big([s,s+\Delta t]\times\mathbb{R}^2\big)}{\Delta t}=\lim_{\Delta t\to 0+}\frac{\overline{\nu}^-\big([s,s+\Delta t]\times\mathbb{R}^2\big)}{\Delta t}=0. \tag{3.5.7}$$

Noting that for every t such that $\sigma(t)=\sigma^-(t)$ we have $\|\overline{\nu}_t\|_{\mathscr{M}}=\|\overline{\nu}_t^-\|_{\mathscr{M}}$, we can estimate

$$\begin{aligned}\frac{\overline{\nu}\big([s,s+\Delta t]\times\mathbb{R}^2\big)}{\Delta t}&=\Delta t^{-1}\cdot\int_s^{s+\Delta t}\|\overline{\nu}_t\|_{\mathscr{M}}\,\mathrm{d}t\\&=\Delta t^{-1}\cdot\int_{\left\{\substack{t\in(s,s+\Delta t)\\ \sigma^-(t)=\sigma(t)}\right\}}\|\overline{\nu}_t^-\|_{\mathscr{M}}\,\mathrm{d}t+\Delta t^{-1}\cdot\int_{\left\{\substack{t\in(s,s+\Delta t)\\ \sigma^-(t)\neq\sigma(t)}\right\}}\|\overline{\nu}_t\|_{\mathscr{M}}\,\mathrm{d}t\\&\le\Delta t^{-1}\cdot\int_s^{s+\Delta t}\|\overline{\nu}_t^-\|_{\mathscr{M}}\,\mathrm{d}t\\&\quad+\Delta t^{-1}\cdot\|\rho(0,\cdot)\|_{\mathscr{M}}\cdot\mathscr{L}^1\Big\{t\in[s,s+\Delta t]\;\Big|\;\sigma(t)\neq\sigma^-(t)\Big\}.\end{aligned}$$

Then passing to the limit as $\Delta t\to 0$ in the above expression, using Step 1., we obtain the claim.

In light of Step 2, the following steps are dedicated to the proof of

$$\lim_{\Delta t\to 0+}\frac{\overline{\nu}^-\big([s,s+\Delta t]\times\mathbb{R}^2\big)}{\Delta t}=0. \tag{3.5.8}$$

from which the Lemma will follow.

Before moving to the next step, we introduce some notation. We let $\widehat{\lambda}$ be an upper bound for the characteristic speed and define the trapezoids

$$\mathcal{Q}\doteq\Big\{(t,x)\;\Big|\;t\in(s,s+\Delta t),\quad\sigma(s)-\delta+\widehat{\lambda}(t-s)\le x\le\sigma(s)+\delta-\widehat{\lambda}(t-s)\Big\}.$$

where

$$\delta\doteq 3\cdot\widehat{\lambda}\cdot\Delta t.$$

Let $\alpha>0$ sufficiently small (to be chosen later). By Corollary 3.3.15, it is possible to

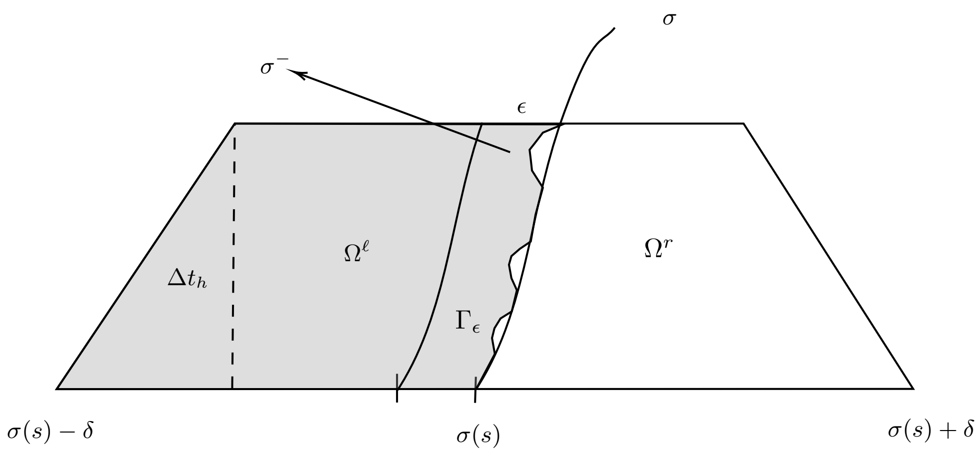

FIGURE 5. Trapezoids used in the proof of Lemma 3.5.5.

choose $\Delta t > 0$ such that

$$u(t, \sigma^-(t)+) \leq u_m + \alpha, \qquad \forall t \in (s, s + \Delta t). \tag{3.5.9}$$

We define the left and right parts of $\mathcal{Q}$ with respect to σ^-:

$$\Omega^\ell \doteq \Big\{(t,x) \in \mathcal{Q} \;\Big|\; x < \sigma^-(t)\Big\}, \qquad \Omega^r \doteq \Big\{(t,x) \in \mathcal{Q} \;\Big|\; x > \sigma^-(t)\Big\}$$

and the separating boundary

$$\Gamma \doteq \Big\{(t,x) \;\Big|\; x = \sigma^-(t), \quad s \leq t \leq s + \Delta t\Big\}.$$

Consider now the measures $\nu^{\ell,p} \in \mathscr{M}\big(\Omega_\ell\big)$, $\nu^{r,p} \in \mathscr{M}\big(\Omega_r\big)$ defined by

$$\nu^{\ell,p} \doteq \lim_{h \to 0} \nu_h^{\ell,p}, \qquad \nu^{r,p} \doteq \lim_{h \to 0} \nu_h^{r,p}$$

where $\nu_h^{\ell,p}$, $\nu_h^{r,p}$ are defined by

$$\nu_h^{\ell,p} \doteq \big[\nu^h_{|\Omega_\ell}\big]^+, \qquad \nu_h^{r,p} \doteq \big[\nu^h_{|\Omega_r}\big]^+.$$

We also define the corresponding projections on the (t,x) coordinates

$$\rho^{\ell,p} = p_\sharp \nu^{\ell,p}, \qquad \rho^{r,p} = p_\sharp \nu^{r,p}$$

with the corresponding disintegrations

$$\nu^{\ell,p} = a_{t,x}^{\ell,p} \otimes \rho^{\ell,p}, \qquad \nu^{r,p} = a_{t,x}^{r,p} \otimes \rho^{r,p}.$$

Notice that, since ν^h is absolutely continuous with respect to $\mathscr{L}^3$, we have

$$\big[\nu^h_{|\mathcal{Q}}\big]^+ = \nu_h^{\ell,p} + \nu_h^{r,p}, \qquad \big[\nu_{|\mathcal{Q}}\big]^+ \leq \nu^{\ell,p} + \nu^{r,p}. \tag{3.5.10}$$

Therefore in particular we have

$$a_{t,x}^+(\{u(t,x-)\}) \leq a_{t,x}^{\ell,p}(\{u(t,x-)\}) + a_{t,x}^{r,p}(\{u(t,x-)\}) \qquad \text{for } \widehat{\rho}\text{-almost } (t,x) \in \Gamma. \tag{3.5.11}$$

3. In this step we prove that

$$a_{t,x}^{r,p}(\{u(t,x-)\}) = 0 \qquad \text{for } \widehat{\rho}\text{-almost } (t,x) \in \Gamma. \tag{3.5.12}$$

Notice that from (3.5.12) and (3.5.11) it follows that

$$a^{+}_{t,x}(\{u(t,x-)\}) \leq a^{\ell,p}_{t,x}(\{u(t,x-)\}) \qquad \text{for } \widehat{\rho}\text{-almost } (t,x) \in \Gamma. \tag{3.5.13}$$

Fix $t \in (s, s+\Delta t)$ and recall that by (3.5.9), we know $u(t, \sigma^-(t)+) \leq u^m + \alpha$. Let $\delta > 0$ be so small that

$$|u(t,y) - u_m| \leq \frac{|u^- - u_m|}{4}, \qquad \forall\, y \in (\sigma^-(t), \sigma^-(t)+\delta)$$

$$|u(t,y) - u^-| \leq \frac{|u^- - u_m|}{4}, \qquad \forall\, y \in (\sigma^-(t)-\delta, \sigma^-(t))$$

Take $\varphi(y), \psi(v)$ positive test functions such that

$$\operatorname{supp}\varphi \subset B_\delta(\sigma^-(t)), \qquad \operatorname{supp}\psi \subset \Big(u_m + \frac{|u^- - u_m|}{4}, +\infty\Big).$$

Then, as in the proof of Lemma 3.5.3, we obtain

$$0 \leq \lim_{h\to 0+} \int_{\mathbb{R}^2} \psi'(v)\varphi(y)\, \mathrm{d}\nu_h^{r,p} \leq \widehat{\rho}_t(\{x\}) \int \psi'(v)\, \mathrm{d}a^{r,p}_{t,x}(v)$$

$$\text{in } \mathscr{D}'_v\Big(u_m + \frac{|u^- - u_m|}{4}, +\infty\Big).$$

It then follows that

$$(a^{p,r}_{t,x})_{|\left(u_m + \frac{|u^- - u_m|}{4}, +\infty\right)} \ll \mathscr{L}^1 \llcorner \Big(u_m + \frac{|u^- - u_m|}{4}, +\infty\Big)$$

is absolutely continuous for $\widehat{\rho}$-almost every $(t,x) \in \Gamma$. In particular this implies

$$a^{r,p}_{t,x}(\{u(t,x-)\}) = 0 \qquad \text{for } \rho^{r,p}\text{-almost } (t,x) \in \Gamma.$$

This proves the claim.

3. *(Transversality argument)* Owing to (3.5.13) if we can show that

$$a^{\ell,p}_{t,x}(\{u(t,x-)\}) = 0 \qquad \text{for } \rho^{\ell,p}\text{-almost } (t,x) \in \Gamma \tag{3.5.14}$$

the Lemma is proved. In this step, will prove the slightly stronger statement

$$\rho^{\ell,p}(\Gamma) = 0 \tag{3.5.15}$$

via a transversality argument. Consider

$$\rho_h^{\ell,p} \doteq \big[(\rho_h)_{|\Omega^\ell}\big]^+$$

and notice that $\rho_h^{\ell,p} \rightharpoonup \rho^{\ell,p}$. Recall that $\rho_h^{\ell,p}$ satisfies a continuity equation with a negative source term $g \in \mathscr{M}^-$

$$\partial_t \rho_h^{\ell,p} + \partial_x\Big(\frac{f(u^h) - f(u)}{u^h - u} \cdot \rho_h^{\ell,p}\Big) = g \qquad \text{in } \mathscr{D}'_{t,x}(\Omega^\ell). \tag{3.5.16}$$

By (3.5.9), $u(t, \sigma^-(t)-) \in u^- + B_\varepsilon(0)$ and Corollary 3.3.15 we find that the speed $\dot{\sigma}^-$ satisfies

$$\dot{\sigma}^-(t) \leq \lambda + \mathcal{O}(1)\cdot(\varepsilon + \alpha) \qquad \forall\, t \in (s, s+\Delta t). \tag{3.5.17}$$

Consider a thin strip of width ϵ, with $\Delta t \gg \epsilon > 0$ expanding to the left of σ^-:

$$\Gamma_\epsilon = \Big\{(t,x) \in \Omega^\ell \;\Big|\; \sigma^-(t) - \epsilon \leq x \leq \sigma^-(t)\Big\}.$$

Notice that by Proposition 3.3.10 there exists $\overline{h} > 0$ such that $u^h(t,x) \leq u(t,x) + \varepsilon$ for all $(t,x) \in \Omega^\ell$. Then, up to choosing α, ε sufficiently small, there exists a positive constant $c_0 > 0$ (depending only on the flux and on the states (u^-, u^+)) such that for every $0 < h < \overline{h}$, using also (3.5.17),

$$\frac{f(u^h(t,x)) - f(u(t,x))}{u^h(t,x) - u(t,x)} \geq c_0 + \dot{\sigma}^-(t) \qquad \text{for a.e. } (t,x) \in \Omega^\ell. \tag{3.5.18}$$

This means that the speed appearing in the continuity equation (3.5.16) is *transversal* to the speed of the curve σ^-. Then, by a standard transversality argument, we have the bound

$$\rho_h^{\ell,p}(\Gamma_\epsilon) \leq \frac{\epsilon}{c_0} \cdot \sup_{h>0} \|\rho_h^{\ell,p}(s,\cdot)\| \qquad \text{for all } 0 < h < \overline{h}. \tag{3.5.19}$$

Passing to the limit $h \to 0$, we obtain

$$\rho^{\ell,p}(\Gamma) \leq \rho^{\ell,p}(\Gamma_\epsilon) = \lim_{h\to 0} \rho_h^{\ell,p}(\Gamma_\epsilon) \leq \mathcal{O}(1) \cdot \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this concludes the proof. □

The following Lemma states that $a_{t,x} \in \mathscr{M}(\mathbb{R})$ must be distributed in such a way that the mean speed of particles is equal to the characteristic speed (or Rankine-Hugoniot speed).

LEMMA 3.5.6. *Let $\{a_{t,x}\}_{t,x}$ be the disintegration associated with an admissible pair $(\widehat{\rho}, \nu)$. Then*

$$\int_{\mathbb{R}} f'(v)\, \mathrm{d}a_{t,x}(v) \;=\; \lambda(t,x) \qquad \text{for } \widehat{\rho}\text{-almost every } (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.5.20}$$

PROOF. By Theorem 3.1.1 ρ solves the continuity equation

$$\partial_t \rho + \partial_x\big(\lambda\rho\big) = 0 \qquad \text{in } \mathscr{D}'_{t,x}$$

Moreover, ν_h satisfies the equation

$$\partial_t \nu_h + \partial_x\big(f'(v)\nu_h\big) = \partial_v\Big(\frac{\mu_h - \mu}{h}\Big) \qquad \text{in } \mathscr{D}'_{t,x,v}$$

Integrating in v and passing to the limit as $h \to 0$ we obtain

$$\partial_t \rho + \partial_x\Big(\int_{\mathbb{R}} f'(v)\, \mathrm{d}\nu(\cdot,\cdot,v)\Big) = 0 \qquad \text{in } \mathscr{D}'_{t,x} \tag{3.5.21}$$

Notice that

$$\rho = \Big(\int_{\mathbb{R}} \mathrm{d}a_{t,x}(v)\Big)\widehat{\rho}$$

so that combining the equation for ρ with (3.5.21) we obtain that it must hold

$$\partial_x\left(\int_{\mathbb{R}} f'(v)\, \mathrm{d}\nu(\cdot,\cdot,v)\right) = \partial_x\big(\lambda\rho\big) \qquad \text{in } \mathscr{D}'_{t,x}$$

or equivalently

$$\int_{\mathbb{R}} f'(v)\, \mathrm{d}\nu(\cdot,\cdot,v) = \lambda\rho + c(t).$$

Since ν, ρ are finite measures, we deduce that $c(t) \equiv 0$, so that

$$\Big(\int_{\mathbb{R}} f'(v)\, \mathrm{d}a_{t,x}(v) \Big) = \lambda(t,x) \Big(\int_{\mathbb{R}} \mathrm{d}a_{t,x}(v) \Big) \qquad \text{for } \widehat{\rho}\text{-a.e. } (t,x) \in J. \tag{3.5.22}$$

□

The following Lemma states that the possible presence of positive masses in $\overline{u}(t,x)$ or negative masses in $\underline{u}(t,x)$ is irrelevant for the balance equation (3.5.23). Define the measures

$$\widetilde{a}_{t,x} \doteq [a_{t,x}]^+_{|[\underline{u}(t,x),\overline{u}(t,x))} - [a_{t,x}]^-_{|(\underline{u}(t,x),\overline{u}(t,x)]} \qquad \text{for } \widehat{\rho}\text{-almost every } (t,x) \in J.$$

LEMMA 3.5.7. *Let $\{a_{t,x}\}_{t,x}$ be the disintegration associated with an admissible pair $(\widehat{\rho}, \nu)$. Then*

$$\int_{\mathbb{R}} f'(v)\ \mathrm{d}\widetilde{a}_{t,x}(v) \ = \ \lambda(t,x) \cdot \int_{\mathbb{R}} \mathrm{d}\widetilde{a}_{t,x}(v) \qquad \textit{for } \widehat{\rho}\textit{-almost every } (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{3.5.23}$$

PROOF. For ρ-almost every $(t,x) \in J$, it holds

$$\max\{u(t,x+), u(t,x-)\} = \overline{u}(t,x), \quad \min\{u(t,x+), u(t,x-)\} = \underline{u}(t,x).$$

We claim that

$$\begin{aligned} &[a_{t,x}]^+(\{\overline{u}(t,x)\}) \cdot f'(\overline{u}(t,x)) = \lambda(t,x) \cdot [a_{t,x}]^+(\{\overline{u}(t,x)\}) \qquad \text{for } \widehat{\rho}\text{-a.e. } (t,x) \in J \\ &[a_{t,x}]^-(\{\underline{u}(t,x)\}) \cdot f'(\underline{u}(t,x)) = \lambda(t,x) \cdot [a_{t,x}]^-(\{\underline{u}(t,x)\}) \qquad \text{for } \widehat{\rho}\text{-a.e. } (t,x) \in J. \end{aligned} \tag{3.5.24}$$

In fact, by Lemma 3.5.5, for $\widehat{\rho}$ a.e. $(t,x) \in J$ such that $f'(\overline{u}(t,x)) \neq \lambda(t,x)$, it holds $a^+_{t,x}(\{\overline{u}(t,x)\}) = 0$, and similarly, for $\widehat{\rho}$ a.e. $(t,x) \in J$ such that $f'(\underline{u}(t,x)) \neq \lambda(t,x)$, it holds $a^-_{t,x}(\{\underline{u}(t,x)\}) = 0$.

To conclude the proof of the lemma, we subtract both equalities in (3.5.24) from (3.5.22), to obtain precisely (3.5.23). □

PROPOSITION 3.5.8. *Let $\{a_{t,x}\}_{t,x}$ be the disintegration associated with a compatible pair $(\widehat{\rho}, \nu)$. Then*

$$|D\,\mathsf{g}_{t,x}|\big(I(u(t,x^-), u(t,x^+)) \setminus \mathscr{K}(t,x)\big) \ = \ 0 \qquad \textit{for } \widehat{\rho}\textit{-a.e. } (t,x).$$

Moreover

$$a_{t,x}(\mathscr{E}(t,x)) \ = \ 0 \qquad \textit{for } \widehat{\rho}\textit{-a.e. } (t,x).$$

PROOF. Consider the set J^- of all $(t,x) \in J$ such that $u(t,x+) < u(t,x-)$. All the following statements hold for $\widehat{\rho}$-a.e. $(t,x) \in J^-$. Recalling Lemma 3.5.3, we can write

$$\widetilde{a}_{t,x} = [\mathsf{g}_{t,x} \cdot \mathscr{L}^1] \llcorner I(u^-, u^+) \ + \ c^+_{t,x} \cdot \delta_{u^+} - c^-_{t,x} \cdot \delta_{u^-} \qquad \text{for } \widehat{\rho}\text{-a.e. } (t,x) \in J^-.$$

Write $\mathsf{g}_{t,x} = \mathsf{g}^+_{t,x} - \mathsf{g}^-_{t,x}$. Since $\mathsf{g}^+_{t,x}$ is nonincreasing, we can decompose it as

$$\mathsf{g}^+_{t,x}(v) = \int_0^{+\infty} \mathbf{1}_{A^+_\vartheta}(v)\, \mathrm{d}\vartheta, \qquad \text{for a.e. } v \in (u^+,\, u^-)$$

where

$$A^+_\vartheta = \Big\{ v \in \mathbb{R}_v \ \mid \ \mathsf{g}^+_{t,x}(v) \geq \vartheta \Big\}$$

and most importantly the level sets are all connected intervals containing u^+ as initial point, say

$$A^+_\vartheta = (u^+,\, v^+_\vartheta), \qquad v_\vartheta \in [u^+, u^-]$$

Analogously, we decompose

$$\mathsf{g}^-_{t,x}(v) = \int_0^{+\infty} \mathbf{1}_{A^-_\vartheta}(v)\,\mathrm{d}\vartheta, \qquad \text{for a.e. } v \in (u^+,\, u^-)$$

where

$$A^-_\vartheta = \Big\{ v \in \mathbb{R}_v \;\mid\; \mathsf{g}^+_{t,x}(v) \geq \vartheta \Big\}$$

The level sets are all connected intervals containing u^- as final point, say

$$A^-_\vartheta = (v^-_\vartheta, u^-), \qquad v^-_\vartheta \in [u^+, u(t, x-)]$$

Then we can write

$$\begin{aligned} 0 = \int (f'(v) - \lambda(t,x))\,\mathrm{d}\widetilde{a}_{t,x}(v) = \int (f'(v) - \lambda(t,x))\mathsf{g}_{t,x}(v)\,\mathrm{d}v \\ + (f'(u^+) - \lambda(t,x)) \cdot c^+_{t,x} - (f'(u^-) - \lambda(t,x)) \cdot c^-_{t,x}. \end{aligned} \tag{3.5.25}$$

We claim that all terms in the right hand side are nonpositive, and therefore they must be all zero. In fact, using Fubini's Theorem we obtain

$$\begin{aligned} \int (f'(v) - \lambda(t,x))\mathsf{g}^+_{t,x}(v)\,\mathrm{d}v &= \int \int_0^{+\infty} (f'(v) - \lambda(t,x))\mathbf{1}_{A^+_\vartheta}(v)\,\mathrm{d}\vartheta\,\mathrm{d}v \\ &= \int_0^{+\infty} \int (f'(v) - \lambda(t,x))\mathbf{1}_{A^+_\vartheta}(v)\,\mathrm{d}v\,\mathrm{d}\vartheta. \end{aligned} \tag{3.5.26}$$

For any $\vartheta > 0$, by the entropy conditions (3.1.14) for the shock u^-, u^+ we find

$$\begin{aligned} \int (f'(v) - \lambda(t,x))\mathbf{1}_{A^+_\vartheta}(v)\,\mathrm{d}v &= \int_{u^+}^{v^+_\vartheta} (f'(v) - \lambda(t,x))\,\mathrm{d}v \\ &= f(v^+_\vartheta) - f(u^+) - \lambda(v^+_\vartheta - u^+) \leq 0. \end{aligned} \tag{3.5.27}$$

With the same calculations, we obtain also

$$\int (f'(v) - \lambda(t,x))\mathsf{g}^-_{t,x}(v)\,\mathrm{d}v \geq 0. \tag{3.5.28}$$

This implies that the first term in the right hand side of (3.5.25) is nonpositive. By the entropy conditions, we also obtain

$$f'(u^-) \geq \lambda(t,x) \geq f'(u^+),$$

therefore, also the second and third terms are nonpositive. But then this implies that all the terms in (3.5.25) are zero. In particular, the vanishing of the second and third term of (3.5.25) together with Lemma 3.5.5 yields the second part of the statement, while from the vanishing of the first term

$$f(v^\pm_\vartheta) - f(u^+) - \lambda(v^\pm_\vartheta - u^+) = 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } \vartheta. \tag{3.5.29}$$

which means

$$v^\pm_\vartheta \in \Big\{ v \in [u^+, u^-] \;\Big|\; f(v) - f(u^+) - \lambda(v - u^+) = 0 \Big\} \qquad \text{for } \mathscr{L}^1\text{-a.e. } \vartheta.$$

By the theory of BV functions (see [**6**]), this implies that $D\mathsf{g}_{t,x}$ is concentrated on $\mathscr{K}(u(t,x^-),u(t,x^+))$. The set J^+ of points (t,x) with $u(t,x^+) > u(t,x^-)$ is treated symmetrically, and this concludes the proof. □

3.6. Structure of Blow-Ups

In this section we study the structure of blow-ups made simultaneously with the limit $h \to 0^+$. The result will be a proof of Theorem 3.1.11.

In the following, we fix a point $(t,x) \in J$ with $t \notin \mathcal{N}$ in which (4) of Proposition 3.3.3 holds for u. We know that the complement of this points is $\mathscr{H}^1$-negligible. Consider the dilation maps centered at (t,x)

$$\alpha_h(s,y) \doteq \big(t + h(s-t),\, x + h(y-x)\big), \qquad h > 0$$

and the functions

$$V_h(s,\, y) \doteq u^h\big(\alpha_h(s,y)\big)$$

together with the time sections at $s = t$:

$$U_h(y) \doteq V_h(t,y).$$

By Proposition 3.3.10 every limit points $\overline{V} = \lim_{k\to\infty} V_{h_k}$, $\overline{U} = \lim_{k\to\infty} U_{h_k}$ satisfy

$$\operatorname{Im}\overline{V} \subset \overline{I}(u^-,u^+), \qquad \operatorname{Im}\overline{U} \subset \overline{I}(u^-,u^+). \tag{3.6.1}$$

The following Lemma relates the structure of the level sets of $\overline{U}$ with the probability measure $a_{t,x}$ of Theorem 3.1.6, in particular with its absolutely continuous part $\mathsf{g}_{t,x}$.

Lemma 3.6.1. *Let $\overline{U} : \mathbb{R} \to \mathbb{R}$ be such that there exists a sequence $h_k \to 0$ with*

$$\overline{U} = \lim_{k\to+\infty} U_{h_k} \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R}).$$

Then for a.e. $v \in \mathbb{R}$ it holds

$$\mathscr{L}^1\Big(\big\{y \in \mathbb{R} \;\big|\; \overline{u}(y) \le v \le \overline{U}(y)\big\}\Big) - \mathscr{L}^1\Big(\big\{y \in \mathbb{R} \;\big|\; \overline{U}(y) \le v \le \overline{u}(y)\big\}\Big) = \widehat{\rho}_t(\{x\})\, \mathsf{g}_{t,x}(v) \tag{3.6.2}$$

Proof. Consider the region delimited by the graph of U_{h_k} and the graph of $u(t, h_k(\cdot - x))$:

$$H_{h_k} \doteq \Big\{(y,v) \in \mathbb{R}^2 \,\big|\, u(t, x + h_k(y-x)) \le v \le U_{h_k}(y)\Big\}.$$
$$\widetilde{H}_{h_k} \doteq \Big\{(y,v) \in \mathbb{R}^2 \,\big|\, U_{h_k}(y) \le v \le u(t, x + h_k(y-x))\Big\}.$$

Let $p_v : \mathbb{R}_x \times \mathbb{R}_v \to \mathbb{R}_v$ be the canonical projection the second variable. We make the following trivial but key observation: for every $C > 0$ and for every $k \in \mathbb{N}$, one has

$$\begin{aligned} &(p_v)_\sharp\left(\nu^h_t \llcorner (x - Ch, x + Ch) \times \mathbb{R}\right) \\ &\quad = (p_v)_\sharp\Big((\mathbf{1}_{H_{h_k}} - \mathbf{1}_{\widetilde{H}_{h_k}})\mathscr{L}^2 \llcorner (x - C, x + C) \times \mathbb{R}\Big) \qquad \text{in } \mathscr{M}(\mathbb{R}_v) \end{aligned}$$

which holds because by definition one has

$$\mathbf{1}_{H_{h_k}}\mathscr{L}^2 - \mathbf{1}_{\widetilde{H}_{h_k}}\mathscr{L}^2 = (\beta_{h_k})_\sharp\, \nu^h_t$$

where β_h is the scaling only in the y-variable

$$\beta_h(y,v) = (x + h^{-1}(y-x), v) \qquad \forall y, v$$

so that in particular the distribution of mass in the variable v is unchanged. Next, by Lemma 3.3.11, for every $\varepsilon > 0$ there is a constant $\overline{C} > 0$ such that for every $C > \overline{C}$ it holds

$$\lim_{k\to\infty} \nu_t^{h_k} \llcorner \big(x - Ch_k,\, x + Ch_k\big) \times \big(u^+ + \varepsilon,\, u^- - \varepsilon\big) \;=\; \rho_t \llcorner \{x\} \,\times\, a_{t,x}\big(u^+ + \varepsilon,\, u^- - \varepsilon\big).$$

Then, we have

$$\begin{aligned}(p_v)_\sharp\, \rho_t \llcorner \{x\} \times a_{t,x} \llcorner \big(u^+ + \epsilon,\, u^- - \epsilon\big) &= (p_v)_\sharp \lim_{k\to\infty} \nu_t^{h_k} \llcorner \big(x - Ch_k,\, x + Ch_k\big) \times \big(u^+ + \epsilon,\, u^- - \epsilon\big) \\ &= \lim_{k\to\infty} (p_v)_\sharp \Big((\mathbf{1}_{H_{h_k}} - \mathbf{1}_{\widetilde{H}_{h_k}}) \mathscr{L}^2 \llcorner (x - C, x + C) \times \big(u^+ + \epsilon,\, u^- - \epsilon\big) \Big).\end{aligned}$$

Letting

$$H \;\doteq\; \Big\{ (y, v) \,\big|\, \overline{u}(y) \le v \le \overline{U}(y) \Big\}, \quad \widetilde{H} \;\doteq\; \Big\{ (y, v) \,\big|\, \overline{U}(y) \le v \le \overline{u}(y) \Big\}$$

we thus have obtained

$$\widehat{\rho}_t(\{x\}) \cdot a_{t,x} \llcorner \big(u^+ + \epsilon,\, u^- - \epsilon\big) = (p_v)_\sharp \Big((\mathbf{1}_H - \mathbf{1}_{\widetilde{H}}) \mathscr{L}^2 \llcorner (x - C, x + C) \times \big(u^+ + \epsilon,\, u^- - \epsilon\big) \Big)$$

This holds for every $\epsilon > 0$ and $C > \overline{C}$, therefore we have obtained

$$\widehat{\rho}_t(\{x\}) \cdot a_{t,x} \llcorner \big(u^+,\, u^-\big) \;=\; (p_v)_\sharp\, (\mathbf{1}_H - \mathbf{1}_{\widetilde{H}}) \mathscr{L}^2.$$

But this is equivalent to the statement in the Lemma. □

Corollary 3.6.2. *Let $\overline{U}$ be as in Lemma 3.6.1. Then*

$$\overline{U}(y) \;\in\; \overline{I}(u^-, u^+) \setminus \mathscr{I} \qquad \textit{for } \mathscr{L}^1\textit{-a.e. } y \in \mathbb{R}. \tag{3.6.3}$$

Proof. As observed in (3.6.1) we already know that $\operatorname{Im} \overline{U} \subset \overline{I}(u^-, u^+)$. Using (3.2.1), we deduce that if $E \subset I(u^-, u^+) \setminus \mathscr{K}(u^+, u^+)$,

$$\begin{aligned}\mathscr{L}^1\Big(\big\{ y \in \mathbb{R} \;\big|\; \overline{U}(y) \in E \big\} \Big) &= - \int_E \mathrm{d}\, D_v\, \mathscr{L}^1\Big(\Big\{ y \in \mathbb{R}^+ \;\big|\; v \le \overline{U}(y) \Big\} \Big) \\ &\quad + \int_E \mathrm{d}\, D_v\, \mathscr{L}^1\Big(\Big\{ y \in \mathbb{R}^- \;\big|\; v \ge \overline{U}(y) \Big\} \Big) \\ &= -\widehat{\rho}_t(\{x\}) \int_E \mathrm{d}\, D\mathtt{g}_{t,x}(v) = 0\end{aligned}$$

where the penultimate inequality holds by Lemma 3.6.1, and the last equality holds by Proposition 3.5.8. This proves the Corollary. □

Lemma 3.6.3. *Assume that u^-, u^+ are connected by an entropy admissible shock with speed λ. Assume that $\overline{V}(s, y)$, with $\overline{U}(y) = \overline{V}(0, y)$, is an eternal entropy solution in $\mathbb{R}^2$ such that*

(1) $\operatorname{Im} \overline{V} \subset \overline{I}(u^-, u^+)$;
(2) $\operatorname{Im} \overline{U} \subset \overline{I}(u^-, u^+) \setminus \mathscr{I}(u^-, u^+)$;
(3) $\liminf_{y\to-\infty} |\overline{U}(y) - u^-| = 0$ *and* $\liminf_{y\to+\infty} |\overline{U}(y) - u^+| = 0$.

Then $\overline{U}$ is monotone nondecreasing if $u^- < u^+$ and monotone nonincreasing if $u^- > u^+$ and

$$\overline{V}(s, y) \;=\; \overline{U}(y - \lambda\, s).$$

PROOF. Assume that $u^- > u^+$, being the opposite case entirely analogous. For any point $y \in \mathbb{R}$ the left/right limits $U_\pm \doteq \overline{U}(y\pm)$ exist by Proposition 3.3.3. We define γ^- as the minimal backward admissible boundary passing through y in the following way:

$$\gamma^- = \max\Big\{\gamma \;\Big|\; \exists\,(\gamma, \mathsf{w}) \in \mathcal{K}\,\text{with}\,\gamma(0) < y\Big\}$$

The maximum exists being $\mathcal{K}$ a closed set. In particular, γ^- can be obtained as a uniform limit $\gamma^- = \lim_n \gamma_n$, with $(\gamma_n, \mathsf{w}_n) \in \mathcal{K}$ and $\gamma_n(0) \to y^-$. Since $\overline{U}(y-) = U_-$, one necessarily has that $\mathsf{w}_n \to U_-$, and therefore by Proposition 3.3.7, (γ^-, U_-) is an admissible boundary for $\overline{V}$. Now choose $\widetilde{y} < y$ and consider any admissible boundary $(\widetilde{\gamma}, \widetilde{\mathsf{w}})$ with $\gamma(0) = \widetilde{y}$. By contradiction, assume that there exists some $T > 0$ such that $\gamma(-T) = \widetilde{\gamma}(-T)$. Notice that in this case, Proposition 3.3.8 gives the explicit solution in the region

$$\Omega \doteq \Big\{(t, x) \;\Big|\; \widetilde{\gamma}(t) \le x \le \gamma(t),\ -T \le t \le 0\Big\}.$$

If $\widetilde{\mathsf{w}} < U_-$, it follows from Proposition 3.3.8 that either $\overline{U}$ is constant in $(\widetilde{\gamma}(0), \gamma(0))$ or that $\operatorname{Im}\overline{U} \not\subset \overline{I}(u^-, u^+) \setminus \mathscr{I}(u^-, u^+)$ The second case cannot happen by our assumption, while the first case is excluded by Lemma 3.3.13. Then assume that $\widetilde{\mathsf{w}} \ge \overline{U}_-$. By monotonicity of the family of boundaries and by (2), there exists another boundary $(\overline{\gamma}, \overline{\mathsf{w}})$ such that $\overline{\mathsf{w}} \in \overline{I}(u^-, u^+) \setminus \mathscr{I}(u^-, u^+)$, $\overline{\gamma}(0) \in (\widetilde{\gamma}(0), \gamma(0))$ and such that for some $0 \le \overline{T} \le T$ it holds $\overline{\gamma}(-\overline{T}) = \gamma(-\overline{T})$. Then Proposition 3.3.8 readily implies that $\overline{U}$ is constant in the interval $(\overline{\gamma}(0), \gamma(0))$. Again by Lemma 3.3.13 this is not possible.

We thus have proved the following: for every $(\widetilde{\gamma}, \mathsf{w})$ with $\widetilde{\gamma}(0) < \gamma$, it holds $\widetilde{\gamma}(t) < \gamma(t)$ for all $t < 0$. On the other hand, since γ is the minimal boundary passing through $(0, y)$, every sequence γ_n with $\gamma_n(0) \to y-$ must converge locally uniformly to γ. Then, by (Lemma 4.1, [**37**]), γ is a straight line with speed λ. The exact same argument works for γ^+, so that in particular we obtain

$$\gamma^-(t) = \gamma^+(t) = y + t\,\lambda \qquad \forall\, t \le 0.$$

This shows that it must hold $\overline{U}_- \ge \overline{U}_+$. Moreover, this also shows that every admissible boundary is a straight line with speed λ for $t < 0$, which proves the second part of the Lemma. □

PROOF OF THEOREM 3.1.11. Any limit points $\overline{U}$, $\overline{V}$ of U_h, V_h satisfy (3.6.1), therefore (1) of Lemma 3.6.3 is satisfied. Moreover, (2) of Lemma 3.6.3 holds by Corollary 3.6.2, while (3) holds because $\overline{U} - \overline{u}$ is in $\mathbf{L}^1$. Therefore by Lemma 3.6.3, using also Lemma 3.6.1, we deduce that $\overline{V} = U_{\rho(\{x\})}[\mathsf{g}_{t,x}]$, as wanted. □

CHAPTER 4

Intermediate Domains for Scalar Conservation Laws

4.1. A Family of Metric Interpolation Spaces

Consider an open set $\Omega \subseteq \mathbb{R}^n$ and let X be a Banach space contained in the set $\mathbf{L}^0(\Omega)$ of Lebesgue measurable functions $f : \Omega \mapsto \mathbb{R}$. Let $0 < \alpha < 1$ be given. A distance function $d(\cdot,\cdot) : \mathbf{L}^0(\Omega) \times \mathbf{L}^0(\Omega) \to [0, +\infty]$ can be defined as follows. For any $\lambda \in]0,1]$, we begin by setting

$$d^\lambda(f,g) \doteq d^\lambda(f-g,0), \tag{4.1.1}$$

$$\begin{aligned} d^\lambda(f,0) \doteq \inf\Big\{C \geq 0 \;\Big|\; &\text{there exists } \widetilde{f} \in X \text{ such that} \\ &\|\widetilde{f}\|_X \leq C\,\lambda^{\alpha-1}, \quad \mathscr{L}^n\Big\{x \in \Omega\,;\, f(x) \neq \widetilde{f}(x)\Big\} \leq C\,\lambda^\alpha\Big\}. \end{aligned} \tag{4.1.2}$$

Finally, we define

$$d(f,g) \doteq \sup_{0<\lambda\leq 1} d^\lambda(f,g). \tag{4.1.3}$$

By possibly identifying couples of functions f, g, such that $d(f,g) = 0$, we claim that (4.1.3) yields a distance on the set of Lebesgue measurable functions $f \in \mathbf{L}^0(\Omega)$ for which $d(f,0) < \infty$. This is usually a strictly larger set than X.

Lemma 4.1.1. *Let $0 < \alpha < 1$ be given, and let $d(\cdot,\cdot)$ be as in (4.1.1)–(4.1.3). Then the following properties hold.*

(i) $d(f,g) = d(g,f) \geq 0$.
(ii) If $f \in X$, then $d(f,0) \leq \|f\|_X$.
(iii) $d(f,h) \leq d(f,g) + d(g,h)$.

Proof. **1.** Part (i) is trivial. If $f \in X$, choosing $\widetilde{f} = f$ in (4.1.2), we see that $d^\lambda(f,0) \leq \|f\|_X$ for every $\lambda > 0$. This yields (ii).

2. We now check that each $d^\lambda(\cdot,\cdot)$, $0 < \lambda \leq 1$, satisfies the triangle inequality. Toward this goal, let f_1, f_2 be measurable functions such that

$$d^\lambda(f_i, 0) = C_i, \qquad i = 1,2.$$

We then need to show that

$$d^\lambda(f_1, f_2) \doteq d^\lambda(f_1 - f_2, 0) \leq C_1 + C_2\,. \tag{4.1.4}$$

Given $\varepsilon > 0$, by assumption there exist functions $\widetilde{f}_1, \widetilde{f}_2 \in X$ such that

$$\|\widetilde{f}_i\|_X \leq C_i\lambda^{\alpha-1} + \varepsilon, \qquad \mathscr{L}^n\big\{x \in \Omega \;\mid\; \widetilde{f}_i(x) \neq f_i(x)\big\} \leq C_i\lambda^\alpha + \varepsilon, \qquad i = 1,2. \tag{4.1.5}$$

Then the function $\widetilde{f}_1 - \widetilde{f}_2$ satisfies

$$\|\widetilde{f}_1 - \widetilde{f}_2\|_X \leq (C_1 + C_2)\lambda^{\alpha-1} + 2\varepsilon,$$

$$\mathscr{L}^n\Big\{x \in \Omega \;\big|\; \widetilde{f}_1(x) - \widetilde{f}_2(x) \neq f_1(x) - f_2(x)\Big\} \le (C_1 + C_2)\lambda^\alpha + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves (4.1.4).

3. In turn, the triangle inequality (iii) follows from

$$\begin{aligned} d(f-g,0) &= \sup_{\lambda \in]0,1]} d^\lambda(f-g,0) \le \sup_{\lambda \in]0,1]} \Big(d^\lambda(f,0) + d^\lambda(g,0)\Big) \\ &\le \sup_{\lambda \in]0,1]} d^\lambda(f,0) + \sup_{\lambda \in]0,1]} d^\lambda(g,0) = d(f,0) + d(g,0). \end{aligned}$$

□

REMARK 4.1.2. One should keep in mind that, in general, the balls $\{g \in \mathbf{L}^0(\Omega)\,;\, d(g,f) \le r\}$ are not convex. Moreover, the function $f \mapsto d(f,0)$ is not a norm.

REMARK 4.1.3. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathbf{L}^0(\Omega)$ be a sequence of measurable functions and assume that

$$\sum_{n=1}^{\infty} f_n(x) \qquad \text{converges for a.e. } x \in \Omega.$$

Then $f = \sum_{n\in\mathbb{N}} f_n \in \mathbf{L}^0(\Omega)$ and it holds

$$d\,(f,0) \le \sum_{n=1}^{+\infty} d\,(f_n,0)\,.$$

In fact, it is sufficient to prove that

$$d^\lambda\,(f,0) \le \sum_{n=1}^{+\infty} d^\lambda\,(f_n,0)$$

for every $\lambda \in]0,1]$. Assume that $\sum_{n=1}^{+\infty} d^\lambda\,(f_n,0)$ is finite, and fix $\varepsilon > 0$. For every $n \in \mathbb{N}$ we choose a function $\widetilde{f}_n$ such that

$$\|\widetilde{f}_n\|_X \le \left(d^\lambda\,(f_n,0) + \frac{\varepsilon}{2^n}\right)\lambda^{\alpha-1}, \qquad \text{meas}\Big\{x \in \Omega\,;\; f_n(x) \neq \widetilde{f}_n(x)\Big\} \;\le\; \left(d^\lambda\,(f_n,0) + \frac{\varepsilon}{2^n}\right)\lambda^{\alpha} \tag{4.1.6}$$

Define $\widetilde{f} = \sum_{n=1}^{+\infty} \widetilde{f}_n$. Notice that this sum is well defined: in fact, one has

$$\sum_{n=1}^{+\infty} \widetilde{f}_n = \sum_{n=1}^{+\infty} (\widetilde{f}_n - f_n) + f_n$$

and $\sum_n f_n$ converges by assumption, while $\sum_n (\widetilde{f}_n - f_n)$ converges because, using that $\sum_n d^\lambda(f_n,0)$ is finite, we deduce from the second inequality in (4.1.6) that almost every $x \in \Omega$ belongs to at most a finite number of sets $\{f_n \neq \widetilde{f}_n\}$. Therefore we have:

$$\|\widetilde{f}\|_X \le \sum_{n=1}^{+\infty} \|\widetilde{f}_n\|_X \le \left(\sum_{n=1}^{+\infty} d^\lambda\,(f_n,0) + \varepsilon\right)\lambda^{\alpha-1},$$

$$\text{meas}\Big\{x \in \Omega\,;\; f(x) \neq \widetilde{f}(x)\Big\} \le \sum_{n=1}^{+\infty} \text{meas}\Big\{x \in \Omega\,;\; f_n(x) \neq \widetilde{f}_n(x)\Big\} \le \left(\sum_{n=1}^{+\infty} d^\lambda\,(f_n,0) + \varepsilon\right)\lambda^{\alpha},$$

and the claim follows by letting $\varepsilon \to 0$.

In connection with $(\mathbf{P}_\alpha)$ (recall (19), (20)), for $0 < \alpha < 1$ we consider the distances $d^\lambda(f,g)$ as in (4.1.1), where now

$$d^\lambda(f,0) \doteq \inf\Big\{C \geq 0\,;\ \text{there exists } \widetilde{f} \in \mathbf{L}^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R}) \text{ such that} \\ \text{Tot.Var.}\{\widetilde{f}\} \leq C\,\lambda^{\alpha-1}, \quad \text{meas}\{x \in \mathbb{R}\,;\ f(x) \neq \widetilde{f}(x)\} \leq C\,\lambda^\alpha\Big\}. \tag{4.1.7}$$

We notice that this is consistent with the definition in (4.1.2) for the Banach space $X = \mathbf{L}^1(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$ endowed with the norm Tot.Var.$\{\cdot\}$. Finally, given $\overline{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, we define

$$\|\overline{u}\|_{\mathcal{P}_\alpha} \doteq \sup_{0<\lambda\leq 1} d^\lambda(\overline{u},0) \tag{4.1.8}$$

and write $\overline{u} \in \mathcal{P}_\alpha$ if $\|\overline{u}\|_{\mathcal{P}_\alpha} < +\infty$. Notice that this holds provided that $\overline{u}$ satisfies the condition $(\mathbf{P}_\alpha)$. Throughout the following, we shall use $\|\overline{u}\|_{\mathcal{P}_\alpha}$ as a convenient notation. However, as already pointed out in Remark 4.1.2, one should be aware that $\|\cdot\|_{\mathcal{P}_\alpha}$ is not a norm.

Lemma 4.1.4. *Given $\overline{u} \in \mathbf{L}^0(\mathbb{R})$ we have that $\|\overline{u}\|_{\mathcal{P}_\alpha} < +\infty$ if and only if $\overline{u}$ satisfies the condition $(\mathbf{P}_\alpha)$. Moreover, it holds*

$$\inf\big\{C\ :\ \overline{u} \textit{ satisfies } (\mathbf{P}_\alpha) \textit{ with constant } C\big\} = \|\overline{u}\|_{\mathcal{P}_\alpha}.$$

Proof. Let $\overline{u}$ satisfy $(\mathbf{P}_\alpha)$ with a constant $C \geq 0$. We show that

$$d^\lambda(\overline{u},0) \leq C \qquad \forall\,\lambda \in]0,1]. \tag{4.1.9}$$

Let $V(\lambda) \subset \mathbb{R}$ be an open set satisfying (19)-(20). The open set $V(\lambda)$ is a countable union of disjoint open intervals

$$V(\lambda) = \bigcup_{k\geq 1}]a_k, b_k[\,,$$

we define a new function $\widetilde{f}$ by replacing $\overline{u}$ with an affine function on each interval $[a_j, b_j]$, namely

$$\widetilde{f}(x) = \begin{cases} \overline{u}(x) & \text{if} \quad x \notin \cup_k [a_k, b_k], \\ \dfrac{(b_j - x)\overline{u}(a_j) + (x - a_j)\overline{u}(b_j)}{b_j - a_j} & \text{if} \quad x \in [a_j, b_j]\,. \end{cases}$$

This implies

$$\text{Tot.Var.}\{\widetilde{f}\} = \text{Tot.Var.}\big\{\overline{u}\,;\ \mathbb{R} \setminus V(\lambda)\big\} \leq C\,\lambda^{\alpha-1}.$$

Using also (19), we deduce (4.1.9), so that in particular it holds

$$\|\overline{u}\|_{\mathcal{P}_\alpha} = \sup_{0<\lambda\leq 1} d^\lambda(\overline{u},0) \leq \inf\big\{C\ :\ \overline{u} \text{ satisfies } (\mathbf{P}_\alpha) \text{ with constant } C\big\}.$$

On the other hand, assume that $\overline{u} \in \mathcal{P}_\alpha$. Then, for every $0 < \lambda \leq 1$ fixed, letting $\widetilde{f}$ be as in (4.1.7) (with $f = \overline{u}$), and

$$V(\lambda) \doteq \{x \in \mathbb{R}\ :\ \widetilde{f}(x) \neq \overline{u}(x)\}$$

it follows that $\text{meas}\big(V(\lambda)\big) \ \le\ \|\overline{u}\|_{\mathcal{P}_\alpha}\,\lambda^\alpha$ and

$$\text{Tot.Var.}\big\{\overline{u}\,;\ \mathbb{R}\setminus V(\lambda)\big\} \le \text{Tot.Var.}\big\{\widetilde{f}\,;\mathbb{R}\big\}\ \le\ \|\overline{u}\|_{\mathcal{P}_\alpha}\,\lambda^{\alpha-1}. \tag{4.1.10}$$

Therefore $C = \|\overline{u}\|_{\mathcal{P}_\alpha}$ is an admissible constant in the definition of $(\mathbf{P}_\alpha)$, so that

$$\inf\big\{C\ :\ \overline{u} \text{ satisfies } (\mathbf{P}_\alpha) \text{ with constant } C\big\} \le \|\overline{u}\|_{\mathcal{P}_\alpha}$$

and this concludes the proof. □

4.2. Examples

We present here some examples, to motivate the results proved in later sections. We consider Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{4.2.1}$$

Throughout the following, we use the semigroup notation $t \mapsto S_t\overline{u}$ to denote the solution of (4.2.1) with initial data (12).

EXAMPLE 4.2.1. Fix $\beta > 0$ and consider the decreasing sequence of points $x_n = n^{-\beta}$, $n \ge 1$. As shown in Fig. 1, define the piecewise affine function

$$\overline{u}(x) = \begin{cases} 0 & \text{if} \quad x \notin\ [0,1]\,, \\ \frac{x-x_{n+1}}{x_n-x_{n+1}} & \text{if} \quad x_{n+1} < x < x_n\,. \end{cases} \tag{4.2.2}$$

We claim that this initial data lies in some of the subdomains $\widetilde{\mathcal{D}}_\alpha$, depending on the exponent β. Indeed, fix a time $t \in]0,1]$. Consider the position $x_k(t)$ of the shock which is initially located at x_k. By Oleinik's inequality, the total variation of the solution $u(t,\cdot)$ can be estimated by

$$\text{Tot.Var.}\big\{u(t,\cdot)\,;\ [0,x_k(t)]\big\}\ \le\ 2\frac{x_k(t)}{t}\,. \tag{4.2.3}$$

On the other hand, for $x > x_k(t)$, we can calculate explicitly the solution, to obtain the estimate

$$\begin{aligned} \text{Tot.Var.}\big\{u(t,\cdot)\,;\ [x_k(t),x_1(t)]\big\} &\le 2\frac{x_1(t)-x_k(t)}{(x_{k-1}-x_k)+t} + u(t,x_k(t)) - u(t,x_1(t)) \\ &\le 2\frac{x_1(t)-x_k(t)}{(x_{k-1}-x_k)+t} + 1\,. \end{aligned} \tag{4.2.4}$$

Observing that

$$x_k(t)\ \le\ x_k + t, \qquad\qquad x_1(t) - x_k(t) \le 1 + t,$$

from (4.2.3)-(4.2.4) we deduce

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \le 2\frac{x_k+t}{t} + 2\frac{1+t}{(x_{k-1}-x_k)+t} + 1 \le 5 + 2\frac{x_k}{t} + \frac{2}{(x_{k-1}-x_k)+t}\,. \tag{4.2.5}$$

Since we are assuming $x_k = k^{-\beta}$, the previous estimate yields

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \le 4 + \frac{2k^{-\beta}}{t} + \frac{2}{\beta k^{-\beta-1}+t}\,.$$

Here $k \geq 1$ is arbitrary. Choosing $k \approx t^{-\gamma}$, we obtain

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \leq \mathcal{O}(1)\cdot\left(\frac{t^{\beta\gamma}}{t} + \frac{1}{t^{\gamma(\beta+1)}+t}\right) = \mathcal{O}(1)\cdot\Big(t^{\beta\gamma-1} + t^{-\gamma(\beta+1)}\Big).$$

Here and throughout the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity. The two terms on the right hand side have similar magnitude if $\gamma = (1+2\beta)^{-1}$. With this choice, we obtain

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \leq \mathcal{O}(1)\cdot t^{-\frac{\beta+1}{2\beta+1}},$$

hence

$$\overline{u} \in \widetilde{\mathcal{D}}_\alpha, \qquad \text{with} \qquad \alpha = 1 - \frac{\beta+1}{2\beta+1} = \frac{\beta}{2\beta+1}$$

where we recall the definition of $\widetilde{\mathcal{D}}_\alpha$ in (15).

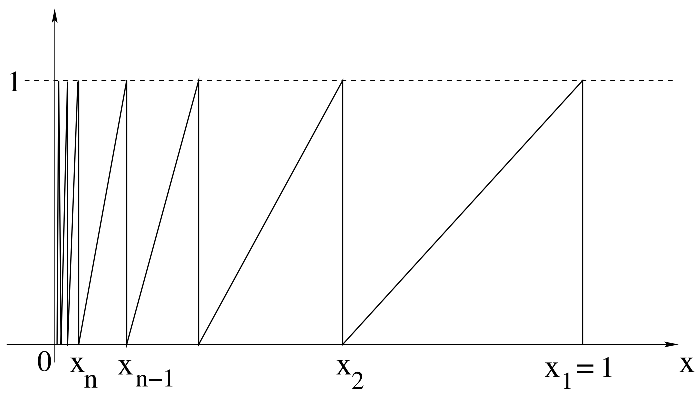

FIGURE 1. The initial data considered in Example 4.2.1.

In the next examples we consider initial data consisting of one or more triangular blocks. As shown in Fig. 2, left, the most elementary case is

$$w(0,x) = \overline{w}(x) = \begin{cases} h - \frac{2h}{\ell}\cdot|x-\ell/2| & \text{if } x \in [0,\ell], \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.6}$$

At time $t = \ell/(2h)$, a shock is created in the solution at the point $\ell > 0$. Characteristics originating from points $0 < x < \ell/2$ start impinging on the shock, and the solution has a right triangle shape:

$$w(t,x) = \begin{cases} \frac{2hx}{2ht+\ell} & \text{if } x \in [0, L(t)], \\ 0 & \text{otherwise} \end{cases} \tag{4.2.7}$$

Conservation of mass implies that the shock at time $t \geq \ell/(2h)$ is located at

$$L(t) \;=\; \sqrt{\frac{\ell}{2}\,(2\,h\,t+\ell)}. \tag{4.2.8}$$

Always for $t \geq \ell/(2h)$, we thus have

$$\text{Tot.Var.}\{S_t\overline{w}\} \;=\; 2\,p(t), \qquad p(t) \doteq h\sqrt{\frac{2\ell}{2ht+\ell}}. \tag{4.2.9}$$

We notice for later use that the (decreasing) function $p(t)$ satisfies the lower bound

$$p(t) \geq \sqrt{\frac{h\,\ell}{2t}} \qquad \text{for all} \quad t \geq \ell/2h. \tag{4.2.10}$$

and that the (increasing) function $L(t)$ satisfies the upper bound

$$L(t) \leq \sqrt{2\,\ell\,h\,t} \qquad \text{for all} \quad t \geq \ell/2h. \tag{4.2.11}$$

We also consider initial data containing packets of triangular blocks, shifted by different amounts so they do not overlap with each other. See Fig. 2, right.

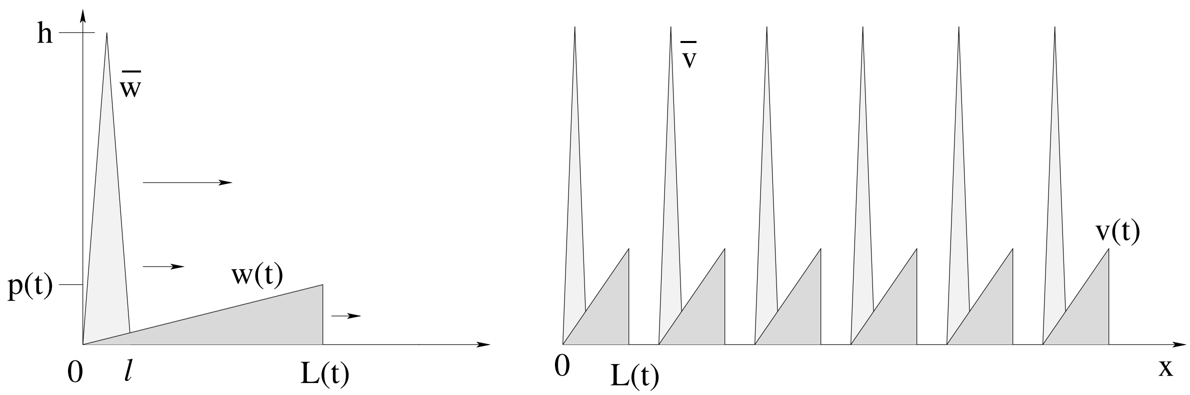

FIGURE 2. Left: the elementary solution to Burgers' equation considered at (4.2.6)-(4.2.7). Right: the superposition of several shifted copies of the same solution.

In the following proposition we denote by $\mathcal{C}^{0,\sigma}(\mathbb{R})$ the space of Holder functions $\overline{u} : \mathbb{R} \to \mathbb{R}$ with exponent $0 < \sigma < 1$, equipped with the norm

$$\|\overline{u}\|_{\mathcal{C}^{0,\sigma}} \doteq \|\overline{u}\|_{\mathcal{C}^0} + |\overline{u}|_{\mathcal{C}^{0,\sigma}}$$

$$|\overline{u}|_{\mathcal{C}^{0,\sigma}} \doteq \sup_{x<y} \frac{|\overline{u}(y) - \overline{u}(x)|}{|y-x|^\sigma}.$$

PROPOSITION 4.2.2. *There exists a compactly supported function $\overline{u} : \mathbb{R} \to \mathbb{R}$ with the following properties:*

(1) $\overline{u} \in \mathcal{P}_\alpha$ for every $0 < \alpha \leq 1/2$;
(2) $\overline{u} \in \mathbf{C}^{0,\sigma}(\mathbb{R})$ for every $0 < \sigma < 1$;
(3) $\overline{u} \notin \widetilde{\mathcal{D}}_\beta$ for any $0 < \beta < 1$. Namely:

$$\limsup_{t\to 0+} t^{1-\beta} \cdot \mathit{Tot.Var.}\{S_t\overline{u}\} = +\infty \qquad \forall\, 0 < \beta < 1. \tag{4.2.12}$$

REMARK 4.2.3. The function $\overline{u}$ constructed in Proposition 4.2.2 does not belong to any $\mathcal{P}_\alpha$ if $1/2 < \alpha < 1$. This suggests that $\alpha = 1/2$ is a critical exponent for the decay of solution with initial data in $\mathcal{P}_\alpha$. In fact, this surprising behavior will be later confirmed by Theorem 4.5.1.

REMARK 4.2.4. By part 2. of Proposition 4.2.2 and by the embedding $\mathbf{C}^{0,\sigma}(\mathbb{R}) \hookrightarrow W^{s,p}_{loc}(\mathbb{R})$ for every $0 < s < \sigma < 1$, $p \geq 1$, we obtain that

$$W^{s,p}(\mathbb{R}) \not\subset \widetilde{\mathcal{D}}_\beta \qquad \text{for every } 0 < s < 1,\ 1 \leq p < \infty \text{ and } 0 < \beta < 1.$$

On the other hand, the inclusion $W^{\alpha,1}(\mathbb{R}) \subset \mathcal{D}_\alpha$ does hold for every $0 < \alpha < 1$, as proved in Proposition 4.3.2.

The proof of Proposition 4.2.2 is based on the following lemma.

LEMMA 4.2.5. *For every fixed $t \in (0,1)$, there exists a function $\widehat{u} : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\begin{cases} \widehat{u}(x) \in [0,1] & \text{if } x \in [0,1], \\ \widehat{u}(x) = 0 & \text{if } x \notin [0,1], \end{cases} \tag{4.2.13}$$

$$\|\widehat{u}\|_{\mathcal{P}_\alpha} \leq C_0\,, \qquad \forall\, 0 < \alpha \leq 1/2, \qquad\qquad \text{Tot.Var.}\{S_t \widehat{u}\} \geq \frac{1}{C_0\, t}\,. \tag{4.2.14}$$

$$\|\widehat{u}\|_{\mathcal{C}^{0,\sigma}} \leq C_\sigma \doteq \left(e^{2\left(\frac{1}{1-\sigma}\right)}\right)^{(1-\sigma)e^{-1}} \qquad \text{for all } \sigma \in]0,1[. \tag{4.2.15}$$

Here C_0 is a constant independent of t.

We postpone the proof of the lemma, and begin by showing that it implies the previous proposition.

