

UNIVERSITÀ DI PADOVA – DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA

Scuole di Dottorato in Matematica Pura e Matematica Computazionale

Seminario Dottorato 2006/07



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Presentazione

Questo documento è il resoconto dell'attività del Seminario Dottorato 2006/07. Svolto in forma regolare per la prima volta durante quest'anno accademico, il Seminario Dottorato fornisce un'opportunità a dottorandi, assegnisti e borsisti in Matematica Pura e Computazionale di comunicare le proprie ricerche in modo interessante per un pubblico matematicamente ben istruito ma non specialista, traendone giovamento in termini di autocoscienza di ciò che si sta studiando. La cura con la quale i seminari sono stati preparati (di cui è testimonianza anche la redazione di queste note riassuntive), e la generalmente buona partecipazione dei compagni di corso e di altri colleghi di volta in volta interessati, ci autorizzano ad archiviare in modo soddisfacente questo ciclo, e a programmarne un proseguimento anche per l'avvenire.

Un grazie va a tutti coloro che hanno contribuito alla riuscita dell'iniziativa, in primo luogo naturalmente a chi ha accettato di svolgere i seminari e di stendere queste note.

Padova, 2 luglio 2007

Corrado Marastoni, Tiziano Vargiolu

Sunti dei seminari (tratti dalla pagina web del Seminario Dottorato)

17 gennaio 2007

Rappresentazioni Integrali del Propagatore di Schrödinger LORENZO ZANELLI (dottorato in Matematica Pura – XIX ciclo)

Per l'equazione di Schrödinger legata all'operatore Hamiltoniano $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$ introduco una nuova classe di rappresentazioni integrali del relativo operatore di evoluzione. In questo contesto risulta centrale l'utilizzo di integrali oscillanti costruiti con funzioni di fase reali e globali nel tempo. In particolare viene utilizzata la classe delle funzioni generatrici debolmente quadratiche della famiglia di sottovarietà Lagrangiane $\Lambda_t \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ legate al flusso di trasformazioni canoniche classiche $\phi_H^t : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ risolvente le equazioni di Hamilton per $H = \frac{p^2}{2m} + V(x)$.

31 gennaio 2007

Soluzione Numerica di EDP per la Modellistica di Semiconduttori ROBERTO BERTELLE (dottorato in Matematica Computazionale – XIX ciclo)

I calcolatori elettronici hanno una grande importanza nello sviluppo e nello studio di nuovi dispositivi elettronici. Ogni dispositivo elettronico è, semplicemente, una regione dello spazio sede di campi e di correnti elettriche la cui conoscenza è di capitale importanza per la comprensione del comportamento fisico del dispositivo. Il seminario è una introduzione e può essere suddiviso in più parti. Una prima parte riguarda la scrittura dei modelli matematici idonei a rappresentare comiutamente un dispositivo elettronico. Nella seconda parte, le equazioni del modello sono risolte numericamente mediante discretizzazione con elementi finiti lineari. Infine, la terza parte illustra, mediante esempi numerici riferentesi a modelli mono-dimensional, l'efficacia e i vantaggi delle tecniche di risoluzione numerica descritte. (Parole chiave: modellistica per semiconduttori, giunzione p-n, elementi finiti.)

14 febbraio 2007

Dualità di Tannaka per il Gruppoide di Lie GIORGIO TRENTINAGLIA (dottorato in Matematica Pura – XIX ciclo)

La teoria dei gruppoide di Lie unifica concettualmente nozioni geometriche alquanto disparate: varietà differenziabili, gruppi di Lie, azioni lisce di gruppi di Lie su varietà, foliazioni su varietà, orbifolds ... Nel corso degli ultimi anni gli studiosi si sono scontrati spesso con le difficoltà legate al concetto di rappresentazione, senza tuttavia fornire una spiegazione soddisfacente dei motivi e del carattere di tali difficoltà (non era in particolare chiaro se queste ultime fossero davvero insormontabili). Il mio lavoro fornisce una risposta chiara e definitiva a questo annoso problema

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e apre nuove prospettive d'indagine. Nel mio discorso limiterò la teorizzazione astratta al minimo necessario, cercando di procedere per mezzo di esempi chiave. (Parole chiave: teoria della rappresentazione, gruppoidi di Lie.)

7 marzo 2007

Tecniche di Accelerazione della Convergenza per Soluzioni di EDP per la Modellistica di Semiconduttori

MARIA ROSARIA RUSSO (assegnista in Matematica Computazionale)

Nel seminario verranno presentati alcuni metodi di estrapolazione per accelerare la convergenza di successioni vettoriali. I metodi presi in analisi, sebbene derivino da filosofie differenti, sono in grado di accelerare il processo iterativo di convergenza di successioni vettoriali, anche senza nessuna informazione sul modo in cui esse vengono generate; questo li rende applicabili a diverse tipologie di problemi, lineari e non lineari. Verranno mostrati alcuni risultati relativi alla applicazione delle tecniche di accelerazione per la risoluzione di sistemi lineari di grandi dimensioni, e per accelerare il processo di convergenza di algoritmi iterativi non lineari di tipo Gummel (mappa di Gummel), solitamente usati per disaccoppiare le equazioni del modello Drift Diffusion, nella modellistica per semiconduttore. (Parole chiave: estrapolazione vettoriale, modellistica per semiconduttori.)

14 marzo 2007

Funzioni Separatamente Olomorfe e CR

RAFFAELLA MASCOLO (dottorato in Matematica Pura – XIX ciclo)

Esistono varie caratterizzazioni equivalenti per funzioni olomorfe definite su aperti di \mathbb{C}^n , la prima delle quali è la proprietà di essere rappresentate localmente come somme di serie di potenze convergenti. È ovvio che una funzione olomorfa in più variabili è olomorfa separatamente in ciascuna variabile. È proprio separando le variabili che molte delle proprietà a noi familiari delle funzioni olomorfe di una variabile complessa, come la formula integrale di Cauchy, hanno una versione corrispondente in più variabili complesse. D'altra parte è un fatto notevole che una funzione separatamente olomorfa sia in realtà di classe C^1 , e quindi olomorfa nel complesso delle sue variabili (Teorema di Hartogs, 1906). L'esposizione intende affrontare il problema della separata analiticità ed estendere la discussione al caso di funzioni separatamente CR su varietà CR, attraverso una rivisitazione ed una generalizzazione di un risultato di Henkin e Tumanov del 1983. (Parole chiave: Teorema di Hartogs, funzioni CR, separata analiticità)

28 marzo 2007

Problemi con Preferenze ed Incertezza

MARIA SILVIA PINI (dottorato in Matematica Computazionale – XIX ciclo)

Molti problemi della vita reale presentano dei vincoli, cioè delle richieste che devono essere soddisfatte totalmente. A volte però risulta più naturale esprimere questi vincoli in maniera meno stringente tramite delle preferenze. Oltre alle preferenze, molti problemi reali sono caratterizzati da incertezza, cioè dalla presenza di eventi incerti che non possono essere controllati dall'utente. In alcuni casi l'utente può avere un'informazione di tipo probabilistico o possibilistico riguardo al verificarsi di questi eventi incerti, altre volte può non avere alcuna informazione. In questo seminario presenteremo dei formalismi che modellano problemi con vari tipi di preferenze e l'incertezza, analizzeremo le proprietà di questi formalismi e considereremo il caso in cui le preferenze sono espresse da più utenti. In questo contesto considereremo l'aggregazione di preferenze e analizzeremo proprietà desiderabili come la fairness e la non-manipolabilità, estendendo risultati ben noti nella teoria dei voti. Infine esamineremo scenari in cui alcuni utenti decidono di non rivelare tutte le loro preferenze su un certo insieme di alternative, per esempio per ragioni di privacy, e analizzeremo la complessità computazionale di calcolare le alternative che sono comunque ottime.

18 aprile 2007

Fasci ed Equazioni Differenziali Lineari

GIOVANNI MORANDO (dottore di ricerca in Matematica Pura)

Le equazioni differenziali lineari sono tra gli oggetti matematici più trasversali. I metodi per studiarle così come le loro applicazioni possono essere di natura fisica, geometrica, analitica, numerica, algebrica... In questo seminario presenteremo l'approccio algebrico sviluppato dalla scuola giapponese di M. Sato sin dagli anni '60, basato sulla nozione di "D-modulo" (la generalizzazione algebrica dei sistemi di equazioni differenziali lineari). Uno dei risultati più significativi di questo approccio è la "corrispondenza di Riemann-Hilbert", una generalizzazione del 21mo problema di Hilbert (esiste un'equazione differenziale lineare le cui soluzioni olomorfe abbiano un gruppo di monodromia prescritto?). Cercheremo di spiegare, attraverso numerosi semplici esempi, la corrispondenza di Riemann-Hilbert in quanto equivalenza tra le categorie delle equazioni differenziali lineari a singolarità regolari (oggetti di natura analitica) e degli spazi delle loro soluzioni olomorfe (oggetti di natura topologica). Tale corrispondenza sottolinea quindi come oggetti di natura e utilizzo diversi siano in realtà strettamente legati.

16 maggio 2007

Numeri p -adici e Studio Coomologico delle Varietà Algebriche sopra un Anello di Valutazione Discreta

DANIELE CHINELLATO (dottorato in Matematica Pura – XIX ciclo)

Sia p un primo. I numeri p -adici possono essere pensati come completamento dei razionali \mathbb{Q} rispetto ad una distanza che permetta di codificare proprietà aritmetiche relative al primo p . Ciò permette di introdurre tecniche analoghe all'analisi reale nello studio di problemi diofantei. D'altro canto tali problemi (ad esempio la congettura di Mordell, o il teorema di Fermat) trovano una loro

naturale collocazione nella teoria degli schemi sopra un anello (in generale...). In tale teoria nel corso degli anni sono state sviluppate adeguate tecniche coomologiche nella speranza di ricondurre lo studio di problemi aritmetici (e di altro genere) nell'alveo dell'"algebra lineare" (o semilineare). In una prima parte del seminario verranno introdotti i numeri p-adici e alcune loro proprietà aritmetiche e "metriche". In seguito si passerà ad illustrare brevemente il significato dello studio di un argomento diofanteo dal punto di vista della coomologia di una varietà (funzione zeta...). Infine, nell'ottica così introdotta esporrà brevemente la ricerca compiuta nella tesi di dottorato.

23 maggio 2007

Accoppiamenti Completamente Monotoni per Processi di Markov
IDA MINELLI (assegnista in Matematica Computazionale)

Verranno discussi i concetti di monotonia e monotonia completa per processi di Markov a valori in uno spazio degli stati finito e parzialmente ordinato. La monotonia è una proprietà della matrice di transizione (o del generatore infinitesimale, nel caso di tempi continui) del processo. La monotonia completa è un concetto più forte del precedente ed è utile in molte applicazioni, ad esempio quando si vogliono ottenere simulazioni dalla misura stazionaria di una catena di Markov utilizzando algoritmi di simulazione perfetta. Al contrario di quanto accade per la monotonia, non esiste un criterio semplice per verificare la monotonia completa. Per questo motivo è naturale domandarsi per quali insiemi parzialmente ordinati i due concetti sono equivalenti. Questo problema è stato completamente risolto nel caso di processi a tempi discreti, ma si è rivelato più complesso nel caso di processi a tempi continui.

6 giugno 2007

Un Modello Dinamico di Contagio: Interpretazione Finanziaria di un Modello di Particelle Interagenti a Campo Medio
ELENA SARTORI (dottorato in Matematica Computazionale – XIX ciclo)

Un gruppo di aziende attive sul mercato può essere rappresentato da un sistema di N particelle con interazione a campo medio. Definiremo per esse una dinamica tale per cui il modello risultante sarà non reversibile: di questo modello sarà interessante studiare il comportamento per tempi "lunghi" trovandone le equazioni della dinamica a volume infinito e le soluzioni stazionarie, ed in seguito le approssimazioni a volume finito (N grande, ma finito). Quantificheremo infine le perdite sofferte da un'istituzione finanziaria che possiede un portafoglio le cui posizioni sono date dalle N aziende che affrontano il rischio di credito. Grazie a questo approccio, che tiene conto dell'interazione tra aziende (a livello microscopico), saremo in grado di spiegare il fenomeno delle crisi di credito, ossia di periodi in cui molte aziende si ritrovano improvvisamente in uno stato di forte stress finanziario.

Integral Representations of the Schrödinger Propagator

LORENZO ZANELLI ^(*)

Abstract. For the Schrödinger equation related to the Hamiltonian operator $\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$ I introduce a new class of integral representations for the evolution operator. The main feature is the use of oscillatory integrals constructed with real and global phase functions. In particular it is used the class of global weakly quadratic generating functions of the Lagrangian submanifolds $\Lambda_t \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ related to the flow of classical canonical transformations $\phi_H^t : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ solving Hamilton's equations for $H = \frac{p^2}{2m} + V(x)$. [Keywords: Schrödinger equation, semigroups of linear operators, oscillatory integrals, symplectic geometry, path integrals.]

Sunto. *Rappresentazioni Integrali dell'operatore di Schrödinger.* Per l'equazione di Schrödinger legata all'operatore Hamiltoniano $\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$ introduco una nuova classe di rappresentazioni integrali del relativo operatore di evoluzione. In questo contesto risulta centrale l'utilizzo di integrali oscillanti costruiti con funzioni di fase reali e globali nel tempo. In particolare viene utilizzata la classe delle funzioni generatrici debolmente quadratiche della famiglia di sottovarietà Lagrangiane $\Lambda_t \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ legate al flusso di trasformazioni canoniche classiche $\phi_H^t : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ risolvente le equazioni di Hamilton per $H = \frac{p^2}{2m} + V(x)$.

The Schrödinger equation

$$(1) \quad \begin{cases} i\hbar\partial_t\psi(t, x) = \left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)\psi(t, x), \\ \psi(0, x) = \varphi(x) \in H^2(\mathbb{R}^n), \end{cases}$$

modelizes the behaviour of a non relativistic particle immersed in an electrostatic potential V . As known the solution of this Cauchy problem can be represented by the formula

$$\psi(t) = U(t)\varphi,$$

where $U(t) = e^{-\frac{i}{\hbar}t\widehat{H}}$ is called Schrödinger Propagator, that corresponds to the group of unitary linear operators generated from the quantistic Hamiltonian $\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$. It turns out that it can be constructed a class of integral representations for a suitable

^(*)Dottorato in Matematica Pura, XIX ciclo. Seminario tenuto il 17 gennaio 2007.

ε -regularization $U_\varepsilon(t)$, showing afterwards the expected convergence $U_\varepsilon(t)\varphi \xrightarrow[\varepsilon \rightarrow 0^+]{\longrightarrow} U(t)\varphi$. More precisely, it is introduced the regularized Propagator,

$$U_\varepsilon(t) := e^{-\frac{i+\varepsilon}{\hbar} t \widehat{H}_\varepsilon},$$

where the generator of this semigroup is defined as $\widehat{H}_\varepsilon := -\frac{\hbar^2}{2m}\Delta - \frac{V}{(i+\varepsilon)^2}$, and for which it is proved the following property of strong convergence:

$$(2) \quad U(t)\varphi \stackrel{L^2(\mathbb{R}^n)}{=} \lim_{\varepsilon \rightarrow 0^+} U_\varepsilon(t)\varphi, \quad \forall \varphi \in H^2(\mathbb{R}^n), \quad \forall t \geq 0.$$

The integral representations for this type of operators are:

$$U_\varepsilon(t)\varphi(x) = \int_{\mathbb{R}^n} U_\varepsilon(t, x, y)\varphi(y) dy,$$

where the kernel is constructed through a class of oscillating integrals:

$$(3) \quad U_\varepsilon(t, x, y) = \int_{\mathbb{R}^k} e^{\frac{i}{(1+\varepsilon^2)\hbar} S(t, x, y, u)} \rho_\varepsilon^\hbar(t, x, y, u) du.$$

In this type of representation it is important the use of the class of generating function S (weakly) quadratic at infinity relative to the family of Lagrangian submanifolds, graph of Canonical Transformation,

$$(4) \quad \begin{aligned} \Lambda_t &:= \{(y, \xi; x, p) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : (x, p) = \phi_H^t(y, \xi)\} \\ &= \{(y, \xi; x, p) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : p = \nabla_x S, \quad \xi = -\nabla_y S, \quad 0 = \nabla_u S\}, \end{aligned}$$

These generating functions, introduced and used by Chaperon, Laudenbach, Sikorav e Viterbo, are typical objects of *symplectic topology* and arises in many questions of Calculus of Variations, Morse Theory and Lusternik-Schnirelman Theory. Their frequent use in the actual framework (Schrödinger, Oscillating Integrals) is new, even respect to the near theory of Fourier Integral Operator.

In (4) ϕ_H^t is the group of canonical transformations from $T^*\mathbb{R}^n$ in $T^*\mathbb{R}^n$ that solves the related Hamilton's equations for $H(x, p) = \frac{p^2}{2m} + V(x)$. In other words, $(x, p)(t) := \phi_H^t(y, \xi)$ solves:

$$(5) \quad \begin{cases} \dot{x} = \nabla_p H(x, p), \\ \dot{p} = -\nabla_x H(x, p). \end{cases}$$

The construction of this type of integrals it is realized by series of operators strongly convergent to the regularized Propagator,

$$(6) \quad U_\varepsilon(t)\varphi = \sum_{j=0}^{\infty} B_{\varepsilon, j}(t)\varphi,$$

where the terms admits integral representations with kernels

$$B_{\varepsilon,j}(t, x, y) = \int_{\mathbb{R}^k} e^{\frac{i}{(1+\varepsilon^2)\hbar} S(t, x, y, u)} \rho_{\varepsilon,j}^\hbar(t, x, y, u) du,$$

and where the functions $\rho_{\varepsilon,j}^\hbar \in L^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k; \mathbb{C})$, $\forall \varepsilon > 0$, $\forall j \geq 0$, define an L^1 -convergent series:

$$\sum_{j=0}^{\infty} \rho_{\varepsilon,j}^\hbar = \rho_\varepsilon^\hbar \in L^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k; \mathbb{C}).$$

These considerations permit us to realize that the kernel of the regularized Propagator $U_\varepsilon(t)$ can be represented by an oscillating integral of phase function S and complex amplitude ρ_ε^\hbar depending on the choice of S , obtaining in this way the representation (3) announced above.

An important consequence of the existence of this type of representations consists in a finite dimensional Path Integral formulation of the Propagator. More precisely in [2] it has been proved that the classical mechanical Action functional

$$A[\gamma] = \int_0^t \frac{1}{2} m |\dot{\gamma}(s)|^2 - V(\gamma(s)) ds,$$

evaluated on an suitable space of curves $\Gamma(t, x, y) \subset H^1([0, t]; \mathbb{R}^n)$, that it is in fact a finite dimensional manifold, corresponds to a weakly quadratic global generating function S for Λ_t . Applying the general results described above in this particular case of generating function we prove the following representation:

$$U_\varepsilon(t, x, y) = \int_{\Gamma(t, x, y)} e^{\frac{i}{(1+\varepsilon^2)\hbar} A[\gamma]} P_\varepsilon^\hbar(d\gamma)$$

Where $P_\varepsilon^\hbar(d\gamma)$ is well defined, as the complex image measure of $\rho_\varepsilon^\hbar(t, x, y, u) du$ on $\Gamma(t, x, y)$. It is important to point out that the construction (3) is global in time, overcoming the problem related to the occurrence of caustics. This is the well known problem of the non-transversality of the Lagrangian submanifolds Λ_t that occurs after a critical time, and this behaviour does not permit the existence of a global generating function $S(t, x, y)$ (without auxiliary parameters). It is possible to overtake this problem by using the direct generalizations of this function, namely the class of global generating functions $S(t, x, y, u)$ involving auxiliary parameters $u \in \mathbb{R}^k$ that permit, as shown in (4), to generate the entire Lagrangian submanifold even in the presence of non-transversality.