PROOF OF PROPOSITION 4.2.2. Let $(t_j)_{j\geq 1}$ be a sequence decreasing to 0 sufficiently fast (to be specified later). Let

$$x_0 = 0, \quad x_j \doteq \sum_{k=0}^{j-1} 2 \cdot 2^{-k}, \qquad I_j \doteq [x_j, x_{j+1}], \qquad j = 0,1,2\dots$$

Let $\widehat{u}_j$ be a function satisfying the properties (4.2.13)-(4.2.14) in Lemma 4.2.5 for $t = t_j$. Consider the rescaled functions

$$\overline{u}_j(x) \doteq 2^{-j} \widehat{u}_j\big(2^j(x - x_j)\big), \qquad j = 0,1,2\dots$$

Notice that $\|\overline{u}_j\|_{\mathcal{P}_\alpha} = 2^{-j}\|\widehat{u}_j\|_{\mathcal{P}_\alpha}$. By Lemma 4.2.5, the corresponding rescaled solution of Burgers' equation satisfies

$$\text{Tot.Var.}\{S_{t_j}\overline{u}_j\} = 2^{-j}\, \text{Tot.Var.}\{S_{t_j}\widehat{u}_j\} \geq \frac{1}{C_0\, 2^j\, t_j}\,.$$

Here we used the fact that if u is a solution to Burgers' equation then the functions defined by

$$u_\lambda(t,x) \doteq \frac{1}{\lambda} u(t, \lambda x)$$

solve Burgers' equation as well. Define the initial datum

$$\overline{u} = \sum_{j=0}^{\infty} \overline{u}_j. \tag{4.2.16}$$

Notice that

$$\text{supp}\, \overline{u}_j(t,\cdot) \subseteq [x_j, x_j + 2^{-j}(1+t)] \subset I_j \qquad\qquad \text{for } t \in [0,1].$$

In particular for every $i \neq j$ and $t < 1$ the supports of $S_t\overline{u}_i$ and $S_t\overline{u}_j$ remain disjoint. Choosing $t_j < 2^{-j}$ we thus obtain

$$\big(S_{t_j}\overline{u}\big)(x) = \big(S_{t_j}\overline{u}_j\big)(x) \qquad \forall x \in I_j\,.$$

Since every $\widehat{u}_j$ satisfies (4.2.14), for $0 < \alpha \leq 1/2$, by the triangle inequality of Remark 4.1.3 it now follows

$$\|\overline{u}\|_{\mathcal{P}_\alpha} \leq \sum_{j=0}^{\infty} \|\overline{u}_j\|_{\mathcal{P}_\alpha} \leq C_0 \sum_{j=0}^{\infty} 2^{-j} < +\infty.$$

This implies that $\overline{u}$ satisfies assumption 1. of Proposition 4.2.2. By (4.2.15) of Lemma 4.2.5, we infer that

$$\|\overline{u}_j\|_{\mathcal{C}^{0,\sigma}} \leq 2^{-(1-\sigma)j} \, \|\widehat{u}_j\|_{\mathcal{C}^{0,\sigma}} \leq C_\sigma \, 2^{-(1-\sigma)j}.$$

Therefore since the supports of $\overline{u}_j$ are all disjoint, the function $\overline{u}$ belongs to all the Hölder spaces $\mathcal{C}^{0,\sigma}$ for $0 < \sigma < 1$. Finally, choosing for example $t_j = \exp(-2^j)$, for any $0 < \beta < 1$ we obtain

$$\begin{aligned} \limsup_{t \to 0+} t^\beta \, \text{Tot.Var.}\{S_t \overline{u}\} &\geq \lim_{j \to +\infty} t_j^\beta \, \text{Tot.Var.}\{S_{t_j} \overline{u}\} \geq \lim_{j \to +\infty} t_j^\beta \, \text{Tot.Var.}\{S_{t_j} \overline{u}_j\} \\ &\geq \frac{1}{C_0 \, 2^j \, t_j^{1-\beta}} \to +\infty, \end{aligned}$$

completing the proof of 3. of Proposition 4.2.2. □

Proof of Lemma 4.2.5. **1.** Let $t > 0$ be a fixed positive time. Given a sequence of positive numbers $\{\ell_k\}_k$ satisfying $\ell_k \leq 2^{-k}$ (to be chosen later), and an integer $k \geq 1$, we construct a packet of triangular waves by setting

$$v_k = \sum_{j=1}^{N_k} w_k^j \,, \tag{4.2.17}$$

where $w_k^j = \overline{w}_k(x - j\, L_k)$ are translations by $j\, L_k > 0$ of elementary triangular blocks as in (4.2.6), with width ℓ_k and height

$$h_k \doteq 2^k \ell_k \leq 1 \,. \tag{4.2.18}$$

The distance L_k between the supports of two blocks is chosen large enough so that the supports of the corresponding solutions remain disjoint up to the given time t. By (4.2.11), it is sufficient to separate these elementary blocks by a distance $L_k \doteq \sqrt{2h_k \, \ell_k \, t}$, see Figure 3. With this choice, the support of v_k is contained inside an interval I_k with length

$$\mathscr{L}^1(I_k) \leq N_k L_k = \sqrt{2t}\, 2^{k/2} N_k \ell_k \,. \tag{4.2.19}$$

Moreover, since the elementary blocks do not interact with each other up to time t, assuming $\ell_k/(2h_k) \leq t$, by (4.2.10), one has

$$\text{Tot.Var.}\{S_t v_k\} = 2N_k p_k(t) \geq \sqrt{2} N_k \cdot \sqrt{\frac{h_k \ell_k}{t}} = \frac{\sqrt{2} \cdot 2^{k/2} \, \ell_k}{\sqrt{t}} \, N_k \,. \tag{4.2.20}$$

We now choose

$$\ell_k \doteq 2^{-1} \cdot 2^{-k/2} \cdot k^{-k}, \qquad\qquad N_k \doteq 2^{-1} \cdot 2^{-k/2} \cdot \ell_k^{-1} = k^k \tag{4.2.21}$$

and notice that, by (4.2.18), this implies

$$\text{Tot.Var.}\{v_k\} = 2N_k h_k = 2^{k/2}.$$

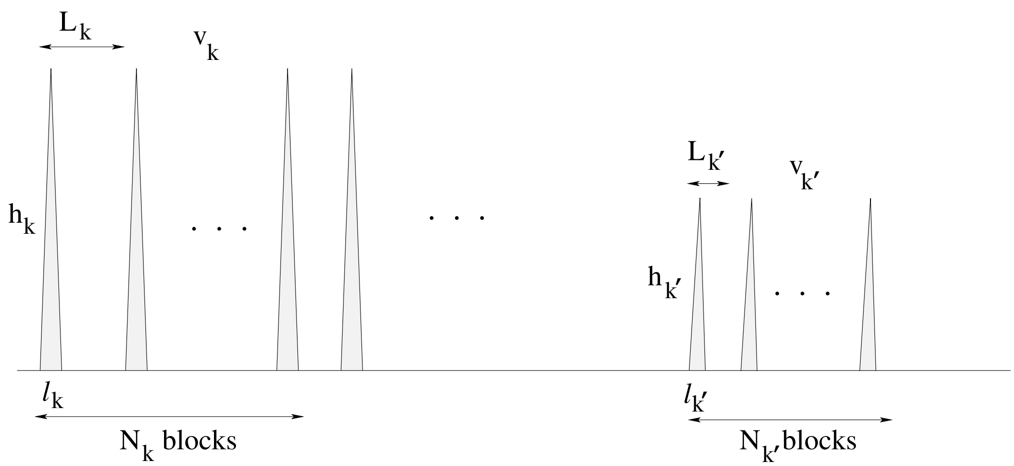

FIGURE 3. A family of triangular wave packets.

2. We put next to each other all the wave packets v_k, for $k_1 \le k \le k_2$, where $k_1, k_2 \in \mathbb{N}$ will be chosen later (see Figure 3), and define

$$\widehat{u} \doteq \sum_{k=k_1}^{k_2} v_k(x - \widetilde{L}_k)\,. \tag{4.2.22}$$

where $\widetilde{L}_k$

$$\widetilde{L}_k = \sum_{j=k}^{+\infty} L_j N_j.$$

Recalling (4.1.7)-(4.1.8), we now show that, for $\alpha = \frac{1}{2}$, the above construction yields a uniform bound

$$\|\widehat{u}\|_{\mathcal{P}_\alpha} \le C, \tag{4.2.23}$$

with a constant C independent of k_1, k_2. As a consequence, the same bound holds for $\alpha \in]0, \frac{1}{2}[$.

To prove (4.2.23), let $\lambda \in]0,1]$ and choose $\bar{k}$ such that $\lambda \in]2^{-\bar{k}}, 2^{-\bar{k}+1}]$. Set $\boldsymbol{V}(\lambda)$ in (19)-(20) to be the support of $\sum_{k=\bar{k}}^{k_2} v_k$. By (4.2.21) the following estimates hold:

$$\mathscr{L}^1(\boldsymbol{V}(\lambda)) \le \sum_{k=\bar{k}}^{k_2} N_k \ell_k = \frac{1}{2} \sum_{k=\bar{k}}^{k_2} 2^{-k/2} < \frac{1}{2} \frac{\sqrt{2}}{\sqrt{2}-1} \cdot 2^{-\bar{k}/2} < C_0\,\lambda^{1/2},$$

$$\text{Tot.Var.} \left\{ \sum_{k=k_1}^{\bar{k}-1} v_k;\, \mathbb{R} \right\} \le \sum_{k=k_1}^{\bar{k}-1} N_k 2^k \ell_k < \frac{1}{2} \frac{2^{\bar{k}/2}}{\sqrt{2}-1} < C_0\,\lambda^{-1/2}.$$

This proves (4.2.23). A more general result will be obtained in Proposition 4.4.3, to which we refer for additional details.

3. In view of (4.2.19), (4.2.21), the support of $\widehat{u}$ is contained in an interval I whose length is

$$\mathscr{L}^1(I) \leq \sum_{k_1}^{k_2} \mathscr{L}^1(I_k) = \sum_{k_1}^{k_2} \sqrt{2t}2^{k/2} N_k \ell_k = \sqrt{\frac{t}{2}}(k_2 - k_1 + 1). \tag{4.2.24}$$

We now compute the total variation of the corresponding solution $S_t\widehat{u}$ at a given time $t > 0$. If $t \geq 2^{-1} \cdot 2^{-k_1}$, i.e., if $k_1 \geq \log_2(1/2t)$, then at time t all the elementary solutions appearing in the blocks v_k, $k \geq k_1$, have a right triangle shape and we can use the estimate (4.2.20). Since these blocks do not interact with each other, we have

$$\text{Tot.Var.}\{S_t\widehat{u}\} = \sum_{k_1}^{k_2} \text{Tot.Var.}\{S_t v_k\} \geq \sqrt{\frac{2}{t}} \sum_{k_1}^{k_2} 2^{k/2} N_k \ell_k = \sqrt{\frac{1}{2t}}(k_2 - k_1 + 1). \tag{4.2.25}$$

Using the notation $\lceil a \rceil$ to denote the smallest integer $\geq a$, we now choose $k_1 = \lceil \log_2(1/t) \rceil$ and $k_2 = k_1 - 2 + \lceil \sqrt{\frac{2}{t}} \rceil$. By (4.2.24) we deduce that the support of $\widehat{u}$ is contained in the interval I whose length is

$$\mathscr{L}^1(I) \leq \sqrt{\frac{t}{2}}(k_2 - k_1 + 1) = \sqrt{\frac{t}{2}} \left(\left\lceil \sqrt{\frac{2}{t}} \right\rceil - 1 \right) \leq 1.$$

By (4.2.25), $S_t\widehat{u}$ has total variation

$$\text{Tot.Var.}\{S_t\widehat{u}\} \geq \sqrt{\frac{1}{2t}}(k_2 - k_1 + 1) = \sqrt{\frac{1}{2t}} \left(\left\lceil \sqrt{\frac{2}{t}} \right\rceil - 1 \right) \geq \sqrt{\frac{1}{2t}} \left\lceil \frac{\sqrt{2} - 1}{\sqrt{t}} \right\rceil \geq \frac{1}{c} \cdot \frac{1}{t}$$

where $c > 0$ is an absolute constant.

Finally, by (4.2.18), (4.2.21), the function $\widehat{u}$ satisfies

$$\sup_{x<y} \frac{|\widehat{u}(y) - \widehat{u}(x)|}{|y - x|^\sigma} \leq \sup_k \left(k^{-(1-\sigma)k} \cdot 2^k \right) \leq \left(e^{2^{\frac{1}{1-\sigma}}} \right)^{(1-\sigma)e^{-1}},$$

since the function $k \mapsto k^{-(1-\sigma)k} \cdot 2^k$ attains its maximum in $\mathbb{R}$ at $k = 2^{\frac{1}{1-\sigma}} e^{-1}$. □

4.3. The Intermediate Domains $\mathcal{D}_\alpha$

We consider entropy solutions to the scalar conservation law (11) and let $S : \mathbf{L}^1(\mathbb{R}) \times \mathbb{R}_+ \to \mathbf{L}^1(\mathbb{R})$ be the corresponding semigroup of entropy weak solutions. Notice that in this section the convexity assumption of the flux f is not necessary. The goal of this section is to study the subdomains $\mathcal{D}_\alpha \subset \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ defined by

$$\mathcal{D}_\alpha \doteq \left\{ \overline{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R}) \, ; \, \sup_{0<t\leq 1} t^{-\alpha} \|S_t\overline{u} - \overline{u}\|_{\mathbf{L}^1} < +\infty \right\}. \tag{4.3.1}$$

Since the semigroup S_t is nonlinear, the domains $\mathcal{D}_\alpha$ are not vector spaces. However, we can ask if they contain some classical linear spaces, such as fractional Sobolev spaces.

Let $1 \leq p < +\infty$ and $0 < \alpha \leq 1$ be given, together with an open set $\Omega \subset \mathbb{R}$. The fractional Sobolev space $W^{\alpha,p}(\Omega)$ is defined by (see for example [**58**])

$$W^{\alpha,p}(\Omega) \doteq \left\{ u \in \mathbf{L}^p(\Omega); \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{p}+\alpha}} \in \mathbf{L}^p(\Omega \times \Omega) \right\}, \tag{4.3.2}$$

equipped with the norm

$$\|u\|_{W^{\alpha,p}} \doteq \|u\|_{\mathbf{L}^p} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{1+\alpha p}}\,dxdy \right)^{\frac{1}{p}}. \tag{4.3.3}$$

As it is well known, functions in Sobolev spaces can be approximated by smooth functions by taking mollifications. Let $\eta : \mathbb{R} \mapsto [0,1]$ be a symmetric, $\mathcal{C}^\infty$ mollifier with compact support, so that

$$\begin{cases} \eta(s) = \eta(-s) \in [0,1] & \text{if} \quad s \in [-1,1], \\ \eta(s) = 0 & \text{if} \quad |s| \geq 1, \\ \big|\eta'(s)\big| \leq 2 & \forall s \in \mathbb{R}, \end{cases} \qquad \int \eta(s)\,ds = 1. \tag{4.3.4}$$

Here and in the sequel, the prime $'$ denotes a derivative. For $h > 0$, define the rescaled kernels by setting

$$\eta_h(s) = \frac{1}{h}\eta\left(\frac{s}{h}\right). \tag{4.3.5}$$

For $u \in \mathbf{L}^1_{loc}$, consider the convolution $u_h = u \star \eta_h$. The rate of convergence of these mollifications depends on the regularity properties of the function u.

LEMMA 4.3.1. *Assume $u \in W^{\alpha,1}(\mathbb{R})$ for some $0 < \alpha \leq 1$. Then, for every $h > 0$, the convolution $u_h = u \star \eta_h$ satisfies*

$$\|u - u_h\|_{\mathbf{L}^1} \leq \|u\|_{W^{\alpha,1}} \cdot h^\alpha, \qquad \|u_h'\|_{\mathbf{L}^1} \leq C\|u\|_{W^{\alpha,1}} \cdot \frac{1}{h^{1-\alpha}}, \tag{4.3.6}$$

for some constant C independent of u.

PROOF. A direct computation yields

$$\begin{aligned} \int |u(x) - u_h(x)|\,dx &\leq \iint |u(x) - u(y)|\eta_h(y-x)\,dxdy \\ &= \int \frac{1}{h}\eta(s/h) \int |u(x+s) - u(x)|\,dxds \\ &\leq \int_{-h}^{h} \left(\frac{1}{|s|} \int \big|u(x+s) - u(x)\big|\,dx \right) ds \\ &\leq h^\alpha \iint \frac{|u(x+s) - u(x)|}{|s|^{1+\alpha}}\,dxds \\ &= \|u\|_{W^{\alpha,1}} \cdot h^\alpha. \end{aligned}$$

Moreover, the total variation of u_h is bounded by

$$\begin{aligned}
\int_{\mathbb{R}} |u_h'(x)|\,dx &= \int \left| \int u(y)\eta_h'(y-x)dy \right| dx \\
&= \int \left| \int u(x+s)\eta_h'(s)ds \right| dx \\
&= \int \frac{1}{h^2} \left| \int u(x+s)\eta'(s/h)ds \right| dx \\
&= \int \frac{1}{h^2} \left| \int_0^h \eta'(s/h)(u(x+s)-u(x-s))ds \right| dx \\
&\leq C \int \frac{1}{h^{1-\alpha}} \int_0^h \frac{|u(x+s)-u(x-s)|}{s^{1+\alpha}} ds dx \\
&\leq C \|u\|_{W^{\alpha,1}} \frac{1}{h^{1-\alpha}},
\end{aligned}$$

Notice that the constant C depends only on the mollifying kernel η.

□

PROPOSITION 4.3.2. *Let (11) be any conservation law with continuously differentiable flux. For every $\alpha \in]0,1]$ we have the inclusion $\mathbf{L}^\infty(\mathbb{R}) \cap W^{\alpha,1}(\mathbb{R}) \subseteq \mathcal{D}_\alpha$.*

PROOF. Let $\overline{u} \in \mathbf{L}^\infty(\mathbb{R}) \cap W^{\alpha,1}(\mathbb{R})$ and consider the mollifications $u_h \doteq \eta_h \star \overline{u}$. By Lemma 4.3.1 it follows

$$\|S_h u_h - u_h\|_{\mathbf{L}^1} \leq \|f'(u_h)\|_{\mathbf{L}^\infty}\, h \cdot \text{Tot.Var.}\{u_h\} \leq C\,\|f'\|_{\mathbf{L}^\infty} \|\overline{u}\|_{W^{\alpha,1}} h^\alpha,$$

where the $\mathbf{L}^\infty$ norm of f' is taken on the interval $[-\|\overline{u}\|_{\mathbf{L}^\infty}, \|\overline{u}\|_{\mathbf{L}^\infty}]$. Therefore

$$\begin{aligned}
\|S_h\overline{u} - \overline{u}\|_{\mathbf{L}^1} &\leq \|S_h\overline{u} - S_h u_h\|_{\mathbf{L}^1} + \|S_h u_h - u_h\|_{\mathbf{L}^1} + \|u_h - \overline{u}\|_{\mathbf{L}^1} \\
&\leq (2 + C\,\|f'\|_{\mathbf{L}^\infty})\, \|\overline{u}\|_{W^{\alpha,1}} h^\alpha.
\end{aligned}$$

□

The second result in this section is formulated in terms of the property $(\mathbf{P}_\alpha)$.

PROPOSITION 4.3.3. *Let (11) be a conservation law with continuously differentiable flux. For any $0 < \alpha < 1$, if $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ satisfies $(\mathbf{P}_\alpha)$, then $\overline{u} \in \mathcal{D}_\alpha$.*

PROOF. Let $\overline{u}$ satisfy $(\mathbf{P}_\alpha)$. Given $t \in]0,1]$, set $\lambda = t$ and let $V(\lambda) \subset \mathbb{R}$ be an open set satisfying (19)-(20). Observing that this open set $\boldsymbol{V}(t)$ is a countable union of disjoint open intervals

$$\boldsymbol{V}(t) = \bigcup_{k \geq 1}]a_j, b_k[\,,$$

we define a new function $\overline{v}$ by replacing $\overline{u}$ with an affine function on each interval $[a_j, b_j]$. Namely,

$$\overline{v}(x) \;=\; \begin{cases} \overline{u}(x) & \text{if} \quad x \notin \cup_k [a_k, b_k], \\ \frac{(b_j - x)\overline{u}(a_j) + (x - a_j)\overline{u}(b_j)}{b_j - a_j} & \text{if} \quad x \in [a_j, b_j]\,. \end{cases}$$

This implies

$$\text{Tot.Var.}\{\overline{v}\} \leq \text{Tot.Var.}\big\{\overline{u}\,;\mathbb{R}\setminus \boldsymbol{V}(t)\big\} \leq C\,t^{\alpha-1},$$

$$\|\overline{v}-\overline{u}\|_{\mathbf{L}^1} \leq 2\|\overline{u}\|_{\mathbf{L}^\infty}\cdot \mathscr{L}^1(\boldsymbol{V}(t)) \leq 2\|\overline{u}\|_{\mathbf{L}^\infty}\cdot Ct^\alpha.$$

We thus obtain

$$\begin{aligned}\|S_t\overline{u}-\overline{u}\|_{\mathbf{L}^1} &\leq \|S_t\overline{u}-S_t\overline{v}\|_{\mathbf{L}^1}+\|S_t\overline{v}-\overline{v}\|_{\mathbf{L}^1}+\|\overline{u}-\overline{v}\|_{\mathbf{L}^1}\\ &\leq \|\overline{v}-\overline{u}\|_{\mathbf{L}^1}+t\cdot\|f'\|_{\mathbf{L}^\infty}\cdot\text{Tot.Var.}\{\overline{v}\}+\|\overline{v}-\overline{u}\|_{\mathbf{L}^1}\\ &\leq 4\,\|\overline{u}\|_{\mathbf{L}^\infty}\,C\,t^\alpha+\|f'\|_{\mathbf{L}^\infty}\,Ct^\alpha,\end{aligned}$$

where the $\mathbf{L}^\infty$ norm of f' is taken on the interval $[-\|\overline{u}\|_{\mathbf{L}^\infty}, \|\overline{u}\|_{\mathbf{L}^\infty}]$. Since the same constant C is valid for all $t\in\,]0,1]$, this proves that $\overline{u}\in\mathcal{D}_\alpha$. □

4.4. A Decomposition Property for Functions $\overline{u}\in\mathcal{P}_\alpha$

In this section we study properties of functions that lie in the metric space $\mathcal{P}_\alpha$ introduced at (4.1.8). These are functions that satisfy the property $(\mathbf{P}_\alpha)$ at (19)-(20). Our main result provides a decomposition of a function $\overline{u}\in\mathcal{P}_\alpha$, as the sum of countably many components with different degrees of regularity.

THEOREM 4.4.1. *Let $\overline{u}:\mathbb{R}\mapsto\mathbb{R}$ be a measurable function and let $0<\alpha<1$ be given. Then $\overline{u}\in\mathcal{P}_\alpha$ if and only if it can be decomposed as*

$$\overline{u}(x)=\sum_{k=0}^\infty v_k(x)\qquad \textit{for a.e. } x\in\mathbb{R}, \tag{4.4.1}$$

where the v_k satisfy the following properties. For some constant $C=\mathcal{O}(1)\cdot\|\overline{u}\|_{\mathcal{P}_\alpha}$ one has

(i) Bounds on the support and on the total variation:

$$\text{Tot.Var.}\{v_0\}\leq C, \tag{4.4.2}$$

$$\text{Tot.Var.}\{v_k\}\;\leq\;C\cdot 2^{(1-\alpha)k},\qquad \mathscr{L}^1\big(\{v_k\neq 0\}\big)\;\leq\;C\cdot 2^{-\alpha k}\quad\text{for all } k\geq 1. \tag{4.4.3}$$

(ii) One-sided Lipschitz bound:

$$v_k(x_2)-v_k(x_1)\leq 2^k\cdot(x_2-x_1)\qquad \forall x_1<x_2\,. \tag{4.4.4}$$

(iii) A further decomposition:

For each $k\geq 1$ we can further decompose

$$v_k\;=\;\sum_{p=1}^\infty v_k^p$$

so that the following conditions hold: the functions v_k^p satisfy (4.4.4), their supports have disjoint interiors, and setting $\ell_k^p\doteq\text{meas}(\text{supp}\,v_k^p)$ it holds

$$\big|v_k^p(x)\big|\leq h_k^p\doteq 2^k\ell_k^p,\qquad \forall\,x\in\text{supp}\,v_k^p\,,$$

and

$$\sum_{p\geq 1}\ell_k^p\leq C\cdot 2^{-\alpha k}. \tag{4.4.5}$$

The proof of Theorem 4.4.1 will be achieved in three steps. We first show in Lemma 4.4.3 that the existence of a decomposition as in (4.4.1) which satisfies property (i) is equivalent to the statement that $\overline{u} \in \mathcal{P}_\alpha$. Next, in Lemma 4.4.6 we show that this decomposition can be refined so to satisfy also property (ii), still with some constant C of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$. Finally, Lemma 4.4.7 shows that (iii) is an easy consequence of (i) and (ii) .

Before starting the proof of Theorem 4.4.1, we provide a remark about the embedding and scaling properties of the space $\mathcal{P}_\alpha$.

REMARK 4.4.2. Let $\overline{u} \in \mathcal{P}_\alpha$ and $K \subset \mathbb{R}$ be a fixed compact set and let $u_m \doteq |K|^{-1}\int_K \overline{u}$. We can estimate

$$\|\overline{u} - u_m\|_{\mathbf{L}^p(K)} \leq \sum_{k=0}^{+\infty} \|v_k - v_{k,m}\|_{\mathbf{L}^p(K)} \leq \sum_{k=1}^{+\infty} \big(\|v_k\|_{\mathbf{L}^p(K)} + \|v_{k,m}\|_{\mathbf{L}^p(K)}\big) + \|v_0 - v_{0,m}\|_{\mathbf{L}^p(K)},$$

where $v_{k,m}$ is the average of v_k on K. We can estimate for $k \geq 1$

$$\begin{aligned} \|v_k\|_{\mathbf{L}^p(K)} &\leq \|v_k\|_{\mathbf{L}^\infty} \mathrm{meas}(\{v_k \neq 0\})^{\frac{1}{p}} \leq \frac{1}{2}\mathrm{Tot.Var.}\{v_k;\mathbb{R}\} \cdot \mathrm{meas}(\{v_k \neq 0\})^{\frac{1}{p}} \\ &\leq \frac{C}{2} \cdot C^{1/p} 2^{k(1-\alpha-\alpha p^{-1})}, \end{aligned} \tag{4.4.6}$$

$$\|v_{k,m}\|_{\mathbf{L}^p(K)} \leq |K|^{\frac{1-p}{p}} \|v_k\|_{\mathbf{L}^1(K)} \leq \|v_k\|_{L^p(K)} \leq \frac{C}{2} \cdot C^{1/p} 2^{k(1-\alpha-\alpha p^{-1})},$$

where $C = \mathcal{O}(1) \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}$. Instead, for $k = 0$

$$\|v_0 - v_{0,m}\|_{\mathbf{L}^p(K)} \leq \mathrm{Tot.Var.}\{v_0;\mathbb{R}\} \cdot |K|^{1/p} \leq C \cdot |K|^{1/p}.$$

Then, provided $1 - \alpha - \frac{\alpha}{p} < 0$, namely if $p < \frac{\alpha}{1-\alpha}$, it holds

$$\|\overline{u} - u_m\|_{\mathbf{L}^p(K)} \leq c(K) \cdot \max\{\|\overline{u}\|_{\mathcal{P}_\alpha}, \|\overline{u}\|_{\mathcal{P}_\alpha}^{1+p^{-1}}\} \frac{1}{1 - 2^{(1-\alpha-\alpha p^{-1})}}$$

so that we have the embedding $\mathcal{P}_\alpha \hookrightarrow \mathbf{L}^p_{loc}$, up to subtracting the average. In particular $\mathcal{P}_\alpha \hookrightarrow \mathbf{L}^1_{loc}$ if $\alpha > 1/2$. Notice that this is consistent with the scaling property

$$\|u\|_{\mathcal{P}_\alpha} = \|u_\mu\|_{\mathcal{P}_\alpha} \qquad \text{with} \quad u_\mu(x) \doteq \mu^{\frac{1-\alpha}{\alpha}} u(\mu x) \qquad \text{for } \mu \geq 1.$$

More generally, if $p < \frac{\alpha}{1-\alpha}$, and we have a sequence u_n bounded in $\mathcal{P}_\alpha$ with equi-bounded averages on a compact set K, then $(u_n)_{n\in\mathbb{N}}$ admits a convergent subsequence in $\mathbf{L}^p(K)$.

Indeed, denoting by $\{v_k^n\}_{k\in\mathbb{N}}$ the functions appearing in the decomposition of u_n, by a diagonal argument using Helly's compactness theorem one extracts a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that $v_k^{n_i}$ converges in $\mathbf{L}^p(K)$, for every k. By (4.4.6), for every $\varepsilon > 0$ we can choose $N \in \mathbb{N}$ sufficiently large so that

$$\left\| \sum_{k=N}^{\infty} v_k^{n_i} \right\|_{\mathbf{L}^p(K)} < \varepsilon \qquad \forall i \in \mathbb{N},$$

therefore u_{n_i} is converging in $\mathbf{L}^p(K)$.

LEMMA 4.4.3. *Let $\bar{u} : \mathbb{R} \mapsto \mathbb{R}$ be a measurable function. Then $\overline{u} \in \mathcal{P}_\alpha$ if and only if it can be decomposed as (4.4.1), where the v_k satisfy (4.4.2) and (4.4.3). The smallest constant C for which (4.4.2) and (4.4.3) hold for every $k \geq 1$ is of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$.*

PROOF. **1.** Assume that $\overline{u}$ admits a decomposition as in (4.4.1)-(4.4.3). We show that

$$\|\overline{u}\|_{\mathcal{P}_\alpha} = \mathcal{O}(1) \cdot C.$$

Consider the case where $\lambda \in]0, 1]$ is of the form $\lambda = 2^{-q}$ for some integer $q \geq 1$. Then we set

$$\widetilde{v} \doteq \sum_{k=0}^{q} v_k$$

and estimate

$$\text{Tot.Var.}\{\widetilde{v}\} \leq \sum_{k=0}^{q} \text{Tot.Var.}\{v_k\} \leq C \cdot \sum_{k=0}^{q} 2^{(1-\alpha)k} = \mathcal{O}(1) \cdot C \cdot 2^{(1-\alpha)q}.$$

Moreover

$$\mathscr{L}^1\big(\{\overline{u} \neq \widetilde{v}\}\big) \leq C \cdot \sum_{k=q+1}^{+\infty} 2^{-\alpha k} = \mathcal{O}(1) \cdot C \cdot 2^{-\alpha q}.$$

This proves

$$\sup_{q \geq 1} d^{(2^{-q})}(\overline{u}, 0) = \mathcal{O}(1) \cdot C.$$

A simple argument now shows that the estimate holds when the supremum is taken over all $0 < \lambda \leq 1$.

2. Assume now that $\overline{u} \in \mathcal{P}_\alpha$. By the definition of $\|\cdot\|_{\mathcal{P}_\alpha}$ at (4.1.7)-(4.1.8), choosing $\lambda = 2^{-k}$, for every $k \geq 0$ we obtain a function u_k such that

$$\text{Tot.Var.}\{u_k\} \;\leq\; \|\overline{u}\|_{\mathcal{P}_\alpha} \cdot 2^{(1-\alpha)k}, \qquad \text{meas}\Big(\{\overline{u} \neq u_k\}\Big) \;\leq\; \|\overline{u}\|_{\mathcal{P}_\alpha} \cdot 2^{-\alpha k}. \tag{4.4.7}$$

We define the functions $\{v_k\}_{k \geq 1}$ by setting

$$v_0 \doteq u_0, \qquad v_k \doteq u_k - u_{k-1} \qquad \forall\, k \geq 1. \tag{4.4.8}$$

By the second inequality in (4.4.7) it follows that $\text{meas}\Big(\{\overline{u} \neq u_k\}\Big)$ is summable in k and therefore

$$\lim_{k \to +\infty} u_k(x) = \overline{u}(x), \qquad \text{pointwise for a.e. } x \in \mathbb{R}.$$

Using a telescopic sum one obtains

$$\overline{u}(x) \;=\; \lim_{N \to +\infty} u_N(x) \;=\; \lim_{N \to +\infty} \sum_{k=0}^{N} v_k(x) \qquad \text{pointwise for a.e. } x \in \mathbb{R}.$$

We conclude by observing that

$$\text{Tot.Var.}\{v_k\} \;\leq\; \text{Tot.Var.}\{u_k\} + \text{Tot.Var.}\{u_{k-1}\} \;\leq\; 2 \cdot \|\overline{u}\|_{\mathcal{P}_\alpha} \cdot 2^{(1-\alpha)k}$$

and

$$\text{meas}\big(\{u_k \neq u_{k-1}\}\big) \;\le\; \text{meas}\big(\{\overline{u} \neq u_{k-1}\}\big) + \text{meas}\big(\{\overline{u} \neq u_k\}\big) \;\le\; (2^\alpha + 1)\cdot \|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{-\alpha k}.$$

□

We will show in Lemma 4.4.6 that one can choose the decomposition in Lemma 4.4.3 in such a way that all functions v_k are one-sided Lipschitz. In order to do so, we introduce a notion of lower one-sided p-Lipschitz envelope and we prove some of its properties: given $f \in BV(\mathbb{R})$ and $p \in \mathbb{R}$ with $p > 0$ let us denote by

$$\mathcal{L}_p f(x) \;\doteq\; \sup\Big\{u(x)\,;\; u:\mathbb{R}\to\mathbb{R},\; u(y)\le f(y)\,,\; u(y')-u(y)\le p(y'-y)\;\forall\, y,y'\in\mathbb{R},\; y<y'\Big\}.$$

The function $\mathcal{L}_p f$ is the largest one-sided p-Lipschitz function whose graph lies below the graph of f. Denoting by Tot.Var.$^+\{g\}$ the positive variation of a function g

$$\text{Tot.Var.}^+\{g\} \doteq \sup \sum_{x_0\le\cdots\le x_n} \big[g(x_i)-g(x_{i-1})\big]^+, \quad \big[g(x_i)-g(x_{i-1})\big]^+ = \max\big\{0,\; g(x_i)-g(x_{i-1})\big\},$$

we have the following result.

LEMMA 4.4.4. *Let $f \in BV(\mathbb{R})$ and assume that f is continuous from the left, namely $f(x) = \lim_{y\to x^-} f(y)$. The following relations hold:*

$$\begin{cases} \textit{Tot.Var.}^+\{\mathcal{L}_p f\} \;\le\; \textit{Tot.Var.}^+\{f\}, \\ \text{meas}\Big\{x\in\mathbb{R}\;;\; \mathcal{L}_p f(x) < f(x)\Big\} \;\le\; \frac{1}{p}\cdot \textit{Tot.Var.}^+\{f\}, \\ \textit{Tot.Var.}^+\{f-\mathcal{L}_p f\} \;\le\; \textit{Tot.Var.}^+\{f\}. \end{cases} \tag{4.4.9}$$

PROOF. Since f is continuous from the left, then $\mathcal{L}_p f$ is continuous from the left. Denote by $A_p = \{x\in\mathbb{R} : \mathcal{L}_p f(x) < f(x)\}$. We first observe that $x \in A_p$ if and only if there is $y < x : f(x) - f(y) > p(x-y)$. Indeed, since $\mathcal{L}_p f$ is one-sided p-Lipschitz, if there is $y < x : f(x) - f(y) > p(x-y)$, then we have $\mathcal{L}_p f(x) \le \mathcal{L}_p f(y) + p(x-y) \le f(y) + p(x-y) < f(x)$. Viceversa, if for every $y < x$ it holds $f(x) - f(y) \le p(x-y)$ then $f(x) = \mathcal{L}_p f(x)$ by the definition of $\mathcal{L}_p f$.

Since f is continuous from the left, if the above characterization of A_p holds for some x, y with $y < x$ then the same holds for every z, y with z in a left neighborhood of x. In particular A_p is the union of at most countably many intervals of the form $]a,b[$ or $]a,b]$.

By maximality of $\mathcal{L}_p f$, on each connected component of A_p the function $\mathcal{L}_p f$ is affine with derivative equal to p, therefore

$$\big[D(\mathcal{E}_p f)\big]^+(A_p) = p\cdot \text{meas}(A_p),$$

where the left hand side is the positive part of the measure $D(\mathcal{E}_p f)$.

For every connected component I of A_p with endpoints a, b we have

$$f(a) = \mathcal{L}_p f(a) < \mathcal{L}_p f(a) + p(b-a) = \mathcal{L}_p f(b) \le f(b),$$

in particular

$$\text{Tot.Var.}^+\{\mathcal{L}_p f; I\} = p(b-a) \le \text{Tot.Var.}^+\{f; I\}. \tag{4.4.10}$$

In order to estimate

$$\text{Tot.Var.}^+\{\mathcal{L}_p f\} = \sup_{x_0<\ldots<x_n} \sum_{i=1}^n \left[\mathcal{L}_p f(x_i) - \mathcal{L}_p f(x_{i-1})\right]^+,$$

it is not restrictive to assume that x_i does not belong to the interior of A_p for every i, indeed if for some i, the point x_i is in the interior of A_p, we can replace it with the endpoints of the connected component of A_p containing x_i. The first inequality in (4.4.9) follows from this observation, the estimate (4.4.10) and the fact that $f = \mathcal{L}_p f$ on $\mathbb{R} \setminus A_p$. The second inequality in (4.4.9) follows since

$$\text{meas}(A_p) = \frac{1}{p}\big[D(\mathcal{E}_p f)\big]^+(A_p) \leq \frac{1}{p} \cdot \text{Tot.Var.}^+\{\mathcal{L}_p f\} \leq \frac{1}{p} \cdot \text{Tot.Var.}^+\{f\}.$$

Since on each connected component I of A_p, the function $\mathcal{L}_p f$ is affine function with derivative $p > 0$, we have

$$\text{Tot.Var.}^+\{f - \mathcal{L}_p f; I\} \leq \text{Tot.Var.}^+\{f; I\}.$$

Therefore, using again $f = \mathcal{L}_p f$ on $\mathbb{R} \setminus A_p$, we deduce the third inequality in (4.4.9). □

REMARK 4.4.5. The assumption of left continuity of f in the Lemma above is not strictly needed, but it allows to avoid unnecessary pathological cases since we can always assume to take the left-continuous representative of a BV function of a real variable.

We now show that one can choose the decomposition in Lemma 4.4.3 in such a way that all functions v_k are one-sided Lipschitz.

LEMMA 4.4.6. *Consider any function $\overline{u} \in \mathcal{P}_\alpha$. Then, it is possible to choose the functions v_k in (4.4.1)–(4.4.3) in such a way that the additional one-sided Lipschitz bound (4.4.4) holds.*

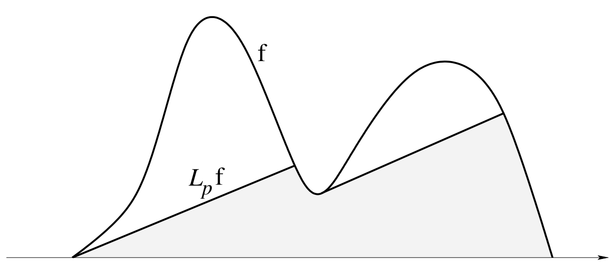

FIGURE 4. The lower one-sided p-Lipschitz envelope $\mathcal{L}_p f$. Here the straight lines have slope p.

PROOF. **1.** Let $\{p_k^i\}_{i,k\in\mathbb{N}}$ be positive numbers (to be chosen later) such that

$$\lim_{i\to+\infty} p_k^i = +\infty \qquad \forall\, k \geq 0, \tag{4.4.11}$$

and consider the decomposition (4.4.1) constructed in Lemma 4.4.3. As an intermediate step, we claim that for every $k \geq 0$, the function v_k can be further decomposed as

$$v_k(x) = \sum_{i=0}^{+\infty} v_k^i(x) \qquad \text{for a.e.} x \in \mathbb{R}, \tag{4.4.12}$$

where each v_k^i is one-sided p_k^i-Lipschitz and satisfies

$$\mathscr{L}^1\big(\{v_k^i \neq 0\}\big) \leq C \cdot \frac{2^{k(1-\alpha)}}{p_k^{i-1}}, \qquad \text{Tot.Var.}^+\{v_k^i\} \leq C \cdot 2^{(1-\alpha)k} \quad \text{if } i+k \neq 0, \qquad \text{Tot.Var.}\{v_0^0\} \leq C, \tag{4.4.13}$$

with $C = \mathcal{O}(1) \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}$, as in Lemma 4.4.3. For $k \geq 0$, we first prove the claim assuming $v_k \geq 0$. Define

$$v_k^0 \doteq \mathcal{L}_{p_k^0}(v_k).$$

By (4.4.9) one has

$$\begin{aligned} &\text{Tot.Var.}^+\{v_k^0\} \leq \text{Tot.Var.}^+\{v_k\} \leq C \cdot 2^{(1-\alpha)k}, \qquad \text{for all } k \geq 0. \\ &\mathscr{L}^1(\{v_k^0 \neq 0\}) \leq \mathscr{L}^1(\{v_k \neq 0\}) \leq C \cdot 2^{-\alpha k} \qquad \text{for all } k \geq 1. \end{aligned}$$

Setting $\psi_0 = v_k - v_k^0$ and using again (4.4.9) we get

$$\mathscr{L}^1(\{\psi_0 \neq 0\}) \leq C \cdot \frac{1}{p_k^0} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{\psi_0\} \leq C \cdot 2^{(1-\alpha)k}.$$

Defining $v_k^1 = \mathcal{L}_{p_k^1}\psi_0$ we obtain

$$\mathscr{L}^1(\{v_k^1 \neq 0\}) \leq C \cdot \frac{1}{p_k^0} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{v_k^1\} \leq C \cdot 2^{(1-\alpha)k}.$$

By induction, assume we are given ψ_{i-2} and $v_k^{i-1} = \mathcal{L}_{p_k^{i-1}}\psi_{i-2}$, $i \geq 2$, both with positive variation $\leq C \cdot 2^{(1-\alpha)k}$. We then define $\psi_{i-1} = \psi_{i-2} - v_k^{i-1}$ and $v_k^i = \mathcal{L}_{p_k^i}\psi_{i-1}$. This yields

$$\mathscr{L}^1(\{\psi_{i-1} \neq 0\}) \leq C \cdot \frac{1}{p_k^{i-1}} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{\psi_{i-1}\} \leq C \cdot 2^{(1-\alpha)k}$$

and hence

$$\mathscr{L}^1(\{v_k^i \neq 0\}) \leq C \cdot \frac{2^{(1-\alpha)k}}{p_k^{i-1}}, \qquad \text{Tot.Var.}^+\{v_k^i\} \leq C \cdot 2^{(1-\alpha)k}. \tag{4.4.14}$$

Therefore, by induction (4.4.14) holds for every i and every k.

Finally, we prove that (4.4.12) holds. Indeed, by the second inequality in (4.4.9) it follows

$$\mathscr{L}^1\left(\left\{v_k \neq \sum_{i=0}^{i^*} v_k^i\right\}\right) \leq \frac{1}{p_k^{i^*}} \cdot \text{Tot.Var.}\, v_k \qquad \forall\, i^* \geq 1.$$

Letting $i^* \to +\infty$, this proves our claim in the case $v_k \geq 0$.

2. Next, we show how to handle the general case where $v_k = v_k^+ - v_k^-$ has a positive and a negative part. We already know how to decompose the positive part v_k^+.

We treat the negative part v_k^- in the same way, but using instead the lower one-sided Lipschitz envelope, defined by

$$\mathcal{L}_p^- f(x) \doteq \sup\Big\{u(x)\,; u(y) \leq f(y)\,, u(y') - u(y) \geq -p(y'-y) \forall\, y, y' \in \mathbb{R}, y < y'\Big\}.$$

This yields a decomposition

$$v_k^- = \sum_{i=0}^{\infty} w_k^i$$

where w_k^i are positive one sided Lipschitz functions which satisfy the same inequalities as in (4.4.13), and whose distributional derivatives satisfy $Dw_i^k \geq -p_k^i$.

Then

$$v_k(x) = v_k^+(x) - v_k^-(x) = \sum_{i=0}^{\infty} v_k^i(x) + \sum_{i=0}^{\infty}\big(-w_k^i(x)\big).$$

is the desired decomposition.

3. We now conclude the proof of the lemma, relying on (4.4.12)-(4.4.13). Defining

$$\widetilde{v}_q = \sum_{i=0}^{q} v_{q-i}^i\,, \tag{4.4.15}$$

we obtain

$$\overline{u} = \sum_{q=0}^{+\infty} \widetilde{v}_q\,. \tag{4.4.16}$$

Next, we choose

$$p_k^i \doteq \frac{6}{\pi^2}\cdot\frac{2^{k+i}}{(i+2)^2}\,.$$

By (4.4.13), for every $q \geq 1$ we obtain

$$\mathscr{L}^1\big(\{\widetilde{v}_q \neq 0\}\big) \leq \frac{\pi^2}{6} C\cdot 2^{-\alpha q}\sum_{i=0}^{q}\frac{2^q\cdot(i+1)^2}{2^{(1-\alpha)i}\cdot 2^{q-1}} \leq \frac{\pi^2}{3}\cdot C\cdot 2^{-\alpha q}\cdot\sum_{i=0}^{q}\frac{(i+1)^2}{2^{(1-\alpha)i}} = \mathcal{O}(1)\cdot C\cdot 2^{-\alpha q}.$$

Moreover, always for $q \geq 1$, the one-sided Lipschitz constant for $\widetilde{v}_q$ is estimated by

$$D\widetilde{v}_q \leq \frac{6}{\pi^2}\sum_{i=0}^{q} p_{q-i}^i \leq 2^q\frac{6}{\pi^2}\sum_{i=0}^{q}\frac{1}{(i+1)^2} \leq 2^q.$$

The one-sided Lipschitz property and the estimate on the support of $\widetilde{v}_q$ readily imply

$$\text{Tot.Var.}\{\widetilde{v}_q\} = \mathcal{O}(1)\cdot C\cdot 2^{(1-\alpha)q}.$$

If $q = 0$, by definition we have $\widetilde{v}_q = v_0^0$, so that by (4.4.13)

$$\text{Tot.Var.}\,\{\widetilde{v}_0\} \,\leq\, C, \qquad D\widetilde{v}_0 \,\leq\, p_0^0 \,\leq\, 1.$$

The conclusion of the lemma is achieved by renaming $v_k \doteq \widetilde{v}_q$, with $q = k$. □

LEMMA 4.4.7. *For every $k \geq 0$, the function v_k constructed in Lemma 4.4.6 satisfies 3 of Theorem 4.4.1.*

PROOF. Since v_k is one-sided Lipschitz, the set $\{v_k \neq 0\}$ has at most countably many connected components, that we denote by $\{I_k^p\}_{p\geq 1}$. Consider then the restrictions $v_k^p \doteq v_k\Big|_{I_k^p}$. On every interval I_k^p, having length ℓ_k^p, since v_k is one-sided 2^k-Lipschitz, one has

$$|v_k(x)| \leq 2^k\ell_k^p = h_k^p, \qquad \forall\, x \in I_k^p$$

Therefore

$$\sum_{p\geq 1}\ell_k^p = \mathscr{L}^1(\{v_k \neq 0\}) \leq C2^{-\alpha k},$$

which proves (4.4.5). □

4.5. Decay Rate of the Total Variation

In this section we prove that if $1/2 < \alpha < 1$, then the conjectured decay of the total variation with rate $t^{\alpha-1}$ holds.

THEOREM 4.5.1. *Consider a bounded, compactly supported initial datum $\overline{u} \in \mathcal{P}_\alpha$, with $1/2 < \alpha < 1$. Then the solution to Burgers' equation (4.2.1) satisfies*

$$\limsup_{t\to 0+} \Big(t^{1-\alpha} \cdot TV\left\{S_t\overline{u}\right\}\Big) \;\le\; C_0\,\frac{\|\overline{u}\|_{\mathcal{P}_\alpha}}{2\alpha-1}\,, \tag{4.5.1}$$

where C_0 is some absolute constant.

PROOF. **1.** Let $u = u(t,x)$ be a solution of Burgers' equation and let $t > 0$ be given. Denote by $J_t \subset \mathbb{R}$ the jump set of $u(t,\cdot)$. By the Lax-Oleinik formula, for every $x \in \mathbb{R}\setminus J_t$ there is a unique backward characteristic from the point (t,x) along which the solution is constant: namely

$$u(t,x) = u(s, x - u(t,x)(t-s)) \qquad \forall\, s \in\,]0,t]. \tag{4.5.2}$$

If (4.5.2) holds we say that the couple

$$(x_0, v) \doteq \big(x - u(t,x)t\,, u(t,x)\big)$$

survives up to time t, and we denote by $\mathcal{Q}(t)$ the set of couples which survive up to time t. The point x_0 is the starting point of the characteristic passing through (t,x) and v is the value of u along the characteristic. Notice however, for example in the case of a centered rarefaction, that we can have $\overline{u}(x_0) \neq v$. Indeed, (4.5.2) does not extend to $t = 0$, in general. We will estimate Tot.Var.$\{S_t\overline{u}\}$ by means of the equality

$$\text{Tot.Var.}\big\{S_t\overline{u}\,;\mathbb{R}\big\} = \text{Tot.Var.}\big\{S_t\overline{u}\,;\mathbb{R}\setminus J_t\big\} = \sup \sum_{\substack{x_1\le\cdots\le x_n,\\ (x_i,v_i)\in\mathcal{Q}(t)}} |v_i - v_{i-1}|, \tag{4.5.3}$$

where in the last sum we assume that if $x_{i-1} = x_i$, then $v_{i-1} \le v_i$. By the Lax formula, the constraint $(x_0, u_0) \in \mathcal{Q}(t)$ is satisfied if and only if

$$\begin{aligned} &\int_{x_0}^{y} \overline{u}(z) - \Big[u_0 - \frac{1}{t}(z-x_0)\Big]\,dz \;\ge\; 0 \qquad \forall\, y \ge x_0\,,\\ &\int_{y}^{x_0} \overline{u}(z) - \Big[u_0 - \frac{1}{t}(z-x_0)\Big]\,dz \;\le\; 0 \qquad \forall\, y \le x_0\,. \end{aligned} \tag{4.5.4}$$

Equivalently:

$$\int_{x_0}^{y} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^+ dz \ge \int_{x_0}^{y} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^- dz \qquad \forall\, y \ge x_0\,, \tag{4.5.5}$$

$$\int_{y}^{x_0} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^+ dz \le \int_{y}^{x_0} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^- dz \qquad \forall\, y \le x_0\,, \tag{4.5.6}$$

where we used the notation

$$[z]^+ \doteq \max\{z, 0\}, \qquad [z]^- \doteq -\min\{z,0\}.$$

The interpretation of (4.5.5) is that for every $y > x_0$ the area of the hypograph of $\overline{u}$ in $[x_0, y]$ that lies above the line passing through (x_0, u_0) with slope $-1/t$ must be bigger than the area of the epigraph lying below the same line. Analogously, the interpretation of (4.5.6) is that for every $y < x_0$ the area of the hypograph of $\overline{u}$ in $[y, x_0]$ that lies above the line passing through (x, u_0) with slope $-1/t$ must be smaller than the area of the epigraph lying below the same line.

2. It suffices to prove the decay estimate (4.5.1) for all times of the form $t = 2^{-k}$, $k \geq 1$. We thus need to estimate the quantity

$$\limsup_{k \to +\infty} \ 2^{(\alpha-1)k} \, \text{Tot.Var.}\big\{S_{2^{-k}}\overline{u}\big\}$$

In the following we fix a time $t = 2^{-k}$ and show that

$$\text{Tot.Var.}\big\{S_{2^{-k}}\overline{u}\big\} \leq c \cdot \frac{2^{(1-\alpha)k}}{2\alpha - 1} \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}\,, \tag{4.5.7}$$

where c is some constant depending only on $\|\overline{u}\|_{\mathbf{L}^\infty}$.

Let $\overline{u} = \sum_{q=0}^{\infty} v_q$ be a decomposition satisfying all the properties listed in Theorem 4.4.1. We write $\overline{u}$ as the sum of two terms:

$$\overline{u} = \sum_{q=0}^{k-1} v_q + \sum_{q=k}^{\infty} v_q \doteq \widetilde{u}_k + \widehat{u}_k. \tag{4.5.8}$$

We regard the function $\widetilde{u}_k$ as the *regular part of* $\overline{u}$, in the sense that it has total variation that is bounded, of size $\mathcal{O}(1) \cdot 2^{(1-\alpha)k}$. In fact, since each v_q is one-sided Lipschitz with constant 2^q and with total variation bounded by $C\,2^{(1-\alpha)q}$, the function $\widetilde{u}_k$ is one-sided Lipschitz with constant 2^k and with total variation bounded by $C2^{(1-\alpha)k}$.

We recall that C is the constant coming from Lemmas 4.4.3 and 4.4.6, of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$.

3. To simplify the exposition, we first give a proof under two additional assumptions:

(H1) the regular part $\widetilde{u}_k$ is zero, i.e.

$$\overline{u} \;=\; \sum_{q=k}^{+\infty} v_q \;=\; \widehat{u}_k. \tag{4.5.9}$$

(H2) All functions v_q are positive:

$$v_q(x) \;\geq\; 0 \qquad \forall\, x \in \mathbb{R}.$$

This implies that all the functions v_q^p constructed in Lemma 4.4.7 are positive as well,

Assuming (H1) and (H2), let $x_0 \leq x_1 \leq \cdots \leq x_n$ and $v_0, \ldots, v_n$, be such that $(x_i, v_i) \in \mathcal{Q}(2^{-k})$. One should not confuse this numbers v_i with the functions v_q in the decomposition (4.5.9). According to (4.5.3), it suffices to estimate the total variation over these points:

$$\sum_{i=1}^{n} |v_i - v_{i-1}|.$$

Since $\overline{u}$ is compactly supported, it is enough to estimate the negative variation, namely

$$\sum_{i=1}^{n} \big[v_i - v_{i-1}\big]^-. \tag{4.5.10}$$

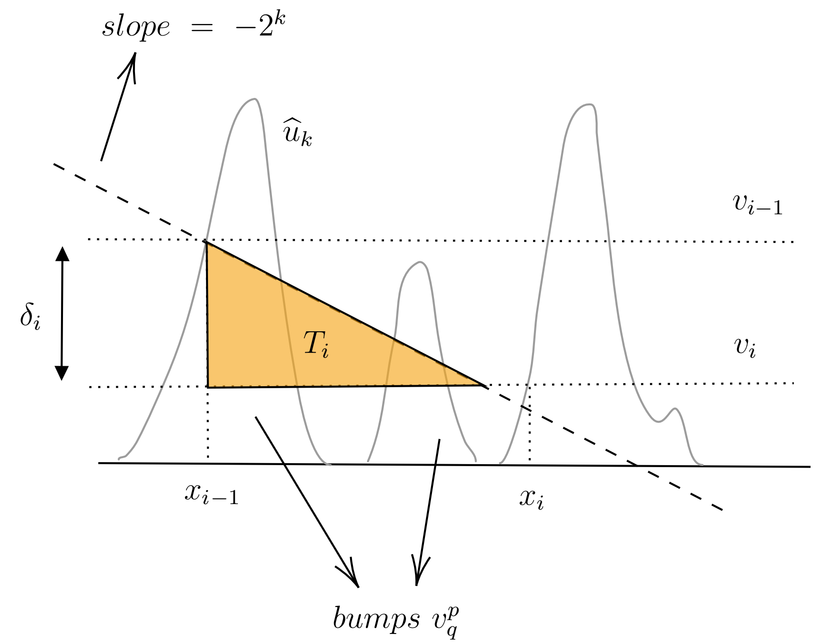

FIGURE 5. The configuration considered in the estimate for $i \in \mathcal{J}$, in step **5** of the proof.

The set of downward jumps

$$\mathcal{N} \doteq \Big\{ i \in \{1, \ldots, n\}\,;\, v_i \,<\, v_{i-1} \Big\}.$$

can be partitioned as $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where

$$\mathcal{I} \doteq \Big\{ i \in \mathcal{N}\,;\,]x_{i-1}, x_i[\subseteq \{\widehat{u}_k \neq 0\} \Big\}, \qquad \mathcal{J} \doteq \mathcal{N} \setminus \mathcal{I}.$$

In the next two steps, the negative variation (4.5.10) will be estimated by considering the terms $i \in \mathcal{I}$ and $i \in \mathcal{J}$ separately.

4. Let $i \in \mathcal{I}$. Since the characteristics starting at x_i, x_{i+1} do not cross up to time $t = 2^{-k}$, this implies

$$v_{i-1} - v_i \leq (x_i - x_{i-1}) \cdot 2^k.$$

Summing over $\mathcal{I}$ one obtains

$$\sum_{i\in\mathcal{I}} \big[v_{i-1} - v_i\big] \leq 2^k \sum_{i\in\mathcal{I}} (x_i - x_{i-1}) \leq 2^k \cdot \mathscr{L}^1\big(\{\widehat{u}_k \neq 0\}\big) \leq 2C \cdot 2^k \cdot 2^{-\alpha k} = c_1\, C \cdot 2^{(1-\alpha)k}, \tag{4.5.11}$$

where we used the inequalities

$$\mathscr{L}^1\big(\{\widehat{u}_k \neq 0\}\big) \leq \sum_{q=k}^{\infty} \mathscr{L}^1\big(\{v_q \neq 0\}\big) \leq C \sum_{q=k}^{\infty} 2^{-\alpha q} \leq \frac{2^\alpha}{2^\alpha - 1} C \cdot 2^{-\alpha k}.$$

5. Next, consider the case where $i \in \mathcal{J}$. As shown in Fig. 5, consider the triangle

$$\mathcal{T}_i \doteq \Big\{ (x, v) \in \mathbb{R}^2\,;\, x_{i-1} < x < x_i, \quad v_i \leq v \leq v_{i-1} - 2^k (x - x_{i-1}) \Big\}.$$

with height $\delta_i \doteq v_{i-1} - v_i$ and base of length $\delta_i 2^{-k}$.

In the following, we denote by $\widehat{U}_k \subset \mathbb{R}^2$ the region below the graph of $\widehat{u}_k$:

$$\widehat{U}_k \doteq \Big\{(x,v) \in \mathbb{R}^2 \,; 0 \leq v \leq \widehat{u}_k(x)\Big\} \subset \mathbb{R}^2.$$

By (4.5.5), the fact that $(x_{i-1}, v_{i-1}) \in \mathcal{Q}(2^{-k})$, implies that the area of the triangle $\mathcal{T}_i$ is bounded by the area of $\widehat{U}_k$ in the strip $[x_{i-1}, x_i] \times \mathbb{R}$:

$$\text{Area}(\mathcal{T}_i) = 2^{-(k+1)}\delta_i^2 \leq \mathscr{L}^2\Big(\widehat{U}_k \cap \big([x_{i-1}, x_i] \times \mathbb{R}\big)\Big).$$

This implies

$$\delta_i \leq 2^{\frac{k+1}{2}} \cdot \mathscr{L}^2\Big(\widehat{U}_k \cap \big([x_{i-1}, x_i] \times \mathbb{R}\big)\Big)^{1/2}. \tag{4.5.12}$$

For each $i \in \mathcal{J}$, we now consider the set of indices $\mathcal{Z}(i) \subset \mathbb{N} \times \mathbb{N}$ defined by

$$\mathcal{Z}(i) \;\doteq\; \Big\{(p,q) \in \mathbb{N}^2 \,; q \geq k, \quad \operatorname{supp} v_q^p \cap]x_{i-1}, x_i[\neq \emptyset\Big\}.$$

This is the set of all functions v_q^p in the decomposition of $\widehat{u}_k$ whose support intersects the open interval $]x_{i-1}, x_i[$.

At this stage, we make an important observation:

- *For any couple (p,q) with $q \geq k$, there can be at most two indices $i \in \mathcal{J}$ such that $(p,q) \in \mathcal{Z}(i)$*

Indeed, assume that this were not the case, i.e. for some $i_1 < i_2 < i_3$ one had $(p,q) \in \mathcal{Z}(i_1) \cap \mathcal{Z}(i_2) \cap \mathcal{Z}(i_3)$. Since the set $\{v_q^p \neq 0\}$ is connected, we would have

$$]x_{i_2 - 1}, x_{i_2}[\subset \{v_q^p \neq 0\} \subseteq \{\widehat{u}_k \neq 0\}.$$

But this is a contradiction because $i \notin \mathcal{I}$.

As in Lemma 4.4.7, call $\ell_q^p \doteq \text{meas}\big(\operatorname{supp} v_q^p\big)$. By (4.5.12) it follows

$$\delta_i \leq \sqrt{2} \cdot 2^{k/2} \cdot \left(\sum_{(p,q) \in \mathcal{Z}(i)} h_q^p \cdot \ell_q^p\right)^{1/2} \leq \sqrt{2} \;\cdot 2^{k/2} \cdot \sum_{(p,q) \in \mathcal{Z}(i)} \Big(h_q^p \cdot \ell_q^p\Big)^{1/2}.$$

Summing over $i \in \mathcal{J}$, and using the fact that each couple (p,q) can appear in the sum at most twice, we obtain

$$\begin{aligned} \sum_{i \in \mathcal{J}} \delta_i &\leq \sqrt{2} \;\cdot 2^{k/2} \cdot \sum_{i \in \mathcal{J}} \sum_{(p,q) \in \mathcal{Z}(i)} \Big(h_q^p \cdot \ell_q^p\Big)^{1/2} \\ &\leq 2^{3/2} \;\cdot 2^{k/2} \cdot \sum_{q=k}^{\infty} \sum_{p \in \mathbb{N}} \Big(h_q^p \cdot \ell_q^p\Big)^{1/2}. \end{aligned} \tag{4.5.13}$$

where we recall that h_q^p are as in (iii) of Theorem 4.4.1. Observing that

$$\Big(\ell_q^p \cdot h_q^p\Big)^{1/2} = 2^{q/2} \ell_q^p$$

and using (4.4.5), from (4.5.13) we obtain

$$\sum_{i\in\mathcal{J}}\delta_i \leq 2^{3/2}\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}\sum_{p\in\mathbb{N}} 2^{q/2}\ell_q^p \leq 2^{3/2}\cdot C\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty} 2^{q/2}2^{-\alpha q}$$
$$\leq c_2\, C\cdot\frac{1}{1-2\alpha}2^{k/2}\cdot 2^{(1/2-\alpha)k} = c_2\, C\cdot\frac{1}{2\alpha-1}\cdot 2^{(1-\alpha)k}, \tag{4.5.14}$$

where c_2 is another absolute constant. Combining (4.5.11) with (4.5.14), we obtain the desired decay rate, under the additional assumptions (H1)-(H2).

6. In the remaining steps we complete the proof of the theorem, removing the assumptions (H1)-(H2).

Recalling the decomposition (4.5.8), we observe that the function $\widetilde{u}_k$ is one-sided 2^k-Lipschitz, because each v_q is one-sided 2^q Lipschitz.

Let $x_0\leq x_1\leq\ldots\leq x_n$ and v_i, $i=0,\ldots,n$ be such that $(x_i,v_i)\in\mathcal{Q}(2^{-k})$. As before, it suffices to estimate the negative variation, i.e.

$$\sum_{i\in\mathcal{N}}\big[v_{i-1}-v_i\big],$$

where

$$\mathcal{N}\doteq\Big\{i\in\{1,\ldots,n\}\,;v_i<v_{i-1}\Big\}.$$

We partition the above set of indices as $\mathcal{N}=\mathcal{I}\cup\mathcal{J}$, where

$$\mathcal{I}\doteq\left\{i\in\mathcal{N}\,;\,]x_{i-1},x_i[\subset\bigcup_{q\geq k}\{v_q\neq 0\}\right\},\qquad \mathcal{J}\doteq\mathcal{N}\setminus\mathcal{I}.$$

Set

$$\delta_i\doteq v_{i-1}-v_i\qquad\forall\, i\in\mathcal{N}.$$

We further partition $\mathcal{J}=\mathcal{J}_1\cup\mathcal{J}_2$, by setting

$$\mathcal{J}_1\doteq\Big\{i\in\mathcal{J}\ \mid\ \widetilde{u}_k(x_{i-1})\geq v_{i-1}-\frac{\delta_i}{3}\quad\text{and}\quad\widetilde{u}_k(x_i)\leq v_i+\frac{\delta_i}{3}\Big\},\qquad\mathcal{J}_2\doteq\mathcal{J}\setminus\mathcal{J}_1\,.$$

We will estimate the quantity

$$\sum_{i\in\mathcal{N}}\delta_i=\sum_{i\in\mathcal{I}}\delta_i+\sum_{i\in\mathcal{J}_1}\delta_i+\sum_{i\in\mathcal{J}_2}\delta_i$$

by providing a bound on each term on the right hand side, in the following three steps.

7. *(Estimate of the sum over $\mathcal{I}$).* With exactly the same argument as in Step 4., we obtain the estimate

$$\sum_{i\in\mathcal{I}}\big[v_{i-1}-v_i\big]\leq c_1\cdot C\cdot 2^{(1-\alpha)k} \tag{4.5.15}$$

where c_1 is an absolute constant.

8. *(Estimate of the sum over $\mathcal{J}_1$).* In this case (see Figure 6), by definition of $\mathcal{J}_1$ the variation of $\widetilde{u}_k$ on $[x_{i-1},x_i]$ is at least one third of δ_i. Therefore the jump δ_i is controlled

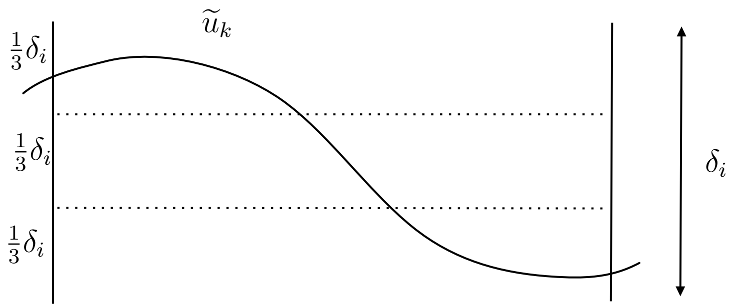

FIGURE 6. Illustration of the case $i \in \mathcal{J}_1$. At least one third of the variation is due to the regular part $\widetilde{u}_k$.

by the variation of $\widetilde{u}_k$, which is the regular part. More precisely, from the definition of $\mathcal{J}_1$ it follows

$$\delta_i \doteq v_{i-1} - v_i \leq \Big(\widetilde{u}_k(x_{i-1}) + \frac{\delta_i}{3}\Big) - \Big(\widetilde{u}_k(x_i) - \frac{\delta_i}{3}\Big) \leq \widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i) + \frac{2\delta_i}{3}\,,$$

and therefore

$$\delta_i \leq 3 \cdot \big(\widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i)\big).$$

Summing over $i \in \mathcal{J}_1$ we obtain

$$\sum_{i\in\mathcal{J}_1} \delta_i \leq 3 \sum_{i\in\mathcal{J}_1} \big(\widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i)\big) \leq 3\,\text{Tot.Var.}\{\widetilde{u}_k\} \leq c_2 \cdot C \cdot 2^{(1-\alpha)k}. \tag{4.5.16}$$

9. *(Estimate of the sum over $\mathcal{J}_2$).* The idea here is that we reduced to a situation where we can proceed as in the simplified case of the previous section, up to a modification of the definition of the triangles $\mathcal{T}_i$ that takes into account the presence of $\widetilde{u}_k$, which is one-sided 2^k-Lipschitz. Since $i \in \mathcal{J}_2$, at least one of the following inequalities is true:

$$\widetilde{u}_k(x_{i-1}) < v_{i-1} - \frac{\delta_i}{3} \qquad \text{or} \qquad \widetilde{u}_k(x_i) > v_i + \frac{\delta_i}{3}. \tag{4.5.17}$$

The proof splits in two cases depending on which one is true. To fix ideas, we assume that the first inequality holds. The second case is entirely similar. It can be handled by the same argument, in connection with the reversed initial datum: $\overline{v}(x) \doteq -\overline{u}(-x)$.

Assuming that the first inequality in (4.5.17) holds, define the triangle:

$$\mathcal{T}_i \doteq \Big\{(x,v) \in \mathbb{R}^2 \;\;|\;\; x \in (x_{i-1}, x_i), \quad v_{i-1} - \frac{\delta_i}{3} + (x - x_{i-1}) \cdot 2^k \leq v \leq v_{i-1} - (x - x_{i-1})2^k\Big\},$$

as shown in Fig. 7. We let $U \subset \mathbb{R}^2$ be the hypograph of $\overline{u}$:

$$U \doteq \Big\{(x,v) \in \mathbb{R}^2\,;\, v \leq \overline{u}(x)\Big\} \subset \mathbb{R}^2,$$

and let $\widetilde{U}_k$ be the hypograph of $\widetilde{u}_k$:

$$\widetilde{U}_k \doteq \Big\{(x,v) \in \mathbb{R}^2\,;\, v \leq \widetilde{u}_k(x)\Big\} \subset \mathbb{R}^2.$$

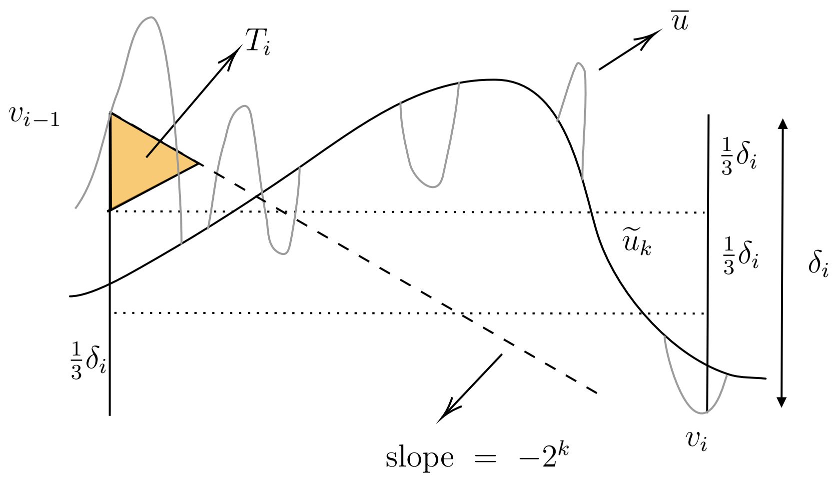

FIGURE 7. The estimate for $i \in \mathcal{J}_2$. From the fact that the value v_{i-1} survives up to time 2^{-k}, we deduce that the area of the yellow triangle can be controlled by the L^1 norm of all the v_q^p in the interval (x_{i-1}, x_i).

The fact that the couple $(x_{i-1},\, v_{i-1})$ survives up to time $t = 2^{-k}$ already implies that

$$\mathscr{L}^2(\mathcal{T}_i \setminus \widetilde{U}_k) \leq \mathscr{L}^2\Big\{(x,v) \in (U \setminus \widetilde{U}_k)\,;\, x \in [x_{i-1}, x_i]\Big\} \doteq A_i\,. \tag{4.5.18}$$

Actually, we claim that $\widetilde{U}_k \cap \mathcal{T}_i = \emptyset$. In fact, $\widetilde{u}_k$ is one-sided 2^k-Lipschitz and satisfies $\widetilde{u}_k(x_{i-1}) \leq v_{i-1} - \frac{\delta_i}{3}$. This implies

$$\widetilde{u}_k(x) \leq v_i - \frac{\delta_i}{3} + (x - x_{i-1}) \cdot 2^k \qquad \forall\, x \in]x_{i-1}, x_i[\,.$$

By definition of the triangle $\mathcal{T}_i$, this means that the hypograph of $\widetilde{u}_k$ lies entirely below the lower side of $\mathcal{T}_i$. Hence $\widetilde{U}_k \cap \mathcal{T}_i = \emptyset$. From (4.5.18) it thus follows

$$\mathscr{L}^2(\mathcal{T}_i) \leq A_i\,. \tag{4.5.19}$$

On the other hand, the area of the triangle $\mathcal{T}_i$ is

$$\mathrm{Area}(\mathcal{T}_i) = \frac{\delta_i^2}{64}\, 2^{-k}. \tag{4.5.20}$$

Combining (4.5.19) with (4.5.20) we obtain

$$\delta_i \leq 8 \cdot 2^{k/2} \cdot A_i^{1/2}. \tag{4.5.21}$$

The area A_i on the right hand side is bounded above by the sum of the areas of the blocks v_q^p whose support intersects the interval $\,]x_i, x_{i+1}[\,$. More precisely, for each $i \in \mathcal{J}_2$, define the set of indices

$$\mathcal{Z}(i) \doteq \Big\{(p,q) \in \mathbb{N}^2 \;\mid\; q \geq k, \quad \mathrm{supp}\, v_q^p \cap]x_{i-1}, x_i[\neq \emptyset\Big\}.$$

With the same argument used in step **5.** we obtain that, for every couple (p,q) with $q \geq k$, there can be at most two indices $i \in \mathcal{J}_2$ such that $(p,q) \in \mathcal{Z}(i)$.

By (4.5.21) we now obtain

$$\delta_i \le 8 \cdot 2^{k/2} \cdot \Bigg(\sum_{(p,q)\in\mathcal{Z}(i)} h^p_q \cdot \ell^p_q \Bigg)^{1/2} \le 8 \cdot 2^{k/2} \cdot \sum_{(p,q)\in\mathcal{Z}(i)} \Big(h^p_q \cdot \ell^p_q \Big)^{1/2}.$$

Summing over i, and using the fact that each (p,q) appears in the sum at most twice, we obtain

$$\sum_{i\in\mathcal{J}_2} \delta_i \le 8 \cdot 2^{k/2} \cdot \sum_{i\in\mathcal{J}_2} \sum_{(p,q)\in\mathcal{Z}(i)} \Big(h^p_q \cdot \ell^p_q \Big)^{1/2} \le 16 \cdot 2^{k/2} \cdot \sum_{q=k}^{\infty} \sum_{p\in\mathbb{N}} \Big(h^p_q \cdot \ell^p_q \Big)^{1/2}. \tag{4.5.22}$$

Observing that $\Big(\ell^p_q \cdot h^p_q\Big)^{1/2} = 2^{q/2}\ell^p_q$ and using (4.4.5), we finally obtain

$$\begin{aligned} \sum_{i\in\mathcal{J}_2} \delta_i &\le 16 \cdot 2^{k/2} \cdot \sum_{q=k}^{\infty} \sum_{p\in\mathbb{N}} 2^{q/2}\ell^p_q \le 16 \cdot 2^{k/2} \cdot C \cdot \sum_{q=k}^{\infty} 2^{q/2} 2^{-\alpha q} \\ &\le c_3 \cdot \frac{1}{2\alpha - 1} \cdot \|\overline{u}\|_{\mathcal{P}_\alpha} 2^{k/2} \cdot 2^{(1/2-\alpha)k} = c_3 \cdot \frac{1}{2\alpha-1} \cdot \|\overline{u}\|_{\mathcal{P}_\alpha} \cdot 2^{(1-\alpha)k}, \end{aligned} \tag{4.5.23}$$

where c_3 is an absolute constant. Combining the three estimates (4.5.15), (4.5.16) and (4.5.23), the proof is completed. □

REMARK 4.5.2. If Burgers' equation is replaced by a general scalar conservation law with a $\mathcal{C}^2$, uniformly convex flux f, so that $f'' \ge c > 0$, from the Hopf-Lax formula we obtain that $(x_0, u_0) \in \mathcal{Q}(t)$ if and only if

$$\int_{x_0}^{y} \left[\overline{u}(z) - (f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \right]^+ dz \ge \int_{x_0}^{y} \left[\overline{u}(z) - (f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \right]^- dz \tag{4.5.24}$$

for all $y \ge x_0$, and

$$\int_{y}^{x_0} \left[\overline{u}(z) - (f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \right]^+ dz \le \int_{y}^{x_0} \left[\overline{u}(z) - (f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \right]^- dz \tag{4.5.25}$$

for all $y \le x_0$. Here

$$f^*(u) \doteq \sup_{v\in\mathbb{R}} \big\{ uv - f(v) \big\}$$

denotes the Legendre transform of f. By a well known property of the Legendre transform (see e.g. [**46, 62**]) we have

$$(f^*)'\big(f'(u_0)\big) = u_0 \,.$$

By uniform convexity it thus follows

$$u_0 - \frac{1}{ct}(z - x_0) \le (f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \qquad \forall\ z > x_0\,,$$

$$(f^*)'\Big(f'(u_0) - \frac{z-x_0}{t}\Big) \le u_0 - \frac{1}{ct}(z - x_0) \qquad \forall\ z < x_0.$$

Using the inequalities above, the above proof remains valid up to minor modifications. Therefore, Theorem 4.4.1 remains valid for any uniformly convex flux f.

Part 2

Structural Properties for Conservation Laws with Discontinuous Flux

CHAPTER 5

Conservation Laws with Discontinuous Flux

5.1. Basic definitions and general setting

In this chapter we study conservation laws with discontinuous flux, see (22) and the following for the general setting and general introduction to the topic.

5.1.1. Connections and AB-entropy solutions. We recall here the definitions and properties of interface connection and of entropy admissible solution introduced in [**4**].

DEFINITION 5.1.1 (**Interface Connection**). Let f be a flux as in (24) satisfying the assumption (25), and let θ_l, θ_r denote the unique critical points of f_l, f_r, respectively. A pair of values $(A, B) \in \mathbb{R}^2$ is called a *connection* if

(1) $f_l(A) = f_r(B)$,
(2) $A \leq \theta_l$ and $B \geq \theta_r$.

We will say that connection (A, B) is *critical* if $A = \theta_l$ or $B = \theta_r$.

Observe that condition (2) is equivalent to: $f_l'(A) \leq 0$, $f_r'(B) \geq 0$. Therefore, if (A, B) is a connection, then the function

$$c^{AB}(x) \doteq \begin{cases} A, & x \leq 0, \\ B, & x \geq 0 \end{cases} \tag{5.1.1}$$

is a weak stationary undercompressive (or marginally undercompressive) solution of (22), since the characteristics diverge from, or are parallel to, the flux-discontinuity interface (see Figure 2). In relation to the function c^{AB} the *adapted entropy* $\eta_{AB}(x, u) = \left|u - c^{AB}(x)\right|$ is introduced in [**27**]. Then, in the spirit of [**26**], the entropy η_{AB} is employed in [**27**] to select a unique solution of the Cauchy problem (22)-(23) that satisfies the

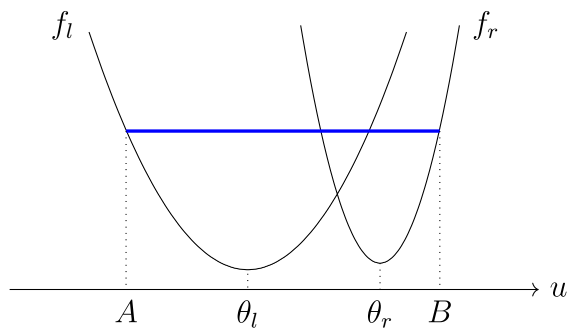

FIGURE 1. An example of connection (A, B) with f_l, f_r strictly convex fluxes

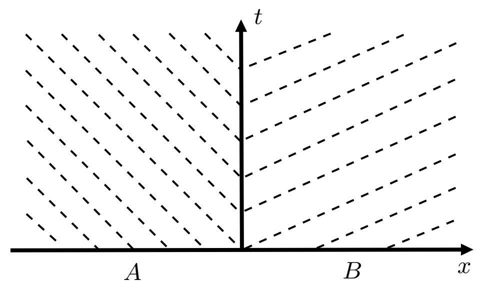

FIGURE 2. The stationary undercompressive solution c^{AB}.

interface entropy inequality

$$\left|u - c^{AB}\right|_t + \left[\operatorname{sgn}(u - c^{AB})(f(x,u) - f(x,c^{AB}))\right]_x \leq 0, \quad \text{in } \mathcal{D}', \tag{5.1.2}$$

in the sense of distributions, which leads to the following definition.

DEFINITION 5.1.2 (*AB*-**entropy solution**). Let (A,B) be a connection and let c^{AB} be the function defined in (5.1.1). A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0,+\infty[\,) \cap \mathcal{C}^0([0,+\infty), \mathbf{L}^1_{loc}(\mathbb{R}))$ is said to be an *AB-entropy solution* of the problem (22),(23) if the following holds:

(1) u is a distributional solution of (22) on $\mathbb{R}\times\,]0,+\infty[$, that is, for all test functions $\phi \in \mathcal{C}^1_c$ with compact support contained in $\mathbb{R}\times\,]0,+\infty[$, it holds true

$$\int_{-\infty}^{\infty}\int_0^{\infty} \left\{u\phi_t + f(x,u)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t = 0\,. \tag{5.1.3}$$

(2) u is a Kruzkov entropy weak solution of (22),(23) on $(\mathbb{R}\setminus\{0\})\times\,]0,+\infty[$, that is the initial condition (23) is satisfied almost everywhere, and:

(2.a) for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $]-\infty,0[\,\times\,]0,+\infty[$, it holds true

$$\int_0^{\infty}\int_{-\infty}^{0} \left\{|u-k|\phi_t + \operatorname{sgn}(u-k)\left(f_l(u) - f_l(k)\right)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall k \in \mathbb{R}\,; \tag{5.1.4}$$

(2.b) for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $]0,+\infty[\,\times\,]0,+\infty[$, it holds true

$$\int_0^{\infty}\int_{0}^{\infty} \left\{|u-k|\phi_t + \operatorname{sgn}(u-k)\left(f_r(u) - f_r(k)\right)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall k \in \mathbb{R}\,. \tag{5.1.5}$$

(3) u satisfies a Kruzkov-type entropy inequality relative to the connection (A,B), that is, for any non-negative test function $\phi \in \mathcal{C}^1_c$ with compact support contained in $\mathbb{R}\times]0,+\infty[$, it holds true

$$\int_{-\infty}^{\infty}\int_0^{\infty} \left\{\left|u - c^{AB}\right|\phi_t + \operatorname{sgn}(u - c^{AB})\left(f(x,u) - f(x,c^{AB})\right)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq 0\,. \tag{5.1.6}$$

REMARK 5.1.3. Since an *AB*-entropy solution u is in particular an entropy weak solution of a scalar conservation law with uniformly convex flux, on $]-\infty,0[\,\times\,]0,+\infty[$, and on $]0,+\infty[\,\times\,]0,+\infty[$ (by property (2) of Definition 5.1.2 and assumption (25)), it follows that $u(\cdot\,,t) \in BV_{\text{loc}}(\mathbb{R}\setminus\{0\})$ for any $t > 0$. Here $BV_{\text{loc}}(\mathbb{R}\setminus\{0\})$ denotes the set of functions that have finite total variation on compact subsets of $\mathbb{R}\setminus\{0\}$. On the other hand, relying on a result in [**92**] (see also [**99**]), we deduce that u admits left and right strong traces at $x = 0$ for a.e. $t > 0$, i.e. that there exist the one-sided limits

$$u_l(t) \doteq u(0-, t), \qquad u_r(t) \doteq u(0+, t), \qquad \text{for a.e. } t > 0\,. \tag{5.1.7}$$

We point out that a consequence of the characterization of attainable profiles provided by our results (Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14) will be that these limits are actually defined at every time $t > 0$ (not only at almost every time). Moreover, since u is also a distributional solution of (22) on $\mathbb{R}\times\,]0, +\infty[$ (by property (1) of Definition 5.1.2), we deduce that u must satisfy the Rankine-Hugoniot condition at the interface $x = 0$:

$$f_l(u_l(t)) = f_r(u_r(t)), \qquad \text{for a.e. } t > 0. \tag{5.1.8}$$

In (5.1.7) and throughout the paper, for the one-sided limits of a function $u(x)$ we use the notation

$$u(x\pm) \doteq \lim_{y\to x\,\pm} u(y). \tag{5.1.9}$$

In relation to a connection (A, B) consider the function

$$I^{AB}(u_l, u_r) \doteq \mathrm{sgn}(u_r - B)\,(f_r(u_r) - f_r(B)) - \mathrm{sgn}(u_l - A)\,(f_l(u_l) - f_l(A))\,, \qquad u_l, u_r \in \mathbb{R}\,, \tag{5.1.10}$$

which is useful to characterize the interface entropy admissibility criterion. In fact, by the analysis in [**27**, Lemma 3.2] and [**25**, Section 4.8], it follows that, because of condition (1) of Definition 5.1.2 and assumption (25), the following holds.

Lemma 5.1.4. *Let $u \in \mathbf{L}^\infty(\mathbb{R}\times[0, +\infty))$ be a function satisfying conditions (1)-(2) of Definition 5.1.2. Then, condition (3) is equivalent to the AB interface entropy condition*

(3)'

$$I^{AB}(u_l(t), u_r(t)) \leq 0 \qquad \textit{for a.e. } t > 0\,. \tag{5.1.11}$$

Lemma 5.1.5. *Let (A, B) be a connection. Then, for any pair $(u_l, u_r) \in \mathbb{R}^2$, the conditions*

$$f_l(u_l) = f_r(u_r), \qquad I^{AB}(u_l, u_r) \leq 0\,, \tag{5.1.12}$$

are equivalent to the conditions

$$\begin{aligned} &f_l(u_l) = f_r(u_r) \geq f_l(A) = f_r(B),\\ &\big(u_l \leq \theta_l, \quad u_r \geq \theta_r\big) \quad\Longrightarrow\quad u_l = A, \quad u_r = B\,. \end{aligned} \tag{5.1.13}$$

The first condition in (5.1.13) tells us that, when we choose a connection (A, B) and we employ the concept of AB-entropy solution, we are imposing a constraint (from below) on the flux at the interface $x = 0$. In order to achieve existence, we need to compensate for this constraint with an additional freedom in the admissibility criteria. In fact, the second condition in (5.1.13) prescribes the admissibility of exactly one undercompressive wave at the interface, given by c^{AB} in (5.1.1). This rule corresponds to the (A, B) *characteristic condition* in [**27**, Definition 1.4].

REMARK 5.1.6. In view of (5.1.8), we can extend the classical concept of genuine characteristic for solutions to conservation laws with continuous fluxes (see [**55**]) by considering also characteristics that are refracted by the discontinuity interface $x = 0$. Thus, we will say that a polygonal line $\vartheta : [0, T] \to \mathbb{R}$ is a *genuine characteristic for an AB-entropy solution u* if one of the following cases occurs:

(i) $\vartheta(t) < 0$ for all $t \in]0, T[$, and ϑ is a characteristic for the restriction of u on $]-\infty, 0[\times]0, T[$;
(ii) $\vartheta(t) > 0$ for all $t \in]0, T[$, and ϑ is a characteristic for the restriction of u on $]0, +\infty[\times]0, T[$;
(iii) there exists $\tau \in]0, T[$, such that:
  - $\vartheta(t) < 0$ for all $t \in]0, \tau[$, and ϑ is a characteristic for the restriction of u on $]-\infty, 0[\times]0, \tau[$,
  - $\vartheta(t) > 0$ for all $t \in]\tau, T[$, and ϑ is a characteristic for the restriction of u on $]0, +\infty[\times]\tau, T[$,
  or viceversa.
  - $f_l(u_l(\tau)) = f_r(u_r(\tau))$ and $I^{AB}(u_l(\tau), u_r(\tau)) \leq 0$,

where we are using the term "characteristic" for a classical genuine characteristic of a solution to the conservation law $u_t + f_l(u)_x = 0$ on $\{x < 0\}$, or to the conservation law $u_t + f_r(u)_x = 0$ on $\{x > 0\}$.

REMARK 5.1.7 (Local solutions). Throughout the paper we say that a function $u \in \mathbf{L}^\infty(\Omega)$ is a (local) *AB-entropy solution* of (22) on a domain

$$\Omega \doteq \big\{(t, x) \mid t \in [a, b], \quad \gamma_1(t) < x < \gamma_2(t)\big\} \subset \mathbb{R}\times]0, +\infty[$$

where $\gamma_1 < \gamma_2 : [a, b] \to \mathbb{R}$ are Lipschitz curves if it satisfies the conditions of Definition 5.1.2 localized on Ω. Namely, if the following holds:

(1) For any test functions $\phi \in \mathcal{C}^1_c(\Omega)$ with compact support contained in Ω, it holds true (5.1.3).
(2) The map $t \mapsto u(\cdot, t)$ is continuous from $I \doteq \{t > 0 : (x, t) \in \Omega\}$ to $\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\Omega_t)$, $\Omega_t \doteq \{x : (x, t) \in \Omega\}$, and it holds:
  (2.a) for any non-negative test function $\phi \in \mathcal{C}^1_c(\Omega)$ with compact support contained in $\Omega \cap \big(]-\infty, 0[\times]0, +\infty[\big)$, it holds true (5.1.4);
  (2.b) or any non-negative test function $\phi \in \mathcal{C}^1_c(\Omega)$ with compact support contained in $\Omega \cap \big(]0, +\infty[\times]0, +\infty[\big)$, it holds true (5.1.5).
(3) For any test functions $\phi \in \mathcal{C}^1_c(\Omega)$ with compact support contained in Ω, it holds true (5.1.6).

We will sometimes implicitly use the following fact: assume that two local AB-entropy solutions u_1, u_2, of (22) are given on two disjoint domains Ω_1, Ω_2, such that $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma$, where Γ is the graph of a Lipschitz curve $\gamma : [a, b] \to \mathbb{R}$, with $\Omega_1 \subset \{x \leq \gamma(t)\}$, $\Omega_2 \subset \{x \geq \gamma(t)\}$. Moreover, assume that $u_1(t, \gamma(t)-) = u_2(t, \gamma(t)+)$ for a.e. $t \in [a, b]$ such that $\gamma(t) \neq 0$, and that

$$f_l(u_1(0-, t)) = f_r(u_2(0+, t)), \qquad I^{AB}(u_1(0-, t), u_2(0+, t)) \leq 0,$$

for a.e. $t \in [a,b]$ such that $\gamma(t) = 0$. Then, by standard arguments one can deduce that the function

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if } (x,t) \in \Omega_1, \\ u_2(x,t) & \text{if } (x,t) \in \Omega_2 \end{cases}$$

is an AB-entropy solution of (22) on Ω.

It was proved in [**4, 27**] (see also [**25, 64**]) that AB-entropy solutions of (22),(23) with bounded initial data are unique and form an $\mathbf{L}^1$-contractive semigroup. Moreover, we will show that they are $\mathbf{L^1}$-stable also with respect to the values A, B of the connection. This type of stability, beside being used to extend our main results from the case of non critical connections to the critical one, has an interest on its own. We will also prove that AB-entropy solutions of (22),(23) are $\mathbf{L^1}$-Lipschitz continuous in time. This property is an immediate consequence of the BV regularity of such solutions in the case of non critical connections. Instead, in the case of critical connections where $A = \theta_l$ or $B = \theta_r$, and $f_l(\theta_l) \neq f_r(\theta_r)$, the total variation of an AB-entropy solution may well blow up in a neighborhood of the flux-discontinuity interface $x = 0$, as shown in [**1**]. However, we recover the $\mathbf{L^1}$-Lipschitz continuity in time also in this case exploiting the BV regularity of the flux of an AB-entropy solution, which is established relying on the analysis pursued in this paper. We collect all these (old and new) results in the following statement:

THEOREM 5.1.8. **(Semigroup of $\boldsymbol{AB}$-Entropy Solutions)** *Let f be a flux as in* (24) *satisfying the assumption* (25)*, and let (A,B) be a connection. Then there exists a map*

$$\mathcal{S}^{[AB]+} : [0,+\infty[\, \times \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R}), \qquad (t,u_0) \mapsto \mathcal{S}_t^{[AB]+} u_0\,, \tag{5.1.14}$$

enjoying the following properties:

(i) For each $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, the function $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$ provides the unique bounded, AB-entropy solution of the Cauchy problem (22), (23)*.*

(ii) $\mathcal{S}_0^{[AB]+} u_0 = u_0$, $\mathcal{S}_s^{[AB]+} \circ \mathcal{S}_t^{[AB]+} u_0 = \mathcal{S}_{s+t}^{[AB]+} u_0$, for all $t, s \geq 0$, $u_0 \in \mathbf{L}^\infty(\mathbb{R})$.

(iii) For any $u_0, v_0 \in \mathbf{L}^\infty(\mathbb{R})$, there exists a constant $L > 0$, depending on f and on $\|u_0\|_{\mathbf{L}^\infty}, \|v_0\|_{\mathbf{L}^\infty}$, such that, for any $R > 0$, it holds:
$\big\|\mathcal{S}_t^{[AB]+} u_0 - \mathcal{S}_t^{[AB]+} v_0\big\|_{\mathbf{L}^1([-R,R])} \leq \big\|u_0 - v_0\big\|_{\mathbf{L}^1([-R-Lt,R+Lt])}$, *for all $t \geq 0$.*

(iv) For any $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, and for any $R > 0$, it holds:
$\big\|\mathcal{S}_t^{[AB]+} u_0 - \mathcal{S}_t^{[A'B']+} u_0\big\|_{\mathbf{L}^1([-R,R])} \leq 2\,t\,\big|f_r(B) - f_r(B')\big|$,
for all connections (A,B), (A',B'), and for all $t \geq 0$.

(v) For any $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, and for any $R > 0$, there exists a constant $C_R > 0$ depending on f, $\|u_0\|_{\mathbf{L}^1}$, R, and on the connection (A,B), such that it holds:
$\big\|\mathcal{S}_t^{[AB]+} u_0 - \mathcal{S}_s^{[AB]+} u_0\big\|_{\mathbf{L}^1([-R,R])} \leq \dfrac{C_R}{t}|s-t|$, *for all $s > t > 0$.*

The proof of the new properties (iv)-(v) is given in Appendix 5.6.

REMARK 5.1.9. Most of the literature on conservation laws with discontinuous flux as in (24) considers fluxes f_l, f_r satisfying

$$f_l(0) = f_r(0), \quad f_l(1) = f_r(1)\,, \qquad \theta_l \geq 0, \quad \theta_r \leq 1. \tag{5.1.15}$$

However the existence of an $\mathbf{L}^1$-contractive semigroup of AB-entropy solutions of the Cauchy problem (22), (23), remains valid also without this assumption as shown for example in [**64**]. On the other hand, by a reparametrization of the fluxes, one can always reduce the problem to the setting where the critical points of f_l, f_r satisfy (5.1.15). In fact, given f_l, f_r, for any pair of invertible affine maps $\phi_l, \phi_r : \mathbb{R} \to \mathbb{R}$, we can observe that a map $u(x,t)$ is an $\phi_l^{-1}(A)\phi_r^{-1}(B)$-entropy solution of (22), (24) with fluxes f_l, f_r if and only if

$$\widetilde{u}(x,t) \doteq \begin{cases} \phi_l^{-1}(u) & \text{if } x < 0, \\ \phi_r^{-1}(u) & \text{if } x > 0, \end{cases}$$

is an AB-entropy solution of (22), (24) with fluxes $f_l \circ \phi_l$, $f_r \circ \phi_r$.

Remark 5.1.10. By the analysis in [**64**, §3.1] (see also [**9**, Remark 4.1]) it follows that, for every $M > 0$, there exists $C_M > 0$ such that, if $\|u_0\|_{\mathbf{L}^\infty} \leq M$, and $A, B \leq M$, then $\|\mathcal{S}_t^{[AB]+} u_0\|_{\mathbf{L}^\infty} \leq C_M$, for all $t > 0$.

Corollary 5.1.11. *Let $\{(A_n, B_n)\}_n$ be a sequence of connections that converges in $\mathbb{R}^2$ to a connection (A, B), and let $\{u_{n,0}\}_n$ be a sequence of functions in $\mathbf{L}^\infty(\mathbb{R})$ that converges in $\mathbf{L}^1_{\mathrm{loc}}$ to $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Let $u_{n,l}, u_{n,r}$ denote, respectively, the left and right traces at $x = 0$ of $u_n(x,t) \doteq \mathcal{S}_t^{[A_n B_n]+} u_{n,0}(x)$, defined as in* (5.1.7)*. Similarly, let u_l, u_r denote the left and right traces at $x = 0$ of $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$. Then, we have:*

$$f_l(u_{n,l}) \rightharpoonup f_l(u_l), \qquad f_r(u_{n,r}) \rightharpoonup f_r(u_r) \quad \textit{weakly in} \quad \mathbf{L}^1(\mathbb{R}^+)\,. \tag{5.1.16}$$

The proof of the Corollary is given in Appendix 5.6.

Remark 5.1.12. We point out that, differently from (5.1.16), in general the $\mathbf{L^1}$-convergence $u_{n,l} \to u_l$ and $u_{n,r} \to u_r$ fails due to the possible formation of stationary boundary layers at the interface $x = 0$, as one can see in the following

Example 5.1.13. Consider a non critical connection (A, B) and the sequence of initial data

$$u_{n,0}(x) = \begin{cases} \overline{A}, & \text{if } x \leq -n^{-1}, \\ A, & \text{if } x \in]-n^{-1}, 0[, \\ \overline{B}, & \text{if } x \geq 0, \end{cases}$$

with

$$\overline{A} \doteq (f_{l|[\theta_l, +\infty[})^{-1} \circ f_l(A)\,, \qquad \overline{B} \doteq (f_{r|]-\infty, \theta_r]})^{-1} \circ f_r(B), \tag{5.1.17}$$

where $f_{|I}$ denotes the restriction of the function f to the interval I. One can immediately check that the AB-entropy solution of (22), (23), with initial datum $u_{n,0}$ is the stationary solution $u_n(t,x) = \mathcal{S}_t^{[AB]+} u_{n,0}(x) = u_{n,0}(x)$ and that u_n converges in $\mathbf{L}^1$ to

$$u(x,t) = \begin{cases} \overline{A} & \text{if} \quad x < 0, \\ \overline{B} & \text{if} \quad x > 0\,. \end{cases}$$

Moreover, one has $u_{n,l}(t) = A$ and $u_l(t) = \overline{A}$ for every $t > 0$.

5.1.2. Backward solution operator. In this section we shall first review quickly the concept of backward solution operator for conservation laws with flux depending only on the state variable, and then we will introduce the definition of backward solution operator associated to a connection (A, B), for spatially discontinuous flux as in (24).

5.1.2.1. *Backward solution operator for conservation laws with space independent flux.* The use of the backward-forward method to characterize the attainable set for conservation laws was first proposed in [**70, 78**] (see also [**60**] in the framework of Hamilton-Jacobi equations). Because of the regularizing effect of the nonlinear dynamics of a conservation law

$$u_t + f(u)_x = 0, \qquad x \in \mathbb{R}\,, \tag{5.1.18}$$

with uniformly convex flux $f(u)$, the only restriction to controllability of (5.1.18) at a fixed time $T > 0$, when one regards as controls the initial data

$$u(x, 0) = u_0(x), \qquad\quad x \in \mathbb{R}\,, \tag{5.1.19}$$

is the decay of positive waves. Therefore it is by now well known the characterization of the attainable set

$$\mathcal{A}(T) = \big\{\omega \ :\ \omega = u(\cdot, T),\ u \text{ entropy weak solution of (5.1.18)-(5.1.19) with } u_0 \in \mathbf{L}^\infty\big\}, \tag{5.1.20}$$

in terms of the Oleinik-type inequality

$$D^+\omega(x) \le \frac{1}{T f''(\omega(x))}, \quad \forall x \in \mathbb{R}\,, \tag{5.1.21}$$

where $D^+\omega$ denotes the upper Dini derivative of ω (see (5.3.9)). Similar results in the case of boundary controllability were obtained in [**3, 16, 17, 73**]).

A different perspective to address this controllability problem was introduced in [**70, 78**], and consists in constructing initial data leading to attainable targets ω at a time horizon $T > 0$, through the definition of an appropriate concept of backward solution to (5.1.18). Namely, letting $\mathcal{S}^+_t u_0(x)$ denote the (forward) entropy weak solution of the Cauchy problem (5.1.18)-(5.1.19) evaluated at (x, t), it was defined in [**78**] an appropriate *backward operator* $\mathcal{S}^-_T : \mathbf{L}^\infty \to \mathbf{L}^\infty$, and proved that a profile ω belongs to $\mathcal{A}(T)$ if and only if $\omega = \mathcal{S}^+_T \circ \mathcal{S}^-_T \omega$, i.e. if and only if it is a fixed point of the backward-forward operator $\mathcal{S}^+_T \circ \mathcal{S}^-_T$ (see [**70**, Corollary 1]). Moreover, for $\omega \in \mathcal{A}(T)$, the solution defined as

$$u^*(x, t) \doteq \mathcal{S}^+_t(\mathcal{S}^-_T \omega)(x), \qquad x \in \mathbb{R},\ t \in [0, T]\,, \tag{5.1.22}$$

is the unique solution to (5.1.18) that is locally Lipschitz on the strip $\mathbb{R}\times]0, T[$, and yields ω at time T. Equivalently,

$$u^*_0 \doteq \mathcal{S}^-_T \omega \tag{5.1.23}$$

is the unique initial datum that produces a solution to (5.1.18) locally Lipschitz on $]0, T[$, yielding ω at time T. The operator $\mathcal{S}^-_t$, for $t \ge 0$, is defined as follows

$$\mathcal{S}^-_t \omega(x) \doteq \mathcal{S}^+_t\big(\omega(-\ \cdot)\big)(-x) \qquad x \in \mathbb{R},\ t \ge 0\,. \tag{5.1.24}$$

In words, we use $\omega(-\ \cdot)$ as initial datum for the forward operator $\mathcal{S}^+_t$, we compute the (forward) solution to (5.1.18), and then we reverse the space variable.

Remark 5.1.14 (Classical solutions). Throughout the paper by a *classical solution* to a conservation law with space independent flux $u_t + f(u)_x = 0$ we mean a locally Lipschitz function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}\times]0,+\infty[$, such that

$$u_t(t,x) + f(u(x,t))_x = 0 \qquad \text{for a.e. } (t,x) \in \Omega.$$

Any classical solution is an entropy admissible weak solution. The function (5.1.22) is a classical solution to (5.1.18). Sometimes in the literature classical solutions are denoted as *strong solutions*.

Remark 5.1.15. One can easily verify that the function $\mathrm{w}(x,t) \doteq \mathcal{S}_t^- \omega(x)$ is the entropy weak solution of the Cauchy problem

$$\begin{cases} \mathrm{w}_t - f(\mathrm{w})_x = 0, & x \in \mathbb{R}, \quad t \geq 0, \\ \mathrm{w}(x,0) = \omega(x), & x \in \mathbb{R}. \end{cases} \tag{5.1.25}$$

In fact, by definition (5.1.24) it follows that $\mathrm{w}(x,t)$ is a distributional solution of (5.1.25), since it is obtained from the distributional solution $\mathcal{S}_t^+\big(\omega(-\ \cdot)\big)(x)$ of (5.1.18) by the change of variable $x \mapsto -x$. On the other hand, since every shock discontinuity of $\mathcal{S}_t^+\big(\omega(-\ \cdot)\big)(x)$, connecting a left state u^- with a right state u^+, must satisfy the Lax condition $u^- > u^+$ (equivalent to the entropy admissibility criterion since the flux $f(u)$ in (5.1.18) is convex, e.g. see [**54, 76**]), it follows that the left and right states u^-, u^+ of every shock discontinuity in $\mathrm{w}(x,t)$ must satisfy the reverse condition $u^- < u^+$, which is the Lax admissibility condition for (5.1.25), since the flux $-f(\mathrm{w})$ is concave. Finally, we can observe that $\mathrm{w}(x,0) = \mathcal{S}_0^+\big(\omega(-\ \cdot)\big)(-x) = \omega(x)$, for all $x \in \mathbb{R}$, which completes the proof of our claim.

This procedure to characterize the attainable profiles is motivated by the following observation. Given a target profile ω, if we know that for any $t \in]0,T[$, the map $x \mapsto v(x,t) \doteq \mathcal{S}_t^+\big(\omega(-\cdot)\big)(x)$ is locally Lipschitz on $\mathbb{R}$, it would follow that $u(x,t) \doteq v(-x, T-t) = \mathcal{S}_{T-t}^- \omega(x)$ is a classical solution of (5.1.18) which attains the target profile ω at time $t = T$, and starts with the initial datum u_0^* in (5.1.23). Since classical solutions of (5.1.18) are entropy admissible, by uniqueness of entropy weak solutions of the Cauchy problem for (5.1.18) it would follow that $u(x,t) = \mathcal{S}_t^+ u_0^*(x) = u^*(x,t)$. However, if v admits shock discontinuities, the function $v(-x, T-t)$ fails to be an entropy admissible solution of (5.1.18), despite still being a weak distributional solution of (5.1.18). The one-sided Lipschitz condition (5.1.21) is precisely equivalent to the property that the map $x \mapsto v(x,t) \doteq \mathcal{S}_t^+\big(\omega(-\cdot)\big)(x)$ is locally Lipschitz on $\mathbb{R}$, for all $t \in]0,T[$ (e.g. see [**10, 16**]), and thus one obtains the characterization of the elements of $\mathcal{A}(T)$ as fixed points of the backward-forward operator.

5.1.2.2. *Backward solution operator in the spatially-discontinuous flux setting.* Given a flux f as in (24) satisfying the assumption (25), and a connection (A,B), let $\mathcal{S}^{[AB]+}$ be the *forward semigroup operator* associated to the connection (A,B), as in Theorem 5.1.8. Observe that, letting $\overline{A}, \overline{B}$ be as in (5.1.17), the pair $(\overline{B}, \overline{A}\,)$ turns out to be a connection for the symmetric flux

$$\overline{f}(x,u) = \begin{cases} f_r(u), & x \leq 0, \\ f_l(u), & x \geq 0\,, \end{cases} \tag{5.1.26}$$

(see Figure 3).

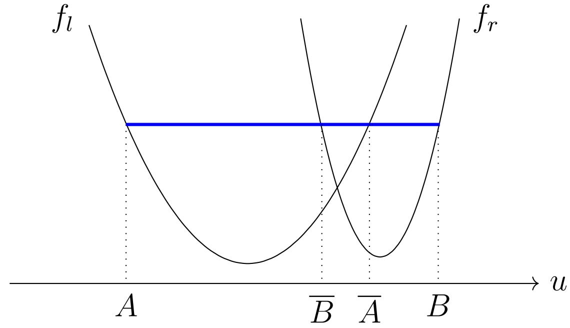

FIGURE 3. The connection $(\overline{B}, \overline{A})$ of the symmetric flux $\overline{f}(x, u)$ defined in (5.1.26).

Then, letting $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+} u_0(x)$ denote the unique $\overline{B}\,\overline{A}$-entropy solution of

$$\begin{cases} u_t + \overline{f}(x,u)_x = 0 & x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{5.1.27}$$

evaluated at (x, t), we shall define the backward solution operator associated to the connection (A, B) in terms of the operator $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}$ as follows.

DEFINITION 5.1.16 (**AB-Backward solution operator**). Given a connection (A, B), the *backward solution operator* associated to (A, B) is the map $\mathcal{S}_{(\cdot)}^{[AB]-} : [0, +\infty) \times \mathbf{L}^\infty(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R})$, defined by

$$\mathcal{S}_t^{[AB]-}\omega(x) \doteq \overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(-x) \qquad x \in \mathbb{R},\ t \geq 0\,. \tag{5.1.28}$$

REMARK 5.1.17. One can show that the function $\mathrm{w}(x,t) \doteq \mathcal{S}_t^{[AB]-}\omega(x)$ is the $\overline{A}\,\overline{B}$-entropy solution of the Cauchy problem

$$\begin{cases} \mathrm{w}_t - f(x, \mathrm{w})_x = 0, & x \in \mathbb{R}, \quad t \geq 0, \\ \mathrm{w}(x,0) = \omega(x), & x \in \mathbb{R}. \end{cases} \tag{5.1.29}$$

Notice that in (5.1.29) the flux is $-f(x, \mathrm{w})$, which is a discontinuous function that coincides with the uniformly strictly concave maps $-f_l(\mathrm{w}), -f_r(\mathrm{w})$, on the left and on the right, respectively, of $x = 0$. As observed in [**9**, §7], in the case of a two-concave flux as $-f(x, \mathrm{w})$, one replaces the $\leq$ sign with the $\geq$ sign, and viceversa, in the Definition 5.1.1 of interface connection. Thus, $(\overline{A}, \overline{B})$ is indeed a connection for the flux $-f(x, \mathrm{w})$. The $\overline{A}\,\overline{B}$ interface entropy admissibility condition for $\mathrm{w}(x,t)$ is formulated as in (5.1.11). In order to verify the claim that $\mathrm{w}(x,t)$ is the $\overline{A}\,\overline{B}$-entropy solution of the Cauchy problem (5.1.29) we proceed as in Remark 5.1.15. We first observe that $\mathrm{w}(x,t)$ is a distributional solution of (5.1.29), and that it is entropy admissible in the regions $\{x < 0\}$, $\{x > 0\}$. In fact, by definition (5.1.28), $\mathrm{w}(x,t)$ is obtained from $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)$ with the change of variable $x \mapsto -x$, and we have $\mathrm{w}(x,0) = \overline{\mathcal{S}}_0^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(-x) = \omega(x)$, for all $x \in \mathbb{R}$. Next, we check that $\mathrm{w}(x,t)$ satisfies the $\overline{A}\,\overline{B}$ entropy condition (5.1.11) for the two-concave flux

$-f(x,\mathrm{w})$, i.e. that, letting $\mathrm{w}_l(t), \mathrm{w}_r(t)$ denote the left and right traces of $\mathrm{w}(x,t)$ at $x=0$, it holds true

$$\operatorname{sgn}(\mathrm{w}_r(t)-\overline{B}\,)\big(-f_r(\mathrm{w}_r(t))+f_r(\,\overline{B}\,)\big)-\operatorname{sgn}(\mathrm{w}_l(t)-\overline{A}\,)\big(-f_l(\mathrm{w}_l(t))+f_l(\,\overline{A}\,)\big)\leq 0 \qquad \text{for a.e. } t>0\,. \tag{5.1.30}$$

Observe that the left and right traces $u_l(t), u_r(t)$ of $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)$ at $x=0$, satisfy the $\overline{B}\,\overline{A}$ entropy condition (5.1.11) for the flux $\overline{f}$ in (5.1.26), that reads

$$\operatorname{sgn}(u_r(t)-\overline{A}\,)\big(f_l(u_r(t))-f_l(\,\overline{A}\,)\big)-\operatorname{sgn}(u_l(t)-\overline{B}\,)\big(f_r(u_l(t))-f_r(\,\overline{B}\,)\big)\leq 0 \qquad \text{for a.e. } t>0\,. \tag{5.1.31}$$

On the other hand, since one obtains $\mathrm{w}(x,t)$ from $\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)$ reversing the space variables, we have $u_l(t)=\mathrm{w}_r(t)$, $u_r(t)=\mathrm{w}_l(t)$ for all $t>0$. Hence, we recover (5.1.30) from (5.1.31), thus completing the proof of the claim.

REMARK 5.1.18. We observe that if ω is an attainable state in $\mathcal{A}^{[AB]}(T)$, it will follow from our results that the solution $v(x,t)\doteq\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)$ related to the backward solution operator may well contain a shock discontinuity exiting from the interface $x=0$ at a time $\tau<T$. As a consequence here, differently from the space-independent flux setting, the map $x\mapsto v(x,t)$ is in general not locally Lipschitz outside the interface $\{x=0\}$. In turn, this implies that, for $\omega\in\mathcal{A}^{[AB]}(T)$, the (forward) AB-entropy solution defined by

$$u^*(x,t)\doteq\mathcal{S}_t^{[AB]+}\big(\mathcal{S}_T^{[AB]-}\omega\big)(x), \qquad x\in\mathbb{R},\ t\in[0,T]\,, \tag{5.1.32}$$

will be in general different from $v(-x,T-t)$ on $\mathbb{R}\times[0,T[$. However, exploiting the duality property enjoyed by the forward and backward solution operators (see § 5.2), one can still prove that $u^*(x,T)=v(-x,0)=\omega(x)$ for all $x\in\mathbb{R}$, which shows that ω is a fixed point of the backward-forward operator $\mathcal{S}_T^{[AB]+}\circ\mathcal{S}_T^{[AB]-}$ as stated in Theorem 9.

5.2. Technical tools for characterization of the near-interface wave structure

In this section we introduce some technical tools needed to characterize the pointwise constraints satisfied by the attainable profiles of (22) in intervals containing the origin. Throughout the section, $f:\mathbb{R}\to\mathbb{R}$ will be a twice continuously differentiable, uniformly convex map, and we let θ be its unique critical point, $f'(\theta)=0$. Set

$$\lambda(u,v)\doteq\frac{f(v)-f(u)}{v-u}, \qquad u,v\in\mathbb{R},\ u\neq v\,, \tag{5.2.1}$$

and observe that, by the convexity of f one has

$$f'(u)<\lambda(u,v)<f'(v) \qquad \forall\ u<v\in\mathbb{R}\,. \tag{5.2.2}$$

5.2.1. Left backward shock (Figure 4, left). For every $B>\theta$, $0<\mathsf{R}<T\cdot f'(B)$, we define here:

- two constants $\boldsymbol{t}[\mathsf{R},B,f]$, $\boldsymbol{u}[\mathsf{R},B,f]$;
- a function $t\mapsto\boldsymbol{y}[\mathsf{R},B,f](t)$, $t\in[\boldsymbol{t}[\mathsf{R},B,f],\,T]$;

which enjoy the following properties that will be justified in the sequel (see § 5.2.4, 5.4.5), but that we highlight here to clarify the purpouse of their introduction. Let (A, B) be a connection for a flux as in (22), and let $\overline{A}, \overline{B}$ be as in (5.1.17). Then, the map $t \mapsto \boldsymbol{y}[\mathsf{R}, B, f_r](t)$ identifies the location of a shock curve in a $\overline{BA}$-entropy solution of $u_t + \overline{f}(x,u)_x = 0$, with $\overline{f}$ as in (5.1.26), defined on some domain $\Omega \subset]-\infty, 0] \times [0, +\infty[$. Since a $\overline{BA}$-entropy solution of (5.1.27) is associated to the backward solution operator (5.1.28), we will say that $\boldsymbol{y}[\mathsf{R}, B, f_r]$ identifies the location of a *left backward shock.*

This curve starts from the interface $\{x = 0\}$ at time $t = \boldsymbol{t}[\mathsf{R}, B, f_r]$, and reaches the point $x = \boldsymbol{y}[\mathsf{R}, B, f_r](T)$ at time $t = T$. Such a shock discontinuity has, at time $t = T$, left state $\boldsymbol{u}[\mathsf{R}, B, f_r]$ and right state $\overline{B}$. The point $(-\mathsf{R}, 0)$ is the center of a rarefaction fan located on the left of the curve $x \mapsto (\boldsymbol{y}[\mathsf{R}, B, f_r](t), t)$.

We proceed to introduce these definitions as follows. Set

$$\boldsymbol{t}[\mathsf{R}, B, f] \doteq \frac{\mathsf{R}}{f'(B)}, \qquad \overline{B} \doteq (f_{|]-\infty,\theta]})^{-1} \circ f(B)\,. \tag{5.2.3}$$

Then, consider the Cauchy problem

$$\begin{cases} y'(t) = \lambda\Big((f')^{-1}\big(\frac{y(t)+\mathsf{R}}{t}\big), \overline{B}\Big), & t \geq \boldsymbol{t}[\mathsf{R}, B, f]\,, \\ y(\boldsymbol{t}[\mathsf{R}, B, f]) = 0. \end{cases} \tag{5.2.4}$$

By (5.2.1), the differential equation in (5.2.4) ensures that, for all $t \geq \boldsymbol{t}[\mathsf{R}, B, f]$, the pair $\big((f')^{-1}(\frac{y(t)+\mathsf{R}}{t}), \overline{B}\big)$ satisfies the Rankine-Hugoniot condition with slope $y'(t)$ for the conservation law $u_t + f(u)_x = 0$. Observe that, since $g(t,y) \doteq \lambda\big((f')^{-1}\big(\frac{y+\mathsf{R}}{t}\big), \overline{B}\big)$ is locally Lipschitz continuous in y, by classical arguments it admits a unique solution $\boldsymbol{y}(t)$ defined on some maximal interval $[t[\mathsf{R}, B, f], \tau[$. On the other hand, because of (5.2.2) we have

$$g(t,y) > f'(\overline{B}) \qquad \forall\, t \in \big[\boldsymbol{t}[\mathsf{R}, B, f], \min\{\tau, T\}\big[\,,\ y > -\mathsf{R} + T \cdot f'(\overline{B})\,, \tag{5.2.5}$$

and hence, since $f'(\overline{B}) < 0$, it follows that

$$\begin{aligned} \boldsymbol{y}(t) &> \big(\min\{\tau, T\} - t[\mathsf{R}, B, f]\big) \cdot f'(\overline{B}) \\ &\geq \min\{\tau, T\} \cdot f'(\overline{B}) \end{aligned} \qquad \forall\, t \in \big[t[\mathsf{R}, B, f], \min\{\tau, T\}\big[\,. \tag{5.2.6}$$

In turn, (5.2.6) implies that $\tau > T$. Then, we will denote by

$$\boldsymbol{y}[\mathsf{R}, B, f] : \big[\boldsymbol{t}[\mathsf{R}, B, f],\, T\big] \;\to\;]-\infty, 0[\,, \qquad t \mapsto \boldsymbol{y}[\mathsf{R}, B, f](t)\,,$$

the unique solution to (5.2.4) defined on the interval $\big[\boldsymbol{t}[\mathsf{R}, B, f],\, T\big]$. Notice that $t \mapsto \frac{d}{dt}\boldsymbol{y}[\mathsf{R}, B, f](t)$ is strictly decreasing, and $\frac{d}{dt}\boldsymbol{y}[\mathsf{R}, B, f](t) \leq 0$ for all $t \in [\boldsymbol{t}[\mathsf{R}, B, f],\, T]$. Hence, by (5.2.6) with $\min\{\tau, T\} = T$, the terminal point satisfies $\boldsymbol{y}[\mathsf{R}, B, f](T) \in]T \cdot f'(\overline{B}), 0[$. Next, we set

$$\boldsymbol{u}[\mathsf{R}, B, f] \doteq (f')^{-1}\left(\frac{\mathsf{R} + \boldsymbol{y}[\mathsf{R}, B, f](T)}{T}\right)\,. \tag{5.2.7}$$

Observe that, by construction, $\boldsymbol{y}[\mathsf{R}, B, f](T)$ and $\boldsymbol{u}[\mathsf{R}, B, f]$ depend continuously on the parameters R and B, and that we have

$$\overline{B} < \boldsymbol{u}[\mathsf{R}, B, f] < B\,. \tag{5.2.8}$$

5.2.2. Right backward shock (Figure 5, right). Symmetrically to § 5.2.1, for every $A < \theta$, $T \cdot f'(A) < \mathsf{L} < 0$, we define here:

- two constants $\boldsymbol{s}[\mathsf{L}, A, f]$, $\boldsymbol{v}[\mathsf{L}, A, f]$;
- a function $t \mapsto \boldsymbol{x}[\mathsf{L}, A, f](t)$, $t \in [\boldsymbol{s}[\mathsf{L}, A, f], T]$;

which enjoy the following properties that we highlight here as in § 5.2.1 to clarify the purpouse of their introduction (but we will justify them in the sequel, see § 5.2.5, 5.4.5). The map $t \mapsto \boldsymbol{x}[\mathsf{L}, A, f_l](t)$ identifies the location of a shock curve in a $\overline{BA}$-entropy solution of $u_t + \overline{f}(x, u)_x = 0$, with $\overline{f}$ as in (5.1.26), defined on some domain $\Omega \subset [0, +\infty[\, \times [0, +\infty[$. Since a $\overline{BA}$-entropy solution of (5.1.27) is associated to the backward solution operator (5.1.28), we will say that $\boldsymbol{x}[\mathsf{L}, A, f_l]$ identifies the location of a *right backward shock.*

This curves starts from the interface $\{x = 0\}$ at time $t = \boldsymbol{s}[\mathsf{L}, A, f_l]$, and reaches the point $x = \boldsymbol{x}[\mathsf{L}, A, f_l](T)$ at time $t = T$. Such a shock discontinuity has, at time $t = T$, left state $\overline{A}$ and right state $\boldsymbol{v}[\mathsf{L}, A, f_l]$. The point $(-\mathsf{L}, 0)$ is the center of a rarefaction fan located on the right of the curve $t \mapsto (\boldsymbol{x}[\mathsf{L}, A, f_l](t), t)$.

We proceed to introduce these definitions as follows. Set

$$\boldsymbol{s}[\mathsf{L}, A, f] \doteq \frac{\mathsf{L}}{f'(A)}, \qquad \overline{A} \doteq (f_{|\,[\theta, +\infty[})^{-1} \circ f(A)\,. \tag{5.2.9}$$

Then, let $\boldsymbol{x}[\mathsf{L}, A, f] : [\boldsymbol{s}[\mathsf{L}, A, f], T] \to\ ]0, +\infty[$ denote the unique solution to the Cauchy problem

$$\begin{cases} x'(t) = \lambda\left((f')^{-1}\left(\frac{x(t)+\mathsf{L}}{t}\right), \overline{A}\right), & t \in \big[\boldsymbol{s}[\mathsf{L}, A, f],\, T\big], \\ x(\boldsymbol{s}[\mathsf{L}, A, f]) = 0. \end{cases} \tag{5.2.10}$$

By (5.2.1), the differential equation in (5.2.10) ensures that, for all $t \geq \boldsymbol{s}[\mathsf{L}, A, f]$, the pair $\big(\overline{A},\, (f')^{-1}(\frac{x(t)+\mathsf{L}}{t})\big)$ satisfies the Rankine-Hugoniot condition with slope $x'(t)$ for the conservation law $u_t + f(u)_x = 0$. The terminal point $\boldsymbol{x}[\mathsf{L}, A, f](T)$ depends continuously on the parameters L, A, and satisfies $\boldsymbol{x}[\mathsf{L}, A, f](T) \in \big]0, T \cdot f'(\overline{A})\big[$. Moreover, the map $t \mapsto \frac{d}{dt}\boldsymbol{x}[\mathsf{L}, A, f](t)$ is strictly increasing. Next, we define the quantity

$$\boldsymbol{v}[\mathsf{L}, A, f] \doteq (f')^{-1}\left(\frac{\mathsf{L} + \boldsymbol{x}[\mathsf{L}, A, f](T)}{T}\right), \tag{5.2.11}$$

which depends continuously on L and A, and satisfies

$$A < \boldsymbol{v}[\mathsf{L}, A, f] < \overline{A}\,. \tag{5.2.12}$$

5.2.3. Duality of forward and backward shocks. The definitions of backward shocks given in § 5.2.1-5.2.2 turn out to be dual one of the other, as clarified by the following:

LEMMA 5.2.1. *With the notations introduced in § 5.2.1-5.2.2, for every $B > \theta$, the following holds. The maps*

$$\begin{aligned} &\boldsymbol{y}[\,\cdot\,, B, f](T) : \,]0, T \cdot f'(B)[\ \to\ ]T \cdot f'(\overline{B}), 0[\,, && \mathsf{R} \mapsto \boldsymbol{y}[\mathsf{R}, B, f](T), \\ &\boldsymbol{x}[\,\cdot\,, \overline{B}, f](T) : \,]T \cdot f'(\overline{B}), 0[\ \to\ ]0, T \cdot f'(B)[\,, && \mathsf{L} \mapsto \boldsymbol{x}[\mathsf{L}, \overline{B}, f](T) \end{aligned} \tag{5.2.13}$$

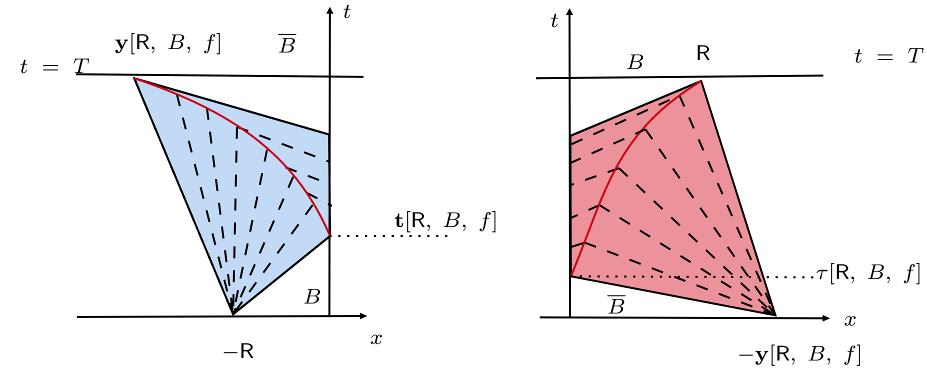

FIGURE 4. The dual solutions $\boldsymbol{y}[\mathsf{R}, B, f](\cdot)$ (left) and $\boldsymbol{x}[\boldsymbol{y}[\mathsf{R}, B, f](T), \overline{B}, f](\cdot)$ (right) of the Cauchy problems (5.2.4), (5.2.10). This represents the statement of Lemma 5.2.1

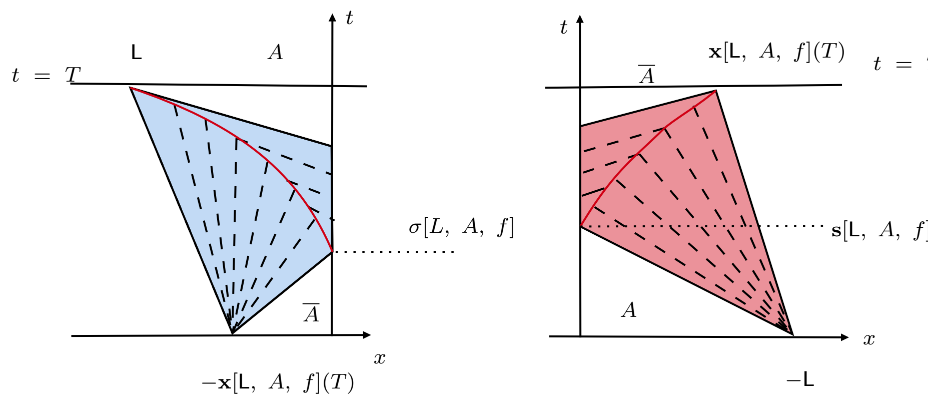

FIGURE 5. The dual solutions $\boldsymbol{x}[\mathsf{L}, A, f](\cdot)$ (right) and $\boldsymbol{y}[\boldsymbol{x}[\mathsf{L}, A, f](T), \overline{A}, f](\cdot)$ (left) of the Cauchy problems (5.2.4), (5.2.10). This represents the statement of Lemma 5.2.1

are increasing, and one is the inverse of the other, i.e. it holds true

$$\begin{aligned} \mathsf{R} &= \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R}, B, f](T), \overline{B}, f\big](T), \qquad && \forall\ \mathsf{R} \in \,]0, T \cdot f'(B)[, \\ \mathsf{L} &= \boldsymbol{y}\big[\boldsymbol{x}[\mathsf{L}, \overline{B}, f](T), B, f\big](T), \qquad && \forall\ \mathsf{L} \in \,]T \cdot f'(\overline{B}), 0[\,. \end{aligned} \tag{5.2.14}$$

Moreover, one has

$$\begin{aligned} &\lim_{\mathsf{R}\to 0+} \boldsymbol{y}[R, B, f](T) = T \cdot f'(\overline{B}), \qquad && \lim_{\mathsf{R}\to T\cdot f'(B)-} \boldsymbol{y}[\mathsf{R}, B, f](T) = 0, \\ &\lim_{\mathsf{L}\to 0-} \boldsymbol{x}[\mathsf{L}, \overline{B}, f](T) = T \cdot f'(B), \qquad && \lim_{\mathsf{L}\to T\cdot f'(\overline{B})+} \boldsymbol{x}[\mathsf{L}, \overline{B}, f](T) = 0. \end{aligned} \tag{5.2.15}$$

PROOF.