Different techniques have been used, by different authors, to find representations of the Schrödinger Propagator. First of all we mention the WKB methods [8] where the solution of (1) is represented as

$$\psi(t, x) = A_\hbar(t, x) e^{\frac{i}{\hbar} S(t, x)}.$$

In this formula the *phase function* S , that it is defined only for $t \in [0, t_0]$ with t_0 small enough (relative to the nonexistence of caustics) solves the Hamilton-Jacobi's equation:

$$\frac{|\nabla S(t, x)|^2}{2m} + V(x) + \partial_t S(t, x) = 0,$$

and the *real amplitude function* admits a formal expansion $A_\hbar(t, x) = \sum_{j=0}^{\infty} \hbar^j A_j(t, x)$, where the terms are obtained through a recursive relation and where the zero order term solves the transport equation:

$$\partial_t A_0(t, x) + \nabla_x S(t, x) \cdot \nabla_x A_0(t, x) + \frac{1}{2} \Delta_x S(t, x) A_0(t, x) = 0.$$

The problem of the solution global in time it has been faced in the theory of Tunnel Canonical Operator [8], and in the theory of Fourier Integral Operators developed by Hörmander [5] and Duistermaat [3] (for more recent results see [7], [4]. Other results have been obtained by the use of Wigner Functions and Wigner Measures, see for example [11]. With respect to this techniques one of the most important differences with this work corresponds to the type of generating functions considered. While in the theory of Fourier Integral Operators are always used conic Lagrangian manifold, that is generated by 1-homogeneous functions, in this approach the generating functions quadratic at infinity have a central role. While in the first approach it is generalized the operation of *Fourier transform*, in the second an heuristic connection can be found in *Fresnel integrals*.

Respect to the problem to represent the Schrödinger Propagator with integrals defined on spaces of curves, known as Path Integrals, there are several approaches (for example [1], [10]. In particular Albeverio and Mazzucchi [1] proved an exact representation by the use of infinite dimensional Fresnel integrals, that provide a rigorous formulation of Feynman integral. We mention the work of Robbin and Salomon [9] that in the case of quadratic Hamiltonians obtained a finite dimensional phase space Path Integral in connection with metaplectic representation.

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Numerical Solution of PDEs for Semiconductor Modeling

ROBERTO BERTELLE (*)

Abstract. The simulation in electronic is the main way to shut down the time to market for new products. This paper is an introduction to the matter. In the first section, we present the basic model for a semiconductor device. Then, we give some numerical procedures to solve it. Finally, a simple example completes the paper. [Keywords: Device modeling, finite element.]

Sunto. *Soluzione numerica di EDP per la modellistica di semiconduttori.* La moderna elettronica usa in modo rilevante la simulazione al fine di contenere il tempo di immissione sul mercato di nuovi prodotti. Il presente lavoro introduce brevemente alla materia, fornendo un modello di dispositivo elettronico, descrivendo un algoritmo idoneo a risolverlo e applicandolo ad un semplice esempio

1 Introduction

The best way to test any new idea is to make it real. This is true also in the electronic industries. When a new, simple, device is fabricated and seriously tested in a laboratory most of the positive and limitations will, probably, be discovered. However, it's a very expensive and time consuming way to do so. Furthermore, it is really impossible to test an electronic device which may contain hundreds of millions of elementary devices in any of its possible working conditions. The modern microelectronic is based on such complex devices. For example, a graphic processor may contain over $300 \cdot 10^6$ elementary devices (mosfet).

Simulations may be used instead of the real device, at least at the beginning of its development stage, to get more inside the new technology, to enhance it and to obtain preliminary results. The simulations are less time consuming and much more less expensive with respect the corresponding tests on a real device (prototype). If, at the end of this simulation step, the new technology still remains a promising one, a real device may be fabricated and, successively, tested. However, the probability to obtain a good product at the first attempt is improved by the, preliminary, simulation step. As a statement, computer simulations greatly enhance the productivity in modern electronics industry.

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2 The Model

An elementary electronic device is a region Ω of the space filled with semiconductor, insulators and conductors materials where the electric variables satisfies the Van Roosbroeck system of equations [5]

$$(1) \quad \begin{cases} \nabla \cdot (-\lambda^2 \nabla \psi) &= p - n + C \\ \nabla \cdot (-\mu_n n \nabla \psi + \mu_n \nabla n) &= R \\ \nabla \cdot (-\mu_p p \nabla \psi - \mu_p \nabla p) &= -R. \end{cases}$$

The first equation is the Poisson equation whereas the last two are, in this order, the continuity equations for electron and hole. In eq. (1), ψ is the electrostatic potential, n and p are the free carrier densities for electron and hole with the corresponding current densities \vec{j}_n and \vec{j}_p , respectively. These latter ones can be written as

$$\begin{cases} \vec{j}_n &= -\mu_n n \nabla \psi + \mu_n \nabla n \\ \vec{j}_p &= -\mu_p p \nabla \psi + \mu_p \nabla p, \end{cases}$$

where μ_n and μ_p are the electron and hole mobilities. C is a, given, doping function, R is the Shockley–Read–Hall recombination function which reads as

$$R = \frac{np - n_i^2}{\tau_p(n + n_i) + \tau_n(p + n_i)},$$

for appropriate minority carrier lifetimes τ_p and τ_n . Finally, λ is the Debye length

$$\lambda = \sqrt{\frac{\varepsilon V_T}{q l_{ref}^2 C_{ref}}},$$

where ε is the permittivity of the material, q is the absolute value of the charge of one electron, $V_T = KT/q$ is the thermal voltage at the device temperature T , l_{ref} and C_{ref} are two references used for the lengths and the concentrations, respectively. K is the Boltzmann constant. The boundary conditions for the system (1) are, mainly, of two kind [2]. For the purpose of this paper, we split the frontier $\partial\Omega$ into a Dirichlet and a Neumann parts. Both are the union of a finite number of regular curves: Ohmic contacts for the first and insulating segments for the second. An Ohmic contact O fixes the values for n , p and ψ as

$$n_O = \ln \left(\frac{C_O + \sqrt{C_O^2 + 4n_i^2}}{2n_i} \right), \quad p_O = \ln \left(\frac{-C_O + \sqrt{C_O^2 + 4n_i^2}}{2n_i} \right), \quad \psi_O = V_O + \ln(n_O/n_i).$$

where n_i is the intrinsic carrier concentration, constant for any temperature T , and V_O is the external applied voltage to the contact O . Now, let $\vec{\nu}$ be the normal outward unit vector to an insulating segment; then we have

$$\vec{j}_n \cdot \vec{\nu} = \vec{j}_p \cdot \vec{\nu} = 0, \quad \nabla \psi \cdot \vec{\nu} = 0.$$

As a last note, we recall the quasi–Fermi potentials ϕ_n and ϕ_p for electron and hole. We have

$$(2) \quad n = n_i \exp(\psi - \phi_n), \quad p = n_i \exp(\phi_p - \psi).$$

This completes the model.

3 Numerical Solution

We may note that the three equations of problem (1) are coupled each other. One way to attempt for its numerical solution is to use a decoupling procedure [1] on the corresponding discretized version. We denote by Ψ , Φ_n , Φ_p the three unknown M –dimensional vectors coming from the discretization procedure which correspond to the electrostatic potential and to the quasi–Fermi potentials. Also, let \mathbf{g}_ψ , \mathbf{g}_{ϕ_n} , \mathbf{g}_{ϕ_p} the maps, from $R^{3 \times M} \rightarrow R^M$, corresponding to, respectively, the discretization of the Poisson and the electron and hole continuity equations. The decoupling procedure find a solution in an iterative fashion where each equation is solved independently from the others. So, chosen appropriate initial guesses $\{\Psi^{(0)}, \Phi_n^{(0)}, \Phi_p^{(0)}\}$ and a stopping criteria based on the error norm,

$$e^{(m)} = \max(\{e_\psi^{(m)}, e_{\phi_n}^{(m)}, e_{\phi_p}^{(m)}\}),$$

where the upper script $^{(m)}$ refers to the m –th iteration and

$$e_\psi^{(m)} = \frac{\|\Psi^{(m+1)} - \Psi^{(m)}\|}{\|\Psi^{(m)}\|}, \quad e_{\phi_n}^{(m)} = \frac{\|\Phi_n^{(m+1)} - \Phi_n^{(m)}\|}{\|\Phi_n^{(m)}\|}, \quad e_{\phi_p}^{(m)} = \frac{\|\Phi_p^{(m+1)} - \Phi_p^{(m)}\|}{\|\Phi_p^{(m)}\|},$$

the algorithm reads as follows.

DECOUPLING APPROACH

```

set  $e_{max}$ 
guess  $\{\Psi^{(0)}, \Phi_n^{(0)}, \Phi_p^{(0)}\}$ 

 $m \leftarrow 1$ 
while ( $e^{(m)} > e_{max}$ )
    solve for  $\Psi^{(m+1)}$  eq.  $\mathbf{g}_\psi(\Psi^{(m+1)}, \Phi_n^{(m)}, \Phi_p^{(m)}) = \mathbf{0}$ 
    solve for  $\Phi_n^{(m+1)}$  eq.  $\mathbf{g}_{\phi_n}(\Psi^{(m+1)}, \Phi_n^{(m+1)}, \Phi_p^{(m)}) = \mathbf{0}$ 
    solve for  $\Phi_p^{(m+1)}$  eq.  $\mathbf{g}_{\phi_p}(\Psi^{(m+1)}, \Phi_n^{(m+1)}, \Phi_p^{(m+1)}) = \mathbf{0}$ 
    compute the error  $e^{(m+1)}$ 
     $m \leftarrow m + 1$ 
endwhile

```

Each equation is used to update one of the three vectors, considering the other two as known.

The Finite Element Method (FEM) discretization for a one dimensional device proceeds as follows. With respect to an x reference axis, let the corresponding domain be $\Omega = [-L, L]$. At $x = -L$ we have an Ohmic contact which supplies the external voltage V_L . In the same manner, at $x = L$ an Ohmic contact supplies for the external voltage V_R . The domain Ω is decomposed into N non overlapping intervals I_k , $k = 1, \dots, N$ using $N + 1$ points x_k , $k = 0, \dots, N$, with $x_0 = -L$, $x_N = L$ and $x_{k-1} < x_k$, $k = 1, \dots, N$. We assume that the k -th interval is $I_k = [x_{k-1}, x_k]$ and $h_k = x_k - x_{k-1}$, $k = 1, \dots, N$ is the corresponding width. We can see a simple example of this decomposition in Fig. 1. Now, we give some notes about the discretization of the equations of the model.

3.1 The Poisson Equation

The FEM solution is a linear piecewise continuous approximation of the true solution, i.e., $\psi(x)|_{I_k}$, $k = 1, \dots, N - 1$ is a polynomial of at most degree one. We denote this space by \mathcal{P}_1 . So, it is completely defined by its nodal values $\Psi = [\psi_0, \psi_1, \dots, \psi_{N-1}, \psi_N]^T$. We assume the same for ϕ_n and ϕ_p with nodal values Φ_n and Φ_p . Note that ψ_0 and ψ_N are given throughout the boundary conditions.



Figure 1. Decomposition of the one dimensional domain $\Omega = [-L, L]$ using $N = 4$ intervals and $N + 1 = 5$ nodes. Note that $x_0 = -L$ and $x_N = L$. The k -th interval I_k has the domain $[x_{k-1}, x_k]$, $k = 1, \dots, N$.

Writing the weak formulation of the Poisson equation in the approximating space \mathcal{P}_1 leads to the non-linear system

$$g_k(\Psi) = \int_{I_k} \lambda^2 \frac{d\psi}{dx} \frac{dv_k}{dx} dx - \int_{I_k} \left\{ n_i e^{[\phi_p(x) - \psi(x)]} - n_i e^{[\psi(x) - \phi_n(x)]} + C(x) \right\} v_k(x) dx = 0,$$

where $k = 1, \dots, N - 1$ and $v_k(x)$, $k = 0, \dots, N$ is a base for \mathcal{P}_1 defined as $v_k(x_j) = \delta_{j,k}$, $\delta_{j,k}$ being the Kronecker symbol. This non-linear system is solved via a damped Newton method where both the Jacobian and the g_k 's may be evaluated using Gaussian quadrature.

3.2 The Continuity Equations

The continuity equations are Diffusion–Advection–Reaction (DAR) equations; so, to avoid numerical instabilities at any useful mesh, they need some stabilization. The Sharfetter Gummel is a classical way to obtain it [4]. Let us show the main idea referring to the simplest case $R = 0$ and constant mobilities. Thus, the current densities are constants all inside the device; we may write $j_n = J_n$ and $j_p = J_p$. Then, we solve the continuity equations inside each interval with respect to the nodal concentrations. For example, for

the electron ones, we have

$$(3) \quad \begin{cases} \mu_n \left(-n \frac{d\psi}{dx} + \frac{dn}{dx} \right) = J_n, & x \in I_k = [x_{k-1}, x_k] \\ n(x_{k-1}) = n_{k-1}, & n(x_k) = n_k. \end{cases}$$

Letting $\Delta\psi_k = \psi_k - \psi_{k-1}$, the solution of this differential equation gets to

$$(4) \quad J_n = \frac{\mu_n}{h_k} [B(\Delta\psi_k)n_k - B(-\Delta\psi_k)n_{k-1}],$$

where $B(t) = t/(\exp(t) - 1)$ is the Bernoulli function. Since j_n is the same constant both inside the k -th and in the $(k+1)$ -th intervals, we lead immediately to the linear system in the nodal unknowns n_k , $k = 1, \dots, N-1$

$$(5) \quad k_{k,k-1}^{(n)} n_{k-1} + k_{k,k}^{(n)} n_k + k_{k,k+1}^{(n)} n_{k+1} = 0, \quad k = 1, \dots, N-1$$

where

$$(6) \quad \begin{cases} k_{k,k-1}^{(n)} &= -\mu_n B(-\Delta\psi_{k-1})/h_{k-1} \\ k_{k,k}^{(n)} &= \mu_n B(\Delta\psi_{k-1})/h_{k-1} + \mu_{n,k} B(-\Delta\psi_k)/h_k \\ k_{k,k+1}^{(n)} &= -\mu_n B(\Delta\psi_k)/h_k. \end{cases}$$

Note that n_0 and n_N are computed using the boundary conditions and so are known values. Finally, the same derivation can be done for the hole continuity equation.

4 Example

In this section we give an application of the previous theory. The implementation uses a matlab 5.3 vectorized code. Another example may be found in the companion paper due to M.R. Russo [3].

We consider a silicon abrupt p-n junction diode with domain $\Omega = [-L, L]$, $L = 1 \mu m$. The doping function is $C(x) = N_0 \operatorname{sign}(x)$ where $N_0 = 10^{22} m^{-3}$. The device temperature is $T = 300 K$ and the Debye length is $\lambda^2 \approx 1.67 \cdot 10^{-3}$. The mobilities are constant and equal to $\mu_n = 0.06 V/(m^2 s)$, $\mu_p = 0.03 V/(m^2 s)$. To the left Ohmic contact we apply an external voltage $V_L = 0 V$ and to the right Ohmic contact we apply an external voltage $V_R = 1 V$. So, the junction is under reverse bias. Although the derivation of the continuity equation was done for a vanishing recombination, in this example, to be more realistic, we have considered R with $\tau_n = \tau_p = 10^{-5} s$. We use a mesh with $N = 100$ points equally spaced in the domain Ω and $n_G = 2$ Gaussian nodes. The stopping criteria uses $e_{max} = 10^{-6}$. The initial guesses $\Psi^{(0)}$, $\Psi_n^{(0)}$ and $\Psi_p^{(0)}$ are chosen as

$$\psi_h^{(k)} = \begin{cases} V_L + V_{bi,L}, & \text{if } x_h \in [-L, 0] \\ V_R + V_{bi,R}, & \text{if } x_h \in [0, L], \end{cases} \quad \phi_{n,h}^{(k)} = \phi_{p,h}^{(k)} = \begin{cases} V_L, & \text{if } x_h \in [-L, 0] \\ V_R, & \text{if } x_h \in [0, L], \end{cases}$$

where x_k is the k -th node of the mesh and $V_{bi,L} \approx -0.347\text{ V}$ and $V_{bi,R} \approx 0.347\text{ V}$ are, respectively, the left and right Ohmic contacts built-in potentials. Fig. 2 shows the potentials (on the right) and the free carrier concentrations (on the left). The algorithm needs three steps to fulfill the stopping criteria. As a final remark, the CPU time is about 0.13 s on a Core Duo Processor laptop with a core frequency of 1.66 GHz (T2300) and 1GB dual Channel RAM memory. For comparison purposes, using 1000 points the CPU time rises to 1.5 s.

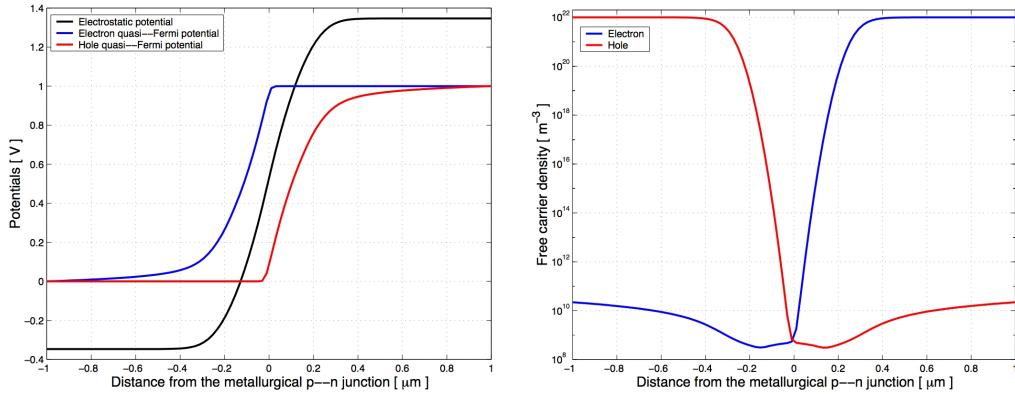


Figure 2. Potentials and free carried densities for an abrupt p–n junction under 1 V reverse bias condition.

5 Conclusions

The present paper is only a very short and simple introduction on the numerical simulation of electronic devices. However, we have to say that the model presented in sec. 2 is the simplest, possible, one. More sophisticated models have been developed and are currently used, such as hydrodynamic or quantistic models. Furthermore, we have considered one dimensional models only. In a commercial simulator we are able to consider, also, two and three dimensional models. Both aspects increases massively the difficult of the problem.

Finally, a note to the interested on the matter reader. More material can be found in the book of Markowich [2] and the references therein. This book, as well as many others, is a suitable, although not easy, starting point.

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Dualità di Tannaka e Gruppoidi di Lie

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Sunto. La teoria della dualità di Tannaka afferma che la struttura di un gruppo di Lie compatto è interamente codificata (a meno di isomorfismo) nella categoria delle rappresentazioni lineari di quest'ultimo provvista della struttura tensoriale standard (il consueto prodotto tensoriale di rappresentazioni). Orbene, un gruppo di Lie è un esempio molto speciale di ‘gruppoide di Lie’, e per i gruppoidi di Lie esiste ancora un concetto naturale di ‘rappresentazione lineare’, che si riduce alla solita nozione nel caso dei gruppi. Ci si chiede: è ancora possibile ottenere un teorema di dualità in questo contesto più generale?

La *teoria dei gruppoidi di Lie* costituisce un linguaggio matematico in grado di unificare concettualmente la trattazione di diverse nozioni geometriche, sia tradizionali che moderne. Per esempio, se da un lato essa generalizza la teoria dei gruppi di Lie e la teoria delle azioni di gruppi di Lie su varietà differenziabili, dall'altro si colloca ormai alla base della teoria delle foliazioni e della teoria degli ‘orbifolds’ [4]. In tempi più recenti, si è avuto modo di apprezzare l'utilità dei gruppoidi di Lie in geometria non-commutativa [1], nell'analisi su spazi singolari, in geometria simplettica e nella teoria della deformazione quantistica.

Dicesi *gruppoide* una qualunque categoria piccola \mathcal{G} i cui morfismi siano tutti invertibili, in altre parole degli isomorfismi. Un gruppoide è dunque individuato da una coppia di insiemi $Ar(\mathcal{G})$ e $Ob(\mathcal{G})$, i cui elementi vengono rispettivamente detti *frecce* e *oggetti*, e da varie applicazioni strutturali: le mappe ‘source’ e ‘target’ $s, t : Ar(\mathcal{G}) \rightarrow Ob(\mathcal{G})$, altrimenti dette ‘dominio’ e ‘codominio’; la *legge di composizione*, che ad ogni coppia (h, g) di frecce componibili, i.e. tali che $sh = tg$, assegna la freccia composta hg ; la *sezione identica* $Ob(\mathcal{G}) \rightarrow Ar(\mathcal{G})$, che all'oggetto x associa la freccia identica $x \rightarrow x$; ed infine la *mappa di inversione* $Ar(\mathcal{G}) \rightarrow Ar(\mathcal{G})$, che trasforma ciascuna freccia $x \rightarrow y$ nella relativa inversa $y \rightarrow x$. Quando gli insiemi $Ar(\mathcal{G})$ e $Ob(\mathcal{G})$ sono entrambi dotati di una struttura di varietà differenziabile di classe C^∞ , qui detta *varietà liscia*, e tutte le applicazioni strutturali appena elencate risultano infinite volte differenziabili (affinché ciò abbia pienamente senso, si postula che la mappa s sia una somersione), si ottiene un gruppoide *di Lie*. Si dirà che la varietà $M = Ob(\mathcal{G})$ è la *base* di \mathcal{G} .