1. We will prove only the first equality in (5.2.14), the proof of the second one being entirely similar. Fix $\mathsf{R} \in \,]0, T \cdot f'(B)[$, and consider the polygonal region (the blue set in

Figure 4) defined by

$$\Delta \doteq \Delta_1 \cup \Delta_2,$$

$$\Delta_1 \doteq \Big\{(x,t) \in]-\infty, 0[\times]0,T[\ : \ \mathsf{L} - (T-t)\cdot f'(\boldsymbol{u}) < x < \mathsf{L} - (T-t)\cdot f'(\bar{B}), \ \boldsymbol{t} < t < T\Big\},$$

$$\Delta_2 \doteq \Big\{(x,t) \in]-\infty, 0[\times]0,T[\ : \ \mathsf{L} - (T-t)\cdot f'(\boldsymbol{u}) < x < (t-\boldsymbol{t})\cdot f'(\,B\,), \ 0 \leq t \leq \boldsymbol{t}\Big\}, \tag{5.2.16}$$

where $\boldsymbol{u} \doteq \boldsymbol{u}[\mathsf{R}, B, f]$ is the constant in (5.2.7), $\boldsymbol{t} \doteq \boldsymbol{t}[\mathsf{R}, B, f]$ is defined as in (5.2.3) and $\mathsf{L} \doteq \boldsymbol{y}[\mathsf{R}, B, f](T)$. Observe that the function $v : \Delta \to \mathbb{R}$ defined by

$$v(x,t) \doteq \begin{cases} \overline{B} & \text{if} \qquad \gamma(t) < x < 0, \\ (f')^{-1}\left(\dfrac{x+\mathsf{R}}{t}\right) & \text{otherwise,} \end{cases} \tag{5.2.17}$$

is locally Lipschitz continuous and satisfies the equation (5.1.18) at every point $(x,t) \in \Delta$ outside the curve $\gamma(\cdot) \doteq \boldsymbol{y}[\mathsf{R}, B, f](\cdot)$. Moreover, because of the construction of $\boldsymbol{y}[\mathsf{R}, B, f](\cdot)$, u satisfies the Rankine-Hugoniot conditions along the curve γ. Therefore $v(x,t)$ is a distributional solution of (5.1.18) on Δ. Hence, applying the divergence theorem to the piecewise smooth vector field $(v, f(v))$ on Δ, and setting $\tau_1 \doteq T - \boldsymbol{y}[\mathsf{R}, B, f]/f'(\overline{B})$, we find

$$\begin{aligned} 0 = &\big(f(\overline{B}) - \overline{B} f'(\overline{B})\big)(T-\tau_1) + \big(\boldsymbol{u}\, f'(\boldsymbol{u}) - f(\boldsymbol{u})\big)T + \\ &\big(f(\,B\,) - B f'(\,B\,)\big)\,\boldsymbol{t} + f(B)(\tau_1 - \boldsymbol{t})\,. \end{aligned}$$

Then, observing that $f(B) = f(\overline{B})$ and that $f'(\,B\,)\boldsymbol{t} = \mathsf{R}$, we find

$$\overline{B}\,\boldsymbol{y}[\mathsf{R}, B, f](T) + B\,\mathsf{R} - \big(\boldsymbol{u}\, f'(\boldsymbol{u}) - f(\boldsymbol{u})\big)T - f(B)T = 0\,. \tag{5.2.18}$$

Since $f'(\boldsymbol{u}) = (\boldsymbol{y}[\mathsf{R}, B, f](T) + \mathsf{R})/T$, and because the Legendre transform f^* of f satisfies the identity

$$f^*(f'(u)) = u\, f'(u) - f(u) \qquad \forall\ u\,,$$

(e.g. see [§A.2][**46**]), we derive from (5.2.18) the identity

$$\overline{B}\,\boldsymbol{y}[\mathsf{R}, B, f](T) + B\,\mathsf{R} - f^*\left(\frac{\boldsymbol{y}[\mathsf{R}, B, f](T) + \mathsf{R}}{T}\right) T - f(B)\,T = 0 \qquad \forall\, \mathsf{R} \in]0, T\cdot f'(B))[\,. \tag{5.2.19}$$

2. Next, consider the polygonal region (the red set in Figure 5 with $A = \overline{B}$ and $\overline{A} = B$) defined by

$$\Gamma \doteq \Gamma_1 \cup \Gamma_2,$$

$$\Gamma_1 \doteq \Big\{(x,t) \in]0,+\infty[\, \times\,]0,T[\ :\ \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) - (T-t)\cdot f'(B) < x < \\ < \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) - (T-t)\cdot f'(\boldsymbol{v}),\ \boldsymbol{s} < t < T\Big\},$$

$$\Gamma_2 \doteq \Big\{(x,t) \in]0,+\infty[\, \times\,]0,T[\ :\ (t-\boldsymbol{s})\cdot f'(\,\overline{B}\,) < x < \\ < \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) - (T-t)\cdot f'(\boldsymbol{v}),\ 0 \leq t \leq \boldsymbol{s}\Big\}, \tag{5.2.20}$$

where $\boldsymbol{v} \doteq \boldsymbol{v}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big]$ is the constant defined as in (5.2.11), with $\mathsf{L} = \boldsymbol{y}[\mathsf{R},B,f](T)$, $A = \overline{B}$ and $\boldsymbol{s} = \boldsymbol{s}[\mathsf{L},A,f]$. Observe that the function $u : \Gamma \to \mathbb{R}$ defined by

$$u(x,t) \doteq \begin{cases} B & \text{if} \quad 0 < x < \gamma(t), \\ (f')^{-1}\left(\dfrac{x + \boldsymbol{y}[\mathsf{R},B,f](T)}{t}\right) & \text{otherwise}, \end{cases} \tag{5.2.21}$$

with $\gamma(t) \doteq \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](t)$, is a distributional solution of (5.1.18) on Γ for the symmetric arguments of the previous point. Then, repeating the same type of analysis of above for the piecewise smooth vector field $(u, f(u))$ on Γ, one finds the identity

$$B\,\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) + \overline{B}\,\boldsymbol{y}[\mathsf{R},B,f](T) + \\ - f^*\left(\frac{\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) + \boldsymbol{y}[\mathsf{R},B,f](T)}{T}\right)T - f(B)\,T = 0 \qquad \forall\, \mathsf{R} \in]0, T\cdot f'(B))[\,. \tag{5.2.22}$$

Notice that, by definition of the function $\boldsymbol{y}[\mathsf{R},B,f](\cdot)$ in § 5.2.1, the terminal value satisfies $\boldsymbol{y}[\mathsf{R},B,f](T) \in]T\cdot f'(\overline{B}), 0[\,$, for all $\mathsf{R} \in]0, T\cdot f'(B))[\,$. In turn, from the definition of $\boldsymbol{x}[\mathsf{L},A,f]$ in §5.2.2, with $A = \overline{B}$, and $\mathsf{L} = \boldsymbol{y}[\mathsf{R},B,f](T)$, it follows that

$$\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T) \in]0, T\cdot f'(B)[\,, \qquad \forall\ \mathsf{R} \in]0, T\cdot f'(B))[\,. \tag{5.2.23}$$

3. We fix now $\mathsf{R} \in]0, T\cdot f'(B))[\,$, and we consider the map $\Upsilon :]0, T\cdot f'(B)[\, \to \mathbb{R}$, defined by

$$\Upsilon(x) \doteq \overline{B}\,\boldsymbol{y}[\mathsf{R},B,f](T) + B\,x - f^*\left(\frac{\boldsymbol{y}[\mathsf{R},B,f](T) + x}{T}\right)T - f(B)\,T\,. \tag{5.2.24}$$

Observe that, by (5.2.19), (5.2.22), (5.2.23), one has

$$\Upsilon(\mathsf{R}) = \Upsilon\Big(\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\overline{B},f\big](T)\Big) = 0\,. \tag{5.2.25}$$

Hence, it is sufficient to show that Υ admits only one zero in the interval $]0, T\cdot f'(B)[$ to conclude the proof of the first equality in (5.2.14). To this end, differentiating Υ and recalling the well known property of the Legendre transform (e.g. see [**46**, §A.2]),

$$(f^*)'(p) = (f')^{-1}(p) \qquad \forall\, p\,,$$

we find

$$\begin{aligned}\Upsilon'(x) &= B - (f')^{-1}\left(\frac{\boldsymbol{y}[\mathsf{R},B,f](T)+x}{T}\right)\\ &= (f')^{-1}\left(\frac{0+Tf'(B)}{T}\right) - (f')^{-1}\left(\frac{\boldsymbol{y}[\mathsf{R},B,f](T)+x}{T}\right).\end{aligned} \tag{5.2.26}$$

Since $\boldsymbol{y}[\mathsf{R},B,f](T) < 0$, $x < T\cdot f'(B)$, and because f' is strictly increasing as f', we deduce from (5.2.26) that $\Upsilon'(x) > 0$ for all $x \in]0, T\cdot f'(B))[$. Therefore Υ is strictly increasing in the interval $]0, T\cdot f'(B))[$, completing the proof of of the first equality in (5.2.14).

4. We show now that the map $\mathsf{R} \mapsto \boldsymbol{y}(\mathsf{R}) \doteq \boldsymbol{y}[\mathsf{R},B,f](T)$ is strictly increasing in the interval $]0, T\cdot f'(B)[$. Differentiating (5.2.19) with respect to R, we obtain

$$\left[\overline{B} - (f')^{-1}\Big(\frac{y(\mathsf{R})+\mathsf{R}}{T}\Big)\right]\boldsymbol{y}'(\mathsf{R}) = (f')^{-1}\Big(\frac{\boldsymbol{y}(\mathsf{R})+\mathsf{R}}{T}\Big) - B \qquad \forall\ \mathsf{R}\in]0, T\cdot f'(B))[\,. \tag{5.2.27}$$

Since $T\cdot f'(\,\overline{B}\,) < \boldsymbol{y}(\mathsf{R}) < 0$ and $0 < \mathsf{R} < T\cdot f'(B)$, because f' is strictly increasing we deduce

$$\overline{B} < (f')^{-1}\Big(\frac{\boldsymbol{y}(\mathsf{R})}{T}\Big) < (f')^{-1}\Big(\frac{\boldsymbol{y}(\mathsf{R})+\mathsf{R}}{T}\Big) < (f')^{-1}\Big(\frac{\mathsf{R}}{T}\Big) < B\,,$$

which, together with (5.2.27), implies that $\boldsymbol{y}'(\mathsf{R}) > 0$ for all $R \in]0, T\cdot f'(B))[$, as wanted. In turn, since $\mathsf{L} \mapsto \boldsymbol{x}[\mathsf{L},\overline{B},f](T)$ is the inverse of $\mathsf{R} \mapsto \boldsymbol{y}[\mathsf{R},B,f](T)$, this implies that $\mathsf{L} \mapsto \boldsymbol{x}[\mathsf{L},\overline{B},f](T)$ is strictly increasing as well in its domain, and that the image of the maps $\boldsymbol{y}[\,\cdot\,,B,f](T)$, $\boldsymbol{x}[\,\cdot\,,\overline{B},f](T)$, in (5.2.13) are the sets $]0, T\cdot f'(B))[$ and $]0, T\cdot f'(B)[$, respectively. This, together with the monotonicity of the maps $\boldsymbol{y}[\,\cdot\,,B,f](T)$, $\boldsymbol{x}[\,\cdot\,,\overline{B},f](T)$, in particular implies the one-sided limits in (5.2.15), thus concluding the proof of the Lemma. □

REMARK 5.2.2. As a consequence of Lemma 5.2.1 and of the monotonicity of f', we find that the maps

$$\mathsf{R} \mapsto \boldsymbol{u}[\mathsf{R},B,f], \qquad\qquad \mathsf{L} \mapsto \boldsymbol{v}[\mathsf{L},A,f], \tag{5.2.28}$$

defined as in (5.2.7) and (5.2.11), are strictly increasing, and that we have

$$\begin{aligned}&\lim_{\mathsf{R}\to 0+} \boldsymbol{u}[\mathsf{R},B,f] = \overline{B}, &&\lim_{\mathsf{R}\to T\cdot f'(B)-} \boldsymbol{u}[\mathsf{R},B,f] = B,\\ &\lim_{\mathsf{L}\to 0-} \boldsymbol{v}[\mathsf{L},A,f] = \overline{A}, &&\lim_{\mathsf{L}\to T\cdot f'(A)+} \boldsymbol{v}[\mathsf{L},A,f] = A.\end{aligned} \tag{5.2.29}$$

This implies that the functions

$$\begin{aligned}&\boldsymbol{u}[\cdot,\cdot,f] :]0, T\cdot f'(B)[\times]\theta,+\infty[\,\to \mathbb{R},\\ &\boldsymbol{v}[\cdot,\cdot,f] :]T\cdot f'(A), 0[\times]\theta,+\infty[\,\to \mathbb{R}\end{aligned}$$

can be extended to continuous function on $[0, T\cdot f'(B)]\times\ ]\theta,+\infty[$ and $[T\cdot f'(A),0]\times\ ]\theta,+\infty[$, setting

$$\begin{aligned} &\boldsymbol{u}[0,B,f]=\overline{B}, &&\boldsymbol{u}[T\cdot f'(B),B,f]=B,\\ &\boldsymbol{v}[0,A,f]=\overline{A}, &&\boldsymbol{v}[T\cdot f'(A),A,f]=A. \end{aligned} \tag{5.2.30}$$

Moreover, one has

$$\begin{aligned} &\boldsymbol{u}[\mathsf{R},B,f]<B &&\forall\ \mathsf{R}\in\,]0,T\cdot f'(B)[\,,\\ &\boldsymbol{v}[\mathsf{L},A,f]>A &&\forall\ \mathsf{L}\in\,]T\cdot f'(A),0[\,. \end{aligned} \tag{5.2.31}$$

5.2.4. Right forward shock-rarefaction wave pattern (Figure 4, right). For every $B>\theta$, $0<\mathsf{R}<T\cdot f'(B)$, we define now:

- a constant $\boldsymbol{\tau}[\mathsf{R},B,f]$;
- a function $(x,t)\mapsto u[\mathsf{R},B,f](x,t)$, $(x,t)\in\Gamma[\mathsf{R},B,f]$;

with the following properties. When $f=f_r$, the function $u[\mathsf{R},B,f](x,t)$ defines a (forward) solution associated to the operator $\mathcal{S}^{[AB]+}$, which contains a shock starting from the interface $\{x=0\}$ at time $t=\boldsymbol{\tau}[\mathsf{R},B,f_r]$. The location of such a shock is given by the map $t\mapsto\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f_r](T),\overline{B},f_r\big](t)$, where $\boldsymbol{y}[\mathsf{R},B,f_r]$ and $\boldsymbol{x}[\mathsf{L},\overline{B},f_r]$ with $\mathsf{L}=\boldsymbol{y}[\mathsf{R},B,f_r](T)$, are the backward shocks of a backward solution associated to the operator $\mathcal{S}^{[AB]-}$ introduced in § 5.2.1-5.2.2. Because of Lemma 5.2.1, the shock $t\mapsto\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f_r](T),\overline{B},f_r\big](t)$ reaches the point $x=R$ at time $t=T$. We can regard $\boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R},B,f_r](T),\overline{B},f_r\big]$ as the "dual shock" of the backward shock $\boldsymbol{y}[\mathsf{R},B,f_r]$.

We proceed to introduce these definitions as follows. With the same notations of § 5.2.1-5.2.2, for every $B>\theta$, $0<\mathsf{R}<T\cdot f'(B)$, we set

$$\boldsymbol{\tau}[\mathsf{R},B,f]\doteq\boldsymbol{s}\big[\boldsymbol{y}[\mathsf{R},B,f](T),\ \overline{B},\ f\big]=\frac{\boldsymbol{y}[\mathsf{R},B,f](T)}{f'(\,\overline{B}\,)}\,. \tag{5.2.32}$$

Notice that, by the construction in § 5.2.1, and because of Lemma 5.2.1, $\boldsymbol{\tau}[\mathsf{R},B,f]$ depends continuously on the parameters R,B, the image of the map $\mathsf{R}\mapsto\boldsymbol{\tau}[\mathsf{R},B,f]$, $\mathsf{R}\in\,]0,T\cdot f'(B)[$, is the set $]0,T[$, and $\mathsf{R}\mapsto\boldsymbol{\tau}[\mathsf{R},B,f]$ is decreasing.

Next, we denote by $\Gamma[\mathsf{R},B,f]\subset(0,T)\times\mathbb{R}$ the polygonal set (the pink set in Figure 4)

$$\Gamma[\mathsf{R},B,f]\doteq\Gamma_1[\mathsf{R},B,f]\cup\Gamma_2[\mathsf{R},B,f]\,, \tag{5.2.33}$$

with

$$\begin{aligned} \Gamma_1[\mathsf{R},B,f]\doteq\Big\{(x,t)\in\,]0,+\infty[\,\times\,]0,T[\ :\ &\mathsf{R}-(T-t)\cdot f'(B)<x<\mathsf{R}-(T-t)\cdot f'(\boldsymbol{u}[\mathsf{R},B,f]),\\ &\boldsymbol{\tau}[\mathsf{R},B,f]<t<T\Big\},\\ \Gamma_2[\mathsf{R},B,f]\doteq\Big\{(x,t)\in\,]0,+\infty[\,\times\,]0,T[\ :\ &-(\boldsymbol{\tau}[\mathsf{R},B,f]-t)\cdot f'(\,\overline{B}\,)<x<\mathsf{R}-(T-t)\cdot f'(\boldsymbol{u}[\mathsf{R},B,f]),\\ &0\le t\le\boldsymbol{\tau}[\mathsf{R},B,f]\Big\}. \end{aligned} \tag{5.2.34}$$

Then, set $\gamma(t) \doteq \boldsymbol{x}\big[\boldsymbol{y}[\mathsf{R}, B, f](T), \overline{B}, f\big](t)$, and denote by $\mathsf{u}[\mathsf{R}, B, f] : \Gamma[\mathsf{R}, B, f] \to \mathbb{R}$ the function defined by

$$\mathsf{u}[\mathsf{R}, B, f](x,t) \doteq \begin{cases} B & \text{if} \qquad 0 < x < \gamma(t), \\ (f')^{-1}\bigg(\dfrac{x - \mathsf{R} + T \cdot f'(\boldsymbol{u}[\mathsf{R}, B, f])}{t}\bigg) & \text{otherwise.} \end{cases} \tag{5.2.35}$$

Notice that, by (5.2.10) and (5.2.14), one has $\gamma(\boldsymbol{\tau}[\mathsf{R}, B, f]) = 0$, $\gamma(T) = \mathsf{R}$. Moreover, by the same arguments of the proof of Lemma 5.2.1 it follows that $u[\mathsf{R}, B, f](x,t)$ is a distributional solution of (5.1.18) on $\Gamma[\mathsf{R}, B, f]$. Furthermore, since $t \mapsto \gamma'(t)$ is strictly increasing as observed in § 5.2.2, it follows that also the map

$$t \mapsto \frac{\gamma(t) - \mathsf{R} + T \cdot f'(\boldsymbol{u}[\mathsf{R}, B, f])}{t}$$

is strictly increasing. Therefore, by virtue of (5.2.14), and relying on (5.2.31), we find

$$\begin{aligned} \lim_{x \to \gamma(t)+} \mathsf{u}[\mathsf{R}, B, f](x,t) &\leq \lim_{x \to \gamma(T)+} \mathsf{u}[\mathsf{R}, B, f](x,T) \\ &= \lim_{x \to \mathsf{R}+} \mathsf{u}[\mathsf{R}, B, f](x,T) \\ &= \boldsymbol{u}[\mathsf{R}, B, f] < B \\ &= \lim_{x \to \gamma(t)-} \mathsf{u}[\mathsf{R}, B, f](x,t) \qquad \forall\, t \in [\boldsymbol{\tau}[\mathsf{R}, B, f], T]\,, \end{aligned} \tag{5.2.36}$$

which shows that the Lax entropy condition is satisfied along the curve $(t, \gamma(t))$, $t \in [\boldsymbol{\tau}[\mathsf{R}, B, f], T]$. Since the flux in (5.1.18) is strictly convex, this proves that $u[\mathsf{R}, B, f](x,t)$ provides an entropy weak solution of (5.1.18) on the region $\Gamma[\mathsf{R}, B, f]$. Notice that, by (5.2.2), from (5.2.36) we deduce in particular that $f'(B) > \lambda\big(\boldsymbol{u}[\mathsf{R}, B, f], B\big) = \gamma'(T)$, which in turn, by the strict monotonicity of $\dot{\gamma}(t)$, yields

$$f'(B) > \gamma'(t) \qquad \forall\, t \in [\boldsymbol{\tau}[\mathsf{R}, B, f], T]\,. \tag{5.2.37}$$

Hence, relying on (5.2.37), we find

$$f'(B) > \frac{\gamma(T) - \gamma(\boldsymbol{\tau}[\mathsf{R}, B, f])}{T - \boldsymbol{\tau}[\mathsf{R}, B, f]} = \frac{\mathsf{R}}{T - \boldsymbol{\tau}[\mathsf{R}, B, f]}\,. \tag{5.2.38}$$

5.2.5. Left forward rarefaction-shock wave pattern (Figure 5, left). Symmetrically to § 5.2.4, for every $A < \theta$, $T \cdot f'(A) < \mathsf{L} < 0$, we define here:

- a constant $\boldsymbol{\sigma}[\mathsf{L}, A, f]$;
- a function $(x,t) \mapsto \mathsf{v}[\mathsf{L}, A, f](x,t)$, $(x,t) \in \Delta[\mathsf{L}, A, f]$;

with the following properties. When $f = f_l$, the function $\mathsf{v}[\mathsf{L}, A, f](x,t)$ defines a (forward) solution associated to the operator $\mathcal{S}^{[AB]+}$, which contains a shock starting from the interface $\{x = 0\}$ at time $t = \boldsymbol{\sigma}[\mathsf{L}, A, f]$. The location of such a shock is given by the map $t \mapsto \boldsymbol{y}\big[\boldsymbol{x}[\mathsf{L}, A, f_l](T),\ \overline{A},\ f_l\big](t)$, where $\boldsymbol{x}[\mathsf{L}, A, f_l]$ and $\boldsymbol{y}[\mathsf{R}, \overline{A}, f_l]$ with $\mathsf{R} = \boldsymbol{x}[\mathsf{L}, A, f_l](T)$, are the backward shocks of a backward solution associated to the operator $\mathcal{S}^{[AB]-}$ introduced in § 5.2.1-5.2.2. Because of Lemma 5.2.1, the shock $t \mapsto \boldsymbol{y}\big[\boldsymbol{x}[\mathsf{L}, A, f_l](T),\ \overline{A},\ f_l\big](t)$ reaches the point $x = L$ at time $t = T$. We can regard $\boldsymbol{y}\big[\boldsymbol{x}[\mathsf{L}, A, f_l](T),\ \overline{A},\ f_l\big]$ as the "dual shock" of the backward shock $\boldsymbol{x}[\mathsf{L}, A, f_l]$.

We proceed to introduce these definitions as follows. With the same notations of § 5.2.1-5.2.2, for every $A < \theta$, $T \cdot f'(A) < \mathsf{L} < 0$ we set

$$\boldsymbol{\sigma}[\mathsf{L}, A, f] \doteq \boldsymbol{t}\big[\boldsymbol{x}[\mathsf{L}, A, f](T),\ \overline{A},\ f\big] = \frac{\boldsymbol{x}[\mathsf{L}, A, f](T)}{f'(\overline{A}\,)}\,. \tag{5.2.39}$$

By the construction in § 5.2.2, and because of Lemma 5.2.1, $\boldsymbol{\sigma}[\mathsf{L}, A, f]$ depends continuously on the parameters L, A, the image of the map $\mathsf{L} \mapsto \boldsymbol{\sigma}[\mathsf{L}, A, f]$, $\mathsf{L} \in]T \cdot f'(A), 0[$, is the set $]0, T[$, and $\mathsf{L} \mapsto \boldsymbol{\sigma}[\mathsf{L}, A, f]$ is increasing.

Next, we denote by $\Delta[\mathsf{L}, A, f] \subset (0, T) \times \mathbb{R}$ the polygonal set (the blue set in Figure 5)

$$\Delta[\mathsf{L}, A, f] \doteq \Delta_1[\mathsf{L}, A, f] \cup \Delta_2[\mathsf{L}, A, f]\,, \tag{5.2.40}$$

with

$$\begin{aligned}\Delta_1[\mathsf{L}, A, f] \doteq \Big\{&(x, t) \in]-\infty, 0[\times]0, T[\ :\ \mathsf{L} - (T-t)\cdot f'(\boldsymbol{v}[\mathsf{L}, A, f]) < x < \mathsf{L} - (T-t)\cdot f'(A),\\ &\boldsymbol{\sigma}[\mathsf{L}, A, f] < t < T\Big\},\\ \Delta_2[\mathsf{L}, A, f] \doteq \Big\{&(x, t) \in]-\infty, 0[\times]0, T[\ :\ \mathsf{L} - (T-t)\cdot f'(\boldsymbol{v}[\mathsf{L}, A, f]) < x < -(\boldsymbol{\sigma}[\mathsf{L}, A, f] - t)\cdot f'(\overline{A}\,),\\ &0 \le t \le \boldsymbol{\sigma}[\mathsf{L}, A, f]\Big\}.\end{aligned} \tag{5.2.41}$$

Then, set $\gamma(t) \doteq \boldsymbol{y}\big[\boldsymbol{x}[\mathsf{L}, A, f](T), \overline{A}, f\big](t)$, and denote by $\mathsf{v}[\mathsf{L}, A, f] : \Delta[\mathsf{L}, A, f] \to \mathbb{R}$ the function defined by

$$\mathsf{v}[\mathsf{L}, A, f](x, t) \doteq \begin{cases} A & \text{if} \qquad \gamma(t) < x < 0,\\ (f')^{-1}\bigg(\dfrac{x - \mathsf{L} + T\cdot f'(\boldsymbol{v}[\mathsf{L}, A, f])}{t}\bigg) & \text{otherwise.}\end{cases} \tag{5.2.42}$$

Observe that, by (5.2.4) and (5.2.14), one has $\gamma(\boldsymbol{\sigma}[\mathsf{L}, A, f]) = 0$, $\gamma(T) = \mathsf{L}$. With the same arguments of § 5.2.4, it follows that $v[\mathsf{L}, A, f](x, t)$ provides an entropy weak solution of (5.1.18) on the region $\Delta[\mathsf{L}, A, f]$, and that we have

$$f'(A) < \frac{\gamma(T) - \gamma(\boldsymbol{\sigma}[\mathsf{L}, A, f])}{T - \boldsymbol{\sigma}[\mathsf{L}, A, f]} = \frac{\mathsf{L}}{T - \boldsymbol{\sigma}[\mathsf{L}, A, f]}\,. \tag{5.2.43}$$

REMARK 5.2.3. The constant $\boldsymbol{u}[\mathsf{R}, B, f]$ defined in § 5.2.1 is crucial to characterize the jump of an attainable profile $\omega \in \mathcal{A}^{[AB]}(T)$ at the point

$$\mathsf{R} \doteq \inf\big\{R > 0\ :\ x - T\cdot f_r'(\omega(x+)) \ge 0 \quad \forall\, x \ge R\big\},$$

when $\mathsf{R} \in]0, T\cdot f_r'(B)[$. The state $\boldsymbol{u}[\mathsf{R}, B, f]$ is constructed so to be the largest right state that one can achieve at (R, T) with a shock that *isolates* the interface $\{x = 0\}$ from the semiaxis $\{x > 0\}$. In fact, the constant $\boldsymbol{u}[\mathsf{R}, B, f]$ with $f = f_r$, identifies a unique state $\boldsymbol{u}$ $< B$ that has the property:

- If $\omega = \mathcal{S}_T^{[AB]+} u_0$, and $u(t, x) = \mathcal{S}_t^{[AB]+} u_0(x)$ admits a shock generated in $\{x \ge 0\}$ at some time $t = \tau$, and reaching the point (R, T), then letting $\gamma(t)$, $t \in [\tau, T]$, denote the location of such a shock, one has

$$u_\gamma \doteq \lim_{t \to T-} u(t, \gamma(t)+) \le (f_r')^{-1}(\mathsf{R}/T) \quad \Longrightarrow \quad u_\gamma \le \boldsymbol{u}\,. \tag{5.2.44}$$

In particular, one has $u_\gamma = \boldsymbol{u}$ in (5.2.44) only in the case where $\mathcal{S}_t^{[AB]+}u_0$ coincides in the polygonal region $\Gamma[\mathsf{R}, B, f]$ with the right forward shock-rarefaction pattern described in section 5.2.4. By definition of u_γ it follows that either $\omega(\mathsf{R}+) = u_\gamma$, or else there is another jump connecting u_γ with $\omega(\mathsf{R}+)$ which must satisfy the Lax entropy condition $\omega(\mathsf{R}+) < u_\gamma$. Therefore, as a consequence of (5.2.44) we find a necessary condition for the attainability of ω at time T given by

$$\omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r]\,, \tag{5.2.45}$$

(see (5.3.13) of Theorem 5.3.3 and the proof in § 5.4.2.3). The interesting fact is that, in the case $\mathsf{R} \in]0, T \cdot f_r'(B)[$, condition (5.2.45), together with the condition

$$\omega(x) \geq B \qquad \forall\ x \in]0, \mathsf{R}[\,, \tag{5.2.46}$$

(see (5.3.14), (5.3.15), of Theorem 5.3.3), is also sufficient to guarantee the existence of an AB-entropy solution $u(x,t)$ that satisfies

$$u(x,T) = \omega(x) \qquad \forall\ x \in]0, \mathsf{R}], \qquad u(\mathsf{R}, T) = \omega(\mathsf{R}+). \tag{5.2.47}$$

To illustrate this claim, in view of the definitions introduced in the previous sections we proceed as follows.

- By solving (5.2.4) one determines the end point $\boldsymbol{y}[\mathsf{R}, B, f_r](T)$ of a "left backward shock" (Figure 4, left). The map $t \mapsto \boldsymbol{y}[\mathsf{R}, B, f_r](t)$ represents the position of a shock in a $\overline{BA}$-entropy solution (which is associated to the backward solution operator $\mathcal{S}^{[AB]-}$, see Definition 5.1.16);
- given the final position $\boldsymbol{y}[\mathsf{R}, B, f_r]$ of the "backward shock" , one considers the solution $t \mapsto \gamma(t) \doteq \boldsymbol{x}[\boldsymbol{y}[\mathsf{R}, B, f_r], \overline{B}, f_r](t)$ to (5.2.10), when $\mathsf{L} = \boldsymbol{y}[\mathsf{R}, B, f_r]$, $A = \overline{B}$, $f = f_r$ (see Figure 4, right). This map represents the position of a shock in a "forward solution", i.e. in an AB-entropy solution associated to the (forward) operator $\mathcal{S}^{[AB]+}$ in (5.1.14). Actually, we will show in §5.4.5, using the results of this section, that $(t, \gamma(t))$ is the location of a shock of $\mathcal{S}_t^{[AB]+}u_0$, with $u_0 = \mathcal{S}_t^{[AB]-}\omega$.
- once determined the point $\boldsymbol{y}[\mathsf{R}, B, f_r](T)$, one defines $\boldsymbol{u}[\mathsf{R}, B, f_r]$ as the state realizing the slope $(\boldsymbol{y}[,B, f_r](T) + \mathsf{R})/T$ (see (5.2.7)):

$$f_r'(\boldsymbol{u}[\mathsf{R}, B, f_r]) = \frac{\boldsymbol{y}[\mathsf{R}, B, f_r](T) + \mathsf{R}}{T};$$

- thanks to Lemma 5.2.1, we know that the final position at time T of the shock $\gamma(t)$ satisfies

$$\gamma(T) = \mathsf{R}.$$

  Using this procedure, if a profile ω satisfies the conditions (5.2.45)-(5.2.46), we will show in § 5.4.4-5.4.5 that we can construct admissible AB-shocks that produce at time T the given jump in the profile ω at position R.

Entirely symmetric considerations hold for the state $\boldsymbol{v}[\mathsf{L}, A, f]$ defined in § 5.2.2 (see Figure 5). As a byproduct of this analysis we will obtain that attainable profiles are fixed points of the backward forward solution operator, as stated in Theorem 9.

5.3. Statement of the main results

Conditions (1), (2) of Theorem 9 will be shown to be equivalent by proving that they are both equivalent to a characterization of the attainable set $\mathcal{A}^{[AB]}(T)$ in (26) via Oleinik-type inequalities and state constraints. To present these results we need to introduce some furhter notations.

Given a flux $f(x,u)$ as in (24), we will use the notations $f_{l,-}^{-1} \doteq (f_{l|(-\infty,\theta_l]})^{-1}$, $f_{r,-}^{-1} \doteq (f_{r|(-\infty,\theta_r]})^{-1}$, for the inverse of the restriction of f_l, f_r to their decreasing part, respectively, and $f_{l,+}^{-1} \doteq (f_{l|[\theta_l,+\infty)})^{-1}$, $f_{r,+}^{-1} \doteq (f_{r|[\theta_r,+\infty)})^{-1}$, for the inverse of the restriction of f_l, f_r to their increasing part, respectively. Then, we set

$$\pi_{l,\pm} \doteq f_{l,\pm}^{-1} \circ f_l\,, \qquad \pi_{r,\pm} \doteq f_{r,\pm}^{-1} \circ f_r\,, \qquad \pi_{l,\pm}^{r} \doteq f_{l,\pm}^{-1} \circ f_r\,, \qquad \pi_{r,\pm}^{l} \doteq f_{r,\pm}^{-1} \circ f_l\,. \tag{5.3.1}$$

Moreover, in connection with a function $\omega : \mathbb{R} \to \mathbb{R}$ we define the quantities

$$\begin{aligned} \mathsf{R}[\omega, f_r] &\doteq \inf\big\{R > 0 \ : \ x - T \cdot f_r'(\omega(x)) \geq 0 \quad \forall\, x \geq R\big\}, \\ \mathsf{L}[\omega, f_l] &\doteq \sup\big\{L < 0 \ : \ x - T \cdot f_l'(\omega(x)) \leq 0 \quad \forall\, x \leq L\big\}, \end{aligned} \tag{5.3.2}$$

and, if $\mathsf{L}[\omega, f_l] \in]T \cdot f_l'(A), 0[$, we set

$$\widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \doteq \big(T - \boldsymbol{\sigma}\big[\mathsf{L}[\omega, f_l], A, f_l\big]\big) \cdot f_r'(B), \tag{5.3.3}$$

while, if $\mathsf{R}[\omega, f_r] \in]0, T \cdot f_r'(B)[$, we set

$$\widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \doteq \big(T - \boldsymbol{\tau}\big[\mathsf{R}[\omega, f_r], B, f_r\big]\big) \cdot f_l'(A). \tag{5.3.4}$$

where $\boldsymbol{\sigma}[\mathsf{L}, A, f_l]$, $\boldsymbol{\tau}[\mathsf{R}, B, f_r]$, denote the shock starting times introduced in § 5.2.4-5.2.5. Recalling (5.2.15), (5.2.32), (5.2.39), we can extend by continuity the definitions (5.3.3), (5.3.4), setting

$$\begin{aligned} \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] &\doteq 0, \qquad \text{if} \qquad \mathsf{L}[\omega, f_l] = 0\,, \\ \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] &\doteq 0, \qquad \text{if} \qquad R[\omega, f_r] = 0\,. \end{aligned} \tag{5.3.5}$$

Such quantities are used to express the pointwise constraints satisfied by ω in intervals containing the origin whenever ω is attainable. Next, to express the Oleinik-type inequalities satisfied by the attainable profiles it is useful to introduce the functions,

$$\begin{aligned} g[\omega, f_l, f_r](x) &\doteq \frac{f_l'(\omega(x))\,\big[f_r' \circ \pi_{r,-}^{l}(\omega(x))\big]^2}{\big[f_r'' \circ \pi_{r,-}^{l}(\omega(x))\big]\,[f_l'(\omega(x))]^2\big(T \cdot f_l'(\omega(x)) - x\big) + x\big[f_r' \circ \pi_{r,-}^{l}(\omega(x))\big]^2 f_l''(\omega(x))}, \\ h[\omega, f_l, f_r](x) &\doteq \frac{f_r'(\omega(x))\big[f_l' \circ \pi_{l,+}^{r}(\omega(x))\big]^2}{\big[f_l'' \circ \pi_{l,+}^{r}(\omega(x))\big]\,[f_r'(\omega(x))]^2\,\big(T \cdot f_r'(\omega(x)) - x\big) + x\big[f_l' \circ \pi_{l,+}^{r}(\omega(x))\big]^2 f_r''(\omega(x))}, \end{aligned} \tag{5.3.6}$$

defined for $x \in]\mathsf{L}[\omega, f_l], 0[$, $\omega(x) \leq A$, and for $x \in]0, \mathsf{R}[\omega, f_r][$, $\omega(x) \geq B$, respectively.

Remark 5.3.1. The definitions of the functions g, h are meaningful in their domains. In fact, the maps $\pi_{r,-}^{l}$, $\pi_{l,+}^{r}$ in (5.3.1) (that appear in the definitions of g, h) are well defined if $\omega(x) \leq A$, and $\omega(x) \geq B$, respectively. Moreover, by definition (5.3.2), we have

$$\begin{aligned} &T f_l'(\omega(x)) - x < 0, \qquad f_l'(\omega(x)) < 0 \qquad \forall\, x \in]\mathsf{L}[\omega, f_l],\, 0[\,, \\ &T f_r'(\omega(x)) - x > 0, \qquad f_r'(\omega(x)) > 0 \qquad \forall\, x \in]0,\, \mathsf{R}[\omega, f_r][\,. \end{aligned}$$

Hence, relying also on (25), we deduce that the denominator of g is strictly negative for $x \in]\mathsf{L}[\omega, f_l], 0[$, while the denominator of h is strictly positive for $x \in]0, \mathsf{R}[\omega, f_r][$. The functions g, h will provide a one-sided upper bound for the derivative of ω only in the interval $]\mathsf{L}[\omega, f_l], 0[$, assuming $\omega(x) \leq A$, and on the interval $]0, \mathsf{R}[\omega, f_r][$, assuming $\omega(x) \geq B$, respectively.

Since by Remark 5.1.3 we know that $\mathcal{A}^{[AB]}(T) \subset BV_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$, we can partition the attainable set as

$$\mathcal{A}^{[AB]}(T) = \bigcup_{\mathsf{L} \leq 0,\, \mathsf{R} \geq 0} \left(\mathcal{A}^{[AB]}(T) \cap \mathscr{A}^{\mathsf{L},\mathsf{R}}\right), \tag{5.3.7}$$

where

$$\mathscr{A}^{\mathsf{L},\mathsf{R}} \doteq \left\{\omega \in (\mathbf{L}^\infty \cap BV_{\mathrm{loc}})(\mathbb{R} \setminus \{0\}) \ : \ \mathsf{L}[\omega, f_l] = \mathsf{L}, \quad \mathsf{R}[\omega, f_r] = \mathsf{R}\right\}. \tag{5.3.8}$$

The characterization of the attainable profiles in $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ will be given in:

- Theorem 5.3.3, if $\mathsf{L} < 0,\ \mathsf{R} > 0$, and (A, B) is non critical;
- Theorem 5.3.9, if $\mathsf{L} < 0,\ \mathsf{R} > 0$, and (A, B) is critical;
- Theorem 5.3.11, if $\mathsf{L} < 0,\ \mathsf{R} = 0$ or $\mathsf{L} = 0,\ \mathsf{R} > 0$;
- Theorem 5.3.14, if $\mathsf{L} = 0,\ \mathsf{R} = 0$.

REMARK 5.3.2. Any element of $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ is an equivalence class of functions that admit one-sided limit at any point $x \in \mathbb{R}$, and that have at most countably many discontinuities. Therefore, for any element of $\mathscr{A}^{\mathsf{L},\mathsf{R}}$, we can always choose a representative which is left or right continuous. For sake of uniqueness, throughout the paper we will consider a representative of ω that is right continuous.

Throughout the following

$$D^-\omega(x) = \liminf_{h \to 0} \frac{\omega(x+h) - \omega(x)}{h}, \qquad D^+\omega(x) = \limsup_{h \to 0} \frac{\omega(x+h) - \omega(x)}{h}, \tag{5.3.9}$$

will denote, respectively, the lower and the upper Dini derivative of a function ω at x.

THEOREM 5.3.3. *In the same setting of Theorem 9, let (A, B) be a non critical connection, let $\mathcal{A}^{[AB]}(T)$, $T > 0$, be the set in (26), and let ω be an element of the set $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ in (5.3.8), with $\mathsf{L} < 0$, $\mathsf{R} > 0$. Then, $\omega \in \mathcal{A}^{[AB]}(T)$ if and only if the limits $\omega(0\pm)$ exist, and there hold:*

(i) the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\ \mathsf{L}[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]\mathsf{R}, +\infty[\,. \end{aligned} \tag{5.3.10}$$

Moreover, letting g, h be the functions in (5.3.6), and letting $\widetilde{\mathsf{L}} \doteq \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B]$, $\widetilde{\mathsf{R}} \doteq \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B]$, be the constants in (5.3.3), (5.3.4), if $\mathsf{R} \in]0, T \cdot f_r'(B)[$, and if $\widetilde{\mathsf{L}} > \mathsf{L}$, then one has

$$D^+\omega(x) \leq g[\omega, f_l, f_r](x) \qquad \forall\, x \in]\mathsf{L},\, \widetilde{\mathsf{L}}\,[\,, \tag{5.3.11}$$

while, if $\mathsf{L} \in]T \cdot f_l'(A), 0[$ *, and if* $\widetilde{\mathsf{R}} < \mathsf{R}$*, then one has*

$$D^+\omega(x) \leq h[\omega, f_l, f_r](x) \qquad \forall\, x \in]\widetilde{\mathsf{R}},\ \mathsf{R}[\,. \tag{5.3.12}$$

(ii) letting $\boldsymbol{u}[\mathsf{R}, B, f_r]$*,* $\boldsymbol{v}[\mathsf{L}, A, f_l]$*, be constants defined as in* (5.2.7)*,* (5.2.11)*, the following pointwise state constraints are satisfied*

$$\begin{aligned} \mathsf{L} \in]T \cdot f_l'(A), 0[\qquad &\Longrightarrow \qquad \omega(\mathsf{L}-) \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq \omega(\mathsf{L}+)\,, \\ \mathsf{R} \in]0, T \cdot f_r'(B)[\qquad &\Longrightarrow \qquad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-)\,. \end{aligned} \tag{5.3.13}$$

$$\begin{aligned} \Big[\mathsf{L} \in]T \cdot f_l'(A), 0[\ \text{ and } \ \mathsf{R} \leq \widetilde{\mathsf{R}}\Big] \quad \text{or} \quad \mathsf{L} \leq T\cdot f_l'(A) \qquad &\Longrightarrow \qquad \omega(x) = B \qquad \forall\, x \in]0, \mathsf{R}[\,, \\ \Big[\mathsf{R} \in]0, T \cdot f_r'(B)[\ \text{ and } \ \widetilde{\mathsf{L}} \leq \mathsf{L}\Big] \quad \text{or} \quad \mathsf{R} \geq T\cdot f_r'(B) \qquad &\Longrightarrow \qquad \omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \end{aligned} \tag{5.3.14}$$

$$\mathsf{L} \in]T \cdot f_l'(A), 0[\ \text{ and } \ \widetilde{\mathsf{R}} < \mathsf{R} \qquad \Longrightarrow \qquad \begin{cases} \omega(x) = B & \forall\, x \in]0, \widetilde{\mathsf{R}}], \\ \omega(\widetilde{\mathsf{R}}+) = B, \\ \omega(x) \geq B & \forall\, x \in]\,\widetilde{\mathsf{R}},\ \mathsf{R}[\,, \end{cases} \tag{5.3.15}$$

$$\mathsf{R} \in]0, T \cdot f_r'(B)[\ \text{ and } \ \mathsf{L} < \widetilde{\mathsf{L}} \qquad \Longrightarrow \qquad \begin{cases} \omega(x) = A & \forall\, x \in [\,\widetilde{\mathsf{L}},\ 0[\,, \\ \omega(\widetilde{\mathsf{L}}-) = A, \\ \omega(x) \leq A & \forall\, x \in]\mathsf{L}, \widetilde{\mathsf{L}}\,[\,, \end{cases} \tag{5.3.16}$$

$$\begin{aligned} \mathsf{L} \leq T \cdot f_l'(A) \qquad &\Longrightarrow \qquad \omega(\mathsf{L}-) \geq \omega(\mathsf{L}+), \\ \mathsf{R} \geq T \cdot f_r'(B) \qquad &\Longrightarrow \qquad \omega(\mathsf{R}-) \geq \omega(\mathsf{R}+). \end{aligned} \tag{5.3.17}$$

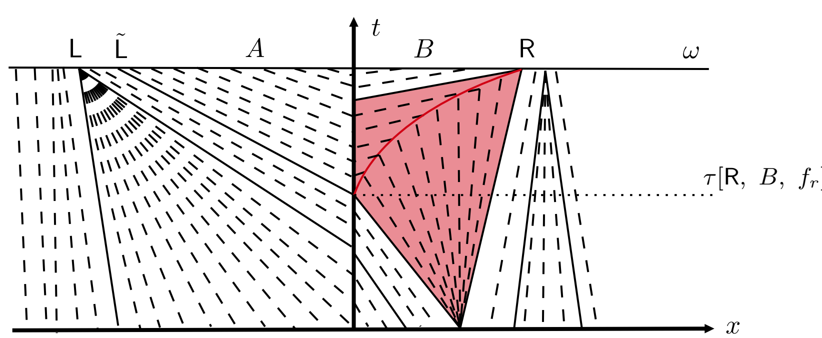

FIGURE 6. Case 1.

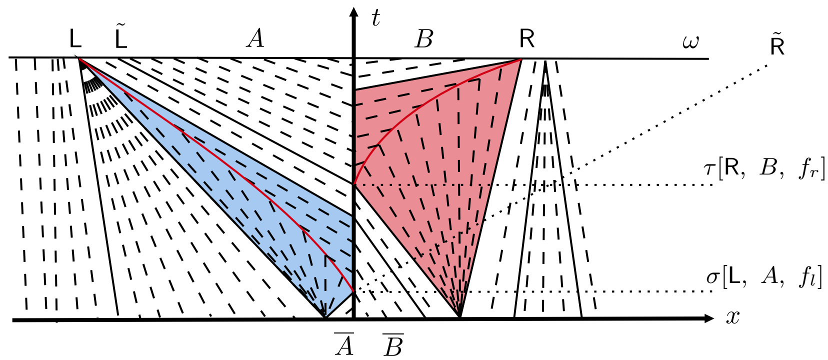

FIGURE 7. Case 2.

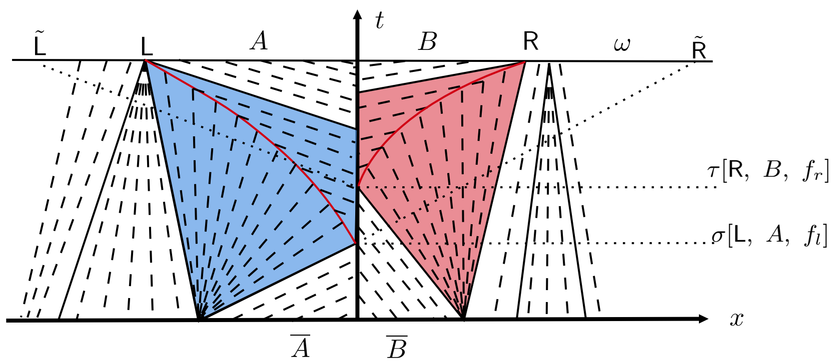

FIGURE 8. Case 3.

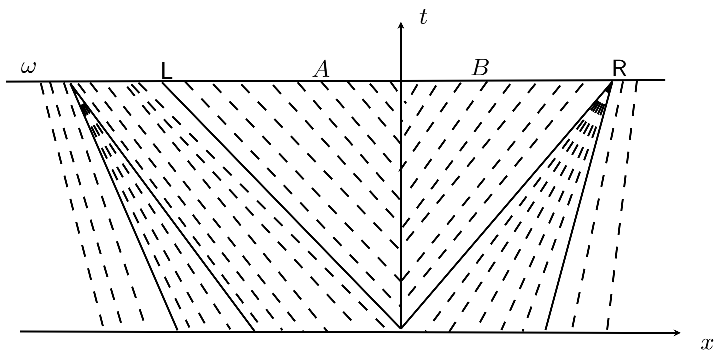

FIGURE 9. Case 4.

REMARK 5.3.4. Notice that conditions (5.3.14), (5.3.15) imply $\omega(\mathsf{R}-) \geq B$. On the other hand, if $\mathsf{R} < T \cdot f_r'(B)$, by virtue of (5.3.13), and because of (5.2.31), we have $\omega(\mathsf{R}+) \leq B$. Hence, because of (5.3.17), it follows that the inequality $\omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)$ is always satisfied. With similar arguments we deduce that also the inequality $\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+)$ is always verified.

REMARK 5.3.5. If $\mathsf{R}[\omega, f_r] \in]0, T \cdot f_r'(B)[$, applying (5.2.38) with f_r in place of f and $\mathsf{R} = \mathsf{R}[\omega, f_r]$, and recalling (5.3.3), we derive

$$\frac{\mathsf{R}[\omega, f_r]}{f_r'(B)} < T - \boldsymbol{\tau}\big[\mathsf{R}[\omega, f_r], B, f_r\big] = \frac{\widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B]}{f_l'(A)} \,. \tag{5.3.18}$$

Similarly, if $\mathsf{L}[\omega, f_l] \in]T \cdot f_l'(A), 0[$, applying (5.2.43) with f_l in place of f and $\mathsf{L} = \mathsf{L}[\omega, f_l]$, and recalling that $f_l'(A) < 0$ we find

$$\frac{\mathsf{L}[\omega, f_l]}{f_l'(A)} < T - \boldsymbol{\sigma}\big[\mathsf{L}[\omega, f_l], A, f_l\big] \,. \tag{5.3.19}$$

Hence, if $\widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \geq \mathsf{L}[\omega, f_l]$, combining (5.3.18), (5.3.19), we deduce

$$\frac{\mathsf{R}[\omega, f_r]}{f_r'(B)} < T - \boldsymbol{\sigma}\big[\mathsf{L}[\omega, f_l], A, f_l\big] \,, \tag{5.3.20}$$

which, in turn, by (5.3.3) yields

$$\mathsf{R}[\omega, f_r] < \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \,. \tag{5.3.21}$$

With entirely similar arguments one can show that, if $\widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \leq \mathsf{R}[\omega, f_r]$, then one has

$$\mathsf{L}[\omega, f_l] > \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \,. \tag{5.3.22}$$

Therefore, when $\mathsf{L}[\omega, f_l] \in]T \cdot f_l'(A), 0[$, and $\mathsf{R}[\omega, f_r] \in]0, T \cdot f_r'(B)[$, we have

$$\begin{aligned} \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \geq \mathsf{L}[\omega, f_l] \quad &\Longrightarrow \quad \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] > \mathsf{R}[\omega, f_r] \,, \\ \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \leq \mathsf{R}[\omega, f_r] \quad &\Longrightarrow \quad \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] < \mathsf{L}[\omega, f_l] \,. \end{aligned} \tag{5.3.23}$$

These implications, in particular, show that it can never occur the case where

$$\widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \geq \mathsf{L}[\omega, f_l] \qquad \text{and} \qquad \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \leq \mathsf{R}[\omega, f_r] \,. \tag{5.3.24}$$

REMARK 5.3.6. Notice that by condition (5.3.14) in Theorem 5.3.3, and because of (5.3.3), it follows that if $\mathsf{L}[\omega, f_l] \in]T \cdot f_l'(A), 0[$, and $\mathsf{R}[\omega, f_r] \leq \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B]$, then one has $\mathsf{R}[\omega, f_r] < T \cdot f_r'(B)$. Therefore, we have

$$\Big[\mathsf{L}[\omega, f_l] \in]T \cdot f_l'(A), 0[\quad \text{and} \quad \mathsf{R}[\omega, f_r] \geq T \cdot f_r'(B)\Big] \quad \Longrightarrow \quad \mathsf{R}[\omega, f_r] > \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B] \,. \tag{5.3.25}$$

Similarly, one can show that, by (5.3.3), (5.3.14), we have

$$\Big[\mathsf{R}[\omega, f_r] \in]0, T \cdot f_r'(B)[\quad \text{and} \quad \mathsf{L}[\omega, f_l] \leq T \cdot f_l'(A)\Big] \quad \Longrightarrow \quad \mathsf{L}[\omega, f_l] < \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B] \,. \tag{5.3.26}$$

Then, relying on (5.3.23), (5.3.25), (5.3.26), we deduce that, for non critical connections, we can distinguish six cases of pointwise constraints prescribed by condition (ii) of Theorem 5.3.3, which depend on the respective positions of the points $\mathsf{L} = \mathsf{L}[\omega, f_l]$, $\mathsf{R} = \mathsf{R}[\omega, f_r]$, and $\widetilde{\mathsf{L}} = \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B]$, $\widetilde{\mathsf{R}} = \widetilde{\mathsf{R}}[\omega, f_l, f_r, A, B]$:

CASE 1: If $\mathsf{L} \leq T \cdot f_l'(A) < 0$, $0 < \mathsf{R} < T \cdot f_r'(B)$ (Figure 6), then $\widetilde{\mathsf{L}} > \mathsf{L}$, and it holds true

$$\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+) \,, \quad \omega(x) \leq A \quad \forall\, x \in]\mathsf{L}, \widetilde{\mathsf{L}}\,[\,, \quad \omega(\,\widetilde{\mathsf{L}}-) = A, \qquad \omega(x) = A \quad \forall\, x \in]\widetilde{\mathsf{L}}, 0[\,, \tag{5.3.27}$$

$$\omega(x) = B \qquad \forall\, x \in]0, \mathsf{R}[\,, \qquad\quad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq B\,; \tag{5.3.28}$$

CASE 2: If $T \cdot f_l'(A) < \mathsf{L} < 0$, $0 < \mathsf{R} < T \cdot f_r'(B)$, and $\widetilde{\mathsf{L}} > \mathsf{L}$, $\widetilde{\mathsf{R}} > \mathsf{R}$ (Figure 7), then it holds true (5.3.28) and

$$\omega(\mathsf{L}-) \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq A\,, \quad \omega(x) \leq A \quad \forall\, x \in]\mathsf{L}, \widetilde{\mathsf{L}}\,[\,, \quad \omega(\widetilde{\mathsf{L}}-) = A, \quad \omega(x) = A \quad \forall\, x \in]\widetilde{\mathsf{L}}, 0[\,; \tag{5.3.29}$$

the symmetric ones:

CASE 1B: If $T \cdot f_l'(A) < \mathsf{L} < 0$, $0 < T \cdot f_r'(B) \leq \mathsf{R}$, then $\widetilde{\mathsf{R}} < \mathsf{R}$ and it holds true that

$$\omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad \omega(\mathsf{L}-) \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq A\,, \tag{5.3.30}$$

$$\omega(x) = B \quad \forall\, x \in]0, \widetilde{\mathsf{R}}\,[\,, \qquad \omega(\widetilde{\mathsf{R}}+) = B, \qquad \omega(x) \geq B \quad \forall\, x \in]\widetilde{\mathsf{R}}, \mathsf{R}[\,; \tag{5.3.31}$$

CASE 2B: If $T \cdot f_l'(A) < \mathsf{L} < 0$, $0 < \mathsf{R} < T \cdot f_r'(B)$, and $\widetilde{\mathsf{L}} < \mathsf{L}$, $\widetilde{\mathsf{R}} < \mathsf{R}$, then it holds true (5.3.30) and

$$\omega(x) = B \quad \forall\, x \in]0, \widetilde{\mathsf{R}}\,[\,, \quad \omega(\widetilde{\mathsf{R}}+) = B, \quad \omega(x) \geq B \quad \forall\, x \in]\widetilde{\mathsf{R}}, \mathsf{R}[\quad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq B\,; \tag{5.3.32}$$

and the remaining ones:

CASE 3: If $T \cdot f_l'(A) < \mathsf{L} < 0$, $0 < \mathsf{R} < T \cdot f_r'(B)$, and $\widetilde{\mathsf{L}} \leq \mathsf{L}$, $\widetilde{\mathsf{R}} \geq \mathsf{R}$ (Figure 8), then it holds true (5.3.28), (5.3.30);

CASE 4: If $\mathsf{L} \leq T \cdot f_l'(A) < 0$ and $\mathsf{R} \geq T \cdot f_r'(B) > 0$ (Figure 9), then it holds true

$$\begin{aligned} &\omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad && \omega(\mathsf{L}-) \geq \omega(\mathsf{L}+)\,, \\ &\omega(x) = B \qquad \forall\, x \in]0, \mathsf{R}[\,, \qquad && \omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)\,. \end{aligned} \tag{5.3.33}$$

The six cases are depicted in Figure 10. One can regard the intervals $]T \cdot f_l'(A), 0[$ and $]0, T \cdot f_r'(B)[$ as "*active zones*" for the presence of shocks in an AB-entropy solution that attains ω at time T: as soon as L belongs to $]T \cdot f_l'(A), 0[$ or R belongs to $]0, T \cdot f_r'(B)[$, it is needed a shock located in $\{x < 0\}$ or in $\{x > 0\}$, respectively, in order to produce the discontinuity occurring in ω at L or R.

REMARK 5.3.7. When the connection is not critical and $\mathsf{L} \doteq \mathsf{L}[\omega, f_l] < 0$, $\mathsf{R} \doteq \mathsf{R}[\omega, f_r] > 0$, the analysis of attainable profiles $\omega \in \mathcal{A}^{AB}(T)$ pursued in [**2**] catches only the profiles described in Cases 3 and 4 of Remark 5.3.6. In fact, the characterization of $\mathcal{A}^{AB}(T)$ established in [**2**, Theorem 6.1] requires that all profiles $\omega \in \mathcal{A}^{AB}(T)$ satisfy the equalities

$$\omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad\qquad \omega(x) = B \qquad \forall\, x \in]0, \mathsf{R}[\,.$$

Therefore, such a characterization in particular excludes all attainable profiles ω that either satisfy conditions (5.3.27) or (5.3.29), of Cases 1 and 2, with

$$\omega(x) < A \qquad \text{for some} \;\; x \in]\mathsf{L}, \widetilde{\mathsf{L}}[\,,$$

or satisfy conditions (5.3.31), (5.3.32), of Cases 1B and 2B, with

$$\omega(x) > B \qquad \text{for some} \;\; x \in]\widetilde{\mathsf{R}}, \mathsf{R}[\,.$$

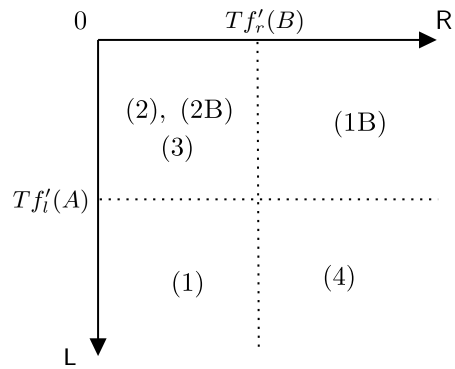

FIGURE 10. The different cases of Remark 5.3.5.

REMARK 5.3.8. Notice that, if $A = \theta_l$, or $\mathsf{R} = 0$, by definition (5.3.4), and because of (5.3.5), it follows that $\widetilde{\mathsf{L}} = 0$. Similarly, if $B = \theta_r$, or $\mathsf{L} = 0$, we have $\widetilde{\mathsf{R}} = 0$. Thus, in the case of critical connections, or whenever $\mathsf{L} = 0$ or $\mathsf{R} = 0$ (for critical and non critical connections), the characterization of the profiles $\omega \in \mathcal{A}^{AB}(T) \cap \mathscr{A}^{\mathsf{L},\mathsf{R}}$ will not involve the constants $\widetilde{\mathsf{L}}$, $\widetilde{\mathsf{R}}$.

THEOREM 5.3.9. *In the same setting of Theorem 5.3.3, let ω be an element of the set $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ in* (5.3.8)*, with $\mathsf{L} < 0$, $\mathsf{R} > 0$, and assume that $(A, B) = (\theta_l, B)$ (connection critical from the left). Then, $\omega \in \mathcal{A}^{[AB]}(T)$ if and only if $B \neq \theta_r$, the limits $\omega(0\pm)$ exist and there hold:*

(i) the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\ \mathsf{L}[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]\mathsf{R}, +\infty[\,. \end{aligned} \tag{5.3.34}$$

Moreover, letting g be the function in (5.3.6)*, then one has*

$$D^+\omega(x) \leq g[\omega, f_l, f_r](x) \qquad \forall\, x \in]\mathsf{L}, 0\,[\,. \tag{5.3.35}$$

(ii) letting $\boldsymbol{u}[\mathsf{R}, B, f_r]$, $\boldsymbol{\tau}[\mathsf{R}, B, f_r]$, be constants defined as in (5.2.7), (5.2.32)*, respectively, the following pointwise state constraints are satisfied*

$$(f_l')^{-1}\left(\frac{x}{T - \boldsymbol{\tau}[\mathsf{R}, B, f_r]}\right) \leq \omega(x) < \theta_l, \qquad \forall\, x \in]\mathsf{L}, 0[\,, \tag{5.3.36}$$

$$\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+), \qquad \omega(0-) = \theta_l\,, \tag{5.3.37}$$

$$\omega(x) = B \quad \forall\, x \in]0, \mathsf{R}[\,, \qquad \mathsf{R} \in]0, T \cdot f_r'(B)[\,, \tag{5.3.38}$$

$$\omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-)\,. \tag{5.3.39}$$

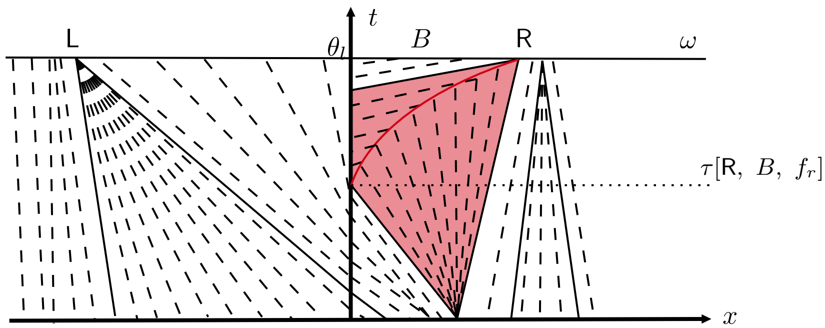

FIGURE 11. Typical profile of Theorem 5.3.9 for connections critical at the left (θ_l, B).

Symmetrycally, assume that $(A, B) = (A, \theta_r)$ *(connection critical from the right). Then,* $\omega \in \mathcal{A}^{[AB]}(T)$ *if and only if* $A \neq \theta_l$*, the limits* $\omega(0\pm)$ *exist and there hold:*
(i)' the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\, \mathsf{L}[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]\mathsf{R}, +\infty[\,. \end{aligned} \tag{5.3.40}$$

Moreover, letting h *be the function in* (5.3.6)*, then one has*

$$D^+\omega(x) \leq h[\omega, f_l, f_r](x) \qquad \forall\, x \in]0, \mathsf{R}[. \tag{5.3.41}$$

(ii)' letting $\boldsymbol{v}[\mathsf{R}, B, f_r]$*,* $\boldsymbol{\sigma}[\mathsf{L}, A, f_l]$*, be constants defined as in* (5.2.11), (5.2.39)*, respectively, the following pointwise state constraints are satisfied*

$$\omega(\mathsf{L}-) \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq \omega(\mathsf{L}+)\,, \tag{5.3.42}$$

$$\omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad \mathsf{L} \in]T \cdot f_l'(A), 0[\,, \tag{5.3.43}$$

$$\omega(0+) = \theta_r\,, \qquad \omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)\,, \tag{5.3.44}$$

$$\theta_r < \omega(x) \leq (f_r')^{-1}\left(\frac{x}{T - \boldsymbol{\sigma}[\mathsf{L}, A, f_l]}\right) \qquad \forall\, x \in]0, \mathsf{R}[\,. \tag{5.3.45}$$

REMARK 5.3.10. For critical connections, whenever $\mathsf{L} < 0 < \mathsf{R}$ we can distinguish two cases of pointwise constraints prescribed by Theorem 5.3.9 on an attainable profile ω, which depend on the side in which the connection is critical.
CASE 1: If $A = \theta_l$, and $\mathsf{L} < 0 < \mathsf{R} < T \cdot f_r'(B)$ (Figure 11), then it holds true

$$(f_l')^{-1}\left(\frac{x}{T - \boldsymbol{\tau}[\mathsf{R}, B, f_r]}\right) \leq \omega(x) < \theta_l, \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad \omega(0-) = \theta_l\,,$$

$$\omega(x) = B \qquad \forall\, x \in]0, \mathsf{R}[\,, \qquad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq B\,;$$

CASE 2: If $B = \theta_r$, and $T \cdot f_l'(A) < \mathsf{L} < 0 < \mathsf{R}$, then it holds true

$$\omega(x) = A \qquad \forall\, x \in]\mathsf{L}, 0[\,, \qquad\qquad A \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq \omega(\mathsf{L}+)\,;$$

$$\theta_r < \omega(x) \leq (f_r')^{-1}\left(\frac{x}{T - \boldsymbol{\sigma}[\mathsf{L}, A, f_l]}\right) \qquad \forall\, x \in]0, \mathsf{R}[\,, \qquad\qquad \omega(0+) = \theta_r\,.$$

In both cases an AB-entropy solution that attains ω at time T must contain a shock located in $\{x > 0\}$ (in CASE 1), or in $\{x < 0\}$ (in CASE 2), in order to produce the discontinuity occurring in ω at R or L.

THEOREM 5.3.11. *In the same setting of Theorem 5.3.3, let ω be an element of the set $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ in (5.3.8), let g, h be the functions in (5.3.6), and let $\boldsymbol{u}[\mathsf{R}, B, f_r]$, $\boldsymbol{v}[\mathsf{L}, A, f_l]$, be constants defined as in (5.2.7), (5.2.11). Then, if $\mathsf{L} < 0$, $\mathsf{R} = 0$, $\omega \in \mathcal{A}^{[AB]}(T)$ if and only if the limits $\omega(0\pm)$ exist, and it holds:*

(i) the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\ \mathsf{L}[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]0, +\infty[\,, \end{aligned} \tag{5.3.46}$$

$$D^+\omega(x) \leq g[\omega, f_l, f_r](x) \qquad \forall\, x \in\ ]\mathsf{L}, 0[\,. \tag{5.3.47}$$

(ii) the following pointwise state constraints are satisfied:

$$\begin{cases} \omega(x) \leq A & \textit{if} \quad A < \theta_l, \\ \omega(x) < A & \textit{if} \quad A = \theta_l, \end{cases} \qquad \forall\, x \in]\mathsf{L}, 0[\,, \tag{5.3.48}$$

$$\omega(0+) \leq \pi^l_{r,-}(\omega(0-))\,, \tag{5.3.49}$$

and

$$\mathsf{L} \in]T \cdot f_l'(A), 0[\qquad \Longrightarrow \qquad \omega(\mathsf{L}-) \geq \boldsymbol{v}[\mathsf{L}, A, f_l] \geq \omega(\mathsf{L}+)\,, \tag{5.3.50}$$

$$\mathsf{L} \leq T \cdot f_l'(A) \qquad \Longrightarrow \qquad \omega(\mathsf{L}-) \geq \omega(\mathsf{L}+). \tag{5.3.51}$$

Symmetrically, if $\mathsf{L} = 0$, $\mathsf{R} > 0$, then $\omega \in \mathcal{A}^{[AB]}(T)$ if and only if it holds true:

(i)′ the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\ 0[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]\mathsf{R}, +\infty[\,, \end{aligned} \tag{5.3.52}$$

$$D^+\omega(x) \leq h[\omega, f_l, f_r](x) \qquad \forall\, x \in\ ]0, \mathsf{R}[\,. \tag{5.3.53}$$

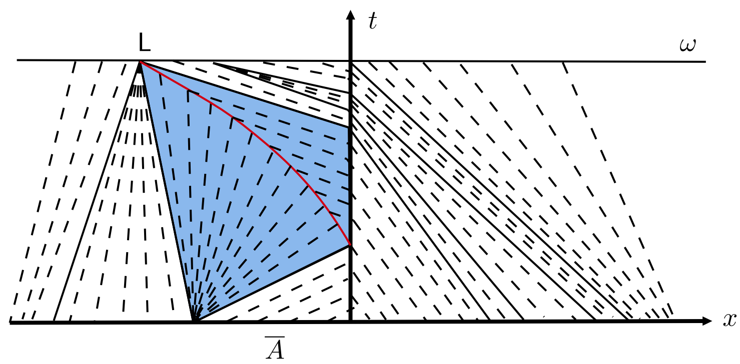

FIGURE 12. Theorem 5.3.11 when $\mathsf{L} < 0$, $\mathsf{R} = 0$ and $\mathsf{L} \in]T \cdot f_l'(A), 0[$.

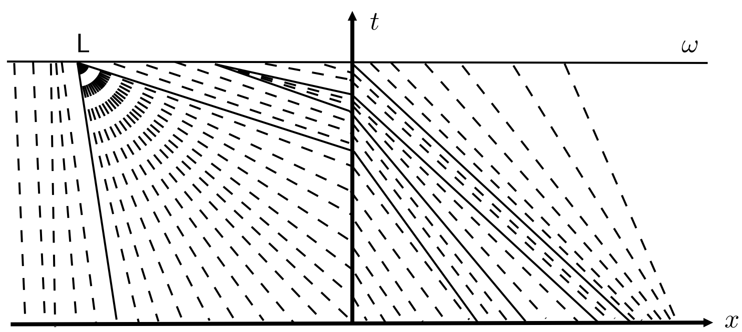

FIGURE 13. Theorem 5.3.11 when $\mathsf{L} < 0$, $\mathsf{R} = 0$ and $\mathsf{L} \leq T \cdot f_l'(A)$.

(ii)′ the following pointwise state constraints are satisfied:

$$\begin{cases} \omega(x) \geq B & \text{if} \quad B > \theta_r, \\ \omega(x) > B & \text{if} \quad B = \theta_r, \end{cases} \qquad \forall\, x \in]0, \mathsf{R}[\,, \tag{5.3.54}$$

$$\omega(0-) \geq \pi^r_{l,+}(\omega(0+))\,, \tag{5.3.55}$$

and

$$\mathsf{R} \in]0, T \cdot f_r'(B)[\qquad \Longrightarrow \qquad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-)\,, \tag{5.3.56}$$

$$\mathsf{R} \geq T \cdot f_r'(B) \qquad \Longrightarrow \qquad \omega(\mathsf{R}+) \leq \omega(\mathsf{R}-). \tag{5.3.57}$$

REMARK 5.3.12. Notice that the implications (5.3.50)-(5.3.51), (5.3.56)-(5.3.57) can be extended to $\mathsf{L} = T \cdot f_l'(A)$ and to $\mathsf{R} = T \cdot f_r'(B)$, respectively. In fact, by definition (5.3.2) of $\mathsf{L} = \mathsf{L}[\omega, f_l]$, one has $f_l'(\omega(\mathsf{L}-)) \geq \mathsf{L}/T$. Hence, if $\mathsf{L} = T \cdot f_l'(A)$ it follows that $f_l'(\omega(\mathsf{L}-)) \geq f_l'(A)$ which yields $\omega(\mathsf{L}-) \geq A$ by the monotonicity of f_l'. Thus, recalling that by (5.2.30) we have $\boldsymbol{v}[T \cdot f'(A), A, f] = A$, we derive

$$\omega(T \cdot f_l'(A)-) \geq \boldsymbol{v}[T \cdot f_l'(A), A, f_l]\,. \tag{5.3.58}$$

On the other hand, since (5.3.48) implies $\omega(T \cdot f_l'(A+) \leq A$, we deduce from (5.3.58) that

$$\omega(T \cdot f_l'(A)-) \geq \omega(T \cdot f_l'(A)+)\,. \tag{5.3.59}$$

With entirely similar arguments one can show that we have

$$\omega(T \cdot f_l'(B)+) \leq \boldsymbol{u}[T \cdot f_l'(B), B, f_r]\,, \tag{5.3.60}$$

$$\omega(T \cdot f_r'(B)+) \leq \omega(T \cdot f_r'(B)-)\,. \tag{5.3.61}$$

Hence, relying on (5.3.48), (5.3.50), (5.3.51), (5.3.54), (5.3.56), (5.3.57), and on (5.3.59), (5.3.61), with the same arguments of Remark 5.3.4 we deduce that the inequalities $\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+)$, $\omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)$ are always satisfied.

Remark 5.3.13. Relying on Remark 5.3.8, we can view the conditions that characterize the pointwise constraints of attainable profiles in Theorem 5.3.11 as limiting cases of the conditions of Theorems 5.3.3, 5.3.9, classified in Remarks 5.3.6, 5.3.10. Namely:

- For non critical connections, the case $\mathsf{L} \in]T \cdot f_l'(A), 0[$, $\mathsf{R} = 0$ (Figure 12), is the limiting situation as $\mathsf{R} \to 0$ of Case 2 in Remark 5.3.6. For critical connections with $A < \theta_l$, $B = \theta_r$, if the constraint (5.3.48) is satisfied with the equality, the case $\mathsf{L} \in]T \cdot f_l'(A), 0[$, $\mathsf{R} = 0$, is the limiting situation as $\mathsf{R} \to 0$ of Case 2 in Remark 5.3.10.
- For non critical connections, the case $\mathsf{L} \leq T \cdot f_l'(A)$, $\mathsf{R} = 0$ (Figure 13), is the limiting situation as $\mathsf{R} \to 0$ of Case 1 in Remark 5.3.6. For critical connections with $A = \theta_l$, $B > \theta_r$, the case $\mathsf{L} \leq T \cdot f_l'(A)$, $\mathsf{R} = 0$, is the limiting situation as $\mathsf{R} \to 0$ of Case 1 in Remark 5.3.10

Symmetrically, we have:

- For non critical connections, the case $\mathsf{L} = 0$, $\mathsf{R} \in]0, T \cdot f_r'(B)[$, is the limiting situation as $\mathsf{L} \to 0$ of Case 2B in Remark 5.3.6. For critical connections with $A = \theta_l$, $B > \theta_r$, if the constraint (5.3.54) is satisfied with the equality, the case $\mathsf{L} = 0$, $\mathsf{R} \in]0, T \cdot f_r'(B)[$ is the limiting situation as $\mathsf{L} \to 0$ of Case 1 in Remark 5.3.10.
- For non critical connections, the case $\mathsf{L} = 0$, $\mathsf{R} \geq T \cdot f_r'(B)$, is the limiting situation as $\mathsf{L} \to 0$ of Case 1B in Remark 5.3.6. For critical connections with $A < \theta_l$, $B = \theta_r$, the case $\mathsf{R} \geq T \cdot f_r'(B)$ is the limiting situation as $\mathsf{L} \to 0$ of Case 2 in Remark 5.3.10.

Notice that, for non critical connections, no limiting situation of Case 3 or of Case 4 in Remark 5.3.6 arises as characterizing the pointwise constraints of attainable profiles in Theorem 5.3.11.
The same type of conditions discussed in Remark 5.3.6 require the presence of shocks in an AB-entropy solution that attains at time T a profile satisfying the conditions of Theorem 5.3.11. In fact, for such profiles it is needed a shock located in $\{x < 0\}$ (in $\{x > 0\}$) to produce the discontinuity in ω at $x = \mathsf{L}$ (at $x = \mathsf{R}$) if and only if $\mathsf{L} \in]T \cdot f_l'(A), 0[$, and $\mathsf{R} = 0$ ($\mathsf{L} = 0$ and $\mathsf{R} \in]0, T \cdot f_r'(B)[$).

Theorem 5.3.14. *In the same setting of Theorem 5.3.3, let ω be an element of the set $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ in* (5.3.8)*, with $\mathsf{L} = 0$, $\mathsf{R} = 0$. Then $\omega \in \mathcal{A}^{[AB]}(T)$ if and only if the limits $\omega(0\pm)$ exist, and it holds true:*

(i) the following Oleinik-type inequalities are satisfied

$$\begin{aligned} D^+\omega(x) &\leq \frac{1}{T \cdot f_l''(\omega(x))} \qquad \forall\, x \in]-\infty,\ 0[\,, \\ D^+\omega(x) &\leq \frac{1}{T \cdot f_r''(\omega(x))} \qquad \forall\, x \in]0, +\infty[\,. \end{aligned} \tag{5.3.62}$$

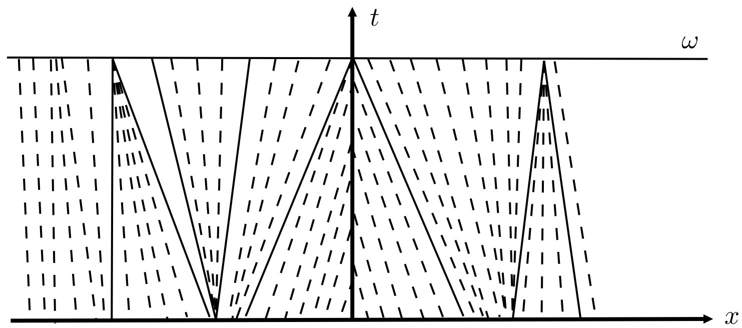

FIGURE 14. Structure of profiles described by Theorem 5.3.14.

(ii) the following pointwise state constraints are satisfied:

$$\omega(0-) \geq \overline{A}, \qquad \omega(0+) \leq \overline{B}\,, \tag{5.3.63}$$

REMARK 5.3.15. Recalling that by (5.2.30) we have $\boldsymbol{v}[0, A, f_l] = \overline{A}$, $\boldsymbol{u}[0, B, f_r] = \overline{B}$, we can rephrase the constraint (5.3.63) as

$$\omega(0-) \geq \ \boldsymbol{v}[0, A, f_l], \qquad \omega(0+) \leq \ \boldsymbol{u}[0, B, f_r]\,. \tag{5.3.64}$$

Any profile ω satisfying the conditions of Theorem 5.3.14 is attainable by AB-entropy solutions that don't contain shocks in $\{x < 0\}$ or in $\{x > 0\}$.

Since by Lemma 5.2.1 we have

$$\lim_{\mathsf{R}\to 0^+} \boldsymbol{u}[\mathsf{R}, f_r, B] = \overline{B}, \qquad \lim_{\mathsf{R}\to 0^-} \boldsymbol{v}[\mathsf{L}, f_l, A] = \overline{A},$$

and because of Remark 4.8, we can recover the conditions that characterize the pointwise constraints of attainable profiles in Theorem 5.3.14 as limiting cases of the conditions of Theorems 5.3.3, 5.3.9, classified in Remarks 5.3.6, 5.3.10. Namely:

- For a non critical connection, the condition (5.3.63) is the limit situation as $\mathsf{L}, \mathsf{R} \to 0$ of the CASE 2 of Remark 5.3.6.
- For a critical connection with $A = \theta_l$, $B > \theta_r$, the second condition of (5.3.63) is is the limiting situation as $\mathsf{R} \to 0$ of CASE 1 in Remark 5.3.10. The first condition of (5.3.63) is trivially satisfied, because $\overline{A} = \theta_l$, and since $\mathsf{L} = 0$ by definition (5.3.2) implies $\omega(0-) \geq \theta_l$. The case of a critical connection with $A < \theta_l$, $B = \theta_r$ is symmetric, and can be recovered as limiting situation as $\mathsf{L} \to 0$ of CASE 2 in Remark 5.3.10.

REMARK 5.3.16. By Remarks 5.3.13, 5.3.15, the conditions that characterize the pointwise constraints of attainable profiles provided by Theorems 5.3.3, 5.3.9 are essentially "dense" in the set of all conditions characterizing the pointwise constraints of any profile $\omega \in \mathcal{A}^{[AB]}(T)$ (in the sense that the further conditions provided by Theorems 5.3.11, 5.3.14 can be recovered via a limiting procedure as the parameters $\mathsf{L}, \mathsf{R} \to 0$).

Combining Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14, with Theorem 9, we obtain:

THEOREM 5.3.17. *In the same setting of Theorem 9, let (A, B) be a connection. Then, for every $T > 0$, and for any $\omega \in \mathbf{L}^\infty(\mathbb{R})$, the following conditions are equivalent.*

(1) $\omega \in \mathcal{A}^{AB}(T)$.

(2) $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}\omega = \omega\,.$

(3) ω *is an element of the set* $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ *in* (5.3.8)*, with* $\mathsf{L} \leq 0$*,* $\mathsf{R} \geq 0$*, that satisfies the conditions of Theorem 5.3.3, 5.3.9, 5.3.11, or 5.3.14.*

Moreover, if (A, B) *is a non critical connection, i.e. if* $A \neq \theta_l, B \neq \theta_r$*, then the conditions (2) and (3) are equivalent to*

(1)' $\omega \in \mathcal{A}_{bv}^{[AB]}(T)$*, where*

$$\mathcal{A}_{bv}^{[AB]}(T) \doteq \left\{\mathcal{S}_T^{[AB]+}u_0 \ : \ u_0 \in BV_{loc}(\mathbb{R})\right\}, \tag{5.3.65}$$

and it holds true

$$\mathcal{A}^{[AB]}(T) = \mathcal{A}_{bv}^{[AB]}(T)\,. \tag{5.3.66}$$

Remark 5.3.18 (Comparison with previous results). Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14 yield the first *complete characterization of the attainable set at time* $T > 0$ *in terms of Oleinik-type inequalities and unilateral constraints*, for critical and non critical connections. Partial results in this direction have been recently obtained for *strict subsets* of $\mathcal{A}^{AB}(T)$. In particular, we refer to:

- the work [**9**], where it is characterized only the subset $\mathcal{A}_{\mathrm{L}}^{AB}(T) \subset \mathcal{A}^{AB}(T)$ given by
$$\mathcal{A}_{\mathrm{L}}^{[AB]}(T) = \{\omega \in \mathcal{A}^{[AB]}(T) \mid \exists\ AB\text{-entropy solution } u \in \mathrm{Lip}_{\mathrm{loc}}((0,T)\times\mathbb{R}\setminus\{0\}) \ : \ u(T,x) = \omega\}.$$
In particular, all the profiles ω for which $\mathsf{L} \in \,]T\cdot f_l'(A), 0[$ or $\mathsf{R} \in \,]0, T\cdot f_r'(B)[$ are missing in the characterization provided in [**9**]. In fact, as observed in Remarks 5.3.6, 5.3.10, 5.3.13, an AB-entropy solutions leading to such profiles at time T must contain a shock located in $\{x < 0\}$ or in $\{x > 0\}$, respectively, in order to produce the discontinuity occurring in ω at L or R.
- the work [**2**], in which, whenever either $\mathsf{L} = 0$, or $\mathsf{R} = 0$, the set $\mathcal{A}^{[AB]}(T)$ is fully characterized in terms of triples (a monotone function and a pair of points) related to the Lax-Oleinik representation formula of solutions (obtained in [**5**] via the Hamiton-Jacobi dual formulation). Instead, in the case of critical connections, all attainable profiles with $\mathsf{L} < 0$ and $\mathsf{R} > 0$ described by Theorem 5.3.9 are missing in [**2**]. On the other hand, when $\mathsf{L} < 0$, $\mathsf{R} > 0$ and (A, B), is a non critical connection, only the profiles of Cases 3, 4, discussed in Remark 5.3.5, are characterized in [**2**], while the ones of Cases 1, 2, 1b, 2b are missing. In fact, the profiles constructed in [**2**] with $\mathsf{L} < 0$, $\mathsf{R} > 0$ for non critical connections, satisfy always the condition $\omega(x) = A$ for all $x \in (\mathsf{L}, 0)$, and $\omega(x) = B$ for all $x \in (0, \mathsf{R})$, which is in general not fulfilled by profiles of Cases 1, 2, 1b, 2b (cfr. Remark 5.3.7).

We point out that, as a byproduct of the characterization of $\mathcal{A}^{AB}(T)$ via Oleinik-type estimates, one can establish uniform BV bounds on solutions to (22), (24) in the case of non critical connections, and on the flux of solutions to (22), (24) for general connections (see. Proposition 5.5.1 in Appendix 5.6). In turn such bounds yield the $\mathbf{L}^1_{loc}$-Lipschitz continuity in time of AB-entropy solutions (see the proof of Theorem 5.1.8-(v)) in Appendix 5.6).

5.4. Proof of Theorem 5.3.17

5.4.1. Proof roadmap. Observe that if (A, B) is a non critical connection, then recalling Definition 5.1.16, and relying on Proposition 5.5.1 in Appendix 5.6, we deduce that $\mathcal{S}_T^{[AB]-}\omega \in BV_{loc}(\mathbb{R})$ for all $\omega \in \mathbf{L}^\infty(\mathbb{R})$. Hence setting $u_0 \doteq \mathcal{S}_T^{[AB]-}\omega$, we deduce immediately the implication (2) ⇒ (1)′. On the other hand, since $\mathcal{A}_{bv}^{[AB]}(T) \subset \mathcal{A}^{[AB]}(T)$, from the implication (1) ⇒ (3), one deduces that (1)′ ⇒ (3) holds as well.

Therefore, in order to establish Theorem 5.3.17 it will be sufficient to prove the equivalence of the conditions (1), (2), (3). We provide here a road map of the proof of (1) ⇒ (2) ⇒ (3) ⇒ (1). There are three main parts, which are somewhat independent one from the other.

Part 1. The case of a non critical connection (1) ⇒ (3). In Sections 5.4.2-5.4.3 we prove the implication (1) ⇒ (3) of Theorem 5.3.17 when (A, B) is a *non critical* connection. The proof has a bootstrap-like structure, and it is divided in two steps. We first prove that (**1**) ⇒ (**3**) under the regularity assumption (H) formulated below, and next we show that this regularity property always holds true.

- *Part 1.a - (1) ⇒ (3) for non critical connections assuming* (H). This is the first fundamental block of our proof. We prove in § 5.4.2 the implication (1) ⇒ (3) for profiles $\omega \in \mathcal{A}^{[AB]}(T)$ that satisfy the BV condition:

$$\exists\, u_0 \in \mathbf{L}^\infty(\mathbb{R}) \ :\ \omega = \mathcal{S}_T^{[AB]+}u_0\,, \qquad \text{and} \qquad \mathcal{S}_t^{[AB]+}u_0 \in BV_{\mathrm{loc}}(\mathbb{R}) \quad \forall\, t>0\,. \tag{H}$$

  The derivation of the conditions of Theorem 5.3.3, 5.3.11, and 5.3.14 is obtained exploiting as in [**9**] the non crossing property of genuine characteristics in the domains $\{x>0,\, t>0\}$, $\{x<0,\, t>0\}$, together with the non existence of rarefactions emanating from the interface (cfr. Appendix 5.7 and [**2**]). Two key novel points of the analysis here are:
  - a blowup argument, possible thanks to assumption (H), to derive the Oleinik-type inequalities satisfied by ω in regions comprising points with characteristics reflected by the interface $x=0$, and points with characteristics refracted by $x=0$.
  - a comparison argument (based on the duality of forward and backward shocks of § 5.2.3, and on the property of the states $\boldsymbol{u}[\mathsf{R}, B, f_r]$, $\boldsymbol{v}[\mathsf{L}, A, f_l]$, defined in § 5.2.1, 5.2.2) to establish the unilateral inequalities satisfied by ω at points of discontinuity generated by shocks that *isolate* the interface $\{x=0\}$ from the semiaxes $\{x<0\}$, $\{x>0\}$ (cfr. Remark 5.2.3).
- *Part 1.b - (1) ⇒ (3) for non critical connections without assuming* (H). We prove in § 5.4.3 the implication (1) ⇒ (3) for every $\omega \in \mathcal{A}^{AB}(T)$ by showing that every $\omega \in \mathcal{A}^{AB}(T)$ actually satisfies condition (H), and then the conclusion follows by Part 1.a. This is achieved: considering a sequence of functions $u_{n,0} \in BV(\mathbb{R})$ that $\mathbf{L}^\mathbf{1}_{\mathrm{loc}}$-converge to $u_0 \in \mathbf{L}^\infty(\mathbb{R})$; observing that $\mathcal{S}_t^{[AB]+}u_{n,0} \in BV(\mathbb{R})$ (see [**1, 64**]); deriving uniform BV bounds on $\mathcal{S}_T^{[AB]+}u_{n,0}$ based on the Oleinik-type inequalities enjoyed by $\mathcal{S}_T^{[AB]+}u_{n,0}$ because of Part 1.a; relying on the $\mathbf{L}^\mathbf{1}_{\mathrm{loc}}$-stability of the semigroup map $u_0 \mapsto \mathcal{S}_T^{[AB]+}u_0$ (see Theorem 5.1.8-(iii)) and on the lower semicontinuity of the total variation with respect to $\mathbf{L}^\mathbf{1}$-convergence.

Part 2. The case of a non critical connection (3) ⇒ (2) ⇒ (1). The implication (2) ⇒ (1) of Theorem 5.3.17 immediately follows observing that, by virtue of (2), one has $\omega = \mathcal{S}_T^{[AB]+} u_0 \in \mathcal{A}^{[AB]}(T)$, with $u_0 \doteq \mathcal{S}_T^{[AB]-}\omega$. Hence, in Sections 5.4.4-5.4.5 we prove only the implication (3) ⇒ (2) of Theorem 5.3.17, in the case of a non critical connection (A, B). This is the <u>second fundamental block</u> of our proof, which consists in first showing that (3) ⇒ (1), and next in proving that (3) ⇒ (2).

- *Part 2.a* - (3) ⇒ (1) *for non critical connections.* Given $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfying the condition of Theorem 5.3.3, we construct explicitly in § 5.4.4 an AB-entropy admissible solution $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$, $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, such that $u(\cdot, T) = \omega$. The case where $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfies the condition of Theorem 5.3.11, or 5.3.14 is entirely similar or simpler. The construction of u_0 and u follows a by now standard procedure (see [**9**], [**16**]) in regions of $\{x < 0\}$ or of $\{x > 0\}$ that are not influenced by waves reflected or refracted by the interface $x = 0$. Namely, in these regions, one construct the solution u along two type of lines that correspond to its characteristics: genuine characteristics ϑ_y ending at points (y, T), where $u = \omega(y)$, in the case ω is continuous at y; compression fronts $\eta_{y,z}$ connecting points $(z, 0)$ and (y, T), where $u = (f_l')^{-1}(\frac{y-x}{T})$, if $y < 0$, and $u = (f_r')^{-1}(\frac{y-x}{T})$, if $y > 0$, in the case ω is discontinuous at y. A key novel point of the analysis here is the construction of u in two polygonal regions around the interface $x = 0$, which relies on the properties of the *shock-rarefaction/rarefaction-shock wave patterns* established in § 5.2.4-5.2.5, which in turn are based on the duality properties of forward/backward shocks derived in 5.2.3. Thanks to this construction, one can in particular explicitly produce AB-entropy solutions that attain at time T the profiles of Cases 1, 2, 1b, 2b discussed in Remark 5.3.5, that are not present in [**2**] (cfr. Remark 5.3.18).
- *Part 2.b* - (3) ⇒ (2) *for non critical connections.* Given $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfying the conditions of Theorem 5.3.3, we show in § 5.4.5 that ω is a fixed point of the backward-forward operator $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}$. The case where $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfies the condition of Theorem 5.3.11, or 5.3.14 is entirely similar. Building on the analysis pursued in the previous part, in order to prove that $\omega = \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}\omega$ it is sufficient to show that, if u_0 is the initial datum of the AB-entropy solution $u(x,t)$ constructed in Part 2.a, then one has $u_0 = \mathcal{S}_T^{[AB]-}\omega$. This is again achieved exploiting the duality properties of forward/backward shocks derived in 5.2.3, and the structural properties of the *shock-rarefaction/rarefaction-shock wave patterns* established in § 5.2.4-5.2.5.

Part 3. The case of a critical connection (1) ⇔ (2) ⇔ (3). In Sections 5.4.6, 5.4.7, 5.4.8 we recover the equivalence of the conditions (1), (2), (3) of Theorem 5.3.17 in the case of critical connections, invoking the validity of this equivalence for non critical connections established in Parts 1-2. The proof is divided in three steps.

- *Part 3.a* - (1) ⇔ (2) *for critical connections.* In § 5.4.6 we prove the implication (1) ⇒ (2), relying on the $\mathbf{L}^1_{\text{loc}}$-stability of the maps $(A, B, u_0) \mapsto \mathcal{S}_T^{[AB]+} u_0$, $(A, B, u_0) \mapsto \mathcal{S}_T^{[AB]-} u_0$ (see Theorem 5.1.8-(iv) and Definition 5.1.16). The reverse implication is immediate as observed in Part 2.

- *Part 3.b* - (1) $\Rightarrow$ (3) *for critical connections.* In § 5.4.7 we prove the implication (1) $\Rightarrow$ (3), relying on the $\mathbf{L^1}$-weak stability of the maps $(A,B) \mapsto f_l(u_l)$, $(A,B) \mapsto f_r(u_r)$, where u_l, u_r denote, respectively the left and right states of $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$ at $x = 0$ (see Corollary 5.1.11), and on the lower/upper $\mathbf{L^1}$-semicontinuity property of solutions to conservation laws with uniformly convex flux (see Lemma 5.8.1 in Appendix 5.8).
- *Part 3.c* - (3) $\Rightarrow$ (1) *for critical connections.* In § 5.4.8 we prove the implication (3) $\Rightarrow$ (1) exploiting again the $\mathbf{L^1}_{\mathsf{loc}}$-stability of the semigroup map of Theorem 5.1.8-(iv), and using a perturbation argument. Namely, given $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfying the conditions of Theorem 5.3.9, 5.3.11, or 5.3.14, we construct a sequence $\{\omega_n\}_n$ of perturbations of ω with the property that $\omega_n \overset{\mathbf{L^1}}{\to} \omega$, and $\omega_n \in \mathcal{A}^{[A_n B_n]}(T)$, for a sequence of non critical connections $\{(A_n, B_n)\}_n$. This is another key point of our analysis, since it provides a general explicit procedure to approximate an attainable profile for a critical connection by attainable profiles for non critical connections.

Remark 5.4.1. In the case of critical connections, one may provide a direct proof of the implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) of Theorem 5.3.17 with similar arguments as the ones used in the case of non critical connections. Only the implication (1) $\Rightarrow$ (3) in the case of critical connections cannot be directly established with the same line of proof followed in § 5.4.2 for non critical connection. The reason is twofold. On one hand we cannot rely on the property of non existence of rarefactions emanating from the interface, since we establish in Appendix 5.7 this property only in the case of non critical connections. On the other hand we cannot exploit the uniform BV_{loc} bounds to perform the blowup argument of § 5.4.2.6, since they are enjoyed by AB-entropy solutions only when the connection is non critical (see § 5.5). An alternative, direct proof of (1) $\Rightarrow$ (3) can be obtained relying on the property of preclusion of rarefactions emanating from the interface derived in [**2**] for general connections. Using this property, it seems reasonable that one may then establish the Oleinik-type estimates that characterize the attainable profiles for critical connections performing a longer, technical analysis of the structure of characteristics that avoids the blow up argument of § 5.4.2.6.

5.4.2. Part 1.a - (1) $\Rightarrow$ (3) **for non critical connections assuming** (H)**.** In this Subsection, given an element ω of the set $\mathcal{A}^{[AB]}(T)$ for a non critical connection (A,B), assuming that ω satisfies (H), we will show that ω fulfills condition (3) of Theorem 5.3.17. Recalling (5.3.7), this is equivalent to show that, letting

$$\mathsf{L} \doteq \mathsf{L}[\omega, f_l]\,, \qquad \mathsf{R} \doteq \mathsf{R}[\omega, f_r], \tag{5.4.1}$$

be quantities defined as in (5.3.2), it holds true that:

2a-i) If $\mathsf{L} < 0$, $\mathsf{R} > 0$, and if ω satisfies (H), then ω satisfies the conditions of Theorem 5.3.3;

2a-ii) If $\mathsf{L} = 0$, $\mathsf{R} > 0$ or viceversa, and if ω satisfies (H), then ω satisfies the conditions of Theorem 5.3.11;

2a-iii) If $\mathsf{L} = 0$, $\mathsf{R} = 0$, then ω satisfies the conditions of Theorem 5.3.14.

We will prove 2a-i) in § 5.4.2.1-5.4.2.6, while 2a-ii) is proven in § 5.4.2.7, and 2a-iii) is discussed in § 5.4.2.8. The further assumption that ω satisfies (H) is needed only to ensure the existence of the one-sided limits $\omega(0\pm)$, and to show that ω satisfies (5.3.11)-(5.3.12) in case 2a-i), and (5.3.47), (5.3.53) in case 2a-ii).

Throughout the subsection we will assume that

$$\omega = \mathcal{S}_T^{[AB]+} u_0, \qquad u_0 \in \mathbf{L}^\infty(\mathbb{R})\,, \tag{5.4.2}$$

and we set $u(x,t) \doteq S_t^{[AB]+} u_0(x)$, $x \in \mathbb{R}$, $t \geq 0$. Under assumption (H) there exist the limits $u(0\pm, t)$, for all $t > 0$. We let $u_l(t), u_r(t)$ denote the left and right traces at $x = 0$ of $u(x,t)$, $t > 0$.

5.4.2.1. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, proof of (5.3.17)). The inequalities $\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+)$, $\omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)$ are the Lax conditions which are satisfied since u is an entropy admissible solution of the conservation law $u_t + f_l(u)_x = 0$, on $x < 0$, and of $u_t + f_r(u)_x = 0$, on $x > 0$, and the fluxes f_l, f_r are convex.

5.4.2.2. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, proof of (5.3.10)). By definition (5.3.2), (5.4.1) of L, R, it follows that backward characteristics for u starting at (x, T), with $x \in]-\infty, 0[\cup]\mathsf{R}, +\infty[$, never crosses the interface $x = 0$. Thus, we recover the Oleinik estimates (5.3.10) as a classical property of solutions to conservation laws with strictly convex flux, which follows from the fact that genuine characteristics never intersect in the interior of the domain (e.g. see [**9**, Lemma 3.2]).

5.4.2.3. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, first part of the proof of (5.3.13)). Letting $\boldsymbol{u}[\mathsf{R}, B, f_r]$ be the constant defined as in (5.2.7) with $f = f_r$, we will prove the implication

$$\mathsf{R} \in]0, T \cdot f'(B)[\quad \Longrightarrow \quad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r], \tag{5.4.3}$$

assuming

$$\mathsf{R} \in]0, T \cdot f'(B)[\,, \qquad \omega(\mathsf{R}+) > \boldsymbol{u}[\mathsf{R}, B, f_r]\,, \tag{5.4.4}$$

and showing that (5.4.4) leads to a contradiction. To complete the proof of (5.3.13) we will show in § 5.4.2.5 that

$$\mathsf{R} \in]0, T \cdot f'(B)[\quad \Longrightarrow \quad \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-). \tag{5.4.5}$$

The proof of the first implication in (5.3.13) is obtained in entirely similar way.

We divide the proof of (5.4.3) in two steps. In the first step we construct the leftmost characteristic curve ξ_{R} that starts on the interface $x = 0$ and reaches the point (R, T), remaining in the region $\{x > 0\}$, with the property that all maximal backward characteristics starting on ξ_{R} don't cross the interface $x = 0$. In the second step, we show that ξ_{R} is located on the left of the shock curve $\boldsymbol{x}$ constructed as in § 5.2.4 that emanates from the interface $x = 0$ and reaches the point (R, T). Thanks to the assumption (5.4.4) this leads to a contradiction in accordance with the characterizing property of $\boldsymbol{u}[\mathsf{R}, B, f_r]$ discussed in Remark 5.2.3.

Step 1 Consider the map $\xi_{\mathsf{R}} : [\tau_{\mathsf{R}}, T] \to [0, +\infty[$ defined by setting

$$\begin{aligned} \xi_{\mathsf{R}}(t) &\doteq \inf\big\{R > 0 \ : \ x - t \cdot f_r'(u(x,t)) \geq 0 \quad \forall\, x \geq R\big\}, \quad t \geq 0\,, \\ \tau_{\mathsf{R}} &\doteq \inf\big\{t \in [0,T] \ : \ \xi_{\mathsf{R}}(s) > 0 \quad \forall\, s \in [t,T]\big\}\,. \end{aligned} \tag{5.4.6}$$

Notice that by definition (5.4.6) we have

$$\xi_{\mathsf{R}}(\tau_{\mathsf{R}}) = 0, \qquad \xi_{\mathsf{R}}(T) = \mathsf{R}, \qquad \xi_{\mathsf{R}}(t) > 0 \quad \forall\, t \in]\tau_{\mathsf{R}}, T], \tag{5.4.7}$$

and that ξ_{R} is a backward characteristic for u starting at (R, T), so that it holds true (e.g. see [**55**])

$$\xi_{\mathsf{R}}'(t) = \begin{cases} f_r'(u(\xi_{\mathsf{R}}(t)\pm, t) & \text{if} \quad u(\xi_{\mathsf{R}}(t)-, t) = u(\xi_{\mathsf{R}}(t)+, t), \\ \lambda_r\big(u(\xi_{\mathsf{R}}(t)-, t),\, u(\xi_{\mathsf{R}}(t)+, t)\big) & \text{if} \quad u(\xi_{\mathsf{R}}(t)-, t) \neq u(\xi_{\mathsf{R}}(t)+, t)\,, \end{cases} \tag{5.4.8}$$

where

$$\lambda_r(u,v) \doteq \frac{f_r(v) - f_r(u)}{v - u}, \qquad u, v \in \mathbb{R},\ u \neq v\,. \tag{5.4.9}$$

We shall provide now a lower bound on the slope of ξ_{R}. Let $t_0 \in]\tau_{\mathsf{R}}, T]$, and observe that by definition (5.4.6) it follows that the minimal backward characteristic starting at $(\xi_{\mathsf{R}}(t_0), t_0)$ must cross the interface $x = 0$ at some non-negative time. Since such a characteristic is genuine and has slope $f_r'(u(\xi_{\mathsf{R}}(t_0)-, t_0)) \geq 0$, and because of the AB-entropy condition (5.1.13), it follows that $f_r(u(\xi_{\mathsf{R}}(t_0)-, t_0)) \geq f_r(B)$ and $u(\xi_{\mathsf{R}}(t_0)-, t_0) \geq \theta_r$. Hence, it holds true

$$u(\xi_{\mathsf{R}}(t_0)-, t_0) \geq B\,. \tag{5.4.10}$$

On the other hand, by definition (5.4.6) we have

$$f_r'(u(\xi_{\mathsf{R}}(t_0)+, t_0)) \leq \xi_{\mathsf{R}}(t_0)/t_0\,. \tag{5.4.11}$$

Thus, letting $\vartheta_{\xi_{\mathsf{R}}(t_0),+}$ denote the maximal backward characteristic starting at $(\xi_{\mathsf{R}}(t_0), t_0)$, because of (5.4.11) it holds true

$$\vartheta_{\xi_{\mathsf{R}}(t_0),+}(0) = \xi_{\mathsf{R}}(t_0) - t_0 \cdot f_r'(u(\xi_{\mathsf{R}}(t_0)+, t_0)) \geq 0\,, \tag{5.4.12}$$

and (5.4.7) implies

$$\vartheta_{\xi_{\mathsf{R}}(t_0),+}(t) > 0 \qquad \forall\, t \in]0, t_0]\,. \tag{5.4.13}$$

Moreover, observe that by the properties of backward characteristics, and by definition (5.4.6), the maximal backward characteristics $\vartheta_{\mathsf{R},+}$ starting at (R, T) satisfies

$$\xi_{\mathsf{R}}(t) \leq \vartheta_{\mathsf{R},+}(t) \qquad \forall\, t \in [\tau_{\mathsf{R}}, T]\,,$$

and, in particular, one has

$$\xi_{\mathsf{R}}(t_0) \leq \vartheta_{\mathsf{R},+}(t_0)\,. \tag{5.4.14}$$

Since maximal backward characteristics cannot intersect in the interior of the domain, it follows from (5.4.14) that

$$\xi_{\mathsf{R}}(t_0) - t_0 \cdot f_r'(u(\xi_{\mathsf{R}}(t_0)+, t_0)) = \vartheta_{\xi_{\mathsf{R}}(t_0),+}(0) \leq \vartheta_{\mathsf{R},+}(0) = \mathsf{R} - T \cdot f_r'(\omega(\mathsf{R}+)) \,. \tag{5.4.15}$$

In turn, (5.4.15) yields

$$\xi_{\mathsf{R}}(t) - \mathsf{R} + T \cdot f_r'(\omega(\mathsf{R}+)) \leq t_0 \cdot f_r'(u(\xi_{\mathsf{R}}(t_0)+, t_0)) \,. \tag{5.4.16}$$

Moreover, one has

$$\frac{\xi_{\mathsf{R}}(t_0) - \vartheta_{\mathsf{R},+}(0)}{t_0} \leq \vartheta'_{\mathsf{R},+} = f_r'(\omega(\mathsf{R}+)) \,. \tag{5.4.17}$$

Since the definition (5.3.2) of R and (5.4.4) imply $f_r'(\omega(\mathsf{R}+)) \leq \mathsf{R}/T < f_r'(B)$, we deduce from (5.4.17) that

$$\frac{\xi_{\mathsf{R}}(t_0) - \mathsf{R} + T \cdot f_r'(\omega(\mathsf{R}+))}{t_0} < f_r'(B). \tag{5.4.18}$$

By the monotonicity of f_r', in turn the estimates (5.4.16), (5.4.18) yield

$$\begin{aligned} &(f_r')^{-1}\left(\frac{\xi_{\mathsf{R}}(t_0) - \mathsf{R} + T \cdot f_r'(\omega(\mathsf{R}+))}{t_0}\right) \leq u(\xi_{\mathsf{R}}(t_0)+, t_0), \\ &(f_r')^{-1}\left(\frac{\xi_{\mathsf{R}}(t_0) - \mathsf{R} + T \cdot f_r'(\omega(\mathsf{R}+))}{t_0}\right) < B. \end{aligned} \tag{5.4.19}$$

Therefore, recalling (5.4.8), (5.4.9), and because of the convexity of f_r, we derive from (5.4.4),(5.4.10), (5.4.19), that

$$\xi'_{\mathsf{R}}(t_0) > \lambda_r\left((f_r')^{-1}\left(\frac{\xi_{\mathsf{R}}(t_0) - \mathsf{R} + T \cdot f_r'(\boldsymbol{u}[\mathsf{R}, B, f_r])}{t_0}\right), B\right) \qquad \forall\; t_0 \in \,]\tau_{\mathsf{R}}, T] \,. \tag{5.4.20}$$

Step 2 (Comparison with an extremal shock). Let $\boldsymbol{y}[\mathsf{R}, B, f_r](\cdot)$ be the function defined in § 5.2.1 with $f = f_r$, set

$$\mathsf{L} \doteq \boldsymbol{y}[\mathsf{R}, B, f_r](T), \tag{5.4.21}$$

and consider the function

$$\boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](t), \qquad\qquad t \in \left[\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r], T\right], \tag{5.4.22}$$

defined as in § 5.2.2, with $A = \overline{B}$ ($\overline{B}$ as in (5.1.17)), and $f = f_r$. By definition (5.2.10), and applying Lemma 5.2.1, it holds true

$$\boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r]) = 0, \qquad\qquad \boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](T) = \mathsf{R}, \tag{5.4.23}$$

and

$$\frac{d}{dt}\boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](t) = \lambda_r\left((f_r')^{-1}\left(\frac{\boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](t) + \mathsf{L}}{t}\right), B\right), \qquad t \in \left[\boldsymbol{s}[\mathsf{L}, \bar{B}, f], T\right]. \tag{5.4.24}$$

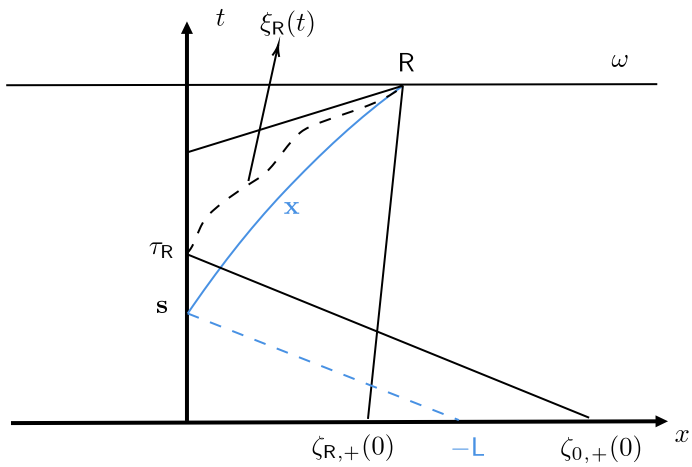

FIGURE 15. Illustration of the proof in § 5.4.2.3. The black lines are characteristics of the solution u, that cross inside the domain and therefore lead to a contradiction. The blue lines are the comparison curves.

Moreover, because of (5.2.7), (5.4.21), we have

$$\mathsf{L} = T \cdot f_r'(\boldsymbol{u}[\mathsf{R}, B, f_r]) - \mathsf{R}\,. \tag{5.4.25}$$

Recall that by (5.4.7), (5.4.23), it holds

$$\xi_{\mathsf{R}}(T) = \mathsf{R} = \boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](T).$$

Then, by virtue of (5.4.20), (5.4.24), a comparison argument yields

$$\xi_{\mathsf{R}}(t) < \boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](t), \qquad \forall\, t \in \big[\max\big\{\tau_{\mathsf{R}},\ \boldsymbol{s}[\mathsf{L}, \overline{B}, f_r]\big\}, T\big[\,. \tag{5.4.26}$$

Notice that, if $\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r] \geq \tau_{\mathsf{R}}$, then because of (5.4.23), (5.4.26), and since $\xi_{\mathsf{R}}(t) \geq 0$, for all $t \in [\tau_{\mathsf{R}}, T]$, we find the contradiction $0 \leq \xi_{\mathsf{R}}(\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r]) < 0$. Hence it must be

$$\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r] < \tau_{\mathsf{R}}. \tag{5.4.27}$$

Next, observe that by definition (5.4.6) and because of (5.4.7), we have $u(0+, \tau_{\mathsf{R}}) \leq \theta_r$. Thus, by virtue of the AB-entropy condition (5.1.13), it follows that $u(0+, \tau_{\mathsf{R}}) \leq \overline{B}$. Then, letting
$\zeta_{0,+} : [0, \tau_{\mathsf{R}}] \to [0, +\infty[$ denote the maximal backward characteristic starting at $(0, \tau_{\mathsf{R}})$, one has

$$\zeta_{0,+}(0) = -\tau_{\mathsf{R}} \cdot f_r'(u(0+, \tau_{\mathsf{R}})) \geq -\tau_{\mathsf{R}} \cdot f_r'(\overline{B}). \tag{5.4.28}$$

On the other hand, by virtue of (5.4.4), (5.4.21), (5.4.27), and recalling the definitions (5.2.7), (5.2.9) of $\boldsymbol{u}[\mathsf{R}, B, f_r]$, $\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r]$, we find that the maximal backward characteristic $\vartheta_{\mathsf{R},+} : [0, T] \to [0, +\infty[$ from (R, T) satisfies

$$\begin{aligned}\vartheta_{\mathsf{R},+}(0) = \mathsf{R} - T \cdot f_r'(\omega(\mathsf{R}+)) &< \mathsf{R} - T \cdot f_r'(\boldsymbol{u}[\mathsf{R}, B, f_r]) \\ &= -\boldsymbol{y}[\mathsf{R}, B, f_r](T) \\ &= -\boldsymbol{s}[\mathsf{L}, \overline{B}, f_r] \cdot f_r'(\overline{B}) \\ &< -\tau_{\mathsf{R}} \cdot f_r'(\overline{B})\,.\end{aligned} \tag{5.4.29}$$

Thus, we deduce from (5.4.28)-(5.4.29) that

$$\vartheta_{\mathsf{R},+}(0) < \zeta_{0,+}(0), \tag{5.4.30}$$

while (5.4.14) yield

$$\vartheta_{\mathsf{R},+}(\tau_{\mathsf{R}}) > 0 = \zeta_{0,+}(\tau_{\mathsf{R}}). \tag{5.4.31}$$

The inequalities (5.4.30)-(5.4.31) imply that the genuine characteristics $\zeta_{0,+}, \vartheta_{\mathsf{R},+}$ intersect each other in the interior of the domain, which gives a contradiction and thus completes the proof of the implication (5.4.3).

5.4.2.4. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, proof of (5.3.15)-(5.3.16)). We will prove only the implication (5.3.16), the proof of (5.3.15) being entirely similar. Let $\widetilde{\mathsf{L}} \doteq \widetilde{\mathsf{L}}[\omega, f_l, f_r, A, B]$ be the constant in (5.3.4), and assume that

$$\mathsf{R} \in \,]0, T \cdot f_r'(B)[\,, \qquad\qquad \mathsf{L} < \widetilde{\mathsf{L}}\,. \tag{5.4.32}$$

Step 1. (proof of: $\omega(x) \leq A$ in $]\mathsf{L}, 0[$).
By definition (5.3.2), (5.4.1) of L, it follows that backward genuine characteristics starting at points (x, T), with $x \in]\mathsf{L}, 0[$ of continuity for ω, must cross the interface $x = 0$ at some non-negative time. Since such characteristics have slope $f_l'(\omega(x)) \leq 0$, and because of the AB-entropy condition (5.1.13), it follows that $f_l(\omega(x)) \geq f_l(A)$ and $\omega(x) \leq \theta_l$ at any point $x \in]\mathsf{L}, 0[$ of continuity for ω. Hence, we have $\omega(x\pm) \leq A$ for all $x \in]\mathsf{L}, 0[$.

Step 2. (proof of: $\omega(x) = A$ in $]\widetilde{\mathsf{L}}, 0[$).
In a similar way to (5.4.6), consider the map $\xi_{\mathsf{L}} : [\tau_{\mathsf{L}}, T] \to]-\infty, 0[$ defined symmetrically by setting

$$\begin{aligned}\xi_{\mathsf{L}}(t) &\doteq \sup\big\{L < 0 \;:\; x - t \cdot f_l'(u(x,t)) \leq 0 \quad \forall\, x \leq L\big\}, \quad t \geq 0\,, \\ \tau_{\mathsf{L}} &\doteq \inf\big\{t \in [0, T] \;:\; \xi_{\mathsf{L}}(s) < 0 \quad \forall\, s \in [t, T]\big\}\,.\end{aligned} \tag{5.4.33}$$

Notice that by definition (5.4.33) we have

$$\xi_{\mathsf{L}}(\tau_{\mathsf{L}}) = 0, \qquad \xi_{\mathsf{L}}(T) = \mathsf{L}, \qquad \xi_{\mathsf{L}}(t) < 0 \quad \forall\, t \in]\tau_{\mathsf{L}}, T].$$

We claim that

$$\tau_{\mathsf{L}} \leq \tau_{\mathsf{R}} \quad \Longrightarrow \quad \tau_{\mathsf{R}} \leq \boldsymbol{\tau}[\mathsf{R}, B, f_r], \tag{5.4.34}$$

where $\boldsymbol{\tau}[\mathsf{R}, B, f_r]$ is the constant defined as in (5.2.32), with $f = f_r$. We will prove the implication (5.4.34) with similar arguments to the proof of (5.3.13) in § 5.4.2.3, assuming

$$\tau_{\mathsf{L}} \leq \tau_{\mathsf{R}}, \qquad \tau_{\mathsf{R}} > \boldsymbol{\tau}[\mathsf{R}, B, f_r], \tag{5.4.35}$$

and showing that (5.4.35) lead to a contradiction.

Since $\tau_{\mathsf{L}} \leq \tau_{\mathsf{R}}$, by definitions (5.4.6), (5.4.33), and by virtue of the AB-entropy condition (5.1.13), it follows

$$(u_l(t), u_r(t)) = (A, B) \qquad \forall\, t \in\,]\tau_{\mathsf{R}}, T], \tag{5.4.36}$$

which in turn implies

$$u(\xi_{\mathsf{R}}(t)-, t) = B \qquad \forall\, t \in\,]\tau_{\mathsf{R}}, T]. \tag{5.4.37}$$

Let $\zeta_{0,+}$, $\vartheta_{\xi_{\mathsf{R}}(t),+}$, be the maximal backward characteristic starting at $(0, \tau_{\mathsf{R}})$, and at $(\xi_{\mathsf{R}}(t), t)$, $t \in\,]\tau_{\mathsf{R}}, T]$, respectively. Relying on (5.4.12), (5.4.28), and since maximal backward characteristics cannot intersect in the interior of the domain, we find

$$-\tau_{\mathsf{R}} \cdot f_r'(\overline{B}) \leq \zeta_{0,+}(0) \leq \vartheta_{\xi_{\mathsf{R}}(t),+}(0) = \xi_{\mathsf{R}}(t) - t \cdot f_r'(u(\xi_{\mathsf{R}}(t)+, t)). \tag{5.4.38}$$

In turn, (5.4.38) together with (5.4.35), yields

$$t \cdot f_r'(u(\xi_{\mathsf{R}}(t)+, t)) \leq \xi_{\mathsf{R}}(t) + \boldsymbol{\tau}[\mathsf{R}, B, f_r] \cdot f_r'(\overline{B}) \qquad \forall\, t \in\,]\tau_{\mathsf{R}}, T], \tag{5.4.39}$$

since $f_r'(\overline{B}) < 0$. By the monotonicity of f_r' we deduce from (5.4.39) that

$$u(\xi_{\mathsf{R}}(t)+, t) \leq (f_r')^{-1}\left(\frac{\xi_{\mathsf{R}}(t) + \boldsymbol{\tau}[\mathsf{R}, B, f_r] \cdot f_r'(\overline{B})}{t}\right). \tag{5.4.40}$$

Therefore, recalling (5.4.8), (5.4.9), and because of the convexity of f_r, we derive from (5.4.37), (5.4.40) that

$$\xi_{\mathsf{R}}'(t) \leq \lambda_r\left((f_r')^{-1}\left(\frac{\xi_{\mathsf{R}}(t) + \boldsymbol{\tau}[\mathsf{R}, B, f_r] \cdot f_r'(\overline{B})}{t}\right), B\right) \qquad \forall\, t \in\,]\tau_{\mathsf{R}}, T]. \tag{5.4.41}$$

On the other hand, letting $\boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](\cdot)$ be the function defined in § 5.2.2, with L as in (5.4.21), $A = \overline{B}$, and $f = f_r$, we have (5.4.23), (5.4.24). Moreover, because of (5.2.32), (5.4.21), it holds true

$$\mathsf{L} = \boldsymbol{\tau}[\mathsf{R}, B, f_r] \cdot f_r'(\overline{B}), \qquad \boldsymbol{s}[\mathsf{L}, \overline{B}, f_r] = \boldsymbol{\tau}[\mathsf{R}, B, f_r]. \tag{5.4.42}$$

Then, by virtue of (5.4.7), (5.4.41), and because of (5.4.23), (5.4.24), (5.4.35), (5.4.42), with a comparison argument we deduce

$$\xi_{\mathsf{R}}(t) \geq \boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](t) \qquad \forall\, t \in [\tau_{\mathsf{R}}, T]. \tag{5.4.43}$$

But (5.4.43), together with (5.4.7), (5.4.35), (5.4.42), and recalling (5.2.10), implies

$$0 = \xi_{\mathsf{R}}(\tau_{\mathsf{R}}) \geq \boldsymbol{x}[\mathsf{L}, \overline{B}, f_r](\tau_{\mathsf{R}}) > x[\mathsf{L}, \overline{B}, f_r](\boldsymbol{\tau}[\mathsf{R}, B, f_r]) = 0, \tag{5.4.44}$$

which gives a contradiction, proving the claim (5.4.34).

Relying on the implication (5.4.34), we show now that $\omega(x) = A$ in $]\widetilde{\mathsf{L}}, 0[$, considering two cases:

CASE 1: $\tau_{\mathsf{R}} < \tau_{\mathsf{L}}$. Then, by definitions (5.4.6), (5.4.33), and by virtue of the AB-entropy condition (5.1.13), it follows

$$(u_l(t), u_r(t)) = (A, B) \qquad \forall\, t \in]\tau_{\mathsf{L}}, T]\,. \tag{5.4.45}$$

Observe that the maximal backward characteristic $\vartheta_{\mathsf{L},+}$ starting at (L, T) crosses the interface $x = 0$ at time $T - \mathsf{L}/f_l'(\omega(\mathsf{L}+)$. Since ξ_{L} is a backward characteristic starting at the same point (L, T) and crossing the interface $x = 0$ at time τ_{L}, one has $\tau_{\mathsf{L}} \leq T - \mathsf{L}/f_l'(\omega(\mathsf{L}+)$. This implies that the backward genuine characteristics from points (x, T), $x \in]\mathsf{L}, 0[$, impact the interface $x = 0$ at times $t_x \geq T - \mathsf{L}/f_l'(\omega(\mathsf{L}+) \geq \tau_{\mathsf{L}}$. Since the value of the solution u is constant along genuine characteristics, we deduce from (5.4.45) that $\omega(x) = A$ for all $x \in]\mathsf{L}, 0[$. Hence, by (5.4.32) in particular it follows that $\omega(x) = A$ for all $x \in]\widetilde{\mathsf{L}}, 0[$.

CASE 2: $\tau_{\mathsf{L}} \leq \tau_{\mathsf{R}}$. Then, because of (5.4.34) we have $\tau_{\mathsf{R}} \leq \boldsymbol{\tau}[\mathsf{R}, B, f_r]$. Observe that by Step 1 we have $\omega(x) \leq A$ for all $x \in]\mathsf{L}, 0[$. Relying on the monotonicity of f_l', this implies that the backward genuine characteristics starting from points (x, T), $x \in]\mathsf{L}, 0[$, impacts the interface $x = 0$ at times

$$\tau(x) \doteq T - \frac{x}{f_l'(\omega(x))} \geq T - \frac{x}{f_l'(A)}\,. \tag{5.4.46}$$

On the other hand, recalling definitions (5.3.4), (5.4.1), we have

$$T - \frac{x}{f_l'(A)} \geq T - \frac{(T - \boldsymbol{\tau}[\mathsf{R}, B, f_r]) \cdot f_l'(A)}{f_l'(A)} = \boldsymbol{\tau}[\mathsf{R}, B, f_r] \geq \tau_{\mathsf{R}}\,, \tag{5.4.47}$$

for all $x \in]\widetilde{\mathsf{L}}, 0[$. Combining (5.4.46), (5.4.47), we deduce that the backward genuine characteristics starting from points (x, T), $x \in]\widetilde{\mathsf{L}}, 0[$, cross the interface $x = 0$ at times $\tau(x) \geq \tau_{\mathsf{R}}$. Hence, relying again on the property that the solution u is constant along genuine characteristics, we infer from (5.4.36) that $\omega(x) = A$ for all $x \in]\widetilde{\mathsf{L}}, 0[$ also in this case, thus completing the proof of Step 2.

Step 3. $\big($proof of: $\omega(\widetilde{\mathsf{L}}-) = A\big)$.

We know by Step 1 and Step 2 that $\omega(\widetilde{\mathsf{L}}-) \leq A$ and $\omega(\widetilde{\mathsf{L}}+) = A$. On the other hand the Lax entropy condition (see § 5.4.2.1) implies $\omega(\widetilde{\mathsf{L}}-) \geq \omega(\widetilde{\mathsf{L}}+) = A$. Therefore one has $A \geq \omega(\widetilde{\mathsf{L}}-) \geq \omega(\widetilde{\mathsf{L}}+) = A$ which yields $\omega(\widetilde{\mathsf{L}}-) = A$. This concludes the proof of (5.3.16).

5.4.2.5. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, proof of (5.3.14) and completion of the proof of (5.3.13)). We will prove only the second implication in (5.3.14), the proof of the first one being entirely symmetric. Assume that

$$\Big[\mathsf{R} \in]0, T \cdot f_r'(B)[\;\text{ and }\; \widetilde{\mathsf{L}} \leq \mathsf{L}\Big] \quad \text{or} \quad \mathsf{R} \geq T \cdot f_r'(B), \tag{5.4.48}$$

and let $\tau_{\mathsf{L}}, \tau_{\mathsf{R}}$ be the constants defined in (5.4.6), (5.4.33), in connection with the characteristics $\xi_{\mathsf{L}}, \xi_{\mathsf{R}}$. As observed in Step 2 of § 5.4.2.4, the fact that ξ_{L} is a backward characteristic starting at (L, T) and crossing the interface $x = 0$ at time τ_{L} implies

$$\tau_{\mathsf{L}} \leq \tau_+(\mathsf{L}) \doteq T - \frac{\mathsf{L}}{f_l'(\omega(\mathsf{L}+)}\,. \tag{5.4.49}$$

We claim that (5.4.48) implies

$$\tau_{\mathsf{R}} \leq \tau_+(\mathsf{L})\,. \tag{5.4.50}$$

Since (5.4.49) clearly implies (5.4.50) when $\tau_{\mathsf{R}} \leq \tau_{\mathsf{L}}$, it will be sufficient to prove the claim under the assumption $\tau_{\mathsf{L}} < \tau_{\mathsf{R}}$. Let's consider first the case that

$$\mathsf{R} \in \,]0, T \cdot f_r'(B)[\quad \text{and} \quad \widetilde{\mathsf{L}} \leq \mathsf{L}\,. \tag{5.4.51}$$

Observe that, because of (5.4.34), $\tau_{\mathsf{L}} < \tau_{\mathsf{R}}$ implies

$$\tau_{\mathsf{R}} \leq \boldsymbol{\tau}[\mathsf{R}, B, f_r]. \tag{5.4.52}$$

Moreover, by Step 1 of § 5.4.2.4, one has $\omega(\mathsf{L}+) \leq A$. Therefore, recalling the definition (5.3.4), and because of the monotonicity of f_l', we deduce from $\widetilde{\mathsf{L}} \leq \mathsf{L}$ that

$$(T - \boldsymbol{\tau}[\mathsf{R}, B, f_r]) \cdot f_l'(\omega(\mathsf{L}+) \leq \mathsf{L}\,, \tag{5.4.53}$$

which, together with (5.4.52), yields (5.4.50), under the assumption (5.4.51). Next, consider the case that

$$\mathsf{R} \geq T \cdot f_r'(B)\,. \tag{5.4.54}$$

Observe that by the analogous argument of Step 1 of § 5.4.2.4 for (5.3.15), one has $\omega(\mathsf{R}-) \geq B$. Moreover, if $\omega(\mathsf{R}-) = B$, by definition (5.3.2) of R it follows that $f_r'(B) \geq \mathsf{R}/T$, which together with (5.4.54), implies $f_r'(B) = \mathsf{R}/T$. In turn, $f_r'(B) = \mathsf{R}/T$ implies that the minimal characteristic starting at (R, T) reaches the interface $x = 0$ at time $t = 0$, and by definition (5.4.6), it coincides with ξ_{R}. Therefore, one has $\tau_{\mathsf{R}} = 0$, which proves (5.4.50). Hence, it remains to consider the case (5.4.54) when $\omega(\mathsf{R}-) > B$. Notice that, if

$$\frac{\mathsf{L}}{f_l'(\omega(\mathsf{L}+))} > \frac{\mathsf{R}}{f_r'(\omega(\mathsf{R}-))}\,, \tag{5.4.55}$$

it follows that the minimal backward characteristic $\vartheta_{\mathsf{R},-}$ from (R, T) crosses the interface $x = 0$ at a time

$$\tau_-(\mathsf{R}) \doteq T - \frac{\mathsf{R}}{f_r'(\omega(\mathsf{R}-))} \tag{5.4.56}$$

strictly greater than the time $\tau_+(\mathsf{L})$ at which the maximal backward characteristic $\vartheta_{\mathsf{L},+}$ from (L, T) crosses the interface $x = 0$. On the other hand, since $\vartheta_{\mathsf{R},-}$ is a genuine characteristic, it follows that $u_r(\tau_-(\mathsf{R})) = \omega(\mathsf{R}-) > B$. Because of the AB-entropy condition (5.1.13) this implies that $u_l(\tau_-(\mathsf{R})) > \theta_l$. Thus we can trace the minimal backward characteristic starting at $(0, \tau_-(\mathsf{R}))$ and lying in $\{x < 0\}$, which has slope $f_l'(u_l(\tau_-(\mathsf{R})) > 0$, and hence it will intersect the characteristic $\vartheta_{\mathsf{L},+}$ at a positive time $t^* \geq \tau_+(\mathsf{L})$, giving a contradiction. Therefore, $\omega(\mathsf{R}-) > B$ implies

$$\frac{\mathsf{L}}{f_l'(\omega(\mathsf{L}+))} \leq \frac{\mathsf{R}}{f_r'(\omega(\mathsf{R}-))}\,. \tag{5.4.57}$$

On the other hand, since ξ_{R} is a backward characteristic starting at (R, T) and crossing the interface $x = 0$ at time τ_{R}, it holds true

$$\tau_{\mathsf{R}} \leq \tau_-(\mathsf{R})\,. \tag{5.4.58}$$

Hence, (5.4.56), (5.4.57), (5.4.58) together yield (5.4.50). This completes the proof of the Claim that (5.4.48) implies (5.4.50). Then, by definitions (5.4.6), (5.4.33), relying on (5.4.49), (5.4.50), and by virtue of the AB-entropy condition (5.1.13), we find that

$$(u_l(t), u_r(t)) = (A, B) \qquad \forall\, t \in]\tau_+(\mathsf{L}),\, T]\,. \tag{5.4.59}$$

Since backward genuine characteristics starting from points (x, T), $x \in]\mathsf{L}, 0[$, cross the interface $x = 0$ at times $t_x \geq \tau_+(\mathsf{L})$, we infer from (5.4.59) that $\omega(x) = A$ for all $x \in]\mathsf{L}, 0[$. This concludes the proof of the second implication in (5.3.14).