Ad esempio, ogni gruppo di Lie G può essere identificato con il gruppoide di Lie \mathcal{G} per cui $Ar(\mathcal{G}) = G$, $Ob(\mathcal{G}) = \{\star\}$ e la composizione di frecce coincide con l'operazione di gruppo. Più in generale, supponiamo che un gruppo di Lie G agisca in modo liscio su una varietà M . L'azione è allora interamente codificata nel relativo *gruppoide delle traslazioni*

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$\mathcal{G} = G \ltimes M$, definito come segue: $Ar(\mathcal{G}) = G \times M$ è la varietà prodotto, $Ob(\mathcal{G}) = M$, s, t sono rispettivamente la proiezione sul secondo fattore e l'azione $G \times M \rightarrow M$, e la composizione di frecce è data dall'equazione $(h, y)(g, x) = (hg, x)$. Come ulteriore esempio, supponiamo di avere un fibrato vettoriale liscio E su una varietà M . Ad esso resta associato un gruppido di Lie di base M , il *gruppido lineare* $GL(E)$, per il quale le frecce $x \rightarrow y$ sono tutti e soli gli isomorfismi lineari $E_x \rightarrow E_y$.

Un *morfismo* di gruppoidi di Lie è un funtore di classe C^∞ . Più esplicitamente, un morfismo $\mathcal{G} \rightarrow \mathcal{H}$ è dato da una coppia di mappe di classe C^∞ , che indichiamo con $Ob(\Phi) : Ob(\mathcal{G}) \rightarrow Ob(\mathcal{H})$ e $Ar(\Phi) : Ar(\mathcal{G}) \rightarrow Ar(\mathcal{H})$, verificanti le familiari condizioni che definiscono un funtore. Una *rappresentazione* di \mathcal{G} è un morfismo di gruppoidi di Lie $\rho : \mathcal{G} \rightarrow GL(E)$ verso il gruppido lineare di un fibrato vettoriale di rango localmente finito su M , tale che $Ob(\rho) = id$.

Quando due anni fa incominciai a interessarmi di gruppoidi di Lie, da tempo gli esperti del settore lamentavano la mancanza di una teoria delle rappresentazioni altrettanto ricca e sviluppata che nel caso classico dei gruppi. Se ciò fosse dovuto alle scarse energie fino ad allora investite nella ricerca, data la relativa giovinezza del soggetto, oppure a delle difficoltà intrinseche, non era dato sapere. Per un verso, si constatabano seri problemi nell'adattamento di alcune costruzioni standard della teoria classica; per esempio, è impossibile estendere ai gruppoidi la familiare rappresentazione aggiunta $ad : G \rightarrow GL(\mathfrak{g})$, che tanta importanza riveste nella teoria dei gruppi di Lie. D'altronde, non era chiaro nemmeno cosa accadesse restringendo il campo d'indagine a gruppoidi dotati di caratteristiche tali da consentire la realizzazione di queste costruzioni. Riprendendo l'esempio della rappresentazione aggiunta, essendo quest'ultima certamente definibile nel caso dei fibrati in gruppi di Lie, a priori non vi sono argomentazioni contro l'esistenza di una teoria delle rappresentazioni ‘soddisfacente’ per questo tipo di gruppoidi. A questo punto non si può proseguire la discussione senza avere prima precisato i requisiti in base ai quali la teoria delle rappresentazioni di un data categoria di gruppoidi di Lie sia da ritenersi ‘soddisfacente’.

Era chiaro agli specialisti quale fosse il criterio da seguire per ottenere una definizione formale del problema: quest'ultimo andava opportunamente enunciato e analizzato nell'ottica della teoria di dualità di Tannaka. È precisamente nella forma seguente che I. Moerdijk mi sottopose inizialmente la questione. Sia \mathcal{G} un gruppido di Lie, e sia $Rep(\mathcal{G})$ la categoria delle sue rappresentazioni, i cui morfismi sono per definizione le mappe \mathcal{G} -equivarianti di fibrati vettoriali. La categoria $Rep(\mathcal{G})$ è dotata di una struttura tensoriale lineare, rigida e additiva [3,2]. Sia $\omega_{\mathcal{G}} : Rep(\mathcal{G}) \rightarrow Vec(M)$ il funtore, a valori nella categoria dei fibrati vettoriali di rango localmente finito sulla base di \mathcal{G} , che alla rappresentazione $R = (E, \rho)$ associa il fibrato E . Trattasi di un funtore tensoriale (*ibid.*), lineare e fedele. Alla coppia di dati $(Rep(\mathcal{G}), \omega_{\mathcal{G}})$ resta associato un gruppido topologico di base M , il *gruppido tannakiano* $\mathcal{T}(\mathcal{G})$, definito come segue: dati $x, y \in M$, le frecce $x \rightarrow y$ sono le trasformazioni naturali λ del funtore $\{R = (E, \rho) \mapsto E_x\}$ verso il funtore $\{R = (E, \rho) \mapsto E_y\}$, compatibili con la struttura tensoriale di $Rep(\mathcal{G})$, vale a dire tali che $\lambda(R \otimes S) = \lambda(R) \otimes \lambda(S)$ e $\lambda(1) = id$; $\mathcal{T}(\mathcal{G})$ è dotato della topologia debole generata dalla collezione di applicazioni $\{\lambda \mapsto \lambda(R)\}$, al variare di R tra le rappresentazioni di \mathcal{G} . Il gruppido tannakiano è legato a \mathcal{G} dal *morfismo di inviluppo* $\pi : \mathcal{G} \rightarrow \mathcal{T}(\mathcal{G})$, definito dall'equazione $\pi(g)(E, \rho) = \rho(g)$.

Ora, in base alla teoria classica di Tannaka, risalente alla fine degli anni '30, si ottiene che tale morfismo è un *omeomorfismo* quando $\mathcal{G} = G$ è un gruppo di Lie *compatto*.

Seguendo la terminologia di [5], diremo che un gruppido di Lie \mathcal{G} è *riflessivo* se il suo morfismo di inviluppo π è un isomorfismo di gruppoidi topologici^(*), in altre parole un omeomorfismo. Sulla base del teorema di riflessività per i gruppi di Lie compatti ricordato poc'anzi, risulterebbe spontaneo formulare la seguente ipotesi: ogni gruppido di Lie *proprio* [5] è riflessivo. O, se conveniamo di chiamare la teoria delle rappresentazioni di una data classe \mathfrak{C} di gruppoidi di Lie ‘soddisfacente’ quando tutti i gruppoidi *propri* appartenenti a \mathfrak{C} sono riflessivi: la teoria delle rappresentazioni dei gruppoidi di Lie è soddisfacente.

Purtroppo, tale ipotesi è decisamente falsa. La non-riflessività, lungi dall'essere un fenomeno patologico nel dominio dei gruppoidi di Lie propri, si manifesta addirittura nello studio dei casi più semplici. Non essendo questo tuttavia il luogo adatto per avviare una discussione di questo interessante fenomeno, rimando il lettore incuriosito a [5], Cap. 4, §1. Mi limito a far presente che un controesempio da me elaborato stabilisce in modo inequivocabile l'impossibilità di ottenere un teorema di dualità tannakiana per i gruppoidi di Lie (propri) servendosi del concetto ‘classico’ di rappresentazione. Una attenta analisi rivela che tale risultato negativo dipende essenzialmente da un fattore, e precisamente dall'eccessiva ‘rigidità’ dei fibrati vettoriali lisci (o anche solo continui), localmente liberi, di rango localmente finito: in generale, non è possibile riprodurre adeguatamente la complicata struttura *non-lineare* di un gruppido finché ci si limita ad usare come supporto per le sue rappresentazioni fibrati *lineari* di tale tipo. Sorge allora spontanea la domanda se sostituendo la categoria $\text{Vec}(M)$ con una opportuna categoria tensoriale $\text{Mod}(M)$ di fasci su M , non necessariamente localmente liberi e quindi più ‘malleabili’, e riformulando in modo appropriato la nozione di rappresentazione, sia possibile estendere il teorema di dualità tannakiana al caso generale dei gruppoidi di Lie propri. In base al lavoro da me svolto, questo obiettivo risulta essenzialmente equivalente a risolvere una famosa congettura di A. Weinstein [6].

Precisamente, si può fornire una descrizione assiomatica delle proprietà che una generica categoria tensoriale $\text{Mod}(M)$ di fasci di \mathcal{O}_M -moduli deve possedere affinché, sulla base del naturale concetto di ‘azione di \mathcal{G} su un fascio tramite bisezioni locali’, sia possibile definire una categoria di rappresentazioni, supportate dagli oggetti di $\text{Mod}(M)$, a partire dalla quale sia poi possibile ricostruire il gruppido \mathcal{G} secondo il metodo di Tannaka. Supponiamo di disporre di una costruzione Mod la quale faccia corrispondere a ciascuna varietà liscia X una categoria tensoriale $\text{Mod}(X)$ di fasci di \mathcal{O}_X -moduli, e a ciascuna mappa liscia $f : X \rightarrow Y$ un funtore tensoriale $f^* : \text{Mod}(Y) \rightarrow \text{Mod}(X)$. Una *azione di \mathcal{G} su $\mathcal{F} \in \text{Mod}(M)$ tramite bisezioni locali* sarà individuata da un morfismo $\rho : s^*\mathcal{F} \rightarrow t^*\mathcal{F}$ in $\text{Mod}(\mathcal{G})$, ove \mathcal{G} indica il gruppido delle bisezioni locali di \mathcal{G} , cf [4] p. 115, e $s, t : \mathcal{G} \rightarrow M$ le rispettive mappe strutturali. Tale morfismo dovrà essere moltiplicativo e preservare gli elementi neutri nel senso appropriato. La categoria delle \mathcal{G} -azioni su $\text{Mod}(M)$ è di gran lunga troppo grande per poter svolgere il ruolo di categoria delle rappresentazioni di \mathcal{G} : quest’ultima categoria andrà invece isolata come sottocategoria piena della prima. Ora,

^(*)Evito di parlare della struttura differenziabile. Risulta dal mio lavoro che la condizione puramente topologica è sufficiente.

si può far vedere come l'enunciato ‘ogni grupperoide proprio è riflessivo’ (usando le azioni coerenti su fasci al posto delle rappresentazioni classiche!) sia equivalente all’enunciato ‘ogni grupperoide proprio è localmente linearizzabile’, che è la congettura a cui mi riferivo sopra. (Attenzione. [6] contiene un errore. L’autore non dimostra la congettura generale, contrariamente a quanto afferma.)

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Acceleration Techniques applied to Semiconductor Device Simulation

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Abstract. We present some extrapolation methods in order to accelerate the convergence of vectorial sequences. Although deriving from different philosophies they are able to accelerate the iterative process of convergence without any information on the way the process itself is generated. So, they are applicable to linear and nonlinear problems such as the Gummel map.

Sunto. *Tecniche di accelerazione applicate alla modellistica di semiconduttori.* Vengono presentati alcuni metodi di estrapolazione per accelerare la convergenza di successioni vettoriali. I metodi presi in analisi, sebbene derivino da filosofie differenti, sono in grado di accelerarne il processo iterativo di convergenza, anche senza nessuna informazione sul modo in cui esse vengono generate. Questo li rende applicabili a diverse tipologie di problemi, lineari e non lineari come ad esempio la mappa di Gummel.

1 Motivations

An important problem that arises in many scientific and engineering applications is to find or approximate limits of infinite sequences (x_n). The elements x_n of such sequences can show up from different problems: partial sums of infinite series, approximations from fixed-point iterations of linear and nonlinear systems of equations, numerical quadrature approximations to finite or infinite range integrals, etc. These sequences may converge slowly, and their direct use to approximate limits is quite limited or expensive.

An effective remedy for these problems is the application of extrapolation methods (or convergence acceleration methods) to the given sequences. Loosely speaking, an extrapolation method takes a finite and, hopefully, small number of terms of x_n and processes them in some way; the initial sequence may be transformed into another sequence converging to the same limit, but faster than the initial one. It is possible to apply these methods to sequences of scalar terms or vector ones, deal with scalar and vector extrapolation methods. The most known extrapolation schemes are the Richardson's extrapolation algorithm and the Aitken's Δ^2 process. During the last thirty years, some books were written on the subject and various comparative studies were done in relation to some important sub-

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classes of convergence acceleration problems. Scalar sequences transformations have also been extensively studied from a theoretical point of view.

The situation is more complicated and more interesting for vector sequences; many iterative processes used in numerical analysis lead in fact to vector sequences. In this case it is always possible to apply a scalar acceleration procedure componentwise, however this strategy does not take into account connections which may exist between the various components, and it also runs the risk that some transformed components fail to exist or may be quite large numerically. These difficulties emphasize the necessity of methods specially built for vector sequences.

We will consider four vector extrapolation methods which can be classified in two categories: the polynomial methods and the ϵ -algorithms. An important property of these methods is that they do not require an explicit knowledge of the sequence generator, so they could be applied directly to the solution of linear and nonlinear systems.

In particular we deal with one step of numerical device simulation which refers to the single, basic, component. This requires the physical model of the component to be based on a system of Partial Differential Equations (PDE). The PDE system is solved via the Gummel map using a nonlinear iterative method. We apply these vector extrapolation methods to the generated vector sequence, to improve the rate of convergence.

It will follow an essential background material on sequence transformations and extrapolation methods for gaining a deeper understanding of applications and numerical examples, the interested reader should consult [3,8] for more details.

2 Vector extrapolation methods

Let us consider a scalar sequence x_n that converges to a limit s , but whose convergence needs to be accelerated. An extrapolation method consists of transforming this sequence into a new one y_n , by a sequence transformation $T : (x_n) \rightarrow (y_n)$. The transformation T is said to accelerate the convergence of the sequence if and only if y_n converges to s faster than x_n , i.e.

$$\lim_{n \rightarrow \infty} (x_n - s)/(y_n - s) = 0$$

Now we take into account sequences of vectors in \mathbb{R}^N , as said before to accelerate the convergence of such sequences it is always possible to apply a scalar algorithm componentwise, however vector sequence transformations specially built for that purpose are more powerful. We give here a survey of the most popular vector extrapolation methods, which can be classified into two categories: the polynomial methods and the ϵ -algorithms.

2.1 Polynomial methods

Let us consider a sequence of vectors in \mathbb{R}^N , $\{x_i\}$, $i = 0, 1, 2, \dots$, with limit s . We suppose that this sequence is generated iteratively by an operator g , which could be linear or non linear, $x_{i+1} = g(x_i)$. Extrapolation methods are capable to accelerate the convergence of the given vector sequence also without no information on the operator g , but only

by a linear combination of the sequence elements. Now we assume that this sequence is generated by a linear process

$$x_{i+1} = g(x_i) = Gx_i + d.$$

The vector sequence may not necessarily converge, but if we assume that G does not have eigenvalue 1, the iteration has a unique fixed point s , which is the limit of the sequence,

$$\lim_{i \rightarrow \infty} x_i = Gs + d = s$$

We can say that, given a sequence of vectors $\{x_i\}$, $i = 0, 1, 2, \dots$, with the property that the unknown operator that generates the vector sequence has a unique fixed point s on a given set S , a vector extrapolation method extrapolates approximations $s_j \in S$ of s using the known elements of the given vector sequence.

By this definition vector extrapolation methods approximate s , and the distance between s and an element of the given vector sequence is referred as the *error*, $e_i = s - x_i$.

If the vector sequence is generated by a linear iteration, the error in step i can easily be connected to the initial error

$$e_i = s - x_i = Gs + d - Gx_{i-1} - d = Ge_{i-1} = G^i e_0 ;$$

then $x_i = s - G^i e_0$.

Considering a linear combination of k elements of the sequences, with k an integer less than N ,

$$(1) \quad \sum_{i=0}^k c_i x_i = \sum_{i=0}^k c_i (s - G^i e_0) = s \sum_{i=0}^k c_i - \sum_{i=0}^k c_i G^i e_0$$

the basic idea of vector extrapolation methods is to find the previous linear combination of certain elements of the vector sequence such that the error term is eliminated. That is find c_i , $i = 0, 1, \dots, k$ such that

$$\sum_{i=0}^k c_i = 1 \quad \text{and} \quad \sum_{i=0}^k c_i G^i e_0 = 0$$

so that the linear combination of the sequence elements (1) reduces to s .

Thus the problem of vector extrapolation with polynomial methods can be reduced to find the minimum value of k for which the second condition holds, thus to finding the minimal polynomial of G with respect to the initial error e_0 . If the sum of the coefficients of the minimal polynomial c_i , $i = 0, 1, \dots, k$ is normalized to 1, the extrapolated limit s is given by

$$s = \sum_{i=0}^k c_i x_i.$$

In a general problem, G is not known, and obviously also e_0 is not known, but it is possible to overcome both these problems with the help of differences vector:

$$u_i = \Delta x_i = g(x_i) - x_i = x_{i+1} - x_i, \quad w_i = \Delta u_i = \Delta^2 x_i.$$

Then u_i can be connected to G as

$$u_i = x_{i+1} - x_i = Gx_i + d - Gx_{i-1} - d = Gu_{i-1} = G(G(u_{i-2})) = G^i u_0.$$

It is possible to prove that the minimal polynomial of G with respect to e_i , $\forall i$, is also the minimal polynomial of G with respect to u_i . After this assertion, we can replace the initial error e_0 with the difference $u_0 = x_1 - x_0$ and find the minimal polynomial of G with respect to u_0 .

$$\sum_{i=0}^k c_i G^i e_0 = \sum_{i=0}^k c_i G^i u_0 = \sum_{i=0}^k c_i u_i = 0.$$

In conclusion the problem can be reduced to find the coefficient c_i , satisfying

$$\sum_{i=0}^k c_i u_i = 0, \quad \sum_{i=0}^k c_i = 1$$

so that the limit is expresses as a linear combination only of the terms of the sequence

$$s = \sum_{i=0}^k c_i x_i.$$

After this briefly presentation of the extrapolation method's idea, we show some more details. In this work we treat with two particular polynomial methods, the MPE and RRE. For these methods the approximation $s_k^{(n)}$ of s is given by solving the linear system

$$\Lambda \zeta = \alpha$$

where Λ is an $(N \times k)$ matrix, α is an N -dimensional column vector, and $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_{k-1})^T$ is the vector of the coefficients. This is an overdetermined system which has to be solved by using linear least squares. The choice of the matrix Λ and of the vector α depend on the particular extrapolation method used.

For the Minimal Polynomial Extrapolation (MPE) method [4], $\Lambda = [u_n, u_{n+1}, \dots, u_{n+k-1}]$ and $\alpha = -u_{n+k}$. After solving the system and setting $\zeta_k = 1$, we compute

$$\gamma_j = \frac{\zeta_j}{\sum_{i=0}^k \zeta_i}, \quad j = 0, 1, \dots, k$$

and the approximation of s is

$$(2) \quad s_k^{(n)} = \sum_{j=0}^k \gamma_j x_{n+j}.$$

For the Reduced Rank Extrapolation (RRE) method [5,6], Λ and α are, respectively, $\Lambda = [w_n, w_{n+1}, \dots, w_{n+k-1}]$ and $\alpha = -u_n$. The least squares approximation of the system leads to

$$(3) \quad s_k^{(n)} = x_n + \sum_{j=0}^{k-1} \zeta_j u_{n+j}.$$

From the previous formulae we could assert that a polynomial algorithm finds the approximation of s as a weighted average of $k+1$ terms, related with the given sequence, where the k independent weights are found by solving a linear system of dimension $(N \times k)$.