Concerning (5.3.13), we prove now the implication (5.4.5). To this end observe that, because of (5.3.14) and (5.3.15) (established in § 5.4.2.4), we have

$$\mathsf{R} \in]0, T\cdot f'(B)[\quad \Longrightarrow \quad \omega(x) \geq B \qquad \forall\, x \in]0, \mathsf{R}[\,,$$

and hence

$$\mathsf{R} \in]0, T\cdot f'(B)[\quad \Longrightarrow \quad \omega(\mathsf{R}-) \geq B\,. \tag{5.4.60}$$

Thus, relying on (5.2.8) with $f = f_r$, we deduce (5.4.5) from (5.4.60), which completes the proof of the second implication in (5.3.13).

5.4.2.6. ($\mathsf{L} < 0$, $\mathsf{R} > 0$, proof of (5.3.11)-(5.3.12)). We will prove only (5.3.11), the proof of (5.3.12) being entirely similar. Then, assume that (5.4.32) holds as in § 5.4.2.4.

Step 1. For every point $x \in]\mathsf{L}, \widetilde{\mathsf{L}}[$ where ω is continuous, consider the map

$$\vartheta_x(t) \doteq \begin{cases} x - (T-t)\cdot f_l'(\omega(x)), & \text{if } \ \tau(x) \leq t \leq T, \\ (t - \tau(x))\cdot f_r' \circ \pi^l_{r,-}(\omega(x)), & \text{if } 0 \leq t < \tau(x), \end{cases} \tag{5.4.61}$$

with

$$\tau(x) \doteq T - \frac{x}{f_l'(\omega(x))}\,, \tag{5.4.62}$$

and set

$$\phi(x) \doteq \vartheta_x(0) = -\tau(x)\cdot f_r' \circ \pi^l_{r,-}(\omega(x))\,. \tag{5.4.63}$$

Observe that

$$\begin{aligned} &\vartheta_x|_{]\tau(x),T]} \quad \text{is a genuine characteristic for u in the halfplane } \ \{x<0\}\,, \\ &\vartheta_x|_{]0,\tau(x)[} \quad \text{is a genuine characteristic for u in the halfplane } \ \{x>0\} \ \text{ if } \ u_r(\tau(x)) \leq \overline{B}\,, \end{aligned} \tag{5.4.64}$$

and thus ϑ_x is a genuine characteristic for u as AB-solution (see Remark 5.1.6) only in the case where $u_r(\tau(x)) \leq \overline{B}$. Note also that $\tau(x)$ is the impact time of ϑ_x with the interface $x = 0$, and that the function τ has at most countably many discontinuity points as ω. Since genuine characteristics cannot intersect in the interior of the domain, it follows that the right continuous extension of τ is a nondecreasing map. On the other hand, because we are assuming that ω satisfies (H) and that (A, B) is a non critical connection, we know by Proposition 5.7.3 in Appendix 5.7 that no pair of genuine characteristics can meet together on the interface $x = 0$. Hence, we deduce that the right continuous extension of the map τ is actually increasing on $]\mathsf{L}, \widetilde{\mathsf{L}}[$.

We will next show that the right continuous extension of the map ϕ is nondecreasing on $]\mathsf{L}, \widetilde{\mathsf{L}}[$.

Step 2. Consider two points $\mathsf{L} < x_1 < x_2 < \widetilde{\mathsf{L}}$ of continuity for ω. By Step 1 we know that $\tau(x_1) < \tau(x_2)$. Moreover, by (5.3.16) (established in § 5.4.2.4) we have $\omega(x) \leq A$ for all $x \in]\mathsf{L}, \widetilde{\mathsf{L}}[$. Then, we shall provide a proof of

$$\phi(x_1) \leq \phi(x_2) \tag{5.4.65}$$

considering different cases according to the fact that $\omega(x_i) = A$ or $\omega(x_i) < A$, $i = 1, 2$.

Case 1: $\omega(x_i) < A$, $i = 1, 2$. Since u is constant along genuine characteristics, and because of the AB-entropy condition (5.1.13), it follows that $u_r(\tau(x_i)) = \pi^l_{r,-}(\omega(x_i)) < \overline{B}$, $i = 1, 2$. Therefore, by (5.4.64) $\vartheta_{x_i}|_{]0,\tau(x_i)[}$, $i = 1, 2$, are genuine characteristics in the half plane $\{x > 0\}$ starting at $(0, \tau(x_i))$, which cannot intersect at positive times. This implies $\phi(x_1) = \vartheta_{x_1}(0) \leq \vartheta_{x_2}(0) = \phi(x_2)$.

Case 2: $\omega(x_i) = A$, $i = 1, 2$. By definition (5.4.61) we know that $\vartheta_{x_i}|_{]0,\tau(x_i)[}$, $i = 1, 2$, are parallel lines (possibly not characteristics for u) with slope $f'_r\big(\pi^l_{r,-}(A)\big) = f'_r(\overline{B})$, starting at $(0, \tau(x_i))$. Hence, $\tau(x_1) < \tau(x_2)$ implies $\phi(x_1) = \vartheta_{x_1}(0) < \vartheta_{x_2}(0) = \phi(x_2)$.

Case 3: $\omega(x_1) = A$, $\omega(x_2) < A$. Notice that, by the monotonicity of f'_l, f'_r, the map

$$\big] -\infty, (f'_l)^{-1}(x_1/T)\big] \ni u \mapsto -\left(T - \frac{x_1}{f'_l(u)}\right) \cdot f'_r \circ \pi^l_{r,-}(u)$$

is decreasing. Then we have

$$\begin{aligned} \phi(x_1) &\leq -\left(T - \frac{x_1}{f'_l(\omega(x_2))}\right) \cdot f'_r \circ \pi^l_{r,-}(\omega(x_2)) \\ &\leq -\left(T - \frac{x_2}{f'_l(\omega(x_2))}\right) \cdot f'_r \circ \pi^l_{r,-}(\omega(x_2)) = \phi(x_2)\,. \end{aligned} \tag{5.4.66}$$

Case 4: $\omega(x_2) = A$, $\omega(x_1) < A$. Since $\omega(x_2) = A$, it follows with the same arguments as above that $u_l(\tau(x_2)) = A$ and that either $u_r(\tau(x_2)) = \overline{B}$ or $u_r(\tau(x_2)) = B$. In the first case, because of (5.4.64) one can proceed as in Case 1 to deduce that $\phi(x_1) \leq \phi(x_2)$. Then, assume $u_r(\tau(x_2)) = B$, and set (see Figure 16)

$$\bar{t} \doteq \inf\Big\{t \leq \tau(x_2) \;\big|\; (u_l(s), u_r(s)) = (A, B) \quad \forall s \in [t, \tau(x_2)]\Big\}. \tag{5.4.67}$$

Notice that since $\tau(x_1) < \tau(x_2)$ and because $u_l(\tau(x_1)) < A$ implies $u_r(\tau(x_1)) < \overline{B}$, it follows that $\bar{t} \in]\tau(x_1), \tau(x_2)]$. We claim that it must hold

$$u_r(\,\bar{t}\,) = \overline{B}\,. \tag{5.4.68}$$

Towards a proof of (5.4.68), notice first that, since $\omega(x) \leq A$ for all $x \in]\mathsf{L}, \widetilde{\mathsf{L}}[$, it follows that $u_l(t) \leq A$ for all $t \in [\tau(x_1), \tau(x_2)]$. Because of the AB-entropy condition (5.1.13) and by definition of $\bar{t}$, this implies that there exists a sequence of times $t_n \uparrow \bar{t}$ such that $u_r(t_n) \leq \overline{B}$. Then, since (A, B) is a non critical connection, we trace the backward characteristics from points $(0, t_n)$, with slope $f'_r(u_r(t_n)) \leq f'_r(\overline{B})$. Using the stability of characteristics with respect to uniform convergence (see for example the proof of Lemma

5.8.1), we thus find that there is a backward characteristic with slope $\leq f_r'(\overline{B})$ starting from $(0,\bar{t})$. This immediately implies that

$$u_r(\,\bar{t}\,) \leq \overline{B}\,. \tag{5.4.69}$$

Then, consider the blow ups

$$u_\rho(x,t) \doteq u(\rho x, \bar{t}+\rho(t-\bar{t})) \qquad x\in\mathbb{R},\ t\geq 0\,, \tag{5.4.70}$$

of u at the point $(0,\bar{t}\,)$, as in the proof of Proposition 5.7.3. When $\rho\downarrow 0$, the blow-ups $u_\rho(\cdot,t)$ converge in $\mathbf{L^1}_{loc}$, up to a subsequence, to a limiting AB-entropy solution $v(\cdot,t)$, for all $t>0$. Moreover, we have

$$v(x,\bar{t}\,) = \begin{cases} u_l(\,\bar{t}\,), & \text{if } x<0,\\ u_r(\,\bar{t}\,), & \text{if } x>0. \end{cases} \tag{5.4.71}$$

By definitions (5.4.67), (5.4.70), it holds true

$$(u_{\rho,l}(t), u_{\rho,r}(t)) = (A,B) \qquad \forall\, t\in \left]\bar{t},\, \bar{t}+\frac{\tau(x_2)-\bar{t}}{\rho}\right[, \tag{5.4.72}$$

where $u_{\rho,l}(t), u_{\rho,r}(t)$ denote the left and right traces of $u_\rho(\cdot,t)$ at $x=0$. Taking the limit as $\rho\downarrow 0$ in (5.4.72), and invoking Corollary 5.1.11 (with $(A_n,B_n)=(A,B)$ for all n), we deduce that

$$v(0-,t)\in\{A,\overline{A}\,\} \qquad v(0+,t)\in\{B,\overline{B}\,\}, \qquad \forall\, t>\bar{t}, \tag{5.4.73}$$

while (5.4.69), (5.4.71) imply

$$v(x,\bar{t}\,) = u_r(\bar{t}\,) \leq \overline{B}, \qquad \forall\, x>0\,. \tag{5.4.74}$$

By a direct inspection we find that, if an AB-entropy solution of a Riemann problem for (22) with initial datum (5.4.71) at time $\bar{t}$, enjoys the properties (5.4.73)-(5.4.74), then the initial datum on $\{(x,\bar{t}\,),\ x>0\}$ must be $v(x,\bar{t}\,)=u_r(\bar{t}\,)=\overline{B}$, thus proving (5.4.68).

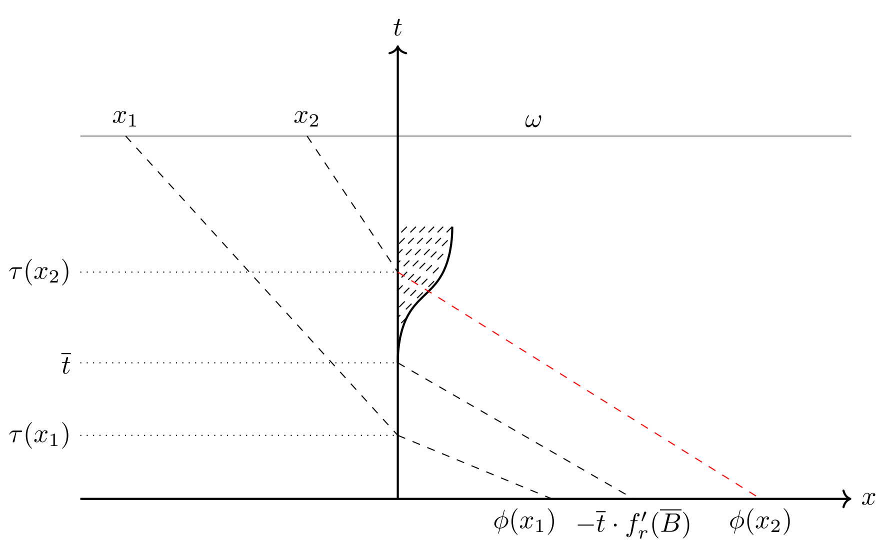

FIGURE 16. The situation described in Case 4.

Relying on (5.4.68) we can now complete the proof of (5.4.65). Since $u_r(\tau(x_1)) < \overline{B}$, we know by (5.4.64) that ϑ_{x_1} is a genuine characteristic in the halfplane $\{x > 0\}$ starting at $(0, \tau(x_1))$. On the other hand, because of (5.4.68) and since (A, B) is a non critical connection, we can trace the maximal backward characteristic from $(0, \bar{t}\,)$ in $\{x > 0\}$, which has slope $f_r'(\overline{B})$ and reaches the x-axis at the point $-\bar{t} \cdot f_r'(\overline{B})$. Such a (genuine) characteristic cannot intersect at a positive time the genuine characteristic ϑ_{x_1}. Therefore, one has

$$\phi(x_1) = \vartheta_{x_1}(0) \leq -\bar{t} \cdot f_r'(\overline{B}). \tag{5.4.75}$$

Moreover, since $\bar{t} \leq \tau(x_2)$, and because $\pi^l_{r,-}(\omega(x_2)) = \pi^l_{r,-}(A) = \overline{B}$, we deduce

$$-\bar{t} \cdot f_r'(\overline{B}) \leq -\tau(x_2) \cdot f_r'(\overline{B}) = \phi(x_2),$$

which together with (5.4.75), yields (5.4.65). This concludes the proof of the nondecreasing monotonicity of ϕ on $]\mathsf{L}, \widetilde{\mathsf{L}}[$. Invoking Lemma 4.4 in [**9**], this is equivalent to the inequality

$$D^+\omega(x) \leq g[\omega, f_l, f_r](x), \qquad \forall\, x \in]\mathsf{L}, \tilde{\mathsf{L}}[\,,$$

where g is the function in (5.3.6). This concludes the proof of (5.3.11), and thus the proof that ω satisfies conditions (i)-(ii) of Theorem 5.3.3 is completed.

5.4.2.7. ($\mathsf{L} < 0$, $\mathsf{R} = 0$ or viceversa, proof of conditions (i)-(ii), or (i)$'$-(ii)$'$, of Theorem 5.3.11). We consider only the case $\mathsf{L} < 0$, $\mathsf{R} = 0$, the other case $\mathsf{L} = 0$, $\mathsf{R} > 0$ being symmetrical. The proofs of (5.3.46), (5.3.47), (5.3.48), (5.3.50), (5.3.51) in this case, are entirely similar to the proofs of (5.3.10), (5.3.11), (5.3.16), (5.3.13), (5.3.17), respectively, in the case $\mathsf{L} < 0$, $\mathsf{R} > 0$. We provide here only the proof of (5.3.49), which is the only new constraint arising in the case $\mathsf{L} < 0$, $\mathsf{R} = 0$, that was not present in the case $\mathsf{L} < 0$, $\mathsf{R} > 0$. Notice first that by (5.3.16) (established in § 5.4.2.4) we know that $\omega(x) \leq A$ for all $x \in]\mathsf{L}, 0[$. Hence, since the connection (A, B) is non critical, tracing the backward characteristics (with negative slope) in the half plane $\{x < 0\}$ from any sequence of points (x_n, T), $x_n \in]\mathsf{L}, 0[$, $x_n \uparrow 0$, we deduce that there exists the one-sided limit $u_l(T-)$ and it holds true

$$u_l(T-) = \omega(0-) \leq A\,. \tag{5.4.76}$$

Then, we will distinguish two cases.

Case 1: Assume that $u_r(t) \geq B$ for all $t \in]\tau, T[$, for some $\tau < T$. Then, by the AB-entropy condition (5.1.13), and because of (5.4.76), we deduce that $\omega(0-) = u_l(T-) = A$. On the other hand, since (A, B) is a non critical connection, by definition (5.3.2) it follows that $\mathsf{R} = 0$ implies $f_r'(\omega(0+)) < 0$. Therefore we have $\omega(0+) \leq \overline{B} = \pi^l_{r,-}(A) = \pi^l_{r,-}(\omega(0-))$, proving (5.3.49).

Case 2: Assume that there exists a sequence of times $t_n \uparrow T$ such that $u_r(t_n) \leq \overline{B}$ for all n, and such that $\lim_n u_r(t_n) = u^*$, for some $u^* \leq \overline{B}$. By the AB-entropy condition (5.1.13) we may also assume that $u_r(t_n) = \pi^l_{r,-}(u_l(t_n))$ for all n. Therefore, relying on (5.4.76), we find $u^* = \lim_n \pi^l_{r,-}(u_l(t_n)) = \pi^l_{r,-}(\omega(0-))$. On the other hand we have $\omega(0+) \leq u^*$, since otherwise backward genuine characteristics issuing from points (x_n, T), $x_n \downarrow 0$, would eventually cross backward genuine characteristics in the half plane $\{x > 0\}$ starting from points $(0, t_n)$. In fact, if $\omega(0+) > u^*$ then we can find points (x_n, T), $x_n > 0$ (x_n point of continuity for ω), and $(0, t_n)$, $t_n < T$ (t_n point of continuity for u_r), such that $\omega(x_n) > u_r(t_n)$, which would imply that the backward characteristic starting from (x_n, T)

with negative slope $f_r'(\omega(x_n,T))$ intersect the backward characteristic starting from $(0,t_n)$ with slope $f_r'(u_r(t_n)) < f_r'(\omega(x_n,T))$. Therefore it must be $\omega(0+) \leq u^*$, which together with $u^* = \pi^l_{r,-}(\omega(0-))$, yields (5.3.49).

This concludes the proof of (5.3.49), and thus the proof that ω satisfies conditions (i)-(ii)
(or (i)′-(ii)′) of Theorem 5.3.11 is completed.

5.4.2.8. ($\mathsf{L}=0$, $\mathsf{R}=0$, proof of conditions (i)-(ii) of Theorem 5.3.14). The proofs of (5.3.62), (5.3.64), are entirely similar to the proofs of (5.3.10), (5.3.13), in the case $\mathsf{L}<0$, $\mathsf{R}>0$, and of (5.3.49), in the case $\mathsf{L}=\mathsf{R}=0$, respectively. Further, (5.3.63) can be established with the same arguments of the proof of (5.3.13) in the case $\mathsf{L}<0$, $\mathsf{R}>0$, recalling Remark 5.3.15. This completes the proof that ω satisfies conditions (i)-(ii) of Theorem 5.3.14.

5.4.3. Part 1.b - (1) $\Rightarrow$ (3) **for non critical connections without assuming** (H)**.** In this Subsection, given an element ω of the set $\mathcal{A}^{[AB]}(T)$ for a non critical connection (A,B), we will show that ω satisfies (H). In view of the analysis in § 5.4.2, this will imply that ω fulfills condition (3) of Theorem 5.3.17, thus completing the proof of the implication (1) $\Rightarrow$ (3) of Theorem 5.3.17.

Then, given $\omega \in \mathcal{A}^{[AB]}(T)$ with

$$\omega = \mathcal{S}_T^{[AB]+} u_0, \qquad u_0 \in \mathbf{L}^\infty(\mathbb{R}), \tag{5.4.77}$$

set $u(x,t) \doteq S_t^{[AB]+} u_0(x)$, $x \in \mathbb{R}$, $t \geq 0$. Next, let $\{u_{n,0}\}_n$ be a sequence of functions in $BV(\mathbb{R})$ such that

$$u_{n,0} \quad \to \quad u_0 \quad \text{in} \quad \mathbf{L}^1_{\mathsf{loc}}(\mathbb{R}),$$

and define $u_n(x,t) \doteq \mathcal{S}_t^{[AB]+} u_{n,0}(x)$. Then, by Theorem 5.1.8-(iii) it follows

$$u_n(\cdot,t) \ \to \ u(\cdot,t) \qquad \text{in} \quad \mathbf{L}^1_{\mathsf{loc}}(\mathbb{R}) \qquad \forall\, \mathbf{t} \geq \mathbf{0}\,. \tag{5.4.78}$$

Since (A,B) is a non critical connection and because the initial data $u_{n,0}$ are in BV, invoking the BV bounds on AB-entropy solutions provided in [**64**, Lemma 8] (see also [**1**, Theorem 2.13-(iii)]), we deduce that $u_n(\cdot,t) \in BV(\mathbb{R})$ for all $t>0$, and for all n. Therefore,

$$u_n(\cdot,t) \in \mathcal{A}^{[AB]}(t)\,, \quad \text{and} \quad \text{satisfies (H)} \quad \forall\, t>0\,, \quad \forall\, n\,.$$

Hence, relying on the analysis in § 5.4.2, and recalling (5.3.7), we know that, setting

$$\mathsf{L}_n(t) \doteq \mathsf{L}[u_n(\cdot,t), f_l]\,, \qquad \mathsf{R}_n(t) \doteq \mathsf{R}[u_n(\cdot,t), f_r]\,, \tag{5.4.79}$$

each $u_n(\cdot,t)$ satisfies the conditions stated in:

- Theorem 5.3.3 if $\mathsf{L}_n(t)<0$, $\mathsf{R}_n(t)>0$;
- Theorem 5.3.11 if $\mathsf{L}_n(t)<0$, $\mathsf{R}_n(t)=0$ or viceversa;
- Theorem 5.3.14 if $\mathsf{L}_n(t)=0$, $R_n(t)=0$.

Thus, in particular, $u_n(\cdot,t)$ satisfies the Oleinik-type inequalities

$$
\begin{aligned}
&D^+u_n(x,t) \le \frac{1}{t\cdot f_l''(u_n(x,t))} && \text{in}\quad]-\infty, \mathsf{L}_n(t)[\,,\\
&D^+u_n(x,t) \le g[u_n(\cdot,t), f_l, f_r](x) && \text{in}\quad]\mathsf{L}_n(t),0[\,,\quad \text{if}\quad \mathsf{L}_n(t)<0\,,\\
&D^+u_n(x,t) \le h[u_n(\cdot,t), f_l, f_r](x) && \text{in}\quad]0,\mathsf{R}_n(t)[\,,\quad \text{if}\quad \mathsf{R}_n(t)>0\,,\\
&D^+u_n(x,t) \le \frac{1}{t\cdot f_r''(u_n(x,t))} && \text{in}\quad]\mathsf{R}_n(t),+\infty[\,,
\end{aligned}
\tag{5.4.80}
$$

and the constraints

$$
\begin{aligned}
&u_n(x,t)\le A \qquad \forall\, x\in]\mathsf{L}_n(t),0[\,,\\
&u_n(x,t)\ge B \qquad \forall\, x\in]0,\mathsf{R}_n(t)[\,,
\end{aligned}
\tag{5.4.81}
$$

for all $t>0$. Since (5.4.81) implies $f_r'(u_n(x,t)) \ge f_r'(B)$ for all $x\in]0,\mathsf{R}_n(t)[$, by the monotonicity of f_r', we find

$$
t\cdot f_r'(u_n(x,t)) - x \ge \frac{t\cdot f_r'(B)}{2} \qquad \forall\, x\in \left[0,\, \min\left\{\mathsf{R}_n(t),\, \frac{t\cdot f_r'(B)}{2}\right\}\right[\,. \tag{5.4.82}
$$

Therefore, recalling definition (5.3.6), setting $\overline{\Lambda} \doteq \sup_{|z|\le M} \max\{|f_l'(z)|, |f_r'(z)|\}$, with M being a uniform $\mathbf{L}^\infty$ bound for u_n, and letting a be the lower bound on f_l'', f_r'' given in (25), we deduce from (5.4.82), that, if

$$
\mathsf{R}_n(t) \le \frac{t\cdot f_r'(B)}{2}\,,
$$

then for all n it holds true

$$
h[u_n(\cdot,t), f_l, f_r](x) \le \frac{\overline{\Lambda}^2}{a\, f_r'(B)\big(t\cdot f_r'(u_n(x,t)) - x\big)} \le \frac{2}{a\,t}\cdot\left(\frac{\overline{\Lambda}}{f_r'(B)}\right)^2 \qquad \forall\, x\in [0,\mathsf{R}_n(t)[\,, \tag{5.4.83}
$$

while if

$$
\mathsf{R}_n(t) > \frac{t\cdot f_r'(B)}{2}\,,
$$

then for all n it holds true

$$
h[u_n(\cdot,t), f_l, f_r](x) \le
\begin{cases}
\dfrac{\overline{\Lambda}^2}{a\, f_r'(B)\big(t\cdot f_r'(u_n(x,t)) - x\big)} \le \dfrac{2}{a\,t}\cdot\left(\dfrac{\overline{\Lambda}}{f_r'(B)}\right)^2 & \forall\, x\in \left[0,\, \frac{t\cdot f_r'(B)}{2}\right]\,,\\[2ex]
\dfrac{\overline{\Lambda}}{x\,a} \;\le\; \dfrac{2\,\overline{\Lambda}}{a\,t\cdot f_r'(B)} & \forall\, x\in \left[\frac{t\cdot f_r'(B)}{2},\, \mathsf{R}_n(t)\right[\,.
\end{cases}
\tag{5.4.84}
$$

Hence, we derive from (5.4.80), (5.4.83), (5.4.84), the uniform bounds

$$
\begin{aligned}
&D^+u_n(x,t) \le \frac{2\,\overline{\Lambda}}{a\,t\cdot f_r'(B)}\cdot\max\left\{1,\, \frac{\overline{\Lambda}}{f_r'(B)}\right\} \quad \text{in}\quad]0,\mathsf{R}_n(t)[\,,\qquad \text{if}\quad \mathsf{R}_n(t)>0\,,\\
&D^+u_n(x,t) \le \frac{1}{t\cdot a} \quad \text{in}\quad]\mathsf{R}_n(t),+\infty[\,,
\end{aligned}
\tag{5.4.85}
$$

for all n. Since (A, B) is a non critical connection, for every fixed $\delta > 0$, the one-sided uniform upper bounds provided by (5.4.85) yield uniform bounds on the total increasing variation (and hence on the total variation as well) of $u_n(t)$, $t \geq \delta$, on bounded subsets of $[0, +\infty[$. Thus, by the lower-semicontinuity of the total variation with respect to the $\mathbf{L}^1_{\mathrm{loc}}$ convergence, and because of (5.4.78), we find that

$$u(\cdot, t) \in BV_{\mathrm{loc}}([0, +\infty[\,), \qquad \forall\, t \geq \delta\,. \tag{5.4.86}$$

With the same type of arguments, relying on (5.4.80), (5.4.81), we can show that

$$u(\cdot, t) \in BV_{\mathrm{loc}}(\,]-\infty, 0]), \qquad \forall\, t \geq \delta\,. \tag{5.4.87}$$

Therefore, we deduce from (5.4.86), (5.4.87), that

$$u(\cdot, t) \in BV_{\mathrm{loc}}(\mathbb{R}) \qquad \forall\, t > 0\,, \tag{5.4.88}$$

which shows that the function ω in (5.4.77) satisfies condition (H), thus completing the proof of the implication (1) ⇒ (3) of Theorem 5.3.17 in the case of a non critical connection.

5.4.4. Part 2.a - (3) ⇒ (1) for non critical connections. In this Subsection, given

$$\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}, \qquad \mathsf{L} \doteq \mathsf{L}[\omega, f_l] < 0, \quad \mathsf{R} \doteq \mathsf{R}[\omega, f_r] > 0\,, \tag{5.4.89}$$

($\mathscr{A}^{\mathsf{L},\mathsf{R}}$ being the set in (5.3.8)), assuming that

$$\omega \quad \text{satisfies conditions (i)-(ii) of Theorem 5.3.3}, \tag{5.4.90}$$

we will show that $\omega \in \mathcal{A}^{AB}(T)$ by explicitly constructing an AB-entropy solution attaining ω at time T. With entirely similar arguments one can show that the same conclusion hold assuming that $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$:

- satisfies the conditions of Theorem 5.3.11, if $\mathsf{L} = 0$, $\mathsf{R} > 0$ or viceversa;
- satisfies the conditions of Theorem 5.3.14, if $\mathsf{L} = 0$, $\mathsf{R} = 0$.

Then, consider ω satisfying (5.4.89), (5.4.90). By Remark 5.3.5 we can distinguish six cases of pointwise constraints on ω, prescribed by condition (ii) of Theorem 5.3.3, which depend on the respective positions of the points L, R, and $\widetilde{\mathsf{L}}$, $\widetilde{\mathsf{R}}$, defined in (5.3.2)-(5.3.4). We shall consider here only the Cases 1 and 2 discussed in Remark 5.3.5. The Cases 1b, 2b are symmetrical to Cases 1, 2, up to a change of variables $x \mapsto -x$, while the Cases 3, 4 are entirely similar or simpler.

Notice that, by Remark 5.3.5, in Case 1 it holds true (5.3.27), (5.3.28), and in particular we shall assume that

$$\omega(\mathsf{R}+) < \boldsymbol{u}[\mathsf{R}, B, f_r]\,, \tag{5.4.91}$$

while in Case 2 it holds true (5.3.28), (5.3.29), and we shall assume that (5.4.91) is verified together with

$$\omega(\mathsf{L}-) > \boldsymbol{v}[\mathsf{L}, A, f_l]\,. \tag{5.4.92}$$

The cases in which $\omega(\mathsf{R}+) = \boldsymbol{u}[\mathsf{R}, B, f_r]$ or $\omega(\mathsf{L}-) = \boldsymbol{v}[\mathsf{L}, A, f_l]$ can be treated with entirely similar or simpler arguments. Moreover, in both Cases 1 and 2 we have

$$\widetilde{\mathsf{L}} > \mathsf{L}\,, \qquad \omega(\widetilde{\mathsf{L}}-) = \omega(\widetilde{\mathsf{L}}+)\,. \tag{5.4.93}$$

The construction of the initial datum u_0 so that the corresponding AB-entropy solution solution $u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x)$ attains the value ω at time T follows a by now standard procedure (see [**9**], [**16**]), that we describe in § 5.4.4.4-5.4.4.5 below. To this end we first introduce some technical notations in § 5.4.4.1-5.4.4.3.

5.4.4.1. *Characteristics of compression waves.* We introduce a class of curves connecting two points $(z,0)$, (y,T), that will be treated as characteristics of compression waves generating a shock at the point (y,T). In particular, in the case $y<0<z$, such curves will be characteristics of a compression wave that starts at time $t=0$ on the half plane $\{z \geq 0\}$, and generates a shock at time $t=T$ after being refracted at the discontinuity interface. Given any $y<0$, consider the continuous function

$$]-\infty, (f_l')^{-1}(y/T)] \ni u \mapsto h_y(u) \doteq -\Big(T - \frac{y}{f_l'(u)}\Big) \cdot f_r' \circ \pi_{r,-}^l(u).$$

Notice that, by definition (5.3.1) and since f_l', f_r' are increasing functions, it follows that $u \mapsto -(T - y/f_l'(u))$, $u \mapsto f_r' \circ \pi_{r,-}^l(u)$ are decreasing maps, and hence the map h_y is decreasing as well. On the other hand we have $\lim_{u \to -\infty} h_y(u) = +\infty$, $h_y((f_l')^{-1}(y/T)-) = 0$. Therefore by a continuity and monotonicity argument, it follows that, for every $z>0$, there exists a unique state $u_{y,z} \leq (f_l')^{-1}(y/T)$, such that

$$h_y(u_{y,z}) = z\,. \tag{5.4.94}$$

Moreover, the map $z \mapsto u_{y,z}$, $z>0$ is continuous. Then, for every pair $y<0<z$, we denote by $\eta_{y,z} : [0,T] \mapsto \mathbb{R}$ the polygonal line given by

$$\eta_{y,z}(t) \doteq \begin{cases} y - (T-t) \cdot f_l'(u_{y,z}), & \text{if } \ \tau(y,z) < t \leq T, \\ (t - \tau(y,z)) \cdot f_r' \circ \pi_{r,-}^l(u_{y,z}), & \text{if } \ 0 \leq t \leq \tau(y,z), \end{cases} \tag{5.4.95}$$

where

$$\tau(y,z) \doteq T - \frac{y}{f_l'(u_{y,z})}. \tag{5.4.96}$$

Next, for every pair $y,z<0$, or $y,z>0$, we denote by $\eta_{y,z} : [0,T] \to \mathbb{R}$ the segment

$$\eta_{y,z}(t) \doteq y - (T-t) \cdot \frac{(y-z)}{T} \qquad \forall\, 0 \leq t \leq T\,. \tag{5.4.97}$$

Notice that, in the case $y<0<z$, if we consider a function $u(x,t)$ that assumes the values

$$\begin{aligned} u &= u_{y,z} && \text{on the segment} \quad \eta_{y,z}(t),\ \tau(y,z) < t \leq T, \\ u &= \pi_{r,-}^l(u_{y,z}) && \text{on the segment} \quad \eta_{y,z}(t),\ 0 \leq t \leq \tau(y,z), \end{aligned}$$

then the states $u_l = u_{y,z}$, $u_r = \pi_{r,-}^l(u_{y,z})$ satisfy the interface entropy condition (5.1.13) at time $t = \tau(y,z)$, and $\eta_{y,z}$ enjoys the properties of a (genuine) characteristic for u as an AB-entropy solution (see Remark 5.1.6). Similar observations hold for $\eta_{y,z}$ in the case $y,z<0$, considering a function $u(x,t)$ that assumes the value $(f_l')^{-1}((y-z)/T) = u_{y,z}$ along the segment $\eta_{y,z}$, and in the case $y,z>0$, considering a function $u(x,t)$ that assumes the value $(f_r')^{-1}((y-z)/T) = u_{y,z}$ along the segment $\eta_{y,z}$.

5.4.4.2. *Maximal/minimal backward characteristics.* We introduce a class of curves with end point (y, T) that will be treated as maximal and minimal backward characteristics starting at (y, T). For every $y \in]-\infty, \widetilde{\mathsf{L}}\,] \cup]\mathsf{R}, +\infty[$, we denote by $\vartheta_{y,\pm} : [0, T] \to \mathbb{R}$ the segments or polygonal lines

$$\vartheta_{y,\pm}(t) \doteq \begin{cases} y - (T-t)\cdot f_l'\big(\omega(y\pm)\big), & \text{if } \ y < \mathsf{L}, \quad 0 \le t \le T, \\ y - (T-t)\cdot f_l'\big(\omega(y\pm)\big), & \text{if } \ \mathsf{L} \le y \le \widetilde{\mathsf{L}}, \quad \tau_\pm(y) \le t \le T, \\ \big(t - \tau_\pm(y)\big)\cdot f_r' \circ \pi^l_{r,-}(\omega(y\pm)), & \text{if } \ \mathsf{L} \le y \le \widetilde{\mathsf{L}}, \quad 0 \le t < \tau_\pm(y), \\ y - (T-t)\cdot f_r'\big(\omega(y\pm)\big), & \text{if } \ y > \mathsf{R}, \quad 0 \le t \le T, \end{cases} \tag{5.4.98}$$

where

$$\tau_\pm(y) \doteq T - \frac{y}{f_l'(\omega(y\pm))}. \tag{5.4.99}$$

We will write $\vartheta_y(t) \doteq \vartheta_{y,\pm}(t)$ for all $t \in [0, T]$, whenever $\omega(y-) = \omega(y+)$. In particular, because of (5.4.93), we have $\vartheta_{\widetilde{\mathsf{L}}}(t) \doteq \vartheta_{\widetilde{\mathsf{L}},\pm}(t)$. Further, for $y = \mathsf{R}$, we denote by $\vartheta_{\mathsf{R},+} : [0, T] \to \mathbb{R}$ the segment

$$\vartheta_{\mathsf{R},+}(t) \doteq \mathsf{R} - (T-t)\cdot f_r'\big(\omega(\mathsf{R}+)\big) \qquad \forall\, 0 \le t \le T\,. \tag{5.4.100}$$

Notice that, because of definition (5.3.2), (5.4.89), whenever $y \in]-\infty, \mathsf{L}\,[\cup]\mathsf{R}, +\infty[$, the curves $\vartheta_{y,\pm}$ are segments that never cross the interface $\{x = 0\}$, instead for all $y \in]\mathsf{L}, \widetilde{\mathsf{L}}]$, $\vartheta_{y,\pm}$ are polygonal lines that are refracted at $\{x = 0\}$. Moreover, at every point of discontinuity $y \in]-\infty, \mathsf{L}\,[\cup]\mathsf{R}, +\infty[$ of ω, conditions (5.3.10), (5.3.11) imply the Lax condition $\omega(y-) > \omega(y+)$, which in turn, by the monotonicity of f_l', f_r', implies

$$\vartheta_{y,-}(0) < \vartheta_{y,+}(0) \qquad\quad \forall\, y \in]-\infty, \mathsf{L}\,[\cup]\mathsf{R}, +\infty[\,. \tag{5.4.101}$$

As in § 5.4.4.1, observe that in the case $\mathsf{L} < y \le \widetilde{\mathsf{L}}$, if we consider a function $u(x, t)$ that assumes the values

$$u = \omega(y\pm) \qquad \text{on the segment} \quad \vartheta_{y,\pm}(t),\ \tau_\pm(y) < t \le T,$$
$$u = \pi^l_{r,-}(\omega(y\pm)) \qquad \text{on the segment} \quad \vartheta_{y,\pm}(t),\ 0 \le t \le \tau_\pm(y),$$

than $\vartheta_{y,\pm}$ enjoys the properties of a maximal/minimal backward characteristic for u as an AB-entropy solution that attains the value ω at time T. Similar observations hold for $\vartheta_{y,\pm}$ in the case $y < \mathsf{L}$ or $y \ge \mathsf{R}$, considering a function $u(x, t)$ that assumes the value $\omega(y\pm)$ along $\vartheta_{y,\pm}$.

5.4.4.3. *Partition of* $\mathbb{R}$. The initial datum will be defined in a different way on different intervals of the following partition of $\mathbb{R}$ (see Figure 18):

$$\begin{aligned} \mathcal{I}_{\mathsf{L}} &\doteq \Big\{x \in \mathbb{R} \ \big|\ \vartheta_{\mathsf{L},-}(0) < x < \vartheta_{\mathsf{L},+}(0)\Big\}, \\ \mathcal{I}_{\mathsf{R}} &\doteq \Big\{x \in \mathbb{R} \ \big|\ -\boldsymbol{y}[\mathsf{R}, B, f_r](T) < x < \mathsf{R} - T\cdot f_r'(\omega(\mathsf{R}+))\Big\}, \\ \mathcal{I}_{\mathcal{C}} &\doteq \Big\{x \in \mathbb{R}\setminus(\mathcal{I}_{\mathsf{L}} \cup \mathcal{I}_{\mathsf{R}}) \ \big|\ \nexists\, y \in \mathbb{R} \ :\ \vartheta_{y,+}(0) = x \text{ or } \vartheta_{y,-}(0) = x\Big\}, \\ \mathcal{I}_{\mathcal{R}a} &\doteq \Big\{x \in \mathbb{R}\setminus(\mathcal{I}_{\mathsf{L}} \cup \mathcal{I}_{\mathsf{R}}) \ \big|\ \exists\, y < z \ :\ \vartheta_{y,+}(0) = \vartheta_{z,-}(0) = x\Big\}, \\ \mathcal{I}_{\mathcal{W}} &\doteq \Big\{x \in \mathbb{R}\setminus(\mathcal{I}_{\mathsf{L}} \cup \mathcal{I}_{\mathsf{R}}) \ \big|\ \exists!\, y \in \mathbb{R} \ :\ \vartheta_{y,+}(0) = x \text{ or } \vartheta_{y,-}(0) = x\Big\}, \end{aligned} \tag{5.4.102}$$

where $\boldsymbol{y}[\mathsf{R}, B, f_r](T)$ is defined as in § 5.2.1 with $f = f_r$. Notice that the set $\mathcal{I}_{\mathsf{R}}$ is non empty because the increasing monotonicity of f_r', together with (5.2.7), (5.4.91), implies

$$f_r'(\omega(\mathsf{R}+)) < f_r'\big(\boldsymbol{u}[\mathsf{R}, B, f_r]\big) = \frac{\mathsf{R} + \boldsymbol{y}[\mathsf{R}, B, f_r](T)}{T}\,.$$

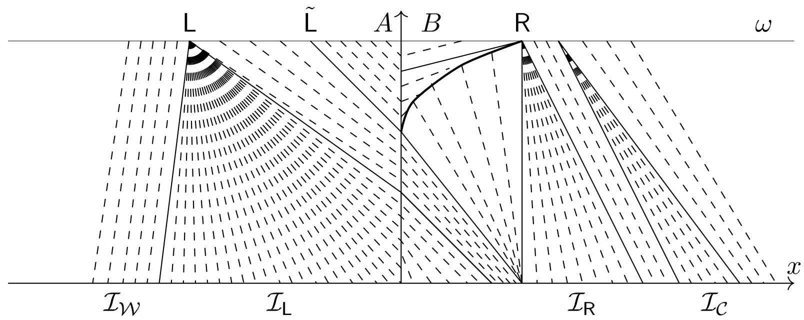

FIGURE 17. Partition of $\mathbb{R}$ in Case 1. The picture displays some connected components in $\mathcal{I}_{\mathsf{L}} \cup \mathcal{I}_{\mathsf{R}} \cup \mathcal{I}_{\mathcal{C}} \cup \mathcal{I}_{\mathcal{W}}$.

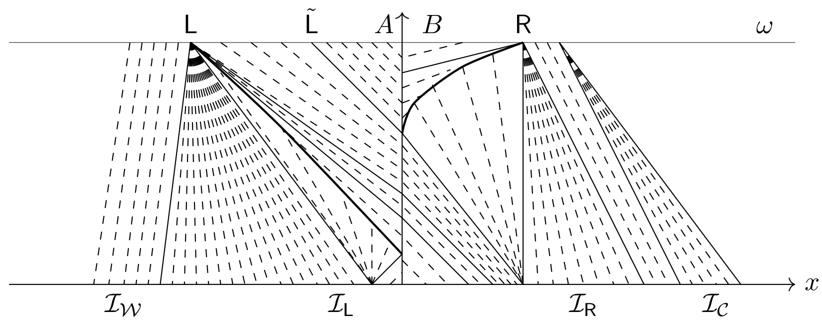

FIGURE 18. Partition of $\mathbb{R}$ in Case 2. The picture displays some connected components in $\mathcal{I}_{\mathsf{L}} \cup \mathcal{I}_{\mathsf{R}} \cup \mathcal{I}_{\mathcal{C}} \cup \mathcal{I}_{\mathcal{W}}$.

The elements of this partition enjoy the following properties.

- In the CASE 1, the set $\mathcal{I}_{\mathsf{L}}$ consists of the starting points of a compression wave that is partly refracted by the interface, and that generates a shock at the point (L, T). In the CASE 2, only the subsets of $\mathcal{I}_{\mathsf{L}}$ given by $]\vartheta_{\mathsf{L},-}(0), -\boldsymbol{\sigma}[\mathsf{L}, A, f_l] \cdot f_l'(\overline{A})[$, and $](\mathsf{L}/f_l'(A) - T) \cdot f_r'(\overline{B}), \vartheta_{\mathsf{L},+}(0)[$, consist of the starting points of compression waves with center at the point (L, T). In the complementary sets of $\mathcal{I}_{\mathsf{L}}$: $]-\boldsymbol{\sigma}[\mathsf{L}, A, f_l] \cdot f_l'(\overline{A}), 0[$ and $]0, (\mathsf{L}/f_l'(A) - T) \cdot f_r'(\overline{B})[$, the initial datum will assume the constant values $\overline{A}$ and $\overline{B}$, respectively. Here $\boldsymbol{\sigma}[\mathsf{L}, A, f_l]$ is the constant defined as in § 5.2.5, with $f = f_l$.
- The set $\mathcal{I}_{\mathsf{R}}$ consists of the starting points of a compression wave that generates a shock at the point (R, T).
- The set $\mathcal{I}_{\mathcal{C}}$ consists of the starting points of compression waves that generate a shock at points (y, T), $y \in]-\infty, \mathsf{L}[\cup]\mathsf{L}, \widetilde{\mathsf{L}}[\cup]\mathsf{R}, +\infty[$. The set $\mathcal{I}_{\mathcal{C}}$ is a disjoint union of at most countably many open intervals of the form

$$\begin{aligned} \mathcal{I}^n_{\mathsf{L}} &=]x_n^-, x_n^+[\,, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \qquad y_n \in]-\infty, \mathsf{L}[\,, \\ \mathcal{I}^n_{\mathsf{R}} &=]x_n^-, x_n^+[\,, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \qquad y_n \in]\mathsf{R}, +\infty[\,, \\ \widetilde{\mathcal{I}}^n_{\mathsf{L}} &=]x_n^-, x_n^+[\,, \quad x_n^\pm = \vartheta_{y_n,\pm}(0), \qquad y_n \in]\mathsf{L}, \tilde{\mathsf{L}}[\,, \end{aligned} \tag{5.4.103}$$

which are non empty because of (5.4.101).

- The set $\mathcal{I}_{\mathcal{R}a}$ consists of at most countably many points that are the centers of rarefaction waves originated at time $t = 0$.
- The set $\mathcal{I}_{\mathcal{W}}$ consists of the starting points of all genuine characteristics reaching points (y, T), $y \in]-\infty, \mathsf{L}[\,\cup\,]\mathsf{L}, \widetilde{\mathsf{L}}[\,\cup\,]\mathsf{R}, +\infty[\,$.

5.4.4.4. *Construction of AB-entropy solution on two regions with vertexes at* (L, T) *and at* (R, T). Consider the two polygonal regions

$$\begin{aligned} \Delta_{\mathsf{L}} &\doteq \Big\{(x,t) \in \mathbb{R} \times [0,T] \ : \ \vartheta_{\mathsf{L},-}(t) < x < \vartheta_{\mathsf{L},+}(t)\Big\}, \\ \Gamma_{\mathsf{R}} &\doteq \Big\{(x,t) \in \mathbb{R} \times [0,T] \ : \ \vartheta_{\tilde{\mathsf{L}}}(t) < x < \vartheta_{\mathsf{R},+}(t)\Big\}. \end{aligned} \tag{5.4.104}$$

In the CASE 2 (see Figure 7), letting $\Delta[\mathsf{L}, A, f_l]$ be the region defined as in § 5.2.5, with $f = f_l$, we can express Δ_{L} as

$$\Delta_{\mathsf{L}} = \Delta[\mathsf{L}, A, f_l] \cup \bigcup_{i=1}^{4} \Delta_{\mathsf{L},i}, \tag{5.4.105}$$

where

$$\begin{aligned} \Delta_{\mathsf{L},1} &\doteq \Big\{(x,t) \in]-\infty, 0[\,\times[0,T] \ : \ \vartheta_{\mathsf{L},-}(t) < x \leq \mathsf{L} - (T-t)\cdot f_l'(\boldsymbol{v}[\mathsf{L}, A, f_l])\Big\}, \\ \Delta_{\mathsf{L},2} &\doteq \Big\{(x,t) \in]-\infty, 0] \times [0,T] \ : \ x \geq (t - \boldsymbol{\sigma}[\mathsf{L}, A, f_l])\cdot f_l'(\,\overline{A}\,)\Big\}, \\ \Delta_{\mathsf{L},3} &\doteq \Big\{(x,t) \in]0, +\infty[\,\times[0,T] \ : \ x < \eta_{\mathsf{L},\,x(A,\overline{B})}(t)\Big\}, \\ \Delta_{\mathsf{L},4} &\doteq \Big\{(x,t) \in \mathbb{R} \times [0,T] \ : \ \eta_{\mathsf{L},\,x(A,\overline{B})}(t) \leq x < \vartheta_{\mathsf{L},+}(t)\Big\}, \end{aligned} \tag{5.4.106}$$

with $\boldsymbol{v}[\mathsf{L}, A, f_l]$ as in (5.2.11) taking $f = f_l$, and

$$x(A, \overline{B}) \doteq \big(\mathsf{L}/f_l'(A) - T\big)\cdot f_r'(\overline{B}) > 0\,. \tag{5.4.107}$$

Similarly, in both CASES 1, 2 (see Figures 6-7), letting $\Gamma[\mathsf{R}, B, f_r]$, be the region defined as in § 5.2.4, with $f = f_r$, we can express Γ_{R} as

$$\Gamma_{\mathsf{R}} = \Gamma[\mathsf{R}, B, f_r] \cup \bigcup_{i=1}^{3} \Gamma_{\mathsf{R},i}, \tag{5.4.108}$$

where

$$
\begin{aligned}
&\Gamma_{\mathsf{R},1} \doteq \Big\{(x,t) \in]-\infty, 0] \times [0,T] \ : \ \vartheta_{\tilde{\mathsf{L}}}(t) < x\Big\},\\
&\Gamma_{\mathsf{R},2} \doteq \Big\{(x,t) \in]0,+\infty[\times [0,T] \ : \ x \leq \mathsf{R} - (T-t)\cdot f_r'(B)\Big\},\\
&\Gamma_{\mathsf{R},3} \doteq \Big\{(x,t) \in]0,+\infty[\times [0,T] \ : \ \mathsf{R} - (T-t)\cdot f_r'(\boldsymbol{u}[\mathsf{R},B,f_r]) \leq x < \vartheta_{\mathsf{R},+}(t)\Big\},
\end{aligned}
\tag{5.4.109}
$$

with $\boldsymbol{u}[\mathsf{R}, B, f_l]$ as in (5.2.7), taking $f = f_r$.

Now, consider the function $u_{\mathsf{L}} : \Delta_{\mathsf{L}} \to \mathbb{R}$ defined by setting for every $(x,t) \in \Delta_{\mathsf{L}}$: in Case 1:

$$
u_{\mathsf{L}}(x,t) = \begin{cases} (f_l')^{-1}\big(\frac{\mathsf{L}-x}{T-t}\big), & \text{if } x \leq 0,\\ \pi^l_{r,-}(u_{\mathsf{L},z}), & \text{if } x = \eta_{\mathsf{L},z}(t), \ \text{for some} \ \ z > 0, \end{cases}
\tag{5.4.110}
$$

where $u_{\mathsf{L},z}$ is defined as in § 5.4.4.1, with $y = \mathsf{L}$;

in Case 2:

$$
u_{\mathsf{L}}(x,t) = \begin{cases} (f_l')^{-1}\big(\frac{\mathsf{L}-x}{T-t}\big), & \text{if } (x,t) \in \Delta_{\mathsf{L},1} \cup \Delta_{\mathsf{L},4}, \ \ x \leq 0,\\ \pi^l_{r,-}(u_{\mathsf{L},z}), & \text{if } (x,t) \in \Delta_{\mathsf{L},4}, \ \ x = \eta_{\mathsf{L},z}(t), \ \text{for some} \ \ z > 0\\ \mathsf{v}[\mathsf{L}, A, f_l](x,t), & \text{if } (x,t) \in \Delta[\mathsf{L}, A, f_l],\\ \overline{A}, & \text{if } (x,t) \in \Delta_{\mathsf{L},2},\\ \overline{B}, & \text{if } (x,t) \in \Delta_{\mathsf{L},3}. \end{cases}
\tag{5.4.111}
$$

where $\mathsf{v}[\mathsf{L}, A, f_l]$ denotes the function defined in (5.2.42), with $f = f_l$.

By construction, because of (25), and relying on the analysis in 5.2.5, it follows that in both Cases 1, 2, the function $u_{\mathsf{L}}(x,t)$:

- is locally Lipschitz continuous on $\big(\Delta_{\mathsf{L}} \setminus \overline{\Delta[\mathsf{L}, A, f_l]}\big) \cap \big((\mathbb{R}\setminus\{0\}) \times]0,T[\big)$, and it is continuous on the boundary $\partial\Delta[\mathsf{L}, A, f_l] \setminus \big(\{0\} \times]0,T[\big)$;
- is a classical solution of $u_t + f_l(u)_x = 0$ on $\big(\Delta_{\mathsf{L}} \setminus \overline{\Delta[\mathsf{L}, A, f_l]}\big) \cap \big(]-\infty,0[\times]0,T[\big)$, and of $u_t + f_r(u)_x = 0$ on $\Delta_{\mathsf{L}} \cap \big(]0,+\infty[\times]0,T[\big)$;
- is an entropy weak solution of $u_t + f_l(u)_x = 0$ on $\Delta[\mathsf{L}, A, f_l]$;
- satisfies the interface entropy condition (5.1.13) at any point $(0,t)$, $t \leq \tau_+(\mathsf{L})$.

Therefore, by Definition 5.1.2, we deduce that u_{L} is an AB-entropy solution of (22) on Δ_{L}.

Next, consider (for both Cases 1, 2) the function $u_{\mathsf{R}} : \Gamma_{\mathsf{R}} \to \mathbb{R}$ defined by setting for every $(x,t) \in \Gamma_{\mathsf{R}}$:

$$
u_{\mathsf{R}}(x,t) = \begin{cases} A, & \text{if } (x,t) \in \Gamma_{\mathsf{R},1},\\ B, & \text{if } (x,t) \in \Gamma_{\mathsf{R},2},\\ \mathsf{u}[\mathsf{R}, A, f_r](x,t), & \text{if } (x,t) \in \Gamma[\mathsf{R}, B, f_r],\\ (f_r')^{-1}\big(\frac{\mathsf{R}-x}{T-t}\big), & \text{if } (x,t) \in \Gamma_{\mathsf{R},3}, \end{cases}
\tag{5.4.112}
$$

where $\mathsf{u}[\mathsf{R}, A, f_r]$ denotes the function defined in (5.2.35), with $f = f_r$. By construction and relying on the analysis in § 5.2.4, we deduce as above that u_R provides an AB-entropy solution of (22) on Γ_R. Moreover, because of (5.3.27), (5.3.28), (5.3.29), we have

$$u_\mathsf{R}(x,T) = \omega(x) \qquad \forall\ x \in]\widetilde{\mathsf{L}}, \mathsf{R}[\,. \tag{5.4.113}$$

5.4.4.5. *Construction of AB-entropy solution on whole* $\mathbb{R}\times[0,T]$. Observing that, because of (5.4.102), (5.4.104), we have $\Delta_\mathsf{L}\cap\{x=0\} = \mathcal{I}_\mathsf{L}$, $\Gamma_\mathsf{R}\cap\{x=0\} = \mathcal{I}_\mathsf{R}$, we define the initial datum on $\mathcal{I}_\mathsf{L}\cup\mathcal{I}_\mathsf{R}$ as

$$u_0(x) = \begin{cases} u_\mathsf{L}(x,0) & \text{if} \quad x\in\mathcal{I}_\mathsf{L}, \\ u_\mathsf{R}(x,0) & \text{if} \quad x\in\mathcal{I}_\mathsf{R}. \end{cases} \tag{5.4.114}$$

where, in Case 1,

$$u_\mathsf{L}(x,0) = \begin{cases} (f_l')^{-1}\big(\frac{\mathsf{L}-x}{T}\big), & \text{if } x\in\mathcal{I}_\mathsf{L},\ \ x\le 0, \\ \pi^l_{r,-}(u_{\mathsf{L},z}), & \text{if } x\in\mathcal{I}_\mathsf{L},\ \ x=\eta_{\mathsf{L},z}(0),\ \text{ for some }\ z>0, \end{cases} \tag{5.4.115}$$

while, in Case 2,

$$u_\mathsf{L}(x,0) = \begin{cases} (f_l')^{-1}\big(\frac{\mathsf{L}-x}{T}\big), & \text{if}\ \ x\in]\vartheta_{\mathsf{L},-}(0),\, \mathsf{L}-T\cdot f_l'(\boldsymbol{v}[\mathsf{L},A,f_l])[, \\ \overline{A}, & \text{if}\ \ x\in]\mathsf{L}-T\cdot f_l'(\boldsymbol{v}[\mathsf{L},A,f_l]),\, 0[\,, \\ \overline{B}, & \text{if}\ \ x\in]0,\, \eta_{\mathsf{L},\,x(A,\overline{B})}(0)[, \\ \pi^l_{r,-}(u_{\mathsf{L},z}), & \text{if}\ \ x\in[\eta_{\mathsf{L},\,x(A,\overline{B})}(0),\, \vartheta_{\mathsf{L},+}(0)[\,,\ \ x=\eta_{\mathsf{L},z}(t),\ \text{ for some }\ z>0, \end{cases} \tag{5.4.116}$$

and, in both Cases 1, 2,

$$u_\mathsf{R}(x,0) = (f_r')^{-1}\big(\tfrac{\mathsf{R}-x}{T}\big), \quad \text{if}\ \ x\in\mathcal{I}_\mathsf{R}. \tag{5.4.117}$$

In view of the observations in § 5.4.4.1-5.4.4.2, the construction of the AB-entropy solution on $\big(\mathbb{R}\times[0,T]\big)\setminus\big(\Delta_\mathsf{L}\cup\Gamma_\mathsf{R}\big)$, and the corresponding definition of the initial datum on $\mathbb{R}\setminus\big(\mathcal{I}_\mathsf{L}\cup\mathcal{I}_\mathsf{R}\big)$, proceed as follows:

- For any $y\in]-\infty,\mathsf{L}[\cup]\mathsf{L},\widetilde{\mathsf{L}}[\cup]\mathsf{R},+\infty[$, we trace the lines $\vartheta_{y,\pm}$ starting at (y,T) until they reach the x-axis at the point $\phi_\pm(y)\doteq\vartheta_{y,\pm}(0)$. Since conditions (5.3.10), (5.3.11) of Theorem 5.3.3 is equivalent to the monotonicity of the map $\phi(y)\doteq\vartheta_y(0)$ (see [**9**, Lemma 4.4]), it follows that $\vartheta_{y,\pm}$ never intersect each other in the region $\mathbb{R}\times]0,T[$. Then, if $y\in]-\infty,\mathsf{L}[\cup]\mathsf{R},+\infty[$, we define a function $u(x,t)$ that is equal to $\omega(y\pm)$ along the segment $\vartheta_{y,\pm}$. Instead if $y\in]\mathsf{L},\widetilde{\mathsf{L}}[$ we define u to be equal to $\omega(y\pm)$ along the segment $\vartheta_{y,\pm}(t)$, $\tau_\pm(y)\le t\le T$, and to be equal to $\pi^l_{r,-}(\omega(y\pm))$ along the segment $\vartheta_{y,\pm}(t)$, $0\le t\le\tau_\pm(y)$.
- For any $z\in\mathcal{I}^n_\mathsf{L}\cup\mathcal{I}^n_\mathsf{R}$, we trace the line $\eta_{y_n,z}$, $y_n\in]-\infty,\mathsf{L}[\cup]\mathsf{R},+\infty[$. By construction the lines $\eta_{y_n,z}$ never cross each other in the region $\mathbb{R}\times]0,T[$. Then, if $y_n\in]-\infty,\mathsf{L}[$, we define $u(x,t)$ to be equal to $(f_l')^{-1}((y-z)/T) = u_{y,z}$ along the segment $\eta_{y_n,z}$, instead if $y_n\in]\mathsf{R},+\infty[$, we define $u(x,t)$ to be equal to $(f_r')^{-1}((y-z)/T) = u_{y,z}$ along the segment $\eta_{y_n,z}$.
- For any $z\in\widetilde{\mathcal{I}}^n_\mathsf{L}$, we trace the polygonal line $\eta_{y_n,z}$, $y_n\in]\mathsf{L},\widetilde{\mathsf{L}}[$. By construction the lines $\eta_{y_n,z}$ never cross each other in the region $\mathbb{R}\times]0,T[$. Then, we define $u(x,t)$ to

be equal to $(f_l')^{-1}((y_n - x)/(T-t)) = u_{y,z}$ along the segment $\eta_{y_n,z}$, $\tau(y,z) < t \leq T$, and to be equal to $\pi^l_{r,-}(u_{y,z})$ along the segment $\eta_{y_n,z}$, $0 \leq t \leq \tau(y,z)$.

Therefore, we define the the function

$$u(x,t) \doteq \begin{cases} \omega(y\pm), & \text{if } \ x = \vartheta_{y,\pm}(t) \ \text{ for some } \ y \in\,]-\infty, \mathsf{L}[\, \cup\,]\mathsf{R}, +\infty[\,, \\ \omega(y\pm), & \text{if } \ x = \vartheta_{y,\pm}(t) < 0 \ \text{ for some } \ y \in\,]\mathsf{L}, \widetilde{\mathsf{L}}[\,, \\ \pi^l_{r,-}(\omega(y\pm)), & \text{if } \ x = \vartheta_{y,\pm}(t) > 0 \ \text{ for some } \ y \in\,]\mathsf{L}, \widetilde{\mathsf{L}}[\,, \\ (f_l')^{-1}\big(\frac{y_n - z}{T}\big), & \text{if } \ x = \eta_{y_n,z}(t) \ \text{ for some } \ z \in \mathcal{I}^n_{\mathsf{L}}, \\ (f_r')^{-1}\big(\frac{y_n - z}{T}\big), & \text{if } \ x = \eta_{y_n,z}(t) \ \text{ for some } \ z \in \mathcal{I}^n_{\mathsf{R}}, \\ (f_l')^{-1}\big(\frac{\mathsf{L} - x}{T-t}\big), & \text{if } \ x = \eta_{y_n,z}(t) < 0 \ \text{ for some } \ z \in \widetilde{\mathcal{I}}^n_{\mathsf{L}}, \\ \pi^l_{r,-}(u_{y_n,z}), & \text{if } \ x = \eta_{y_n,z}(t) > 0 \ \text{ for some } \ z \in \widetilde{\mathcal{I}}^n_{\mathsf{L}}, \\ u_{\mathsf{L}}(x,t), & \text{if } \ (t,x) \in \Delta_{\mathsf{L}}, \\ u_{\mathsf{R}}(x,t), & \text{if } \ (t,x) \in \Gamma_{\mathsf{R}}, \end{cases} \tag{5.4.118}$$

and the initial datum

$$u_0(x) \doteq \begin{cases} \omega(y\pm), & \text{if } \ x \in \mathcal{I}_{\mathcal{W}},\ x = \theta_{y,\pm}(0) \ \text{ for some } \ y \in\,]-\infty, \mathsf{L}[\, \cup\,]\mathsf{R}, +\infty[\,, \\ \pi^l_{r,-}(\omega(y\pm)), & \text{if } \ x \in \mathcal{I}_{\mathcal{W}},\ x = \theta_{y,\pm}(0),\ y \in\,]\mathsf{L}, \tilde{\mathsf{L}}[\,, \\ (f_l')^{-1}\big(\frac{y_n - x}{T}\big), & \text{if } \ x \in \mathcal{I}^n_{\mathsf{L}}, \\ (f_r')^{-1}\big(\frac{y_n - x}{T}\big), & \text{if } \ x \in \mathcal{I}^n_{\mathsf{R}}, \\ \pi^l_{r,-}(u_{y_n,x}), & \text{if } \ x \in \widetilde{\mathcal{I}}^n_{\mathsf{L}}, \\ u_{\mathsf{L}}(x,0), & \text{if } \ x \in \mathcal{I}_{\mathsf{L}}, \\ u_{\mathsf{R}}(x,0), & \text{if } \ x \in \mathcal{I}_{\mathsf{R}}. \end{cases} \tag{5.4.119}$$

Notice that u_0 is not defined on the countable set $\mathcal{I}_{\mathcal{R}a}$ which is of measure zero, and clearly $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. By construction, the function $u(x,t)$:

- is locally Lipschitz continuous on $\big(\mathbb{R}\times\,]0,T[\,\big) \setminus \big(\,\overline{\Delta_{\mathsf{L}} \cup \Gamma_{\mathsf{R}}} \cup (\{0\}\times\,]0,T[\,)\big)$, and it is continuous on the boundary $\partial\big(\Delta_{\mathsf{L}} \cup \Gamma_{\mathsf{R}}\big) \setminus \big(\{0\}\times\,]0,T[\big)$;
- is a classical solution of $u_t + f_l(u)_x = 0$ on $\big(\,]-\infty,0[\,\times\,]0,T[\,\big) \setminus \overline{\Delta_{\mathsf{L}} \cup \Gamma_{\mathsf{R}}}\,$;
- is a classical solution of $u_t + f_r(u)_x = 0$ on $\big(\,]0,+\infty[\,\times\,]0,T[\,\big) \setminus \overline{\Delta_{\mathsf{L}} \cup \Gamma_{\mathsf{R}}}\,$;
- is an AB-entropy solution of (22) on $\Delta_{\mathsf{L}} \cup \Gamma_{\mathsf{R}}$;
- satisfies the interface entropy condition (5.1.13) at any point $(0,t)$, $t \in\,]0,T[\,$.

Thus, by Definition 5.1.2, it follows that the function $u(x,t)$ in (5.4.118) provides an AB-entropy solution to (22) on $\mathbb{R}\times[0,T]$. Moreover, because of (5.4.113), (5.4.118), (5.4.119), we have

$$u(x,0) = u_0(x), \qquad u(x,T) = \omega(x) \qquad \text{for a.e.} \quad x \in \mathbb{R}\,. \tag{5.4.120}$$

This proves that

$$\omega = \mathcal{S}^{[AB]+}_T u_0, \tag{5.4.121}$$

and thus $\omega \in \mathcal{A}^{AB}(T)$, which completes the proof of the implication (3) ⇒ (1) of Theorem 5.3.17 in the case of a non critical connection.

5.4.5. Part 2.b - (3) $\Rightarrow$ (2) **for non critical connections.** As a byproduct of the construction described in § 5.4.4, we show in this Subsection that, if ω satisfies (5.4.89), (5.4.90), then ω verifies condition (2) of Theorem 5.3.17, i.e. ω is a fixed point of the map $\omega \mapsto \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}\omega$. We shall assume that ω satisfies the pointwise constraints of Case 1 discussed in Remark 5.3.5, the other cases being simmetric, or entirely similar, or simpler.

In order to verify that $\mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-}\omega = \omega$, because of (5.4.121) it is sufficient to prove that, letting u_0 be the function defined by (5.4.115), (5.4.117), (5.4.119), it holds true

$$u_0 = \mathcal{S}_T^{[AB]-}\omega\,. \tag{5.4.122}$$

In turn, recalling the definition (5.1.28) of AB backward solution operator, the equality (5.4.122) is equivalent to the equality

$$u_0(-x) = \overline{\mathcal{S}}_T^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x) \qquad x \in \mathbb{R}, \tag{5.4.123}$$

where

$$(x,t) \mapsto \overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x) \tag{5.4.124}$$

denotes the unique $\overline{B}\,\overline{A}$-entropy solution of

$$\begin{cases} v_t + \overline{f}(x,v)_x = 0 & x \in \mathbb{R}, \quad t \geq 0, \\ v(x,0) = \omega(-x) & x \in \mathbb{R}, \end{cases} \tag{5.4.125}$$

$\overline{f}(x,v)$ being the symmetric flux in (5.1.26).

Towards a proof of (5.4.123), we will determine the solution of (5.4.125) on $\mathbb{R} \times [0,T]$ relying on the construction in § 5.4.4 and on the properties of the left forward rarefaction-shock wave pattern derived in § 5.2.5. Observe that the function $u(x,t)$ defined by (5.4.118) for Case 1, with u_{L}, u_{R} defined by (5.4.110), (5.4.112), respectively, is:

- locally Lipschitz continuous in the region

$$\mathscr{L} \doteq \big(\mathbb{R}\times\,]0,T[\,\big) \setminus \big((\{0\}\times\,]0,T[\,) \cup \Gamma[\mathsf{R},B,f_r]\big)$$

  where $\Gamma[\mathsf{R},B,f_r]$, is defined as in § 5.2.4, with $f = f_r$ (the region $\mathscr{L}$ is the complement of the pink region and of the axis $\{x = 0\}$ in Figure 6);
- a classical solution of $u_t + f_l(u)_x = 0$ on $]-\infty,0[\,\times\,]0,T[\,$;
- a classical solution of $u_t + f_r(u)_x = 0$ on $\big(]0,+\infty[\,\times\,]0,T[\,\big) \setminus \overline{\Gamma[\mathsf{R},B,f_r]}$;
- satisfies the interface entropy condition (5.1.13) at any point $(0,t)$, $t \in\,]0,T[\,$.

Therefore, if we define the transformation $(x,t) \mapsto \alpha(x,t) \doteq (-x, T-t)$, the function

$$v(x,t) \doteq u(-x, T-t), \qquad (x,t) \in \alpha(\,\overline{\mathscr{L}}) \setminus (\{0\}\times\,]0,T[\,), \tag{5.4.126}$$

is:

- an entropy weak solution of $v_t + f_r(v)_x = 0$ in the open set $\alpha(\mathscr{L}) \cap \{x < 0\}$;
- an entropy weak solution of $v_t + f_l(v)_x = 0$ in the open set $\alpha(\mathscr{L}) \cap \{x > 0\}$.

On the other hand, letting $\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]$ denote the region defined in (5.2.40) with $\mathsf{L} = y[\mathsf{R},B,f_r]$, $A = \overline{B}$, and $f = f_r$. one can directly verify that

$$\alpha\big(\Gamma[\mathsf{R},B,f_r]\big) = \Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r] \subset\,]-\inf,0[\times\,]0,T[\,. \tag{5.4.127}$$

Notice that $(\mathbb{R}\times[0,T])\setminus(\{0\}\times]0,T[\,)$ is the disjoint union of $\alpha(\overline{\mathscr{L}})\setminus(\{0\}\times]0,T[\,)$ and of $\alpha\big(\Gamma[\mathsf{R},B,f_r]\big)$. Then, letting $\mathsf{v}[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r](x,t)$ denote the function defined in (5.2.42), with $\mathsf{L}=y[\mathsf{R},B,f_r]$, $A=\overline{B}$, and $f=f_r$, consider the function $v:\mathbb{R}\times[0,T]\to\mathbb{R}$ defined by setting

$$v(x,t)\doteq\begin{cases}u(-x,T-t), & \text{if}\quad (x,t)\in\alpha(\overline{\mathscr{L}})\setminus(\{0\}\times]0,T[\,),\\ \mathsf{v}[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r](x,t), & \text{if}\quad (x,t)\in\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]\,.\end{cases}\tag{5.4.128}$$

By construction and because of the analysis in § 5.2.5, the function $v(x,t)$:

- is locally Lipschitz continuous on $\big(\mathbb{R}\times]0,T[\,\big)\setminus\big(\,\overline{\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]}\cup(\{0\}\times]0,T[\,)\big)$, and it is continuous on the boundary $\partial\big(\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]\big)\setminus\big(\{0\}\times]0,T[\big)$;
- is a classical solution of $v_t+f_r(v)_x=0$ on $\big(\,]-\infty,0[\,\times]0,T[\,\big)\setminus\overline{\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]}\,$;
- is an entropy weak solution of $v_t+f_r(v)_x=0$ on $\Delta[\boldsymbol{y}[\mathsf{R},B,f_r],\overline{B},f_r]$;
- is a classical solution of $v_t+f_l(v)_x=0$ on $]0,+\infty[\,\times]0,T[$;
- satisfies the $\overline{BA}$ interface entropy condition, namely, setting $v_l(t)\doteq v(0-,t)$, $v_r(t)\doteq v(0+,t)$, and considering the function

  $$I^{\overline{BA}}(v_l,v_r)\doteq\operatorname{sgn}(v_r-\overline{A}\,)\,\big(f_l(v_r)-f_l(\,\overline{A}\,)\big)\;-\operatorname{sgn}(v_l-\overline{B}\,)\,\big(f_r(v_l)-f_r(\,\overline{B}\,)\big)\,,$$

  it holds true

$$f_r(v_l(t))=f_l(v_r(t)),\qquad\quad I^{\overline{BA}}\big(v_l(t),v_r(t)\big)\le 0,\qquad \text{for a.e. } t\in]0,T[\,.\tag{5.4.129}$$

Notice also that, because of (5.4.120), (5.4.127), (5.4.128), it follows

$$v(x,0)=u(-x,T)=\omega(-x)\qquad\text{for a.e.}\quad x\in\mathbb{R}\,.\tag{5.4.130}$$

Therefore, by Definition 5.1.2, we deduce that the function $v(x,t)$ in (5.4.128) provides the $\overline{B}\,\overline{A}$-entropy solution to (5.4.125) on $\mathbb{R}\times[0,T]$, and hence we have

$$v(x,t)=\overline{\mathcal{S}}_t^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)\qquad x\in\mathbb{R},\ t\in[0,T]\,.\tag{5.4.131}$$

Moreover, by (5.4.120), (5.4.128), it holds true

$$\overline{\mathcal{S}}_T^{[\overline{B}\,\overline{A}]+}\big(\omega(-\,\cdot\,)\big)(x)=u(-x,0)=u_0(-x)\qquad x\in\mathbb{R}\,,$$

which proves (5.4.123), and thus concludes the proof of the implication (3) $\Rightarrow$ (2) of Theorem 5.3.17 in the case of a non critical connection.

5.4.6. Part 3.a - (1) $\Leftrightarrow$ (2) **for critical connections.** In this Subsection we rely on the fact that the equivalence of conditions (1), (2) of Theorem 5.3.17 holds for connections which are non critical (by the proofs in § 5.4.2, 5.4.3, 5.4.4, 5.4.5), and we will show that it remains true also for critical connections. To fix the ideas, throughout this and the following subsections we shall assume that the connection (A,B) is critical at the left, i.e. that

$$A=\theta_l\,,\tag{5.4.132}$$

the case where one assumes that $B=\theta_r$ being symmetric. Notice that the assumption $A=\theta_l$ does not prevent the connection to be critical also at the right, i.e. $B=\theta_r$: it might or might not happen. Notice that there exists a sequence $\{A_n, B_n\}_n$ of non critical connections that satisfy

$$\lim_n (A_n, B_n) = (A, B)\,. \tag{5.4.133}$$

We will show only that (1) ⇒ (2), since the reverse implication is clear (see § 5.4.1). Then, given $\omega \in \mathcal{A}^{[AB]}(T)$ with

$$\omega = \mathcal{S}_T^{[AB]+} u_0, \qquad u_0 \in \mathbf{L}^\infty(\mathbb{R})\,, \tag{5.4.134}$$

set

$$\omega_n \doteq \mathcal{S}_T^{[A_n B_n]+} u_0\,. \tag{5.4.135}$$

Hence, since $\omega_n \in \mathcal{A}^{[A_n B_n]}(T)$, by the validity of Theorem 5.3.17 in the non critical case it holds

$$\omega_n = \mathcal{S}_T^{[A_n B_n]+} \circ \mathcal{S}_T^{[A_n B_n]-} \omega_n \qquad \forall\, n\,. \tag{5.4.136}$$

Notice that by definition (5.1.28) it follows that the $\mathbf{L}^1_{\rm loc}$ stability property (iv) of Theorem 5.1.8 holds also for the AB-backward solution operator $\mathcal{S}_T^{[AB]-}$, so that we have

$$\mathcal{S}_T^{[A_n B_n]-} \omega_n \quad \to \quad \mathcal{S}_T^{[AB]-} \omega \qquad \text{in} \quad \mathbf{L}^1_{\rm loc}(\mathbb{R})\,. \tag{5.4.137}$$

Hence, we deduce that

$$\omega \overset{[\text{Thm 5.1.8-(iv)}]}{=} \lim_n \omega_n \overset{[(5.4.136)]}{=} \lim_n \mathcal{S}_T^{[A_n B_n]+} \circ \mathcal{S}_T^{[A_n B_n]-} \omega_n \overset{[\text{Thm 5.1.8-(iv) and (5.4.137)}]}{=} \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-} \omega\,,$$

which proves (1) ⇒ (2). □

5.4.7. Part 3.b - (1) ⇒ (3) for critical connections. In this Subsection we rely on the fact that the implication (1) ⇒ (3) of Theorem 5.3.17 holds for connections which are non critical, and in particular we know (by § 5.4.2, 5.4.3, 5.4.4) that Theorems 5.3.3, 5.3.11, 5.3.14, are verified for non critical connections. We will prove that, for a critical connection (A, B), any element $\omega \in \mathcal{A}^{AB}(T)$ satisfies the conditions of Theorem 5.3.9, or of Theorem 5.3.11, or of Theorem 5.3.14. We divide the proof in nine steps.

Step 1. Let $\{A_n, B_n\}_n$ be a sequence of non critical connections as in Part 3.a. Given $\omega \in \mathcal{A}^{AB}(T)$ as in (5.4.134), and ω_n, as in (5.4.135), set

$$u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x), \qquad t \geq 0, \quad x \in \mathbb{R}\,, \tag{5.4.138}$$

and consider the sequence u_n of A_nB_n-entropy weak solutions defined by

$$u_n(x,t) \doteq \mathcal{S}_t^{[A_n B_n]+} u_0(x), \qquad t \geq 0, \quad x \in \mathbb{R}\,. \tag{5.4.139}$$

Let $u_{n,l}, u_{n,r}$ denote, respectively, the left and right traces of u_n at $x = 0$ defined as in (5.1.7), and let u_l, u_r be the left and right traces of u at $x = 0$ (whose existence is derived in Steps 5, 8). Then, by Theorem 5.1.8 and Corollary 5.1.11, and because of (5.4.133), it follows

$$u_n(\cdot,t) \quad \to \quad u(\cdot,t) \qquad \text{in} \quad \mathbf{L}^1_{loc}(\mathbb{R}) \qquad \forall\, t \in [0,T]\,, \tag{5.4.140}$$

$$f_l(u_{n,l}) \quad \rightharpoonup \quad f_l(u_l) \qquad \text{in} \quad \mathbf{L}^1([0,T])\,, \tag{5.4.141}$$

$$f_r(u_{n,r}) \quad \rightharpoonup \quad f_r(u_r) \qquad \text{in} \quad \mathbf{L}^1([0,T])\,, \tag{5.4.142}$$

and hence, in particular, we have

$$\omega_n \quad \to \quad \omega \qquad \text{in} \quad \mathbf{L}^1_{\rm loc}(\mathbb{R})\,. \tag{5.4.143}$$

In order to prove that ω satisfies condition (3) of Theorem 5.3.17, letting

$$\mathsf{L} \doteq \mathsf{L}[\omega, f_l]\,, \qquad \mathsf{R} \doteq \mathsf{R}[\omega, f_r], \tag{5.4.144}$$

be quantities defined as in (5.3.2), we need to show that:

- If $\mathsf{L} = 0$, $\mathsf{R} > 0$ or viceversa, then ω satisfies the conditions of Theorem 5.3.11;
- If $\mathsf{L} = 0$, $\mathsf{R} = 0$, then ω satisfies the conditions of Theorem 5.3.14.
- If $\mathsf{L} < 0$, $\mathsf{R} > 0$, then ω satisfies the conditions of Theorem 5.3.9.

We shall first address the two cases $\mathsf{L} = 0, \mathsf{R} > 0$, and $\mathsf{L} = 0$, $\mathsf{R} = 0$ (the analysis of the case $\mathsf{L} < \mathsf{R} = 0$ being entirely similar to the one of $\mathsf{L} = 0 < \mathsf{R}$). Next, we shall analyze the third case $\mathsf{L} < 0 < \mathsf{R}$. Throughout the subsection we let L_n,. R_n, denote the objects defined as in (5.4.144) for ω_n:

$$\mathsf{L}_n \doteq \mathsf{L}[\omega_n, f_l]\,, \qquad \mathsf{R}_n \doteq \mathsf{R}[\omega_n, f_r]\,. \tag{5.4.145}$$

Observe that by Remark 5.1.10, and because of (5.4.143), the functions ω_n have a uniform bound

$$\|\omega_n\|_{\mathbf{L}^\infty} \leq \overline{C} \qquad \forall\ n, \tag{5.4.146}$$

for constant $\overline{C} > 0$. Hence, by definition (5.3.2), the constant $|\mathsf{L}_n|$ are bounded by $T \cdot \sup_{|u|\leq \overline{C}} |f_l'(u)|$, and the constant R_n are bounded by $T \cdot \sup_{|u|\leq \overline{C}} |f_r'(u)|$. Thus, up to a subsequence, we can define the limits

$$\widehat{\mathsf{L}} \doteq \lim_{n\to\infty} \mathsf{L}_n\,, \qquad \widehat{\mathsf{R}} \doteq \lim_{n\to\infty} \mathsf{R}_n\,. \tag{5.4.147}$$

We claim that

$$\widehat{\mathsf{L}} \leq \mathsf{L}, \qquad \widehat{\mathsf{R}} \geq \mathsf{R}\,. \tag{5.4.148}$$

By definition (5.3.2), (5.4.144) of R, in order to prove the second inequality in (5.4.148), it is sufficient to show that

$$\widehat{\mathsf{R}} - T \cdot f_r'(\omega(\widehat{\mathsf{R}}+)) \geq 0\,. \tag{5.4.149}$$

Observe that by Definition 5.1.2 u_n and u are entropy weak solutions of

$$u_t + f_r(u)_x = 0 \qquad x > 0, \quad t \in [0, T], \tag{5.4.150}$$

that, because of (5.4.140), (5.4.142), satisfy the assumptions (5.8.2), (5.8.3) of Lemma 5.8.1 in Appendix 5.8. Thus, applying (5.8.4), and recalling (5.4.134), (5.4.135), we find

$$\omega(\widehat{\mathsf{R}}+) \leq \liminf_{\substack{n\to\infty \\ y\to\widehat{\mathsf{R}},\, y>0}} \ \omega_n(y+)\,. \tag{5.4.151}$$

Since (5.4.147) and the liminf property imply

$$\liminf_{\substack{n\to\infty \\ y\to\widehat{\mathsf{R}},\, y>0}} \ \omega_n(y+) \leq \liminf_n \omega_n(\mathsf{R}_n+)\,, \tag{5.4.152}$$

we derive from (5.4.151) that

$$\omega(\widehat{\mathsf{R}}+) \leq \liminf_n \omega_n(\mathsf{R}_n+)\,. \tag{5.4.153}$$

On the other hand, by definition (5.3.2), (5.4.145) of R_n, it holds

$$\omega_n(\mathsf{R}_n+) \leq (f_r')^{-1}\left(\frac{\mathsf{R}_n}{T}\right)\,. \tag{5.4.154}$$

Hence, from (5.4.153), (5.4.154) and (5.4.147) we deduce

$$\omega(\widehat{\mathsf{R}}+) \leq \lim_{n\to\infty} (f_r')^{-1}\left(\frac{\mathsf{R}_n}{T}\right) = (f_r')^{-1}\left(\frac{\widehat{\mathsf{R}}}{T}\right), \tag{5.4.155}$$

which yields (5.4.149). This completes the proof of the second inequality in (5.4.148), while the proof of the first one is entirely similar.

Relying on (5.4.148), we will show in Steps 2-7 the existence of $\omega(0\pm)$, and that ω satisfies the conditions (i)', (ii)' of Theorem 5.3.11 in the case $\mathsf{L} = 0$, $\mathsf{R} > 0$. Namely, in Step 2 we prove (5.3.54), in Step 3 we prove (5.3.56), in Step 4 we prove (5.3.57), in Step 5 we prove (5.3.53) and the existence of $\omega(0\pm)$, while in Step 6 we prove (5.3.55). Finally, in Step 7 we prove (5.3.52), concluding the proof of conditions (i)', (ii)' of Theorem 5.3.11. The proof of the existence of $\omega(0\pm)$, and that ω satisfies conditions (i), (ii) of Theorem 5.3.11 in the case $\mathsf{L} < 0$, $\mathsf{R} = 0$ is entirely similar to the case $\mathsf{L} = 0$, $\mathsf{R} > 0$, although the symmetry is broken (because of assumption (5.4.132)), and it is briefly discussed in Step 8. Next, in Step 9 we will show that ω satisfies conditions (i), (ii) of Theorem 5.3.14 in the case $\mathsf{L} = 0$, $\mathsf{R} = 0$. Finally, in Steps 10-13 we will show that ω satisfies conditions (i), (ii) of Theorem 5.3.9.

Step 2. $\big(\mathsf{L} = 0,\ \mathsf{R} > 0$, proof of (5.3.54): $\omega(x) \geq B$ in $]0, \mathsf{R}[\,\big)$.
Applying (5.3.14), (5.3.15) of Theorem 5.3.3-(ii) or (5.3.54) of Theorem 5.3.11-(ii)' for ω_n, in the case of the non critical connections (A_n, B_n), we deduce that

$$\omega_n(x) \geq B_n \qquad \forall\, x \in]0, \mathsf{R}^n[\,, \quad \forall\, n\,. \tag{5.4.156}$$

On the other hand, by virtue of (5.4.143), (5.4.148), we can extract a subsequence of $\{\omega_n\}$ that converges to ω for almost every $x \in]0, \mathsf{R}[$. Then, taking the limit in (5.4.156), relying on (5.4.133), and because of the normalization of ω as a right continuous function (see Remark 5.3.2), we derive (5.3.54).

Step 3. $\big(\mathsf{L} = 0,\ \mathsf{R} > 0$, proof of (5.3.56): $\mathsf{R} \in]0, T\cdot f_r'(B)[\ \Rightarrow\ \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-)\big)$.
Observe first that, in the case $\widehat{\mathsf{R}} = \mathsf{R}$, by the continuity of the function $\boldsymbol{u}[\mathsf{R}, B, f_r]$ with respect to R, B (see § 5.2.1), and because of (5.4.133), we find

$$\lim_{n\to\infty} \boldsymbol{u}[\mathsf{R}_n, B_n, f_r] = \boldsymbol{u}[\mathsf{R}, B, f_r]\,. \tag{5.4.157}$$

On the other hand, if $\mathsf{R} = \widehat{\mathsf{R}} \in]0, T\cdot f_r'(B)[\,$, we may assume that $\mathsf{R}_n \in]0, T\cdot f_r'(B_n)[\,$, for n sufficiently large. Hence, applying either (5.3.13) of Theorem 5.3.3-(ii), or (5.3.56) of Theorem 5.3.11-(ii)' for the corresponding ω_n in the case of the non critical connections (A_n, B_n), we deduce

$$\liminf_{n\to\infty} \omega_n(\mathsf{R}_n+) \leq \lim_{n\to\infty} \boldsymbol{u}[\mathsf{R}_n, B_n, f_r]\,. \tag{5.4.158}$$

Then, combining (5.4.157), (5.4.158), with (5.4.153), and recalling (5.2.8) with $f = f_r$, we derive

$$\omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] < B\,, \tag{5.4.159}$$

which, together with (5.3.54) (established in Step 2), proves (5.3.56) in the case $\widehat{\mathsf{R}} = \mathsf{R}$.

Thus, because of (5.4.148), it remains to analyze the case $\widehat{\mathsf{R}} > \mathsf{R}$. Let ϑ_n^- denote the minimal backward characteristic for u_n starting from (R_n, T) and lying in the domain

$x > 0$. Recalling the definition (5.3.2), (5.4.145) of R_n this is a map $\vartheta_n^- :]\tau_n, T] \to]0, +\infty[$, $\tau_n \geq 0$, with the property that $\lim_{t\to\tau_n} \vartheta_n^-(t) = 0$. By possibly taking a subsequence, we may assume that $\{\tau_n\}_n$ converges to some $\overline{\tau} \geq 0$. Observe that ϑ_n^- are genuine characteristics which, up to a subsequence, converge to a genuine characteristic $\vartheta^- :]\overline{\tau}, T] \to]0, +\infty[$ for u, starting from $(\vartheta^-(T), T) = (\widehat{\mathsf{R}}, T)$ (see proof of Lemma 5.8.1 in Appendix 5.8). The trajectory of ϑ^- is a segment with slope $f_r'(u(\widehat{\mathsf{R}}-, T)) = f_r'(\omega(\widehat{\mathsf{R}}-))$. Therefore, if $\overline{\tau} > 0$, it follows that $f_r'(\omega(\widehat{\mathsf{R}}-)) > \widehat{\mathsf{R}}/T$, which by definition of R implies $\mathsf{R} \geq \widehat{\mathsf{R}}$, contradicting the assumption $\widehat{\mathsf{R}} > \mathsf{R}$. Hence, it must be $\overline{\tau} = 0$, $\lim_{t\to 0} \vartheta^-(t) = 0$, and the trajectory of ϑ^- is a segment joining the point $(\widehat{\mathsf{R}}, T)$ with the origin $(0, 0)$. Since $\mathsf{R} < \widehat{\mathsf{R}}$ and because backward characteristics cannot intersect in the domain $x > 0, t > 0$, this in turn implies that the slope $f_r'(\omega(\mathsf{R}+))$ of the maximal backward characteristic for u starting at (R, T) must be greater or equal than R/T. On the other hand, by definition (5.3.2), (5.4.144) of R, we have $f_r'(\omega(\mathsf{R}+)) \leq \mathsf{R}/T$, and hence it follows that

$$f_r'(\omega(\mathsf{R}+)) = \mathsf{R}/T. \tag{5.4.160}$$

Observe now that, applying Theorem 5.3.3-(ii) or Theorem 5.3.11-(ii)' for ω_n, in the case of the non critical connections (A_n, B_n), we know that (5.4.156) is verified. Moreover, because of $\mathsf{R} < \widehat{\mathsf{R}} = \lim_n \mathsf{R}_n$ we may assume that $\mathsf{R} < \mathsf{R}_n$ for n sufficiently large. Hence, by virtue of (5.4.143), we can extract a subsequnce of $\{\omega_n\}$ that converges to ω for almost every $x \in]0, \mathsf{R}[$, and thus we derive from (5.4.133), (5.4.156) that $\omega(\mathsf{R}+) \geq B$. This inequality, together with (5.4.160), yields

$$\mathsf{R} \geq T \cdot f_r'(B)\,, \tag{5.4.161}$$

proving the implication (5.3.56) also in the case $\widehat{\mathsf{R}} > \mathsf{R}$.

Step 4. $\big(\mathsf{L} = 0,\ \mathsf{R} > 0$, proof of (5.3.57): $\mathsf{R} > T \cdot f_r'(B) \quad \Rightarrow \quad \omega(\mathsf{R}+) \leq \omega(\mathsf{R}-)\big)$.
By virtue of (5.4.133), (5.4.148), we may assume that

$$\mathsf{R}_n > T \cdot f_r'(B_n) \qquad \forall\, n\,. \tag{5.4.162}$$

Then, applying (5.3.17) of Theorem 5.3.3-(ii) or (5.3.57) of Theorem 5.3.11-(ii)' for ω_n and the non critical connections (A_n, B_n), we derive

$$\omega_n(\mathsf{R}_n-) \geq \omega_n(\mathsf{R}_n+) \qquad \forall\, n\,. \tag{5.4.163}$$

On the other hand, if $\widehat{\mathsf{R}} = \mathsf{R}$, invoking (5.8.4), (5.8.5) of Lemma 5.8.1 in Appendix 5.8, we deduce as in Step 3 that

$$\omega(\mathsf{R}-) \geq \limsup_n \omega_n(\mathsf{R}_n-)\,, \qquad \omega(\mathsf{R}+) \leq \liminf_n \omega_n(\mathsf{R}_n+)\,. \tag{5.4.164}$$

Then, (5.4.163)-(5.4.164) together yield $\omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)$, proving (5.3.57) in the case $\widehat{\mathsf{R}} = \mathsf{R}$. Instead, if $\widehat{\mathsf{R}} > \mathsf{R}$, we can assume that $\mathsf{R}_n > \mathsf{R}$ for all n sufficiently large. Then, observe that applying (5.3.12), (5.3.14), (5.3.15), of Theorem 5.3.3, or (5.3.53) of Theorem 5.3.11, for ω_n and the non critical connections (A_n, B_n), we deduce

$$\omega_n(\mathsf{R}-) \geq \omega_n(\mathsf{R}+) \qquad \forall\, n\,. \tag{5.4.165}$$

Hence, with the same arguments of above we find that

$$\omega(\mathsf{R}-) \geq \limsup_n \omega_n(\mathsf{R}-)\,, \qquad \omega(\mathsf{R}+) \leq \liminf_n \omega_n(\mathsf{R}+)\,, \tag{5.4.166}$$

which, together with (5.4.165), yields $\omega(\mathsf{R}-) \geq \omega(\mathsf{R}+)$, completing the proof of (5.3.57).

Step 5. $\big(\mathsf{L}=0,\ \mathsf{R}>0$, proof of (5.3.53): $D^+\omega(x) \leq h[\omega, f_l, f_r](x)$ in $]0,\mathsf{R}[$, and of the existence of $\omega(0\pm)\,\big)$.
Applying Theorem 5.3.3-(i) or Theorem 5.3.11-(i)' for ω_n in the case of the non critical connections (A_n, B_n), we know that

$$D^+\omega_n(x) \leq h[\omega_n, f_l, f_r](x) \qquad \forall\ x \in]0, \mathsf{R}_n[\,. \tag{5.4.167}$$

As shown in [**9**, Lemma 4.4], the inequality (5.4.167) is equivalent to the fact that the maps

$$\phi_n(x) \doteq -\tau_n(x)\cdot f_l' \circ \pi_{l,+}^r(\omega_n(x)), \qquad \tau_n(x) \doteq T - \frac{x}{f_r'(\omega_n(x))}\,, \qquad x \in]0, \mathsf{R}_n[\,, \tag{5.4.168}$$

are, respectively, nondecreasing and decreasing. Since by (5.4.148) it holds $\lim_n \mathsf{R}_n \geq \mathsf{R}$, relying on (5.4.143) we deduce that, up to a subsequence, $\{\omega_n\}_n$ converges to ω for almost every $x \in]0, \mathsf{R}[$. In turn, this implies that the sequences $\{\phi_n\}_n$, $\{\tau_n\}_n$, converges for almost every $x \in]0, \mathsf{R}[$ to the maps

$$\phi(x) \doteq -\tau(x)\cdot f_l' \circ \pi_{l,+}^r(\omega(x)), \qquad \tau(x) \doteq T - \frac{x}{f_r'(\omega(x))}\,, \qquad x \in]0, \mathsf{R}[\,. \tag{5.4.169}$$

Then, the monotonicity of each map $\phi_n(x)$ and $\tau_n(x)$, imply the same monotonicity of the maps $\phi(x), \tau(x)$ defined in (5.4.169). Namely, ϕ is a nondecreasing map and τ is a decreasing map. But this is equivalent to the inequality (5.3.53), relying again on [**9**, Lemma 4.4]. Next, we observe that the monotonicity of the maps $x \mapsto \phi(x)$, $x \mapsto \tau(x)$, readily implies the existence of the one-sided limit $\omega(0+)$. In fact, since ϕ and τ are monotone, it follows that the limits $\phi(0+)$, $\tau(0+)$ do exist. On the other hand, observing that the map $\omega \mapsto f_l' \circ \pi_{l,+}^r(\omega)$, $\omega \geq B$, is invertible, by (5.4.169) we can write

$$\omega(x) = \big[f_l' \circ \pi_{l,+}^r\big]^{-1}\bigg(-\frac{\phi(x)}{\tau(x)}\bigg) \qquad \forall\ x \in]0, \mathsf{R}[\,.$$

Therefore, since the limit for $x \to 0+$ of the right hand side exists, it follows that the limit $\omega(0+)$ exists as well. Finally, concerning the existence of $\omega(0-)$, given any sequence $\{x_n\}_n \subset]-\infty, 0[$ of points of continuity for ω such that $x_n \to 0$, consider the backward genuine characteristics ϑ_n for u starting at (x_n, T). Because of the assumption $\mathsf{L} = 0$, and by definition (5.3.2), (5.4.144) of L, it follows that ϑ_n never cross the interface $x = 0$. Observe that $\{\vartheta_n\}_n$ is a sequence of Lipschitz continuous functions with a uniform Lipschitz constant, defined on $[0, T]$ and lying in the semiplane $\{x < 0\}$. Hence, by Ascoli-Arzelà Theorem, we can assume that, up to a subsequence, $\{\vartheta_n\}_n$ converges uniformly to some Lipschitz continuous function $\vartheta :\ [0, T] \to\ ]-\infty, 0[$. Therefore, with the same arguments of the proof of Lemma 5.8.1 in Appendix 5.8, since uniform limit of genuine characteristics is a genuine characteristic, and because genuine characteristics cannot intersect in $\{x < 0\}$, we deduce that ϑ is the minimal backward characteristic for u in $\{x \leq 0\}$ starting at $(0, T)$. Moreover, ϑ has slope $\vartheta' = \lim_n \vartheta_n' = \lim_n f_l'(\omega(x_n))$. In turn, this implies that $\lim_n \omega(x_n) = (f_l')^{-1}(\theta')$. Since this limit is independent on the choice of x_n we deduce that the one-sided limit $\omega(0-)$ exists and $\omega(0-) = (f_l')^{-1}(\theta')$.

Step 6. $\big(\mathsf{L}=0,\ \mathsf{R}>0$, proof of (5.3.55): $\omega(0-)\geq\pi^r_{l,+}(\omega(0+))\big)$.
Let $x\in\,]0,\mathsf{R}[$ be a point of continuity for ω, and consider the backward genuine characteristics for u starting at (x,T), defined by

$$\vartheta_x(t)\doteq x-(T-t)\cdot f_r'((\omega(x))\qquad t\in\,]\tau(x),T]\,,\tag{5.4.170}$$

with

$$\tau(x)\doteq T-\frac{x}{f_r'(\omega(x))}\,,\tag{5.4.171}$$

so that one has $\lim_{t\to\tau(x)}\vartheta_x(t)=0$. Observe that the inequality (5.3.53) (established at **Step 3**) implies that the function $\tau(x)$ is decreasing. On the other hand, because of (5.4.146), the slopes of ϑ_x are uniformly bounded by $\sup_{|u|\leq\overline{C}}|f_r'(u)|$. Therefore, letting $\{x_n\}_n\subset\,]0,\mathsf{R}[$ be a sequence of points of continuity for ω, such that $x_n\to 0$, it follows that

$$\lim_n\tau(x_n)=T\,.\tag{5.4.172}$$

Notice that, since ϑ_{x_n} are genuine characteristics, we have

$$\omega(x_n)=u_r(\tau(x_n))\qquad\forall\ n\,.\tag{5.4.173}$$

Since, by definition (5.3.1) one has $\pi^r_{l,+}(u)\geq\theta_l$ for any u, we may assume that, up to a subsequence, either

$$\pi^r_{l,+}(\omega(x_n))=\theta_l\qquad\forall\ n\,,\tag{5.4.174}$$

or

$$\pi^r_{l,+}(\omega(x_n))>\theta_l\qquad\forall\ n\,.\tag{5.4.175}$$

In the first case (5.4.174) we deduce that

$$\pi^r_{l,+}(\omega(0+))=\lim_n\pi^r_{l,+}(\omega(x_n))=\theta_l\,,\tag{5.4.176}$$

which yields (5.3.55) observing that, by definition (5.3.2), (5.4.144), $\mathsf{L}=0$ implies $f_l'(\omega(0-))\geq 0$, which is equivalent to $\omega(0-)\geq\theta_l$. In the second case (5.4.175) observe that, since the map τ in (5.4.171) is decreasing, and because x_n are points of continuity for ω, then $\tau(x_n)$ are points of continuity for u_r. Hence, by the interface entropy condition (5.1.13) it follows that $\tau(x_n)$ are points of continuity also for u_l. Then, we can trace the backward genuine characteristics for u starting at $(0,\tau(x_n))$, that, because of (5.4.173), are defined by

$$\vartheta_n(t)\doteq(t-\tau(x_n))\cdot f_l'\circ\pi^r_{l,+}(\omega(x_n)),\qquad t\in[0,\tau(x_n)]\,.\tag{5.4.177}$$

Notice that $\{\vartheta_n\}_n$ is a sequence of Lipschitz continuous functions with a uniform Lipschitz constant, defined on uniformly bounded intervals $[0,\tau(x_n)]$. Hence, by Ascoli-Arzelà Theorem, and because of (5.4.172), we can assume that, up to a subsequence, $\{\vartheta_n\}_n$ converges uniformly to some Lipschitz continuous function $\vartheta:\ [0,T]\to\ [0,+\infty[$. Therefore, with the same arguments of the proof of Lemma 5.8.1 in Appendix 5.8, since uniform limit of genuine characteristics is a genuine characteristic we deduce that ϑ is a backward genuine characteristic starting at $(0,T)$, that has slope $f_l'\circ\pi^r_{l,+}(\omega(0+))$. On the other hand the minimal backward characteristic starting at $(0,T)$ has slope $f_l'(\omega(0-))$. Since the slope of the minimal backward characteristic is larger than the slope of any other backward characteristic passing through the same point, it follows that $f_l'(\omega(0-))\geq f_l'\circ\pi^r_{l,+}(\omega(0+))$, which implies (5.3.55). This concludes the proof of this step.

Step 7. $\big(\mathsf{L} = 0,\ \mathsf{R} > 0$, proof of (5.3.52): $D^+\omega(x) \leq \frac{1}{T\cdot f_l''(\omega(x))}$ in $]-\infty, 0[$, and $D^+\omega(x) \leq \frac{1}{T\cdot f_r''(\omega(x))}$ in $]\mathsf{R}, +\infty[\big)$.
Observe that, by definition (5.3.2), (5.4.144) of L, R, and since $\mathsf{L} = 0$, backward characteristics starting at points (x, T), with $x \in]-\infty, 0[\cup]\mathsf{R}, +\infty[$ do no intersect the interface $x = 0$. Hence, we recover the Oleinik estimates (5.3.52) as a classical property of solutions to conservation laws with strictly convex flux, which follows from the fact that genuine characteristics never intersect at positive times. This completes the proof of the existence of $\omega(0\pm)$ and that ω satisfies conditions (i)'-(ii)' of Theorem 5.3.11.

Step 8. $\big(\mathsf{L} < 0,\ \mathsf{R} = 0$, proof of the existence of $\omega(0\pm)$, and of conditions (i)-(ii) of Theorem 5.3.11$\big)$.
The Oleinik-type inequalities (5.3.46), (5.3.47), and the existence of $\omega(0\pm)$ can be established with the same arguments of Steps 5, 7. The proofs of (5.3.48), (5.3.49), are entirely similar to the proofs of (5.3.54), (5.3.55), in Steps 2 and 6, respectively. Since $A = \theta_l$, and hence $f_l'(A) = 0$, the implication (5.3.50) is trivially verified. Finally, with the same arguments of the proof of (5.3.57) in Step 4 one can recover the inequality $\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+)$, thus proving the implication (5.3.51). Therefore the proof of the existence of $\omega(0\pm)$ and that ω satisfies the conditions (i)-(ii) of Theorem 5.3.11 is completed.

Step 9. $\big(\mathsf{L} = 0, \mathsf{R} = 0$, proof of (5.3.62): $D^+\omega(x) \leq \frac{1}{T\cdot f_l''(\omega(x))}$ in $]-\infty, 0[$, and $D^+\omega(x) \leq \frac{1}{T\cdot f_r''(\omega(x))}$ in $]0, +\infty[$, of the existence of $\omega(0\pm)$: $\omega(0-) \geq \pi^r_{l,+}(\omega(0+)$, and of (5.3.63): $\omega(0-) \geq \overline{A}$, $\omega(0+) \leq \overline{B}\big)$.
Since $\mathsf{L} = 0$, $\mathsf{R} = 0$, by definition (5.3.2) it follows that backward characteristics starting at (x, T), $x \in]-\infty, 0[\cup]0, +\infty[$, never intersect the interface $x = 0$. Thus, as observed in Step 7, the Oleinik estimates in (5.3.62) are a classical property of solutions. Moreover, with the same arguments of Step 5 one deduces the existence of $\omega(0\pm)$. Furhter, the inequality $\omega(0-) \geq \pi^r_{l,+}(\omega(0+)$ can be established with the same proof of (5.3.55) in Step 6. Finally, the inequality $\omega(0+) \leq \overline{B}$ is obtained with the same arguments of the proof of (5.3.56) in Step 3, observing that by Remark 5.2.2 we have $\boldsymbol{u}[0, B, f_r] = \boldsymbol{u}[0+, B, f_r] = \overline{B}$. The other inequality $\omega(0-) \geq \overline{A}$ can derived in entirely similar way. Therefore the proof of the existence of $\omega(0\pm)$ and that ω satisfies the conditions (i)-(ii) of Theorem 5.3.14 is completed.

Step 10. $\big(\mathsf{L} < 0 < \mathsf{R}$, proof of condition (i) of Theorem 5.3.9 and of the existence of $\omega(0-)\big)$.
The Oleinik inequalities (5.3.34) are a classical property of solutions to conservation laws with strictly convex flux as observed in Step 7. The proof of the Oleinik type inequality (5.3.35) can be recovered with the same limiting procedure of Step 5, passing to the limit the monotonicity of the maps

$$\phi_n(x) \doteq -\tau_n(x)\cdot f_r'\circ\pi^l_{r,-}(\omega_n(x)), \qquad \tau_n(x) \doteq T - \frac{x}{f_l'(\omega_n(x))}, \qquad x \in]\mathsf{L}_n, 0[\,, \tag{5.4.178}$$

ensured by the Oleinik type inequalities satisfied by $\omega_n \in \mathcal{A}^{[A_n B_n]}(T)$, and relying on (5.4.143), (5.4.148). This also shows that the one-sided limit $\omega(0-)$ exists, using the monotonicity of the limiting maps $\phi(x) = \lim_n \phi_n(x)$, $\tau(x) = \lim_n \tau_n(x)$, $x \in]\mathsf{L}, 0[$, as in Step 5.

Step 11. $\big(\mathsf{L} < 0 < \mathsf{R}$, proof of (5.3.37): $\omega(\mathsf{L}-) \geq \omega(\mathsf{L}+),\ \ \omega(0-) = \theta_l\big)$.
The proof of the first constraint in (5.3.37) can be obtained with the same procedure of Step 4 (with L in place of R). Concerning the second constraint in (5.3.37), notice first that we have $\omega(0-) \leq \theta_l$, since otherwise we may consider a sequence $\{x_n\}_n$ of continuity points for ω, such that $x_n \uparrow 0$, and for n sufficiently large the backward characteristics for u from (x_n, T) would intersect in $\{x < 0\}$ the maximal backward characteristic for u from the point (L, T), which gives a contradiction. Next, assume that the strict inequality $\omega(0-) < \theta_l$ holds. Then, let $x < 0$ be a continuity point of ω sufficiently close to 0 so that $\omega(x) < \theta_l$, and consider the time

$$\tau(x) = T - \frac{x}{f_l'(\omega(x))}$$

at which the backward (genuine) characteristic for u from (x, T) impacts the interface $x = 0$. Since x is a continuity point for ω, by the strict monotonicity of the map τ on $]\mathsf{L}, 0[$ (derived in Step 9 as in Step 5) it follows that $u_l(\tau(x)) = \omega(x) < \theta_l$. Using also the AB-entropy conditions (5.1.13) we then deduce that $u_r(\tau(x)) = \pi^l_{r,-}(\omega(x)) < \theta_r$. But this implies that the maximal backward characteristic for u from $(0, \tau(x))$ intersects in $\{x > 0\}$ the minimal backward characteristic for u from (R, T), which again gives a contradiction, and thus completes the proof of (5.3.37).

Step 12. $\big(\mathsf{L} < 0 < \mathsf{R}$, proof of (5.3.38): $\omega(x) = B$ in $]0, \mathsf{R}[\,,\ \ \mathsf{R} \in]0, T \cdot f_r'(B)[\,$, of $B \neq \theta_r$, and of (5.3.39): $\omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r] \leq \omega(\mathsf{R}-)\big)$.
Towards a proof of (5.3.38), first notice that the maximal backward characteristic for u from (L, T) must intersect the interface $x = 0$ at a time

$$\tau_{\mathsf{L}} \doteq T - \frac{\mathsf{L}}{f_l'(\mathsf{L}+)} < T - \frac{\mathsf{R}}{f_r'(\mathsf{R}-)}. \tag{5.4.179}$$

In fact otherwise, we could consider a point $x > \mathsf{L}$ of continuity for ω sufficiently close to L, so that $\tau(x) > T - \mathsf{R}/f_r'(\mathsf{R}-)$. But then, by the analysis in Step 11, the maximal backward characteristic for u from $(0, \tau(x))$ would intersect in $\{x > 0\}$ the minimal backward characteristic for u from (R, T), thus giving a contradiction. Next, observe that at any point $x \in]0, \mathsf{R}[$ of continuity for ω we have $\omega(x) \geq \theta_r$, since otherwise the backward characteristic for u from (x, T) would intersect in $\{x > 0\}$ the minimal backward characteristic for u from the point (R, T), which gives a contradiction. By the AB-entropy conditions, and because of the strict monotonicity of the map

$$\tau(x) = T - \frac{x}{f_r'(\omega(x))}, \qquad x \in]0, \mathsf{R}[\,,$$

(that can be established with the same limiting procedure of Steps 5, 10), it then follows that $\omega(x) \geq B$ for all $x \in]0, \mathsf{R}[\,$. Assume now that $\omega(x) > B$ for some $x \in]0, \mathsf{R}[$ continuity point for ω. Observe that, because of the non crossing property of characteristics in $\{x > 0\}$, and by (5.4.179), the backward characteristic for u from (x, T) impacts the interface $x = 0$ at time

$$\tau(x) \geq T - \frac{\mathsf{R}}{f_r'(\mathsf{R}-)} > \tau_{\mathsf{L}}. \tag{5.4.180}$$

Then, relying on the strict monotonicity of the map τ, and using the AB-entropy conditions (5.1.13), we deduce that $u_r(\tau(x)) = \omega(x) > B$, $u_l(\tau(x)) = \pi^r_{l,+}(\omega(x)) > \theta_l$. But, because of (5.4.180), this implies that the minimal backward characteristic for u from $(0, \tau(x))$ intersects in $\{x < 0\}$ the maximal backward characteristic for u from (L, T) which again gives a contradiction. Hence we have shown that $\omega(x) = B$ in $]0, \mathsf{R}[$. By definition (5.3.2) of R, and because of (5.4.180), this implies that $\mathsf{R} \in]0, T \cdot f_r'(B)[$, and thus completes the proof of (5.3.38). This also shows that we must have $B \neq \theta_r$. Furthermore, we can derive (5.3.39) with exactly the same arguments contained in the proof of (5.3.56) in Step 3.

Step 13. $\big(\mathsf{L} < 0 < \mathsf{R}$, proof of (5.3.36): $(f_l')^{-1}(\frac{x}{T - \boldsymbol{\tau}[\mathsf{R}, B, f_r]}) \leq \omega(x) < \theta_l$ in $]\mathsf{L}, 0[\ \big)$.
Let $\mathsf{L}_n, \mathsf{R}_n$ be the constants defined at (5.4.145) as in (5.3.2) and, according with (5.3.4), define

$$\widetilde{\mathsf{L}}_n \doteq \big(T - \boldsymbol{\tau}[\mathsf{R}_n, B_n, f_r]\big) \cdot f_l'(A_n). \tag{5.4.181}$$

Since (5.4.132), (5.4.133) imply $\lim_n f_l'(A_n) = 0$, and recalling that R_n are bounded (see Step 1), it follows that $\lim_n \widetilde{\mathsf{L}}_n = 0$. Thus, recalling also that the limit $\widehat{\mathsf{R}}$ of a subsequence of $\{\mathsf{R}_n\}_n$ satisfies (5.4.148), we may assume that $\mathsf{L}_n < \widetilde{\mathsf{L}}_n < 0 < \mathsf{R}_n$ for n sufficiently large. Then, applying (5.3.14) or (5.3.16) of Theorem 5.3.3 for ω_n in the case of the non critical connections (A_n, B_n), we deduce that

$$\omega_n(x) \leq A_n \qquad \forall\, x \in]\widetilde{\mathsf{L}}_n, \mathsf{L}_n[\,, \qquad\qquad \omega_n(\widetilde{\mathsf{L}}_n \pm) = A_n\,. \tag{5.4.182}$$

Therefore, by (5.4.181) the backward characteristic for u_n starting at $(\widetilde{\mathsf{L}}_n, T)$ reaches the interface $x = 0$ at time $\boldsymbol{\tau}_n \doteq \boldsymbol{\tau}[\mathsf{R}_n, B_n, f_r]$. In turn, this implies that, for every $x \in]\mathsf{L}_n, \widetilde{L}_n[$ point of continuity for ω_n, the backward characteristic starting at (x, T) must cross the interface $x = 0$ at a time smaller or equal than $\boldsymbol{\tau}_n \doteq \boldsymbol{\tau}[\mathsf{R}_n, B_n, f_r]$, since otherwise it would intersect the backward characteristic for u_n starting at $(\widetilde{\mathsf{L}}_n, T)$ in the domain $\{x < 0\}$, which gives a contradiction. Thus we have

$$T - \frac{x}{f_l'(\omega_n(x))} \leq \boldsymbol{\tau}_n\,, \tag{5.4.183}$$

for every $x \in]\mathsf{L}_n, \widetilde{L}_n[$ point of continuity for ω_n. On the other hand, recall that by Lemma 5.2.1 the map $\mathsf{R} \mapsto \boldsymbol{y}[\mathsf{R}, B, f_r](T) < 0$ is strictly increasing, and hence by (5.2.32) the map $\mathsf{R} \to \boldsymbol{\tau}[\mathsf{R}, B, f_r]$ is strictly decreasing. Therefore, since $\boldsymbol{\tau}[\mathsf{R}, B, f_r]$ depends continuously on the parameters R, B (see § 5.2.4), and because of (5.4.148), we deduce that

$$\lim_n \boldsymbol{\tau}_n = \boldsymbol{\tau}[\,\widehat{\mathsf{R}}, B, f_r] \leq \boldsymbol{\tau}[\mathsf{R}, B, f_r]\,. \tag{5.4.184}$$

Hence, taking the limit in (5.4.183) as $n \to \infty$, and relying again on (5.4.143), (5.4.148), we derive

$$T - \frac{x}{f_l'(\omega(x))} \leq \boldsymbol{\tau}[\mathsf{R}, B, f_r] \qquad \text{for a.e. } x \in]\mathsf{L}, 0[\,. \tag{5.4.185}$$

In turn, the inequality (5.4.185) yields the first inequality in (5.3.36). On the other hand, if $\omega(x) \geq \theta_l$ for some $x \in]\mathsf{L}, 0[$, with the same arguments of Step 11 one deduces that the backward characteristic for u starting from (x, T) must intersect in $\{x < 0\}$ the maximal backward characteristic for u from the point (L, T), which gives a contradiction. This shows that also the second inequality in (5.3.36) is satisfied. Therefore, the proof of the existence of $\omega(0\pm)$ and that ω satisfies the conditions (i)-(ii) of Theorem 5.3.9 is completed. This concludes the proof of the implication (1) $\Rightarrow$ (3) for critical connections.

5.4.8. Part 3.c - (3) $\Rightarrow$ (1) for critical connections. In this Subsection we rely on the fact that Theorem 5.3.17 holds for connections which are non critical, and in particular we know (by § 5.4.2, 5.4.3, 5.4.4) that Theorems 5.3.3, 5.3.11, 5.3.14, are verified for non critical connections. We will prove that if $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ satisfies the conditions of Theorem 5.3.9, 5.3.11, or of Theorem 5.3.14, then $\omega \in \mathcal{A}^{[AB]}(T)$ also for a critical connection (A, B) satisfying (5.4.132).

Step 1. Given an element ω of the set $\mathscr{A}^{\mathsf{L},\mathsf{R}}$ in (5.3.8), assuming that:

- if $\mathsf{L} < 0$, $\mathsf{R} > 0$ or viceversa, ω satisfies the conditions of Theorem 5.3.9;
- if $\mathsf{L} = 0$, $\mathsf{R} > 0$ or viceversa, ω satisfies the conditions of Theorem 5.3.11;
- if $\mathsf{L} = 0$, $\mathsf{R} = 0$, ω satisfies the conditions of Theorem 5.3.14;

we will construct a sequence $\{\omega_n\}_n$ of suitable perturbations of ω with the property that:

$$\omega_n \in \mathcal{A}^{[A_n B_n]}(T) \quad \forall\, n, \qquad\qquad \omega_n \to \omega \quad \text{in} \;\; \mathbf{L}^{\mathbf{1}}_{\text{loc}}(\mathbb{R}), \tag{5.4.186}$$

for a sequence of non critical connections $\{(A_n, B_n)\}_n$ satisfying (5.4.133) and

$$A_n < A, \qquad\qquad B_n > B \qquad\quad \forall\, n\,. \tag{5.4.187}$$

The conditions in (5.4.186) in turn will imply that $\omega \in \mathcal{A}^{[AB]}(T)$. In fact, by the validity of Theorem 5.3.17 in the non critical case, and because of (5.4.186), it holds

$$\omega_n = \mathcal{S}_T^{[A_n B_n]+} \circ \mathcal{S}_T^{[A_n B_n]-} \omega_n \qquad \forall\, n\,. \tag{5.4.188}$$

On the other hand, relying on the stability property (iv) of Theorem 5.1.8, and thanks to (5.4.186), one finds as in § 5.4.6 that

$$S_T^{[A_n B_n]+} \circ \mathcal{S}_T^{[A_n B_n]-} \omega_n \quad \to \quad \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-} \omega \qquad \text{in} \quad \mathbf{L}^1_{\text{loc}}(\mathbb{R})\,. \tag{5.4.189}$$

Hence, combining together (5.4.186), (5.4.188), (5.4.189), we derive

$$\omega = \mathcal{S}_T^{[AB]+} \circ \mathcal{S}_T^{[AB]-} \omega\,,$$

which clearly yields $\omega \in \mathcal{A}^{[AB]}(T)$. Therefore, to establish the implication (3) $\Rightarrow$ (1) of Theorem 5.3.17 for non critical connections, it remains to produce a family $\{\omega_n\}_n$ that satisfies (5.4.186). We shall construct such perturbations of $\omega \in \mathscr{A}^{\mathsf{L},\mathsf{R}}$ as suitable "*(A_n, B_n) admissible envelopes*" of ω.

We will first consider in Steps 2-8 the case $\mathsf{L} = 0$, $\mathsf{R} \geq 0$, while the symmetric case $\mathsf{L} < 0$, $\mathsf{R} = 0$ is entirely similar. Next, we will consider separately the case $\mathsf{L} < 0, \mathsf{R} > 0$, in Step 9.

Step 2. We shall assume throughout Steps 2-8 that

$$\mathsf{L} = \mathsf{L}[\omega, f_l] = 0, \qquad\qquad \mathsf{R} = \mathsf{R}[\omega, f_r] \geq 0\,, \tag{5.4.190}$$

and that: ω satisfies the conditions of Theorem 5.3.11 if $\mathsf{R} > 0$; ω satisfies the conditions of Theorem 5.3.14 if $\mathsf{R} = 0$.

We will perturb ω to obtain an attainable profile ω_n for the (A_n, B_n) connection by:

- shifting ω on the right of $x = 0$ by a size $\delta_{1,n}$, and on the left of $x = 0$ by a size $\delta_{2,n}$;
- choosing $\delta_{1,n}$ so to satisfy the admissibility condition (5.3.56) at $x = \mathsf{R} + \delta_{1,n}$ (if $\mathsf{R} + \delta_{1,n} > 0$);
- lifting ω to the value B_n of the connection when it is below, in the interval $]0, \mathsf{R} + \delta_{1,n}[$, so to satisfy the admissibility condition (5.3.54) (if $\mathsf{R} + \delta_{1,n} > 0$);

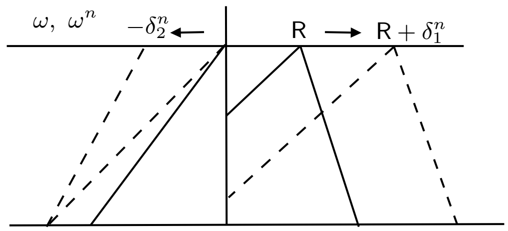

FIGURE 19. The "candidate" characteristics of ω and of its admissible A^nB^n envelopes ω^n (dashed).

- inserting a profile of a rarefaction in the interval $]-\delta_{2,n},0[$, so to satisfy the Lax-type admissibility condition (5.3.55) at $x=0$.

Namely, consider the function

$$\omega_n(x) \doteq \begin{cases} \omega(x-\delta_{1,n}), & \text{if} \quad x \geq \mathsf{R}+\delta_{1,n}, \\ \max\{\omega(\mathsf{R}+), B_n\}, & \text{if} \quad x \in]\mathsf{R}, \mathsf{R}+\delta_{1,n}[, \\ \max\{\omega(x), B_n\}, & \text{if} \quad x \in]0, \mathsf{R}[, \\ (f_l')^{-1}\bigg(\dfrac{x+T\cdot f_l'\big(\max\{\omega(0-),\ \overline{A}_n\}\big)}{T}\bigg), & \text{if} \quad x \in]-\delta_{2,n},0[, \\ \omega(x+\delta_{2,n}), & \text{if} \quad x \leq -\delta_{2,n}. \end{cases} \tag{5.4.191}$$

with

$$\begin{aligned} \delta_{1,n} &\doteq \inf\Big\{\delta \geq 0 \quad : \quad \text{either} \quad \mathsf{R}+\delta \geq T\cdot f_r'(B_n), \\ &\qquad\qquad \text{or} \quad \mathsf{R}+\delta < T\cdot f_r'(B_n) \quad \text{and} \quad \omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}+\delta, B_n, f_r]\Big\}, \\ \delta_{2,n} &\doteq T\cdot f_l'\big(\max\{\omega(0-),\ \overline{A}_n\}\big) - T\cdot f_l'(\omega(0-)), \end{aligned} \tag{5.4.192}$$

where $\overline{A}_n$ is defined as in (5.1.17). Recalling the definitions (5.3.2), and because of (5.4.190), we deduce that

$$\mathsf{L}_n \doteq \mathsf{L}[\omega_n, f_l] = 0, \qquad \mathsf{R}_n \doteq \mathsf{R}[\omega_n, f_r] = \mathsf{R}+\delta_{1,n} \qquad \forall\ n\,. \tag{5.4.193}$$

Notice that the assumption that ω satisfies conditions (ii)' of of Theorem 5.3.11 or conditions (ii) of Theorem 5.3.14, together with (5.4.133), imply that

$$\lim_{n\to\infty} \delta_{1,n} = \lim_{n\to\infty} \delta_{2,n} = 0\,. \tag{5.4.194}$$

In fact, if $\mathsf{R} \leq T\cdot f_r'(B)$, relying on conditions (5.3.56), (5.3.60) of Theorem 5.3.11 or on condition (5.3.64) of Theorem 5.3.14 we deduce that $\omega(\mathsf{R}+) \leq \boldsymbol{u}[\mathsf{R}, B, f_r]$. Moreover, we know by Remark 5.2.2 that $\boldsymbol{u}[\cdot,\cdot,f_r]$ is continuous in the first two entries. Therefore, because of (5.4.133), we derive from definition (5.4.192) that, if $\mathsf{R} \leq T\cdot f_r'(B)$, then $\lim_n \delta_{1,n} = 0$. On the other hand, if $\mathsf{R} > T\cdot f_r'(B)$, then it follows from definition (5.4.192) and (5.4.133), that $\delta_{1,n} = 0$ for sufficiently large n. Next, observe that, since $\mathsf{L} = 0$, by definition (5.3.2) one has $\omega(0-) \geq \theta_l$. On the other hand by assumptions (5.4.132), (5.4.133) it follows that $\lim_n \overline{A}_n = \theta_l$. Therefore, by definition (5.4.192) we deduce that $\lim_n \delta_{2,n} = 0$.