2.2 The ε -algorithms

Let (x_n) be a scalar sequence which converges to s . A simple recursive formula to implement the *Shanks transformation* [7], is the scalar ε -algorithm proposed by [9], defined as

$$(4) \quad \varepsilon_{-1}^{(n)} = 0, \quad \varepsilon_0^{(n)} = x_n, \quad n = 0, 1, \dots$$

$$(5) \quad \varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + [\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}]^{-1} \quad k, n = 0, 1, \dots$$

In order to generalize the scalar ε -algorithm to the vector case, Wynn suggested the interpretation of the 'inverse' of a column vector z as the *Samelson inverse*

$$z^{-1} = z / \|z\|^2,$$

and the vector version of (4)-(5) is called the *Vector Epsilon Algorithm* (VEA). This algorithm suffers the theoretical defect that it is difficult to find its kernel, that is the set of sequences for which the exact limit is obtained. Due to this drawback, another generalization of ε -algorithm for vector sequences, the *Topological Epsilon Algorithm* (TEA), has been proposed later [2]. It can be expressed as a ratio of determinants, and computed by the following recursive algorithm:

$$\begin{aligned} \varepsilon_{-1}^{(n)} &= 0, \quad \varepsilon_0^{(n)} = x_n, \quad n = 0, 1, \dots \\ \varepsilon_{2k+1}^{(n)} &= \varepsilon_{2k-1}^{(n+1)} + y / (\Delta \varepsilon_{2k}^{(n)}) \\ \varepsilon_{2k+2}^{(n)} &= \varepsilon_{2k}^{(n+1)} + \Delta \varepsilon_{2k}^{(n)} / (\Delta \varepsilon_{2k+1}^{(n)}, \Delta \varepsilon_{2k}^{(n)}) \end{aligned}$$

where y is a chosen fixed vector, (\cdot, \cdot) is the Euclidian inner product, and Δ is the usual forward difference operator which, in our case, acts on the upper indexes.

3 Applications

The numerical approximation of nonlinear partial differential equations requires the computation of large nonlinear systems, that are typically solved by iterative schemes. At each step of the iterative process, a large and sparse linear system has to be solved, and the amount of time elapsed per step grows with the dimensions of the problem. As a consequence, the convergence rate may become very slow, requiring massive cpu-time to compute the solution. In all such cases, it is important to improve the rate of convergence of the iterative scheme. This can be achieved, for instance, by vector extrapolation methods. We apply some vector extrapolation methods to the electronic device simulation in order to improve the rate of convergence of the family of Gummel decoupling algorithms.

For an introduction to the Gummel map, as well as for a short review on physical models for electronics devices, we refer to the companion paper due to R. Bertelle [1].

3.1 The MOS capacitor

A MOS capacitor is a thin layer of insulator between two, distinct, (semi-) conductor materials. One contact is a conductor sheet made on one face of the insulator. The other contact, made on the opposite face, is a thick semiconductor layer. In this example, the insulator has a thickness $t_i = 0.1 \mu m$ and a relative dielectric constant $\kappa_i = 3.9$. The semiconductor has a constant doping $C = -10^{22} m^{-3}$, a relative dielectric constant $\kappa_s = 11.7$ and is $t_s = 1 \mu m$ thick. Two external voltages V_i and V_s are applied to the capacitor throughout the two contacts, one on the insulator and another one on the semiconductor. The total voltage applied V is defined as $V = V_i - V_s$. For a steady-state solution, the insulator insures $J_n = J_p = 0$ and $R = 0$. So, the physical model reduces to the Poisson equation only. This equation is discretized using the Finite Element Method with linear elements and leads to a non-linear system of equations which is solved via the pure Newton iterative method. The stopping criteria is based on the relative norm of the difference between two consecutive iterations, i.e. the algorithm stops if $\|\psi^{(k+1)} - \psi^{(k)}\| / \|\psi^{(k)}\| < \varepsilon$ where $\varepsilon = 10^{-4}$ and $\psi^{(k)}$ is the k -th step approximation of the electrostatic potential. The starting guess of the potential is $\psi^{(0)} = V_s$ inside the semiconductor and is linearly varying inside the insulator between V_i and V_s . The mesh has 200 points and is finely tuned to the problem to get a better solution. The first 50 points are equally spaced in the insulator domain. The second 100 points are equally spaced in the first $0.2 \mu m$ of the silicon domain. Finally, the last 50 points are equally spaced in the remaining part of the silicon domain.

A typical convergence progress is shown in Figure 1, which refers to an applied voltage $V = 2$ Volt, the x -axis refers to the Gummel iterations, whereas the y -axis reports ε_ψ . The stopping criteria is fulfilled with more than 60 iterations without any acceleration technique. Using RRE and VEA to accelerate ψ , with $N_\psi = 10$, the convergence progress is greatly improved. Similar results are obtained changing the mesh size, N_ψ and V . Furthermore, RRE and TEA lead to similar results as MPE and VEA, respectively. Depending on the mesh and on the specific acceleration technique used, the CPU time using the acceleration is reduced by a factor of three to four. In Figure 2, we can see some numerical results of the simulation; in particular on the right the inversion on the p-silicon

occurs for $V = 2$ Volt.

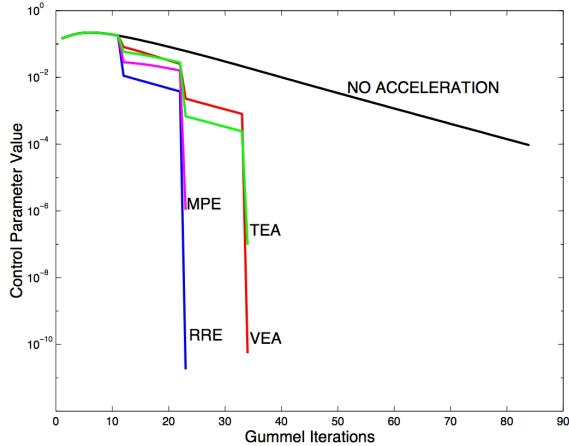


Figure 1. A typical MOS capacitor convergence progress.

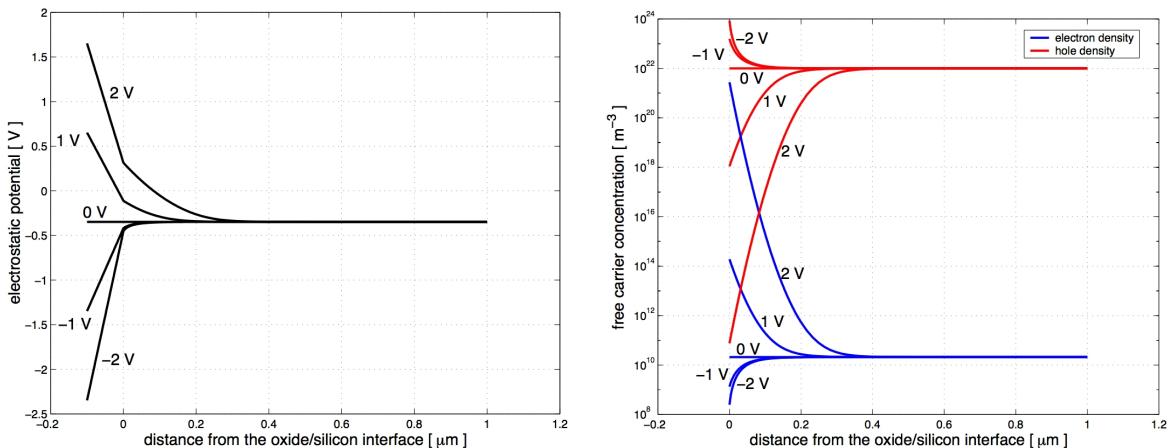


Figure 2. Behaviour of electrostatic potential and free carrier densities.

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Separately Holomorphic and CR functions

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Abstract. The aim of the following lecture is to introduce the problem of separate analyticity for functions defined on open sets of \mathbb{C}^n . The fundamental result in this field is Hartogs Theorem (1906), which states that a separately holomorphic function is C^1 , and therefore holomorphic. We extend the discussion to the case of separately CR functions, defined on CR manifolds; we regain and generalize in this way former results by Henkin and Tumanov (1983) and by Hanges and Treves (1983), through different techniques. [Keywords: Hartogs Theorem, CR functions, separate analyticity.]

Sunto. *Funzioni separatamente olomorfe e CR.* Obiettivo della seguente trattazione è introdurre il problema della separata analiticità per funzioni definite su aperti di \mathbb{C}^n . Risultato fondamentale in questo ambito di ricerca è il Teorema di Hartogs (1906), il quale afferma che una funzione separatamente olomorfa è C^1 , e dunque olomorfa. Estendiamo la discussione al caso di funzioni separatamente CR, definite su varietà CR; riotteniamo e generalizziamo in questo modo alcuni risultati precedenti di Henkin e Tumanov (1983) e di Hanges e Treves (1983), attraverso differenti tecniche di dimostrazione.

1 Introduction

This short note is divided into three parts: the first part introduces the problem of separate analyticity, just giving the basic definitions and properties of holomorphic and separately holomorphic functions on domains of \mathbb{C}^n and, next, defining CR functions on CR submanifolds of \mathbb{C}^n . The second part describes Hartogs Theorem (1906): using a lemma on subharmonic functions, it proves that a separately holomorphic function is C^1 and therefore jointly holomorphic, without requiring any hypothesis on the function. Then, the third part presents the problem of separately CR functions and shows how our result in [4] is related to the former literature; in fact, it generalizes a result by Henkin and Tumanov (1983), using the technique of approximation.

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2 Basic properties of holomorphic functions and CR functions

Let us consider (z, \bar{z}) as coordinates in \mathbb{C}^n , through the identification of \mathbb{R}^{2n} with the diagonal of $\mathbb{C}^n \times \bar{\mathbb{C}}^n$.

Definition 2.1 A function f , defined on a domain Ω in \mathbb{C}^n , is said to be holomorphic if it is C^1 and satisfies the differential system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, \dots, n$.

Definition 2.2 A function f , defined on a domain Ω in \mathbb{C}^n , is separately holomorphic if it satisfies the differential system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, \dots, n$.

This differential system is the well-known Cauchy-Riemann system. It is obvious that a holomorphic function is separately holomorphic; what we want to show is that the hypothesis of separate analyticity suffices to conclude that the function is C^1 , and then holomorphic.

Let us recall some properties of holomorphic functions of several complex variables that give us two results on separate analyticity with the hypothesis of continuity and boundedness on compacts.

Theorem 2.3 [Cauchy integral formula on polydiscs] Let f be a continuous function on the closure of a polydisc $P = D_1 \times \dots \times D_n \subset \mathbb{C}^n$, which is, for any j , a holomorphic function of z_j , when the other variables z_k , for $k \neq j$, are fixed. Then, for any $z \in P$, we have

$$f(z) = (2\pi i)^{-n} \int_{\partial_0 P} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge \dots \wedge d\zeta_n,$$

where $\partial_0 P = \partial D_1 \times \dots \times \partial D_n$.

This is the generalization of Cauchy formula on the complex plane, which is a consequence of Stokes formula.

Corollary 2.4 If f is $C^0(\Omega)$ and separately holomorphic in each z_j , when the other variables are fixed, then f is $C^\infty(\Omega)$. (In particular, f is holomorphic on Ω .)

It suffices to consider Cauchy formula on a polydisc contained in Ω ; the integrand is C^∞ and analytic in z when $(\zeta, z) \in \partial_0 P \times P$, so we can derive under the integral.

Theorem 2.5 [Cauchy inequalities] Let f be holomorphic on P and continuous on \bar{P} . Then

$$|f^\alpha(z_o)| \leq \frac{\alpha!}{r^\alpha} \sup_{\partial_0 P(z_o, r)} |f|.$$

Corollary 2.6 [Stjelties-Vitali] Let $\{f_n\}$ be a sequence of holomorphic functions on Ω , uniformly bounded on compact sets of Ω ; then, there exists a subsequence $\{f_{n_k}\}$ uniformly convergent on compact sets of Ω , and its limit is holomorphic.

Because of Cauchy inequalities, for holomorphic functions, uniform boundedness is equivalent to equicontinuity, so we get

Proposition 2.7 *If f is separately holomorphic and bounded on compact sets of Ω , then f is holomorphic on Ω .*

We want to introduce now CR manifolds and CR functions as generalizations of complex manifolds and holomorphic functions. Let M be a smooth submanifold of \mathbb{C}^n .

Definition 2.8 *For a point $z \in M$, the complex tangent space of M at z is the vector space $T_z^\mathbb{C}M = T_zM \cap iT_zM$.*

Definition 2.9 *M is called a CR submanifold of \mathbb{C}^n if the dimension of $T_z^\mathbb{C}M$ is constant.*

Examples 2.10 In \mathbb{C}^n , any complex submanifold is a CR submanifold. Real hypersurfaces are CR submanifolds, while the manifold $M = \{z \in \mathbb{C}^n : |z| = 1, \operatorname{Im} z_1 = 0\}$, which is the equator of the unit sphere in \mathbb{C}^n , is not a CR submanifold, because $\dim T_z^\mathbb{C}M$ is not constant.

Definition 2.11 *M is called totally real if $T_z^\mathbb{C}M = \{0\}$, for every $z \in M$; M is called generic if $T_zM + \mathcal{J}T_zM = T_z\mathbb{C}^n$, for every $z \in M$.*

We define $T^{1,0}(\mathbb{C}^n) := \operatorname{Span}\{\partial_{z_j}\}$ and $T^{0,1}(\mathbb{C}^n) := \operatorname{Span}\{\partial_{\bar{z}_j}\}$; we also define $T^{1,0}M := T^{1,0}\mathbb{C}^n \cap (\mathbb{C} \otimes TM)$ and $T^{0,1}M := T^{0,1}\mathbb{C}^n \cap (\mathbb{C} \otimes TM)$. For a CR submanifold M of \mathbb{C}^n , we give the notion of CR function on M .

Definition 2.12 *A C^1 function $f : M \rightarrow \mathbb{C}$ is CR if $\bar{L}f = 0$, for every $\bar{L} \in T^{0,1}M$.*

The restriction of a holomorphic function to a CR submanifold is a CR function; in particular, if $M = \mathbb{C}^n$, holomorphic functions are CR, while, for the converse, we need M and f to be C^ω . In general, the class of CR functions is strictly larger than the class of restrictions of holomorphic functions. In other terms, CR functions not always extend as holomorphic functions.

Examples 2.13 $M = \mathbb{R} \times \mathbb{R}$ is a totally real and generic submanifold of \mathbb{C}^2 and $T^{0,1}M = \{0\}$, so all the functions $f(x_1, x_2)$ of class C^1 on M are CR. $M = \mathbb{R} \times \mathbb{C}$ is a generic submanifold of \mathbb{C}^2 , such that $T^{0,1}M$ is spanned by the vector field $\partial_{\bar{z}_2}$. Thus, every function $f(x_1, z_2)$ of class C^1 on M , which satisfies $\partial_{\bar{z}_2}f = 0$, is CR. These functions are separately holomorphic in z_2 and the holomorphic extension needs f to be C^ω . Finally, $M = \mathbb{C} \times \mathbb{C}$ is a complex submanifold and $T^{0,1}M$ is spanned by $\partial_{\bar{z}_1}$ and $\partial_{\bar{z}_2}$; thus, CR functions and holomorphic functions coincide in this case.

3 Hartogs Theorem

Lemma 3.1 [Hartogs lemma] *Let v_k be a sequence of subharmonic functions, which are uniformly bounded on any compact subset of Ω . Let $\limsup_{k \rightarrow \infty} v_k(z) \leq C$, $\forall z \in \Omega$; then, $\forall \epsilon > 0$ and $\forall K \subset\subset \Omega$, there is k_0 such that*

$$\sup_{z \in K} v_k(z) \leq C + \epsilon, \quad \forall k \geq k_0.$$

The use of the previous lemma yields the strongest result on separate analyticity

Theorem 3.2 [Hartogs, 1906] *If $f : \Omega \rightarrow \mathbb{C}$ is separately holomorphic, then it is holomorphic. (We do not need any hypothesis on the initial regularity of f ; this is C^1 as a consequence.)*

Proof. The statement is local and can be proved by adding, one by one, the directions of separate analyticity: so we can consider the bidisc $\bar{\Delta} \times \bar{\Delta} \subset \subset \Omega$ in \mathbb{C}^2 and prove these two steps:

STEP 1: Analyticity on $\Delta_\epsilon \times \Delta$

We prove that f , which is separately holomorphic in z_1 and z_2 , is holomorphic on $\Delta_\epsilon \times \Delta$. Let us define $E_l := \{z_1 \in \Delta : \sup_{z_2 \in \Delta} |f(z_1, z_2)| \leq l\}$. E_l is closed and $\cup_l E_l = \Delta$. By Baire Theorem, there exists l_0 such that $\text{Int}(E_{l_0}) \neq \emptyset$; so f is holomorphic on $\text{Int}(E_{l_0}) \times \Delta$ and, repeating the same construction with different sets E_l on any open subset of Δ , we can say that f is holomorphic on $B \times \Delta$, for an open dense subset $B \subset \Delta$. Also, we can assume, without loss of generality, that $0 \in B$, so f is holomorphic on the strip $\Delta_\epsilon \times \Delta$.

STEP 2: Analyticity on $\Delta \times \Delta$

At this point, we can even forget that f is separately holomorphic in z_2 , when z_1 is outside Δ_ϵ . So we prove that, if f is holomorphic on $\Delta_\epsilon \times \Delta$, and separately holomorphic for $z_1 \in \Delta$ when z_2 is fixed, then f is holomorphic on $\Delta \times \Delta$.

We consider the Taylor series of f with respect to z_1 , at $z_1 = 0$:

$$f(z_1, z_2) = \sum_k \frac{\partial_{z_1}^k f(0, z_2)}{k!} z_1^k;$$

it converges uniformly for $z_2 \in \Delta$ and normally for $z_1 \in \Delta_\epsilon$. We define

$$v_k(z_2) := \left(\frac{|\partial_{z_1}^k f(0, z_2)|}{k!} \right)^{\frac{1}{k}}.$$

Cauchy inequalities yield, by the assumption of separate analyticity of f in $z_1 \in \Delta$, for fixed z_2 , $\limsup_{k \rightarrow \infty} v_k(z_2) \leq 1$; on the other hand, by the assumption of analyticity of f on $\Delta_\epsilon \times \Delta$, they yield $\limsup_k \sup_{z_2 \in \Delta} v_k \leq \epsilon^{-1}$. By Hartogs lemma, the pointwise estimate in z_2 becomes uniform

$$\limsup_k \sup_{z_2 \in \Delta} v_k \leq 1.$$

Hence, the power series in z_1 , with holomorphic coefficients in z_2 , converges normally for $z_1 \in \Delta$ and uniformly for $z_2 \in \Delta$, so the sum is a holomorphic function on $\Delta \times \Delta$.

Remark 3.3 Hartogs Theorem consists, in fact, only in Step 2; when we forget that f is separately holomorphic in z_2 , if z_1 is outside Δ_ϵ , it is not easy to prove the analyticity on the bidisc, even if we suppose that f is continuous: the problem has become a problem of propagation.

4 Separately CR functions

Remark 4.1 Let f be separately holomorphic on $\Delta^+ \times \Delta = \{(z_1, z_2) \in \Delta \times \Delta : \operatorname{Re} z_1 > 0\}$ and holomorphic on $\Delta_\epsilon^+ \times \Delta = \{(z_1, z_2) \in \Delta^+ \times \Delta : |z_1| < \epsilon, \operatorname{Re} z_1 > 0\}$; then, f is holomorphic on $\Delta^+ \times \Delta$.

We can prove the previous result through iteration of Hartogs Theorem, just “doubling” the radius of convergence, while, if the leaves of the foliation are complex curves, it becomes a problem of propagation for a non-holomorphic foliation. Without any other hypothesis on the function f , it is not known if we have the same result; if f is C^0 , we can say

Let $\{\gamma_\lambda\}_{\lambda \in \Lambda}$ be a foliation of $\Delta^+ \times \Delta$ by complex curves, such that $\gamma_\lambda \cap (\Delta_\epsilon^+ \times \Delta) \neq \emptyset$, $\forall \lambda \in \Lambda$. Let f be a C^0 function on $\Delta^+ \times \Delta$, such that f is holomorphic on $\Delta_\epsilon^+ \times \Delta$ and $f|_{\gamma_\lambda}$ is holomorphic, $\forall \lambda \in \Lambda$; then, f is holomorphic on $\Delta^+ \times \Delta$.

This statement has been generalized in many directions: first, in replacing the open set $\Delta^+ \times \Delta$ by a CR manifold M , and $\Delta_\epsilon^+ \times \Delta$ by an open set $M_\epsilon \subset M$ (that can even be shrunk to a proper submanifold N), and also in replacing the foliation $\{\gamma_\lambda\}$ of complex curves by a foliation $\{\gamma_\lambda\}$ of CR manifolds of CR dimension 1.

Theorem 4.2 [Mascolo] *Let M be a CR connected manifold with boundary N , foliated by a family $\{\gamma_\lambda\}$ of CR manifolds of CR dimension 1, issued from N , with $T^\mathbb{C}\gamma_\lambda$ transversal to $T^\mathbb{C}N$ at any common point of $\gamma_\lambda \cap N$. Let f be a C^0 function on M , which is CR along N , CR and C^1 along each γ_λ . Then, f is CR all over M .*

We have used the technique of polynomial approximation, as was first exploited by Tumanov in [5], to prove that the function f is CR in a neighbourhood of N in M ; then, we have repeated the same method to get the global theorem, using a connectedness argument.

Theorem 4.2 is a generalization of

Theorem 4.3 [Henkin-Tumanov, 1983] *It is the previous theorem if the $\{\gamma_\lambda\}$ are complex curves.*

Theorem 4.3 is also related to the following theorem, in which the presence of a single complex curve, instead of a full foliation, suffices, also if Theorem 4.4 gets extension only for holomorphic functions, and not for CR functions. (Cf. [1] for a more detailed account on propagation.)

Theorem 4.4 [Hanges-Treves, 1983] *Let M be a hypersurface of \mathbb{C}^n , Ω one side of M , γ a complex curve of M , z_o a point of γ , f a holomorphic function on Ω . Then, if f extends across M at z_o , it also extends at any other point $z_1 \in \gamma$.*

Example 4.5 In \mathbb{C}^4 we consider $M = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : y_1 \geq -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$, with boundary $N = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : y_1 = -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$. Let

$(\gamma_{a,b,c})_{a,b,c \in \mathbb{R}}$ be manifolds in \mathbb{C}^4 defined by

$$\begin{cases} y_2 = |z_1|^2 + a \\ y_3 = |z_1|^2 + b \\ y_4 = |z_1|^2 + c. \end{cases}$$

Here, we can use Theorem 4.2 and not Theorem 4.3, because the strict pseudoconvexity of the $\gamma_{a,b,c}$ makes it impossible for them to be foliated by complex curves.

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Problems with Preferences and Uncertainty

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Abstract. Many real-life problems are characterized by preferences and uncertain events, which cannot be controlled by the agent. We present formalisms that model problems with various kinds of preferences and with uncertainty, we study the properties of such formalism, and we consider scenarios where several agents express simultaneously their preferences. [Keywords: preferences, uncertainty, multi-agent systems.]

Sunto. *Problemi con preferenze ed incertezza.* Molti problemi reali sono caratterizzati da preferenze e da eventi incerti che non possono essere controllati dall'utente. Presentiamo dei formalismi che modellano problemi con vari tipi di preferenze e l'incertezza, studiamo le proprietà di questo formalismo e consideriamo scenari in cui più utenti esprimono contemporaneamente le loro preferenze.

1 Motivation

Preferences are ubiquitous in real life. In fact, most problems are over-constrained and would not be solvable if we insist that all their requirements are strictly met, hence it is more reasonable to express their requirements in a soft way, i.e., via preferences. Moreover, many problems are more naturally described via preferences rather than hard statements.

Preferences come in many kinds. In some cases it could be natural to express preferences in quantitative terms, while in other situations it could be better to use qualitative statements. Moreover, preferences can be unconditional or conditional. Finally, preferences can model priorities, rankings, different levels of importance, desires or rejection levels.

Preferences can help whenever the task involves decision making and/or knowledge representation. They are essential to treat reasoning about action and time, planning diagnosis and configuration. Preferences are the key to understand the non-crisp aspect of many human behaviors. For example, in mathematical decision theory, preferences (often expressed as utilities) are used to model people's economic behavior. In Artificial Intelligence (AI), preferences help to capture agents' goals. In philosophy, preferences are used to reason about values, desires, and duties. Thus, the representation and handling

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of preferences should be available and efficient in any sophisticated automated reasoning tool. Preferences are gaining more and more attention in AI, in particular in the Constraint Programming (CP) area also in connection with Operations Research (OR). AI permits complex preference representations and thus allows to reason with and about preferences, providing a new perspective for formalizing preference information in qualitative and quantitative way, that is essential for many decision making problems.

Preferences are one type of soft information present in real-life problems. Another important feature, which arises in many real world problems, is uncertainty. In fact, many problems are characterized by uncertain parameters which are not under the user's direct control, but that can be decided only by Nature. An example of an uncertain parameter is, in the context of satellite scheduling or weather prediction, the time when clouds will disappear, which can be decided only by Nature. Another example in which uncertainty occurs is a scheduling problem, which constrains the order of execution of various activities, where the duration of some activity is uncertain. In this case the goal is to define a schedule which is the most robust with respect to uncertainty.

Uncertainty can be represented in several ways. In some problems the user can be completely ignorant about the occurrence of the uncertain events, in others he can have additional information, which can be more or less precise regarding the occurrence of uncertain events [4,5].

Preferences and uncertainty often coexist in real-word problems. Consider for example a scheduling problem with uncertain durations, which is over-constrained. It would be impossible to solve it if we insist that all its requirements are strictly met. Therefore, it is more reasonable to express (at least some of) its requirements as preferences rather than hard statements. Doing so, we obtain a problem defined by preferences and uncertainty, where the solutions are schedules with different levels of desirability. The goal is then to find solutions with the highest level of desirability which are also robust with respect to uncertain durations.

Since preferences and uncertainty are very often the core of real-life problems, it is important to model faithfully these two aspects, both for problems involving a single agent and for problems regarding multiple agents. While there are several formalisms to handle some notions of preferences and/or uncertainty, much work still needs to be done to handle them in a general and efficient way. We give a contribution in this direction.

2 Objectives

There are many issues to be addressed about preferences and uncertainty. The main issue is preference and uncertainty specification and representation, i.e., which formalisms can be used to model the preferences of an agent and the uncertainty of the problem. In this respect, contributions have been brought from studying the axiomatic properties of preferences, as well as logics of preferences or their topological and algebraic structures [1] and from defining formalisms for representing various kinds of uncertainty [5]. In a multi-agent scenario, instead, core issues are preference composition, merging and aggregation, as well as preference elicitation, conflict resolution and belief revision.

Our goal is to define and study formalisms that can model problems with many kinds of

preferences and/or uncertainty, to study properties of such formalisms, and to develop tools to solve such problems. Moreover, we want to deal also with scenarios where preferences are expressed by several agents and where preference aggregation is therefore needed to find the optimal solutions.

In order to achieve this objective, we start by defining formalisms for expressing preferences of a single agent in presence of uncertainty. We start considering problems where preferences are expressed in a quantitative non conditional way and where uncertainty is characterized by lack of data or imprecise knowledge. In some formalisms for dealing with preferences and uncertainty, uncertainty is expressed in terms of probability theory [5]. We consider a different form of uncertainty, less precise than the probabilistic one, since we intend to model scenarios where probabilistic estimates are not available. We define a formalism for handling preferences and this kind of uncertainty, and algorithms for solving them. To achieve this goal, we exploit two formalisms: the semiring-based soft constraint formalism [1] to deal with preferences, and possibility theory [4] to reason with uncertainty.

Generally speaking, a soft constraint is just a classical constraint plus a way to associate, either to the entire constraint or to each assignment of variables, a certain element, which is usually interpreted as a level of preference or importance. Such levels are usually ordered and the order reflects the idea that some levels are better than others. Moreover, one has also to say, via suitable combination operators, how to obtain the level of preference of a global solution from the preferences in the constraint.

Many formalisms have been developed to describe one or more classes of soft constraints. For instance, consider fuzzy CSPs, where the crisp constraints are extended with a level of preference represented by a real number between 0 and 1, or probabilistic CSPs, where the probability to be in the real problem is assigned to each constraint. We choose to use one of the most general frameworks to deal with soft constraints [1]. The framework is based on a semiring structure that is equipped with the operations needed to combine the constraints present in the problem and to choose the best solutions. According to the choice of the semiring, this framework is able to model all the specific soft constraint notions mentioned above. The semiring-based soft constraint framework provides a structure capable of representing in a compact way problems with preferences.

For handling uncertainty we consider possibility theory, which is a mathematical theory for dealing with a certain type of uncertainty. This theory is an alternative to probability theory. It can be seen as an imprecise probability theory. Possibility theory has been introduced as an extension of the theory of fuzzy sets and fuzzy logic and many contributions to its development have been presented, for example, in [4].

Another issue that we consider is the representation of bipolarity. Bipolarity is an important focus of research in several domains, e.g. psychology, multi-criteria decision making, and more recently in AI (argumentation and qualitative reasoning). Preferences on a set of possible choices are often expressed in two forms: positive and negative statements. In fact, in many real-life situations agents express what they like and what they dislike, thus often preferences are bipolar. Starting from this observation, we define a formalism for handling quantitative (unconditional) preferences, which is able to represent positive and negative statements, and also to deal with uncertainty. Starting from an ex-

isting formalism for handling negative preferences, i.e., soft constraints [1], we extend it to handle also positive preferences. The aim is to handle bipolar preferences in a way which is as similar as possible to what naturally happens in real-life scenarios. That is, combining two negative statements should be even worse, combining two positive statements should be even better, and combining a positive with a negative statement should be positive if the positive statement is stronger than the negative one, and negative otherwise. Moreover, we want to able to express indifference, i.e., a preference which is neither positive nor negative.

In many situations, we need to represent and reason about simultaneous preferences of several agents. To aggregate agents' preferences, which in general express a partial order over the possible outcomes, we can query each agent in turn and collect together the results. Hence, we can see preference aggregation in terms of voting, which is a topic widely studied in Operations Research. In this context, we study classical properties such as fairness and non-manipulability, and we consider classical results on fairness of social [6] welfare functions as Arrow's impossibility theorem and results on non-manipulability of social choice functions as Gibbard-Satterthwaite's theorem [6]. The main difference is that, in contrast to what is assumed in social welfare scenarios, our agents describe their preference using partial orders and not total orders, i.e., they can consider incomparable pairs of outcomes, which are too dissimilar to be compared. We study if results similar to the ones of social choice and social welfare settings still hold, by suitably adapting some of their assumptions to deal with incomparability.

Finally, we consider uncertainty in a multi-agent scenario. We consider a multi-agent setting where agents can hide some of their preferences. In a preference ordering, the relationship between some pairs of outcomes may not be specified. For example, agents may have privacy concerns about revealing their complete preference ordering or, as in the context of preference elicitation, preferences have not been fully elicited [3]. In this context it is interesting to determine the complexity of computing the outcomes which are always optimal, or optimal in at least one way in which incompleteness is resolved. Moreover, if this computation is difficult, it is useful to find cases where this computation is easy. Regarding this topic, we investigate such complexity results both in general and for specific preference aggregation systems, and we analyze the issue of manipulation in this context.

3 Results

We have followed the research lines outlined in the previous section, and we have obtained the following main results.

We have started by considering a special case of quantitative preferences, i.e., fuzzy preferences, and we have considered an existing technique to integrate such preferences with uncertainty, which uses possibility theory [4]. We have shown that the integration provided by this technique is too tight since the resulting ordering over solutions does not allow one to discriminate between solutions which are highly preferred but assume unlikely events and solutions which are not preferred but robust with respect to uncertainty. Thus, while following the same basic idea of translating uncertainty into fuzzy constraints, we

have proposed an algorithm which allows us to observe separately the preference and the robustness of the solutions. Moreover, we have defined suitable semantics for ordering the solutions in a more or less risky way with respect to uncertainty. Then, for finding optimal solutions according to the different semantics, we have developed a solver which exploits branch and bound techniques [8]. Moreover, we have defined a more general formalism for handling different kinds of quantitative preference, proving that some desirable properties continue to hold [9]. This has allowed us to handle the coexistence of preferences and uncertainty in a more general setting.

We have also defined a formalism to handle positive and negative preferences, which reflects the natural behaviour that the combination of positive and negative statements has in real-life scenarios. For doing so, we have first shown that the negative preferences are handled by the semiring-based formalism of soft constraints. Then, we have introduced a new algebraic structure for handling positive preferences, which has properties similar to semirings. Hence, we have defined a new mathematical structure for handling both positive and negative preferences by linking the positive and the negative structures in a suitable way, so that combination of positive preferences produces a better positive preference, the combination of negative preferences produces a worse negative preference, and the combination of positive and negative preferences produces a preference which is better than or equal to the negative preference and worse than or equal to the positive one. We have studied the properties of this formalism and we have defined a solver to solve such problems [2]. Moreover, we have generalized this solver to handle also uncertainty.

We have then considered scenarios where several agents express their preferences via a partial order over the possible outcomes. We have seen each agent as voting if an outcome dominates another one. Thus, we have considered preference aggregation in terms of voting, analyzing some of the main results concerning fairness and non-manipulability [6]. We have shown that they can be generalized to preference aggregation systems, where agents can express also incomparability between pair of outcomes [10,11].

We have finally considered scenarios where agents, for example for privacy reasons, decide to hide some of their preferences. We have determined the computational complexity of computing optimal outcomes, where optimality has the meaning of being always the best outcome (regardless of how incompleteness is resolved), or in at least one possible complete world. We have shown that computing such outcomes is in general difficult, and we have determined cases where such a problem is tractable. Moreover, we have shown how the computation of such outcomes can be useful for deciding when preference elicitation is over, which is in general a difficult problem [12]. Finally, we have investigated other tractability and intractability results for a specific voting rule, i.e, the sequential majority voting. Such a rule performs a sequence of pairwise comparisons between two candidates along a binary tree, and the winner depends on the chosen sequence. We have focused on candidates that will win in some sequences or in all sequences and we have shown that in general it is easy to find them, while it is difficult if we insist that the tree is balanced. We have interpreted this difficulty in terms of difficulty for the chair to manipulate [7].

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Sheaves and Linear Differential Equations

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Abstract. The "Riemann-Hilbert correspondence" is a generalization of the 21th Hilbert's problem (does there exist a differential equation whose holomorphic solutions have a prescribed monodromy group?). Using several simple examples, we shall try to explain the Riemann-Hilbert correspondence in terms of equivalence between the categories of linear differential equations with regular singularities (objects of analytic nature) and the spaces of their holomorphic solutions (objects of topological nature). Such a correspondence proves that these objects, of seemingly different types, are in fact closely related to each other.

Sunto. *Fasci ed Equazioni Differenziali Lineari.* La "corrispondenza di Riemann-Hilbert" è una generalizzazione del 21mo problema di Hilbert (esiste un'equazione differenziale lineare le cui soluzioni olomorfe abbiano un gruppo di monodromia prescritto?). Cercheremo di spiegare, attraverso numerosi semplici esempi, la corrispondenza di Riemann-Hilbert in quanto equivalenza tra le categorie delle equazioni differenziali lineari a singolarità regolari (oggetti di natura analitica) e degli spazi delle loro soluzioni olomorfe (oggetti di natura topologica). Tale corrispondenza sottolinea quindi come oggetti di natura e utilizzo diversi siano in realtà strettamente legati.

Introduction and basic examples

Let

$$P := \sum_{\substack{J=\{j_1, \dots, j_n\} \\ |J| \leq m_1}} a_J(z) \frac{d^J}{dz^J}, \quad Q := \sum_{\substack{J=\{j_1, \dots, j_n\} \\ |J| \leq m_2}} b_J(z) \frac{d^J}{dz^J}$$

be two linear differential operators with holomorphic coefficients on \mathbb{C}^n . We say that P and Q are equivalent if, for any functional space \mathcal{F} on which P and Q act, the solutions of P and Q with values in \mathcal{F} are isomorphic. e.g. \mathcal{F} can be the space of distributions, holomorphic functions, polynomials (if P and Q have polynomial coefficients), hyperfunctions ...

Such a definition of equivalence relation is ambiguous, since it is not clear which kind of solutions we consider and which isomorphisms we allow. The language of categories will make the statement precise.

Throughout the paper, even if not specified, all the differential operators and equations will be linear.

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The main property of an operator that we will consider in this paper is the regularity. In dimension 1 the notion of regular operator is defined as follows. Let $X \subset \mathbb{C}$ be an open neighbourhood of 0, let z be the holomorphic coordinate on X . Set $\partial_z := \frac{d}{dz}$. An operator $P = \sum_{j=0}^n a_j(z) \frac{d^j}{dz^j}$, $a_n \neq 0$, defined on X is said *regular at 0* if

$$n - \text{ord}_0(a_n) \geq j - \text{ord}_0(a_j) \quad \text{for any } j = 0, \dots, n.$$

The simplest example of singular regular operator is

$$P_\alpha := z\partial_z - \alpha ,$$

for $\alpha \in \mathbb{C}$. The operator $z^2\partial_z - 1$ is irregular.

Given P a differential operator as above and $\mathcal{F}(U)$ the elements of the functional space defined on an open set U ov \mathbb{C}^n , we set

$$\mathcal{S}(P, \mathcal{F}(U)) := \{u \in \mathcal{F}(U); P u = 0\} ,$$

in case \mathcal{F} is a sheaf on \mathbb{C}^n we denote by $\mathcal{S}(P, \mathcal{F})$ the sheaf obtained from $\mathcal{S}(P, \mathcal{F}(\cdot))$.

Example. Let $\alpha \in \mathbb{C}$. Consider the following differential operators

$$P_\alpha := z\partial_z - \alpha ,$$

$$P_{\alpha+1} := z\partial_z - \alpha - 1 .$$

Let \mathcal{F} be a functional space on which such operators act. Let us consider the solutions of P_α and $P_{\alpha+1}$ with values in \mathcal{F} .

If $(z\partial_z - \alpha)u = 0$, then $(z\partial_z - \alpha - 1)(zu) = 0$. Similarly, since $z\partial_z^2 = \partial_z z\partial_z - \partial_z$ (as operators), if $(z\partial_z - \alpha - 1)v = 0$ and $\alpha \neq -1$ then $(z\partial_z - \alpha)(\frac{1}{\alpha+1}\partial_z v) = 0$. Hence, the morphisms

$$\begin{array}{ccc} \mathcal{S}(P_\alpha, \mathcal{F}) & \xrightarrow{z \cdot} & \mathcal{S}(P_{\alpha+1}, \mathcal{F}) \\ & \xleftarrow{\frac{1}{\alpha+1}\partial_z} & \end{array}$$

are well defined. Further, remark that, if $(z\partial_z - \alpha)u = 0$, then $\frac{1}{\alpha+1}\partial_z(zu) = u$. Similarly, if $(z\partial_z - \alpha - 1)v = 0$, then $z\frac{1}{\alpha+1}\partial_z v = v$. In particular, the morphisms above are inverse one to the other. It follows that the operators P_α and $P_{\alpha+1}$ are equivalent if $\alpha \neq -1$.

The content of this paper is subdivided as follows.

- In the first section we consider the \mathbb{C} -vector spaces $\mathcal{S}(P, \mathcal{F}(U))$, for \mathcal{F} the sheaf of holomorphic functions, distributions, meromorphic functions. By means of $\mathcal{S}(P, \mathcal{F}(U))$, we prove the following two easy results.
 - If $h, k \in \mathbb{Z}_{>0}$ then P_k is not equivalent to P_{-h} .
 - If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $k \in \mathbb{Z}$ then P_α is not equivalent to P_k .

Further, using local holomorphic solutions of the non-homogenous equations, we prove that

- the operators ∂_z and $z\partial_z$ are not equivalent.

Further we remark that P_{-1} and $z^2\partial_z - 1$ have not isomorphic local meromorphic solutions. Hence, they are not equivalent. But such a result cannot be achieved by the only use of holomorphic solutions.