Because of (5.4.133), and relying on conditions (ii) of Theorem 5.3.11 or of Theorem 5.3.14, we deduce that the limit (5.4.194) implies the $\mathbf{L}^1_{\mathrm{loc}}$ convergence of ω_n to ω as $n \to \infty$. Hence, in order to show that ω_n satisfy (5.4.186) it remains to prove that $\omega_n \in \mathcal{A}^{[A_n B_n]}(T)$ for all n. Since we are assuming in particular the validity of the implication (2) $\Rightarrow$ (1) of Theorem 5.3.17 for non critical connections, in order to establish $\omega_n \in \mathcal{A}^{[A_n B_n]}(T)$ it will be sufficient to show that: if $\mathsf{R}_n = 0$ then ω_n satisfies conditions (i)-(ii) of Theorem 5.3.14; if $\mathsf{R}_n > 0$ then ω_n satisfies conditions (i)'-(ii)' of Theorem 5.3.11. This is established in the steps below distinguishing the cases where $\mathsf{R} = 0$ or $\mathsf{R} > 0$ and $\mathsf{R}_n = 0$ or $\mathsf{R}_n > 0$. Notice that, by definition (5.4.191), we always have

$$\omega_n(\mathsf{R}_n-) \geq \omega_n(\mathsf{R}_n+)\,. \tag{5.4.195}$$

Step 3. $\big(\mathsf{R}_n > 0$, $\mathsf{R} = 0$, proof that ω_n satisfies condition (ii)' of Theorem 5.3.11$\big)$.
By definition (5.4.191) one has

$$\omega_n(0-) = \max\{\omega(0-), \overline{A}_n\}\,, \qquad \omega_n(0+) = \max\{\omega(0+), B_n\}, \qquad \omega_n(\mathsf{R}_n+) = \omega(0+)\,, \tag{5.4.196}$$

and

$$\omega_n(x) \geq B_n \qquad \forall x \in \,]0, \mathsf{R}_n[\,, \tag{5.4.197}$$

while definition (5.4.192), together with (5.4.193), (5.4.196), (5.4.197), and (5.2.8) with $f = f_r$, yield

$$\mathsf{R}_n \in \,]0, T \cdot f_r'(B_n)[\qquad \Longrightarrow \qquad \omega_n(\mathsf{R}_n+) \leq \boldsymbol{u}[\mathsf{R}_n, B_n, f_r] \leq \omega_n(\mathsf{R}_n-)\,. \tag{5.4.198}$$

Since $\pi^r_{l,+}(B_n) = \overline{A}_n$, from (5.4.196) we deduce

$$\pi^r_{l,+}(\omega_n(0+)) = \max\{\pi^r_{l,+}(\omega(0+)), \overline{A}_n\}. \tag{5.4.199}$$

Hence (5.4.196), (5.4.199), imply

$$\omega_n(0-) \geq \pi^r_{l,+}(\omega_n(0+))\,. \tag{5.4.200}$$

Therefore, if $\mathsf{R}_n > 0$ and $\mathsf{R} = 0$, then conditions (5.4.195), (5.4.197), (5.4.198), (5.4.200) show that ω_n satisfies condition (ii)' of Theorem 5.3.11.

Step 4. $\big(\mathsf{R}_n > 0$, $\mathsf{R} > 0$, proof that ω_n satisfies condition (ii)' of Theorem 5.3.11$\big)$.
By definition (5.4.191) one has

$$\omega_n(0-) = \max\{\omega(0-), \overline{A}_n\}\,, \qquad \omega_n(0+) = \max\{\omega(0+), B_n\}, \qquad \omega_n(\mathsf{R}_n+) = \omega(\mathsf{R}+)\,, \tag{5.4.201}$$

and

$$\omega_n(x) \geq B_n \qquad \forall x \in \,]0, \mathsf{R}_n[\,, \tag{5.4.202}$$

while definition (5.4.192), together with (5.4.193), (5.4.201), (5.4.202), and (5.2.8) with $f = f_r$, yield the implication (5.4.198). Since we are assuming that ω satisfies condition (5.3.55) of Theorem 5.3.11, relying on (5.4.201) we deduce as in Step 4 that (5.4.199), (5.4.200) hold. Therefore, if $\mathsf{R}_n > 0$ and $\mathsf{R} > 0$, then (5.4.195), (5.4.198), (5.4.200), (5.4.202) show that ω_n satisfies condition (ii)' of Theorem 5.3.11.

Step 5. $\big(\mathsf{R}_n = \mathsf{R} = 0$, proof that ω_n satisfies condition (ii) of Theorem 5.3.14$\big)$.
By definition (5.4.191), we have

$$\omega_n(0-) = \max\{\omega(0-),\ \overline{A}_n\} \geq \overline{A}_n, \qquad \omega_n(0+) = \omega(0+)\,, \tag{5.4.203}$$

while definition (5.4.192) yields $\omega(0+) \leq \boldsymbol{u}[0, B_n, f_r]$. Since by Remark 5.2.2 we have $\boldsymbol{u}[0, B_n, f_r] = \overline{B}_n$ ($\overline{B}_n$ as in (5.1.17)), it follows that

$$\omega_n(0+) \leq \overline{B}_n. \tag{5.4.204}$$

Moreover, by virtue of (5.4.203) we deduce (5.4.200). Hence, if $\mathsf{R}_n = 0$ and $\mathsf{R} = 0$, then conditions (5.4.200), (5.4.203), (5.4.204) show that ω_n satisfies condition (ii) of Theorem 5.3.14.

Step 6. $\big(\mathsf{R}_n > 0$, $\mathsf{R} \geq 0$, proof that ω_n satisfies (5.3.52) of Theorem 5.3.11$\big)$.
Since we are assuming that ω satisfies either the Oleinik estimates (5.3.62) of Theorem 5.3.14 (in case $\mathsf{R} = 0$), or the Oleinik estimates (5.3.52) of Theorem 5.3.11 (in case $\mathsf{R} > 0$), computing the Dini derivative of ω_n in (5.4.191), we find

$$\begin{aligned}
&D^+\omega_n(x) = D^+\omega(x - \delta_{1,n}) \leq \frac{1}{T \cdot f_r''(\omega(x - \delta_{1,n}))} = \frac{1}{T \cdot f_r''(\omega_n(x))} \qquad \forall\, x > \mathsf{R}_n\,,\\
&D^+\omega_n(x) = D^+\omega(x + \delta_{2,n}) \leq \frac{1}{T \cdot f_l''(\omega(x + \delta_{2,n}))} = \frac{1}{T \cdot f_l''(\omega_n(x))} \qquad \forall\, x < -\delta_{2,n}\,,\\
&D^+\omega_n(x) = \frac{1}{T \cdot f_l''(\omega_n(x))} \qquad \forall\, x \in [-\delta_{2,n}, 0[\,.
\end{aligned} \tag{5.4.205}$$

Observing that ω_n is continuous at $x = -\delta_{2,n}$, we deduce from (5.4.205) that ω_n satisfies the Oleinik estimates (5.3.52) of Theorem 5.3.11.

Step 7. $\big(\mathsf{R}_n > 0$, $\mathsf{R} \geq 0$, proof that ω_n satisfies (5.3.53) of Theorem 5.3.11$\big)$.
Observe that by definition (5.4.191) ω_n is constant in $]\mathsf{R}, \mathsf{R}_n[$. Therefore, since it holds (5.4.195), in order to show that ω_n satisfies the estimate (5.3.53) on $]0, \mathsf{R}_n[$ it will be sufficient to show that (5.3.53) is verified on $]0, \mathsf{R}[$, assuming that $\mathsf{R} > 0$.

As observed in Step 5 of § 5.4.7, the assumption that ω satisfies condition (5.3.53) of Theorem 5.3.11 is equivalent to the fact that the maps

$$\phi(x) \doteq -\tau(x) \cdot f_l' \circ \pi_{l,+}^r(\omega(x)), \qquad \tau(x) \doteq T - \frac{x}{f_r'(\omega(x))}\,, \qquad x \in]0, \mathsf{R}[\,. \tag{5.4.206}$$

are, respectively, nondecreasing and decreasing. Then consider the corresponding maps for ω_n

$$\phi_n(x) \doteq -\tau_n(x) \cdot f_l' \circ \pi_{l,+}^r(\omega_n(x)), \qquad \tau_n(x) \doteq T - \frac{x}{f_r'(\omega_n(x))}\,, \qquad x \in]0, \mathsf{R}[\,, \tag{5.4.207}$$

and compare their values in two points $0 < x_1 < x_2 < \mathsf{R}$, of continuity for ω and ω_n:
- if $\omega_n(x_i) = \omega(x_i)$ for $i = 1, 2$, then one clearly has that $\phi_n(x_1) = \phi(x_1) \leq \phi(x_2) = \phi_n(x_2)$, $\tau_n(x_1) = \tau(x_1) > \tau(x_2) = \tau_n(x_2)$;
- if $\omega_n(x_i) \neq \omega(x_i)$ for $i = 1, 2$, then by definition (5.4.191) we have $\omega_n(x_i) = B_n$ for $i = 1, 2$ and therefore one has $\phi_n(x_1) < \phi_n(x_2)$, $\tau_n(x_1) > \tau_n(x_2)$;
- if $\omega_n(x_1) = \omega(x_1)$ and $\omega_n(x_2) \neq \omega(x_2)$, then by definition (5.4.191) we have $\omega_n(x_1) \geq B_n$, $\omega_n(x_2) = B_n$, which implies $f_r'(\omega_n(x_2)) \leq f_r'(\omega_n(x_1))$. Moreover, since $\omega(x) \geq \theta_r$, $\omega_n(x) \geq \theta_r$, it follows that $f_l' \circ \pi_{l,+}^r(\omega_n(x_1)) \geq f_l' \circ \pi_{l,+}^r(\omega_n(x_2))$. Hence, we derive that $\phi_n(x_1) \leq \phi_n(x_2)$ $\tau_n(x_1) > \tau_n(x_2)$;
- if $\omega_n(x_1) \neq \omega(x_1)$ and $\omega_n(x_2) = \omega(x_2)$, then by definition (5.4.191) we have $\omega_n(x_1) =$

$B_n > \omega(x_1)$, which implies $f_r'(\omega_n(x_1)) > f_r'(\omega(x_1))$. Notice that by definition (5.3.2) of R it follows that $\omega(x_1) \geq \theta_r$. Since also $\omega_n(x_1) \geq \theta_r$, we deduce that $f_l' \circ \pi_{l,+}^r(\omega_n(x_1)) > f_l' \circ \pi_{l,+}^r(\omega(x_1))$. Thus, it follows that $\phi_n(x_1) < \phi(x_1) \leq \phi(x_2) = \phi_n(x_2)$, $\tau_n(x_1) > \tau(x_1) > \tau(x_2) = \tau_n(x_2)$.
Hence, extending the above estimates to the right limits of ω_n in its points of discontinuity, we have shown that it holds true

$$\phi_n(x_1) \leq \phi_n(x_2), \qquad \tau_n(x_1) > \tau_n(x_2) \qquad \forall\ 0 < x_1 < x_2 < \mathsf{R}. \tag{5.4.208}$$

In turn, the monotonicity (5.4.208) of ϕ_n, τ_n is equivalent to the fact that ω_n satisfies (5.3.53), by the same arguments of Step 5 of § 5.4.7.

Step 8. $\big(\mathsf{R}_n = \mathsf{R} = 0$, proof that ω_n satisfies (5.3.62) of Theorem 5.3.14$\big)$.
The proof is entirely similar to the one of Step 6, under the assumption that ω satisfies the Oleinik estimates (5.3.62) of Theorem 5.3.14.

Step 9. Finally, let us assume

$$\mathsf{L} = \mathsf{L}[\omega, f_l] < 0, \qquad \mathsf{R} = \mathsf{R}[\omega, f_r] > 0\,, \tag{5.4.209}$$

and (because of (5.4.132)) that ω satisfies the conditions (i), (ii) of Theorem 5.3.9. Hence, by virtue of (5.4.133) we may assume also that, for n sufficiently large there hold

$$\mathsf{L} < T \cdot f_l'(A_n), \qquad \mathsf{R} < T \cdot f_r'(B_n)\,. \tag{5.4.210}$$

In a similar way to what is done in Step 2, we will perturb ω to obtain an attainable profile ω_n for the (A_n, B_n) connection by:

- shifting ω on the right of $x = 0$ by a size $\delta_{1,n}$
- choosing $\delta_{1,n}$ so to satisfy the admissibility condition (5.3.13) at $x = \mathsf{R} + \delta_{1,n}$;
- dropping ω to the value A_n of the connection when it is above, in the interval $]\mathsf{L}, 0[$, so to satisfy the admissibility condition (5.3.16).

Namely, consider the function

$$\omega_n(x) \doteq \begin{cases} \omega(x - \delta_{1,n}), & \text{if} \quad x \geq \mathsf{R} + \delta_{1,n}, \\ B_n & \text{if} \quad x \in]0, \mathsf{R} + \delta_{1,n}[, \\ \min\{A_n, \omega(x)\} & \text{if} \quad x \in]\mathsf{L}, 0[, \\ \omega(x) & \text{if} \quad x \leq \mathsf{L}, \end{cases} \tag{5.4.211}$$

with

$$\delta_{1,n} \doteq \inf\Big\{\delta \in \mathbb{R} \quad : \quad \boldsymbol{\tau}[\mathsf{R} + \delta, B_n, f_r] = \boldsymbol{\tau}[\mathsf{R}, B, f_r]\Big\}\,. \tag{5.4.212}$$

Notice that the definition (5.4.212) is meaningful since the map $\mathsf{R} \mapsto \boldsymbol{\tau}[\mathsf{R}, B_n, f_r]$ is strictly monotone and continuous, and because the image of the maps

$$\boldsymbol{\tau}[\,\cdot\,, B, f_r] \,:\,]0, T \cdot f'(B)[\;\to\;]0, +\infty[\,,$$
$$\boldsymbol{\tau}[\,\cdot\,, B_n, f_r] \,:\,]0, T \cdot f'(B_n)[\;\to\;]0, +\infty[\,,$$

is the set $]0, T[$ (see § 5.2.4). Then, recalling the definitions (5.3.2), (5.3.4), and because of (5.4.210), we deduce that

$$\mathsf{L}_n \doteq \mathsf{L}[\omega_n, f_l] = \mathsf{L}, \qquad \mathsf{R}_n \doteq \mathsf{R}[\omega_n, f_r] = \mathsf{R} + \delta_{1,n}, \tag{5.4.213}$$

and

$$\widetilde{\mathsf{L}}_n \doteq \widetilde{\mathsf{L}}[\omega_n, f_l] = (T - \boldsymbol{\tau}_n) \cdot f_l'(A_n) = (T - \boldsymbol{\tau}) \cdot f_l'(A_n), \tag{5.4.214}$$

where

$$\boldsymbol{\tau}_n \doteq \boldsymbol{\tau}[\mathsf{R} + \delta_{1,n}, B_n, f_r], \qquad \boldsymbol{\tau} \doteq \boldsymbol{\tau}[\mathsf{R}, B, f_r]. \tag{5.4.215}$$

Relying on (5.4.133) and since $\boldsymbol{\tau}[\mathsf{R}, B, f_r]$ depends continuously on the parameters R, B (see § 5.2.4), one deduces that $\lim_n \delta_{1,n} = 0$, that $\lim_n \widetilde{\mathsf{L}}_n = 0$, and that ω_n converges to ω in $\mathbf{L}^1_{\mathrm{loc}}$ as $n \to \infty$. Hence, as in Step 2 above we conclude that, in order to show the validity of (5.4.186) it remains to prove that ω_n satisfies conditions (i)-(ii) of Theorem 5.3.3.

Assuming that $\widetilde{\mathsf{L}}_n \in]\mathsf{L}, 0[$ for n sufficiently large, in order to show that ω_n satisfies (5.3.16) of Theorem 5.3.3 it will be sufficient to prove that

$$A_n \leq \omega(x) \qquad \forall\, x \in]\widetilde{\mathsf{L}}_n, 0[\,, \qquad A_n \leq \omega(\widetilde{\mathsf{L}}_n-)\,. \tag{5.4.216}$$

To this end observe that, by definition (5.4.214), we have $(f_l')^{-1}(\frac{x}{T-\boldsymbol{\tau}[\mathsf{R},B,f_r]}) > A_n$ for all $x \in]\widetilde{\mathsf{L}}_n, 0[\,$. Thus, the first inequality in (5.3.36) satisfied by ω implies that $\omega(x) > A_n$ for all $x \in]\widetilde{\mathsf{L}}_n, 0[\,$, which proves the first condition in (5.4.216). Next observe that, since $\widetilde{\mathsf{L}}_n \in]\mathsf{L}, 0[$, from the first inequality in (5.3.36) and by definition (5.4.214) it follows

$$f_l'(\omega(\widetilde{\mathsf{L}}_n-)) \geq \frac{\widetilde{\mathsf{L}}_n}{T - \boldsymbol{\tau}} \geq f_l'(A_n),$$

which implies $\omega(\widetilde{\mathsf{L}}_n-) \geq A_n$. This completes the proof of (5.4.216) and thus that ω_n satisfies (5.3.16). The verification that ω_n satisfies the remaining conditions in (i)-(ii) of Theorem 5.3.3 is entirely similar to the one performed in Steps 4, 6, 7 above, and is accordingly omitted. This concludes the proof of the implication (3) ⇒ (1) for critical connections.

REMARK 5.4.2. Whenever Tot.Var.$(\omega) < +\infty$, the perturbed profiles ω_n approximating ω constructed in Step 2 and in Step 9 of § 5.4.8 may possibly have larger total variation than the one of ω. However, ω_n have always local bounded variation, even in the case where Tot.Var.$(\omega) = +\infty$. In fact, assuming (5.4.190) and that ω satisfies the conditions of Theorem 5.3.14, suppose that ω has unbounded total variation on a right neighborhood of $x = 0$. Then, letting $\{(A_n, B_n)\}_n$ be a sequence of non critical connections satisfying (5.4.133), (5.4.187), there should exist a sequence of positive values $\rho_n \downarrow 0$, so that

$$\omega(x) \leq B_n, \qquad \forall\, x \in]0, \rho_n]\,, \tag{5.4.217}$$

for all n sufficiently large. If this is not the case, then there should exist $\overline{\rho} > 0$ and $\overline{n}$ so that $\omega(x) \geq B_{\overline{n}} > \theta_r$ for all $x \in]0, \overline{\rho}]$. But this in turn would yield uniform upper bounds on $D^+\omega$ (and hence on the total variation of ω as well) on bounded subsets K of $[0, +\infty[$, with the same type of analysis of § 5.4.3. Therefore, because of (5.4.217), by definition (5.4.191) we have

$$\omega_n(x) = B_n, \qquad \forall\, x \in]0, \rho_n]\,, \tag{5.4.218}$$

for all n sufficiently large. The property (5.4.218) has precisely the effect to cut the possible large oscillation of ω occurring in a right neighborhood of $x = 0$, and hence to ensure that Tot.Var.$(\omega_n,\, K) < +\infty$ for all n large. Clearly, we will have that $\lim_n$ Tot.Var.$(\omega_n,\, K) = +\infty$. With entirely similar arguments one can show that, if ω satisfies the conditions (i),

(ii) of Theorem 5.3.9, then the profile ω_n defined by (5.4.211) has always local bounded variation.

5.5. BV bounds for AB-entropy solutions

We collect in this section the BV bounds for solutions, and for the flux of the solutions, that arise as a corollary of our analysis.

PROPOSITION 5.5.1. *In the same setting of Theorem 5.1.8, for every $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, and for any bounded set $K \subset \mathbb{R}$, the following properties are verified.*

(i) For any non critical connection (A,B), there exists a constant $C_1 = C_1(A,B,\|u_0\|_{\mathbf{L}^\infty},K) > 0$ such that it holds true

$$\text{Tot.Var.}\big(\mathcal{S}_t^{[AB]+}u_0,\, K\big) \leq \frac{C_1}{t} \qquad \forall\, t>0\,. \tag{5.5.1}$$

In particular, any attainable profile $\omega \doteq \mathcal{S}_T^{[AB]+}u_0$, $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, $T>0$, enjoy the property (H) *stated in § 5.4.1-***Part 1***.*

(ii) There exists a constant $C_2 = C_2(\|u_0\|_{\mathbf{L}^\infty},K) > 0$ such that, for any connection (A,B), it holds true

$$\begin{aligned}&\text{Tot.Var.}\big(f_l\circ\mathcal{S}_t^{[AB]+}u_0,\, K\cap\,]-\infty,0]\big) \leq \frac{C_2}{t},\\ &\text{Tot.Var.}\big(f_r\circ\mathcal{S}_t^{[AB]+}u_0,\, K\cap[0,+\infty[\big) \leq \frac{C_2}{t},\end{aligned}\qquad \forall\, t>0\,, \tag{5.5.2}$$

where the inequalities are understood to be verified whenever $K\cap\,]-\infty,0]\neq\emptyset$, or $K\cap[0,+\infty[\,\neq\emptyset$, respectively.

PROOF. Since $\mathcal{S}_t^{[AB]+}u_0 \in \mathcal{A}^{[AB]}(t)$ and thanks to the implication (1) ⇒ (3) of Theorem 5.3.17, we know that $\mathcal{S}_t^{[AB]+}u_0$ satisfies the conditions stated in Theorem 5.3.3, 5.3.9, 5.3.11, or 5.3.14, that cover all possible cases. We divide the proof in four steps.

Step 1. *(proof of (i)).*
In the case of a non critical connection (A,B), it is well known that for initial data $u_0 \in BV(\mathbb{R})$, one has $\mathcal{S}_t^{[AB]+}u_0 \in BV(\mathbb{R})$ for all $t>0$ (see [**64**, Lemma 8] and [**1**, Theorem 2.13-(iii)]). On the other hand, for initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, we know that $\mathcal{S}_t^{[AB]+}u_0$ satisfies the Oleinik-type inequalities stated in Theorem 5.3.3, 5.3.11, or 5.3.14. Thus, since (A,B) is a non critical connection, by the analysis in § 5.4.3 we deduce that $D^+(\mathcal{S}_t^{[AB]+}u_0)$ satisfies one-sided uniform upper bounds as the ones provided by (5.4.85). In turn, such bounds yield the existence of uniform bounds on the total increasing variation (and hence on the total variation as well) of $S_t^{[AB]+}u_0$ on bounded subsets K of $[0,+\infty[$, which depend on the connection (A,B), on the set K, and on $\|u_0\|_{\mathbf{L}^\infty}$. By similar arguments we derive bounds on the total variation of $S_t^{[AB]+}u_0$ on bounded subsets of $]-\infty,0]$, which yields (5.5.1), completing the proof of (i).

Step 2. *(proof of (ii) when $S_t^{[AB]+}u_0$ satisfies the conditions of Theorem 5.3.14).*
Since $S_t^{[AB]+}u_0$ satisfies the Oleinik-type inequalities (5.3.62) of Theorem 5.3.14, we immediately deduce a uniform bound on the total increasing variation of $S_t^{[AB]+}u_0$ on bounded

sets, which does not depend on the values $f_l'(A), f_r'(B)$. In turn, such bounds yield the existence of uniform bounds on the total increasing variation (and hence on the total variation as well) of $S_t^{[AB]+}u_0$ on bounded subsets K of $[0,+\infty[$, which depend on the set K and on $\|u_0\|_{\mathbf{L}^\infty}$. By similar arguments we derive bounds on the total variation of $S_t^{[AB]+}u_0$ on bounded subsets of $]-\infty,0]$, which yields (5.5.1), with a constant C_1 that depends only on the set K and on $\|u_0\|_{\mathbf{L}^\infty}$. In turn, (5.5.1) yields (5.5.2) relying on the Lipschitzianity of f_l, f_r on the set $[-M,M]$, with $M \doteq \|u_0\|_{\mathbf{L}^\infty}$. This completes the proof of (ii) in the case where $S_t^{[AB]+}u_0$ satisfies the conditions stated in Theorem 5.3.14.

Step 3. *(proof of (ii) when $S_t^{[AB]+}u_0$ satisfies the conditions of Theorem 5.3.11).*
To fix the ideas, we assume that $\omega \doteq S_t^{[AB]+}u_0$ satisfies the inequalities (i)' and the pointwise constraints-(ii)' stated in Theorem 5.3.11. Notice that, by the same arguments of above, (5.3.52) yields the estimate (5.5.1) (and hence also (5.5.2)) for bounded set $K \subset]-\infty,0]$ or $K \subset [\mathsf{R},+\infty[$. Then consider a set $K \subset [0,\mathsf{R}]$, with $\mathsf{R} = \mathsf{R}[\omega, f_r]$ defined as in (5.3.2), and assume that the inequalities (5.3.53), (5.3.54), are satisfied. Observe that, by the uniform convexity (25) of f_l, f_r, we have

$$\frac{f_l'' \circ \pi_{l,+}^r(u)}{\big[f_l' \circ \pi_{l,+}^r(u)\big]^2} \geq c_1, \qquad f_r''(u) \geq c_1, \qquad \forall\ |u| \leq \|\omega\|_{\mathbf{L}^\infty}\,, \tag{5.5.3}$$

for some constant $c_1 > 0$ depending on $\|\omega\|_{\mathbf{L}^\infty}$. Moreover, by definition (5.3.2) of R it holds true

$$t \cdot f_r'(\omega(x)) > x \qquad \forall\ x \in [0,\mathsf{R}[\,, \quad t > 0\,. \tag{5.5.4}$$

Hence, recalling the definition (5.3.6) of the function h, and relying on (5.3.53), (5.3.54), (5.5.3), (5.5.4), we derive

$$\begin{aligned} D^+(f_r \circ \omega)(x) &= f_r'(\omega(x))\, D^+\omega(x) \\ &\leq f_r'(\omega(x))\, h[\omega, f_l, f_r](x) \\ &\leq \frac{[f_r'(\omega(x))]^2}{c_1\,[f_r'(\omega(x))]^2\big(t\cdot f_r'(\omega(x)) - x\big) + c_1\, x} \qquad \forall\ x \in [0,\mathsf{R}[\,. \quad t > 0\,. \end{aligned} \tag{5.5.5}$$

Towards an estimate of (5.5.5), consider the map

$$\Phi(x,t,u) \doteq \begin{cases} \dfrac{[f_r'(u)]^2}{[f_r'(u)]^2\big(t\cdot f_r'(u) - x\big) + x}, & \text{if} \quad u > \theta_r, \\ 0, & \text{if} \quad u = \theta_r, \end{cases} \qquad x \in [0,\mathsf{R}[\,, \quad t > 0\,. \tag{5.5.6}$$

By direct computations one finds

$$\Phi_u(x,t,u) = \frac{f_r'(u)\, f_r''(u)\big(2x - t\,[f_r'(u)]^3\big)}{\Big([f_r'(u)]^2\big(t\cdot f_r'(u) - x\big) + x\Big)^2}\,.$$

Hence, since $f_r'(u) \geq 0$ for all $u \geq \theta_r$, and because $f_r''(u) > 0$ for all u, we deduce that, setting

$$u_{x,t} \doteq (f_r')^{-1}\bigg(\sqrt[3]{\frac{2x}{t}}\bigg), \tag{5.5.7}$$

for all $x, t > 0$ it holds true

$$u_{x,t} > \theta_r, \qquad \Phi_u(x,t,u) \begin{cases} \geq 0 & \text{if} \quad u \in [\theta_r, u_{x,t}], \\ \leq 0 & \text{if} \quad u \geq u_{x,t} \,. \end{cases} \tag{5.5.8}$$

In turn, (5.5.7), (5.5.8) imply that $u_{x,t}$ is a point of global maximum for the map $u \mapsto \Phi(x,t,u)$, $u \geq \theta_r$. On the other hand, because of (5.5.8) we have

$$t \cdot f_r'(u_{x,t}) > x \quad \Longrightarrow \quad x < \sqrt{2}\, t \,. \tag{5.5.9}$$

Thus we find

$$\Phi(x,t,u) \leq \Phi(x,t,u_{x,t}) = \frac{1}{\sqrt[3]{x}\Big(3\big(\frac{t}{2}\big)^{\frac{2}{3}} - x^{\frac{2}{3}}\Big)} < \frac{1}{\sqrt[3]{x}}\left(\frac{2}{t}\right)^{\frac{2}{3}}, \qquad \forall\, x < \sqrt{2}\, t, \quad u \geq \theta_r \,, \tag{5.5.10}$$

and

$$\Phi(x,t,u) \leq \frac{[f_r'(u)]^2}{x} \leq \frac{c_2}{t}\,, \qquad \forall\, x \geq \sqrt{2}\, t, \quad \theta_r \leq u \leq \|\omega\|_{\mathbf{L}^\infty} \,, \tag{5.5.11}$$

for some constant c_2 depending on $\|\omega\|_{\mathbf{L}^\infty}$. Then, relying on (5.5.4), (5.5.5), (5.5.6), (5.5.9), (5.5.10), (5.5.11), we derive

$$D^+(f_r \circ \omega)(x) \leq \begin{cases} \dfrac{c_3}{\sqrt[3]{x}\ t^{\frac{2}{3}}} & \text{if} \quad x < \sqrt{2}\, t, \quad x \in [0, \mathsf{R}[\,, \\ \dfrac{c_3}{t} & \text{if} \quad x \geq \sqrt{2}\, t, \quad x \in [0, \mathsf{R}[\,, \end{cases} \tag{5.5.12}$$

for some other constant c_3 depending on $\|\omega\|_{\mathbf{L}^\infty}$. Hence, recalling Remark 5.1.10, we deduce that, given a bounded set $K \subset [0, \mathsf{R}]$, we have

$$\int_K D^+\big(f_r \circ \mathcal{S}_t^{[AB]+} u_0\big)(x)\ dx \leq \frac{C}{t},$$

for some constant C depending only on $\|u_0\|_{\mathbf{L}^\infty}, K$, which yields (5.5.2). This completes the proof of (ii) in the case where $S_t^{[AB]+} u_0$ satisfies the conditions stated in Theorem 5.3.11.

Step 4. *(proof of (ii) when $S_t^{[AB]+} u_0$ satisfies the conditions of Theorem 5.3.3 or of Theorem 5.3.9).*
Since $S_t^{[AB]+} u_0$ satisfies the Oleinik-type inequalities (5.3.10) of Theorem 5.3.3 (or (5.3.34), (5.3.40) of Theorem 5.3.9), with the same analysis of Step 2 we deduce the uniform bound in (5.5.2) for bounded subset K of $]-\infty, \mathsf{L}]$ or of $[\mathsf{R}, +\infty[$. Next, for sets $K \subset [0, \mathsf{R}]$ or $K \subset [\mathsf{L}, 0]$, relying on the Oleinik-type inequalities (5.3.11), (5.3.12), of Theorem 5.3.3 (or (5.3.35), (5.3.41) of Theorem 5.3.9), we recover the bound in (5.5.2) performing the same analysis of Step 3. This completes the proof of (ii) in the case where $S_t^{[AB]+} u_0$ satisfies the conditions stated in Theorem 5.3.3 or in Theorem 5.3.9, and concludes the proof of the proposition. □

5.6. Appendix A: Stability of solutions with respect to connections and BV bounds

We provide here a proof of Properties (iv)-(v) of Theorem 5.1.8, which seems to be absent in the literature. To this end we first recall a by now classical technical lemma, useful for the analysis of stability of discontinuous conservation laws (e.g. see [**26**], [**64**, Proposition 1]). For sake of completeness we provide a proof below.

LEMMA 5.6.1. *Fix a connection* (A, B) *and let* I^{AB} *be the map in* (5.1.10). *Then, for any couple of pairs* (u_l, u_r), $(v_l, v_r) \in \mathbb{R}^2$ *that verify*

$$I^{AB}(u_l, u_r) \leq 0, \qquad I^{AB}(v_l, v_r) \leq 0\,, \tag{5.6.1}$$

and

$$f_l(u_l) = f_r(u_r), \qquad f_l(v_l) = f_r(v_r)\,, \tag{5.6.2}$$

setting

$$\alpha(u_l, u_r, v_l, v_r) \doteq \operatorname{sgn}(u_r - v_r) \cdot (f_r(u_r) - f_r(v_r)) - \operatorname{sgn}(u_l - v_l) \cdot (f_l(u_l) - f_l(v_l))\,, \tag{5.6.3}$$

it holds true

$$\alpha(u_l, u_r, v_l, v_r) \leq 0\,. \tag{5.6.4}$$

PROOF. Observe that, if $u_r = v_r$ or $u_l = v_l$, then the left hand side of (5.6.4) is zero and (5.6.4) is verified. Hence, without loss of generality, we may assume that $u_r > v_r$ and $u_l \neq v_l$. If $u_l > v_l$, the left hand side of (5.6.4) is again zero, because of (5.6.2). Thus, assuming that $u_l < v_l$, we have

$$\alpha(u_l, u_r, v_l, v_r) = 2(f_r(u_r) - f_r(v_r))\,. \tag{5.6.5}$$

If we suppose, by contradiction, that (5.6.4) is not verified, it would follow by (5.6.5), that $f_r(u_r) > f_r(v_r)$. Moreover, because of assumptions (5.6.1)-(5.6.2), and applying Lemma 5.1.5, we know that $f_r(u_r), f_r(v_r) \geq f_r(B)$. Since $u_r > v_r$, these inequalities together imply that $u_r \geq B$. On the other hand, by (5.6.2), it also holds $f_l(u_l) > f_l(v_l)$. Relying again on (5.6.1)-(5.6.2) and Lemma-5.1.5, we deduce that $f_l(u_l), f_l(v_l) \geq f_l(A)$, which, coupled with $u_l < v_l$, $f_l(u_l) > f_l(v_l)$, implies $u_l \leq A$. Hence, by Lemma 5.1.5 it follows that $u_r = B$ and $u_l = A$, and then we would have

$$\begin{aligned} \alpha(u_l, u_r, v_l, v_r) &= \alpha(A, B, v_l, v_r) \\ &= \operatorname{sgn}(B - v_r) \cdot (f_r(B) - f_r(v_r)) - \operatorname{sgn}(A - v_l) \cdot (f_l(A) - f_l(v_l)) \\ &= I^{AB}(v_l, v_r) \leq 0 \end{aligned} \tag{5.6.6}$$

which is a contradiction. Therefore (5.6.4) is satisfied, and the proof is concluded. □

In order to obtain stability with respect to perturbations of the connection, the following quantitative version of Lemma 5.6.1 will be useful. A general version of this Lemma can be found in [**25**, Proposition 3.21] (see also [**24**, Proposition 2.10] for the case $f_l = f_r$).

LEMMA 5.6.2. *Let* (A, B), (A', B') *be two connections. Then, for any couple of pairs* (u_l, u_r), $(v_l, v_r) \in \mathbb{R}^2$ *that verify*

$$I^{AB}(u_l, u_r) \leq 0, \qquad I^{A'B'}(v_l, v_r) \leq 0\,, \tag{5.6.7}$$

and (5.6.2), *it holds true*

$$\alpha(u_l, u_r, v_l, v_r) \leq 2\,|f_r(B') - f_r(B)|\,. \tag{5.6.8}$$

Proof. With no loss of generality assume that $B' > B$. Then, applying Lemma 5.1.5, one deduces that $B' > B$, together with (5.6.2) and $I^{A'B'}(v_l, v_r) \leq 0$, implies that one of the following two holds:

(1) $I^{AB}(v_l, v_r) \leq 0$,
(2) $(v_l, v_r) = (A', B')$.

If (1) holds, then by Lemma 5.6.1 we have $\alpha(u_l, u_r, v_l, v_r) \leq 0$, and therefore (5.6.8) is verified. Otherwise, (2) holds. In this case, we can add and subtract the non positive quantity $I^{AB}(u_l, u_r)$, and rewrite α as

$$\alpha(u_l, u_r, A', B') = \alpha_r(u_r) - \alpha_l(u_l) + I^{AB}(u_l, u_r) \leq \alpha_r(u_r) - \alpha_l(u_l), \tag{5.6.9}$$

where

$$\alpha_r(u_r) \doteq \operatorname{sgn}(u_r - B') \cdot (f_r(u_r) - f_r(B')) - \operatorname{sgn}(u_r - B) \cdot (f_r(u_r) - f_r(B)),$$
$$\alpha_l(u_l) \doteq \operatorname{sgn}(u_l - A') \cdot (f_l(u_l) - f_l(A')) - \operatorname{sgn}(u_l - A) \cdot (f_l(u_l) - f_l(A)).$$

We provide separately an estimate on $\alpha_r(u_r)$ and on $\alpha_l(u_l)$. We consider first the term α_r, and we distinguish three cases.

(1) $u_r > B'$. Then one has
$$\alpha_r(u_r) = f_r(u_r) - f_r(B') - f_r(u_r) + f_r(B) = f_r(B) - f_r(B').$$
(2) $u_r \in [B, B']$. Observe that, applying Lemma 5.1.5 and relying on (5.6.2) and $I^{AB}(u_l, u_r) \leq 0$, we deduce $f_r(u_r) \geq f_r(B)$. Then one has
$$\begin{aligned}\alpha_r(u_r) &= -f_r(u_r) + f_r(B') - f_r(u_r) + f_r(B)\\ &= (f_r(B) + f_r(B') - 2f_r(u_r)) \leq (f_r(B') - f_r(B)).\end{aligned}$$
(3) $u_r < B$. Then one has
$$\alpha_r(u_r) = -f_r(u_r) + f_r(B') + f_r(u_r) - f_r(B) \leq f_r(B') - f_r(B).$$

In every case, we obtain

$$\alpha_r(u_r) \leq |f_r(B') - f_r(B)|\,. \tag{5.6.10}$$

Analogously, and thanks to (5.6.2), we can prove that

$$\alpha_l(u_l) \geq -|f_l(A') - f_l(A)| = -|f_r(B') - f_r(B)| \tag{5.6.11}$$

which in turn, together with (5.6.10), implies

$$\alpha(u_l, u_r, A', B') \leq 2\,|f_r(B') - f_r(B)|\,, \tag{5.6.12}$$

and this concludes the proof of the lemma. □

Proof of Theorem 5.1.8-(*iv*)-(*v*). Set

$$u(x,t) \doteq \mathcal{S}_t^{[AB]+} u_0(x), \qquad v(x,t) \doteq \mathcal{S}_t^{[A'B']+} u_0(x). \tag{5.6.13}$$

Relying on property (2) of Definition 5.1.2, with standard doubling of variable arguments (e.g. see [**41**, §6.3]) one obtains that, for every non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $]-\infty, 0[\times]0, +\infty[$, it holds true

$$\int_{-\infty}^{0}\int_{0}^{\infty} \left\{|u-v|\phi_t + \operatorname{sgn}(u-v)\,(f_l(u) - f_l(v))\,\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq 0, \tag{5.6.14}$$

and, for for every non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $]0,+\infty[\times]0,+\infty[$, it holds true

$$\int_0^\infty \int_0^\infty \left\{|u-v|\phi_t + \mathrm{sgn}(u-v)\left(f_r(u)-f_r(v)\right)\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq 0. \tag{5.6.15}$$

Hence, with the same arguments, one deduces that, for every non-negative test function $\phi \in \mathcal{C}_c^1$ with compact support contained in $\mathbb{R}\times\,]0,+\infty[$, it holds true

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{\,|u-v|\,\phi_t + \mathrm{sgn}(u-v)(f(x,u)-f(x,v))\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq -E\,, \tag{5.6.16}$$

where E is the extra boundary term at $x=0$ (due to the fact that, differently from (5.6.14)-(5.6.15), ϕ will not vanish in general at $x=0$) given by

$$E = \int_0^{+\infty} \left[\mathrm{sgn}(u(x,t)-v(x,t))(f(x,u(x,t))-f(x,v(x,t)))\right]_{x=0-}^{x=0+} \phi(0,t)\,\mathrm{d}t\,,$$

with $[\cdot]_{x=0-}^{x=0+}$ denoting the limit from the right minus the limit from the left at $x=0$. Observe that, letting u_l, u_r denote the one-sided limit of u in $x=0$ as in (5.1.7), and denoting v_l, v_r, the corresponding ones for v, recalling (5.6.3) we can rewrite the quantity E as

$$E = \int_0^{+\infty} \alpha(u_l(t), u_r(t), v_l(t), v_r(t))\phi(0,t)\,\mathrm{d}t\,. \tag{5.6.17}$$

On the other hand, since u_l, u_r, and v_l, v_r satisfy the Rankine-Hugoniot condition (5.1.8), together with the inequality (5.1.11) related to the (A,B), and (A',B') connection, respectively, applying Lemma 5.6.2 we deduce that it holds true

$$\alpha\big(u_l(t(,u_r(t),v_l(t),v_r(t)\big) \leq 2\,|f_r(B')-f_r(B)| \qquad \text{for a.e. } t>0\,. \tag{5.6.18}$$

Thus, combining (5.6.16) with (5.6.17), (5.6.8), we find

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{\,|u-v|\,\phi_t + \mathrm{sgn}(u-v)(f(x,u)-f(x,v))\phi_x\right\} \mathrm{d}x\,\mathrm{d}t \geq -2|f_r(B)-f_r(B')| \int_0^{+\infty} \phi(0,t)\,\mathrm{d}t\,. \tag{5.6.19}$$

Now fix $\tau > \tau_0 > 0$, $R>0$, and consider the trapezoid $\Omega \doteq \{(x,t)\ :\ \tau_0 \leq t \leq \tau,\ |x| \leq R + L(\tau-t)\}$, where $L \doteq \sup_{|z|\leq M} \max\{|f_l'(z)|, |f_r'(z)|\}$, with M being a uniform $\mathbf{L}^\infty$ bound for u and v. Then, by a standard technique (e.g. see [**41**, §6.3]), one can construct a sequence of test functions $\phi_n \in \mathcal{C}_c^1$, with compact support contained in $\mathbb{R}\times\,]0,+\infty[$, that approximate the characteristic function of Ω when $n\to\infty$. Employing (5.6.19) with ϕ_n, and letting $n\to\infty$ we obtain

$$\int_{|x|\leq R} \big|u(x,\tau)-v(x,\tau)\big|\,\mathrm{d}x \leq \int_{|x|\leq R+L(\tau-\tau_0)} \big|u(x,\tau_0)-v(x,\tau_0)\big|\,\mathrm{d}x + 2(\tau-\tau_0)\big|\big|f_r(B)-f_r(B')\big|\,. \tag{5.6.20}$$

Relying on the $\mathbf{L^1}$-continuity of u and v at $\tau_0 = 0$ (property (2) of Definition 5.1.2), and letting $R\to\infty$ in (5.6.20), we obtain the estimate of Theorem 5.1.8-(iv) for $t=\tau$.

To establish property (v) of Theorem 5.1.8 observe that, if (A,B) is a non critical connection, then by Lemma 5.5.1-(i) one has $\mathcal{S}_t^{AB} u_0 \in BV_{\mathrm{loc}}(\mathbb{R})$ for all $t>0$, and for any $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Therefore, in this case, relying on this property we immediately recover the $\mathbf{L}^1_{\mathrm{loc}}$-Lipschitz continuity of

$t \mapsto \mathcal{S}_t^{AB} u_0$ by standard arguments (e.g. see [**41**, proof of Theorem 9.4]). On the other hand, in the case of a critical connection (A, B), we derive the $\mathbf{L}^1_{loc}$-Lipschitz continuity of $t \mapsto \mathcal{S}_t^{AB} u_0$ applying Lemma 5.5.1-(ii) and following the same arguments in [**54**, proof of Theorem 4.3.1]. □

Proof of Corollary 5.1.11. Relying on Theorem 5.1.8-(iv) we deduce that

$$u_n(\cdot, t) \quad \to \quad u(\cdot, t) \qquad \text{in} \quad \mathbf{L}^1_{\text{loc}}(\mathbb{R}) \qquad \forall\, t \geq 0\,, \tag{5.6.21}$$

which in turn implies that there exists $\overline{x} > 0$ such that

$$f_r(u_n(\overline{x}, \cdot)) \quad \to \quad f_r(u(\overline{x}, \cdot)) \qquad \text{in} \quad \mathbf{L}^1_{\text{loc}}([0, +\infty[)\,. \tag{5.6.22}$$

Then, observe that by Definition 5.1.2 u_n and u are entropy weak solutions of $u_t + f_r(u)_x = 0$ on $]0, +\infty[\, \times [0, +\infty[$. Hence, by a general property of weak solutions (e.g. see [**41**, Remark 4.2]), for every fixed $s > 0$ one has

$$\begin{aligned} \int_0^s f_r(u_r(s))ds &= \int_0^s f_r(u(\overline{x}, \cdot))ds + \int_0^{\overline{x}} u(z, T)dz - \int_0^{\overline{x}} u(z, 0)(z)dz\,, \\ \int_0^s f_r(u_{n,r}(s))ds &= \int_0^s f_r(u_n(\overline{x}, \cdot))ds + \int_0^{\overline{x}} u_n(z, T)dz - \int_0^{\overline{x}} u_n(z, 0)dz \qquad \forall\, n\,. \end{aligned} \tag{5.6.23}$$

Since we are assuming that $\{u_n(\cdot, 0)\}_n$ converges to $u(\cdot, 0)$ in $\mathbf{L}^1_{\text{loc}}$, taking the limit as $n \to \infty$ in (5.6.23) we deduce from (5.6.21), (5.6.22), (5.6.23) by standard arguments that

$$f_r(u_{n,r}) \quad \rightharpoonup \quad f_r(u_r) \quad \text{weakly in} \quad \mathbf{L}^1(\mathbb{R}^+). \tag{5.6.24}$$

With entirely similar arguments one derives also the other convergence in (5.1.16). □

5.7. Appendix B: Preclusion of rarefactions emanating from the interface

A distinctive feature of the structure of AB-entropy solutions is the fact that no rarefaction wave can emerge at positive times from the interface $x = 0$. This property was established in [**2**] exploiting an explicit representation formula for AB-entropy solutions a la Lax-Oleinik. A different, rather technical proof, based on a detailed analysis of the structure of AB-entropy solutions was derived in [**9**], under the additional assumption that the traces of the solution at $x = 0$ admit one sided limits. Here, we provide a much simpler proof that establishes this fact in the case of a non critical connection (A, B), and for a BV_{loc} AB-entropy solution. The proof relies on the properties of solutions of Riemann problems and on a blow-up argument. Namely, the key point is to show that Riemann-type initial data from which rarefaction waves emerge are not attainable by an AB-entropy solution at any positive time $t > 0$. Next, by contradiction and performing a blow-up analysis, we prove that if a rarefaction emerges from an AB-entropy solution at some time $\overline{t} > 0$, then there exists a Riemann-type datum $\overline{u}$ that generates a rarefaction and which is attainable by an AB-entropy solution at time $\overline{t}$.

One can recover this property of preclusion of rarefactions emanating from the interface (for any AB-entropy solution and general connections) as a byproduct of the characterization of attainable profiles $\omega \in \mathcal{A}^{[AB]}(T)$ provided by Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14 (see Remark 5.7.4).

DEFINITION 5.7.1. We say that an AB-entropy solution $u(x,t)$ to (22) has a *rarefaction fan emerging at the right (at the left) from the interface* $x=0$ at time $\bar{t}$, if there exists $\delta>0$ and two continuity points $0<x_1<x_2$ for $u(\cdot,\bar{t}+\delta)$ such that

$$x_1-\delta f_r'(u(x_1,\bar{t}+\delta)) = x_2-\delta f_r'(u(x_2,\bar{t}+\delta)) = 0.$$

Notice that Definition B.1 does no require to know that the solution u admits one-sided limits at $x=0$, and it is invariant with respect to the scaling $(x,t)\to(\rho x,\bar{t}+\rho(t-\bar{t}))$, $\rho>0$. This definition is equivalent to say that there exists an outgoing rarefaction fan emerging at time $\bar{t}$, at the right, if there exist two distinct genuine characteristics located in $\{x>0\}$ for times $t\in\,]\bar{t},\bar{t}+\delta]$, $\delta>0$, that emerge from the point $(0,\bar{t})$.

PROPOSITION 5.7.2. *Let (A,B) be a connection, consider a Riemann data*

$$\overline{u}=\begin{cases} u^-, & x<0,\\ u^+, & x>0\,,\end{cases} \tag{5.7.1}$$

and assume that the solution $\mathcal{S}_t^{[AB]+}\overline{u}(x)$ contains a rarefaction wave located in the left halfplane $\{x\le 0\}$, or in the right one $\{x\ge 0\}$. Then for every $T>0$ it holds $\overline{u}\notin\mathcal{A}^{[AB]}(T)$.

PROOF. By contradiction, suppose that $\mathcal{S}_t^{[AB]+}\overline{u}(x)$ contains a rarefaction wave located in $\{x\ge0\}$, and assume that $\overline{u}\in\mathcal{A}^{[AB]}(T)$, i.e. that there exists an AB-entropy solution solution $u(x,t)$ of (22),(23), such that $u(\cdot\,,T)=\overline{u}$. Then, by uniqueness, one has $u(x,T+t)=\mathcal{S}_t^{[AB]+}\overline{u}(x)$ for all $x\in\mathbb{R}$, $t\ge0$. Since $\mathcal{S}_t^{[AB]+}\overline{u}(x)$ is a solution of a Riemann problem containing a rarefaction with nonnegative characteristic speeds, and because of the admissibility conditions (5.1.13), it then follows that the right trace u_r of u at $x=0$ satisfies $B\le u_r(t)<u(T,x)=u^+$ for every $t>T$, $x>0$. Tracing the backward characteristics from points (T,x), $x>0$, we find that $u_r(t)=u^+>B$ for every $t\in\,]0,T[\,$. Therefore, because of the admissibility conditions (5.1.13), one has $u_l(t)=\pi^r_{l,+}(u^+)$ (with $\pi^r_{l,+}$ defined as in (5.3.1)), for every $t\in\,]0,T[\,$. Then, letting ξ_-,ξ_+ denote the minimal and maximal backward characteristics starting at $(0,T)$, we deduce that $f_l'(u^-)=\dot{\xi}_-(T)>\dot{\xi}_+(T)=f_l'(\pi^l_{r,+}(u^+))$, which in turn implies $u^->\pi^r_{l,+}(u^+)$, $\pi^l_{r,+}(u^-)>u^+$. Observe now that the AB-entropy solution of a Riemann problem with initial data (5.7.1) satisfying $u^->\pi^l_{r,+}(u^+)$, and $u^+>B$, consists of a single shock located in the halfplane $\{x\ge0\}$, and connecting the left state $\pi^l_{r,+}(u^-)$ with the right state u^+. This is in contrast with the assumption made on $\mathcal{S}_t^{[AB]+}\overline{u}(x)$, thus completing the proof.

□

PROPOSITION 5.7.3. *Let (A,B) be a non critical connection, and let u be an AB-entropy solution to* (22) *that satisfies $u(\cdot,t)\in BV_{\mathrm{loc}}(\mathbb{R})$, for all $t>0$. Then u does not contain rarefaction waves emerging from the interface $x=0$ at times $\bar{t}>0$.*

PROOF. Assume by contradiction that the solution u has a rarefaction wave, say located in $\{x\ge0\}$, which emerges from the interface at some time $\bar{t}>0$. Let $0<\overline{\rho}<\bar{t}/3$, and for any $\rho>0$, set

$$I_\rho \doteq \{x\in\mathbb{R}\,:\,|x|\le\rho\}\,. \tag{5.7.2}$$

Observe that the domain of dependence of $u(x,t)$, for $(x,t) \in I_{\overline{\rho}} \times [\overline{t}-\overline{\rho}, \overline{t}+\overline{\rho}]$, is the trapezoid $\Omega \doteq \{(x,t) \,:\, |x| \leq \overline{\rho} + \Lambda \cdot (\overline{t}+\overline{\rho}-t),\, t \in [\overline{t}-2\overline{\rho}, \overline{t}+\overline{\rho}]\}$, where $\Lambda \doteq \sup_{|z|\leq M} \max\{|f_l'(z)|, |f_r'(z)|\}$, with M being a uniform $\mathbf{L}^\infty$ bound for u. Therefore, since the total variation of $u(t,\cdot)$ on I_{l_t}, $l_t \doteq |x| \leq \overline{\rho} + \Lambda \cdot (\overline{t}+\overline{\rho}-t)$, is bounded, and because (A,B) is a non critical connection, we can invoke the uniform BV bounds on AB-entropy solutions established in [**64**, Lemma 8] (see also [**1**, Theorem 2.13-(iii)]) to deduce that

$$\text{Tot.Var.}(u(\cdot,t),\, I_{\overline{\rho}}) \leq \overline{C}\big(M + \text{Tot.Var.}(u(\cdot,\overline{t}-2\overline{\rho}), I_{(1+2\Lambda)\overline{\rho}})\big) \qquad \forall\, t \in \overline{t} + I_{\overline{\rho}}\,, \tag{5.7.3}$$

for some constant $\overline{C} > 0$. Next, consider the blow-up of u at the point $(0, \overline{t}\,)$:

$$u_\rho(x,t) \doteq u(\rho x, \overline{t} + \rho(t-\overline{t})) \qquad x \in \mathbb{R},\ t \geq 0\,, \tag{5.7.4}$$

with $0 < \rho < \overline{\rho}/\overline{t}$, and observe that it holds true

$$\text{Tot.Var.}(u_\rho(\cdot,t), I_{\overline{\rho}/\rho}) \leq \sup_{\tau\,\in\,\overline{t}+I_{\overline{\rho}}} \text{Tot.Var.}(u(\cdot,\tau), I_{\overline{\rho}}) \qquad \forall\, 0 \leq t < \overline{t} + \frac{\overline{\rho}}{\rho}\,. \tag{5.7.5}$$

Combining (5.7.3), (5.7.5), we find a uniform bound on the total variation of $u_r(\cdot,t)$ on the interval $I_{\overline{\rho}/\rho}$, for all $t < \overline{t} + \overline{\rho}/\rho$, and $0 < \rho < \overline{\rho}/\overline{t}$. Moreover, observe that because of the finite speed of propagation Λ, by standard arguments (e.g. see [**41**, §7.4]) one deduces that

$$\|u_\rho(\cdot,t) - u_\rho(\cdot,s)\|_{\mathbf{L^1}(I_{\overline{\rho}/\rho})} \leq \overline{\Lambda} \cdot (t-s) \qquad \forall\, 0 \leq s < t < \overline{t} + \frac{\overline{\rho}}{\rho}\,, \tag{5.7.6}$$

for all $0 < \rho < \overline{\rho}/\overline{t}$, and for some constant $\overline{\Lambda}$. Notice that the sets $I_{\overline{\rho}/\rho} \times [0, \overline{t}+\overline{\rho}/\rho[$ invade $\mathbb{R} \times [0,+\infty[$ as $\rho \to 0$. Therefore we can apply Helly's compactness theorem [**41**, Theorem 2.4] to the sequence $\{u_\rho\}_{0<\rho<\overline{\rho}/\overline{t}}$, and deduce the existence of a function $v \in \mathbf{L}^\infty(\mathbb{R} \times [0,+\infty[)$, so that, up to a subsequence, $u_\rho(\cdot,t)$ converges to $v(\cdot,t)$ in $\mathbf{L^1}_{\text{loc}}$, as $\rho \to 0$, for all $t > 0$. By Definition 5.1.2 it follows that also v is an AB-entropy solution of (22)-(23), with $u_0 \doteq v(\cdot,0)$. Notice that

$$\lim_{\rho\to 0} u_\rho(x,\overline{t}\,) = \overline{u}(x) \doteq \begin{cases} u(0+,\overline{t}\,) & \text{if} \quad x > 0\,, \\ u(0-,\overline{t}\,) & \text{if} \quad x < 0\,, \end{cases} \tag{5.7.7}$$

and thus we find

$$v(\cdot,\overline{t}\,) = \overline{u}\,,$$

which implies

$$\overline{u} \in \mathcal{A}^{[AB]}(\,\overline{t}\,)\,. \tag{5.7.8}$$

On the other hand, observe that the rarefaction wave which emerges in the solution u at time $\overline{t}$ is preserved by the blow-ups u_ρ in (5.7.4), because it is self similar for the scaling $(x,t) \mapsto (\rho x, \overline{t} + \rho(t-\overline{t}))$. Therefore there will be a rarefaction wave emerging at time $\overline{t}$, and located in $\{x \geq 0\}$, also in the solution v. This in turn implies that the solution $\mathcal{S}_t^{AB}\overline{u}(x)$ to the Riemann problem with initial datum $\overline{u}$ contains a rarefaction emerging at $t = 0$ and located in $\{x \geq 0\}$, since $\mathcal{S}_t^{AB}\overline{u}(x) = v(\overline{t}+t, x)$ for all $x \in \mathbb{R}$, $t \geq 0$. This, together with (5.7.8), is in contradiction with Proposition 5.7.2, thus completing the proof. $\square$

Remark 5.7.4. In the case of a general connection (A, B), relying on the characterization of $\mathcal{A}^{[AB]}(t)$, $t > 0$, provided by Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14, we can show that no rarefaction can emerge from the interface $x = 0$ at any time $\bar{t} > 0$ for any AB-entropy solution u to (22), as follows. Suppose, by contradiciton, that a rarefaction is generated in a time interval $[\,\bar{t}, \bar{t}+\delta]$, for some $\delta > 0$, and that lies in the semiplane $\{x \geq 0\}$. In particular this means that there exist two genuine characteristics $\xi_1, \xi_2 : [\,\bar{t}, \bar{t}+\delta\,] \to\ [0, +\infty[\,$, such that $\xi_1(\,\bar{t}\,) = \xi_2(\,\bar{t}\,) = 0$, $\overline{x}_1 \doteq \xi_1(\,\bar{t}+\delta) < \overline{x}_2 \doteq \xi_2(\,\bar{t}+\delta)$. We may also assume that $\xi_i' = f_r'(\omega(\overline{x}_i))$, $i = 1, 2$. Let $\omega \doteq u(\cdot, \bar{t}+\delta)$, $\mathsf{R} \doteq \mathsf{R}[\omega, f_r]$ (see def (5.3.2)), and consider the time $\tau(x) = (\bar{t}+\delta) - x/f_r'(\omega(x))$, $x \in\,]0, \mathsf{R}[\,$, at which the characteristic starting at $(x, \bar{t}+\delta)$, with slope $f_r'(\omega(x))$ impacts the interface $x = 0$. Notice that $\tau(\overline{x}_1) = \tau(\overline{x}_2) = \bar{t}$. Moreover, thanks to Lemma 4.4 in [**9**], the Oleinik estimates satisfied by ω (because of condition (i) or (i)' of Theorems 5.3.3, 5.3.9, 5.3.11, 5.3.14) imply the strict monotonicity of the map $x \to \tau(x)$, $x \in\,]0, \mathsf{R}[\,$. In turn, the strict monotonicity of τ implies $\tau(\overline{x}_1) \neq \tau(\overline{x}_2)$, thus contradicting the assumption $\tau(\overline{x}_1) = \tau(\overline{x}_2) = \bar{t}$.

5.8. Appendix C: Semicontinuity properties of solutions to convex conservation laws

Solutions to conservation laws with convex flux enjoy a lower and upper semicontinuity property with respect to the $\mathbf{L}^1$ convergence as stated in the following

Lemma 5.8.1. *Given a uniformly convex map f, and $T > 0$, let $\{u_n\}_n$ be a sequence of entropy weak solutions of*

$$u_t + f(u)_x = 0 \qquad x > 0, \quad t \in [0, T], \tag{5.8.1}$$

that admit a strong trace $u_n(0+, t) = \lim_{x\to 0+} u_n(x, t)$ at $x = 0$, for all $t \in [0, T]$, and let u be an entropy weak solution of (5.8.1) *that admits a strong trace $u(0+, t) = \lim_{x\to 0+} u(x, t)$ at $x = 0$, for all $t \in [0, T]$. Assume that $\{u_n\}_n$ are uniformly bounded in $\mathbf{L}^\infty$, that*

$$u_n(\cdot, t) \quad \to \quad u(\cdot, t) \qquad in \quad \mathbf{L}^1_{\mathsf{loc}}(\,]0, +\infty[), \qquad \forall\, t \in [0, T]\,, \tag{5.8.2}$$

and that

$$f(u_n(0+, \cdot)) \quad \rightharpoonup \quad f(u(0+, \cdot)) \qquad weakly \;\; in \quad \mathbf{L}^1([0, T])\,. \tag{5.8.3}$$

Then, for every $x \geq 0$, it holds true

$$u(x+, T) \leq \liminf_{\substack{n\to\infty \\ y\to x,\, y>0}} \; u_n(y+, T)\,. \tag{5.8.4}$$

If we assume that u_n, u, are entropy weak solutions of $u_t + f'(u)_x = 0$ on $x < 0$, $t \in [0, T]$, and that the convergences (5.8.2), (5.8.3)*, hold in $\mathbf{L}^1_{\mathsf{loc}}(\,]-\infty, 0[)$, and for the left traces in $x = 0$, respectively, then for every $x \leq 0$, it holds true*

$$u(x-, T) \geq \limsup_{\substack{n\to\infty \\ y\to x,\, y<0}} \; u_n(y-, T)\,. \tag{5.8.5}$$

Proof. We will establish only the inequality (5.8.4), the proof of (5.8.5) being entirely similar. Given $x \geq 0$, $T > 0$, consider a sequence $\{y_n\}_n$, $y_n > 0$, converging to x, and such that

$$\lim_n u_n(y_n+, T) = \liminf_{\substack{n\to\infty \\ y\to x}} \; u_n(y+, T)\,. \tag{5.8.6}$$

Let $\vartheta_n^+ : \,]\tau_n, T] \to \,]0, +\infty[\,$, $\tau_n \geq 0$, denote the maximal backward characteristic for u_n starting from (y_n, T), with the property that either $\tau_n = 0$, or $\lim_{t\to\tau_n} \vartheta_n^+(t) = 0$. By possibly taking a subsequence, we can assume that either $\tau_n = 0$ for all n, or that $\lim_{t\to\tau_n} \vartheta_n^+(t) = 0$ for all n. We recall that a maximal backward characteristic for u_n passing through (y_n, T), $y > 0$, is a genuine (shock free) characteristics whose trajectory is a segment with constant slope $f'(u_n(y_n+, T))$ (e.g. see [**54**]). Notice that $\{\vartheta_n^+\}_n$ is a sequence of Lipschitz continuous functions with a uniform Lipschitz constant $\sup_{|u|\leq M} f'(u)$ (M being a uniform $\mathbf{L}^\infty$ bound on u_n), defined on uniformly bounded intervals $]\tau_n, T]$. Hence, by Ascoli-Arzelà Theorem we can assume that, up to a subsequence, $\{\vartheta_n^+\}_n$ converges uniformly to some Lipschitz continuous function $\vartheta : \,]\tau, T] \to \,]0, +\infty[\,$, such that

$$\begin{aligned} &\tau = 0, \qquad \text{if} \qquad \tau_n = 0 \quad \forall\, n\,, \\ &\tau = \lim_{n\to\infty} \tau_n\,, \quad \lim_{t\to\tau} \vartheta(t) = 0, \qquad \text{if} \quad \lim_{t\to\tau_n} \vartheta_n^+(t) = 0 \quad \forall\, n\,, \end{aligned} \tag{5.8.7}$$

and such that $\vartheta(T) = x$. By a general property of characteristics, the uniform limit of genuine characteristics is also a genuine characteristic. This can be easily verified in this context observing that the trajectory of a genuine characteristic passing through a point (y, t), $y > 0$, is a segment connecting (y, t) with the point $(0, \tau(y, t))$ or with the point $(z(y, t), 0)$, where $\tau(y, t)$ and $z(y, t)$ denotes the points of minimum for the functionals involved in the Lax-Oleinik representation formula of solutions for the boundary value problem (see [**77**]), and that such functionals are $\mathbf{L}^1$ continuous with respect to the initial datum and and weakly continuous in $\mathbf{L}^1$ with respect to the flux-trace of the solution at $x = 0$. Therefore it follows that ϑ is a genuine characteristic with constant slope ϑ' satisfying

$$\vartheta' = \lim_{n\to\infty} (\vartheta_n^+)' = \lim_{n\to\infty} f'(u_n(y_n+, T)), \qquad\qquad \vartheta' \geq f'(u(x+, T))\,. \tag{5.8.8}$$

Since f' is increasing, we deduce from (5.8.8) that

$$\lim_n u_n(y_n+, T) \geq u(x+, T)\,, \tag{5.8.9}$$

which, together with (5.8.6), yields (5.8.4).

□

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