- In the second section, we introduce sheaves, categories and locally constant sheaves. We consider the locally constant sheaves $\mathcal{S}(P_\alpha, \mathcal{O}_{\mathbb{C}^\times})$, where $\mathcal{O}_{\mathbb{C}^\times}$ is the sheaf of holomorphic functions on \mathbb{C}^\times . We prove that if P_α is equivalent to P_β then $\alpha - \beta \in \mathbb{Z}$.
- In Section 3, we introduce the subcategory of regular ordinary differential equations in order to state a weak form of the Riemann-Hilbert correspondence in complex dimension 1. In order to remain in an introductory framework to differential equations and sheaves, we state a weak version of the Riemann-Hilbert correspondence which, roughly speaking, corresponds to the following statement. Two regular differential operators are equivalent if and only if the holomorphic solutions sheaves are isomorphic.
- In the end, in Section 4, we consider tempered holomorphic solutions of differential equations. Such objects are no longer sheaves on a topological space but they are subanalytic sheaves. We show how the use of tempered holomorphic solutions allows to distinguish irregular non-equivalent operators. For example, we prove that P_{-1} and $z^2\partial_z - 1$ are not equivalent.

1 Solutions as \mathbb{C} -vector spaces

Let us consider the operators

$$z\partial_z - \alpha \quad (\alpha \in \mathbb{C}).$$

Their solutions (for the moment just as symbols) are $c \cdot z^\alpha$ ($c \in \mathbb{C}$).

Let us study the solutions of the operators $z\partial_z - \alpha$ with values in the functional spaces of holomorphic functions $\mathcal{O}_{\mathbb{C}}$, of meromorphic functions \mathcal{M} and of distributions $\mathcal{D}b$ on the following open sets

- (a) U : an open disc centered at the origin,
- (b) V : an open disc which does not contain the origin,
- (c) W : an open ring centered at the origin.

We have the following isomorphisms of \mathbb{C} -vector spaces.

$$\begin{aligned}\mathcal{S}(z\partial_z - \alpha, \mathcal{O}_{\mathbb{C}}(V)) &\simeq \mathbb{C} \cdot z^\alpha \quad \forall \alpha \in \mathbb{C} \\ \mathcal{S}(z\partial_z - \alpha, \mathcal{O}_{\mathbb{C}}(W)) &\simeq \begin{cases} \mathbb{C} \cdot z^\alpha & \alpha \in \mathbb{Z} \\ 0 & \alpha \in \mathbb{C} \setminus \mathbb{Z} \text{ (since } z^\alpha \notin \mathcal{O}(W)\text{)} \end{cases} \\ \mathcal{S}(z\partial_z - \alpha, \mathcal{O}_{\mathbb{C}}(U)) &\simeq \begin{cases} \mathbb{C} \cdot z^\alpha & \alpha \in \mathbb{Z}_{\geq 0} \\ 0 & \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0} \text{ (since } z^\alpha \notin \mathcal{O}(U)\text{)} \end{cases}\end{aligned}$$

It follows that considering solutions as holomorphic functions on an open set, we can say that an element in one of the following three families,

$$\{z\partial_z - \alpha, \alpha \in \mathbb{Z}_{\geq 0}\}, \{z\partial_z - \alpha, \alpha \in \mathbb{Z}_{< 0}\}, \{z\partial_z - \alpha, \alpha \in \mathbb{C} \setminus \mathbb{Z}\},$$

is not equivalent to an element in the two others.

Let us now consider the operator ∂_z . Its solutions are the locally constant functions. We have the following isomorphisms of \mathbb{C} -vector spaces.

$$\begin{aligned}\mathcal{S}(z\partial_z, \mathcal{O}_{\mathbb{C}}(U)) &\simeq \mathcal{S}(\partial_z, \mathcal{O}(U)) \simeq \mathbb{C} \\ \mathcal{S}(z\partial_z, \mathcal{D}b(U)) &\simeq \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot \text{Heaviside} \\ \mathcal{S}(\partial_z, \mathcal{D}b(U)) &\simeq \mathbb{C} \cdot 1\end{aligned}$$

Hence, considering solutions in the space of distributions over opens sets, we can state that the operators ∂_z are $z\partial_z$ not equivalent. Further, it seems that the local holomorphic solutions are not enough to prove it.

Now, consider the following morphisms of \mathbb{C} -vector spaces

$$\mathcal{O}(U) \xrightarrow{z\partial_z} \mathcal{O}(U)$$

$$\mathcal{O}(U) \xrightarrow{\partial_z} \mathcal{O}(U).$$

It is easy to see that the cokernels of such morphisms are not isomorphic. Indeed, for $f \in \mathcal{O}(U)$, the equation $z\partial_z u = f$ has solutions if and only if $f(0) = 0$, the equation $\partial_z u = f$ has always solutions.

Hence, it is sufficient to consider the local holomorphic solutions of homogeneous and non-homogeneous equations to prove that the operators $z\partial_z$ and ∂_z are not equivalent.

Let us now consider the operator

$$z^2\partial_z + 1,$$

whose solutions are the complex multiples of $c \cdot \exp(\frac{1}{z})$ ($c \in \mathbb{C}$).

We have the following isomorphisms of \mathbb{C} -vector spaces.

$$\begin{aligned}\mathcal{S}(z\partial_z + 1, \mathcal{M}(U)) &\simeq \mathbb{C} \cdot \frac{1}{z}, \\ \mathcal{S}(z^2\partial_z + 1, \mathcal{M}(U)) &\simeq 0 \quad (\text{since } \exp(\frac{1}{z}) \notin \mathcal{M}(U)).\end{aligned}$$

Hence, using local meromorphic solutions, we prove that the two operators are not equivalent. Indeed, it is not sufficient to use holomorphic solutions to prove it, we need the meromorphic functions at the origin.

Remark that local holomorphic solutions do not distinguish between $z\partial_z + 1$ and $z^2\partial_z + 1$, or between $z\partial_z + i$ and $z\partial_z + 2i$.

In the next section we will introduce locally constant sheaves. Considering holomorphic solutions as locally constant sheaves, we will be able to prove that $z\partial_z + i$ and $z\partial_z + 2i$ are not equivalent, thus sharpening the results obtained with local holomorphic solutions. On the contrary holomorphic solutions as locally constant sheaves will still not be enough to distinguish between $z\partial_z + 1$ and $z^2\partial_z + 1$. Indeed, it will be clear in the third section that the complex (i.e. considering ker and coker) of sheaves of holomorphic solutions can distinguish all the non-equivalent regular differential equations. The operator $z^2\partial_z + 1$ is one of the simplest examples of an irregular operator.

2 Solutions as locally constant sheaves

2.1 The language of categories

We do not enter in the detail of universes when dealing with categories.

Definition 1 A category \mathcal{C} consists of a set $Ob(\mathcal{C})$, said the set of objects of \mathcal{C} , and for any $X, Y \in Ob(\mathcal{C})$, a set of morphisms $Hom_{\mathcal{C}}(X, Y)$ with an associative composition law satisfying natural axioms of composition and identity.

Examples.

- (a) **Set:** the objects are the sets and the morphisms are maps between sets.
- (b) **Mod(\mathbb{C}):** the objects are the \mathbb{C} -vector spaces and the morphisms are linear maps.

In Section 1 we have considered the local solutions of operators as objects of $Mod(\mathbb{C})$.

An application between categories is said a *functor*.

Definition 2

- (a) Given two categories $\mathcal{C}, \mathcal{C}'$, a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is the data of a map of sets

$$F_o : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{C}')$$

and for any $X, Y \in Ob(\mathcal{C})$, a map

$$F_m : Hom_{\mathcal{C}}(X, Y) \rightarrow Hom_{\mathcal{C}'}(F_o X, F_o Y)$$

commuting with composition.

- (b) Two categories are said equivalent if there exists a functor F between them such that F_o is a bijection on isomorphism classes of objects and such that F_m is a bijection.

2.2 Locally constant sheaves

Let X be a topological space.

Definition 3

(a) A sheaf in \mathbb{C} -vector spaces on X consists of

$$\begin{aligned} \{\text{Open sets of } X\} &\longrightarrow \text{Mod}(\mathbb{C}) \\ U &\longmapsto F(U) \\ (U \subset V) &\longmapsto (F(V) \xrightarrow{\cdot|_U} F(U)) \quad (\text{restriction}) \end{aligned}$$

satisfying the following gluing property.

Given an open set U , an open covering $U = \cup_{j \in J} U_j$ and local sections $s_j \in F(U_j)$ such that $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$, there exists a unique $s \in F(U)$ such that $s|_{U_j} = s_j$, for any $j \in J$.

(b) Given two sheaves F and G , a morphism of sheaves $\phi : F \rightarrow G$ is a family $\phi_U : F(U) \rightarrow G(U)$ ($U \subset X$ open) commuting with restrictions.

Examples. (1) The following functional spaces give rise to sheaves on appropriate spaces (topological, complex analytic ...): continuous functions, holomorphic functions, \mathcal{C}^∞ , distributions.

(2) On the contrary, bounded functions or L^2 -functions do not give rise to sheaves. Indeed gluing bounded (square-integrable) functions may give a non globally bounded (square-integrable) function.

(3) The map

$$U \longmapsto \{\text{locally constant functions with values in } \mathbb{C}^n\},$$

defines the *constant sheaf* $\underline{\mathbb{C}}_X^n$, with fiber \mathbb{C}^n .

Example. Let X be a complex analytic manifold (locally \mathbb{C}^n), we have the following injective morphisms of sheaves on X ,

$$\underline{\mathbb{C}}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{C}_X^\infty \rightarrow \mathcal{C}_X^0.$$

The simplest generalization of constant sheaves are the locally constant sheaves.

Definition 4 A sheaf F on a topological space X is said to be *locally constant* if there exists an open covering $X = \cup_{j \in J} U_j$ such that F restricted to each of the open sets U_j is a constant sheaf on U_j .

Example. Let P be a differential operator on X (open subset of \mathbb{C}) with holomorphic coefficients.

$$P = a_n(z)\partial_z^n + \dots + a_1(z)\partial_z + a_0(z) .$$

The map $U \mapsto \mathcal{S}(P, \mathcal{O}_X(U)) = \{u \in \mathcal{O}_X(U); Pu = 0\}$ defines a sheaf on X .

The Cauchy Theorem of local existence and uniqueness of solutions implies that such a sheaf is locally constant with fiber \mathbb{C}^n on the open sets which do not contain any zeros of $a_n(z)$. In particular, the sheaves $\mathcal{S}(P_\alpha, \mathcal{O}_{\mathbb{C}^\times})$ are locally constant on \mathbb{C}^\times .

Let us recall now a classical fundamental result stating that the category of locally constant sheaves on X is completely determined by the fundamental group of X (the group of homotopy classes of closed paths), denoted $\pi_1(X)$.

Theorem 5 *Let X be a non empty, connected, locally connected and locally simply connected topological space. The category of locally constant sheaves on X with values in $\text{Mod}(\mathbb{C})$ and the category of representations of $\pi_1(X)$ with values in $\text{Mod}(\mathbb{C})$ with finite rank are equivalent.*

Example. Let $X = \mathbb{C} \setminus \{0\}$. Giving a locally constant sheaf on $\mathbb{C} \setminus \{0\}$ is equivalent to give a representation of $\pi_1(\mathbb{C} \setminus \{0\}) \simeq \mathbb{Z}$. In particular, locally constant sheaves with fiber \mathbb{C} are equivalent to representations of \mathbb{Z} with values in \mathbb{C} of rank 1. Such representations are obviously identified with \mathbb{C}^\times . Let us see an analytic realization of such an equivalence of categories.

We have already said that the holomorphic solutions of the operator $z\partial_z - \alpha$ form a locally constant sheaf on \mathbb{C}^\times . Which is the correspondent representation of $\pi_1(\mathbb{C}^\times) \simeq \mathbb{Z}$?

The solution z^α , defined in a neighbourhood of $z_0 \neq 0$, can be analytically extended around the origin. After a loop, the function turns out to be a multiple of itself, the multiplicative constant being $e^{2\pi i \alpha} \in \mathbb{C}^\times$. It follows that

$$\mathcal{S}(z\partial_z - \alpha, \mathcal{O}_{\mathbb{C}^\times}) \simeq \mathcal{S}(z\partial_z - \beta, \mathcal{O}_{\mathbb{C}^\times})$$

in the category of locally constant sheaves on $\mathbb{C} \setminus \{0\}$, if and only if

$$\alpha - \beta \in \mathbb{Z} .$$

3 Regular differential equations and the Riemann-Hilbert correspondence in dimension 1

Let $X \subset \mathbb{C}$ be an open neighbourhood of the origin, $P = \sum_{j=0}^n a_j(z)\partial_z^j$, with $a_j \in \mathcal{O}_X(X)$ and $a_n(z) \neq 0$ if $z \neq 0$.

Proposition 6 *The following conditions are equivalent.*

- (a) *For any $j \leq n$, it holds $n - \text{ord}_0(a_n) \geq j - \text{ord}_0(a_j)$.*

(b) The equation $Pu = 0$ has n linearly independent solutions of the form $u(z) = h(z)z^\alpha \log(z)^r$, for $r \in \mathbb{Z}_{\geq 0}$, h holomorphic at 0 and $\alpha \in \mathbb{C}$.

(c) Let $\mathbb{C}\{z\}$ (resp. $\mathbb{C}[[z]]$) the ring of convergent (resp. formal) power series.

The natural morphism $\mathbb{C}\{z\} \rightarrow \mathbb{C}[[z]]$ induces an isomorphism between the kernels and the cokernels of the following morphisms,

$$\mathbb{C}\{z\} \xrightarrow{P} \mathbb{C}\{z\},$$

$$\mathbb{C}[[z]] \xrightarrow{P} \mathbb{C}[[z]].$$

i.e. formal solutions are convergent.

Definition 7 If P verifies one of the above conditions, P is said to have a regular singularity at 0.

Theorem 8 [Weak Riemann-Hilbert correspondence] Let P and Q be regular operators. The following conditions are equivalent.

- (a) P and Q are equivalent (i.e. for any possible sheaves of functions \mathcal{F} , the complex of solutions sheaves of P and Q with values in \mathcal{F} are isomorphic).
- (b) There exist $\phi, \psi : \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $Q \circ \phi = \psi \circ P$ and ϕ (resp. ψ) induces an isomorphism between the kernels (resp. cokernels) of the morphisms

$$\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X,$$

$$\mathcal{O}_X \xrightarrow{Q} \mathcal{O}_X.$$

In particular the sheaves of holomorphic solutions are sufficient to say whether two regular operators are equivalent or not.

4 Subanalytic sheaves

The classification of equivalent regular operators is completely achieved by the mean of complexes of holomorphic solutions. It is not the case for irregular operators. In this section we study tempered holomorphic solutions of irregular operators proving that they are useful to distinguish non-equivalent irregular operators.

Definition 9 Let $U \subset \mathbb{C}$ be a subanalytic relatively compact open set, $f \in \mathcal{O}(U)$ is said to be tempered if there exist $M, C > 0$ such that

$$|f(z)| \leq \frac{C}{\text{dist}(z, \partial U)^M}.$$

The family of tempered holomorphic functions on U is denoted ${}_{[0,T]}(U)$.

Lemma 10 [Kashiwara] *The function $\exp(1/z)$ is tempered on U if and only if there exists $A > 0$ such that U is contained in $\mathbb{C} \setminus \overline{B(A, 0), A}$.*

Using a gluing theorem for tempered distributions due to S. Łojasiewicz, one can obtain the following gluing property for tempered holomorphic functions. Let U_1, U_2 be subanalytic relatively compact open sets. Let $s_j \in {}_{[0,T]}(U_j)$ ($j = 1, 2$) be such that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$. Then there exists a unique $s \in {}_{[0,T]}(U_1 \cup U_2)$ such that $s|_{U_j} = s_j$ ($j = 1, 2$). It follows that ${}_{[0,T]}$ is a sheaf when one considers just subanalytic open sets and finite coverings, i.e. ${}_{[0,T]}$ is a subanalytic sheaf.

The sheaves on a complex analytic manifold are naturally subanalytic sheaves (since they satisfy the gluing condition for a larger family of coverings). In particular the category of subanalytic sheaves is strictly bigger than the category of sheaves.

Let us recall that P is regular at the origin if and only if its holomorphic solutions are tempered. In particular, tempered holomorphic solutions of P_{-1} and $z\partial_z^2 + 1$ are not isomorphic.

Theorem 11 [Hukuhara-Turrittin] *Let P be a linear ordinary differential operator. Let S be an open sector of sufficiently small amplitude. The \mathbb{C} -vector space of holomorphic solutions of P on S is spanned by $h_j(z)\exp(\phi_j)$, ($j = 1, \dots, n$), for $h_j, h_j^{-1} \in {}_{[0,T]}$ and $\phi_j \in z^{-1/l}\mathbb{C}[z^{-1/l}]$. Further the ϕ_j do not depend upon the sector.*

The functions of the kind $\exp(\phi)$, for $\phi \in z^{-1}\mathbb{C}[z^{-1}]$, are the basic difference between solutions of regular operators and irregular ones.

Let us consider the operators $z^2\partial_z + 1$ and $z^3\partial_z + 2$ whose solutions are respectively $\exp(1/z)$ and $\exp(1/z^2)$.

Theorem 12 *There exists an open subanalytic set U such that $\exp(1/z) \in {}_{[0,T]}(U)$ and $\exp(1/z^2) \notin {}_{[0,T]}(U)$.*

Corollary 13 *There exists an open subanalytic set U such that*

$$\mathcal{S}ol(z^2\partial_z + 1, {}_{[0,T]}(U)) \simeq \mathbb{C} \cdot \exp(1/z) ,$$

$$\mathcal{S}ol(z^3\partial_z + 2, {}_{[0,T]}(U)) \simeq 0 .$$

In particular, tempered holomorphic solutions are useful to determine whether two irregular differential equations are equivalent or not.

The natural algebraic generalization of systems of linear differential equations is the theory of \mathcal{D} -modules. For a brief introduction to \mathcal{D} -module theory we refer to [3]. For a complete exposition of \mathcal{D} -module theory we refer to [1] and [4]. For the basic knowledge

about categories, we refer to the first chapter of [5]. For a detailed study of \mathcal{D} -modules in dimension 1, we refer to [2] and [8]. In the end, for a complete exposition of subanalytic sheaves and tempered holomorphic solutions of \mathcal{D} -modules we refer to [6] and [7].

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Accoppiamenti completamente monotoni per Processi di Markov

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Sunto. Si discutono i concetti di monotonia e monotonia completa per processi di Markov a valori in uno spazio degli stati finito e parzialmente ordinato. La monotonia è una proprietà della matrice di transizione (o del generatore infinitesimale, nel caso di tempi continui) del processo. La monotonia completa è un concetto più forte del precedente ed è utile in molte applicazioni, ad esempio quando si vogliono ottenere simulazioni dalla misura stazionaria di una catena di Markov utilizzando algoritmi di simulazione perfetta. Al contrario di quanto accade per la monotonia, non esiste un criterio semplice per verificare la monotonia completa. Per questo motivo è naturale domandarsi per quali insiemi parzialmente ordinati i due concetti sono equivalenti. Questo problema è stato completamente risolto nel caso di processi a tempi discreti, ma si è rivelato più complesso nel caso di processi a tempi continui.

Un processo stocastico è una famiglia di variabili aleatorie $(X_t)_{t \in \mathbb{T}}$ indicizzata in un insieme di tempi \mathbb{T} e a valori in uno spazio misurabile (E, \mathcal{E}) ,

$$X_t : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (E, \mathcal{E}) \text{ misurabile } \forall t.$$

Esso descrive l'evoluzione temporale di un certo fenomeno che prende valori in E . L'insieme E si dice *spazio degli stati* del processo. D'ora in poi supporremo che l'insieme E sia finito. Una *catena di Markov* è un particolare processo stocastico, dove l'insieme dei tempi è discreto, cioè $\mathbb{T} = \mathbb{N}$ o \mathbb{Z} . Da un punto di vista intuitivo, una catena di Markov è un modello per un sistema che, nei tempi $n \in \mathbb{N}$, passa da uno stato all'altro di E in modo aleatorio secondo lo schema seguente: se x_0 è lo stato al tempo t_0 (per semplicità supporremo $t_0 = 0$), al tempo 1 il sistema "sceglie" lo stato successivo secondo una certa distribuzione di probabilità su E che indicheremo con $P(x_0, \cdot)$ e che è assegnata a priori. Supporremo dunque assegnata una famiglia di misure di probabilità $\{P(x, \cdot); x \in E\}$ indicizzata nell'insieme di tutti i possibili stati iniziali. Supponiamo ora che al tempo $n \geq 1$ il sistema si trovi nello stato y . Allora, al tempo $n + 1$ esso "si dimentica del passato" (cioè di *come* è arrivato in y) e sceglie lo stato successivo secondo la probabilità $P(y, \cdot)$.

Per ciascuna coppia x, y di elementi E , il numero $P(x, \{y\}) = p_{xy} \geq 0$ rappresenta la probabilità di passare dallo stato x allo stato y in un intervallo di tempo unitario e si ha, ovviamente, $P(x, E) = \sum_{y \in E} p_{xy} = 1$. I numeri p_{xy} , al variare di x, y in E , individuano così una matrice, che indicheremo con P e che si dice *matrice di transizione*.

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Definizione 1 Sia E un insieme finito. Un processo stocastico $(X_n)_{n \in \mathbb{N}}$, definito su uno spazio probabilizzato $(\Omega, \mathcal{F}, \mathbb{P})$ e a valori in $(E, \mathcal{P}(E))$, è una Catena di Markov uscente da $x_0 \in E$ e con matrice di transizione $P = (p_{xy})_{x,y \in E}$ se

$X_0 \equiv x_0$ \mathbb{P} -quasi certamente

$\mathbb{P}(X_1 = x_1, X_2 = x_2 \dots, X_n = x_n) = p_{x_0 x_1} p_{x_1 x_2} \dots p_{x_{n-1} x_n}$ per ogni $n \in \mathbb{N}$ e per ogni scelta di $x_1, x_2, \dots, x_n \in E$.

Osserviamo che secondo la definizione $\mathbb{P}(X_1 = x_1) = p_{x_0 x_1}$, mentre $p_{x_0 x_1} p_{x_1 x_2} \dots p_{x_{n-1} x_n}$ è la probabilità che, partendo da x_0 al tempo 0, al tempo n si arrivi ad x_n percorrendo il cammino x_1, \dots, x_{n-1}, x_n . Allora, se facciamo la somma di queste probabilità al variare di tutti i possibili cammini che in n passi portano ad x_n , otteniamo la probabilità di raggiungere x_n dopo un tempo n :

$$\mathbb{P}(X_n = x_n) = \sum_{x_1, \dots, x_{n-1} \in E} p_{x_0 x_1} p_{x_1 x_2} \dots p_{x_{n-1} x_n} = (P^n)_{x_0 x_n}$$

dove $(P^n)_{xy} = p_{xy}(n)$ è l'elemento di posto x, y della matrice P^n . Rappresentiamo ciascuna misura di probabilità su $(E, \mathcal{P}(E))$ con un vettore riga μ (dove l'elemento di posto x rappresenta $\mu(\{x\})$). In particolare, indichiamo con δ_x il vettore che ha tutte le componenti uguali a zero, tranne quella di indice x . La relazione sopra descritta ci dice allora che la distribuzione di X_n è data dalla misura $\delta_{x_0} P^n$.

Osservazione 2 Nella definizione di catena di Markov, si può anche supporre che X_0 sia una variabile aleatoria con una distribuzione assegnata ν_0 . In questo caso, la probabilità del cammino x_1, x_2, \dots, x_n diventa $\sum_{x \in E} \nu_0(x) p_{xx_1} p_{x_1 x_2} \dots p_{x_{n-1} x_n}$ e la distribuzione di X_n sarà data da $\nu_0 P^n$.

Definizione 3 Sia P una matrice di transizione su E . Una misura di probabilità μ su E si dice invariante per P se $\mu P = \mu$.

Se μ è invariante per P , allora $\mu P^n = \mu$ per ogni n , cioè se $(X_n)_{n \in \mathbb{N}}$ è la catena di Markov associata a P e con distribuzione iniziale μ , la distribuzione di X_n è la stessa per ogni n . Le misure invarianti rappresentano gli stati di equilibrio di una dinamica markoviana.

Esempio 4 Consideriamo l'insieme $E = \{1, 2, 3, 4, 5\}$. La matrice

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

è la matrice di transizione di una passeggiata aleatoria con riflessione su E . Se la catena si trova nello stato x al tempo n , al tempo $n+1$ si troverà con probabilità $\frac{1}{2}$ nello stato

$(x + 1) \wedge 5$ e con probabilità $\frac{1}{2}$ nello stato $(x - 1) \vee 1$. La misura $\mu = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ è invariante per P .

Una matrice di transizione si dice *irriducibile* se per ogni $x, y \in E$ esiste un cammino $x, x_1, \dots, x_{n-1}, y$ tale che $p_{xx_1} \dots p_{x_{n-1}y} \neq 0$. Se $x \in E$, il *periodo* d_x di x è il massimo comun divisore dell'insieme $\{n : p_{xx}(n) > 0\}$. Una catena di Markov si dice *aperiodica* se $d_x = 1$ per ogni $x \in E$. Vale il seguente

Teorema 5 *Se P è irriducibile e aperiodica, esiste un'unica misura μ invariante per P e si ha, per ogni $x_0, x \in E$,*

$$\lim_{n \rightarrow +\infty} p_{x_0 x}(n) = \mu(x).$$

Il Teorema ci dice che, sotto certe ipotesi sulla matrice P , dopo che è trascorso un tempo sufficientemente lungo, il processo si dimentica da dove è partito e la distribuzione di X_n si avvicina sempre più a μ . Questo risultato è utilissimo in molte applicazioni. Ad esempio, se si vuole simulare una misura μ in molti casi è più semplice simulare una catena di Markov che abbia μ come misura invariante e prendere il valore di X_n (con n sufficientemente grande) come simulazione di μ .

Consideriamo ora la catena dell'esempio 4. Un modo per farne una simulazione può essere il seguente: si considera una famiglia $(\xi_n)_{n \in \mathbb{N}}$ di variabili aleatorie a valori in $\{0, 1\}$ e indipendenti tra loro con $\mathbb{P}(\xi_n = 0) = \mathbb{P}(\xi_n = 1) = \frac{1}{2}$ per ogni n (una famiglia di questo tipo si può realizzare, ad esempio, lanciando una moneta equilibrata in ogni tempo $n \in \mathbb{N}$). Si sceglie uno stato iniziale $X_0^{x_0} \equiv x_0 \in \{1, \dots, 5\}$ e, per $n \geq 1$ si pone $X_{n+1}^{x_0} = f(\xi_{n+1}, X_n^{x_0})$, dove $f : \{0, 1\} \times E \rightarrow E$ e

$$f(0, x) = (x - 1) \vee 1, \quad f(1, x) = (x + 1) \wedge 5.$$

In analogia con quanto si fa con i sistemi dinamici deterministici, il processo si può definire mediante l'iterazione di una mappa, ma in questo caso la mappa è "casuale", cioè viene scelta a caso tra $f(0, \cdot)$ e $f(1, \cdot)$.

Sfruttando la successione $(\xi_n)_n$, possiamo considerare il processo $\{(X_n^1, \dots, X_n^5)\}_{n \in \mathbb{N}}$. Questo è un processo di Markov e una rappresentazione di questo tipo la chiameremo un *accoppiamento completo*.

Definizione 6 *Sie $E = \{1, \dots, N\}$ un insieme finito e P una matrice di transizione su E . Diremo che P ammette un accoppiamento completo se esiste uno spazio probabilizzato $(\Omega, \mathcal{F}, \mathbb{P})$, una famiglia $\{\xi_n\}_{n \geq 1}$ di variabili aleatorie indipendenti e identicamente distribuite a valori in un opportuno spazio misurabile (Θ, \mathcal{A}) e una funzione $f : \Theta \times E \rightarrow E$ tali che, per ogni $x \in E$ il processo definito da $X_0^x \equiv x$, $X_{n+1}^x = f(\xi_{n+1}, X_n^x)$ è una catena di Markov uscente da x con matrice di transizione P e $\{(X_n^1, \dots, X_n^N)\}_{n \in \mathbb{N}}$ è una catena di Markov a valori in E^N .*

Si può dimostrare che, per ogni P esiste sempre un accoppiamento completo. D'ora in poi supporremo che E sia un insieme parzialmente ordinato (e per E utilizzeremo la locuzione *poset*).

Definizione 7 Un accoppiamento completamente monotono è un accoppiamento completo tale che, per ogni $x, y \in E$ si abbia

$$x \leq y \Rightarrow X_n^x \leq X_n^y \text{ per ogni } n \geq 0.$$

In altre parole, un accoppiamento è completamente monotono se ciascuna delle funzioni $f(\theta, \cdot) : E \rightarrow E$, con $\theta \in \Theta$, che intervengono nella definizione 6 è crescente.

Osservazione 8 L'accoppiamento dell'esempio precedente è completamente monotono.

Abbiamo visto che si può utilizzare una catena di Markov (opportunamente definita) per simulare una misura su E . Il metodo descritto sopra fornisce però solo una approssimazione di μ . Esiste invece un altro algoritmo che fornisce una simulazione perfetta di μ . Esso consiste nel considerare un accoppiamento completo per una catena di Markov avente μ come unica misura invariante e osservare, per ogni $x \in E$, e per $n \in \mathbb{N}$

$$F_{-n}^x := f(\xi_{-1}, f(\dots f(\xi_{-n+1}, f(\xi_{-n}, x)) \dots))$$

cioè si fa partire il processo al tempo $-n$ e si osserva lo stato al tempo 0. Se $F_{-n}^1 = \dots = F_{-n}^N$ per un certo n , l'algoritmo si ferma e il valore che si ottiene è una simulazione perfetta di μ .

Teorema 9 Se $T = \inf\{n : F_{-n}^1 = \dots = F_{-n}^N\} < +\infty$ \mathbb{P} -q.c., allora F_T^i ha distribuzione μ .

Osserviamo che, perché l'algoritmo funzioni, deve essere $T < +\infty$ q.c.. Inoltre bisogna simulare N catene di Markov e, se la cardinalità di E è molto grande, questo può essere un grosso inconveniente. Se l'accoppiamento è completamente monotono questi due problemi non sussistono. Infatti, se ad esempio E ha un elemento massimale x^* ed un elemento minimale x_* si ha $T = \inf\{n : F_{-n}^{x^*} = F_{-n}^{x_*}\}$, cioè occorre simulare solo due processi. Vale inoltre il seguente

Teorema 10 Se P ammette un accoppiamento completamente monotono, allora T è q.c. finito.

Dunque, data una matrice di transizione, è importante stabilire se essa ammette un accoppiamento completamente monotono. In generale, non esiste un metodo semplice per verificarlo, però in Probabilità esiste un concetto più debole di monotonia che è una condizione necessaria per l'esistenza di un accoppiamento completamente monotono. Nella definizione che segue, rappresentiamo le funzioni $f : E \rightarrow E$ mediante vettori di \mathbb{R}^N .

Definizione 11 Una matrice di transizione P è monotona se, per ogni funzione $f : E \rightarrow E$ crescente, si ha che Pf è una funzione crescente.

Vi sono diverse formulazioni equivalenti della monotonia. Una di esse consiste in un semplice algoritmo mediante il quale si può verificare la monotonia in termini degli elementi della matrice P .

Allora è naturale domandarsi sotto quali condizioni una matrice di transizione monotona

ammette un accoppiamento completamente monotono, cioè sotto quali condizioni i due concetti di monotonia sono equivalenti. Si scopre che l'equivalenza dipende dalla struttura di ordine parziale su E . Vale il seguente risultato:

Teorema 12 *Una matrice di transizione monotona P è completamente monotona se e solo se l'insieme E è aciclico, cioè non contiene cicli $x_0, x_1, \dots, x_n, x_{n+1} = x_0$ tali che, per $i = 0, 1, \dots, n$*

- i. $x_i \neq x_{i+1}$.
- ii. $x_i < x_{i+1}$ oppure $x_i > x_{i+1}$.
- iii. $x_i \leq y \leq x_{i+1}$ or $x_i \geq y \geq x_{i+1} \Rightarrow y = x_i$ o $y = x_{i+1}$.

Ci domandiamo ora che cosa succede se il processo è a tempi continui. La definizione di processo di Markov a tempi continui è analoga a quella in tempi discreti, con l'unica differenza che in questo caso le transizioni da uno stato all'altro non avvengono in intervalli di tempo unitari, per cui occorre assegnare a priori una famiglia $(P_t)_{t \in \mathbb{R}^+}$ di matrici di transizione con le proprietà $P_0 = \text{Id}$, $P_{t+s} = P_t P_s$. L'elemento $(P_t)_{xy} = p_{xy}(t)$ è la probabilità di passare dallo stato x allo stato y in un intervallo di tempo t .

Definizione 13 *$(X_t)_{t \in \mathbb{R}^+}$ è un processo di Markov (regolare) uscente da x_0 se*

$X_0 \equiv x_0$ \mathbb{P} -quasi certamente

$$\mathbb{P}(X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n) = p_{x_0 x_1}(t_1) p_{x_1 x_2}(t_2 - t_1) \dots p_{x_{n-1} x_n}(t_n - t_{n-1})$$

per ogni $t_1, \dots, t_n \in \mathbb{R}^+$ e per ogni scelta di $x_1, x_2, \dots, x_n \in E$.

Si può dimostrare che, sotto certe ipotesi su $(P_t)_t$, esiste una matrice L tale che, per ogni t

$$P_t = e^{tL} = \sum_{n=0}^{+\infty} \frac{(tL)^n}{n!}.$$

La matrice L si dice *generatore infinitesimale* del processo e determina completamente $(P_t)_t$.

Definizione 14 *Un generatore infinitesimale si dice completamente monotono, se ammette un accoppiamento completamente monotono, cioè se esiste uno spazio probabilizzato $(\Omega, \mathcal{F}, \mathbb{P})$ e un processo di Markov $\{(X_t^1, \dots, X_t^N)\}_{t \in \mathbb{R}^+}$ a valori in E^N tale che, per ogni i $\{X_t^i\}_{t \in \mathbb{R}^+}$ è un processo di Markov uscente da i con generatore L e vale*

$$x \leq y \Rightarrow X_t^x \leq X_t^y \text{ per ogni } t \geq 0.$$

Un generatore infinitesimale si dice monotono se, data $f : E \rightarrow E$ crescente, per ogni t la funzione $P_t f$ è crescente.

Anche per processi a tempi continui, le due nozioni di monotonia non sono equivalenti (la monotonia completa implica la monotonia "semplice", ma non è vero il viceversa),

ma in questo caso una classificazione degli insiemi per cui non c'è tale equivalenza è più complicata. Vi sono insiemi che contengono cicli (e dunque, nel caso a tempi discreti per essi non c'è equivalenza) per cui si ha equivalenza ed altri per cui non c'è equivalenza. Valgono in ogni caso i seguenti risultati:

Teorema 15 *Se E è un insieme per cui c'è equivalenza in tempi discreti, allora per E c'è equivalenza anche in tempi continui.*

Proposizione 16 *Per tutti i poset di 4 punti c'è equivalenza. Gli unici poset di 5 punti per cui non c'è equivalenza sono i seguenti:*

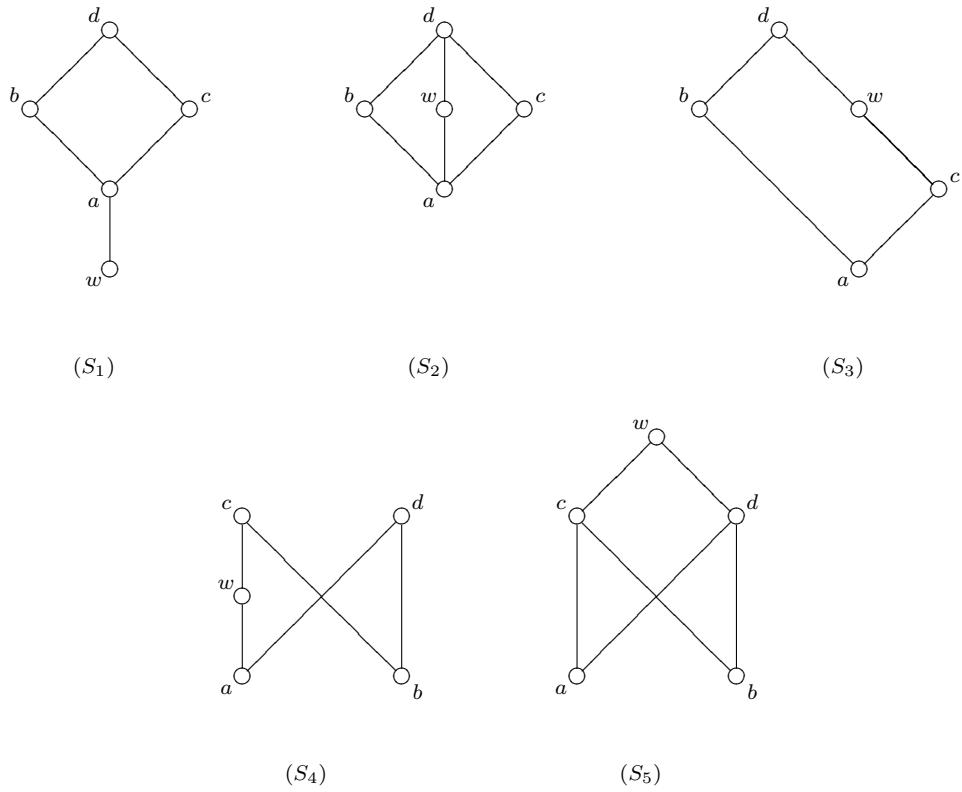


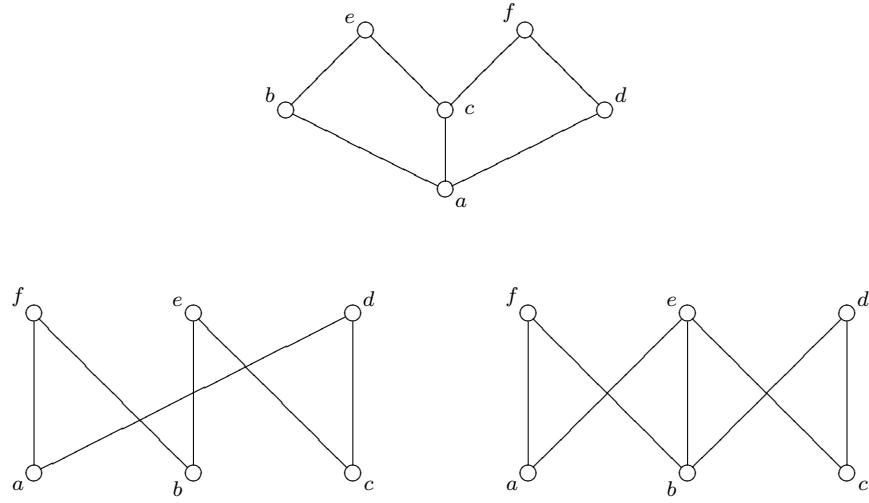
Figura 1. Posets di 5 punti per cui non c'è equivalenza in tempi continui.

Ricordiamo la seguente definizione.

Definizione 17 *S' è un sottoposet indotto di E se, per ogni $x, y \in S'$, $x \leq y$ in S' è equivalente a $x \leq y$ in E .*

Teorema 18 *Se un insieme E ha come sottoposet indotto uno dei posets di 5 punti rappresentati nella Figura 1, allora per E non c'è equivalenza.*

Proposizione 19 *Gli unici posets di 6 punti per cui non c'è equivalenza sono, oltre a quelli determinati dal teorema precedente, i seguenti*



Teorema 20 *Se E ha come sottoposet indotto uno dei poset di sei punti della figura, allora per E non c'è equivalenza.*

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A Dynamic Contagion Model: Financial Interpretation of a Mean-Field Interacting Particle Model

ELENA SARTORI (*)

Abstract. N firms, active on the market, are represented by a system of N mean-field interacting particles. We shall study its behavior for “long” times. Then we shall quantify the losses suffered by a bank, that holds a portfolio with positions issued by the N firms, which face credit risk. Moreover we shall explain qualitatively the phenomenon of credit crises. [Keywords: Mean-field interacting particle systems, non-reversible Markov processes, credit contagion, credit risk.]

Sunto. *Un Modello Dinamico di Contagio: Interpretazione Finanziaria di un Modello di Particelle Interagenti a Campo Medio.* N aziende attive sul mercato vengono rappresentate da un sistema di N particelle interagenti a campo medio. Studieremo il suo comportamento per tempi “lunghi”. Quantificheremo poi le perdite sofferte da una banca, che possiede un portafoglio le cui posizioni sono date dalle N aziende, che affrontano il rischio di credito. Spiegheremo inoltre qualitativamente il fenomeno delle crisi di credito.

1 Preliminary notions

Let (Ω, \mathcal{F}, P) be a *probability space*.

On discrete time...

Let $(X_n)_{n \in \mathbb{N}}$ be a discrete time, finite state *Markov chain*, with state space (E, \mathcal{E}) measurable and finite, starting at $x_0 \in E$.

μ , probability measure on (E, \mathcal{E}) , is an *invariant* (or *stationary*) distribution of $(X_n)_n$ with transition matrix \mathbf{P} if

$$\mu \mathbf{P} = \mu .$$

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(The chain does not stand still in a point, because it is random, but it has always the same distribution).

Theorem 1.1 *Under certain hypotheses on \mathbf{P} , there is a unique stationary measure μ and*

$$P_{x_0x}(n) \xrightarrow{n \rightarrow \infty} \mu(x),$$

where $P_{x_0x}(n) = (P^n)_{x_0x}$ is the element x_0x of the n -step transition matrix P^n .

(Waiting a sufficiently large time the chain forgets the past and its distribution becomes the stationary μ).

It is often difficult to solve the stationary condition $\mu\mathbf{P} = \mu$, i.e. $\sum_{x \in E} \mu(x)\mathbf{P}_{xy} = \mu(y)$, $\forall y \in E$. Then let consider the following sufficient condition for stationarity, (*detailed balance condition*):

$$\mu(x)\mathbf{P}_{xy} = \mu(y)\mathbf{P}_{yx}, \quad \forall x, y \in E.$$

A chain satisfying the detailed balance condition is *reversible*.

On continuous time...

Let $(X_t)_{t \geq 0}$ be a continuous time, finite state, E finite, *Markov process*, starting at x_0 . μ , probability measure on (E, \mathcal{E}) , is a *stationary* distribution of the Markov process $(X_t)_t$ with semigroup $(S_t)_t$, or infinitesimal generator L , if for every $t \geq 0$, $\mu S_t = \mu$, or equiv. $\mu L = 0$.

Theorem 1.2 *Under certain hypotheses on S_t , or equiv. on L , there is a unique stationary μ and, for every initial distribution ν_0 ,*

$$\nu_0 S_t \xrightarrow{t \rightarrow \infty} \mu.$$

This distribution describes the behavior of the process for “long” times, meaning that, given any function $f : E \rightarrow \mathbb{R}$ and any initial distribution,

$$\lim_{t \rightarrow \infty} E(f(X_t)) = \sum_{x \in E} f(x)\mu(x).$$

Example 1.3 In a time interval $[0, T]$, let consider N *interacting* spins (*particles* taking values ± 1). Let describe the dynamics of this system through Markov processes.

Let denote by X_i the state of the i -th particle. The generic state of the system is represented by $X = (X_1, \dots, X_N) \in \{-1, +1\}^N$.

If $f : \{-1, +1\}^N \rightarrow \mathbb{R}$ is a function of X ,

$$Lf(X) = \sum_{i=1}^N c_N(i, X) [f(X^i) - f(X)],$$

with $X^i = (X_1, \dots, X_{i-1}, -X_i, X_{i+1}, \dots, X_N)$.

$c_N(i, X)$ is the *transition intensity (or rate)* and it is the probability that the i -th particle jumps in a time interval δt , very short.

$$c_N \sim \frac{\text{Prob.}}{\delta t}.$$

According to the particle system to describe, an intensity transition is defined, which describes the type of interaction there is.

Given the initial distribution λ , $X(0) \sim \lambda$, and the infinitesimal generator L , $(X(t))_{t \in [0, T]}$ is a continuous time, $t \in [0, T]$, finite state, $E = \{-1, +1\}^N$, Markov process.

2 Introduction

The financial problem: *credit risk*, i.e. the risk faced by a financial institution, which holds a large portfolio, whose positions are issued by firms, which may default and therefore may not be able to honor their financial obligations.

The problem is compounded by the facts that:

- i) *Default* may be *contagious*.
- ii) There may be a *clustering of defaults*.

(When we talk about default we mean a state of *financial distress*, i.e. a probability > 0 of reacting). From a financial point of view:

- We describe the propagation of financial distress (*financial contagion*) in a network of firms linked by business relationships.
- We investigate the phenomenon of *credit crises* (periods in which many firms fall to financial distress in a short time).
- We quantify the impact of contagion on the *losses* suffered by a bank that holds a *large portfolio* of firms facing credit risk (deterioration of credit quality). (*It's important being able to compute the probability of a global loss exceeding a given threshold*).

Within the class of the reduced form model we consider a *model of interacting particles* (firms=particles): Contrary to models relying only on macroeconomic factors (exogenous), we have direct interaction (once fixed the realization of macroeconomic factors, we do not suppose the independence of the events of default!).

This approach allows to explain the phenomenon of “clustering of default”.

3 The model

We choose a mean-field interacting model of the Curie-Weiss type (each particle interacts with all the others in the same way).

Since we are interested in describing dynamically the credit quality of firms, we use a simple model, where the “credit state” of each firm is identified by two variables (σ, ω) .

So let consider N firms active on the market.

1. The *credit quality* itself of firm i is described by a binary variable $\sigma_i \in \{-1, +1\}$. View it as a two rating classes model with σ a *rating indicator*.
 - A low value of σ reflects a bad rating indicator, i.e. a higher probability of not being able to pay back obligations. It is an informational indicator.
2. The *strength* of firm i is described by a binary variable $\omega_i \in \{-1, +1\}$. View it as a *liquidity indicator* or *sign of cash balances*.
 - It is a more fundamental indicator that is assumed not to be directly observable from the market.

For the time evolution on a generic interval $[0, T]$ of the “state” of the particle system, i.e. $(\sigma_i(t), \omega_i(t))_{i=1, \dots, N} \in \mathcal{D}^{2N}[0, T]$, we need to specify the rates/intensities for the transitions $\sigma_i \rightarrow -\sigma_i, \omega_i \rightarrow -\omega_i$. ($\mathcal{D}[0, T]$: space of right continuous, piecewise constant $f: [0, T] \rightarrow \{-1, +1\}$ with Skorohod topology).

The mean-field assumption leads to letting the interaction depend on the global health indicator (endogenous global factor) $m_N^\sigma(t) := 1/N \sum_{i=1}^N \sigma_i(t)$. The vehicle of contagion is given by

$$\begin{array}{ccccccc} \omega_i & \rightarrow & \sigma_i & \rightarrow & m_N^\sigma & \rightarrow & \omega_j \\ \text{fundam. indic.} & & \text{rating class} & & \text{global health indic.} & & \end{array}$$

Then the *transition intensities* are given by

$$\begin{aligned} \sigma_i \rightarrow -\sigma_i &\quad \text{with intensity } \lambda_i := e^{-\beta \sigma_i \omega_i}, \quad \beta > 0. \\ \omega_j \rightarrow -\omega_j &\quad \text{with intensity } \mu_j := e^{-\gamma \omega_j m_N^\sigma}, \quad \gamma > 0. \end{aligned}$$

- The parameters β, γ indicate the strength of the interaction.
- The transition from $\sigma = -1$ to $\sigma = +1$ is higher when $\omega = +1$ (positive cash balances make the firm strong).
- The transition from $\omega = -1$ to $\omega = +1$ is higher when m_N^σ is higher (high values of the global health indicator improve the fundamental indicator ω_i of each firm).

The state variables form a *continuous-time Markov chain* on the configuration space $\{-1, +1\}^{2N}$ with *infinitesimal generator* L acting on $f: \{-1, +1\}^{2N} \rightarrow \mathbb{R}$ as

$$Lf(\sigma, \omega) = \sum_{i=1}^N \lambda_i \nabla_i^\sigma f(\sigma, \omega) + \sum_{j=1}^N \mu_j \nabla_j^\omega f(\sigma, \omega),$$

with $\nabla_i^\sigma f(\sigma, \omega) = f(\sigma^i, \omega) - f(\sigma, \omega)$; $\nabla_j^\omega f(\sigma, \omega) = f(\sigma, \omega^j) - f(\sigma, \omega)$, $\sigma^i = (\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_N)$; anal. for ω^j .

- Unlike most mean-field models in Statistical mechanics our model can be shown to be non-reversible. A direct study of a stationary distribution and its $N \rightarrow \infty$ asymptotics is thus not reasonable. A dynamic approach to understand the long time behavior of the system is thus necessary.

In order to study the behavior on “long” times in *reversible models*...

- since often the stationarity condition is difficult to solve and an explicit formula for invariant μ may not be determined, it can be useful to solve the detailed balance condition, sufficient condition for stationarity, i.e. for $t \rightarrow \infty$,
- then it may be studied the limit when $N \rightarrow \infty$.

In this case this approach cannot be possible, since the *model* is *non-reversible*, so...

- We study the $N \rightarrow \infty$ limit of the dynamics of the system: we compute the limiting distribution on the path space of the particle system, obtaining *limit evolution equations*.
- We study the *equilibria* of the limiting dynamics.
- We detect *phase transition*: for certain values of the parameters β, γ , the limiting dynamics may have multiple stable equilibria as $N \rightarrow \infty$, and this fact allows us to characterize clustering of defaults.
- We describe “*finite volume approximations*” (N large, but finite) of the limiting distribution, studying the fluctuations around the limit.

4 Limit of the dynamics

A crucial object is the *empirical measure*

$$\rho_N = \frac{1}{N} \sum_{i=1}^N \delta_{\{\sigma_{[0,T]}(i), \omega_{[0,T]}(i)\}} ,$$

$\rho_N \in \mathcal{M}_1(\mathcal{D}[0, T] \times \mathcal{D}[0, T])$. $\rho_N(A) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\{(\sigma_i[0,T], \omega_i[0,T]) \in A\}}$, where $A \subset \mathcal{D}[0, T] \times \mathcal{D}[0, T]$. For a generic function f of the trajectory one is interested in the asymptotic behavior of empirical averages of the form

$$\frac{1}{N} \sum_{i=1}^N f(\sigma_i[0, T], \omega_i[0, T]) =: \int f d\rho_N$$

if $f \equiv \sigma_t \Rightarrow \int f d\rho_N = m_N^\sigma(t)$, if $f \equiv \omega_t \Rightarrow \int f d\rho_N = m_N^\omega(t)$, if $f \equiv \sigma_t \omega_t \Rightarrow \int f d\rho_N = m_N^{\sigma\omega}(t)$.

Theorem 4.1 Let $(\sigma(t), \omega(t))_{t \in [0, T]}$ be the Markov process with generator $Lf(\sigma, \omega)$ and initial distribution such that $(\sigma_i(0), \omega_i(0))$, $i = 1, \dots, N$, are i.i.d. with law λ ,

- i) then there exists $Q^* \in \mathcal{M}_1(\mathcal{D}[0, T] \times \mathcal{D}[0, T])$ such that, if $N \rightarrow \infty$, $\rho_N \rightarrow Q^*$ a.s. in the weak topology (Law of Large Numbers);
- ii) if $q_t \equiv \Pi_t Q^*$ (“photograph” of Q^* at time t), then q_t is the unique solution of the McKean-Vlasov equation (MKV)

$$\begin{cases} \frac{\partial q_t}{\partial t} = \mathcal{L}q_t, & t \in [0, T] \\ q_0 = \lambda \end{cases}$$

with $\mathcal{L}q(\sigma, \omega) = \nabla^\sigma [e^{-\beta\sigma\omega} q(\sigma, \omega)] + \nabla^\omega [e^{-\gamma\omega m_q^\sigma} q(\sigma, \omega)]$,
 m_q^σ : expected global health under the measure $q \in \mathcal{M}_1(\{-1, +1\})^2$.

The MKV equation describes the dynamics of our particle system in the limit when $N \rightarrow \infty$. Using Theorem 4.1, one can easily describe also the *dynamics of the aggregate variables*:

$$\begin{cases} m_{q_t}^\sigma = \int \sigma dq_t \\ m_{q_t}^\omega = \int \omega dq_t \end{cases}$$

Write $m_t^\sigma = m_{q_t}^\sigma$ (analogously for m_t^ω).

Actually (MKV) can be rewritten as follows:

$$(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega) \quad (\text{mkv}) \quad \text{with}$$

$$V(x, y) := (2 \sinh(\beta)y - 2 \cosh(\beta)x, 2 \sinh(\gamma x) - 2y \cosh(\gamma x)).$$

To analyze the equilibria of the limiting dynamics, let consider (mkv) instead of (MKV).

Theorem 4.2

- i) Suppose $\gamma \leq \frac{1}{\tanh(\beta)}$. Then equation (mkv) has $(0, 0)$ as a unique equilibrium solution, which is globally asymptotically stable, i.e. for every initial condition (m_0^σ, m_0^ω) , we have

$$\lim_{t \rightarrow +\infty} (m_t^\sigma, m_t^\omega) = (0, 0).$$

- ii) For $\gamma < \frac{1}{\tanh(\beta)}$ the equilibrium $(0, 0)$ is linearly stable.

For $\gamma = \frac{1}{\tanh(\beta)}$ the linearized system has a neutral direction, i.e. $DV(0, 0)$ (the Jacobian matrix) has one zero eigenvalue.

- iii) For $\gamma > \frac{1}{\tanh(\beta)}$ the point $(0, 0)$ is still an equilibrium for (mkv), but it is a saddle point for the linearized system, i.e. the matrix $DV(0, 0)$ has two nonzero real eigenvalues of opposite sign. Moreover (mkv) has two linearly stable solutions (m_*^σ, m_*^ω) and $(-m_*^\sigma, -m_*^\omega)$, where m_*^σ is the unique strictly positive solution of the equation

$$x = \tanh(\beta) \tanh(\gamma x), \quad \text{and} \quad m_*^\omega = \frac{1}{\tanh(\beta)} m_*^\sigma.$$

- iv) For $\gamma > \frac{1}{\tanh(\beta)}$, the phase space $[-1, +1]^2$ is bi-partitioned by a smooth curve Γ containing $(0, 0)$ such that $[-1, +1]^2 \setminus \Gamma$ is the union of two disjoint sets D^+, D^- that are open in the induced topology of $[-1, +1]^2$.

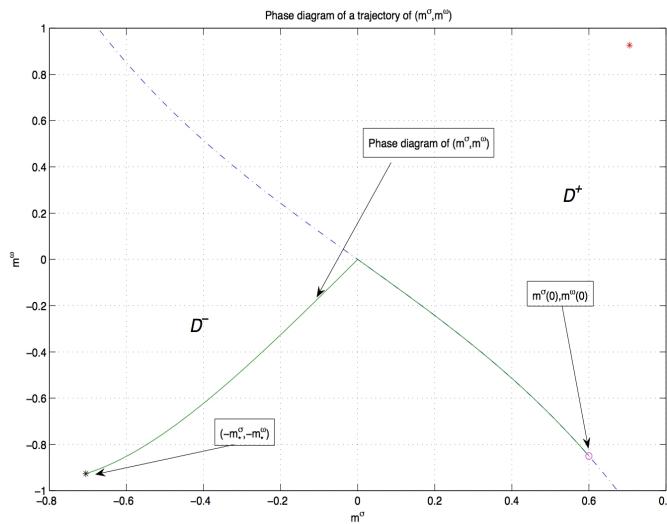
Moreover

$$\lim_{t \rightarrow +\infty} (m_t^\sigma, m_t^\omega) = \begin{cases} (m_*^\sigma, m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in D^+ \\ (-m_*^\sigma, -m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in D^- \\ (0, 0) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma. \end{cases}$$

The fact that the limiting dynamics ($N \rightarrow \infty$) may have multiple stable equilibria implies that our system exhibits *phase transition*. One obtains different *domains of attraction* corresponding to each of the stable equilibria.

For certain values of the initial conditions (see a simulation result in the next figure):

- the system is driven towards a “neutral” equilibrium, $(0, 0)$, in which half of the firms are in good financial health.
 - After a time depending on the initial state, the system is “captured” by an unstable direction of this neutral equilibrium and moves towards a stable asymmetric equilibrium.
- During the transition the *volatility* of the system, defined later, increases sharply before decaying to a stationary value (later other simulation results).
- All this occurs on a time-scale independent on N , quite realistic for financial applications.
- This phenomenon can be interpreted as a *credit crisis* and may account for *clustering of defaults*.



Having established a Law of Large Numbers $\rho_N \rightarrow Q^*$, we analyze fluctuations around the limit and the asymptotic distribution of $\rho_N - Q^*$. So

Theorem 4.3 *Let*

$$\begin{cases} x_N(t) &= \sqrt{N}(m_N^\sigma(t) - m_t^\sigma) \\ y_N(t) &= \sqrt{N}(m_N^\omega(t) - m_t^\omega) \\ z_N(t) &= \sqrt{N}(m_N^{\sigma\omega}(t) - m_t^{\sigma\omega}) \end{cases} .$$

Then $(x_N(t), y_N(t), z_N(t)) \xrightarrow{N \rightarrow \infty} (x(t), y(t), z(t))$ in the sense of weak convergence of stochastic processes, where $(x(t), y(t), z(t))$ is a centered Gaussian process, unique solution of the linear stochastic differential equation

$$\begin{pmatrix} dx(t) \\ dy(t) \\ dz(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} dt + D(t) \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{pmatrix}$$

where B_1, B_2, B_3 are independent Brownian motions and $A(t), D(t)$ are 3×3 appropriate matrices depending on β, γ and $m_t^\sigma, m_t^\omega, m_t^{\sigma\omega}$ and $(x(0), y(0), z(0))$ is a centered Gaussian.

The *asymptotic* ($N \rightarrow \infty$) distribution of $(x_N(t), y_N(t), z_N(t))$ is then, for each fixed t , a centered Gaussian with *covariance matrix* Σ_t , the volatility mentioned before, satisfying

$$\frac{d\Sigma_t}{dt} = A(t)\Sigma_t + \Sigma_t A'(t) + D(t)D'(t).$$

$$(\Sigma_\sigma(t)) = (\Sigma_t)_{11}).$$

Corollary 4.4 *As $N \rightarrow \infty$ we have for the global health indicator*

$$\sqrt{N}(m_N^\sigma(t) - m_q^\sigma(t)) \rightarrow \mathcal{N}(0, \Sigma_\sigma(t)).$$

Moreover, for $\alpha \in \mathbb{R}$ we have

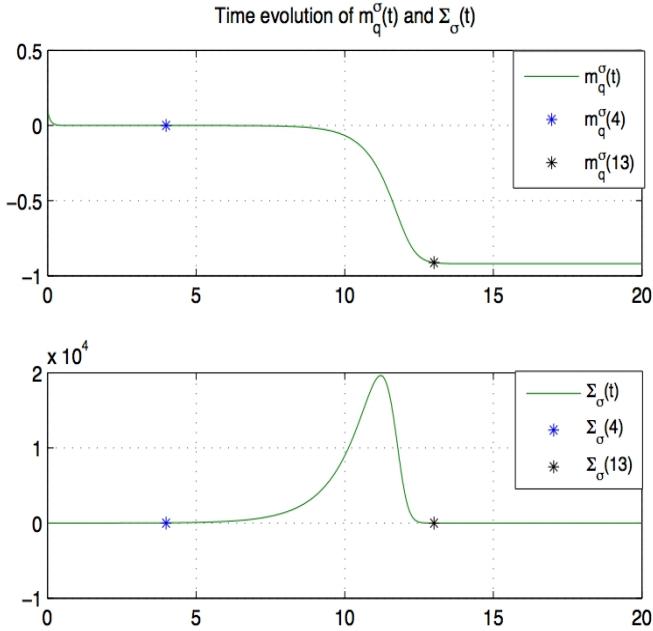
$$P(m_N^\sigma(t) \geq \alpha) \approx \Phi \left(\frac{\sqrt{N}m_t^\sigma - \sqrt{N}\alpha}{\sqrt{\Sigma_\sigma(t)}} \right), \text{ where } \Phi(c) = P(\mathcal{N}(0, 1) \geq c).$$

5 Portfolio losses

Let define the *total loss* at time t that a bank may suffer due to a risky portfolio of positions issued by the N firms,

$$L^N(t) := \sum_{i=1}^N L_i(t),$$

where $L_i \in \mathbb{R}^+$ are the marginal losses, $i = 1, \dots, N$. Once specified how $L_i(t)$ are distributed, Corollary 4.4 allows to compute, in some cases, approximately for a large number



N of firms, for an appropriate deterministic function $L(t)$,

$$P(L^N(t) \geq \alpha) \approx \Phi \left(\frac{NL(T) - \alpha}{\sqrt{N} \sqrt{\Sigma_\sigma(T)}} \right).$$

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